AE4301 Automatic Flight Control System Design
Part I: Control Theory

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Lecture 3

What we learned by now

- Laplace transform: From differential equations to transfer function
- Transfer function for computing output in Laplace domain
- Effects of poles and zeros of transfer functions on system behavior
- Pole locations for various gains with root-locus method

Main objectives

- Being able to derive state-space representation (in time domain)
- Being able to compute transfer function from state-space representation and vice versa
- Being able to determine controllability and observability of dynamical systems

Control design methods

1. Based on Laplace transform:
   - Based on input-output relationship (transfer function)
   - Valid for linear time-invariant systems
   - Works in frequency domain
2. Based on state-space representation:
   - Based on **1st-order** differential equation (vector & matrix) representation
   - Valid for various (linear or nonlinear, time-invariant or time-varying) systems
   - Works in time domain

Preliminary definitions

- **State variables:** Variables of a dynamical system describing system behavior for all time instants
- **State:** Smallest set of state variables such that knowing value of these variables at $t = 0$ & knowing input for $t \geq 0$ completely determines system behavior for $t \geq 0$
- **State vector:** Vector consisting of all state variables
- **State space:** Space of n dimensions (for n^{th} order system) composing of n coordinate axis for n state variables (any state of system can be represented as a point in state space)

State-space equations

- Three types of variables are used:
  - Input variables
  - Output variables
  - State variables

- State-space representation of a given system is not unique

- For linear time-invariant systems:
  - State equation: $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$
  - Output equation: $\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$

Example: 2nd-order system

$$\ddot{y} + 2\dot{y} + y = u$$

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$$\dot{x_1} = x_2$$
$$\dot{x_2} = -2x_2 - x_1 + u$$

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- For a 2nd-order system a set of 2 1st-order equations
- Use $x_1 = y$ and $x_2 = \dot{y}$:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_2 - x_1 + u$$

- Rewrite in vector-matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

State space to transfer function

State-space representation for SISO system:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + Bu(t)$$
$$y(t) = C\boldsymbol{x}(t) + Du(t)$$

Applying Laplace transform:

$$s\boldsymbol{X}(s) = A\boldsymbol{X}(s) + B\boldsymbol{U}(s) \Rightarrow$$
$$\boldsymbol{X}(s) = (sI - A)^{-1} B\boldsymbol{U}(s)$$

Determine output in frequency domain:

$$\boldsymbol{Y}(s) = \left[C\left(sI - A\right)^{-1} B + D\right]\boldsymbol{U}(s)$$

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Transfer function: $\boxed{G(s) = C\,(sI - A)^{-1} B + D}$

Note: Eigenvalues of A = poles of $G(s)$

Transfer function to state space

There may be infinitely many possible state-space representations for a transfer function

Example: For $\frac{Y(s)}{U(s)} = \frac{1}{(s+10)(s^2+4s+16)}$

Option 1: $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top + 0u$$

Option 2: $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top + 0u$$

Example: Controllable canonical form

Consider transfer function $\dfrac{Y(s)}{U(s)} = \dfrac{b_0 s^n + b_1 s^{n-1} + \ldots b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$

Controllable cannonical form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdots & b_1 - a_1 b_0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + b_0 u$$

