

Inlet hydraulics

Depth at upstream end of superlink

$$Q_{uk} = C_{uk} A_{uk} \sqrt{2g \Delta H_{uk}} \quad (1)$$

$$\Delta H_{uk} = H_{juk} - h_{uk} - z_{inv,uk} \quad (2)$$

$$Q_{uk}^2 = C_{uk}^2 A_{uk}^2 g (H_{juk} - h_{uk} - z_{inv,uk}) \quad (3)$$

$$|Q_{uk}^t| Q_{uk}^{t+\Delta t} = C_{uk}^2 A_{uk}^2 g (H_{juk} - h_{uk} - z_{inv,uk}) \quad (4)$$

$$h_{uk} = -\frac{|Q_{uk}^t| Q_{uk}^{t+\Delta t}}{C_{uk}^2 A_{uk}^2 g} + H_{juk} - z_{inv,uk} \quad (5)$$

Depth at downstream end of superlink

$$Q_{dk} = C_{dk} A_{dk} \sqrt{2g \Delta H_{dk}} \quad (6)$$

$$\Delta H_{dk} = h_{dk} + z_{inv,dk} - H_{jdk} \quad (7)$$

$$Q_{dk}^2 = C_{dk}^2 A_{dk}^2 g (h_{dk} + z_{inv,dk} - H_{jdk}) \quad (8)$$

$$|Q_{dk}^t| Q_{dk}^{t+\Delta t} = C_{dk}^2 A_{dk}^2 g (h_{dk} + z_{inv,dk} - H_{jdk}) \quad (9)$$

$$h_{dk} = \frac{|Q_{dk}^t| Q_{dk}^{t+\Delta t}}{C_{dk}^2 A_{dk}^2 g} + H_{jdk} - z_{inv,dk} \quad (10)$$

Superlink boundary conditions

From recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k}h_{uk}^{t+\Delta t} + Y_{1k} + Z_{1k}h_{dk}^{t+\Delta t} \quad (11)$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}h_{dk}^{t+\Delta t} + V_{Nk} + W_{Nk}h_{uk}^{t+\Delta t} \quad (12)$$

From depth boundary conditions:

$$h_{uk} = \gamma_{uk}Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk} \quad (13)$$

$$h_{dk} = \gamma_{dk}Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk} \quad (14)$$

Where:

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{C_{uk}^2 A_{uk}^2 g} \quad (15)$$

$$\gamma_{dk} = \frac{|Q_{dk}^t|}{C_{dk}^2 A_{dk}^2 g} \quad (16)$$

Substituting into the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k}(\gamma_{uk}Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) + Y_{1k} + Z_{1k}(\gamma_{dk}Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) \quad (17)$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}(\gamma_{dk}Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) + V_{Nk} + W_{Nk}(\gamma_{uk}Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) \quad (18)$$

Expanding:

$$Q_{uk}^{t+\Delta t} = X_{1k}\gamma_{uk}Q_{uk}^{t+\Delta t} + X_{1k}H_{juk}^{t+\Delta t} - X_{1k}z_{inv,uk} + Y_{1k} + Z_{1k}\gamma_{dk}Q_{dk}^{t+\Delta t} + Z_{1k}H_{jdk}^{t+\Delta t} - Z_{1k}z_{inv,dk} \quad (19)$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}\gamma_{dk}Q_{dk}^{t+\Delta t} + U_{Nk}H_{jdk}^{t+\Delta t} - U_{Nk}z_{inv,dk} + V_{Nk} + W_{Nk}\gamma_{uk}Q_{uk}^{t+\Delta t} + W_{Nk}H_{juk}^{t+\Delta t} - W_{Nk}z_{inv,uk} \quad (20)$$

Rearranging:

$$0 = (X_{1k}\gamma_{uk} - 1)Q_{uk}^{t+\Delta t} + Z_{1k}\gamma_{dk}Q_{dk}^{t+\Delta t} + X_{1k}H_{juk}^{t+\Delta t} + Z_{1k}H_{jdk}^{t+\Delta t} + \pi_1 \quad (21)$$

$$0 = W_{Nk}\gamma_{uk}Q_{uk}^{t+\Delta t} + (U_{Nk}\gamma_{dk} - 1)Q_{dk}^{t+\Delta t} + W_{Nk}H_{juk}^{t+\Delta t} + U_{Nk}H_{jdk}^{t+\Delta t} + \pi_2 \quad (22)$$

