Control Engineering (SC42095)

Lecture 5, 2020

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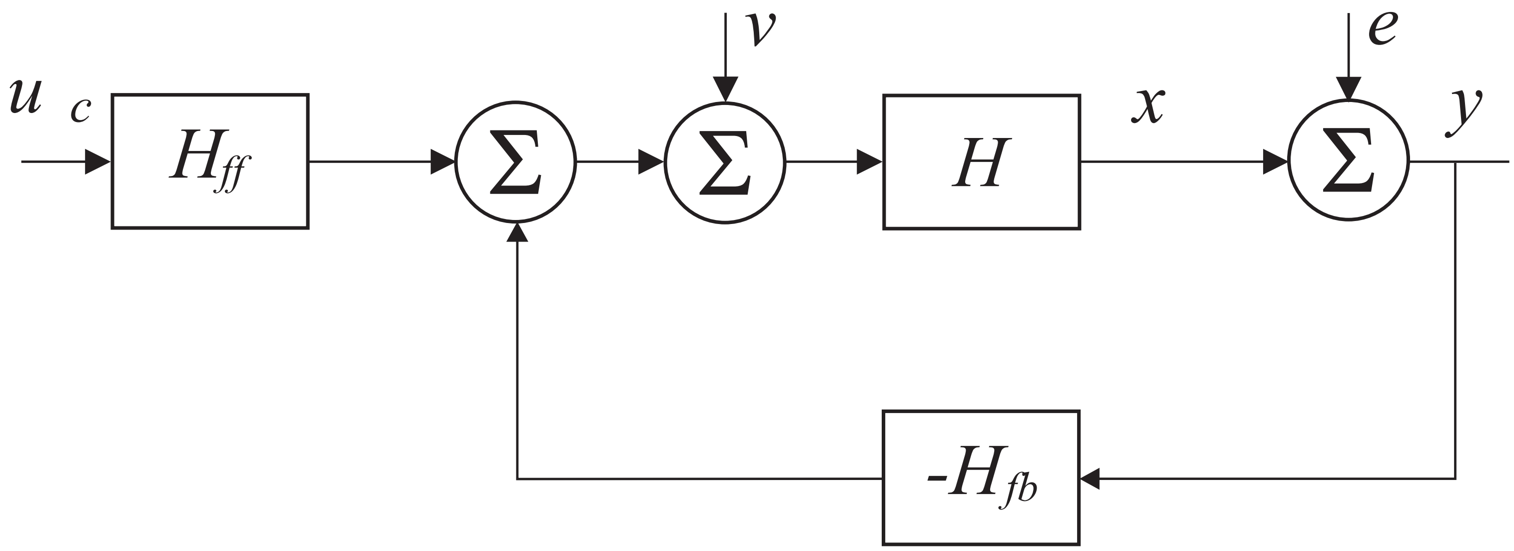
Lecture outline

- State-feedback control
- State estimation – observers
- Output-feedback control

Taxonomy of Controllers

- Presence of feedback: feedforward, feedback, 2-DOF
- Type of feedback: output, state
- Presence of dynamics: static, dynamic
- Dependence on time: fixed, adaptive
- Use of models: model-free, model-based

Control design



- Attenuation of load disturbances (regulation problem).
- Effect of measurement noise.
- Reference tracking (servo problem).
- Uncertain dynamics / process parameters (robust control).
  Section 3.3 in CCS book on sensitivity and robustness.

State feedback: problem formulation

- $Model$: $x(k+1) = \Phi x(k) + \Gamma u(k)$
- $Admissible\ controls$: linear controllers

$$u(k) = -Lx(k)$$

- $Disturbances$: widely spread pulses ($x(0) = x_0$)
- $Criterion$: $x(k) \to 0$ reasonably quickly with reasonable inputs u (choose closed-loop poles)

State feedback: problem formulation

- *Design parameters*: closed-loop poles, sampling interval
- *Evaluation*: compare $x(k)$ and $u(k)$ with specifications
  (trade-off between control magnitude and speed of response)

More complicated (and realistic) problems later . . .

Example - double integrator

$$x(k+1) = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} h^2/2 \\ h \end{pmatrix} u(k)$$

Linear state-feedback controller

$$u(k) = -Lx(k) = -l_1 x_1(k) - l_2 x_2(k)$$

Closed-loop system becomes

$$x(k+1) = (\Phi - \Gamma L)x(k) = \begin{pmatrix} 1 - l_1 h^2/2 & h - l_2 h^2/2 \\ -l_1 h & 1 - l_2 h \end{pmatrix} x(k)$$

Double integrator - cont'd

Desired characteristic equation: $z^2 + p_1 z + p_2 = 0$

Closed-loop characteristic equation:

$$z^2 + \left(\frac{l_1 h^2}{2} + l_2 h - 2\right) z + \left(\frac{l_1 h^2}{2} - l_2 h + 1\right) = 0$$

Linear equations for l_1 and l_2:

$$\begin{aligned} \frac{l_1 h^2}{2} + l_2 h - 2 &= p_1 \\ \frac{l_1 h^2}{2} - l_2 h + 1 &= p_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} l_1 &= \tfrac{1}{h^2}(1 + p_1 + p_2) \\ l_2 &= \tfrac{1}{2h}(3 + p_1 - p_2) \end{aligned}$$