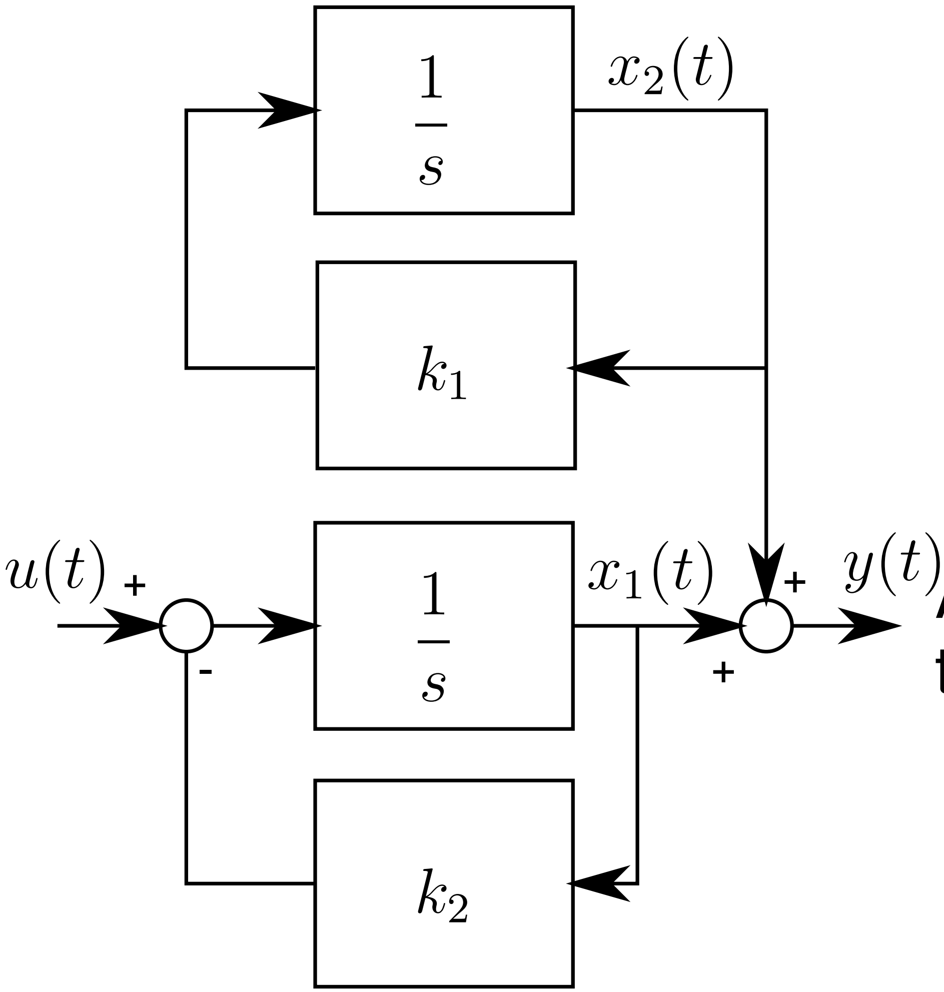
Exercise:

Convert the following transfer function into a state-space representation (the input and output of the state-space representation are the same as those of the transfer function)

$$G(s) = \frac{2s^2}{s^2 + 2s + 1}$$

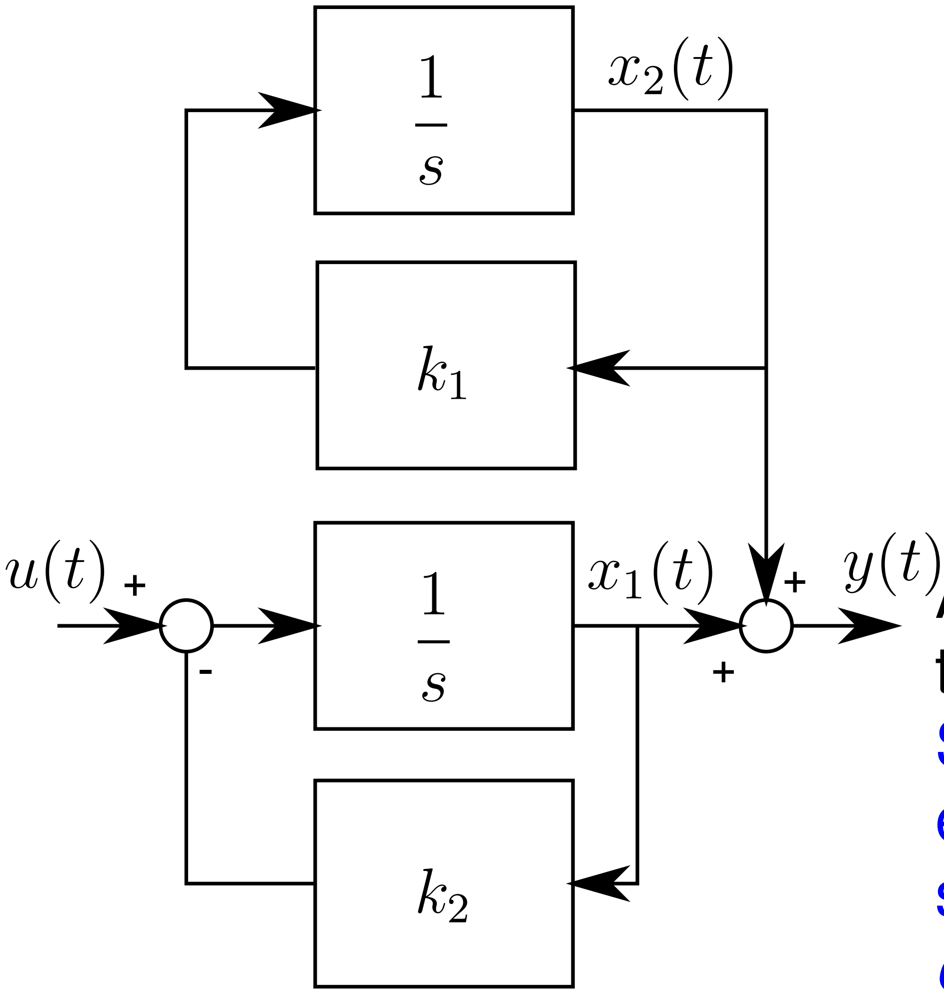
Solve at home & ask for help from teaching assistants

