AE4301 Automatic Flight Control System Design
Part I: Control Theory

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Lecture 1

Main objectives

- Being able to determine transfer function of linear time-invariant (LTI) systems
- Being able to analyze (transient and steady-state) response of 1st & 2nd-order LTI systems

Overview of today's lecture

- Laplace transform
- Transfer function
  - how output and input of a system are related
  - poles and zeros & response to external stimuli
- First and second-order systems
- Time-domain transient response

Material

- Slides on Brightspace
- Homework assignments on Brightspace
- Discussions during lectures

Laplace transform

- To analyze real-life systems and to design controllers for them these systems are modeled mathematically
- Dynamics of real-life systems evolves in time thus corresponding mathematical models appear as a set of differential equations
- With Laplace transform LTI systems can be analyzed within frequency domain instead of time domain
- With Laplace transform ordinary differential equations are transformed into algebraic equations
- With Laplace transform convolution integral is transformed into multiplication

Laplace transform

- Laplace transform $F(s)$ of function $f(t)$:

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$$

- Laplace transform of function $f(t)$ exists if Laplace integral converges
- See next slide for table of Laplace transforms you will frequently need
- Solve enough examples to remember simple Laplace transforms by heart

Table of Laplace transforms

Unit step function $1(t)$	$\frac{1}{s}$
Unit impulse (Dirac delta) function $\delta(t)$	1
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Two **important** rules about Laplace transform

- Multiplication of $f(t)$ by e^{-at}:
  If $f(t)$ is Laplace transformable with Laplace transform $F(s)$ then
  $$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)$$
- Change of time scale:
  If $f(t)$ is Laplace transformable with Laplace transform $F(s)$ then
  $$\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\} = aF(as)$$
- with these 2 rules you can find Laplace transform of many different functions

Differentiation theorem

For function $f(t)$ of exponential order:

- For 1st order derivative we have:

$$\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = sF(s) - f(0)$$

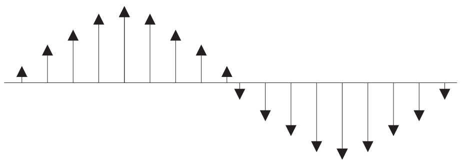
- For 2nd order derivative we have:

$$\mathcal{L}\left\{\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right\} = s^2F(s) - sf(0) - \frac{\mathrm{d}f}{\mathrm{d}t}(0)$$

Note: Partial integration: $\int_{v_0}^{v_f} u\mathrm{d}v = u_f v_f - u_0 v_0 - \int_{u_0}^{u_f} v\mathrm{d}u$

Convolution integral

For linear systems input functions can be formulated as a sum of impulses of proper strength:



If the unit impulse response of a linear system is $h(t)$ the response of the linear system to input function $u(t)$ is:

$$y(t) \approx \sum_{n=0}^{n=t/(\Delta t)} u(n\Delta t)h(t - n\Delta t)\Delta t$$

This turns into “convolution integral” for $\Delta t \rightarrow 0$:

$$y(t) = \int_0^t u(\tau)h(t - \tau)\mathrm{d}\tau$$

Laplace transform of convolution

Laplace transform of convolution integral is multiplication of the Laplace transforms of the two functions:

$$\mathcal{L}\left\{\int_0^t u(\tau)h(t-\tau)\mathrm{d}\tau\right\} = U(s)H(s)$$

Transfer function

- Transfer function characterizes systems that are linear & time-invariant
- Transfer function: ratio of Laplace transform of output $Y(s)$ and Laplace transform of input $U(s)$ assuming all initial conditions are zero. Thus Transfer function is Laplace transform of unit impulse response
- For input $U(s)$ the output in Laplace domain is $Y(s) = H(s)U(s)$ and in time domain is

$$y(t) = \int_0^t u(\tau)h(t-\tau)d\tau$$

(Multiplication in Laplace domain - Convolution integral in time domain)

