Control Engineering (SC42095)

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Lecture outline

- Analysis of discrete-time systems
  - Stability (definitions, tests, Lyapunov method)
  - Controllability and reachability
  - Observability and detectability
  - Simulation

Stability: basic notions

Defined with respect to change in initial conditions.

The solution $x^0(k)$ of

$$x(k+1) = f(x(k), k)$$

is *stable* if for a given $\epsilon > 0$, there exists a $\delta(\epsilon, k_0) > 0$

such that all solutions $x(k)$ with $\|x(k_0) - x^0(k_0)\| < \delta$

are such that $\|x(k) - x^0(k)\| < \epsilon$ for $k \geq k_0$.

Definition - Asymptotic stability

The solution $x^0(k)$ is *asymptotically stable*

if it is stable and δ can be chosen such that

$\|x(k_0) - x^0(k_0)\| < \delta$ implies that

$\|x(k) - x^0(k)\| \to 0$ when $k \to \infty$.

Linear discrete-time systems

$$x(k+1) = \Phi x(k), \quad x(0) = a$$

Stability of $one\ solution$ implies stability of all solutions, i.e., stability of the $system$.

A linear discrete-time system is asymptotically stable if and only if all eigenvalues of Φ are strictly inside the unit disc $\big(|\lambda_i| < 1\big)$.

(Solutions $x(k) = \Phi^k x(0)$ are combinations of terms λ_i^k or $p_i(k)\lambda_i^k$.)

BIBO stability

A linear time-invariant system is BIBO stable if a bounded input gives a bounded output for all initial values.

Asymptotic stability is the strongest concept.

Asymptotically stable $\Rightarrow$ BIBO stable,

but

stable $\not\Rightarrow$ BIBO stable.

Stability tests

- Direct numerical or algebraic computation of $\lambda(\Phi)$ in MATLAB: `eig(Phi)`.

- Methods based on properties of the characteristic polynomial (Jury's criterion).

$$A(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0$$

- Root-locus method, Nyquist criterion, Bode plot.

- Lyapunov method.

Lyapunov theory

$$x(k+1) = f(x(k)), \quad f(0) = 0$$

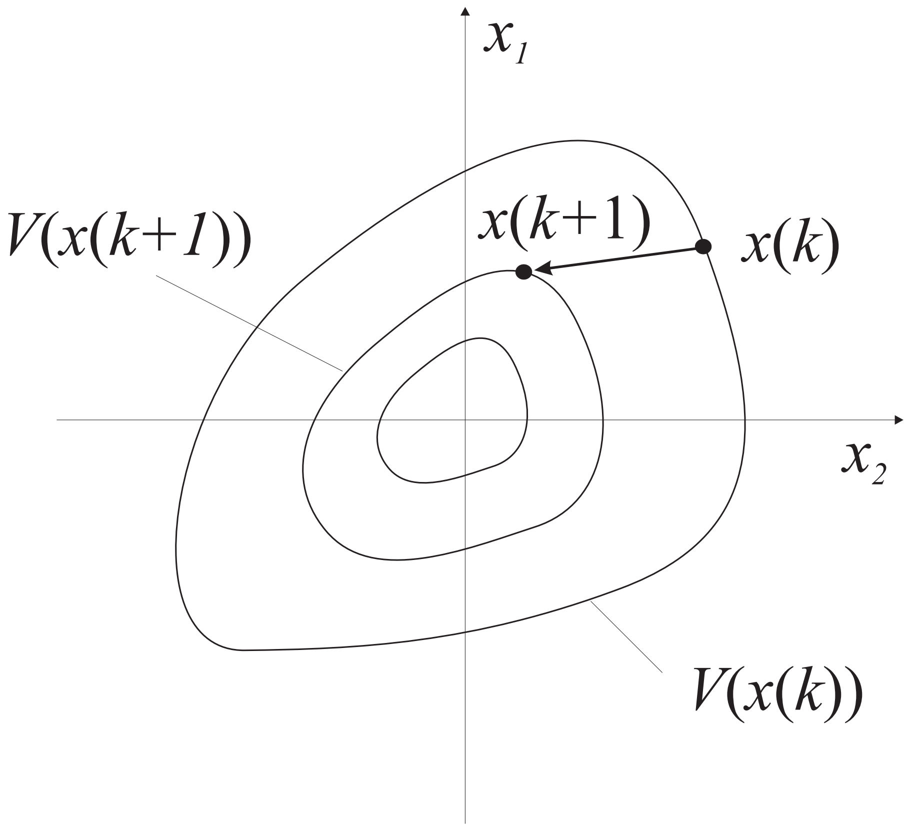
Monotonic convergence $\|x(k+1)\| < \|x(k)\|$ – too strong condition for stability. Another measure: Lyapunov function:

