Statistical Science

Chapter 3.4 Unit Conversion and Rigid Rescaling

ReCap. Quantitative Reasoning(Ch 1)
Quantities (Ch2)
Re-Scaling (Ch3)
3.1 Logical Re-scaling
3.2 Operations on Ratio Scale Quantities
3.3 Descriptive Statistics and Rescaling
3.4 Unit Conversion and Rigid Rescaling

on chalk board

Organizational Details

Recap Chapter 1
Quantitative reasoning: Example of scallops, which combined stats and models
Recap Chapter 2
Quantities: Five part definition
Measurements made on four types of scale: nominal, ordinal, interval, ratio
Recap Chapter 3. Re-scaling
Logical rescaling (from one type of unit to another).
Re-scaling is a common technique in quantitative biology.
Operations on measured quantities differ from operations on numbers.
-the rules differ
-physically interpretable, not just abstract mathematical procedures
Renormalization occurs when we convert a quantity to a ratio with no units by rescaling it to a quantity with the same units. We can renormalize to the maximum value, resulting in a ratio between zero and one. We can renormalize to the minimum value, resulting in a scope. Statistical renormalization results from scaling to a statistic such as a sum, a mean, a range, or a standard deviation.

Today: Apply rules for operations on quantities to develop concepts of unit conversion and rigid rescaling.

Wrap-up:
Rigid (1 to 1) rescaling replaces one unit with another, either by remeasurement, or by calculation based on calibration factors.
Rigid rescaling is the basis of unit conversion via unit cancellation.

Rigid Rescaling

Rigid rescaling replaces one unit with another, either by remeasurement, or by calculation based on calibration factors. Rescaling via remeasurement can be visualized as lining up small units into larger units or dividing large units into equal subunits. An example is using a 10 meter wire to mark off plots 100 m on a side, then using 1 meter paces to find locations along the perimeter of the plot. Rescaling from 1 decameter to 1 m units can be viewed as breakage of the wire into smaller units. Another example is unit replacement of areas: an area of one hectare can be broken into exactly 100^2 squares each being a meter on a side.

Rigid conversion factors consist of a fixed ratio between a large and a small unit. If the smaller unit occurs in the numerator, the factor represents the operation of breaking large units into smaller units. If the larger unit occurs in the numerator, the rigid factor represents the operation of aligning small units into a larger unit.

Here is an example of rigid rescaling. In this example units "cancel out" because any unit scaled to itself is one: $\cancel{\text{m}}/\cancel{\text{m}} = 1$.

$$700\ \cancel{\text{yards}} \cdot \frac{0.9144\ \cancel{\text{m}}}{\cancel{\text{yard}}} \cdot \frac{1\ \text{km}}{1000\ \cancel{\text{m}}} \quad => \quad 0.64\ \text{km}$$

The symbol => is read "calculated as." It indicates that the quantity on the right Q_{new} is calculated from the quantity Q_{old} and two conversion factors on the left.
This example demonstrates unit conversion via unit cancellation.
Units disappear because any unit scaled to itself = 1 (no units):

$\cancel{\text{m}}/\cancel{\text{m}}$ is notation for metre/metre = 1.
hectare/km^2 has no units, it a dimensionless ratio
kcal/Joules is a number with no units
hp/Watt has no units: it is just the number of watts in 1 hp
$\text{km}^{1.2}/\text{m}^{1.2}$ has no units: it is the number of crooked m per crooked km
can you think of other examples?

Unit cancellation is used to convert from one unit to a similar unit.
Example: convert kilometres to metres.
Use ratio (1000 m/1km) that cancels units
This ratio (1000 m / 1 km) = 1
So multiply by this ratio to cancels units:
2 km (1000m/1km) = 2000 m (km/km = 1)

It is better to go by cancellation, than by intuition. This procedure will be familiar to many readers.

Rigid Rescaling – Generic Procedure

Here is the same rescaling, this time aligned with a generic formula for rigid rescaling of one quantity Q_{old} to another Q_{new}

Q_{old}	·	k_1	·	k_2	=	Q_{new}
Q_{old}	·	$\frac{k_1\text{new units}}{\text{old unit}}$	·	$\frac{k_2\text{newer units}}{\text{new units}}$	=	Q_{newer}
700 yards	·	$\frac{0.9144\text{ m}}{\text{yard}}$	·	$\frac{1\text{ km}}{1000\text{ m}}$	=>	0.64 km

Table 3.3 lists a general recipe for rigid rescaling. The equation in Table 3.3 will not be familiar, so to explain the expression it has been aligned with a specific calculation in Box 3.3.

