AE4301 Automatic Flight Control System Design
Part I: Control Theory

Anahita Jamshidnejad
Aerospace Engineering

Lecture 5

What we learned so far

- Modeling dynamical systems:
  - transfer functions (frequency domain)
  - state-space representation (time domain)
- Pole placement
- Proportional, integral, and derivative controllers

Main objectives

- Being able to analyze frequency response (steady-state response of LTI systems to sine inputs)
- Being able to plot and analyze Bode diagrams: magnitude and phase of sinusoidal transfer functions for the entire frequency spectrum

Frequency-response: Introduction

- *Frequency response:* Steady-state response of a system to a sinusoidal input $u(t) = \bar{u} \sin \omega t$
- Change the input frequency over a certain range and analyze the response
- Steady-state response to sine input of frequency ω is

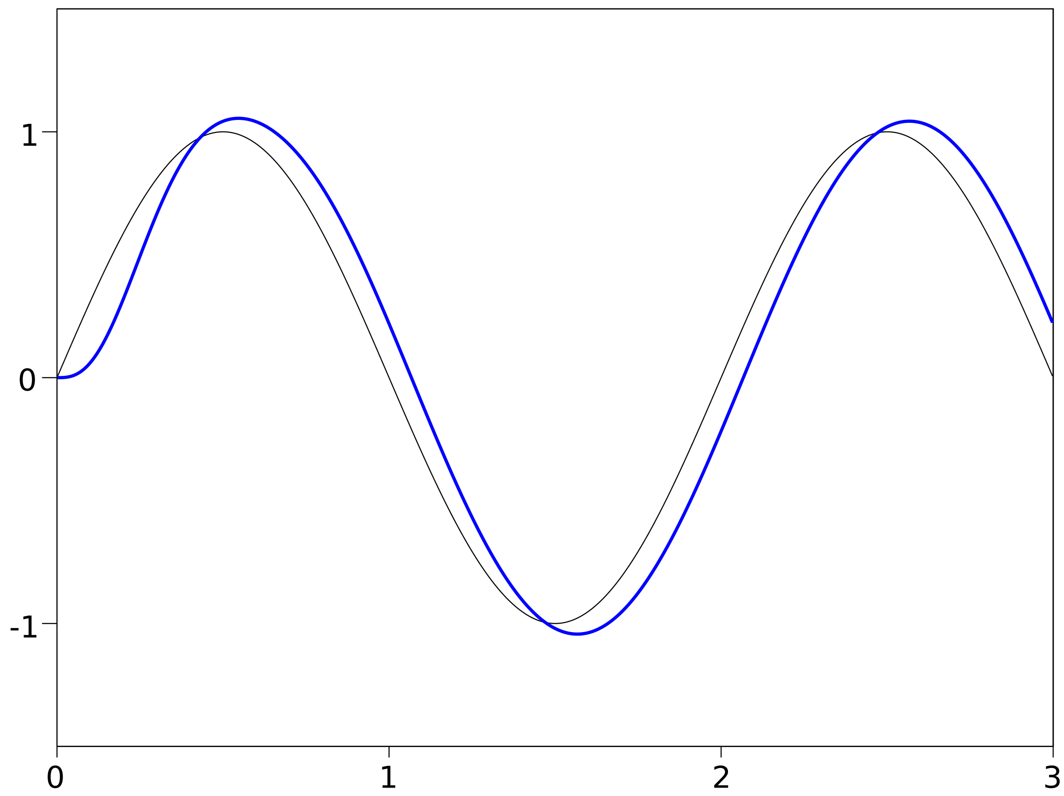
$$y_{\text{ss}} = \bar{u}|G(j\omega)| \sin\left(\omega t + \angle G(j\omega)\right)$$

- $G(j\omega)$ is *sinusoidal transfer function*
- Output of a *stable, linear, time-invariant* system to a sinusoidal input $u(t) = \bar{u} \sin \omega t$ is sinusoidal of same frequency and different amplitude and phase:

amplitude = $\bar{u} \cdot |G(j\omega)|$
phase = $\angle G(j\omega)$

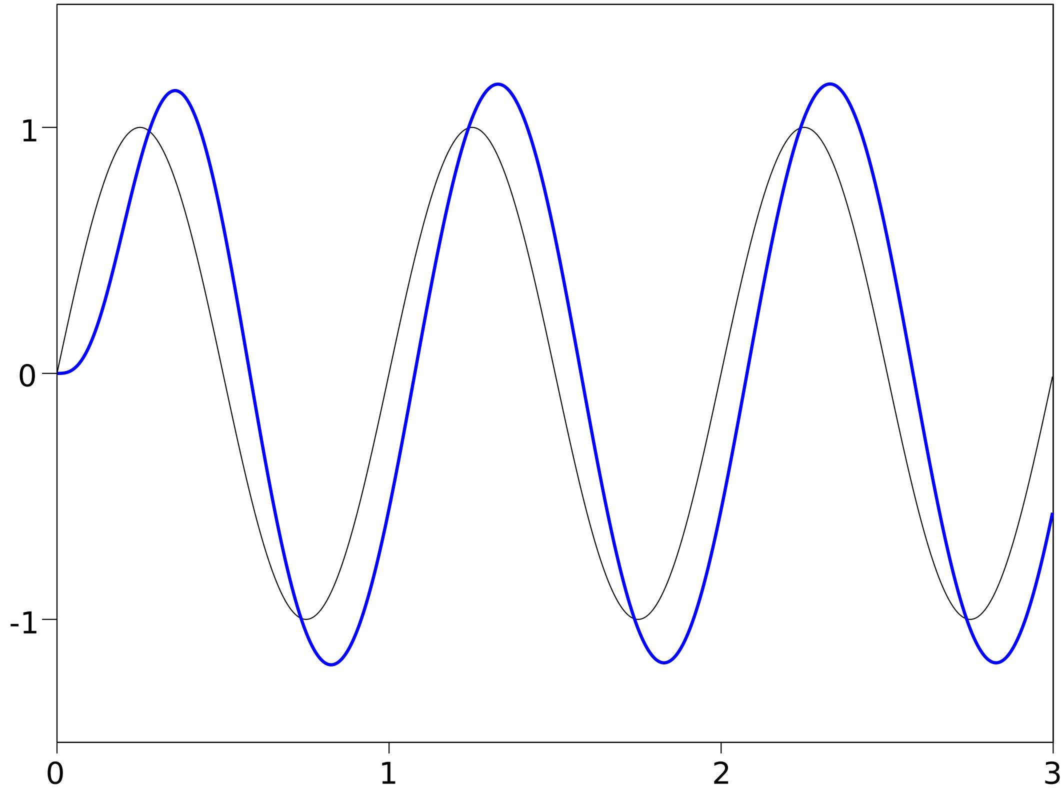
Response to a 0.5 Hz sine

Response of $\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$, $\zeta = 0.4$, $\omega_n = 4\pi$



