Control Engineering (SC42095)

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Lecture outline

- Poles and zeros
- Selection of sampling interval
- z–transform, relation to shift operator
- Computation of $H(z)$ from $G(s)$
- Obtaining a process model
- Linearization

Interpretation of poles and zeros

Poles:

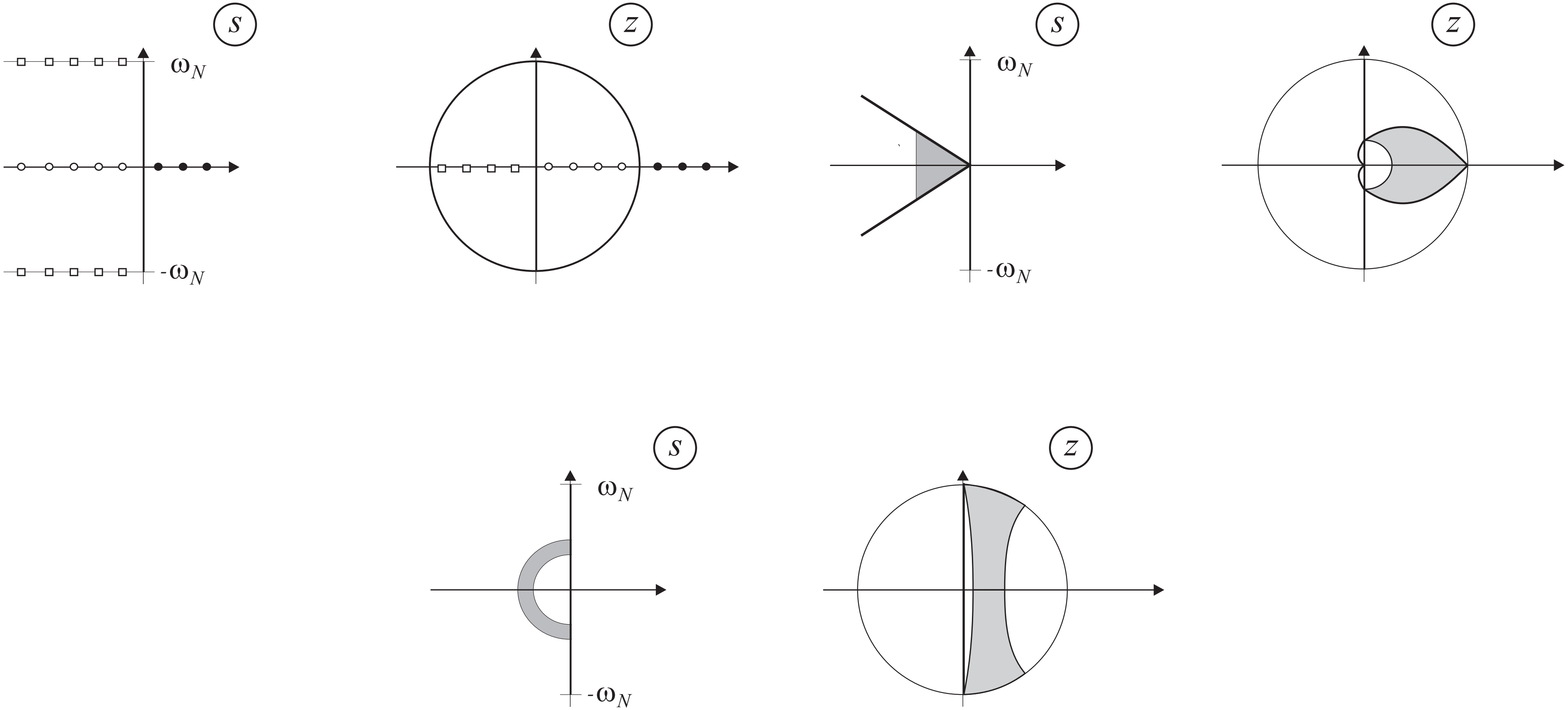
- A pole is an eigenvalue of Φ
- A pole $z = a$ corresponds to a free mode $y(k) = a^k$

Zeros:

- A zero $z = a$ implies that the transmission of the input $u(k) = a^k$ is blocked by the system
- A zero is related to how inputs and outputs are coupled to the states

Transformation of poles

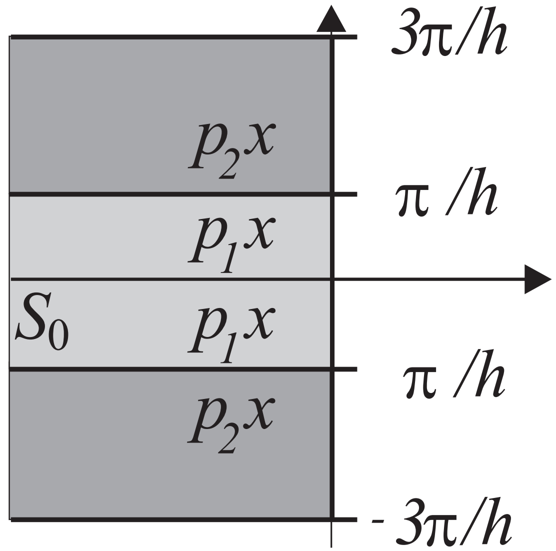
$$\lambda_i(\Phi) = e^{\lambda_i(A)h} \quad \text{(property of matrix functions)}$$

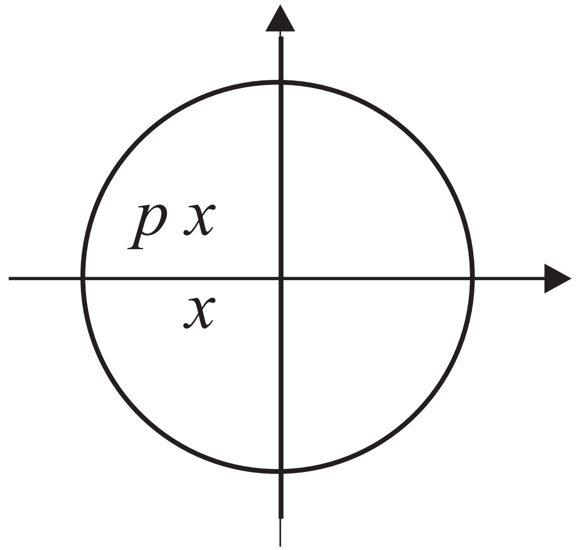


Another evidence of alias problem

$$z = e^{sh}$$

Several points in the s-plane are mapped into the same point in the z-plane.





