Covariant derivative and its meaning

Let us consider a vectorfield in polar coordinates, namely:

$$A(r,\varphi) = A^r(r,\varphi)\frac{\partial}{\partial r}\Big)_{r,\varphi} + A^\varphi(r,\varphi)\frac{\partial}{\partial \varphi}\Big)_{r,\varphi}.$$

We will choose the following components:

$$A^r(r,\varphi) = 0$$
$$A^\varphi(r,\varphi) = \Omega, \qquad (*)$$

from which our intuition tells us, that the vectorfield has only a constant factor in φ-direction.

We want to compute the change of the component $A^y(x,y) = dy(A(x,y))$ along the direction $\frac{\partial}{\partial x}\Big)_{x,y}$ for the vector field in $(*)$. There are several ways to compute this.

1) Just calculate

With our knowledge of differentials and basis-vectors, we can easily compute A^x and A^y:

$$A^x(x,y) = dx(A) = \left(\frac{dx}{\partial\varphi}d\varphi + \frac{\partial x}{\partial r}dr\right)A$$

$$= \left[-r\sin\varphi\, d\varphi + \cos\varphi\, dr\right](A)$$

$$= \left[-y\, d\varphi + \frac{x}{\sqrt{x^2+y^2}}\, dr\right](A)$$

$$\Rightarrow A^x(x,y) = -y\, A^{\varphi}(r(x,y), \varphi(x,y)) + \frac{x}{\sqrt{x^2+y^2}} \underbrace{A^r(r(\ldots),\varphi(\ldots))}_{=0}$$

$$= -y\Omega$$

Similarily: $A^y(x,y) = x\Omega$.

So in total: $A(x,y) = -y\Omega \frac{\partial}{\partial x} + x\Omega \frac{\partial}{\partial y}$.

As $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are complex mathematical objects in tangent space, we will only consider the components of A (which carry the physical information).

$$\underline{A}(x,y) = \begin{pmatrix} -y\Omega \\ x\Omega \end{pmatrix}$$

with $\partial_y(A^x) = -\Omega$ and $\partial_x(A^y) = \Omega$.

2) ~~Use~~ Go to (r,φ)-space

We can also compute $\partial_x(A^y)$ by transforming into (r,φ) coordinates:

$$\underbrace{\partial_x}A^y = \underbrace{\left[\frac{\partial \varphi}{\partial x}\frac{\text{[redacted]}}{\partial \varphi} + \frac{\partial r}{\partial x}\frac{\text{[redacted]}}{\partial r}\right]}\underbrace{dy(A)}$$

$$= \left[\qquad -\,''\,- \qquad\right] \underbrace{\frac{\partial y}{\partial r}\underbrace{dr(A)}_{=A^r=0} + \frac{\partial y}{\partial \varphi}\underbrace{d\varphi(A)}_{=A^{\varphi}=\Omega}}$$

$$= \left[\frac{\partial \varphi}{\partial x}\frac{\text{[redacted]}}{\partial \varphi} + \frac{\partial r}{\partial x}\frac{\text{[redacted]}}{\partial r}\right]\underbrace{\frac{\partial y}{\partial \varphi}}_{=r\,\cancel{\sin}\cos\varphi}\Omega$$

$$= -\underbrace{\frac{\partial \varphi}{\partial x}}_{=-\frac{y}{x^2+y^2}} r\sin\varphi\,\Omega + \underbrace{\frac{\partial r}{\partial x}}_{=\frac{x}{\sqrt{x^2+y^2}}}\cos\varphi\,\Omega$$

$$= +\frac{y\sin\varphi}{\sqrt{x^2+y^2}}\,\Omega + \frac{x\cos\varphi}{\sqrt{x^2+y^2}}\,\Omega \quad ; \quad \frac{x}{r} = \cos\varphi, \quad \frac{y}{r} = \sin\varphi$$

$$= \Omega$$

□

Notice that it is crucial for this procedure to not forget about the transformation $\frac{\partial y}{\partial\varphi}$.

