A TREATISE ON PROBABILITY

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to the remarkable regularity of comparatively *rare* events. Von Bortkiewicz has enlarged Quetelet's catalogue with modern instances out of the statistical records of bureaucratic Germany. The classic instance, perhaps, is the number of Prussian cavalrymen killed each year by the kick of a horse. The table is worth giving as a statistical curiosity. (The period is from 1875 to 1894; G stands for the Corps of Guards, and I.-XV. for the 15 Army Corps.)

	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	9
G.	..	2	2	1	..	..	1	1	..	3	..	2	1	..	..	1	..	1	..	1
I.	..	..	..	2	..	3	..	2	..	..	..	1	1	1	..	2	..	3	1	.
II.	..	..	..	2	..	2	..	..	1	1	..	..	2	1	1	..	..	2	..	.
III.	..	..	..	1	1	1	2	..	2	..	..	..	1	..	1	2	1	..	..	.
IV.	..	1	..	1	1	1	1	..	..	..	..	1	..	..	..	..	1	1	..	.
V.	..	..	..	..	2	1	..	..	1	..	..	1	..	1	1	1	1	1	1	.
VI.	..	..	1	..	2	..	..	1	2	..	1	1	3	1	1	1	..	3	..	.
VII.	1	..	1	..	..	..	1	..	1	1	..	..	2	..	..	2	1	..	2	.
VIII.	1	..	..	..	1	..	..	1	..	..	..	..	1	..	..	..	1	1	..	1
IX.	..	..	..	..	..	2	1	1	1	..	2	1	1	..	1	2	..	1	.	.
X.	..	..	1	1	..	1	..	2	..	2	..	..	..	..	2	1	3	..	1	1
XI.	..	..	..	..	2	4	..	1	3	..	1	1	1	1	2	1	3	1	3	1
XIV.	1	1	2	1	1	3	..	4	..	1	..	3	2	1	..	2	1	1	..	..
XV.	..	1	..	..	..	..	..	1	..	1	1	..	..	..	2	2·	..	..	..	..

The agreement of this table with the theoretical results of a random distribution of the total number of casualties is remarkably close:[1]

Casualties in a Year.	Number of Occasions on which the Annual Casualties in a Corps reach the Figure in Column 1.	
	Actual.	Theoretical.
0	144	143·1
1	91	92·1
2	32	33·3
3	11	8·9
4	2	2·0
5 and more	..	0·6

Other instances are furnished by the numbers of child suicides in Prussia, and the like.

It is Von Bortkiewicz's thesis that these observed regularities

[1] Bortkiewicz, *op. cit.* p. 24.

have a good theoretical explanation behind them, which he dignifies with the name of the *Law of Small Numbers*.

The reader will recall that, according to the theory of Lexis, his measure of stability Q is, in the more general case, made up of two components r and p, combined in the expression $\sqrt{r^2+p^2}$, of which one is due to fluctuations from the average of the conditions governing all the members of a series, which furnishes us with one of our observed frequencies, and of which the other is due to fluctuations in the individual members of the series about the true norm of the series. Bortkiewicz carries the same analysis a little further, and shows that Lexis's Q is of the form $\sqrt{1+(n-1)c^2}$, where n is the number of times that the event occurs in each series.[1] That is to say, Q increases with n, and, when n is small, Q is likely to exceed unity to a less extent than when n is large. To postulate that n is small, is, when we are dealing with observations drawn from a wide field, the same thing as to say that the event we are looking for is a comparatively rare one. This, in brief, is the mathematical basis of the Law of Small Numbers.

In his latest published work on these topics,[2] Von Bortkiewicz builds his mathematical structure considerably higher, without, however, any further underpinning of the logical foundations of it. He has there worked out further statistical constants, arising out of the conceptions on which Lexis's Q is based (the precise bearing of which is not made any clearer by his calling them *coefficients of syndromy*), which are explicitly dependent on the value of n; and he elaborately compares the theoretical value of the coefficients with the observed value in certain actual statistical material. He concludes with the thesis, that Homogeneity and Stability (defined as he defines them) are opposed conceptions, and that it is not correct to premise, that the larger statistical mass is as a rule more stable than the smaller, unless

[1] I refer the reader to the original, *op. cit.* pp. 29-31, for the interpretation of c (which is a function of the mean square errors arising in the course of the investigation) and for the mathematical argument by which the above result is justified.

