Glossary

Variable	Description
Q_{uk}	Discharge at upstream end of superlink k
C_{uk}	Coefficient of discharge at upstream end of superlink k
A_{uk}	Cross-sectional area of flow at upstream end of superlink k
ΔH_{uk}	Head difference at upstream end of superlink k
H_{juk}	Head at junction upstream of superlink k (ground elevation + water depth)
h_{uk}	Water depth at upstream of superlink k
$z_{inv,uk}$	Invert elevation at upstream end of superlink k
Q_{dk}	Discharge at downstream end of superlink k
C_{dk}	Coefficient of discharge at downstream end of superlink k
A_{dk}	Cross-sectional area of flow at downstream end of superlink k
ΔH_{dk}	Head difference at downstream end of superlink k
H_{jdk}	Head at junction downstream of superlink k (ground elevation + water depth)
h_{dk}	Water depth at downstream of superlink k
$z_{inv,dk}$	Invert elevation at downstream end of superlink k
$NBDj$	Number of superlinks with downstream end attached to superjunction j
$NBUj$	Number of superlinks with upstream end attached to superjunction j
H_j	Head at junction j
U_{Ik}, V_{Ik}, W_{Ik}	Coefficients
X_{Ik}, Y_{Ik}, Z_{Ik}	Coefficients

Basic equations

Continuity:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_0 \tag{1}$$

Conservation of momentum:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(Qu) + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f + S_L\right) = 0 \tag{2}$$

Discretization of momentum

Discretizing the momentum equation:

$$
\begin{aligned}
(Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + u_{I+1k}Q_{I+1k}^{t+\Delta t} - u_{Ik}Q_{Ik}^{t+\Delta t} \\
+gA(h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) - gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(S_{f,ik} + S_{L,ik})\Delta x = 0
\end{aligned} \tag{3}
$$

This equation can be written in terms of the following coefficient equation:

$$
a_{ik}Q_{i-1k}^{t+\Delta t} + b_{ik}Q_{ik}^{t+\Delta t} + c_{ik}Q_{i+1k}^{t+\Delta t} = P_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \tag{4}
$$

Where:

$$
a_{ik} = -\max(u_{Ik}, 0) \tag{5}
$$

$$
c_{ik} = -\max(-u_{I+1k}, 0) \tag{6}
$$

$$
b_{ik} = \frac{\Delta x_{ik}}{\Delta t} + \frac{gn_{ik}^2|Q_{ik}^t|\Delta x_{ik}}{A_{ik}R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2C_{ik}^2} - a_{ik} - c_{ik} \tag{7}
$$

$$
P_{ik} = Q_{ik}\frac{\Delta x_{ik}}{\Delta t} + gA_{ik}S_{o,ik}\Delta x_{ik} \tag{8}
$$

Substituting the coefficients:

$$
\begin{aligned}
-\max(u_{Ik}, 0)Q_{i-1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{gn_{ik}^2|Q_{ik}^t|\Delta x_{ik}}{A_{ik}R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2C_{ik}^2} + \max(u_{Ik}, 0) + \max(-u_{I+1k}, 0)\right)Q_{ik}^{t+\Delta t} \\
-\max(-u_{I+1k}, 0)Q_{i+1k}^{t+\Delta t} \\
= Q_{ik}\frac{\Delta x_{ik}}{\Delta t} + gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t})
\end{aligned} \tag{9}
$$

Assuming $u_{ik} > 0$ and $u_{i-1k} \approx u_{ik} \approx u_{i+1k}$:

$$
\begin{aligned}
-u_{Ik}Q_{i-1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{gn_{ik}^2|Q_{ik}^t|\Delta x_{ik}}{A_{ik}R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2C_{ik}^2} + u_{Ik}\right)Q_{ik}^{t+\Delta t} \\
= Q_{ik}^t\frac{\Delta x_{ik}}{\Delta t} + gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t})
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
(Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t})u_{Ik} \\
+gA_{ik}(\frac{n_{ik}^2|Q_{ik}^t|Q_{ik}^{t+\Delta t}}{A_{ik}^2 R_{ik}^{4/3}} + \frac{|Q_{ik}^t|Q_{ik}^{t+\Delta t}}{gC_{ik}^2 A_{cik}^2 \Delta x_{ik}})\Delta x_{ik} \\
= gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t})
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
(Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t})u_{Ik} + gA_{ik}(S_{f,ik} + S_{L,ik})\Delta x_{ik} \\
= gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t})
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
(Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t})u_{Ik} \\
+gA_{ik}(h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) + gA_{ik}(S_{f,ik} - gA_{ik}S_{o,ik}\Delta x_{ik} + S_{L,ik})\Delta x_{ik}
\end{aligned}
\tag{13}
$$