Sampling of a second order system

$$\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

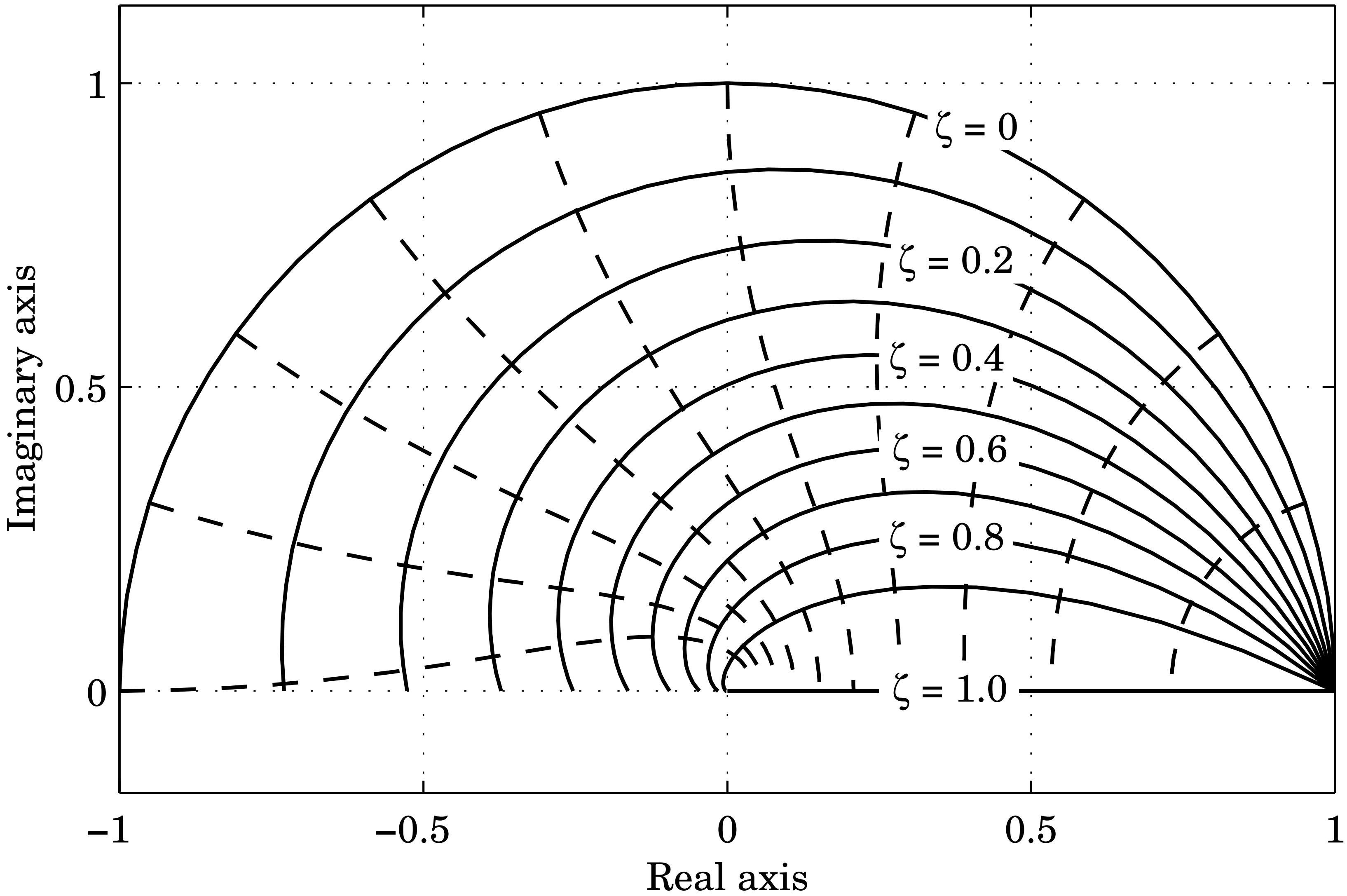
Poles of the discrete-time system are given by

$$z^2 + a_1 z + a_2 = 0$$

where

$$\begin{aligned} a_1 &= -2e^{-\zeta\omega_0 h} \cos\left(\sqrt{1-\zeta^2}\omega_0 h\right) \\ a_2 &= e^{-2\zeta\omega_0 h} \end{aligned}$$

Sampling of a second-order system



Transformation of zeros

No simple formula for the mapping of zeros
For short sampling periods ($h \to 0$)

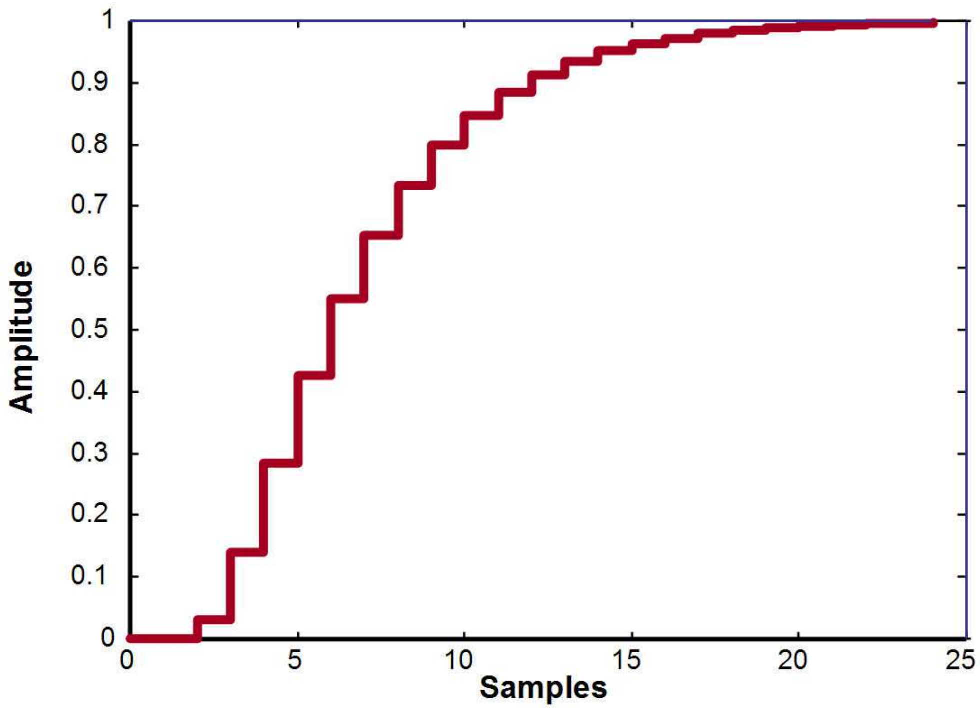
$$z_i \approx e^{s_i h}$$

plus $r = d - 1$ zeros introduced by sampling

d	Z_d
1	1
2	$z + 1$
3	$z^2 + 4z + 1$
4	$z^3 + 11z^2 + 11z + 1$
5	$z^4 + 26z^3 + 66z^2 + 26z + 1$