Unique solution (in this example), L depends on h

Solution in the general SISO case

Transform $x(k+1) = \Phi x(k) + \Gamma u(k)$ into the reachable canonical form: $z(k+1) = \tilde{\Phi} z(k) + \tilde{\Gamma} u(k)$ with:

$$\tilde{\Phi} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \qquad \tilde{\Gamma} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Coefficients of the characteristic polynomial of $\tilde{\Phi} - \tilde{\Gamma}\tilde{L}$:

$$\left(-a_1 - \tilde{l}_1 \quad - a_2 - \tilde{l}_2 \quad \dots \quad - a_n - \tilde{l}_n\right) \quad \Rightarrow \quad \tilde{L}$$

Transform back: $u = -\tilde{L} z = -\tilde{L} T x = -L x$

How to find the transformation matrix T

Recall the reachability matrix for a transformed system:

$$\begin{aligned}
\tilde{W}_c &= \begin{pmatrix} \tilde{\Gamma} & \tilde{\Phi}\tilde{\Gamma} & \dots & \tilde{\Phi}^{n-1}\tilde{\Gamma} \end{pmatrix} \\
&= \begin{pmatrix} T\Gamma & T\Phi T^{-1}T\Gamma & \dots & T\Phi^{n-1}T^{-1}T\Gamma \end{pmatrix} \\
&= TW_c
\end{aligned}$$

Can be solved for T if the system is reachable: $T = \tilde{W}_c W_c^{-1}$

Ackermann's formula

After some derivations (book on page 127) ...

$$L = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix} W_c^{-1} P(\Phi)$$

where $P(\Phi)$ is the desired characteristic polynomial (in Φ!)

in Matlab:

```
L = acker(Phi,Gamma,Po)
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(SISO, numerical problems)

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L = place(Phi,Gamma,Po)
```
(MISO, more robust)

Example I - double integrator

Desired characteristic polynomial: $P(z) = z^2 + p_1 z + p_2$

Ackermann's formula: $L = \begin{pmatrix} 0 & 1 \end{pmatrix} W_c^{-1} P(\Phi)$

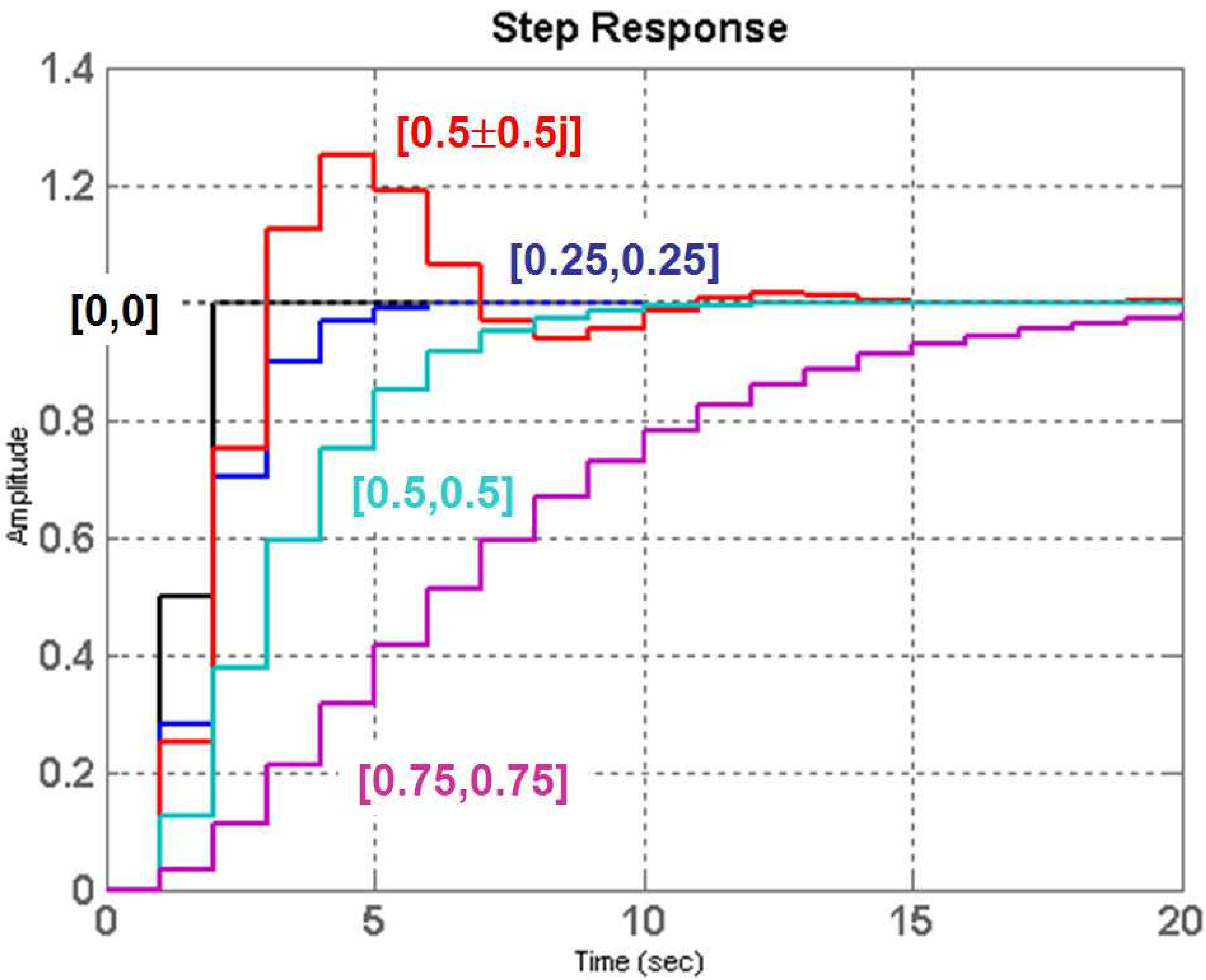
$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma \end{pmatrix} = \begin{pmatrix} h^2/2 & 3h^2/2 \\ h & h \end{pmatrix}, \quad W_c^{-1} = \begin{pmatrix} -1/h^2 & 1.5/h \\ 1/h^2 & -0.5/h \end{pmatrix}$$