Alternatively, assuming $u_{ik} < 0$ and $u_{i-1k} \approx u_{ik} \approx u_{i+1k}$:

$$
\begin{aligned}
u_{I+1k}Q_{i+1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{gn_{ik}^2|Q_{ik}^t|\Delta x_{ik}}{A_{ik}R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} - u_{I+1k}\right)Q_{ik}^{t+\Delta t} \\
= Q_{ik}^t\frac{\Delta x_{ik}}{\Delta t} + gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t})
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
(Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t})u_{I+1k} \\
+gA_{ik}(\frac{n_{ik}^2|Q_{ik}^t|Q_{ik}^{t+\Delta t}}{A_{ik}^2 R_{ik}^{4/3}} + \frac{|Q_{ik}^t|Q_{ik}^{t+\Delta t}}{gC_{ik}^2 A_{cik}^2 \Delta x_{ik}})\Delta x_{ik} \\
= gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t})
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
(Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t})u_{I+1k} + gA_{ik}(S_{f,ik} + S_{L,ik})\Delta x_{ik} \\
= gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t})
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
(Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t})u_{I+1k} \\
+gA_{ik}(h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) + gA_{ik}(S_{f,ik} - gA_{ik}S_{o,ik}\Delta x_{ik} + S_{L,ik})\Delta x_{ik}
\end{aligned}
\tag{17}
$$

Recurrence relationships

Backward recurrence

Starting at the downstream end of superlink k:

$$Q_{nk}^{t+\Delta t} - Q_{nk-1}^{t+\Delta t} + E_{Nk} h_{Nk}^{t+\Delta t} = D_{Nk} \tag{18}$$

$$a_{nk} Q_{nk-1}^{t+\Delta t} + b_{nk} Q_{nk}^{t+\Delta t} + c_{nk} Q_{nk+1}^{t+\Delta t} = P_{nk} + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{19}$$

Assuming $Q_{nk}^{t+\Delta t} = Q_{nk+1}^{t+\Delta t}$:

$$a_{nk}(Q_{nk}^{t+\Delta t} + E_{Nk} h_{Nk}^{t+\Delta t} - D_{Nk}) + b_{nk} Q_{nk}^{t+\Delta t} + c_{nk} Q_{nk}^{t+\Delta t} = P_{nk} + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{20}$$

$$(a_{nk} + b_{nk} + c_{nk}) Q_{nk}^{t+\Delta t} = -E_{Nk} a_{nk} h_{Nk}^{t+\Delta t} + (P_{nk} + a_{nk} D_{Nk}) + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{21}$$

$$(a_{nk} + b_{nk} + c_{nk}) Q_{nk}^{t+\Delta t} = (g A_{nk} - E_{Nk} a_{nk}) h_{Nk}^{t+\Delta t} + (P_{nk} + D_{Nk} a_{nk}) - g A_{nk} h_{Nk+1}^{t+\Delta t} \tag{22}$$

$$Q_{nk}^{t+\Delta t} = \frac{(g A_{nk} - E_{Nk} a_{nk}) h_{Nk}^{t+\Delta t} + (P_{nk} + D_{Nk} a_{nk}) - g A_{nk} h_{Nk+1}^{t+\Delta t}}{(a_{nk} + b_{nk} + c_{nk})} \tag{23}$$

Thus for the downstream end of superlink k:

$$\boxed{Q_{nk}^{t+\Delta t} = X_{Nk} h_{Nk}^{t+\Delta t} + Y_{Nk} + Z_{Nk} h_{Nk+1}^{t+\Delta t}} \tag{24}$$