Converting a differential equation

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_1\dot{u}(t) + b_0 u(t)$$

- Input: $u(t)$
- Laplace transform of input: $U(s)$
- Output: $y(t)$
- Laplace transform of output: $Y(s)$
- Assuming zero initial conditions: $y(0) = 0$, $\dot{y}(0) = 0$, $u(0) = 0$
- Laplace transform of derivatives:
  $\mathcal{L}\{\ddot{y}(t)\} = s^2 Y(s)$, $\mathcal{L}\{\dot{y}(t)\} = sY(s)$, $\mathcal{L}\{\dot{u}(t)\} = sU(s)$
- In Laplace domain:

$$s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = b_1 s U(s) + b_0 U(s)$$

$$Y(s)(s^2 + a_1 s + a_0) = U(s)(b_1 s + b_0)$$

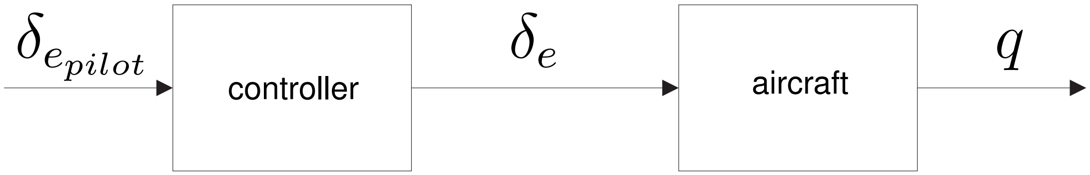
- Transfer function: $H(s) = \frac{Y(s)}{U(s)} = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$

Exercise

- Consider the differential equation $\ddot{y} + 4\dot{y} + 3y = u$
- Convert this into a transfer function
- Determine the output (in Laplace domain) to an impulse input with size 2
- Determine the response in the time domain (**Hint:** Use partial fraction expansion)

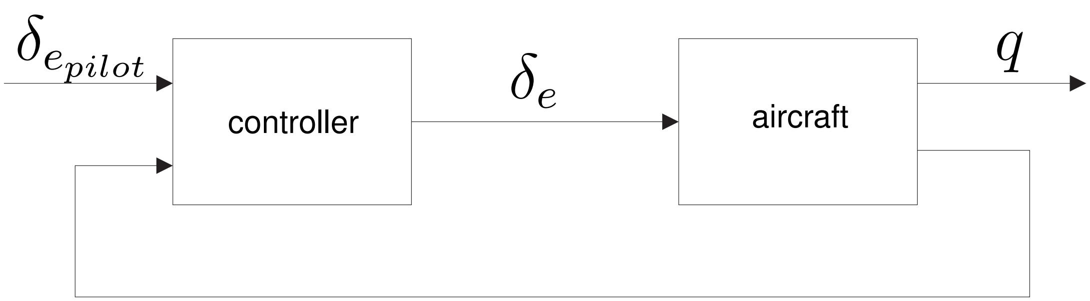
Control: Modifying the response

Open-loop control



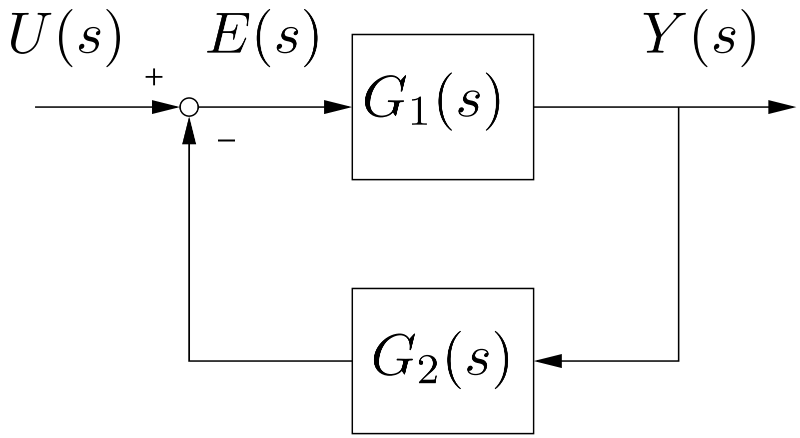
- Does not compensate for disturbances
- Cannot stabilize unstable systems

Closed-loop control



- Uses output of the system for control
- Can stabilize unstable systems

Closed-loop transfer function



- Equivalent transfer function for closed-loop system:

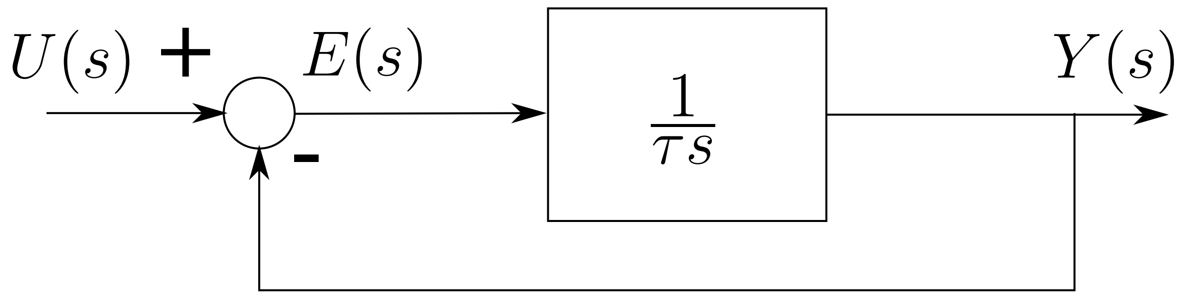
$$H(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

- *zeros*: roots of numerator
- *poles*: roots of denominator

System's response

- First step for analyzing dynamical systems is to derive a mathematical model
- Next we can analyze the system's response to particular test input functions
- **Typical test input functions:** unit-step, unit-impulse, unit-ramp, sinusoidal functions, ...
- Generally the time response of a dynamical system consists of two components:
  (1) transient response (2) steady-state response

First-order systems



Closed-loop transfer function (suppose that $\tau \neq 0$):

$$H_c(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{\tau s}}{1 + \frac{1}{\tau s}} = \frac{1}{\tau s + 1}$$

Closed-loop pole: $s = -\frac{1}{\tau}$

1st-order system: Unit-step response

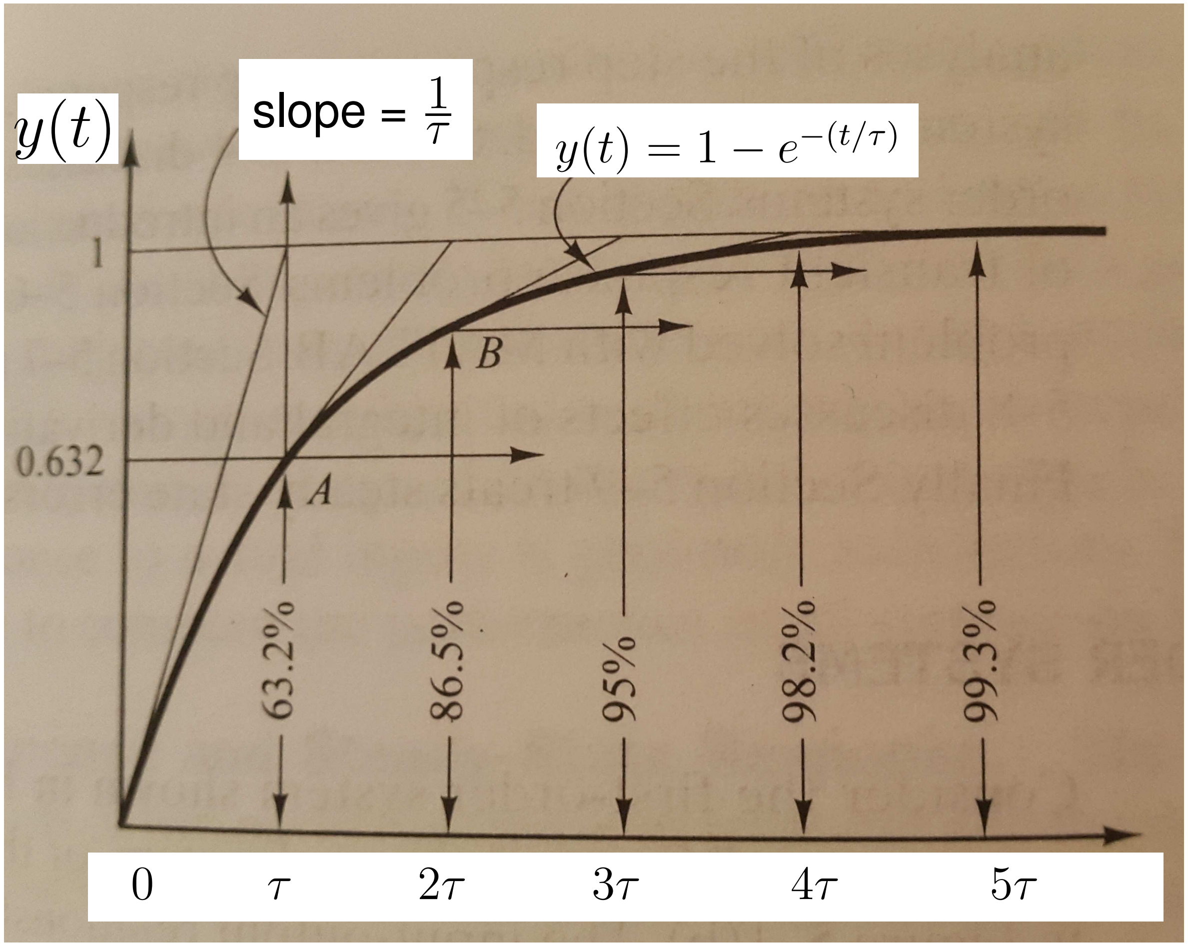
- Laplace transform of unit-step function: $U(s) = \frac{1}{s}$
- Multiply the transfer function and the Laplace transform of input function to get the output:

$$Y(s) = \frac{1}{\tau s + 1} \cdot \frac{1}{s}$$

- Perform partial fraction expansion and use Laplace table:

$$Y(s) = \frac{1}{s} - \frac{\tau}{\tau s + 1} = \frac{1}{s} - \frac{1}{s + 1/\tau} \Rightarrow \boxed{y(t) = 1 - e^{-t/\tau}}$$

For $t = 0$ output is 0
For $t \to \infty$ output becomes 1 when $\tau > 0$ and output goes to $-\infty$ when $\tau < 0$

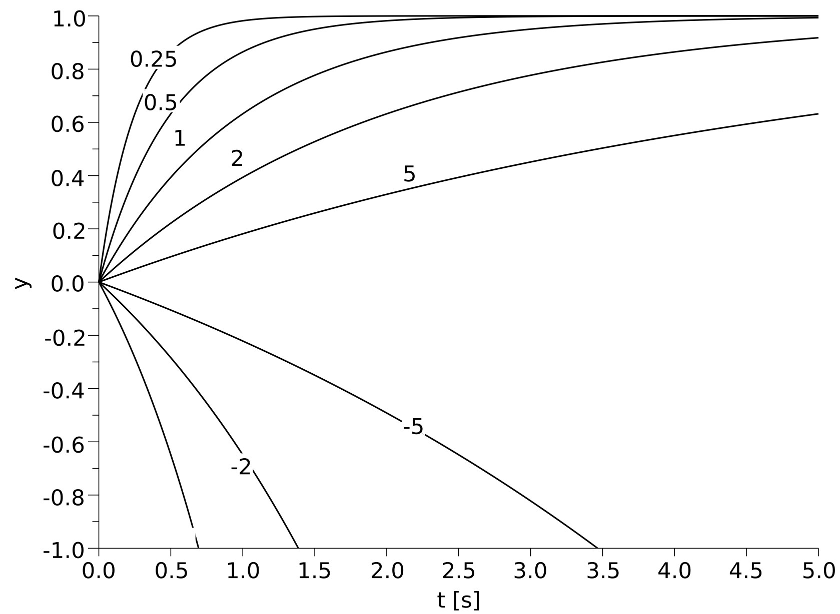


For $t = \tau$ the output is $y(t) = 1 - e^{-1} = 0.632$

Characteristic 1. At $t = \tau$ the response of the system independent of τ is 0.632

Characteristic 2. $\dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \dfrac{1}{\tau}e^{-t/\tau}$; at $t = 0$: slope = $\dfrac{1}{\tau}$

Characteristic 3. Time constant τ determines the speed of the response: the smaller the value of τ. the faster the response



- A negative pole close to 0 (corresponding to **large positive** τ) results in **slow** response
- A negative pole far from 0 (corresponding to **small positive** τ) results in **fast** response
- A positive pole (corresponding to **negative** τ) results in **unstable** behavior

1st-order system: Unit-ramp response

- Unit-ramp function: $f(t) = t$ for $t \geq 0$
- Laplace transform of unit-ramp function: $F(s) = \frac{1}{s^2}$
- Multiply the transfer function and the Laplace transform of input function to get the output:

$$Y(s) = \frac{1}{\tau s + 1} \cdot \frac{1}{s^2}$$

- Perform partial fraction expansion and use Laplace table:

$$Y(s) = \frac{1}{s^2} - \frac{\tau}{s} - \frac{\tau^2}{\tau s + 1} \Rightarrow \boxed{y(t) = t - \tau + \tau e^{-t/\tau}}$$

