Robust Stability for MIMO systems

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Uncertainty

Previous lecture we introduced an uncertainty block Δ.

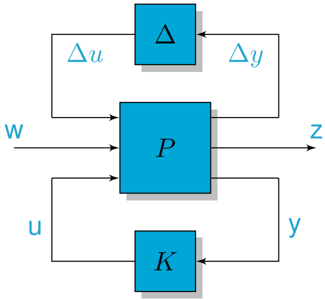
For MIMO systems it is of interest to consider several Δ_i's

Approach: We collect all the uncertainties in a big block diagonal uncertainty block:

$$\Delta = \begin{bmatrix} \Delta_1 & & & \\ & \ddots & & \\ & & \Delta_i & \\ & & & \ddots \end{bmatrix}$$

If we now pull out all the uncertainties and controller:

$\Downarrow$



Useful for controller synthesis

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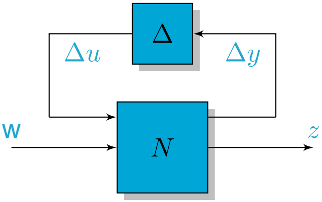

Useful for **RP**

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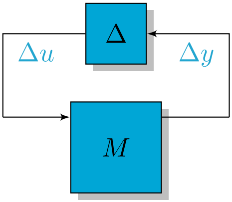


Useful for **RS**

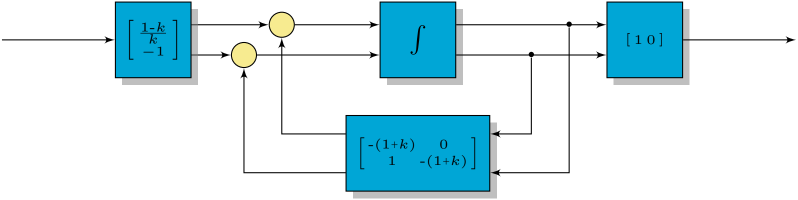
Example

Suppose we have the following SISO system:

$$\dot{x} = \underbrace{\begin{bmatrix} -(1+k) & 0 \\ 1 & -(1+k) \end{bmatrix}}_{A_p} x + \underbrace{\begin{bmatrix} \frac{1-k}{k} \\ -1 \end{bmatrix}}_{B_p} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where $k = 0.5 + 0.1\delta$ where $|\delta| < 1$.



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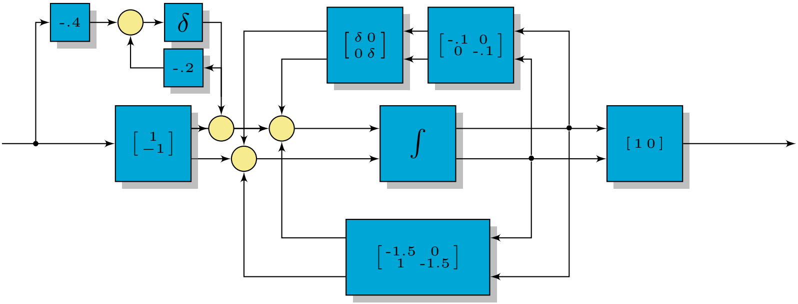
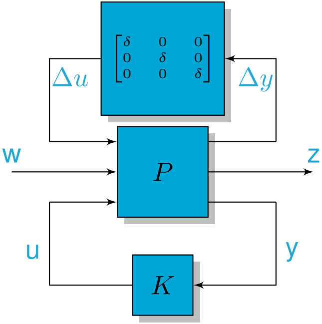


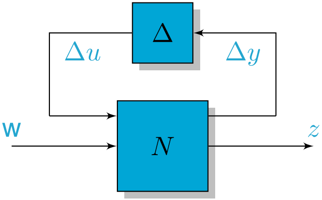
Example (cont'd)



Observe:

- We have a diagonal block uncertainty
- We have considered a SISO system
- There is more structure all the $\delta' s$ are the same

