Multiterminal Josephson Junctions (MTJJs): non-hermiticity, topology and reflectionless modes

Marco Coraiola
Tommaso Antonelli
Deividas Sabonis
Fabrizio Nichele
Sofieke C. ten Kate
IBM Research | Zurich

David Christian Ohnmacht
Wolfgang Belzig
Valentin Wilhelm
Hannes Weisbrich
Aleksandr E. Svetogorov
SFB 1432
Universität Konstanz

Juan Carlos Cuevas Juan
UAM Universidad Autónoma de Madrid

Erik Cheah
Filip Krizek
Rüdiger Schott
Werner Wegscheider
ETH zürich

26. May 2025

Multiterminal Josephson Junctions (MTJJs): non-hermiticity, topology and reflectionless modes

1) MTJJs are an excellent platform to study engineered (non-hermitian) topology

2) Refectionless scattering modes are a source of topology in MTJJs

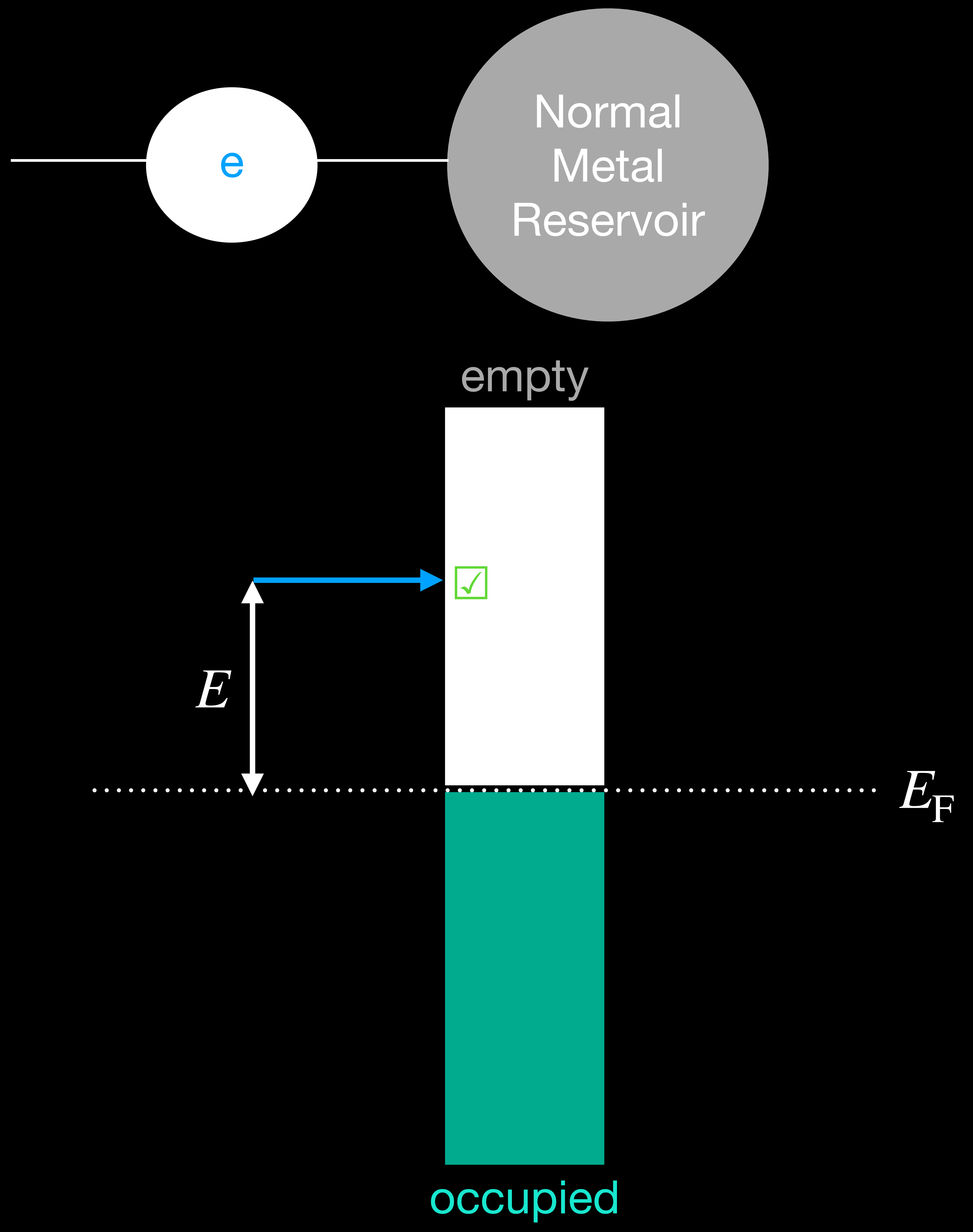
Erik Cheah
Filip Krizek
Rüdiger Schott
Werner Wegscheider

Marco Coraiola
Tommaso Antonelli
Deividas Sabonis
Fabrizio Nichele
Sofieke C. ten Kate

David Christian Ohnmacht
Wolfgang Belzig
Valentin Wilhelm
Hannes Weisbrich
Aleksandr E. Svetogorov

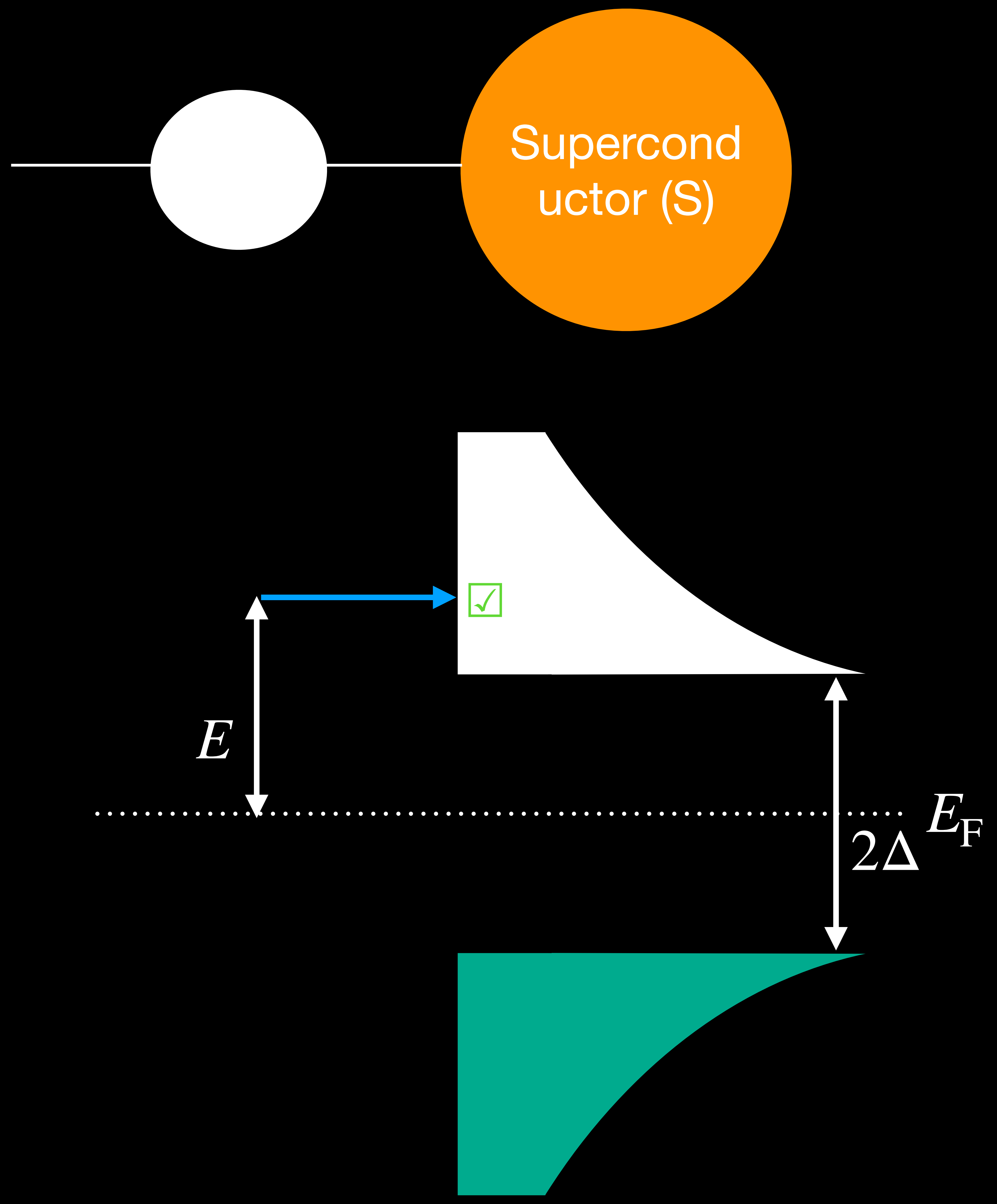
Juan Carlos Cuevas Juan

What's special about superconducting junctions?



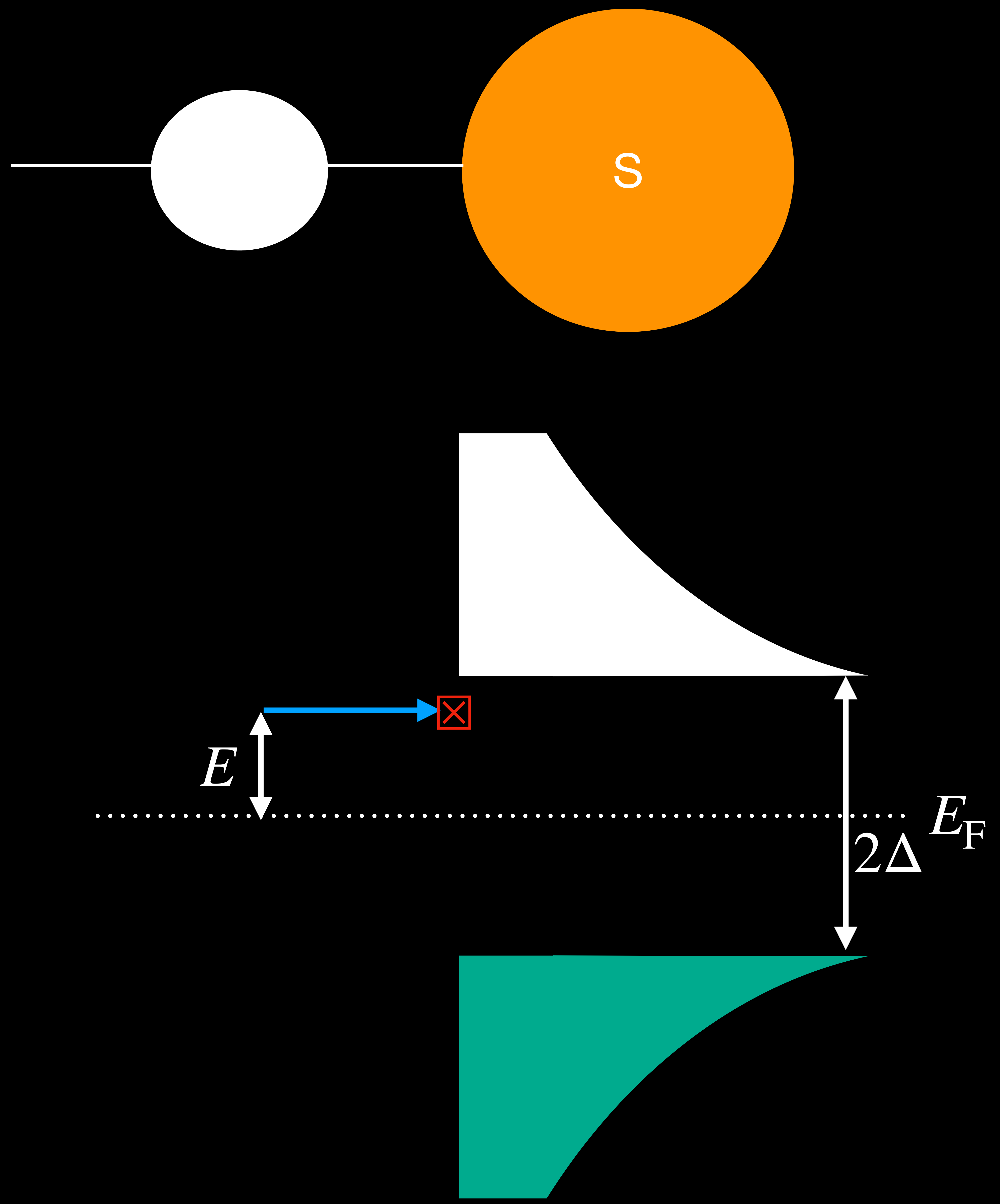
☑ Quasiparticle (QP) tunneling

Andreev reflection



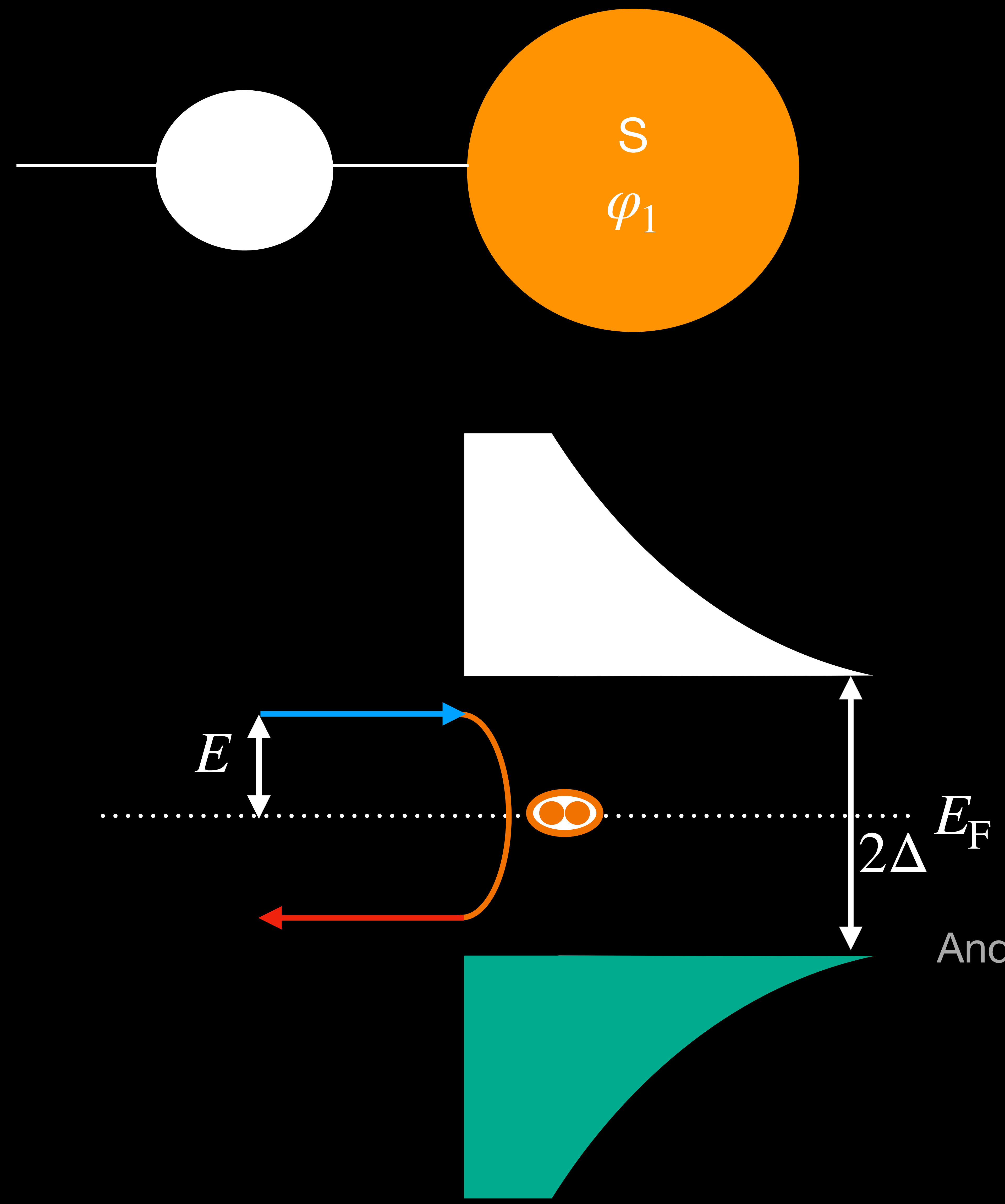
☑ Quasiparticle (QP) tunneling

Andreev reflection



☒ No QP tunneling

Andreev reflection

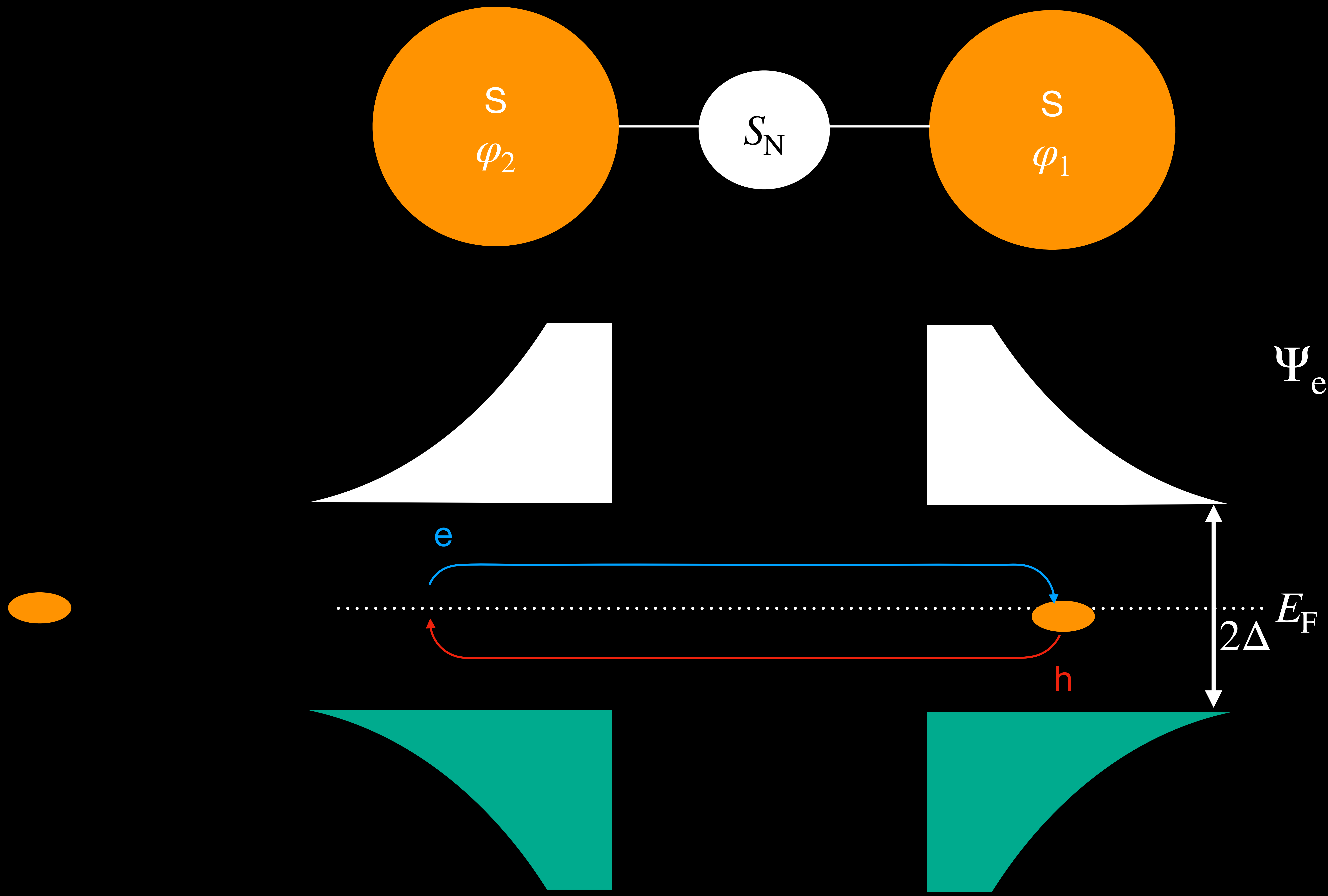


☑ Andreev reflection (AR)

Andreev, A. F., Sov. Phys. JETP. 19: 1228 (1964)

$$\Psi_h \propto \alpha(E) e^{\imath\varphi_1} \Psi_e$$

Supercurrent

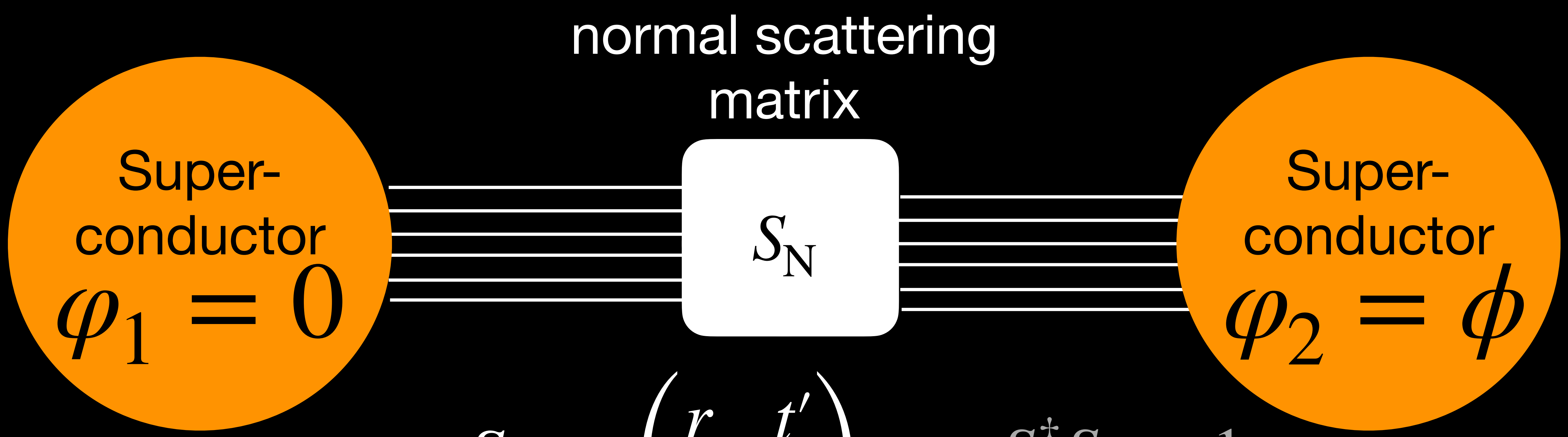


$$\Psi_e = \alpha^2 e^{-\imath\vec{\varphi}} S_N^*(-E) e^{\imath\vec{\varphi}} S_N(E) \Psi_e$$

$$S_N = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

$$e^{\imath\vec{\varphi}} = \begin{pmatrix} e^{\imath\varphi_2} & \\ & e^{\imath\varphi_1} \end{pmatrix}$$

2-terminal Josephson junction (JJ): Andreev bound states



$$S_{\mathrm{N}} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \qquad S_{\mathrm{N}}^{\dagger} S_{\mathrm{N}} = 1$$

$r,\ t,\ t',\ r'$ $\quad n \times n$ **matrices**

Transmission matrix with *n* transmission eigenvalues

$$T = tt^{\dagger} \qquad \{T_i\}_{i=1,\ldots,n}, \quad 0 \leq T_i \leq 1$$

Andreev Bound state spectrum (ABS):

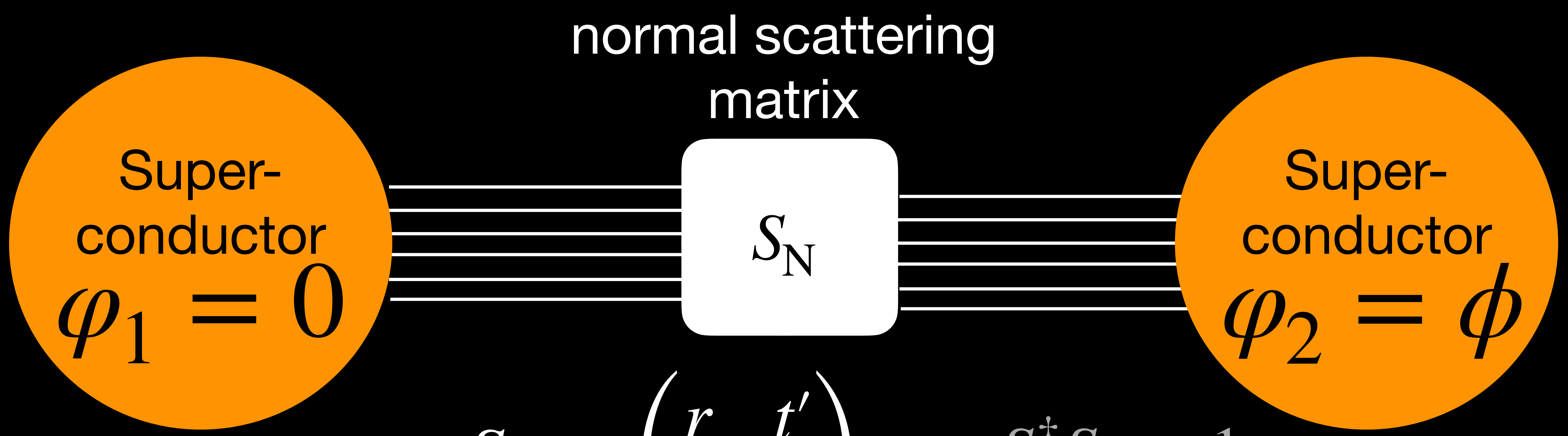
$$E^i_{\mathrm{ABS}}(\phi) = \pm |\Delta| \sqrt{1 - T_i \sin^2(\phi/2)}$$

Supercurrent:

$$I \propto \partial_{\phi} E_{\mathrm{ABS}}$$

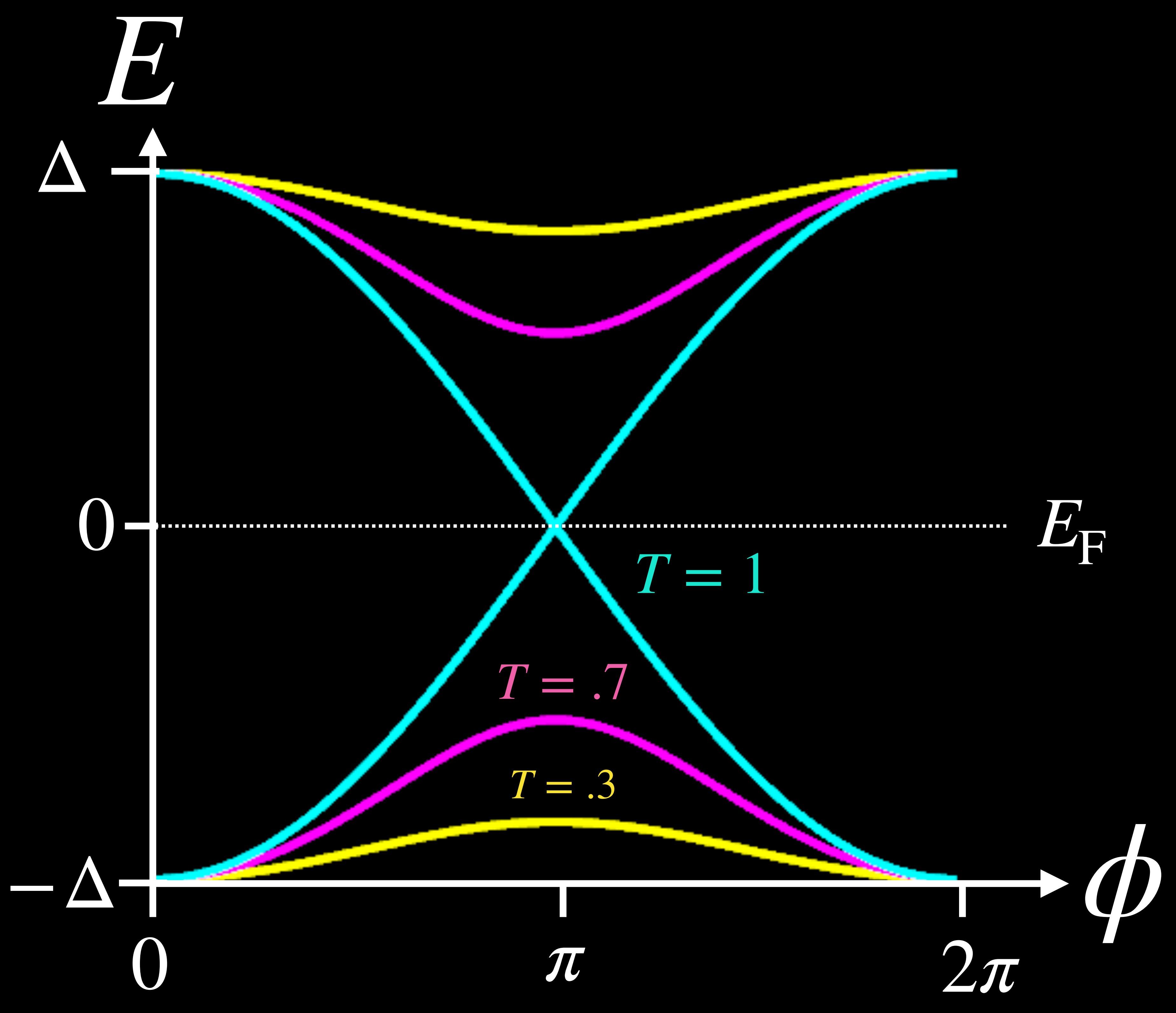
C. W. J. Beenakker and H. van Houten, Phys. Rev. Lett. 66, 3056 (1991)

2-terminal Josephson junction (JJ): Andreev bound states



$$S_N = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \qquad S_N^\dagger S_N = 1$$

$r,\ t,\ t',\ r'$ $\quad n \times n$ **matrices**



Transmission matrix with *n* transmission eigenvalues

$$T = tt^\dagger \qquad \{T_i\}_{i=1,\ldots,n}, \quad 0 \le T_i \le 1$$

Andreev Bound state spectrum (ABS):

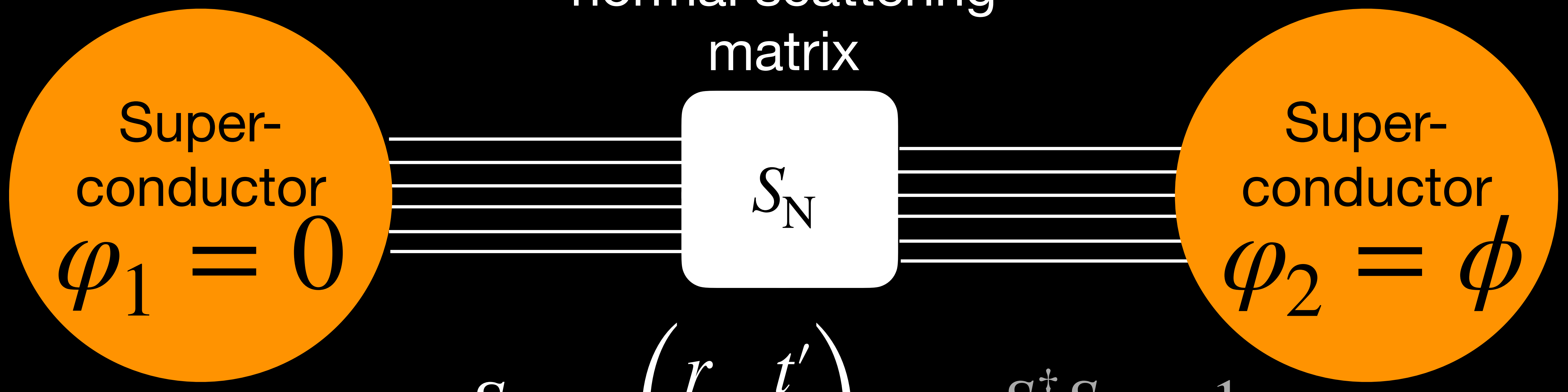
$$E^i_{\mathrm{ABS}}(\phi) = \pm|\Delta|\sqrt{1 - T_i \sin^2(\phi/2)}$$

Supercurrent:

$$I \propto \partial_\phi E_{\mathrm{ABS}}$$

C. W. J. Beenakker and H. van Houten, Phys. Rev. Lett. 66, 3056 (1991)

2-terminal Josephson junction (JJ): Andreev bound states



$$S_N = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \qquad S_N^\dagger S_N = 1$$

$r,\ t,\ t',\ r'$ $\quad n \times n$ **matrices**

Transmission matrix with *n* transmission eigenvalues

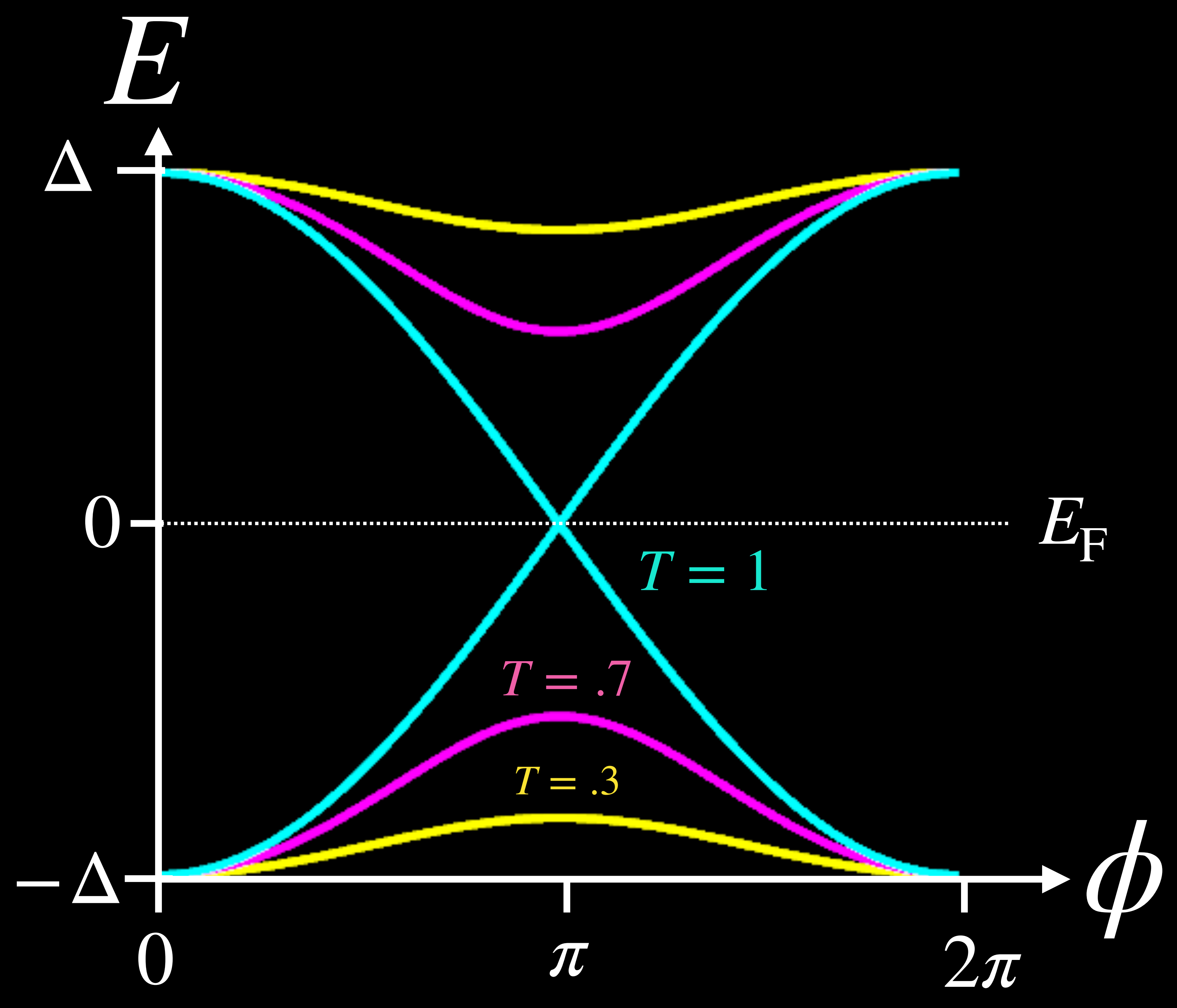
$$T = tt^\dagger \qquad \{T_i\}_{i=1,\ldots,n}, \quad 0 \le T_i \le 1$$

Andreev Bound state spectrum (ABS):

$$E^i_{\text{ABS}}(\phi) = \pm |\Delta| \sqrt{1 - T_i \sin^2(\phi/2)}$$

Supercurrent:

$$I \propto \partial_\phi E_{\text{ABS}}$$



Zero energy $\Leftrightarrow T_i = 1$

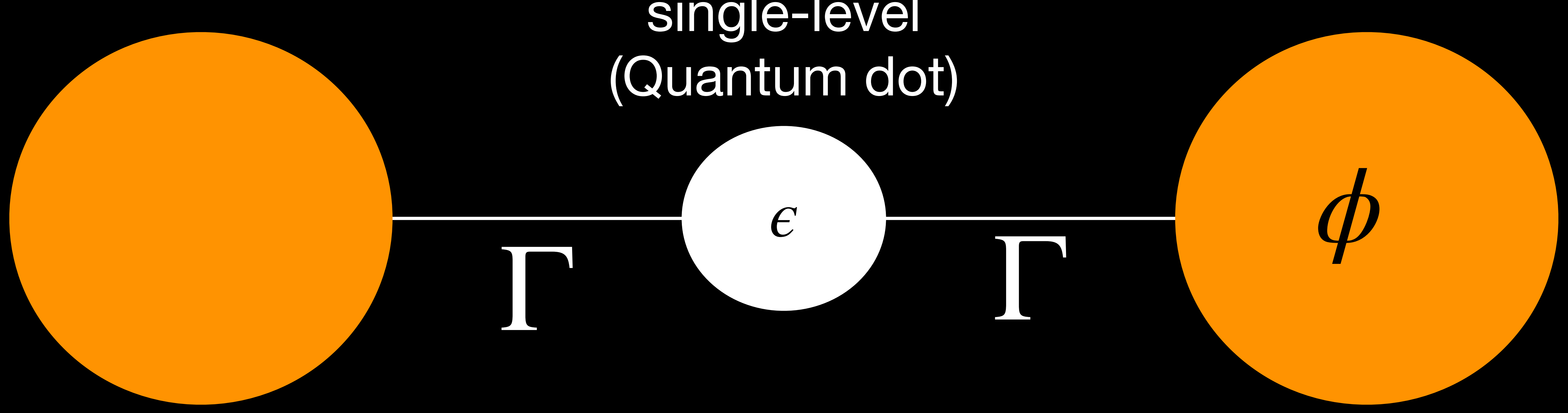
OR: $\det(r) = 0$

"Reflectionless scattering mode"

C. W. J. Beenakker and H. van Houten, Phys. Rev. Lett. 66, 3056 (1991)

2-terminal Josephson junction (JJ): Andreev bound states

Example: 1-dot 2-terminal JJ



normal S-matrix

$$S_{\mathrm{N}} = 1 - \frac{\Gamma}{E - \epsilon + 2\imath\Gamma} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Transmission (Breit-Wigner)

$$T(E) = \frac{4\Gamma^2}{(E-\epsilon) + 4\Gamma^2}$$

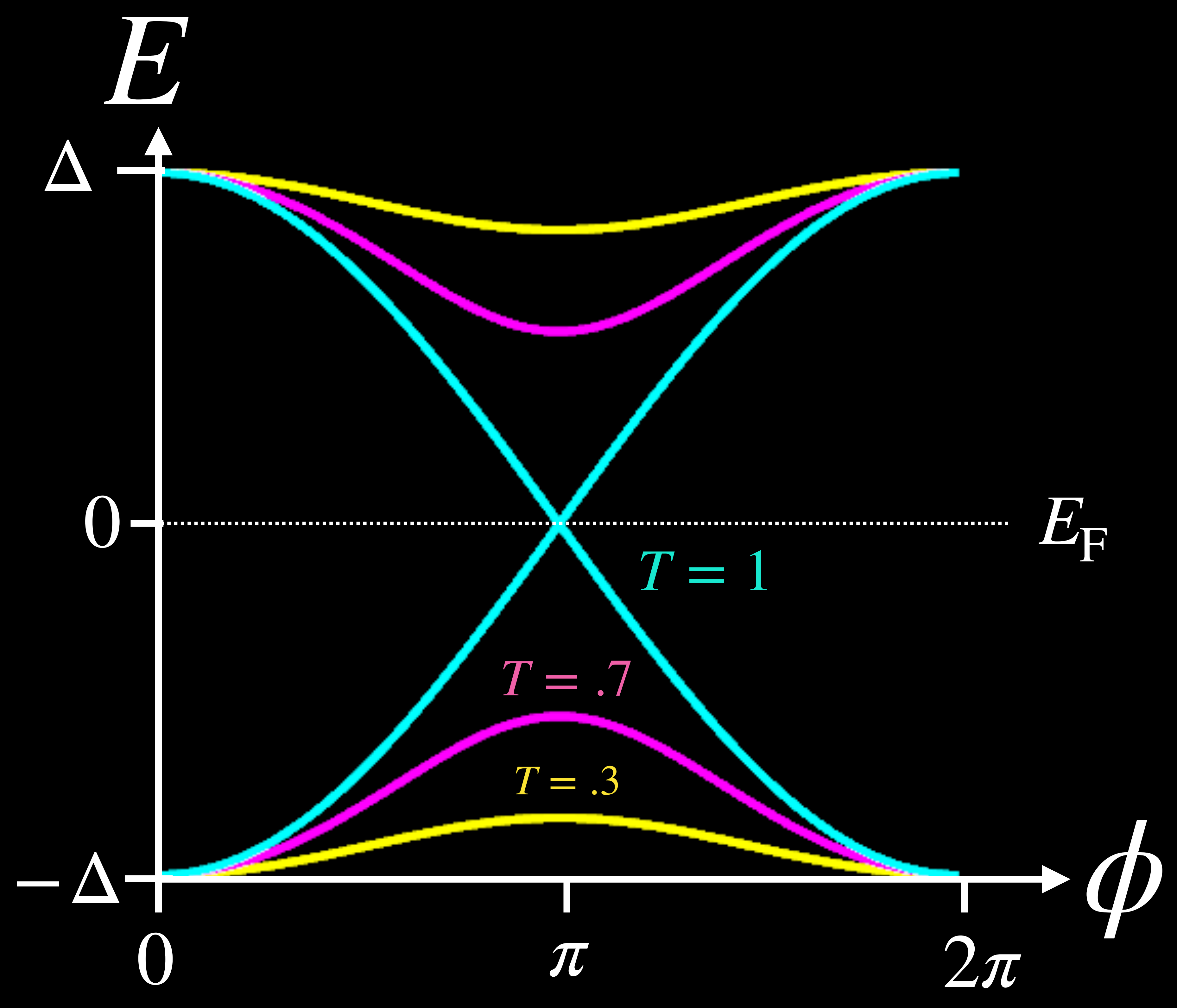
equal to 1…

Reflection

$$\det(r) = 1 - \frac{2\Gamma}{2\Gamma + \imath\epsilon}$$

equal to 0…

…for $\epsilon = 0$!



Zero energy $\Leftrightarrow T_i = 1$

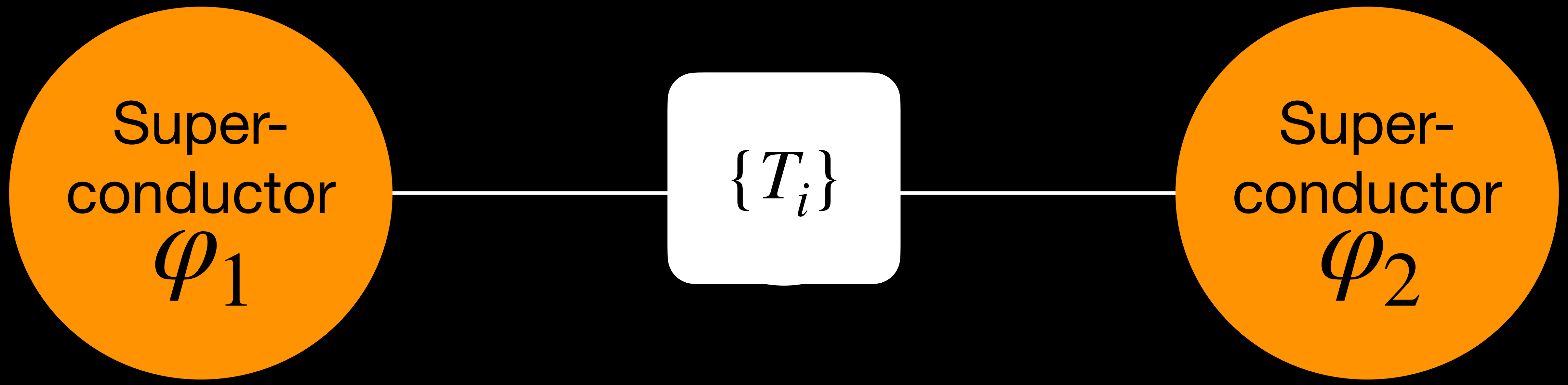
OR: $\det(r) = 0$

“Reflectionless scattering mode”

C. W. J. Beenakker and H. van Houten, Phys. Rev. Lett. 66, 3056 (1991)

Multiterminal Josephson junctions (MTJJs)

2-terminal Josephson junctions



1-to-1 correspondence between ABSs and transmission eigenvalues

$$E^i_{\mathrm{ABS}} \leftrightarrow T_i$$

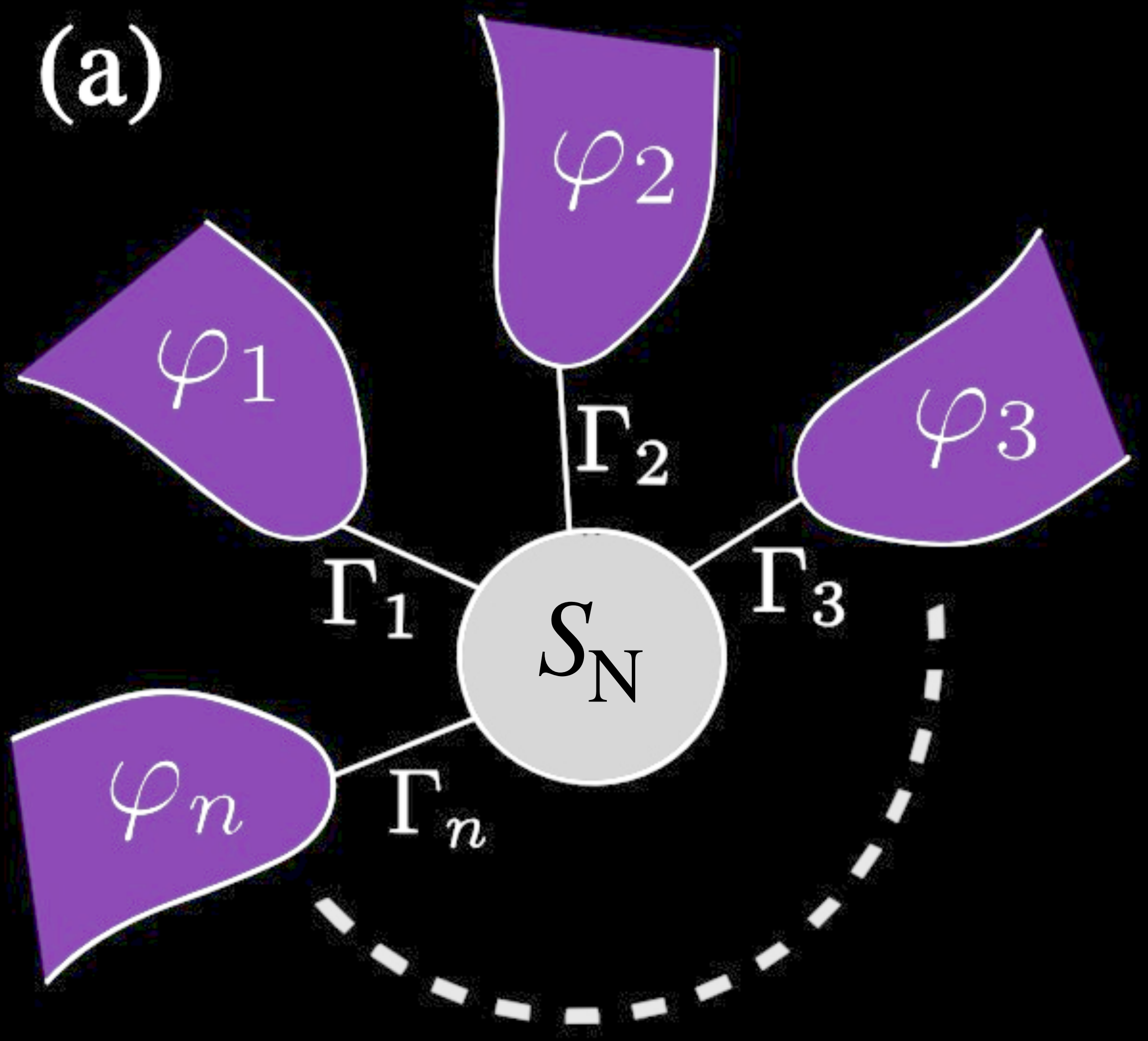
One superconducting phase ϕ

$$E_{\mathrm{ABS}}(T_i, \phi)$$

Transmissions are Kramers degenerate (without TRS breaking in normal state)

B. van Heck, et. al., Phys. Rev. B 90, 155450 (2014)

MTJJs



no 1-to-1 correspondence!

