Uncertainty and robustness for SISO systems

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How do we define robustness?

A control system is robust if it is insensitive to differences between the actual system and the model used for controller synthesis

The difference between the model and the real system is called: model/plant mismatch or model uncertainty

In the $\mathcal{H}_\infty$ robust control paradigm we check if the system is **RS** and **RP** for the worst-case scenario

In this lecture we introduce the tools/framework to check **RS** and **RP** for SISO systems (for $N = W_p S$)

Approach:

1. Determine the uncertainty set (structured and unstructured uncertainty)
2. Check **RS** for all the systems in the uncertainty set
3. If **RS** we check **RP** for all the systems in the uncertainty set

Comments:

a. Will not always achieve optimal performance

b. We will not consider faulty sensors/actuators, or robustness of the optimization algorithm

c. We assume a fixed linear controller

How do we define an uncertainty set?

Π - set of all possible perturbed models (Uncertainty set)

$G(s) \in \Pi$ - Nominal plant

$G_p(s) \in \Pi$ or $G'(s) \in \Pi$- Particular plant models

In this course we use a norm-bounded uncertainty description. For example, this will look like $\Pi = G + Gw_I\Delta$ where w_I is used for scaling and $||\Delta||_\infty < 1$ is the set of all normalized perturbations.

Why not add the uncertainty as additional fictitious disturbances

Suppose we have a nominal plant: $y = Gu + G_d d$

And let $G_p = G + E$ (where E is an additive uncertainty)

If we design a controller, K, for: $y = Gu + \underbrace{Eu}_{d_1} + \underbrace{G_d d}_{d_2}$

For the closed-loop we get: $S = \left(I + (G + E)\,K\right)^{-1}$ can be unstable for some $E \neq 0$ so no **RS**

Where is the uncertainty coming from?

- There are always unknown parameters in the model
- Parameter may enter the system in a nonlinear way
- Imperfections measurement device
- At high frequencies the model is not accurate
- It can be preferable to have a low order model

Model uncertainty can be grouped in two classes:

1. **Parametric (real) uncertainty**: uncertain parameters in a known model structure
2. **Dynamic (frequency-dependent) uncertainty**: error due to missing dynamics

Parametric uncertainty vs Dynamic uncertainty

Parametric uncertainty:

We assume a parameter is bounded within a certain interval $[k_{min}, k_{max}]$

This can be written as: $k_p = \bar{k}\,(1 + r_k \Delta)$. Where $\bar{k}$ is the mean value, $r_k = (k_{max} - k_{min})\,/\,(k_{max} + k_{min})$ is the relative uncertainty, and Δ is any *real scalar* $|\Delta| \leq 1$

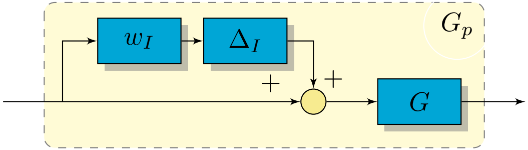
Dynamic uncertainty:

Little bit more difficult to grasp but typically we will use the frequency domain to quantify this uncertainty

Typically this leads to normalize *complex perturbations*: $||\Delta||_\infty \leq 1$

Main topic of this lecture

Multiplicative uncertainty



Often lump dynamic uncertainties into multiplicative uncertainty of the form:

$$\Pi_I : \quad G_p(s) = G(s)\left(1 + w_I(s)\Delta_I(s)\right); \qquad \underbrace{|\Delta_I(j\omega)| \leq 1 \forall\omega}_{||\Delta_I||_\infty \leq 1}$$

Remark 1: Δ_I can be any stable transfer function which is norm bounded by 1

Remark 2: I stands for input but for SISO systems we can also rewrite an output uncertainty as an input uncertainty

Remark 3: You also have the inverse multiplicative uncertainty:

$$\Pi_I : \quad G_p(s) = G(s)\left(1 + w_{iI}(s)\Delta_{iI}(s)\right)^{-1}; \qquad ||\Delta_{iI}||_\infty \leq 1$$