System with unstable inverse

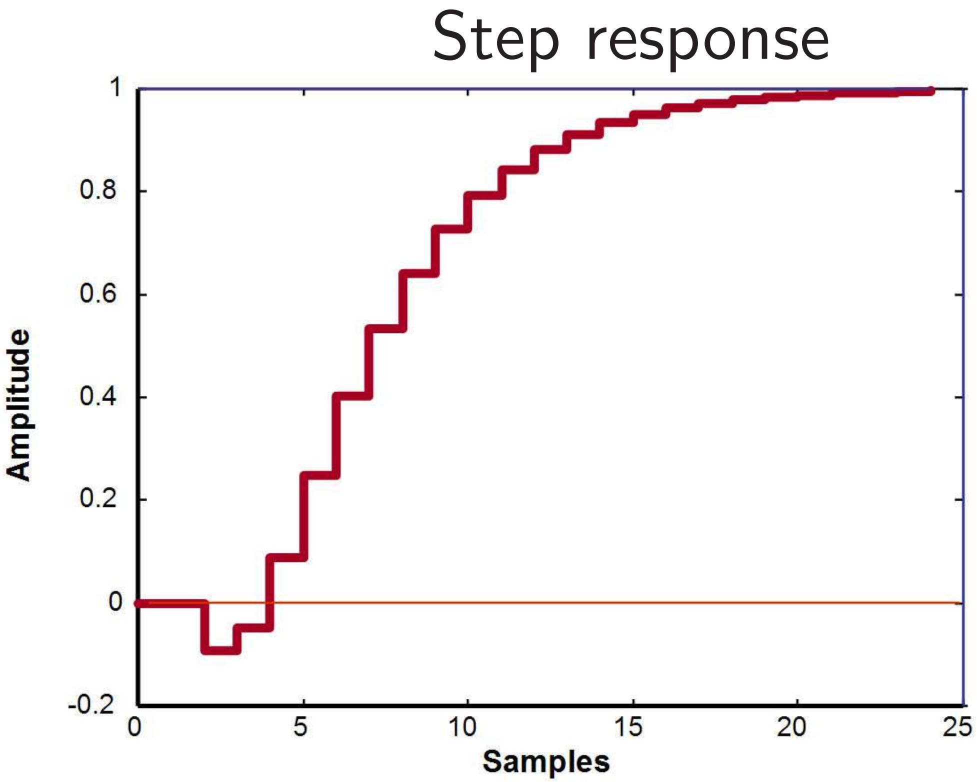
Step response



$$H(z) = K \frac{z+2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)\left(z-\frac{3}{4}\right)}$$

Zero outside unit circle
(unstable inverse)

Other system with unstable inverse

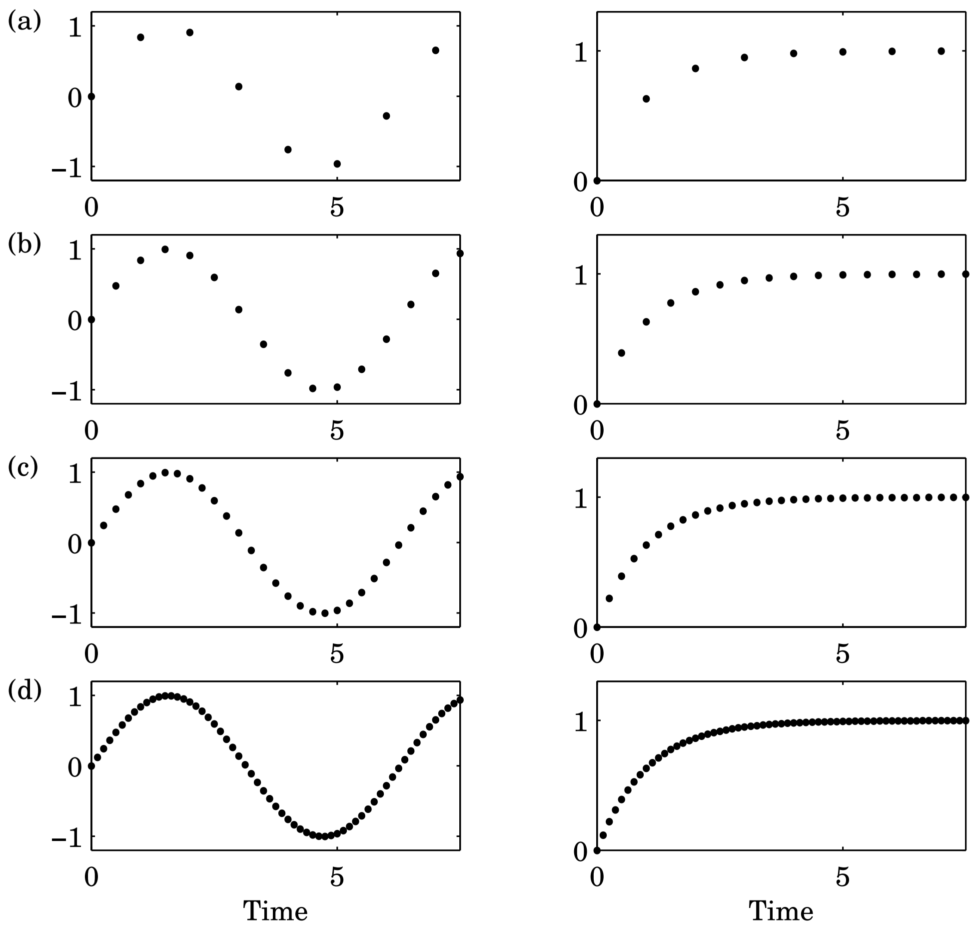


$$H(z) = K\frac{z-2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)\left(z-\frac{3}{4}\right)}$$

Zero outside unit circle
(unstable inverse)

