Control Engineering (SC42095)

Lecture 2, 2020

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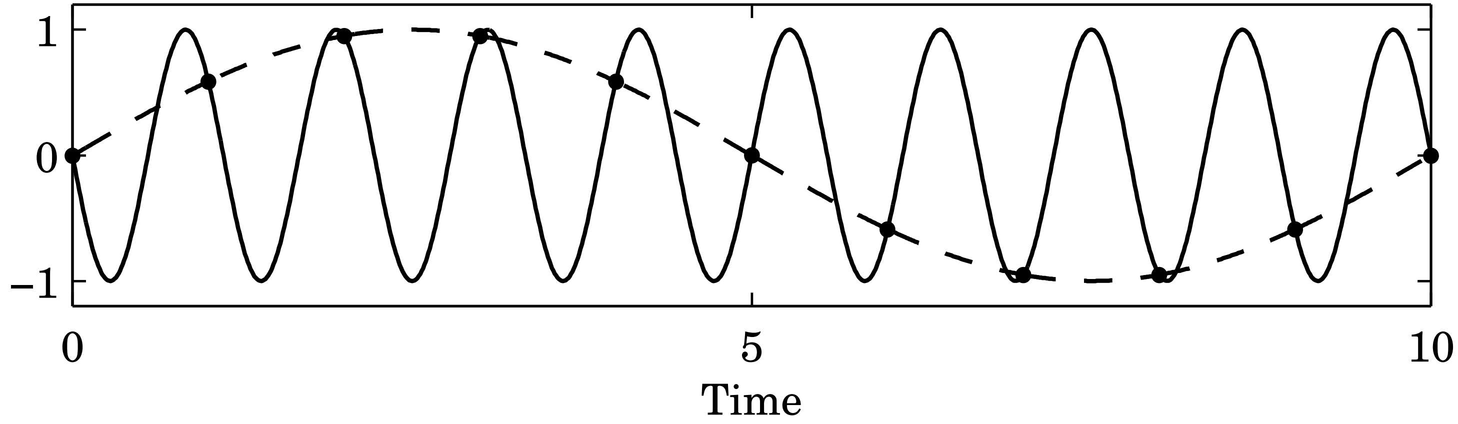
Lecture outline

- Shannon's sampling theorem, frequency folding.
- Sampling of continuous-time systems.
- State-space models.
- Input-output models.
- Shift-operator calculus.
- Pulse-transfer function.
- Poles and zeros.

Aliasing (recap)



Sampling of two or more harmonic signals with *different* frequencies may result in *the same* discrete time signal. This is called aliasing.

$$\omega_{sampled} = |\omega \pm n\omega_s|, \quad n \text{ integer}$$

with

ω ... frequency of continuous-time signal

$\omega_s = 2\pi/h$ ... sampling frequency

Aliasing – cont'd

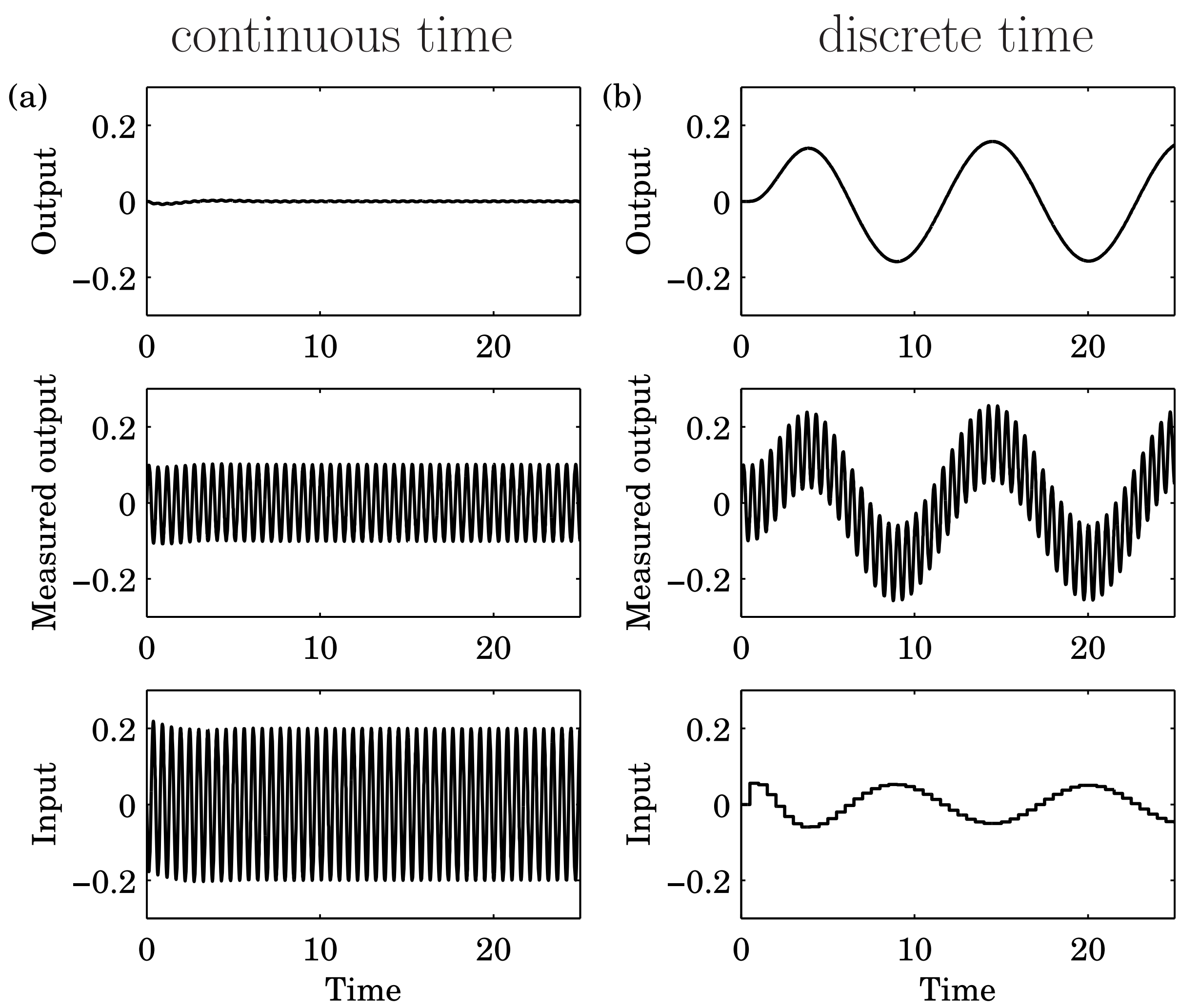
Proof:

$$e^{j2\pi(f+n/h)t} = e^{j2\pi ft} \cdot e^{j2\pi nt/h} = e^{j2\pi ft}$$

as nt/h is an integer for $t \in \{0, h, 2h, 3h, \ldots\}$.

Sampling of measurement noise

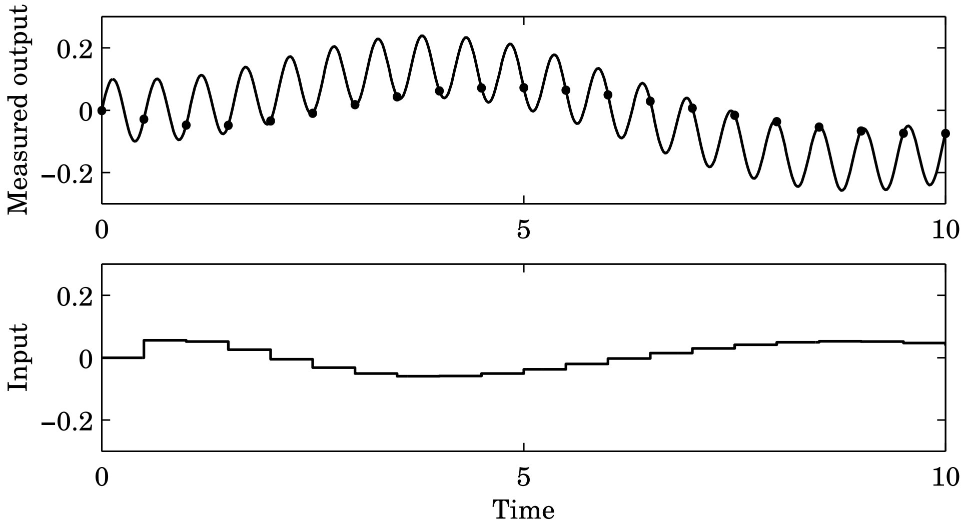
Double integrator, $h = 0.5$, measurement noise $0.1\sin(12t)$



Sampling creates signals with new frequencies!

