AE4301 Automatic Flight Control System Design
Part I: Control Theory

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Lecture 2

Main objectives

- Understanding characteristics of time-domain transient response & being able to compute them
- Being able to develop and use root locus for pole placement

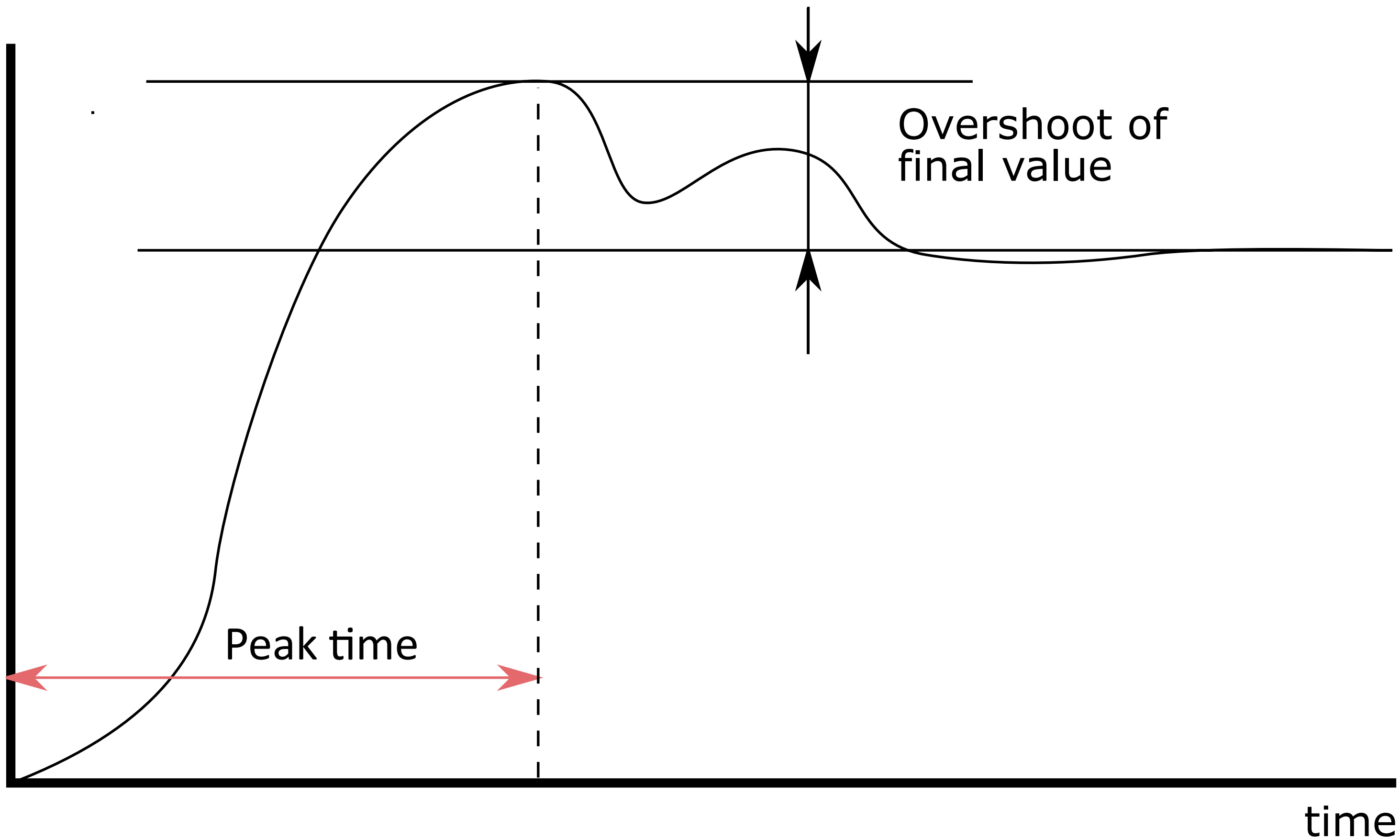
Material

- Slides on Brightspace
- Homework assignments on Brightspace
- Discussions during lectures

