Control Engineering (SC42095)

Lecture 8, 2020

Tamás Keviczky, Azita Dabiri

Delft Center for Systems and Control
Faculty of Mechanical Engineering
Delft University of Technology
The Netherlands

e-mail: t.keviczky@tudelft.nl, a.dabiri@tudelft.nl

Lecture outline

- Internal model principle
- Repetitive control
- Disturbance models

Generating polynomial

Assume reference signal or disturbance $d(t)$ satisfies the differential equation:

$$
\begin{aligned}
\frac{d^{n_d}}{dt^{n_d}}d(t) + \gamma_{n_d-1}\frac{d^{n_d-1}}{dt^{n_d-1}}d(t) + \cdots + \gamma_1\frac{d}{dt}d(t) + \gamma_0 d(t) \;&=\; 0 \\
\underbrace{\left(s^{n_d} + \gamma_{n_d-1}s^{n_d-1} + \cdots + \gamma_0\right)}_{\Gamma_d(s)} D(s) \;&=\; f(0,s)
\end{aligned}
$$

- $\Gamma_d(s)$ is called disturbance generating polynomial
- $f(0,s)$ is a polynomial in s (due to initial conditions)

Disturbance generating polynomial examples

$$d(t) = d_0 \text{ constant} \quad \rightarrow \quad \Gamma_d(s) = s$$

$$d(t) = \sin(\omega t) \quad \rightarrow \quad \Gamma_d(s) = s^2 + \omega^2$$

$$d(t) = e^{at} \quad \rightarrow \quad \Gamma_d(s) = s - a$$

$$d(t) = d_0 + d_1 e^{at} \quad \rightarrow \quad \Gamma_d(s) = s(s - a)$$

Internal model principle (IMP)

Assume a standard one-degree-of-freedom control architecture.

If $d_i(t), d_o(t), r(t)$ has $\Gamma_d(s)$ as their generating polynomial, then the controller

$$C(s) = \frac{P(s)}{\Gamma_d(s)\bar{L}(s)}$$

can asymptotically reject the effect of input-, output-disturbance, and track the reference.

Note: Only the generating polynomial is needed, not the magnitude of the disturbance / reference.

Why does it work?

Recall: For step reference tracking, step disturbance rejection, we needed an integrator in the controller to get zero steady state error.

Describe plant as $G(s) = \frac{B(s)}{A(s)}$, and assume that $\Gamma_d(s)$ is not a factor of $B(s)$.

Closed-loop transfer functions:

$$S = \frac{\Gamma_d \bar{L} A}{\Gamma_d \bar{L} A + PB}, \qquad S_i = \frac{\Gamma_d \bar{L} B}{\Gamma_d \bar{L} A + PB}$$
$$T = \frac{PB}{\Gamma_d \bar{L} A + PB}$$

Why does it work? (cont'd)

Suppose $\bar{L}, P$ chosen such that the closed-loop characteristic equation

$$A_{cl}(s) = \Gamma_d(s)\bar{L}(s)A(s) + P(s)B(s)$$

has roots with negative real parts.

E.g., by pole-placement, input-output design (using Diophantine equation or Sylvester matrix).

Why does it work? (cont'd)

Response of system to output disturbance $d_o(t)$ with generating polynomial $\Gamma_d(s)$:

$$Y(s) = S(s)D_o(s) = S(s)\frac{f(0,s)}{\Gamma_d(s)} = \frac{\bar{L}(s)A(s)}{A_{cl}(s)}$$

Notice that cancelation of $\Gamma_d(s)$ occurs and $y(t \to \infty) = 0$ since A_{cl} is stable.
For input disturbance $d_i(t)$:

$$Y(s) = S_i(s)D_i(s) = \frac{\bar{L}(s)B(s)}{A_{cl}(s)}$$

For reference $r(t)$ $(e(t) = r(t) - y(t))$:

$$E(s) = (1 - T(s))\frac{f(0,s)}{\Gamma_d(s)} = S(s)\frac{f(0,s)}{\Gamma_d(s)}$$

IMP in state-space

Create a disturbance exo-system

$$\begin{aligned}\dot{x}_d &= A_d x_d \\ d &= C_d x_d\end{aligned}$$

Estimate $d(t)$ using an observer (see in Lecture 6).

The controller will have the eigenvalues of A_d as poles.

$$\begin{aligned}D(s) &= C_d(sI - A_d)^{-1} x_d(0) \\ \Gamma_d(s) &= \det(sI - A_d)\end{aligned}$$

IMP in discrete-time

$d(k)$ satisfies:

$$
\begin{aligned}
d(k) + \gamma_1 d(k-1) + \cdots + \gamma_N d(k-N) &= 0 \\
\underbrace{\left(1 + \gamma_1 q^{-1} + \cdots + \gamma_N q^{-N}\right)}_{\Gamma_d(q^{-1})} d(k) &= 0
\end{aligned}
$$

The rest of the story is the same as in continuous-time...

Repetitive control

Goal is to eliminate the effect of periodic disturbance,
or to track periodic reference input.
(Need to "learn" the disturbance $\rightarrow$ special IMP controller.)

Repetitive control

Consider N-periodic disturbances: $d(k-N) = d(k)$

This means that $d(k)$ satisfies $q^{-N}d(k) = d(k)$

$$\left(1 - q^{-N}\right) d(k) = 0 \quad \rightarrow \quad \Gamma_d(q^{-1}) = 1 - q^{-N}$$

Let's try to use IMP in discrete-time:

$$C(q^{-1}) = \frac{P(q^{-1})}{\Gamma_d(q^{-1})\bar{L}(q^{-1})}$$

Problem: Disturbance generating polynomial is of very high order, difficult to assign poles.

Prototype controller

Assume $G(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})}$ is stable, minimum phase.