Measurement noise - cont'd

Zoom in:



$$\omega_{sampled} = |\omega \pm n\omega_s|$$

$\omega = 12$ rad/s

$\omega_s = 2\pi/h = 4\pi = 12.57$ rad/s

$\omega_{sampled} = 0.57$ rad/s $\longrightarrow$ period of 11 s

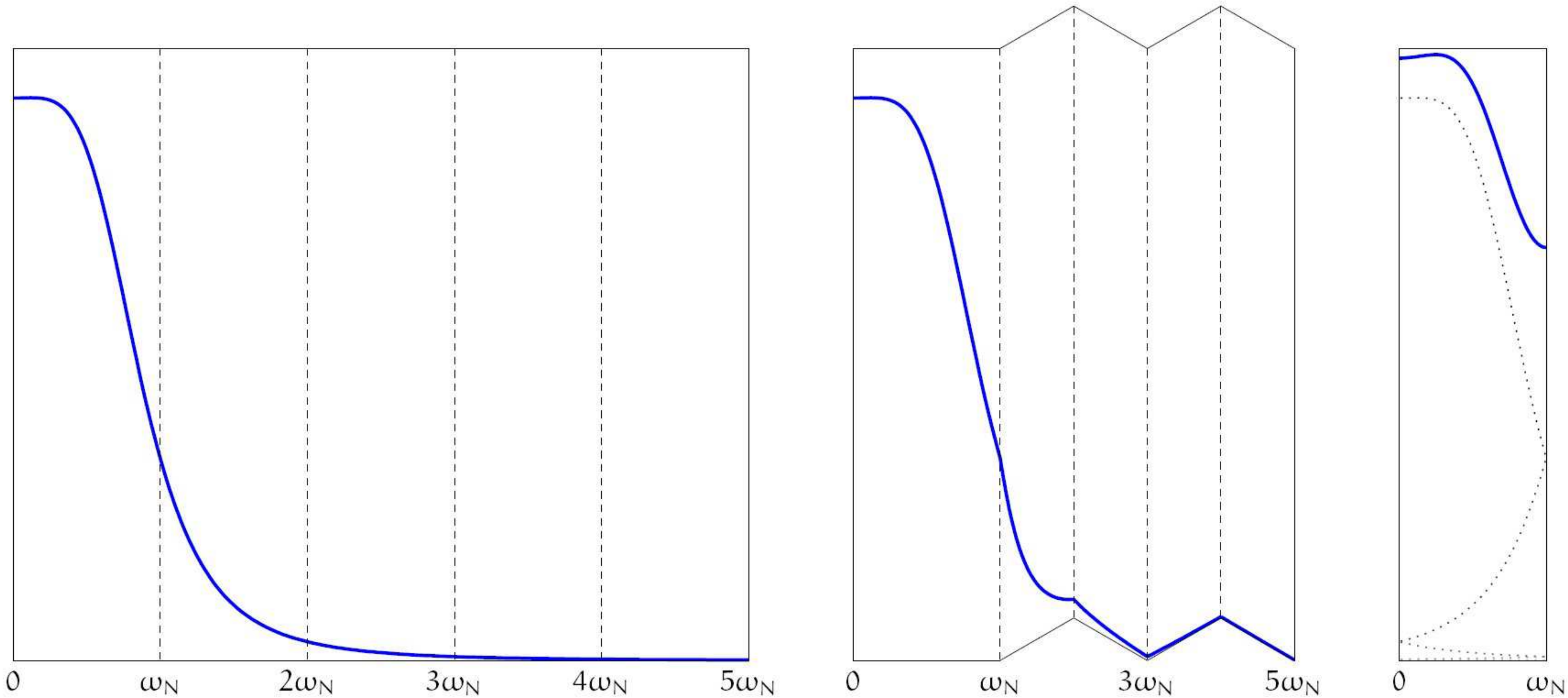
The sampling period is too long with respect to the noise.
It is important to filter before sampling!

Shannon's sampling theorem

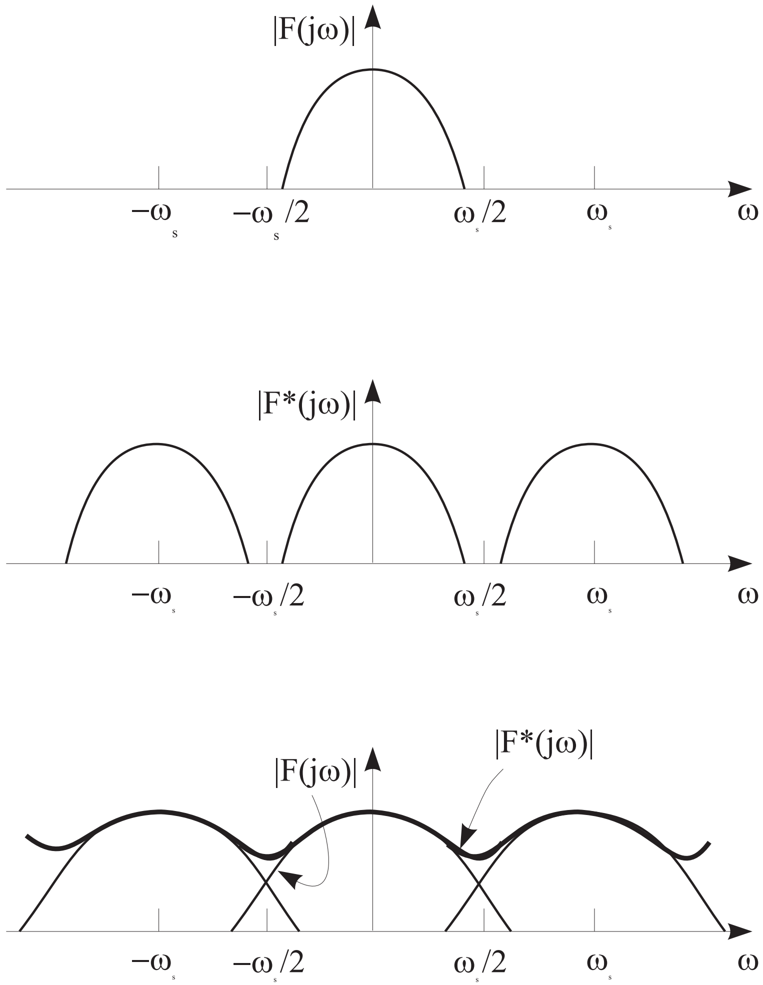
A continuous-time signal which contains no frequency components greater than ω_0 is uniquely determined by its values in equidistant points if the sampling frequency ω_s is higher than $2\omega_0$.

$\omega_N = \omega_s/2$ is called the *Nyquist frequency*.

Frequency Folding



Frequency Folding



$$|F^*(j\omega)| = \frac{1}{h}\sum_{k\in\mathbb{Z}} |F(j\omega + j\omega_s k)|$$

$$\omega_s = \frac{2\pi}{h}$$

Anti-aliasing filters (filter before sampling!)

Quiz question: aliasing

Q: The phenomenon that sampling ($\omega_s = 2\pi/h$) of two or more harmonic signals with different frequencies may result in the same discrete time signal is called aliasing.

