Introduction to multi-variable control

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SC42145, 2024/25

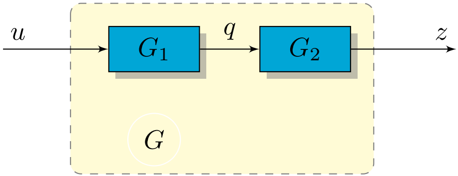
Introduction

We consider:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_\ell \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & \dots & G_{1m} \\ G_{21} & G_{22} & \dots & G_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ G_{\ell 1} & G_{\ell 2} & \dots & G_{\ell m} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

- Interaction, one input can affect multiple outputs
- Main difference with SISO, input and output have a direction
- Main tool: Singular Value Decomposition (SVD)
- $GK \neq KG$
- No generalization of Bode's stability condition

Transfer functions for MIMO systems

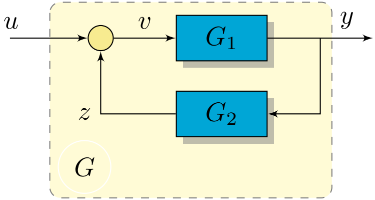


Cascade rule:

$$z = G_2 q \qquad \text{and} \qquad \mathrm{q} = \mathrm{G}_1\mathrm{u}$$

$$z = \underbrace{G_2 G_1}_{G} u$$

Transfer functions for MIMO systems

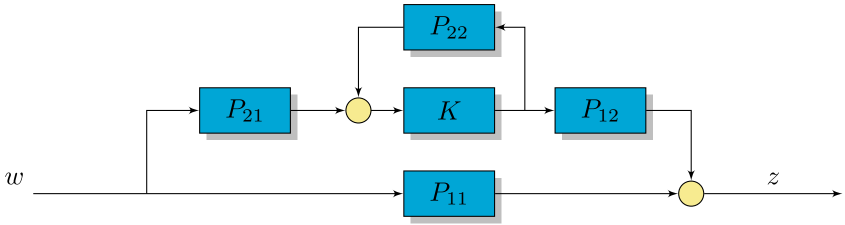


Feedback Rule: $v = (I - L)^{-1} u$ with $L = G_2 G_1$

The plant: $y = \underbrace{G_1 (I - L)^{-1}}_{G} u$

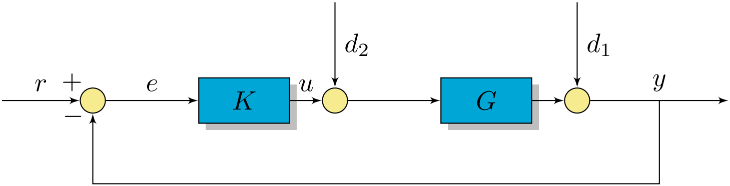
Push through rule: $G_1 (I - G_2 G_1)^{-1} = (I - G_1 G_2)^{-1} G_1$

Transfer functions for MIMO systems



Quiz: can you derive the transfer function?

Transfer functions for MIMO systems



For the output side:

$$\mathrm{S_O}\colon \frac{y}{d_1} = (I+GK)^{-1} = S = \frac{e}{r} \quad \text{and} \quad \mathrm{T_O}\colon \frac{y}{r} = GK(I+GK)^{-1} = T$$

For the input side:

$$\mathrm{S_I}\colon \frac{u+d_2}{d_2} = (I+KG)^{-1} \quad \text{and} \quad \mathrm{T_I}\colon -\frac{u}{d_2} = KG(I+KG)^{-1}$$

Gain of a system

SISO (independent of magnitude):

$$\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|$$

How can we do this for MIMO? Sum up the magnitudes?

$$\frac{||y(\omega)||_2}{||d(\omega)||_2} = \frac{\sqrt{\sum_j |y_j(\omega)|^2}}{\sqrt{\sum_j |d_j(\omega)|^2}} = \frac{||G(j\omega)d(\omega)||_2}{||d(\omega)||_2}$$

This gain measure depends on direction of input

Max. gain given by:
$$\max_{d\neq 0} \frac{||G(j\omega)d(\omega)||_2}{||d(\omega)||_2} = \max_{||d||_2=1} ||G(j\omega)d(\omega)||_2 = \overline{\sigma}(G(j\omega))$$

Min. gain given by:
$$\min_{d\neq 0} \frac{||G(j\omega)d(\omega)||_2}{||d(\omega)||_2} = \min_{||d||_2=1} ||G(j\omega)d(\omega)||_2 = \underline{\sigma}(G(j\omega))$$