- $V(x)$ is continuous in x and $V(0) = 0$
- $V(x)$ is positive definite
- $\Delta V(x) = V(f(x)) - V(x)$ is negative definite

Existence of a Lyapunov function implies asymptotic stability for the solution $x = 0$

Geometric interpretation

$$x(k+1) = f(x(k)), \quad f(0) = 0$$



Linear system

$$x(k+1) = \Phi x(k), \qquad V(x) = x^T P x, \quad P > 0$$

$$\Delta V(k) = V(\Phi x) - V(x) = x^T \Phi^T P \Phi x - x^T P x$$

$$= x^T (\Phi^T P \Phi - P) x$$

V is a Lyapunov function iff there exists a $P > 0$ that satisfies the *Lyapunov equation*

$$\Phi^T P \Phi - P = -Q, \qquad Q > 0$$

Controllability and reachability

Controllability: Possible to find a control sequence such that the origin can be reached from any initial state in finite time.

Reachability: Possible to find a control sequence such that an arbitrary state can be reached from any initial state in finite time.

With continuous-time systems, controllability implies reachability.

Not for discrete-time systems...

Controllability and reachability (cont'd)

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k), \quad x(0) \text{ given} \\ y(k) &= C x(k) \end{aligned}$$

At time n ($n =$ order of the system):

$$\begin{aligned} x(n) &= \Phi^n x(0) + \Phi^{n-1}\Gamma u(0) + \ldots + \Gamma u(n-1) \\ x(n) &= \Phi^n x(0) + W_c U \end{aligned}$$

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma & \ldots & \Phi^{n-1}\Gamma \end{pmatrix}, \quad U = \begin{pmatrix} u^T(n-1) & \ldots & u^T(0) \end{pmatrix}^T$$

System is reachable iff W_c has rank n

W_c is called the controllability matrix

Reachability and coordinate transformation

$$
\begin{aligned}
\tilde{W}_c &= \begin{pmatrix} \tilde{\Gamma} & \tilde{\Phi}\tilde{\Gamma} & \dots & \tilde{\Phi}^{n-1}\tilde{\Gamma} \end{pmatrix} \\
&= \begin{pmatrix} T\Gamma & T\Phi T^{-1}T\Gamma & \dots & T\Phi^{n-1}T^{-1}T\Gamma \end{pmatrix} \\
&= TW_c
\end{aligned}
$$

$\Rightarrow$ reachability is not influenced by change in coordinates

Controllable canonical form

If W_c is nonsingular and Φ has the characteristic polynomial:

$$\det[\lambda I - \Phi] = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$$

then there exists a transformation $T = \tilde{W}_c W_c^{-1}$ such that

$$z(k+1) = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u(k)$$

$$y(k) = (b_1 \quad \dots \quad b_n)\, z(k)$$

Observability and detectability

The system

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) \end{aligned}$$

is observable if there is a finite k such that the knowledge of $u(0), \ldots, u(k-1)$ and $y(0), \ldots, y(k-1)$ is sufficient to determine the initial state $x(0)$ of the system.