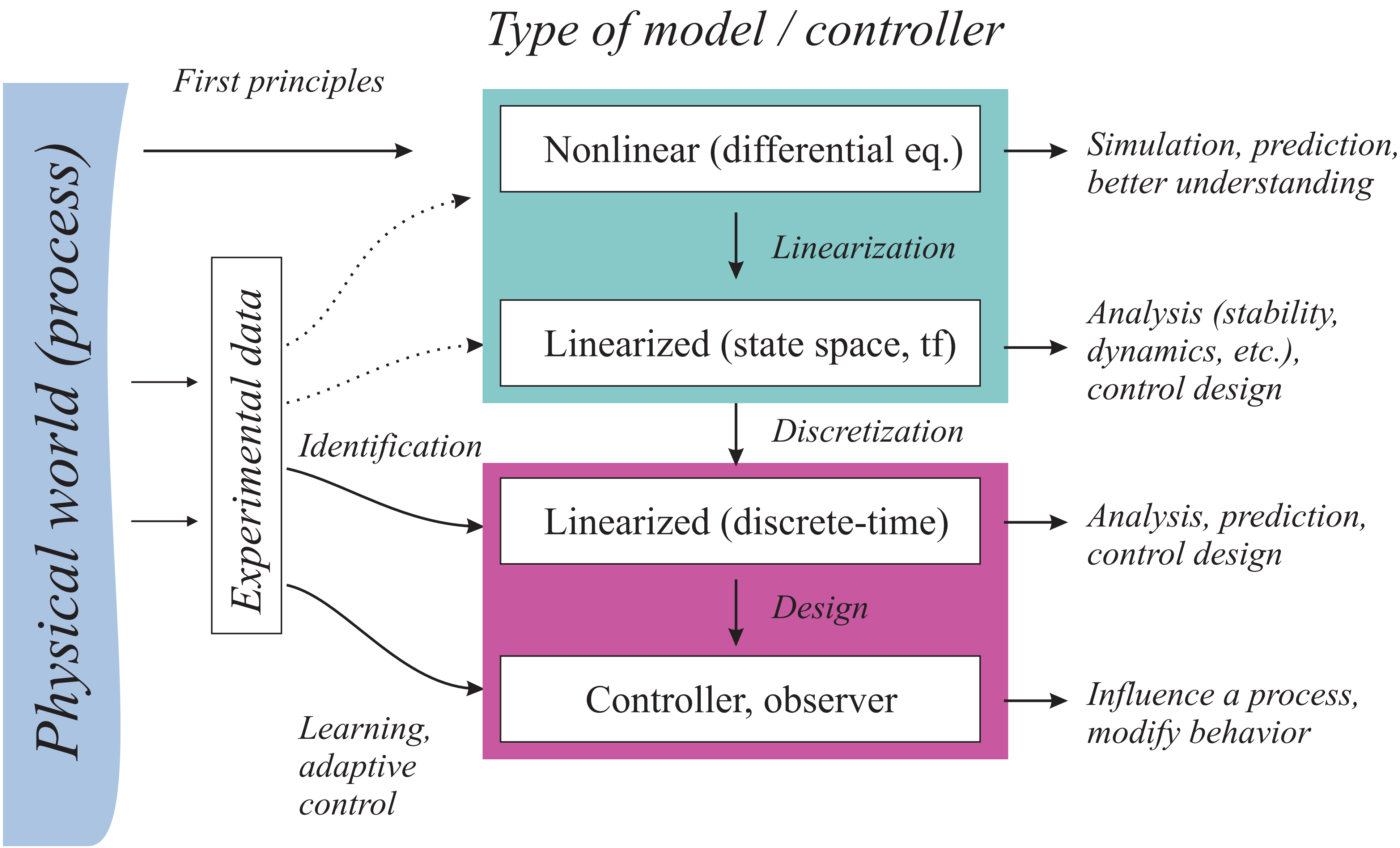
Assume that you are going to sample a system that has frequency content up to ω_{max}. Should you sample with sample frequency

1. $\omega_s \geq 2\omega_{max}$

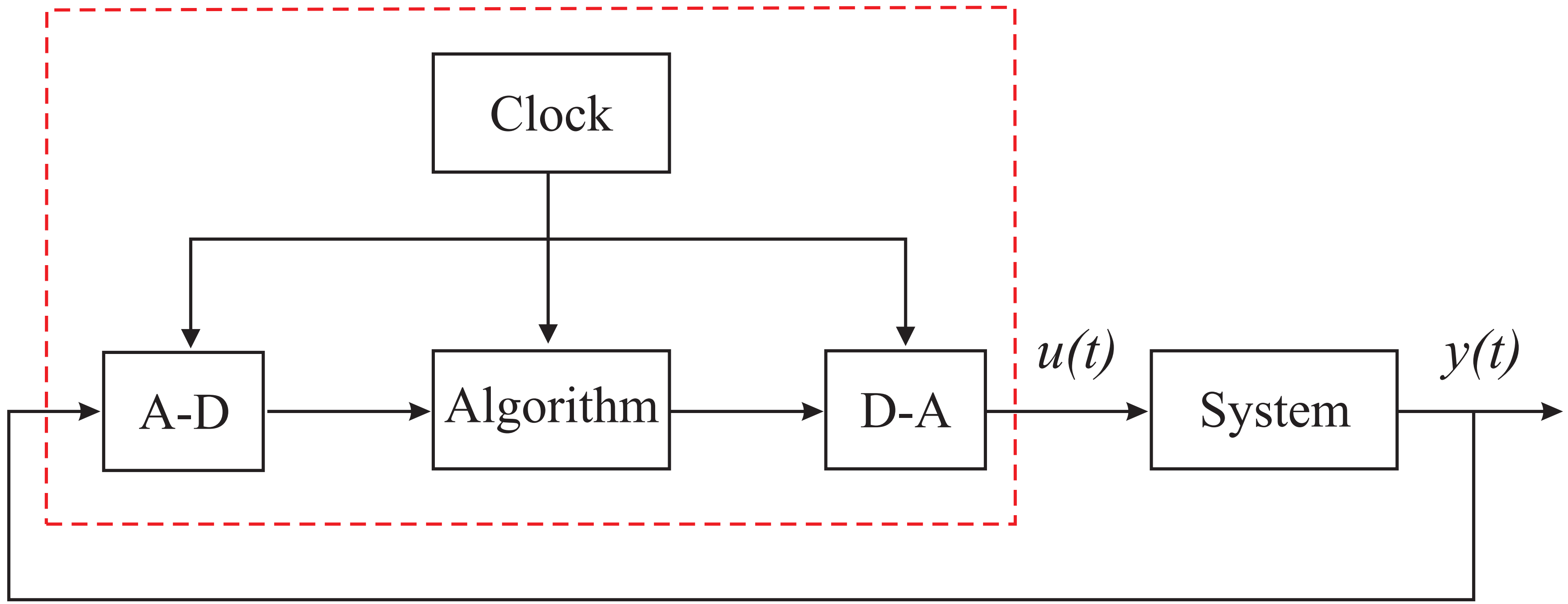
   or

2. $\omega_s \geq \frac{1}{2}\omega_{max}$?