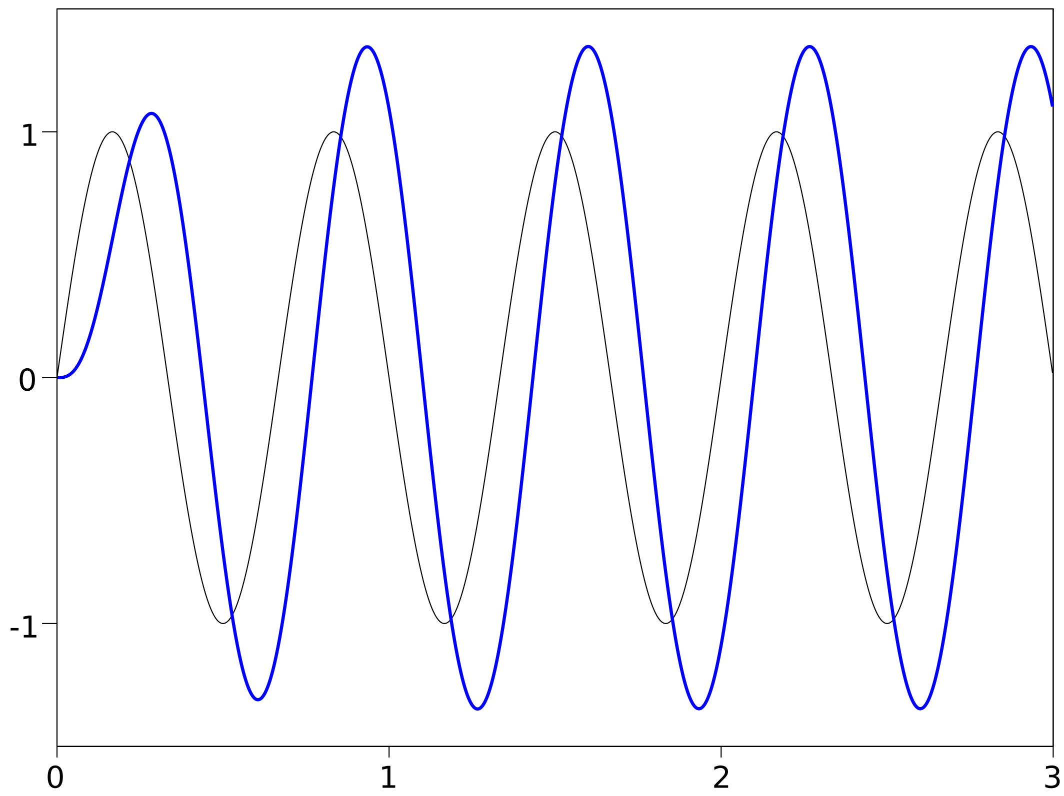
Response to a 1 Hz sine

Response of $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, $\zeta = 0.4$, $\omega_n = 4\pi$



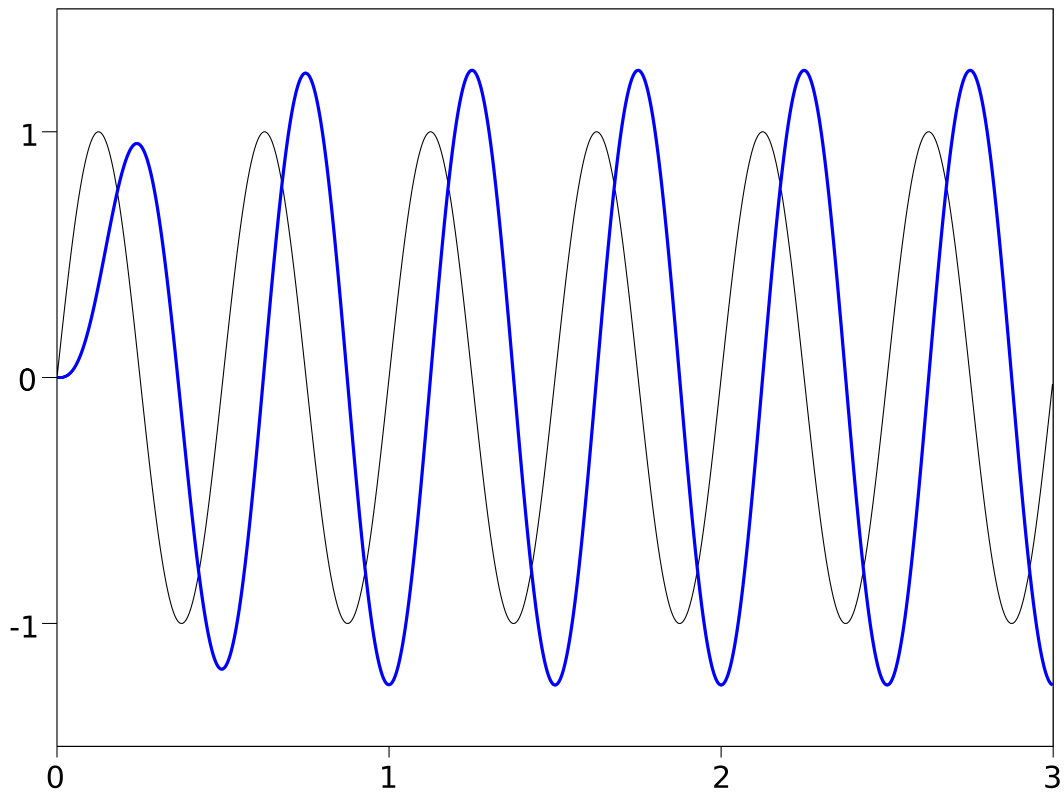
Response to a 1.5 Hz sine

Response of $\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$, $\zeta = 0.4$, $\omega_n = 4\pi$



