Superlink derivations

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1 Glossary

Variable	Description
Q_{uk}	Discharge at upstream end of superlink k
C_{uk}	Coefficient of discharge at upstream end of superlink k
A_{uk}	Cross-sectional area of flow at upstream end of superlink k
ΔH_{uk}	Head difference at upstream end of superlink k
H_{juk}	Head at junction upstream of superlink k (ground elevation + water depth)
h_{uk}	Water depth at upstream of superlink k
$z_{inv,uk}$	Invert elevation at upstream end of superlink k
Q_{dk}	Discharge at downstream end of superlink k
C_{dk}	Coefficient of discharge at downstream end of superlink k
A_{dk}	Cross-sectional area of flow at downstream end of superlink k
ΔH_{dk}	Head difference at downstream end of superlink k
H_{jdk}	Head at junction downstream of superlink k (ground elevation + water depth)
h_{dk}	Water depth at downstream of superlink k
$z_{inv,dk}$	Invert elevation at downstream end of superlink k
$NBDj$	Number of superlinks with downstream end attached to superjunction j
$NBUj$	Number of superlinks with upstream end attached to superjunction j
H_j	Head at junction j
U_{Ik}, V_{Ik}, W_{Ik}	Coefficients
X_{Ik}, Y_{Ik}, Z_{Ik}	Coefficients

2 Basic equations

The two governing equations for SUPERLINK are continuity and conservation of momentum.

Continuity:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_0 \tag{1}$$

Conservation of momentum:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(Qu) + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f + S_L\right) = 0 \tag{2}$$

3 Discretization of momentum

Starting with the equation for conservation of momentum:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(Qu) + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f + S_L\right) = 0 \tag{3}$$

The following discretization scheme can be applied to link ik:

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + u_{I+1k}Q_{I+1k}^{t+\Delta t} - u_{Ik}Q_{Ik}^{t+\Delta t} \\ +gA(h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) - gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(S_{f,ik} + S_{L,ik})\Delta x = 0 \end{aligned} \tag{4}$$

This equation can be written in terms of the following coefficient equation:

$$\boxed{a_{ik}Q_{i-1k}^{t+\Delta t} + b_{ik}Q_{ik}^{t+\Delta t} + c_{ik}Q_{i+1k}^{t+\Delta t} = P_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t})} \tag{5}$$

Where:

$$\boxed{a_{ik} = -\max(u_{Ik}, 0)} \tag{6}$$

$$\boxed{c_{ik} = -\max(-u_{I+1k}, 0)} \tag{7}$$

$$\boxed{b_{ik} = \frac{\Delta x_{ik}}{\Delta t} + \frac{gn_{ik}^2|Q_{ik}^t|\Delta x_{ik}}{A_{ik}R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} - a_{ik} - c_{ik}} \tag{8}$$

$$\boxed{P_{ik} = Q_{ik}\frac{\Delta x_{ik}}{\Delta t} + gA_{ik}S_{o,ik}\Delta x_{ik}} \tag{9}$$

This coefficient equation can be verified by substituting the expressions for the coefficients:

$$\begin{aligned} -\max(u_{Ik}, 0)Q_{i-1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{gn_{ik}^2|Q_{ik}^t|\Delta x_{ik}}{A_{ik}R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} + \max(u_{Ik}, 0) + \max(-u_{I+1k}, 0)\right)Q_{ik}^{t+\Delta t} \\ -\max(-u_{I+1k}, 0)Q_{i+1k}^{t+\Delta t} \\ = Q_{ik}\frac{\Delta x_{ik}}{\Delta t} + gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{10}$$

Assuming $u_{ik} > 0$ and $u_{i-1k} \approx u_{ik} \approx u_{i+1k}$:

$$\begin{aligned} -u_{Ik} Q_{i-1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{g n_{ik}^2 |Q_{ik}^t| \Delta x_{ik}}{A_{ik} R_{ik}^{4/3}} + \frac{A_{ik} |Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} + u_{Ik} \right) Q_{ik}^{t+\Delta t} \\ = Q_{ik}^t \frac{\Delta x_{ik}}{\Delta t} + g A_{ik} S_{o,ik} \Delta x_{ik} + g A_{ik} (h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{11}$$

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t) \frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t}) u_{Ik} \\ + g A_{ik} \left(\frac{n_{ik}^2 |Q_{ik}^t| Q_{ik}^{t+\Delta t}}{A_{ik}^2 R_{ik}^{4/3}} + \frac{|Q_{ik}^t| Q_{ik}^{t+\Delta t}}{g C_{ik}^2 A_{cik}^2 \Delta x_{ik}} \right) \Delta x_{ik} \\ = g A_{ik} S_{o,ik} \Delta x_{ik} + g A_{ik} (h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{12}$$

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t) \frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t}) u_{Ik} + g A_{ik} (S_{f,ik} + S_{L,ik}) \Delta x_{ik} \\ = g A_{ik} S_{o,ik} \Delta x_{ik} + g A_{ik} (h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{13}$$

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t) \frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t}) u_{Ik} \\ + g A_{ik} (h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) + g A_{ik} (S_{f,ik} - g A_{ik} S_{o,ik} \Delta x_{ik} + S_{L,ik}) \Delta x_{ik} \end{aligned} \tag{14}$$

Alternatively, assuming $u_{ik} < 0$ and $u_{i-1k} \approx u_{ik} \approx u_{i+1k}$:

$$\begin{aligned} u_{I+1k} Q_{i+1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{g n_{ik}^2 |Q_{ik}^t| \Delta x_{ik}}{A_{ik} R_{ik}^{4/3}} + \frac{A_{ik} |Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} - u_{I+1k} \right) Q_{ik}^{t+\Delta t} \\ = Q_{ik}^t \frac{\Delta x_{ik}}{\Delta t} + g A_{ik} S_{o,ik} \Delta x_{ik} + g A_{ik} (h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{15}$$

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t) \frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t}) u_{I+1k} \\ + g A_{ik} \left(\frac{n_{ik}^2 |Q_{ik}^t| Q_{ik}^{t+\Delta t}}{A_{ik}^2 R_{ik}^{4/3}} + \frac{|Q_{ik}^t| Q_{ik}^{t+\Delta t}}{g C_{ik}^2 A_{cik}^2 \Delta x_{ik}} \right) \Delta x_{ik} \\ = g A_{ik} S_{o,ik} \Delta x_{ik} + g A_{ik} (h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{16}$$

$$\begin{aligned} (Q_{ik}^{t+\Delta t} - Q_{ik}^t) \frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t}) u_{I+1k} + g A_{ik} (S_{f,ik} + S_{L,ik}) \Delta x_{ik} \\ = g A_{ik} S_{o,ik} \Delta x_{ik} + g A_{ik} (h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \end{aligned} \tag{17}$$

$$
\begin{aligned}
&(Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t})u_{I+1k} \\
&+gA_{ik}(h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) + gA_{ik}(S_{f,ik} - gA_{ik}S_{o,ik}\Delta x_{ik} + S_{L,ik})\Delta x_{ik}
\end{aligned} \tag{18}
$$

4 Discretization of continuity

Starting with the continuity equation:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_0 \tag{19}$$