Generalization

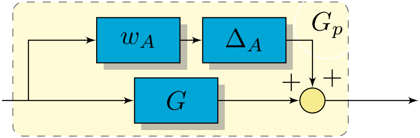


- For every Δ_i we have $\overline{\sigma}(\Delta_i(j\omega)) \leq 1 \forall \omega$
- Due to diagonal structure $||\Delta||_\infty \leq 1$
- Remember that Δ has structure, so we don't allow all Δ's
- Therefore we will introduce in this lecture the structured singular value, μ
- μ can directly be used for analysis
- μ can be used for synthesis (solving a couple of scaled H_∞ problems)

SISO vs MIMO uncertainty descriptions

The representation of parametric uncertainty carries straight over to MIMO systems

Unstructured perturbations are often used to get a simple uncertainty model. One uncertainty source is considered to be full. For example:



Additive uncertainty: Π_A : $\quad G_p = G + w_A \Delta_A$

For SISO lump uncertainties in a single complex uncertainty.
For MIMO?

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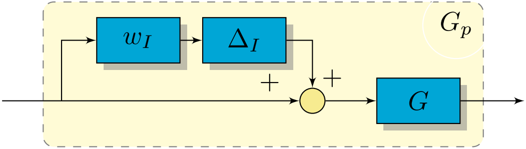

Multiplicative input uncertainty: $\Pi_I : \quad G_p = G(I + w_I \Delta_I)$

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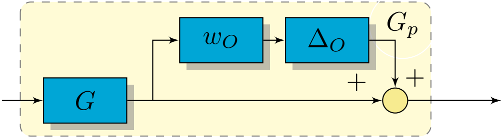

Multiplicative output uncertainty: $\Pi_O: \quad G_p = (I + w_O\Delta_O)G$

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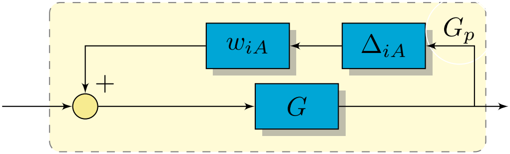

Inv. additive uncertainty: $\Pi_{iA}:\quad G_p = G\left(I - w_{iA}\Delta_{iA}G\right)^{-1}$

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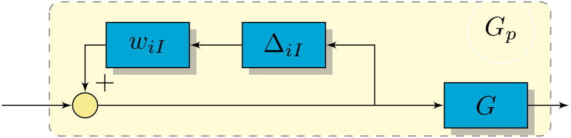

Inv. multiplicative input uncertainty: Π_{iI} : $\quad G_p = G\left(I - w_{iI}\Delta_{iI}\right)^{-1}$

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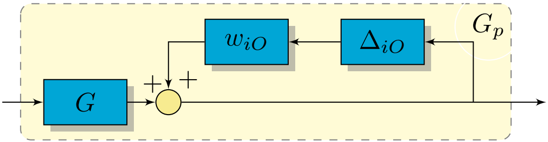


Inv. multiplicative output uncertainty: Π_{iO} : $\quad G_p = (I - w_{iO}\Delta_{iO})^{-1}G$
For SISO lump uncertainties in a single complex uncertainty.

For MIMO?

SISO vs MIMO uncertainty descriptions (cont'd)

You can but, uncertainty set becomes **bigger**

Example: We consider the following unstructured input uncertainty:
$G_p = G(I + E_I)$

We can simply apply the SISO approach to find a multiplicative input uncertainty description:
$l_I(\omega) = \max_{E_I} \overline{\sigma}\left(G^{-1}\left(G_p - G\right)\right) = \max_{E_I} \overline{\sigma}\left(E_I\right)$

We can simply apply the SISO approach to find a multiplicative output uncertainty description:
$l_O(\omega) = \max_{E_I} \overline{\sigma}\left(\left(G_p - G\right) G^{-1}\right) = \max_{E_I} \overline{\sigma}\left(G E_I G^{-1}\right)$

For SISO $l_O = l_I$ for MIMO $l_O \sim \gamma l_I$ where γ is the condition number.

Note that if $l_I(\omega)$ or $l_O(\omega)$ is above 1 the system can basically not be controlled at that frequency.