$$P(\Phi) = \Phi^2 + p_1\Phi + p_2 I = \begin{pmatrix} 1 + p_1 + p_2 & 2h + p_1 h \\ 0 & 1 + p_1 + p_2 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 1 \end{pmatrix} W_c^{-1} P(\Phi) = \begin{pmatrix} 1/h^2 & -0.5/h \end{pmatrix} P(\Phi)$$

$$= \begin{pmatrix} \dfrac{1 + p_1 + p_2}{h^2} & \dfrac{3 + p_1 - p_2}{2h} \end{pmatrix}$$

Example I - double integrator



Example II – an unreachable system

$$x(k+1) = \begin{pmatrix} 0.5 & 1 \\ 0 & 0.3 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k)$$

not reachable because $\det W_c = \det \begin{pmatrix} 1 & 0.5 \\ 0 & 0 \end{pmatrix} = 0$

Control law $u(k) = -l_1 x_1(k) - l_2 x_2(k)$ gives:

$$\det(zI - \Phi + \Gamma L) = 0$$

$$(z - 0.5 + l_1)(z - 0.3) = 0$$

How to place the poles?

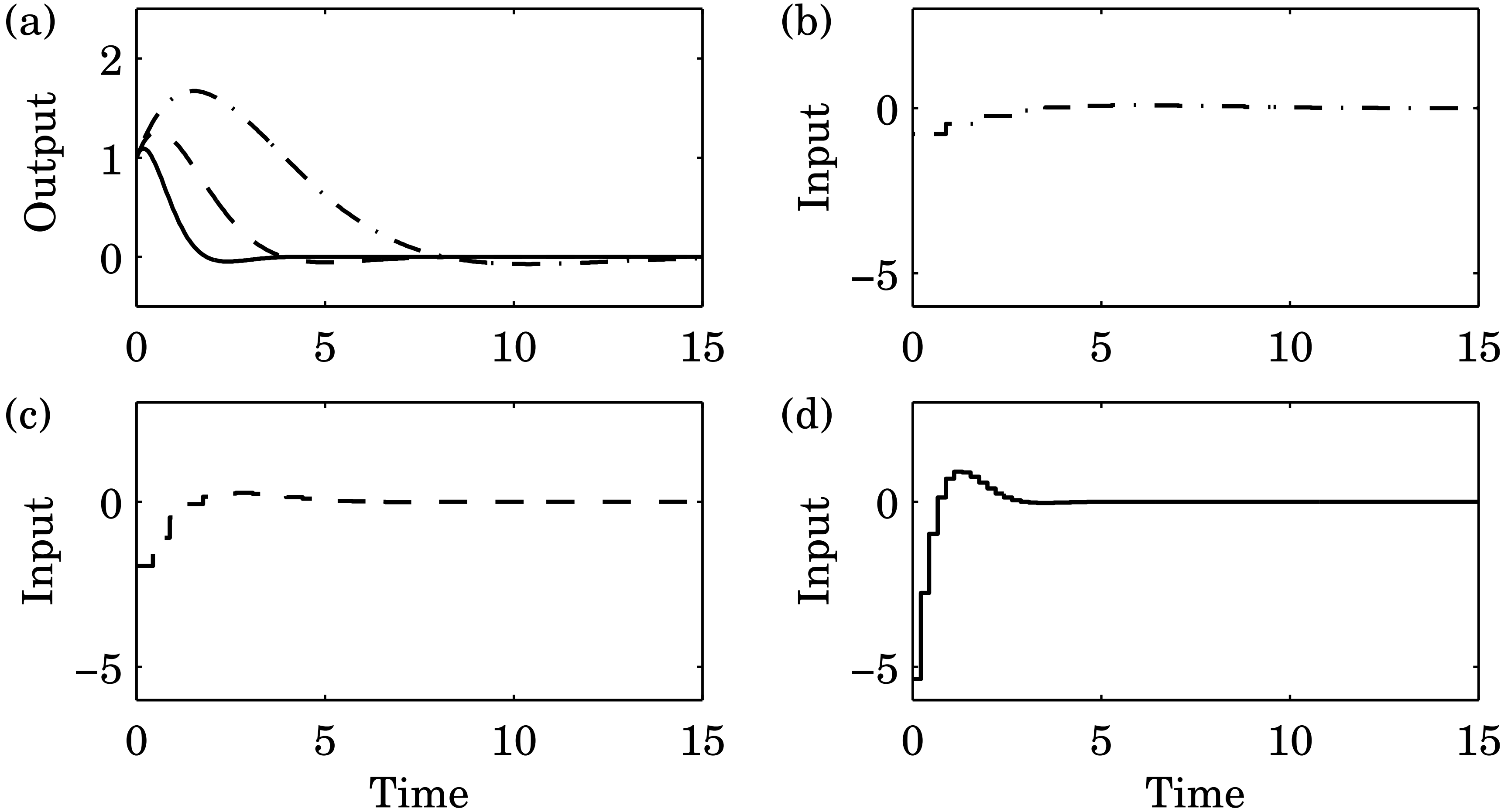
Using the continuous-time counterpart (2nd order)