Gain of a system: Example

Define $d = \begin{bmatrix} d_{10} \\ d_{20} \end{bmatrix}$ with the following 5 inputs:

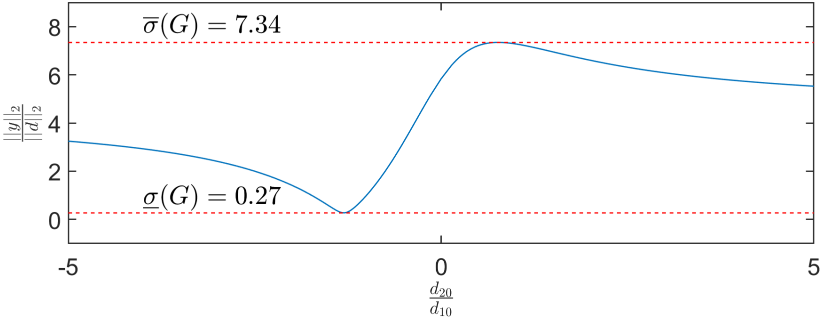
$$d_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad d_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \quad d_4 = \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}, \quad d_5 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix},$$

We consider: $G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$

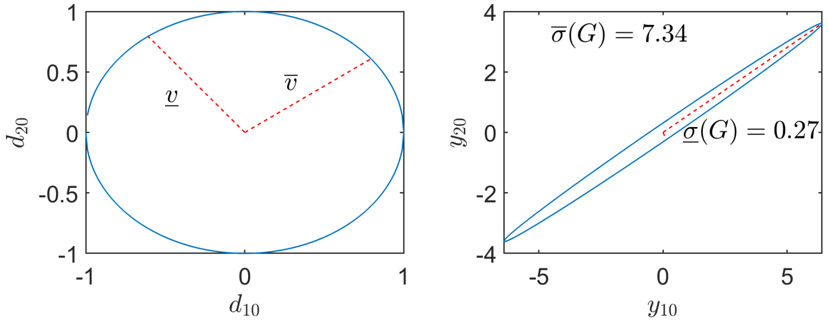
$$y_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 6.36 \\ 3.54 \end{bmatrix}, \quad y_4 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \quad y_5 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix},$$

Note that $||d||_2 = 1$ and we have:
$||y_1||_2 = 5.83$, $||y_2||_2 = 4.47$, $||y_3||_2 = 7.30$, $||y_4||_2 = 1.00$, $||y_5||_2 = 0.28$

Gain of a system: Example (cont'd)



Gain of a system: Example (cont'dd)



We can use the SVD

Eigenvalues as measure?

- Can only be computed for square matrices
- Can be really misleading

Let's consider:

$$G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}$$

From eigenvalues one might conclude that the gain is zero

Note that: $d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ results in $y = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$

Eigenvalues are a poor measure and can not be captured in a matrix norm

The Singular Value Decomposition

Let's consider a fixed frequency ω_o. Then, every matrix can be decomposed in:

$$G(\omega_o) = U\Sigma V^H$$

where:

- $\Sigma \in \ell \times m$
- $U \in \ell \times \ell$ Unitary matrix Output singular vectors
- $V \in m \times m$ Unitary matrix Input singular vectors

Structure for real-valued 2×2 matrix:

$$G = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cos\theta_2 & \pm\sin\theta_2 \\ -\sin\theta_2 & \pm\cos\theta_2 \end{bmatrix}^T}_{V^T}$$

The Singular Value Decomposition (cont'd)

Note that since V is unitary we have $GV = U\Sigma$ or:

$$G\begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & & 0 \\ 0 & \sigma_2 & & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix}$$

where $k = \min(m, \ell)$.

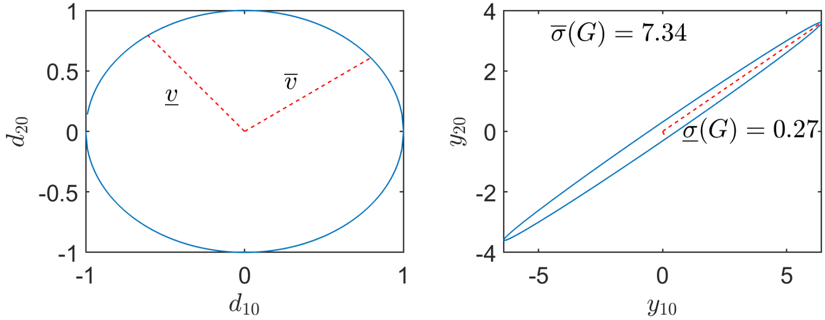
Max. gain given by:

$$\overline{\sigma}(G) \equiv \sigma_1(G) = \max_{d\neq 0} \frac{||Gd||_2}{||d||_2} = \frac{||Gv_1||_2}{||v_1||_2}$$