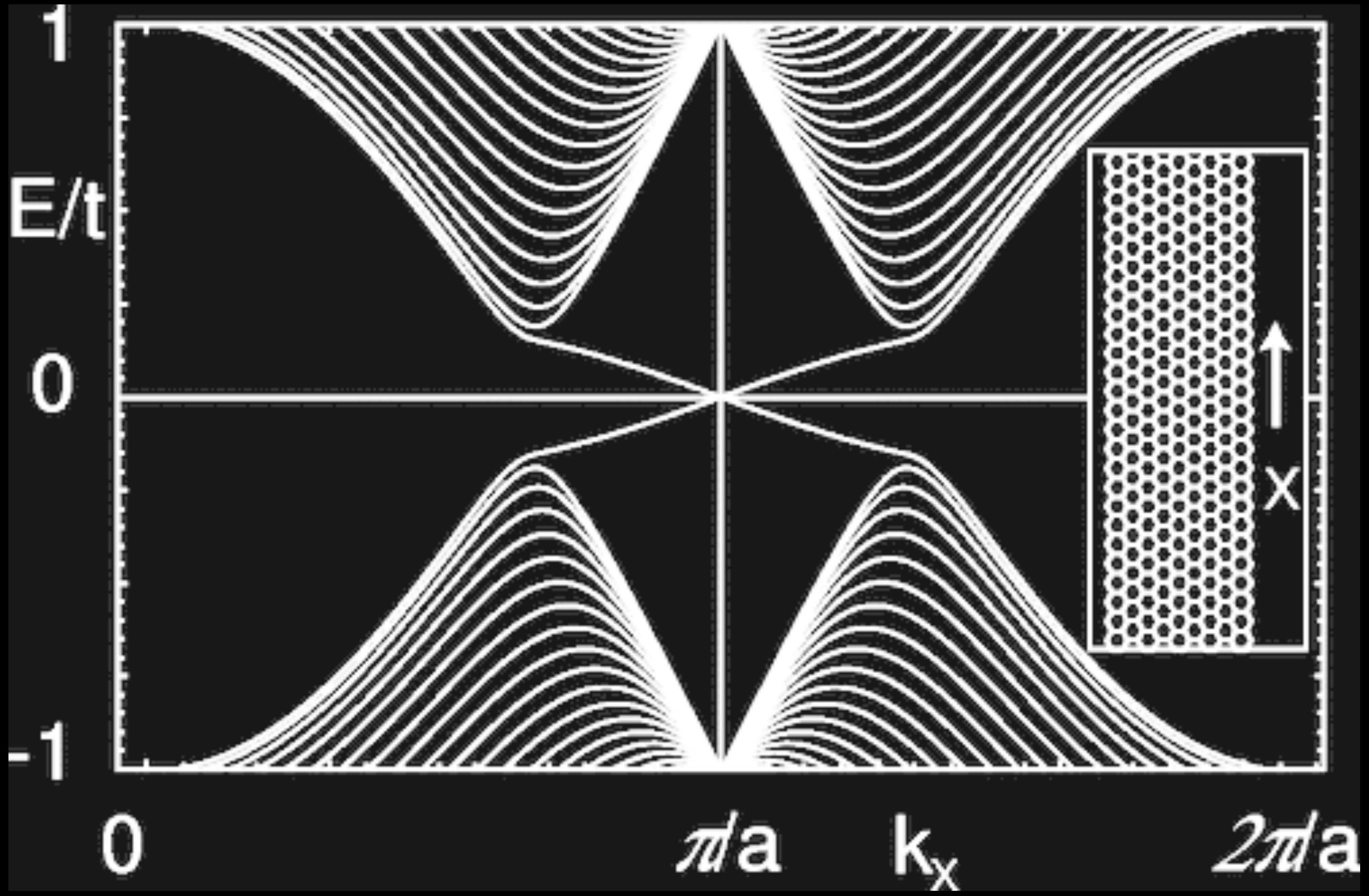
$$\cancel{E^i_{\mathrm{ABS}} \leftrightarrow T_i}$$

$n-1$ superconducting phases $\{\phi_i\}$

$$E_{\mathrm{ABS}}(S, \vec{\phi})$$

Why care about topology?

Topological Phases of Matter



C. L. Kane and E. J. Mele Phys. Rev. Lett. 95, 226801 (2005)
König, M. et al. Science 318, 766–770 (2007)

Fundamental Insights into Symmetry and Topology

Class	T	C	S	1	2	3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	0	0	0
BDI	1	1	1	$\mathbb{Z}$	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Altland and M. R. Zirnbauer, Phys. Rev. B 55, 1142 (1997)
Andreas P. Schnyder, et. al., Phys. Rev. B 78, 195125 (2008)

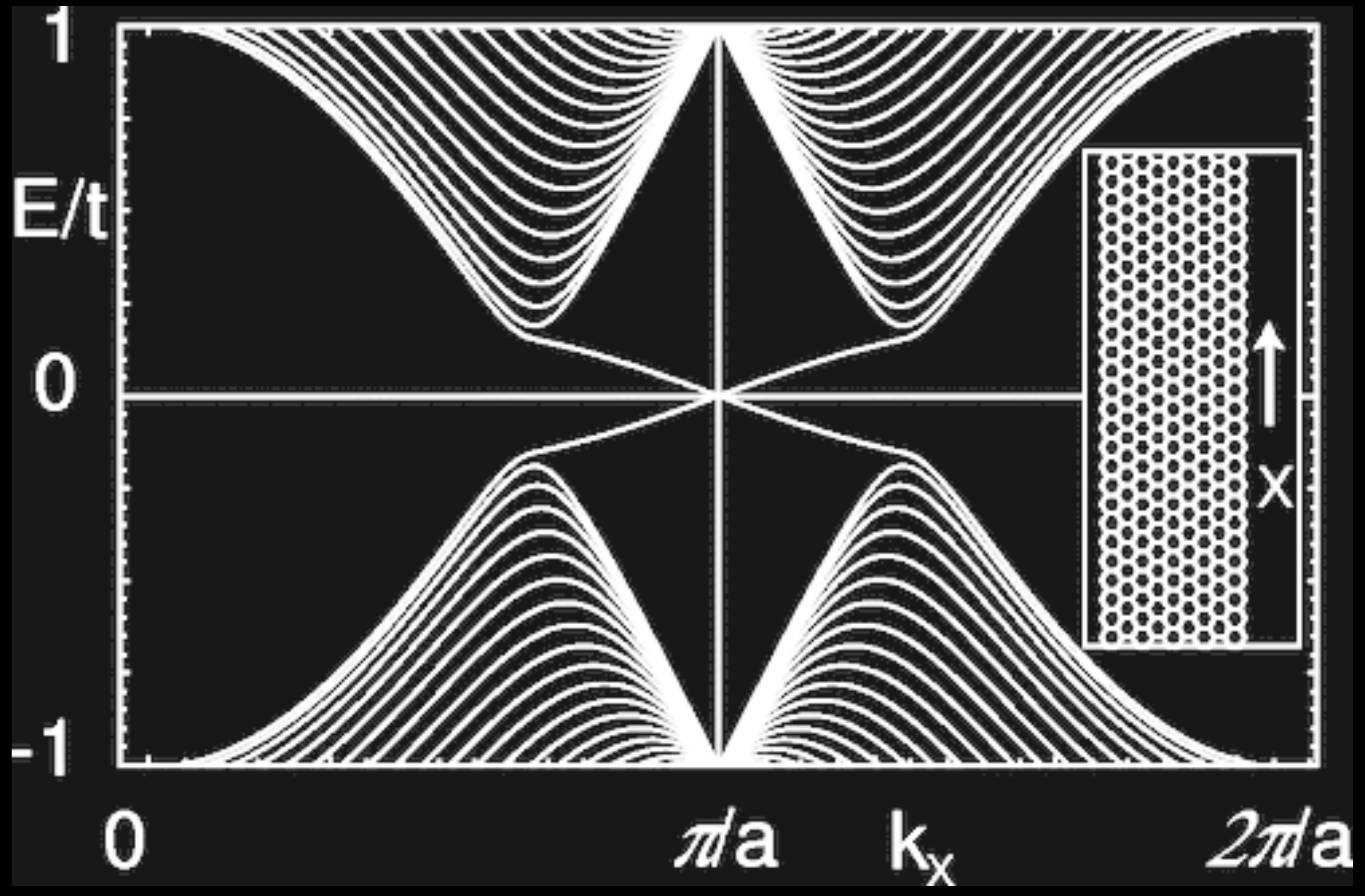
Stability Against Perturbations

Applications in Quantum Computing and Electronics

...

Why care about topology?

Topological Phases of Matter



C. L. Kane and E. J. Mele Phys. Rev. Lett. 95, 226801 (2005)
König, M. et al. Science 318, 766–770 (2007)

Fundamental Insights into Symmetry and Topology

Class	T	C	S	1	2	3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	0	0	0
BDI	1	1	1	$\mathbb{Z}$	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Altland and M. R. Zirnbauer, Phys. Rev. B 55, 1142 (1997)
Andreas P. Schnyder, et. al., Phys. Rev. B 78, 195125 (2008)

Stability Against Perturbations

Applications in Quantum Computing and Electronics

...

How to define topology (in MTJJs)?

Effective Hamiltonian with eigenenergies and eigenvectors

$$H_{\text{eff}} = \vec{d} \cdot \vec{\sigma} / \ H_{\text{eff}} = -G_{\text{C}}^{-1}(E=0)$$

$$E_i(\vec{k}) / |v_i(\vec{k})\rangle \qquad E_i(\vec{\phi}) / |v_i(\vec{\phi})\rangle$$

Define the Chern number of a band E_i like
for topological insulators

$$C_i^{jk} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} B_i^{jk}(\vec{k}) dk_j dk_k \qquad C_i^{jk} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} B_i^{jk}(\vec{\phi}) d\phi_j d\phi_k$$

Berry curvature

$$B_i^{jk} = -2\mathrm{Im}\langle \partial_{k_j} v_i | \partial_{k_k} v_i \rangle \qquad B_i^{jk} = -2\mathrm{Im}\langle \partial_{\phi_j} v_i | \partial_{\phi_k} v_i \rangle$$

Intuition: Chern Numer

Xiao-Liang Qi, et. al., Phys. Rev. B 74, 085308 (2006)

Example: Qi-Wu-Zhang Modell

$$H_{\text{eff}} = \vec{d} \cdot \vec{\sigma} \qquad \vec{d} = \begin{pmatrix} \sin k_x \\ \sin k_y \\ m + \cos k_x + \cos k_y \end{pmatrix}$$

Algebraic

$$\vec{e} = \vec{d}/|\vec{d}|$$

Spectral

Topological phase transition for

$E_\pm = |\vec{d}| = 0$:

Weyl/Dirac point

Topological

Chern number as function of m

$$C = \begin{cases} 1, & 2 > m > 0 \\ -1, & 0 > m > -2 \\ 0, & \text{else} \end{cases}$$

Intuition: Chern Numer

Xiao-Liang Qi, et. al., Phys. Rev. B 74, 085308 (2006)

Example: Qi-Wu-Zhang Modell

$$H_{\text{eff}} = \vec{d} \cdot \vec{\sigma} \qquad \vec{d} = \begin{pmatrix} \sin k_x \\ \sin k_y \\ m + \cos k_x + \cos k_y \end{pmatrix}$$

Algebraic

$$\vec{e} = \vec{d} / |\vec{d}|$$

Spectral

Topological phase transition for

$E_\pm = |\vec{d}| = 0$:

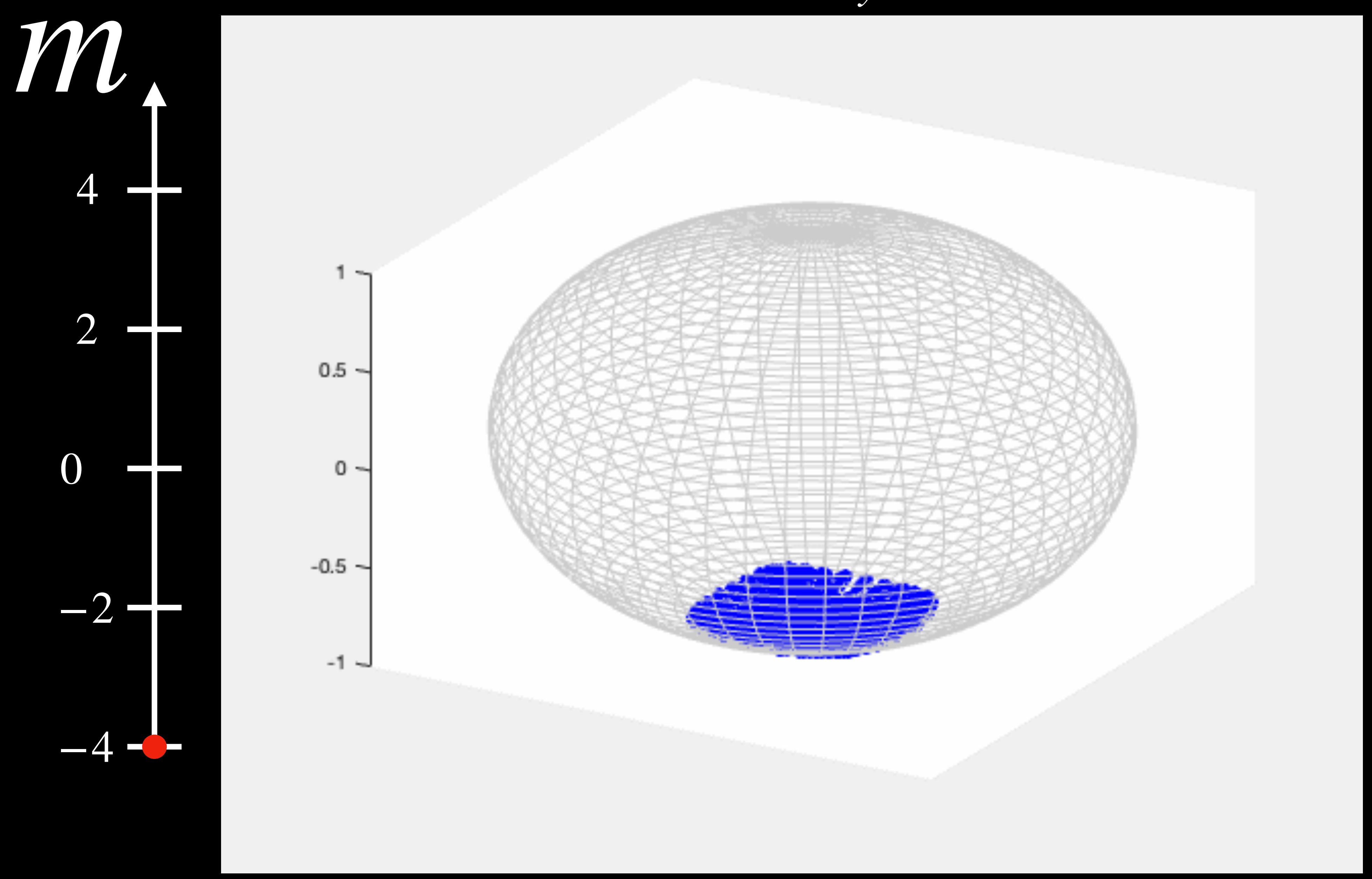
Weyl/Dirac point

Topological

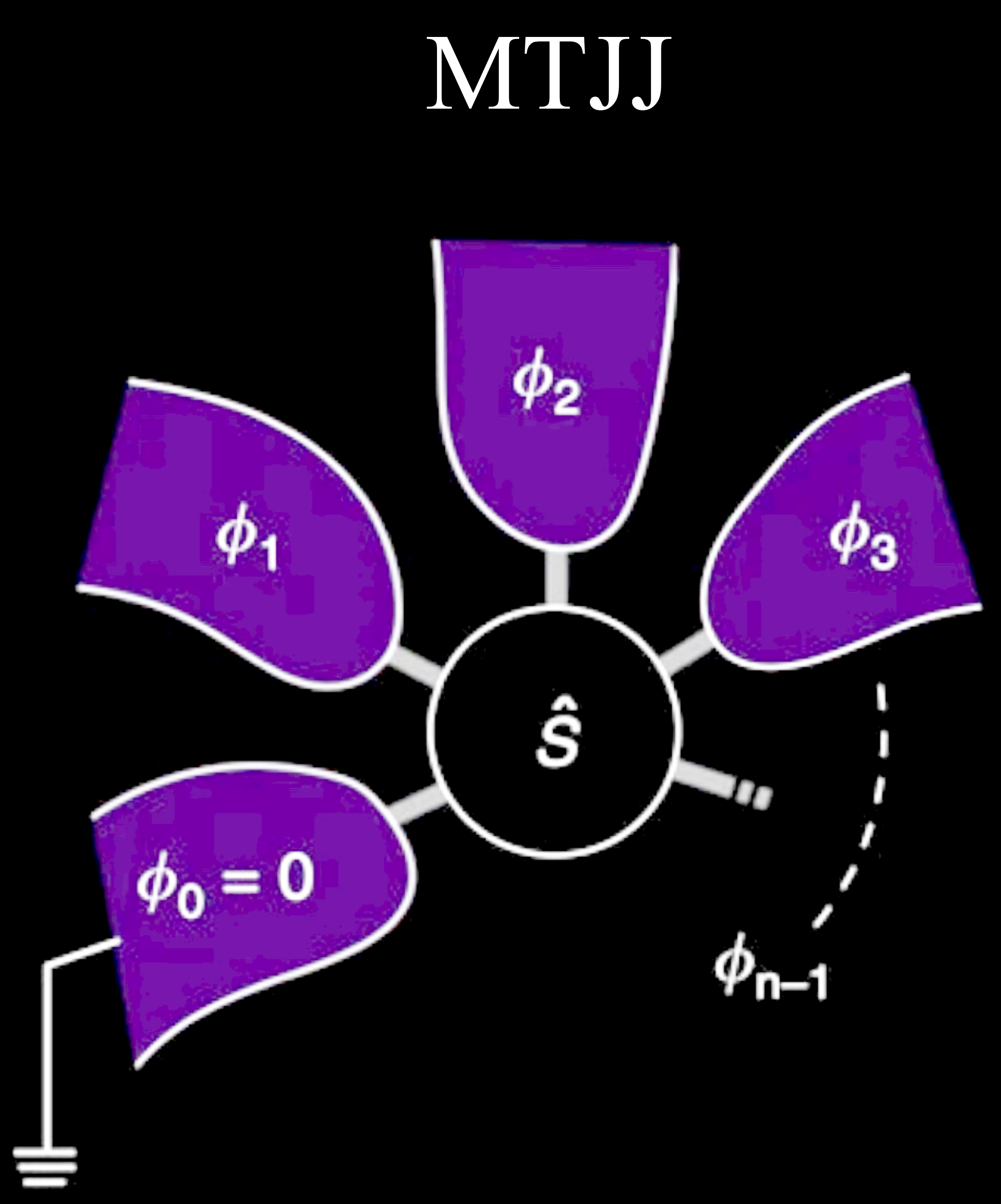
Chern number as function of m

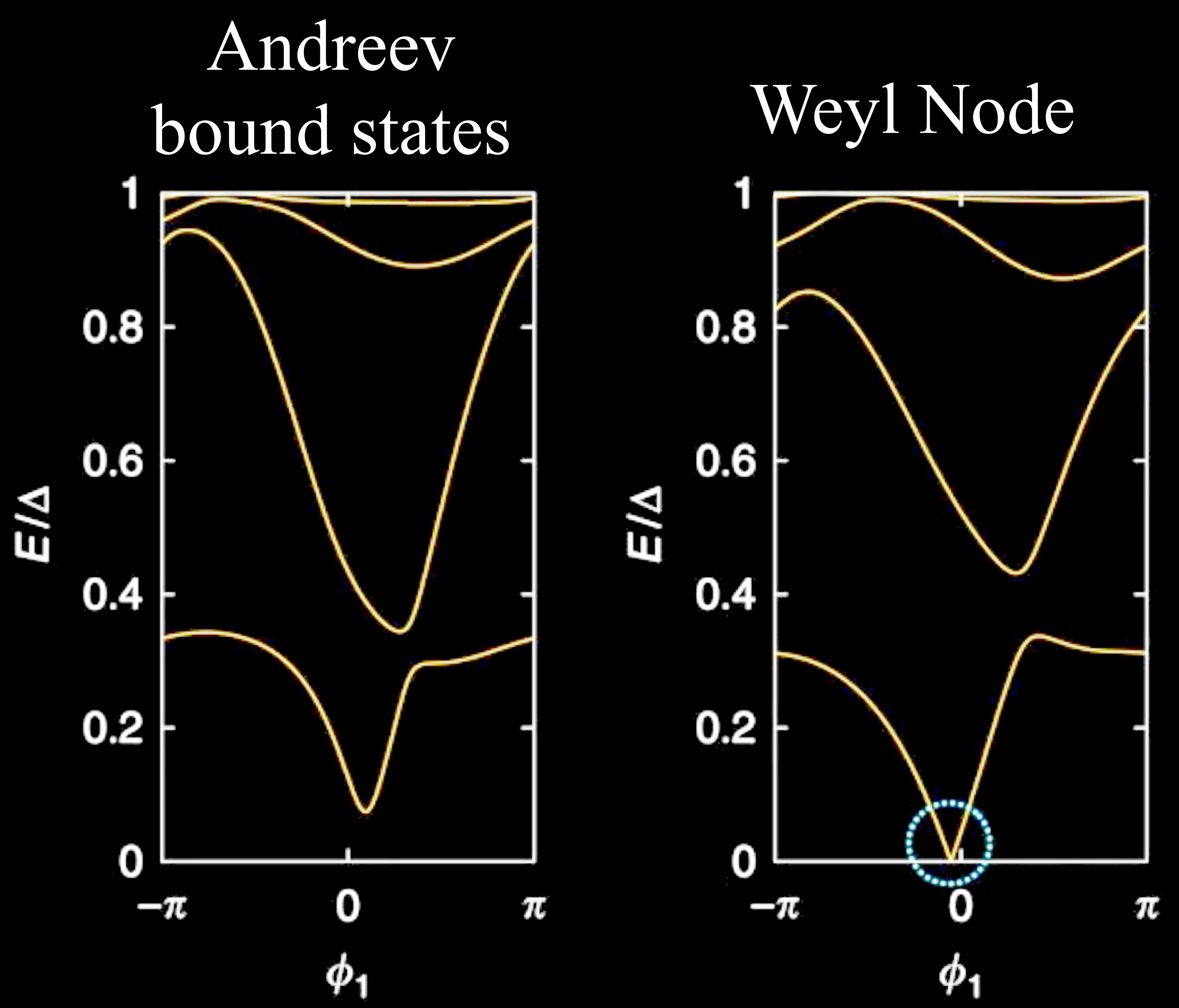
$$C = \begin{cases} 1, & 2 > m > 0 \\ -1, & 0 > m > -2 \\ 0, & \text{else} \end{cases}$$

$\vec{e}(k_x, k_y)\,|_{k_x, k_y \in [0, 2\pi)}$



Topology in Multiterminal Josephson junctions (MTJJs)



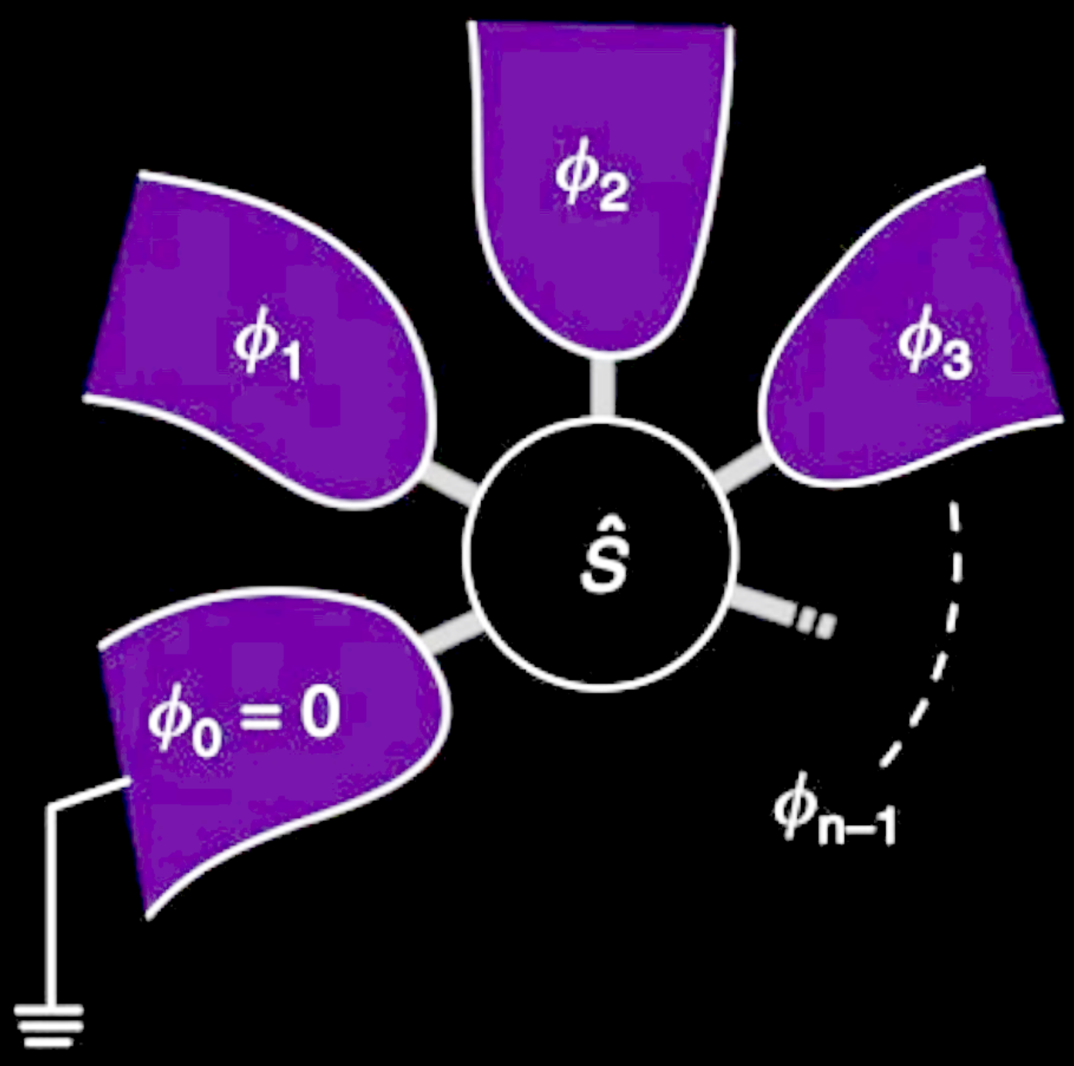


Superconducting phases function as pseudo momenta

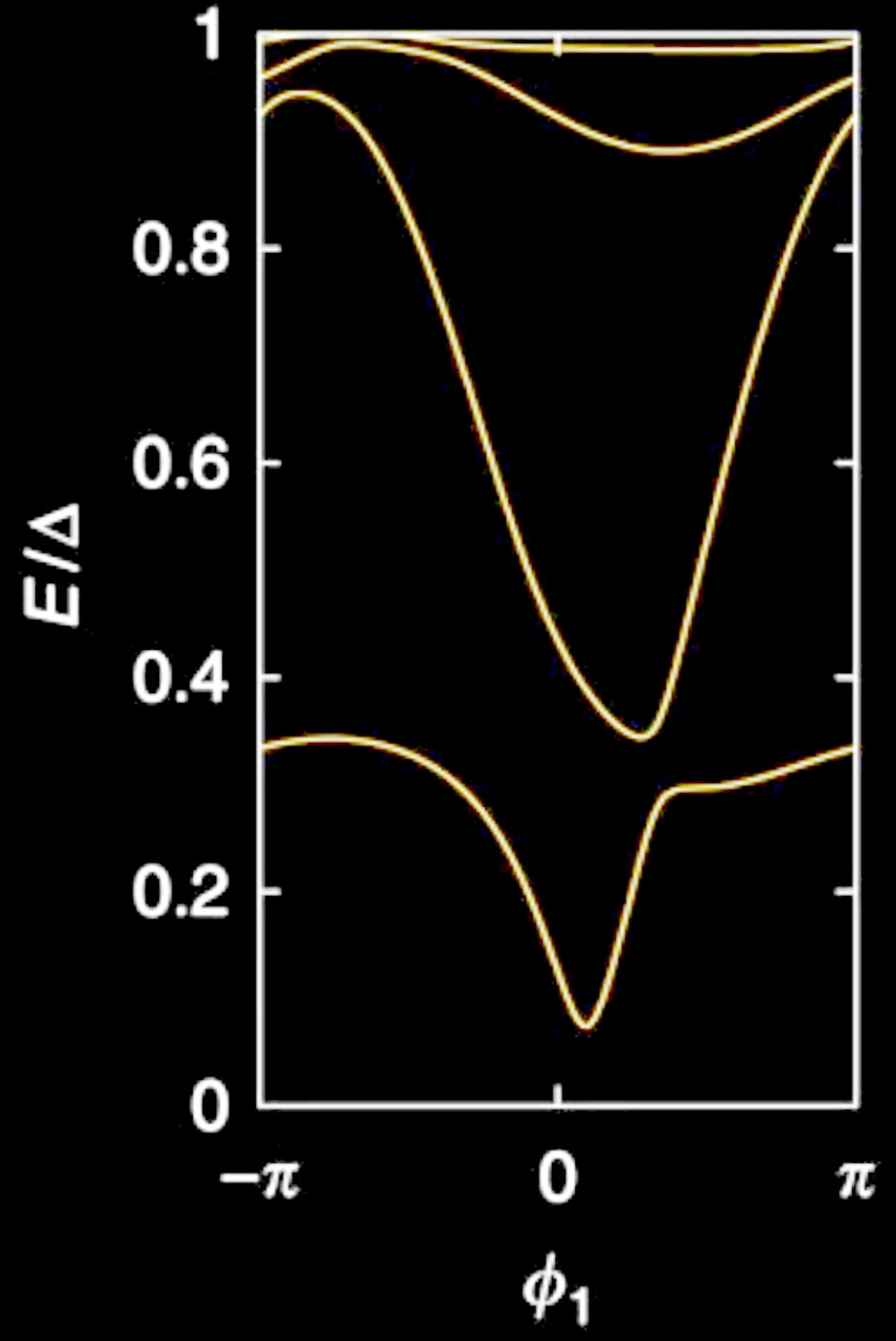
R.-P. Riwar, et. al., Nature Commun. 7, 1 (2016)
E. Eriksson, et. al., Phys. Rev. B 95, 075417 (2017)
H.-Y. Xie, et. al., Phys. Rev. B 96, 161406(R) (2017)
R. L. Klees, et. al., Phys. Rev. Lett. 124, 197002 (2020)
…

Topology in Multiterminal Josephson junctions (MTJJs)

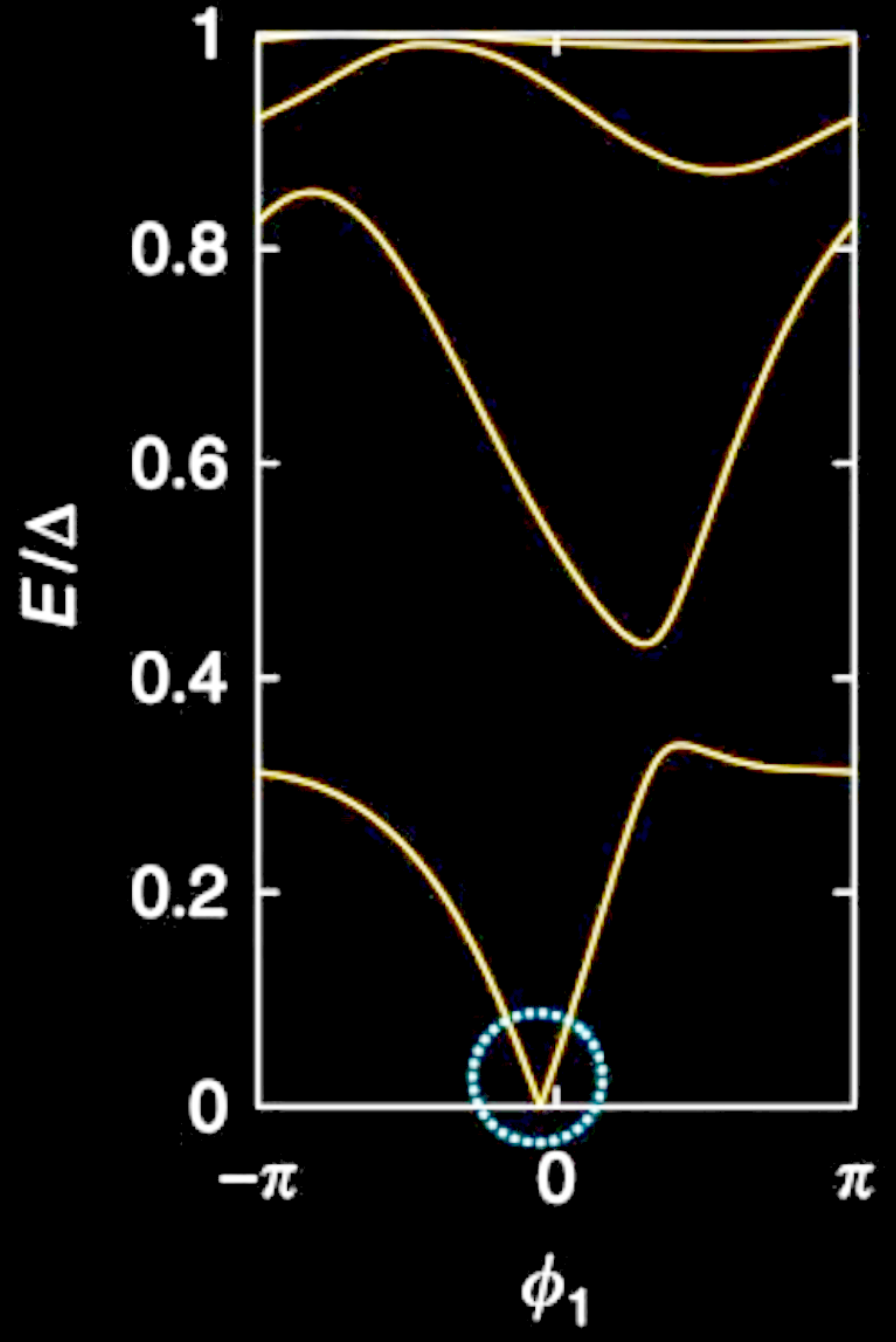
MTJJ



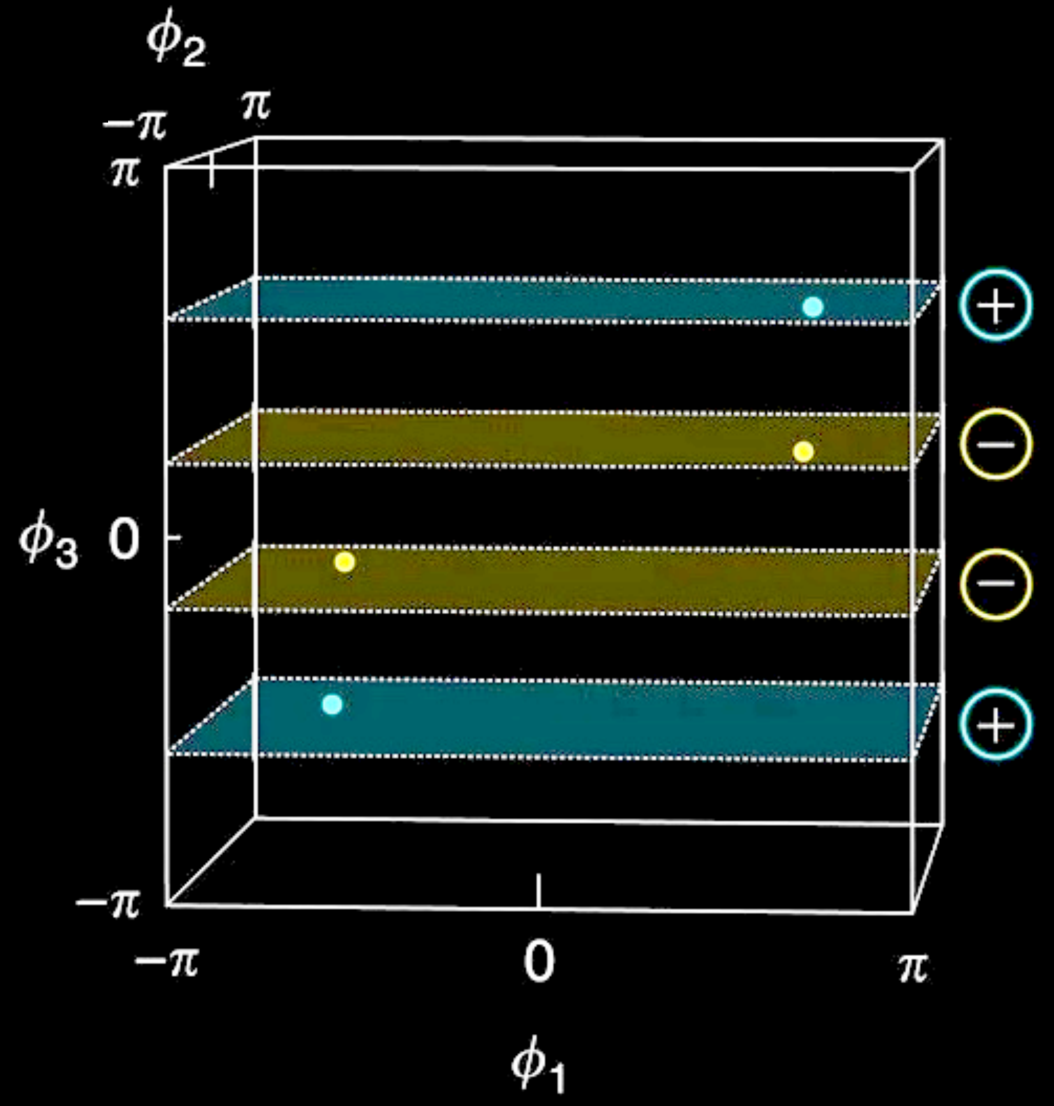
Andreev bound states



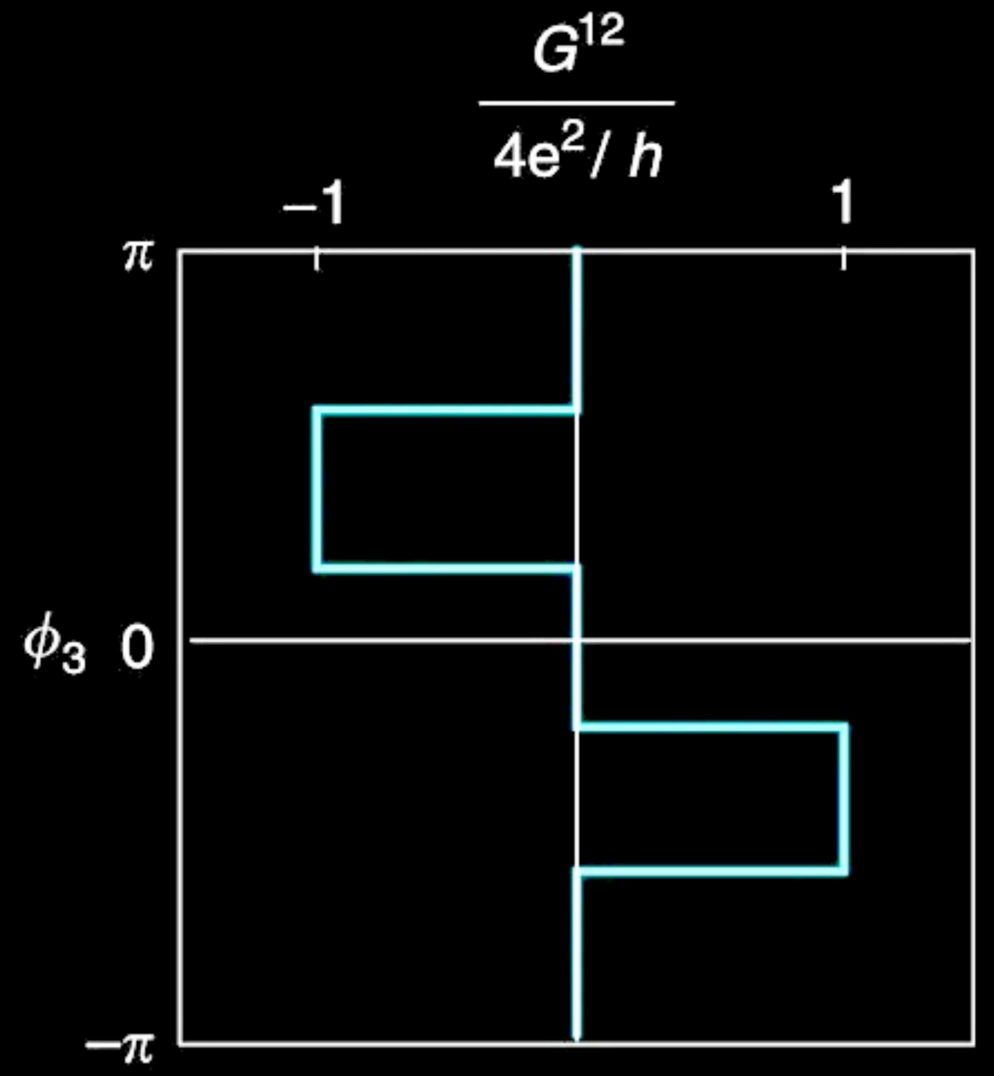
Weyl Node



Weyl Nodes in phase space



Transconductance (Chern number)

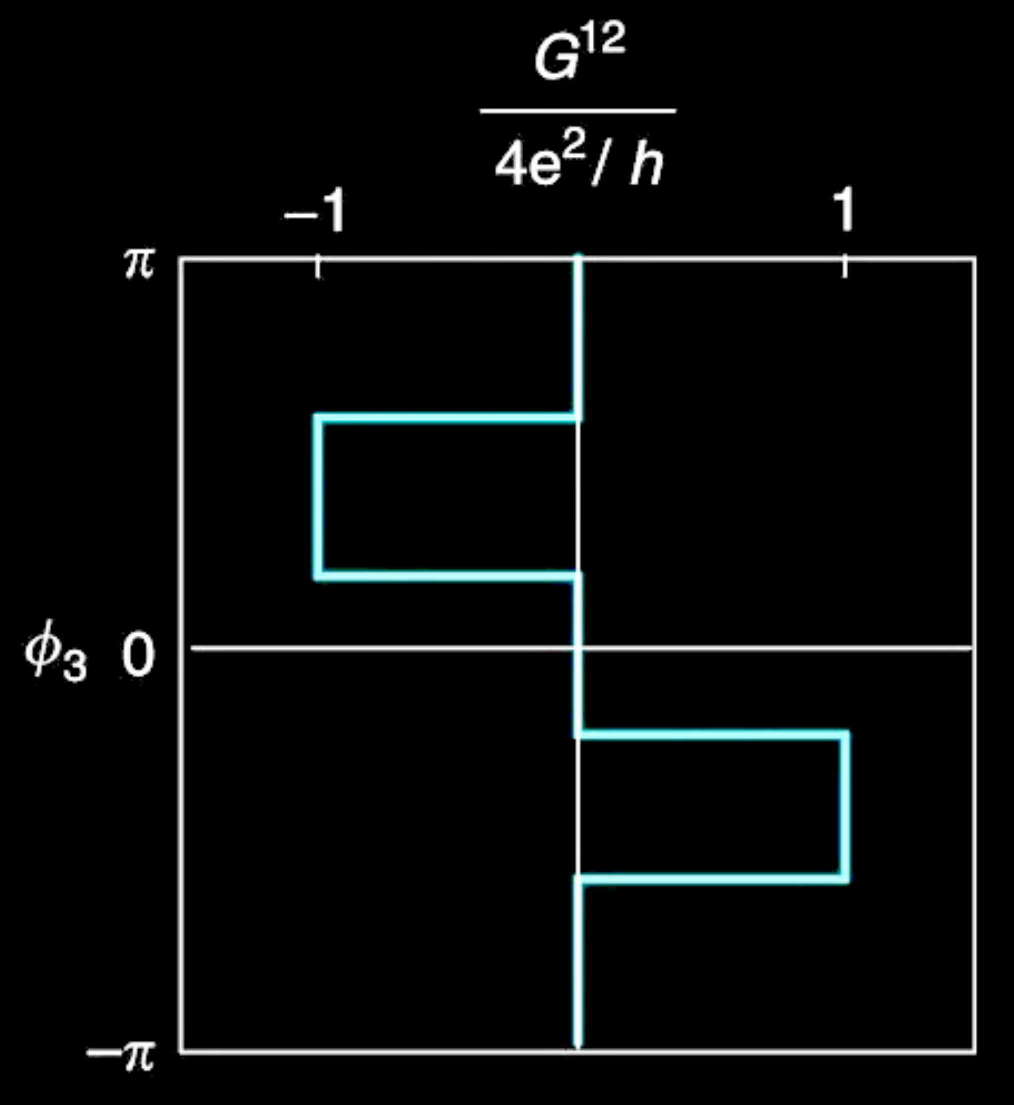


R.-P. Riwar, et. al., Nature Commun. 7, 1 (2016)
E. Eriksson, et. al., Phys. Rev. B 95, 075417 (2017)
H.-Y. Xie, et. al., Phys. Rev. B 96, 161406(R) (2017)
R. L. Klees, et. al., Phys. Rev. Lett. 124, 197002 (2020)
…

Superconducting phases function as pseudo momenta

Topology in Multiterminal Josephson junctions (MTJJs)

Transconductance (Chern number)



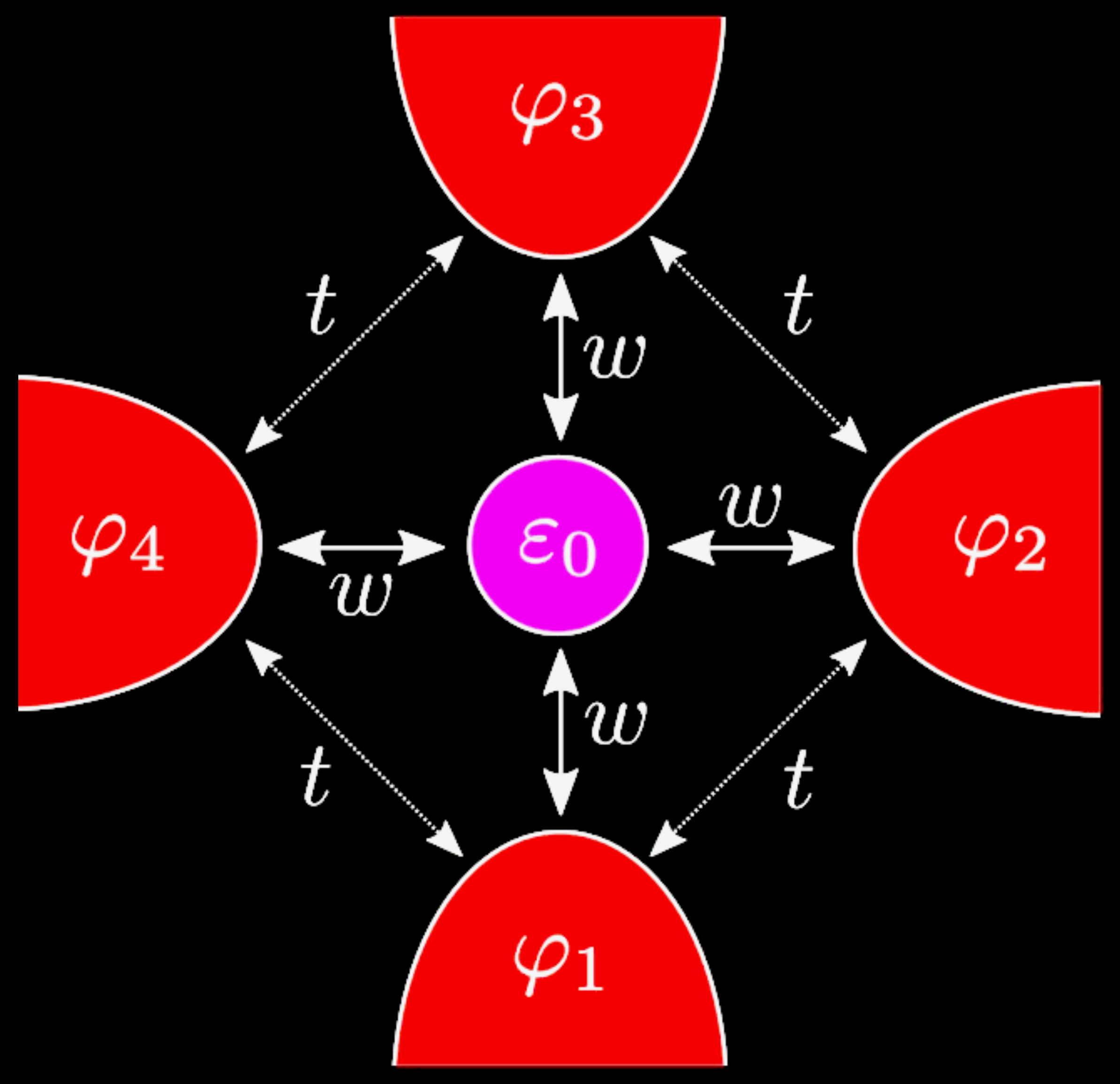
$$I_i(t) = \frac{2e}{\hbar}\partial_{\phi_i} E_{\mathrm{ABS}} - 2e\dot{\phi}_j B^{ij}$$

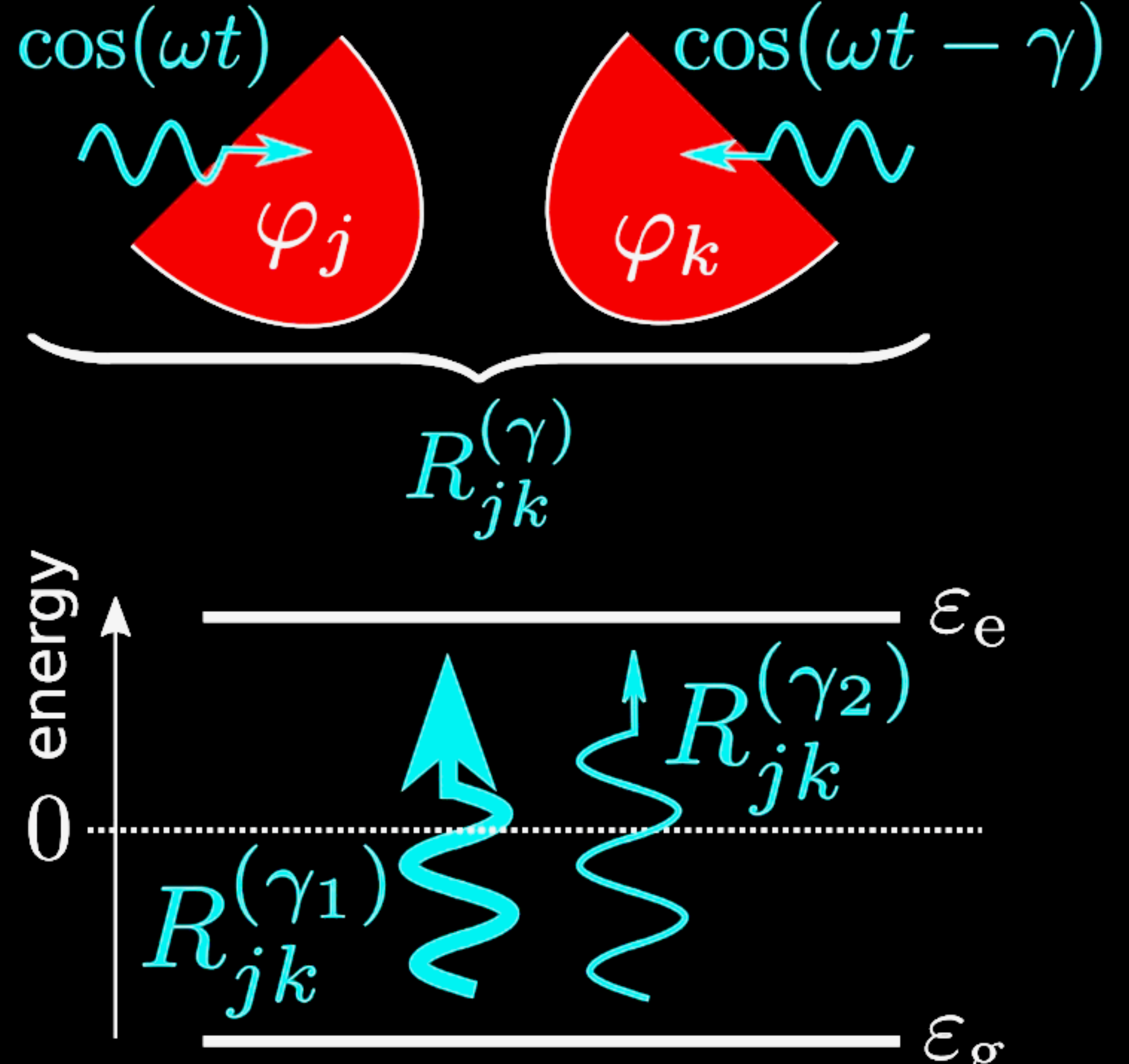
Incommensurate voltages:

$$\langle I_i \rangle = -\frac{4e^2}{h} C^{ij} V_j$$

R.-P. Riwar, et. al., Nature Commun. 7, 1 (2016)

