Physics vs. Math

These are many diffent theories which are used in physics. Mostly, these theories where developed in the most general sense. A recurring notion in physics is the term "Vector."

At first glance, it seems really trivial what a vectar is. What comes to mind are spatial coordinates, velocities, forces, etc. However, these physical quantities are actually not mathematical vectors, they are rather components of vectors, COV.

What is the difference?

COVs are real objects / quantities in the sense that they are based on perception or a measurement.

Vectors are not real objects and are independant on any transformation, perception, etc. They are, by definition, unchangeable.

What are examples of matematical vectors?

One instructive example is the WF $|\psi\rangle \in \mathcal{H}$. It is a vector in Hilbert-Space, which is a vectorspace.

It is nothing more and nothing less.
We can now project this highly complicated vector onto spaces that are more convinient, i.e. on the real space $\mathbb{R}^3$.
This yields the spatial WF, $\psi(\underline{r}) := \langle \underline{r} | \psi \rangle$, which are the <u>components</u> of $|\psi\rangle$ in the $|\underline{r}\rangle$-basis. Equivalently, we can project onto $\underline{k}$-space and obtain $\psi(\underline{k}) := \langle \underline{k} | \psi \rangle$.
These WFs obviously change with a coordinate transformation as they depend on $\underline{r}$ and $\underline{k}$ respectively.

Another example is special relativity where one talks about so-called length contraction, e.g. "the shortening of the <u>measured</u> length of an object moving relative to the observer's frame" [1].
The emphasis is on "measured" because the actual object doesn't change, only the size of the object is <u>perceived</u> as being different.

As an example: Consider a wooden stick of the size of 1m. If this object would actually shorten, it would break. So no, length contraction doesn't affect objects, it just tells us, that thing/objects appear differently when they are moving really fast.

[1] Wikipedia, Length contraction.

Enough talking, let's get to math and define the necessary tools to describe vectors. Note, that this is not meant to replace a corresponding math lecture. Some concepts will be introduced without proof and definitions will not be rigourous. This is only meant to help understand certain topics on a deeper theoretical level.

<u>Def.</u> (Vectorspace) ; short: VS

Let V be a set of objects $\{v_1, v_2, \dots\}$ equipped with an addition $+ : V \times V \to V$; $v_1 + v_2 \in V$ and a multiplication $\cdot : \mathbb{R} \times V \to V$; $c v_1 \in V$; $c \in \mathbb{R}$.

This vectorspace is d-dimensional if there exists a basis $\{e_1, \dots, e_d\}$ such that every $v \in V$ can be written as $v = v^i e_i \overset{(*)}{:=} \sum_i v^i e_i$ where we note the Einstein sum convention. (*)

<u>Note</u>:
- It is not arbitrary that the index of v^i is on top and on the bottom of e_i. This will make sense later.
- v is a mathematical vector. v^i are the components of v. e_i are the basisvectors of V.

<u>Def.</u> (Homomorphism)

$\mathrm{Hom}(V, W)$ denotes the space of all <u>linear</u> functions $f : V \to W$ that map from VS V to VS W.

$$f(v_1 + v_2) \overset{\text{lin.}}{=} f(v_1) + f(v_2)$$

$\underbrace{+}$ in V $\quad$ $\underbrace{+}$ in W

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Def. (Dualspace) ; short: V^* or DS.

The dualspace V^* of a VS V is given by $\mathrm{Hom}(V, \mathbb{R})$ so all linear functions that map a <u>vector</u> to a real number. An element $f \in V^*$ is called dualvector or <u>covector</u>.

Def. (Tensor)

A so-called $\binom{r}{s}$-tensor on the VS V is a map $T: \underbrace{V^* \times \dots \times V^*}_{r\text{-times}} \times \underbrace{V \times \dots \times V}_{s\text{-times}} \to \mathbb{R}$ which

is linear in each argument. So we take r covectors and s vectors and map them to some real number.

For most mathematical objects, it is insightful to consider easy examples to see what we are actually dealing with. It turns out that ~~th~~ it is especially insightful ~~for a~~ to consider $\binom{1}{0}$- and ~~a~~ $\binom{0}{1}$-tensors

<u>Example</u>: $\binom{0}{1}$-tensor & $\binom{1}{0}$-tensor

Following the Def., a $\binom{0}{1}$-tensor is given by a map $T: V \to \mathbb{R}$.

