Robust Controller Synthesis

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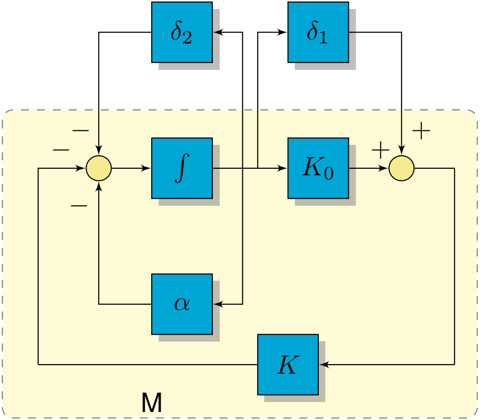
Example

Given:

- Real parametric uncertainty
- $|\delta_1| \leq 1$
- $|\delta_2| \leq 1$
- stable nominal system
- static feedback $K > 0$

Questions:

- Compute $\max_\omega \mu(M(j\omega))$?
- Compute $||M||_\infty$?
- What does it mean?
- What are the real conditions for stability?



$$M = \frac{1}{s+\alpha+K_0K}\begin{bmatrix} -K & -1 \\ -K & -1 \end{bmatrix}$$

example taken from: robust control (lecture notes) by Ad Damen and Siep Weiland

Example: compute μ

First compute $\mu(M(j\omega))$:

$$\begin{aligned}
\det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{1}{\mu}\frac{1}{\mathrm{s}+\alpha+\mathrm{K_0K}}\begin{bmatrix}K & 1\\K & 1\end{bmatrix}\begin{bmatrix}\delta_1 & 0\\0 & \delta_2\end{bmatrix}\right) &= 0\\
\det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{1}{\mu}\frac{1}{\mathrm{s}+\alpha+\mathrm{K_0K}}\begin{bmatrix}K\delta_1 & \delta_2\\K\delta_1 & \delta_2\end{bmatrix}\right) &= 0\\
1 + \frac{1}{\mu}\frac{K\delta_1+\delta_2}{s+\alpha+K_0K} &= 0
\end{aligned}$$

Now we have to find the biggest μ for which the above equality holds:

$$\mu = \frac{-K\delta_1-\delta_2}{s+\alpha+K_0K} = \frac{|K|+1}{s+\alpha+K_0K}$$

Compute $\max_\omega \mu(M(j\omega))$ (note: low pass filter):

$$\max_\omega \mu(M(j\omega)) = \frac{|K|+1}{|\alpha+K_0K|}$$

Example: compute $||M||_\infty$

First remember that $||M||_\infty = \max_\omega \overline{\sigma}(M(\omega))$. So, first find $\overline{\sigma}(M(\omega))$:

$$M = \frac{1}{s+\alpha+K_0K}\begin{bmatrix}-K & -1\\ -K & -1\end{bmatrix}$$

$$= \underbrace{\frac{\begin{bmatrix}1 & *\\ 1 & *\end{bmatrix}}{\sqrt{(2)}}}_{U}\underbrace{\begin{bmatrix}\sqrt{(2)}\frac{1}{|s+\alpha+K_0K|}\sqrt{(K^2+1)} & 0\\ 0 & 0\end{bmatrix}}_{S}\underbrace{\frac{\begin{bmatrix}-K & -1\\ * & *\end{bmatrix}}{\sqrt{(K^2+1)}}}_{V^T}$$

Now compute $||M||_\infty = \max_\omega \overline{\sigma}(M(\omega))$(note: low pass filter):

$$||M||_\infty = \frac{\sqrt{2\,(K^2+1)}}{\sqrt{(\omega^2+(\alpha+K_0K)^2}} = \frac{\sqrt{2\,(K^2+1)}}{|\alpha+K_0K|}$$

Example: What does it mean?

First observe that:

$$\begin{aligned} \mu(M(\omega)) &\leq \overline{\sigma}(M(\omega)) \\ \frac{|K|+1}{\sqrt{\omega^2 + (\alpha + K_0 K)^2}} &\leq \frac{\sqrt{2\,(K^2+1)}}{\sqrt{\omega^2 + (\alpha + K_0 K)^2}} \end{aligned}$$

It means that we know that the system is **RS** if $\max_\omega \mu(M(\omega)) < 1$ or $||M||_\infty < 1$.

Scaling:If $\delta_1 \leq \frac{1}{\gamma}$ and $\delta_2 \leq \frac{1}{\gamma}$ the **RS** turns out to be $\max_\omega \mu(M(\omega)) < \gamma$ or $||M||_\infty < \gamma$.