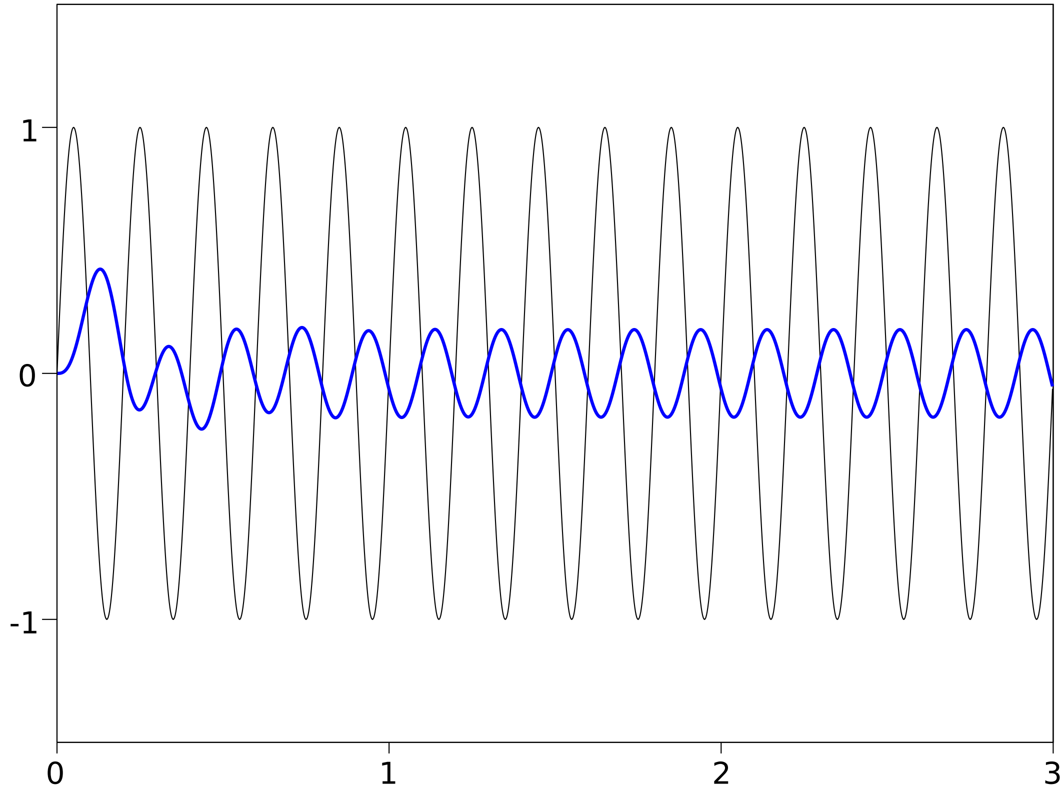
Response to a 2 Hz sine

Response of $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, $\zeta = 0.4$, $\omega_n = 4\pi$



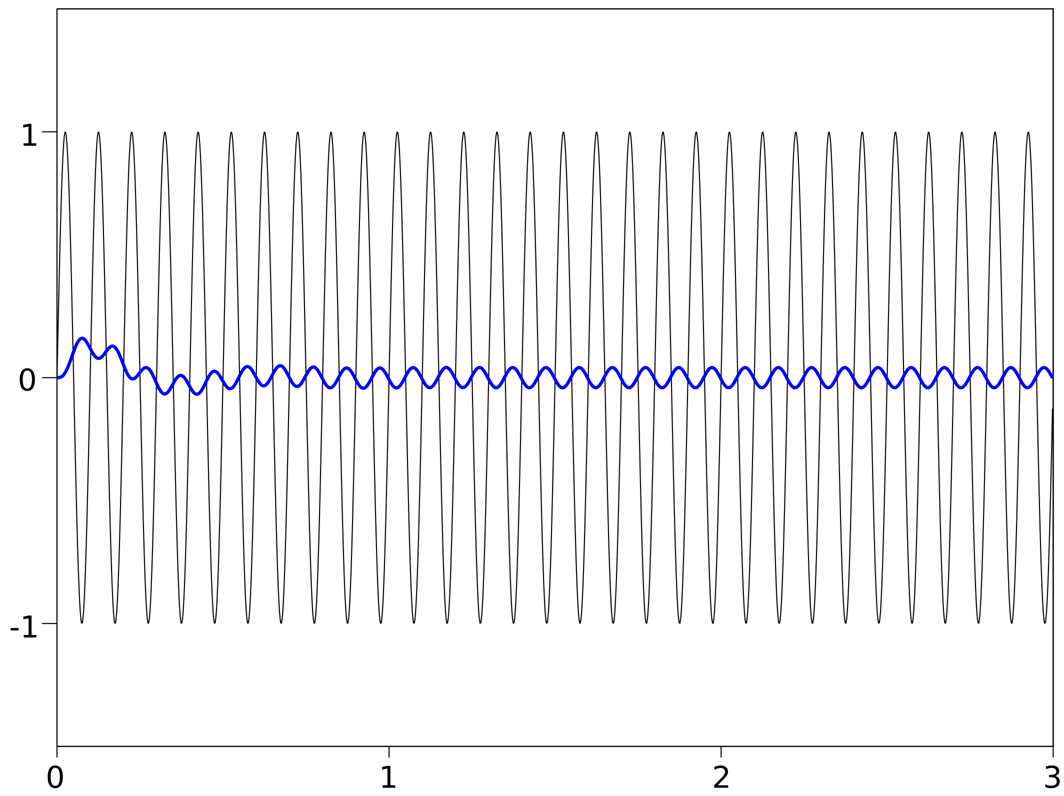
Response to 5 Hz sine

Response of $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, $\zeta = 0.4$, $\omega_n = 4\pi$



Response to a 10 Hz sine

Response of $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, $\zeta = 0.4$, $\omega_n = 4\pi$



Proof

Frequency response to $u(t) = \bar{u}\sin\omega t$:

$$Y(s) = G(s)U(s) = G(s)\frac{\bar{u}\omega}{s^2 + w^2}$$
$$= \frac{\bar{u}\omega\,(s - z_1)\,(s - z_2)\ldots(s - z_m)}{(s + j\omega)\,(s - j\omega)\,(s - p_1)\,(s - p_2)\ldots(s - p_n)}$$

Using partial fraction expansion (assume distinct poles):

$$Y(s) = \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega} + \frac{b_1}{s - p_1} + \frac{b_2}{s - p_2} + \ldots + \frac{b_n}{s - p_n} \qquad (*)$$

Time-domain response (inverse Laplace):

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + b_1 e^{p_1 t} + b_2 e^{p_2 t} + \ldots + b_n e^{p_n t}$$

… Proof

- Stability implies negative real part for poles $p_1, p_2, \ldots$
- Therefore, when $t \to \infty$, $e^{p_1 t} \to 0$, $e^{p_2 t} \to 0, \ldots$
- Steady-state response: $\boxed{y_{\text{ss}} = ae^{-j\omega t} + \bar{a}e^{j\omega t}}$

Q. What if there are poles of multiplicity m_{p}?

- Determining a: multiply both sides of $(*)$ by $(s + j\omega)$ and replace s by $-j\omega$:

$$a = G(s)\frac{\bar{u}\omega}{s^2 + \omega^2}(s + j\omega)\Big|_{s=-j\omega} = -\frac{\bar{u}G(-j\omega)}{2j}$$

- Similarly, $\bar{a} = \dfrac{\bar{u}G(j\omega)}{2j}$

... Proof

- Use polar representation of $G(j\omega)$:

$$G(j\omega) = |G(j\omega)|\, e^{j\phi} \qquad \text{with } \phi = \angle G(j\omega)$$

- Since $G(j\omega)$ and $G(-j\omega)$ are complex conjugates: $|G(j\omega)| = |G(-j\omega)|$
- Rewrite steady-state response:

$$\begin{aligned} y_{\mathrm{ss}} &= \bar{u}\, |G(j\omega)| \, \frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} \\ &= \underbrace{\bar{u}\, |G(j\omega)|}_{\text{amplitude}} \sin(\omega t + \underbrace{\phi}_{\text{phase shift}}) \end{aligned}$$

- Positive phase angle: phase lead $\rightarrow$ lead network
- Negative phase angle: phase lag $\rightarrow$ lag network

Example

For the transfer function $G(s) = \dfrac{K}{\tau s + 1}$ find the steady-state response to sinusoidal input $u(t) = \bar{u}\sin(\omega t)$ and determine whether this is a lead or a lag network (reason based on ω).

$$y_{\text{ss}} = \frac{\bar{u}K}{\sqrt{1+\tau^2\omega^2}} \sin\left(\omega t - \arctan \tau\omega\right)$$

Small ω: amplitude $\approx \bar{u}K$ and phase shift very small

Large ω: amplitude small (almost inversely proportional to ω) and phase shift approaches -90° as ω approaches infinity.

phase-lag network!