Table 3.3

Rigid rescaling of quantities.

The sequence of steps in rigid rescaling is:

1. Write the quantity $(Q_{old})^{\alpha}$ to be rescaled.
2. Apply rigid conversion factors k_1^{α}, k_2^{α}, k_3^{α}, etc. so that units cancel.
3. Complete the calculation of Q_{new}^{α}, with appropriate exponents.

The generic expression for rigid rescaling is:

$$Q_{old}^{\alpha} \cdot k_1^{\alpha} \cdot k_2^{\alpha} = Q_{new}^{\alpha}$$

Rigid rescaling does not change unit exponents.
Hence Q_{old}^{α} and Q_{new}^{α} must have the same exponent.

The rigid conversion factors k_1 and k_2 rescale Q_{old} to a new quantity Q_{new}. Conversion factors were listed in sequence so that the denominator of the first factor cancels the units of Q_{old}, and the denominator of the next factor cancels the numerator of the preceding factor. Box 3.4 shows the derivation of the expression for rigid rescaling from the following general scaling relation.

Extension to more factors: How much food, as a % of body weight, does the Olympic gold medalist Michael Phelps (at 195 lbs), eat in a day during training?
Caloric intake during training is 12,000 Kcal/day = 12,000
670 Kcal/hamburger, 290g/hamburger 2.2 lbs/Kg

How much do you eat in a day, as a % of body weight?
(2000 Kcal/day for women not in training, 2200 kcal/day for men not in training).

Rigid Rescaling – Generic Expression
A scaling relation equates the scope of one quantity *Q(M)* to that of another quantity M:

$$\frac{Q(M)}{Q(M_{ref})} = \left(\frac{M}{M_{ref}}\right)^{\beta} \tag{3.1a}$$

where scope is defined for any object (numerator values) relative to a reference object (denominator values). This scaling relation is rearranged to:

$$Q(M) = \left(\frac{Q(M_{ref})}{{M_{ref}}^{\beta}}\right) M^{\beta} \tag{3.1b}$$

If we take the ratio of Q_{ref} to ${M_{ref}}^{\beta}$ as a fixed value k, equation 3.1b becomes a power law that can serve as a scaling function.

$$Q(M) = k \cdot M^{\beta} \tag{3.1c}$$

Box 3.4. Derivation of generic expression for rigid rescaling from the general scaling relation.

Rigid rescaling arises from the general scaling relation (Eq 2.5a) with exponent $\beta = 1$.

$$\frac{Q(M)}{Q(M_{ref})} = \left(\frac{M}{M_{ref}}\right)^{1}$$

Equivalently

$$Q(M) = Q(M_{ref}) \cdot \left(\frac{M}{M_{ref}}\right)^{1}$$

:
Taking $Q(M)$ as Q_{new}, $Q(M_{ref})$ as Q_{old}, $(M/M_{ref}) = k$, and applying the exponent α, this becomes

$$Q_{new}^{\alpha} = Q_{old}^{\alpha} \cdot k^{\alpha}$$

The reason for stating a generic expression is that it shows how to handle exponents other than $\alpha = 1$.

Working with exponents will require a quick review of logarithms

$\log_2 2^3 = 3$ $\quad \ln e^x = x$

$x^0 = 1$ $\quad 1^x = 1$

$10^b \cdot 10^c = 10^{b+c}$ $\quad km^b \div km^c = km^{b-c}$

Box 3.5 shows rigid rescaling with a familiar exponent of two.

Keep Eq. at top of board, add new example beneath.

Exponents are applied to both units and numbers, not just to the numbers.

$$Q_{old}^{\ 2} = (700 \cdot \text{yard})^2 = 700^2 \text{ yard}^2 \neq 700 \text{ yard}^2$$

Apply the exponent to obtain a conversion factor that will "cancel" units of $Q_{old}^{\ 2}$:

$$k_1^2 = \left(\frac{0.9144 \text{ meter}}{\text{yard}} \right)^2 = \frac{0.9144^2 \text{ m}^2}{\text{yard}^2}$$

Then apply exponents after lining up the conversion factors:

$$\begin{array}{ccccccccc} (700 \text{ yards})^2 & \cdot & \dfrac{0.9144^2 \text{ m}^2}{\text{yard}^2} & \cdot & \dfrac{\text{km}^2}{1000^2 \text{ m}^2} & => & 0.41 \text{ km}^2 \\ Q_{old}^{\ 2} & \cdot & k_1^2 & \cdot & k_2^2 & = & Q_{new}^{\ 2} \end{array}$$

Box 3.5 Rigid rescaling of quantities. Exponent = 2.