Min. gain given by:

$$\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d\neq 0} \frac{||Gd||_2}{||d||_2} = \frac{||Gv_k||_2}{||v_k||_2}$$

Gain of a system: Example (cont'ddd)



We can use the SVD (note that this also works for non-square plants)

$$G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}^T}_{V^T}$$

Singular values for performance

Sensitivity S provides useful information regarding the effectiveness of control.
For SISO systems this is defined as $\left|\frac{e}{r}\right|$

For MIMO we define $\frac{||e||_2}{||r||_2}$ where $||\cdot||_2$ represents the vector 2-norm

This gain depends on direction

We can bound it:

$$\underline{\sigma}(S(j\omega)) \leq \frac{||e||_2}{||r||_2} \leq \overline{\sigma}(S(j\omega))$$

In terms of performance it is reasonable to require that the gain is small for all directions (so look at $\overline{\sigma}(\omega)$)

Singular values for performance (cont'd)

Weighted Sensitivity design:

$\overline{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|, \forall\omega$ or $||w_P S||_\infty < 1$

Bandwidth, ω_B: Frequency where $\overline{\sigma}(S)$ crosses 0.7 from below

Note: the bandwidth is at least ω_B

Measure for directionality

1. Condition Number
   Defined as: $\gamma \triangleq \frac{\overline{\sigma}(G)}{\underline{\sigma}(G)}$
   - If large then ill-conditioned
   - High $\overline{\sigma}(G)$ no issue but low $\underline{\sigma}(G)$ can be an issue
   - Large condition number *may indicate* control issues
2. Relative gain array (RGA)
   See next section

Relative Gain Array (RGA)

The Relative Gain Array is defined as:

$$\mathrm{RGA}(\mathrm{G}) = \Lambda(\mathrm{G}) \triangleq \mathrm{G} \times \left(\mathrm{G}^{-1}\right)^{\mathrm{T}}$$

here $\times$ represents the Hadamard product.

Example for a 2×2 matrix with g_{ij}:

$$\Lambda(G) = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 1-\lambda_{11} \\ 1-\lambda_{11} & \lambda_{11} \end{bmatrix}$$

with $\lambda_{11} = \frac{1}{1-\frac{g_{12}g_{21}}{g_{11}g_{22}}}$

Note that the RGA is frequency dependent.

Interpretation: RGA as an interaction measure

Let's consider: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}}_{G} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\hat{g}_{11}} & \frac{1}{\hat{g}_{12}} \\ \frac{1}{\hat{g}_{21}} & \frac{1}{\hat{g}_{22}} \end{bmatrix}}_{G^{-1}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Look at interaction between u_1 and y_1:

1. Other loops open $u_2 = 0$ and compute $\frac{\partial y_1}{\partial u_1}$.
   g_{11}
2. Other loops closed with perfect control $y_2 = 0$ and compute $\frac{\partial y_1}{\partial u_1}$.
   $\hat{g}_{11}$

RGA$= \begin{bmatrix} \frac{g_{11}}{\hat{g}_{11}} & \frac{g_{12}}{\hat{g}_{21}} \\ \frac{g_{21}}{\hat{g}_{12}} & \frac{g_{22}}{\hat{g}_{22}} \end{bmatrix} = G \times \left(G^{-1}\right)^T$

We prefer to pair variables with an RGA of 1 (unaffected by other loops).

Pairing rules:

1. Prefer pairings such that the rearranged system, with the selected pairings along the diagonal, has an RGA matrix close to identity at the frequencies around the bandwidth
2. Avoid (if possible) pairing on negative steady-state RGA elements.

Other properties:

- RGA independent of scaling
- Rows and columns sum up to 1
- Use pseudo-inverse for non-square plants
- Plants with large RGA elements are ill-conditioned (> 10 difficult to control)

Two step procedure

The easiest way to control MIMO systems is by using a diagonal controller (Decentralized control).

The easiest way to do this is to decouple the system in a diagonal system $\hat{G}(s) = G(s)W_1$. How?