Selection of sampling period

Number of samples per rise time: $N_r = \frac{T_r}{h} \approx 4 - 10$ is reasonable



a) $N_r = 1$

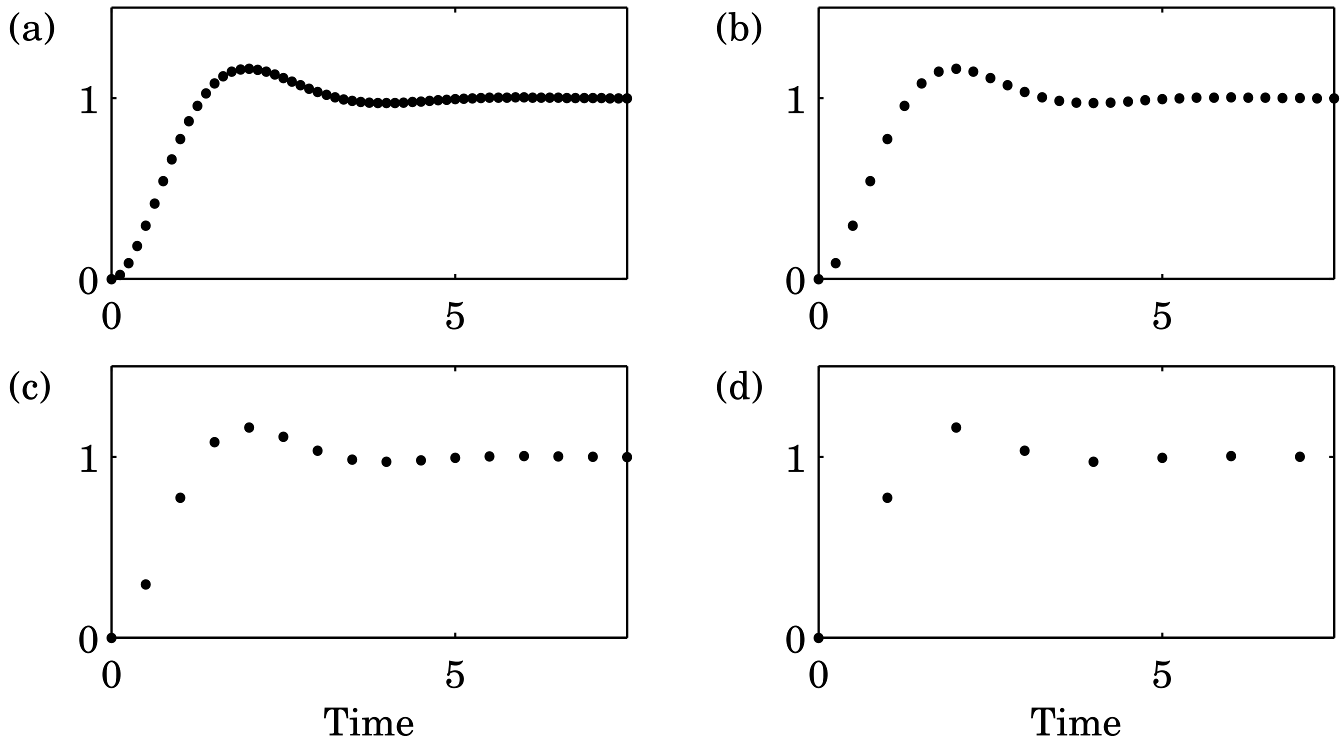
b) $N_r = 2$

c) $N_r = 4$

d) $N_r = 8$

Second order system

$N_r = \frac{T_r}{h} \approx 4 - 10$ corresponds to $\omega_0 h \approx 0.2 - 0.6$



a) $h = 0.125$ ($\omega_0 h = 0.23$), b) $h = 0.250$ ($\omega_0 h = 0.46$),

c) $h = 0.500$ ($\omega_0 h = 0.92$), d) $h = 1.000$ ($\omega_0 h = 1.83$)

z-transform

- Continuous systems: Laplace transform,
  discrete systems: z-transform

Consider discrete-time signal: $\{f(kh) \mid k = 0, 1, \ldots\}$

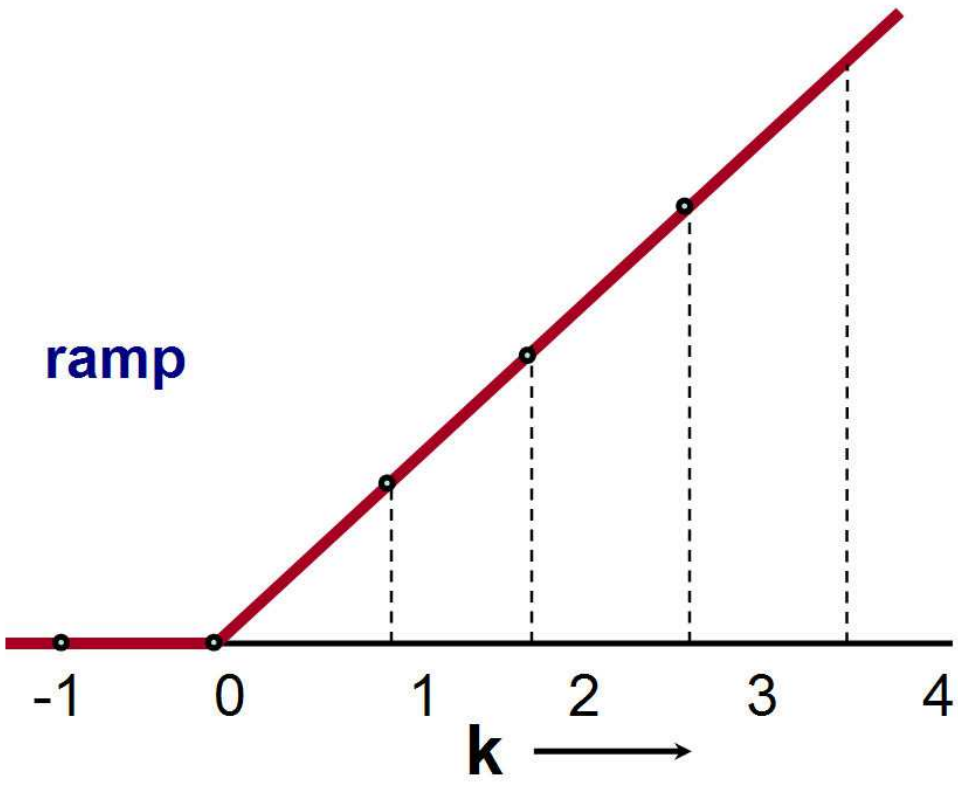
The z-transform of $f(kh)$ is defined as

$$Z\{f(kh)\} = F(z) = \sum_{k=0}^{\infty} f(kh) z^{-k}$$

where z is a complex variable.