Box 3.6 shows rigid rescaling for a fractal quantity, a length with non-integral exponent ($\text{km}^{1.2}$).

Non-integer exponents are handled the same way as integer exponents.

$$\begin{array}{ccccccccc} (700 \text{ yards})^{1.2} & \cdot & \dfrac{0.9144^{1.2} \text{ m}^{1.2}}{\text{yard}^{1.2}} & \cdot & \dfrac{\text{km}^{1.2}}{1000^{1.2} \text{ m}^{1.2}} & => & 0.59 \text{ km}^{1.2} \\ Q_{old}^{\ 1.2} & \cdot & k_1^{1.2} & \cdot & k_2^{1.2} & = & Q_{new}^{\ 1.2} \end{array}$$

Box 3.6 Rigid rescaling of quantities. Exponent = 1.2.

Rigid factors are written as a symbol representing a ratio of units:

$$\frac{1 \text{ Joule}}{4.187 \text{ cal}} = k_{\text{Joule/cal}}$$

The reason for writing a rigid factor as a ratio is that units are converted by multiplication, not by substitution. It might seem that the relation of calories to Joules could be written 1 Joule = 4.187 cal, but this can leads to error by encouraging substitution rather than multiplication to rescale quantities. The secret of success in rigid rescaling is to line up factors that cancel units, and to make sure that the exponents allow units to cancel.

Rigid factors have several sources. They often arise by defining a unit at one resolution as a multiple of a unit at a finer resolution. Thus a rigid factor $k_{g/Mg}$ is by definition:

$$\frac{10^6 \text{ gram}}{1 \text{ Megagram}} = k_{g/Mg}$$

Rigid factors also arise from definition of measurement units. The definition of a Watt is

$$\text{Watt} = \text{Joule s}^{-1}$$

and consequently the rigid conversion factor is:

$$\frac{1 \text{ Joule} \cdot \text{s}}{1 \text{ Watt}} = k_{\text{Joule-s/Watt}}$$

Some rigid factors are precisely measured and considered to hold regardless of circumstance. For example, 1 unit of heat energy (1 cal) is equal to 4.187 units of mechanical energy, where each unit of mechanical energy is a Newton · m:

$$\frac{4.187 \text{ Newton} \cdot \text{m}}{1 \text{ cal}} = k_{\text{work/heat}}$$

Many rigid factors are estimated from data. These factors are completely empirical, with applicability that depends on circumstance. An example is the calibration of a satellite image to a measure of vegetation cover. The calibration could be used in similar circumstances, but could not be applied to a satellite image from anywhere in the world.

Some conversion factors are universally true. Some are not. The symbol := is used to indicate an equality that is true under limited conditions. The symbol := is read "conditionally equal to." An example of a factor that holds conditionally is the density (mass per unit volume) of living organisms:

$$\frac{1 \text{ m}^3}{1000 \text{ kg}} := k_{\text{vol/mass}}$$

Most organisms have densities close to this value, even though many do not have exactly this value. This ratio is so useful that it is worth keeping the few exceptions in mind in order to be able to use it.

Another ratio that has a narrow enough scope to be worth using as a rigid scaling factor is the ratio of biomass consumed by a population M_{in} to the biomass transferred to higher trophic levels M_{out}:

$$\frac{M_{in}}{10\, M_{out}} := k_{in/out}$$

The transfer efficiency between trophic levels is not fixed at 10%, but the scope of this ratio is small enough (ca. 20% / ca. 5% = 4) so that it is useful in making order of magnitude calculations of production at one trophic level from measurements at another level.

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In the next few days, look
around for examples of fractal
rather than Euclidean shapes.
Ie shapes somewhere between a
line and a plane, which is to
say somewhere between m¹ and m².
Or somewhere between a plane
and a volume, which is to say,
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Another look at Chapter 3.4.

Write out the generic expression in Table 3.3, for rigid rescaling. Then write out the sequence of conversion factors required to compute the number of eagle nests per $\text{km}^{1.3}$ of coastline, if there are to 5 per nautical $\text{mile}^{1.3}$. Nautical mile = 1.8652 km.