Trajectory-Based Mutual Information in a Markovian System

Introduction

In the context of the generalized Jarzynski equality under non-equilibrium feedback control, the mutual information I_c quantifies the information gained through measurements over entire trajectories. Here, we consider a system where the true positions (system trajectory) are denoted by $\{x_k\}$ and the corresponding measurement outcomes (measurement trajectory) by $\{m_k\}$. In a Markovian system, both the system dynamics and the measurement process are memoryless. This allows us to factorize the conditional probabilities.

Key Equations

1. Definition of the Mutual Information I_c

The mutual information is defined as the ensemble average of the logarithm of the ratio between the conditional probability of a measurement trajectory given a system trajectory and its marginal probability:

$$I_c = \left\langle \ln \left(\frac{P[\{m_k\} \mid \{x_k\}]}{P[\{m_k\}]} \right) \right\rangle . \tag{1}$$

2. Factorization of the Conditional Probability in a Markovian System

For a Markovian system, the conditional probability of the measurement trajectory given the system trajectory factorizes as:

$$P[\{m_k\} \mid \{x_k\}] = \prod_{k=1}^{T} P(m_k \mid x_k). \tag{2}$$

Assuming a Gaussian noise model for the measurements, the conditional probability at each time step is given by:

$$P(m_k \mid x_k) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(m_k - x_k)^2}{2\sigma^2} \right), \tag{3}$$

where σ is the noise standard deviation.

For the i-th trajectory, the overall conditional probability is:

$$P[\{m_k^{(i)}\} \mid \{x_k^{(i)}\}] = \prod_{k=1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(m_k^{(i)} - x_k^{(i)})^2}{2\sigma^2} \right). \tag{4}$$

3. Estimation of the Marginal Probability

The marginal probability for a given measurement trajectory $\{m_k^{(i)}\}$ is estimated by averaging the conditional probabilities over an ensemble of M trajectories:

$$P[\{m_k^{(i)}\}] \approx \frac{1}{M} \sum_{j=1}^{M} \prod_{k=1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(m_k^{(i)} - x_k^{(j)})^2}{2\sigma^2} \right). \tag{5}$$

4. Final Expression for the Mutual Information

Combining the above results, the mutual information is given by:

$$I_c \approx \frac{1}{M} \sum_{i=1}^{M} \ln \left(\frac{\prod_{k=1}^{T} P(m_k^{(i)} \mid x_k^{(i)})}{P[\{m_k^{(i)}\}]} \right). \tag{6}$$

This expression quantifies the information gain per measurement trajectory, averaged over all trajectories in the ensemble.

Conclusion

By collecting the full trajectories $\{x_k\}$ (true positions) and $\{m_k\}$ (measurement outcomes) for each simulation run, and assuming a Markovian system with Gaussian measurement noise, the mutual information I_c can be computed using the factorized probabilities shown above.