The following discretization scheme can be applied to junction Ik:

$$Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t} + \left(\frac{B_{ik}\Delta x_{ik}}{2} + \frac{B_{i-1k}\Delta x_{i-1k}}{2} + A_{s,Ik}\right) \cdot \frac{h_{Ik}^{t+\Delta t} - h_{Ik}^{t}}{\Delta t} = Q_{0,Ik} \tag{20}$$

Through substitution, the discretized continuity equation can be represented as follows:

$$\boxed{Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t} + E_{Ik}h_{Ik}^{t+\Delta t} = D_{Ik}} \tag{21}$$

Where:

$$\boxed{E_{Ik} = \frac{1}{\Delta t}\left(\frac{B_{ik}\Delta x_{ik}}{2} + \frac{B_{i-1k}\Delta x_{i-1k}}{2} + A_{s,Ik}\right)} \tag{22}$$

$$\boxed{D_{Ik} = Q_{0,Ik} + \frac{h_{Ik}^{t}}{\Delta t}\left(\frac{B_{ik}\Delta x_{ik}}{2} + \frac{B_{i-1k}\Delta x_{i-1k}}{2} + A_{s,Ik}\right)} \tag{23}$$

5 Recurrence relationships

5.1 Forward recurrence

Starting at the upstream end of superlink k:

$$Q_{2k}^{t+\Delta t} - Q_{1k}^{t+\Delta t} + E_{2k}h_{2k}^{t+\Delta t} = D_{2k} \tag{24}$$

$$a_{1k}Q_{0k}^{t+\Delta t} + b_{1k}Q_{1k}^{t+\Delta t} + c_{1k}Q_{2k}^{t+\Delta t} = P_{1k} + gA_{1k}(h_{1k}^{t+\Delta t} - h_{2k}^{t+\Delta t}) \tag{25}$$

Assuming $Q_{0k}^{t+\Delta t} = Q_{1k}^{t+\Delta t}$:

$$a_{1k}Q_{1k}^{t+\Delta t} + b_{1k}Q_{1k}^{t+\Delta t} + c_{1k}(Q_{1k}^{t+\Delta t} - E_{2k}h_{2k}^{t+\Delta t} + D_{2k}) = P_{1k} + gA_{1k}(h_{1k}^{t+\Delta t} - h_{2k}^{t+\Delta t}) \tag{26}$$

$$(a_{1k} + b_{1k} + c_{1k})Q_{1k}^{t+\Delta t} = E_{2k}c_{2k}h_{2k}^{t+\Delta t} + (P_{1k} + c_{1k}D_{2k}) + gA_{1k}(h_{1k}^{t+\Delta t} - h_{2k}^{t+\Delta t}) \tag{27}$$

$$(a_{1k} + b_{1k} + c_{1k})Q_{1k}^{t+\Delta t} = (E_{2k}c_{1k} - gA_{1k})h_{2k}^{t+\Delta t} + (P_{1k} - D_{2k}c_{1k}) + gA_{1k}h_{1k}^{t+\Delta t} \tag{28}$$

$$Q_{1k}^{t+\Delta t} = \frac{(E_{2k}c_{1k} - gA_{1k})h_{2k}^{t+\Delta t} + (P_{1k} - D_{2k}c_{1k}) + gA_{1k}h_{1k}^{t+\Delta t}}{a_{1k} + b_{1k} + c_{1k}} \tag{29}$$

Thus for the upstream end of superlink k:

$$\boxed{Q_{1k}^{t+\Delta t} = U_{1k}h_{2k}^{t+\Delta t} + Y_{1k} + Z_{1k}h_{1k}^{t+\Delta t}} \tag{30}$$

Where:

$$\boxed{T_{1k} = a_{1k} + b_{1k} + c_{1k}} \tag{31}$$

$$\boxed{U_{1k} = \frac{E_{2k}c_{1k} - gA_{1k}}{T_{1k}}} \tag{32}$$

$$\boxed{V_{1k} = \frac{P_{1k} - D_{2k}c_{1k}}{T_{1k}}} \tag{33}$$

$$\boxed{W_{1k} = \frac{gA_{1k}}{T_{1k}}} \tag{34}$$

For the next element downstream:

$$Q_{3k}^{t+\Delta t} - Q_{2k}^{t+\Delta t} + E_{3k}h_{3k}^{t+\Delta t} = D_{3k} \tag{35}$$

$$a_{2k}Q_{1k}^{t+\Delta t} + b_{2k}Q_{2k}^{t+\Delta t} + c_{2k}Q_{3k}^{t+\Delta t} = P_{2k} + gA_{2k}(h_{2k}^{t+\Delta t} - h_{3k}^{t+\Delta t}) \tag{36}$$

Substituting:

$$\begin{aligned} a_{2k}(Q_{2k}^{t+\Delta t} + E_{2k}h_{2k}^{t+\Delta t} - D_{2k}) + (b_{2k})Q_{2k}^{t+\Delta t} + c_{2k}(Q_{2k} - E_{3k}h_{3k}^{t+\Delta t} + D_{3k}) \\ -P_{2k} - gA_{2k}(h_{2k}^{t+\Delta t} - h_{3k}^{t+\Delta t}) = 0 \end{aligned} \tag{37}$$

$$\begin{aligned} (a_{2k} + b_{2k} + c_{2k})Q_{2k}^{t+\Delta t} + (E_{2k}a_{2k} - gA_{2k})h_{2k}^{t+\Delta t} \\ +(gA_{2k} - c_{2k}E_{3k})h_{3k}^{t+\Delta t} + (-D_{2k}a_{2k} + D_{3k}c_{2k} - P_{2k}) = 0 \end{aligned} \tag{38}$$

Multiplying $h_{2k}^{t+\Delta t}$ by $(U_{1k} - E_{2k})/(U_{1k} - E_{2k})$ and rearranging:

$$\begin{aligned} (a_{2k} + b_{2k} + c_{2k})Q_{2k}^{t+\Delta t} + \frac{(E_{2k}a_{2k} - gA_{2k})(U_{1k} - E_{2k})}{(U_{1k} - E_{2k})}h_{2k}^{t+\Delta t} \\ +(gA_{2k} - c_{2k}E_{3k})h_{3k}^{t+\Delta t} + (-P_{2k} - D_{2k}a_{2k} + D_{3k}c_{2k}) = 0 \end{aligned} \tag{39}$$

Note that:

$$U_{1k}h_{2k}^{t+\Delta t} = (Q_{1k}^{t+\Delta t} - V_{1k} - W_{1k}h_{1k}^{t+\Delta t}) \tag{40}$$

$$E_{2k}h_{2k}^{t+\Delta t} = (D_{2k} - Q_{2k}^{t+\Delta t} + Q_{1k}^{t+\Delta t}) \tag{41}$$

Thus:

$$\begin{aligned} (a_{2k} + b_{2k} + c_{2k})Q_{2k}^{t+\Delta t} \\ +\frac{(E_{2k}a_{2k} - gA_{2k})}{(U_{1k} - E_{2k})}[(Q_{1k}^{t+\Delta t} - V_{1k} - W_{1k}h_{1k}^{t+\Delta t}) - (D_{2k} - Q_{2k}^{t+\Delta t} + Q_{1k}^{t+\Delta t})] \\ +(gA_{2k} - c_{2k}E_{3k})h_{3k}^{t+\Delta t} + (-P_{2k} - D_{2k}a_{2k} + D_{3k}c_{2k}) = 0 \end{aligned} \tag{42}$$