Transient response: Time domain criteria

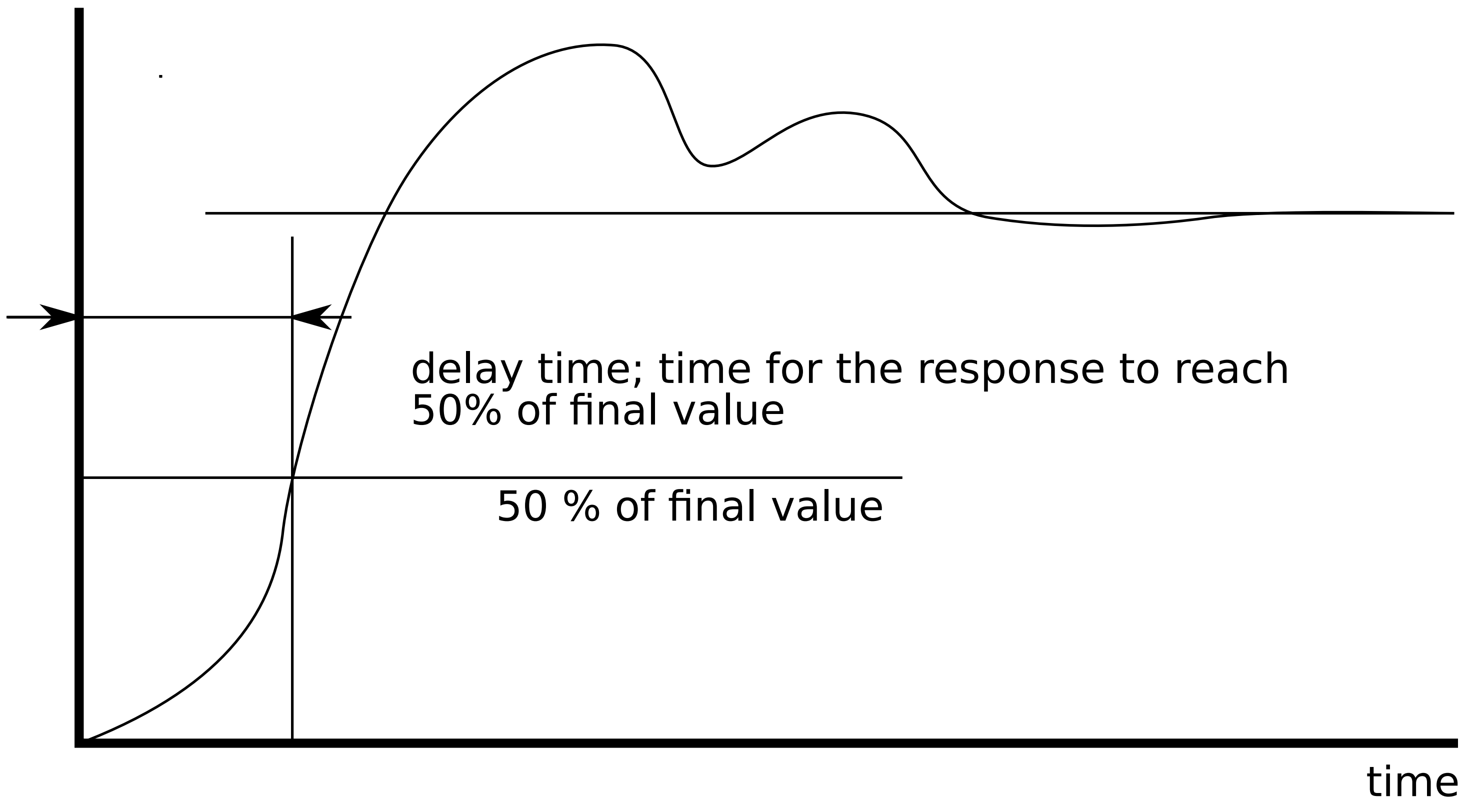
- We characterize a system's transient response based on unit-step input
- Transient response characteristics are based on assumption of zero initial conditions
- Characteristics for transient response to unit-step input:
  - Overshoot
  - Delay time
  - Rise time
  - Settling time
  - Peak time
- Time domain criteria can be deduced via pole locations

Overshoot & peak time



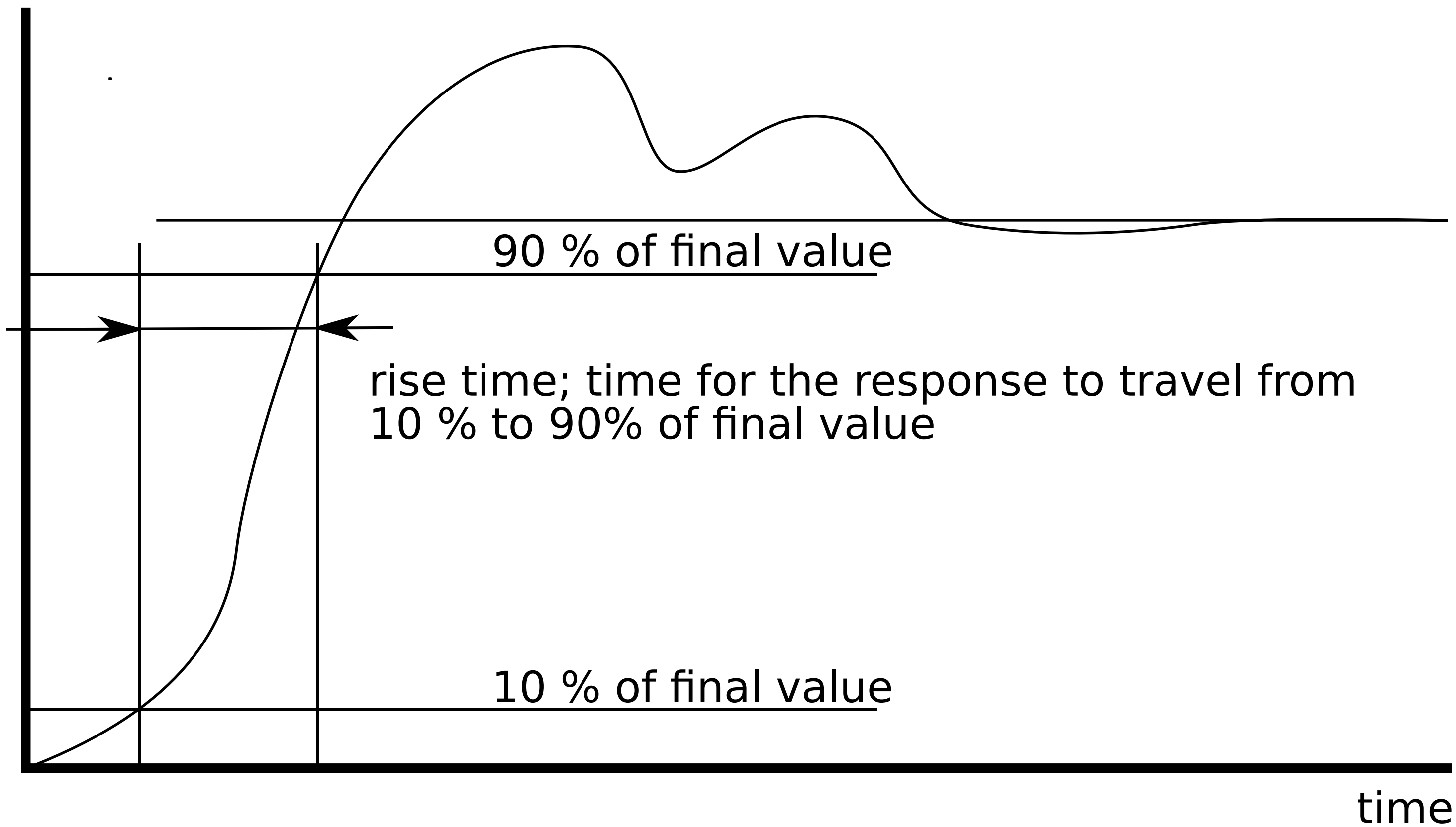
Overshoot is often given in % of final value