Microwave spectroscopy





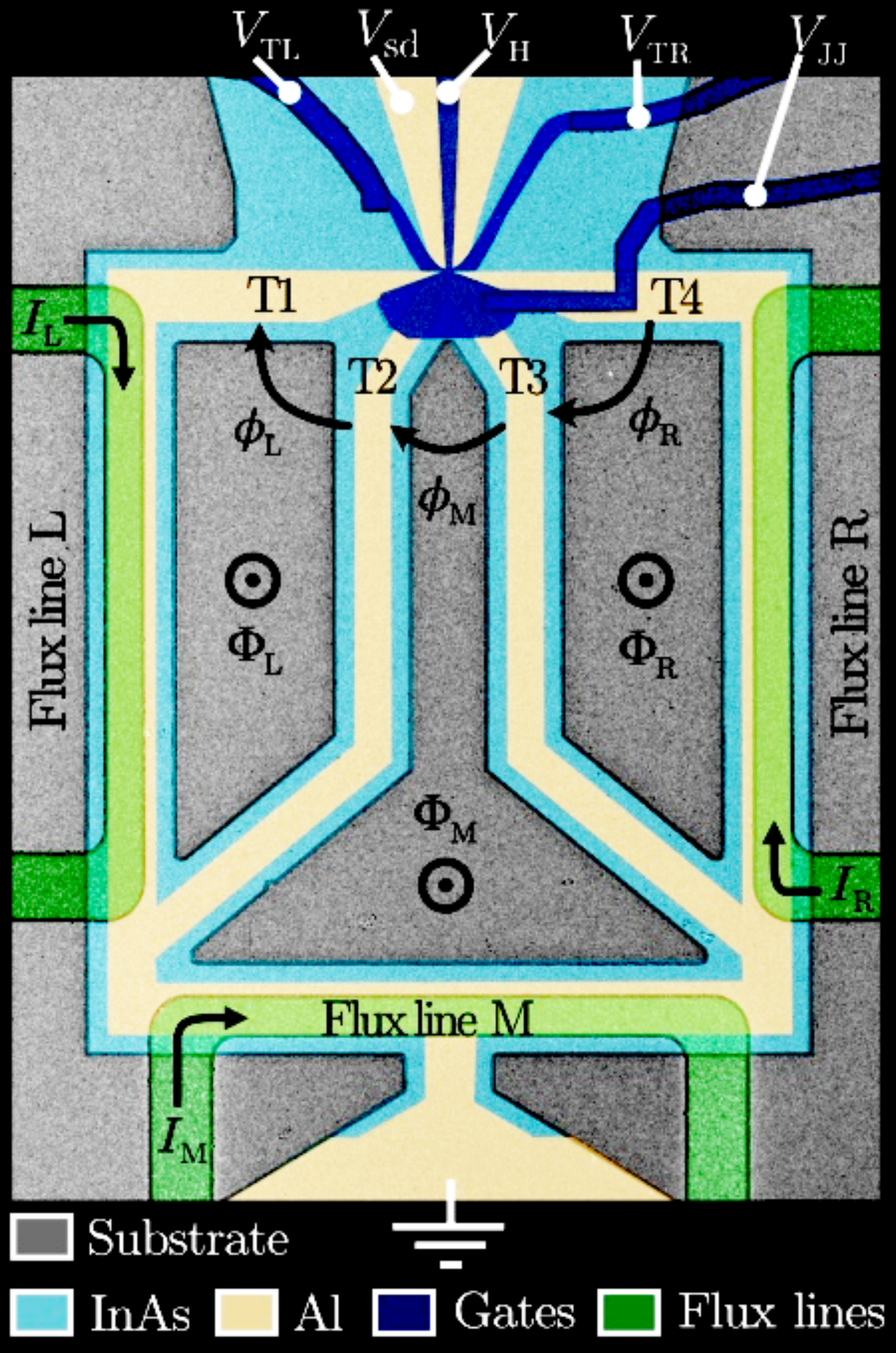
Access to the quantum geometric tensor

$$\chi_{jk} = \langle \partial_j g | (1 - |g\rangle\langle g|) | \partial_k g \rangle$$

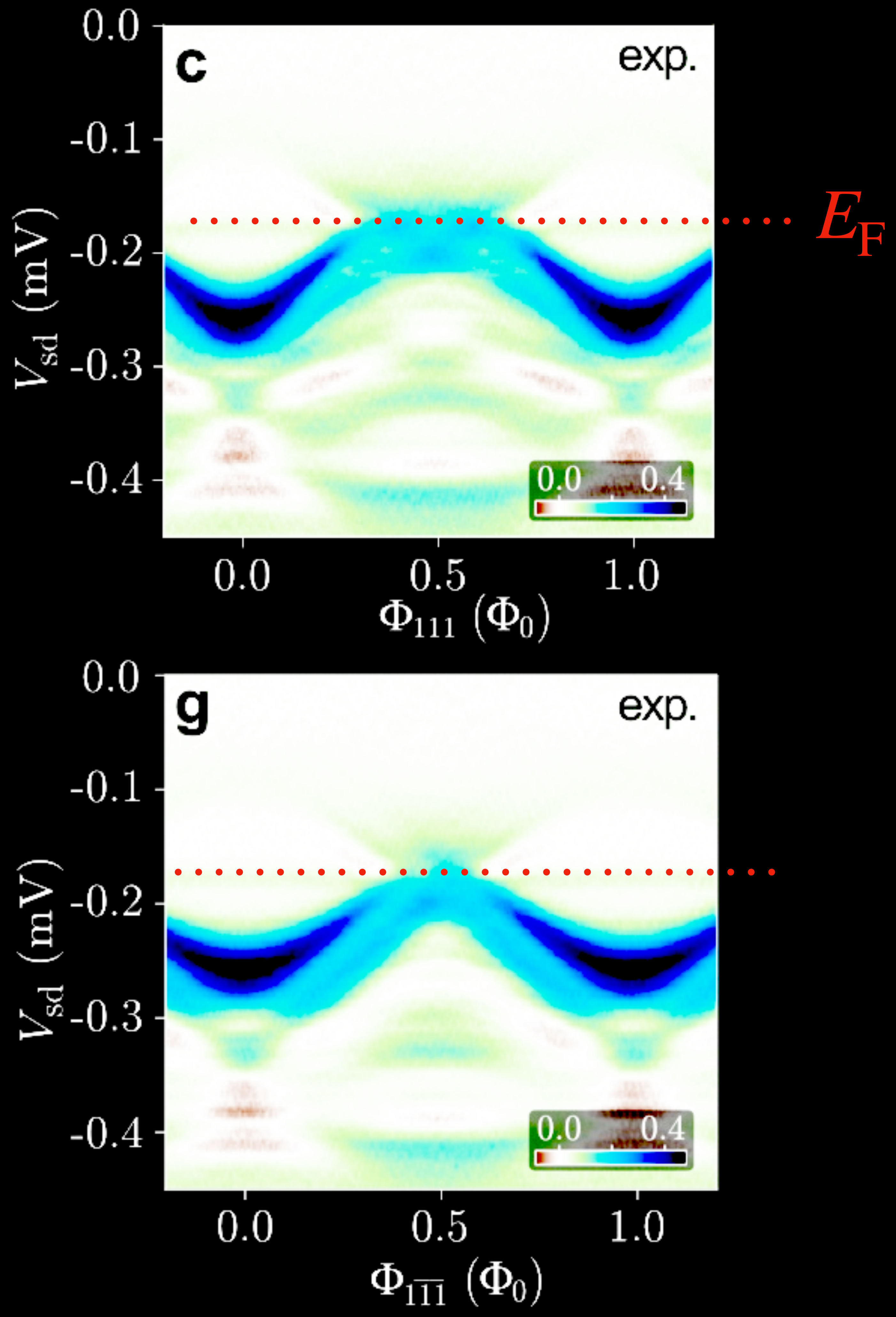
R. L. Klees, et. al., Phys. Rev. Lett. 124, 197002 (2020)

Experiments on MTJJs: Topology?

Topography of
4-terminal MTJJ



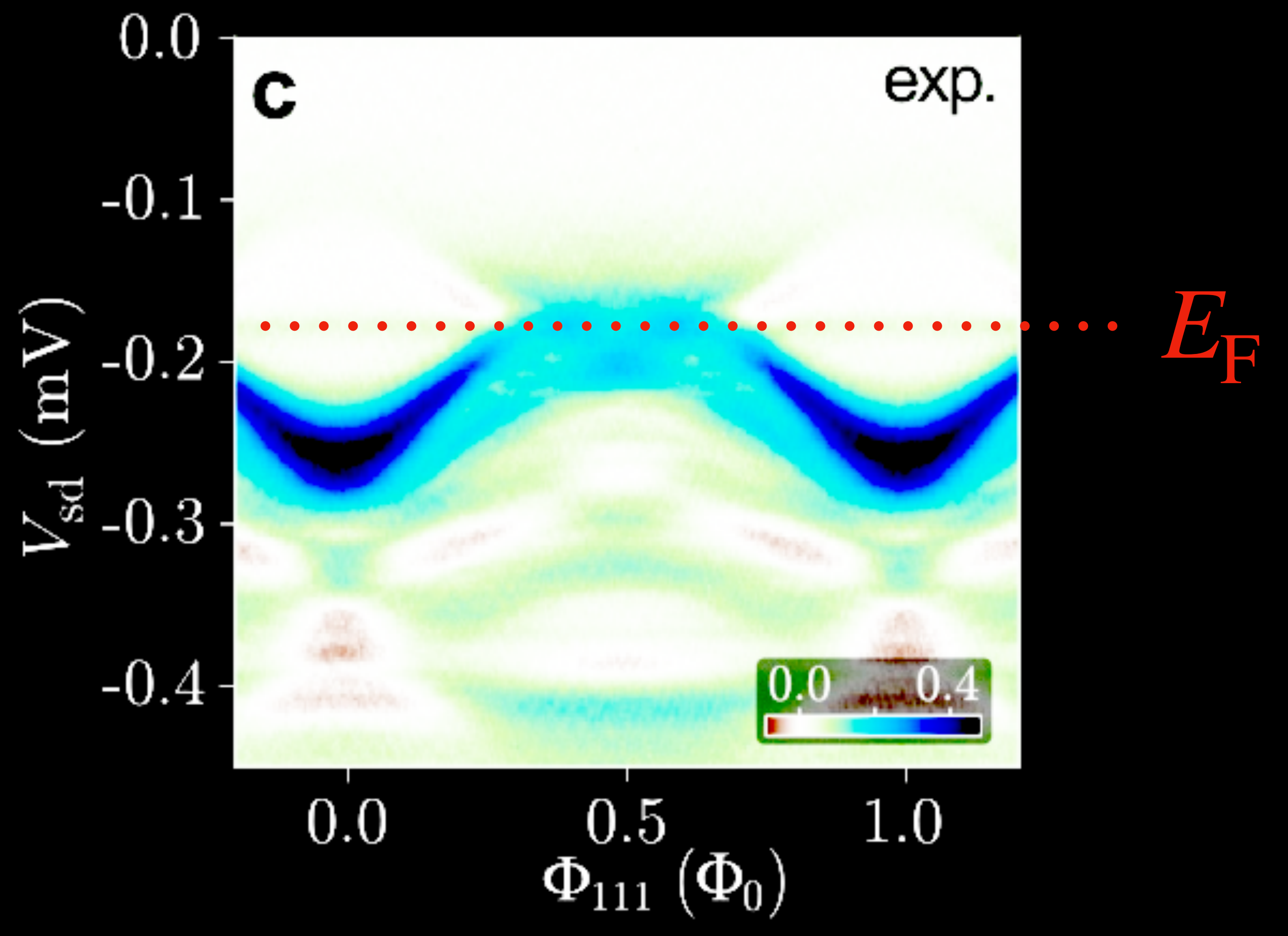
Tunneling spectroscopy

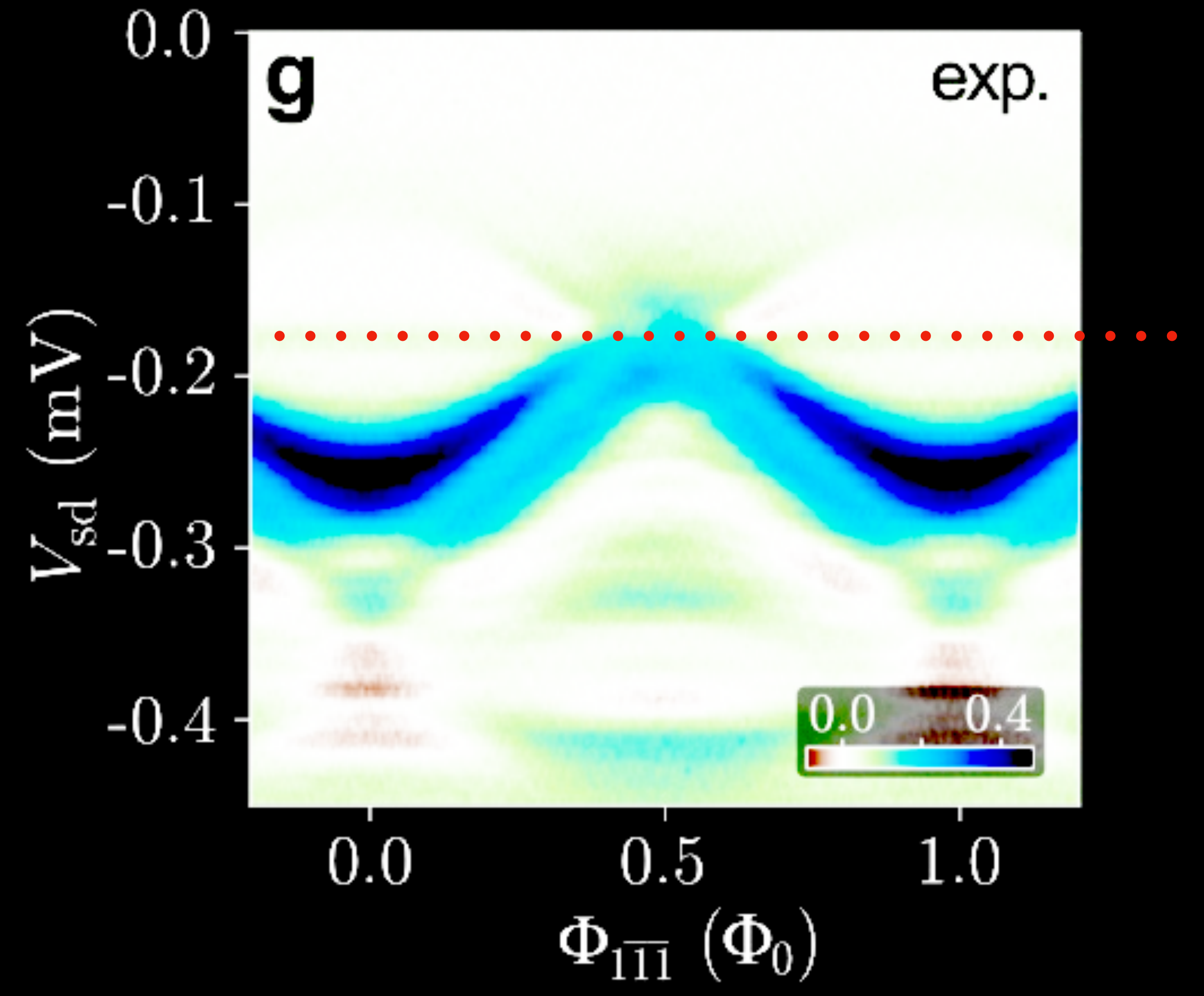


T. Antonelli, et. al., arXiv:2408.01281 (2025).

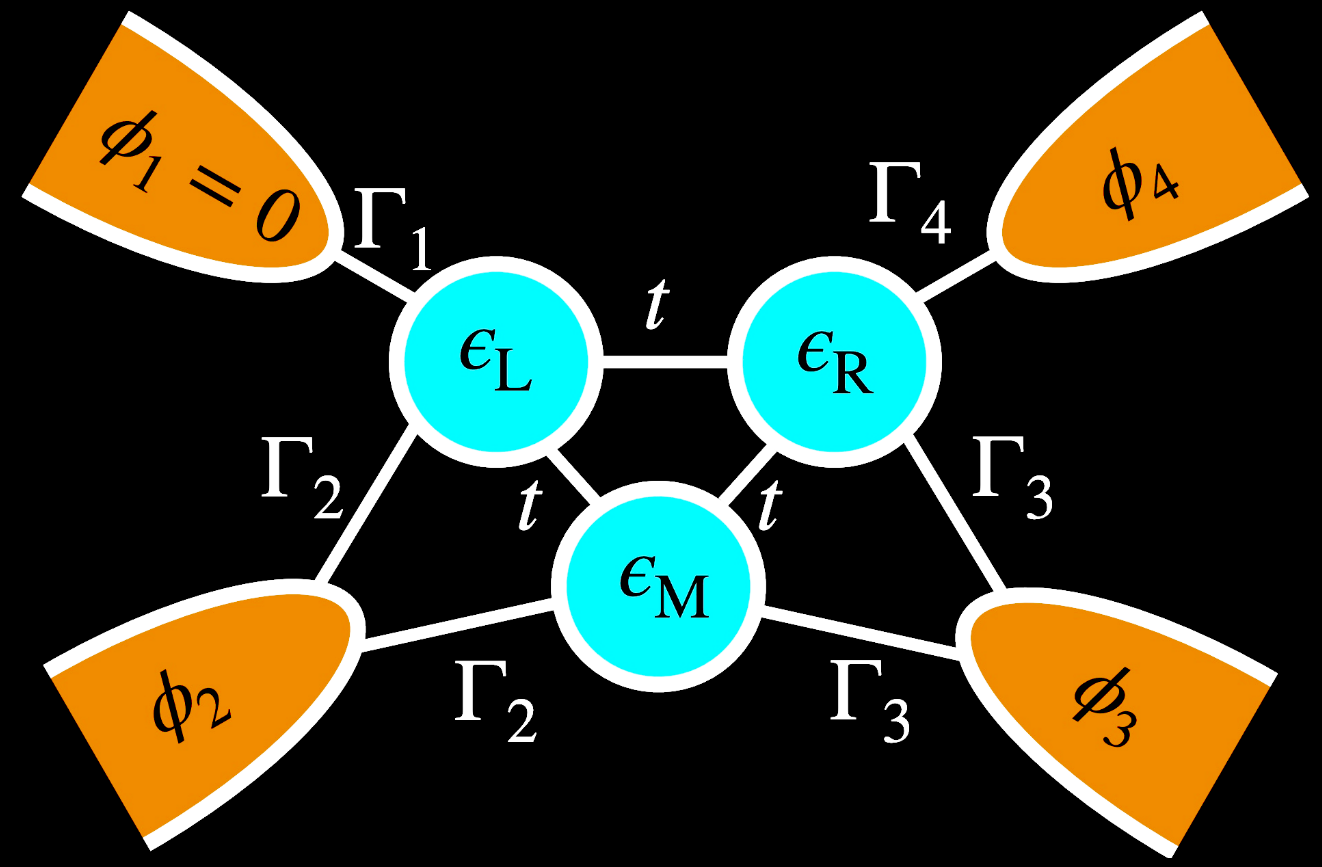
Experiments on MTJJs: Topology?

Tunneling spectroscopy





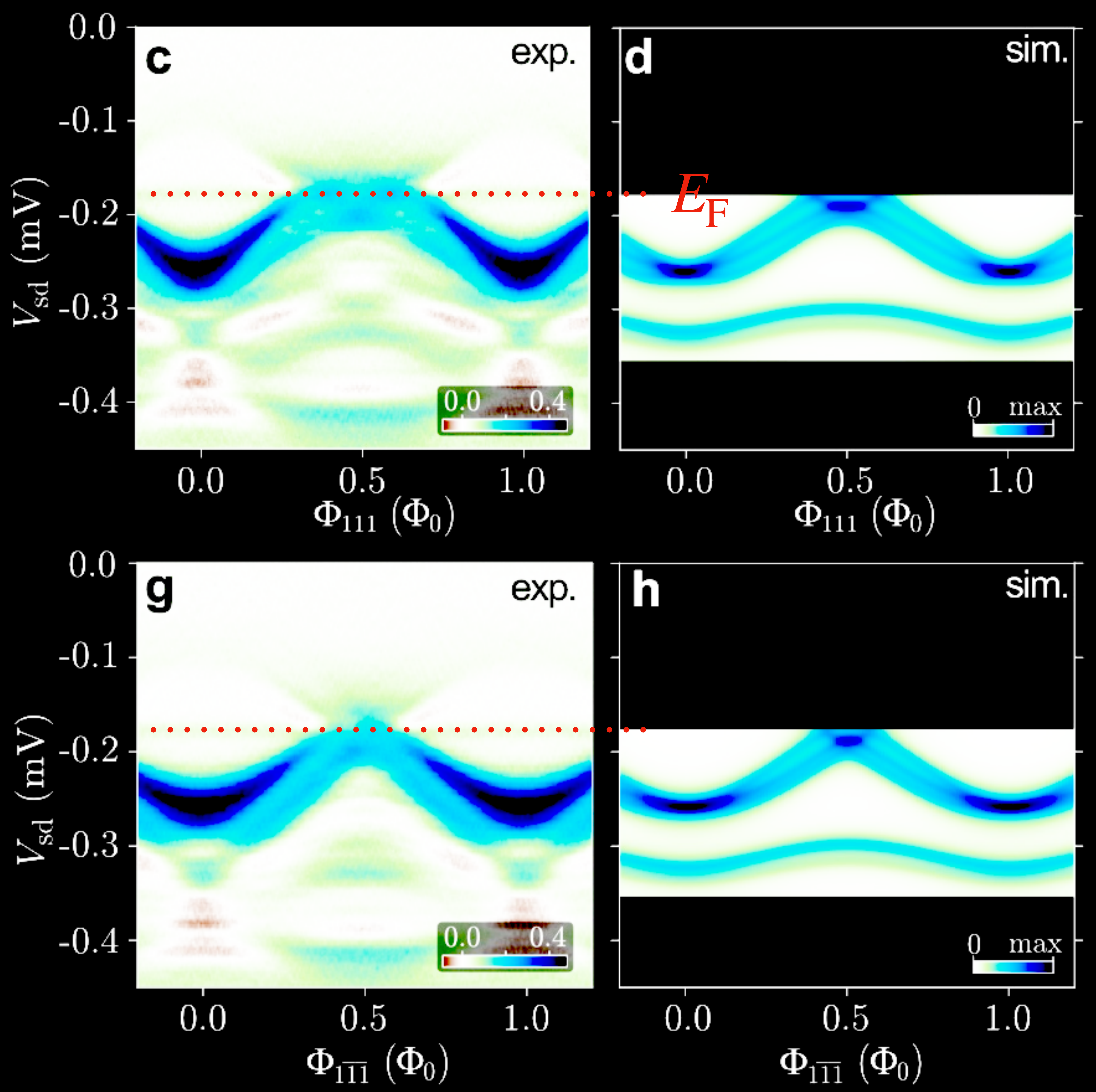
Three state Andreev molecule model



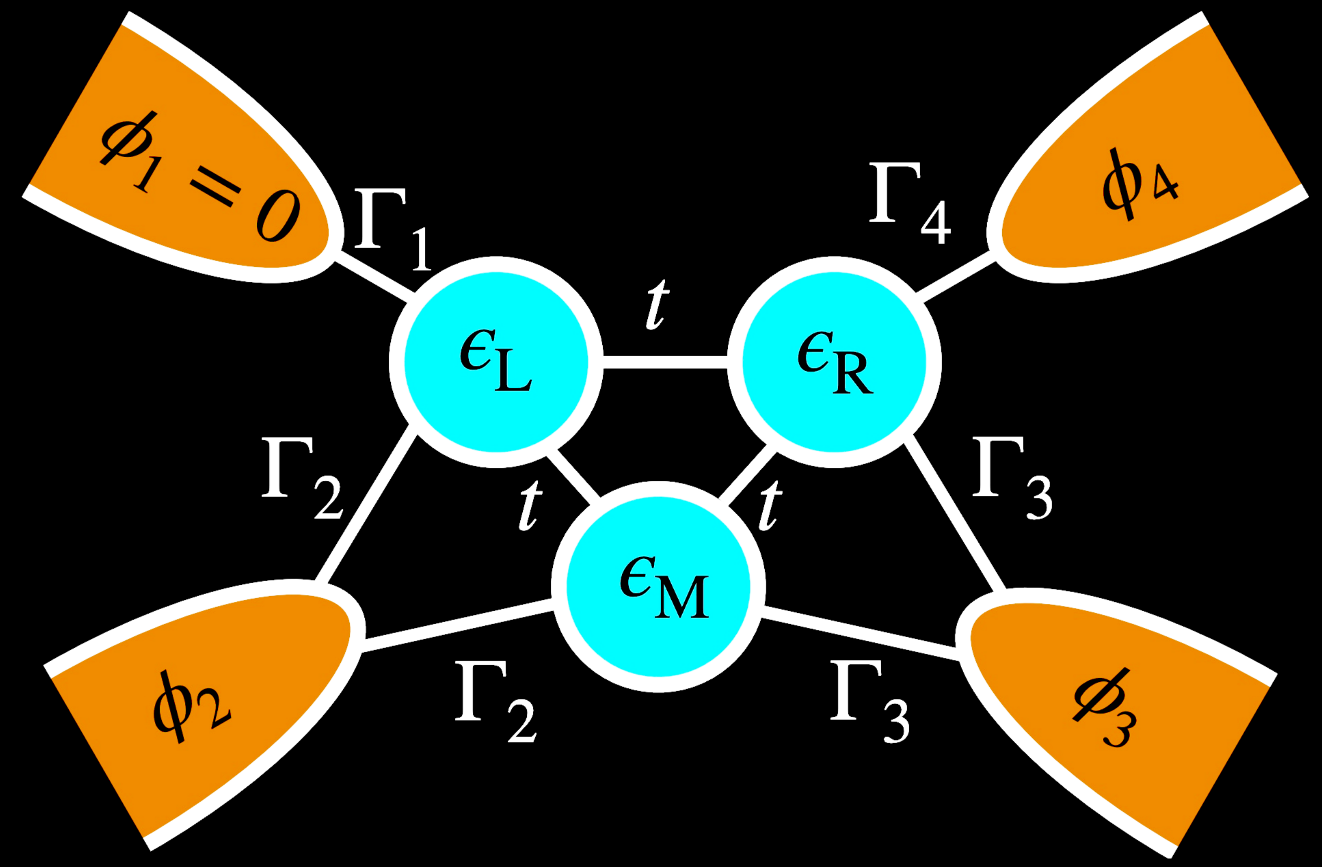
T. Antonelli, et. al., arXiv:2408.01281 (2025).

Experiments on MTJJs: Topology?

Tunneling spectroscopy



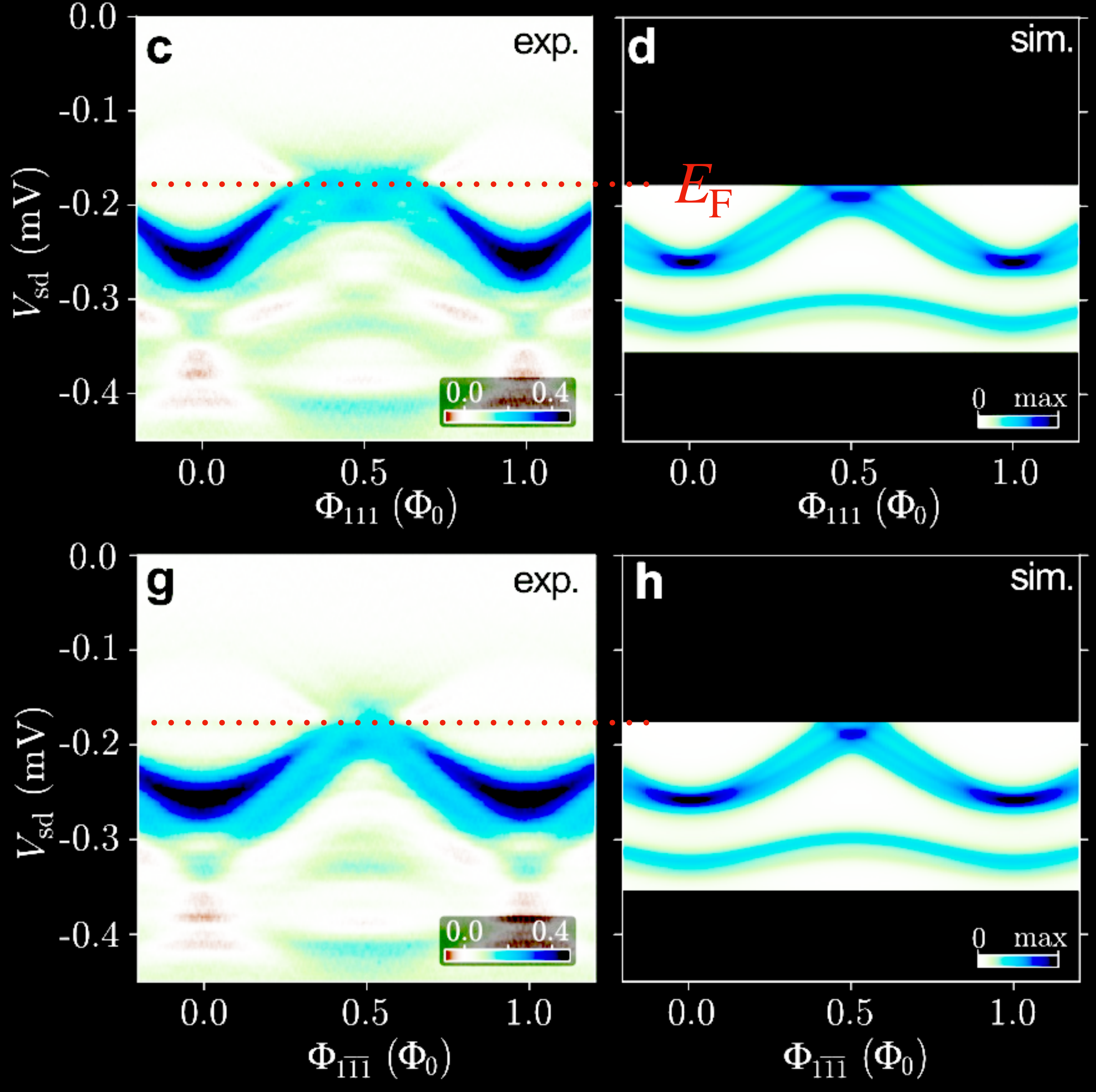
Three state Andreev molecule model



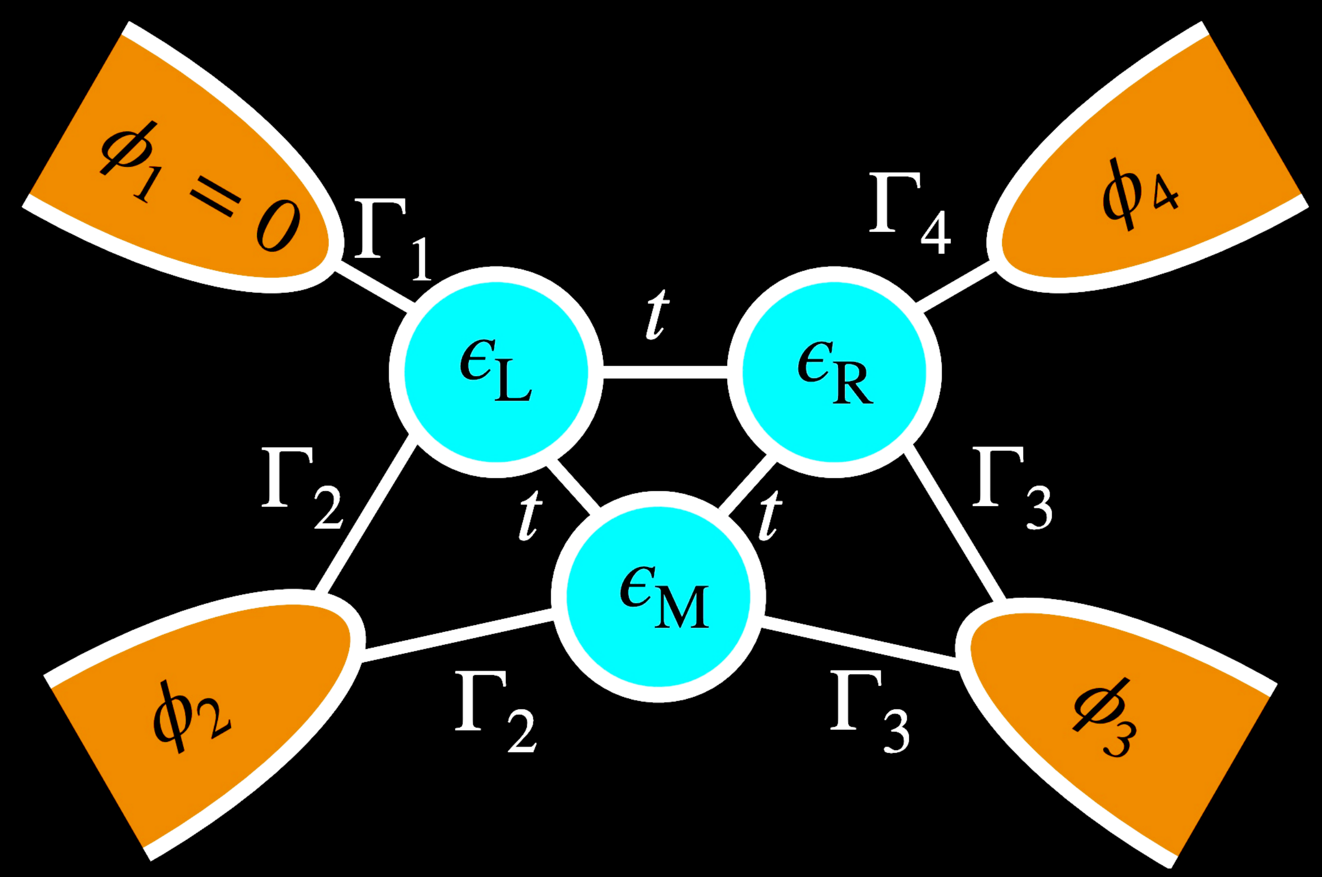
T. Antonelli, et. al., arXiv:2408.01281 (2025).

Experiments on MTJJs: Topology?

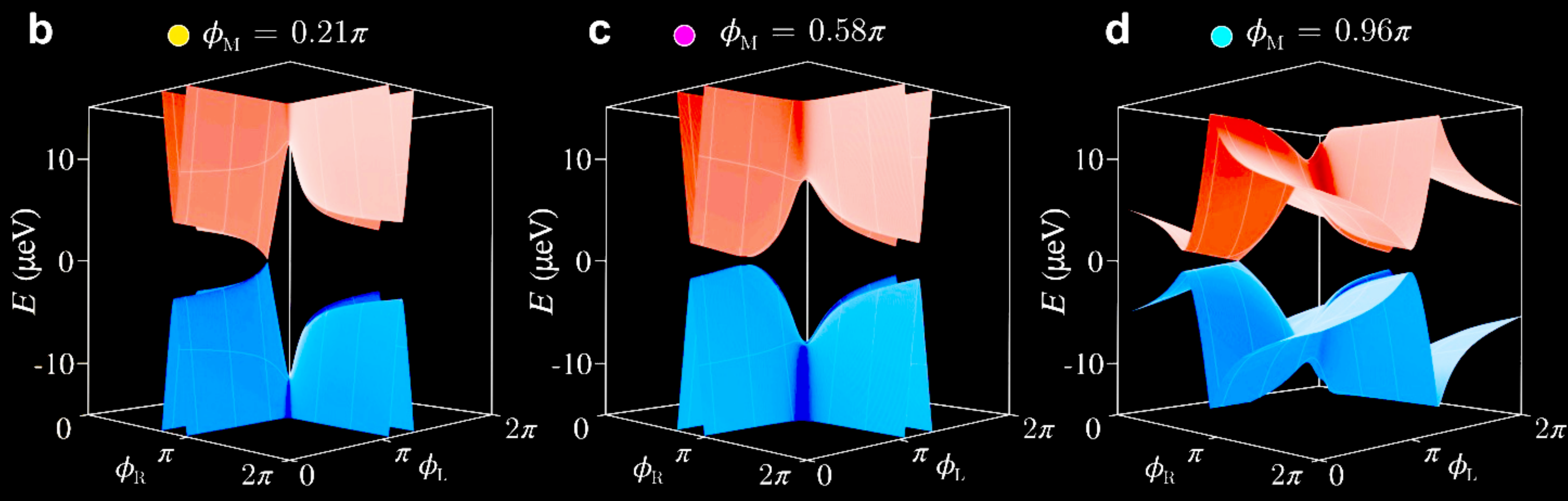
Tunneling spectroscopy



Three state Andreev molecule model



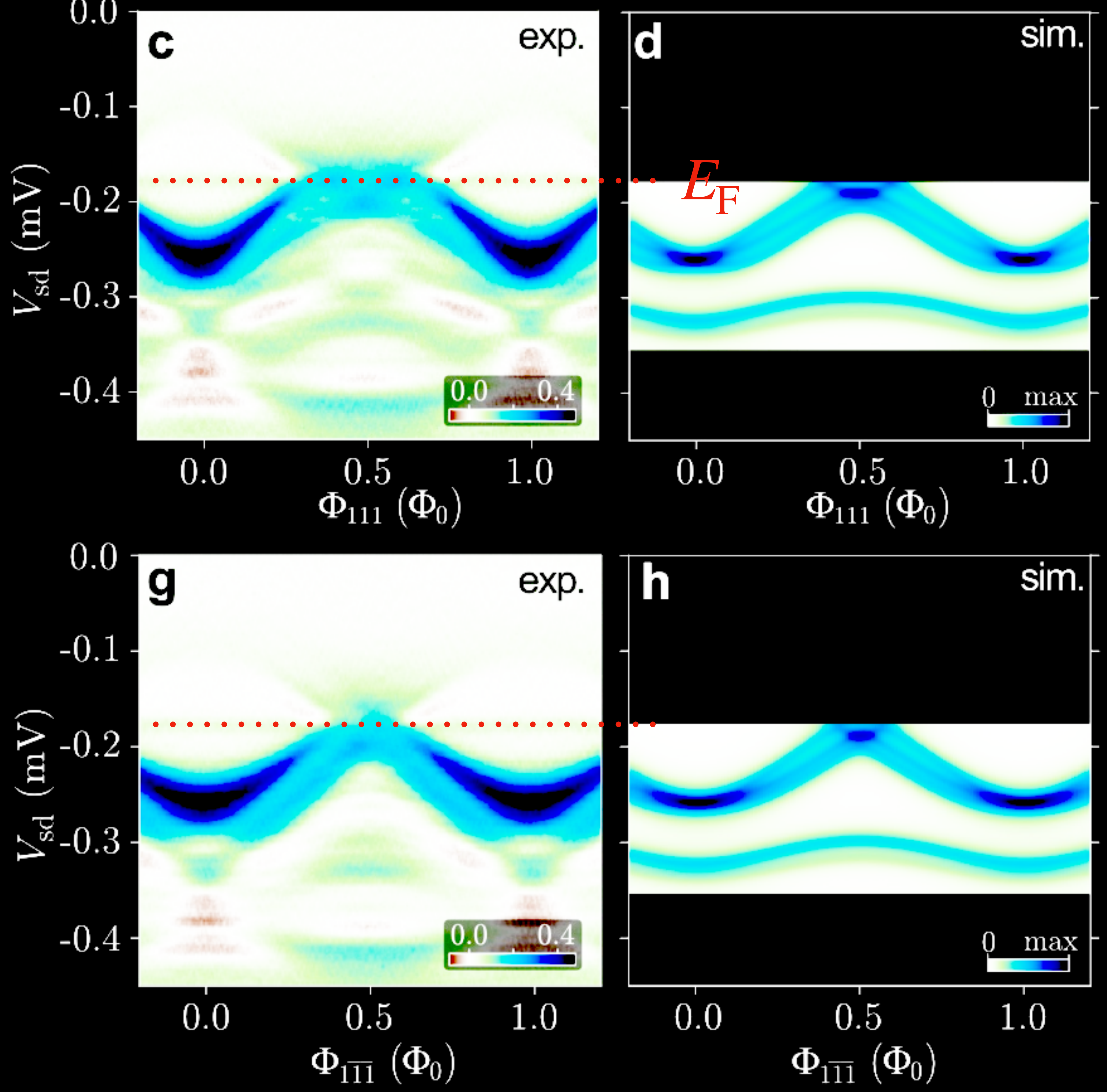
Model predicts Weyl points and non-trivial topology



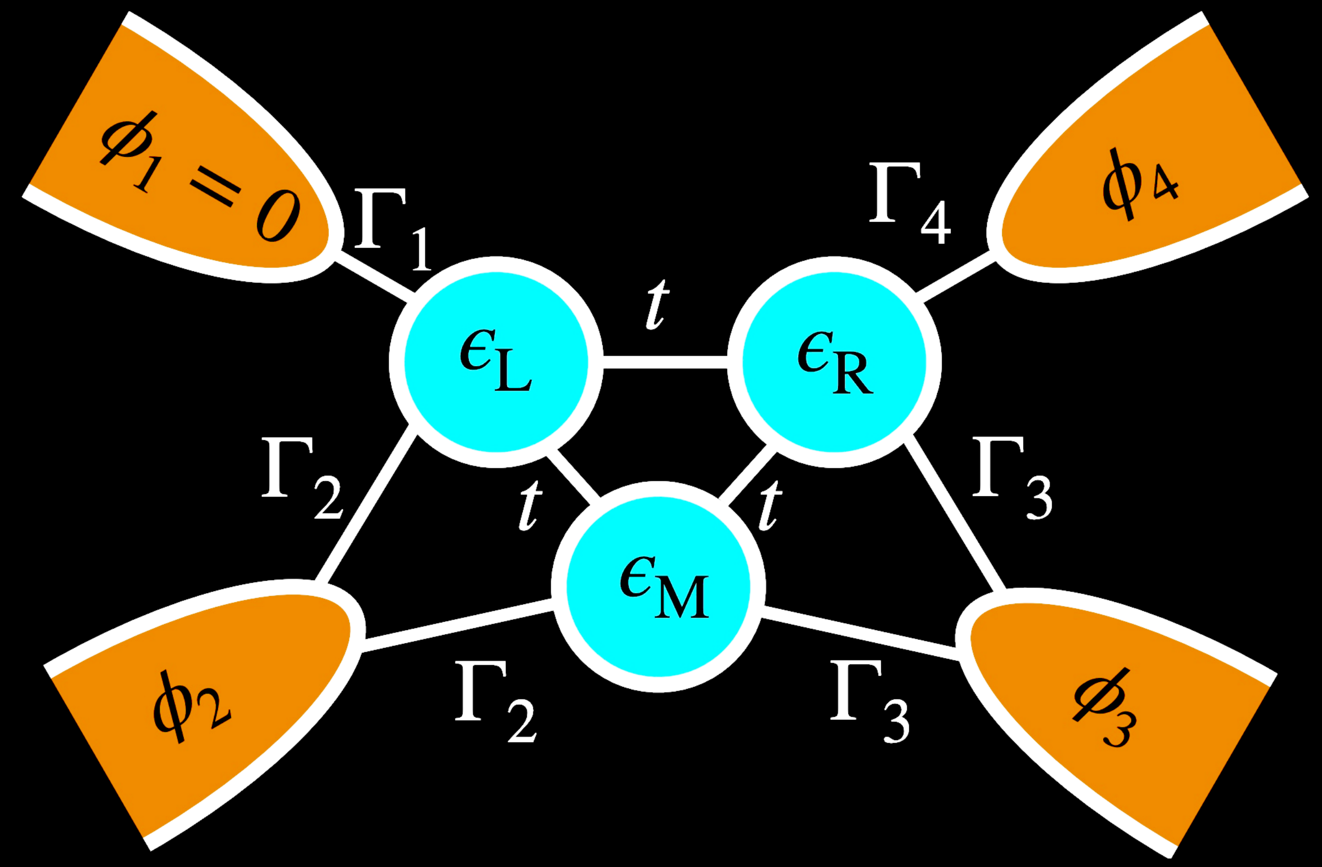
T. Antonelli, et. al., arXiv:2408.01281 (2025).

Experiments on MTJJs: Topology?

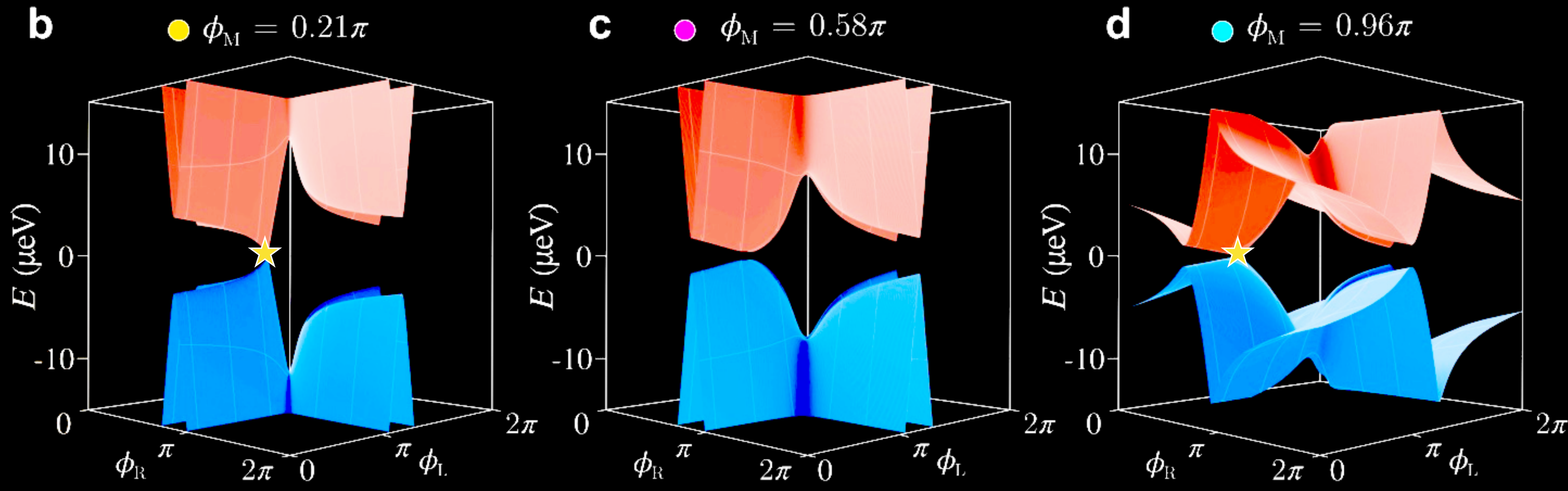
Tunneling spectroscopy



Three state Andreev molecule model



Model predicts Weyl points and non-trivial topology

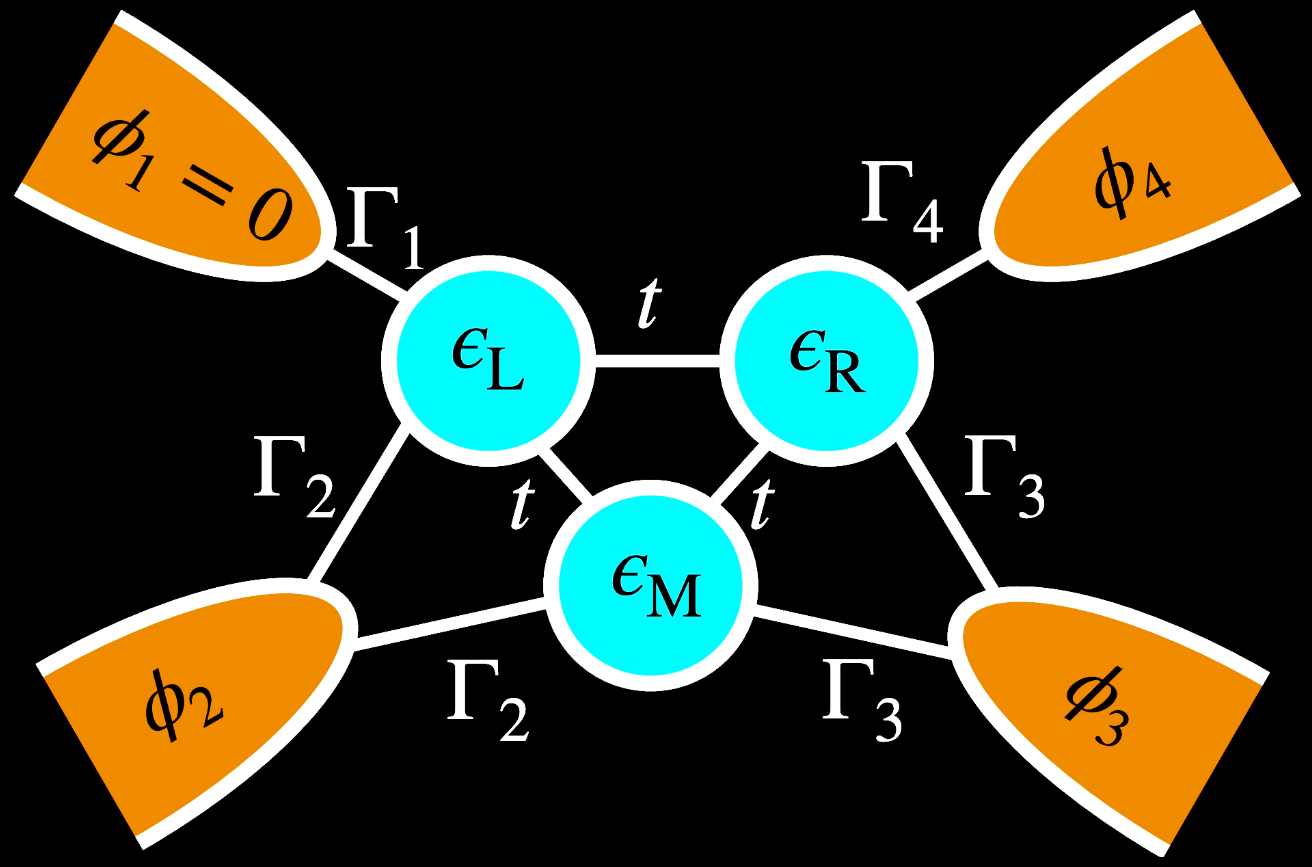


Weyl points

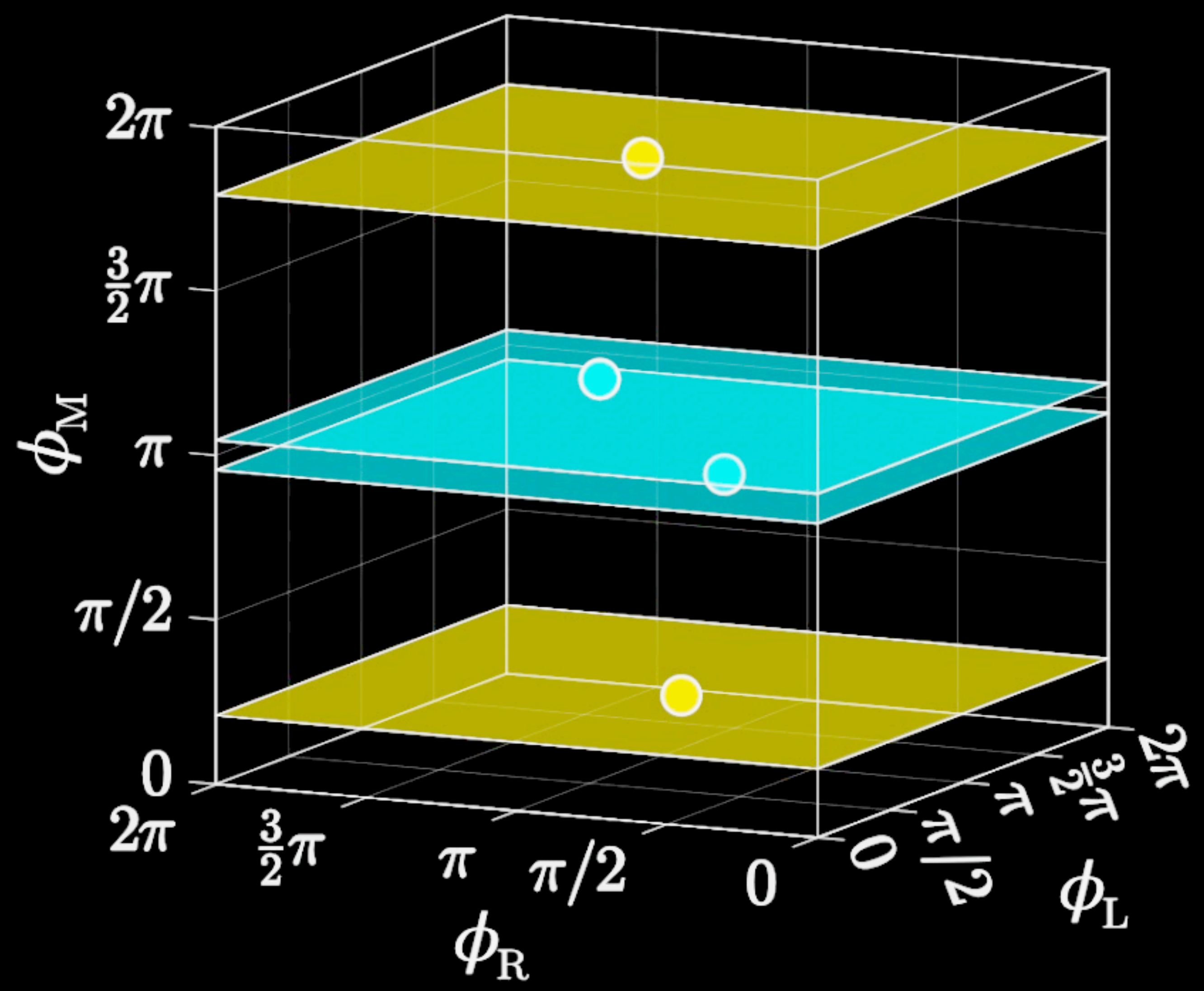
T. Antonelli, et. al., arXiv:2408.01281 (2025).