Exercise

For the transfer function $G(s) = \dfrac{s + \dfrac{1}{\tau_1}}{s + \dfrac{1}{\tau_2}}$ determine based on τ_1 and τ_2 whether this network is a lead or a lag network.

Why sine input? Fourier series

- Any periodic function can be decomposed into a sum of sinusoidal components.
- Fourier series expansion:
  $$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\omega_n t\right) + b_n \sin\left(\omega_n t\right)\right]$$
- If we know the response to each sine frequency *and* the system is linear, we know the response to any periodic signal.

Bode plot

- When you subject a sine wave input to an LTI system (mathematically speaking, including linear operations only) it does not change shape $\rightarrow$ It won't change its frequency ω!!

A sine wave input to an LTI system will always generate a sine wave output of the same frequency

- Two possible changes only:
  - Amplitude (height of sine wave) $\rightarrow$ corresponds to magnitude of sinusoidal transfer function
  - Phase (shifting sine wave in time) $\rightarrow$ corresponds to phase angle of sinusoidal transfer function

response to a single frequency (what we did analytically) ...
response for frequency spectrum (what Bode plots do graphically)

Bode plots

- A Bode diagram consists of two graphs:
  - Magnitude of sinusoidal transfer function in dB , i.e., $20\log|G(j\omega)|$, versus $\log\omega$
  - Phase angle of sinusoidal transfer function in degrees, i.e., $\angle G(j\omega)$, versus $\log\omega$
- Main advantage of Bode plots: multiplication of magnitudes can be converted into addition.
- Basic factors of $G_1(j\omega)G_2(j\omega)\dots G_\ell(j\omega)$:
  - Gain K
  - Integral and derivative factors $(j\omega)^{\pm1}$
  - First-order factors $(1+j\omega\tau)^{\pm1}$
  - Quadratic factors $(1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2)^{\pm1}$

Bode plot: Gain K

- A number greater than unity has a positive value in dB.
- A number smaller than unity has a negative value in dB.
- Log-magnitude curve for a constant gain K is a horizontal straight line at $20 \log K$ dB.
- Phase angle of constant gain is zero.
- Effect of varying K:
  - Raises or lowers log-magnitude curve
  - No effect on phase curve
- **Reminder:** In dB reciprocal of a number differs from its value in sign only:

$$\boxed{20 \log K = -20 \log \frac{1}{K}}$$

Bode plot: Integral & derivative

- Logarithmic magnitude of $1/j\omega$ and $j\omega$ in dB:

$$20\log|1/j\omega| = -20\log\omega \quad \text{dB}, \qquad 20\log|j\omega| = 20\log\omega \quad \text{dB}$$

- Plot of magnitude versus $\log\omega$: straight line with slope $-20\frac{\text{dB}}{\text{decade}}$ for integral and slope $20\frac{\text{dB}}{\text{decade}}$ for derivative.

  Note: A decade is a frequency band from ω to 10ω.

  On logarithmic scale papers distance between $\omega = 1$ and $\omega = 10$, and between $\omega = 10$ and $\omega = 100$, and between $\omega = 5$ and $\omega = 50$ are same (a decade).

- Phase angle of $1/j\omega$ is -90°, and phase angle of $j\omega$ is 90°.

Bode plot: First-order factors

- We use *asymptotes* to approximate Bode plots.
- Log-magnitude of $1/\left(1+j\omega\tau\right)$ in dB:

$$20\log\left|1/\left(1+j\omega\tau\right)\right| = -20\log\sqrt{1+\omega^2\tau^2} \quad \text{dB}$$

  - $\omega\tau \ll 1$: $-20\log\sqrt{1+\omega^2\tau^2} \approx 0 \quad \text{dB}$
  - $\omega\tau \gg 1$: $-20\log\sqrt{1+\omega^2\tau^2} \approx -20\log\omega\tau \quad \text{dB}$

- Starting from small $\omega\tau$ log-magnitude curve is 0-dB line. When $\omega\tau$ is very large log-magnitude curve decreases value by 20 dB per decade $\rightarrow$ log-magnitude curve becomes straight line with slope $-20\frac{\text{dB}}{\text{decade}}$.
- Corner frequency: $\omega_{\text{C}} = 1/\tau \rightarrow \tau$ shifts ω_{C} to left & right

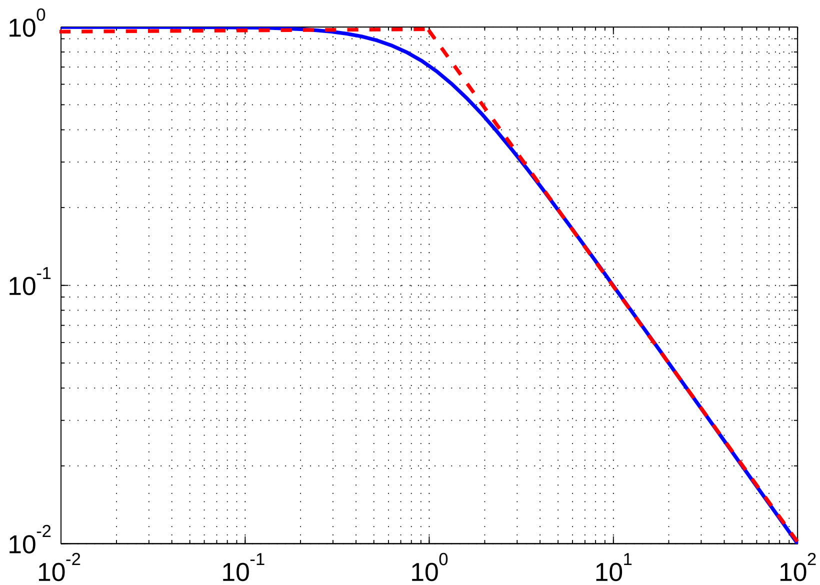
- Phase of $1/\left(1+j\omega\tau\right)$: $\phi = -\arctan\omega\tau$.
  $\omega\tau = 0 \Rightarrow \phi = 0°$, $\omega\tau = 1 \Rightarrow \phi = -45°$, $\omega\tau \rightarrow \infty \Rightarrow \phi = -90°$