We notice that this is exactly the Def. for a covector, hence: $T \in V^*$.

It is sligtly more complicated for a $\binom{1}{0}$-tensor. It is given by the map: $T: V^* \to \mathbb{R}$.

Following the Def. of a dualspace, we can see that T is an element of the dualspace of the original dualspace, so

$$T \in (V^*)^*$$

But what does that mean? We are looking for functions that map a function to a real number.

For this consider a vector $v \in V$ and construct a map: $[v] : V^* \to \mathbb{R}$ by def. the action of $[v]$ on f by: $[v](f) := f(v) \in \mathbb{R}$ for all $f \in V^*$.

It can be shown that every $T \in (V^*)^*$ can be mapped to exactly one $v \in V$, meaning that

$$V \cong (V^*)^* \iff T = T_{(v)}$$

T is uniquely defined by $v \in V$.

(this means that V is isomorphic to V^{**}).

We can also see that the algebraic structure of $(V^*)^*$ is the same as V's:

+ in $\mathbb{R}$

$$
\begin{aligned}
([v_1] + [v_2])(f) &= [v_1](f) + [v_2](f) \\
&= f(v_1) + f(v_2) \\
&= f(v_1 + v_2) \\
&=: [v_1 + v_2](f)
\end{aligned}
$$

+ in $(V^*)^*$

+ in V

which means that it doesn't really matter if we talk about a T or the corresponding v.

Hence, we can write: $T \in V$ for a $\binom{1}{0}$-tensor.

Consequently, we can find out what a $\binom{1}{1}$-tensor is by first defining the tensor product.

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Def. (Tensorproduct)

Let T be an $\binom{r}{s}$-tensor and T' be an $\binom{r'}{s'}$-tensor. Then $\bar{T} := T \otimes T'$ is a new $\binom{r+r'}{s+s'}$-tensor which is defined by the property of factorization, meaning that:

$$\bar{T}(f_1, \dots, f_r, f_1', \dots, f_{r'}', v_1, \dots, v_s, v_1', \dots, v_{s'}')$$
$$= T(f_1, \dots, f_r, v_1, \dots, v_s) \cdot T'(f_1', \dots, f_{r'}', v_1', \dots, v_{s'}')$$

where $f_i, f_j' \in V^*$ are covectors for $i = 1, \dots, r$, $j = 1, \dots, s$

and $v_{i'}, v_{j'}' \in V$ are vectors for $i' = 1, \dots, r'$, $j' = 1, \dots, s'$

This definition is too complicated. What is even happening? Again, easy examples are instructive to show what is actually happening. We know how ~~know~~ a $\binom{1}{0}$-tensor looks like ($\hat{=}$ vector in V) and how a $\binom{0}{1}$-tensor looks like ($\hat{=}$ covector in V^*).

Let $T_{\binom{1}{0}} = v \in V$ be a vector and

$T_{\binom{0}{1}} = \varphi \in V^*$ be a covector and

$T = T_{\binom{1}{0}} \otimes T_{\binom{0}{1}}$ be the tensor product of the two tensors. Then:

$$T = v \otimes \varphi, \quad \text{so} \quad T \in V \otimes V^* \text{ is a } \binom{1}{1}\text{-tensor!}$$

Yes, that is it. However, the important thing is the factorization property, meaning that:

$$T(f, w) = (v \otimes \varphi)(f, w) = v(f) \cdot \varphi(w) =: f(v)\varphi(w). \quad \text{for } f \in V^*,\ w \in V.$$

As a result, we found that a $\binom{1}{1}$-tensor is a tensor product of a $\binom{1}{0}$- and $\binom{0}{1}$ tensor.

<u>Def.</u> (Dualbasis)

We know that we can find a basis in V, namely $\{e_1, \ldots, e_d\}$. This basis actually introduces a basis in our dualspace, namely $\{\varepsilon^1, \ldots, \varepsilon^d\}$ ~~which~~ with the property: $\varepsilon^i(e_j) \overset{!}{=} \delta^i{}_j$.