Example: What are the real conditions for stability?

Note that the closed loop pole is given by: $-\left(\alpha+\delta_{2}\right)-\left(K_{0}+\delta_{1}\right)K$
Or: $-\alpha-K_{0}K-\delta_{2}-\delta_{1}K$

The system is **NS** and consequently $\alpha+K_{0}K>0$

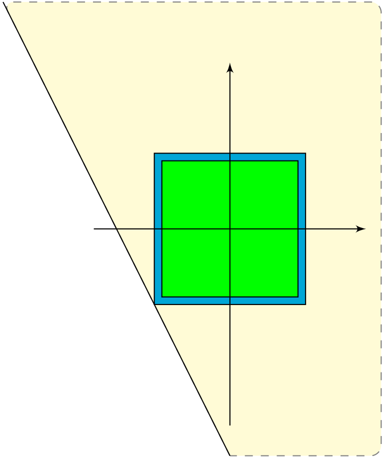
This directly implies for **RS** that $\alpha+K_{0}K>-\delta_{2}-\delta_{1}K$

Numerical example: $K_{0}=\alpha=1$ and $K=2$ for **RS** it should hold that:

- True: $2\delta_{1}+\delta_{2}>-3$
- According μ: $|\delta_{1}|<1$ and $|\delta_{2}|<1$
- According ∞: $|\delta_{1}|<\frac{1}{\sqrt{\frac{10}{9}}}$ and $|\delta_{2}|<\frac{1}{\sqrt{\frac{10}{9}}}$

Example: Graphical representation

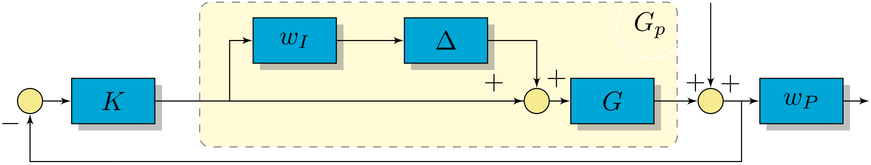
- True
- μ
- ∞



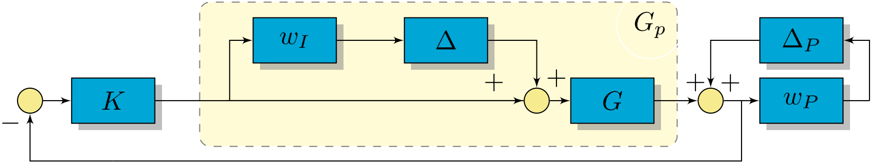
Why didn't we work with: $M = \frac{1}{s+\alpha+K_0K}\begin{bmatrix}-K & -1\end{bmatrix}$

The similarity between RS and RP (SISO example)

Consider:

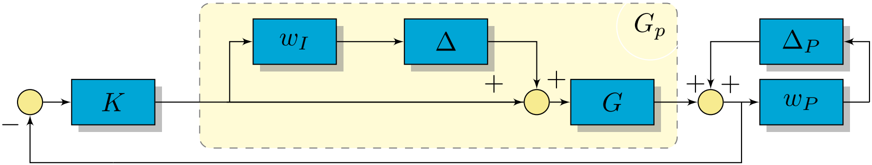


Remember: **RS:** $|w_I T| < 1 \; \forall\omega$ and **RP:** $|w_I T| + |w_P S| < 1 \; \forall\omega$



Main idea: Check **RS:** for the settings above

The similarity between RS and RP (SISO example)



Assume: Stable L_p and **NS** (nyquist doesn't encircle -1)

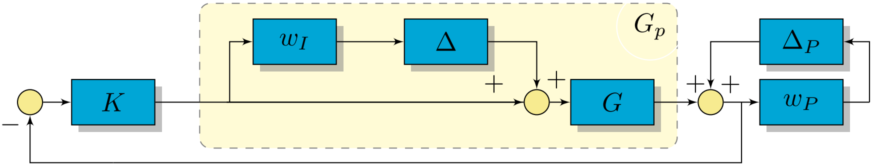
$$\begin{aligned}
\text{RS} \quad &\Leftrightarrow \quad |1+L_p| > 0, \qquad \forall L_p, \forall\omega \\
&\Leftrightarrow \quad |1+L(1+w_I\Delta)(1-w_P\Delta_P)^{-1}| > 0, \forall\Delta, \forall\Delta_P, \forall\omega \\
&\Leftrightarrow \quad |1+L+w_I L\Delta - w_P\Delta_P| > 0, \forall\Delta, \forall\Delta_P, \forall\omega
\end{aligned}$$