Delay time



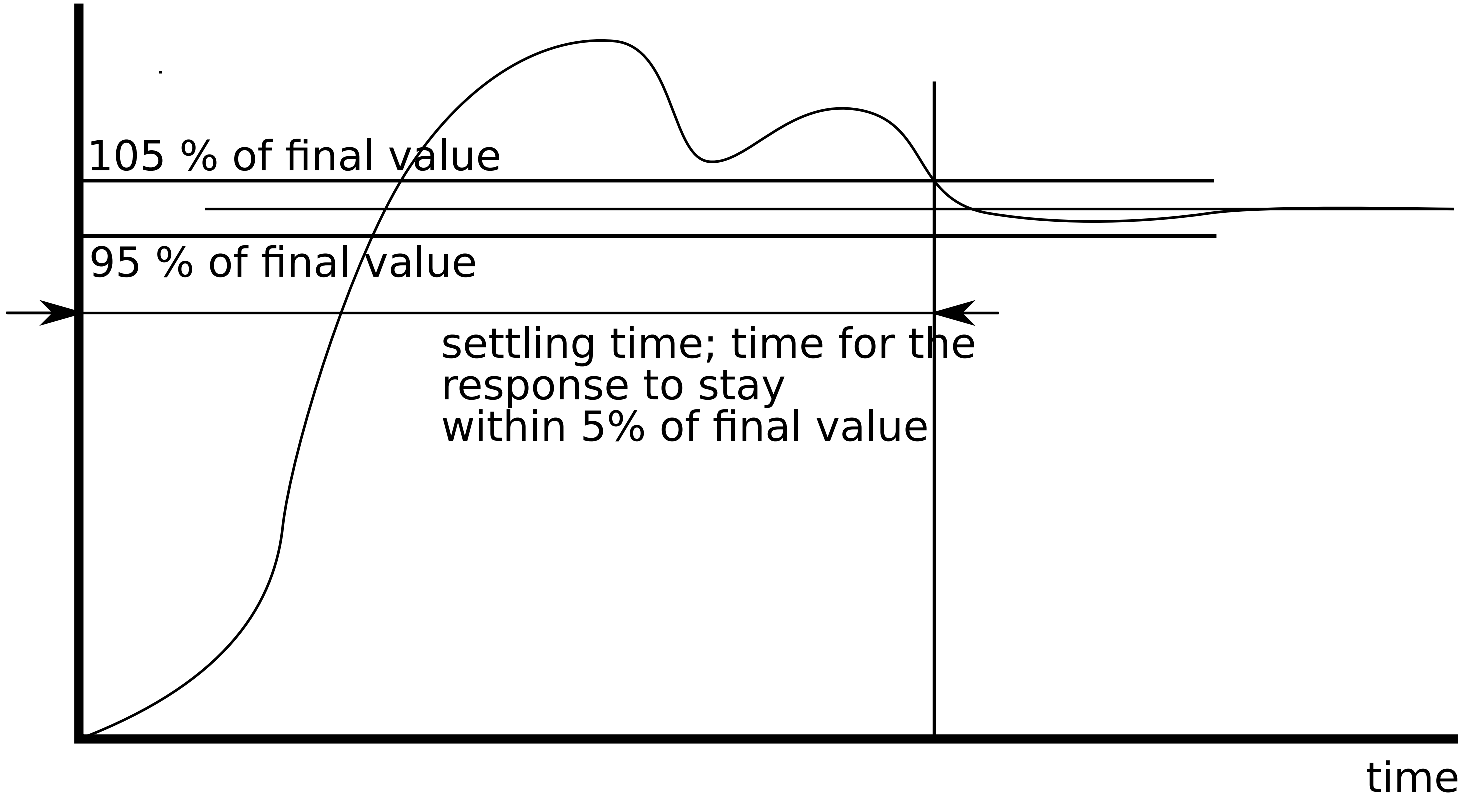
Other criteria (e.g. 10%) than the 50% limit may also be used