- Dynamic decoupling, $\hat{G}(s) = G(s)G(s)^{-1}$ where the controller is given by $\frac{k}{s}I$ (inversed based control)
- Steady state decoupling, $\hat{G}(s) = G(s)G(0)^{-1}$ which is a constant pre-compensator.
- Approximate decoupling, $\hat{G}(s) = G(s)G(j\omega_o)^{-1}$ typically ω_o is chosen close to the bandwidth
- Approximate decoupling with post-compensator, $\hat{G}(s) = \underbrace{U(j\omega_o)^T}_{W_2} G(s) \underbrace{V(j\omega_o)}_{W_1}$ typically ω_o is chosen close to the bandwidth

Direct synthesis of MIMO controller

Typical approach:

$$||N||_\infty = \max_\omega \overline{\sigma}(N(j\omega)) < 1; \qquad N = \begin{bmatrix} W_p S \\ W_u K S \end{bmatrix}$$

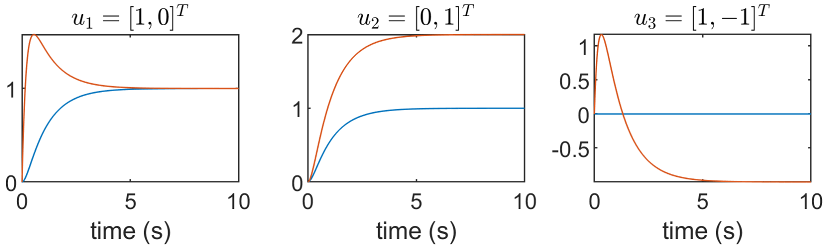
- Design W_p, common choice $W_p = diag(w_{p,i})$ with $w_{p,i} = \frac{s/M_i + \omega_{Bi}}{s + \omega_{Bi} A_i}$ with $A_i \ll 1$
- Design W_u, common I or $W_u = s/(s + \omega_l)$ (so low penalty at low frequencies)
- To find suitable values, design a controller by hand (decoupling)
- Add more transfer functions to the objective function

Example

Let's consider:

$$G(s) = \frac{1}{(0.2s+1)(s+1)} \begin{bmatrix} 1 & 1 \\ 1+2s & 2 \end{bmatrix}$$

Step response (y_1 and y_2):

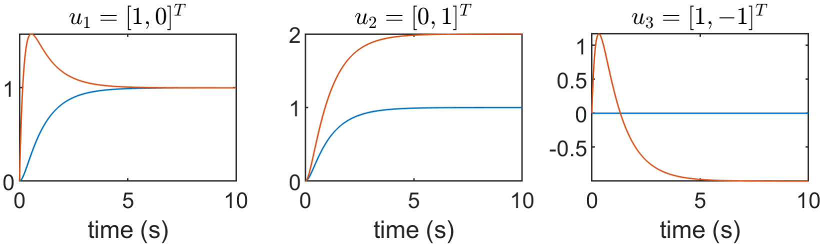


Looking at the TF's there is no reason to assume that we have a RHP-zero. However, there is one at $z = 0.5$

Example (cont'd)

$$G(0.5) = \frac{1}{1.65}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.45 & 0.89 \\ 0.89 & -0.45 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 1.92 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.71 & 0.71 \\ 0.71 & -0.71 \end{bmatrix}^T}_{V^T}$$

The $blue$ elements represent the input and output direction corresponding to a RHP-zero



MIMO zeros

MIMO zeros: z_i is a zero of G(s) if the rank of $G(z_i)$ is less than the normal rank of $G(z_i)$

Compute MIMO zeros:

1. $$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$
2. $$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \end{bmatrix}$$
3. $$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \\ \frac{s-1}{s+2} & \frac{s-2}{s+1} \end{bmatrix}$$

What is the effect of a MIMO RHP-zero?

Example (cont'dd)

We solve:

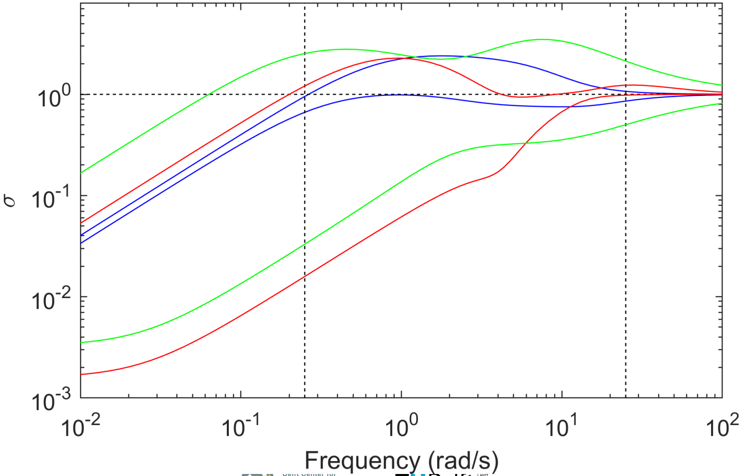
$$||N||_\infty = \max_\omega \overline{\sigma}(N(j\omega)) < 1; \qquad N = \begin{bmatrix} W_P S \\ W_u K S \end{bmatrix}$$