Allowing $Q_{1k}^{t+\Delta t}$ to be eliminated:

$$
\begin{aligned}
(a_{2k}+b_{2k}+c_{2k})Q_{2k}^{t+\Delta t}+\frac{(E_{2k}a_{2k}-gA_{2k})}{U_{1k}-E_{2k}}Q_{2k}^{t+\Delta t}\\
+\frac{(E_{2k}a_{2k}-gA_{2k})(-W_{1k})}{U_{1k}-E_{2k}}h_{1k}^{t+\Delta}+(gA_{2k}-c_{2k}E_{3k})h_{3k}^{t+\Delta t}\\
+(-P_{2k}-D_{2k}a_{2k}+D_{3k}c_{2k}+(E_{2k}a_{2k}-gA_{2k})\frac{(-V_{1k}-D_{2k})}{(U_{1k}-E_{2k})})=0
\end{aligned}
\tag{43}
$$

Rearranging:

$$
\begin{aligned}
\left(a_{2k}+b_{2k}+c_{2k}-\frac{gA_{2k}-E_{2k}a_{2k}}{U_{1k}-E_{2k}}\right)Q_{2k}^{t+\Delta t}\\
+\frac{(gA_{2k}-E_{2k}a_{2k})W_{1k}}{U_{1k}-E_{2k}}h_{1k}^{t+\Delta}+(gA_{2k}-c_{2k}E_{3k})h_{3k}^{t+\Delta t}\\
+\left(-P_{2k}-D_{2k}a_{2k}+D_{3k}c_{2k}+(gA_{2k}-E_{2k}a_{2k})\frac{V_{1k}+D_{2k}}{U_{1k}-E_{2k}}\right)=0
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\left(a_{2k}+b_{2k}+c_{2k}-\frac{gA_{2k}-E_{2k}a_{2k}}{U_{1k}-E_{2k}}\right)Q_{2k}^{t+\Delta t}\\
=(E_{3k}c_{2k}-gA_{2k})h_{3k}^{t+\Delta t}\\
+\left(P_{2k}+D_{2k}a_{2k}-D_{3k}c_{2k}-(gA_{2k}-E_{2k}a_{2k})\frac{V_{1k}+D_{2k}}{(U_{1k}-E_{2k})}\right)\\
-\frac{(gA_{2k}-E_{2k}a_{2k})W_{1k}}{U_{1k}-E_{2k}}h_{1k}^{t+\Delta}
\end{aligned}
\tag{45}
$$

Generalizing for $i=2$, $I=2$:

$$
\begin{aligned}
\left(a_{ik}+b_{ik}+c_{ik}-\frac{gA_{ik}-E_{Ik}a_{ik}}{U_{I-1k}-E_{Ik}}\right)Q_{ik}^{t+\Delta t}\\
=(E_{I+1k}c_{ik}-gA_{ik})h_{I+1k}^{t+\Delta t}\\
+\left(P_{ik}+D_{Ik}a_{ik}-D_{I+1k}c_{ik}-(gA_{ik}-E_{ik}a_{ik})\frac{V_{I-1k}+D_{Ik}}{U_{I-1k}-E_{Ik}}\right)\\
-\frac{(gA_{ik}-E_{Ik}a_{ik})W_{I-1k}}{U_{I-1k}-E_{Ik}}h_{1k}^{t+\Delta}
\end{aligned}
\tag{46}
$$

Condensing in terms of coefficients:

$$
\boxed{Q_{ik}^{t+\Delta t}=U_{Ik}h_{I+1k}^{t+\Delta t}+V_{Ik}+W_{Ik}h_{1k}^{t+\Delta t}}
\tag{47}
$$

Where:

$$\boxed{U_{Ik} = \frac{E_{I+1k}c_{ik} - gA_{ik}}{T_{ik}}} \tag{48}$$

$$\boxed{V_{Ik} = \frac{P_{ik} + D_{Ik}a_{ik} - D_{I+1k}c_{ik} - (gA_{ik} - E_{Ik}a_{ik})\frac{V_{I-1k}+D_{Ik}}{U_{I-1k}-E_{Ik}}}{T_{ik}}} \tag{49}$$

$$\boxed{W_{Ik} = -\frac{(gA_{ik} - E_{Ik}a_{ik})W_{I-1k}}{U_{I-1k} - E_{Ik}}} \tag{50}$$

$$\boxed{T_{ik} = \left(a_{ik} + b_{ik} + c_{ik} - \frac{gA_{ik} - E_{Ik}a_{ik}}{U_{I-1k} - E_{Ik}}\right)} \tag{51}$$

5.2 Backward recurrence

Starting at the downstream end of superlink k:

$$Q_{nk}^{t+\Delta t} - Q_{nk-1}^{t+\Delta t} + E_{Nk} h_{Nk}^{t+\Delta t} = D_{Nk} \tag{52}$$

$$a_{nk} Q_{nk-1}^{t+\Delta t} + b_{nk} Q_{nk}^{t+\Delta t} + c_{nk} Q_{nk+1}^{t+\Delta t} = P_{nk} + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{53}$$

Assuming $Q_{nk}^{t+\Delta t} = Q_{nk+1}^{t+\Delta t}$:

$$a_{nk}(Q_{nk}^{t+\Delta t} + E_{Nk} h_{Nk}^{t+\Delta t} - D_{Nk}) + b_{nk} Q_{nk}^{t+\Delta t} + c_{nk} Q_{nk}^{t+\Delta t} = P_{nk} + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{54}$$

$$(a_{nk} + b_{nk} + c_{nk}) Q_{nk}^{t+\Delta t} = -E_{Nk} a_{nk} h_{Nk}^{t+\Delta t} + (P_{nk} + a_{nk} D_{Nk}) + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{55}$$

$$(a_{nk} + b_{nk} + c_{nk}) Q_{nk}^{t+\Delta t} = (g A_{nk} - E_{Nk} a_{nk}) h_{Nk}^{t+\Delta t} + (P_{nk} + D_{Nk} a_{nk}) - g A_{nk} h_{Nk+1}^{t+\Delta t} \tag{56}$$

$$Q_{nk}^{t+\Delta t} = \frac{(g A_{nk} - E_{Nk} a_{nk}) h_{Nk}^{t+\Delta t} + (P_{nk} + D_{Nk} a_{nk}) - g A_{nk} h_{Nk+1}^{t+\Delta t}}{(a_{nk} + b_{nk} + c_{nk})} \tag{57}$$

Thus for the downstream end of superlink k:

$$\boxed{Q_{nk}^{t+\Delta t} = X_{Nk} h_{Nk}^{t+\Delta t} + Y_{Nk} + Z_{Nk} h_{Nk+1}^{t+\Delta t}} \tag{58}$$