Controllability



Are all state variables of this system influenced by $u(t)$?

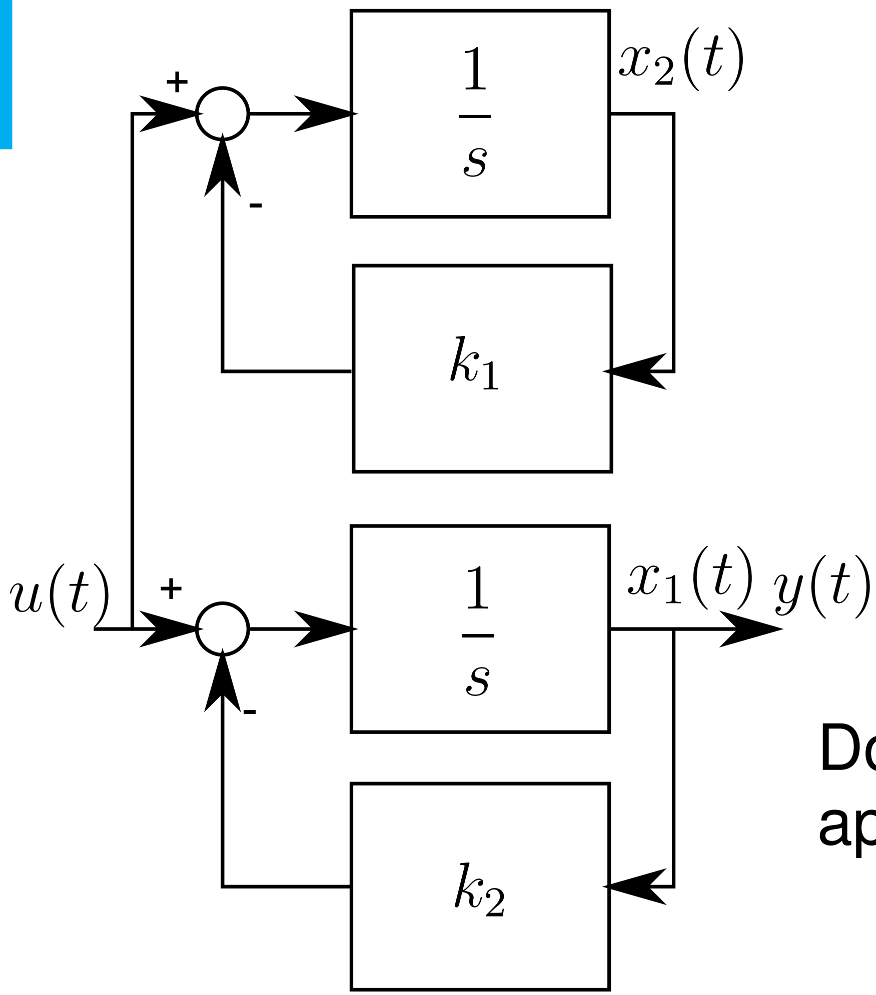
Controllability



Are all state variables of this system influenced by $u(t)$?

State variable $x_2(t)$ is not influenced by control input, i.e., the system is not completely state *controllable*.