Input uncertainty

Input uncertainty comes from: amplifier dynamics, signal converter

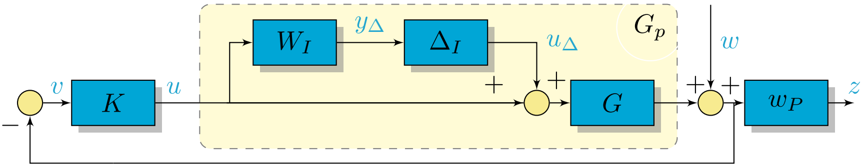
The physical input to the system is $m_i = h_i(s)u_i$ (typically decoupled for every channel i)

The known dynamics is typically absorbed in the plant model but the uncertainty can easily be presented by a multiplicative uncertainty:
$h_{pi}(s) = h_i(s)(1 + w_{Ii}(s)\delta_s(s)), \quad |\delta_i(j\omega)| \leq 1, \forall\omega$

Combining this leads to: $G_p = G(I + W_I\Delta_I)$ where both W_I and Δ_I are diagonal matrices (Typically w_{Ii} is given by $\frac{\tau s + r_o}{\frac{\tau}{r_\infty}s+1}$)

Diagonal input uncertainty should always be considered:
1. It is always present in a real system
2. It often restricts the performance with MIMO control

Obtaining P, N, and M



$$P: \quad \begin{bmatrix} y_\Delta \\ z \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 & W_I \\ W_P G & W_p & W_P G \\ -G & -I & -G \end{bmatrix} \begin{bmatrix} u_\Delta \\ w \\ u \end{bmatrix}$$

$$N: \quad \begin{bmatrix} y_\Delta \\ z \end{bmatrix} = \begin{bmatrix} -W_I KG(I+KG)^{-1} & -W_I K(I+GK)^{-1} \\ W_P G(I+KG)^{-1} & W_P(I+GK)^{-1} \end{bmatrix} \begin{bmatrix} u_\Delta \\ w \end{bmatrix}$$

$$M: \quad \begin{bmatrix} y_\Delta \end{bmatrix} = \begin{bmatrix} -W_I KG(I+KG)^{-1} \end{bmatrix} \begin{bmatrix} u_\Delta \end{bmatrix}$$

Use `N=lft(P,K)` and `M=lft(Δ,N)` to generate systems

Question

Given: $\dfrac{y}{u} = \dfrac{s+a}{s+b} = G(s)$

Find P such that $F_l\left(P, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = G(s)$

Definition robust stability and performance

We have the $N\Delta$ structure (unstructured Δ block):

$$N: \quad \begin{bmatrix} y_\Delta \\ z \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} u_\Delta \\ w \end{bmatrix}$$

$$F_u(N,\Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \quad \text{(e.g. } F_u(N,\Delta) = W_P S_p\text{)}$$

NS: N is internally stable

NP: $||N_{22}||_\infty < 1$ and **NS**

RS: $F_u(N,\Delta)$ is stable for all Δ, $||\Delta||_\infty \leq 1$ and **NS**

RP: $||F_u(N,\Delta)||_\infty < 1$ for all Δ, $||\Delta||_\infty \leq 1$ and **NS**

Robust stability of the $M\Delta$ structure

$F_u(N,\Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$ we assume **NS** (N_{11}, N_{21}, N_{12}, and N_{22} stable)

We also assume Δ to be stable

We have **RS** if $(I - M\Delta)^{-1}$ is stable with $M = N_{11}$

Remember generalized Nyquist theorem:
- The $\det(\mathrm{I} - \mathrm{M}\Delta)$ will not encircle the point 0, $\forall\Delta$
- $\det(\mathrm{I} - \mathrm{M}\Delta) \neq 0, \forall\Delta, \forall\omega$
- $\lambda_i(M\Delta) \neq 1, \forall i, \Delta, \forall\omega$

Remember: $\det(I\lambda_i - M\Delta) = 0$ characteristic polynominal $M\Delta$ and λ_i eigenvalues of $M\Delta$

RS: $\rho(M\Delta(j\omega)) < 1, \forall\omega, \forall\Delta$ (or $\max_\Delta \rho(M\Delta(j\omega)) < 1, \quad \forall\omega$)