Where:

$$\boxed{O_{nk} = a_{nk} + b_{nk} + c_{nk}} \tag{59}$$

$$\boxed{X_{Nk} = \frac{(g A_{nk} - E_{Nk} a_{nk})}{O_{nk}}} \tag{60}$$

$$\boxed{Y_{Nk} = \frac{P_{nk} + D_{Nk} a_{nk}}{O_{nk}}} \tag{61}$$

$$\boxed{Z_{Nk} = -\frac{gA_{nk}}{O_{nk}}} \tag{62}$$

For the next element upstream:

$$Q_{nk-1}^{t+\Delta t} - Q_{nk-2}^{t+\Delta t} + E_{Nk-1}h_{Nk-1}^{t+\Delta t} = D_{Nk-1} \tag{63}$$

$$a_{nk-1}Q_{nk-2}^{t+\Delta t} + b_{nk-1}Q_{nk-1}^{t+\Delta t} + c_{nk-1}Q_{nk}^{t+\Delta t} = P_{nk-1} + gA_{nk-1}(h_{Nk-1}^{t+\Delta t} - h_{Nk}^{t+\Delta t}) \tag{64}$$

$$\begin{aligned} a_{nk-1}(Q_{nk-1}^{t+\Delta t} + E_{Nk-1}h_{Nk-1}^{t+\Delta t} - D_{Nk-1}) + (b_{nk-1})Q_{nk-1}^{t+\Delta t} + c_{nk-1}(Q_{nk-1} - E_{Nk}h_{Nk}^{t+\Delta t} + D_{Nk}) \\ -P_{nk-1} - gA_{nk-1}(h_{Nk-1}^{t+\Delta t} - h_{Nk}^{t+\Delta t}) = 0 \end{aligned} \tag{65}$$

$$\begin{aligned} (a_{nk-1} + b_{nk-1} + c_{nk-1})Q_{nk-1}^{t+\Delta t} + (-E_{Nk}c_{nk-1} + gA_{nk-1})h_{Nk}^{t+\Delta t} \\ +(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} + (-D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1} - P_{nk-1}) = 0 \end{aligned} \tag{66}$$

Multiplying $h_{Nk}^{t+\Delta t}$ by $(X_{Nk} + E_{Nk})/(X_{Nk} + E_{Nk})$ and rearranging:

$$\begin{aligned} (a_{nk-1} + b_{nk-1} + c_{nk-1})Q_{nk-1}^{t+\Delta t} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})(X_{Nk} + E_{Nk})}{(X_{Nk} + E_{Nk})}h_{Nk}^{t+\Delta t} \\ +(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} + (-P_{nk-1} - D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1}) = 0 \end{aligned} \tag{67}$$

Note that:

$$X_{Nk}h_{Nk}^{t+\Delta t} = (Q_{nk}^{t+\Delta t} - Y_{Nk} - Z_{Nk}h_{Nk+1}^{t+\Delta t}) \tag{68}$$

$$E_{Nk}h_{Nk}^{t+\Delta t} = (D_{Nk} - Q_{nk}^{t+\Delta t} + Q_{nk-1}^{t+\Delta t}) \tag{69}$$

Thus:

$$\begin{aligned} (a_{nk-1} + b_{nk-1} + c_{nk-1})Q_{nk-1}^{t+\Delta t} \\ +\frac{(gA_{nk-1} - E_{Nk}c_{nk-1})}{(X_{Nk} + E_{Nk})}[(Q_{nk}^{t+\Delta t} - Y_{Nk} - Z_{Nk}h_{Nk+1}^{t+\Delta t}) + (D_{Nk} - Q_{nk}^{t+\Delta t} + Q_{nk-1}^{t+\Delta t})] \\ +(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} + (-P_{nk-1} - D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1}) = 0 \end{aligned} \tag{70}$$

Allowing $Q_{nk}^{t+\Delta t}$ to be eliminated:

$$
\begin{aligned}
&(a_{nk-1}+b_{nk-1}+c_{nk-1})Q_{nk-1}^{t+\Delta t} \\
&+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})}{(X_{Nk}+E_{Nk})}Q_{nk-1}^{t+\Delta t}+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})(-Z_{Nk})}{(X_{Nk}+E_{Nk})}h_{Nk+1}^{t+\Delta t} \\
&+(E_{Nk-1}a_{nk-1}-gA_{nk-1})h_{Nk-1}^{t+\Delta t} \\
&+\left(-P_{nk-1}-D_{Nk-1}a_{nk-1}+D_{Nk}c_{nk-1}+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})(-Y_{Nk}+D_{Nk})}{(X_{Nk}+E_{Nk})}\right)=0
\end{aligned}
\tag{71}
$$

Rearranging:

$$
\begin{aligned}
&\left(a_{nk-1}+b_{nk-1}+c_{nk-1}+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})}{(X_{Nk}+E_{Nk})}\right)Q_{nk-1}^{t+\Delta t} \\
&+(E_{Nk-1}a_{nk-1}-gA_{nk-1})h_{Nk-1}^{t+\Delta t}-\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})Z_{Nk}}{(X_{Nk}+E_{Nk})}h_{Nk+1}^{t+\Delta t} \\
&+\left(-P_{nk-1}-D_{Nk-1}a_{nk-1}+D_{Nk}c_{nk-1}+(gA_{nk-1}-E_{Nk}c_{nk-1})\frac{(D_{Nk}-Y_{Nk})}{(X_{Nk}+E_{Nk})}\right)=0
\end{aligned}
\tag{72}
$$

$$
\begin{aligned}
&\left(a_{nk-1}+b_{nk-1}+c_{nk-1}+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})}{(X_{Nk}+E_{Nk})}\right)Q_{nk-1}^{t+\Delta t} \\
&=(gA_{nk-1}-E_{Nk-1}a_{nk-1})h_{Nk-1}^{t+\Delta t} \\
&+\left(P_{nk-1}+D_{Nk-1}a_{nk-1}-D_{Nk}c_{nk-1}-(gA_{nk-1}-E_{Nk}c_{nk-1})\frac{(D_{Nk}-Y_{Nk})}{(X_{Nk}+E_{Nk})}\right) \\
&+\frac{(gA_{nk-1}-E_{Nk}c_{nk-1})Z_{Nk}}{(X_{Nk}+E_{Nk})}h_{Nk+1}^{t+\Delta t}
\end{aligned}
\tag{73}
$$

Generalizing for $i=nk-1$, $I=Nk-1$:

$$
\begin{aligned}
&\left(a_{ik}+b_{ik}+c_{ik}+\frac{(gA_{ik}-E_{I+1k}c_{ik})}{(X_{I+1k}+E_{I+1k})}\right)Q_{ik}^{t+\Delta t} \\
&=(gA_{ik}-E_{Ik}a_{ik})h_{Ik}^{t+\Delta t} \\
&+\left(P_{ik}+D_{Ik}a_{ik}-D_{I+1k}c_{ik}-(gA_{ik}-E_{I+1k}c_{ik})\frac{(D_{I+1k}-Y_{I+1k})}{(X_{I+1k}+E_{I+1k})}\right) \\
&+\frac{(gA_{ik}-E_{I+1k}c_{ik})Z_{I+1k}}{(X_{I+1k}+E_{I+1k})}h_{Nk+1}^{t+\Delta t}
\end{aligned}
\tag{74}
$$