Last condition is most easily violated if $|\Delta| = |\Delta_P| = 1$ and if $1+L$ and $w_I L\Delta$ and $w_P\Delta_P$ have opposite signs

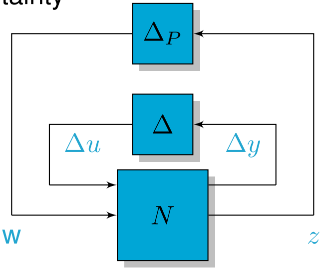
$$|1+L| - |w_I L| - |w_P| > 0, \quad \forall\omega$$

RS $\Leftrightarrow |w_I T| + |w_P S| < 1, \forall\omega$

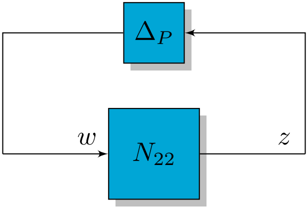
The similarity between RS and RP (SISO example)



Conclusion: We can reformulate the **RP** problem into an **RS** problem with structured uncertainty



The similarity between RS and NP (General)



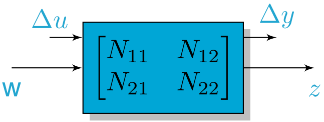
In the $\mathcal{H}_\infty$ framework we have NP if $||N_{22}||_\infty < 1$ (given NS).

Lets consider the following feedback loop with a full Δ_P block (for all stable $||\Delta_P||_\infty \leq 1$)

Apply Generalized Nyquist theorem: $\det(\mathrm{I} - \mathrm{N}_{22}(\mathrm{j}\omega)\Delta_\mathrm{P}(\mathrm{j}\omega))$ shouldn't encircle the origin

Same condition RS $M\Delta$-structure (see last lecture).
So, NP can be represented by a full uncertainty structure.

General conditions (still no controller synthesis!)



NS: N internally stable

NP: $\overline{\sigma}(N_{22}) < 1 \quad \forall\omega$ (or $\mu_{\Delta_P}(N_{22}) < 1$) and **NS**

RS: $\mu_{\Delta}(N_{11}) < 1 \quad \forall\omega$ and **NS**

RP: $\mu_{\hat{\Delta}}(N) < 1 \quad \forall\omega,\ \hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}$ and **NS**

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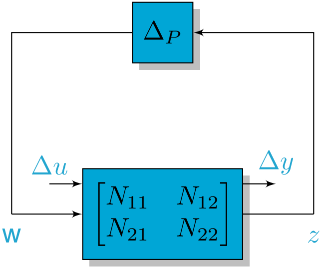

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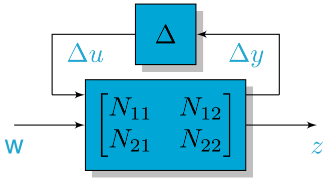

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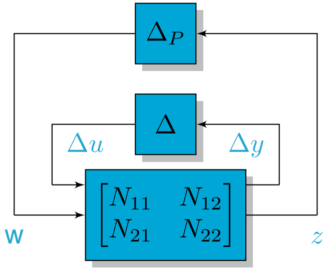

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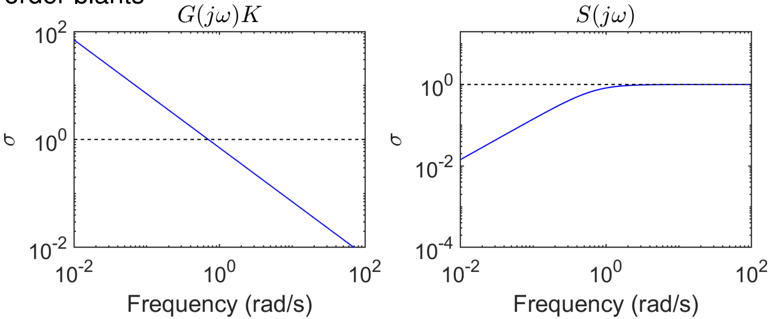
RP: $\mu_{\hat{\Delta}}(N) < 1 \quad \forall\omega,\ \hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}$ and **NS**

Old Example 2: Distillation column

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \text{ with RGA } \forall\omega \begin{bmatrix} 35.1 & -34.1 \\ -34.1 & 35.1 \end{bmatrix}$$