Where:

$$\boxed{O_{nk} = a_{nk} + b_{nk} + c_{nk}} \tag{25}$$

$$\boxed{X_{Nk} = \frac{(g A_{nk} - E_{Nk} a_{nk})}{O_{nk}}} \tag{26}$$

$$\boxed{Y_{Nk} = \frac{P_{nk} + D_{Nk}a_{nk}}{O_{nk}}} \tag{27}$$

$$\boxed{Z_{Nk} = -\frac{gA_{nk}}{O_{nk}}} \tag{28}$$

For the next element upstream:

$$Q_{nk-1}^{t+\Delta t} - Q_{nk-2}^{t+\Delta t} + E_{Nk-1}h_{Nk-1}^{t+\Delta t} = D_{Nk-1} \tag{29}$$

$$a_{nk-1}Q_{nk-2}^{t+\Delta t} + b_{nk-1}Q_{nk-1}^{t+\Delta t} + c_{nk-1}Q_{nk}^{t+\Delta t} = P_{nk-1} + gA_{nk-1}(h_{Nk-1}^{t+\Delta t} - h_{Nk}^{t+\Delta t}) \tag{30}$$

$$\begin{aligned} a_{nk-1}(Q_{nk-1}^{t+\Delta t} + E_{Nk-1}h_{Nk-1}^{t+\Delta t} - D_{Nk-1}) + (b_{nk-1})Q_{nk-1}^{t+\Delta t} + c_{nk-1}(Q_{nk-1} - E_{Nk}h_{Nk}^{t+\Delta t} + D_{Nk}) \\ -P_{nk-1} - gA_{nk-1}(h_{Nk-1}^{t+\Delta t} - h_{Nk}^{t+\Delta t}) = 0 \end{aligned} \tag{31}$$

$$\begin{aligned} (a_{nk-1} + b_{nk-1} + c_{nk-1})Q_{nk-1}^{t+\Delta t} + (-E_{Nk}c_{nk-1} + gA_{nk-1})h_{Nk}^{t+\Delta t} \\ +(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} + (-D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1} - P_{nk-1}) = 0 \end{aligned} \tag{32}$$

Multiplying $h_{Nk}^{t+\Delta t}$ by $(X_{Nk} + E_{Nk})/(X_{Nk} + E_{Nk})$ and rearranging:

$$\begin{aligned} (a_{nk-1} + b_{nk-1} + c_{nk-1})Q_{nk-1}^{t+\Delta t} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})(X_{Nk} + E_{Nk})}{(X_{Nk} + E_{Nk})}h_{Nk}^{t+\Delta t} \\ +(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} + (-P_{nk-1} - D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1}) = 0 \end{aligned} \tag{33}$$

Note that:

$$X_{Nk}h_{Nk}^{t+\Delta t} = (Q_{nk}^{t+\Delta t} - Y_{Nk} - Z_{Nk}h_{Nk+1}^{t+\Delta t}) \tag{34}$$

$$E_{Nk}h_{Nk^{t+\Delta t}} = (D_{Nk} - Q_{nk}^{t+\Delta t} + Q_{nk-1}^{t+\Delta t}) \tag{35}$$

Thus:

$$
\begin{aligned}
&(a_{nk-1} + b_{nk-1} + c_{nk-1})Q_{nk-1}^{t+\Delta t} \\
&+\frac{(gA_{nk-1} - E_{Nk}c_{nk-1})}{(X_{Nk} + E_{Nk})}[(Q_{nk}^{t+\Delta t} - Y_{Nk} - Z_{Nk}h_{Nk+1}^{t+\Delta t}) - (D_{Nk} - Q_{nk}^{t+\Delta t} + Q_{nk-1}^{t+\Delta t})] \\
&+(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} + (-P_{nk-1} - D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1}) = 0
\end{aligned}
\tag{36}
$$

Allowing $Q_{nk}^{t+\Delta t}$ to be eliminated:

$$
\begin{aligned}
&(a_{nk-1} + b_{nk-1} + c_{nk-1})Q_{nk-1}^{t+\Delta t} \\
&+\frac{(gA_{nk-1} - E_{Nk}c_{nk-1})}{(X_{Nk} + E_{Nk})}Q_{nk-1}^{t+\Delta t} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})(-Z_{Nk})}{(X_{Nk} + E_{Nk})}h_{Nk+1}^{t+\Delta t} \\
&+(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} \\
&+\left(-P_{nk-1} - D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})(-Y_{Nk} + D_{Nk})}{(X_{Nk} + E_{Nk})}\right) = 0
\end{aligned}
\tag{37}
$$