Condensing in terms of coefficients:

$$\boxed{Q_{ik}^{t+\Delta t} = X_{ik} h_{Ik}^{t+\Delta t} + Y_{Ik} + Z_{Ik} h_{Nk+1}^{t+\Delta t}} \tag{75}$$

Where:

$$\boxed{X_{Ik} = \frac{gA_{ik} - E_{Ik} a_{ik}}{O_{ik}}} \tag{76}$$

$$\boxed{Y_{Ik} = \frac{P_{ik} + D_{Ik} a_{ik} - D_{I+1k} c_{ik} - (gA_{ik} - E_{I+1k} c_{ik}) \frac{(D_{I+1k} - Y_{I+1k})}{(X_{I+1k} + E_{I+1k})}}{O_{ik}}} \tag{77}$$

$$\boxed{Z_{Ik} = \frac{(gA_{ik} - E_{I+1k} c_{ik}) Z_{I+1k}}{(X_{I+1k} + E_{I+1k}) O_{ik}}} \tag{78}$$

$$\boxed{O_{ik} = \left(a_{ik} + b_{ik} + c_{ik} + \frac{gA_{ik} - E_{I+1k} c_{ik}}{X_{I+1k} + E_{I+1k}} \right)} \tag{79}$$

6 Inlet hydraulics

6.1 Depth at upstream end of superlink

The discharge at the upstream end of a superlink is given by:

$$Q_{uk} = C_{uk} A_{uk} \sqrt{2g \Delta H_{uk}} \tag{80}$$

Where:

$$\Delta H_{uk} = H_{juk} - h_{uk} - z_{inv,uk} \tag{81}$$

Squaring and rearranging provides the depth boundary condition at the upstream end:

$$Q_{uk}^2 = 2 C_{uk}^2 A_{uk}^2 g (H_{juk} - h_{uk} - z_{inv,uk}) \tag{82}$$

$$|Q_{uk}^t| Q_{uk}^{t+\Delta t} = 2 C_{uk}^2 A_{uk}^2 g (H_{juk} - h_{uk} - z_{inv,uk}) \tag{83}$$

$$h_{uk} = -\frac{|Q_{uk}^t| Q_{uk}^{t+\Delta t}}{2 C_{uk}^2 A_{uk}^2 g} + H_{juk} - z_{inv,uk} \tag{84}$$

6.2 Depth at downstream end of superlink

The discharge at the downstream end of a superlink is given by:

$$Q_{dk} = C_{dk} A_{dk} \sqrt{2g \Delta H_{dk}} \tag{85}$$

Where:

$$\Delta H_{dk} = h_{dk} + z_{inv,dk} - H_{jdk} \tag{86}$$

Squaring and rearranging provides the depth boundary condition at the downstream end:

$$Q_{dk}^2 = 2 C_{dk}^2 A_{dk}^2 g (h_{dk} + z_{inv,dk} - H_{jdk}) \tag{87}$$

$$|Q_{dk}^t|Q_{dk}^{t+\Delta t} = 2C_{dk}^2 A_{dk}^2 g(h_{dk} + z_{inv,dk} - H_{jdk}) \tag{88}$$

$$h_{dk} = \frac{|Q_{dk}^t|Q_{dk}^{t+\Delta t}}{2C_{dk}^2 A_{dk}^2 g} + H_{jdk} - z_{inv,dk} \tag{89}$$

6.3 Superlink boundary conditions

From the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k} h_{uk}^{t+\Delta t} + Y_{1k} + Z_{1k} h_{dk}^{t+\Delta t} \tag{90}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk} h_{dk}^{t+\Delta t} + V_{Nk} + W_{Nk} h_{uk}^{t+\Delta t} \tag{91}$$

From the depth boundary conditions at the ends of each superlink:

$$h_{uk} = \gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk} \tag{92}$$

$$h_{dk} = \gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk} \tag{93}$$

Where:

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{2C_{uk}^2 A_{uk}^2 g} \tag{94}$$

$$\gamma_{dk} = \frac{|Q_{dk}^t|}{2C_{dk}^2 A_{dk}^2 g} \tag{95}$$

Substituting into the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k}(\gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) + Y_{1k} + Z_{1k}(\gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) \tag{96}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}(\gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) + V_{Nk} + W_{Nk}(\gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) \tag{97}$$

Expanding:

$$Q_{uk}^{t+\Delta t} = X_{1k}\gamma_{uk}Q_{uk}^{t+\Delta t} + X_{1k}H_{juk}^{t+\Delta t} - X_{1k}z_{inv,uk} + Y_{1k} + Z_{1k}\gamma_{dk}Q_{dk}^{t+\Delta t} + Z_{1k}H_{jdk}^{t+\Delta t} - Z_{1k}z_{inv,dk} \quad (98)$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}\gamma_{dk}Q_{dk}^{t+\Delta t} + U_{Nk}H_{jdk} - U_{Nk}z_{inv,dk} + V_{Nk} + W_{Nk}\gamma_{uk}Q_{uk}^{t+\Delta t} + W_{Nk}H_{juk} - W_{Nk}z_{inv,uk} \quad (99)$$

Rearranging:

$$0 = (X_{1k}\gamma_{uk} - 1)Q_{uk}^{t+\Delta t} + Z_{1k}\gamma_{dk}Q_{dk}^{t+\Delta t} + X_{1k}H_{juk}^{t+\Delta t} + Z_{1k}H_{jdk}^{t+\Delta t} + \pi_1 \quad (100)$$

$$0 = W_{Nk}\gamma_{uk}Q_{uk}^{t+\Delta t} + (U_{Nk}\gamma_{dk} - 1)Q_{dk}^{t+\Delta t} + W_{Nk}H_{juk} + U_{Nk}H_{jdk} + \pi_2 \quad (101)$$

Where:

$$\pi_1 = Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk} \quad (102)$$

$$\pi_2 = V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk} \quad (103)$$

$$\quad (104)$$

Writing as a matrix equation:

$$\begin{bmatrix} (X_{1k}\gamma_{uk} - 1) & Z_{1k}\gamma_{dk} \\ W_{Nk}\gamma_{uk} & (U_{Nk}\gamma_{dk} - 1) \end{bmatrix} \begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \begin{bmatrix} -X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \quad (105)$$

Taking the matrix inverse:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk} - 1) & -Z_{1k}\gamma_{dk} \\ -W_{Nk}\gamma_{uk} & (X_{1k}\gamma_{uk} - 1) \end{bmatrix} \begin{bmatrix} -X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \quad (106)$$

Where:

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \quad (107)$$

Expanding:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk} - 1)(-X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1) + (-Z_{1k}\gamma_{dk})(-W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2) \\ (-W_{Nk}\gamma_{uk})(-X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1) + (X_{1k}\gamma_{uk} - 1)(-W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2) \end{bmatrix} \tag{108}$$