Rise time



Different rise-time criteria is possible (e.g. 0% to 100% or 5% to 95%)

Settling time



Common settling time criteria: 10%; 5%; 2%

1st-order system: Design

- Unit-step response: $y(t) = 1 - e^{-t/\tau}$
- Design τ for desired delay time ($y\left(t^{\text{delay}}\right) = 0.5$): $\tau = -\dfrac{t^{\text{delay}}}{\ln 0.5}$
- Design τ for desired rise time (10% to 90%): $\tau = \dfrac{t^{\text{rise}}}{\ln 9}$
- Design τ for desired 2% settling time, i.e., $y\left(t^{\text{settle}}\right) = 0.98$: $\tau = -\dfrac{t^{\text{settle}}}{\ln 0.02}$
- **Question:** What is the overshoot?
- **Conclusion:** For 1st-order systems time domain criteria of response can be adjusted via τ

2nd-order system: Design

- Unit-step response (under-damped case):

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\left(\omega_d t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t\right)\right)$$

- Desired peak time: $t^{\text{peak}} = \pi/\omega_d$

- **Question:** What is the overshoot? ... $e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

- Desired rise time (0% to 100%):

$$t^{\text{rise}} = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\arctan\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

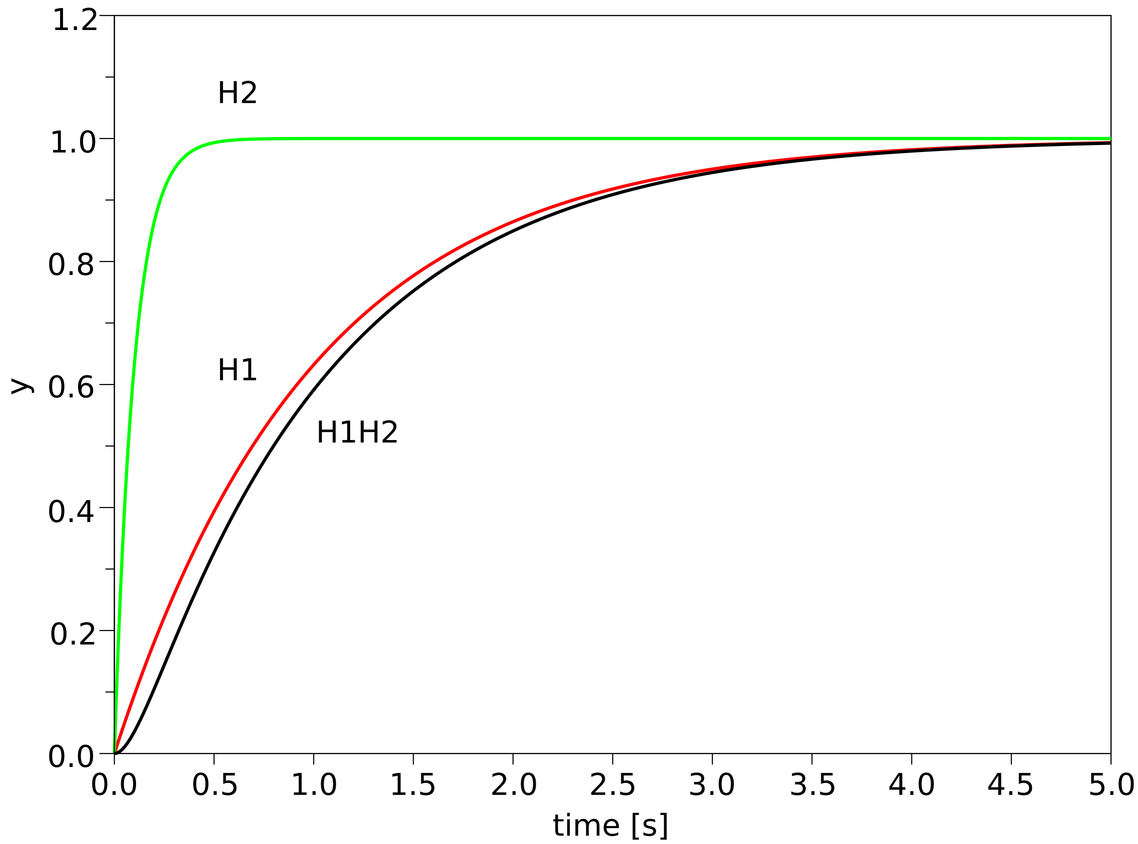
- For 2nd-order systems time domain criteria of response can be adjusted via ω_n and ζ (i.e., ω_d for under-damped systems)

Pole placement

- Pole-placement design: decide on closed-loop pole locations
- Control effort is related to how far open-loop poles are moved by feedback
- When a zero is near a pole, system may be nearly uncontrollable (moving such poles needs large control gains/effort)
- **Dominant poles**: Poles closest to imaginary axis in s-plane give rise to longest lasting terms of system's transient response. These poles are called dominant poles.