Experiments on MTJJs: Topology?

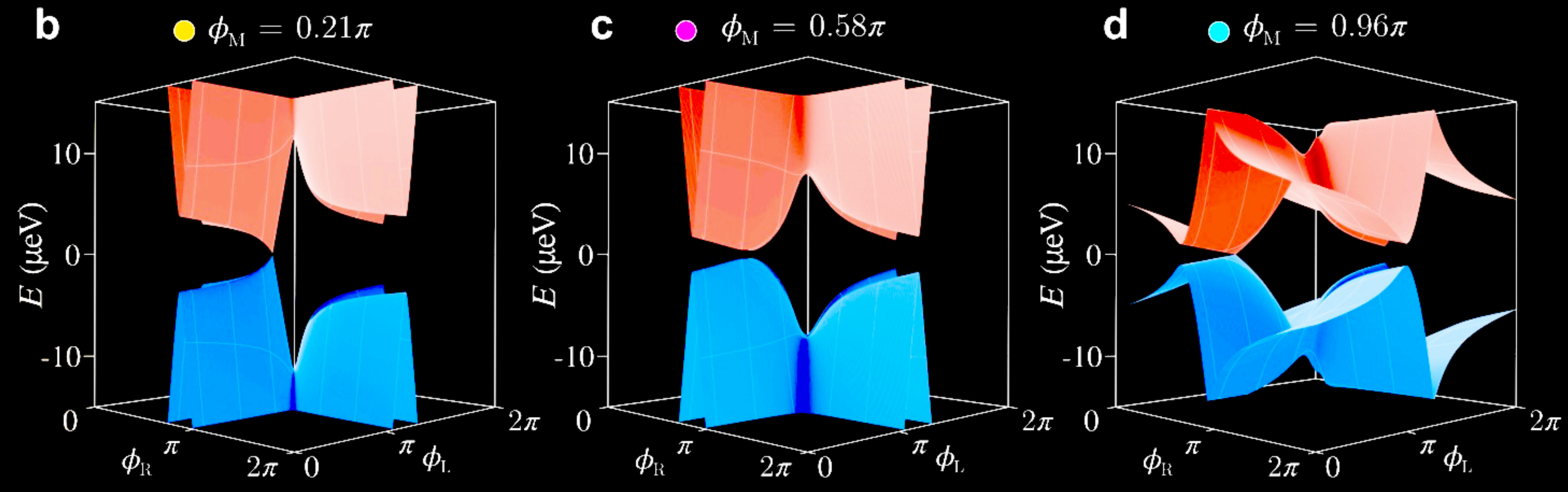
Three state Andreev molecule model



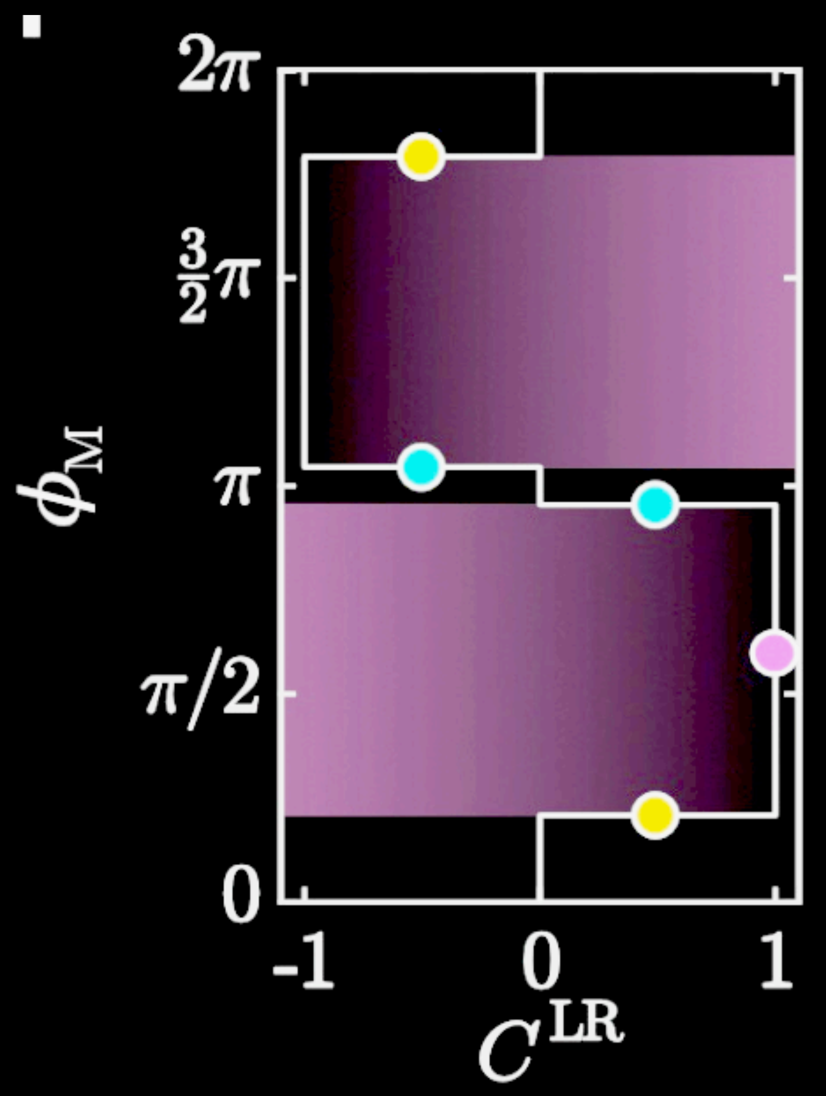
Position of Weyl points



Model predicts Weyl points and non-trivial topology



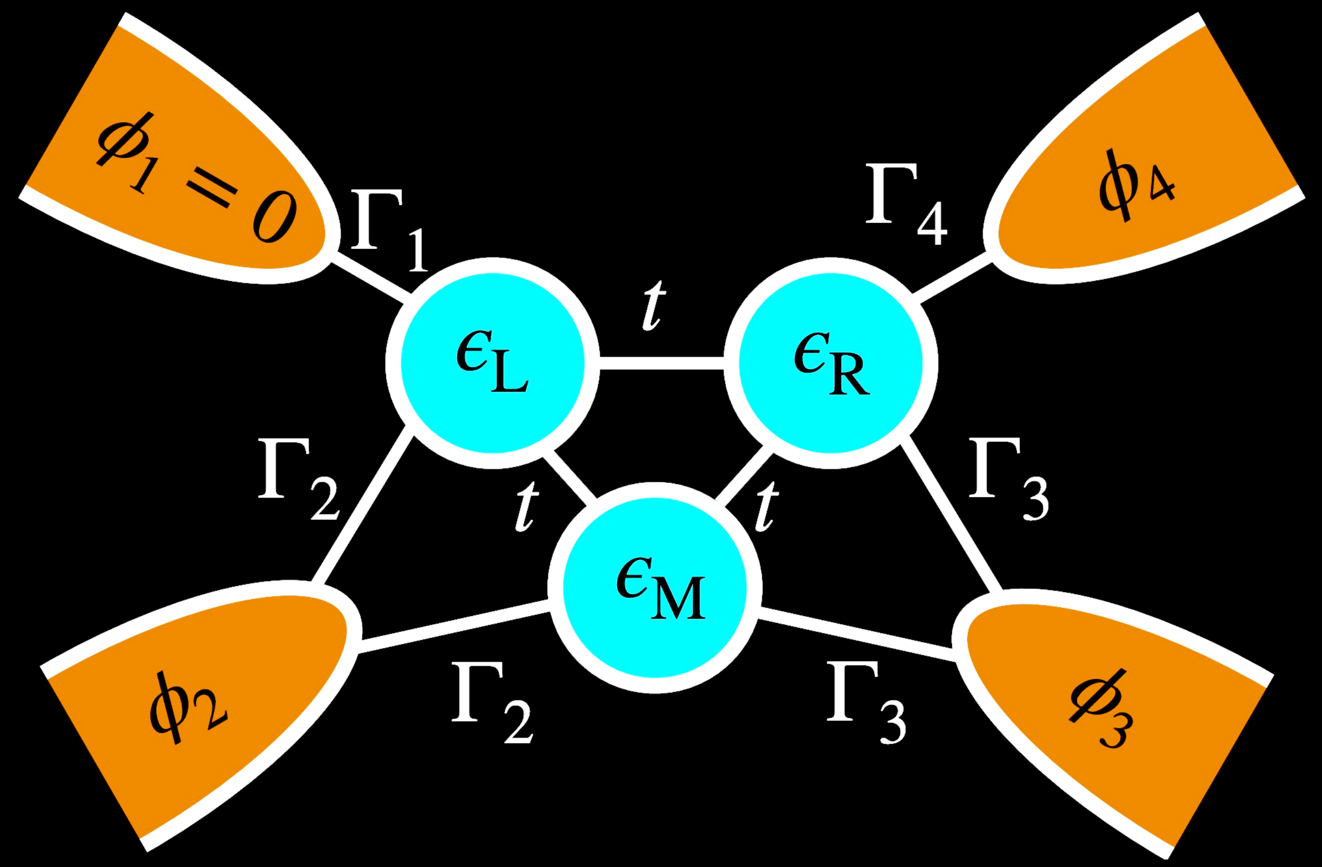
Phase transition



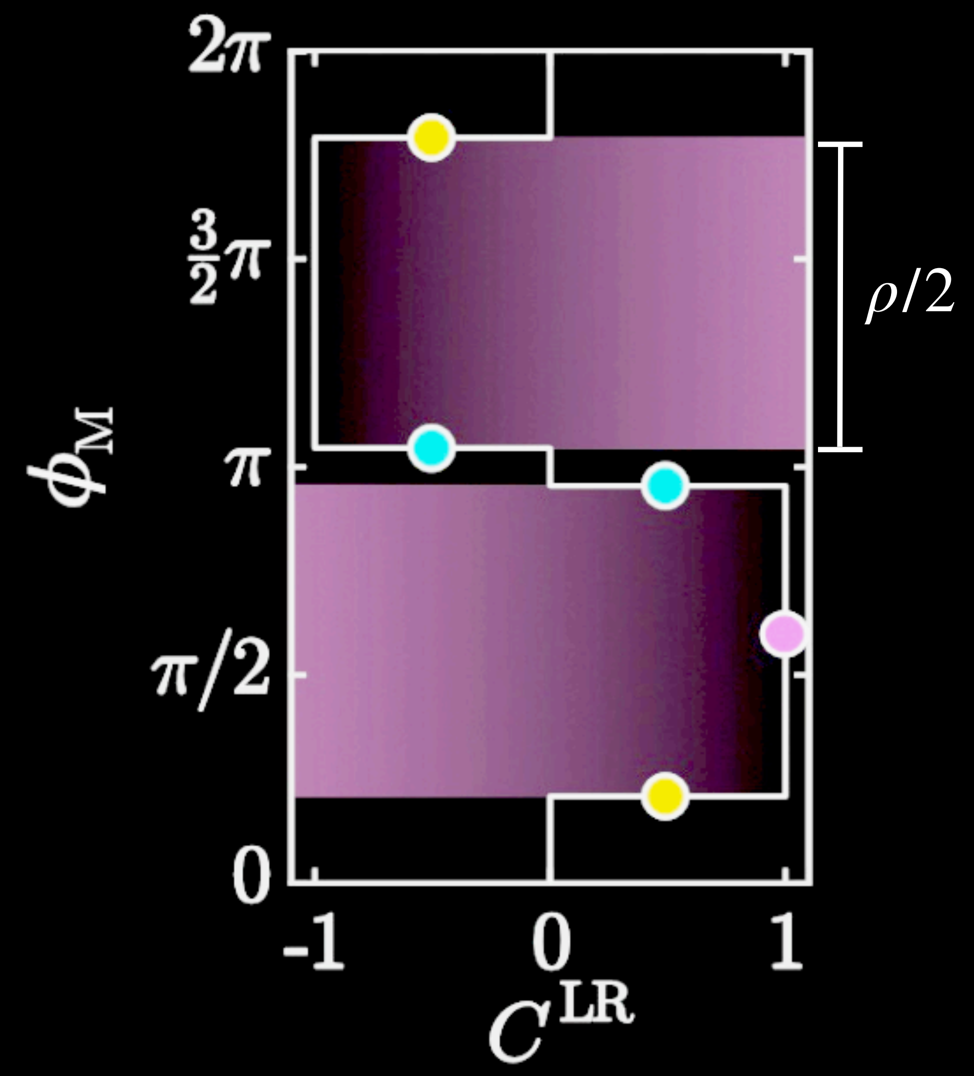
T. Antonelli, et. al., arXiv:2408.01281 (2025).

Experiments on MTJJs: Topology?

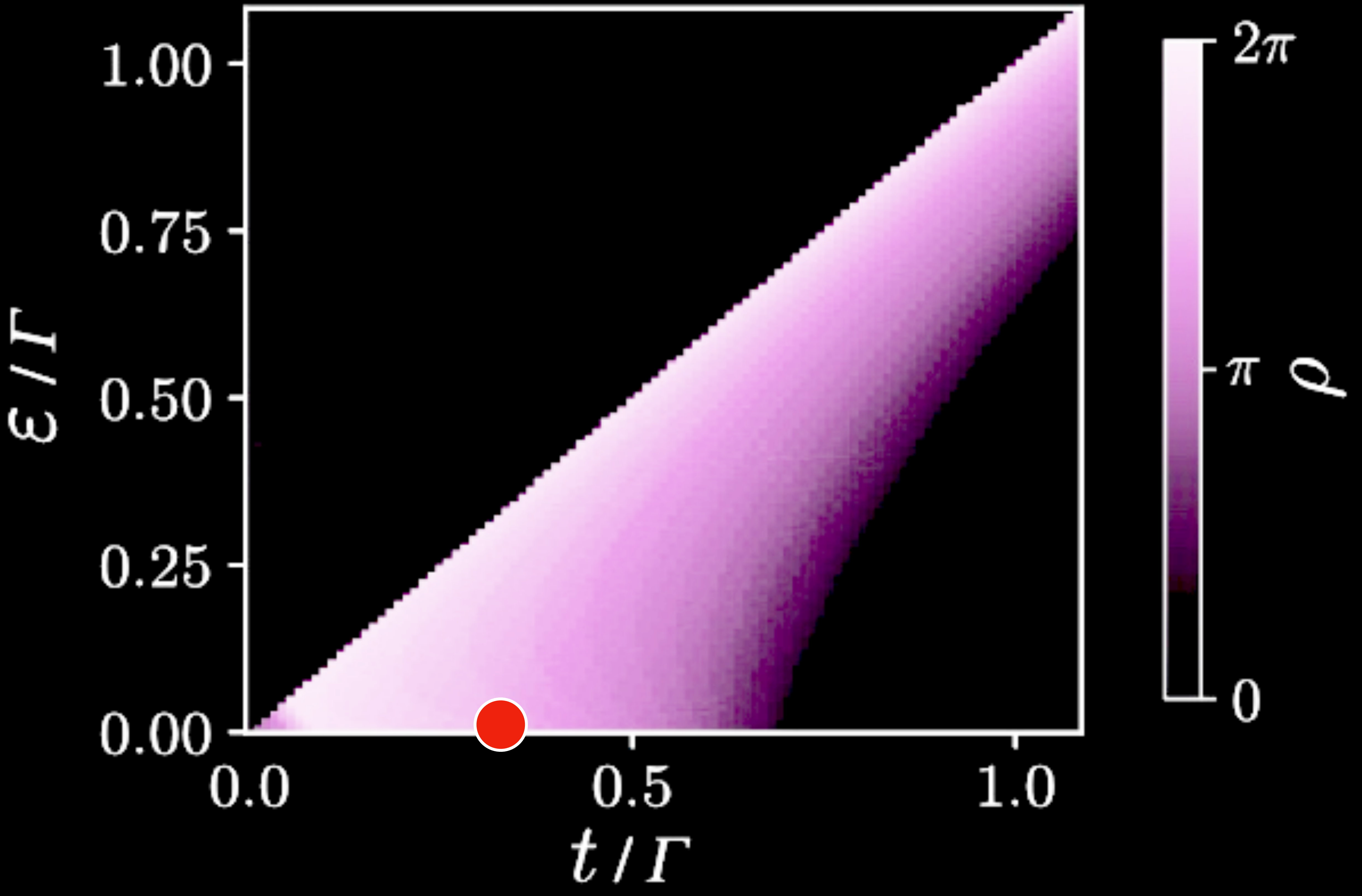
Three state Andreev molecule model



Phase transition



Stability in parameter space
"Transmission" vs Hybridization



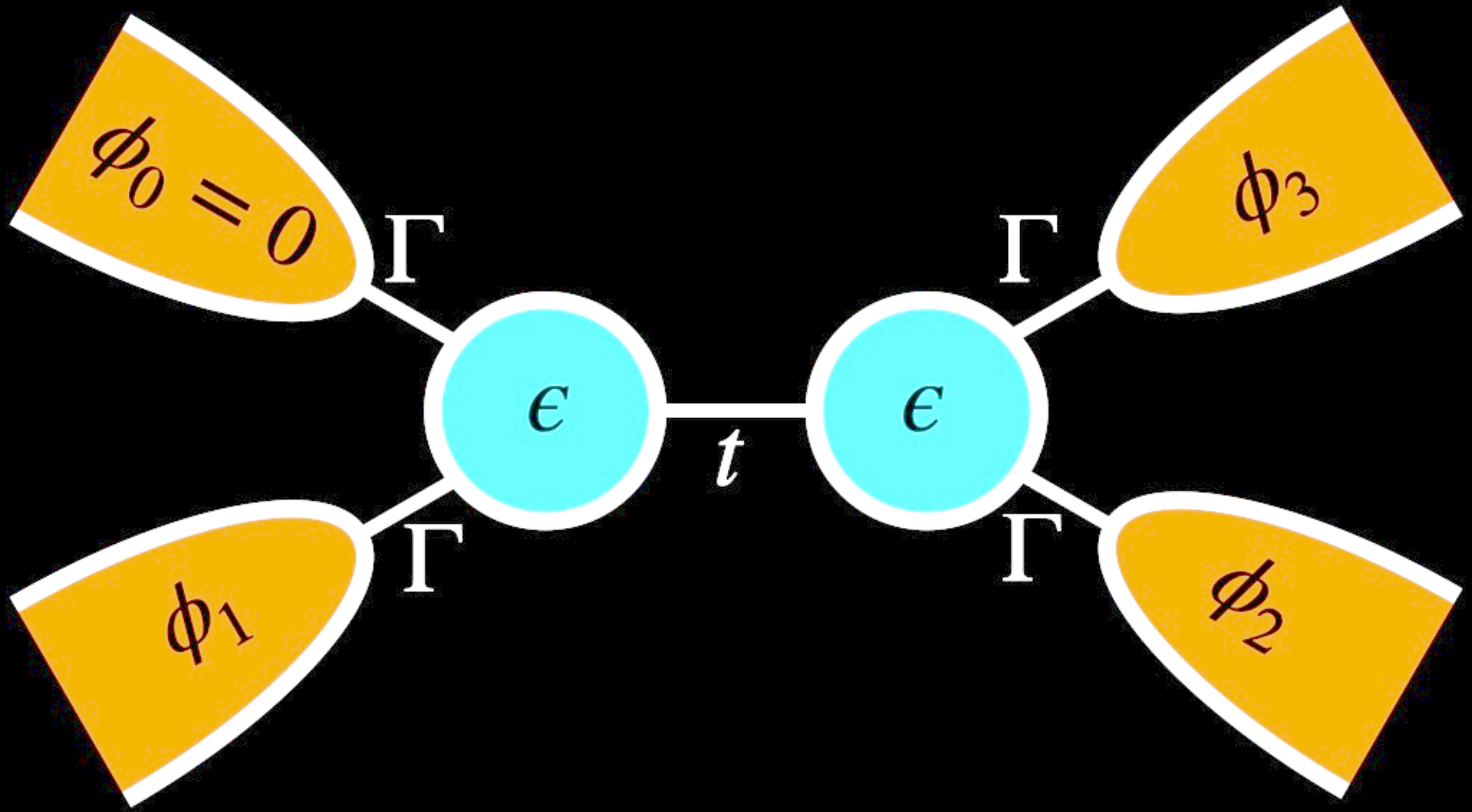
Is there any insight theory can provide for experiment?

T. Antonelli, et. al., arXiv:2408.01281 (2025).

MTJJ considered here

Effective quantum dot models

2-dot modell



L. Teshler, et.al., SciPost Phys. 15, 214 (2023)

Effective low energy Hamiltonian

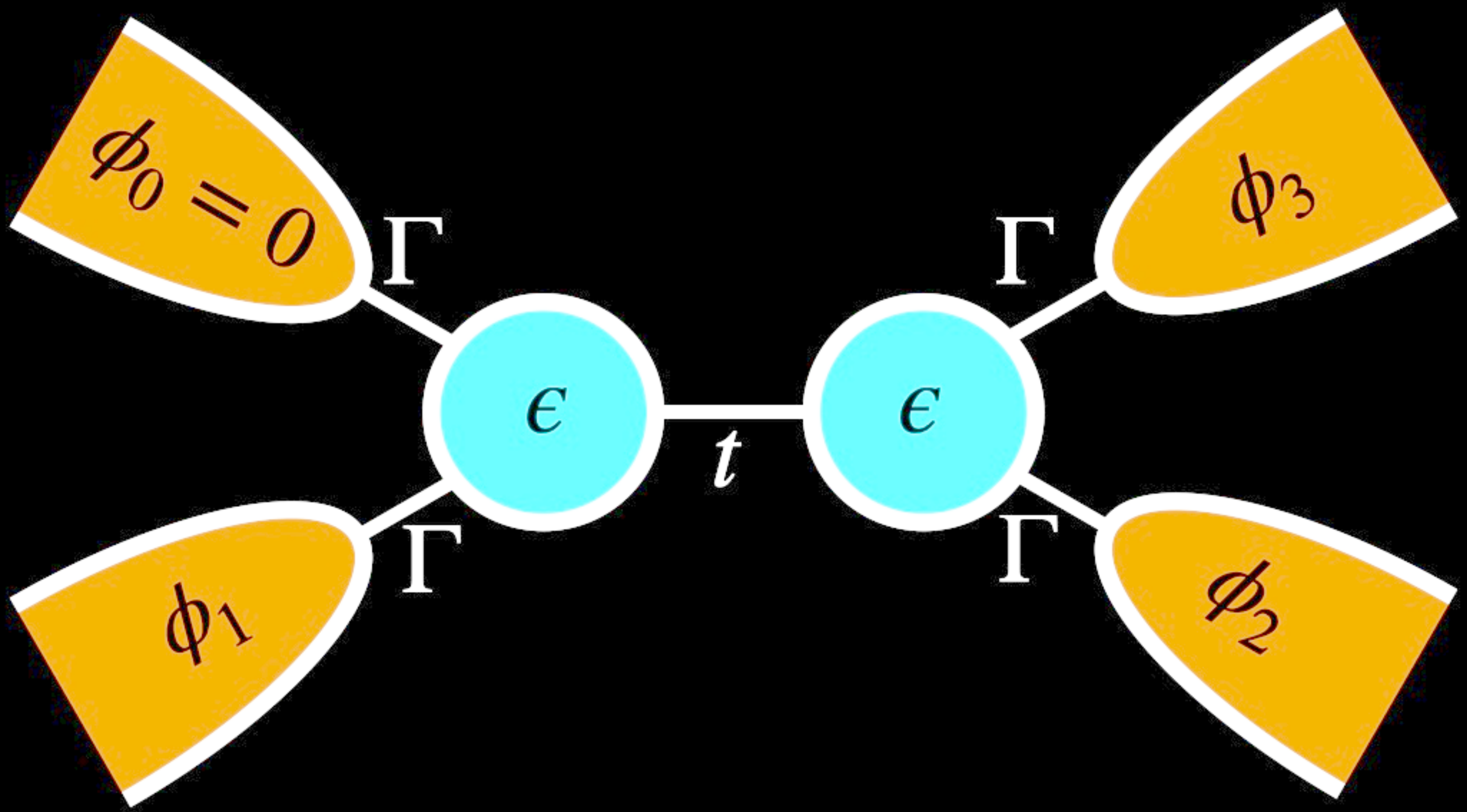
$$H_{\text{eff}} = \begin{pmatrix} H_{\text{N}} & \Sigma \\ \Sigma^* & -H_{\text{N}} \end{pmatrix}$$

$$H_{\text{N}} = \begin{pmatrix} \epsilon & t \\ t & \epsilon \end{pmatrix}, \ \sigma(H_{\text{N}}) = \{\epsilon \pm t\}$$

$$\Sigma = \begin{pmatrix} \Gamma + \Gamma e^{\imath\phi_1} & 0 \\ 0 & \Gamma e^{\imath\phi_2} + \Gamma e^{\imath\phi_3} \end{pmatrix}$$

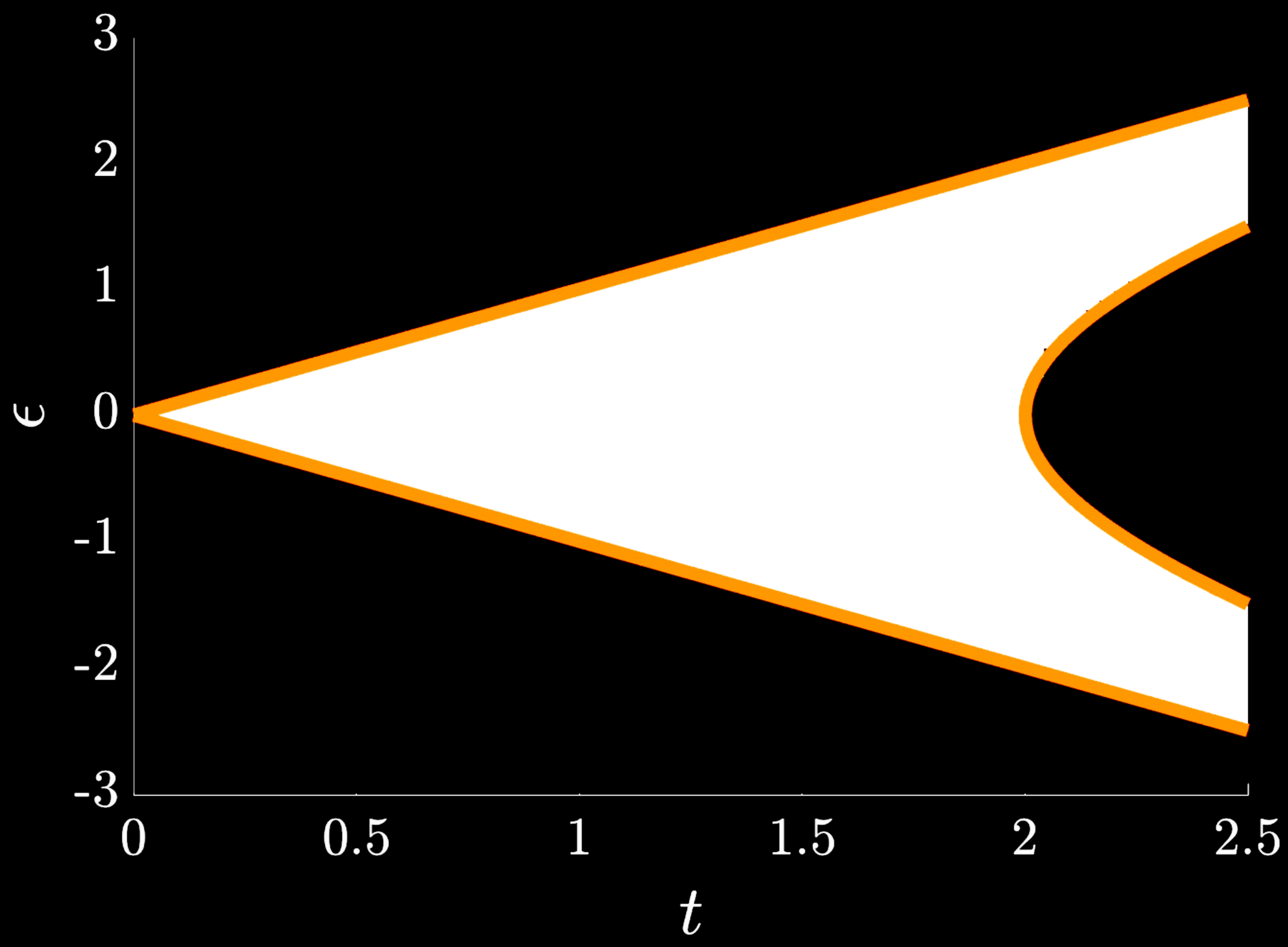
What's the source of topology in MTJJs?

4-terminal MTJJ with two dots



L. Teshler, et.al., SciPost Phys. 15, 214 (2023)

Topological phase diagram



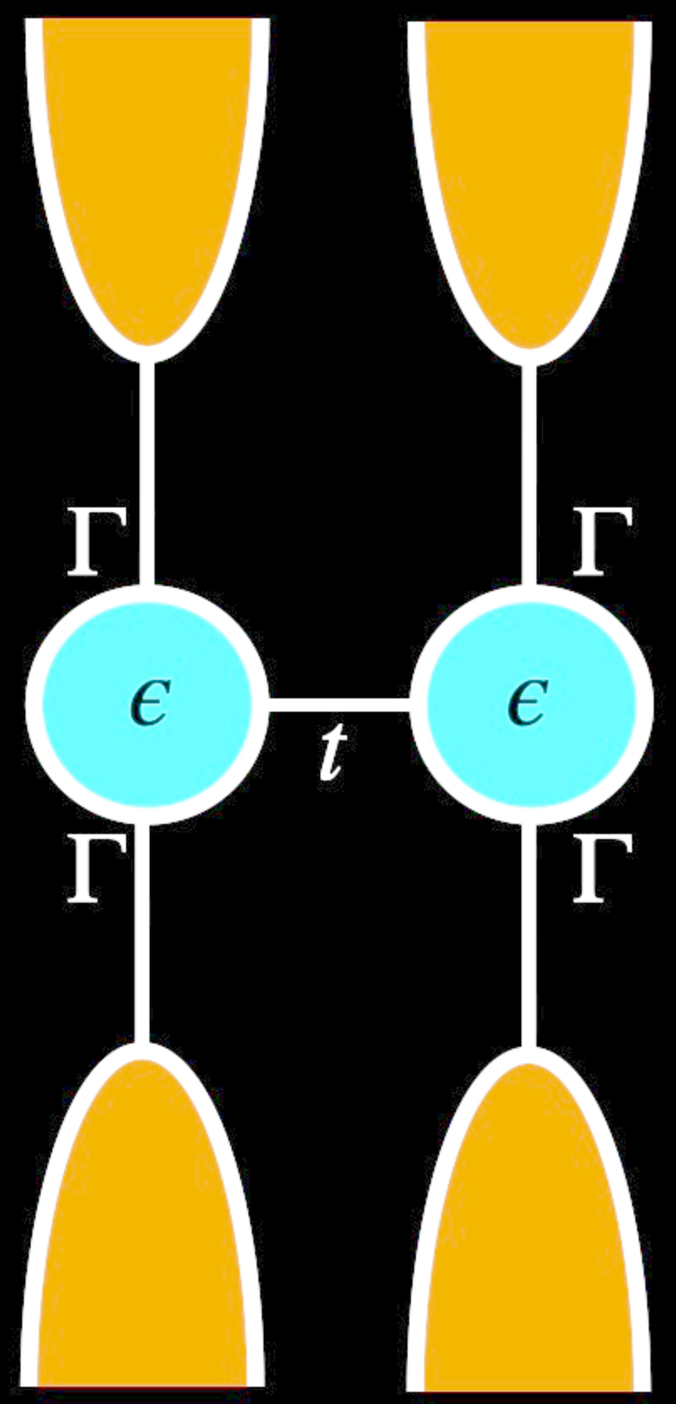
non-trivial Chern number

in superconducting phase space $\left\{ \vec{\phi} \right\}$

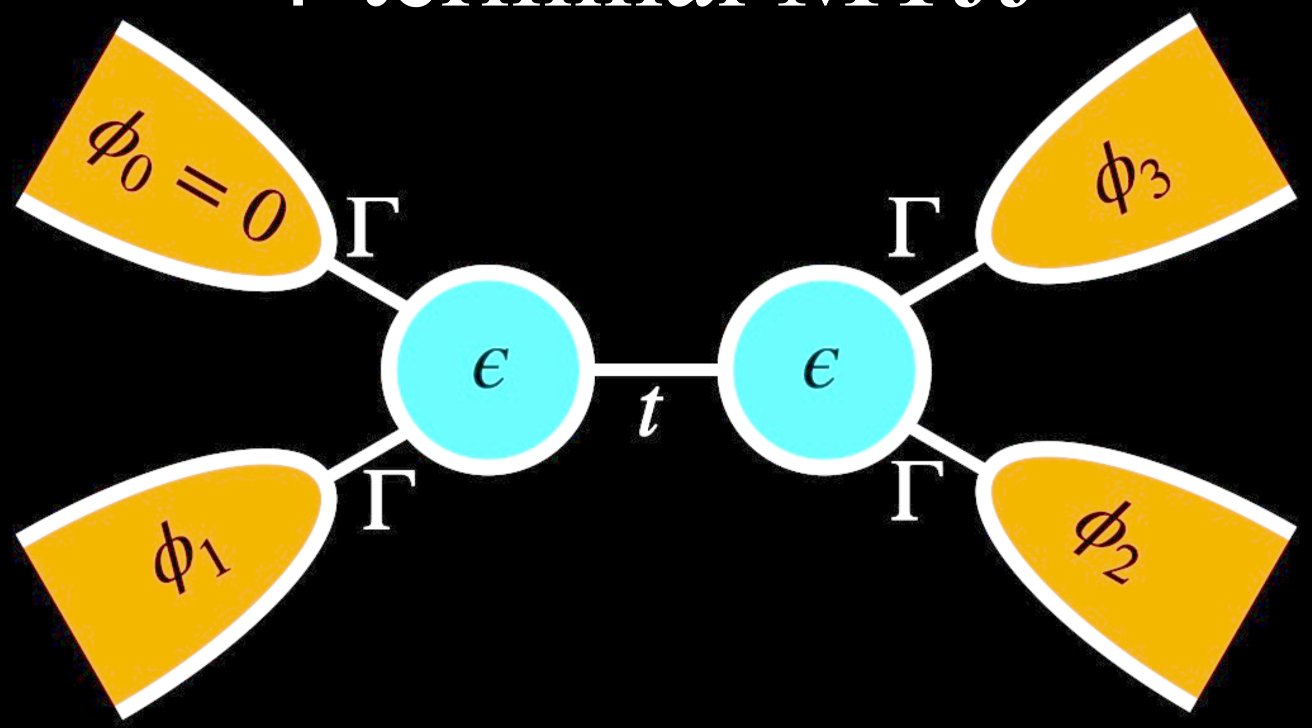
Reflectionless modes as source of Weyl Nodes

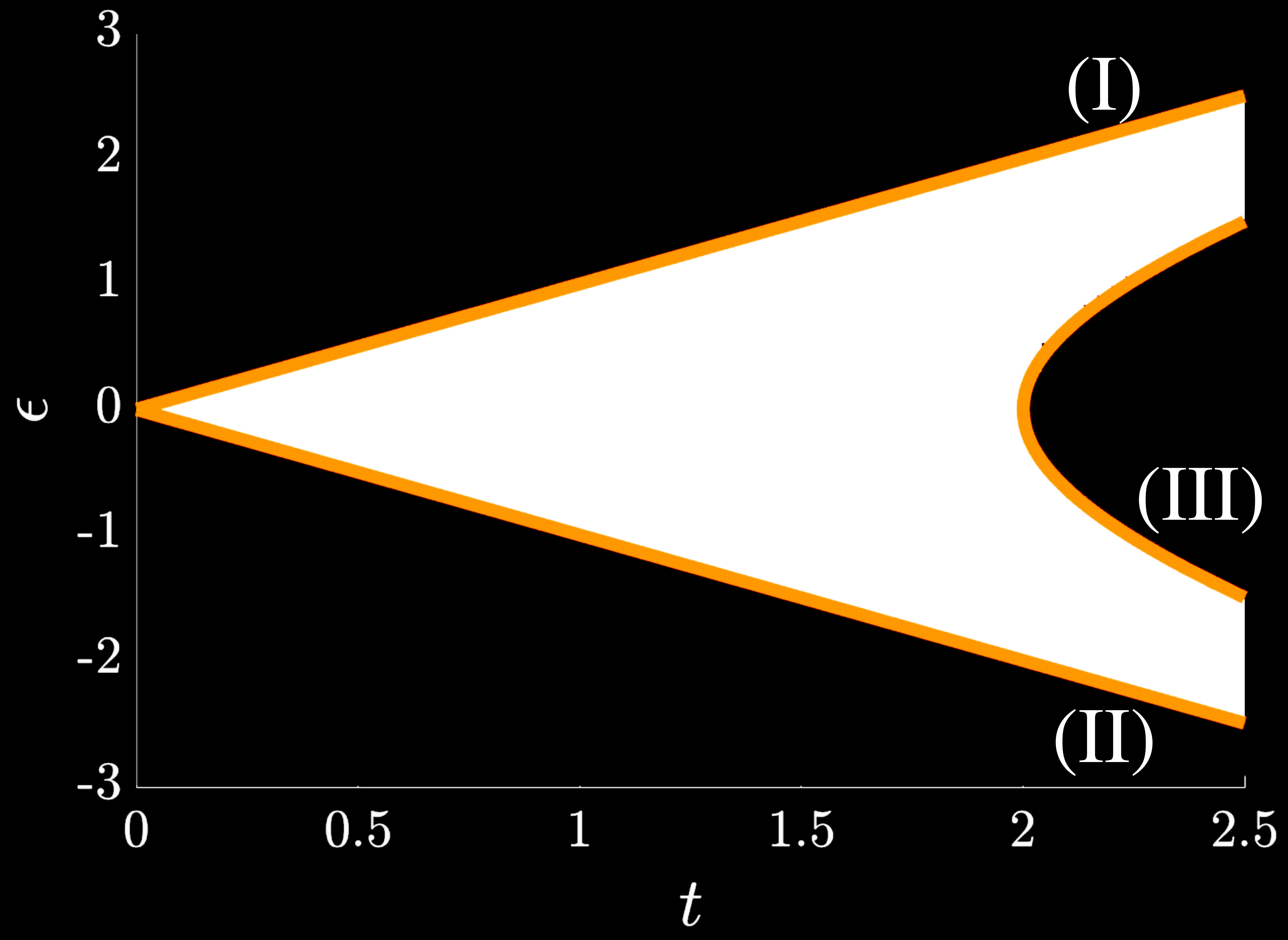
D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

effective 2-terminal JJ



4-terminal MTJJ

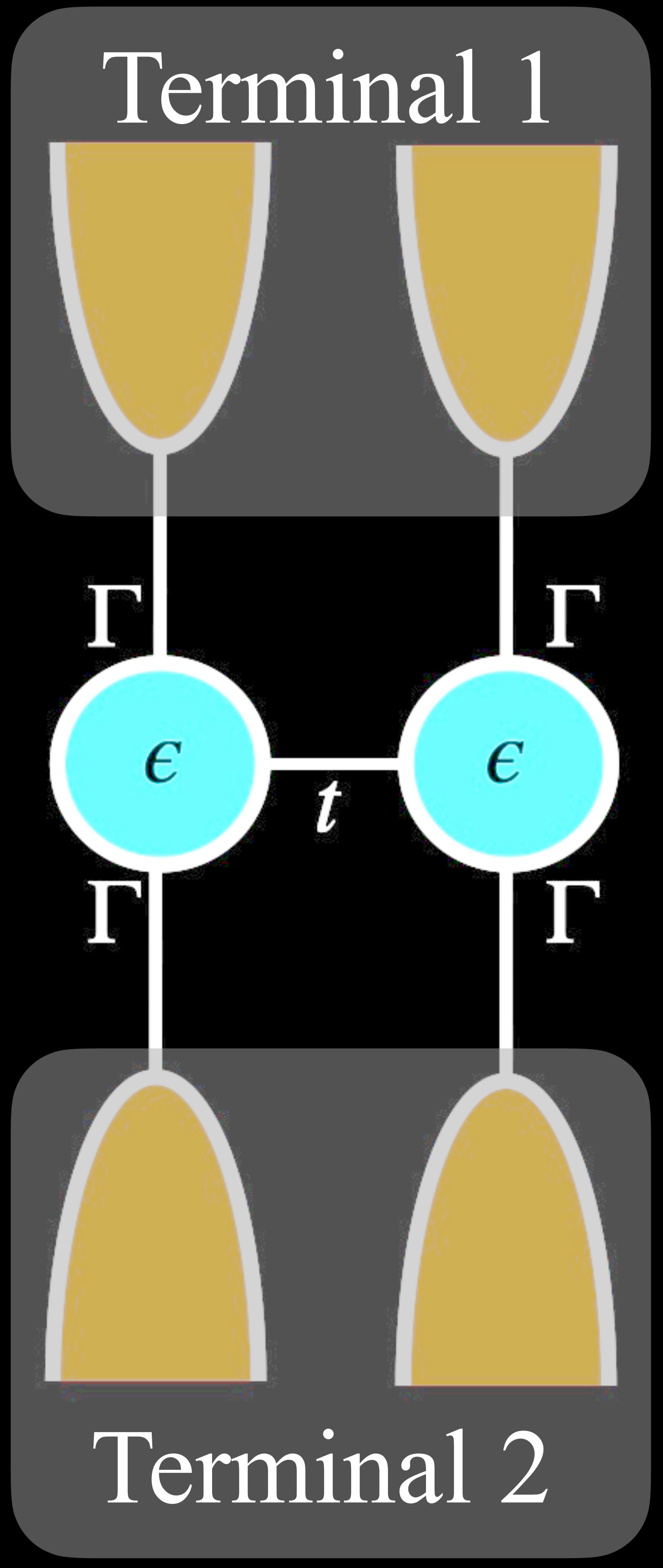




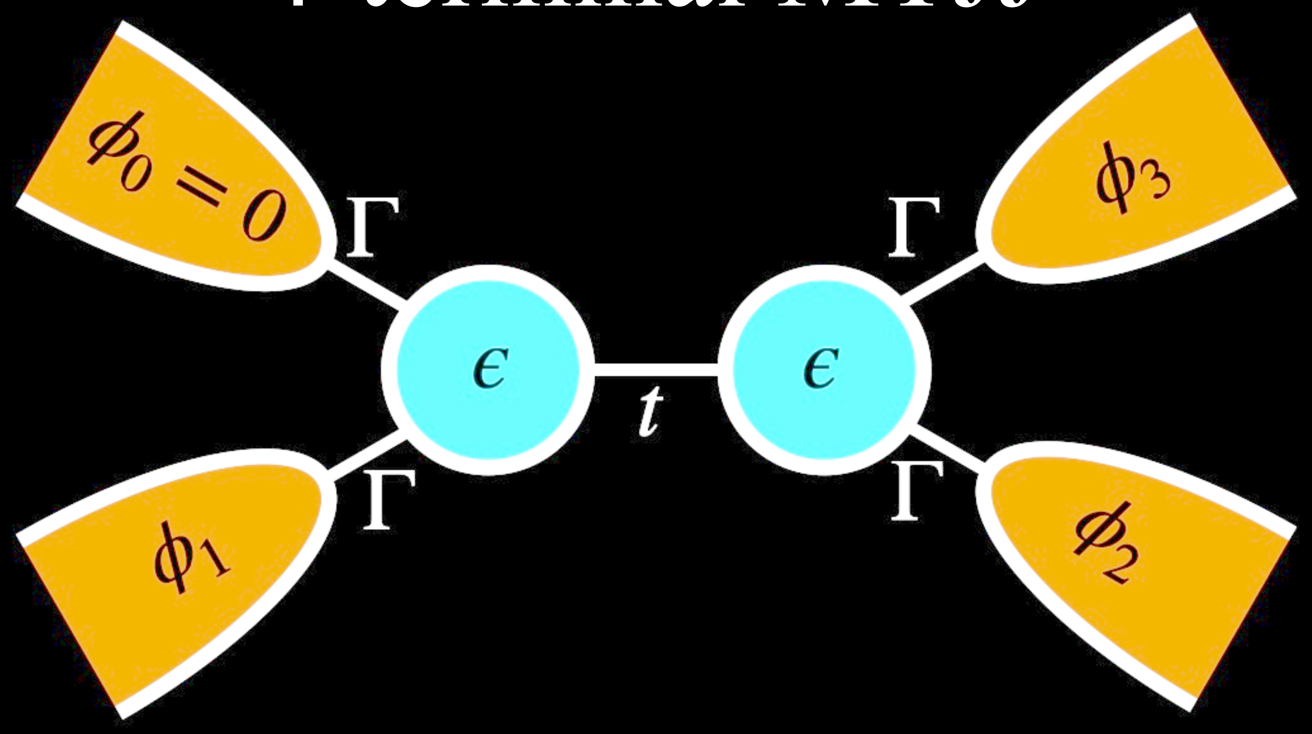
Reflectionless modes as source of Weyl Nodes