with $W_u = I$, $W_p = diag(w_{Pi})$ with $w_{Pi} = \frac{s/M_i + \omega_{Bi}}{s + \omega_{Bi} A_i}$ with $A_i = 10^{-4}$

```
>>s=tf('s');
>>G=1/(0.2*s+1)/(s+1)*[ 1 1; 1+2*s 2];
>>wB1=0.25; % desired closed-loop bandwidth
>>wB2=0.25; % desired closed-loop bandwidth
>>A=1/1000; % desired disturbance attenuation inside bandwidth
>>M=1.5 ; % desired bound on hinfnorm(S)
>>Wp=[(s/M+wB1)/(s+wB1*A) 0; 0 (s/M+wB2)/(s+wB2*A)]; %
Sensitivity weight
>>Wu=eye(2); % Control weight
>>Wt=[] % Empty weight
>>[K,CL,GAM,INFO]=mixsyn(G,Wp,Wu,Wt);
```

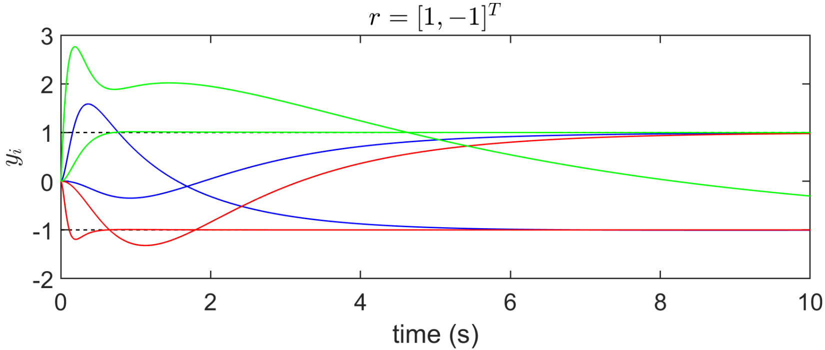
Example (cont'ddd)

Design 1: $M_1 = M_2 = 1.5$, $\omega_{B1} = \omega_{B2} = 0.25$. $||N||_\infty = 2.80$
Design 2: $M_1 = M_2 = 1.5$, $\omega_{B1} = 0.25$, $\omega_{B2} = 25$. $||N||_\infty = 2.92$
Design 3: $M_1 = M_2 = 1.5$, $\omega_{B1} = 25$, $\omega_{B2} = 0.25$. $||N||_\infty = 6.70$



Example (cont'dddd)

Design 1: $M_1 = M_2 = 1.5$, $\omega_{B1} = \omega_{B2} = 0.25$. $||N||_\infty = 2.80$
Design 2: $M_1 = M_2 = 1.5$, $\omega_{B1} = 0.25$, $\omega_{B2} = 25$. $||N||_\infty = 2.92$
Design 3: $M_1 = M_2 = 1.5$, $\omega_{B1} = 25$, $\omega_{B2} = 0.25$. $||N||_\infty = 6.70$

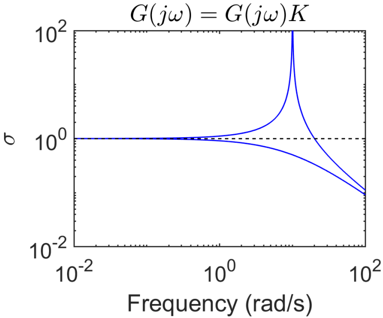


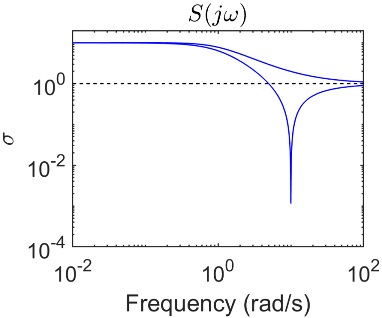
Example 1: Spinning satellite

$$G(s) = \frac{1}{s^2+100}\begin{bmatrix} s-100 & 10s+10 \\ -10s-10 & s-100 \end{bmatrix}$$ system has poles at $s = \pm j10$

Apply negative unity, I, feedback.