Example: a ramp signal

Ramp: $y(kh) = kh$ for $k \geq 0$



$$
\begin{aligned}
Y(z) &= \sum_{k=0}^{\infty} f(kh) z^{-k} \\
&= 0 + hz^{-1} + 2hz^{-2} + \cdots \\
&= h(z^{-1} + 2z^{-2} + \cdots) \\
&= \frac{hz}{(z-1)^2}
\end{aligned}
$$

Properties of z-transform

1. Linearity:

$$Z\{\alpha f + \beta g\} = \alpha Z f + \beta Z g$$

2. Time-shift:

$$Z\{q^{-n} f\} = z^{-n} F$$

$$Z\{q^{n} f\} = z^{n} \left(F - F_1\right)$$

where

$$F_1(z) = \sum_{j=0}^{n-1} f(jh) z^{-j}$$

Properties of z-transform (cont'd)

3. Initial value theorem:

$$f(0) = \lim_{z \to \infty} F(z)$$

4. Final value theorem:

$$\lim_{k \to \infty} f(kh) = \lim_{z \to 1} (1 - z^{-1}) F(z).$$

(if $(1 - z^{-1})F(z)$ has no poles on or outside the unit circle)

5. Convolution:

$$Z(f * g) = Z\left\{ \sum_{n=0}^{k} f(n) g(k-n) \right\} = (Zf)\,(Zg)$$

Solving difference equations

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Take the z-transform of both sides

$$\sum_{k=0}^{\infty} z^{-k} x(k+1) = \sum_{k=0}^{\infty} \Phi z^{-k} x(k) + \sum_{k=0}^{\infty} \Gamma z^{-k} u(k)$$

$$z \left(\sum_{k=0}^{\infty} z^{-k} x(k) - x(0) \right) = \sum_{k=0}^{\infty} \Phi z^{-k} x(k) + \sum_{k=0}^{\infty} \Gamma z^{-k} u(k)$$

Solving difference equations

$$
\begin{aligned}
z\Big(X(z) - x(0)\Big) &= \Phi X(z) + \Gamma U(z) \\
X(z) &= (zI - \Phi)^{-1}(zx(0) + \Gamma U(z))
\end{aligned}
$$

$$
Y(z) = C(zI - \Phi)^{-1}zx(0) + (C(zI - \Phi)^{-1}\Gamma + D)U(z)
$$

Pulse-transfer function (the same as with q):

$$
H(z) = C(zI - \Phi)^{-1}\Gamma + D
$$

Why both z and q ?

- Both can be used to manipulate difference equations, sometimes, the same notation used for both.
- Formal difference: q is a shift operator, z is a complex variable.
- Also a system-theoretic reason: example

$$y(k+1) + ay(k) = u(k+1) + au(k) \tag{1}$$

$$H(z) = \frac{z+a}{z+a} = 1 \quad \text{(z is a complex variable)}$$

Solution of (1): $y(k) = (-a)^k y(0) - a^k u(0) + u(k)$

Calculation of $H(z)$ from $G(s)$

Method 1: State-space model, calculate Φ and Γ, find $H(z)$

1. Matrices Φ and Γ

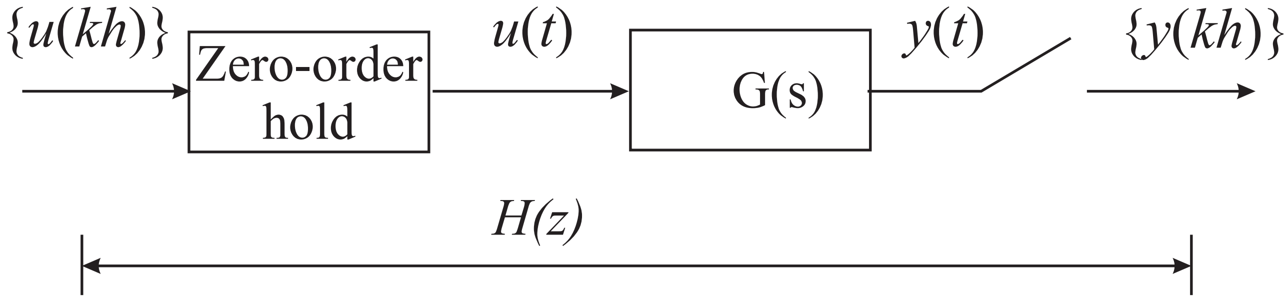
$$\Phi = e^{Ah} \quad , \quad \Gamma = \int_0^h e^{As} ds B$$

2. Convert $\{\Phi, \Gamma, C, D\}$ to pulse transfer function:

$$H(z) = C(zI - \Phi)^{-1}\Gamma + D$$

Calculation of $H(z)$ from $G(s)$

Method 2:



1. Determine the step response of the system: $Y(s) = G(s)/s$.
2. Determine the corresponding z-transform of the step response using the table: $\tilde{Y}(z) = Z(L^{-1}Y)$.
3. Divide by the z-transform of the step function:

$$H(z) = \frac{z-1}{z}\tilde{Y}(z)$$

Example: double integrator

$$1) \quad Y(s) = G(s) \cdot \frac{1}{s} = \frac{1}{s^3}$$

$$2) \quad Y(z) = \frac{h^2}{2}\frac{z(z+1)}{(z-1)^3} \qquad \text{(Table 2.3)}$$

$$3) \quad H(z) = \frac{Y(z)}{U(z)} = \frac{h^2}{2}\frac{z(z+1)}{(z-1)^3} \cdot \frac{z-1}{z} = \frac{h^2}{2}\frac{(z+1)}{(z-1)^2}$$

Compare with the use of state-space representation.

z-transform table

$f(kh)$	L_f	Z_f
$\delta(k)$ (pulse)	-	1
1 $k \geq 0$ (step)	$\frac{1}{s}$	$\frac{z}{z-1}$
kh	$\frac{1}{s^2}$	$\frac{hz}{(z-1)^2}$
$\frac{1}{2}(kh)^2$	$\frac{1}{s^3}$	$\frac{h^2 z(z+1)}{2(z-1)^3}$
$e^{-kh/T}$	$\frac{T}{1+sT}$	$\frac{z}{z-e^{-h/T}}$

Use the table correctly! Z_f in the table *does not* give the zero-order-hold sampling of a system with the transfer function L_f.

Modified z-transform

$$\tilde{F}(z,m) = \sum_{k=0}^{\infty} z^{-k} f(kh - h + mh), \quad 0 \leq m \leq 1$$

Can be used to determine intersample behavior

Controller's computational delay less than h

Using Matlab to change system representation

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}$$

$$\Updownarrow$$

`ss2tf` and `tf2ss`

$$\Updownarrow$$

$$\frac{dx(t)}{dt} = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} b_{n-1} & b_{n-2} & \cdots & b_1 & b_0 \end{pmatrix} x(t)$$

How to obtain process model?