Choose

$$C(q^{-1}) = k_r \frac{Aq^{\delta - N}}{B(1 - q^{-N})}$$

which leads to $e(k) = (1 - k_r)e(k - N)$ and it

1. inverts the plant
2. has $(1 - q^{-N})$ in denominators because of IMP
3. delays the control by one cycle (q^{-N})

Remarks

- $A(q^{-1})$ should be stabilized first.
- $B(q^{-1})$ unstable factors shouldn't be canceled.
- Further modification necessary (gain mod., zero phase comp.).
- Robustness problems using $\Gamma_d = (1 - q^{-N})$, which implies controller with very high gain at *all* harmonics of the disturbance frequency.

  Solution: limit bandwidth by modifying $1 - q^{-N}$ to

$$1 - Q(q, q^{-1})q^{-N}$$

  where Q is a unity gain zero phase filter, e.g.:

$$Q(q, q^{-1}) = 0.1q^2 + 0.15q + 0.5 + 0.15q^{-1} + 0.1q^{-2}$$

  It smoothes out the generating polynomial and reduces gain at high order harmonics.

Example

Consider a single integrator

$$G(q^{-1}) = \frac{hq^{-1}}{1 - q^{-1}}$$

thus $A = 1 - q^{-1}$, $B = h$, $\delta = 1$. The controller

$$C(q^{-1}) = \frac{k_r}{h} \frac{q^{-(N-1)}(1 - q^{-1})}{1 - q^{-N}}$$

leads to

$$u(k) = u(k - N) - \frac{k_r}{h}\left(e(k + 1 - N) - e(k + 2 - N)\right)$$

which shows that the control action is updated with the error in the previous cycle (the control is “learned”).

Repetitive control in state-space

Disturbance generating exo-system:

$$\begin{pmatrix} x_{d1}(k+1) \\ \vdots \\ x_{dN}(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_{d1}(k) \\ \vdots \\ x_{dN}(k) \end{pmatrix}$$

$$d(k) = x_{d1}(k)$$

Repetitive control in state-space (cont'd)

We then proceed with the disturbance-estimate feedback approach to IMP:

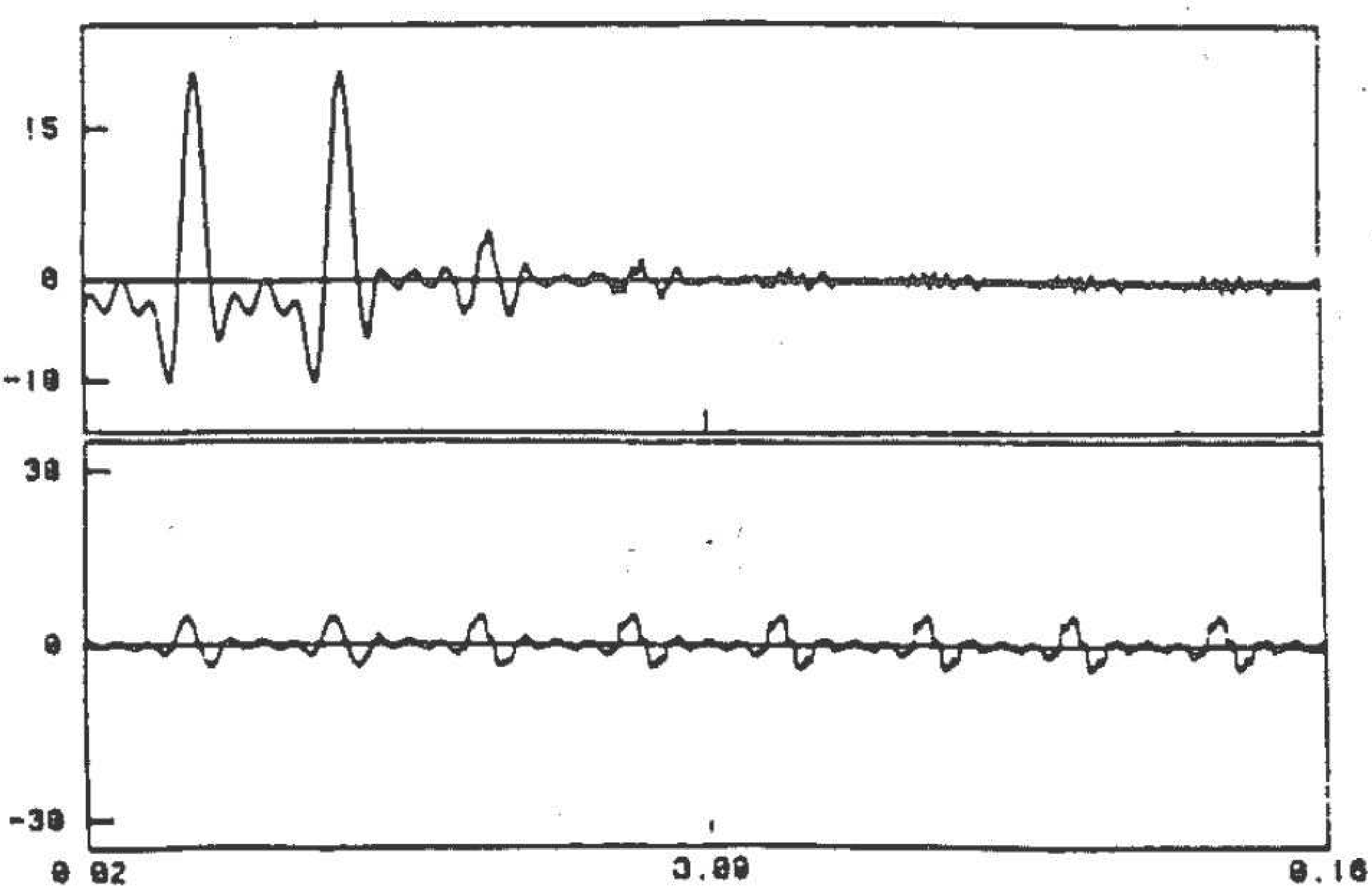
$$u(k) = -Kx(k) - \hat{d}(k)$$

$$\begin{pmatrix} \hat{x}(k+1) \\ \hat{x}_d(k+1) \end{pmatrix} = \overbrace{\begin{pmatrix} A & BC_d \\ 0 & A_d \end{pmatrix}}^{\tilde{A}} \begin{pmatrix} \hat{x}(k) \\ \hat{x}_d(k) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u(k) - L\left(y(k) - C\hat{x}(k)\right)$$

$$\hat{d}(k) = C_d \hat{x}_d(k)$$

Choose L such that $\tilde{A} - LC$ is stable.