- **NS:** The closed loop system has two poles at $s = -1$
- **NP:**



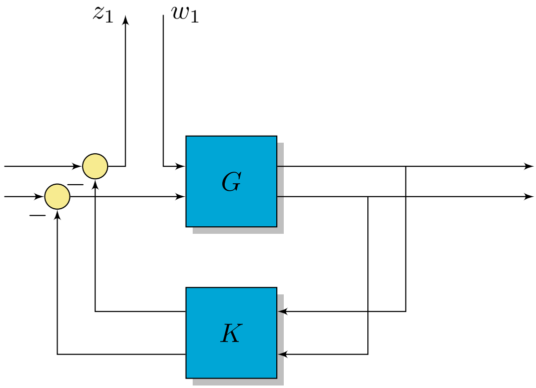


- **RS:** See next slide

Example 1: Spinning satellite (cont'd)

RS: We will consider diag. input uncertainty (present in every plant)

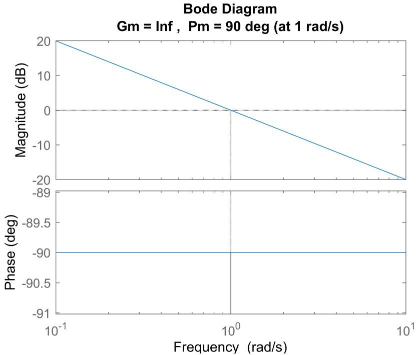
Consider opening one loop:



Example 1: Spinning satellite (cont'd)

RS: We will consider diag. input uncertainty (present in every plant)

Always stable (for all input perturbations)

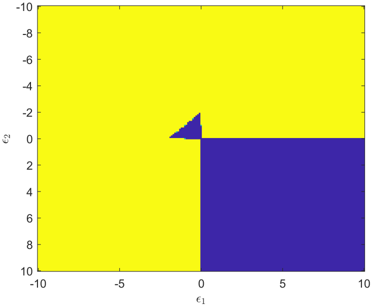


Similar for other loop

Example 1: Spinning satellite (cont'dd)

Uncertainty in input is given by: $u_1' = (1+\epsilon_1)\,u_1, \quad u_2' = (1+\epsilon_2)\,u_2$

It is easy to show that the system is stable for $-1 < \epsilon_1 < \infty, \epsilon_2 = 0$ and $\epsilon_1 = 0, -1 < \epsilon_2 < \infty$



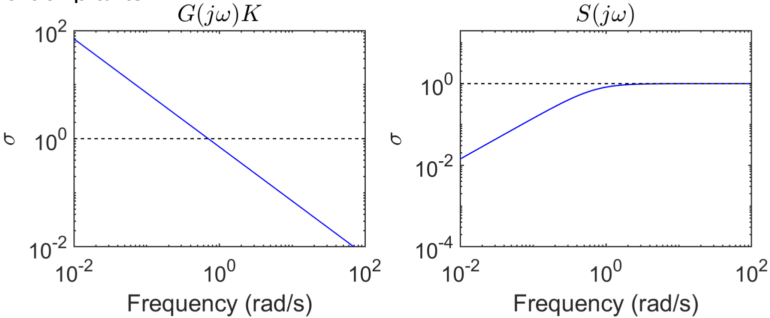
RS: For MIMO GM and PM do not provide RS information. Large $\overline{\sigma}(S)$ indicate robustness issues.

Example 2: Distillation column

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & 86.4 \\ 108.2 & -109.6 \end{bmatrix} \text{ with RGA } \forall\omega \begin{bmatrix} 35.1 & -34.1 \\ -34.1 & 35.1 \end{bmatrix}$$

Due to large elements in RGA difficult to control

Controller: inverse with integral action $K_{inv} = \frac{0.7}{s}G(s)^{-1}$

- **NS:** With inverse control you end up with decoupled two first order plants
- 

- **RS:** No high $\overline{\sigma}(S)$ but high RGA values cause for concern ($\Rightarrow$)

Example 2: Distillation column (cont'd)

RS: We will consider diag. input uncertainty
(typically 20% for process applications)

Uncertainty in input is given by: $u_1' = (1 + \epsilon_1)\, u_1, \quad u_2' = (1 + \epsilon_2)\, u_2$