$$s^2 + 2\zeta\omega s + \omega^2$$

gives $z^2 + p_1 z + p_2$ with

$$\begin{aligned} p_1 &= -2e^{-\zeta\omega h}\cos\left(\omega h\sqrt{1-\zeta^2}\right) \\ p_2 &= e^{-2\zeta\omega h} \end{aligned}$$

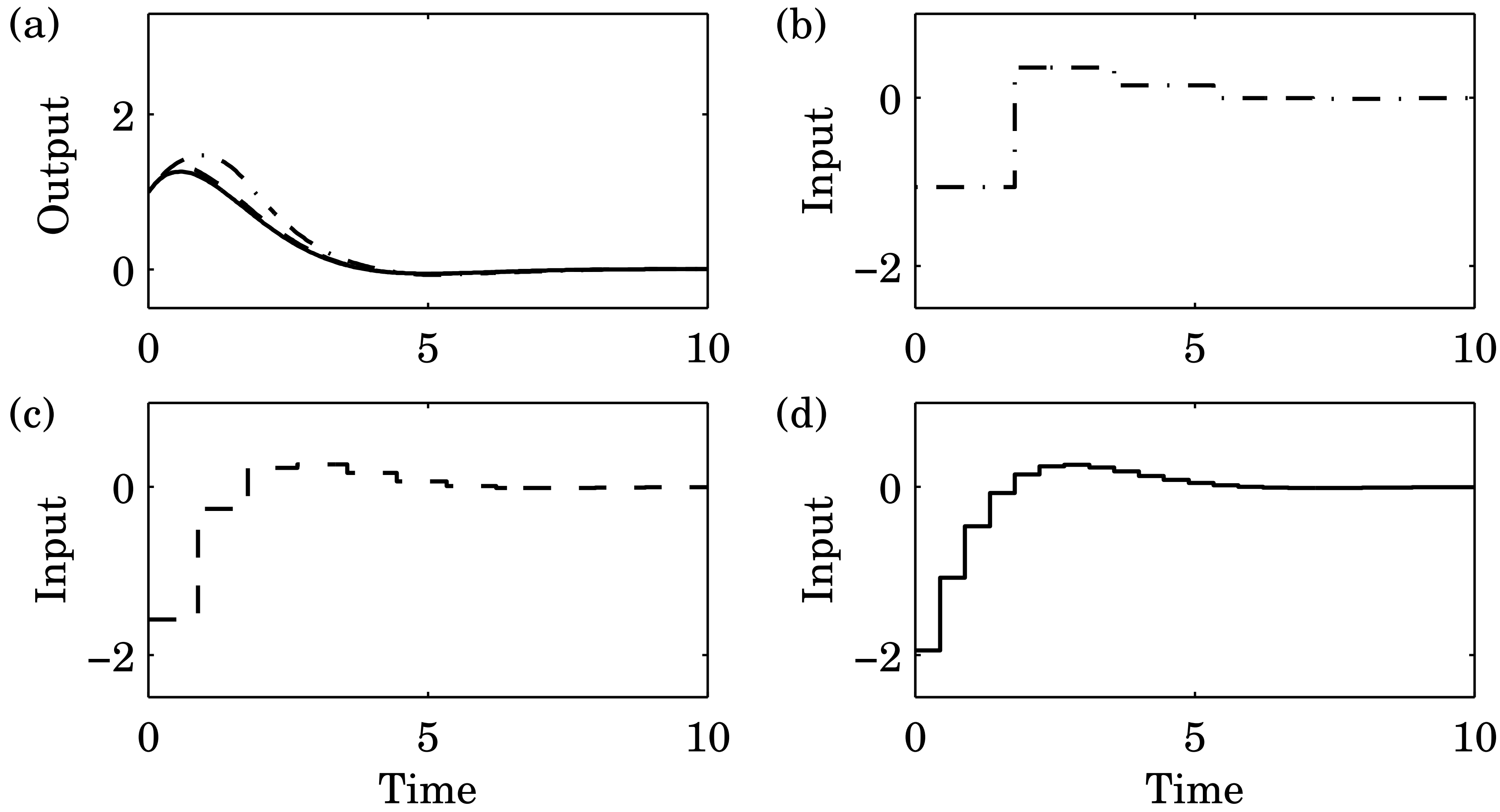
Response for different ω



$x(0) = [1 \quad 1]^T$ b) $\omega = 0.5$, c) $\omega = 1$, d) $\omega = 2$

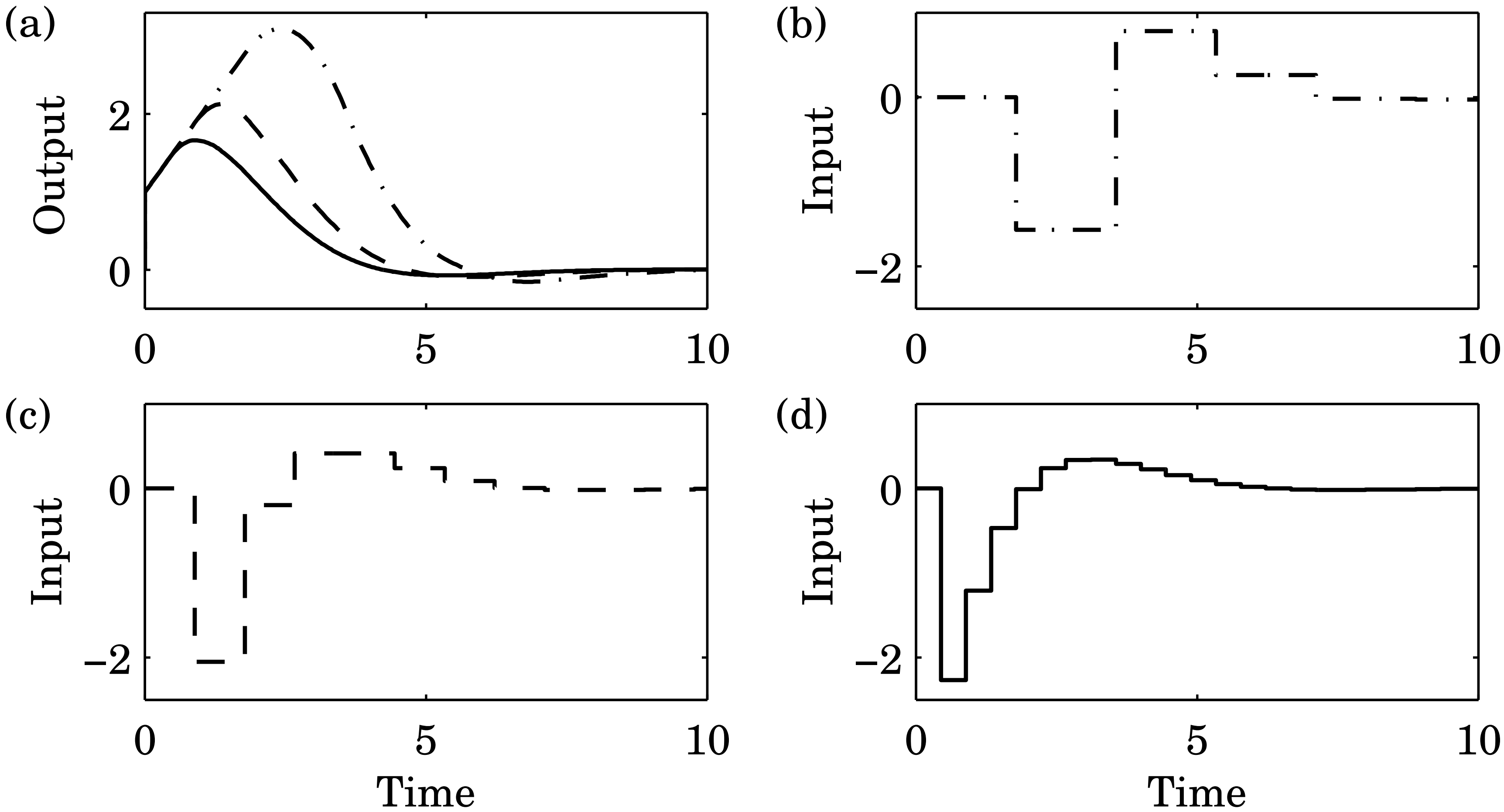
Response for different sampling times

Change of the number of samples within $2\pi/\omega_0$



$x(0) = [1 \quad 1]^T$ b) $N = 5$, c) $N = 10$, d) $N = 20$

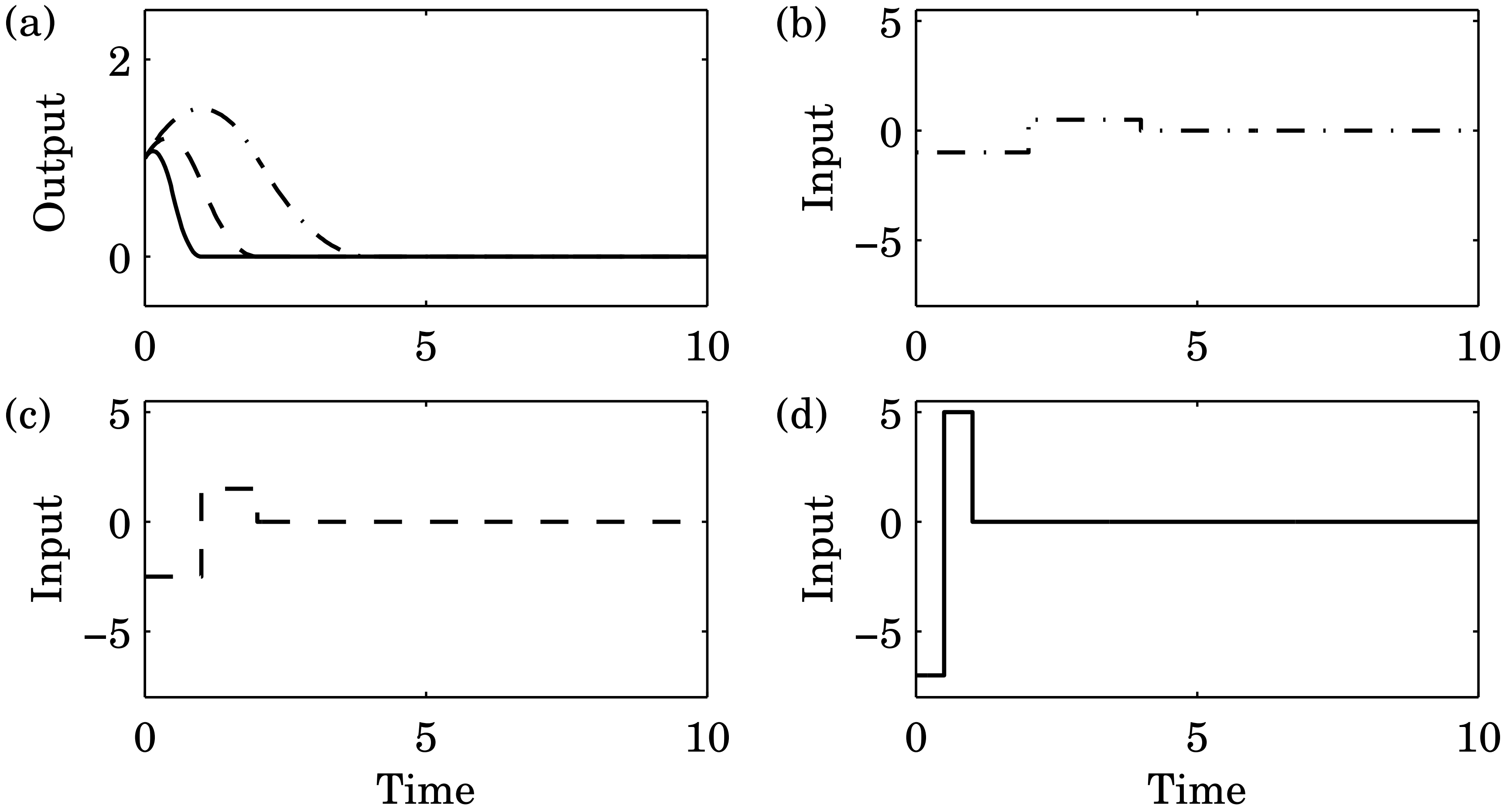
Another initial condition



$x^T(0^+) = [1 \quad 1]$ b) $N = 5$, c) $N = 10$, d) $N = 20$

Deadbeat control

Choose $P(z) = z^n$ (the remaining design parameter is h)



b) $h = 2$, c) $h = 1$, d) $h = 0.5$