Arranging in terms of the unknown heads:

$$\begin{aligned} Q_{uk}^{t+\Delta t} = [(U_{Nk}\gamma_{dk} - 1)(-X_{1k}) + (-Z_{1k}\gamma_{dk})(-W_{Nk})]H_{juk}^{t+\Delta t} + \\ [(U_{Nk}\gamma_{dk} - 1)(-Z_{1k}) + (-Z_{1k}\gamma_{dk})(-U_{Nk})]H_{jdk}^{t+\Delta t} + \\ [(U_{Nk}\gamma_{dk} - 1)(-\pi_1) + (-Z_{1k}\gamma_{dk})(-\pi_2)] \end{aligned} \tag{109}$$

$$\begin{aligned} Q_{dk}^{t+\Delta t} = [(-W_{Nk}\gamma_{uk})(-X_{1k}) + (X_{1k}\gamma_{uk} - 1)(-W_{Nk})]H_{juk}^{t+\Delta t} + \\ [(-W_{Nk}\gamma_{uk})(-Z_{1k}) + (X_{1k}\gamma_{uk} - 1)(-U_{Nk})]H_{jdk}^{t+\Delta t} + \\ [(-W_{Nk}\gamma_{uk})(-\pi_1) + (X_{1k}\gamma_{uk} - 1)(-\pi_2)] \end{aligned} \tag{110}$$

Finally, the upstream and downstream flows can be expressed as:

$$Q_{uk}^{t+\Delta t} = \alpha_{uk}H_{juk}^{t+\Delta t} + \beta_{uk}H_{jdk}^{t+\Delta t} + \chi_{uk} \tag{111}$$

$$Q_{dk}^{t+\Delta t} = \alpha_{dk}H_{juk}^{t+\Delta t} + \beta_{dk}H_{jdk}^{t+\Delta t} + \chi_{dk} \tag{112}$$

Where:

$$\alpha_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})X_{1k} + Z_{1k}\gamma_{dk}W_{Nk}}{D_k^*} \tag{113}$$

$$\beta_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})Z_{1k} + Z_{1k}\gamma_{dk}U_{Nk}}{D_k^*} \tag{114}$$

$$\chi_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk}) + (Z_{1k}\gamma_{dk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk})}{D_k^*} \tag{115}$$

$$\alpha_{dk} = \frac{(1 - X_{1k}\gamma_{uk})W_{Nk} + W_{Nk}\gamma_{uk}X_{1k}}{D_k^*} \tag{116}$$

$$\beta_{dk} = \frac{(1 - X_{1k}\gamma_{uk})U_{Nk} + W_{Nk}\gamma_{uk}Z_{1k}}{D_k^*} \tag{117}$$

$$\chi_{dk} = \frac{(1 - X_{1k}\gamma_{uk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk}) + (W_{Nk}\gamma_{uk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk})}{D_k^*} \tag{118}$$

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \tag{119}$$

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{2C_{uk}^2 A_{uk}^2 g} \tag{120}$$

$$\gamma_{dk} = \frac{|Q_{dk}^t|}{2C_{dk}^2 A_{dk}^2 g} \tag{121}$$

7 Forming the solution matrix

The equations for the flows at the ends of each superlink are given by:

$$\sum_{l=1}^{NBDj} Q_{dk_l}^{t+\Delta t} - \sum_{m=1}^{NBUj} Q_{uk_m}^{t+\Delta t} + Q_{o,j} = \frac{A_{sj}(H_j^{t+\Delta t} - H_j)}{\Delta t} \tag{122}$$

Substituting the linear expressions for the upstream and downstream flows:

$$\begin{aligned} \frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} = & \sum_{l=1}^{NBDj} (\alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \beta_{dk_l} H_{jdk_l}^{t+\Delta t} + \chi_{dk_l}) \\ & - \sum_{m=1}^{NBUj} (\alpha_{uk_m} H_{juk_m}^{t+\Delta t} + \beta_{uk_m} H_{jdk_m}^{t+\Delta t} + \chi_{uk_m}) + Q_{o,j} \end{aligned} \tag{123}$$

Because $H_{jdk_l} = H_j$ and $H_{juk_m} = H_j$:

$$\begin{aligned} \frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} = & \sum_{l=1}^{NBDj} (\alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \beta_{dk_l} H_j^{t+\Delta t} + \chi_{dk_l}) \\ & - \sum_{m=1}^{NBUj} (\alpha_{uk_m} H_j^{t+\Delta t} + \beta_{uk_m} H_{jdk_m}^{t+\Delta t} + \chi_{uk_m}) + Q_{o,j} \end{aligned} \tag{124}$$

Rearranging:

$$\left(\frac{A_{sj}}{\Delta t} + \sum_{m=1}^{NBUj} \alpha_{uk_m} - \sum_{l=1}^{NBDj} \beta_{dk_l}\right) H_j^{t+\Delta t} - \sum_{l=1}^{NBDj} \alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \sum_{m=1}^{NBUj} \beta_{uk_m} H_{jdk_m}^{t+\Delta t} \\ = \frac{A_{sj}(H_j^t)}{\Delta t} + \sum_{l=1}^{NBDj} \chi_{dk_l} - \sum_{m=1}^{NBUj} \chi_{uk_m} + Q_{o,j} \tag{125}$$

The continuity equation for each superjunction can thus be redefined in terms of the following coefficients.

$$\boxed{F_{j,j} H_j^{t+\Delta t} + \sum_{\ell=1}^{NBDj} \Phi_{j,juk_\ell} H_{juk_\ell}^{t+\Delta t} + \sum_{m=1}^{NBUj} \Psi_{j,jdk_m} H_{jdk_m}^{t+\Delta t} = G_j} \tag{126}$$

Where:

$$\boxed{F_{j,j} = \frac{A_{sj}}{\Delta t} - \sum_{\ell=1}^{NBDj} \beta_{dk_\ell} + \sum_{m=1}^{NBUj} \alpha_{uk_m}} \tag{127}$$

$$\boxed{\Phi_{j,juk_\ell} = -\alpha_{dk_\ell}} \tag{128}$$

$$\boxed{\Psi_{j,jdk_m} = \beta_{uk_m}} \tag{129}$$

$$\boxed{G_j = \frac{A_{sj}}{\Delta t} H_j^t + Q_{0,j} - \sum_{\ell=1}^{NBDj} \chi_{uk_\ell} + \sum_{m=1}^{NBUj} \chi_{dk_m}} \tag{130}$$

For the example network in Ji (1998), the sparse matrix equation is given as:

$$Ax = b \tag{131}$$

$$
\begin{bmatrix}
F_{1,1} & \Psi_{1,2} & 0 & 0 & 0 & 0 \\
\Phi_{2,1} & \Psi_{2,2} & \Psi_{2,3} & 0 & \Psi_{2,5} & 0 \\
0 & \Phi 3,2 & F_{3,3} & \Psi_{3,4} & \Phi_{3,5} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & \Phi_{5,2} & \Psi_{5,3} & 0 & F_{5,5} & \Psi_{5,6} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
H_1^{t+\Delta t} \\ H_2^{t+\Delta t} \\ H_3^{t+\Delta t} \\ H_4^{t+\Delta t} \\ H_5^{t+\Delta t} \\ H_6^{t+\Delta t}
\end{bmatrix}
=
\begin{bmatrix}
G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ G_6
\end{bmatrix}
\tag{132}
$$