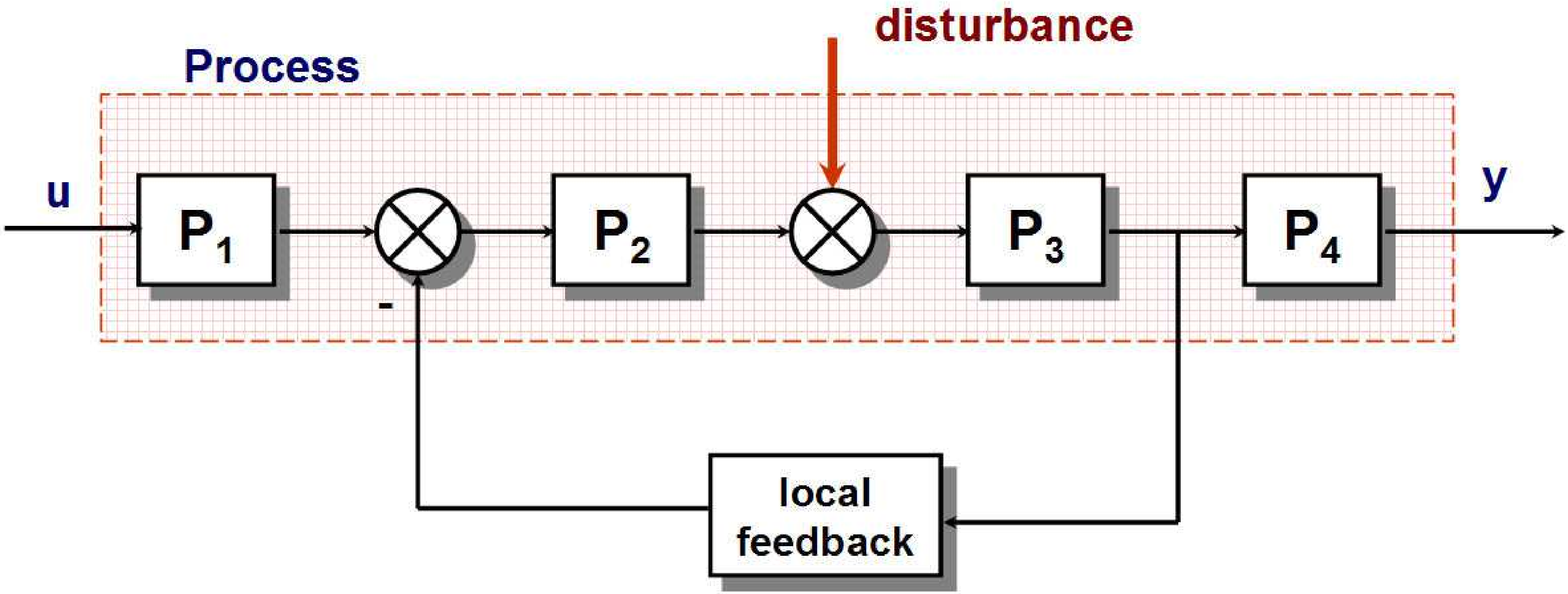
Repetitive control result example



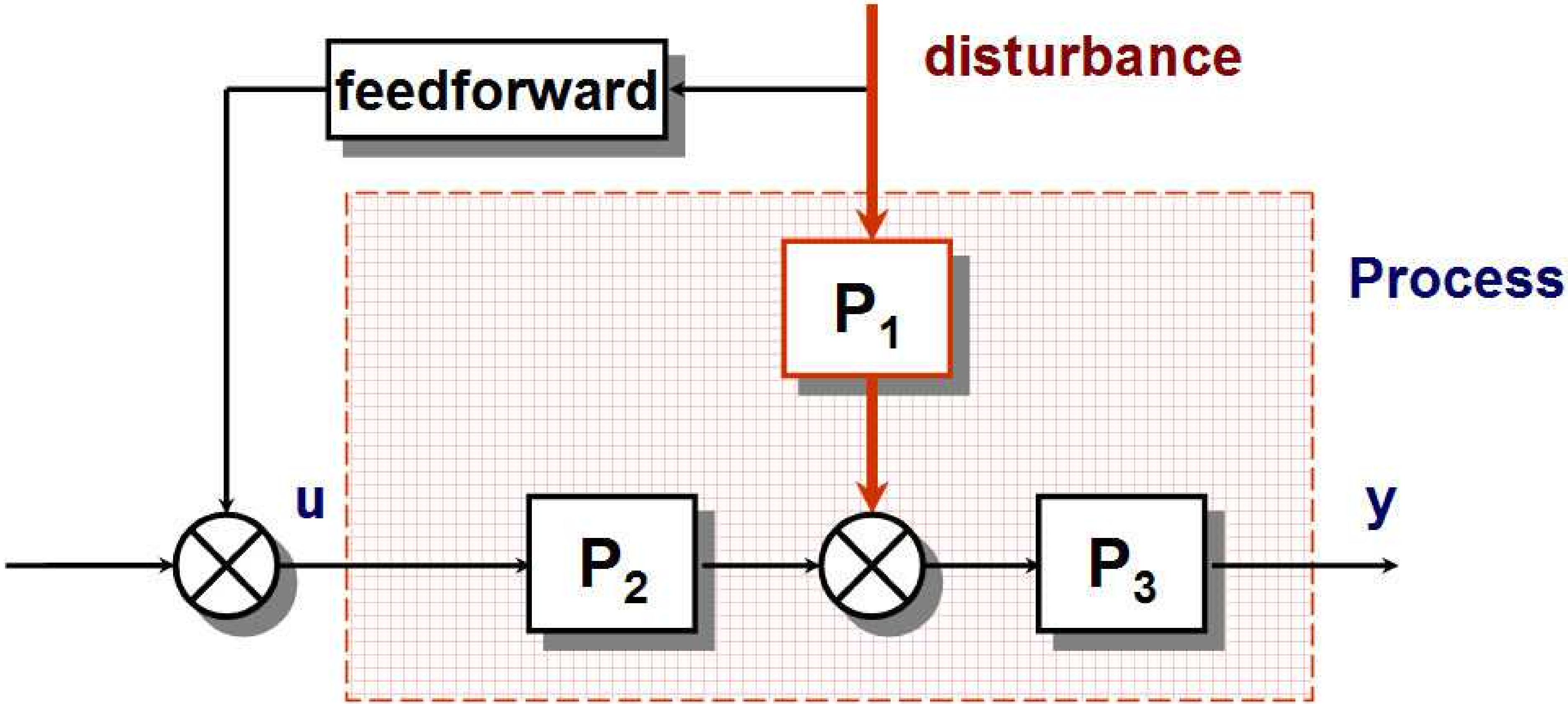
Reduction of effects of disturbances

- Reduction at the source
- Reduction by local feedback
- Reduction by feedforward
- Reduction by prediction

Disturbance compensation with local feedback



Disturbance compensation with feedforward



Disturbance compensation using prediction

Signal $y(k)$ is assumed to be generated by:

$$\begin{aligned} x(k+1) &= \Phi x(k) + v(k) \\ y(k) &= Cx(k) \end{aligned}$$

where $v(k)$ is assumed to be zero except at isolated points.

$$x(k-n+1) = W_o^{-1} \begin{pmatrix} y(k-n+1) & \cdots & y(k-1) & y(k) \end{pmatrix}^T$$

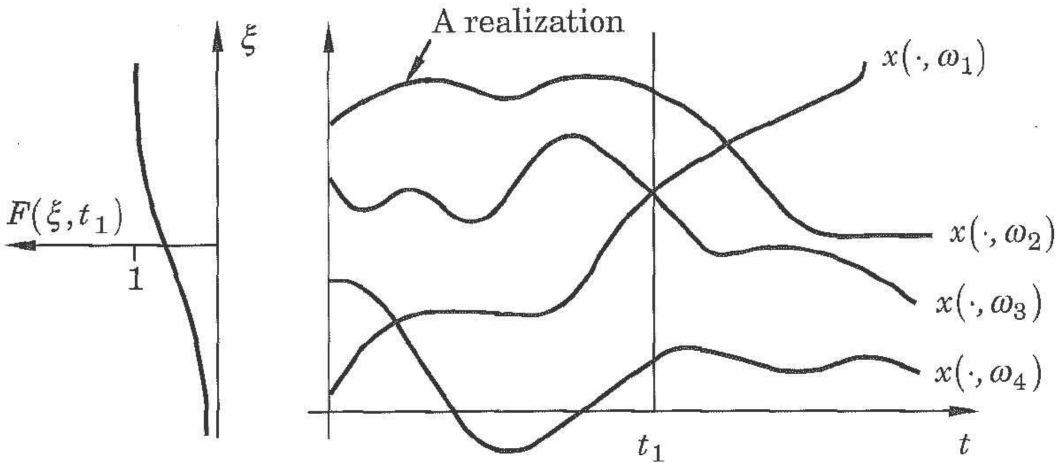
$$\hat{x}(k+m|k) = \Phi^{m+n-1} W_o^{-1} \begin{pmatrix} y(k-n+1) & \cdots & y(k-1) & y(k) \end{pmatrix}^T$$

This gives the predictor as a polynomial of degree $n-1$

$$\hat{y}(k+m|k) = P^*(q^{-1})y(k)$$

Stochastic processes and disturbance models

Stochastic (random) process: $\{x(t,\omega), t \in T, \omega \in W\}$
$\sim$ indexed family of random variables.



Finite-dimensional distribution function:

$$F(\xi_1, \ldots, \xi_n; t_1, \ldots, t_n) = P\{x(t_1) \le \xi_1, \ldots, x(t_n) \le \xi_n\}$$

If all finite-dimensional distributions are shift-invariant:
$stationary$ stochastic process

Stochastic processes and disturbance models

Mean-value function (constant for stationary processes):

$$m(t) = \mathrm{E}\left(x(t)\right) = \int_{-\infty}^{\infty} \xi dF(\xi; t)$$

Covariance function (function of only $s-t$ for stationary processes):