From Parametric to multiplicative uncertainty

Some parametric uncertainties can be rewritten as multiplicative uncertainties

Gain uncertainty:
Suppose $G_p = k_p G_o$ with $k_{min} \leq k_p \leq k_{max}$ this can be rewritten as:

$$G_p = \underbrace{\overline{k} G_o}_{G} \left(1 + \underbrace{r_k}_{w_I} \Delta \right) \text{ with } |\Delta| \leq 1$$

Time constant uncertainty:
Suppose $G_p = \frac{1}{\tau_p s + 1} G_o$ with $\tau_{min} \leq \tau_p \leq \tau_{max}$. By rewriting τ_p as $\overline{\tau}(1 + r_\tau \Delta)$ we get: $G_p = \frac{G_o}{1 + \overline{\tau}s + r_\tau \overline{\tau} s \Delta} = \underbrace{\frac{G_o}{1 + \overline{\tau}s}}_{G} \frac{1}{1 + w_{i1}\Delta}$ with

$w_{i1} = \frac{r_\tau \overline{\tau} s}{1 + \overline{\tau} s}$

Pole uncertainty:
See time constant uncertainty

Example

Lets consider the following uncertain plant:

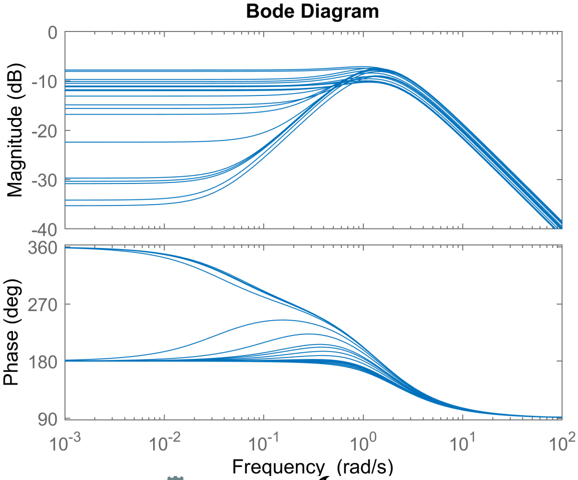
$$\dot{x} = \begin{bmatrix} -(1+k) & 0 \\ 1 & -(1+k) \end{bmatrix} x + \begin{bmatrix} \frac{1-k}{k} \\ -1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & \alpha \end{bmatrix} x$$

where $k = 0.5 + 0.1\delta_1$, $|\delta_1| \leq 1$ and $\alpha = 1 + 0.2\delta_2$ with $|\delta_2| \leq 1$
Code:

```
 Making use of the robust control toolbox
>>k = ureal('k', 0.5, 'Range',[0.4 0.6]); % uncertain parameter
>>alpha = ureal('alpha', 1, 'Range', [0.8 1.2]);
>>A = [-(1+k) 0; 1 -(1+k)];
>>B = [(1/k -1), -1]';
>>C = [0 alpha];
>>Gp = ss(A,B,C,0);
```

Example (cont'd)

```
Making use of the robust control toolbox
>>bode(Gp);
```



Uncertainty in frequency domain

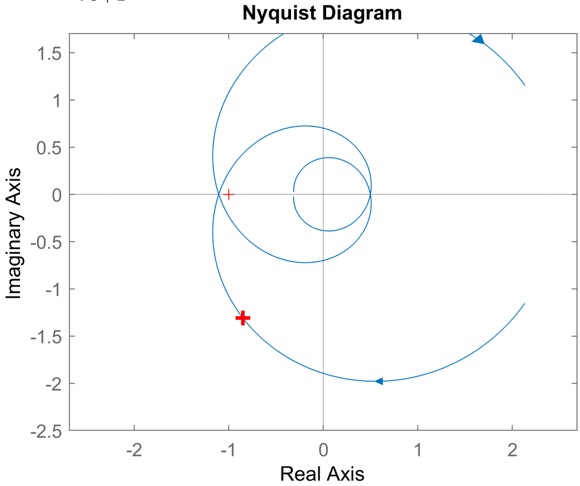
The design of feedback controllers in the presence of non-parametric and unstructured uncertainty ... is the *raison d'etre* for $\mathcal{H}_\infty$ feedback optimization, for parametric uncertainty there are other methods

Parametric uncertainty is often replaced by complex uncertainty:
$|\Delta| \leq 1 \quad \Rightarrow |\Delta(j\omega)| \leq 1$