Observability and detectability (cont'd)

Assume $u(k) = 0$, iterate the system equation:

$$\begin{aligned} y(0) &= Cx(0) \\ y(1) &= Cx(1) = C\Phi x(0) \\ &\vdots \\ y(n-1) &= C\Phi^{n-1}x(0) \end{aligned}$$

$$\begin{pmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{pmatrix} = \begin{pmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{pmatrix} x(0)$$

Observability and detectability (cont'd)

The system is observable iff the *observability matrix*

$$W_o = \begin{pmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{pmatrix}$$

has rank n.

A system is detectable if the unobservable states decay to the origin (the corresponding eigenvalues are stable).

Observability - Example

$$x(k+1) = \begin{pmatrix} 1.1 & -0.3 \\ 1 & 0 \end{pmatrix} x(k)$$

$$y(k) = (1 \quad -0.5)x(k)$$

Observability matrix

$$W_o = \begin{pmatrix} C \\ C\Phi \end{pmatrix} = \begin{pmatrix} 1 & -0.5 \\ 0.6 & -0.3 \end{pmatrix}$$

The rank of W_o is 1

The Kalman decomposition

It is possible introduce coordinates that lead to this partitioning

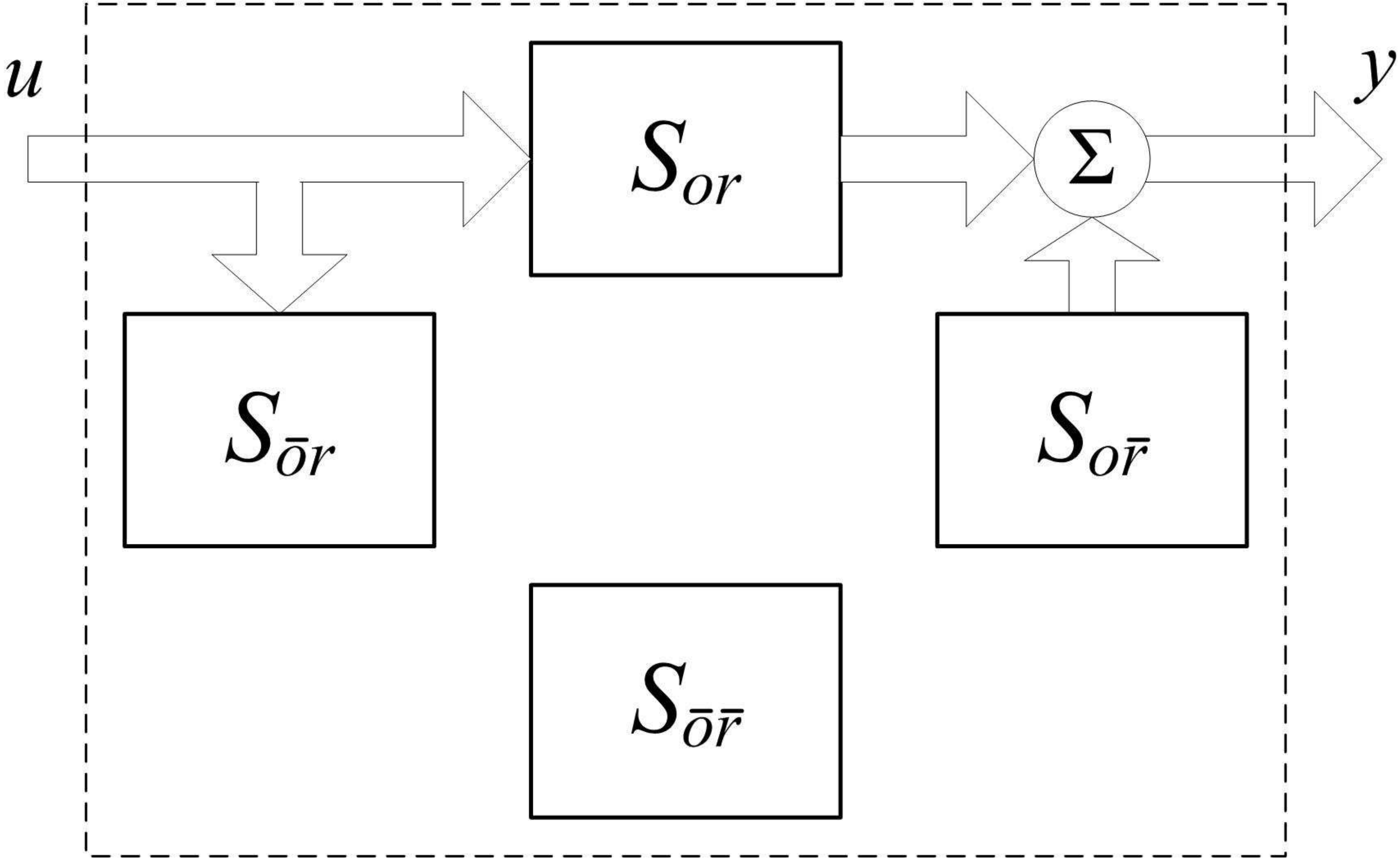
$$x(k+1) = \begin{pmatrix} \Phi_{11} & \Phi_{12} & 0 & 0 \\ 0 & \Phi_{22} & 0 & 0 \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} \\ 0 & \Phi_{42} & 0 & \Phi_{44} \end{pmatrix} x(k) + \begin{pmatrix} \Gamma_1 \\ 0 \\ \Gamma_3 \\ 0 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} C_1 & C_2 & 0 & 0 \end{pmatrix} x(k)$$