Expanding the coefficients:

$$
A =
\begin{bmatrix}
(\frac{A_{s1}}{\Delta t} + \alpha_{u1}) & \beta_{u1} & 0 & 0 & 0 & 0 \\
-\alpha_{d1} & (\frac{A_{s2}}{\Delta t} + \alpha_{u2} + \alpha_{u4} - \beta_{d1}) & \beta_{u2} & 0 & \beta_{u4} & 0 \\
0 & -\alpha_{d2} & (\frac{A_{s3}}{\Delta t} + \alpha_{u3} - \beta_{d2} - \beta_{d6}) & \beta_{u3} & -\alpha_{d6} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -\alpha_{d4} & \beta_{u6} & 0 & (\frac{A_{s5}}{\Delta t} + \alpha_{u5} + \alpha_{u6} - \beta_{d4}) & \beta_{u5} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\tag{133}
$$

$$
b =
\begin{bmatrix}
\frac{A_{s1} H_1^t}{\Delta t} - \chi_{u1} + Q_{0,1} \\
\frac{A_{s2} H_2^t}{\Delta t} + \chi_{d1} - (\chi_{u2} + \chi_{u4}) + Q_{0,2} \\
\frac{A_{s3} H_3^t}{\Delta t} + (\chi_{d2} + \chi_{d6}) - \chi_{u3} + Q_{0,3} \\
H_{4,bc} \\
\frac{A_{s5} H_5^t}{\Delta t} + \chi_{d4} - (\chi_{u5} + \chi_{u6}) + Q_{0,5} \\
H_{6,bc}
\end{bmatrix}
\tag{134}
$$

8 Representing orifices

For orifices, six different flow cases are possible:

- Side-mounted orifice with both sides submerged
- Side-mounted orifice with one side submerged
- Side-mounted orifice with weir-like flow
- Bottom-mounted orifice with both sides submerged
- Bottom-mounted orifice with one side submerged
- No-flow condition

The governing equations for each condition are presented here:

Side-mounted orifice with both sides submerged

This flow regime occurs when both of the following conditions are met:

- $\max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) > z_o + uy_{max,o}$
- $\min(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) > z_o + \frac{uy_{max,o}}{2}$

The effective head is computed as:

$$H_{e,o} = |H_{uo} - H_{do}| \tag{135}$$

And the flow is computed as:

$$Q_o = \mathrm{sgn}(H_{uo} - H_{do}) \cdot C_o A_o \sqrt{2gH_{e,o}} \tag{136}$$

Side-mounted orifice with one side submerged

This flow regime occurs when both of the following conditions are met:

- $\max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) > z_o + uy_{max,o}$
- $\min(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) < z_o + \frac{uy_{max,o}}{2}$

The effective head is computed as:

$$H_{e,o} = \cdot\big[\max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,do}) - (z_o + \frac{uy_{max,o}}{2})\big] \tag{137}$$

And the flow is computed as:

$$Q_o = \mathrm{sgn}(H_{uo} - H_{do}) \cdot C_o A_o \sqrt{2gH_{e,o}} \tag{138}$$

Side-mounted orifice with weir-like flow

This flow regime occurs when both of the following conditions are met:

- $\max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) > z_o$
- $\max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) < z_o + uy_{max,o}$

The effective head is computed as:

$$H_{e,o} = \max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) - z_o \tag{139}$$

And the flow is computed as:

$$Q_o = \frac{C_o A_o \sqrt{g}}{uy_{max,o}} \sqrt{H_{e,o}} \tag{140}$$

Bottom-mounted orifice with both sides submerged

This flow regime occurs when both of the following conditions are met:

- $\max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) > z_o$
- $\min(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) > z_o$

The effective head is computed as:

$$H_{e,o} = |H_{uo} - H_{do}| \tag{141}$$

And the flow is computed as:

$$Q_o = \mathrm{sgn}(H_{uo} - H_{do}) \cdot C_o A_o \sqrt{2gH_{e,o}} \tag{142}$$

Bottom-mounted orifice with one side submerged

This flow regime occurs when both of the following conditions are met:

- $\max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) > z_o$
- $\min(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) < z_o$

The effective head is computed as:

$$H_{e,o} = \cdot\big[\max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,do}) - z_o\big] \tag{143}$$

And the flow is computed as:

$$Q_o = \text{sgn}(H_{uo} - H_{do}) \cdot C_o A_o \sqrt{2gH_{e,o}} \tag{144}$$

No-flow condition

This flow regime occurs when the following condition is met:

- $\max(H_{uo} - z_{inv,uo}, H_{do} - z_{inv,uo}) \leq z_o$

In this case, the effective head and flow are both zero:

$$H_{e,o} = 0 \tag{145}$$

$$Q_o = 0 \tag{146}$$

8.1 Representing orifice equations in the solution matrix

Orifices can be represented in the solution matrix as follows.

Define the following indicator functions:

$$\Omega(H_{juo}, H_{jdo}) = \begin{cases} 1, & H_{juo} \geq H_{jdo} \\ 0, & o/w \end{cases} \tag{147}$$

$$\tau(o) = \begin{cases} 1, & \text{orifice } o \text{ is side-mounted} \\ 0, & \text{orifice } o \text{ is bottom-mounted} \end{cases} \tag{148}$$

Similarly, define boolean-valued functions to represent the following flow conditions:

Submerged on high-head side

$$\Theta_{o,1} = \begin{cases} 1, & \Omega H_{uo} + (1-\Omega)H_{do} > z_o + z_{inv,juo} + \tau u y_{max,o} \\ 0, & o/w \end{cases} \tag{149}$$

Submerged on low-head side

$$\Theta_{o,2} = \begin{cases} 1, & (1-\Omega)H_{uo} + \Omega H_{do} > z_o + z_{inv,juo} + \frac{\tau u y_{max,o}}{2} \\ 0, & o/w \end{cases} \tag{150}$$

Above bottom rim on high-head side

$$\Theta_{o,3} = \begin{cases} 1, & \Omega H_{uo} + (1-\Omega)H_{do} > z_o + z_{inv,juo} \\ 0, & o/w \end{cases} \tag{151}$$

The flow through an orifice can now be represented using the following linearized coefficient equation:

$$Q_o^{t+\Delta t} = \alpha_o H_{uo}^{t+\Delta t} + \beta_o H_{do}^{t+\Delta t} + \chi_o \tag{152}$$