Rearranging:

$$
\begin{aligned}
&\left(a_{nk-1} + b_{nk-1} + c_{nk-1} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})}{(X_{Nk} + E_{Nk})}\right)Q_{nk-1}^{t+\Delta t} \\
&+(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} - \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})Z_{Nk}}{(X_{Nk} + E_{Nk})}h_{Nk+1}^{t+\Delta t} \\
&+\left(-P_{nk-1} - D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1} + (gA_{nk-1} - E_{Nk}c_{nk-1})\frac{(D_{Nk} - Y_{Nk})}{(X_{Nk} + E_{Nk})}\right) = 0
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
&\left(a_{nk-1} + b_{nk-1} + c_{nk-1} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})}{(X_{Nk} + E_{Nk})}\right)Q_{nk-1}^{t+\Delta t} \\
&= (gA_{nk-1} - E_{Nk-1}a_{nk-1})h_{Nk-1}^{t+\Delta t} \\
&+\left(P_{nk-1} + D_{Nk-1}a_{nk-1} - D_{Nk}c_{nk-1} - (gA_{nk-1} - E_{Nk}c_{nk-1})\frac{(D_{Nk} - Y_{Nk})}{(X_{Nk} + E_{Nk})}\right) \\
&+\frac{(gA_{nk-1} - E_{Nk}c_{nk-1})Z_{Nk}}{(X_{Nk} + E_{Nk})}h_{Nk+1}^{t+\Delta t}
\end{aligned}
\tag{39}
$$

Generalizing for $i = nk - 1$, $I = Nk - 1$:

$$
\begin{aligned}
\left(a_{ik}+b_{ik}+c_{ik}+\frac{(gA_{ik}-E_{I+1k}c_{ik})}{(X_{I+1k}+E_{I+1k})}\right)Q_{ik}^{t+\Delta t} \\
=(gA_{ik}-E_{Ik}a_{ik})h_{Ik}^{t+\Delta t} \\
+\left(P_{ik}+D_{Ik}a_{ik}-D_{I+1k}c_{ik}-(gA_{ik}-E_{I+1k}c_{ik})\frac{(D_{I+1k}-Y_{I+1k})}{(X_{I+1k}+E_{I+1k})}\right) \\
+\frac{(gA_{ik}-E_{I+1k}c_{ik})Z_{I+1k}}{(X_{I+1k}+E_{I+1k})}h_{Nk+1}^{t+\Delta t}
\end{aligned}
\tag{40}
$$

Condensing in terms of coefficients:

$$
\boxed{Q_{ik}^{t+\Delta t}=X_{ik}h_{Ik}^{t+\Delta t}+Y_{Ik}+Z_{Ik}h_{Nk+1}^{t+\Delta t}}
\tag{41}
$$

Where:

$$
\boxed{X_{Ik}=\frac{gA_{ik}-E_{Ik}a_{ik}}{O_{ik}}}
\tag{42}
$$

$$
\boxed{Y_{Ik}=\frac{P_{ik}+D_{Ik}a_{ik}-D_{I+1k}c_{ik}-(gA_{ik}-E_{I+1k}c_{ik})\frac{(D_{I+1k}-Y_{I+1k})}{(X_{I+1k}+E_{I+1k})}}{O_{ik}}}
\tag{43}
$$

$$
\boxed{Z_{Ik}=\frac{(gA_{ik}-E_{I+1k}c_{ik})Z_{I+1k}}{(X_{I+1k}+E_{I+1k})O_{ik}}}
\tag{44}
$$

$$
\boxed{O_{ik}=\left(a_{ik}+b_{ik}+c_{ik}+\frac{gA_{ik}-E_{I+1k}c_{ik}}{X_{I+1k}+E_{I+1k}}\right)}
\tag{45}
$$