Robust stability of the $M\Delta$ structure: complex unstructured uncertainty

Now we assume that Δ is allowed to be any (full) complex transfer function with $||\Delta||_\infty \leq 1$ (Unstructured uncertainty). For this case the following equality holds:

$$\max_{\Delta} \rho(M\Delta) = \max_{\Delta} \overline{\sigma}(M\Delta) = \max_{\Delta} \overline{\sigma}(\Delta)\overline{\sigma}(M) = \overline{\sigma}(M)$$

Sketch of proof: We know that:

$\max_\Delta \rho(M\Delta) \leq \max_\Delta \overline{\sigma}(M\Delta) \leq \max_\Delta \overline{\sigma}(\Delta)\overline{\sigma}(M) = \overline{\sigma}(M)$

$\Rightarrow$For every M there exists a Δ' such that $\rho(M\Delta') = \overline{\sigma}(M)$

$\Rightarrow \Delta' = VU^H$ where $M = U\Sigma V^H$

$\Rightarrow \max_\Delta \rho(U\Sigma U^H) = \max_\Delta \overline{\sigma}(U\Sigma U^H) = 1 \times \overline{\sigma}(U\Sigma V^H) = \overline{\sigma}(U\Sigma V^H)$

RS: $\overline{\sigma}(M) < 1, \forall\omega$ (or $||M||_\infty < 1$)

Application of the unstructured RS condition

Remember all the six uncertainty structures and redefine $E = W_2 \Delta W_1$, with $||\Delta||_\infty \leq 1$.

We now isolate the perturbations to obtain $M = W_1 M_o W_2$

We now have:

- $G_p = G + E_A$: $\quad M_o = K(I+GK)^{-1} = KS$
- $G_p = G(I + E_I)$: $\quad M_o = K(I+GK)^{-1}G = T_I$
- $G_p = (I + E_O)G$: $\quad M_o = GK(I+GK)^{-1} = T$

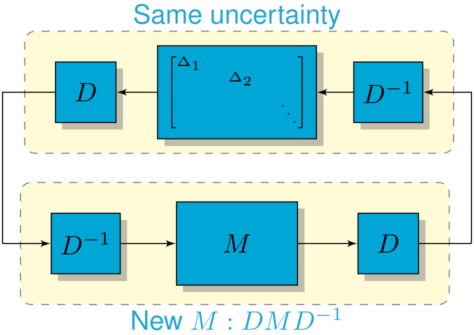
RS: $||W_1 M_o W_2||_\infty < 1$

- Multiplicative Input uncertainty (scalar weight): $||w_I T_I||_\infty < 1$

Robust stability with structured uncertainty

Structured uncertainty: **RS** if $\overline{\sigma}(M) < 1$, $\forall\omega$ (note if and not iff)

We introduce scaling: $D = \mathrm{diag}\{\mathrm{d_i I_i}\}$



The following should also hold: **RS** if $\overline{\sigma}(DMD^{-1}) < 1$, $\forall\omega$

To obtain the least conservative **RS** condition:
RS if $\min_{D(\omega)\in\mathcal{D}} \overline{\sigma}(D(\omega)M(j\omega)D(\omega)^{-1}) < 1$, $\forall\omega$

μ-The structured singular value

The structured singular value is a generalization of the maximum singular value (also μ, Mu, mu, SSV)

Find the smallest structured Δ (measured in terms of $\overline{\sigma}(\Delta)$) which makes the matrix $I - M\Delta$ singular; then $\mu(M) = 1/\overline{\sigma}(\Delta)$

Or: $$\frac{1}{\mu(M)} \triangleq \min_{\Delta}\{\overline{\sigma}(\Delta)|\det(\mathrm{I}-\mathrm{M}\Delta)=0 \quad \text{for} \quad \text{structured} \quad \Delta\}$$

Or:
$$\frac{1}{\mu(M)} \triangleq \min\{k_m|\det(\mathrm{I}-\mathrm{k_m}\mathrm{M}\Delta)=0 \quad \text{for} \quad \text{structured} \quad \overline{\sigma}(\Delta)\leq 1\}$$

Small μ is good, large μ bad

The scalar μ gives a measure instead of yes/no condition.