State estimation: Observers

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) \\
y(k) &= C x(k)
\end{aligned}
$$

Input and output available, reconstruct the state:

- Direct calculation (requires differentiation)
- Luenberger observer (model-based)
- Kalman filter (optimal in presence of noise)

the terms "observer", "estimator", "filter" are in this context used synonymously

Direct calculation of state variables

Starting from initial state $x(k-n+1)$ and applying the control sequence $U(k-1) = (u(k-n+1) \;\; u(k-n+2) \;\; \cdots \;\; u(k-1))^T$, the resulting output sequence $Y(k) = (y(k-n+1) \;\; y(k-n+2) \;\; \cdots \;\; y(k))^T$ can be expressed as

$$Y(k) = W_o x(k-n+1) + W_u U(k-1)$$

$$W_o = \begin{pmatrix} C \\ C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^{n-1} \end{pmatrix} \qquad W_u = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ C\Gamma & 0 & \cdots & 0 \\ C\Phi\Gamma & C\Gamma & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C\Phi^{n-2}\Gamma & C\Phi^{n-3}\Gamma & \cdots & C\Gamma \end{pmatrix}$$

Direct calculation of state variables

$$Y(k) = W_o x(k-n+1) + W_u U(k-1)$$

If the states are observable then

$$x(k-n+1) = W_o^{-1} Y(k) - W_o^{-1} W_u U(k-1)$$

and propagating the past states leads to

$$\begin{aligned} x(k) &= \Phi^{n-1} x(k-n+1) + \Phi^{n-2} \Gamma u(k-n+1) + \cdots + \Gamma u(k-1) \\ x(k) &= A_y Y(k) + B_u U(k-1) \end{aligned}$$

where

$$A_y = \Phi^{n-1} W_o^{-1} \qquad B_u = \begin{pmatrix} \Phi^{n-2}\Gamma & \Phi^{n-3}\Gamma & \cdots & \Gamma \end{pmatrix} - \Phi^{n-1} W_o^{-1} W_u$$