Pulse-transfer operator:

$$H(q) = C_1 \left(qI - \Phi_{11}\right)^{-1} \Gamma_1$$

The Kalman decomposition



Sampling $\rightarrow$ reachability and observability

- Φ and Γ depend on h.

- To get reachable discrete-time system it is necessary that the continuous-time system is controllable.

- May happen that reachability is lost for some h

- Sampled-data system may be unobservable even if the continuous-time system is observable (hidden oscillations).

Hidden oscillations

Inter-sample ripple

Two cases:

1. Oscillations seen in continuous-time output of an open-loop / closed-loop system.
   Loss of observability purely by sampling.

2. Oscillations seen in the control input, but not in the sampled output. Loss of observability by feedback, cancellation of poorly damped zeros by the controller.

Hidden oscillation in open-loop system

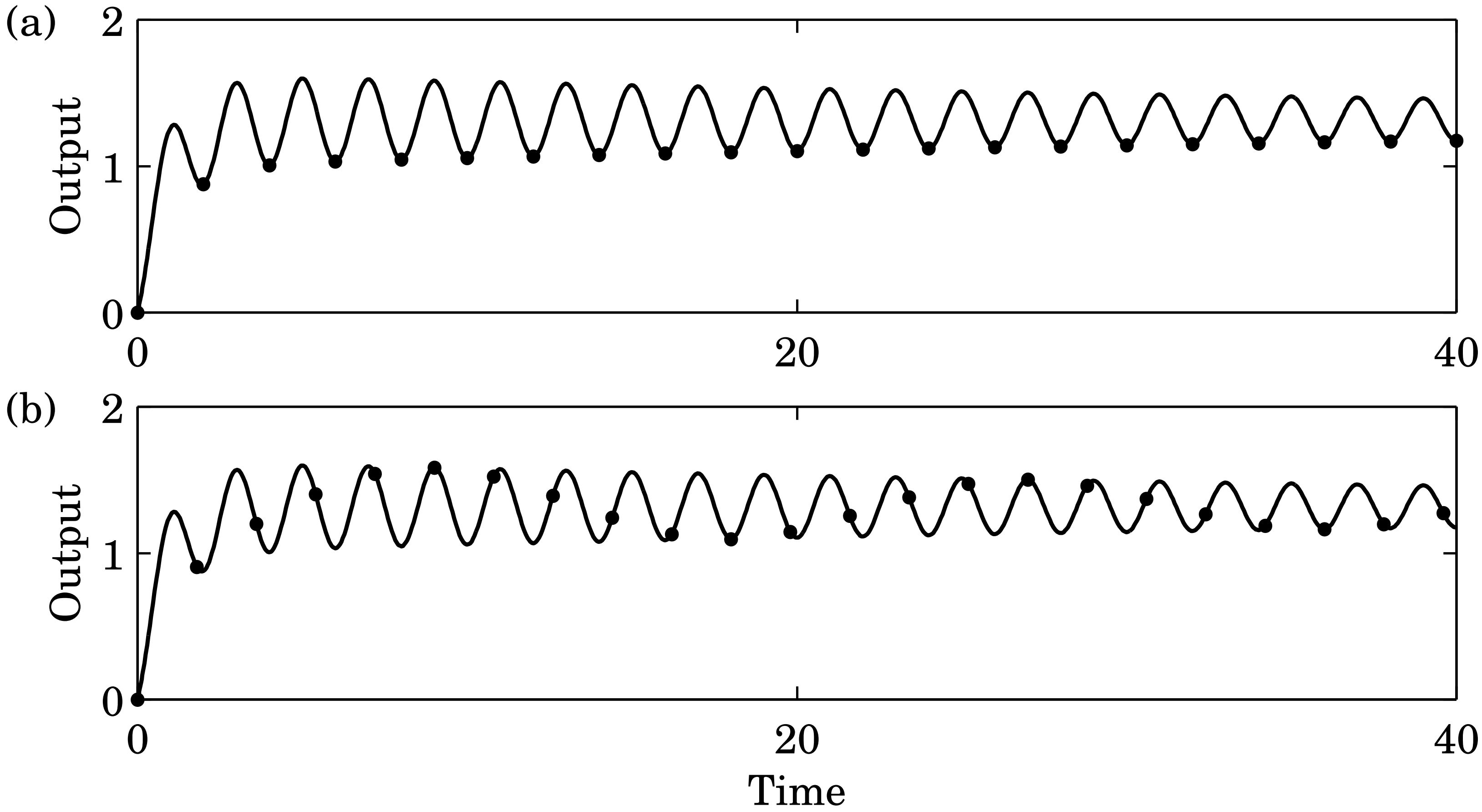
$$G(s) = \frac{1}{s+1} + \frac{\pi}{(s+0.02)^2 + \pi^2}$$

