Glossary

Variable	Description
Q_{uk}	Discharge at upstream end of superlink k
C_{uk}	Coefficient of discharge at upstream end of superlink k
A_{uk}	Cross-sectional area of flow at upstream end of superlink k
ΔH_{uk}	Head difference at upstream end of superlink k
H_{juk}	Head at junction upstream of superlink k (ground elevation + water depth)
h_{uk}	Water depth at upstream of superlink k
$z_{inv,uk}$	Invert elevation at upstream end of superlink k
Q_{dk}	Discharge at downstream end of superlink k
C_{dk}	Coefficient of discharge at downstream end of superlink k
A_{dk}	Cross-sectional area of flow at downstream end of superlink k
ΔH_{dk}	Head difference at downstream end of superlink k
H_{jdk}	Head at junction downstream of superlink k (ground elevation + water depth)
h_{dk}	Water depth at downstream of superlink k
$z_{inv,dk}$	Invert elevation at downstream end of superlink k
$NBDj$	Number of superlinks with downstream end attached to superjunction j
$NBUj$	Number of superlinks with upstream end attached to superjunction j
H_j	Head at junction j
U_{Ik}, V_{Ik}, W_{Ik}	Coefficients
X_{Ik}, Y_{Ik}, Z_{Ik}	Coefficients

Inlet hydraulics

Depth at upstream end of superlink

The discharge at the upstream end of a superlink is given by:

$$Q_{uk} = C_{uk} A_{uk} \sqrt{2g \Delta H_{uk}} \tag{1}$$

Where:

$$\Delta H_{uk} = H_{juk} - h_{uk} - z_{inv,uk} \tag{2}$$

Squaring and rearranging provides the depth boundary condition at the upstream end:

$$Q_{uk}^2 = 2C_{uk}^2 A_{uk}^2 g(H_{juk} - h_{uk} - z_{inv,uk}) \tag{3}$$

$$|Q_{uk}^t| Q_{uk}^{t+\Delta t} = 2C_{uk}^2 A_{uk}^2 g(H_{juk} - h_{uk} - z_{inv,uk}) \tag{4}$$

$$h_{uk} = -\frac{|Q_{uk}^t| Q_{uk}^{t+\Delta t}}{2C_{uk}^2 A_{uk}^2 g} + H_{juk} - z_{inv,uk} \tag{5}$$

Depth at downstream end of superlink

The discharge at the downstream end of a superlink is given by:

$$Q_{dk} = C_{dk} A_{dk} \sqrt{2g \Delta H_{dk}} \tag{6}$$

Where:

$$\Delta H_{dk} = h_{dk} + z_{inv,dk} - H_{jdk} \tag{7}$$

Squaring and rearranging provides the depth boundary condition at the downstream end:

$$Q_{dk}^2 = 2C_{dk}^2 A_{dk}^2 g(h_{dk} + z_{inv,dk} - H_{jdk}) \tag{8}$$

$$|Q_{dk}^t| Q_{dk}^{t+\Delta t} = 2C_{dk}^2 A_{dk}^2 g(h_{dk} + z_{inv,dk} - H_{jdk}) \tag{9}$$

$$h_{dk} = \frac{|Q_{dk}^t| Q_{dk}^{t+\Delta t}}{2C_{dk}^2 A_{dk}^2 g} + H_{jdk} - z_{inv,dk} \tag{10}$$

Superlink boundary conditions

From the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k} h_{uk}^{t+\Delta t} + Y_{1k} + Z_{1k} h_{dk}^{t+\Delta t} \tag{11}$$
$$Q_{dk}^{t+\Delta t} = U_{Nk} h_{dk}^{t+\Delta t} + V_{Nk} + W_{Nk} h_{uk}^{t+\Delta t} \tag{12}$$

From the depth boundary conditions at the ends of each superlink:

$$h_{uk} = \gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk} \tag{13}$$
$$h_{dk} = \gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk} \tag{14}$$

Where:

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{C_{uk}^2 A_{uk}^2 g} \tag{15}$$
$$\gamma_{dk} = \frac{|Q_{dk}^t|}{C_{dk}^2 A_{dk}^2 g} \tag{16}$$

Substituting into the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k}(\gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) + Y_{1k} + Z_{1k}(\gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) \tag{17}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}(\gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) + V_{Nk} + W_{Nk}(\gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) \tag{18}$$

Expanding:

$$Q_{uk}^{t+\Delta t} = X_{1k}\gamma_{uk} Q_{uk}^{t+\Delta t} + X_{1k} H_{juk}^{t+\Delta t} - X_{1k} z_{inv,uk} + Y_{1k} + Z_{1k}\gamma_{dk} Q_{dk}^{t+\Delta t} + Z_{1k} H_{jdk}^{t+\Delta t} - Z_{1k} z_{inv,dk} \tag{19}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}\gamma_{dk} Q_{dk}^{t+\Delta t} + U_{Nk} H_{jdk} - U_{Nk} z_{inv,dk} + V_{Nk} + W_{Nk}\gamma_{uk} Q_{uk}^{t+\Delta t} + W_{Nk} H_{juk} - W_{Nk} z_{inv,uk} \tag{20}$$