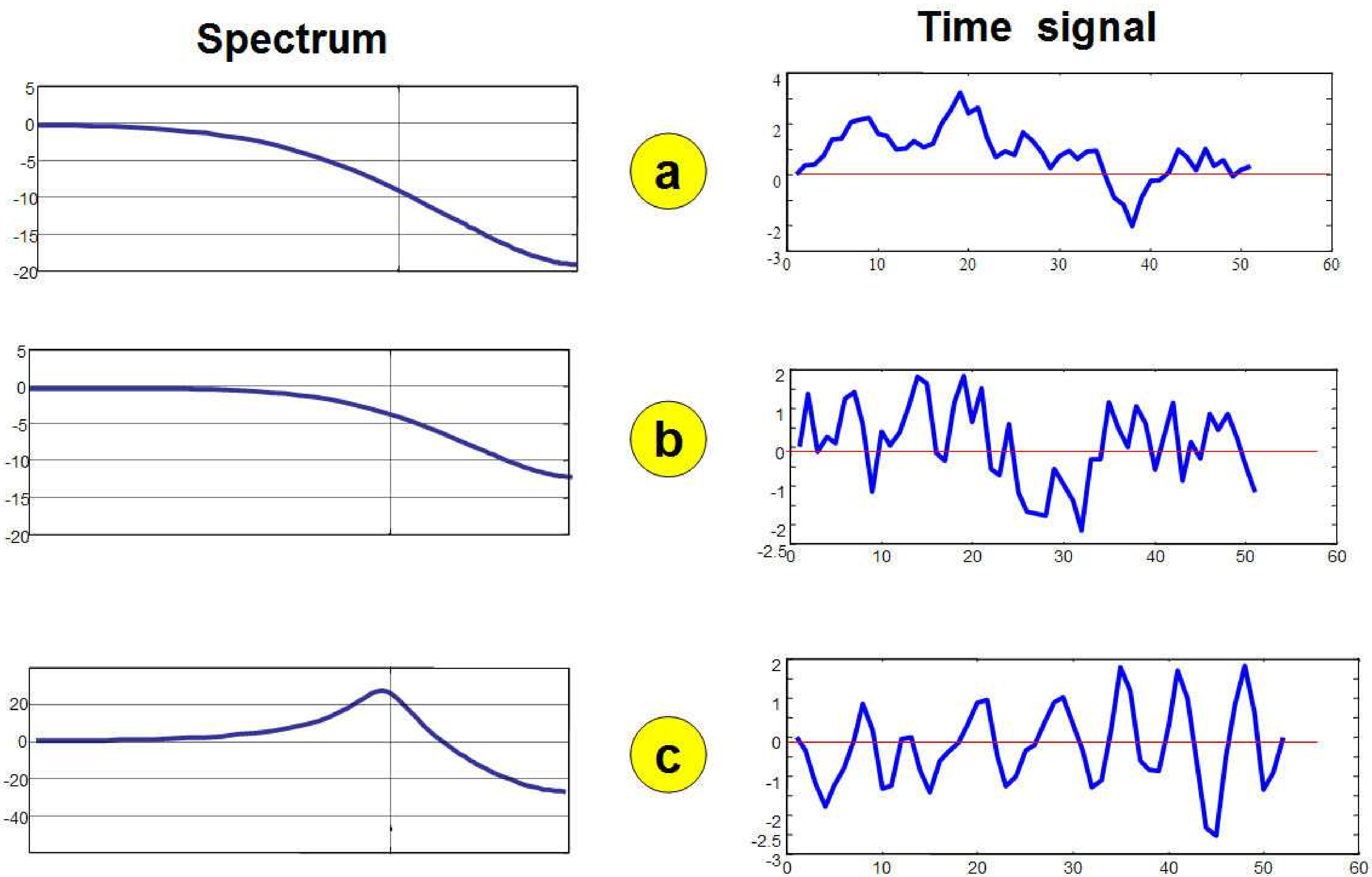
$$\begin{aligned} r_{xx}(s,t) &= \operatorname{cov}\left(x(s), x(t)\right) = \mathrm{E}\left((x(s)-m(s))(x(t)-m(t))^T\right) \\ &= \int\int (\xi_1 - m(s))(\xi_2 - m(t))^T dF(\xi_1, \xi_2; s, t) \end{aligned}$$

Process variance is $r_x(0)$ (how large fluctuations are).

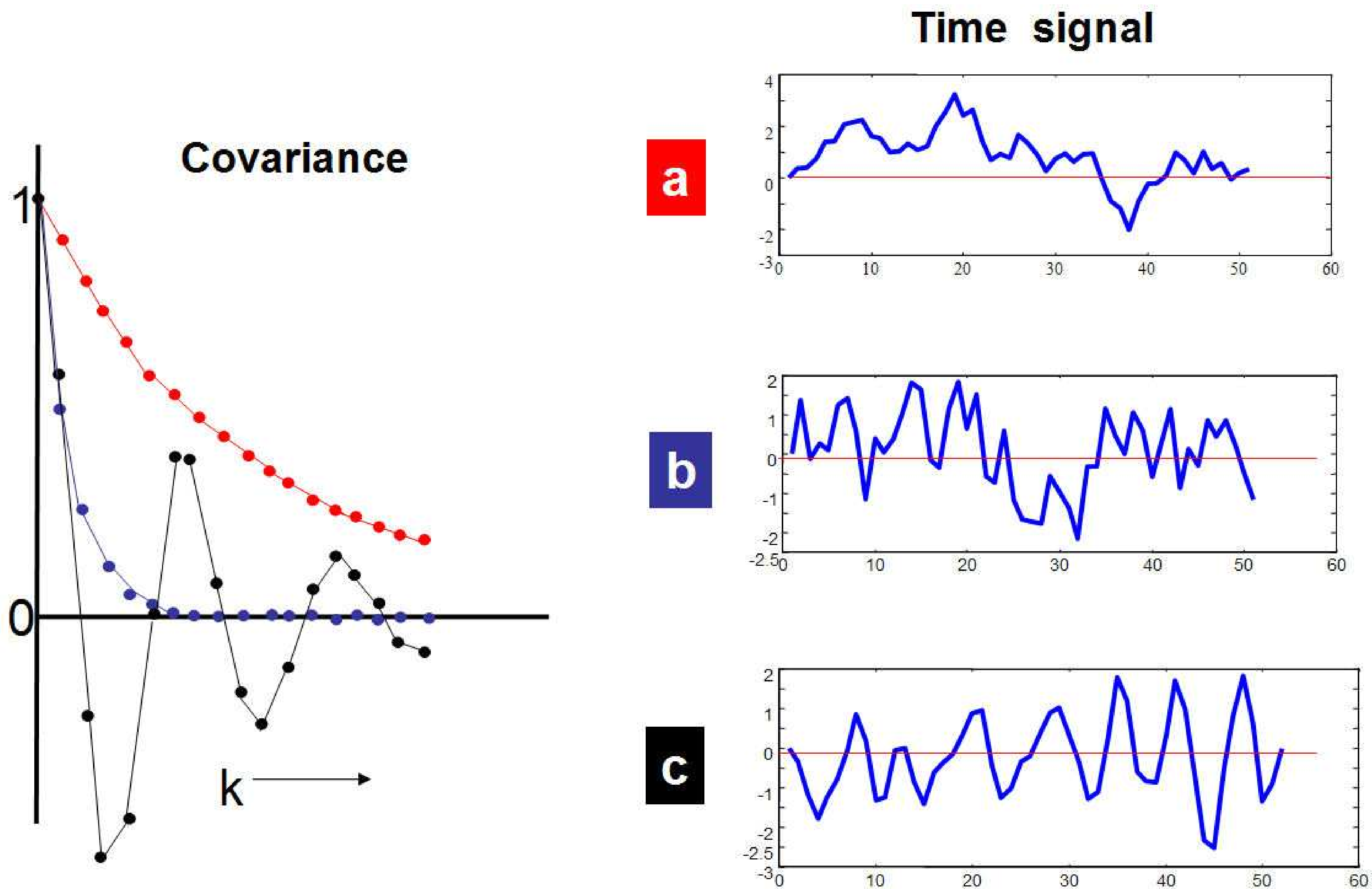
Spectral density:

$$\phi_x(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r_x(k) e^{-jk\omega}$$

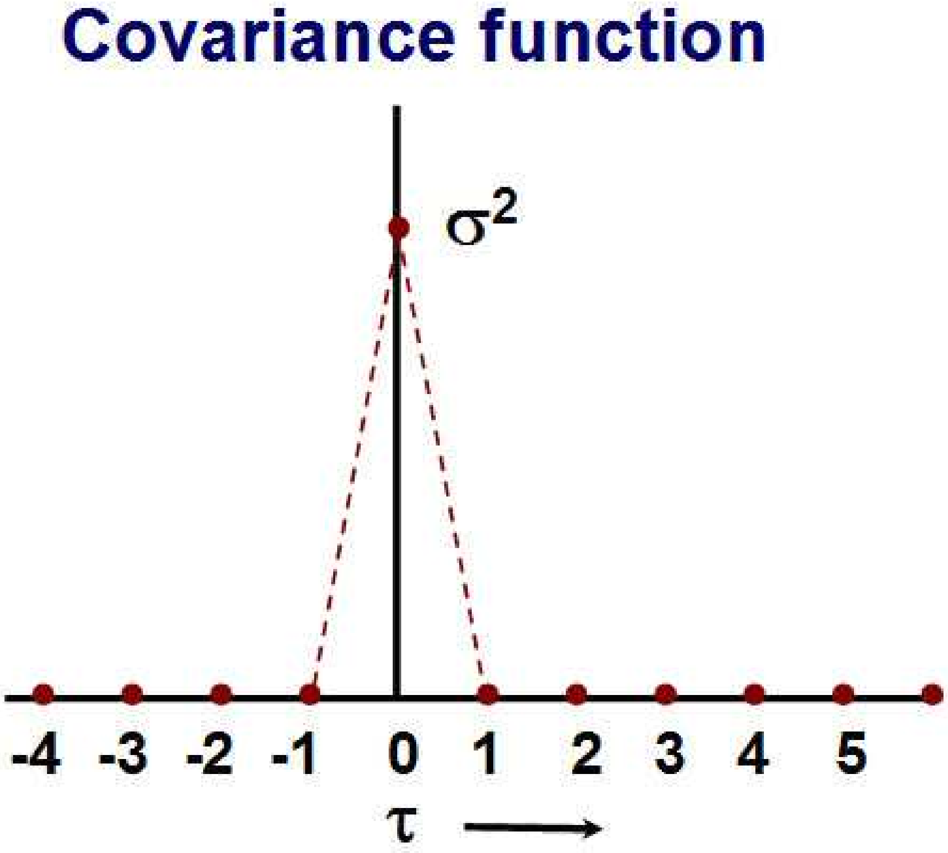
Interpretation of spectrum



Interpretation of covariance

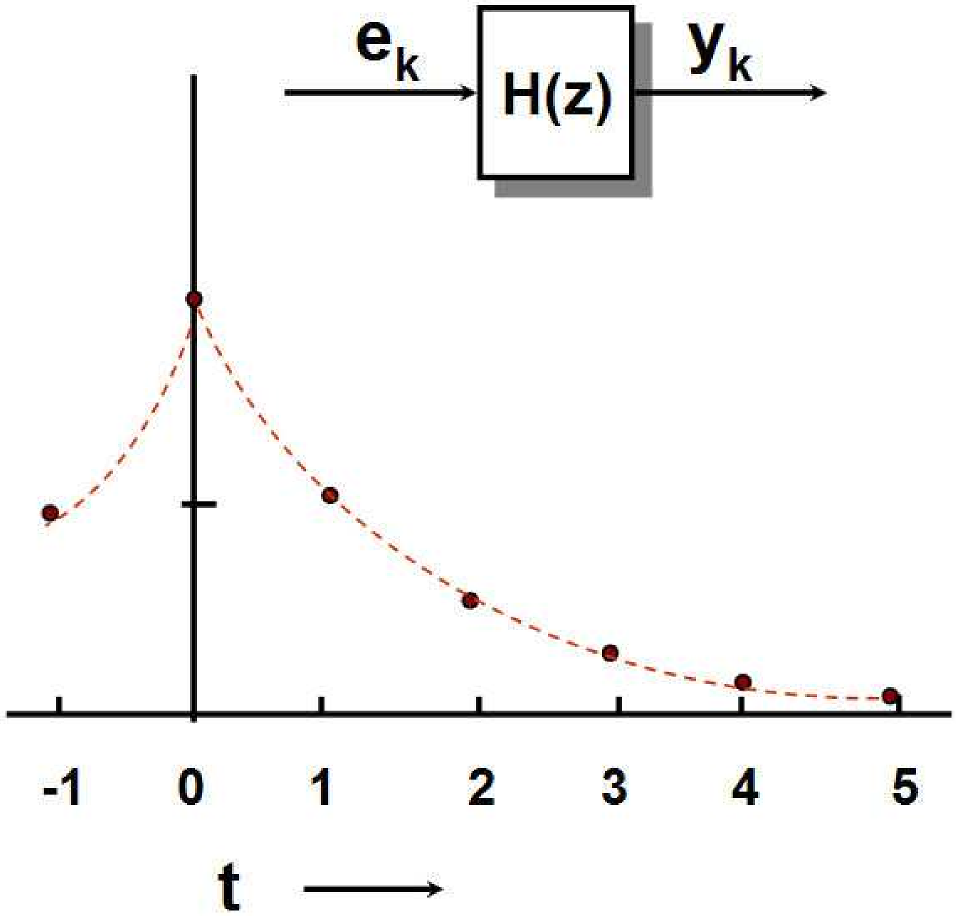


Discrete-time white noise



$$\phi(\omega) = \frac{\sigma^2}{2\pi}$$

Stochastic disturbance model



$$H(z) = \frac{1}{z-a}$$

$$\Downarrow$$

$$y(k+1) = ay(k) + e(k)$$

$$\Downarrow$$

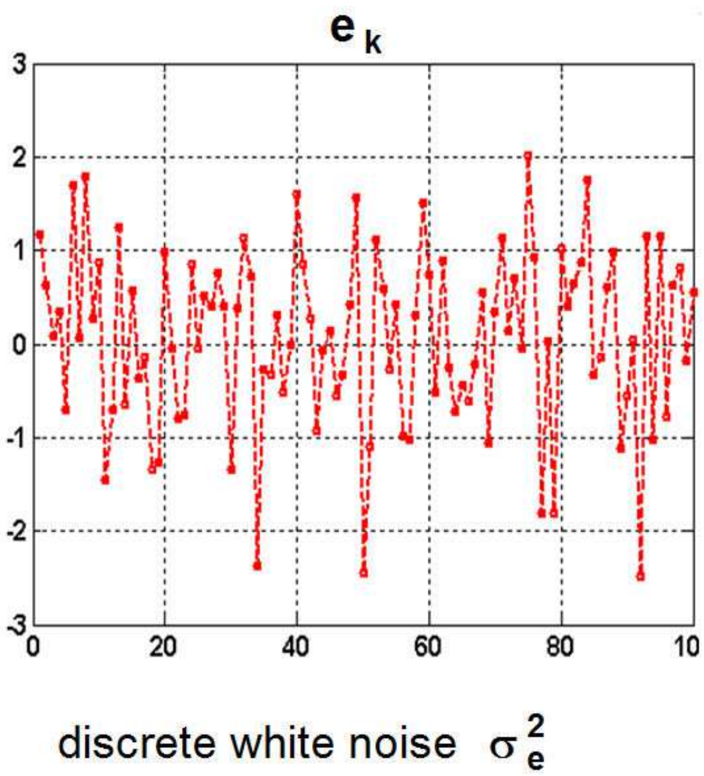
$$\sigma_y^2 = \frac{1}{1-a^2}\sigma_e^2$$