This becomes beneficial when there are multiple uncertain parameters and then we can lump them in one single complex uncertainty

Example

$$G_p = \frac{k}{\tau s+1}e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3, \, \omega = 0.5$$



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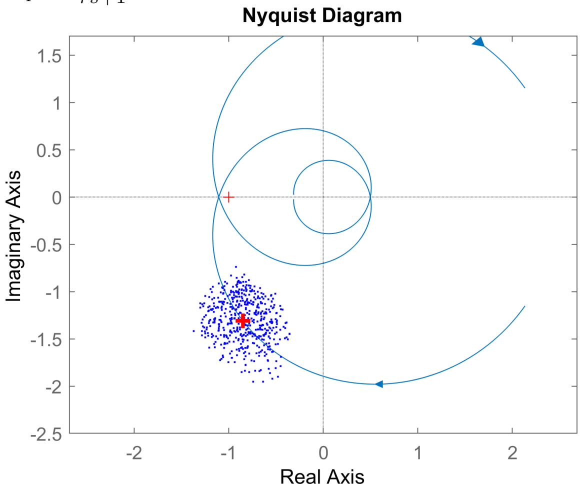

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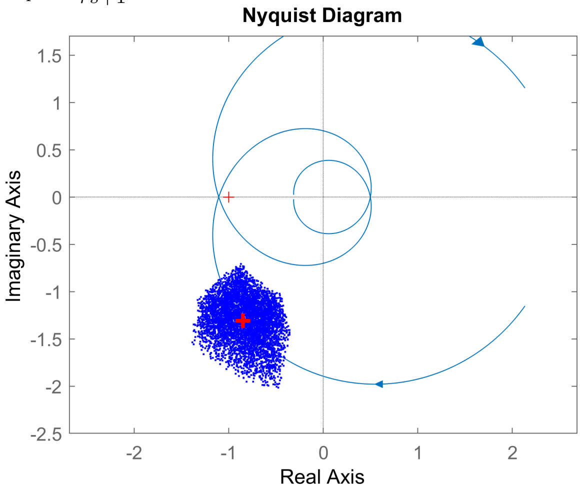

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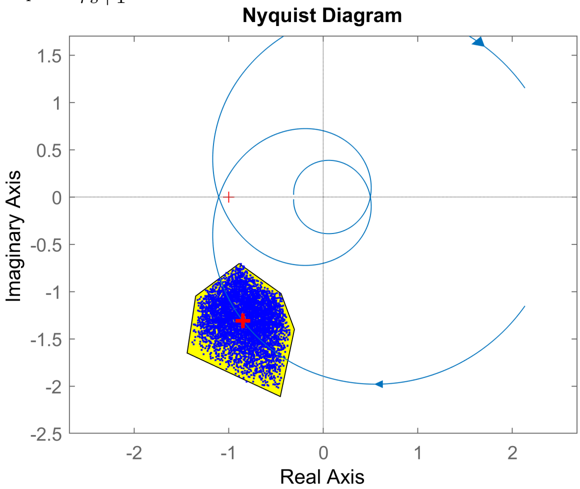
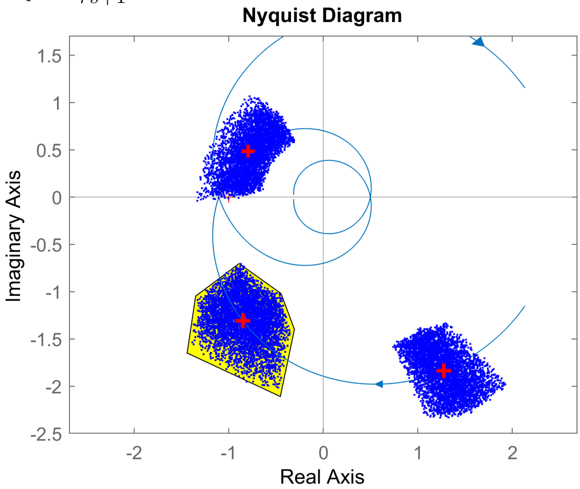