Bode plot: First-order factors

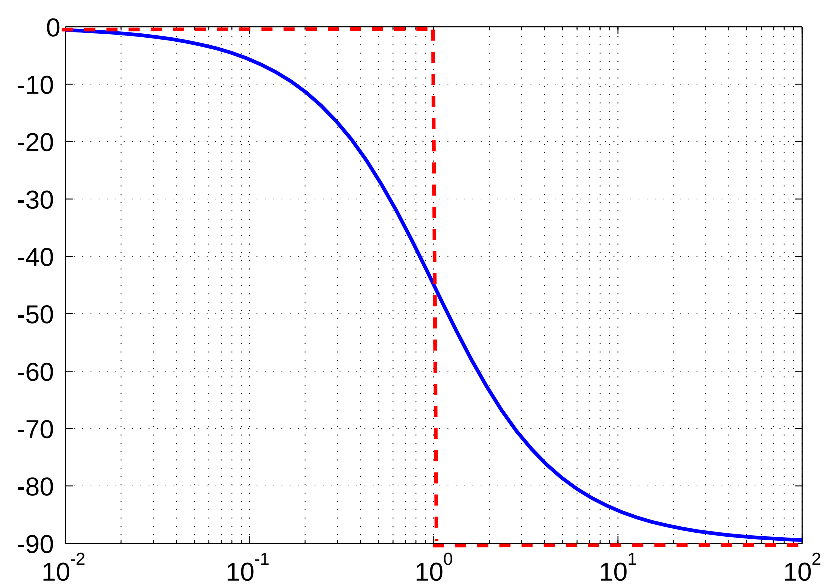
- In Bode plot for reciprocal factors log-magnitude and phase angle curves change in sign only.
- Change signs obtained for $1/\left(1+j\omega\tau\right)$ to get curves for $1+j\omega\tau$.
- Corner frequencies for $1/\left(1+j\omega\tau\right)$ and $1+j\omega\tau$ are same.
- Slope of high-frequency asymptote for $1+j\omega\tau$ is $20\frac{\mathrm{dB}}{\mathrm{decade}}$.
- Phase angle for $1+j\omega\tau$ varies between 0° and 90° as ω varies from 0 to ∞.

Bode plots for $1/(1+j\omega\tau)$

Magnitude plot



Argument plot



Asymptotes

For $\omega\tau \ll 1 \;\rightarrow\; \frac{1}{1+j\omega\tau} \approx 1$

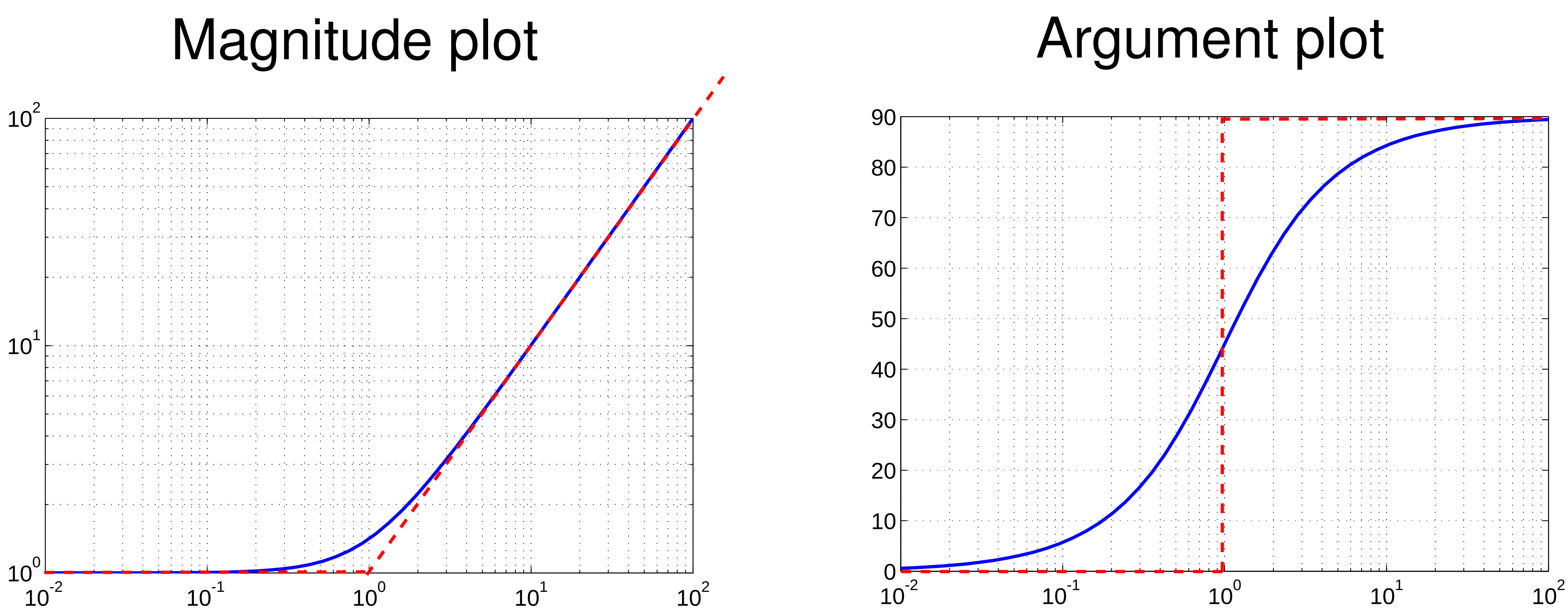
For $\omega\tau \gg 1 \;\rightarrow\; \frac{1}{1+j\omega\tau} \approx \frac{1}{j\omega\tau}$

Asymptotes

For $\omega\tau \ll 1 \;\rightarrow\; \angle(\frac{1}{1+j\omega\tau}) \approx 0$

For $\omega\tau \gg 1 \;\rightarrow\; \angle(\frac{1}{1+j\omega\tau}) \approx -90^\circ$

Bode plots for $1 + j\omega\tau$



Asymptotes

For $\omega\tau \ll 1 \;\rightarrow\; 1 + j\omega\tau \approx 1$

For $\omega\tau \gg 1 \;\rightarrow\; 1 + j\omega\tau \approx j\omega\tau$

Asymptotes

For $\omega\tau \ll 1 \;\rightarrow\; \angle(1 + j\omega\tau) \approx 0$

For $\omega\tau \gg 1 \;\rightarrow\; \angle(1 + j\omega\tau) \approx 90^\circ$

Bode plot: Second-order factors

- $G(j\omega) = \dfrac{1}{1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2}$
- Magnitude: $20 \log \left| \dfrac{1}{1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2} \right| =$
  $-20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}$
- $\frac{\omega}{\omega_n} \ll 1 \Rightarrow$ Magnitude ≈ 0 dB : Low-frequency asymptote is a horizontal line at 0 dB.
- $\frac{\omega}{\omega_n} \gg 1 \Rightarrow$ Magnitude $\approx -20 \log \frac{\omega^2}{\omega_n^2} = -40 \log \frac{\omega}{\omega_n}$ dB : High-frequency asymptote is a straight line of slope -40 dB/decade.
  What is ω at intersection of asymptotes? ω_n
- For quadratic factor in transfer function, corner frequency is same as natural frequency $\omega_C = \omega_n$.