Due to large elements in RGA difficult to control

Controller: inverse with integral action $K_{inv} = \frac{0.7}{s}G^{-1}$

- **NS:** With inverse control you end up with decoupled two first order plants
- 

- **RS:** No high $\overline{\sigma}(S)$ but high RGA values cause for concern ($\Rightarrow$)

Old Example 2: Distillation column (cont'd)

RS: We will consider diag. input uncertainty
(typically 20% for process applications)

Uncertainty in input is given by: $u_1' = (1+\epsilon_1)\,u_1, \quad u_2' = (1+\epsilon_2)\,u_2$

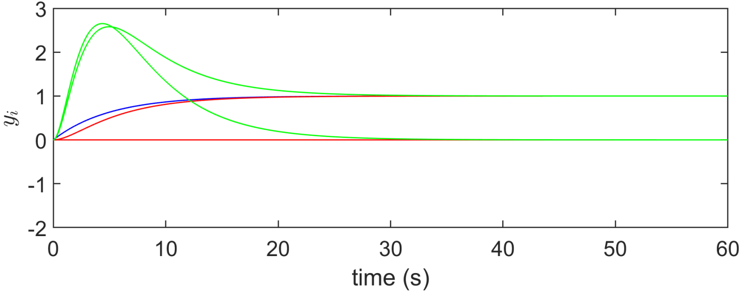
We have: $L(s) = \frac{0.7}{s}\begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix}$

Compute poles: $\det(I+L) = (s+0.7\,(1+\epsilon_1))\,(s+0.7\,(1+\epsilon_2))$. We can have up to 100% error in all the input channels

RP: Consider $u_1' = 1.2u_1, \quad u_2' = 0.8u_2$

Old Example 2: Distillation column (cont'dd)

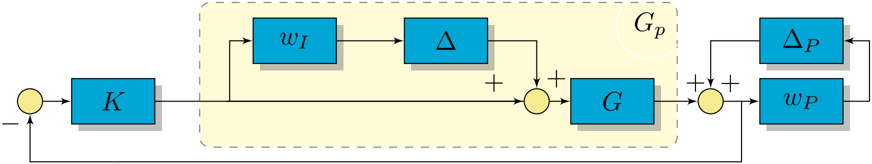
RP: Consider $u_1' = 1.2u_1, \quad u_2' = 0.8u_2$



Reference, Nominal control, Uncertain Input

RP From the response we can conclude that we don't have RP

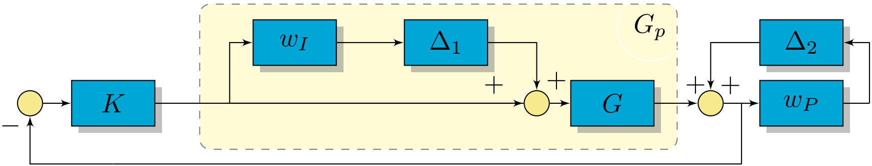
Old Example 2: **NS:** N internally stable



With: $w_I = \frac{s+0.2}{0.5s+1}$ and $w_P = \frac{\frac{s}{2}+0.05}{s}$

```
>>systemnames = 'G Wp Wi';
>>inputvar ='[udel(2); w(2); u(2)]';
>>outputvar='[Wi; Wp; -G-w]';
>>input_to_G='[u+udel]';
>>input_to_Wp='[G+w]';
>>input_to_Wi='[u]';
>>sysoutname='P'; cleanupsysic= 'yes'; sysic;
>>N=lft(P,Kinv); max(real(eig(N))); % -1E-6
```

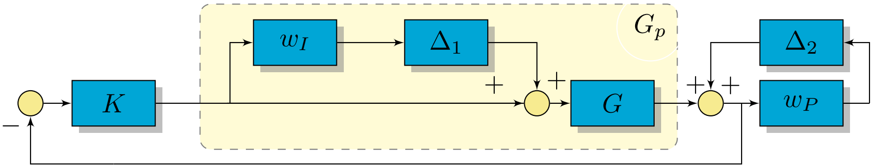
Old Example 2: **NP:** $\mu_{\Delta_P}(N_{22}) < 1$



With: $w_I = \frac{s+0.2}{0.5s+1}$ and $w_P = \frac{\frac{s}{2}+0.05}{s}$

```
>>omega=logspace(-3,3,61);
>>Nf=frd(N,omega);
>>blk=[ 2 2]; % Full complex uncertainty block
>>[mubnds,muinfo]=mussv(Nf(3:4,3:4),blk,'c');
>>muNP=mubnds(:,1);
>>[muNPinf, muNPw]=norm(muNP,inf); % bound = 0.5
```