Effect dominant pole 1

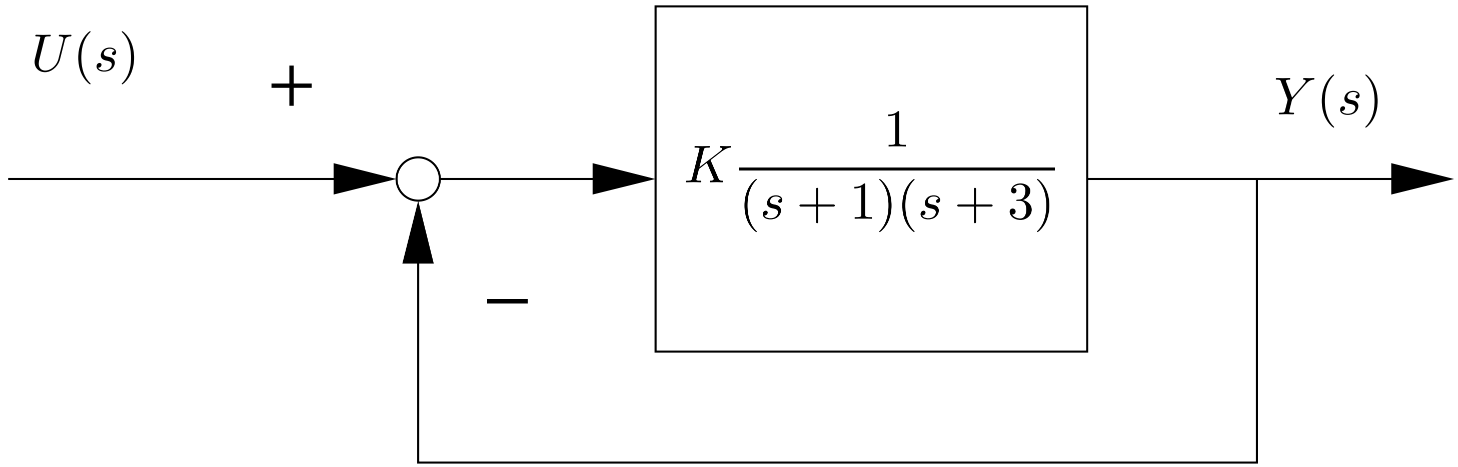
$$H_1 = 1/(s+1),\, H_2 = 10/(s+10),$$



Control: Selecting pole locations

- Characteristics of transient response of closed-loop systems is closely related to location of closed-loop poles
- When a variable gain is in closed-loop system, location of closed-loop poles depend on gain value
  Design problem: selection of gain value to, e.g.:
  - Stability (all poles of transfer function must be in LHP)
  - Tracking (force output to follow a reference input as closely as possible)
  - Regulation (keep response's error small in presence of disturbances)