The big picture

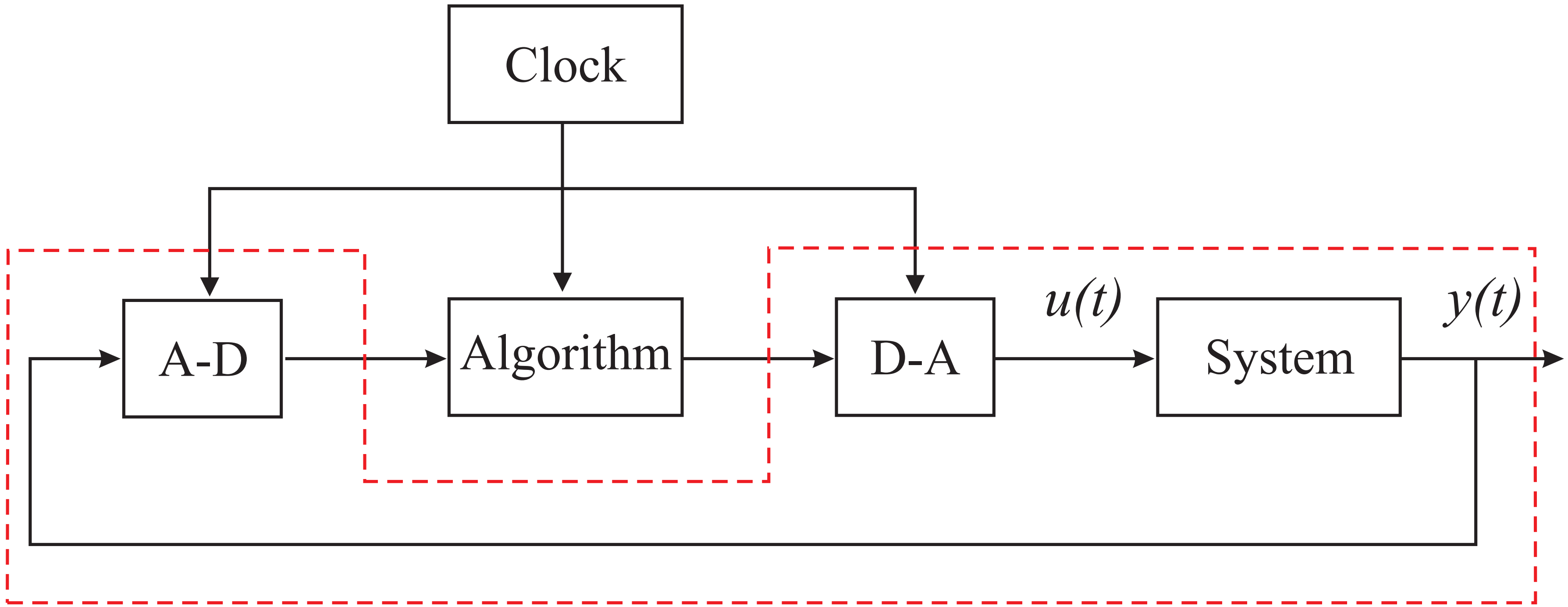


Computer-controlled system: approach 1



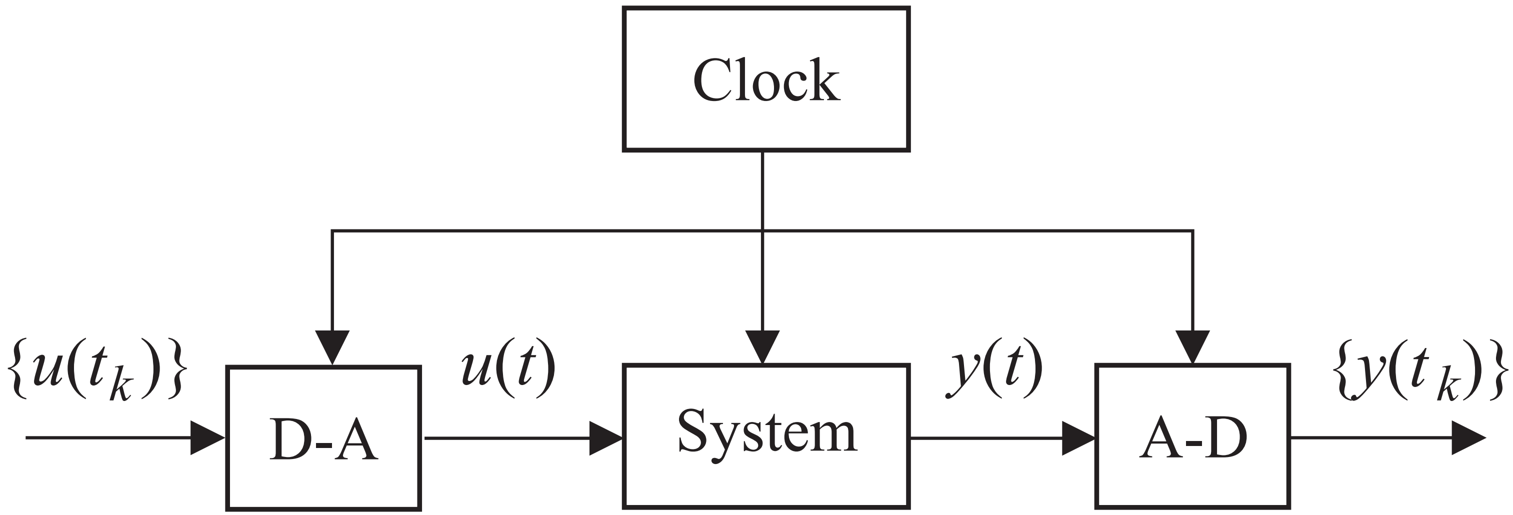
Design a continuous-time controller, make sure that the computer implementation resembles the continuous-time controller as well as possible.

Computer-controlled system: approach 2



Describe the system from the computer's viewpoint and design directly a discrete-time controller.

System from the computer's viewpoint



Zero-order hold sampling of systems

System description:

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

- Let the input be piecewise constant (ZOH).
- Look at the behavior at sampling points only.
- Use linearity and calculate step responses.

Solving the system equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

$$\begin{aligned}
x(t) &= e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-s')}Bu(s')ds' \\
&= e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-s')}ds'Bu(t_k) \\
&= e^{A(t-t_k)}x(t_k) + \int_{0}^{t-t_k} e^{As}dsBu(t_k) \\
&= \Phi(t,t_k)x(t_k) + \Gamma(t,t_k)u(t_k)
\end{aligned}$$

General discrete-time model

$$
\begin{aligned}
x(t_{k+1}) &= \Phi(t_{k+1}, t_k)x(t_k) + \Gamma(t_{k+1}, t_k)u(t_k) \\
y(t_k) &= Cx(t_k) + Du(t_k)
\end{aligned}
$$

with

$$\Phi(t_{k+1}, t_k) = e^{A(t_{k+1}-t_k)} \quad , \quad \Gamma(t_{k+1}, t_k) = \int_0^{t_{k+1}-t_k} e^{As} ds B$$

linear difference equation

Periodic sampling

$$\Phi(t_{k+1}, t_k) = e^{A(t_{k+1} - t_k)} \quad , \quad \Gamma(t_{k+1}, t_k) = \int_0^{t_{k+1} - t_k} e^{As} ds B$$

$t_k = k \cdot h \quad \rightarrow \quad t_{k+1} - t_k = h$ (sampling period)

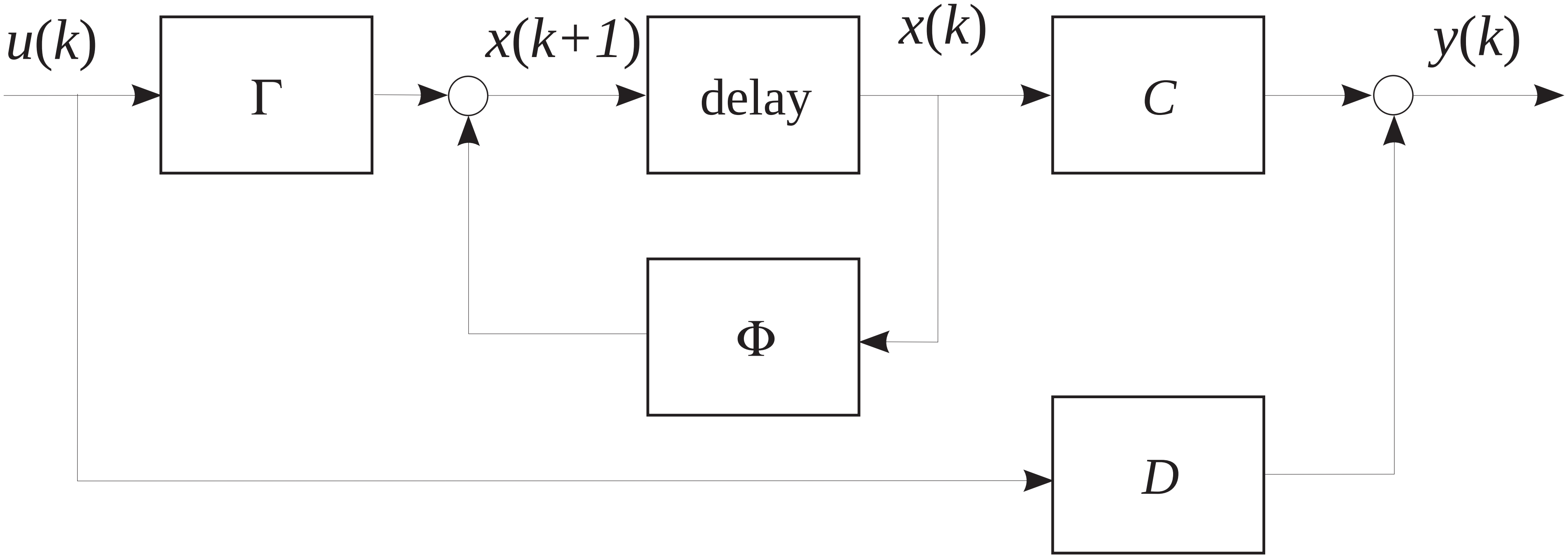
$$\begin{aligned} x(kh + h) &= \Phi x(kh) + \Gamma u(kh) \\ y(kh) &= C x(kh) + D u(kh) \end{aligned}$$

with

$$\Phi = e^{Ah} \quad , \quad \Gamma = \int_0^h e^{As} ds B$$

Discrete-time state-space system

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) \\
y(k) &= Cx(k) + Du(k)
\end{aligned}
$$



Example: first-order system

$$\frac{dx(t)}{dt} = ax(t) + bu(t)$$

System matrices (parameters):

$$\begin{aligned}\Phi &= e^{ah} \\ \Gamma &= \int_0^h e^{as} ds \cdot b = \frac{b}{a}\left(e^{ah} - 1\right)\end{aligned}$$

Discrete-time system:

$$x(kh + h) = e^{ah} x(kh) + \frac{b}{a}(e^{ah} - 1)u(kh)$$

Computing Φ and Γ

- Series expansion of the matrix exponential.