- physical (mechanistic) modeling
  1. first principles $\rightarrow$ differential equations (nonlinear)
  2. linearization around an operating point
  3. discretization

- system identification
  1. measure input–output data (around an operating point)
  2. define model structure (order)
  3. estimate model parameters from data (least squares)

Linearization of nonlinear models

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= g(x,u)\end{aligned}$$

Choose x_0, y_0 or u_0 such that $0 = f(x_0, u_0)$ and $y_0 = g(x_0, u_0)$ (equilibrium). Linearize the above model:

$$\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\u=u_0}} (x - x_0) \;+\; \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_0\\u=u_0}} (u - u_0)$$

$$y - y_0 = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_0\\u=u_0}} (x - x_0) \;+\; \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_0\\u=u_0}} (u - u_0)$$

Linearization of nonlinear models

$$\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\u=u_0}} (x - x_0) \quad + \quad \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_0\\u=u_0}} (u - u_0)$$

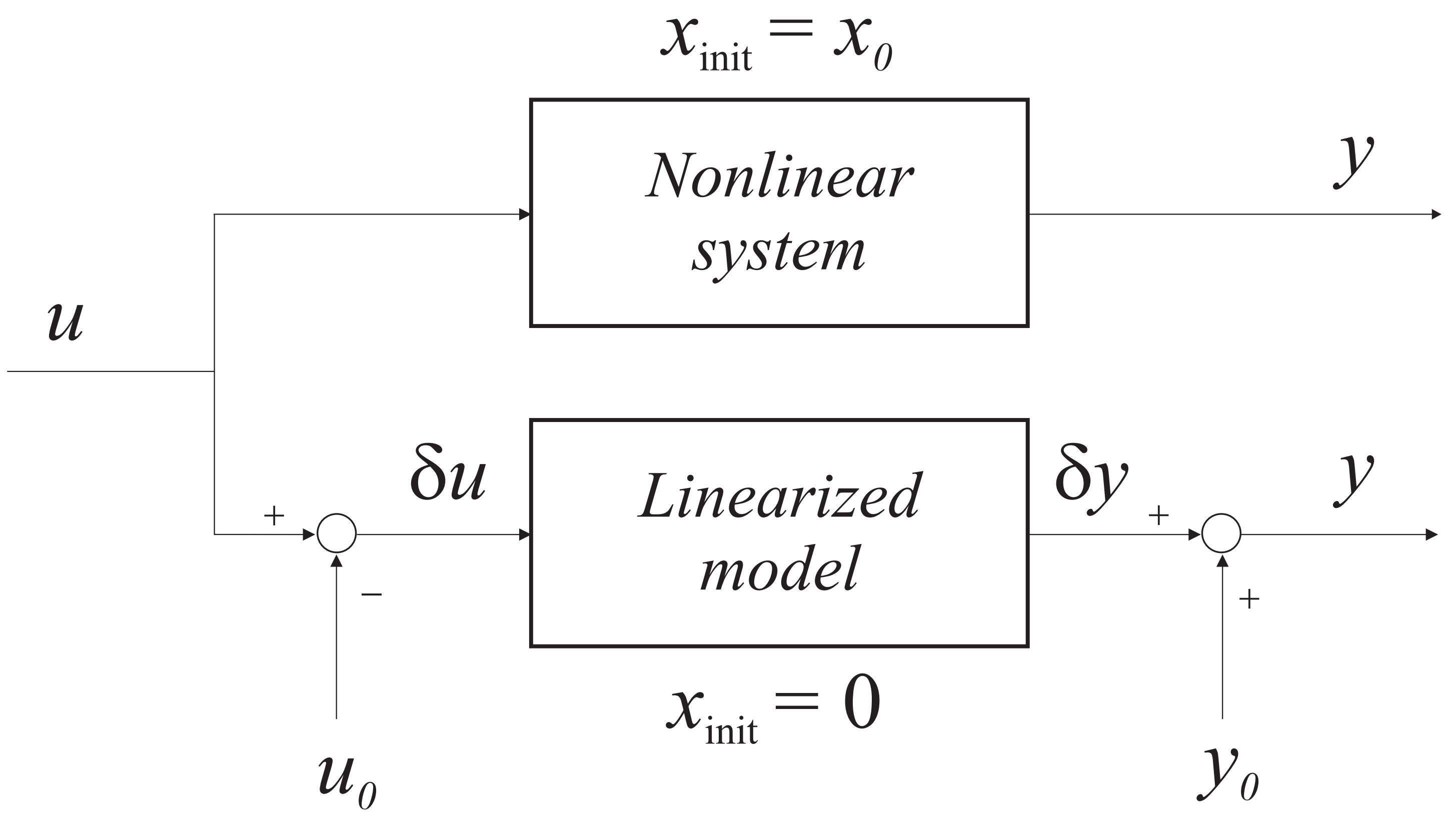
$$y - y_0 = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_0\\u=u_0}} (x - x_0) \quad + \quad \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_0\\u=u_0}} (u - u_0)$$