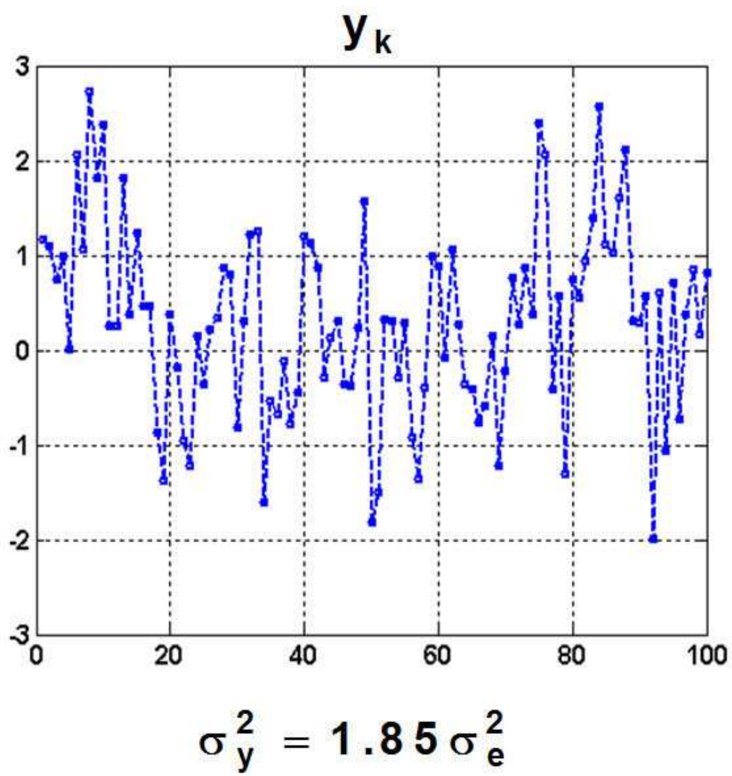
$$\Downarrow$$

$$r_y(\tau) = \frac{1}{1-a^2}a^{|\tau|}$$

Calculation of variances - Example 1

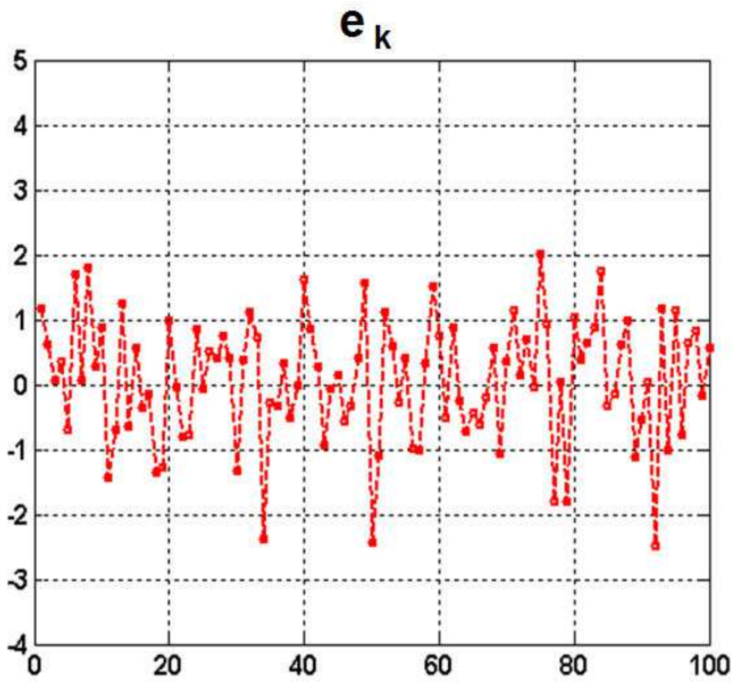
$$H(q) = \frac{q - 0.5}{q - 0.9} \quad \text{with } \sigma_e^2 = 1$$



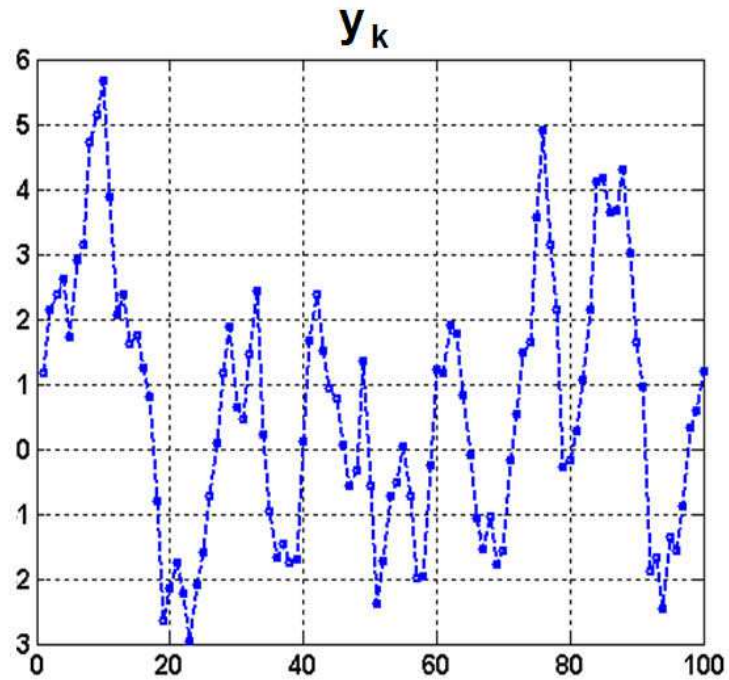


Calculation of variances - Example 2

$$H(q) = \frac{1}{q^2 - 1.3q + 0.4} \quad \text{with } \sigma_e^2 = 1$$



discrete white noise σ_e^2



$\sigma_y^2 = 8.6\sigma_e^2$

ARMA processes

Moving Average:

$$y(k) = e(k) + b_1 e(k-1) + \cdots + b_n e(k-n)$$

Auto Regressive:

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = e(k)$$

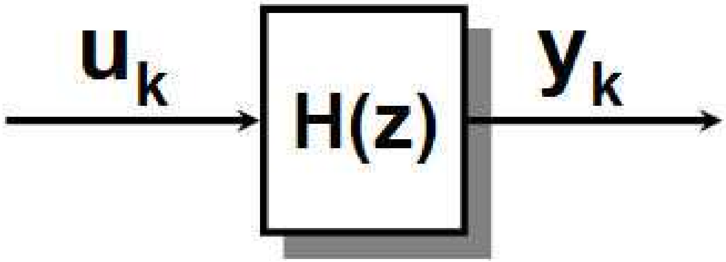
ARMA:

$$y(k)+a_1y(k-1)+\cdots+a_ny(k-n) = e(k)+b_1e(k-1)+\cdots+b_ne(k-n)$$

ARMAX:

$$\begin{aligned} y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) &= b_0 u(k-d) + \cdots + b_m u(k-d-m) \\ &+ e(k) + c_1 e(k-1) + \cdots + c_n e(k-n) \end{aligned}$$