Where:

$$\alpha_o = \begin{cases} \gamma_o u^2, & \Theta_{o,1} \wedge \Theta_{o,2} \\ \gamma_o \Omega (-1)^{1-\Omega} u^2, & \Theta_{o,1} \wedge \neg\Theta_{o,2} \\ \frac{\gamma_o}{2y_{max,o}^2} \Omega (-1)^{1-\Omega}, & \neg\Theta_{o,1} \wedge \Theta_{o,3} \\ 0, & \neg\Theta_{o,3} \end{cases} \tag{153}$$

$$\beta_o = \begin{cases} -\gamma_o u^2, & \Theta_{o,1} \wedge \Theta_{o,2} \\ \gamma_o (1-\Omega)(-1)^{1-\Omega} u^2, & \Theta_{o,1} \wedge \neg\Theta_{o,2} \\ \frac{\gamma_o}{2y_{max,o}^2} (1-\Omega)(-1)^{1-\Omega}, & \neg\Theta_{o,1} \wedge \Theta_{o,3} \\ 0, & \neg\Theta_{o,3} \end{cases} \tag{154}$$

$$\chi_o = \begin{cases} 0, & \Theta_{o,1} \wedge \Theta_{o,2} \\ \gamma_o (-1)^{1-\Omega} (-z_{inv,uo} - z_o - \frac{\tau u y_{max,o}}{2}), & \Theta_{o,1} \wedge \neg\Theta_{o,2} u^2 \\ \frac{\gamma_o}{2y_{max,o}^2} (-z_{inv,uo} - z_o), & \neg\Theta_{o,1} \wedge \Theta_{o,3} \\ 0, & \neg\Theta_{o,3} \end{cases} \tag{155}$$

$$\gamma_o = \frac{2gC_o^2 A_o^2}{|Q_o^t|} \tag{156}$$

These equations can be added to the solution matrix in much the same way as the linearized superlink coefficients ($\alpha_{uk}, \beta_{uk}, \chi_{uk}, \alpha_{dk}, \beta_{dk}, \chi_{dk}$).

Representing weirs

This section discusses the governing equations for weirs, and explains how weirs can be incorporated into the solution matrix. Only transverse weirs will be considered.

First, without loss of generality, assume all weirs can be represented as trapezoidal weirs (given that both rectangular and triangular weirs are special cases of the trapezoidal weir).

The flow through a trapezoidal weir is the sum of the flow through the rectangular and triangular sections:

$$Q_w = Q_{wR} + Q_{wT} \tag{157}$$

$$= C_{wR} L_w H_{e,w}^{3/2} + C_{wT} s_w H_{e,w}^{5/2} \tag{158}$$

The flow at the next time step can thus be computed as:

$$Q_w^{t+\Delta t} = \frac{C_{wR} L_w H_{e,w} + C_{wT} s_w (H_{e,w}^t)^2}{|Q_w^t|} H_{e,w}^{t+\Delta t} \tag{159}$$

Representing weirs in the solution matrix

Define the following indicator function:

$$\Omega(H_{juw}, H_{jdw}) = \begin{cases} 1, & H_{juw} \geq H_{jdw} \\ 0, & o/w \end{cases} \tag{160}$$

Similarly, define boolean-valued functions to represent the following flow conditions:

Submerged on high-head side

$$\Theta_{w,1} = \begin{cases} 1, & \Omega H_{uw} + (1-\Omega)H_{dw} > z_w + z_{inv,juw} + (1-u)y_{max,w} \\ 0, & o/w \end{cases} \tag{161}$$

Submerged on low-head side

$$\Theta_{w,2} = \begin{cases} 1, & (1-\Omega)H_{uw} + \Omega H_{dw} > z_o + z_{inv,juw} + (1-u)y_{max,w} \\ 0, & o/w \end{cases} \tag{162}$$

The flow through a weir can now be represented using the following linearized coefficient equation:

$$Q_w^{t+\Delta t} = \alpha_w H_{uw}^{t+\Delta t} + \beta_w H_{dw}^{t+\Delta t} + \chi_w \tag{163}$$

Where:

$$\alpha_w = \begin{cases} \gamma_w, & \Theta_{w,1} \wedge \Theta_{w,2} \\ \gamma_w \Omega (-1)^{1-\Omega}, & \Theta_{w,1} \wedge \neg\Theta_{w,2} \\ 0, & \neg\Theta_{w,1} \end{cases} \tag{164}$$

$$\beta_o = \begin{cases} -\gamma_w, & \Theta_{w,1} \wedge \Theta_{w,2} \\ \gamma_w (1-\Omega)(-1)^{1-\Omega}, & \Theta_{w,1} \wedge \neg\Theta_{w,2} \\ 0, & \neg\Theta_{w,1} \end{cases} \tag{165}$$

$$\chi_o = \begin{cases} 0, & \Theta_{w,1} \wedge \Theta_{w,2} \\ \gamma_w (-1)^{1-\Omega}(-z_{inv,uw} - z_o - (1-u)y_{max,w}), & \Theta_{w,1} \wedge \neg\Theta_{w,2} u^2 \\ 0, & \neg\Theta_{w,3} \end{cases} \tag{166}$$

$$\gamma_w = \frac{C_{wR} L_w H_{e,w} + C_{wT} s_w (H_{e,w}^t)^2}{|Q_w^t|} \tag{167}$$

9 Representing pumps

The relationship between flow and head in a pump is usually defined by a pump curve. For this implementation, we assume that the flow/head relationship can be approximated by an ellipse centered at the origin.

First, define the effective head for the pump as follows:

$$\Delta H_{e,p} = \begin{cases} \Delta H_{max,p}, & H_{dp} - H_{up} > \Delta H_{max,p} \\ H_{dp} - H_{up}, & \Delta H_{min,p} < H_{dp} - H_{up} < \Delta H_{max,p} \\ \Delta H_{min,p}, & H_{dp} - H_{up} < \Delta H_{min,p} \end{cases} \tag{168}$$

Then, using the elliptical approximation, the flow through the pump can be represented as:

$$Q_p = u\sqrt{a_q^2(1 - \frac{\Delta H_{e,p}^2}{a_h^2})} \tag{169}$$

Define boolean-valued functions to represent the following flow conditions:

Submerged inlet

$$\Theta_{p,1} = \begin{cases} 1, & H_{up} \geq z_{inv,up} + z_p \\ 0, & o/w \end{cases} \tag{170}$$

Head in pump curve range

$$\Theta_{p,2} = \begin{cases} 1, & \Delta H_{min,p} < H_{dp} - H_{up} < \Delta H_{max,p} \\ 0, & o/w \end{cases} \tag{171}$$

The flow through a pump can now be represented using the following linearized coefficient equation:

$$Q_p^{t+\Delta t} = \alpha_p H_{up}^{t+\Delta t} + \beta_p H_{dp}^{t+\Delta t} + \chi_p \tag{172}$$

$$\alpha_p = \begin{cases} \gamma_p u^2, & \Theta_{p,1} \wedge \Theta_{p,2} \\ 0, & \Theta_{p,1} \wedge \neg\Theta_{p,2} \\ 0, & \neg\Theta_{p,1} \end{cases} \tag{173}$$

$$\beta_p = \begin{cases} -\gamma_p u^2, & \Theta_{p,1} \wedge \Theta_{p,2} \\ 0, & \Theta_{p,1} \wedge \neg\Theta_{p,2} \\ 0, & \neg\Theta_{p,1} \end{cases} \tag{174}$$

$$\chi_p = \begin{cases} \frac{a_q^2}{|Q_p^t|}, & \Theta_{p,1} \wedge \Theta_{p,2} \\ u\sqrt{a_q^2\big(1 - \frac{(\Delta H_e^t)^2}{a_h^2}\big)}, & \Theta_{p,1} \wedge \neg\Theta_{p,2} \\ 0, & \neg\Theta_{p,1} \end{cases} \tag{175}$$

$$\gamma_p = \frac{a_q^2|H_{dp}^t - H_{up}^t|}{a_h^2|Q_p^t|} \tag{176}$$