732A96/TDDE15 Advanced Machine Learning

Reinforcement Learning

Jose M. Peña
IDA, Linköping University, Sweden

Lectures 8: Q-Learning Algorithm

Contents

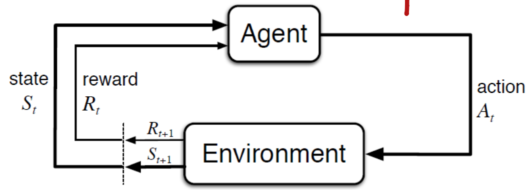
- Learning through Interaction
- Markov Decision Processes
- Bellman Equations
- Value Iteration
- Policy Iteration
- Q-Learning
- Example: Grid Worlds

Literature

- Main source
  - Sutton, R. S. and Barto, A. G. *Reinforcement Learning: An Introduction*. The MIT Press, 2018. Chapters 1-7.
- Additional source
  - Russel, S. and Norvig, P. *Artifical Intelligence: A Modern Approach*. Pearson, 2010. Chapters 16, 17 and 21.

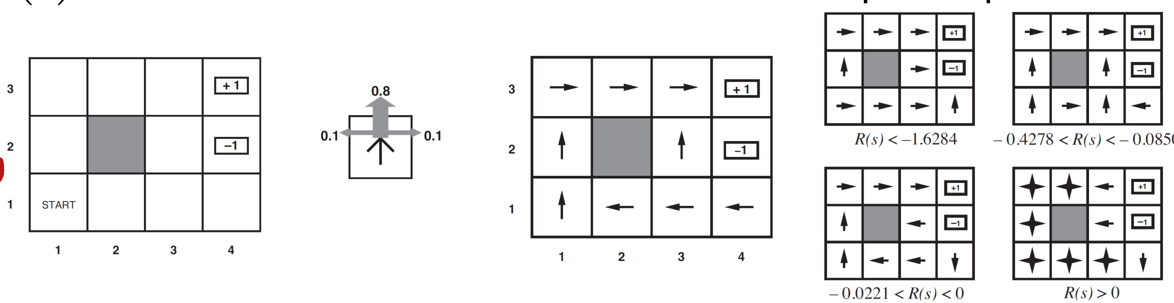
Learning through Interaction

Car for example



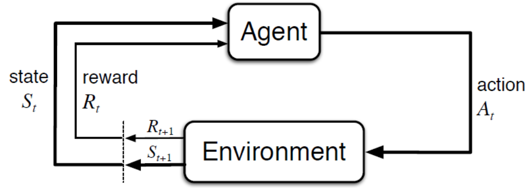
- **Agent**: The learner and decision maker.
- **Environment**: The agent interacts with it.
  - S ▸ **State**: State of the agent and the environment.
  - A ▸ **Action**: The agent decides the next action on the basis of the current state.
  - R ▸ **Reward**: Numerical response to the action chosen by the agent. The agent aims to learn how to act so as to maximize the cumulative reward.
  - **Trajectory**: $S_0, A_0, R_1, S_1, A_1, R_2, S_2, A_2, R_3, S_3, \ldots$
- **Policy**: Probability of doing an action in a state. The agent acts according to it. The agent aims to learn an optimal one.
- Example: A robot moves with probability 0.8 in the intended direction, and at the right angles of it otherwise. The reward for non-terminal states is $R(s) = -0.04$. All this is unknown to the robot. Optimal policies shown.

Do learning while collecting data



- RL is neither supervised nor unsupervised learning.

Markov Decision Processes



- We assume that the agent-environment interaction follows a finite **Markov decision process**:

  We know where robot is

  - Finite sets of states, actions and rewards. Fully observable state.
  - Markovian and stationary transition model:

    prev. state & action

    $$p(\underset{\text{state}}{s_t}, \underset{\text{reward}}{r_t} | s_{0:t-1}, a_{0:t-1}, r_{1:t-1}) = p(s_t, r_t | s_{t-1}, a_{t-1}) = p(s', r | s, a).$$

- The **transition model** is typically unknown to the agent. Note the randomness of the next state and reward.
- The objective of the agent is to learn a policy $\pi(a|s)$ that maximizes the **expected discounted return**:

  Probability of an action given the state

  Sum of all rewards → Return

  $$E_\pi[G_t] = E_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}\right] = E_\pi[R_{t+1} + \gamma G_{t+1}]$$

  $\gamma = 1$: All rewards weighted the same

  where $0 < \gamma \leq 1$ describes our preference between present and future rewards. Note the infinite horizon. However, the expectation is finite if $\gamma < 1$. For episodic tasks, $\gamma = 1$ and $R_{t+k+1} = 0$ for all $t + k + 1 > T$.