The scalar μ depends on M and the structure of Δ.

μ-The structured singular value (cont'd)

Remember: **RS** if $\det(I - M\Delta(j\omega)) \neq 0$, $\forall\omega$, $\forall\Delta$, $\overline{\sigma}(\Delta(j\omega)) \leq 1$

Note that this is a yes or no condition

Find the smallest k_m such that $\det(I - k_m M\Delta(j\omega)) = 0$

From the definition of μ we have $\mu = \frac{1}{k_m}$ and allowing structured uncertainty

RS iff $\mu(M(j\omega)) < 1, \quad \forall\omega$

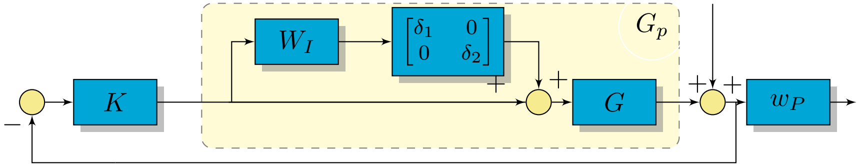
μ-The structured singular value (cont'dd)

Some properties (assuming complex perturbations):

1. $\mu(M) = \max_{\Delta, \overline{\sigma}(\Delta) \leq 1} \rho(M\Delta)$
2. For a scalar α it holds that $\alpha\mu(M) = \mu(\alpha M)$
3. For $\Delta = I\delta$ (δ a complex scalar) it holds that $\mu(M) = \rho(M)$
4. For a full complex Δ it holds that $\mu(M) = \overline{\sigma}(M)$
5. In general it holds that $\rho(M) \leq \mu(M) \leq \overline{\sigma}(M)$
6. Consider $\Delta D = D\Delta$ it holds that $\mu(DMD^{-1}) = \mu(M)$
7. Upper bound: $\mu(M) \leq \overline{\sigma}(DMD^{-1})$ (using 5 and 6)

Remarks: In practice a really tight bound and a convex problem

Example 8.9:



$$G = \frac{1}{\tau s+1}\begin{bmatrix}-87.8 & 1.4\\ -108.2 & -1.4\end{bmatrix},\ K = \frac{1+\tau s}{s}\begin{bmatrix}-0.0015 & 0\\ 0 & -0.075\end{bmatrix},\ w_I = \frac{s+0.2}{0.5s+1}$$

RS if $\sigma(M(j\omega)) < 1, \quad \forall\omega$ or for this example if $\overline{\sigma}(T_I) < \frac{1}{|w_I|} \quad \forall\omega$

RS iff $\mu(M(j\omega)) < 1, \quad \forall\omega$ or for this example iff $\mu(T_I) < \frac{1}{|w_I|} \quad \forall\omega$

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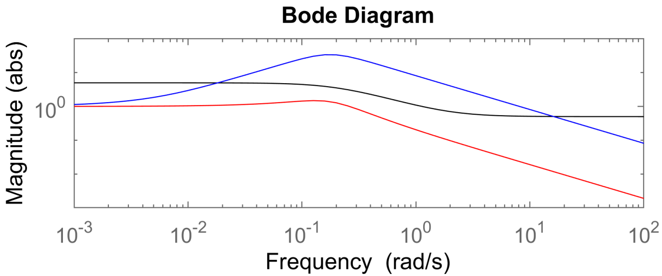
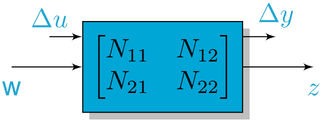


$\frac{1}{|w_I|}$ $\overline{\sigma}(T_I)$ $\mu(T_I)$

Summary:



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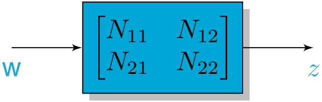

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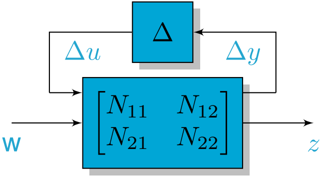

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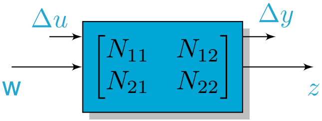

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