- The Laplace transform – the Laplace transform of $\exp(At)$ is $(sI - A)^{-1}$.

- Cayley-Hamilton's theorem (computation of matrix functions, see Appendix B).

- Numerical calculation in MATLAB or other SW.

- Symbolic computer algebra, using programs such as Maple and Mathematica, or Symbolic Toolbox in MATLAB.

Series expansion

$$\Phi = e^{Ah} \qquad \Gamma = \int_0^h e^{As} ds B$$

$$\Phi = e^{Ah} = I + Ah + \frac{A^2h^2}{2!} + \frac{A^3h^3}{3!} + \cdots$$

$$\Psi = \int_0^h e^{As} ds = Ih + \frac{Ah^2}{2!} + \frac{A^2h^3}{3!} + \cdots$$

$$\Gamma = \Psi B$$

Example: sampling of double integrator

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

$$\Phi = e^{Ah} = I + Ah + A^2h^2/2 + \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

Example: sampling of double integrator

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

$$\Gamma = (Ih + Ah^2/2 + A^2h^3/3! + \cdots)B$$

$$= \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{h^2}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{h^2}{2} \\ h \end{pmatrix}$$

Matrix exponential

$$
\begin{aligned}
\Phi &= e^{Ah} \\
\Gamma &= \int_0^h e^{As} ds B
\end{aligned}
$$

Differentiate

$$
\begin{aligned}
\frac{d\Phi(t)}{dt} &= A\Phi(t) = \Phi(t)A \\
\frac{d\Gamma(t)}{dt} &= \Phi(t)B
\end{aligned}
$$

$$
\frac{d}{dt}\begin{pmatrix} \Phi(t) & \Gamma(t) \\ 0 & I \end{pmatrix} = \begin{pmatrix} \Phi(t) & \Gamma(t) \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}
$$

Matrix exponential

$$\frac{d}{dt}\begin{pmatrix} \Phi(t) & \Gamma(t) \\ 0 & I \end{pmatrix} = \begin{pmatrix} \Phi(t) & \Gamma(t) \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

$\Phi(h)$ and $\Gamma(h)$ can be obtained from the block matrix

$$\begin{pmatrix} \Phi(h) & \Gamma(h) \\ 0 & I \end{pmatrix} = \exp\left\{ \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} h \right\}$$

Laplace transform

$$\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = e^{At}$$

Proof from $h(t) = \mathcal{L}^{-1}\left\{H(s)\right\}$, where

$$\begin{aligned} H(s) &= C(sI-A)^{-1}B + D \\ h(t) &= Ce^{At}B + D \end{aligned}$$

Then use

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s')}Bu(s')ds'$$

with $x(0) = 0$ and $u(t) = \delta(t)$.

Laplace transform

- To get $\Phi = e^{Ah}$ use the inverse Laplace transform of $(sI - A)^{-1}$ and substitute $t = h$.

- The function e^{At} can be used also to calculate $\Gamma = \int_0^h e^{As} B ds$.

Laplace transform example

DC motor model:

$$H(s) = \frac{1}{s(s+1)} \qquad \Rightarrow \qquad \frac{dx(t)}{dt} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

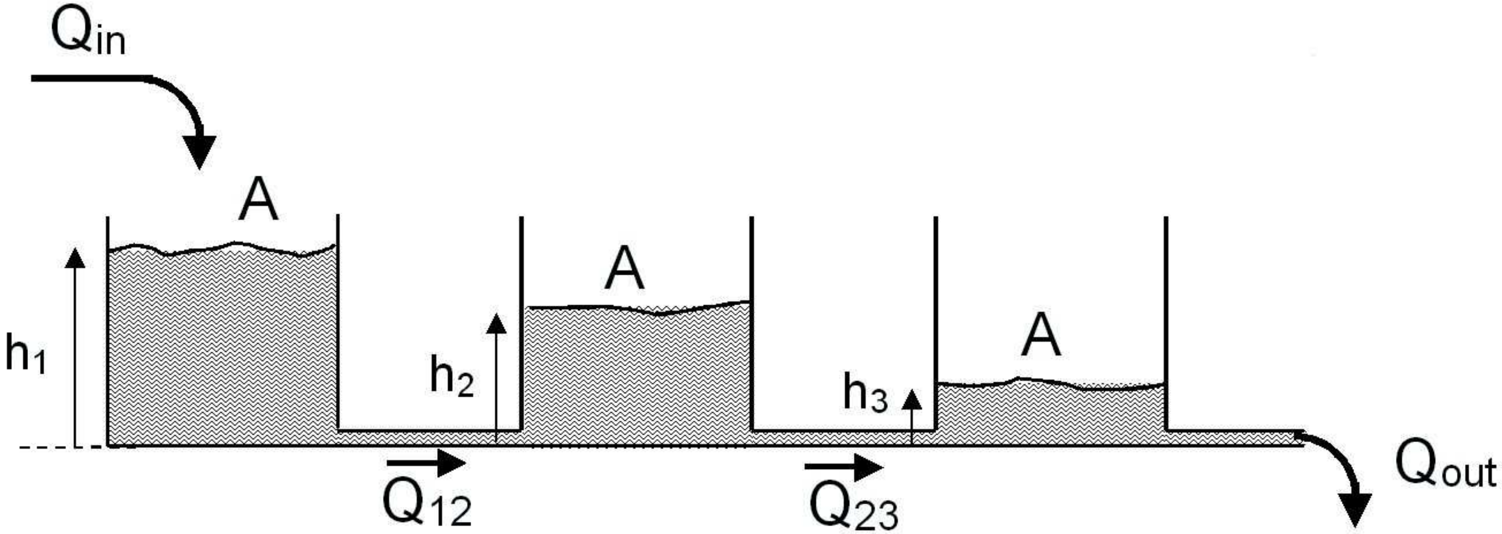
$$y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)$$

$$(sI - A)^{-1} = \begin{pmatrix} s+1 & 0 \\ -1 & s \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s(s+1)} & \frac{1}{s} \end{pmatrix}$$

$$\Phi = e^{Ah} = \begin{pmatrix} e^{-h} & 0 \\ 1 - e^{-h} & 1 \end{pmatrix} \qquad \Rightarrow \qquad \Gamma = \int_0^h \begin{pmatrix} e^{-s'} \\ 1 - e^{-s'} \end{pmatrix} ds' = \begin{pmatrix} 1 - e^{-h} \\ h - 1 + e^{-h} \end{pmatrix}$$

Discretization in MATLAB

Water system:



From linearized balance equations:

$$
\begin{aligned}
\dot{h}_1 &= \frac{1}{A} Q_{in} - \frac{k}{A}(h_1 - h_2) \\
\dot{h}_2 &= \frac{k}{A}(h_1 - h_2) - \frac{k}{A}(h_2 - h_3) \\
\dot{h}_3 &= \frac{k}{A}(h_2 - h_3) - \frac{k}{A} h_3
\end{aligned}
$$

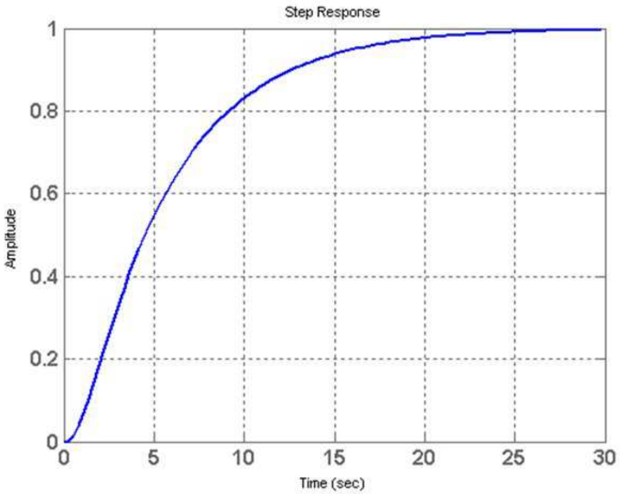
Discretization in MATLAB

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ Discretize using `c2d` command.

