We have seen hints that our charts actually seem to define a basis for our manifold on $\mathbb{R}^n$. We can make this more specific:

<u>Def.</u> (Chart induced basis)
Let $\mathcal{M}$ be a manifold and $\varphi : \mathcal{M} \to \mathbb{R}^n$ a chart. We define the following for $f \in C^\infty(\mathcal{M})$

$$\left(\frac{\partial}{\partial \varphi^i}\right)_p f \equiv \left(\frac{\partial f}{\partial \varphi^i}\right)_p := \left[\partial_{\varphi^i}(f \circ \varphi^{-1})\right](\varphi(p)),$$

which is the <u>i-th basis vector</u> of the tangentspace of the manifold with respect to chart φ.

Note, that $\left(\frac{\partial}{\partial \varphi^i}\right)_p$ is <u>not</u> ~~just~~ a partial derivative in the standard sense. Of course, as we write it this way, it is very similar to a partial derivative. However, as $\left(\frac{\partial}{\partial \varphi^i}\right)_p$ acts on f which is a function from $\mathcal{M} \to \mathbb{R}$, it makes no sense for $\left(\frac{\partial}{\partial \varphi^i}\right)_p$ to be a partial derivative.

Note however that ∂_{φ^i} is infact a partial derivative. This makes sense, as

$$f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$$

is a standard function (scalar field) from $\mathbb{R}^n$ to $\mathbb{R}$.

In later chapters, we will write $\partial_{\varphi^i} \Leftrightarrow \frac{\partial}{\partial \varphi^i}$, because it will be obvious with what objects we are dealing with.

Again, the f-function let's us input the charts to extract the components. In the case of $\left(\frac{\partial}{\partial y^i}\right)_p$ we find:

γ-th component in y-basis of $\left(\frac{\partial}{\partial y^i}\right)_p$ $\hat{=} \left(\frac{\partial}{\partial y^i}\right)_p y^\gamma \equiv \left(\frac{\partial y^\gamma}{\partial y^i}\right)_p := \underbrace{\left[\partial_{y^i}\left(y^\gamma \circ y^{-1}\right)\right] y(p)}_{= \delta^\gamma_i}$ (this is not super obvious),

That property is very similar to standard partial derivatives.

Notice that the j-th component of $\left(\frac{\partial}{\partial y^i}\right)_p$ is δ^i_j, meaning that $\left(\frac{\partial}{\partial y^i}\right)_p$'s are the basis vectors themselves!

Note: By now, it wasn't clear that $\left(\frac{\partial}{\partial y^i}\right)_p$ is a ~~[illegible]~~ vector in T_pM (see Def. on 5.2/5.3).

Proof. (that $\left(\frac{\partial}{\partial y^i}\right)_p$ are basis of all $v_{\gamma,p}$'s)

$$
\begin{aligned}
v_{\gamma,p}(f) &= (f \circ \gamma)'(0) \\
&= \Big(\underbrace{(f \circ y^{-1})}_{\mathbb{R}^n \to \mathbb{R}} \circ \underbrace{(y \circ \gamma)}_{\mathbb{R} \to \mathbb{R}^d}\Big)'(0) \\
&= \underbrace{\partial_{y^i}\left(f \circ y^{-1}\right)\Big|_{y(\gamma(0))}}_{=: \left(\frac{\partial f}{\partial y^i}\right)_p} \underbrace{(y^i \circ \gamma)'(0)}_{=: \dot{\gamma}^i \in \mathbb{R}} \\
&= \dot{\gamma}^i \left(\frac{\partial f}{\partial y^i}\right)_p
\end{aligned}
$$

$$\Rightarrow \quad v_{\gamma,p} = \dot{\gamma}^i \left(\frac{\partial}{\partial y^i}\right)_p$$

where $\dot{\gamma}^i$ are the <u>components</u> of $v_{\gamma,p}$ and $\left(\frac{\partial}{\partial y^i}\right)_p$ are the <u>basis vectors</u>. Hence the basis for <u>any</u> vector on M are the "partial derivatives"!