Poles for various gains

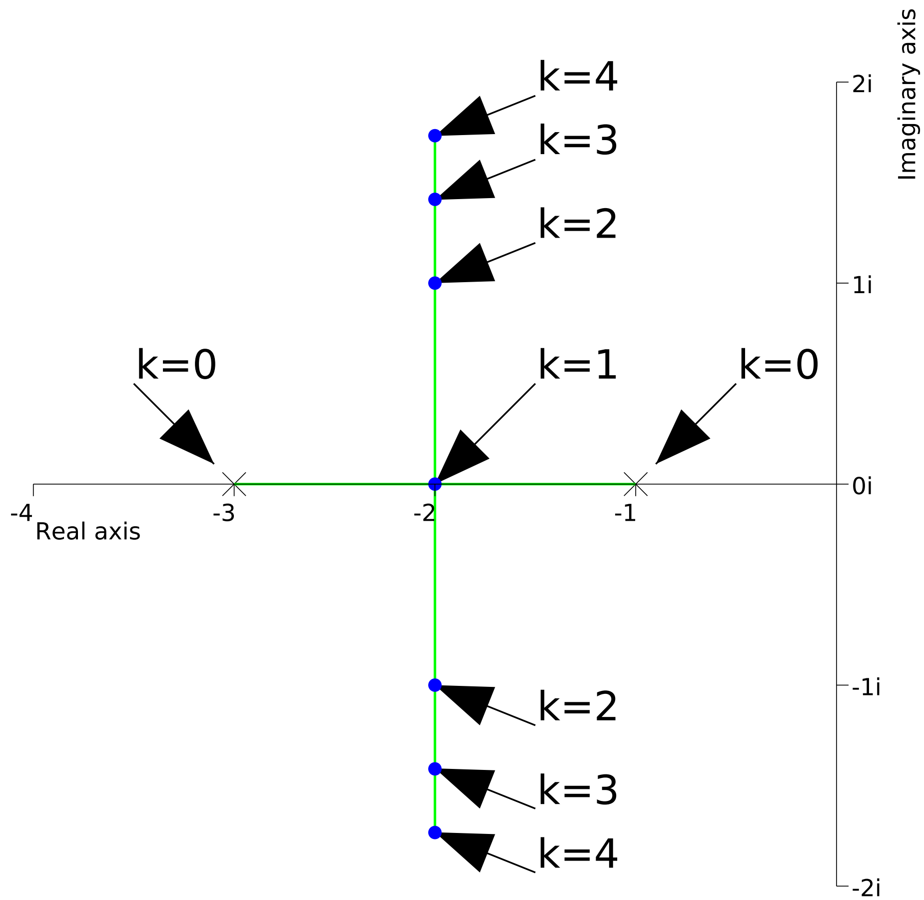


Closed loop transfer function: $H_c(s) = \frac{K}{(s+1)(s+3)+K}$

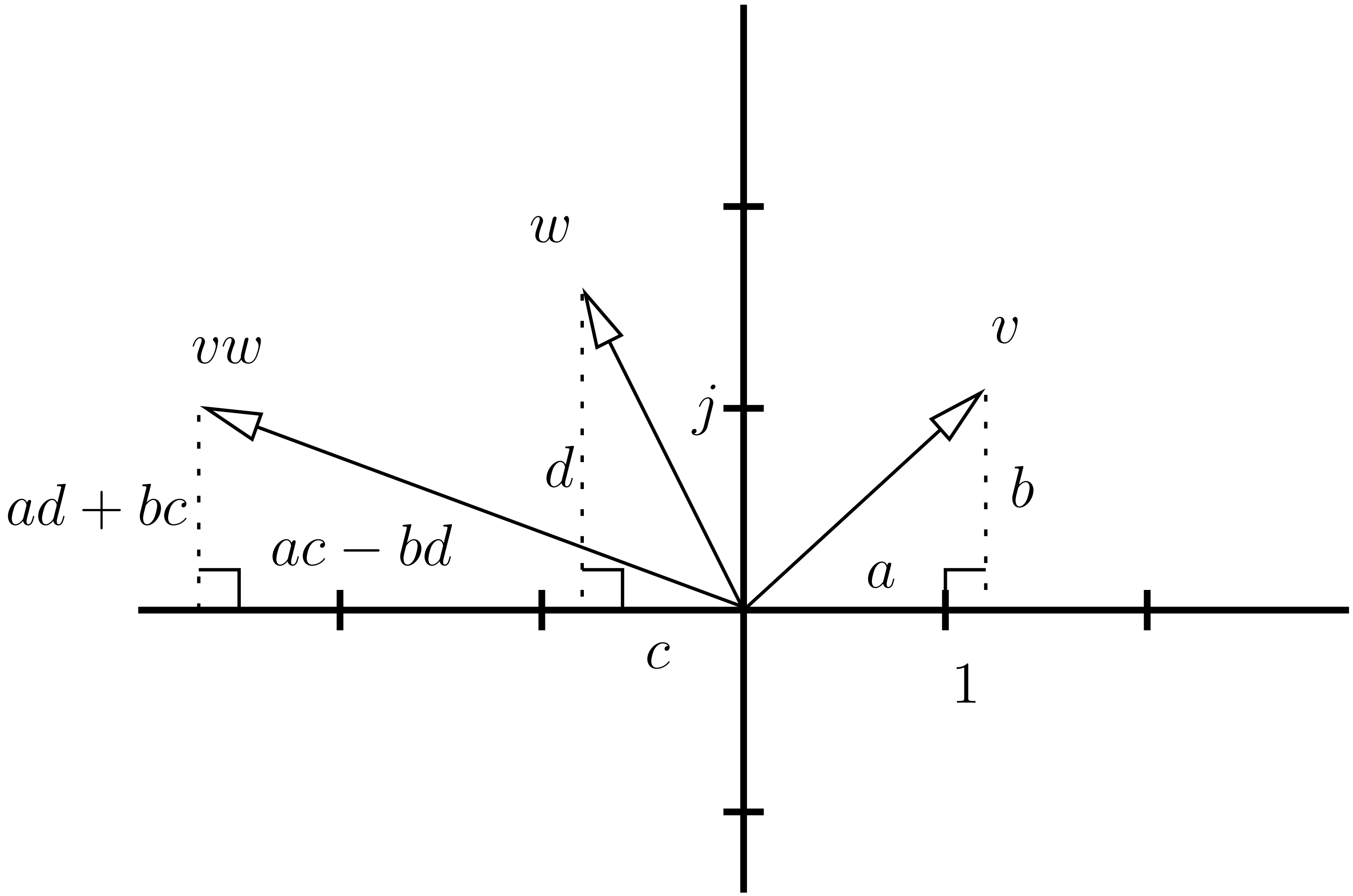
Characteristic equation: $s^2 + 4s + K + 3 = 0$