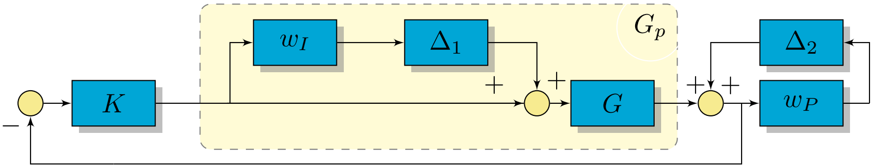
Old Example 2: **RS:** $\mu_\Delta(N_{11}) < 1$



With: $w_I = \frac{s+0.2}{0.5s+1}$ and $w_P = \frac{\frac{s}{2}+0.05}{s}$

```
>>omega=logspace(-3,3,61);
>>Nf=frd(N,omega);
>>blk=[ 1 1; 1 1]; % structured uncertainty
>>[mubnds,muinfo]=mussv(Nf(1:2,1:2),blk,'c');
>>muRS=mubnds(:,1);
>>[muRSinf, muRSw]=norm(muRS,inf); % bound = 0.5242
```

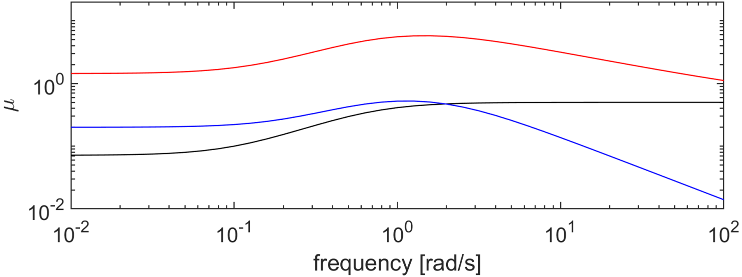
Old Example 2: **RP:** $\mu_{\hat{\Delta}}(N) < 1$



With: $w_I = \frac{s+0.2}{0.5s+1}$ and $w_P = \frac{\frac{s}{2}+0.05}{s}$

```
>>omega=logspace(-3,3,61);
>>Nf=frd(N,omega);
>>blk=[ 1 1; 1 1; 2 2]; % structured uncertainty and ΔP
>>[mubnds,muinfo]=mussv(Nf(1:4,1:4),blk,'c');
>>muRP=mubnds(:,1);
>>[muRPinf, muRPw]=norm(muRP,inf); % bound = 5.77
```

Old Example 2: The different SSV's



$$\underbrace{\mu_{\Delta_P}(N_{22}(j\omega))}_{NP} \qquad \underbrace{\mu_{\Delta}(N_{11}(j\omega))}_{RS} \qquad \underbrace{\mu_{\hat{\Delta}}(N(j\omega))}_{RP}$$

D-K iterations

For MIMO systems we know how to check for **NS**, **NP**, **RS**, **RP**:
μ-analysis

Seek a controller that minimizes a certain μ-condition:
the μ-synthesis problem

There is no direct method to synthesize a μ-optimal controller

However, we have an upperbound:
$\mu(N(K)) \leq \min_{D \in \mathcal{D}} \overline{\sigma}(DN(K)D^{-1})$

Seek a controller that: $\min_K(\min_{D \in \mathcal{D}} ||DN(K)D^{-1}||_\infty)$

Alternate, between minimizing $||DND^{-1}||_\infty$ using D or K.
(D-K iterations)

D-K iterations (cont'd)

The D-K iterations (start with $D = I$):

1. **K-step:** Synthesis a controller that minimizes the scaled problem: $\min_K ||DN(K)D^{-1}||_\infty$.
2. **D-step:** Find $D(j\omega)$ to minimize at each frequency $\overline{\sigma}(D(j\omega)N(j\omega)D^{-1}(j\omega))$ with fixed N (number of frequency points).
3. Fit a transfer function on top of $D(j\omega)$ and return to step-1.