This might seem overwhelming, so let us make an example, namely polar coordinates.

Consider $S^1 = \{ x^2+y^2=1 \,/\, x, y \in \mathbb{R} \}$ and the chart:

$$\psi \begin{pmatrix} x \\ y \end{pmatrix} = (r, \varphi) = (\psi^1, \psi^2)\begin{pmatrix} x \\ y \end{pmatrix}$$

with $r = \sqrt{x^2+y^2}$ and $\varphi = \arctan(y/x)$.

Let us compute $\left(\frac{\partial}{\partial \psi^1}\right)_p = \left(\frac{\partial}{\partial r}\right)_p$ and $\left(\frac{\partial}{\partial \psi^2}\right)_p = \left(\frac{\partial}{\partial \varphi}\right)_p$!

$$\left(\frac{\partial}{\partial r}\right)_p (f) \equiv \left.\frac{\partial f}{\partial r}\right)_p = \left(\partial_r f(\psi^{-1})\right)\Big|_{\psi(p)}$$

Notice, that $\psi^{-1} : (r, \varphi) \longmapsto (x, y) = (r\cos\varphi, r\sin\varphi)$.

$$= \left(\partial_r f(x(r,\varphi), y(r,\varphi))\right)\Big|_{\psi(p)}$$

Chain rule for standard functions as $M = S^1 \subseteq \mathbb{R}^2$

$$= \left[\partial_x f(x,y)\frac{\partial x(r,\varphi)}{\partial r} + \partial_y f(x,y)\frac{\partial y(r,\varphi)}{\partial r}\right]\Big|_{\psi(p)}$$

$= (r(x,y), \varphi(x,y))$

$= (\sqrt{x^2+y^2}, \arctan(\frac{y}{x}))$

$$= \left[\partial_x f(x,y)\cos\varphi + \partial_y f(x,y)\sin\varphi\right]\Big|_{\psi(p)}$$

$$= \partial_x f(x,y)\cos(\arctan(\tfrac{y}{x})) + \partial_y f(x,y)\sin(\arctan(y/x))$$

$$= \partial_x f \frac{x}{\sqrt{x^2+y^2}} + \partial_y f \frac{y}{\sqrt{x^2+y^2}}$$

$x^2+y^2=1$

$$= (x\,\partial_x + y\,\partial_y) f$$

Similarily: $\left(\frac{\partial}{\partial \varphi}\right)_p f = (x\,\partial_y - y\,\partial_x) f$

Notice that, as $M \subseteq \mathbb{R}^2$, we ~~[illegible]~~ wrote standard partial derivatives ∂_x, ∂_y instead of $\left(\frac{\partial}{\partial x}\right)_p$ and $\left(\frac{\partial}{\partial y}\right)_p$.

Proof. (that $\partial_x \equiv \left(\frac{\partial}{\partial x}\right)_p$ on $\mathbb{R}^n$)

$$\left(\frac{\partial}{\partial x}\right)_p f = \left(\partial_x f(\mathcal{Y}^{-1})\right)\Big|_{\mathcal{Y}(p)}$$

with the identity chart $\mathcal{Y} : \binom{x}{y} \longmapsto \binom{x}{y}$, so:

$$\left(\frac{\partial}{\partial x}\right)_p f = \left(\partial_x f(x,y)\right). \qquad \square$$

So, to conclude this, we can write:

$$\left(\frac{\partial}{\partial r}\right)_p = x\left(\frac{\partial}{\partial x}\right)_p + y\left(\frac{\partial}{\partial y}\right)_p \quad ; p = (x,y)$$

and

$$\left(\frac{\partial}{\partial \varphi}\right)_p = x\left(\frac{\partial}{\partial y}\right)_p - y\left(\frac{\partial}{\partial x}\right)_p .$$

Note: Does this seem familiar?

This relation can be shown in a more general sense.