One could easily make a mistake here. For example if one were to compute the partial derivative of A (complete vector) instead of its components:

$$\partial_x A = \left[\frac{\partial\varphi}{\partial x}\partial_\varphi + \frac{\partial r}{\partial x}\partial_r\right]\left[A^\varphi \frac{\partial}{\partial\varphi} + \underbrace{A^r}_{=0}\frac{\partial}{\partial r}\right]$$

$$= \dots\, \partial_\varphi \frac{\partial}{\partial\varphi} + \dots\, \underbrace{\partial_r \frac{\partial}{\partial\varphi}}_{=\,?},$$

However, it is not defined what $\partial_\varphi \frac{\partial}{\partial\varphi}$ means in the context of differential geometry. (∂_φ can only act on functions)

Note: $\partial_\varphi \neq \frac{\partial}{\partial\varphi}$

Nonetheless, we as physicist do have an idea: Remember how in 7.6 we identified

$$e_r = \frac{\partial}{\partial r}$$

$$\text{and} \quad e_\varphi = \frac{1}{r}\frac{\partial}{\partial\varphi}.$$

What if we can translate

$$\partial_\varphi \frac{\partial}{\partial\varphi} \Rightarrow \partial_\varphi\left[r e_\varphi\right] \quad ?$$

We remember that basis vectors can indeed depend on coordinates. This is implicitly included in the framework of differential geometry as we write $\left.\frac{\partial}{\partial z^i}\right)_p$, so evaluated at point p. However, as we only deal with components of vectors in diff. geo., we don't care about that. Or do we? Let us first translate the above problem into standard basis vectors:

3) Compute as embedding in $\mathbb{R}^2$

We can write the vectorfield $A = A^r \frac{\partial}{\partial r} + A^\varphi \frac{\partial}{\partial \varphi}$ as $A = A^r \underline{e}_r + A^\varphi r\, \underline{e}_\varphi$ with

$$\underline{e}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

and

$$\underline{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

Only holds here!

Now, we will perform the partial derivative $\partial_x =: \frac{\partial}{\partial x}$ in the standard way:

$$\frac{\partial A}{\partial x} = \left[\frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\right]\left(\underbrace{A^r}_{=0} \underline{e}_r + A^\varphi r\, \underline{e}_\varphi\right)$$

$$= \frac{\partial \varphi}{\partial x} A^\varphi r \frac{\partial \underline{e}_\varphi}{\partial \varphi} + \frac{\partial r}{\partial x} \underline{e}_\varphi \frac{\partial}{\partial r}(r) A^\varphi$$

$$= \frac{\partial \varphi}{\partial x} A^\varphi r\,(-\underline{e}_r) + \frac{\partial r}{\partial x} \underline{e}_\varphi A^\varphi$$

$$= \Omega\left[-r\frac{\partial \varphi}{\partial x}\underline{e}_r + \frac{\partial r}{\partial x}\underline{e}_\varphi\right],$$

where we note that we <u>explicitly</u> calculated the change of the basis vectors w.r.t. the coordinates.

8.4.

Use the following identities ~~[illegible]~~

$$e_r = \cos\varphi\, e_x + \sin\varphi\, e_y = \frac{x}{\sqrt{\dots}}\, e_x + \frac{y}{\sqrt{\dots}}\, e_y$$

$$e_\varphi = -r\sin\varphi\, e_x + r\cos\varphi\, e_y = -y\, e_x + x\, e_y$$

as well as $\frac{\partial \varphi}{\partial x} r = y$; $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{\dots}}$

to arrive at:

$$\Rightarrow \frac{\partial A}{\partial x} = \Omega\, e_y$$ which translates to:

$$\left(\frac{\partial A}{\partial x}\right)^y = \Omega$$ in the diff. geo. way.

Note: We computed $\left(\frac{\partial A}{\partial x}\right)^y$ not $\partial_x (A^y)$!

Notice:

1) $\frac{\partial A}{\partial x}$ only makes sense in the context of embedding A completely in $\mathbb{R}^3$, namely we write the components in the usual way (as they are numbers), but the unit vectors are translated into vectors on $\mathbb{R}^2$.

2) The derivative is only non-zero because the basis vectors change in a non-trivial manner when changing coordinates. This was implicitly included in ~~[illegible]~~ 2) ~~[illegible]~~ to compute this in the coordinate dependence of $\frac{\partial y}{\partial \varphi}$. However, how can we generalize this circumstance?

We don't want to perform every transformation all the time by hand.

3) Notice that we compute $(\partial x\, A)^Y$ in third way whereas we calculated $\partial x(A^Y)$ in the previous two ways. Notice, that in this case, it holds that

$$(\partial x\, A)^Y = \partial x(A^Y).$$

This is <u>not</u> obvious.

8.6.