Fourier-space Form of the Biot-Savart Law

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Introduction

Magnetostatics deals with magnetic fields in the presence of steady currents or magnetization. The magnetic field due to such sources can be determined using the Biot-Savart law. We limit considerations to the case of steady currents.

Biot-Savart Law in Real Space

The Biot-Savart law for a steady current distribution $\mathbf{J}(\mathbf{r})$ in real space is:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3\mathbf{r}'$$

where μ_0 is the permeability of free space. The integration is over the entire space where the current distribution is non-zero, $\mathbf{r}$ is the position vector of the point where the magnetic field $\mathbf{B}(\mathbf{r})$ is to be determined, and $\mathbf{r}'$ is the position vector of the current element.

Fourier Transformation

The convention for the physical Fourier transform used is:

$$g(k) = \int G(x) \exp(-ikx) \, dx$$
$$G(x) = \frac{1}{2\pi} \int g(k) \exp(ikx) \, dk$$

where k is the wave-vector. In DFT, the cyclic frequency f is used and is related to the wave-vector as:

$$k = 2\pi f$$

Biot-Savart Law in Fourier Space

In Fourier space, the Biot-Savart law becomes:

$$\hat{\mathbf{B}}(k) = i\mu_0 k \times \hat{\mathbf{J}}(k)$$

To find the magnetic field in real space, one can take the inverse Fourier transform:

$$\mathbf{B}(\mathbf{r}) = \frac{1}{2\pi} \int e^{ik\mathbf{r}} \hat{\mathbf{B}}(k) \, dk$$

Two-dimensional Fourier Transform

When dealing with planar distributions, it is convenient to use two-dimensional Fourier transform. Assuming distributions in the $x - y$ plane, the Fourier transform is:

$$\hat{g}(k_x, k_y) = \int \int G(x, y) \exp(-i(k_x x + k_y y)) \, dx \, dy \tag{1}$$

Conclusion

By using the Fourier-space representation of the Biot-Savart law and the specific conventions for the Fourier transform, we can efficiently compute the magnetic field from a given current distribution. The approach is extended for two-dimensional distributions and can be adapted for various physical situations.