Poles as function of gain K: $p_{1,2}(K) = -2 \pm \sqrt{1-K}$

Pole locations for various k



Complex numbers: Cartesian representation

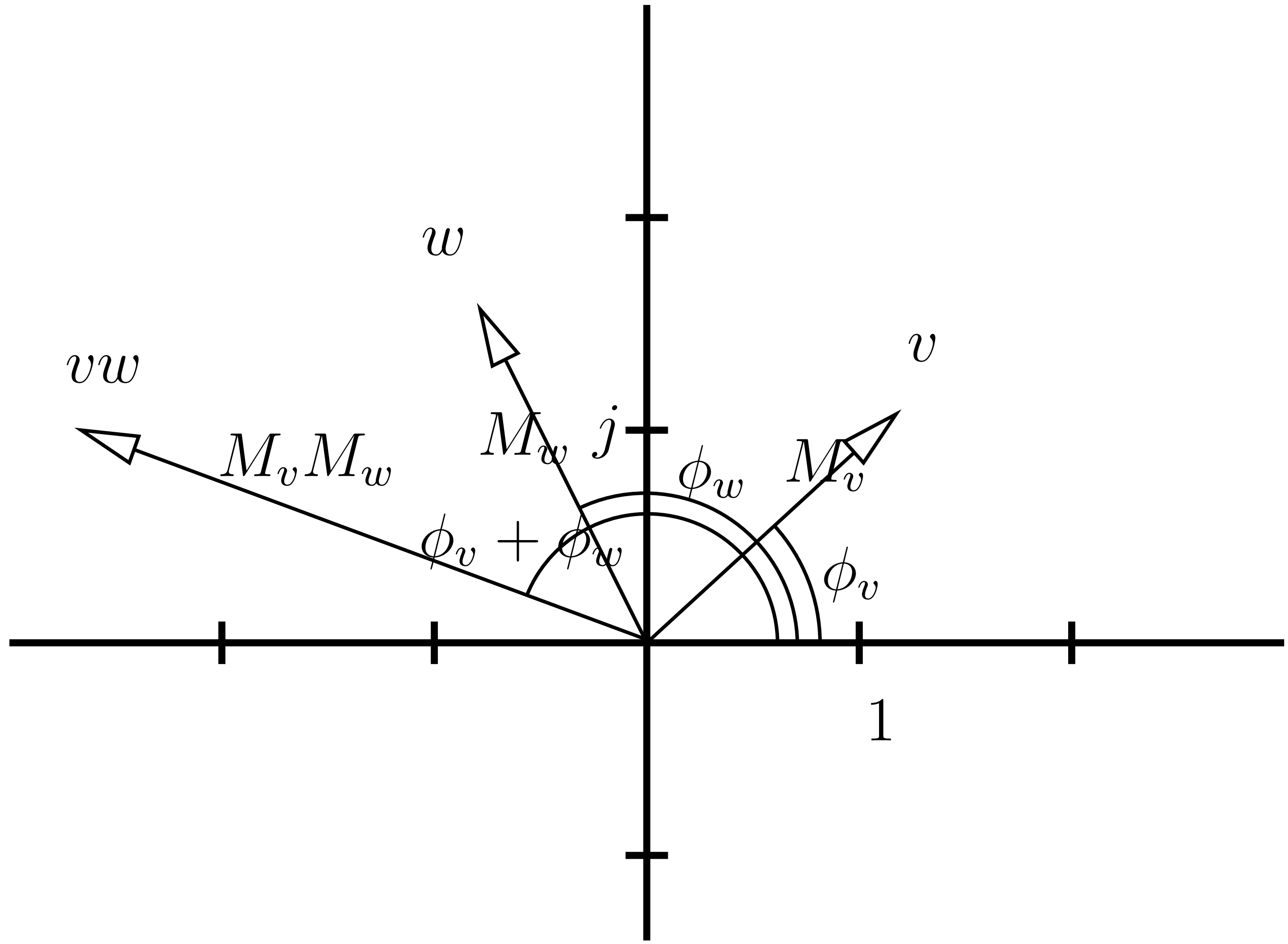


Multiplication & division: Cartesian representation

$$vw = (a + jb)(c + jd) = (ac - bd) + j(ad + bc)$$

$$\frac{v}{w} = \frac{a + jb}{c + jd} = \frac{a + jb}{c + jd} \cdot \frac{c - jd}{c - jd} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

Complex numbers: Polar representation



Multiplication & division: Polar representation

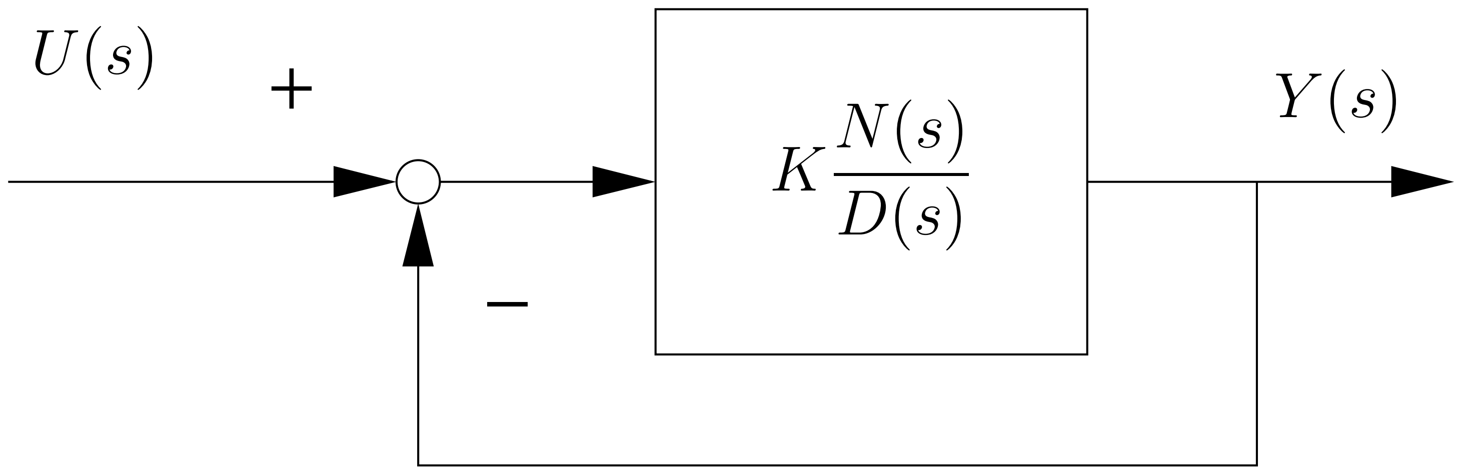
$$
\begin{aligned}
v &= M_v e^{j\phi_v} \\
w &= M_w e^{j\phi_w} \\
vw &= M_v e^{j\phi_v} M_w e^{j\phi_w} = M_v M_w e^{j(\phi_v + \phi_w)} \\
\frac{v}{w} &= \frac{M_v e^{j\phi_v}}{M_w e^{j\phi_w}} = \frac{M_v}{M_w} e^{j(\phi_v - \phi_w)}
\end{aligned}
$$