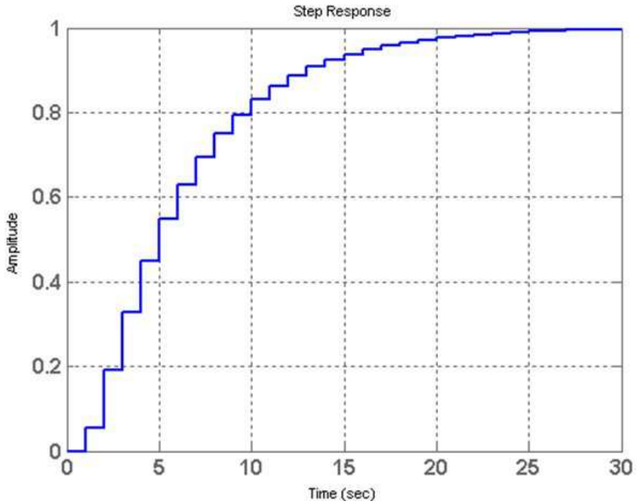
With $h = 1$: $$\Phi = \begin{pmatrix} 0.5235 & 0.3070 & 0.1138 \\ 0.3070 & 0.3304 & 0.1931 \\ 0.1138 & 0.1931 & 0.2165 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 0.7018 \\ 0.2253 \\ 0.0557 \end{pmatrix}$$

With $h = 0.1$: $$\Phi = \begin{pmatrix} 0.9092 & 0.0864 & 0.0042 \\ 0.0864 & 0.8271 & 0.0821 \\ 0.0042 & 0.0821 & 0.8228 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 0.0953 \\ 0.0045 \\ 0.0001 \end{pmatrix}$$

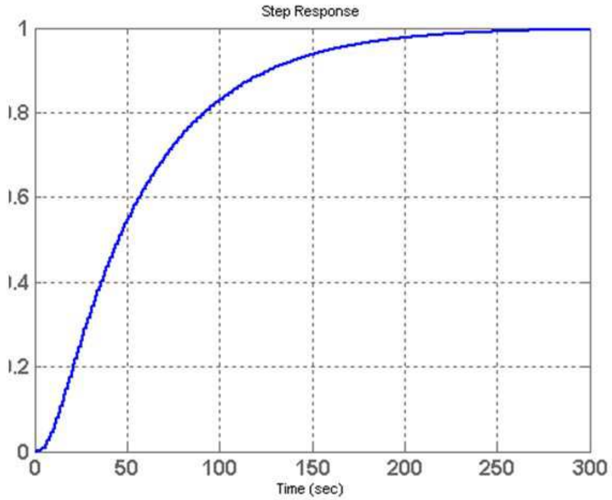
Open-loop step responses of water system



continuous



$h = 1$ sec



$h = 0.1$ sec

Sampling systems with a time delay

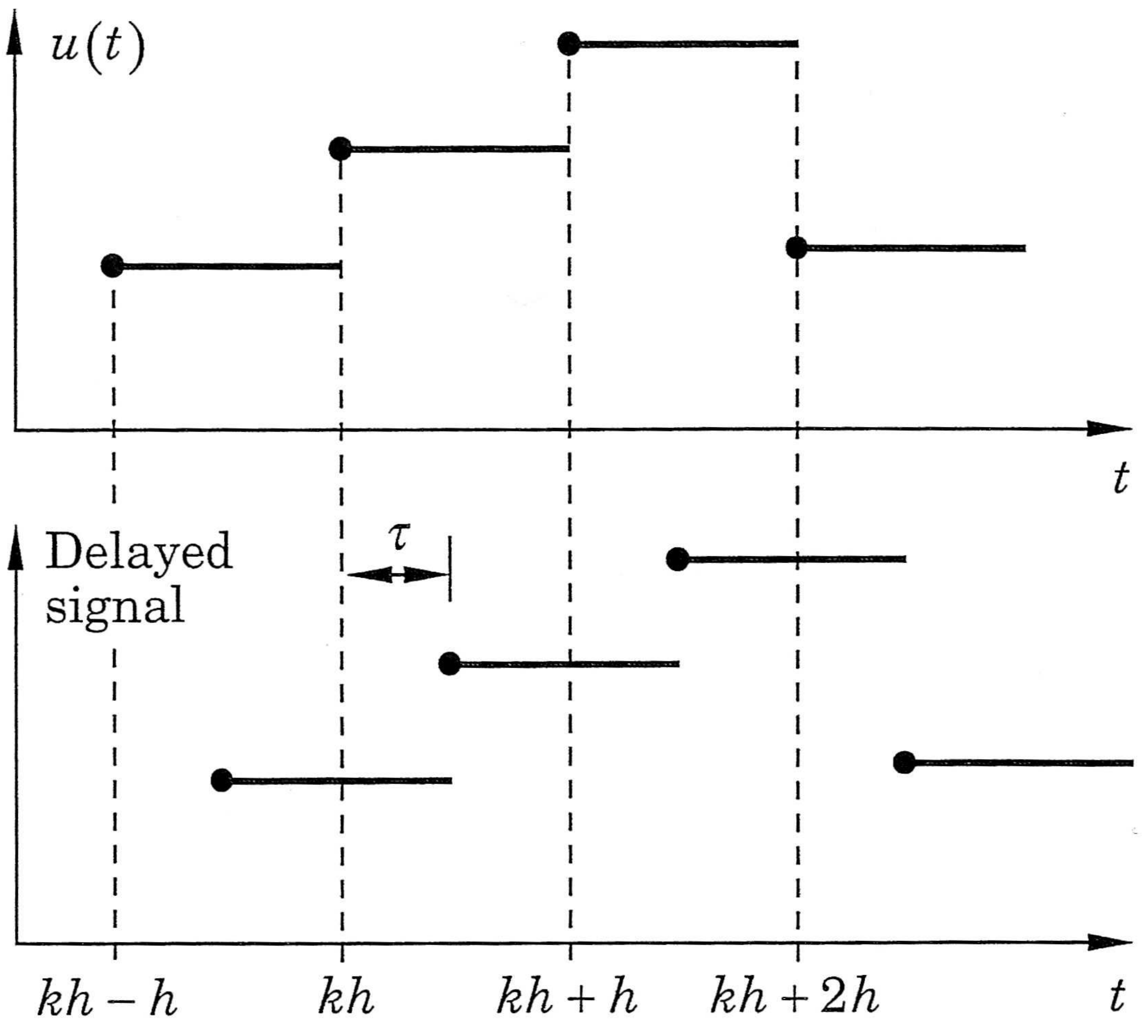
$$\frac{dx(t)}{dt} = Ax(t) + Bu(t-\tau)$$

$$x(kh+h) = e^{Ah}x(kh) + \int_{kh}^{kh+h} e^{A(kh+h-s')}Bu(s'-\tau)ds'$$

$$\int_{kh}^{kh+h} e^{A(kh+h-s')}Bu(s'-\tau)ds' = \int_{kh}^{kh+\tau} e^{A(kh+h-s')}Bds'u(kh-h) +$$

$$+ \int_{kh+\tau}^{kh+h} e^{A(kh+h-s')}Bds'u(kh) = \Gamma_1 u(kh-h) + \Gamma_0 u(kh)$$

Sampling systems with a time delay



Sampling systems with a time delay

$$x(kh + h) = \Phi x(kh) + \Gamma_0 u(kh) + \Gamma_1 u(kh - h)$$

State-space model:

$$\begin{pmatrix} x(kh+h) \\ u(kh) \end{pmatrix} = \begin{pmatrix} \Phi & \Gamma_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(kh) \\ u(kh-h) \end{pmatrix} + \begin{pmatrix} \Gamma_0 \\ I \end{pmatrix} u(kh)$$

r additional states introduced ($r =$ number of inputs)

Double integrator with time delay

$$\Phi = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

$$\Gamma_1 = \begin{pmatrix} 1 & h-\tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\tau^2}{2} \\ \tau \end{pmatrix} = \begin{pmatrix} \tau\left(h - \frac{\tau}{2}\right) \\ \tau \end{pmatrix}$$

$$\Gamma_0 = \begin{pmatrix} \frac{(h-\tau)^2}{2} \\ h-\tau \end{pmatrix}$$

Longer time delays

In continuous time, time delay can be approximated by infinite series of first-order systems:

$$e^{-\tau s} = \lim_{n \to \infty} \left(\frac{1}{1 + \frac{\tau}{n} s} \right)^n$$

In discrete time we know exactly what the order of the approximation is:

If $\tau = dh + \tau'$ with $0 < \tau' < h$,
then additional $(d + 1) \cdot r$ states are needed.