Rearranging:

$$0 = (X_{1k}\gamma_{uk} - 1) Q_{uk}^{t+\Delta t} + Z_{1k}\gamma_{dk} Q_{dk}^{t+\Delta t} + X_{1k} H_{juk}^{t+\Delta t} + Z_{1k} H_{jdk}^{t+\Delta t} + \pi_1 \tag{21}$$
$$0 = W_{Nk}\gamma_{uk} Q_{uk}^{t+\Delta t} + (U_{Nk}\gamma_{dk} - 1) Q_{dk}^{t+\Delta t} + W_{Nk} H_{juk} + U_{Nk} H_{jdk} + \pi_2 \tag{22}$$

Where:

$$\pi_1 = Y_{1k} - X_{1k} z_{inv,uk} - Z_{1k} z_{inv,dk} \tag{23}$$

$$\pi_2 = V_{Nk} - W_{Nk} z_{inv,uk} - U_{Nk} z_{inv,dk} \tag{24}$$

$$\tag{25}$$

Writing as a matrix equation:

$$\begin{bmatrix} (X_{1k}\gamma_{uk} - 1) & Z_{1k}\gamma_{dk} \\ W_{Nk}\gamma_{uk} & (U_{Nk}\gamma_{dk} - 1) \end{bmatrix} \begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \begin{bmatrix} -X_{1k} H_{juk}^{t+\Delta t} - Z_{1k} H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk} H_{juk}^{t+\Delta t} - U_{Nk} H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \tag{26}$$

Taking the matrix inverse:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk} - 1) & -Z_{1k}\gamma_{dk} \\ -W_{Nk}\gamma_{uk} & (X_{1k}\gamma_{uk} - 1) \end{bmatrix} \begin{bmatrix} -X_{1k} H_{juk}^{t+\Delta t} - Z_{1k} H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk} H_{juk}^{t+\Delta t} - U_{Nk} H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \tag{27}$$

Where:

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \tag{28}$$

Expanding:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk} - 1)(-X_{1k} H_{juk}^{t+\Delta t} - Z_{1k} H_{jdk}^{t+\Delta t} - \pi_1) + (-Z_{1k}\gamma_{dk})(-W_{Nk} H_{juk}^{t+\Delta t} - U_{Nk} H_{jdk}^{t+\Delta t} - \pi_2) \\ (-W_{Nk}\gamma_{uk})(-X_{1k} H_{juk}^{t+\Delta t} - Z_{1k} H_{jdk}^{t+\Delta t} - \pi_1) + (X_{1k}\gamma_{uk} - 1)(-W_{Nk} H_{juk}^{t+\Delta t} - U_{Nk} H_{jdk}^{t+\Delta t} - \pi_2) \end{bmatrix} \tag{29}$$

Arranging in terms of the unknown heads:

$$\begin{aligned} Q_{uk}^{t+\Delta t} = {} & [(U_{Nk}\gamma_{dk} - 1)(-X_{1k}) + (-Z_{1k}\gamma_{dk})(-W_{Nk})] H_{juk}^{t+\Delta t} + \\ & [(U_{Nk}\gamma_{dk} - 1)(-Z_{1k}) + (-Z_{1k}\gamma_{dk})(-U_{Nk})] H_{jdk}^{t+\Delta t} + \\ & [(U_{Nk}\gamma_{dk} - 1)(-\pi_1) + (-Z_{1k}\gamma_{dk})(-\pi_2)] \end{aligned} \tag{30}$$

$$\begin{aligned} Q_{dk}^{t+\Delta t} = {} & [(-W_{Nk}\gamma_{uk})(-X_{1k}) + (X_{1k}\gamma_{uk} - 1)(-W_{Nk})] H_{juk}^{t+\Delta t} + \\ & [(-W_{Nk}\gamma_{uk})(-Z_{1k}) + (X_{1k}\gamma_{uk} - 1)(-U_{Nk})] H_{jdk}^{t+\Delta t} + \\ & [(-W_{Nk}\gamma_{uk})(-\pi_1) + (X_{1k}\gamma_{uk} - 1)(-\pi_2)] \end{aligned} \tag{31}$$

Finally, the upstream and downstream flows can be expressed as:

$$Q_{uk}^{t+\Delta t} = \alpha_{uk} H_{juk}^{t+\Delta t} + \beta_{uk} H_{jdk}^{t+\Delta t} + \chi_{uk} \tag{32}$$

$$Q_{dk}^{t+\Delta t} = \alpha_{dk} H_{juk}^{t+\Delta t} + \beta_{dk} H_{jdk}^{t+\Delta t} + \chi_{dk} \tag{33}$$