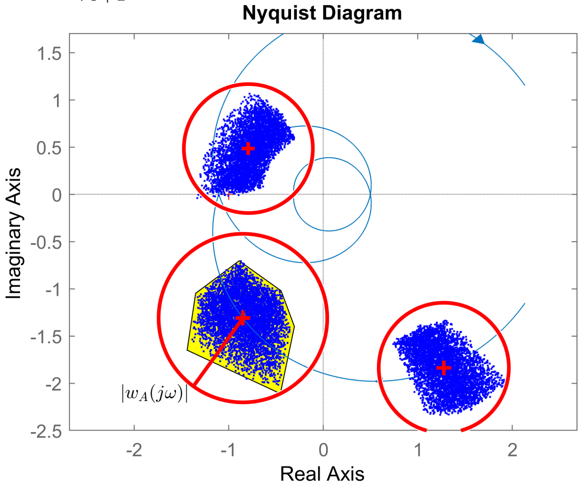
Example

$G_p = \frac{k}{\tau s+1} e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3, \omega = 0.5$, and $\omega = 1$ and $\omega = 0.2$



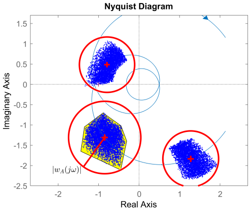
Example

$$G_p = \frac{k}{\tau s+1} e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3, \omega = 0.5, \text{ and } \omega = 1 \text{ and } \omega = 0.2$$



Covered by a disc (complex (additive) uncertainty)

Additive uncertainty and Multiplicative uncertainty



Additive uncertainty: $\Pi_A: \quad G_p = G + w_A\Delta_A$ with $|\Delta_A(j\omega)| \leq 1 \forall\omega$

Multiplicative uncertainty: $\Pi_I: \quad G_p = G\left(1 + w_I\Delta_I\right)$ with $\Delta_I(j\omega) \leq 1\forall\omega$

With $|w_I(j\omega)| = \frac{|w_A(j\omega)|}{|G(j\omega)|}$ these two representations are equivalent

Remember: The Δ's are **any** stable transfer function

How can we construct Δ_A or Δ_I?

1. Select nominal model G
2. Additive uncertainty find the smallest radius that includes all possible plants: $l_A(\omega)$ ($\forall\omega$) and then fit a transfer function $|w_A(j\omega)| \geq l_A(\omega) \quad \forall\omega$
3. Multiplicative uncertainty find the smallest l_I that satisfies:

$$l_I(\omega) = \max_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G(j\omega)}{G(j\omega)} \right|$$

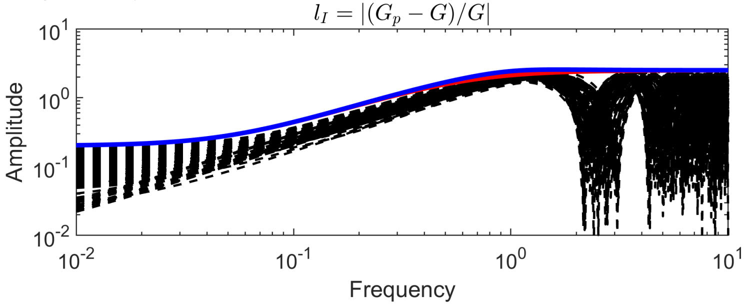
and then fit a transfer function $|w_I(j\omega)| \geq l_I(\omega) \quad \forall\omega$

Example: How can we construct Δ_I?

Again we consider: $G_p = \frac{k}{\tau s+1} e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3$

We can use *usample* from robust control toolbox but can not deal with delay

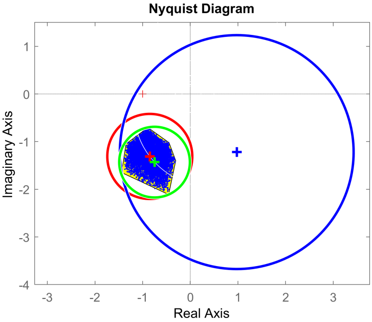
We just randomly sample k, τ, θ around nominal value 2.5, 2.5 and 0 respectively