Inlet hydraulics

Depth at upstream end of superlink

The discharge at the upstream end of a superlink is given by:

$$Q_{uk} = C_{uk} A_{uk} \sqrt{2g \Delta H_{uk}} \tag{46}$$

Where:

$$\Delta H_{uk} = H_{juk} - h_{uk} - z_{inv,uk} \tag{47}$$

Squaring and rearranging provides the depth boundary condition at the upstream end:

$$Q_{uk}^2 = 2 C_{uk}^2 A_{uk}^2 g (H_{juk} - h_{uk} - z_{inv,uk}) \tag{48}$$

$$|Q_{uk}^t| Q_{uk}^{t+\Delta t} = 2 C_{uk}^2 A_{uk}^2 g (H_{juk} - h_{uk} - z_{inv,uk}) \tag{49}$$

$$h_{uk} = - \frac{|Q_{uk}^t| Q_{uk}^{t+\Delta t}}{2 C_{uk}^2 A_{uk}^2 g} + H_{juk} - z_{inv,uk} \tag{50}$$

Depth at downstream end of superlink

The discharge at the downstream end of a superlink is given by:

$$Q_{dk} = C_{dk} A_{dk} \sqrt{2g \Delta H_{dk}} \tag{51}$$

Where:

$$\Delta H_{dk} = h_{dk} + z_{inv,dk} - H_{jdk} \tag{52}$$

Squaring and rearranging provides the depth boundary condition at the downstream end:

$$Q_{dk}^2 = 2 C_{dk}^2 A_{dk}^2 g (h_{dk} + z_{inv,dk} - H_{jdk}) \tag{53}$$

$$|Q_{dk}^t|Q_{dk}^{t+\Delta t} = 2C_{dk}^2 A_{dk}^2 g(h_{dk} + z_{inv,dk} - H_{jdk}) \tag{54}$$

$$h_{dk} = \frac{|Q_{dk}^t|Q_{dk}^{t+\Delta t}}{2C_{dk}^2 A_{dk}^2 g} + H_{jdk} - z_{inv,dk} \tag{55}$$

Superlink boundary conditions

From the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k} h_{uk}^{t+\Delta t} + Y_{1k} + Z_{1k} h_{dk}^{t+\Delta t} \tag{56}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk} h_{dk}^{t+\Delta t} + V_{Nk} + W_{Nk} h_{uk}^{t+\Delta t} \tag{57}$$

From the depth boundary conditions at the ends of each superlink:

$$h_{uk} = \gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk} \tag{58}$$

$$h_{dk} = \gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk} \tag{59}$$

Where:

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{2C_{uk}^2 A_{uk}^2 g} \tag{60}$$

$$\gamma_{dk} = \frac{|Q_{dk}^t|}{2C_{dk}^2 A_{dk}^2 g} \tag{61}$$

Substituting into the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k}(\gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) + Y_{1k} + Z_{1k}(\gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) \tag{62}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}(\gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) + V_{Nk} + W_{Nk}(\gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) \tag{63}$$

Expanding:

$$Q_{uk}^{t+\Delta t} = X_{1k}\gamma_{uk}Q_{uk}^{t+\Delta t} + X_{1k}H_{juk}^{t+\Delta t} - X_{1k}z_{inv,uk} + Y_{1k} + Z_{1k}\gamma_{dk}Q_{dk}^{t+\Delta t} + Z_{1k}H_{jdk}^{t+\Delta t} - Z_{1k}z_{inv,dk} \quad (64)$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}\gamma_{dk}Q_{dk}^{t+\Delta t} + U_{Nk}H_{jdk} - U_{Nk}z_{inv,dk} + V_{Nk} + W_{Nk}\gamma_{uk}Q_{uk}^{t+\Delta t} + W_{Nk}H_{juk} - W_{Nk}z_{inv,uk} \quad (65)$$

Rearranging:

$$0 = (X_{1k}\gamma_{uk} - 1)Q_{uk}^{t+\Delta t} + Z_{1k}\gamma_{dk}Q_{dk}^{t+\Delta t} + X_{1k}H_{juk}^{t+\Delta t} + Z_{1k}H_{jdk}^{t+\Delta t} + \pi_1 \quad (66)$$

$$0 = W_{Nk}\gamma_{uk}Q_{uk}^{t+\Delta t} + (U_{Nk}\gamma_{dk} - 1)Q_{dk}^{t+\Delta t} + W_{Nk}H_{juk} + U_{Nk}H_{jdk} + \pi_2 \quad (67)$$