No hidden state needed:

$$
\begin{tikzcd}
z^0 \arrow[r] \arrow[d] & z^1 \arrow[r] \arrow[d] & z^2 \arrow[d] \\
\cancel{x^0} & \cancel{x^1} & \cancel{x^2}
\end{tikzcd}
$$

Don't need this!
Fully observable.

We don't have a goal.

→ Implicit in the model.

Markov Decision Processes

- The **state value** function $v_\pi(s)$ is the expected return of following policy π starting from state s:

$$v_\pi(s) = E_\pi[G_t|S_t = s] = \sum_a \pi(a|s)E_\pi[G_t|S_t = s, A_t = a] = \sum_a \pi(a|s)q_\pi(s,a).$$

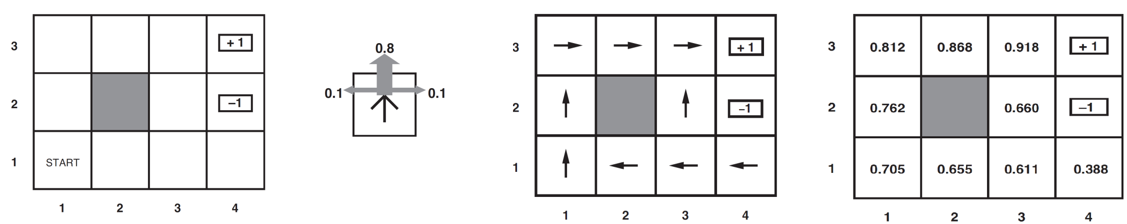
sum over all actions

- The **action value** function $q_\pi(s,a)$ is the expected return of doing action a in state s and then following policy π:

$$q_\pi(s,a) = E_\pi[G_t|S_t = s, A_t = a] = E_\pi[R_{t+1}+\gamma G_{t+1}|s,a] = \sum_{s',r} p(s',r|s,a)(r+\gamma v_\pi(s')).$$

Transition model   State value function

- Example: Environment, policy and state values.



- We can define the objective of the agent as learning a policy π_* such that

Value of all states given that I follow policy π_*

$$v_*(s) \geq v_\pi(s) \text{ for all } \pi, s.$$

For MDPs, there is always **at least one** such optimal policy.

Bellman Equations

- The state value function satisfies a recursive relationship known as **Bellman equation**:

$$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)(r + \gamma v_\pi(s')).$$

- Moreover, v_π is the **unique solution** to the equations. Note that there are as many equations as unknowns. Since the equations are linear, they can be solved by linear algebra methods in $O(n^3)$. But this requires knowing the transition model.
- Likewise for the action value function:

Next states

$$q_\pi(s,a) = \sum_{s',r} p(s',r|s,a)(r + \gamma \sum_{a'} \pi(a'|s') q_\pi(s',a')).$$

Next action

- The Bellman equations of an **optimal policy** are

Policy π_*

$$v_*(s) = \max_a \sum_{s',r} p(s',r|s,a)(r + \gamma v_*(s'))$$

Not linear equation

and

$$q_*(s,a) = \sum_{s',r} p(s',r|s,a)(r + \gamma \max_{a'} q_*(s',a')).$$

- As before, v_* and q_* are the **unique solutions** to these equations. Note that the equations are now non-linear due to the max operator and, thus, harder to solve. Again, this requires knowing the transition model.

Bellman Equations

- Once we have v_* or q_*, it is easy to determine an **optimal policy**:

$$\pi_*(a|s) = \arg\max_a q_*(s, a)$$

  or

$$\pi_*(a|s) = \arg\max_a \sum_{s',r} p(s', r|s, a)(r + \gamma v_*(s')).$$

- Note that the optimal policy is **deterministic**. So, we can consider only deterministic policies without loss of generality, i.e. $\pi(s)$ instead of $\pi(a|s)$.

Value Iteration

- We can avoid solving the Bellman equations for the state values of an optimal policy by turning them into update rules.