Proof.

$$\left(\frac{\partial}{\partial \mathcal{Y}^i}\right)_p f = \left(\partial_{\mathcal{Y}^i}\left(f \circ \mathcal{Y}^{-1}\right)\right)\left[\mathcal{Y}(p)\right]$$

$$= \left(\partial_{\mathcal{Y}^i}\left(\left(f \circ \Lambda^{-1}\right) \circ \left(\Lambda \circ \mathcal{Y}^{-1}\right)\right)\right)\mathcal{Y}(p)$$

$$= \underbrace{\partial_{\Lambda^\gamma}\left(f \circ \Lambda^{-1}\right)\Big|_{\underset{=\Lambda(p)}{\Lambda(\mathcal{Y}^{-1}(\mathcal{Y}(p)))}}}_{=:\left(\frac{\partial}{\partial \Lambda^\gamma}\right)_p f} \; \underbrace{\partial_{\mathcal{Y}^i}\left(\Lambda^\gamma \circ \mathcal{Y}^{-1}\right)\Big|_{\mathcal{Y}(p)}}_{=:\left(\frac{\partial}{\partial \mathcal{Y}^i}\right)_p \Lambda^\gamma \;\equiv\; \left(\frac{\partial \Lambda^\gamma}{\partial \mathcal{Y}^i}\right)_p}$$

$$\Rightarrow \left(\frac{\partial}{\partial \mathcal{Y}^i}\right)_p = \left(\frac{\partial \Lambda^\gamma}{\partial \mathcal{Y}^i}\right)_p \left(\frac{\partial}{\partial \Lambda^\gamma}\right)_p \quad (*) \qquad \square$$

Let us now revisit the example on 5.5 and 5.6, where we obtained:

1) $v_{\gamma|p}(\varphi_0) = 1$

2) $v_{\gamma|p}(\underline{\varphi}_{\mathbb{R}}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and notice that for the case 1), we have one basis vector $\left(\frac{\partial}{\partial \varphi_0}\right)_p$ and for the case 2), we have two basis vectors $\left(\frac{\partial}{\partial \varphi^1_{\mathbb{R}^2}}\right)_p$ and $\left(\frac{\partial}{\partial \varphi^2_{\mathbb{R}^2}}\right)_p$, @ $p = (1,0)$

We can use the relation (*):

$$v_{\gamma|p} = 1 \cdot \left(\frac{\partial}{\partial \varphi_0}\right)_p$$

$$= \left(\frac{\partial \varphi^1_{\mathbb{R}^2}}{\partial \varphi_0}\right)_p \left(\frac{\partial}{\partial \varphi^1_{\mathbb{R}^2}}\right)_p + \left(\frac{\partial \varphi^2_{\mathbb{R}^2}}{\partial \varphi_0}\right)_p \left(\frac{\partial}{\partial \varphi^2_{\mathbb{R}^2}}\right)_p$$

where we note that

$$\left(\frac{\partial \varphi^1_{\mathbb{R}^2}}{\partial \varphi_0}\right)_p = \partial_{\varphi_0}\left(\varphi^1_{\mathbb{R}^2} \circ \varphi_0^{-1}\right)\Big|_{\varphi_0(p)}$$

with $\varphi_0(p) = \varphi_0(x, y) = \arctan(y/x)$

and $(\varphi^1_{\mathbb{R}^2} \circ \varphi_0^{-1})(\lambda)$

$$= \varphi^1_{\mathbb{R}^2}(\cos \lambda, \sin \lambda) = \cos \lambda$$

$$\Rightarrow \partial_{\varphi_0}\left(\varphi^1_{\mathbb{R}^2} \circ \varphi_0^{-1}\right)\Big|_{\varphi_0(p)}$$

$$= \partial_\lambda \cos \lambda = -\sin \lambda \Big|_{\arctan(0)}$$

$$= 0$$

Similarily:

$$\left(\frac{\partial z^2_{\mathbb{R}^2}}{\partial z_0}\right)_p = 1.$$

$$\Rightarrow v_{\gamma,p} = \left(\frac{\partial}{\partial z_0}\right)_p = \left(\frac{\partial}{\partial z^2_{\mathbb{R}^2}}\right)_p . \quad \square$$

Hence, we were just expressing $v_{\gamma,p}$ in diffent basis!