$$\dot{x} = A\,\delta x \quad + \quad B\,\delta u$$

$$\delta y = C\,\delta x \quad + \quad D\,\delta u$$

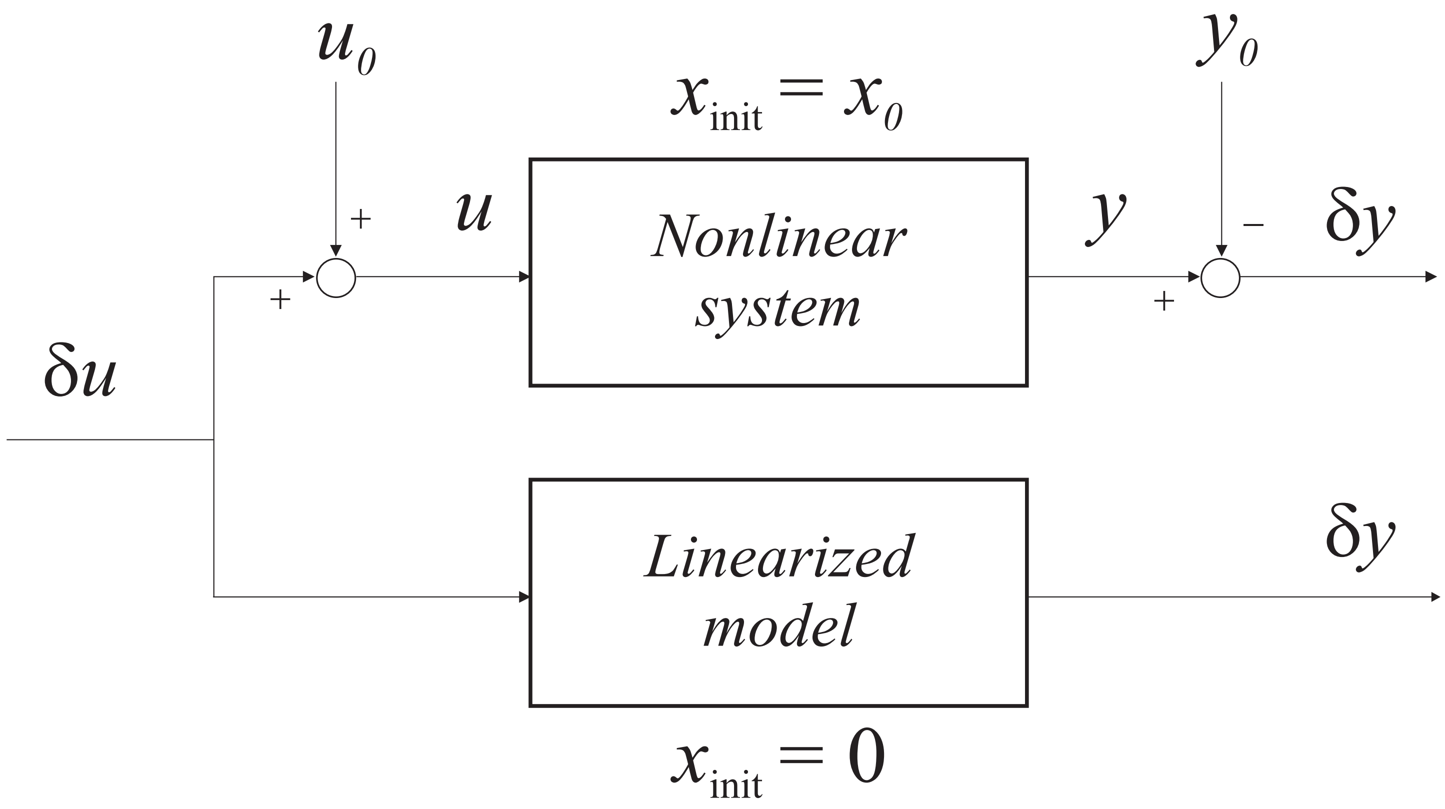
with $\delta x = x - x_0$, $\delta u = u - u_0$, $\delta y = y - y_0$, etc.

Comparison of models through simulation



$\delta u = u - u_0$ $\qquad\qquad$ $y = \delta y + y_0$

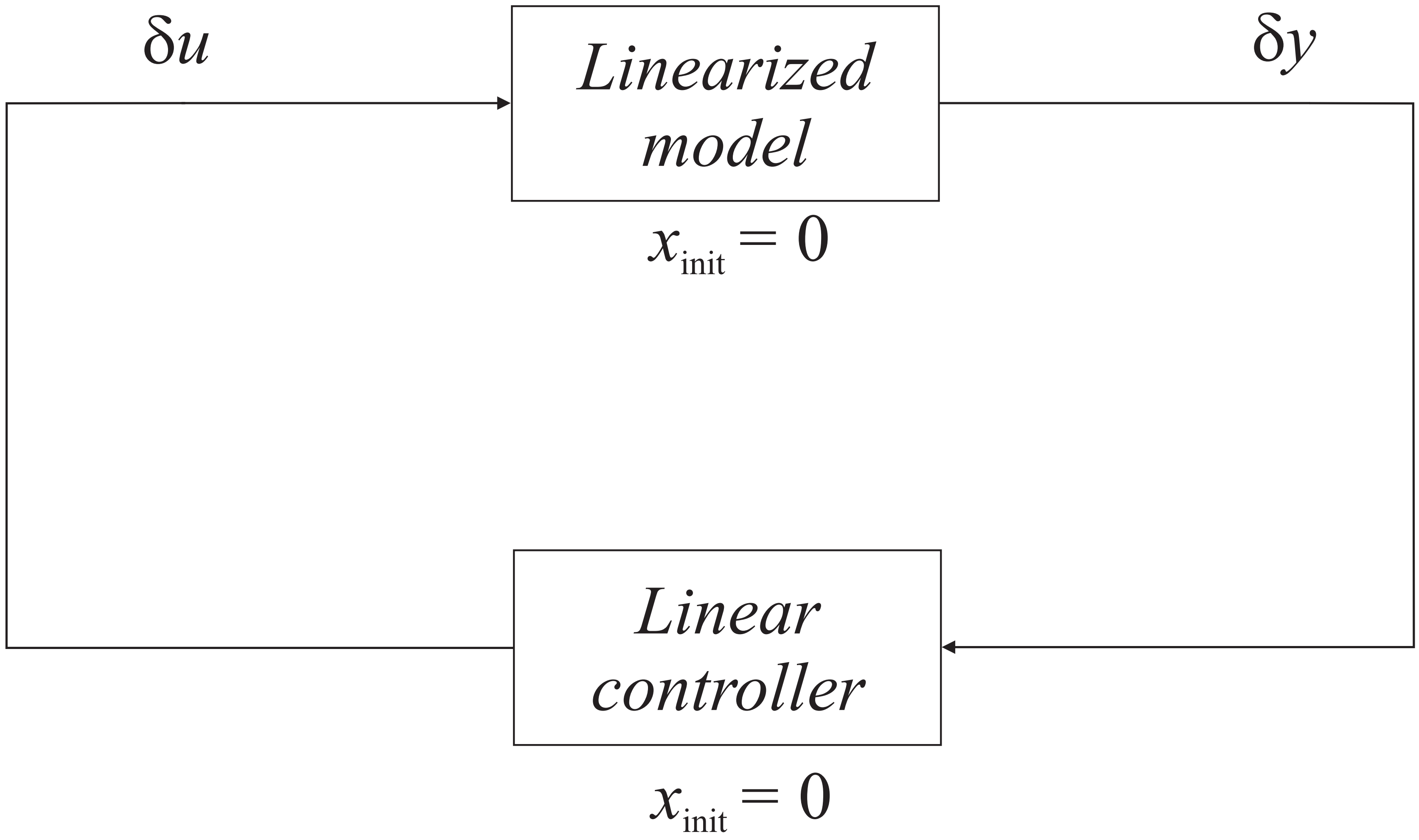
Alternatively . . .