Where:

$$\alpha_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})X_{1k} + Z_{1k}\gamma_{dk}W_{Nk}}{D_k^*} \tag{34}$$

$$\beta_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})Z_{1k} + Z_{1k}\gamma_{dk}U_{Nk}}{D_k^*} \tag{35}$$

$$\chi_{uk} = \frac{(1 - U_{Nk}\gamma_{dk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk}) + (Z_{1k}\gamma_{dk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk})}{D_k^*} \tag{36}$$

$$\alpha_{dk} = \frac{(1 - X_{1k}\gamma_{uk})W_{Nk} + W_{Nk}\gamma_{uk}X_{1k}}{D_k^*} \tag{37}$$

$$\beta_{dk} = \frac{(1 - X_{1k}\gamma_{uk})U_{Nk} + W_{Nk}\gamma_{uk}Z_{1k}}{D_k^*} \tag{38}$$

$$\chi_{dk} = \frac{(1 - X_{1k}\gamma_{uk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk}) + (W_{Nk}\gamma_{uk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk})}{D_k^*} \tag{39}$$

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \tag{40}$$

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{C_{uk}^2 A_{uk}^2 g} \tag{41}$$

$$\gamma_{dk} = \frac{|Q_{dk}^t|}{C_{dk}^2 A_{dk}^2 g} \tag{42}$$

Forming the solution matrix

The equations for the flows at the ends of each superlink are given by:

$$\sum_{l=1}^{NBDj} Q_{dk_l}^{t+\Delta t} - \sum_{m=1}^{NBUj} Q_{uk_m}^{t+\Delta t} + Q_{o,j} = \frac{A_{sj}(H_j^{t+\Delta t} - H_j)}{\Delta t} \tag{43}$$

Substituting the linear expressions for the upstream and downstream flows:

$$\begin{aligned} \frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} &= \sum_{l=1}^{NBDj} (\alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \beta_{dk_l} H_{jdk_l}^{t+\Delta t} + \chi_{dk_l}) \\ &- \sum_{m=1}^{NBUj} (\alpha_{uk_m} H_{juk_m}^{t+\Delta t} + \beta_{uk_m} H_{jdk_m}^{t+\Delta t} + \chi_{uk_m}) + Q_{o,j} \end{aligned} \tag{44}$$

Because $H_{jdk_l} = H_j$ and $H_{juk_m} = H_j$:

$$\begin{aligned} \frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} &= \sum_{l=1}^{NBDj} (\alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \beta_{dk_l} H_j^{t+\Delta t} + \chi_{dk_l}) \\ &- \sum_{m=1}^{NBUj} (\alpha_{uk_m} H_j^{t+\Delta t} + \beta_{uk_m} H_{jdk_m}^{t+\Delta t} + \chi_{uk_m}) + Q_{o,j} \end{aligned} \tag{45}$$

Rearranging:

$$\begin{aligned} &\frac{A_{sj}(H_j^t)}{\Delta t} + \sum_{l=1}^{NBDj} \chi_{dk_l} - \sum_{m=1}^{NBUj} \chi_{uk_m} + Q_{o,j} \\ &= \left(\frac{A_{sj}}{\Delta t} + \sum_{m=1}^{NBUj} \alpha_{uk_m} - \sum_{l=1}^{NBDj} \beta_{dk_l} \right) H_j^{t+\Delta t} \\ &- \sum_{l=1}^{NBDj} \alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \sum_{m=1}^{NBUj} \beta_{uk_m} H_{jdk_m}^{t+\Delta t} \end{aligned} \tag{46}$$

For the example network in Ji (1998):

$$Ax = b \tag{47}$$

$$A = \begin{bmatrix} (\frac{A_{s0}}{\Delta t} + \alpha_{u0}) & \beta_{u0} & 0 & 0 & 0 & 0 \\ -\alpha_{d0} & (\frac{A_{s1}}{\Delta t} + \alpha_{u1} + \alpha_{u3} - \beta_{d0}) & \beta_{u1} & 0 & \beta_{u3} & 0 \\ 0 & -\alpha_{d1} & (\frac{A_{s2}}{\Delta t} + \alpha_{u2} - \beta_{d1} - \beta_{d5}) & \beta_{u2} & -\alpha_{d5} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\alpha_{d3} & \beta_{u5} & 0 & (\frac{A_{s4}}{\Delta t} + \alpha_{u4} + \alpha_{u5} - \beta_{d3}) & \beta_{u4} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{48}$$

$$b = \begin{bmatrix} \frac{A_{s0} H_0^t}{\Delta t} - \chi_{u0} + Q_{o0} \\ \frac{A_{s1} H_1^t}{\Delta t} + \chi_{d0} - (\chi_{u1} + \chi_{u3}) + Q_{o1} \\ \frac{A_{s2} H_2^t}{\Delta t} + (\chi_{d1} + \chi_{d5}) - \chi_{u2} + Q_{02} \\ H_{3,bc} \\ \frac{A_{s4} H_4^t}{\Delta t} + \chi_{d3} - (\chi_{u4} + \chi_{u5}) + Q_{04} \\ H_{5,bc} \end{bmatrix} \tag{49}$$