1st-order system: Unit-impulse response

- Laplace transform of unit-impulse function: $F(s) = 1$
- Multiply the transfer function and the Laplace transform of input function to get the output:

$$Y(s) = \frac{1}{\tau s + 1} = \frac{1}{\tau} \cdot \frac{1}{s + 1/\tau}$$

- Use Laplace table:

$$\boxed{y(t) = \frac{1}{\tau} e^{-t/\tau}}$$

1st-order systems: recap

- Unit-ramp response: $y(t) = t - \tau + \tau e^{-t/\tau}$
- Unit-step response: $y(t) = 1 - e^{-t/\tau}$
- Unit-impulse response: $y(t) = \frac{1}{\tau} e^{-t/\tau}$
- Note:
  Unit-step function is the derivative of unit-ramp function
  Unit-impulse function is the derivative of unit-step function
- Important characteristic of linear time-invariant systems:
  Response of the system to the derivative of an input function is the same as the derivative of the response of the system to the input function itself

Second-order system

$$\boxed{H_c(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

$\omega_n = \sqrt{k/m}$: undamped natural frequency, $\zeta = \frac{c}{2\sqrt{km}}$: damping ratio

No zeros and 2 poles:

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

- $0 < \zeta < 1$ (under-damped): closed-loop poles are complex conjugates in left-hand s-plane
- $\zeta = 1$ (critically damped): closed-loop poles lie on each other in left-hand s-plane
- $\zeta > 1$ (over-damped): closed-loop poles are real negative values in the s-plane

2nd-order system: Unit-step response

Case 1: Under-damped $0 < \zeta < 1$:

Note: $\omega_d = \omega_n \sqrt{1-\zeta^2} > 0$: *damped natural frequency*

- Multiply $H_c(s)$ and $U(s) = \frac{1}{s}$ to obtain $Y(s)$:

$$Y(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- Use partial fraction expansion:

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

... use Laplace table

- First rewrite the output as:

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s - p_1)(s - p_2)}$$

- Recall $p_1 = -\zeta\omega_n + j\omega_d$ and $p_2 = -\zeta\omega_n - j\omega_d$:

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- Following final reformulation allows direct use of Laplace table:

$$Y(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Unit-step response of 2nd-order system in time domain

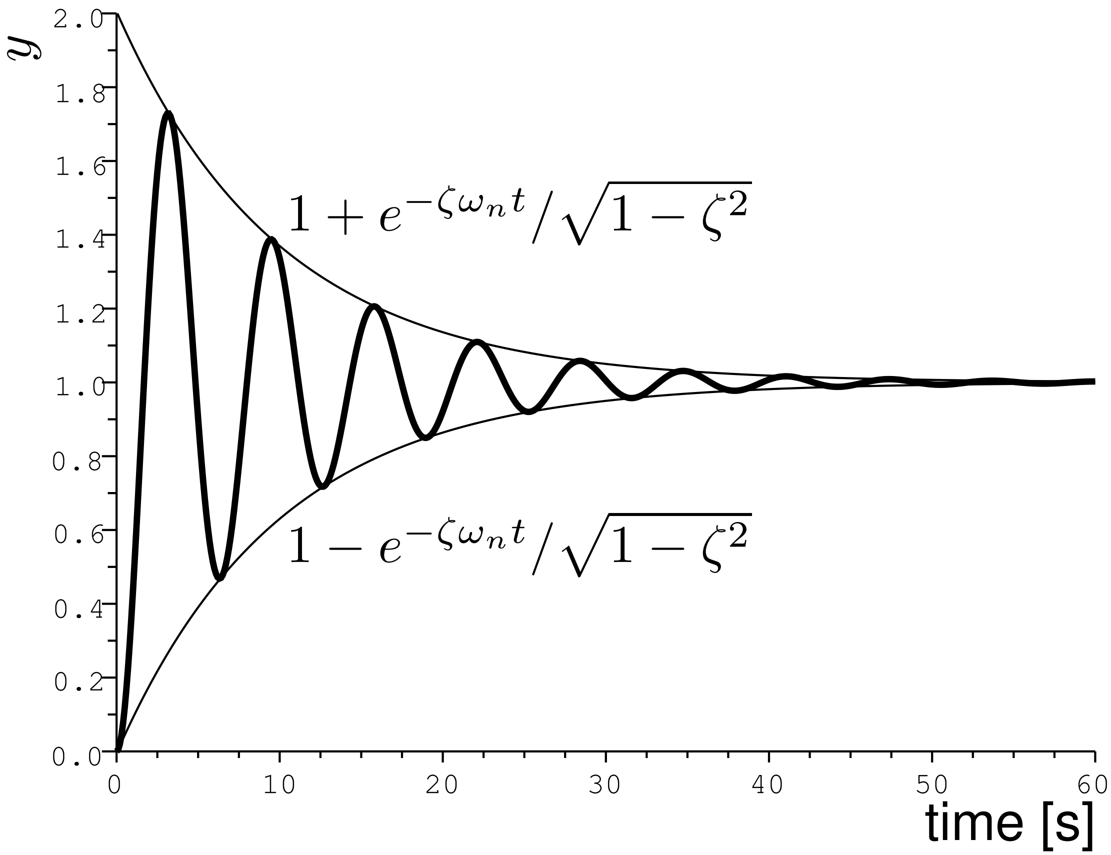
$$\mathcal{L}^{-1}\left\{Y(s)\right\} = y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\left(\omega_d t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t\right)\right)$$