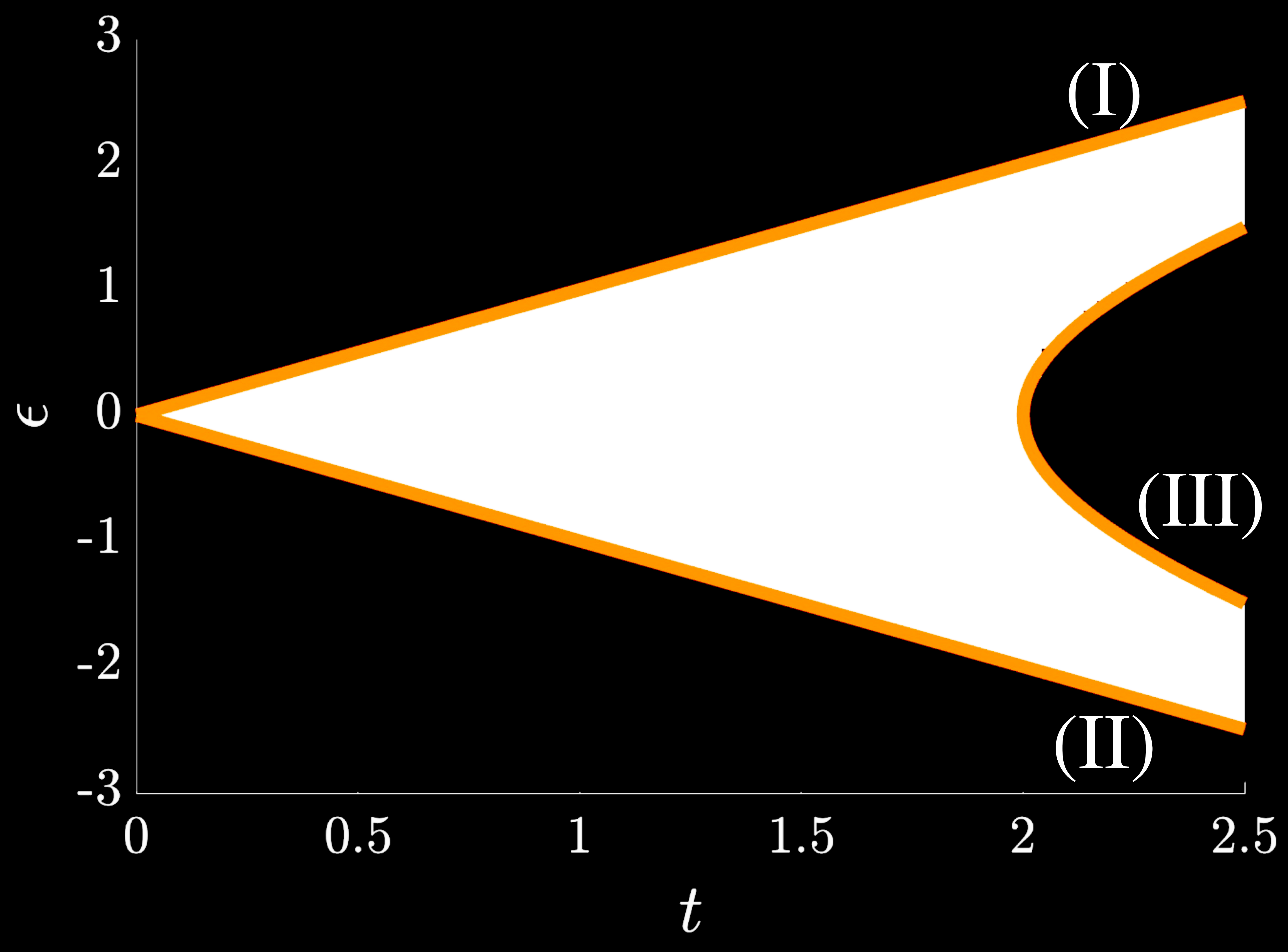
D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

effective 2-terminal JJ



4-terminal MTJJ

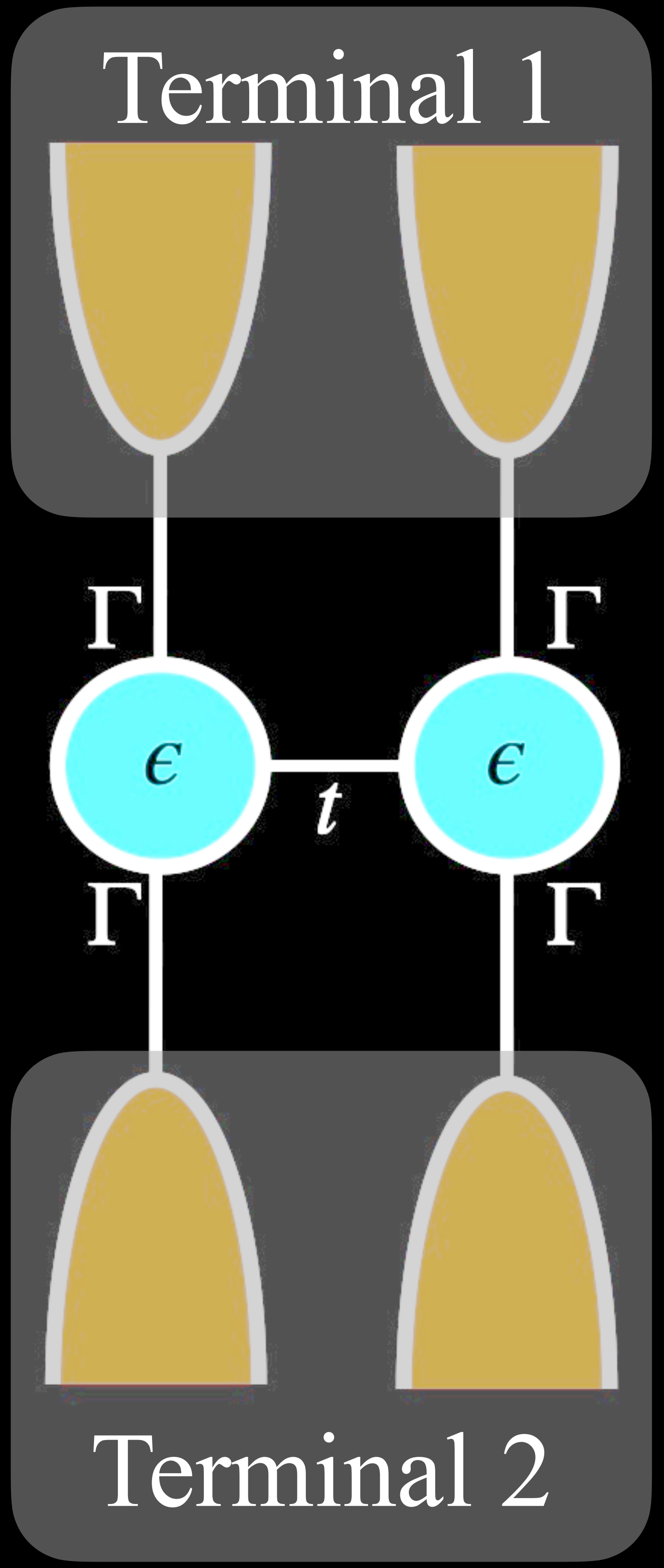




Reflectionless modes as source of Weyl Nodes

D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

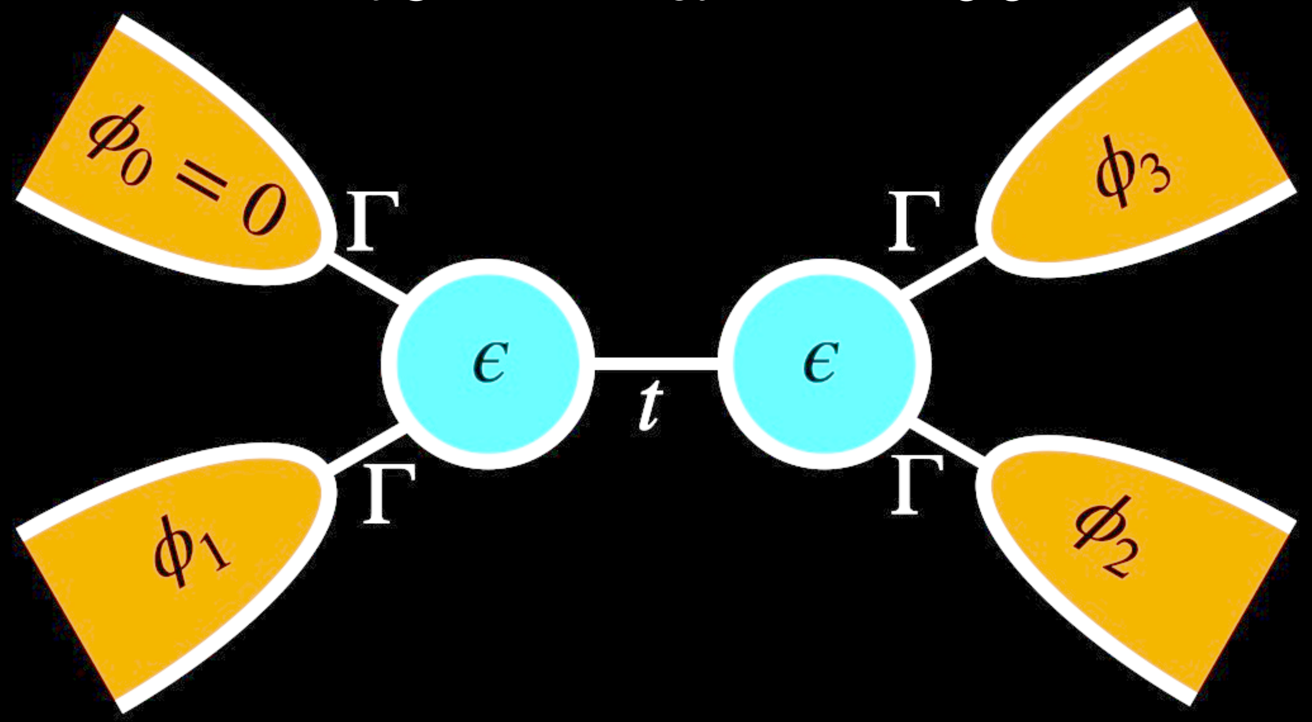
effective 2-terminal JJ

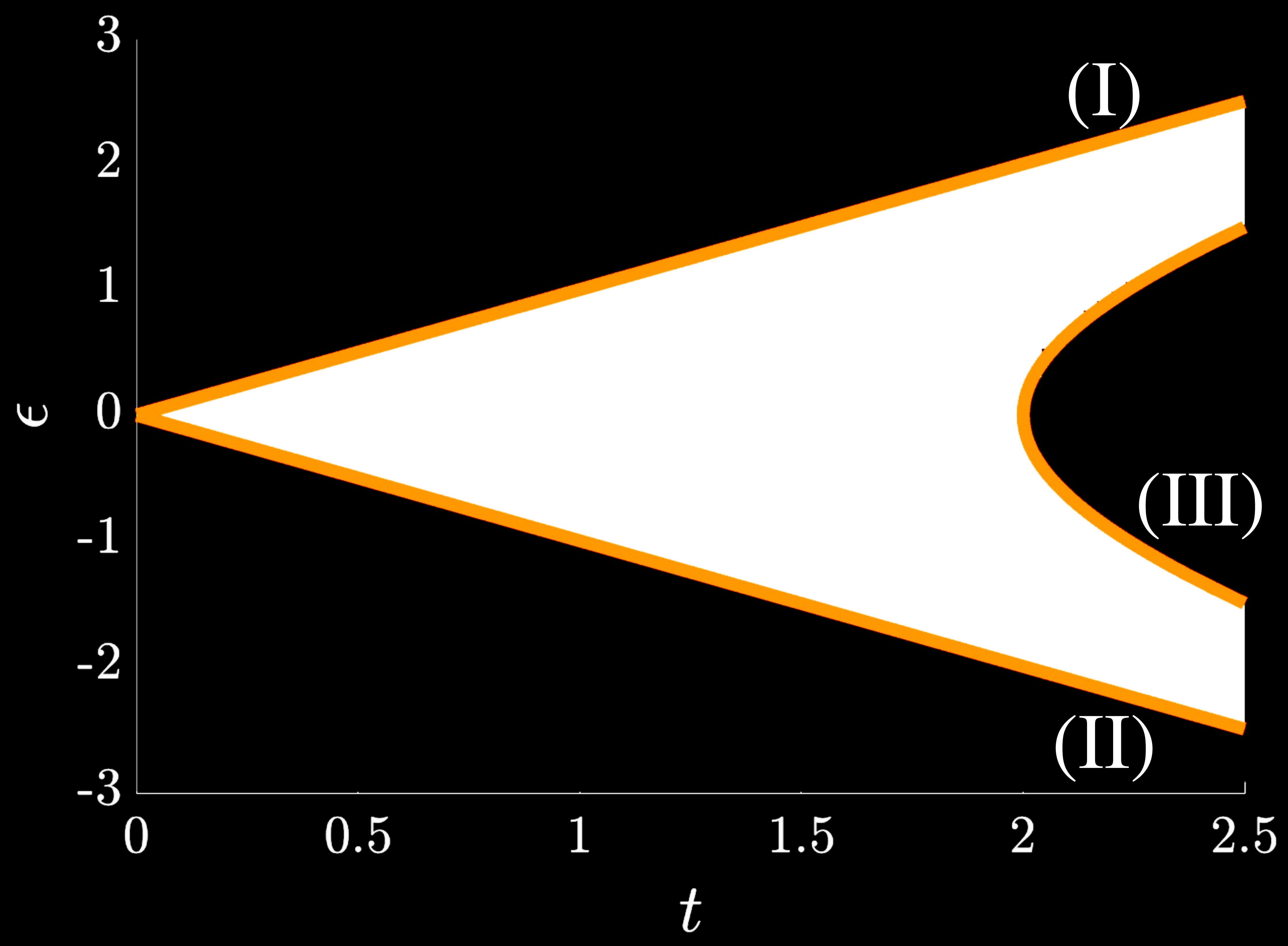


Normal state Scattering matrix

$$S_{\mathrm{N}} = \begin{pmatrix} r_{2\times 2} & t_{2\times 2} \\ t'_{2\times 2} & r'_{2\times 2} \end{pmatrix}$$

4-terminal MTJJ

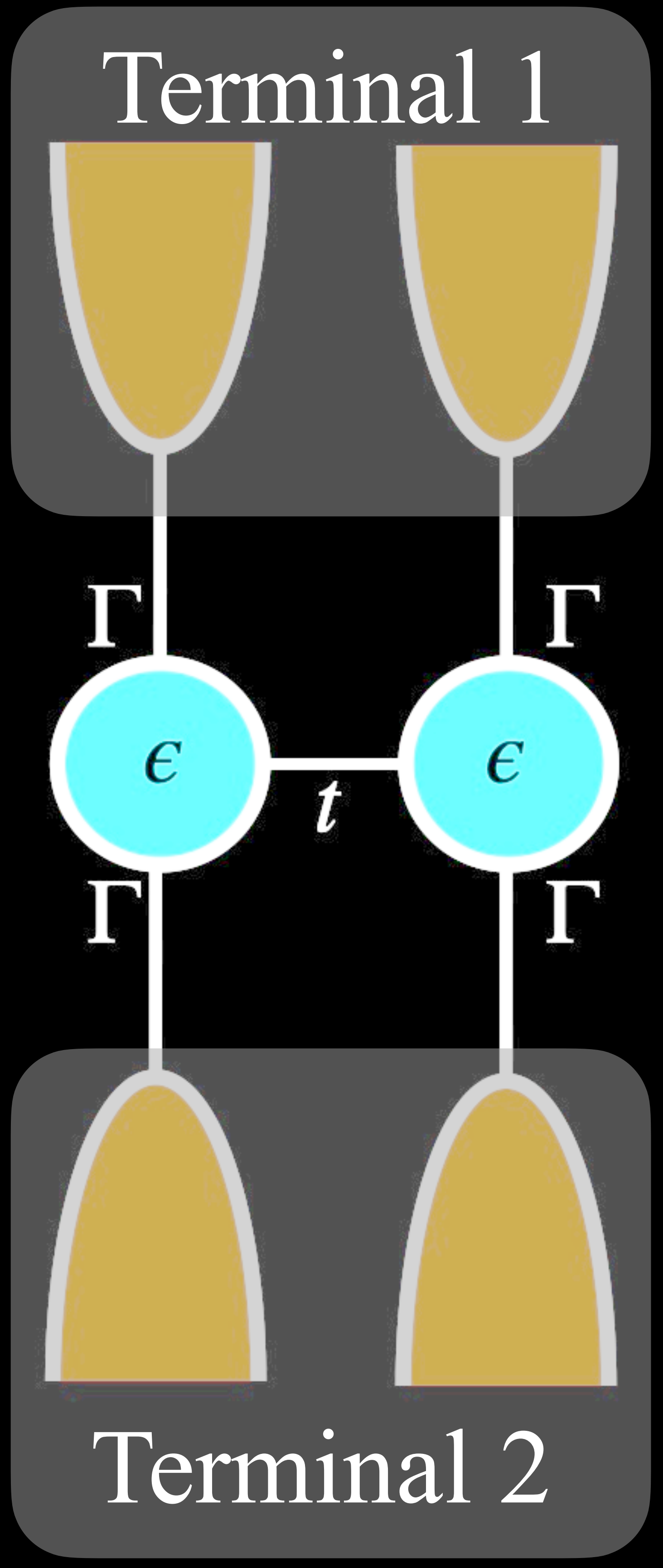




Reflectionless modes as source of Weyl Nodes

D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

effective 2-terminal JJ



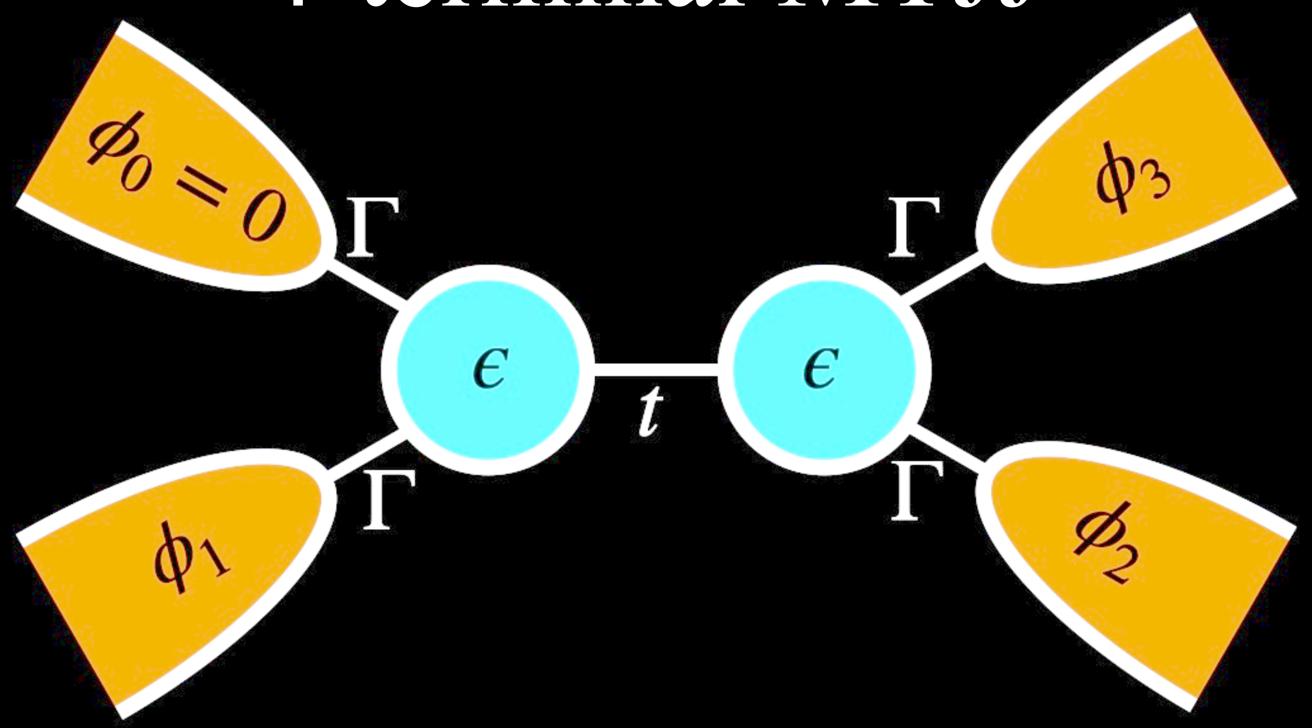
Normal state Scattering matrix

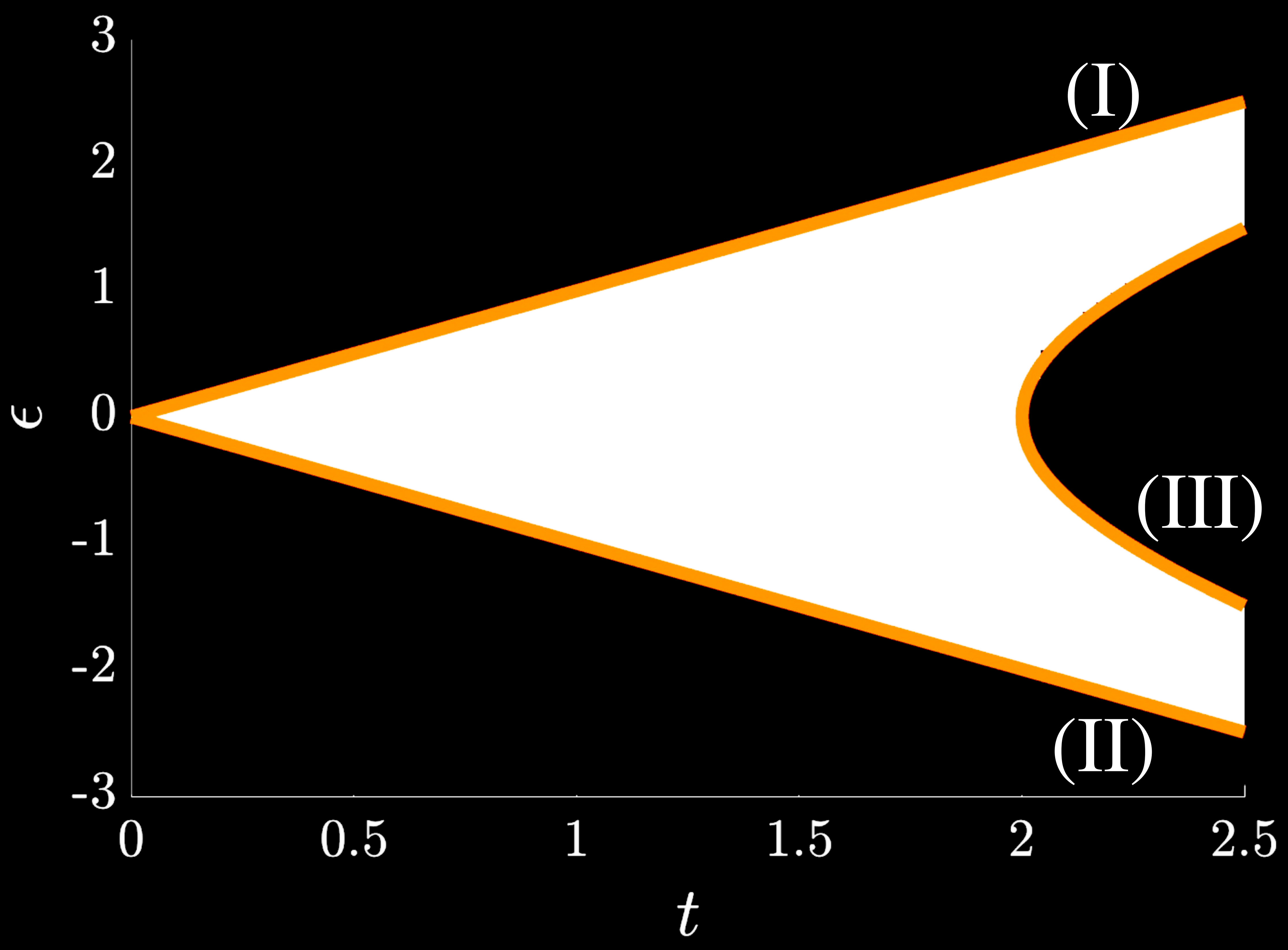
$$S_N = \begin{pmatrix} r_{2\times2} & t_{2\times2} \\ t'_{2\times2} & r'_{2\times2} \end{pmatrix}$$

Diagonalized refl. matrix

$$D_{r_{2\times2}} = \begin{pmatrix} \frac{E-(\epsilon-t)}{E-(\epsilon-t-2\imath\Gamma)} & 0 \\ 0 & \frac{E-(\epsilon+t)}{E-(\epsilon+t-2\imath\Gamma)} \end{pmatrix}$$

4-terminal MTJJ

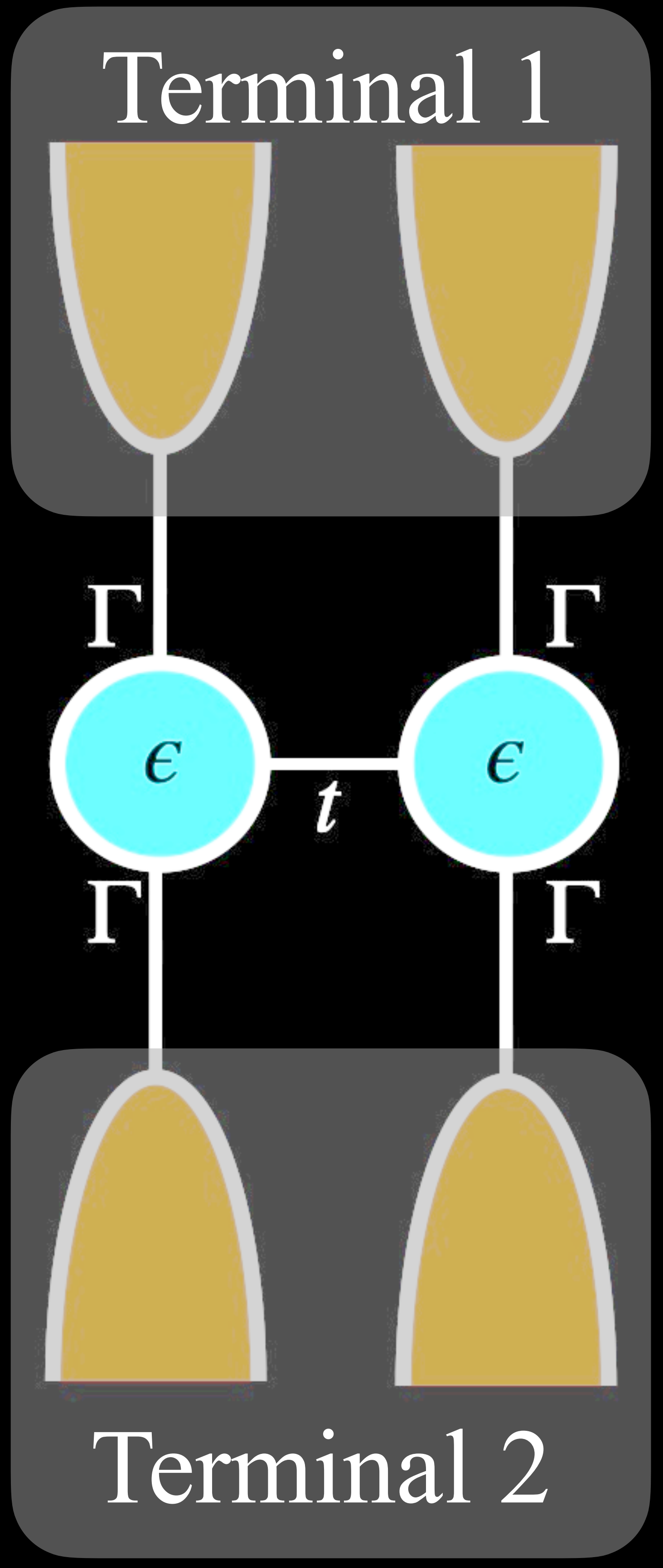




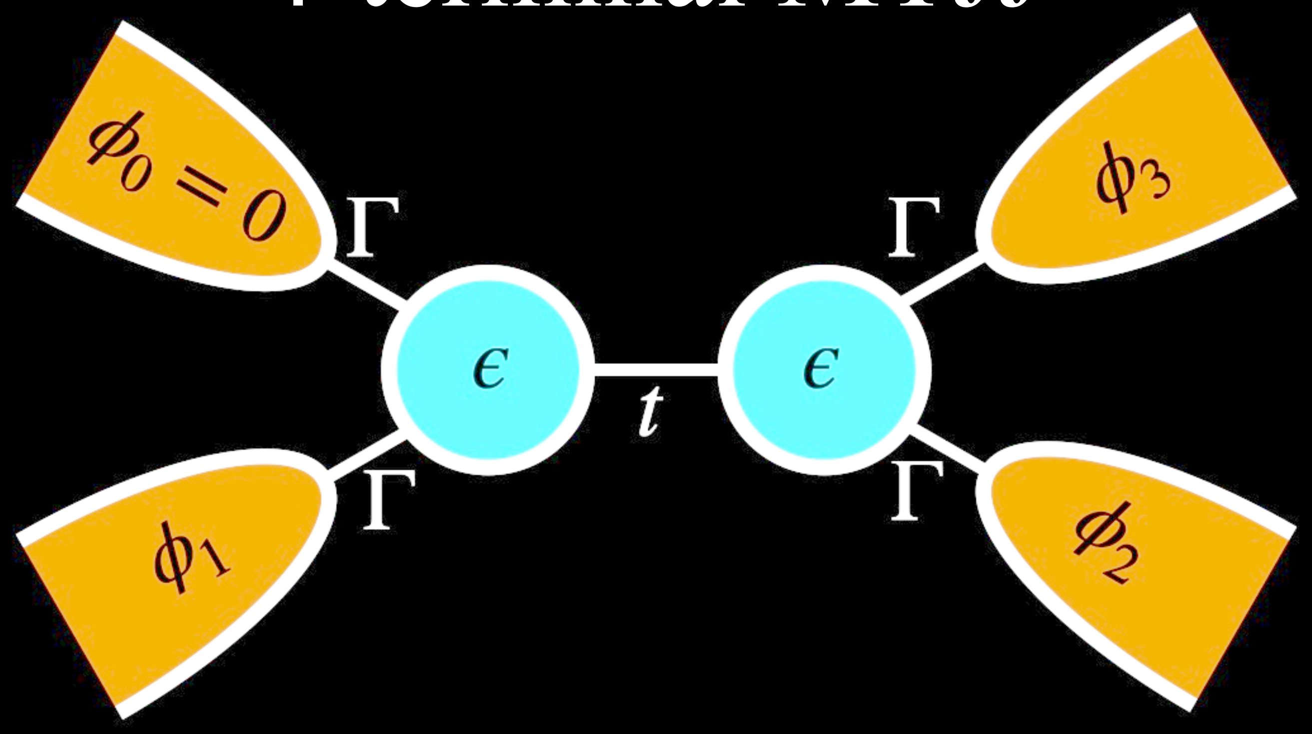
Reflectionless modes as source of Weyl Nodes

D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

effective 2-terminal JJ



4-terminal MTJJ

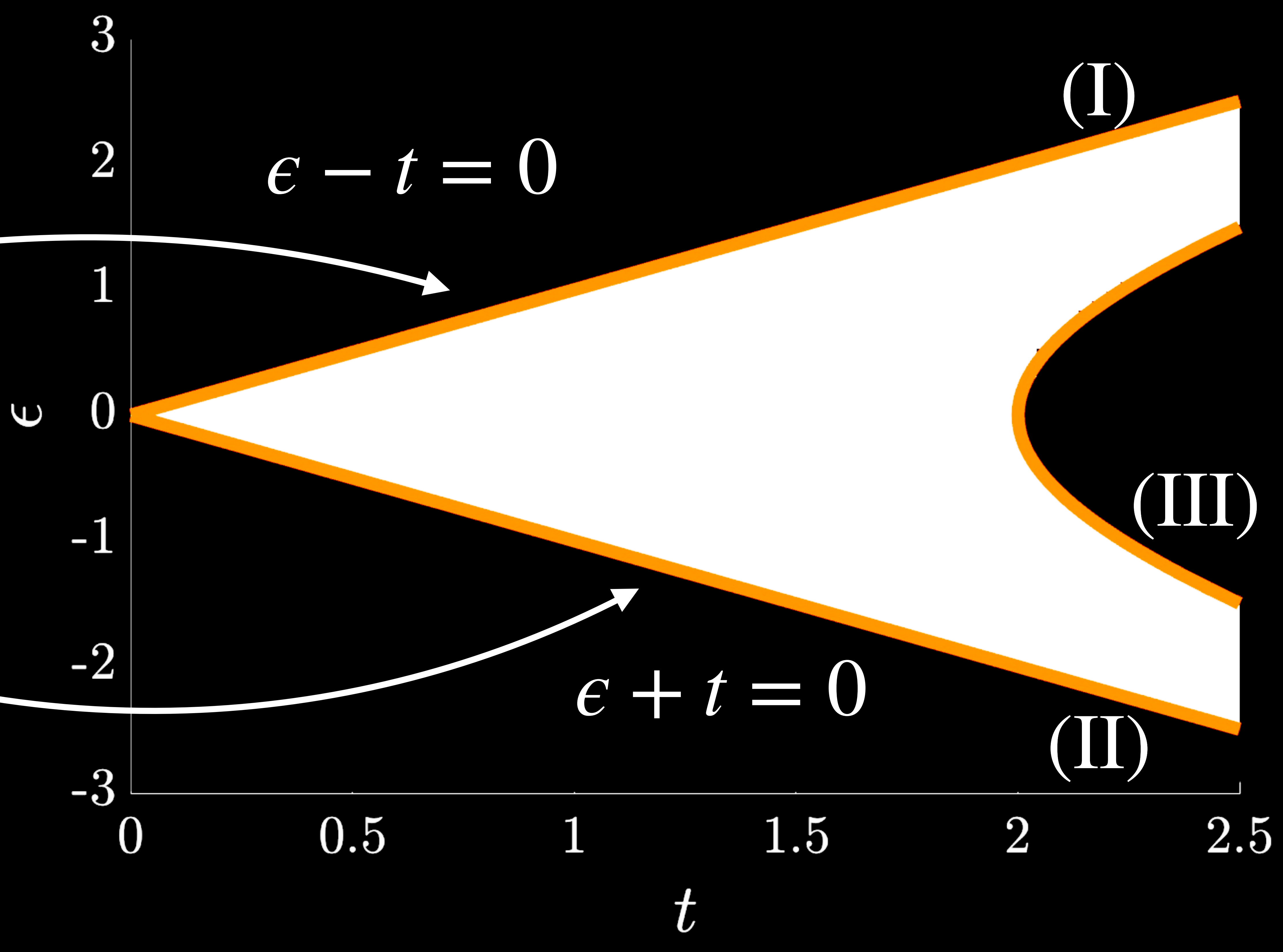


Normal state
Scattering matrix

$$S_{\mathrm{N}} = \begin{pmatrix} r_{2\times2} & t_{2\times2} \\ t'_{2\times2} & r'_{2\times2} \end{pmatrix}$$

Diagonalized refl. matrix

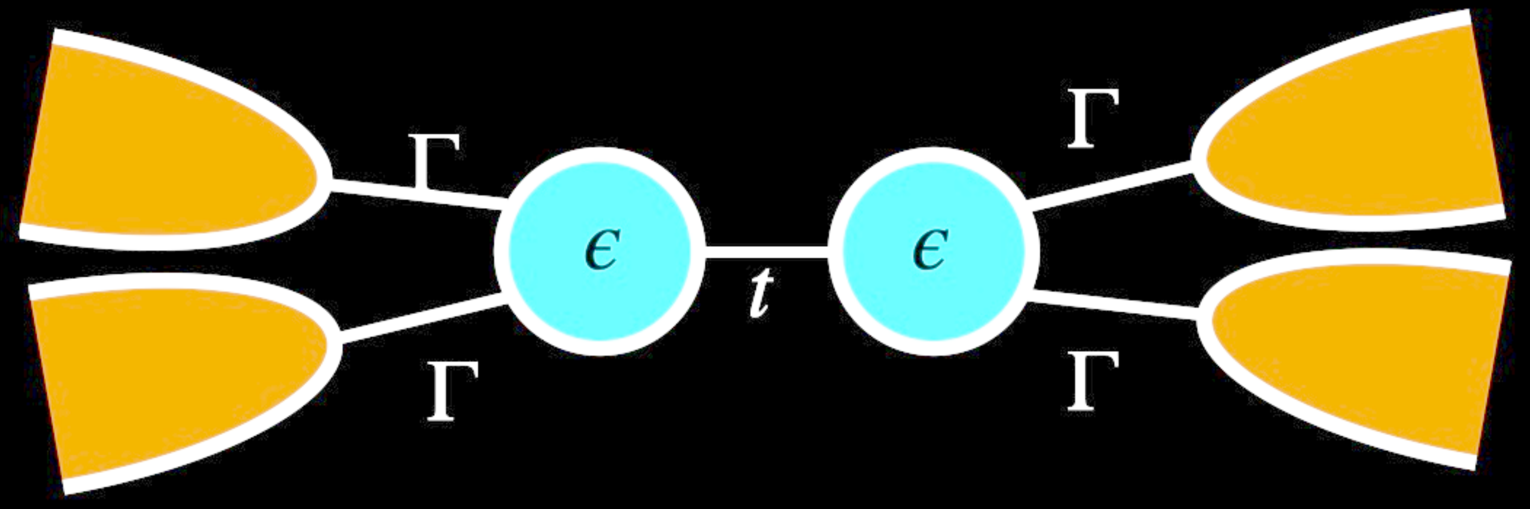
$$D_{r_{2\times2}} = \begin{pmatrix} \dfrac{E-(\epsilon-t)}{E-(\epsilon-t-2\imath\Gamma)} & 0 \\ 0 & \dfrac{E-(\epsilon+t)}{E-(\epsilon+t-2\imath\Gamma)} \end{pmatrix}$$



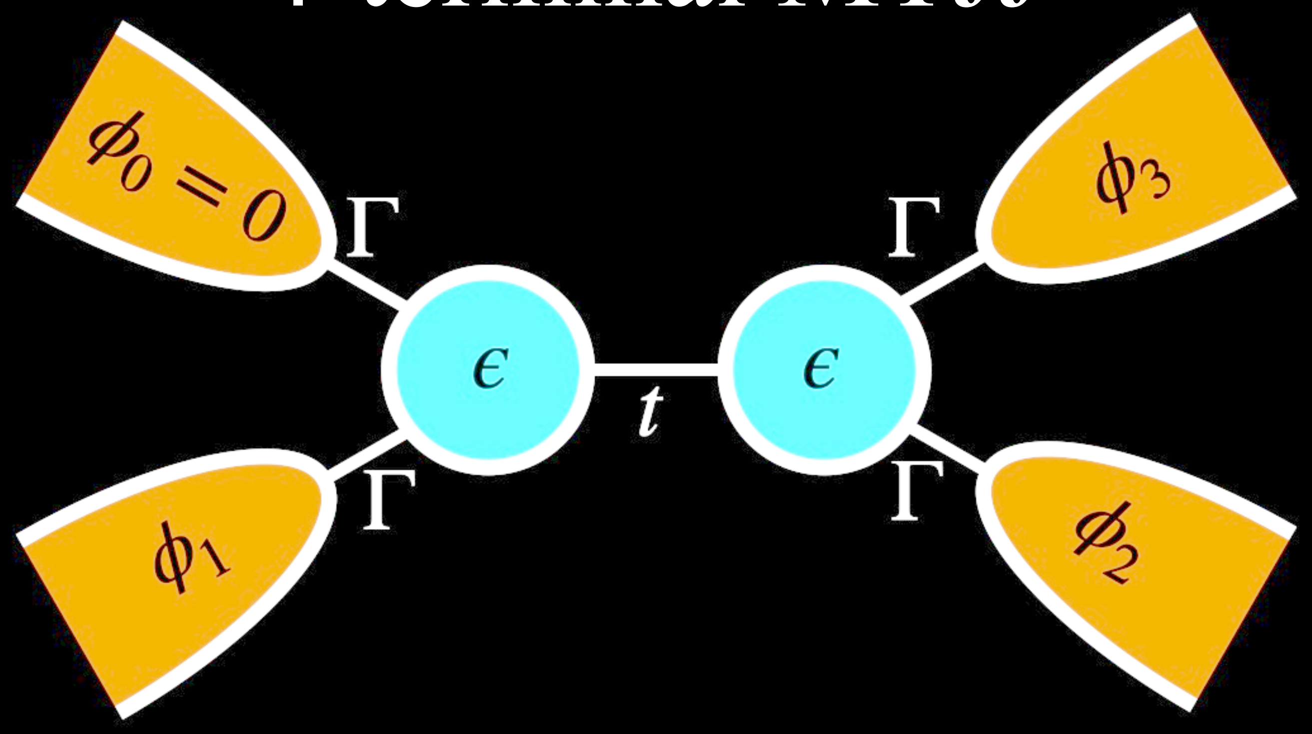
Reflectionless modes as source of Weyl Nodes

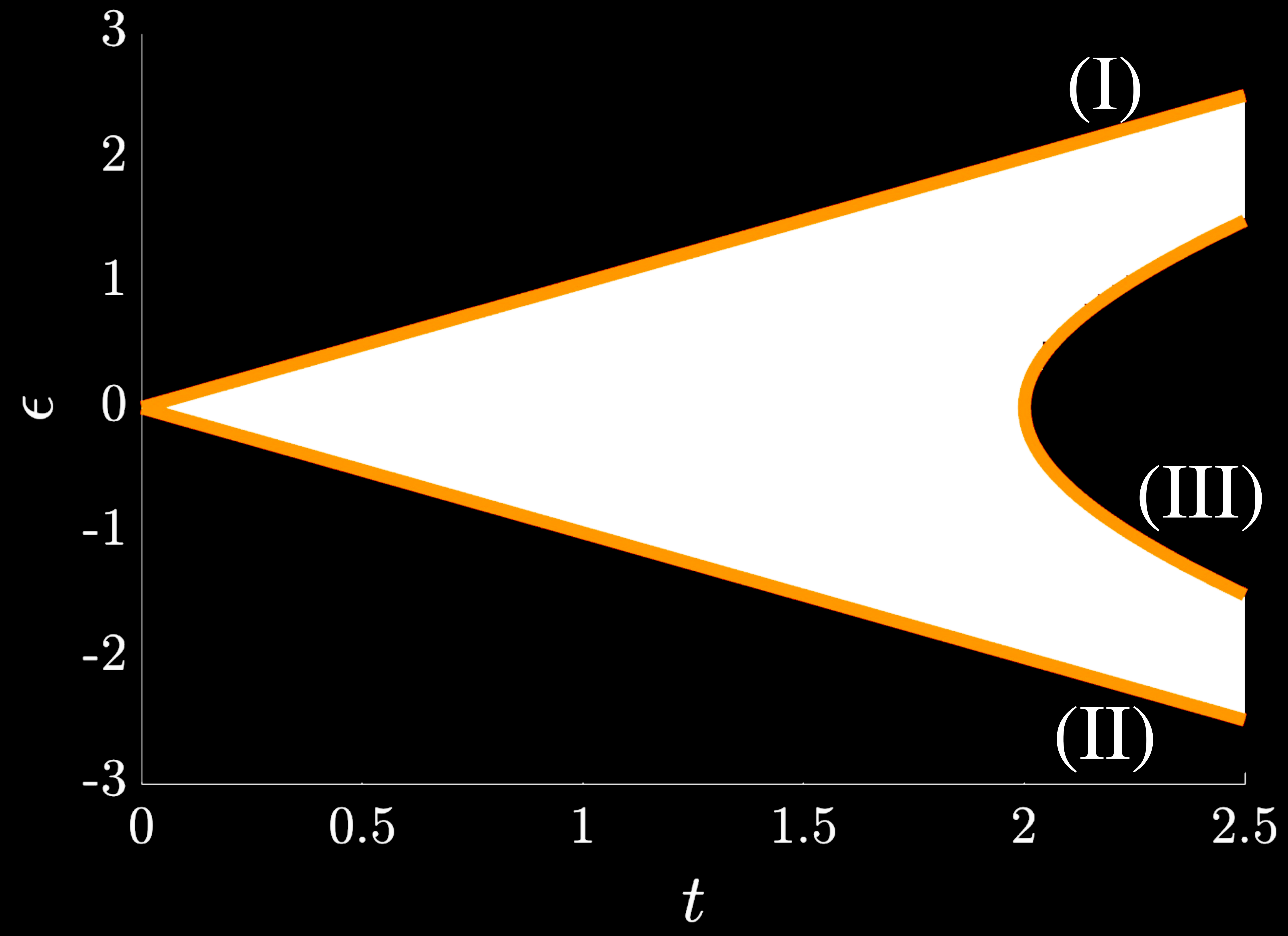
D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

effective 2-terminal JJ



4-terminal MTJJ

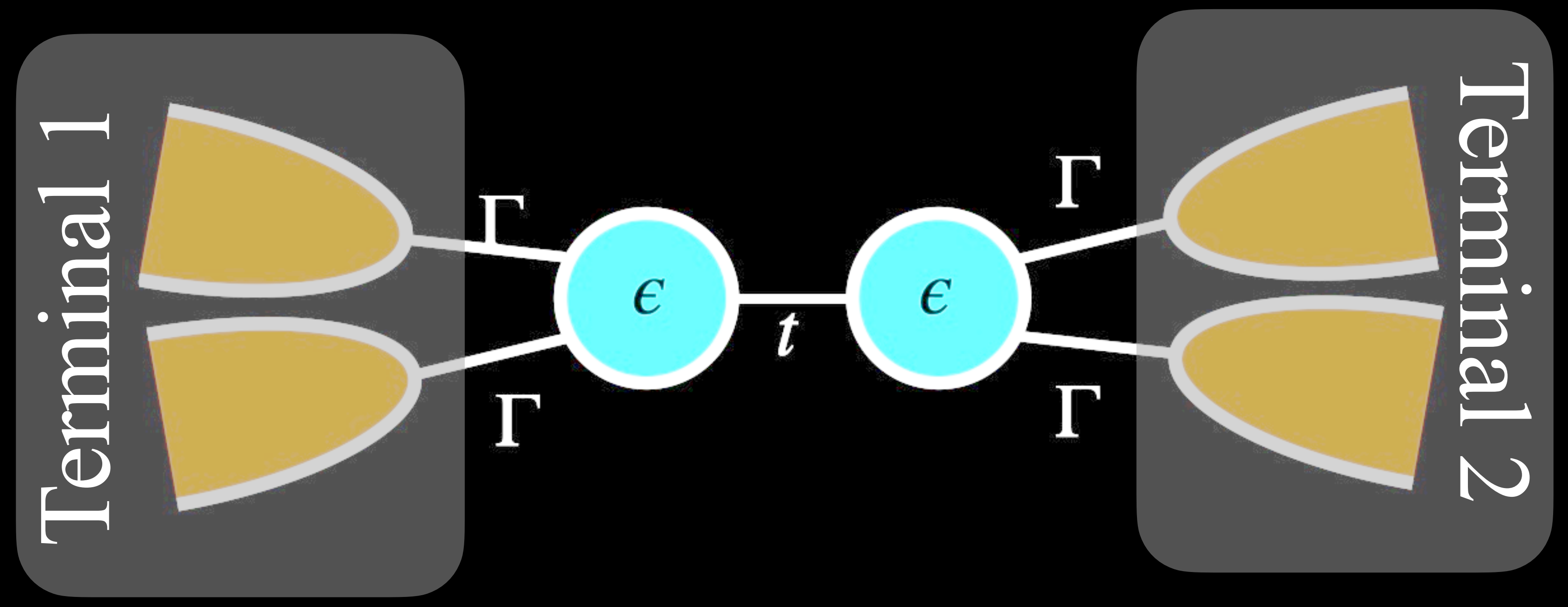




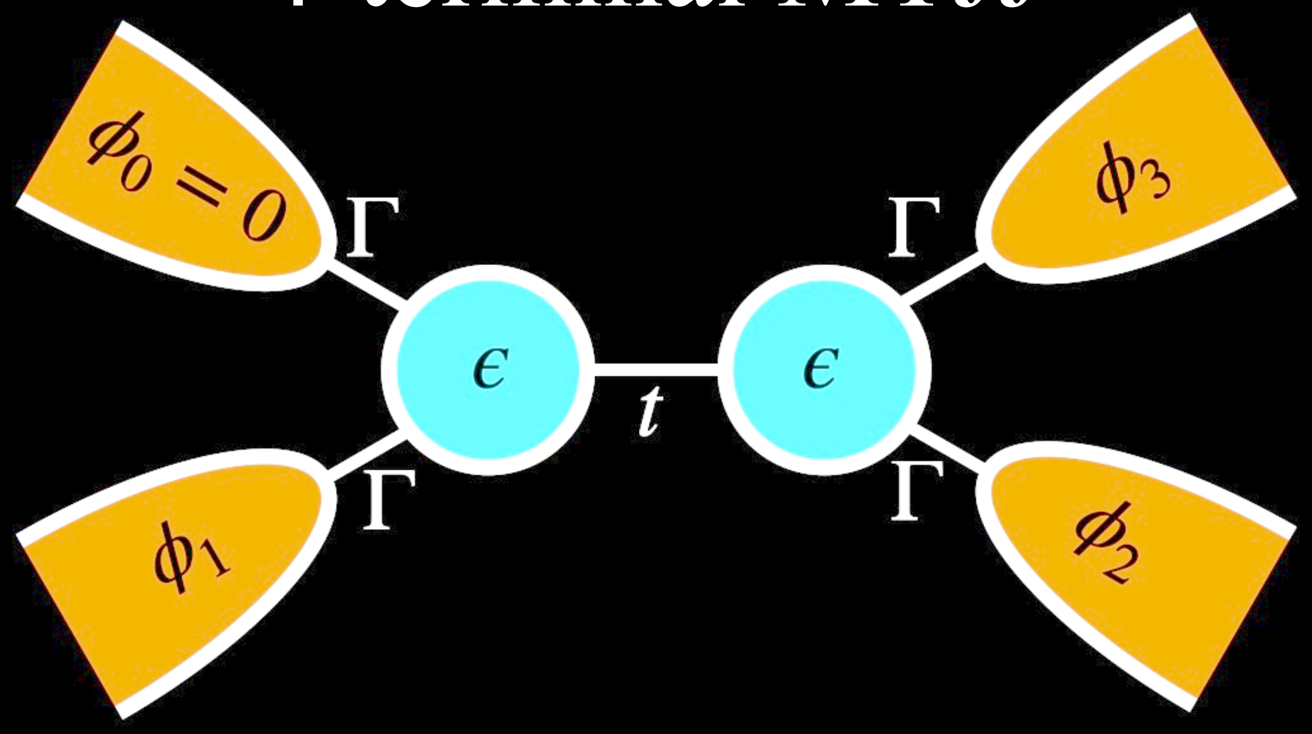
Reflectionless modes as source of Weyl Nodes