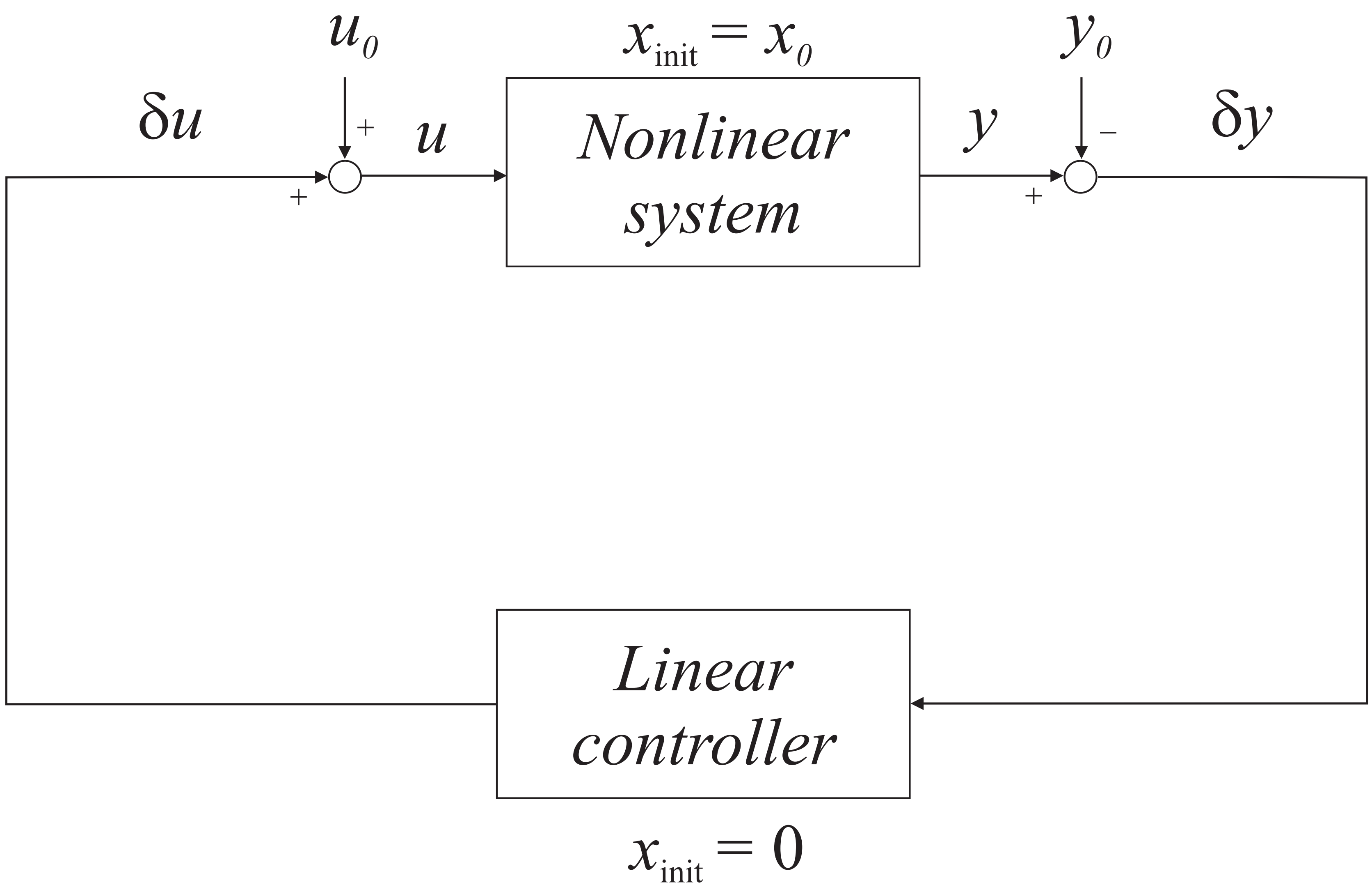
$u = \delta u + u_0$

$\delta y = y - y_0$

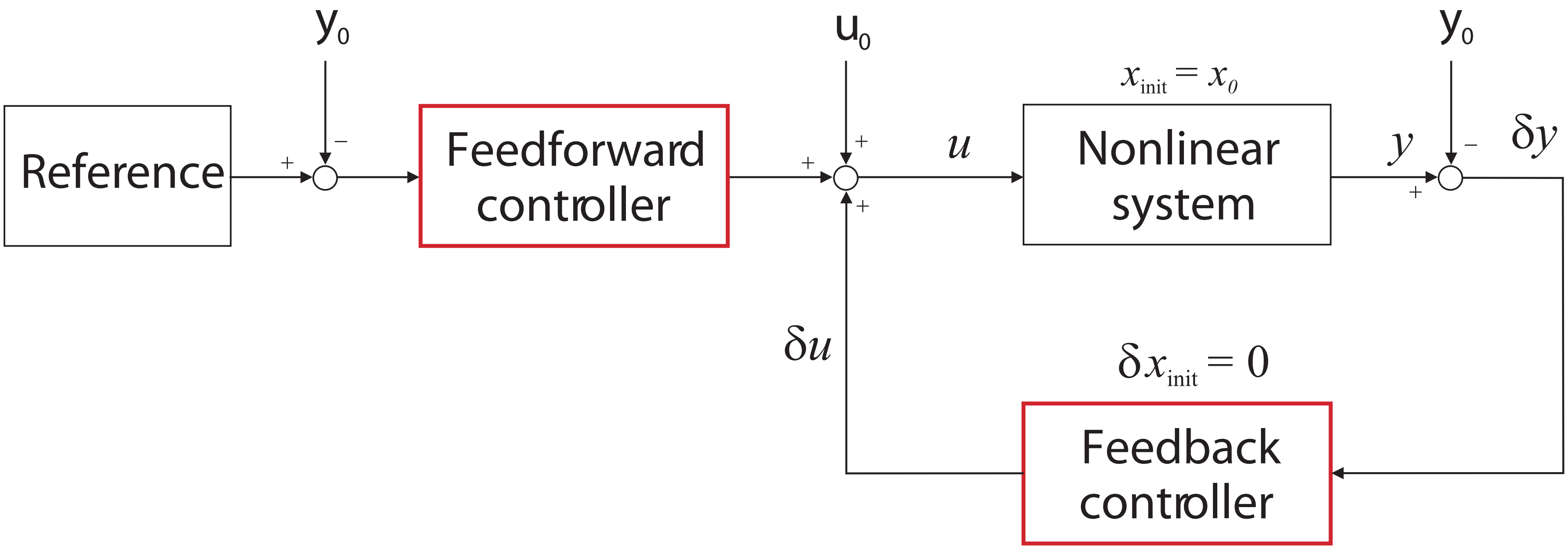
Simulation of a linear controller



Control by linear controller



Two-degree-of-freedom control



Linear controllers must use signals δu, δy, δx !

Summary

- Poles and zeros in discrete time
- Selection of sampling interval
- z–transform
- How to obtain $H(z)$ from $G(s)$
- Linearization of nonlinear models