MA5678 Assignment: Stage 2

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0.1 Advection (Transport) Equation

The advecton equation is as follows:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{1}$$

Defined on the domain $\mathbb{R} \times [0, \infty]$ with initial condition $u(x, 0) = f(x)$. A solution to this equation is $u(x, t) = f(x - ct)$:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial f(x-ct)}{\partial t} = -cf'(x-ct)$$
$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial f(x-ct)}{\partial x} = f'(x-ct)$$

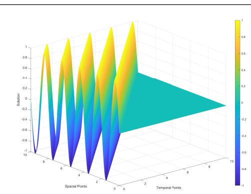
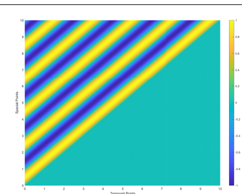
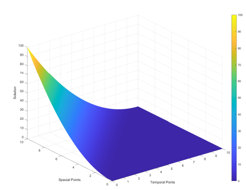
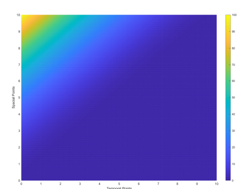
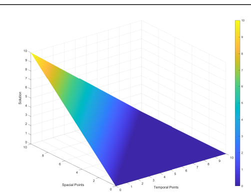
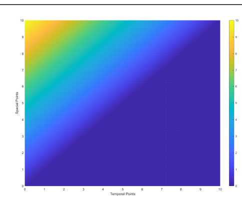
$$-cf'(x-ct) + cf'(x-ct) = 0$$

$$u(x,0) = u(x,0) = f(x-(0)t) = f(x)$$

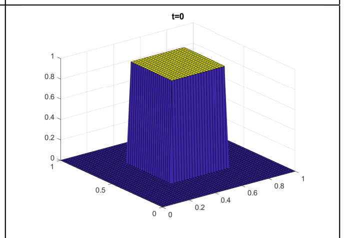
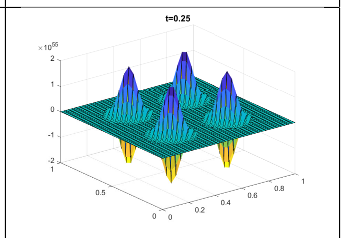
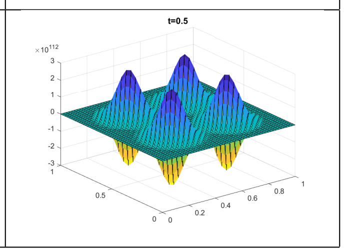
A fully commented program to solve this equation numerically using finite differences on a fixed interval $[x_{min}, x_{max}] \subset \mathbb{R}$ can be found in `transport.m`.

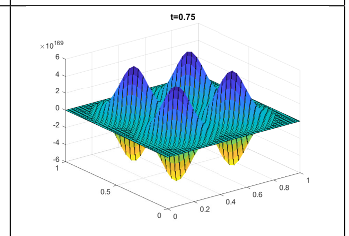
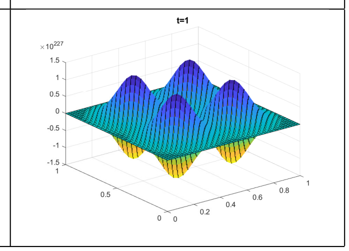
The script `TransportPlot.m` has been modified. Instead of plotting 4 different cross-sectional plots, it plots a 3D graph to illustrate all the data on one plot. You can run this by running the file `main.m`. You can change the function to one of the three variables `f0`, `f1` or `f2`.

Below are some plots for each of the functions `f0`, `f1` and `f2`.