[2] "Homogeneität und Stabilität in der Statistik," published in the *Skandinavisk Aktuarietidskrift*, 1918. Those readers, who look up my references, will, I think, agree with me that Von Bortkiewicz does not get any less obscure as he goes on. The mathematical argument is right enough, and often brilliant. But what it is all really about, what it all really amounts to, and what the premisses are, it becomes increasingly perplexing to decide.

we also assume that the larger mass is less homogeneous. At this point, it would have helped, if Von Bortkiewicz, excluding from his vocabulary homogeneity, paradromy, γ'_M, and the like, had stopped to tell in plain language where his mathematics had led him, and also whence they had started. But like many other students of Probability he is eccentric, preferring algebra to earth.

9. Where, then, though an admirer, do I criticise all this? I think that the argument has proceeded so far from the premisses, that it has lost sight of them. If the limitations prescribed by the premisses are kept in mind, I do not contest the mathematical accuracy of the results. But many technical terms have been introduced, the precise signification and true limitations of which will be misunderstood if the conclusion of the argument is allowed to detach itself from the premisses and to stand by itself. I will illustrate what I mean by two examples from the work of Von Bortkiewicz described above.

Von Bortkiewicz enunciates the seeming paradox that the larger statistical mass is only, as a rule, more stable if it is less homogeneous. But an illustration which he himself gives shows how misleading his aphorism is. The opposition between stability and homogeneity is borne out, he says, by the judgment of practical men. For actuaries have always maintained that their results average out better, if their cases are drawn from a wide field subject to *variable* conditions of risk, whilst they are chary of accepting too much insurance drawn from a single homogeneous area which means a concentration of risk. But this is really an instance of Von Bortkiewicz's own distinction between a general probability p and special probabilities p_1 etc., where

$$p = \frac{z_1}{z}p_1 + \frac{z_2}{z}p_2 + \ . \ . \ .$$

If we are basing our calculations on p and do not know p_1, p_2, etc., then these calculations are more likely to be borne out by the result if the instances are selected by a method which spreads them over all the groups 1, 2, etc., than if they are selected by a method which concentrates them on group 1. In other words, the actuary does not like an undue proportion of his cases to be drawn from a group which may be subject to a common relevant influence *for which he has not allowed.* If the *à priori* calculations are based on the average over a field which is not homogeneous in all its parts, greater stability of result will be obtained if the instances are drawn from all parts of the non-homogeneous total field, than if they are drawn now from one homogeneous sub-field and now from another. This is not at all paradoxical. Yet I believe, though with hesitation, that this is all that Von Bortkiewicz's elaborately supported mathematical conclusion really amounts to.

My second example is that of the Law of Small Numbers. Here also we are presented with an apparent paradox in the statement that the regularity of occurrence of rare events is more stable than that of commoner events. Here, I suspect, the paradoxical result is really latent in the particular measure of stability which has been selected. If we look back at the figures, which I have quoted above, of Prussian cavalrymen killed by the kick of a horse, it is evident that a measure of stability could be chosen according to which exceptional instability would be displayed by this particular material; for the frequency varies from 0 to 4 round a mean somewhat less than unity, which is a very great *percentage* fluctuation. In fact, the particular measure of stability which Von Bortkiewicz has adopted from Lexis has about it, however useful and convenient it may be, especially for mathematical manipulation, a great deal that is arbitrary and conventional. It is only one out of a great many possible formulae which might be employed for the numerical measurement of the conception of stability, which, quantitatively at least, is not a perfectly precise one. The so-called Law of Small Numbers is, therefore, little more than a demonstration that, where rare events are concerned, the Lexian measure of stability does not lead to satisfactory results. Like some other formulae which involve a use of Bernoullian methods in an approximative form, it does not lead to reliable results in all circumstances. I should add that there is one other element which may contribute to the total psychological reaction of the reader's mind to the Law of Small Numbers, namely, the surprising and *piquant* examples which are cited in support of it. It is startling and even amusing to be told that horses kick cavalrymen with the same sort of regularity as characterises the rainfall. But our surprise at this particular example's fulfilling the Law of Great Numbers has little or nothing to do with the exceptional stability about which the Law of Small Numbers purports to concern itself.