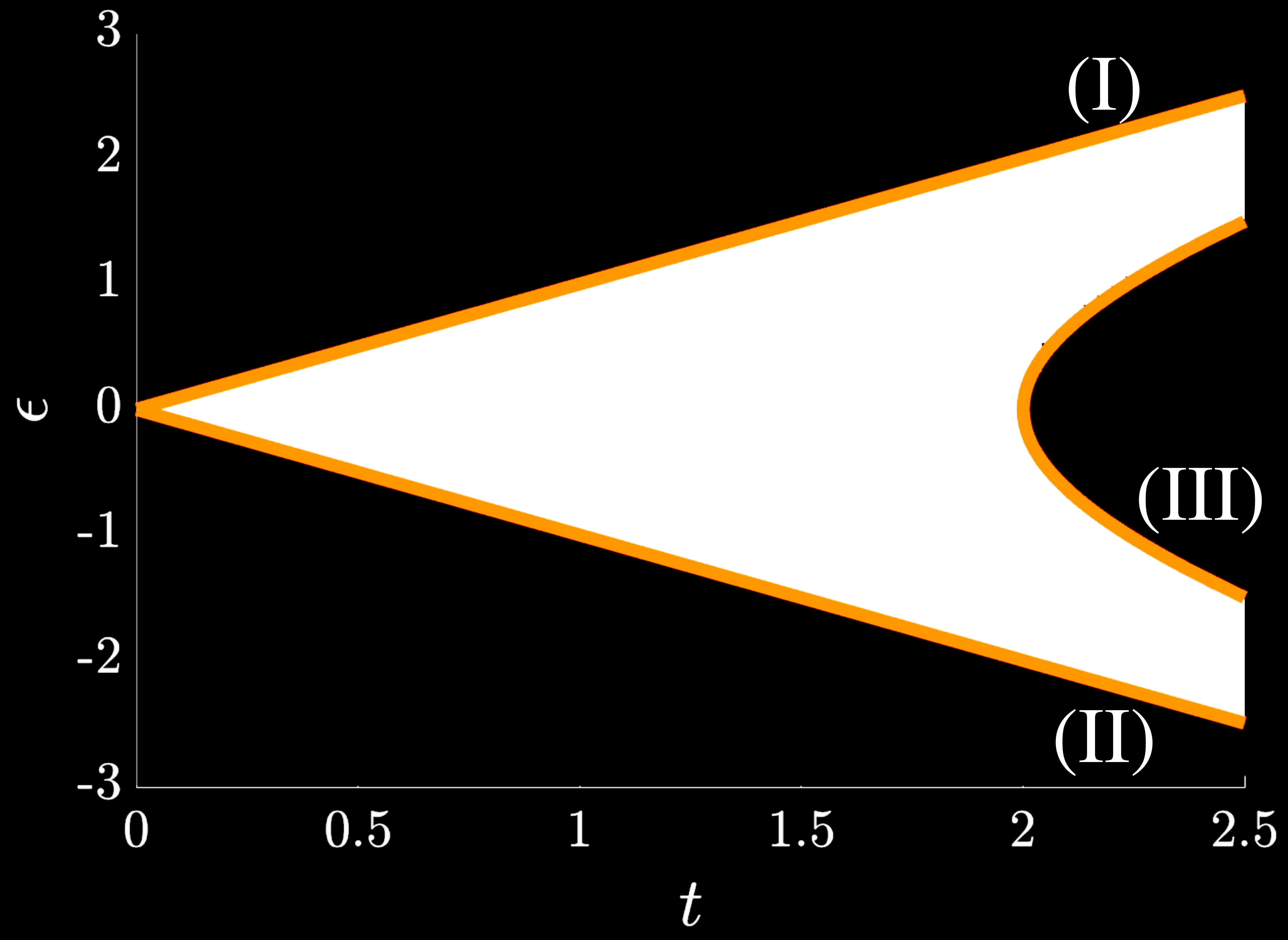
D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

effective 2-terminal JJ



4-terminal MTJJ

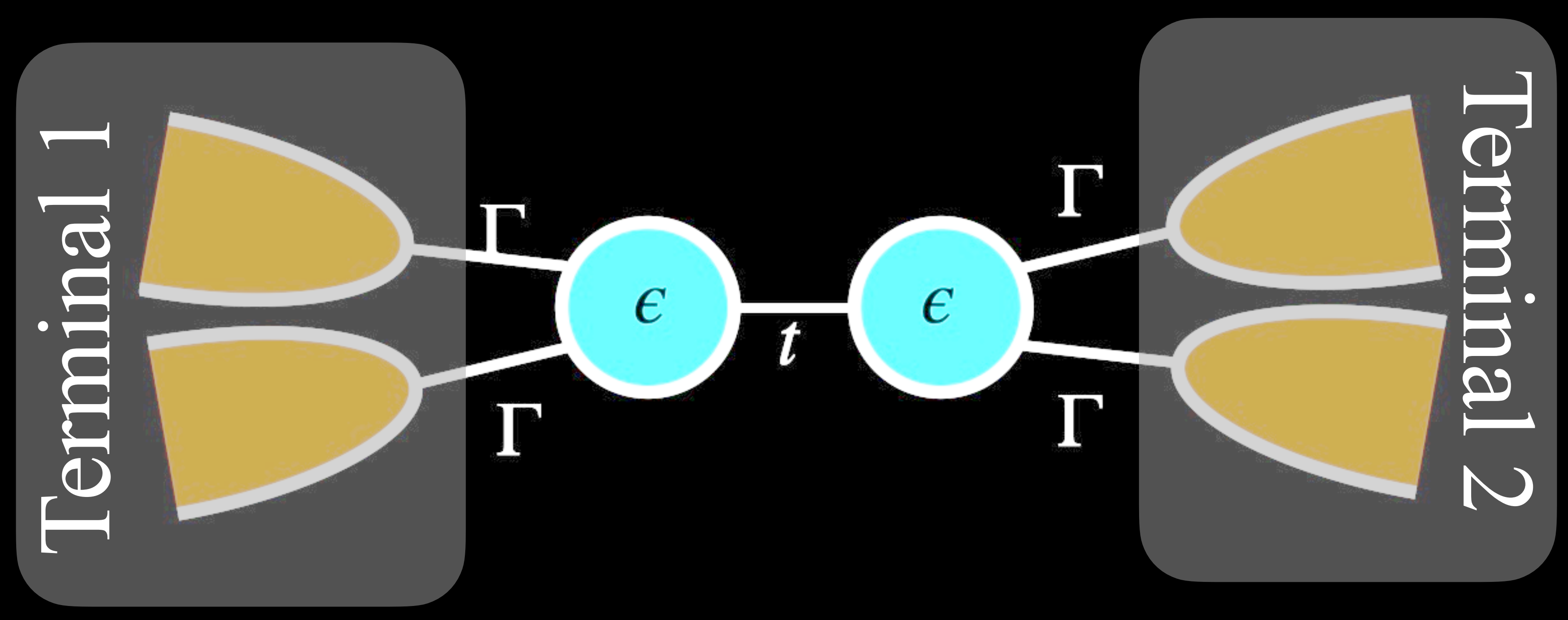




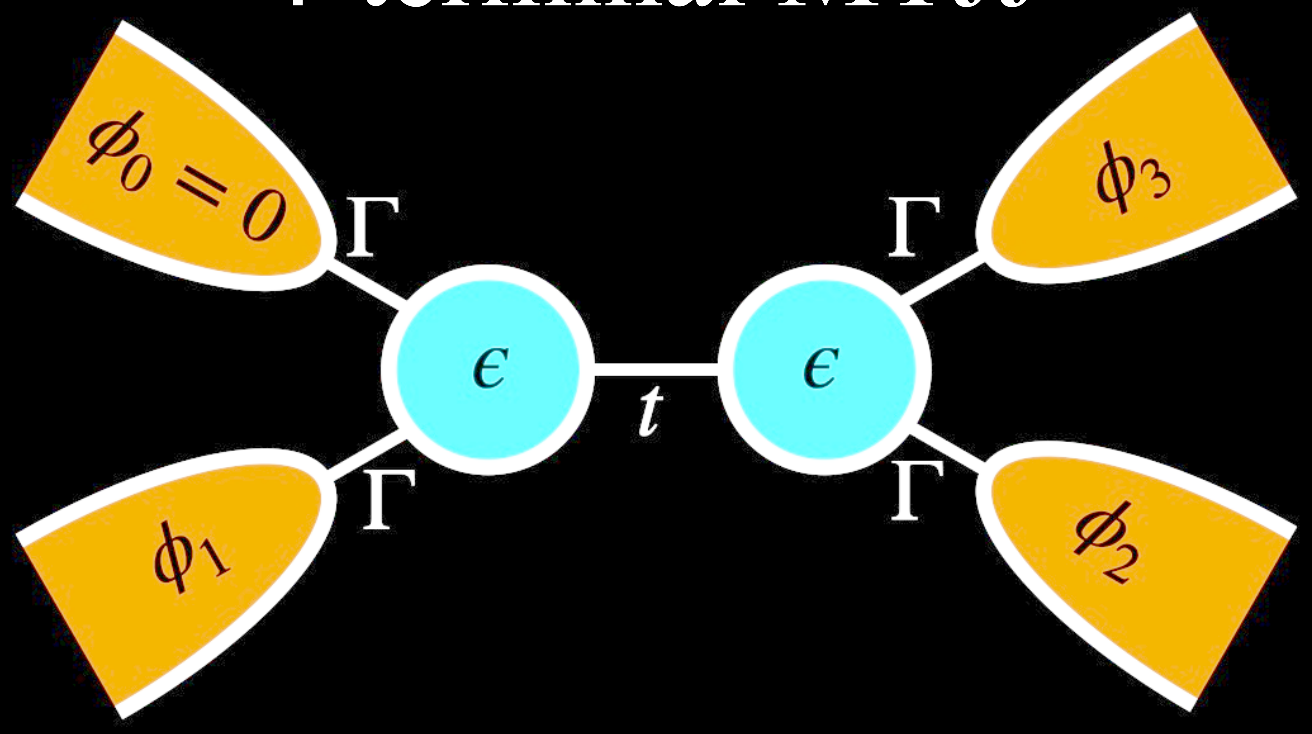
Reflectionless modes as source of Weyl Nodes

D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

effective 2-terminal JJ

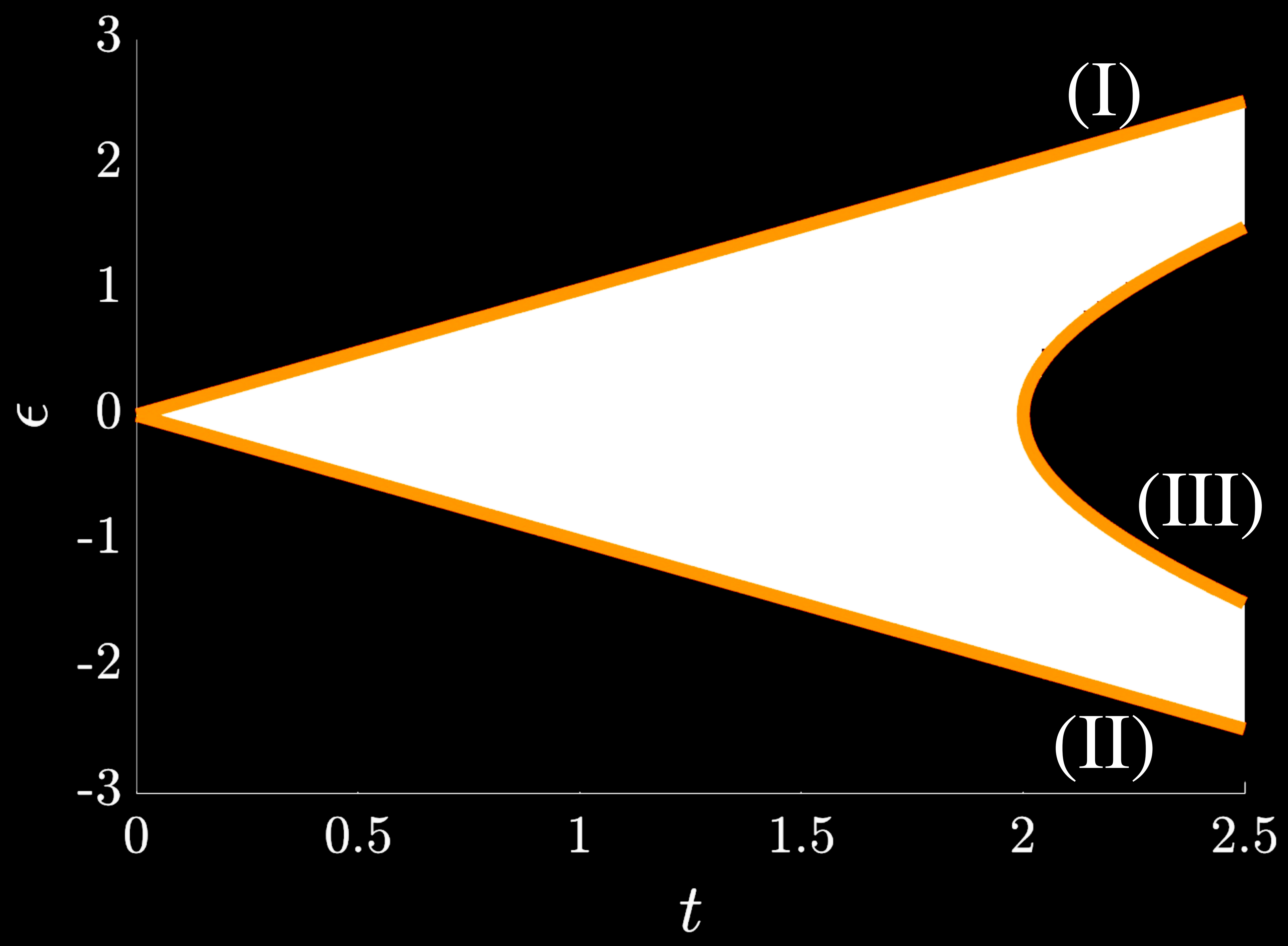


4-terminal MTJJ



Scattering matrix

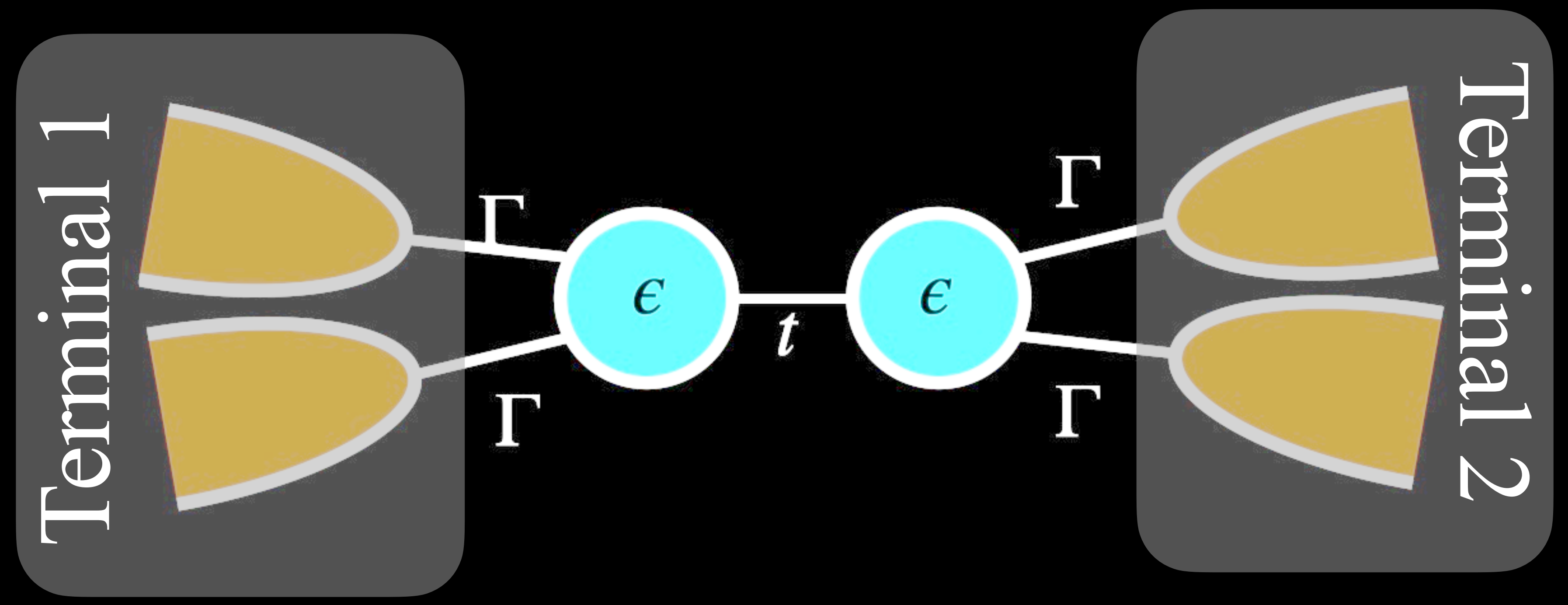
$$S_{\mathrm{N}} = \begin{pmatrix} r_{2\times2} & t_{2\times2} \\ t'_{2\times2} & r'_{2\times2} \end{pmatrix}$$



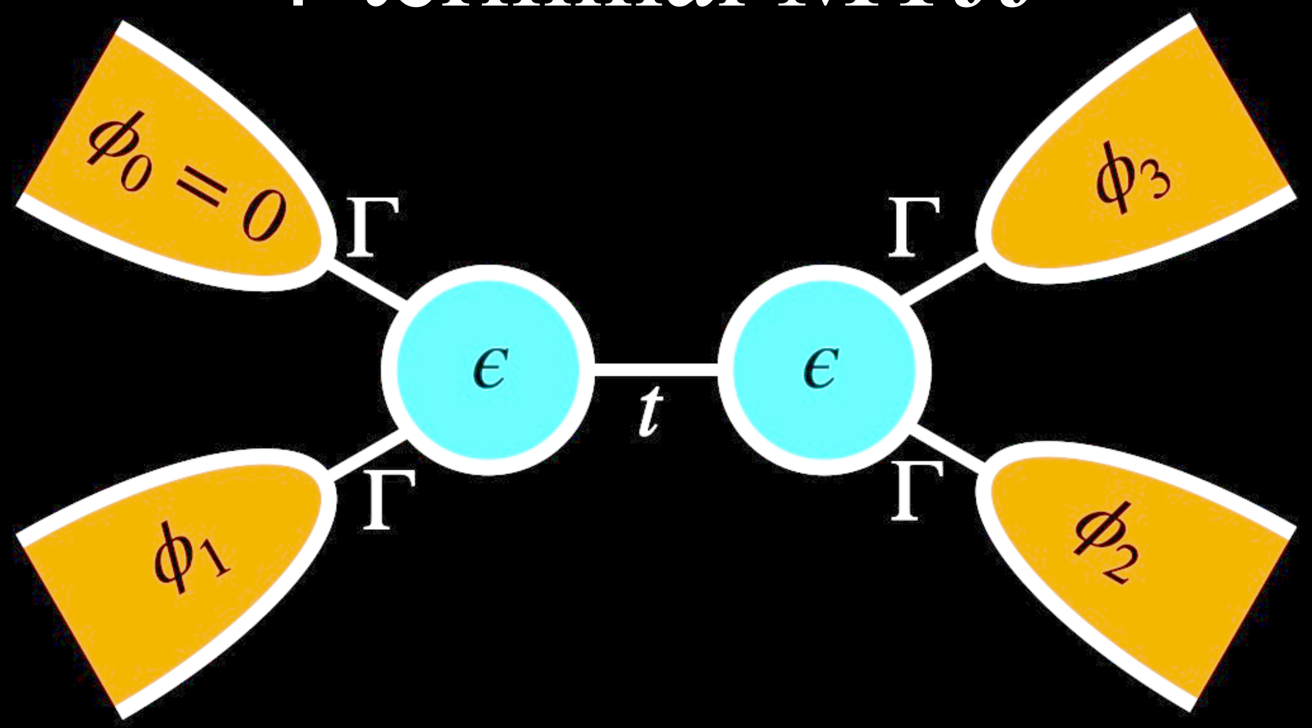
Reflectionless modes as source of Weyl Nodes

D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

effective 2-terminal JJ



4-terminal MTJJ

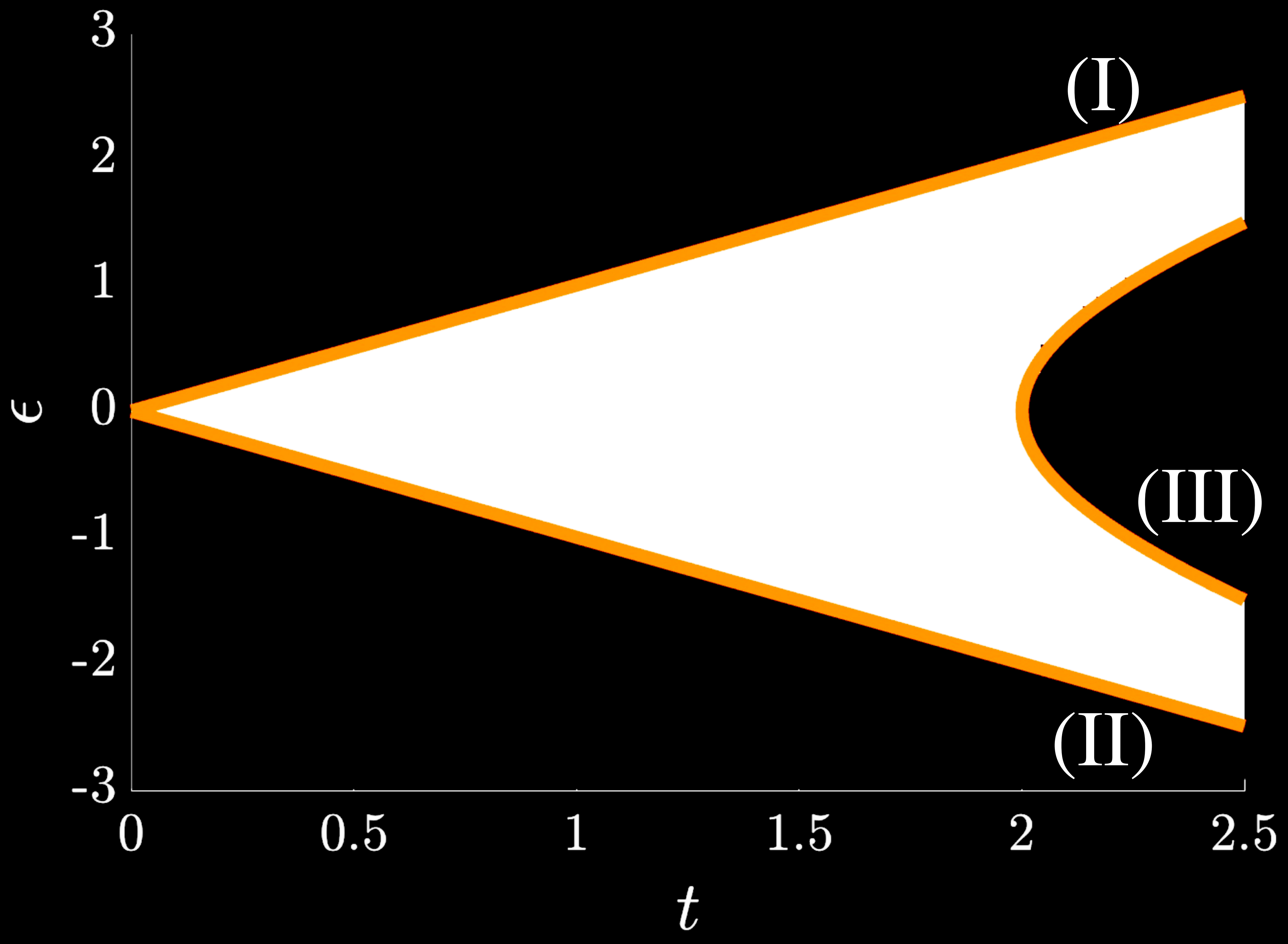


Scattering matrix

$$S_{\mathrm{N}} = \begin{pmatrix} r_{2\times 2} & t_{2\times 2} \\ t'_{2\times 2} & r'_{2\times 2} \end{pmatrix}$$

Diagonalized refl. matrix

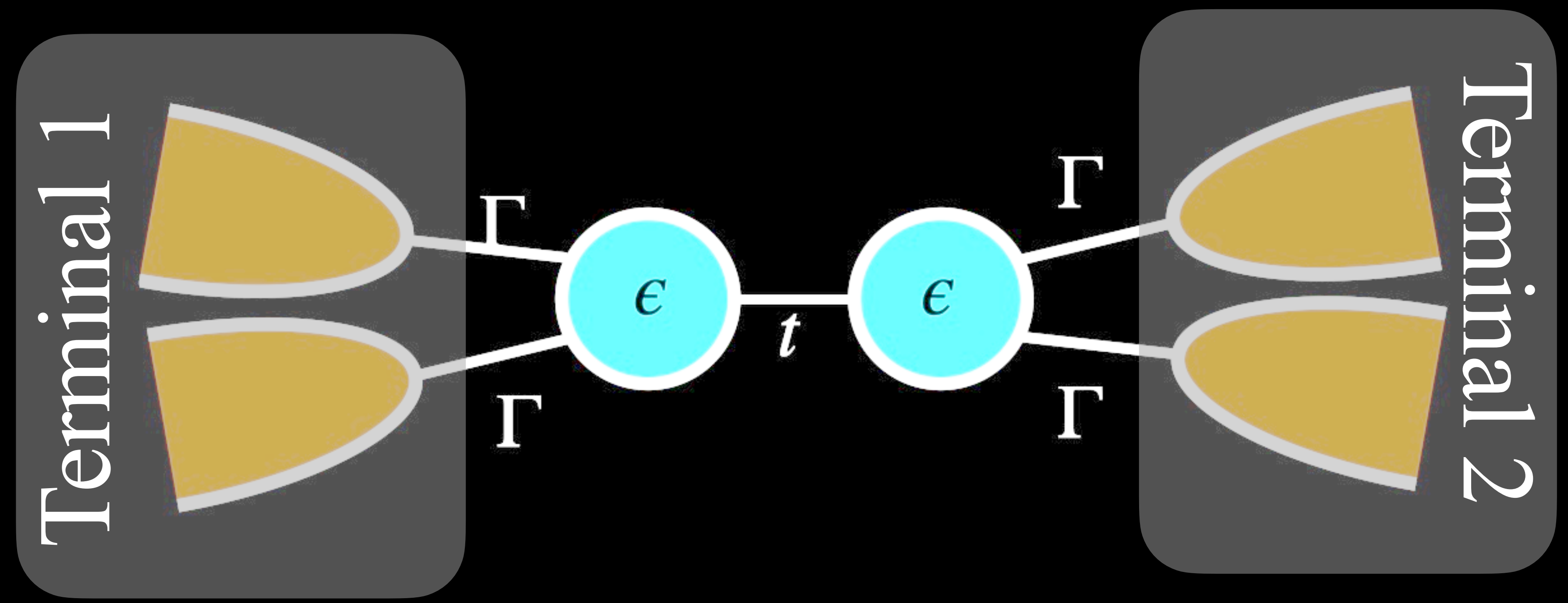
$$D_{r_{2\times 2}} = \begin{pmatrix} \left[E(E-e) - \left(t^2 - \epsilon^2 - 4\Gamma^2\right)\right]/D(E) & 0 \\ 0 & 1 \end{pmatrix}$$



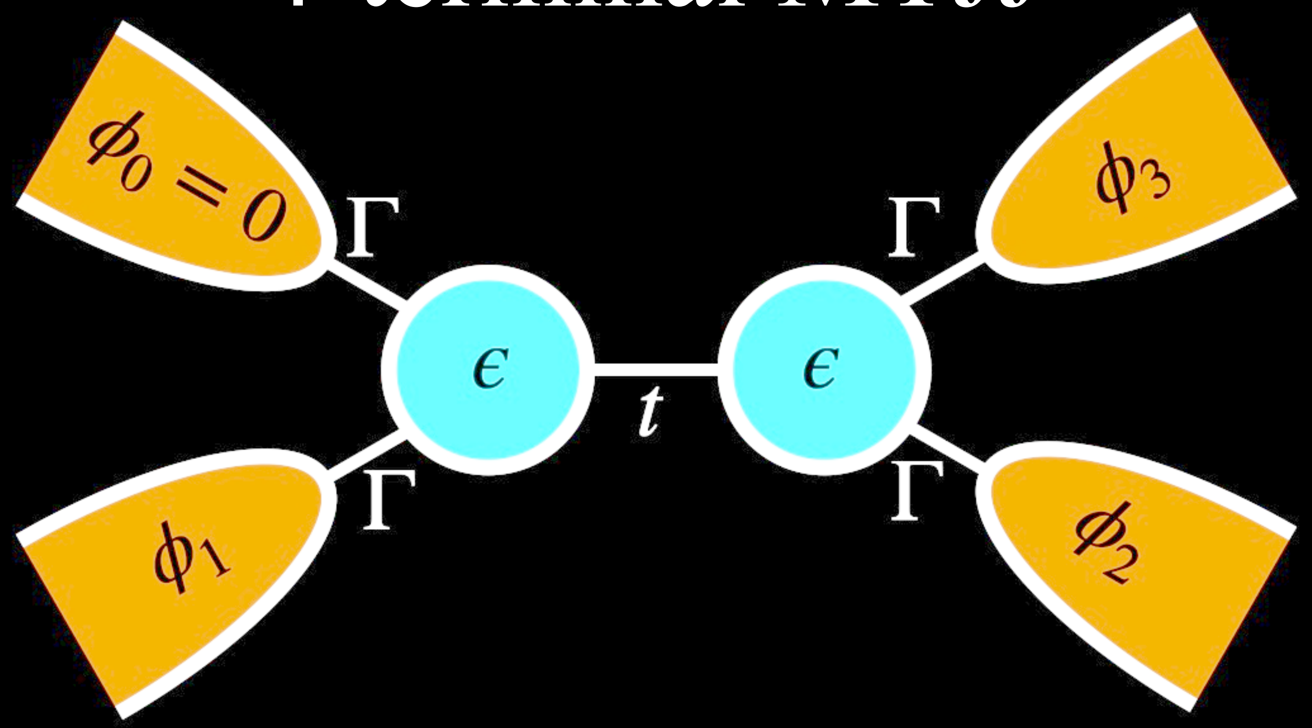
Reflectionless modes as source of Weyl Nodes

D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

effective 2-terminal JJ



4-terminal MTJJ

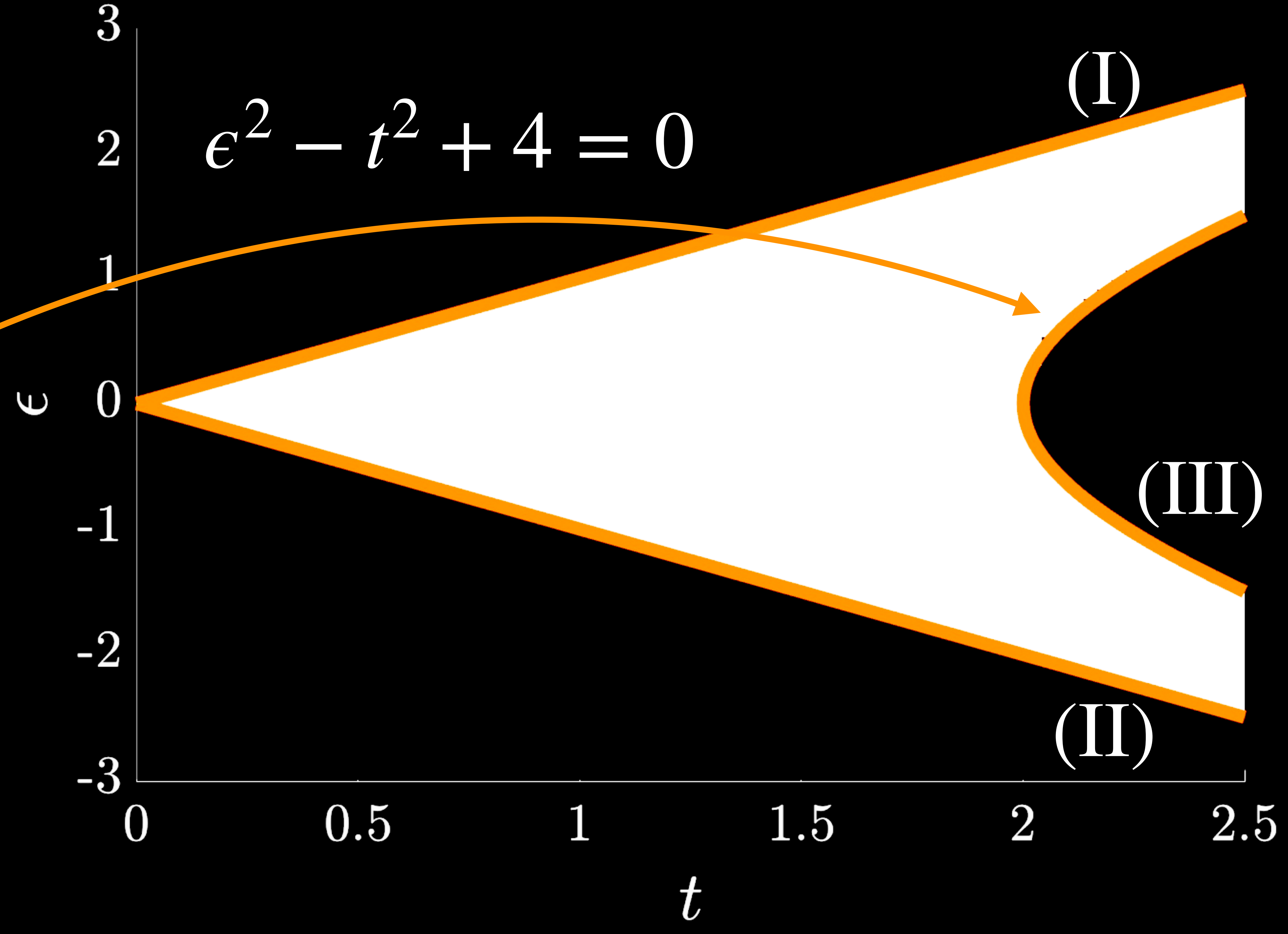


Scattering matrix

$$S_{\mathrm{N}} = \begin{pmatrix} r_{2\times2} & t_{2\times2} \\ t'_{2\times2} & r'_{2\times2} \end{pmatrix}$$

Diagonalized refl. matrix

$$D_{r_{2\times2}} = \begin{pmatrix} \left[E(E-e) - \left(t^2 - \epsilon^2 - 4\Gamma^2\right)\right]/D(E) & 0 \\ 0 & 1 \end{pmatrix}$$



Conclusion I

1) MTJJs are an excellent platform to study engineered ~~(non-hermitian)~~ topology

Topology in three state Andreev Molecule
T. Antonelli, et. al., arXiv:2501.07982 (2025)

2) Refectionless scattering modes are a source of topology in MTJJs

Reflectionless Modes lead to Weyl nodes in MTJJs
D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

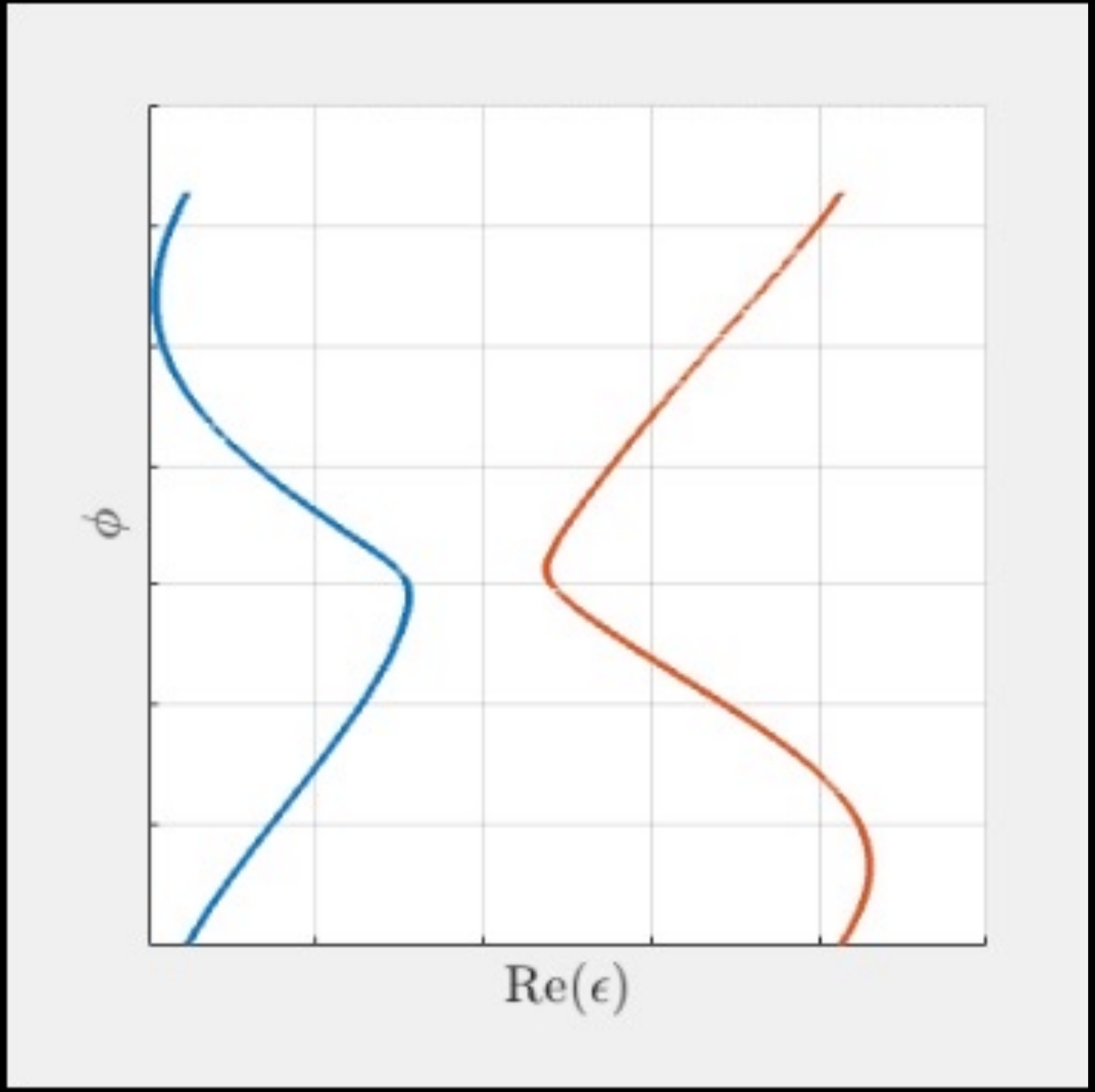
Hermitian physics

Hermitian

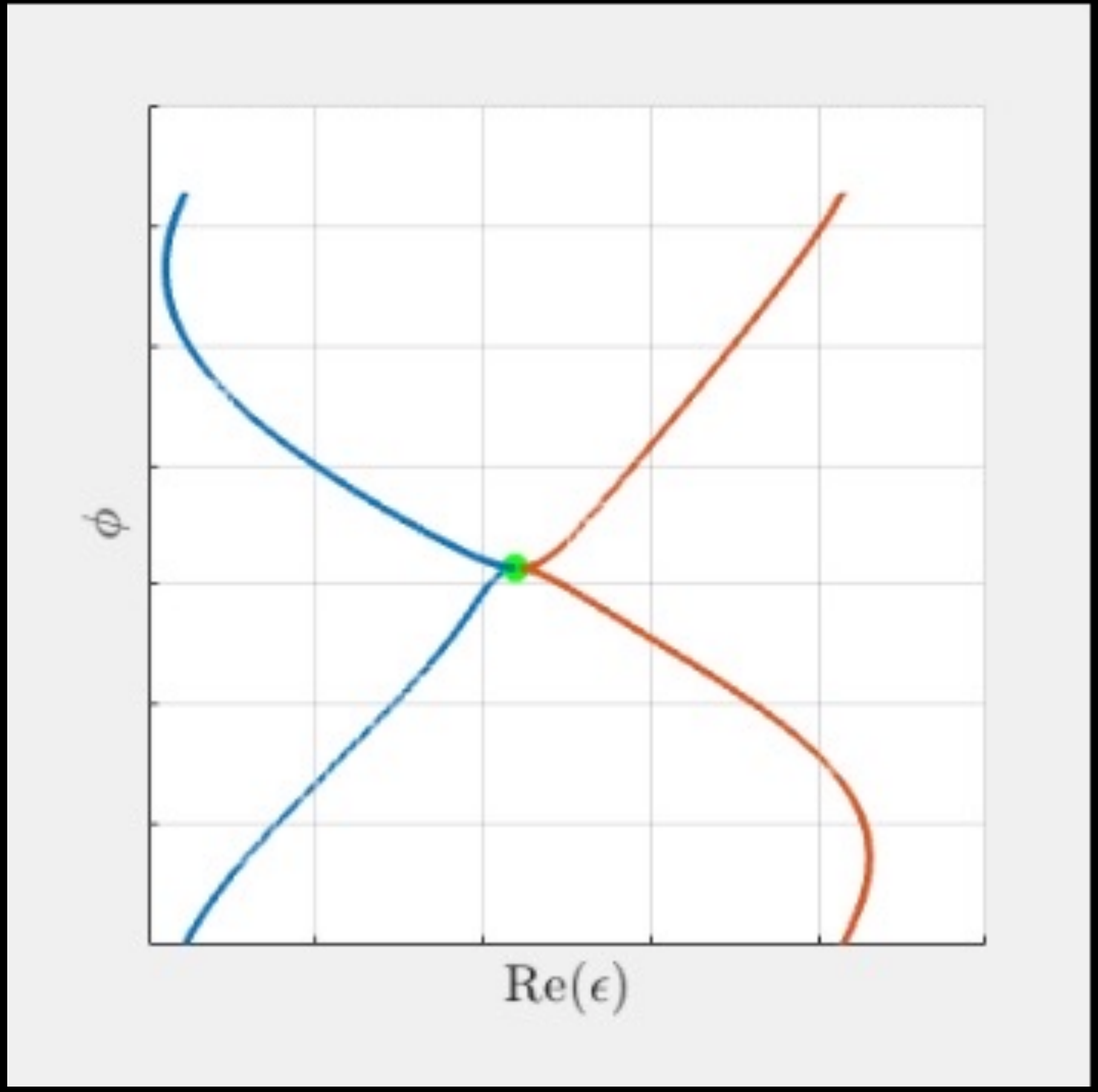
$$H = H^\dagger$$

$$\epsilon_\alpha \in \mathbb{R}$$

Gapped



Ungapped



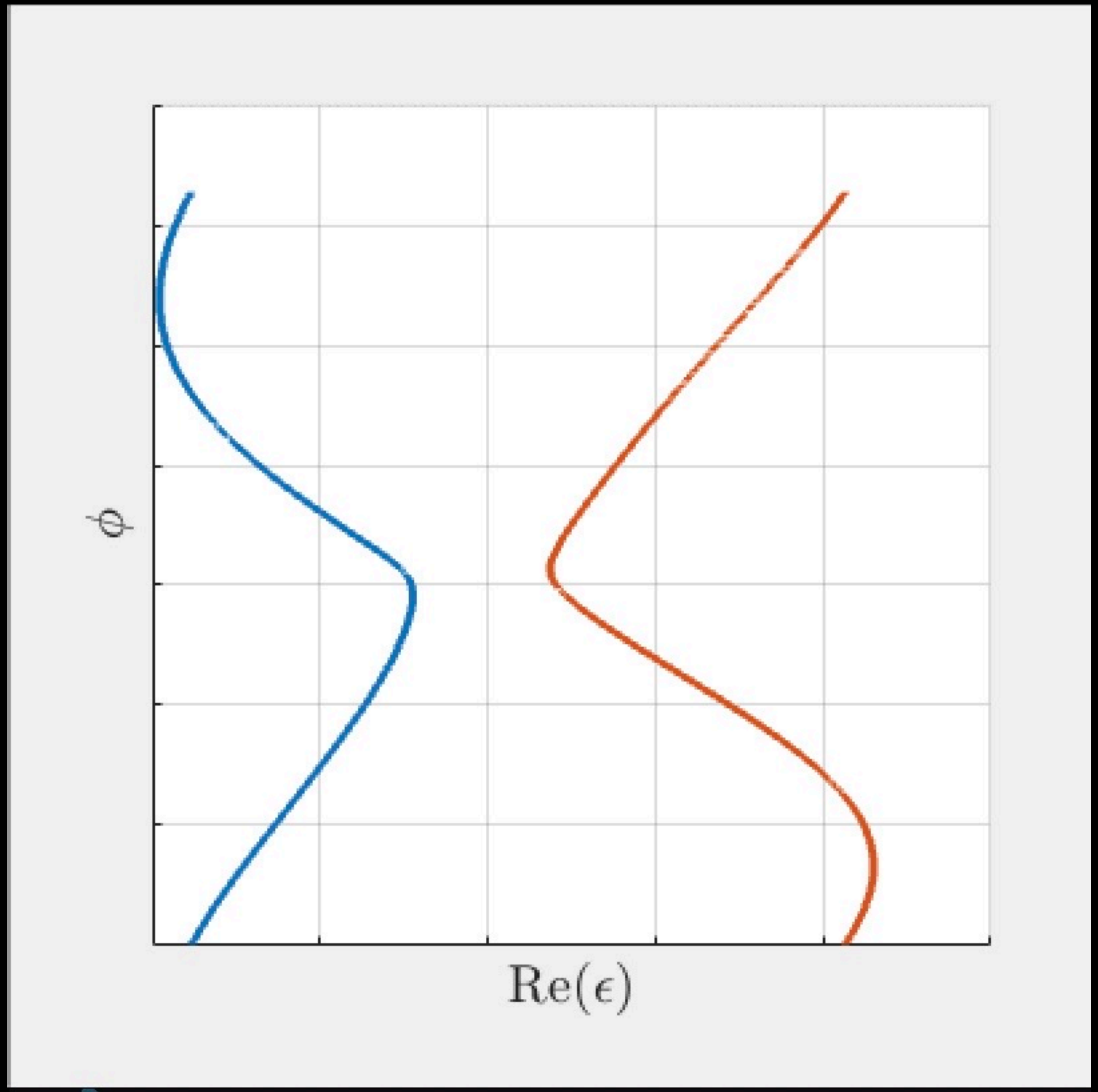
Non-hermitian physics

Kohei Kawabata, et. al., Phys. Rev. X 9, 041015 (2019)

non-Hermitian

$$H \neq H^\dagger$$

$$\epsilon_\alpha \in \mathbb{C}$$



Non-hermitian physics

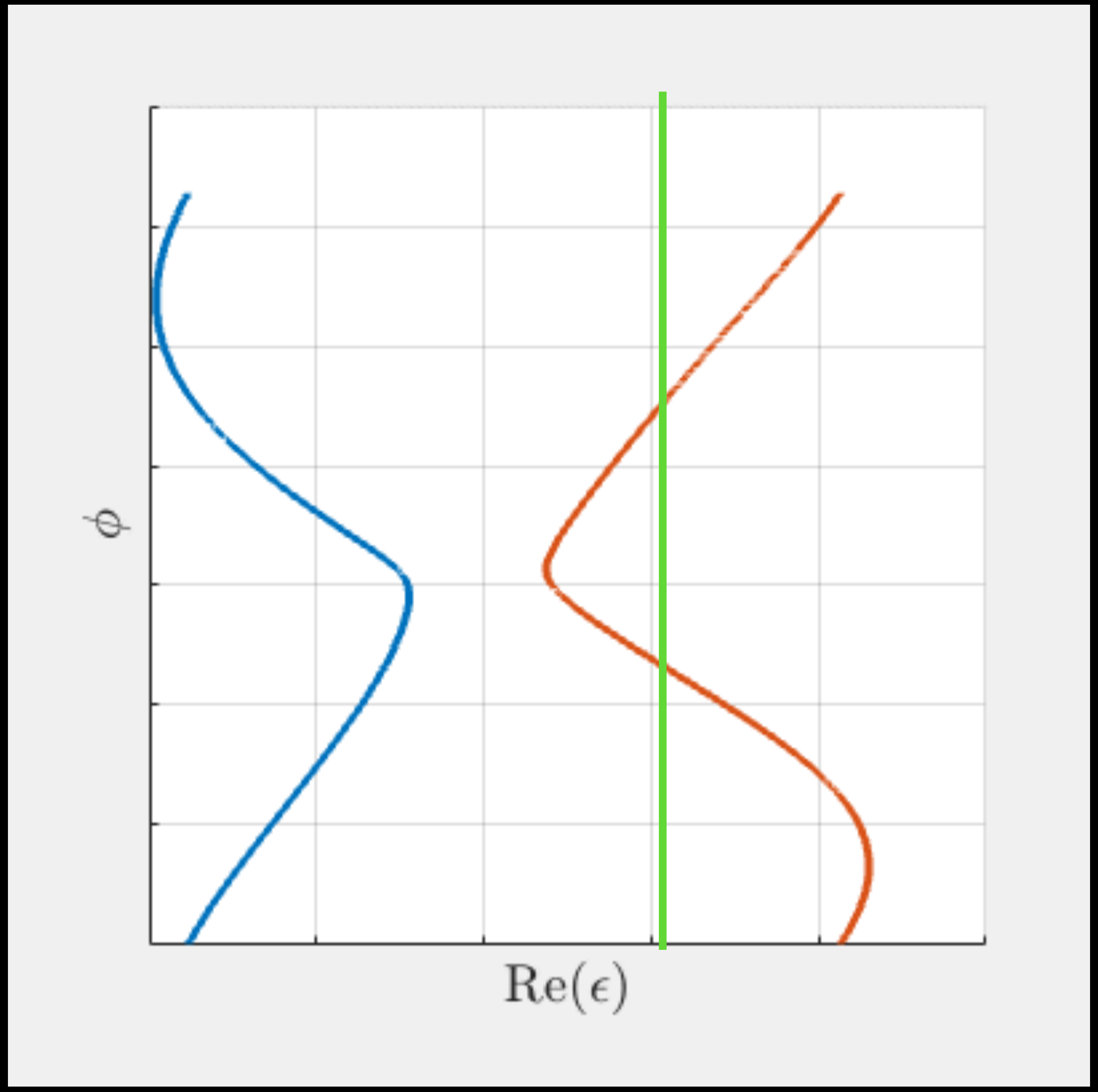
Kohei Kawabata, et. al., Phys. Rev. X 9, 041015 (2019)

non-Hermitian

$$H \neq H^\dagger$$

$$\epsilon_\alpha \in \mathbb{C}$$

Line Gap



Non-hermitian physics

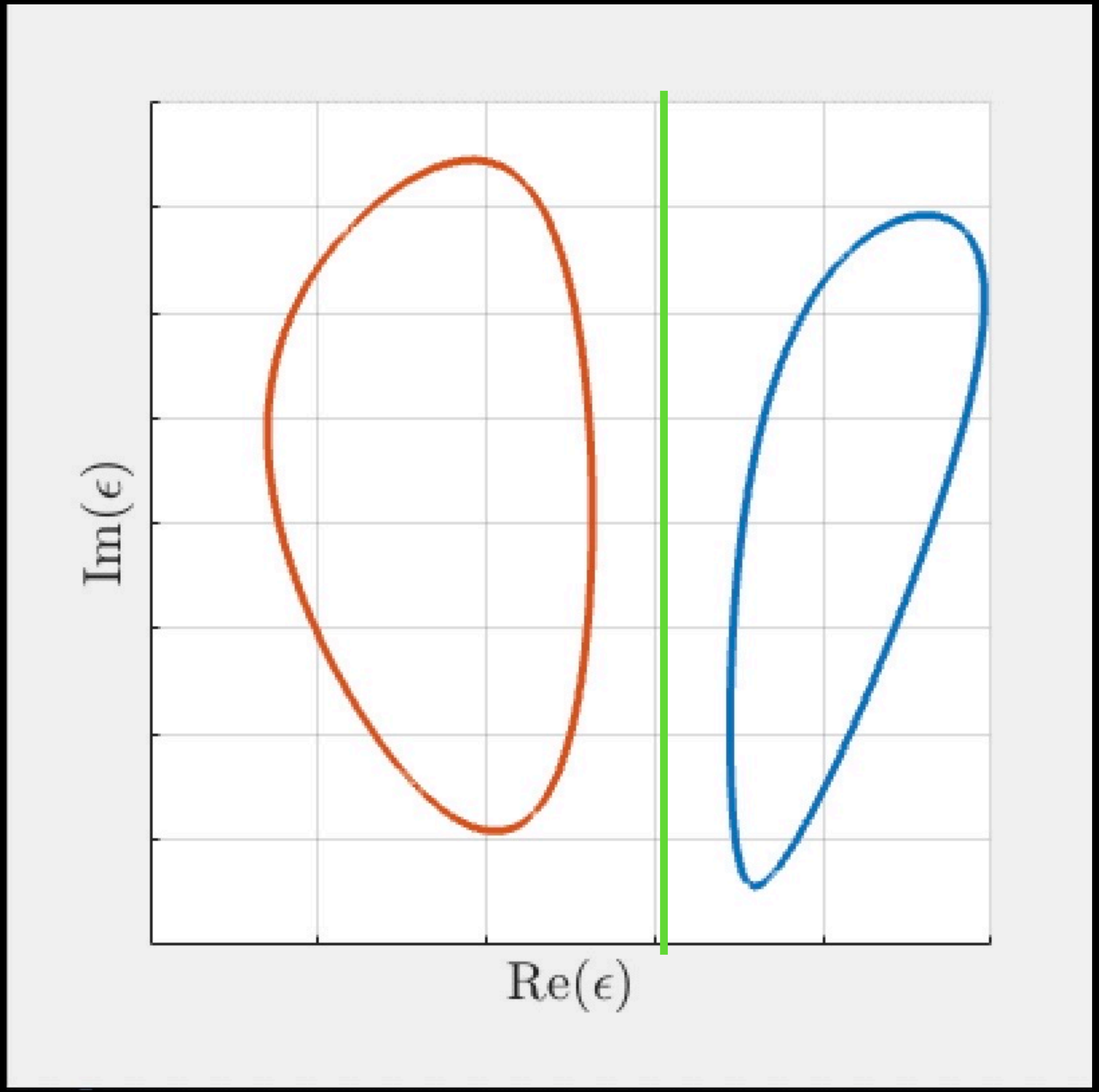
Kohei Kawabata, et. al., Phys. Rev. X 9, 041015 (2019)