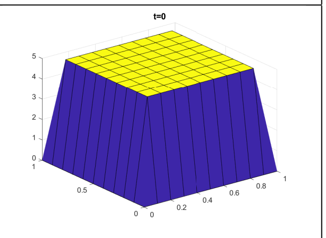
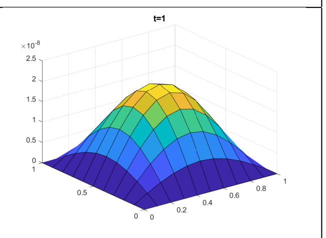
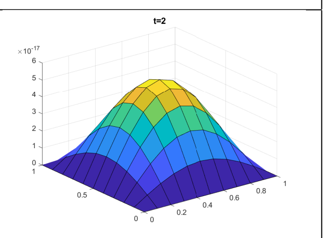
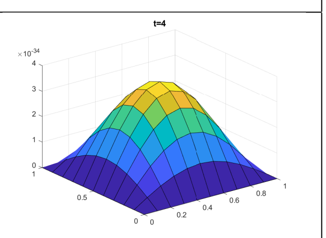
$f(x)$	Side View	Top Down
$f(x) = sin(\pi x)$		
$f(x) = x^2$		
$f(x) = \|x\|$		

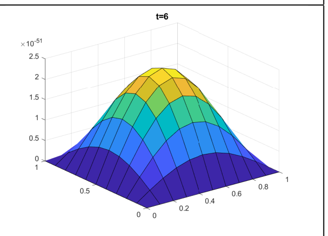
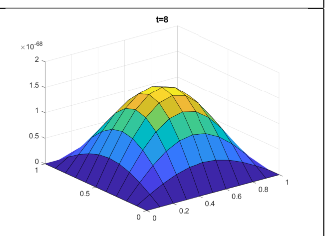
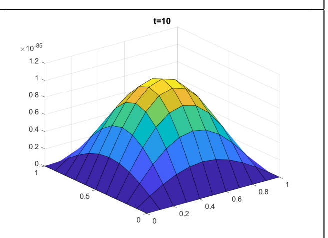
Below are some solutions to the heat equation for the following parameters: `dt = 0.01`, `dx = 0.02`, `dy = 0.02`, `Tmax = 1`, `Tsnap = [0.25, 0.5, 0.75, 1]`, `value = 1` and `bounds = [0.3, 0.7, 0.3, 0.7]`.

time (t)	Solution
t = 0	
t = 0.25	
t = 0.5	

time (t)	Solution
t = 0.75	
t = 1	

Below are some solutions to the heat equation for the following parameters: `dt = 0.001`, `dx = 0.1`, `dy = 0.1`, `Tmax = 10`, `Tsnap = [1, 2, 4, 6, 8, 10]`, `value = 5` and `bounds = [0.1, 0.9, 0.1, 0.9]`.

time (t)	Solution
t = 0	
t = 1	
t = 2	
t = 4	

time (t)	Solution
t = 6	
t = 8	
t = 10	

In `Heat2D_modified.m` you'll find a version of the original code where we have added a heat sink at coordinates $(0.1, 0.1)$. You can modify the value of `alpha` to get varying strengths, a higher alpha value meaning a greater sink of heat.

0.2 Task 3

This section is concerned with the Insect Dispersal Model

$$
\begin{aligned}
n_t &= d_0 \left(\left(\frac{n}{n_0} \right)^m n_x \right)_x && (2)\\
&= \frac{d}{n_0^m} \left(n^m n_x \right)_x && (3)\\
&= \frac{d}{n_0^m} \left(m n^{n-1} (n_x)^2 + n^m n_{xx} \right) && (4)\\
&= \frac{d n^{m-1}}{n_0^m} \left(m (n_x)^2 + n n_{xx} \right) && (5)
\end{aligned}
$$

$$
n_{xx} \approx \frac{n_{i+1,j} + n_{i-1,j} - 2n_{i,j}}{(\Delta x)^2} \tag{6}
$$

$$
n_x \approx \frac{n_{i+1,j} - n_{i-1,j}}{2(\Delta x)} \tag{7}
$$

$$
n_t \approx \frac{n_{i,j+1} - n_{i,j}}{\Delta t} \tag{8}
$$

$$
\begin{aligned}
n_x(-x,t) &= n_x(x,t) = 0 && (9)\\
n_x &= 0\\
n(0,0) &= 0
\end{aligned}
$$

$$
\tag{10}
$$