Direct calculation of state variables – example

$$\Phi = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \frac{h^2}{2} \\ h \end{pmatrix}, \quad C = (1 \quad 0)$$

Observer

$$\begin{aligned}
y(k) &= x_1(k) \\
y(k) &= x_1(k-1) + hx_2(k-1) + \frac{h^2}{2}u(k-1) \\
&= y(k-1) + h(x_2(k) - hu(k)) + \frac{h^2}{2}u(k-1) \\
&\Downarrow \\
x_1(k) &= y(k) \\
x_2(k) &= \frac{y(k) - y(k-1)}{h} + \frac{h}{2}u(k-1)
\end{aligned}$$

requires differentiation

State reconstruction based on a model

Consider the process:

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) \end{aligned}$$

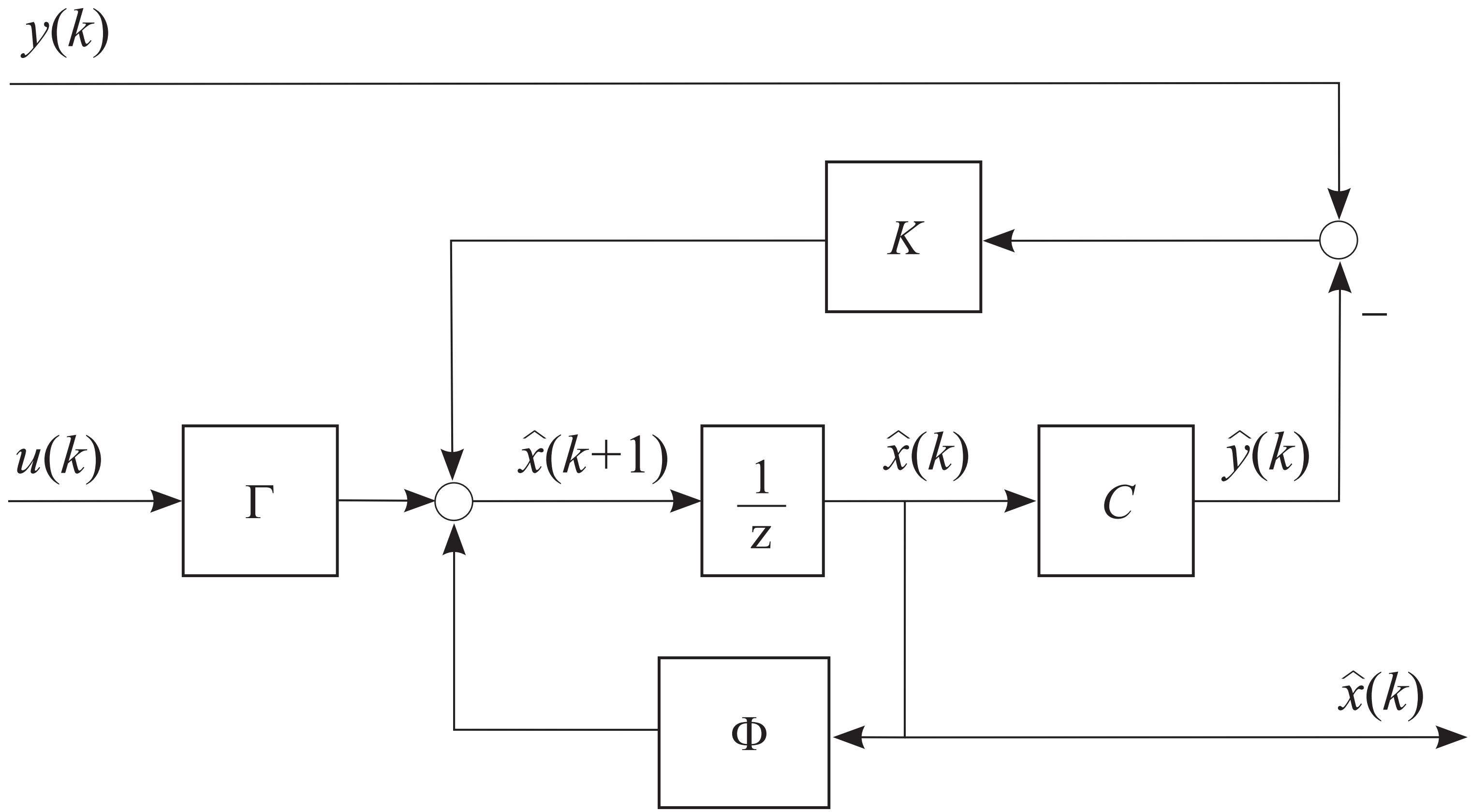
and its model:

$$\begin{aligned} \hat{x}(k+1) &= \Phi \hat{x}(k) + \Gamma u(k) \\ \hat{y}(k) &= C\hat{x}(k) \end{aligned}$$

Introduce “feedback” from measured $y(k)$:

$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + K[y(k) - C\hat{x}(k)]$$

Observer



$$
\begin{aligned}
\hat{x}(k+1) &= \Phi\hat{x}(k) + \Gamma u(k) + K\Big[\,y(k) - \hat{y}(k)\,\Big] \\
\hat{y}(k) &= C\hat{x}(k)
\end{aligned}
$$