Bode plot: Second-order factors

- *Near* corner frequency $\omega_C = \omega_n$ a *resonant peak* occurs.
- Damping ratio ζ determines the magnitude of resonant peak.
- Resonant frequency ω_r (for which $|G(j\omega)|$ is maximum):

$$\boxed{\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \qquad \text{for} \quad 0 \leq \zeta \leq 0.707}$$

- Magnitude of resonant peak M_r:

$$\boxed{M_r = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \qquad \text{for} \quad 0 \leq \zeta \leq 0.707}$$

Bode plot: Second-order factors

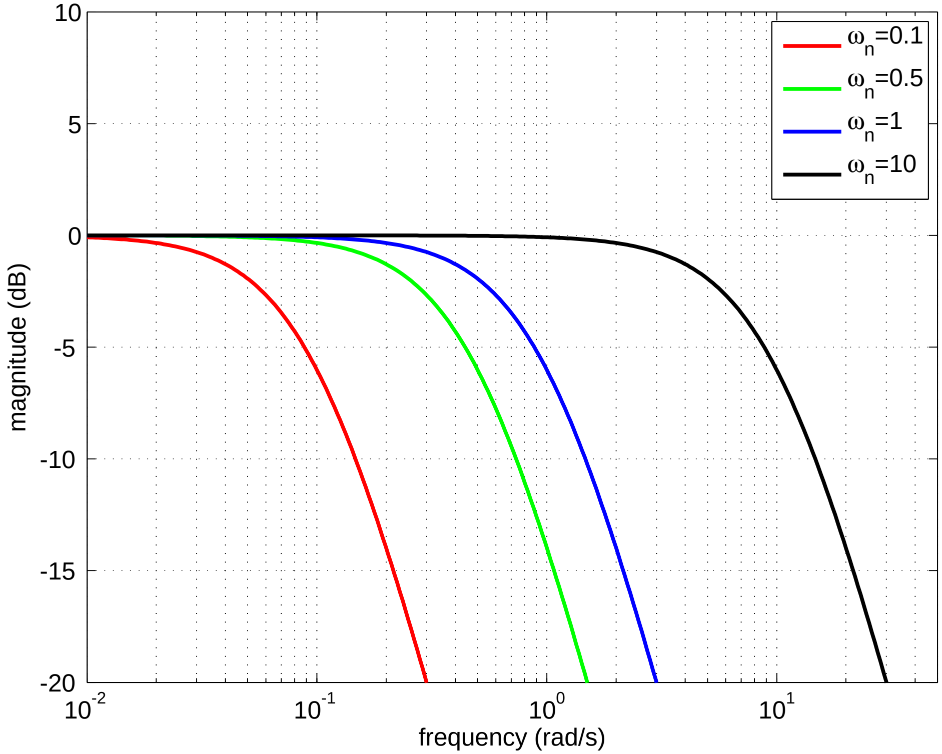
- Phase angle of quadratic factor:

$$\phi = -\arctan \frac{2\zeta \dfrac{\omega}{\omega_n}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}$$

- $\omega = 0 \Rightarrow \phi = 0^\circ$
- $\omega = \omega_C = \omega_n \Rightarrow \phi = -90^\circ$ (regardless of ζ)
- $\omega \to \infty \Rightarrow \phi = -180^\circ$

- **Note:** For second-order factor $1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2$ reverse signs of magnitude and phase Bode plots of $1/\left(1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2\right)$.

Influence of ω_n

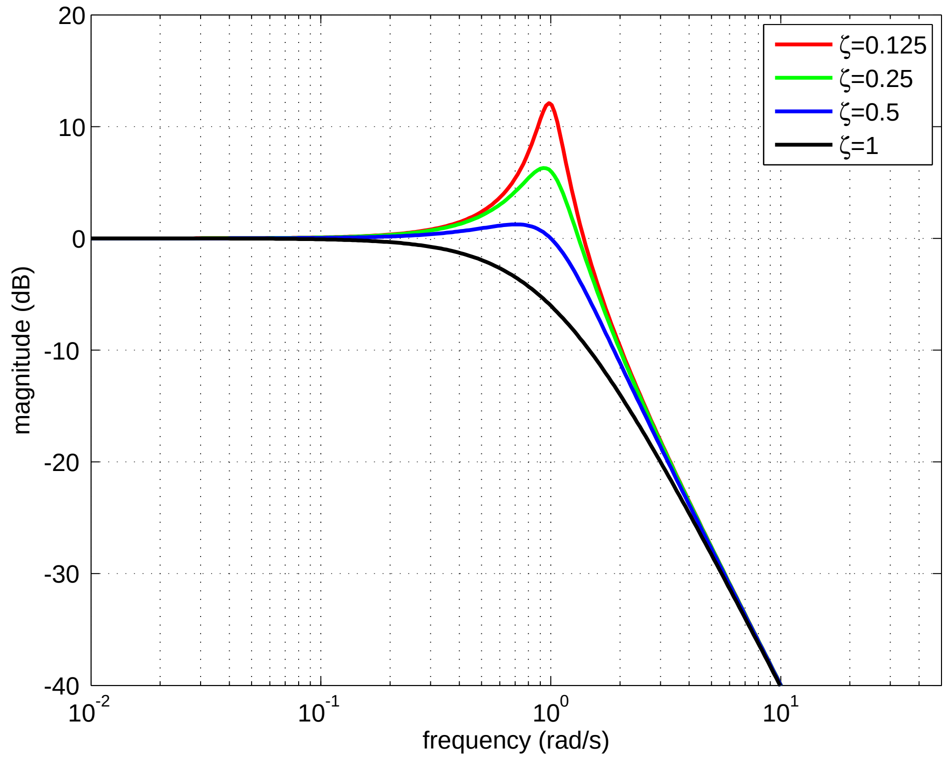


$$G(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

Conclusion:

$\omega_n \uparrow \Rightarrow$ "bandwidth" increases

Influence of ζ



$$G(s) = \frac{1}{s^2 + 2\zeta s + 1}$$

Conclusion:

$\zeta \uparrow \Rightarrow$ "resonant peak" decreases

Bode plots: Rewrite transfer functions

- Now we are familiar with logarithmic Bode plots of basic factors of transfer functions: gain, integral and derivative, 1st-order and 2nd-order factors.
- We can use these to construct a composite logarithmic plot for any general form $G_1(s)G_2(s)\ldots G_\ell(s)$.
- Because adding logarithms of gains corresponds to multiplying them, sketching curves for individual factors and adding them gives Bode plot of original transfer function.
- We should first rewrite transfer functions to obtain these basic factors.