Longer time delays

$$x(k+1) = \Phi x(k) + \Gamma_0 u(k-d) + \Gamma_1 u(k-d-1)$$

In state-space form:

$$\begin{pmatrix} x(k+1) \\ u(k-d) \\ \vdots \\ u(k-1) \\ u(k) \end{pmatrix} = \begin{pmatrix} \Phi & \Gamma_1 & \Gamma_0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u(k-d-1) \\ \vdots \\ u(k-2) \\ u(k-1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{pmatrix} u(k)$$

Inverse of sampling

Is it always possible to find a continuous-time system that corresponds to a discrete-time system? Two problems:
existence, uniqueness.

Existence (example):

$$x(kh+h) = \Phi x(kh) + \Gamma u(kh)$$

$$e^{ah} = \Phi \quad \longrightarrow \quad a = \frac{1}{h}\ln\Phi$$

$$\frac{b}{a}(e^{ah}-1) = \Gamma \quad \longrightarrow \quad b = \frac{1}{h}\ln\Phi\cdot\frac{\Gamma}{\Phi-1}$$

Continuous-time system with real coefficients obtained only for positive Φ.

Solution of discrete-time system equations

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) \\
y(k) &= Cx(k) + Du(k)
\end{aligned}
$$

Iterate the equations:

$$
\begin{aligned}
x(k_0+1) &= \Phi x(k_0) + \Gamma u(k_0) \\
x(k_0+2) &= \Phi x(k_0+1) + \Gamma u(k_0+1) \\
&= \Phi^2 x(k_0) + \Phi\Gamma u(k_0) + \Gamma u(k_0+1) \\
&\vdots \\
x(k) &= \Phi^{k-k_0} x(k_0) + \Phi^{k-k_0-1}\Gamma u(k_0) + \cdots + \Gamma u(k-1)
\end{aligned}
$$

Solution of discrete-time system equations

$$
\begin{aligned}
x(k) &= \Phi^{k-k_0}x(k_0) + \Phi^{k-k_0-1}\Gamma u(k_0) + \cdots + \Gamma u(k-1) \\
&= \Phi^{k-k_0}x(k_0) + \sum_{j=k_0}^{k-1} \Phi^{k-j-1}\Gamma u(j)
\end{aligned}
$$

$$
y(k) = C\Phi^{k-k_0}x(k_0) + \sum_{j=k_0}^{k-1} C\Phi^{k-j-1}\Gamma u(j) + Du(k)
$$

Initial value + influence of input signal

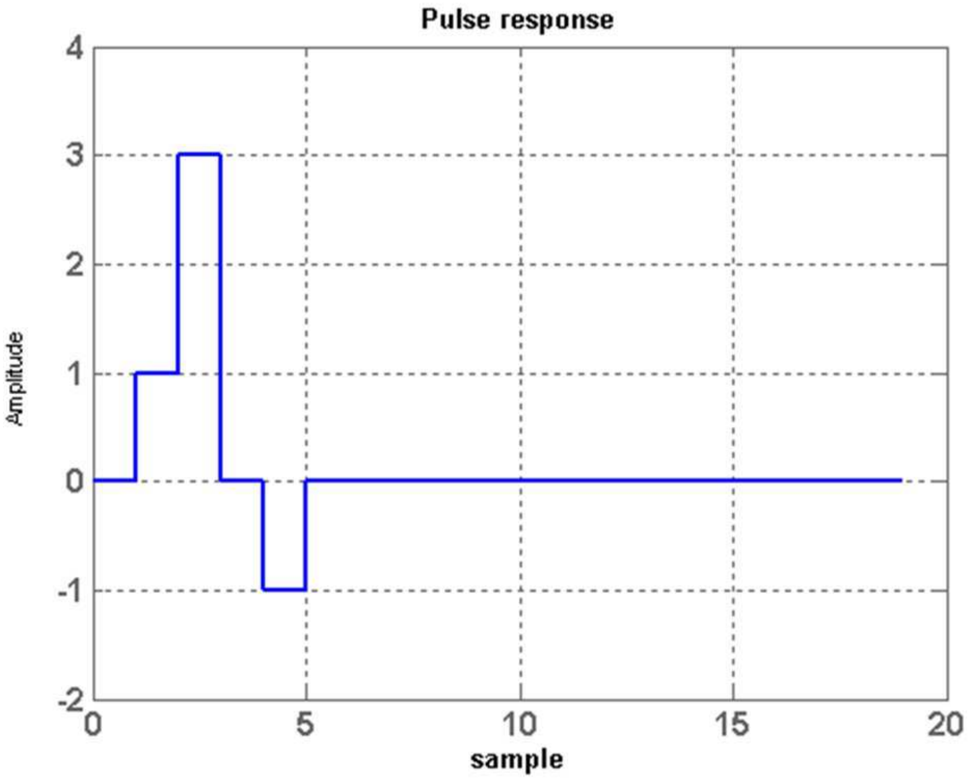
Input-output models

$$
\begin{aligned}
y(k) &= C\Phi^{k-k_0}x(k_0) + \sum_{j=k_0}^{k-1} C\Phi^{k-j-1}\Gamma u(j) + Du(k) \\
&= C\Phi^{k-k_0}x(k_0) + \sum_{j=k_0}^{k} h(k-j)u(j)
\end{aligned}
$$

Pulse-response (weighting) function:

$$
h(k) = \begin{cases} 0 & k < 0 \\ D & k = 0 \\ C\Phi^{k-1}\Gamma & k > 0 \end{cases}
$$

Discrete-time convolution example



Pulse response:

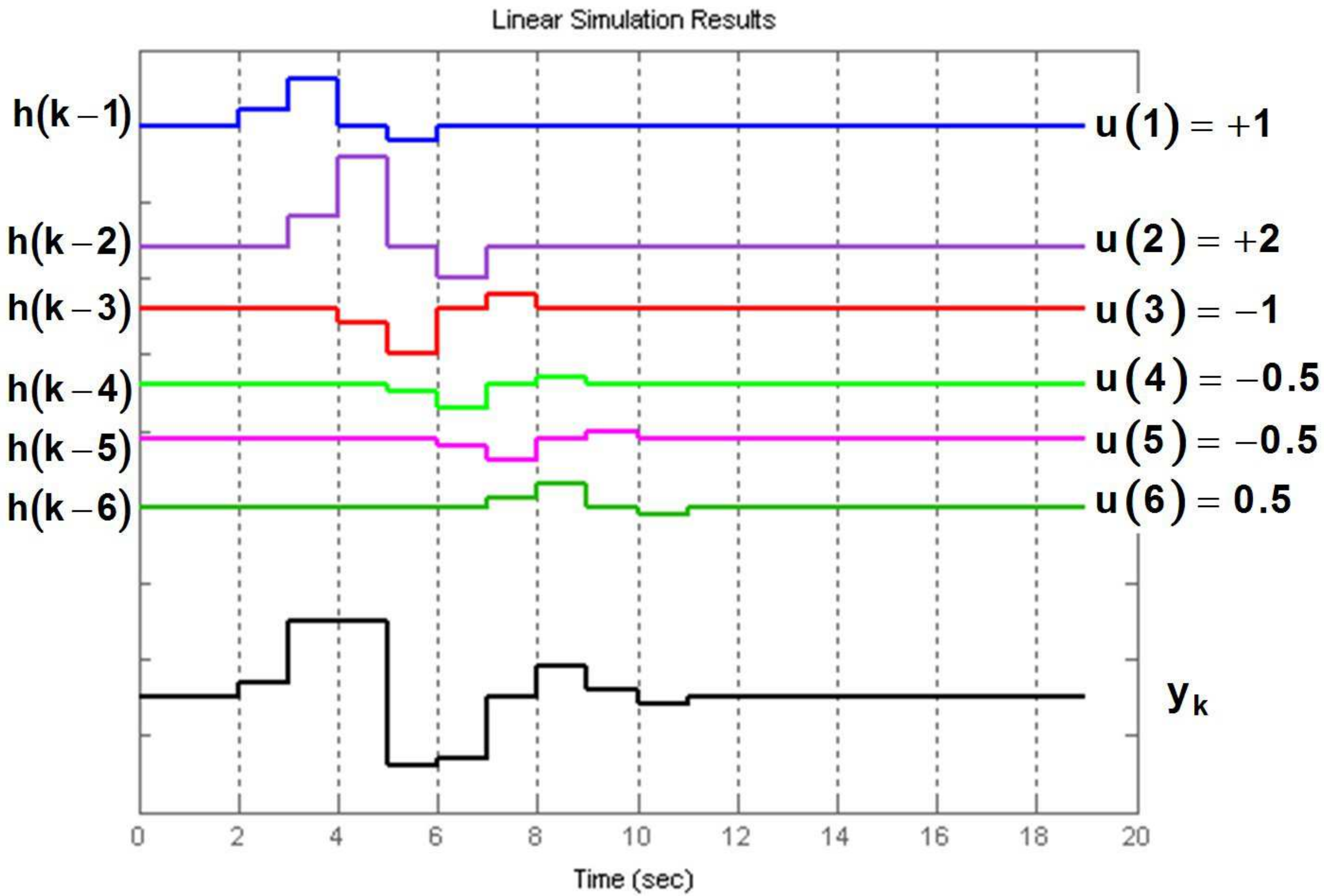
$$h(k) = \begin{cases} 0 \\ 1 \\ 3 \\ 0 \\ -1 \\ 0 \end{cases}$$