Design of observer gain

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ \hat{x}(k+1) &= \Phi \hat{x}(k) + \Gamma u(k) + K[y(k) - C\hat{x}(k)] \end{aligned}$$

Estimation error $e = x - \hat{x}$

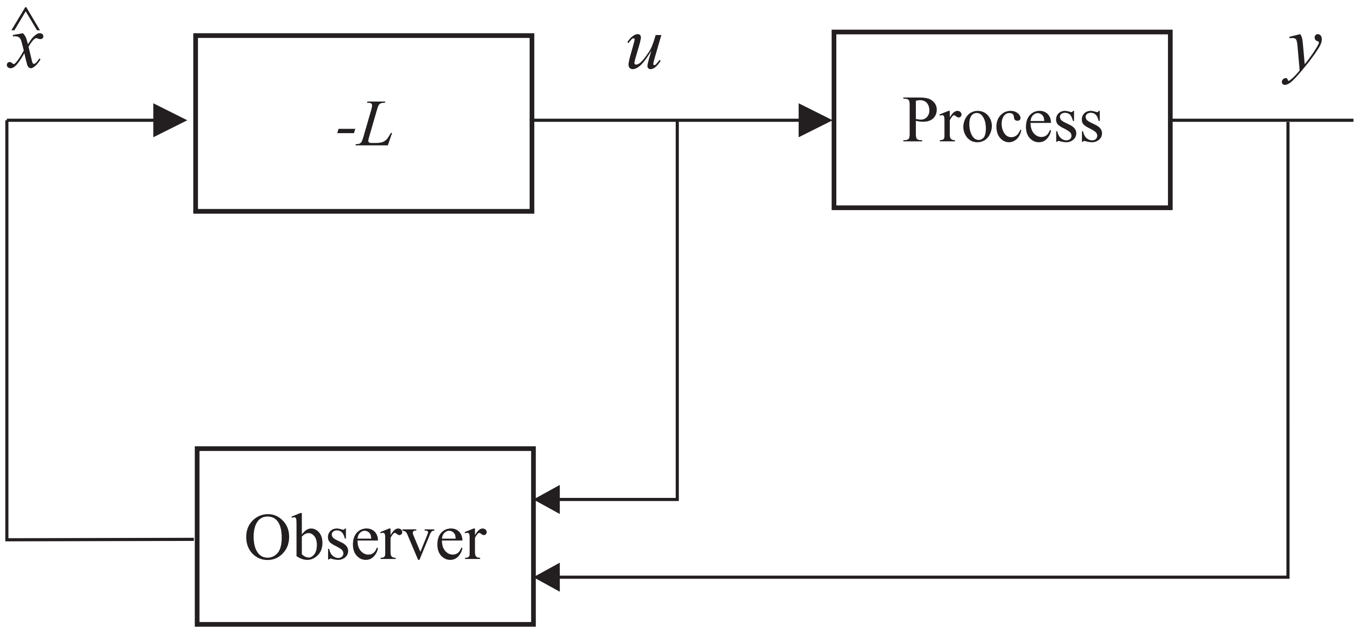
$$e(k+1) = \Phi e(k) - KCe(k) = [\Phi - KC]e(k)$$

- Choose K such that $e(k)$ goes to zero
- Any eigenvalues possible, provided W_o has full rank
- Dual problem to state-feedback design (transpose)

Various types of observers

- Deadbeat observer, see deadbeat controller.
- Observer without delay (estimates $\hat{x}(k \mid k)$ instead of $\hat{x}(k \mid k-1)$).
- Reduced-order observer, MIMO.
- Kalman filter (stochastic disturbances).

Output feedback (observer + state feedback)



$$\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k) + K\Big(y(k) - C\hat{x}(k)\Big)$$

$$u(k) = -L\hat{x}(k)$$

Output feedback controller

$$
\begin{aligned}
\hat{x}(k+1) &= \Phi\hat{x}(k) + \Gamma u(k) + K(y(k) - C\hat{x}(k)) \\
u(k) &= -L\hat{x}(k)
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\hat{x}(k+1) &= (\Phi - KC - \Gamma L)\hat{x}(k) + Ky(k) \\
u(k) &= -L\hat{x}(k)
\end{aligned}
}
$$

Transfer function of the output-feedback controller:

$$
G_r(z) = \frac{U(z)}{Y(z)} = -L(zI - \Phi + \Gamma L + KC)^{-1}K
$$

Poles of the closed-loop system

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ e(k+1) &= x(k+1) - \hat{x}(k+1) = (\Phi - KC)e(k) \\ u(k) &= -L(x(k) - e(k)) \end{aligned}$$

Eliminate $u(k)$:

$$\begin{pmatrix} x(k+1) \\ e(k+1) \end{pmatrix} = \begin{pmatrix} \Phi - \Gamma L & \Gamma L \\ 0 & \Phi - KC \end{pmatrix} \begin{pmatrix} x(k) \\ e(k) \end{pmatrix}$$

Process poles: $A_r(z) = \det(zI - \Phi + \Gamma L)$
Observer poles: $A_o(z) = \det(zI - \Phi + KC)$

Separation principle

Summary

- State-feedback control
- Ackermann's formula
- Choosing closed-loop poles
- Deadbeat control
- Observers and output-feedback control