Where:

$$\pi_1 = Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk} \quad (23)$$

$$\pi_2 = V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk} \quad (24)$$

$$(25)$$

Writing as a matrix equation:

$$\begin{bmatrix} (X_{1k}\gamma_{uk} - 1) & Z_{1k}\gamma_{dk} \\ W_{Nk}\gamma_{uk} & (U_{Nk}\gamma_{dk} - 1) \end{bmatrix} \begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \begin{bmatrix} -X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \quad (26)$$

Taking the matrix inverse:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk} - 1) & -Z_{1k}\gamma_{dk} \\ -W_{Nk}\gamma_{uk} & (X_{1k}\gamma_{uk} - 1) \end{bmatrix} \begin{bmatrix} -X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \quad (27)$$

Where:

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \quad (28)$$

Expanding:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk} - 1)(-X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1) + (-Z_{1k}\gamma_{dk})(-W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2) \\ (-W_{Nk}\gamma_{uk})(-X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1) + (X_{1k}\gamma_{uk} - 1)(-W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2) \end{bmatrix} \quad (29)$$

Arranging in terms of the unknown heads:

$$\begin{aligned} Q_{uk}^{t+\Delta t} &= [(U_{Nk}\gamma_{dk} - 1)(-X_{1k}) + (-Z_{1k}\gamma_{dk})(-W_{Nk})]H_{juk}^{t+\Delta t} + \\ &\quad [(U_{Nk}\gamma_{dk} - 1)(-Z_{1k}) + (-Z_{1k}\gamma_{dk})(-U_{Nk})]H_{jdk}^{t+\Delta t} + \\ &\quad [(U_{Nk}\gamma_{dk} - 1)(-\pi_1) + (-Z_{1k}\gamma_{dk})(-\pi_2)] \end{aligned} \quad (30)$$

$$\begin{aligned}
Q_{dk}^{t+\Delta t} = & [(-W_{Nk}\gamma_{uk})(X_{1k}) + (X_{1k}\gamma_{uk} - 1)(-W_{Nk})]H_{juk}^{t+\Delta t} + \\
& [(-W_{Nk}\gamma_{uk})(-Z_{1k}) + (X_{1k}\gamma_{uk} - 1)(-U_{Nk})]H_{jdk}^{t+\Delta t} + \\
& [(-W_{Nk}\gamma_{uk})(-\pi_1) + (X_{1k}\gamma_{uk} - 1)(-\pi_2)]
\end{aligned} \tag{31}$$

Finally, the upstream and downstream flows can be expressed as:

$$Q_{uk}^{t+\Delta t} = \alpha_{uk}H_{juk}^{t+\Delta t} + \beta_{uk}H_{jdk}^{t+\Delta t} + \chi_{uk} \tag{32}$$

$$Q_{dk}^{t+\Delta t} = \alpha_{dk}H_{juk}^{t+\Delta t} + \beta_{dk}H_{jdk}^{t+\Delta t} + \chi_{dk} \tag{33}$$

Where:

$$\alpha_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})X_{1k} + Z_{1k}\gamma_{dk}W_{Nk}}{D_k^*} \tag{34}$$

$$\beta_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})Z_{1k} + Z_{1k}\gamma_{dk}U_{Nk}}{D_k^*} \tag{35}$$

$$\chi_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk}) + (Z_{1k}\gamma_{dk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk})}{D_k^*} \tag{36}$$

$$\alpha_{dk} = \frac{(1 - X_{1k}\gamma_{uk})W_{Nk} + W_{Nk}\gamma_{uk}X_{1k}}{D_k^*} \tag{37}$$

$$\beta_{dk} = \frac{(1 - X_{1k}\gamma_{uk})U_{Nk} + W_{Nk}\gamma_{uk}Z_{1k}}{D_k^*} \tag{38}$$

$$\chi_{dk} = \frac{(1 - X_{1k}\gamma_{uk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk}) + (W_{Nk}\gamma_{uk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk})}{D_k^*} \tag{39}$$