Where:

$$\pi_1 = Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk} \quad (68)$$

$$\pi_2 = V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk} \quad (69)$$

$$(70)$$

Writing as a matrix equation:

$$\begin{bmatrix} (X_{1k}\gamma_{uk} - 1) & Z_{1k}\gamma_{dk} \\ W_{Nk}\gamma_{uk} & (U_{Nk}\gamma_{dk} - 1) \end{bmatrix} \begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \begin{bmatrix} -X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \quad (71)$$

Taking the matrix inverse:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk} - 1) & -Z_{1k}\gamma_{dk} \\ -W_{Nk}\gamma_{uk} & (X_{1k}\gamma_{uk} - 1) \end{bmatrix} \begin{bmatrix} -X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \quad (72)$$

Where:

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \quad (73)$$

Expanding:

$$
\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk}-1)(-X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1) + (-Z_{1k}\gamma_{dk})(-W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2) \\ (-W_{Nk}\gamma_{uk})(-X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1) + (X_{1k}\gamma_{uk}-1)(-W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2) \end{bmatrix} \tag{74}
$$

Arranging in terms of the unknown heads:

$$
\begin{aligned}
Q_{uk}^{t+\Delta t} = {} & [(U_{Nk}\gamma_{dk}-1)(-X_{1k}) + (-Z_{1k}\gamma_{dk})(-W_{Nk})]H_{juk}^{t+\Delta t} + \\
& [(U_{Nk}\gamma_{dk}-1)(-Z_{1k}) + (-Z_{1k}\gamma_{dk})(-U_{Nk})]H_{jdk}^{t+\Delta t} + \\
& [(U_{Nk}\gamma_{dk}-1)(-\pi_1) + (-Z_{1k}\gamma_{dk})(-\pi_2)]
\end{aligned} \tag{75}
$$

$$
\begin{aligned}
Q_{dk}^{t+\Delta t} = {} & [(-W_{Nk}\gamma_{uk})(-X_{1k}) + (X_{1k}\gamma_{uk}-1)(-W_{Nk})]H_{juk}^{t+\Delta t} + \\
& [(-W_{Nk}\gamma_{uk})(-Z_{1k}) + (X_{1k}\gamma_{uk}-1)(-U_{Nk})]H_{jdk}^{t+\Delta t} + \\
& [(-W_{Nk}\gamma_{uk})(-\pi_1) + (X_{1k}\gamma_{uk}-1)(-\pi_2)]
\end{aligned} \tag{76}
$$

Finally, the upstream and downstream flows can be expressed as:

$$
Q_{uk}^{t+\Delta t} = \alpha_{uk}H_{juk}^{t+\Delta t} + \beta_{uk}H_{jdk}^{t+\Delta t} + \chi_{uk} \tag{77}
$$

$$
Q_{dk}^{t+\Delta t} = \alpha_{dk}H_{juk}^{t+\Delta t} + \beta_{dk}H_{jdk}^{t+\Delta t} + \chi_{dk} \tag{78}
$$

Where:

$$
\alpha_{uk} = \frac{(1-U_{Nk}\gamma_{dk})X_{1k} + Z_{1k}\gamma_{dk}W_{Nk}}{D_k^*} \tag{79}
$$

$$
\beta_{uk} = \frac{(1-U_{Nk}\gamma_{dk})Z_{1k} + Z_{1k}\gamma_{dk}U_{Nk}}{D_k^*} \tag{80}
$$

$$
\chi_{uk} = \frac{(1-U_{Nk}\gamma_{dk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk}) + (Z_{1k}\gamma_{dk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk})}{D_k^*} \tag{81}
$$

$$
\alpha_{dk} = \frac{(1-X_{1k}\gamma_{uk})W_{Nk} + W_{Nk}\gamma_{uk}X_{1k}}{D_k^*} \tag{82}
$$

$$\beta_{dk} = \frac{(1 - X_{1k}\gamma_{uk})U_{Nk} + W_{Nk}\gamma_{uk}Z_{1k}}{D_k^*} \tag{83}$$

$$\chi_{dk} = \frac{(1 - X_{1k}\gamma_{uk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk}) + (W_{Nk}\gamma_{uk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk})}{D_k^*} \tag{84}$$