$$w_I = \frac{4s+0.2}{\frac{4}{2.5}s+1} \frac{s^2+1.6s+1}{s^2+1.4s+1}$$

Choice of nominal model

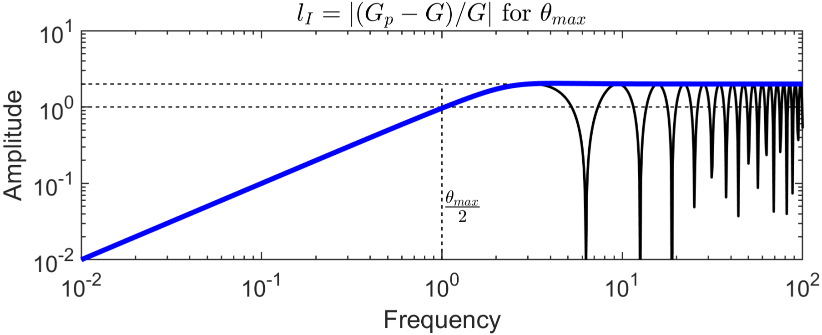
1. Mean parameter values
2. A simplified model (delay free)
3. Smallest discs



Neglected dynamics represented

We consider $G_p = G_o f$ where f represents the neglected dynamics

Neglected delay: $f = e^{-\theta_p s}$ where $0 \leq \theta_p \leq \theta_{max}$



$$w_I = \frac{\left(1 + \frac{r_k}{2}\right)\theta_{max} s + r_k}{\frac{\theta_{max}}{2} s + 1} \cdot \frac{\left(\frac{\theta_{max}}{2.363}\right)^2 s^2 + 2 \cdot 0.838 \cdot \frac{\theta_{max}}{2.363} s + 1}{\left(\frac{\theta_{max}}{2.363}\right)^2 s^2 + 2 \cdot 0.685 \cdot \frac{\theta_{max}}{2.363} s + 1}$$

With relative gain uncertainty r_k

Unmodelled dynamics uncertainty

With unmodelled we refer to dynamics we don't know (different from neglected dynamics)

We typically use the following multiplicative uncertainty:

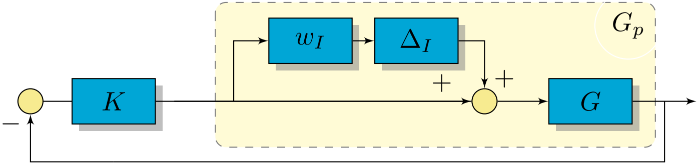
$$w_I = \frac{\tau s + r_0}{\frac{\tau}{r_\infty} s + 1},$$

where:

- r_0 is relative uncertainty at steady state,
- $\frac{1}{\tau}$ the frequency where you have 100% uncertainty,
- r_∞ the weight at high frequencies (> 2)

Selection of these parameters based on application

Robust Stability (RS)



$L_p = G_p K = GK\left(1 + w_I \Delta_I\right) = L + w_I L \Delta_I$, where $|\Delta_I(j\omega)| \leq 1 \quad \forall \omega$

How can we check for **RS**??

Robust Stability (RS): nyquist

$L_p = L + w_I L \Delta_I$,
where $|\Delta_I(j\omega)| < 1 \quad \forall\omega$

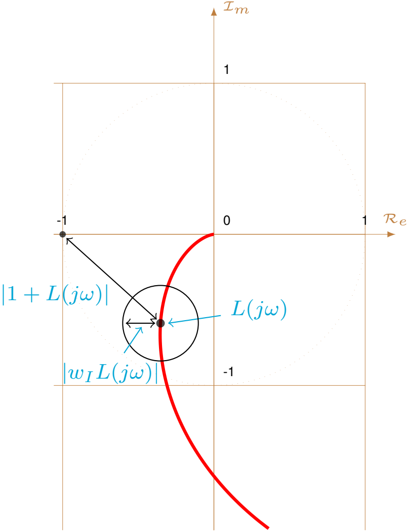
RS $\Leftrightarrow$ stable $\forall L_p$

$\Leftrightarrow$ L_p should not encircle -1

RS $\Leftrightarrow |1 + L(j\omega)| > |w_I L(j\omega)|$

RS $\Leftrightarrow \left|\frac{w_I L(j\omega)}{1+L(j\omega)}\right| < 1, \forall\omega$

RS $\Leftrightarrow ||w_I T||_\infty < 1$



Robust Stability (RS): Algebraic derivation

Assume: Stable L_p and **NS** (nyquist doesn't encircle -1)