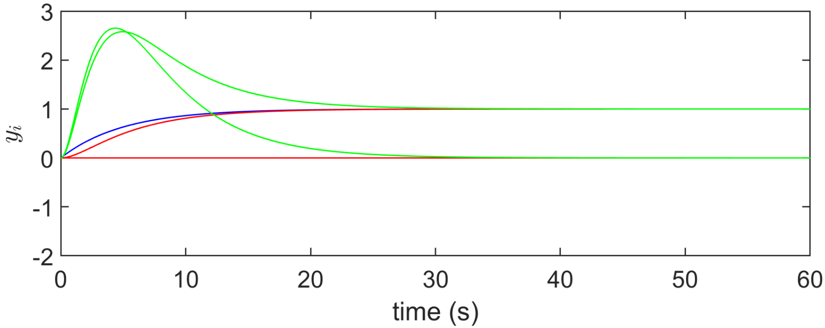
We have: $L(s) = \frac{0.7}{s} \begin{bmatrix} 1 + \epsilon_1 & 0 \\ 0 & 1 + \epsilon_2 \end{bmatrix}$

Compute poles: $\det(I + L) = (s + 0.7\,(1 + \epsilon_1))\,(s + 0.7\,(1 + \epsilon_2))$. We can have up to 100% error in all the input channels

RP: Consider $u_1' = 1.2u_1, \quad u_2' = 0.8u_2$

Example 2: Distillation column (cont'dd)

RP: Consider $u_1' = 1.2u_1, \quad u_2' = 0.8u_2$

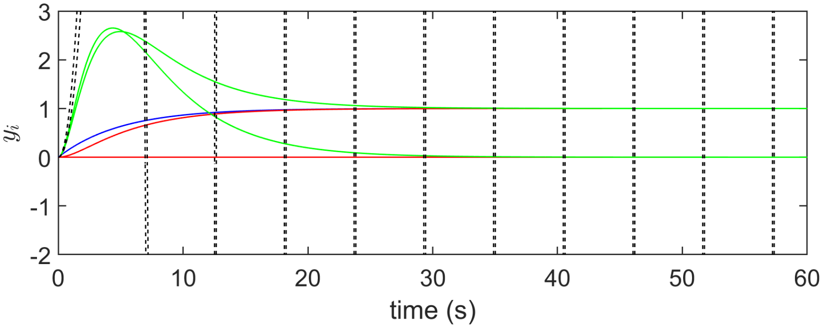


Reference, Nominal control, Uncertain Input, Uncertain I & O

RP From the response we can conclude that we don't have RP

Example 2: Distillation column (cont'ddd)

RS: Consider $u_1' = 1.15u_1$, $u_2' = 0.85u_2$ and $y_1' = 1.15y_1$, $y_2' = 0.85y_2$



Reference, Nominal control, Uncertain Input, Uncertain I & O

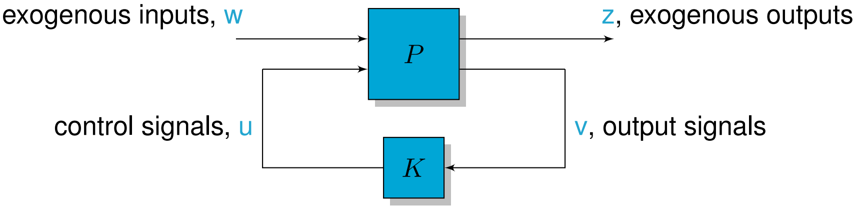
RS With additional output uncertainty also no RS

Conclusions from examples

1. Example 1, excellent PM, GM for single loops but no RS
   Might have been expected from high $\overline{\sigma}(S)$
2. Example 2, good $\overline{\sigma}(S)$ good RS for input uncertainty but no RP
   The RGA can be seen as an indicator for bad diagonal control
3. Example 2, bad RS if we also add output uncertainty

RGA and $\overline{\sigma}(S)$ are indicators for robustness but we need better tools for analysis and synthesis. Later we will develop tools for this within the $\mathcal{H}_\infty$ robust control framework using the structured singular value

Generalized Plant: Definition

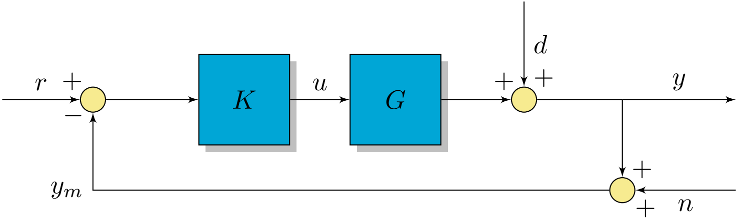


(Note positive feedback)

Find a controller, K, which counteracts the influence of w on z (minimizing a certain norm *e.g.* $\mathcal{H}_\infty$, $\mathcal{H}_2$)