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $v \leftarrow V(s)$
| $V(s) \leftarrow \max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
| $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
  $\pi(s) = \operatorname{argmax}_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$

- VI converges asymptotically for $\gamma < 1$. Since the Bellman optimality equations have a unique solution, VI converges to v_*.
- VI still requires knowing the transition model.

Policy Iteration

- Policy evaluation: Turn the ordinary Bellman equations into update rules.

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} \overbrace{p(s', r \mid s, \pi(s))}^{\text{Transition model}} \big[r + \gamma V(s')\big]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Policy Iteration

- Theorem: If $q_\pi(s, \pi'(s)) \geq q_\pi(s, \pi(s))$ for all s, then $v_{\pi'}(s) \geq v_\pi(s)$ for all s. Thus, we can modify π into a better policy π' by doing

$$\pi'(s) = \arg\max_a q_\pi(s, a) \text{ for all } s.$$

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   - $\Delta \leftarrow 0$
   - Loop for each $s \in \mathcal{S}$:
     - $v \leftarrow V(s)$
     - $V(s) \leftarrow \sum_{s',r} p(s', r \,|\, s, \pi(s))\big[r + \gamma V(s')\big]$
     - $\Delta \leftarrow \max(\Delta, |v - V(s)|)$

   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   - *old-action* $\leftarrow \pi(s)$
   - $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \,|\, s, a)\big[r + \gamma V(s')\big]$
   - If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*

   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Policy Iteration

- PI terminates since each iteration improves the policy and there is a finite number of policies. When PI halts, the Bellman optimality equations hold and, thus, π is optimal. Again, PI requires knowing the transition model.

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   - $\Delta \leftarrow 0$
   - Loop for each $s \in \mathcal{S}$:
     - $v \leftarrow V(s)$
     - $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s))\big[r + \gamma V(s')\big]$
     - $\Delta \leftarrow \max(\Delta, |v - V(s)|)$

   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   - *old-action* $\leftarrow \pi(s)$
   - $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
   - If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*

   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Q-Learning

Transition model not known!

next state

- If the transition model were so that s and a are always followed by s' and r, then $q_*(s,a) = r + \gamma \max_{a'} q_*(s',a')$ by the Bellman optimality equation and, thus, $0 = r + \gamma \max_{a'} q_*(s',a') - q_*(s,a)$. We can try to enforce this constraint by executing π one step from s and a and, then, updating the estimate of $q_*(s,a)$ as

When in state s
I do action a
→ Updating

$$q_*(s,a) \leftarrow q_*(s,a) + \alpha\underbrace{\left(r + \gamma \max_{a'} q_*(s',a') - q_*(s,a)\right)}_{\text{If optimal} = 0}.$$

where $\alpha > 0$ is the learning rate.

Q-table

Q-learning (off-policy TD control) for estimating $\pi \approx \pi_*$

Algorithm parameters: step size $\alpha \in (0,1]$, small $\varepsilon > 0$
Initialize $Q(s,a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, arbitrarily except that $Q(terminal, \cdot) = 0$

Loop for each episode:
    Initialize S
    Loop for each step of episode:
        Choose A from S using policy derived from Q (e.g., ε-greedy)
        Action value function
        Take action A, observe R, S'
        $Q(S,A) \leftarrow Q(S,A) + \alpha\left[R + \gamma \max_a Q(S',a) - Q(S,A)\right]$
        $S \leftarrow S'$
    until S is terminal