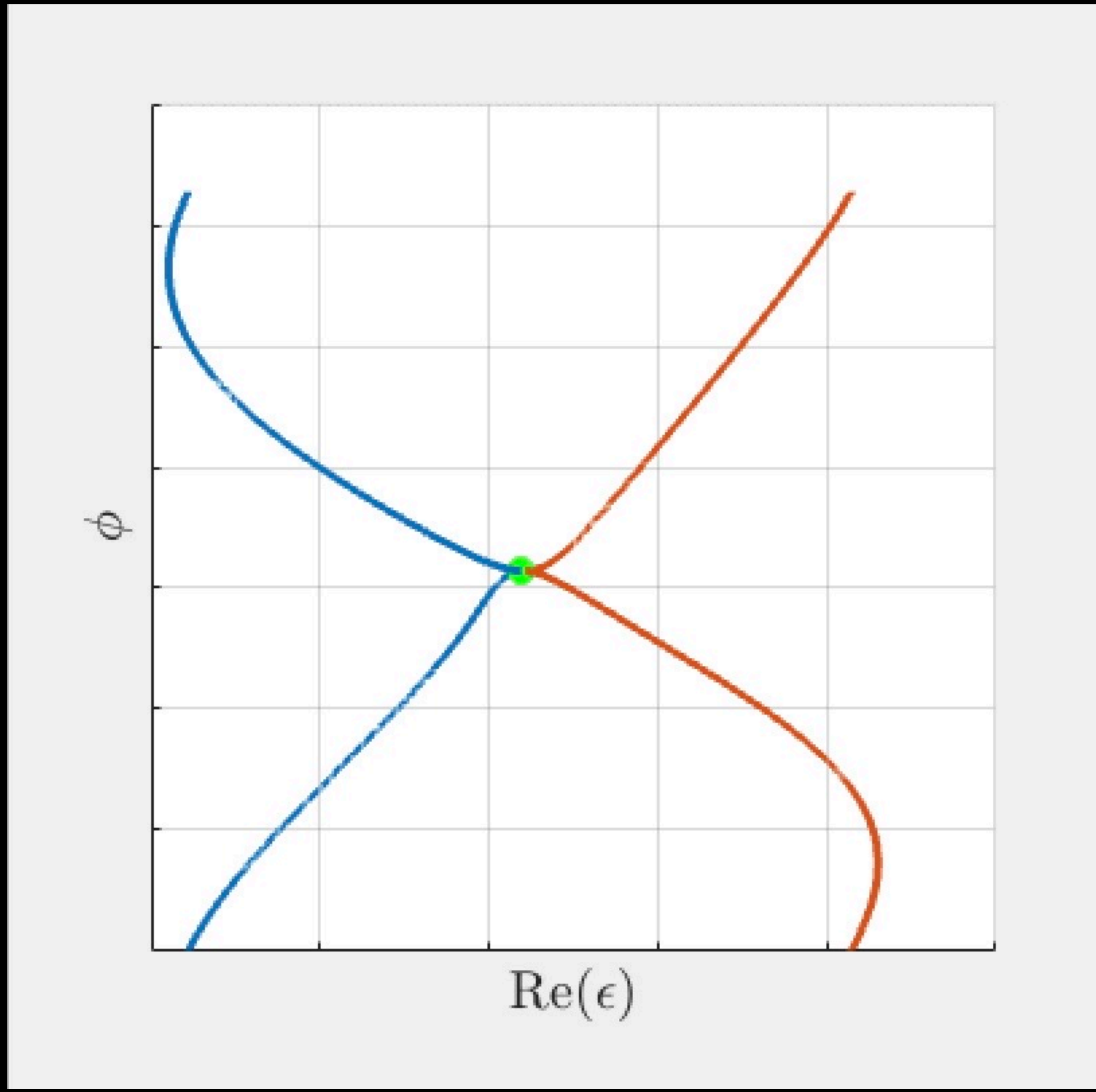
non-Hermitian

$$H \neq H^{\dagger}$$

$$\epsilon_{\alpha} \in \mathbb{C}$$

Line Gap





Non-hermitian physics

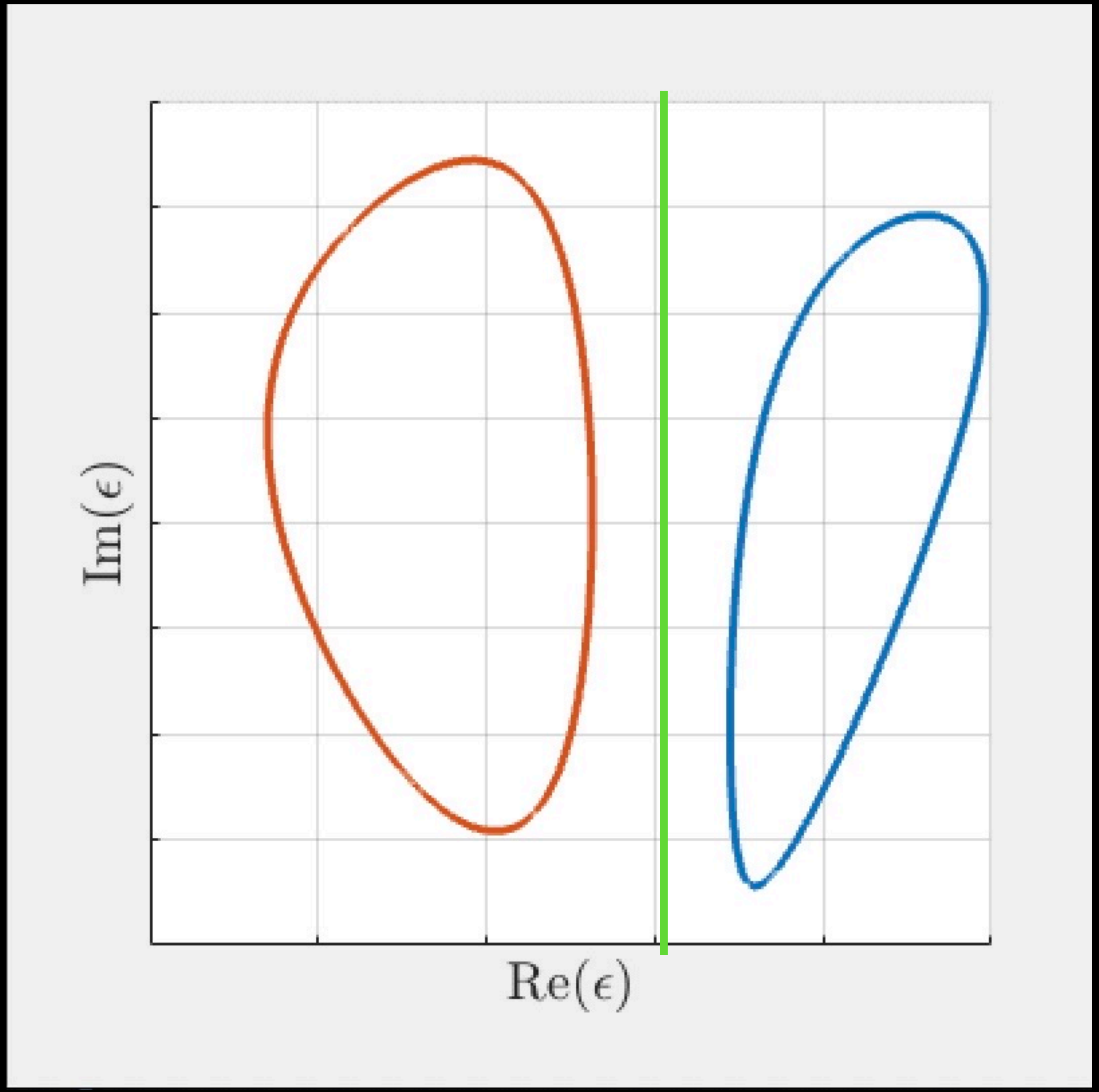
Kohei Kawabata, et. al., Phys. Rev. X 9, 041015 (2019)

non-Hermitian

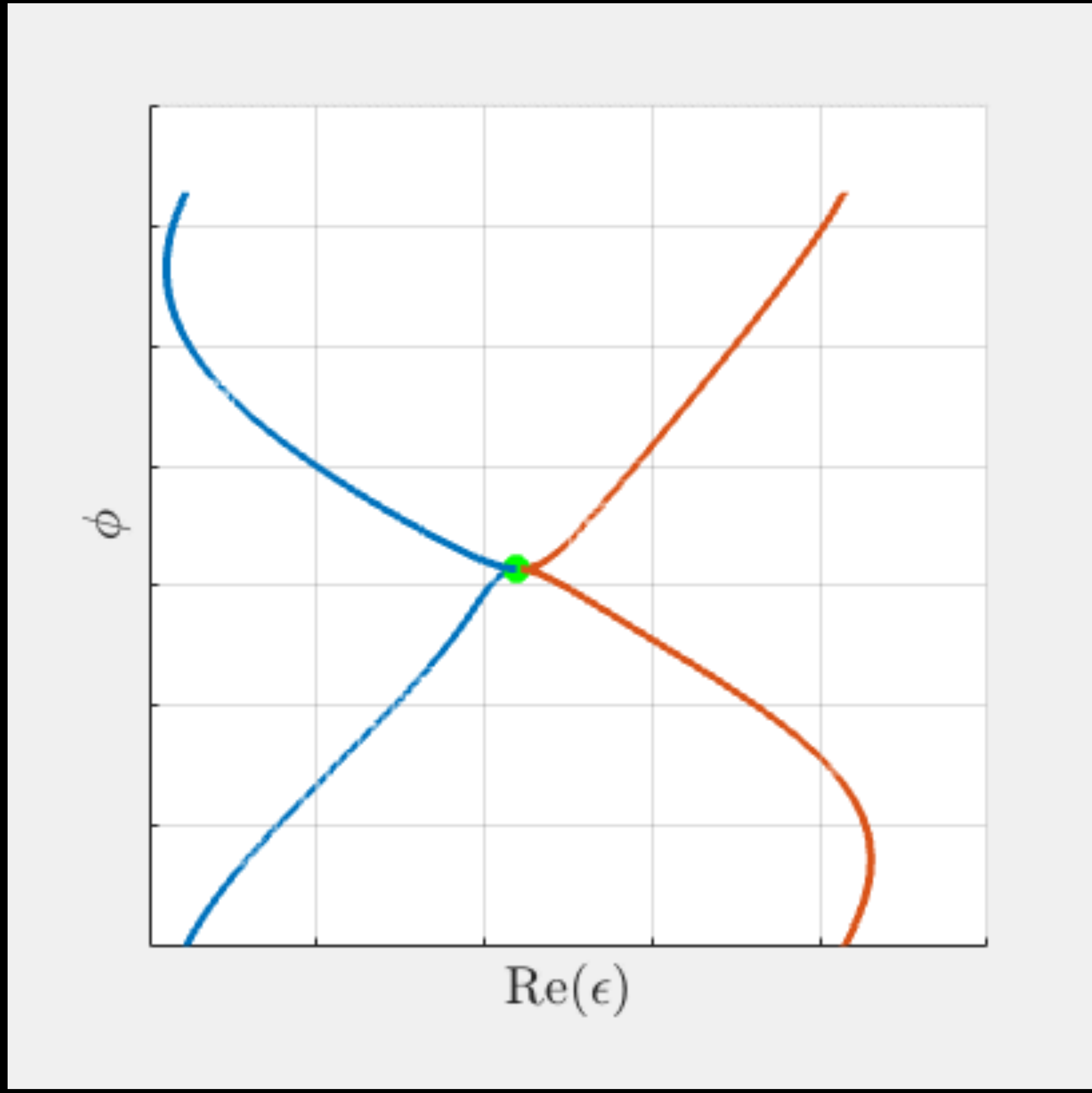
$$H \neq H^{\dagger}$$

$$\epsilon_\alpha \in \mathbb{C}$$

Line Gap



Exceptional Point (EPs)



Non-hermitian physics

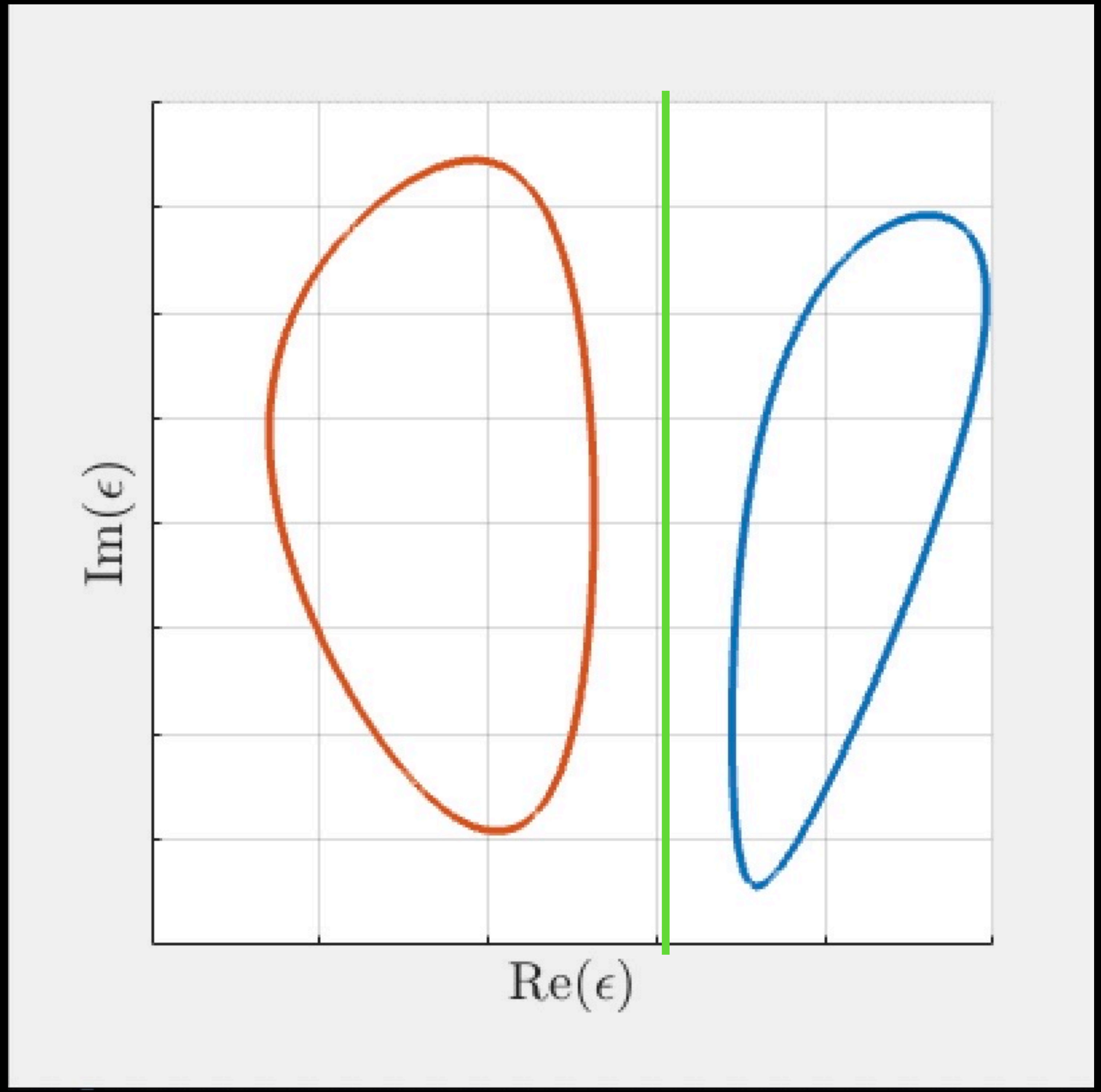
Kohei Kawabata, et. al., Phys. Rev. X 9, 041015 (2019)

non-Hermitian

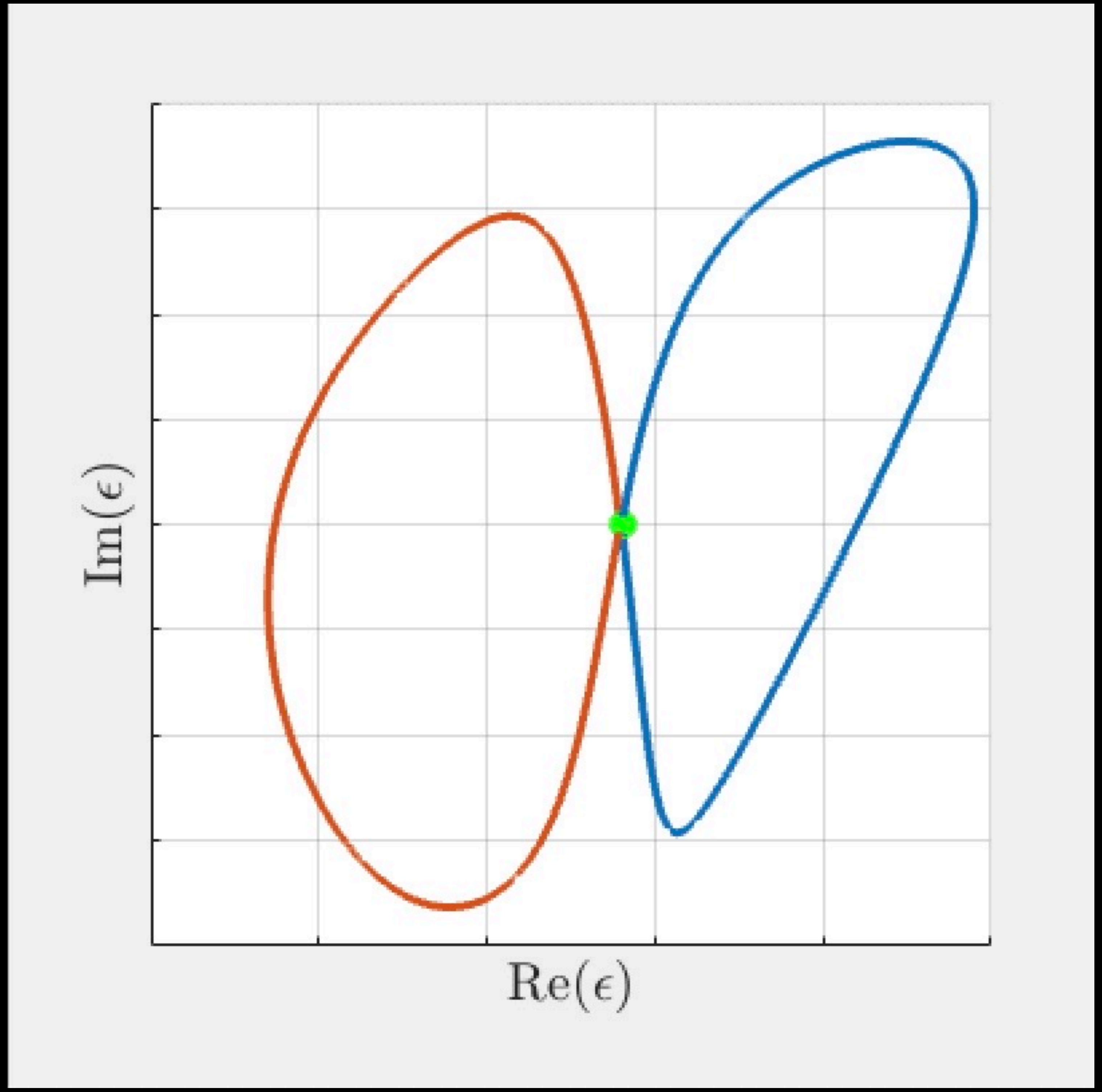
$$H \neq H^\dagger$$

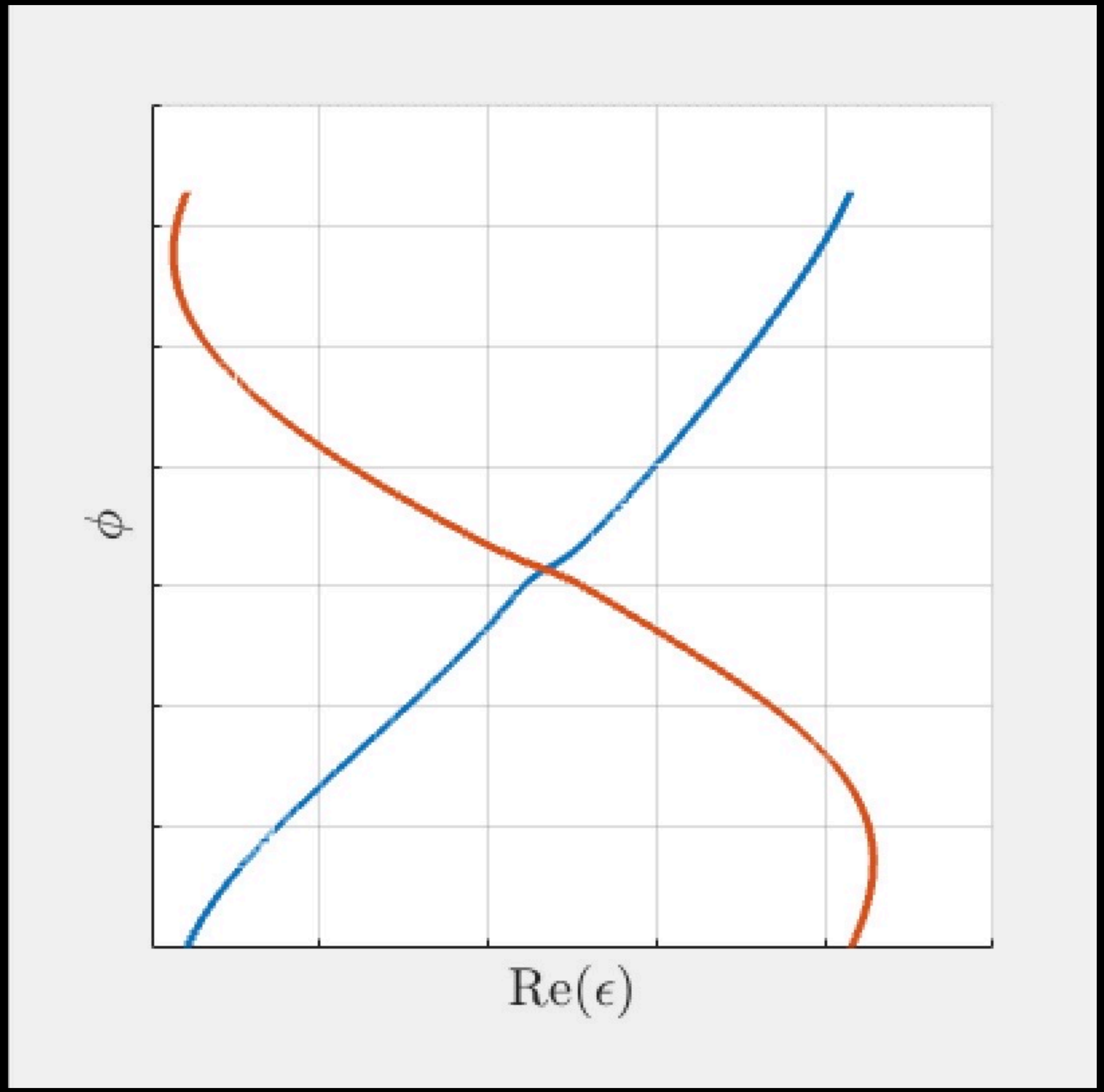
$$\epsilon_\alpha \in \mathbb{C}$$

Line Gap



Exceptional Point (EPs)





Non-hermitian physics

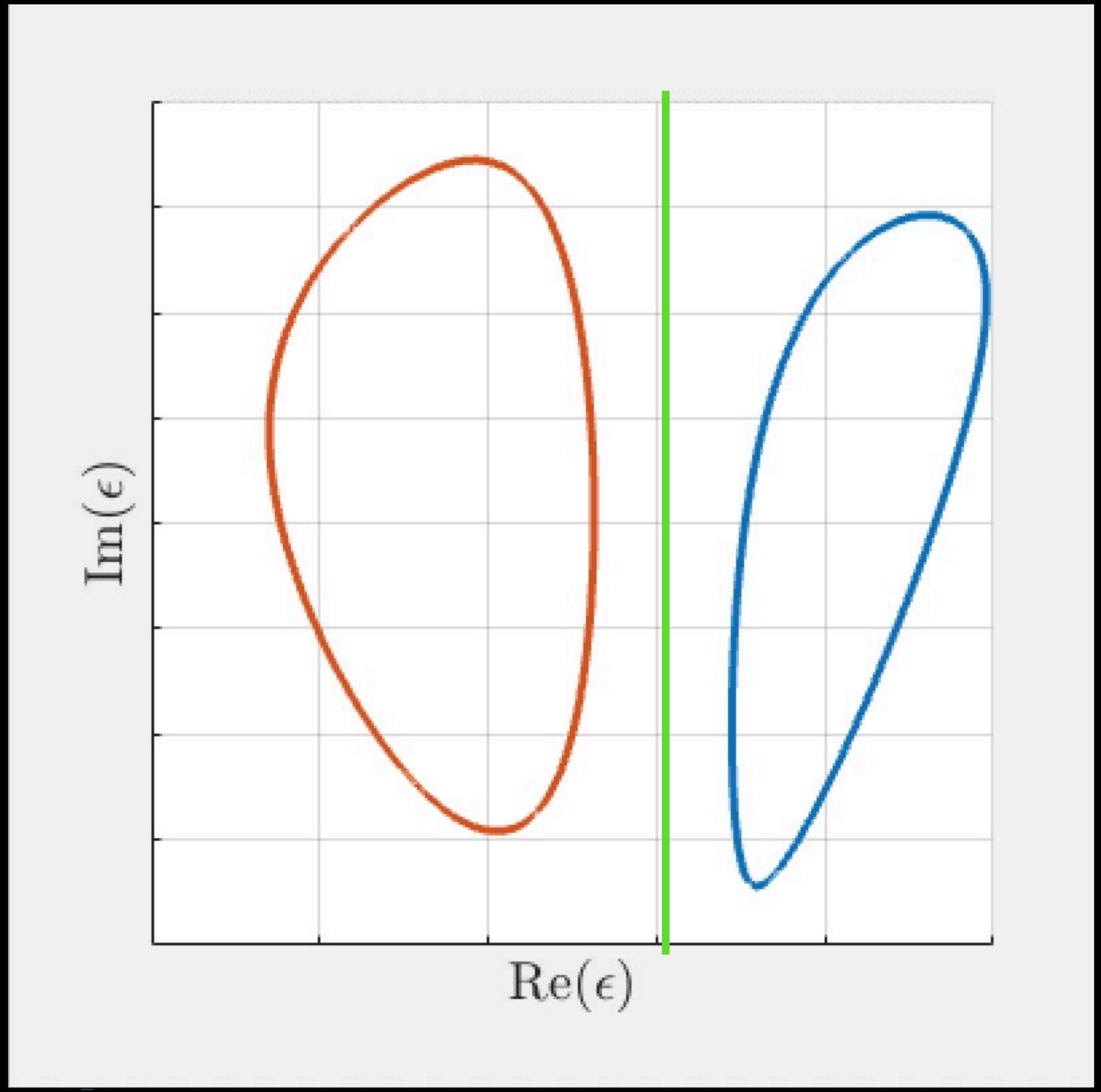
Kohei Kawabata, et. al., Phys. Rev. X 9, 041015 (2019)

non-Hermitian

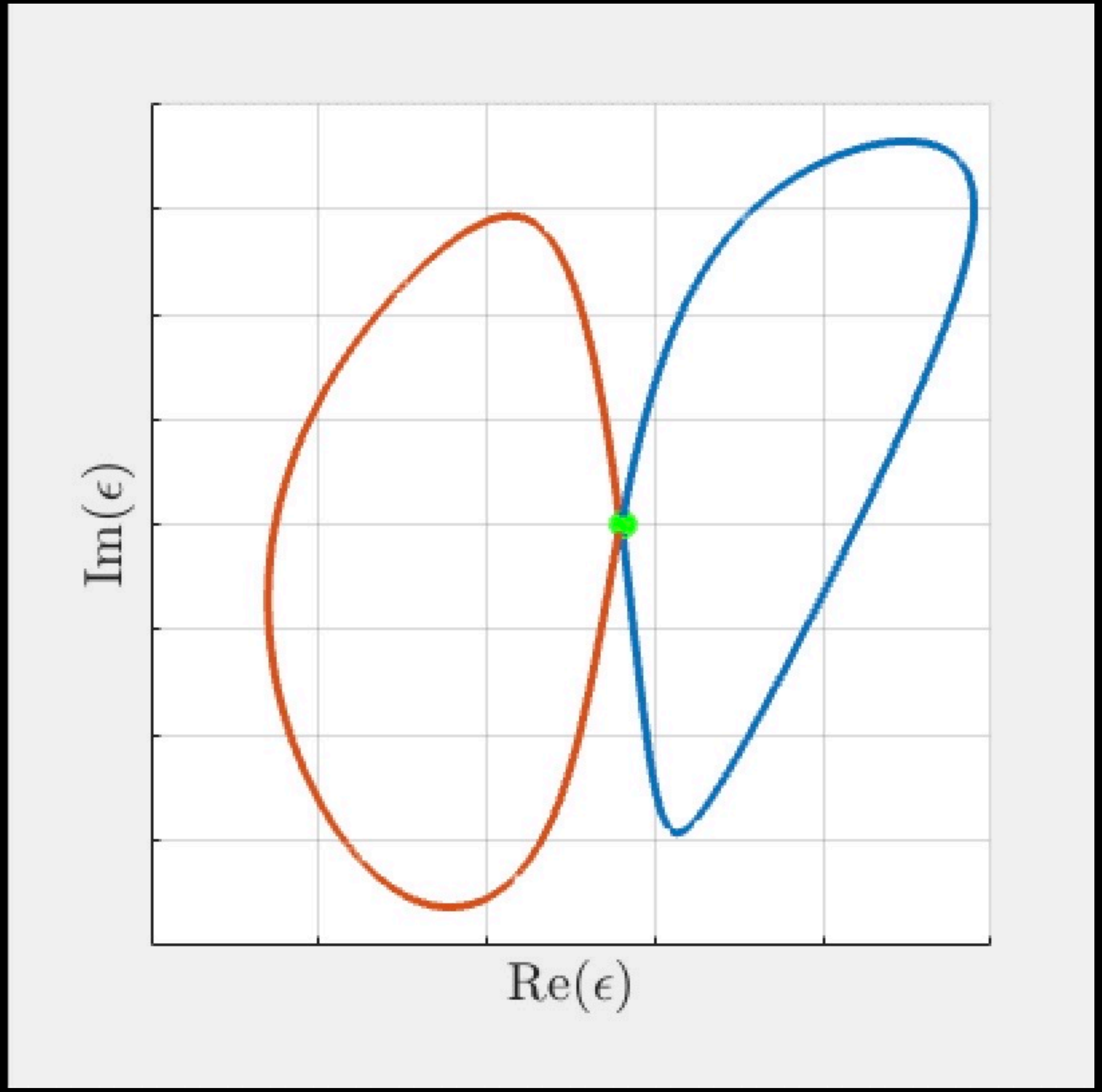
$$H \neq H^\dagger$$

$$\epsilon_\alpha \in \mathbb{C}$$

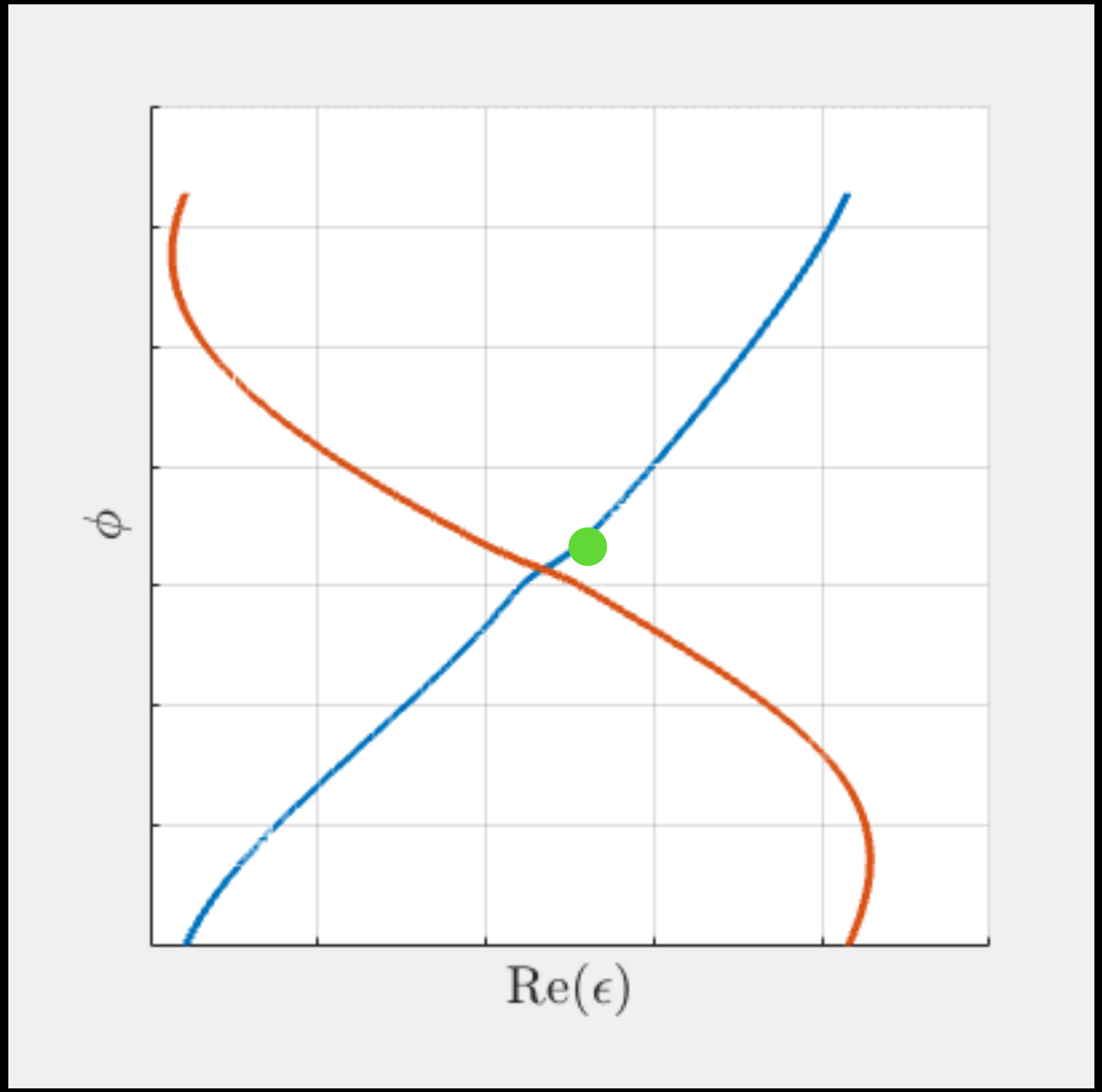
Line Gap



Exceptional Point (EPs)



Point Gap



Why care about non-Hermiticity and (MT)JJs?

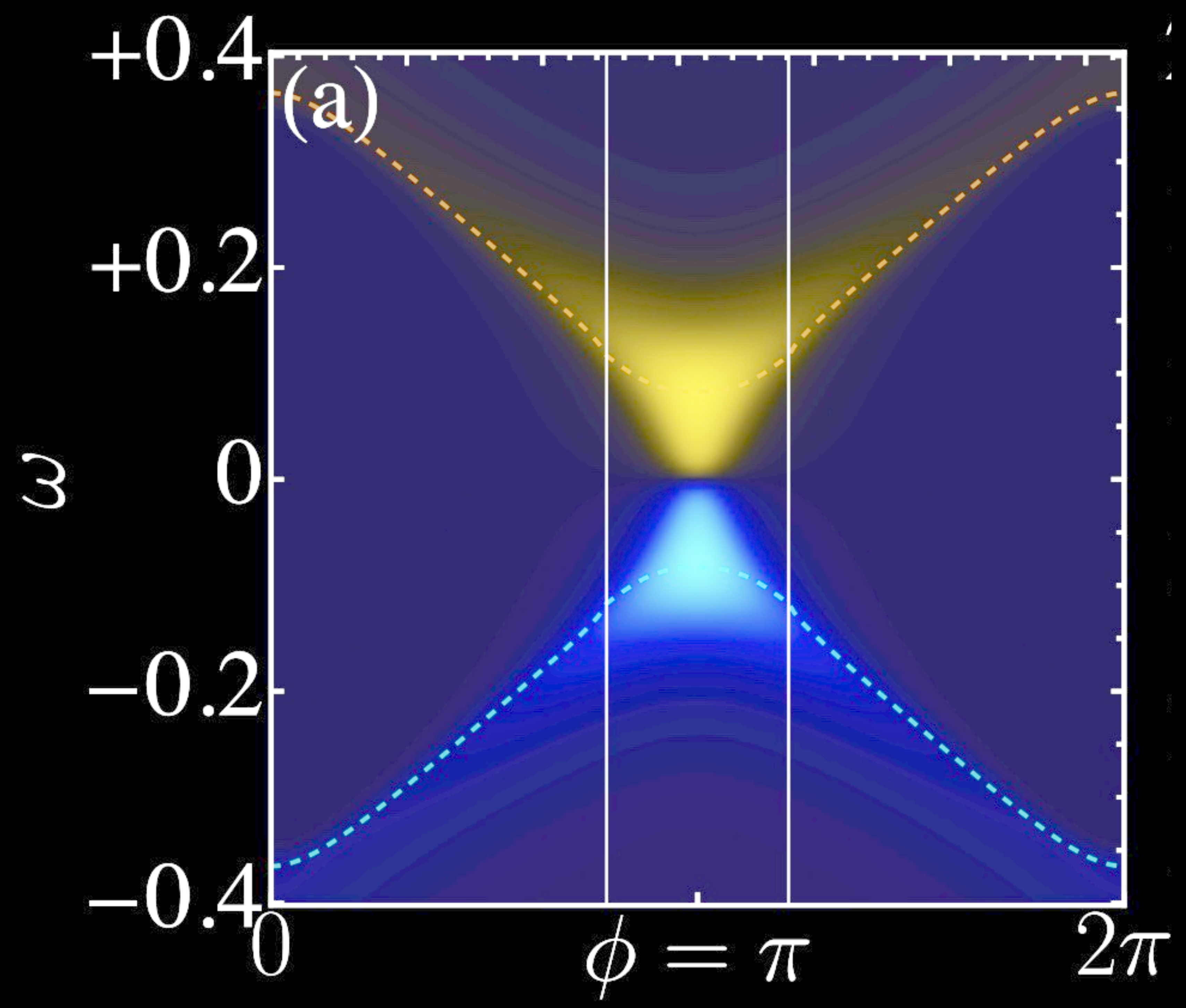
Non-hermitian topology generalisation of hermitian topology

Superconductors offer tuneability via superconducting phases

Enhanced sensing, undirectional laser, bulk-fermi arcs, …

C. Dembowski, et. al., Phys. Rev. Lett. 86, 787 (2001)
J. Doppler, et. al., Nature 537, 76 (2016)
W. Chen, et. al., Nature 548, 192 (2017)
B. Peng, et. al., PNAS 113, 6845 (2016)
H. Zhou, et. al., Science 359, 1009 (2018)
….

Measurability in current susceptibility

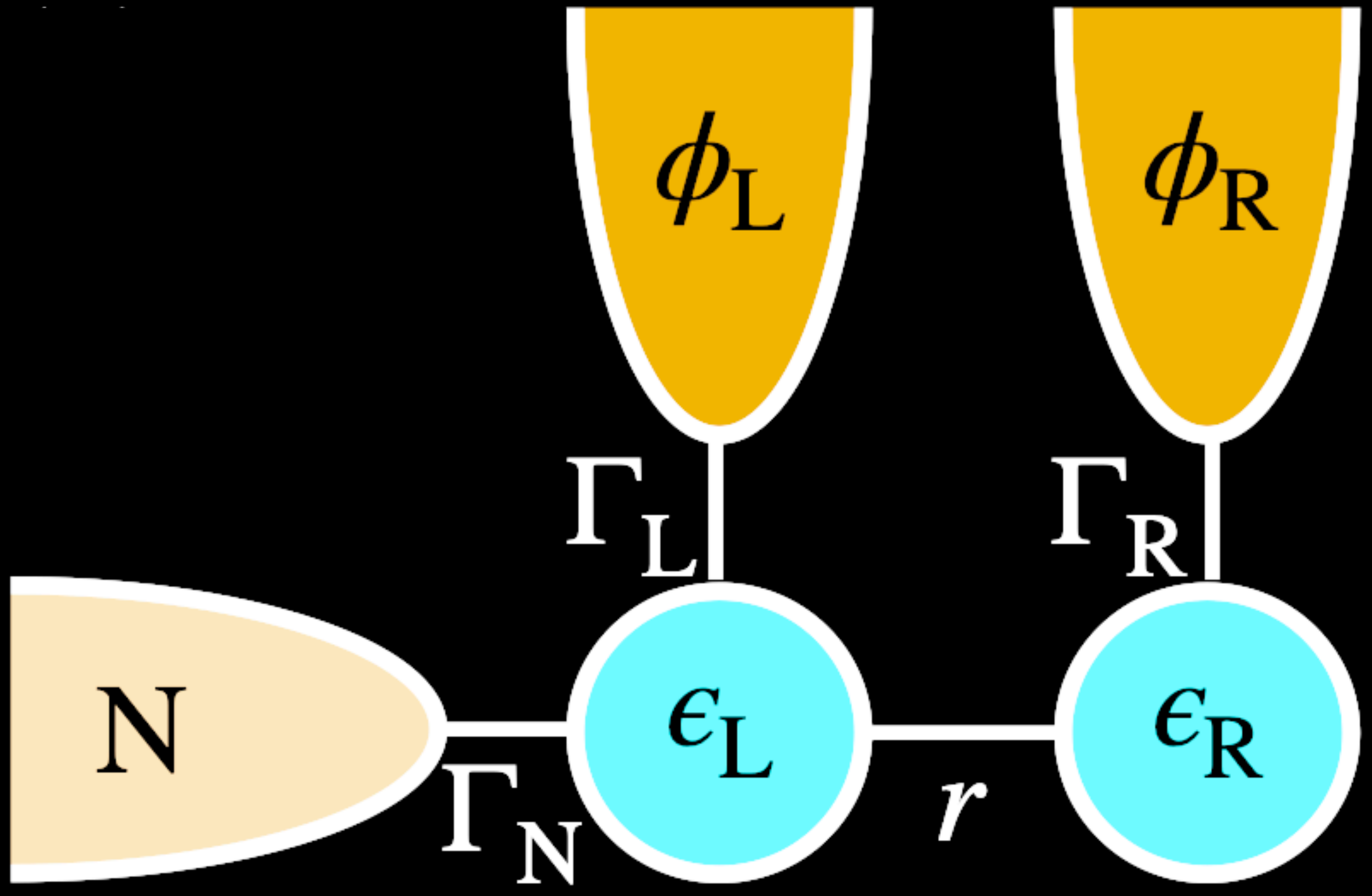


V. Kornich and B. Trauzettel Phys. Rev. Research 4, 033201 (2022)
C. A. Li, et. al. Phys. Rev. B 109, 214514 (2024)
J. Cayao and M. Sato, arXiv:2307.15472 (2024)
C.W.J. Beenakker, Appl. Phys. Lett. 125, 122601 (2024)
P.X. Shen, et. al. Phys. Rev. Lett. 133, 086301 (2024)
J. Cayao and M. Sato, arXiv:2408.17260 (2024)
…

What's non-hermitian about (MT)JJs?

D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

2-terminal junction with added normal metal



Analogous to: J. Cayao and M. Sato, arXiv:2307.15472 (2024)

Effective low-energy Hamiltonian from central Green's function (no-jump limit)

K. Sim, et. al., Phys. Rev. Research 7, 013325 (2025)

$$H_{\mathrm{eff}} = G_{\mathrm{C}}^{-1}(E=0)$$

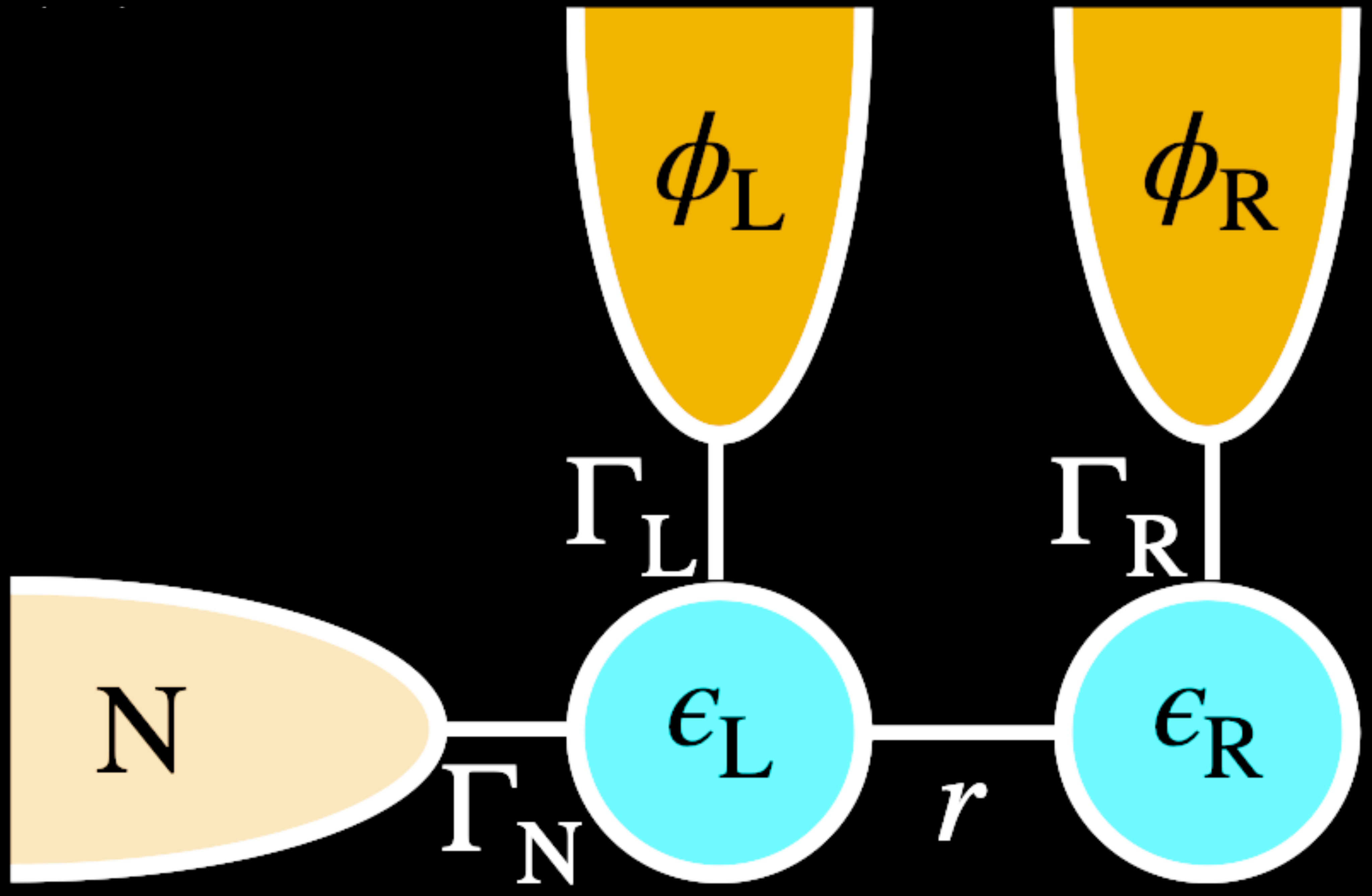
$$= \begin{pmatrix} \epsilon_{\mathrm{L}}\tau_3 + \Gamma_{\mathrm{L}}\tau_1 - i\Gamma_{\mathrm{N}}\tau_0 & r\tau_3 \\ r\tau_3 & \epsilon_{\mathrm{R}}\tau_3 + \Gamma_{\mathrm{R}} e^{i\phi\tau_3}\tau_1 \end{pmatrix}$$

$$\neq H^{\dagger}$$

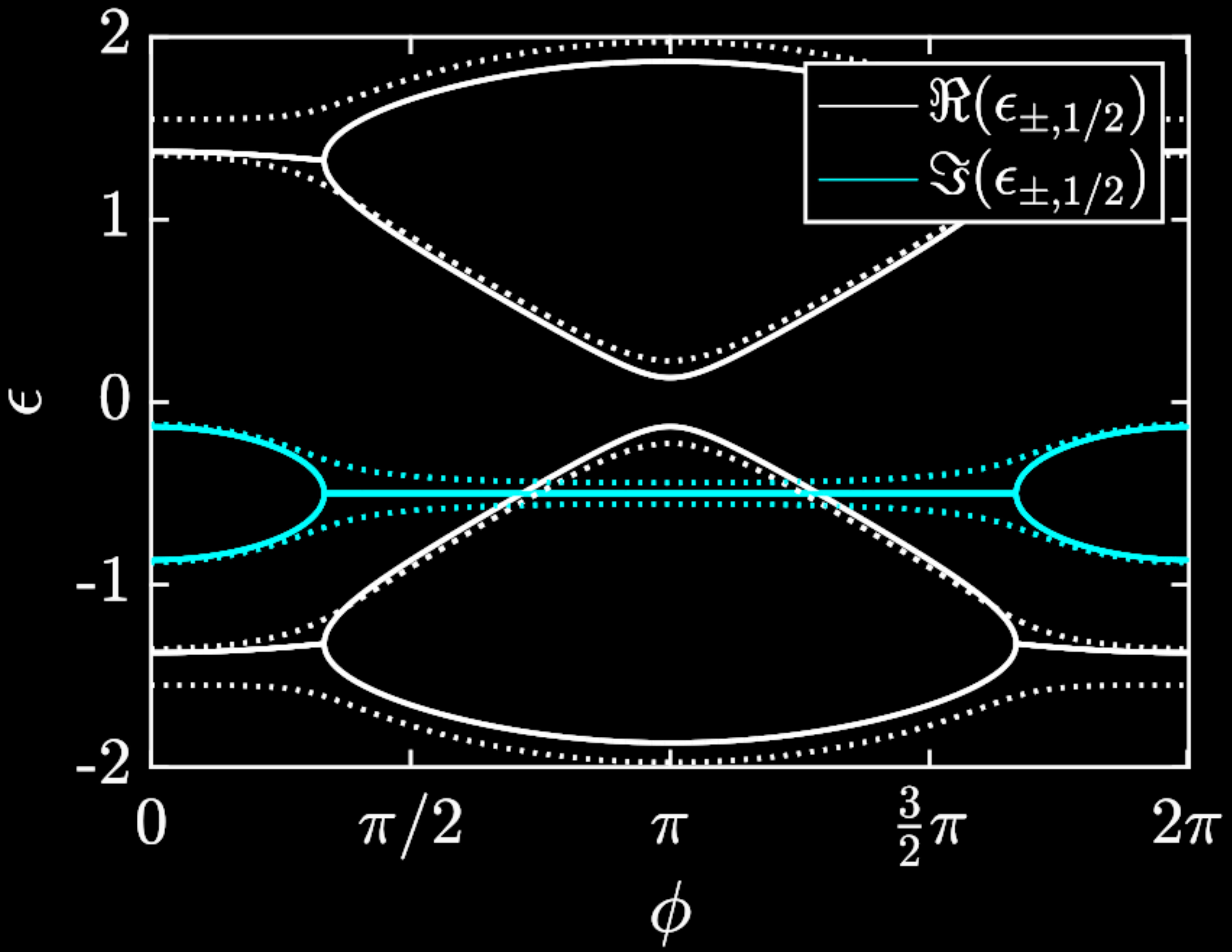
What's non-hermitian about (MT)JJs?