Line of reasoning: If one of the circles contains the point -1 we don't have **RS**

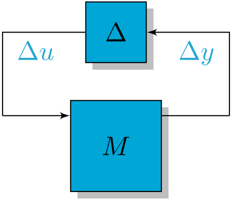
$$\begin{aligned}
\text{RS} \quad &\Leftrightarrow \quad |1 + L_p| \neq 0, && \forall L_p, \forall\omega \\
&\Leftrightarrow \quad |1 + L_p| > 0, && \forall L_p, \forall\omega \\
&\Leftrightarrow \quad |1 + L + w_I L \Delta_I| > 0, && \forall |\Delta_I| < 1, \forall\omega
\end{aligned}$$

Last condition is most easily violated if $\Delta_I = 1$ and if $1 + L$ and $w_I L \Delta_I$ have opposite signs

$$|1 + L| - |w_I L| > 0, \quad \forall\omega$$

RS $\Leftrightarrow \left|\frac{w_I L(j\omega)}{1+L(j\omega)}\right| = |w_I T| < 1, \forall\omega$

Robust Stability (RS): $M\Delta$-structure derivation



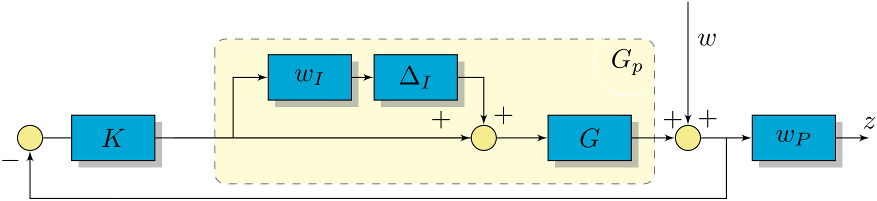
Apply nyquist to this new feedback structure and assume M stable

It should hold that $|1 + M\Delta| > 0 \quad \forall\omega, \quad \forall|\Delta| \leq 1$

$$\begin{aligned} \text{RS} \quad &\Leftrightarrow \quad 1 - |M(j\omega)| > 0, \quad &\forall\omega \\ &\Leftrightarrow \quad |M(j\omega)| < 1, \quad &\forall\omega \end{aligned}$$

In the next lecture we will use the small-gain theorem for this structure

Robust Performance (RP)



We assume **NS**

We can check for **RS**

How can we check for **RP**

Robust Performance (RP): nyquist

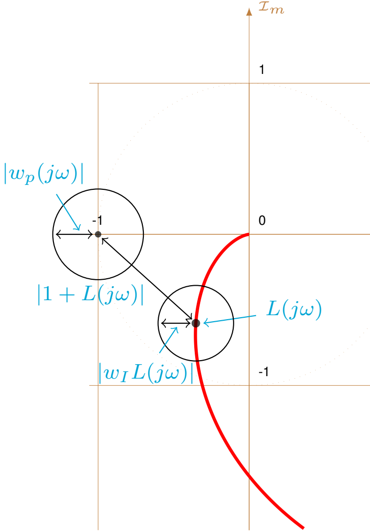
NP: $|w_P S| < 1 \ \forall\omega \Leftrightarrow |w_P| < |1 + L| \ \forall\omega$

RP: $|w_P S_p| < 1 \ \forall\omega \Leftrightarrow |w_P| < |1 + L_p| \ \forall\omega, L_p$

RP: $\Leftrightarrow |w_P| < |1 + L + w_I L \Delta_I| \ \forall\omega, |\Delta_I| < 1$

$$\begin{aligned} \text{RP} \quad &\Leftrightarrow \quad |w_P| + |w_I L| < |1 + L|, \quad \forall\omega \\ &\Leftrightarrow \quad |w_P S| + |w_I T| < 1, \quad \forall\omega \end{aligned}$$

RP: $\Leftrightarrow \max_\omega |w_P S| + |w_I T| < 1$



Summary:

NP: $\Leftrightarrow \quad |w_P S| < 1 \quad \forall\omega$

RS: $\Leftrightarrow \quad |w_I T| < 1 \quad \forall\omega$

RP: $\Leftrightarrow \quad |w_P S| + |w_I T| < 1 \quad \forall\omega$