Example

Rewrite the following transfer function as basic factors to sketch the Bode plots:

$$G(s) = \frac{8s+4}{s(s+3)}$$

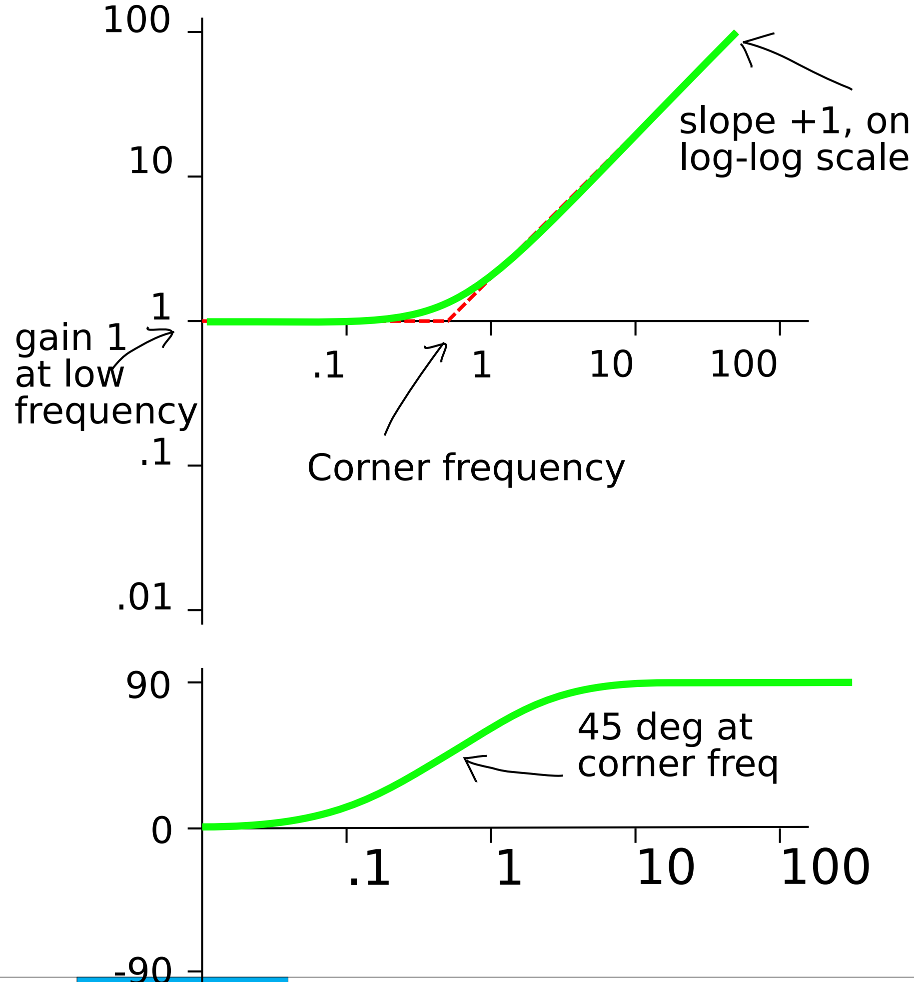
we should obtain these formulations:

- Gain: K
- Integral or derivative: s or $1/s$
- 1st-order factors: $1+\tau s$, $1/(1+\tau s)$

$$G(s) = \frac{4}{3}\,\frac{1+2s}{s\,(1+(1/3)s)}$$

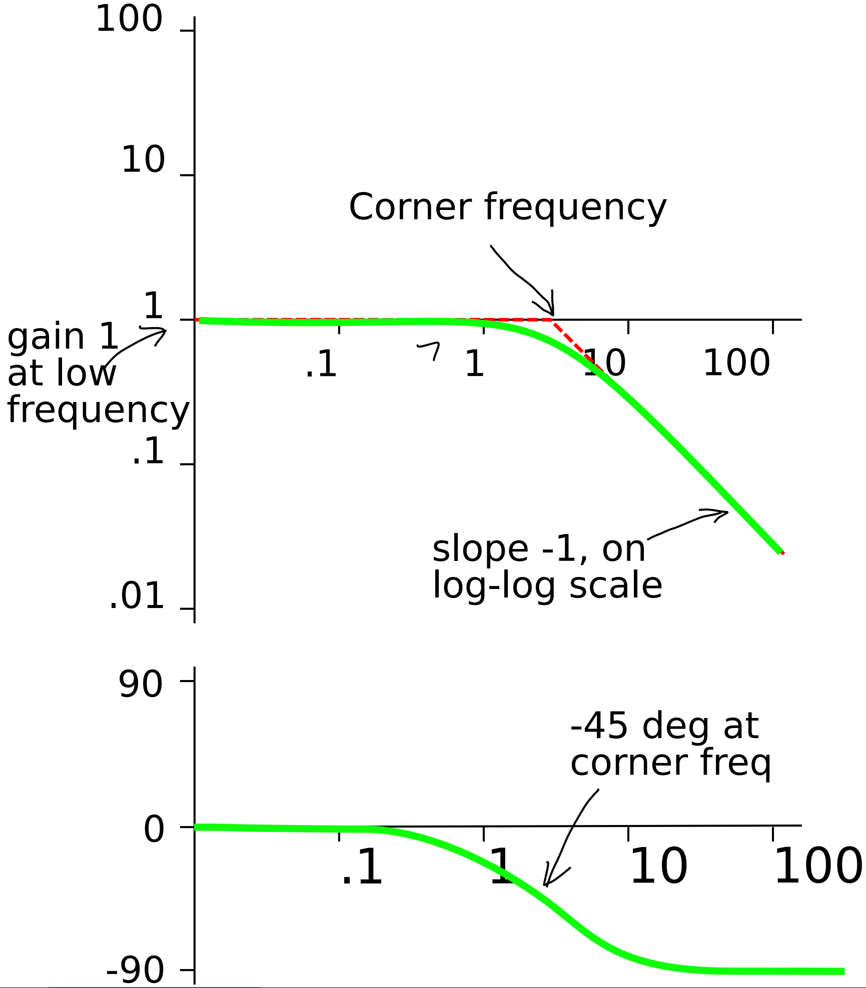
Example: 1st-order factor $1 + 2s$

$G_1(s) = 1 + 2s \Rightarrow$ Sinusoidal transfer function: $G_1(j\omega) = 1 + 2j\omega$