Sample with $h = 2, (\omega_s = \pi)$

$$H(z) = \frac{1-a}{z-a} + \frac{0.0125}{z-\alpha}$$

where $a = e^{-2}$ and $\alpha = e^{-0.04}$

Hidden oscillation in open-loop system



a) $h = 2$, b) $h = 1.8$

Controller-induced hidden oscillation

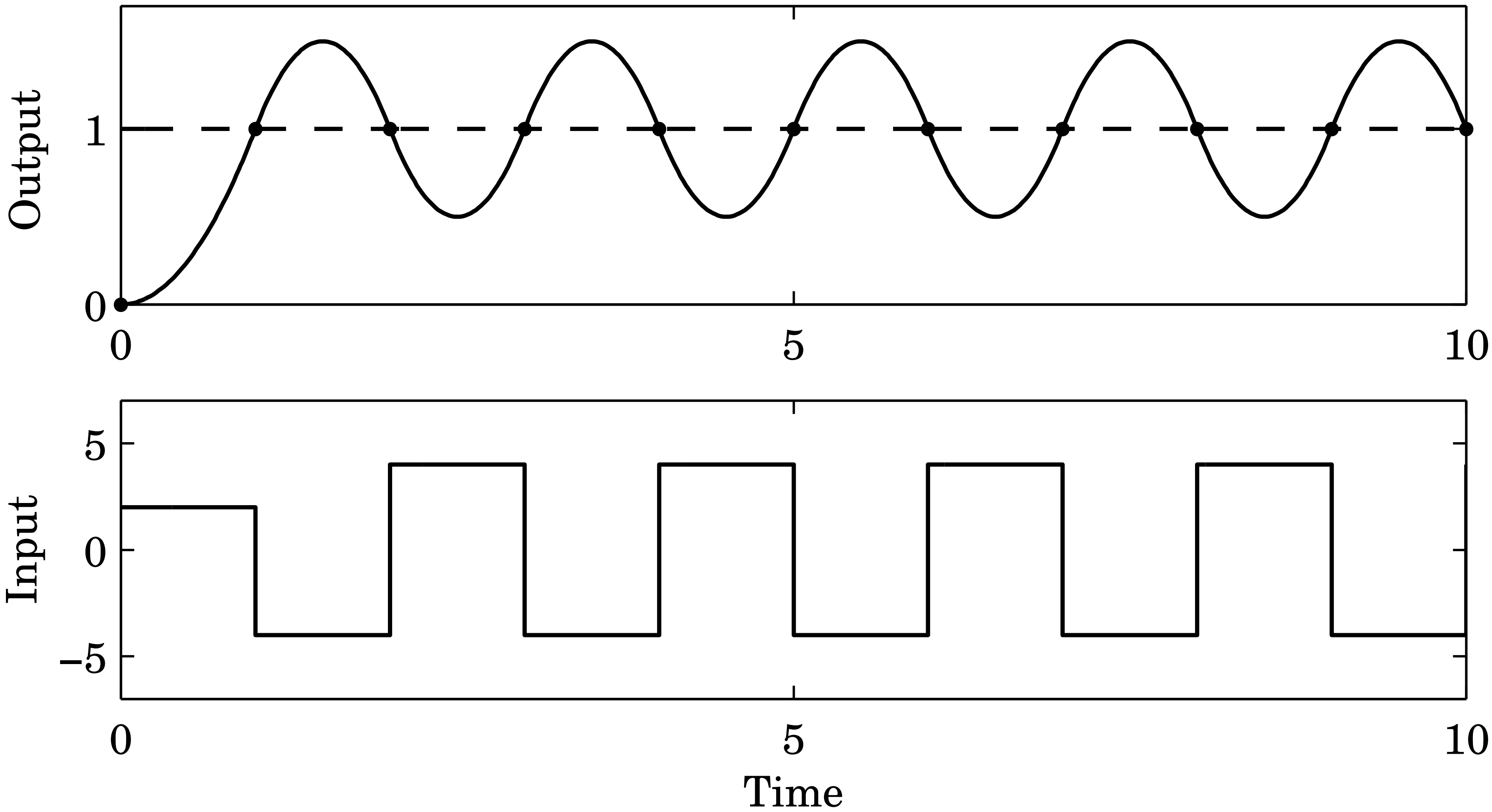
Consider a double integrator with the controller:

$$u(k) = \frac{2q}{q+1} u_c(k) - \frac{2(2q-1)}{q+1} y(k)$$

Closed-loop system

$$\begin{aligned} y(k) &= \frac{q(q+1)}{(q+1)(q^2 - 2q + 1 - (-2q+1))} u_c(k) \\ &= \frac{q(q+1)}{q^2(q+1)} u_c(k) = u_c(k-1) \end{aligned}$$

Controller-induced hidden oscillation



How to detect hidden oscillations?

- Modified z-transform or solve system equation for times between sampling points

- Simulation. Check continuous-time output and control signal

Importance of simulation

- Simpler, cheaper, safer to experiment with a model.

- If the system does not exist.

- Investigate influence of parameter variations.

Simulation can never replace a field test.

Beware of numerical aspects, choice of integration method!

Summary

- Stability
- Controllability and reachability
- Observability and detectability
- Simulation