Remember:

$$\boxed{e^{j\phi} = \cos\phi + j\sin\phi}$$

Root-locus method

Consider transfer function $G(s) = K\dfrac{N(s)}{D(s)}$ in a closed loop:



Equivalent closed-loop transfer function:

$$H_c(s) = \frac{KN(s)}{D(s) + KN(s)}$$

Root-locus method

- Root locus includes location of roots of closed-loop characteristic equation when gain K varies from 0 to ∞
- Root locus plots are always symmetrical w.r.t. real axis $\rightarrow$ we only need to construct upper (lower) half of root loci
- Characteristic equation:

$$D(s) + KN(s) = 0 \quad \Rightarrow \quad \boxed{KN(s)/D(s) = -1}$$

- Location of poles and zeros of $KN(s)/D(s)$ are determined
- Location of roots of $D(s) + KN(s) = 0$ or $KN(s)/D(s) = -1$ when K varies from 0 to ∞ is drawn in complex plane

Rules for root locus

- For $K = 0$ root locus corresponds to open-loop poles
- n (n number of poles) branches of root locus start at poles and m (m number of zeros) branches end at zeros
- For $K \to \infty$, $n - m$ poles follow the aymptotes to infinity
- Break-away points are obtained from $\mathrm{d}K/\mathrm{d}s = 0$
- Root locus includes parts of real axis that are to the left of an odd number of poles and zeros
- Angles of asymptotes with real axis ($l = 0, 1, 2, \ldots$):

$$\Phi = \frac{180^\circ \, (2l + 1)}{n - m}$$

- Intersection point of asymptotes:

$$\sigma^{\mathrm{int}} = \frac{\sum_{j=1}^{n} Re(p_j) - \sum_{i=1}^{m} Re(z_i)}{n - m}$$

Angle & magnitude conditions

- Rewrite characteristic equation as following:

Angle condition:

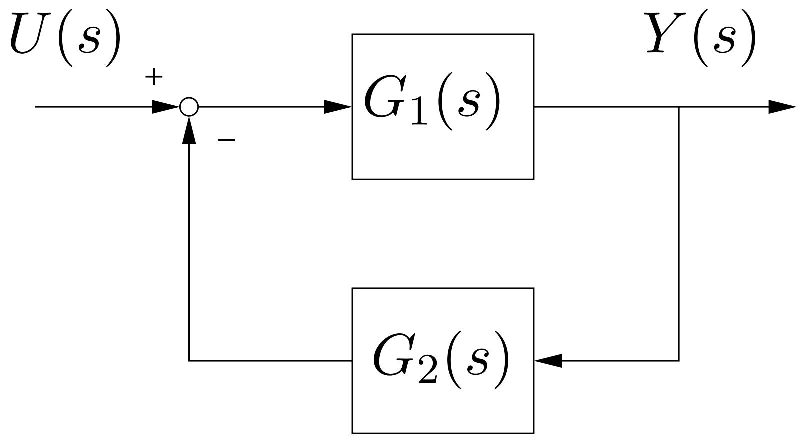
$$\angle\left(K\frac{(s-z_1)}{(s-p_1)(s-p_2)}\right) = \angle -1$$

$$\angle(s-z_1)-\angle(s-p_1)-\angle(s-p_2) = \pm 180^\circ(2l+1) \qquad l = 0, 1, 2, \ldots$$

Magnitude condition:

$$|K|\,\frac{|s-z_1|}{|s-p_1|\,|s-p_2|} = |-1|$$

Example



- Suppose $G_1(s) = \dfrac{K}{s(s+1)(s+2)}$ and $G_2(s) = 1$.
- Sketch root locus plot.

Let's solve this together ...

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  3 asymptotes having angles 60°, -60°, 180°.

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- **Step 4.** Determine intersection of asymptotes with real axis.

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- **Step 4.** Determine intersection of asymptotes with real axis.
- **Step 5.** Determine break-away point.

Break-away points

Use characteristic equation and compute $\frac{\mathrm{d}K}{\mathrm{d}s} = 0$:

$$s(s+1)(s+2) + K = 0$$

Those solutions for s that result in a positive K can be considered as break-away points.

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$$\frac{\mathrm{d}K}{\mathrm{d}s} = -\left(3s^2 + 6s + 2\right) = 0$$

$$\boxed{s = -0.4226} \qquad \Rightarrow K = 0.3849$$
$$s = -1.5774 \qquad \Rightarrow K = -0.3849$$

Summary

- We learned about various characteristics of transient response of 1st and 2nd-order systems via unit-step response
- We learned about dominant poles & root locus