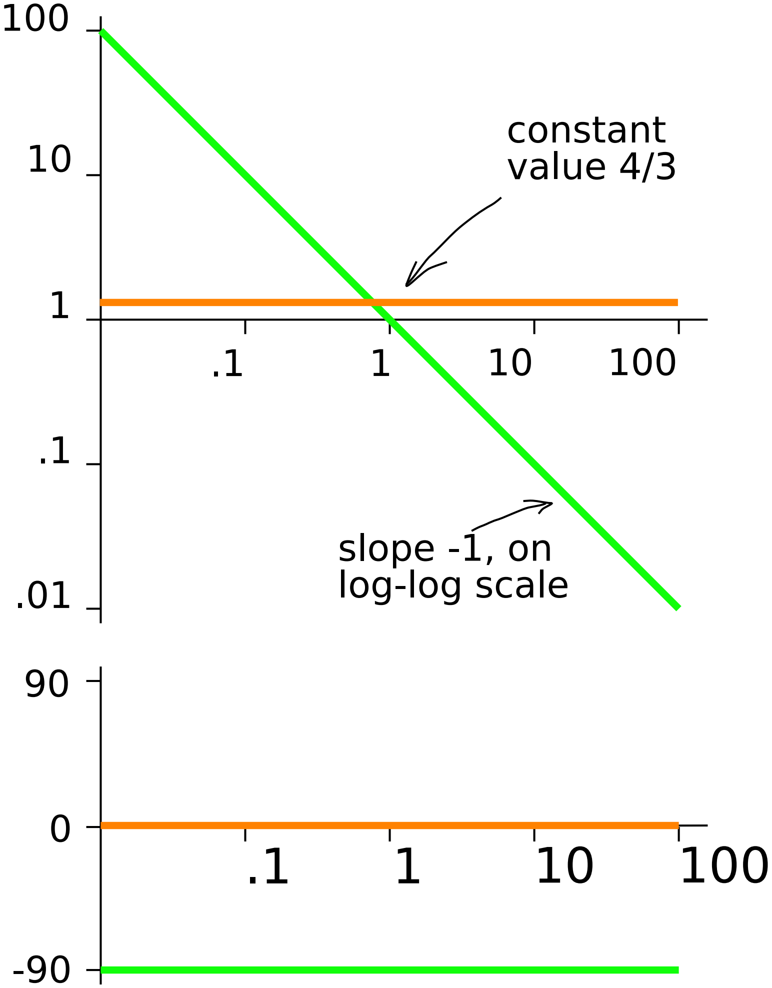
Example: 1st-order factor $1/(1+1/3s)$

$G_2(s) = \frac{1}{1+1/3s} \Rightarrow$ Sinusoidal transfer function: $G_2(j\omega) = \frac{1}{1+1/3j\omega}$

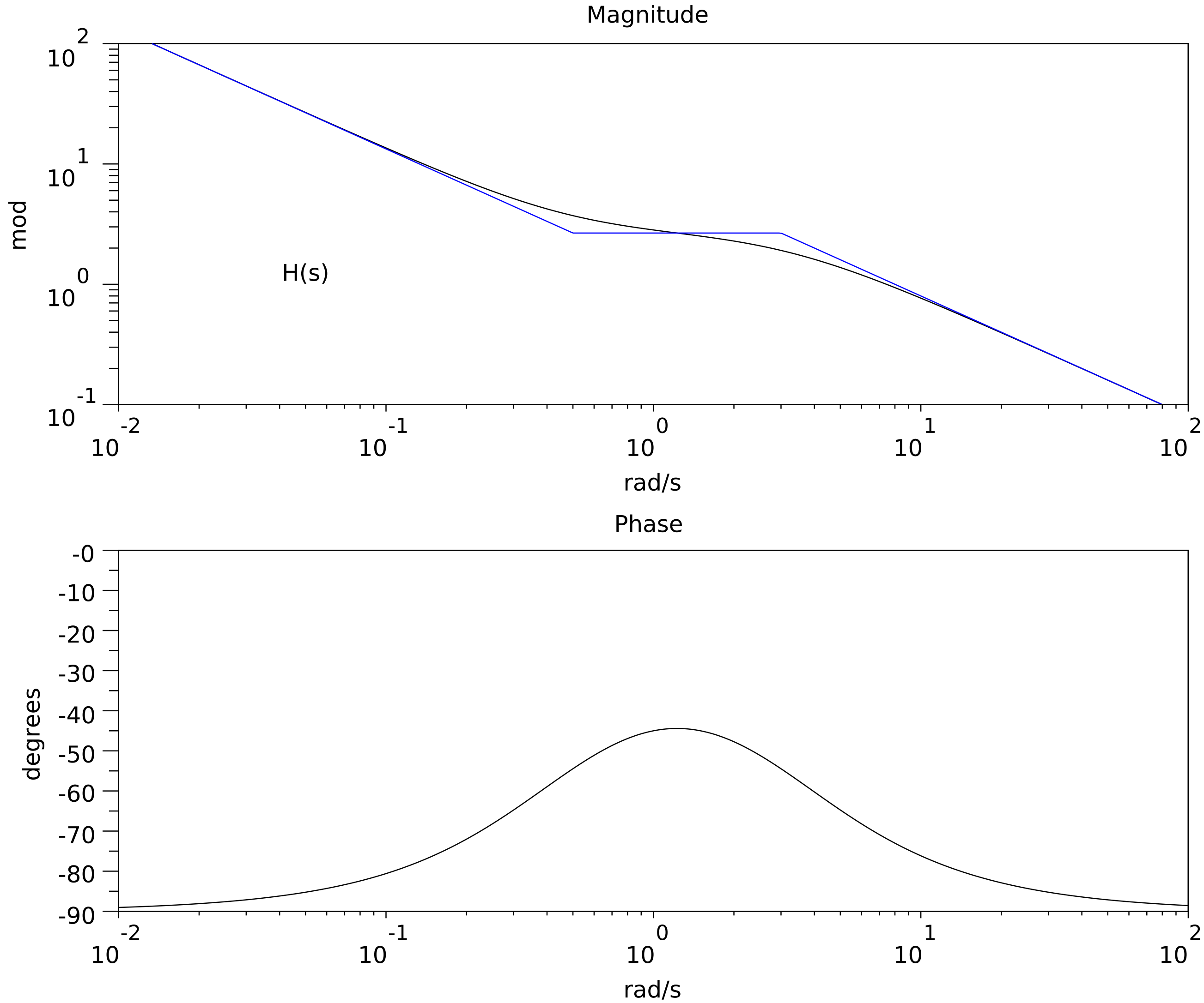


Example: Integral factor $1/s$ and gain

$G_3(s) = \frac{1}{s}$ and $K = 4/3$



Example: Bode plot



Terminology

lead $1 + \tau j\omega$ (1st order) or $1 + 2\zeta j\omega/\omega_n + (j\omega)^2/\omega_n^2$ (2nd order) term in numerator: if, respectively, $\tau > 0$ or $\zeta > 0$, phase contribution is positive; output sine *leads* ahead of the input sine.

lag $1 + \tau j\omega$ (1st order) or $1 + 2\zeta j\omega/\omega_n + (j\omega)^2/\omega_n^2$ (2nd order) in denominator: phase contribution for $\tau > 0$ or $\zeta > 0$ is negative; output sine *lags* behind the input sine.

derivative factor $j\omega$ in numerator gives 90 degrees phase lead.

integral factor $j\omega$ in denominator gives 90 degrees phase lag.

gain Constant factor K has no phase effect, unless $K < 0$, then 180 degrees phase change.

Summary

- To determine frequency response (LTI system's response to sine input function of frequency ω), replace s by $j\omega$ in transfer function $G(s)$.
- $G(j\omega)$ is called sinusoidal transfer function.
- Frequency response of an LTI system is a sine function with same frequency ω as input sine wave, but the amplitude is multiplied by $|G(j\omega)|$ and there is a phase shift of $\angle G(j\omega)$.
- Bode plots sketch $|G(j\omega)|$ and $\angle G(j\omega)$ for the whole frequency spectrum in logarithmic scales.

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