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \tag{85}$$

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{2C_{uk}^2 A_{uk}^2 g} \tag{86}$$

$$\gamma_{dk} = \frac{|Q_{dk}^t|}{2C_{dk}^2 A_{dk}^2 g} \tag{87}$$

Forming the solution matrix

The equations for the flows at the ends of each superlink are given by:

$$\sum_{l=1}^{NBDj} Q_{dk_l}^{t+\Delta t} - \sum_{m=1}^{NBUj} Q_{uk_m}^{t+\Delta t} + Q_{o,j} = \frac{A_{sj}(H_j^{t+\Delta t} - H_j)}{\Delta t} \tag{88}$$

Substituting the linear expressions for the upstream and downstream flows:

$$\begin{aligned} \frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} = & \sum_{l=1}^{NBDj} (\alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \beta_{dk_l} H_{jdk_l}^{t+\Delta t} + \chi_{dk_l}) \\ & - \sum_{m=1}^{NBUj} (\alpha_{uk_m} H_{juk_m}^{t+\Delta t} + \beta_{uk_m} H_{jdk_m}^{t+\Delta t} + \chi_{uk_m}) + Q_{o,j} \end{aligned} \tag{89}$$

Because $H_{jdk_l} = H_j$ and $H_{juk_m} = H_j$:

$$\begin{aligned} \frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} = & \sum_{l=1}^{NBDj} (\alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \beta_{dk_l} H_j^{t+\Delta t} + \chi_{dk_l}) \\ & - \sum_{m=1}^{NBUj} (\alpha_{uk_m} H_j^{t+\Delta t} + \beta_{uk_m} H_{jdk_m}^{t+\Delta t} + \chi_{uk_m}) + Q_{o,j} \end{aligned} \tag{90}$$

Rearranging:

$$\begin{aligned}
&\frac{A_{sj}(H_j^t)}{\Delta t} + \sum_{l=1}^{NBDj} \chi_{dk_l} - \sum_{m=1}^{NBUj} \chi_{uk_m} + Q_{o,j} \\
&= \left(\frac{A_{sj}}{\Delta t} + \sum_{m=1}^{NBUj} \alpha_{uk_m} - \sum_{l=1}^{NBDj} \beta_{dk_l}\right) H_j^{t+\Delta t} \\
&\quad - \sum_{l=1}^{NBDj} \alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \sum_{m=1}^{NBUj} \beta_{uk_m} H_{jdk_m}^{t+\Delta t}
\end{aligned} \tag{91}$$

For the example network in Ji (1998):

$$Ax = b \tag{92}$$

$$A = \begin{bmatrix}
(\frac{A_{s0}}{\Delta t} + \alpha_{u0}) & \beta_{u0} & 0 & 0 & 0 & 0 \\
-\alpha_{d0} & (\frac{A_{s1}}{\Delta t} + \alpha_{u1} + \alpha_{u3} - \beta_{d0}) & \beta_{u1} & 0 & \beta_{u3} & 0 \\
0 & -\alpha_{d1} & (\frac{A_{s2}}{\Delta t} + \alpha_{u2} - \beta_{d1} - \beta_{d5}) & \beta_{u2} & -\alpha_{d5} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -\alpha_{d3} & \beta_{u5} & 0 & (\frac{A_{s4}}{\Delta t} + \alpha_{u4} + \alpha_{u5} - \beta_{d3}) & \beta_{u4} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{93}$$

$$b = \begin{bmatrix}
\frac{A_{s0}H_0^t}{\Delta t} - \chi_{u0} + Q_{o0} \\
\frac{A_{s1}H_1^t}{\Delta t} + \chi_{d0} - (\chi_{u1} + \chi_{u3}) + Q_{o1} \\
\frac{A_{s2}H_2^t}{\Delta t} + (\chi_{d1} + \chi_{d5}) - \chi_{u2} + Q_{02} \\
H_{3,bc} \\
\frac{A_{s4}H_4^t}{\Delta t} + \chi_{d3} - (\chi_{u4} + \chi_{u5}) + Q_{04} \\
H_{5,bc}
\end{bmatrix} \tag{94}$$