Observability



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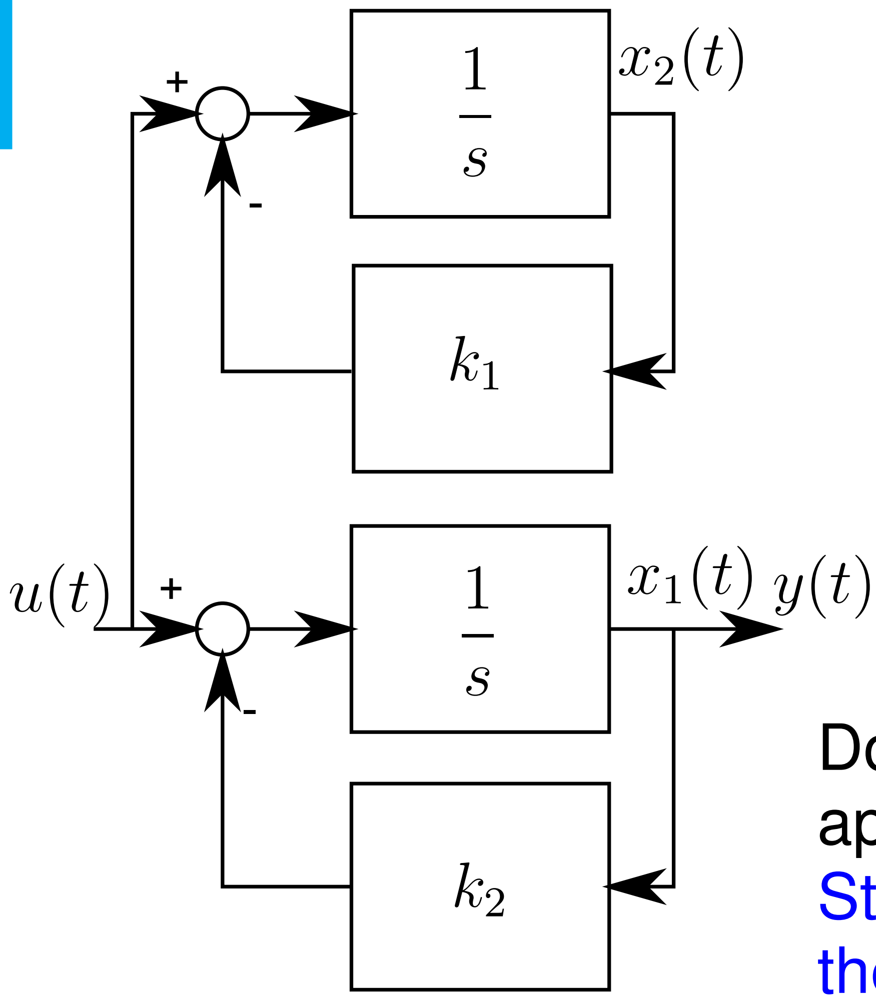

Do all the state variables of this system appear directly via output?
State variable $x_2(t)$ does not end up in the output, so it is not *observable*.

Controllability & observability

- Concepts of controllability and observability play an important role in design of control systems in state space
- A complete solution to control system design problem may not exist if system is not controllable
- When state variables are not observable or accessible for direct measurement state feedback control faces difficulties (estimation of such states will be required)
- It is important to know conditions under which a system is controllable and observable

Controllability condition

Definition: A system is (state) *controllable* at time t_0 if there is a control input $\boldsymbol{u}(t)$ that transfers the system from any initial state $\boldsymbol{x}(t_0)$ to any other state $\boldsymbol{x}(t)$ in a finite interval of time.

Condition for complete state controllability:
The system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ is completely state controllable if for the controllability matrix C_M:

$$C_M = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

the rank of the matrix is n: $\boxed{\text{rank}(C_M) = n}$.

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Is this system controllable? $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

Observability condition

Definition: A system is *observable* at time t_0 if, knowing the initial state $\boldsymbol{x}(t_0)$ of the system, this state can be determined from the observation of the output $\boldsymbol{y}(t)$ over a finite time interval.

Condition for complete state observability:
The system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ and $\boldsymbol{y} = C\boldsymbol{x}$ is observable if for the observability matrix C_O:

$$C_O = \begin{bmatrix} C \\ CA \\ \ldots \\ CA^{n-1} \end{bmatrix}$$

the rank of the matrix is n: $\boxed{\text{rank}(C_O) = n}$

Summary

- State-space representation in time domain is an alternative to transfer function in Laplace domain
- An n^{th}-order dynamical system corresponds to
  - an n^{th}-order differential equation
  - an n^{th}-order transfer function
  - a state-space representation with n state variables
- State-space & transfer function representations for a system may be derived from each other
- Controllability and observability are conditions of systems and can be obtained from state-space representation

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