Spectral densities of filtered signals



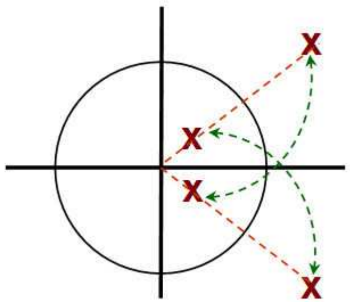
$$\begin{aligned} \phi_{yu}(\omega) &= H(e^{j\omega})\phi_u(\omega) \\ \phi_y(\omega) &= H(e^{j\omega})\phi_u(\omega)H^T(e^{-j\omega}) \end{aligned}$$

Spectral factorization

Given a spectral density $\phi(\omega)$, what is the linear system that gives this as an output, when driven by white noise?

$$F(z) = \frac{1}{2\pi} H(z) H^T(z^{-1})$$

The poles and zeros of $F(z)$ come in pairs:



$$\begin{aligned} z_i z_j &= 1 \\ p_i p_j &= 1 \end{aligned}$$

$$H(z) = K \frac{\prod_i (z - z_i)}{\prod_i (z - p_i)} = \frac{B(z)}{A(z)}, \quad |z_i| < 1, |p_i| < 1$$

All stationary random processes can be thought of as: generated by stable linear systems driven by white noise (special ARMA processes)