Input:

$$u(k) = \begin{cases} 1 \\ 2 \\ -1 \\ -0.5 \\ -0.5 \\ 0.5 \end{cases}$$

Discrete-time convolution example

Discrete convolution is just a summation: $y(k) = \sum_{j=1}^{k} h(k-j)u(j)$



Change of coordinates

Introduce new state vector $z(k) = Tx(k)$

$$
\begin{aligned}
z(k+1) = Tx(k+1) &= T\Phi x(k) + T\Gamma u(k) \\
&= T\Phi T^{-1} z(k) + T\Gamma u(k) \\
&= \tilde{\Phi} z(k) + \tilde{\Gamma} u(k)
\end{aligned}
$$

and

$$
\begin{aligned}
y(k) &= Cx(k) + Du(k) = CT^{-1} z(k) + Du(k) \\
&= \tilde{C} z(k) + \tilde{D} u(k)
\end{aligned}
$$

Change of coordinates

- State space representation depends on the coordinate system
- Invariants:
  - Characteristic equation $\det[\lambda I - \Phi]$
  - Input-output representation
- Simple representations:
  - Diagonal form
  - Jordan form

Diagonal form

Assume that Φ has distinct eigenvalues λ_i, then

$$T\Phi T^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

First-order decoupled difference equations

$$\begin{aligned} z_1(k+1) &= \lambda_1 z_1(k) + \beta_1 u(k) \\ &\vdots \\ z_n(k+1) &= \lambda_n z_n(k) + \beta_n u(k) \end{aligned}$$

Diagonal form – solution

Each mode has the solution

$$z_i(k) = \lambda_i^k z_i(0) + \sum_{j=0}^{k-1} \lambda_i^{k-j-1} \beta_i u(j)$$

Notice the importance of λ_i

Shift operator

Forward shift operator

$$\begin{aligned} qf(k) &= f(k+1) \\ q^2 f(k) &= f(k+2), \text{ etc.} \end{aligned}$$

Backward shift operator

$$\begin{aligned} q^{-1} f(k) &= f(k-1) \\ q^{-2} f(k) &= f(k-2), \text{ etc.} \end{aligned}$$

Compare with the differential operator $p = \frac{d}{dt}$.

Representing difference equations

$$y(k+n_a) + a_1 y(k+n_a-1) + \cdots + a_{n_a} y(k) = b_0 u(k+n_b) + \cdots + b_{n_b} u(k)$$

where $n_a \geq n_b$. Using the shift operator gives

$$(q^{n_a} + a_1 q^{n_a-1} + \cdots + a_{n_a}) y(k) = (b_0 q^{n_b} + \cdots + b_{n_b}) u(k)$$

introduce polynomials:

$$A(q) y(k) = B(q) u(k)$$

or

$$y(k) = \frac{B(q)}{A(q)} u(k)$$

Representing difference equations

Reciprocal polynomial:

$$A^*(q) = 1 + a_1 q + \cdots + a_{n_a} q^{n_a} = q^{n_a} A(q^{-1})$$

by reversing the order of the coefficients.

$$A^*(q^{-1}) y(k) = B^*(q^{-1}) u(k-d), \quad d = n_a - n_b$$

Example:

$$\begin{aligned}
\left(q^2 + a_1 q + a_2\right) y(k) &= (b_0 q + b_1)\, u(k) \\
y(k+2) + a_1 y(k+1) + a_2 y(k) &= b_0 u(k+1) + b_1 u(k) \\
y(k) + a_1 y(k-1) + a_2 y(k-2) &= b_0 u(k-1) + b_1 u(k-2) \\
\left(1 + a_1 q^{-1} + a_2 q^{-2}\right) y(k) &= \left(b_0 q^{-1} + b_1 q^{-2}\right) u(k) \\
&= q^{-1} \left(b_0 + b_1 q^{-1}\right) u(k)
\end{aligned}$$

so $y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k-1) + b_1 u(k-2)$, one time step delay.

Pulse-transfer function operator

$$x(k+1) = qx(k) = \Phi x(k) + \Gamma u(k)$$

$$(qI - \Phi)x(k) = \Gamma u(k)$$
$$x(k) = (qI - \Phi)^{-1}\Gamma u(k)$$

$$\begin{aligned} y(k) &= Cx(k) + Du(k) = \Big(C(qI - \Phi)^{-1}\Gamma + D\Big)\, u(k) \\ y(k) &= H(q)u(k) \end{aligned}$$

Pulse-transfer operator: $H(q) = C(qI - \Phi)^{-1}\Gamma + D$

SISO systems

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D = \frac{B(q)}{A(q)}$$

If no common factors

$$\begin{aligned} \deg A &= n_a \\ A(q) &= \det[qI - \Phi] \end{aligned}$$

and

$$y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k - n_a) = b_0 u(k) + \cdots + b_{n_b} u(k - n_b)$$

where a_i are the coefficients of the characteristic polynomial of Φ (usually $b_0 = 0$ ... causality)

Poles, zeros and system order

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D = \frac{B(q)}{A(q)}$$

Poles: $A(q) = 0$

Zeros: $B(q) = 0$

System order: $\deg A(q)$

$$H^*(q^{-1}) = C(I - q^{-1}\Phi)^{-1}q^{-1}\Gamma + D$$

Example - double integrator

$$H(q) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & -1 \\ 0 & q-1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \frac{1}{2}\frac{q+1}{(q-1)^2}$$

zero at -1, 2 poles at $+1$

$$H^*(q^{-1}) = \frac{1}{2}q^{-1}\frac{1+q^{-1}}{(1-q^{-1})^2}$$

Example - double integrator with a delay

$h = 1$ and $\tau = 0.5$ gives

$$\Phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Gamma_1 = \begin{pmatrix} 0.375 \\ 0.5 \end{pmatrix} \quad \Gamma_0 = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}$$

$$\begin{aligned} H(q) &= C(qI - \Phi)^{-1}(\Gamma_0 + \Gamma_1 q^{-1}) \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\begin{pmatrix} q-1 & 1 \\ 0 & q-1 \end{pmatrix}}{(q-1)^2} \begin{pmatrix} 0.125 + 0.375q^{-1} \\ 0.5 + 0.5q^{-1} \end{pmatrix} \end{aligned}$$

Example - double integrator with a delay

$$H(q) = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\begin{pmatrix} q-1 & 1 \\ 0 & q-1 \end{pmatrix}}{(q-1)^2} \begin{pmatrix} 0.125 + 0.375q^{-1} \\ 0.5 + 0.5q^{-1} \end{pmatrix}$$

$$= \frac{0.125(q^2 + 6q + 1)}{q(q^2 - 2q + 1)} = \frac{0.125(q^{-1} + 6q^{-2} + q^{-3})}{1 - 2q^{-1} + q^{-2}}$$

order: 3, poles: 0, 1 and 1, zeros: $-3 \pm \sqrt{8}$

How to get $H(q)$ from $G(s)$?

Use Table 2.1 in book

$G(s)$	$H(q)$
$\frac{1}{s}$	$\frac{h}{q-1}$
$\frac{1}{s^2}$	$\frac{h^2(q+1)}{2(q-1)^2}$
e^{-sh}	q^{-1}
$\frac{a}{s+a}$	$\frac{1-\exp(-ah)}{q-\exp(-ah)}$

zero-order hold included !

Summary

- Aliasing, filter before sampling.
- Sampling of continuous-time systems.
- State-space and input-output models.
- Shift-operator calculus.
- Pulse-transfer function, poles and zeros.