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \tag{40}$$

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{C_{uk}^2 A_{uk}^2 g} \quad (41)$$

$$\gamma_{dk} = \frac{|Q_{dk}^t|}{C_{dk}^2 A_{dk}^2 g} \quad (42)$$

Forming the solution matrix

The equations for the flows at the ends of each superlink is given by:

$$\sum_{l=1}^{NBDj} Q_{dkl}^{t+\Delta t} - \sum_{m=1}^{NBUj} Q_{ukm}^{t+\Delta t} + Q_{o,j} = \frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} \quad (43)$$

Substituting the linear expressions for the upstream and downstream flow:

$$\begin{aligned} \frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} &= \sum_{l=1}^{NBDj} (\alpha_{dkl} H_{juk_l}^{t+\Delta t} + \beta_{dkl} H_{jdk_l}^{t+\Delta t} + \chi_{dkl}) \\ &\quad - \sum_{m=1}^{NBUj} (\alpha_{ukm} H_{juk_m}^{t+\Delta t} + \beta_{ukm} H_{jdk_m}^{t+\Delta t} + \chi_{ukm}) + Q_{o,j} \end{aligned} \quad (44)$$

Because $H_{jdk_l} = H_j$ and $H_{juk_m} = H_j$:

$$\begin{aligned} \frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} &= \sum_{l=1}^{NBDj} (\alpha_{dkl} H_{juk_l}^{t+\Delta t} + \beta_{dkl} H_j^{t+\Delta t} + \chi_{dkl}) \\ &\quad - \sum_{m=1}^{NBUj} (\alpha_{ukm} H_j^{t+\Delta t} + \beta_{ukm} H_{jdk_m}^{t+\Delta t} + \chi_{ukm}) + Q_{o,j} \end{aligned} \quad (45)$$

Rearranging:

$$\begin{aligned} &\frac{A_{sj}(H_j^t)}{\Delta t} + \sum_{l=1}^{NBDj} \chi_{dkl} - \sum_{m=1}^{NBUj} \chi_{ukm} + Q_{o,j} \\ &= \left(\frac{A_{sj}}{\Delta t} + \sum_{m=1}^{NBUj} \alpha_{ukm} - \sum_{l=1}^{NBDj} \beta_{dkl} \right) H_j^{t+\Delta t} \\ &\quad - \sum_{l=1}^{NBDj} \alpha_{dkl} H_{juk_l}^{t+\Delta t} + \sum_{m=1}^{NBUj} \beta_{ukm} H_{jdk_m}^{t+\Delta t} \end{aligned} \quad (46)$$

For the example network in Ji (1998):

$$Ax = b \quad (47)$$

$$A = \begin{bmatrix} (\frac{A_{s0}}{\Delta t} + \alpha_{u0}) & \beta_{u0} & 0 & 0 & 0 & 0 \\ -\alpha_{d0} & (\frac{A_{s1}}{\Delta t} + \alpha_{u1} + \alpha_{u3} - \beta_{d0}) & \beta_{u1} & 0 & \beta_{u3} & 0 \\ 0 & -\alpha_{d1} & (\frac{A_{s2}}{\Delta t} + \alpha_{u2} - \beta_{d1} - \beta_{d5}) & \beta_{u2} & -\alpha_{d5} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\alpha_{d3} & \beta_{u5} & 0 & (\frac{A_{s4}}{\Delta t} + \alpha_{u4} + \alpha_{u5} - \beta_{d3}) & \beta_{u4} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (48)$$

$$b = \begin{bmatrix} \frac{A_{s0}H_0^t}{\Delta t} - \chi_{u0} + Q_{o0} \\ \frac{A_{s1}H_1^t}{\Delta t} + \chi_{d0} - (\chi_{u1} + \chi_{u3}) + Q_{o1} \\ \frac{A_{s2}H_2^t}{\Delta t} + (\chi_{d1} + \chi_{d5}) - \chi_{u2} + Q_{o2} \\ H_{3,bc} \\ \frac{A_{s4}H_4^t}{\Delta t} + \chi_{d3} - (\chi_{u4} + \chi_{u5}) + Q_{o4} \\ H_{5,bc} \end{bmatrix} \quad (49)$$