<u>Note</u>:

- ε^j $(j = 1, \ldots, d)$ are all elements of V^*, meaning that they map vectors to $\mathbb{R}$.
- $\varepsilon^i(e_j) = \delta^i{}_j$ has a meaning. We will find out later that ε^i is the operator that extracts the i-th <u>component</u> out of a <u>vector</u> $v \in V$. Hence, this relation implies that the unit vectors e_j are all orthonormal, meaning that their respective components are always zero except the j-th one.
- As we can write $v = \sum_i v^i e_i = v^i e_i$, we can write $\varphi = \sum_i \varphi_i \varepsilon^i = \varphi_i \varepsilon^i$ for any $\varphi \in V^*$.
- It is <u>not</u> a coincidence that the indices are placed in an <u>opposite way</u> to the <u>ones of a vector</u>. This will make sense once we talk about transformation properties.

With the knowledge of writing $v \in V$ and $\varphi \in V^*$ as supopositions, namely $v = v^i e_i$ and $\varphi = \varphi_i \varepsilon^i$, we can compute the <u>components</u> of a tensor, i.e. for the tensor $T = v \otimes \varphi$ for <u>fixed</u> $v \in V$ and $\varphi \in V^*$.

$$\Rightarrow T = v \otimes \varphi = \textstyle\sum_i v^i e_i \otimes \sum_\gamma \varphi_\gamma \varepsilon^\gamma$$

$$= \textstyle\sum_{i,\gamma} v^i \varphi_\gamma \; e_i \otimes \varepsilon^\gamma$$

$$= \boxed{v^i \varphi_\gamma} \; e_i \otimes \varepsilon^\gamma = \boxed{T^i{}_\gamma} \; e_i \otimes \varepsilon^\gamma$$

where we notice that the components of v, v^i and φ, φ_γ are on equal footing as they appear as a product. Besides their index position, they are the same mathematical objects, mainly a number in $\mathbb{R}$. However, there is a big ~~difference in~~ mathematical difference in <u>e_i</u> and <u>ε^γ</u>, where one is a <u>vector</u>, the other is a <u>covector</u>.

Now we come to the reason of why physics mostly concerns itself with compoments rather than ~~on~~ actuall vectors and covectors.

We define the <u>compoments</u> of the tensor $T = v \otimes \varphi$ by $T^i{}_\gamma := T(e_i, \varepsilon^\gamma) \in \mathbb{R}$. meaning that we get the following expression:

$$T = T^i{}_\gamma \; e_i \otimes \varepsilon^\gamma .$$

Note that these components $T^i{}_\gamma$ <u>uniquely</u> define T, meaning that we don't actually have to think a lot about $e_i \otimes \varepsilon^\gamma \in V \otimes V^*$.

Example: $\binom{0}{2}$ - tensor; metric tensor

The probably most important tensor in Special and General Relativity is the metric tensor:

$$g: V \times V \to \mathbb{R}$$

So g is a $\binom{0}{2}$ - tensor and $g \in V^* \otimes V^*$. Written out, similarily to the $\binom{1}{1}$ - tensor as before, the metric tensor reads:

$$g = g_{ij} \, \varepsilon^i \otimes \varepsilon^j$$

where $g_{ij} \in \mathbb{R}$ for all $i, j = 1, \dots, d$ and $\varepsilon^i \in V^*$.

With the metric tensor, we can actually now <u>lower</u> indices. For this consider the map

$$\flat: V \to V^*$$
$$v \mapsto g(v, \cdot)$$

which maps a vector $v \in V$ to the covector $v^\flat = g(v, \cdot)$ which takes as input $\cdot$ a vector in V. It holds that:

$$\begin{aligned} v^\flat := g(v, \cdot) &= g_{ij} \, \varepsilon^i(v) \varepsilon^j(\cdot) \\ &= g_{ij} \, \varepsilon^i(v^k e_k) \varepsilon^j(\cdot) \\ &= g_{ij} \, v^k \delta^i{}_k \, \varepsilon^j(\cdot) \\ &= g_{ij} \, v^i \varepsilon^j =: v_j \varepsilon^j \end{aligned}$$

where v_j is the <u>co-component</u> of the component v^j of the vector v.

Notice, that v^j and v_j are the <u>same</u> mathematical object at first glance; they are both just numbers. However, on the greater scheme, v^j belongs to a vector and v_j belongs to a covector.

This has <u>major</u> implications for their respective propoties under a symmetry transformation.

<u>Note</u>:
- The compoments of g (metric tensor) are defined by $g_{ij} = g(e_i, e_j)$ in the basis $\{e_1, \dots, e_d\}$.