$$\Rightarrow \boxed{y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}$$

Note: Assume $\sin\theta = \sqrt{1-\zeta^2}$ and $\cos\theta = \zeta$.

Question: What happens for $\zeta = 0$?

Under-damped transient response



2nd-order system: Unit-step response

Case 2: Critically damped $\zeta = 1$:

$$y(t) = 1 - e^{-\omega_n t}\left(1 + \omega_n t\right)$$

2nd-order system: Unit-step response

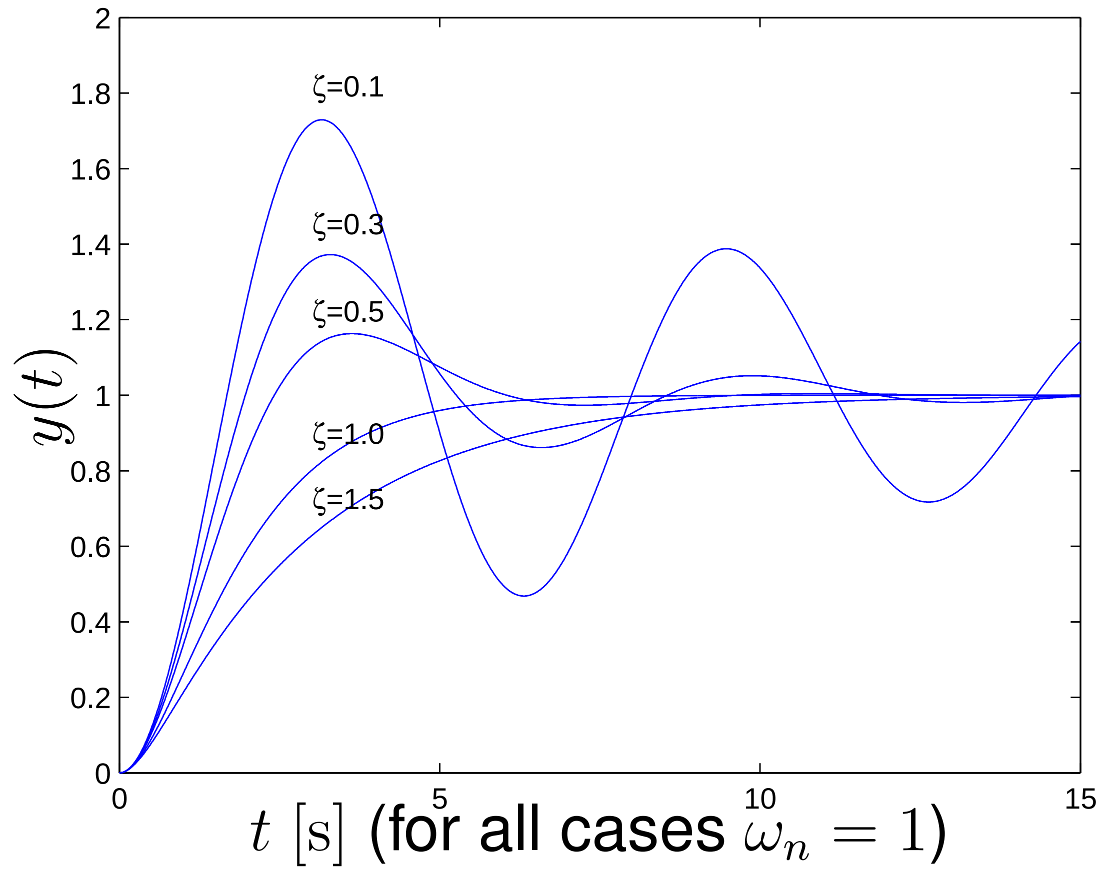
Case 3: Over-damped $\zeta > 1$:

$$y(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right)$$

With

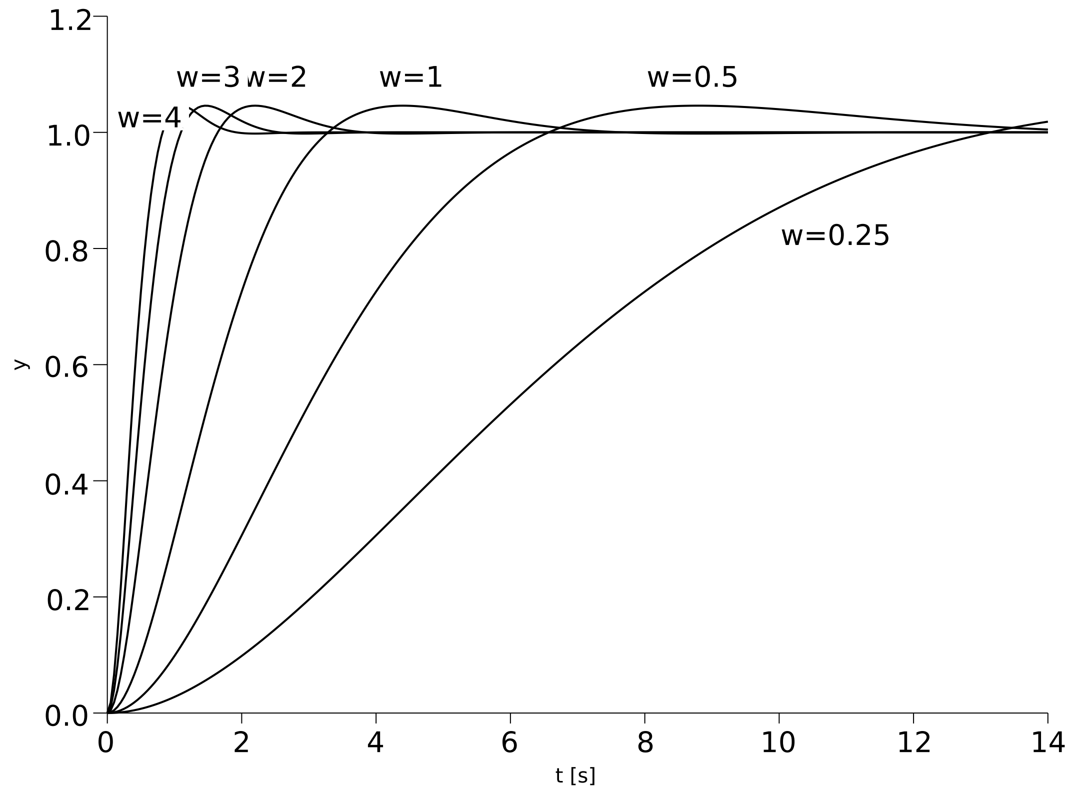
$$s_1 = \left(\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n, \qquad s_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right) \omega_n$$

2nd-order system: Effect of ζ on unit-step response



Shape of the response function is determined by ζ.

2nd-order system: Effect of ω_n on unit-step response



Speed of the response function is determined by ω_n ($\zeta = 0.7$).

Summary

- Ordinary differential equations describe the behavior of linear time-invariant dynamic systems
- Using Laplace transform an ODE is represented by an algebraic function that for a system gives the transfer function
- We learned about 1st and 2nd-order dynamical systems & their response to various standard inputs

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