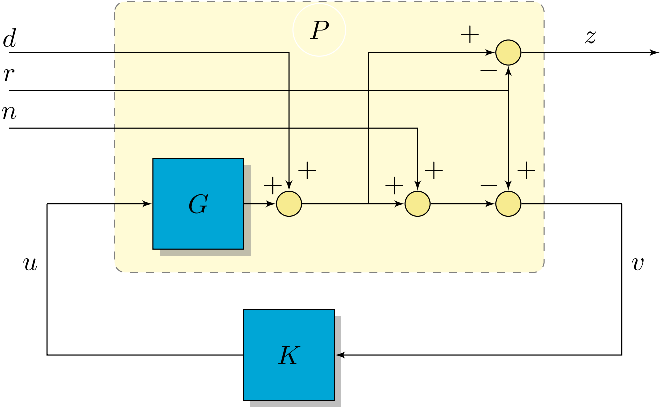
Generalized Plant: Example

Consider the following feedback structure:



$$w = \begin{bmatrix} d \\ r \\ n \end{bmatrix}, \; v = r - y_m \,, \; z = y - r$$

Generalized Plant: Example (cont'd)

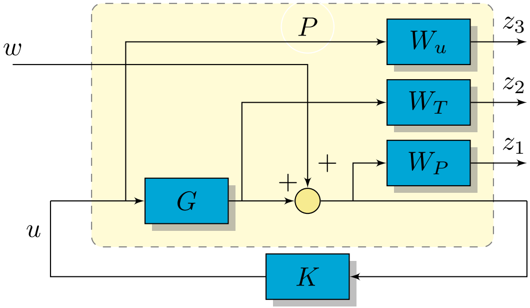


$$P = \begin{bmatrix} I & -I & 0 & G \\ -I & I & -I & -G \end{bmatrix}$$

Generalized Plant: Mixed sensitivity

Can we also embed the Mixed sensitivity problem in this framework?

$$\min_K ||N||_\infty \quad N = \begin{bmatrix} W_P S \\ W_T T \\ W_u K S \end{bmatrix} \qquad P = \left[\begin{array}{c|c} W_P & W_P G \\ 0 & W_T G \\ 0 & W_u \\ \hline I & G \end{array}\right] = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right]$$



Generalized Plant: Mixed sensitivity (cont'd)

Code: (you need code from Section MIMO Design to run)

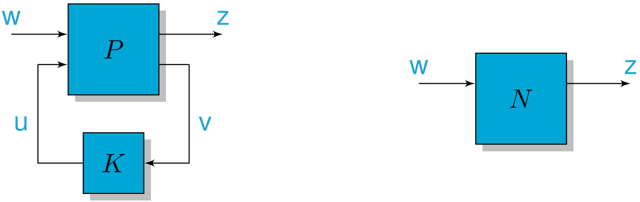
```
 Uses the robust control toolbox
>>systemnames ='G Wp Wu Wt'; % Define systems
>>inputvar ='[w(2); u(2)]'; % Input generalized plant
>>input_to_G= '[u]';
>>input_to_Wu= '[u]';
>>input_to_Wt= '[G]';
>>input_to_Wp= '[w+G]';
>>outputvar= '[Wp; Wt; Wu; G+w]'; % Output generalized plant
>>sysoutname='P';
>>sysic;
>>[K2,CL2,GAM2,INFO2] = hinfsyn(P,2,2); % Hinf design
```

Generalized Plant: Closing the loop

We have:

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right]$$

We have $z = Nw$ if we close the loop $u = Kv$:

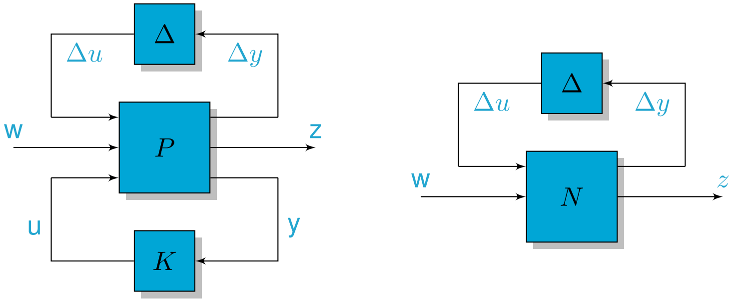


$$N = P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1}P_{21} \triangleq F_l(P, K)$$

$F_l(P, K)$ is called the lower Linear Fractional Transformation (LFT)

Generalized Plant: Robust control

What we are going to do later: Synthesis for robust control (Direct synthesis for RS and RP).



where Δ is a block diagonal matrix containing all the perturbations (typically $||\Delta||_\infty \leq 1$)