- The metric tensor introduces a natural <u>direct product</u> onto our vectorspace V, namely the scalar product. s:

$$s(v, w) := v \bullet w := g(v, w) = \underline{g(e_i, e_j)} v^i w^j$$
$$= \underline{g_{ij}} v^i w^j$$
$$= v_j w^j \qquad (v_j = g_{ij} v^i)$$
$$= v^i w_i \qquad (w_i = g_{ij} w^j)$$

in particular: $s(v,v) = v \bullet v =: v^2$
$$= v_j v^j = v^i v_i$$

- For $\mathbb{R}^3$, the metric tensor is given by:
$$g_{ij} = \delta_{ij}$$

- For $\mathbb{R}^4$ in SRT, the metric tensor is given by:

$$g_{ij} = \begin{cases} 1\, \delta_{i0}\, \delta_{j0} \\ -1\, \delta_{ij} \quad (i \neq 0, j \neq 0) \end{cases}$$

Transformation rules

In every vector space V, it is possible to perform basis transformations, namely by applying a lin. map to the basis vectors:

$$\tilde{e}_i = \sum_\gamma \Lambda_i{}^\gamma e_\gamma \quad \text{for } \Lambda_i{}^\gamma \in \mathbb{R} \text{ for all } i,\gamma$$

which is similar to a matrix multiplication except that $\tilde{e}_i$ and e_γ are general vectors, so they can be anything from a "vector" $(1\ 0\ 0)$ to a wavefunction $|\psi\rangle$.

The transformation property of ~~components~~ basis vectors is often called covariant (for some reason). The word has actually no meaning and also leads to more confusion than relevation. We will omit the term covariant and replace it with "transforms as a (basis) vector".

We can also backtransform via $e_\gamma = (\Lambda^{-1})_\gamma{}^i \, \tilde{e}_i$ which is also covariant as the placement of the indices is similar, but it's an inverse transformation.

But how does the dualbasis transform?

Consider:

$$\begin{aligned}
\varepsilon^i(\tilde{e}_k) &= \varepsilon^i(\Lambda_k{}^\gamma e_\gamma) \\
&= \Lambda_k{}^\gamma \, \varepsilon^i(e_\gamma) \\
&= \Lambda_k{}^\gamma \, \delta^i_\gamma \\
&= \Lambda_k{}^i \\
&= \Lambda_\gamma{}^i \, \delta_k{}^\gamma \\
&= \Lambda_\gamma{}^i \, \tilde{\varepsilon}^\gamma(\tilde{e}_k)
\end{aligned}$$

where $\tilde{\varepsilon}^\gamma$ is the dualbasis induced by the transformed basisvectors $\tilde{e}_k$.

It follows that: $\varepsilon^i = \Lambda_\gamma{}^i \, \tilde{\varepsilon}^\gamma$ (1)

and equivalently: $\tilde{\varepsilon}^\gamma = (\Lambda^{-1})_i{}^\gamma \, \varepsilon^i$

where it is seen that the summation is not done over the same index. This type of transformation property is called contravariant or "transforms like a covector."

But! We are mainly interested in components of vectors and covectors. We saw with the metric tensor that the components are ~~the~~ significant, not the basis.
So how do the components transform?

For a $v \in V$, write: $v = v^i e_i$.
We extract the k-th component out of v via ε^k:

$$\varepsilon^k(v) = \varepsilon^k(v^i e_i) = v^i \, \varepsilon^k(e_i) = v^i \, \delta^k{}_i = v^k.$$

$$\Rightarrow v^k = \varepsilon^k(v) \overset{(1)}{=} \Lambda_\gamma{}^k \, \tilde{\varepsilon}^\gamma(v) = \Lambda_\gamma{}^k \, \tilde{v}^\gamma$$

which means that the components of a vector transform contravariantly or like a covector.

For a $\varphi \in V^*$, write $\varphi = \varphi_\gamma \varepsilon^\gamma$
We extract the k-th component by e_k

$$\Rightarrow \varphi(e_k) = \varphi_\gamma \, \varepsilon^\gamma(e_k) = \varphi^k$$

3.2 $$\Rightarrow \varphi^k = \varphi_\gamma \, \Lambda_i{}^\gamma \, \tilde{\varepsilon}^i(e_k) = \Lambda_k{}^\gamma \varphi_\gamma.$$