Q-Learning

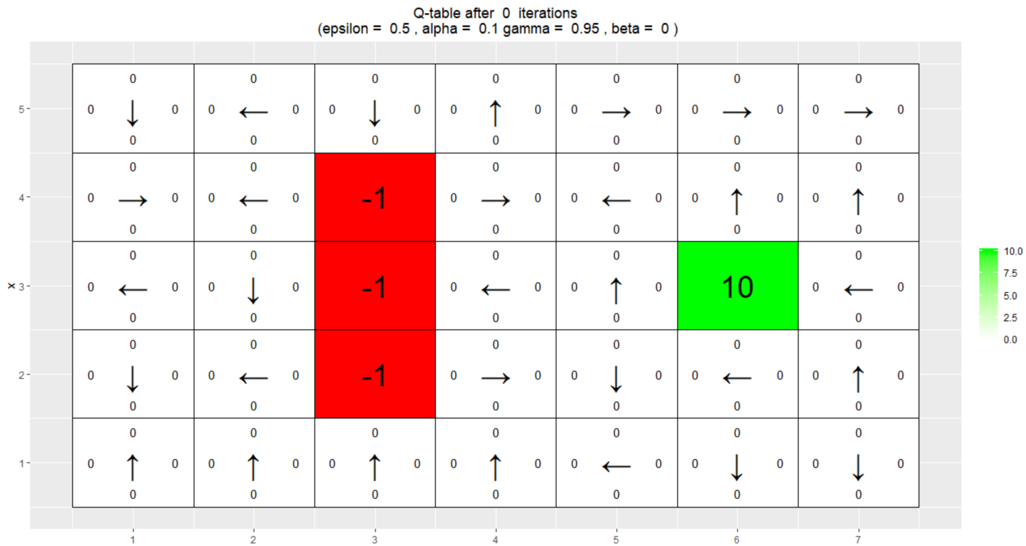
Q-learning (off-policy TD control) for estimating $\pi \approx \pi_*$

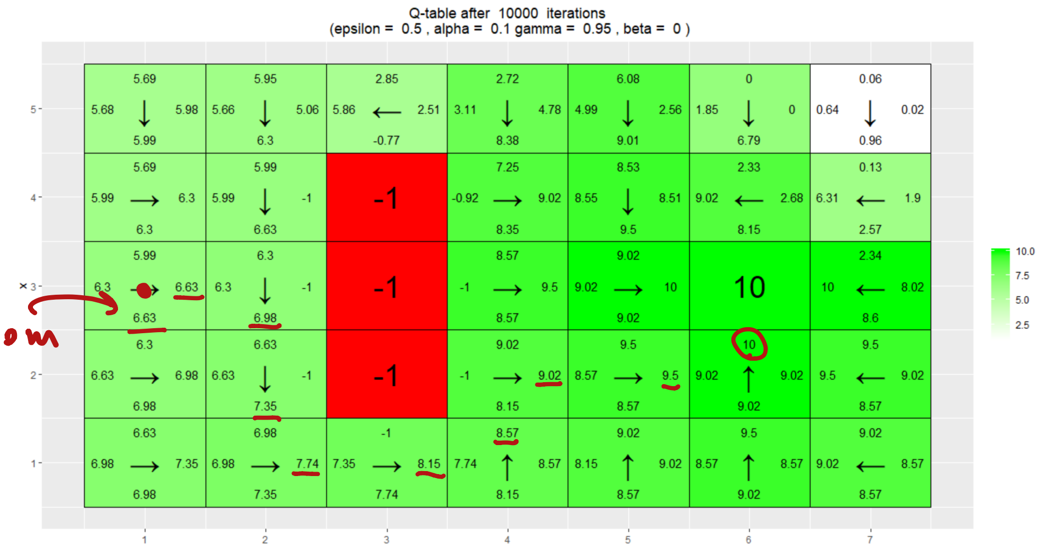
Algorithm parameters: step size $\alpha \in (0, 1]$, small $\varepsilon > 0$
Initialize $Q(s, a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, arbitrarily except that $Q(terminal, \cdot) = 0$

Loop for each episode:
  Initialize S
  Loop for each step of episode:
    Choose A from S using policy derived from Q (e.g., ε-greedy)
    Take action A, observe R, S'
    $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma \max_a Q(S', a) - Q(S, A)\big]$
    $S \leftarrow S'$
  until S is terminal

- Q-learning converges asymptotically if e.g. $\alpha(t) = O(1/N(s, t))$. Constant but small
- Q-learning converges asymptotically to q_* if e.g. an **ϵ-greedy** policy is used to keep updating all the state-action pairs: Choose the action with maximal estimated value with probability $1 - \epsilon$, and a random one with probability ϵ.
- Q-learning also works for stochastic transition models, since the number of times that s and a are followed by s' and r in the sampled episodes is proportional to the transition probability.
- Q-learning does **not** require knowing the transition model.

Example: Grid Worlds





Picked at random

Summary

- Learning through Interaction
- Markov Decision Processes
- Bellman Equations
- Value Iteration
- Policy Iteration
- Q-Learning
- Example: Grid Worlds
- Interested in more ? Check out AlphaGo - The Movie.

Thank you

Questions:

- Transition model, what does it mean here?
- Value iteration
- Policy iteration