Example (cont'dd)

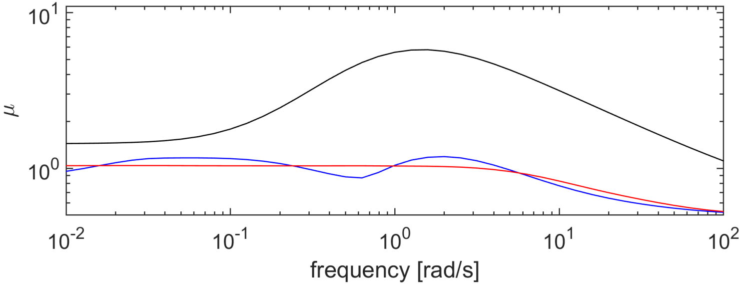
Back to the distillation column: $G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$

Automatic D-K iterations:

```
%% D-K iterations auto-tuning
>>Delta=[ultidyn('D_1',[1,1]) 0; 0 ultidyn('D_2',[1,1])];
>>Punc=lft(Delta,P);
>>opt=dkitopt('FrequencyVector', omega,'DisplayWhileAutoIter','on')
>>[K,clp,bnd,dkinfo]=dksyn(Punc,2,2,opt);
```

Example (cont'dd)

Back to the distillation column: $G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$

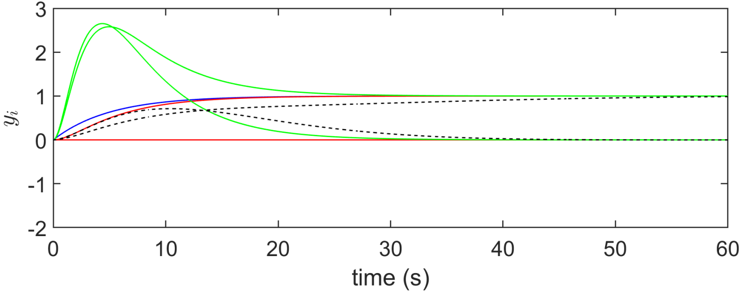


Decentralized design, μ-design iteration 1 , μ-design iteration 2

	K_{inv}	μ it. 1	μ it. 2
Peak μ-value	5.77	1.215	1.048
D-order	-	0	12
K-order	6	6	18

Example (cont'ddd)

RP: Consider $u_1' = 1.2u_1, \quad u_2' = 0.8u_2$



Reference, Nominal control, Uncertain Input, Robust

RP From μ we know we almost have **RP**.

D-scalings (example)

Given $M = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$ compute μ with $\overline{\sigma}(\Delta) \leq 1$:

1. where Δ is a full block (compute SVD)
2. where $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$ $(\det(..) = 1 + \frac{\delta_1}{\mu} - 3\frac{\delta_2}{\mu})$
3. where $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix}$ $(\det(..) = 1 - 2\frac{\delta_1}{\mu})$

Answers: 1) $\sqrt{20}$, 2) 4 3) 2

D-scalings (example, cont'd)

Given $M = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$ compute D: not by hand

1. where Δ is a full block D=I
2. where $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$ $(D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix})$
3. where $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix}$ (D full)

```
%% code to find D-scaling
>>M=[-1 -1; 3 3];
%% 1) full block
>>blk=[2 2];
>>[mubnds,muinfo]=mussv(M,blk); mubnds(2)
>>[VDelta,VSigma,VLmi] = mussvextract(muinfo);
>>D=VSigma.DLeft
%% 2) Two diagonal element
>>blk=[1 0; 1 0];
>>[mubnds,muinfo]=mussv(M,blk); mubnds(2)
>>[VDelta,VSigma,VLmi] = mussvextract(muinfo);
>>D=VSigma.DLeft
%% 3) Repeated block
>>blk=[2 0];
>>[mubnds,muinfo]=mussv(M,blk); mubnds(2)
>>[VDelta,VSigma,VLmi] = mussvextract(muinfo);
>>D=VSigma.DLeft
```

```
%% doing mu synthesis using hinfsyn
>> blk=[ 1 1; 1 1; 2 2];
>>omega=logspace(-3,3,61);
>>[K2,CL,GAM,INFO] = hinfsyn(P,2,2);
>>
>>i=1:1:10
>>Nf=frd(lft(P,K2),omega);
>>[mubnds,muinfo]=mussv(Nf(1:4,1:4),blk,'c');
>>muRP=mubnds(:,1); [muRPinf, muRPw]=norm(muRP,inf);
>>[VDelta,VSigma,VLmi] = mussvextract(muinfo);
>>D=VSigma.DLeft;
>>dd1 = fitmagfrd((D(1,1)/D(3,3)),6);
>>dd2 = fitmagfrd((D(2,2)/D(3,3)),6);
>>Dscale=minreal(append(dd1, dd2,tf(eye(4))));
>>[K2,CL,GAM2,INFO] = hinfsyn(minreal(Dscale*P*inv(Dscale)),2,2);
>>end
```