D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

2-terminal junction with added normal metal



Real- and imaginary part of spectrum



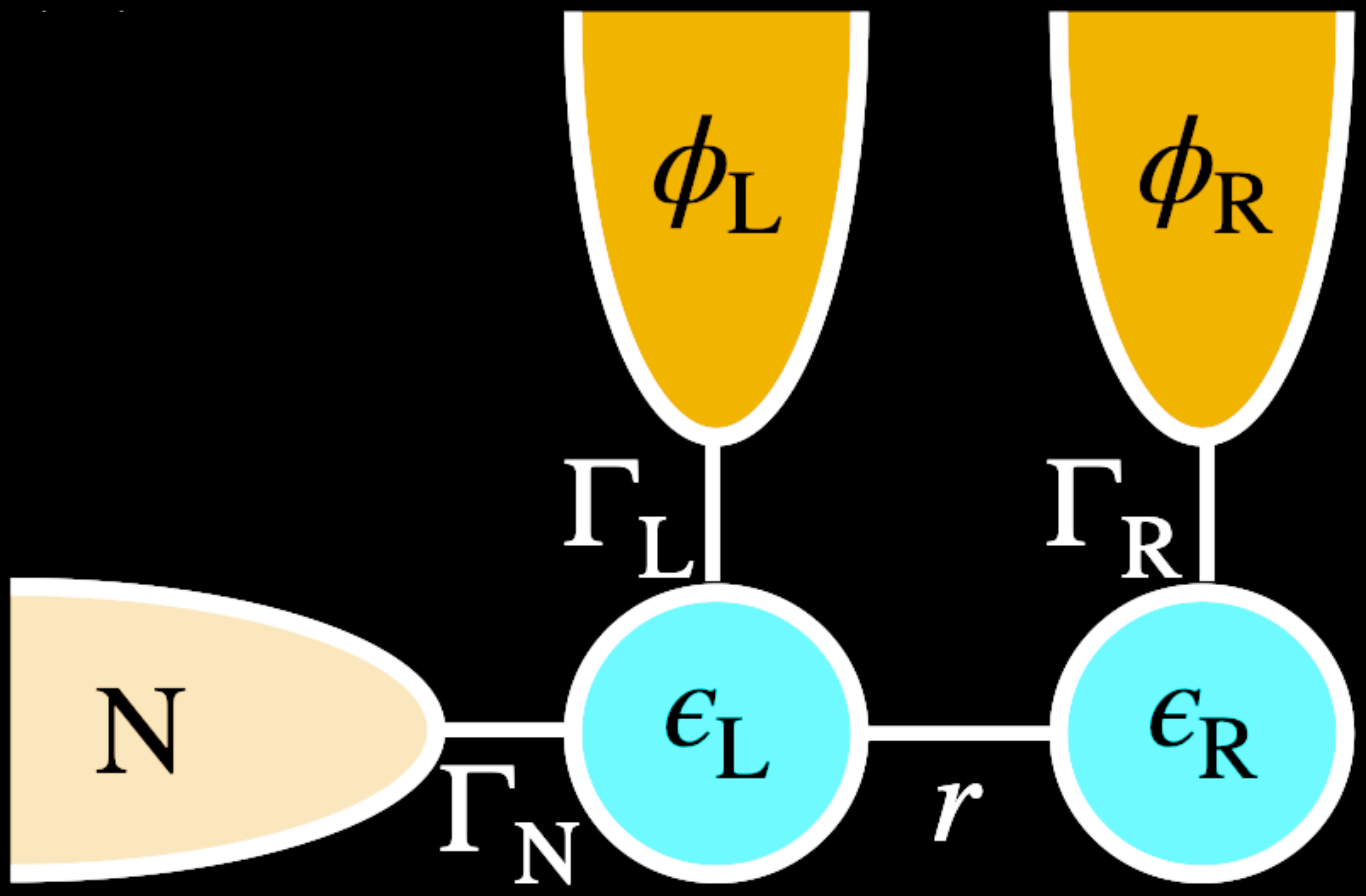
— $\Gamma_{L/R} = r = 1, \Gamma_N = 1, \epsilon_{L/R} = 0$

······ $\Gamma_R = 1, \Gamma_L = 1.2, \Gamma_N = 1, r = 1, \epsilon_{L/R} = 0$

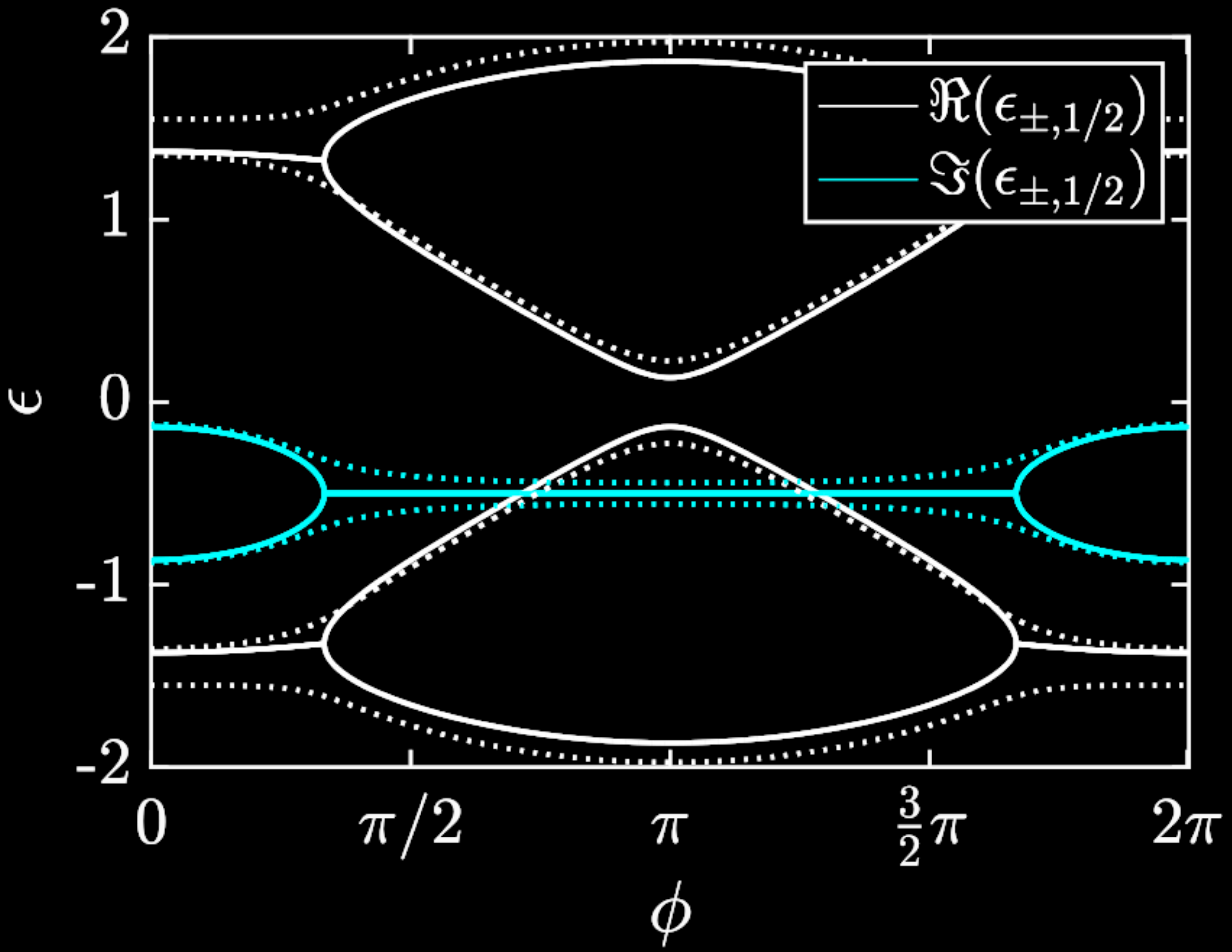
What's non-hermitian about (MT)JJs?

D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

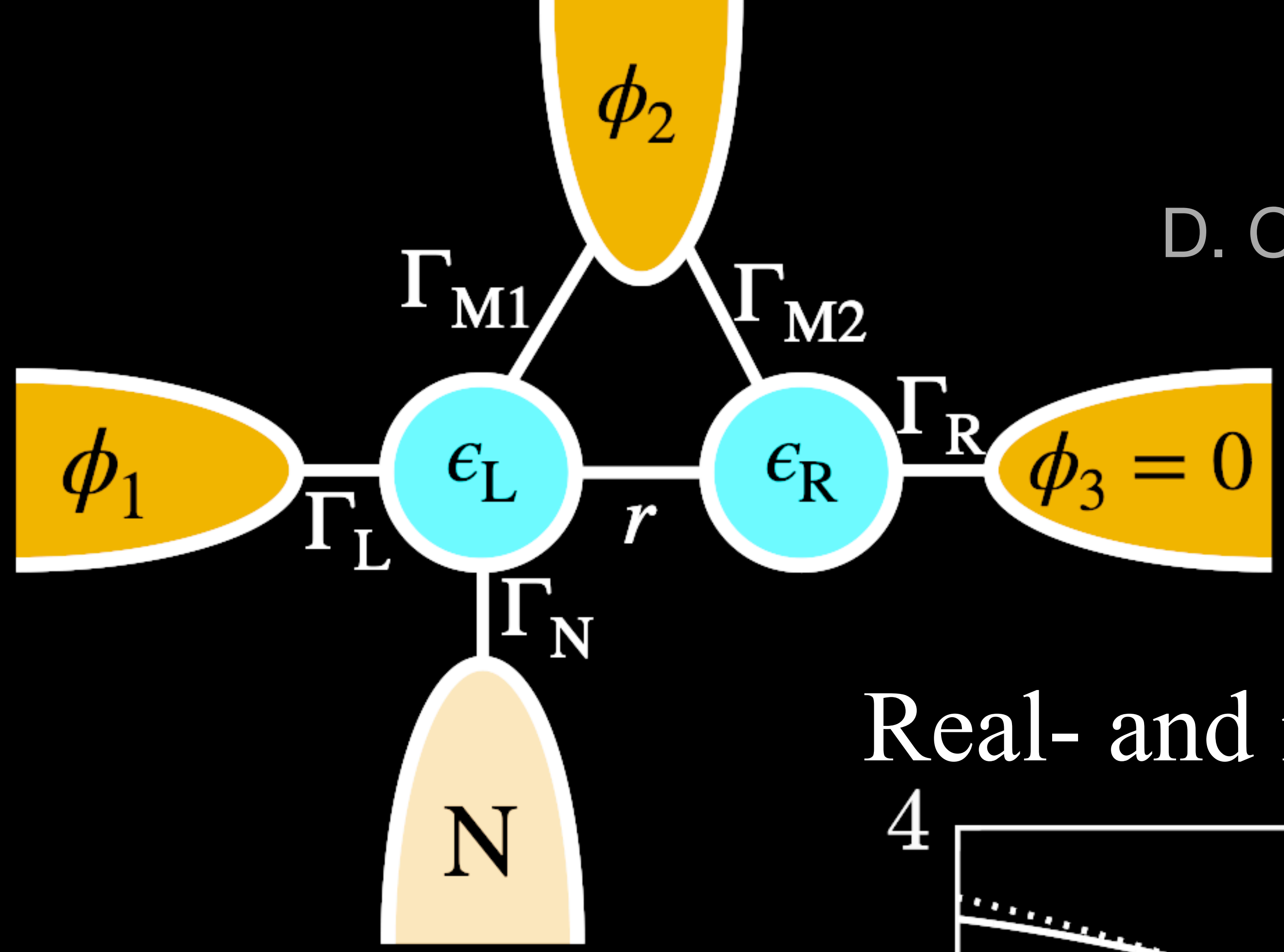
2-terminal junction with added normal metal



Real- and imaginary part of spectrum



EPs are **not** stable (protected by symmetry)

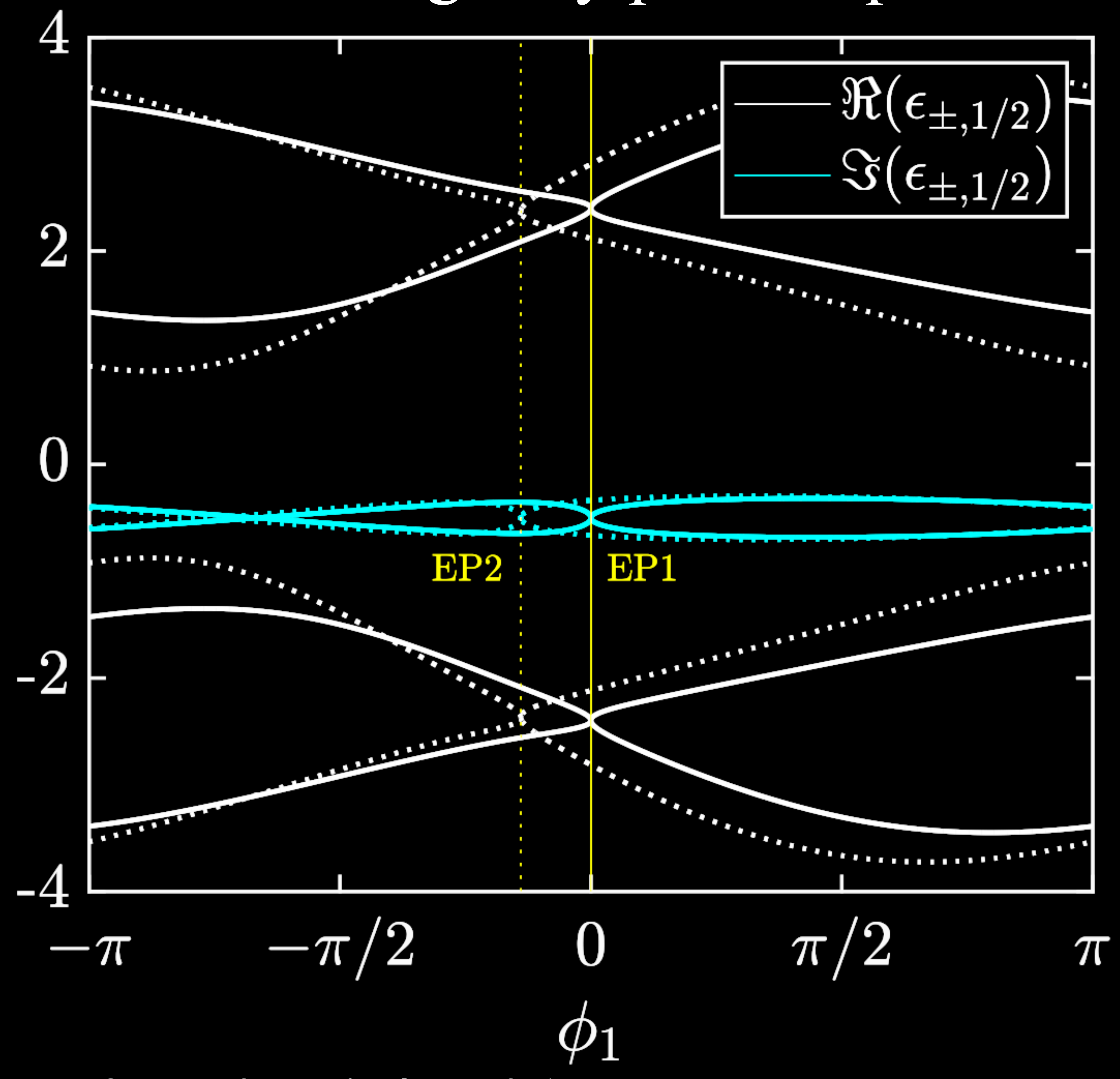


3-terminal model

D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

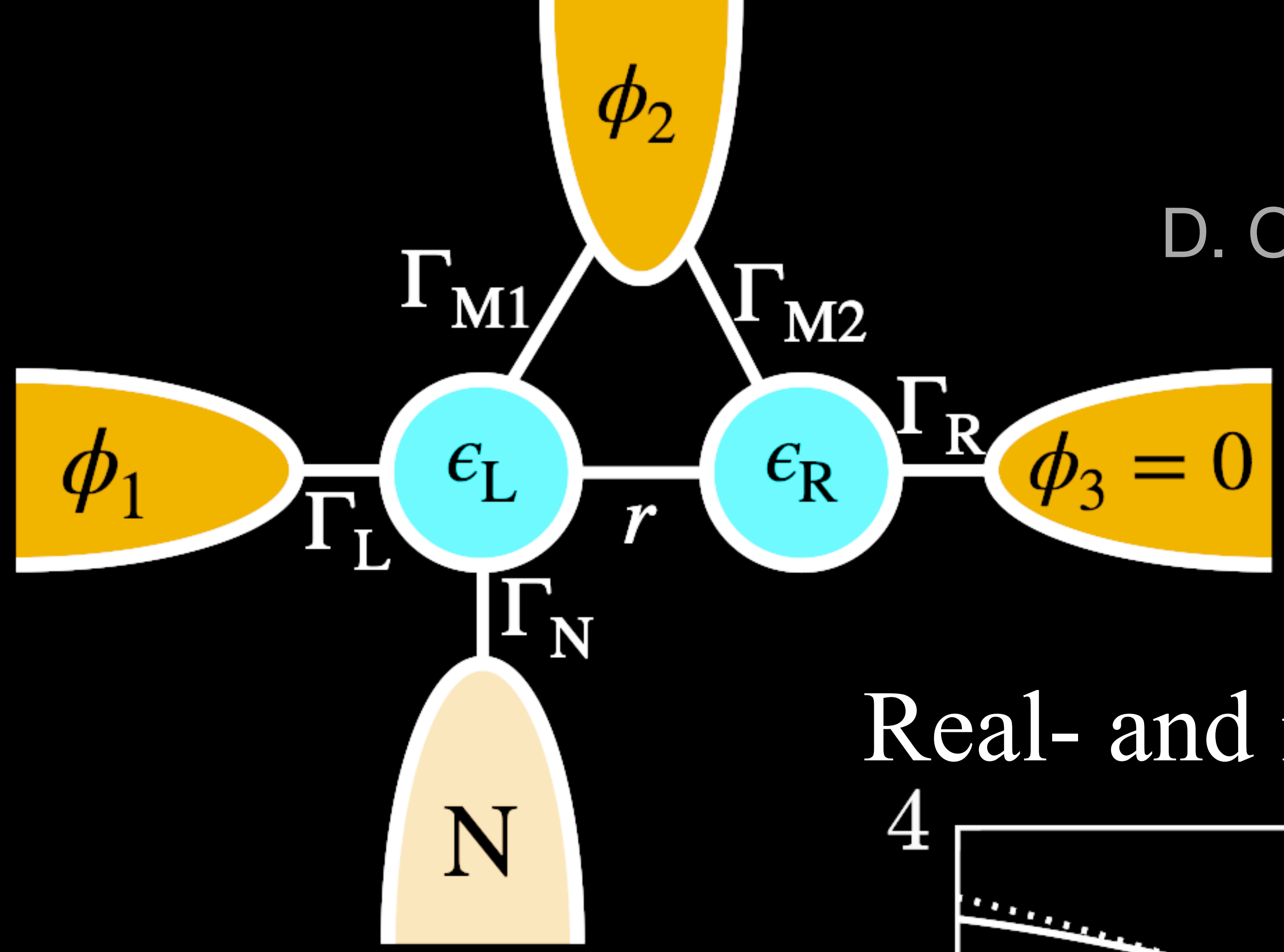
Real- and imaginary part of spectrum

$\phi_2 \neq 0$



$\Gamma_{L/R/M1/M2} = \Gamma_N = 1, \epsilon_{1/2} = 0, r = 2$ and $\phi_2 = 2.1$

$\Gamma_L = 1.4, \Gamma_R = 1.1, \Gamma_{M1} = 1.0, \Gamma_{M2} = 0.5, \Gamma_N = 1, \epsilon_1 = 0.1, \epsilon_2 = 0.3, r = 2,$ and $\phi_2 = 1.75$

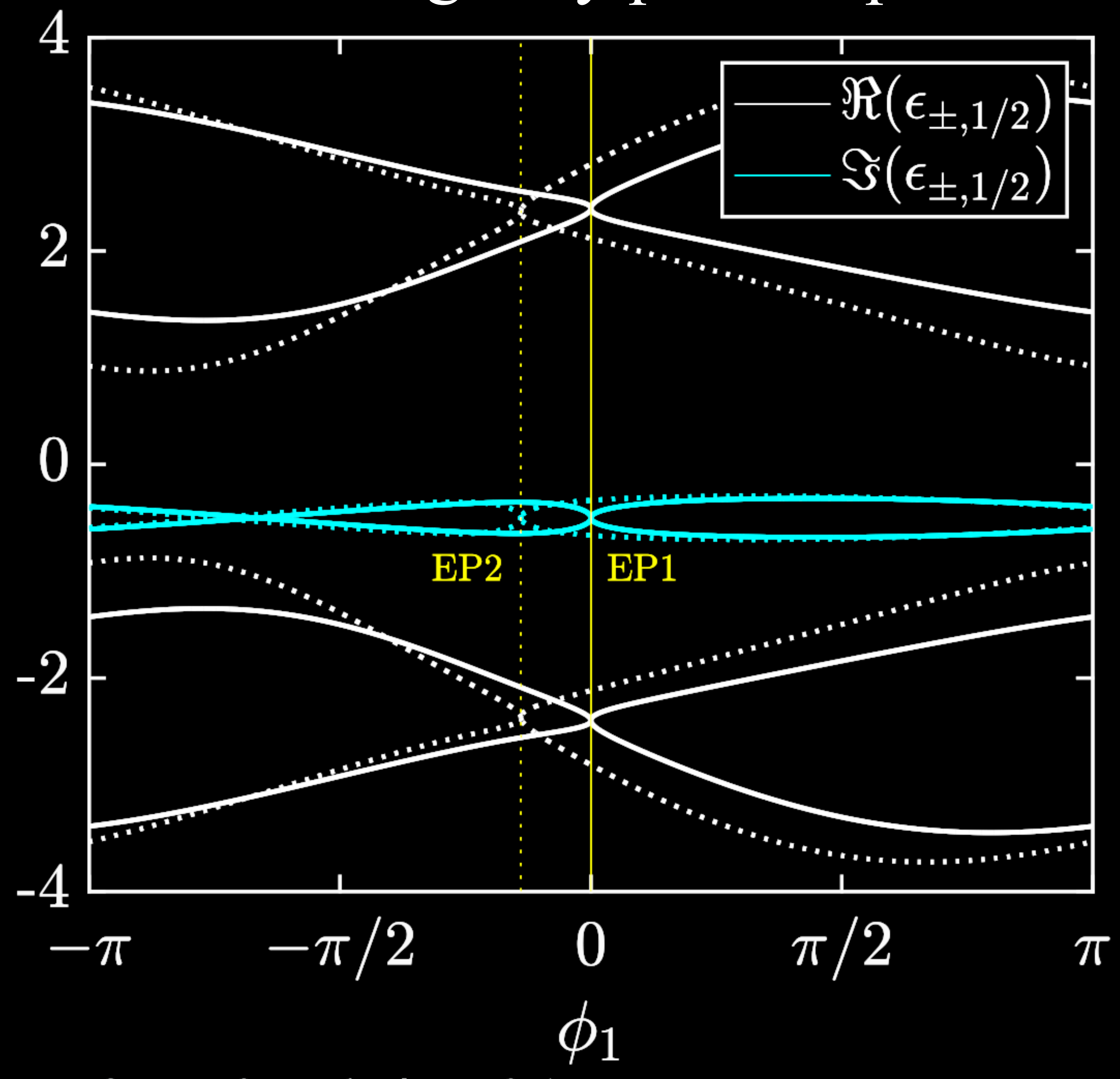


3-terminal model

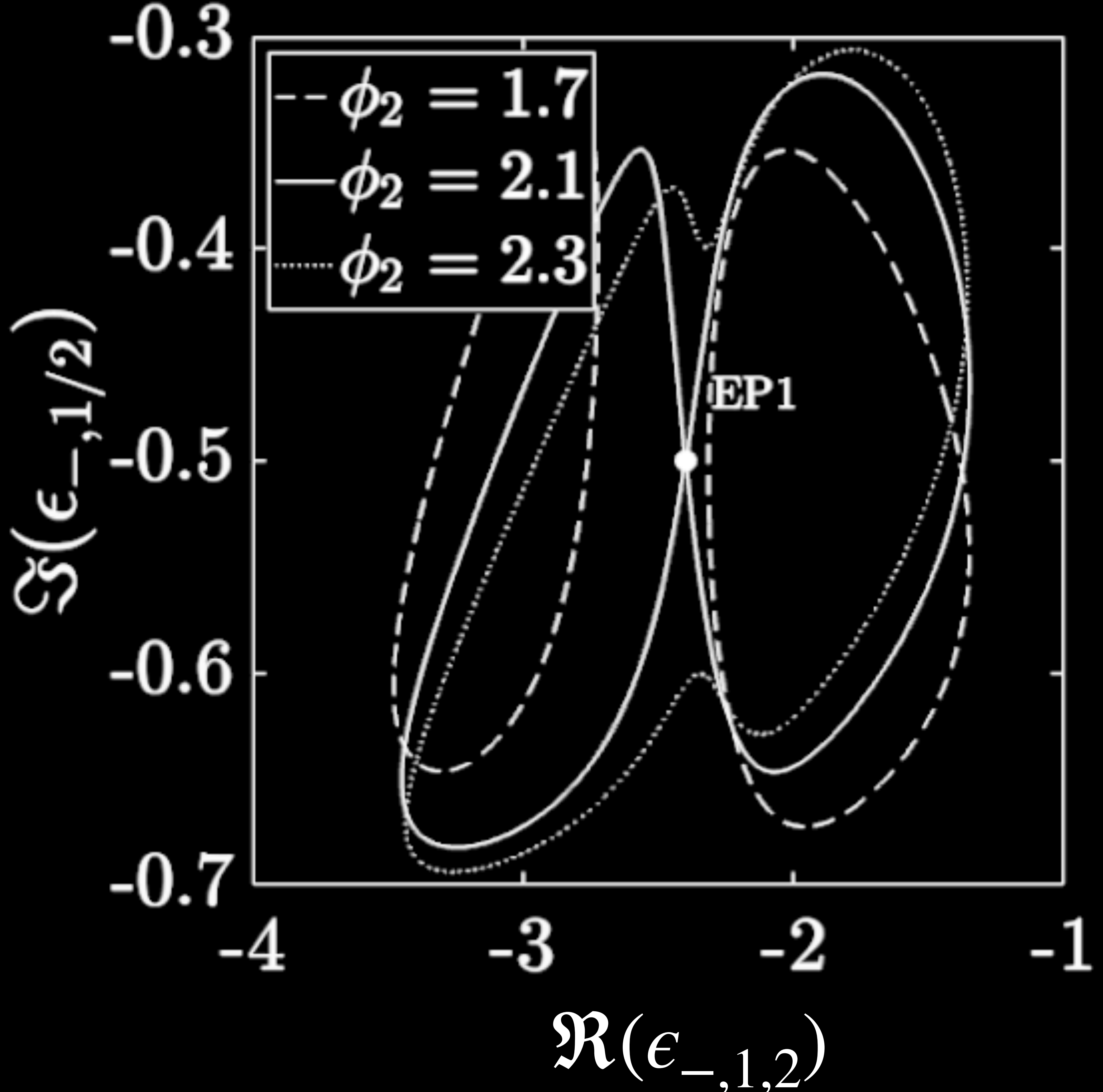
D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

$\phi_2 \neq 0$

Real- and imaginary part of spectrum



Projected spectrum

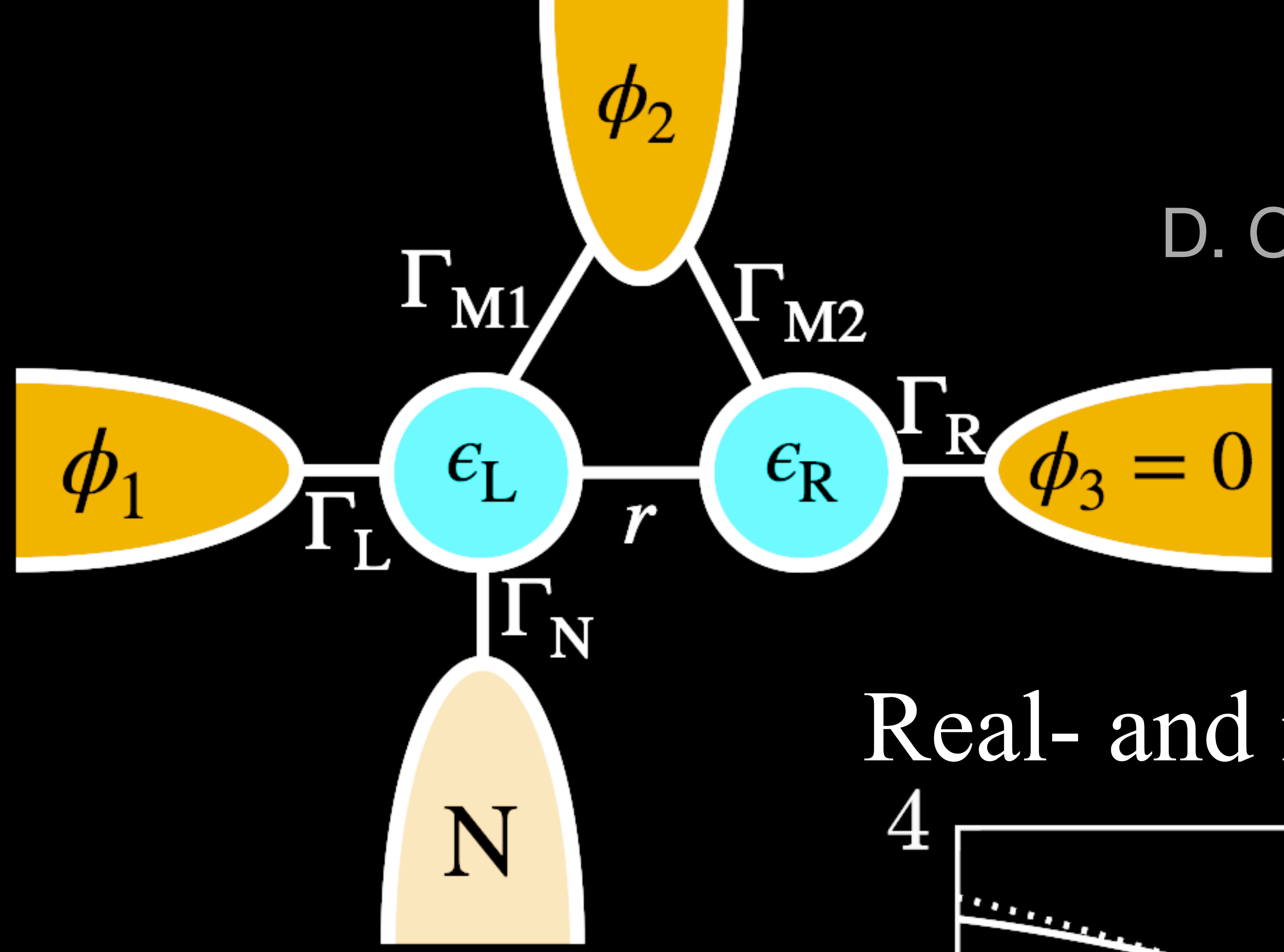


—— $\Gamma_{\mathrm{L/R/M1/M2}} = \Gamma_{\mathrm{N}} = 1, \epsilon_{1/2} = 0, r = 2$ and $\phi_2 = 2.1$

······ $\Gamma_{\mathrm{L}} = 1.4, \Gamma_{\mathrm{R}} = 1.1, \Gamma_{\mathrm{M1}} = 1.0, \Gamma_{\mathrm{M2}} = 0.5, \Gamma_{\mathrm{N}} = 1, \epsilon_1 = 0.1, \epsilon_2 = 0.3, r = 2,$ and $\phi_2 = 1.75$

3-terminal model

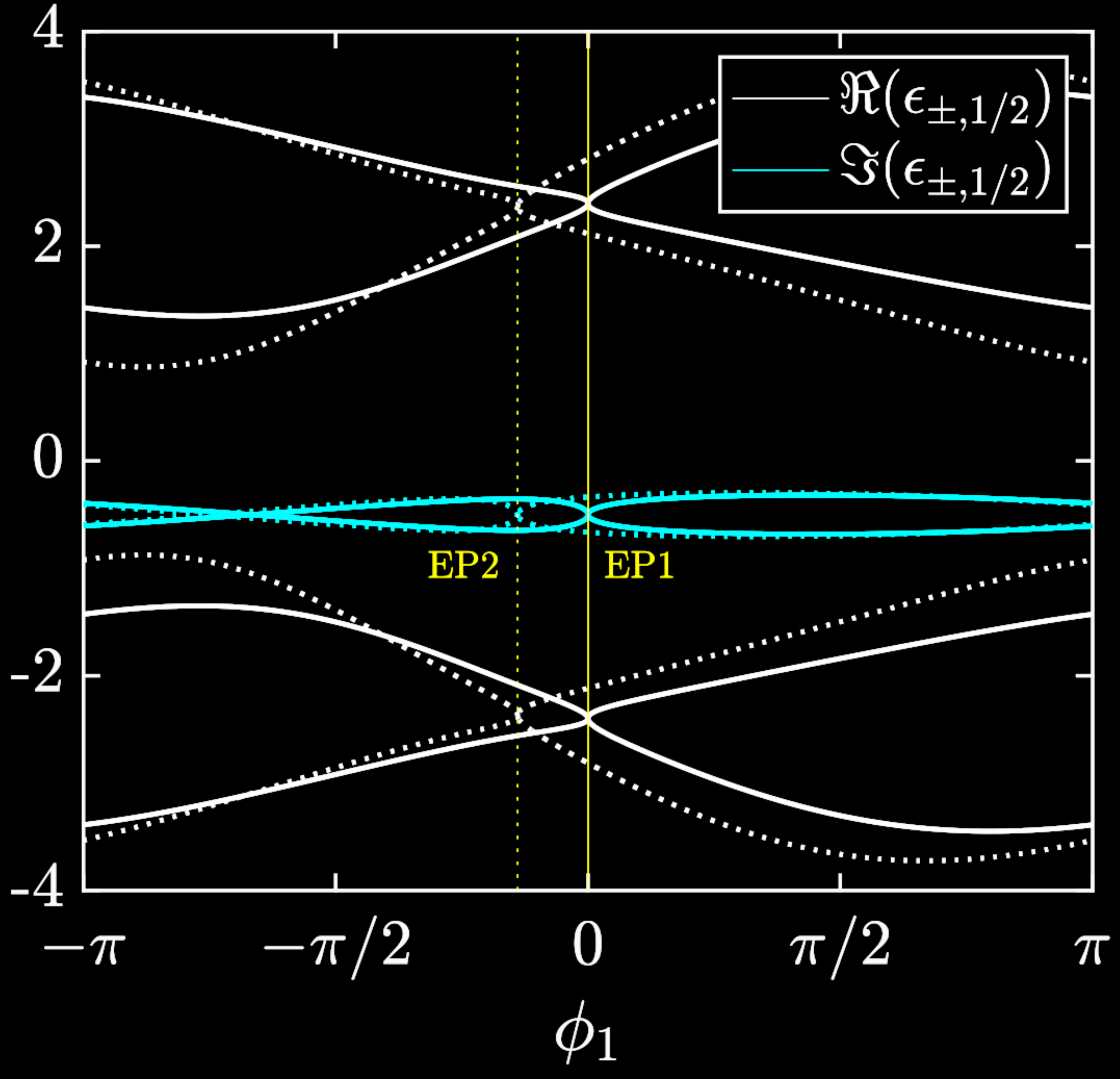
D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)



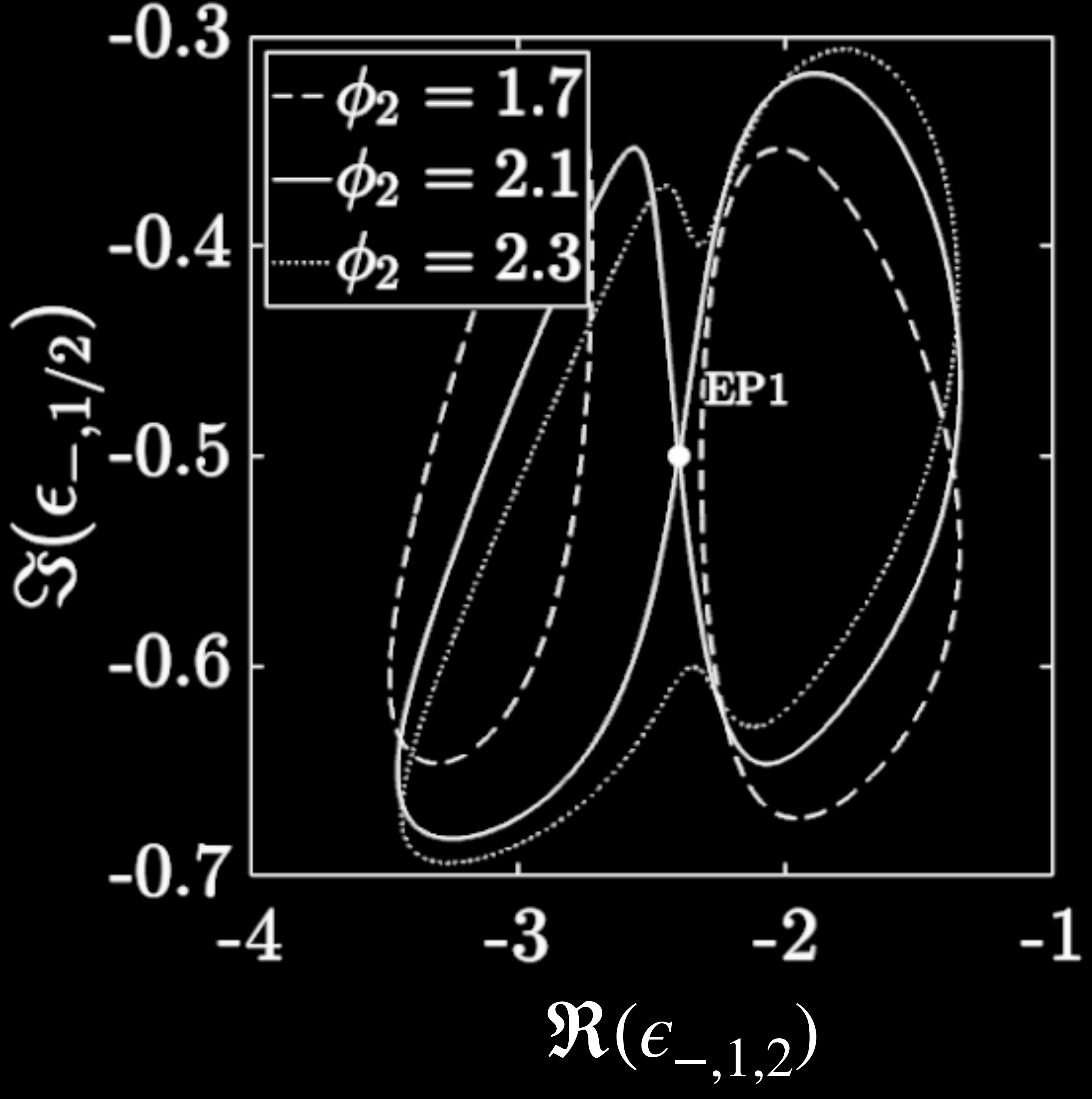
EPs are stable!

$\phi_2 \neq 0$

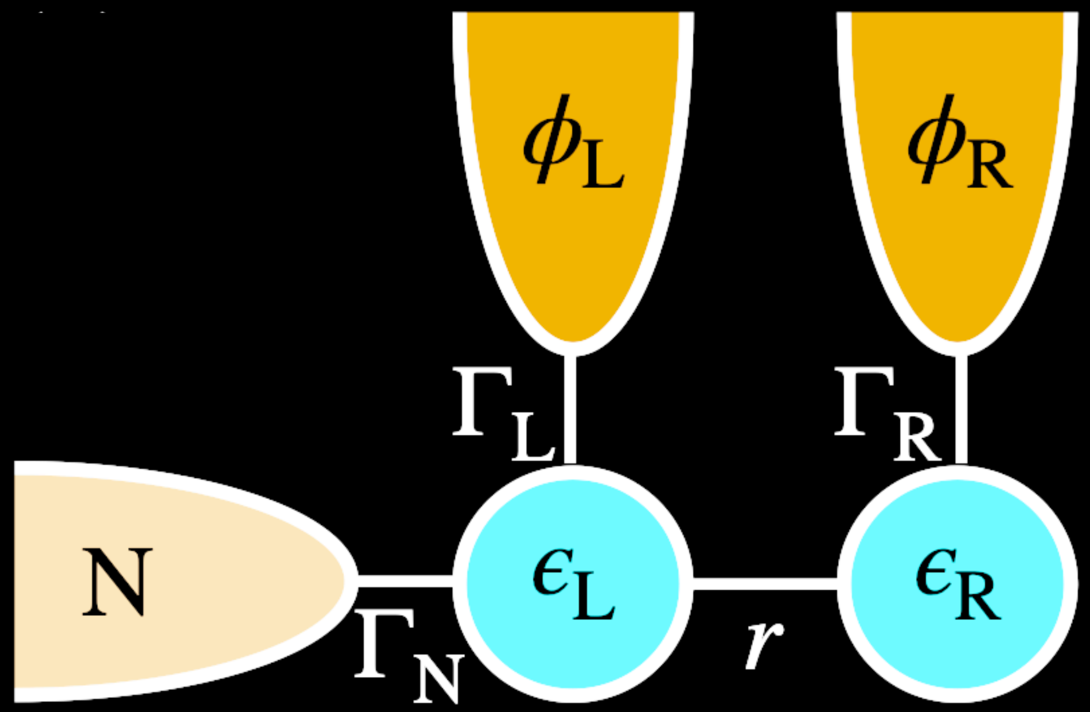
Real- and imaginary part of spectrum



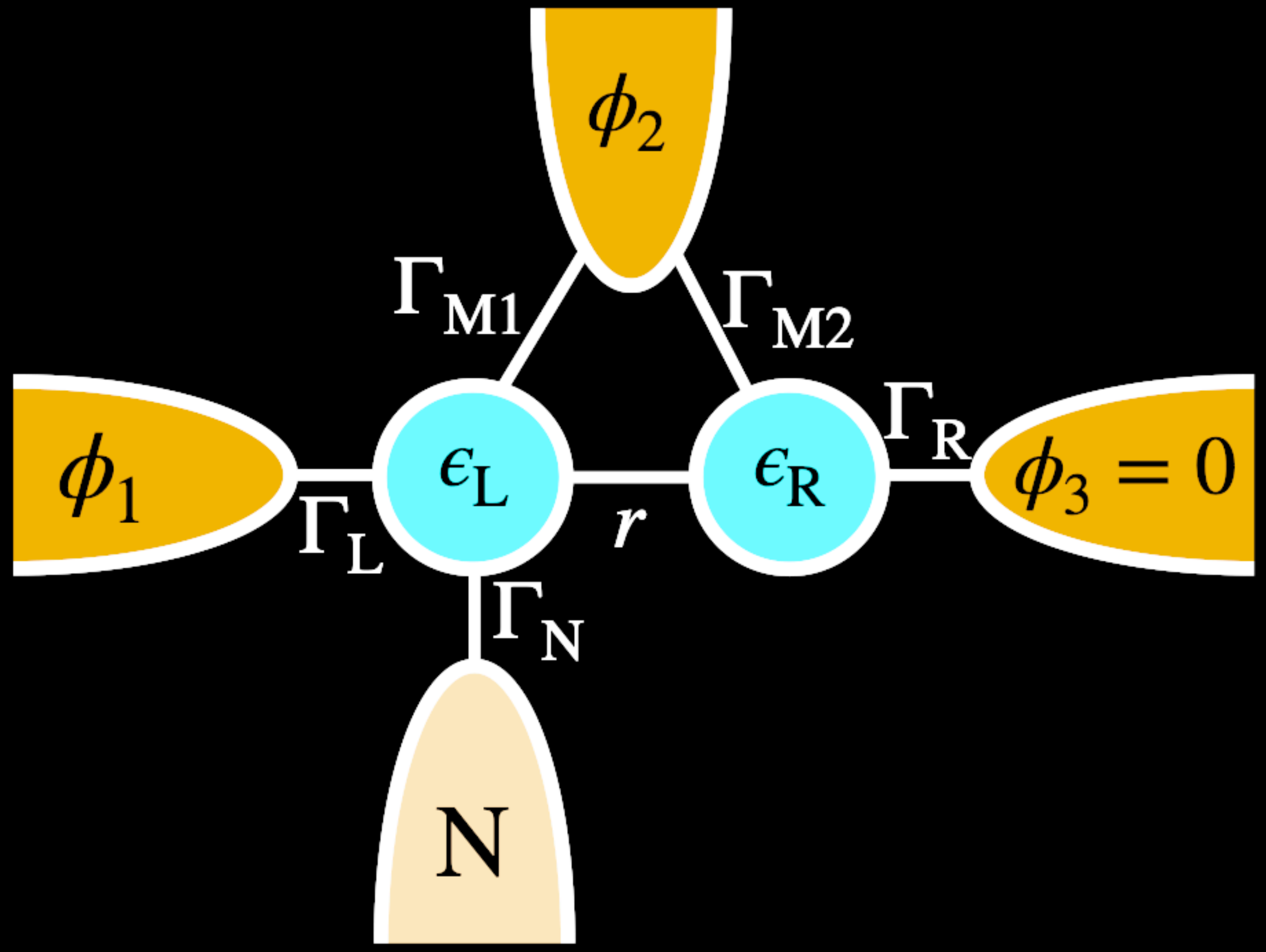
Projected spectrum



Why the difference?



VS.



Effective Dimension of gapped phase: $d_\phi = 0$

Effective Dimension of gapped phase: $d_\phi = 1$

How does classification work?

AZ class	Gap	Classifying space	$d=0$	$d=1$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0
AIII	P	$\mathcal{C}_0$	$\mathbb{Z}$	0
	$\mathrm{L_r}$	$\mathcal{C}_1$	0	$\mathbb{Z}$
	$\mathrm{L_i}$	$\mathcal{C}_0 \times \mathcal{C}_0$	$\mathbb{Z} \oplus \mathbb{Z}$	0

How does classification work?

AZ class	Gap	Classifying space	$d=0$	$d=1$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0
AIII	P	$\mathcal{C}_0$	$\mathbb{Z}$	0
	$\mathrm{L_r}$	$\mathcal{C}_1$	0	$\mathbb{Z}$
	$\mathrm{L_i}$	$\mathcal{C}_0 \times \mathcal{C}_0$	$\mathbb{Z} \oplus \mathbb{Z}$	0

What's the symmetry?

Particle-hole symmetry: Class $C^\dagger$

$$U_C H_i^*(\phi) U_C^{-1} = -H_i(\phi)$$

EP @ $E \neq 0$

$\Longrightarrow$

No symmetry: Class A

How does classification work?

AZ class	Gap	Classifying space	$d=0$	$d=1$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0
AIII	P	$\mathcal{C}_0$	$\mathbb{Z}$	0
	L_r	$\mathcal{C}_1$	0	$\mathbb{Z}$
	L_i	$\mathcal{C}_0 \times \mathcal{C}_0$	$\mathbb{Z} \oplus \mathbb{Z}$	0

What's the dimension?

Number of superconducting terminals d_ϕ?

What's the symmetry?

Particle-hole symmetry: Class $C^\dagger$

$$U_C H_i^*(\phi) U_C^{-1} = -H_i(\phi)$$

EP @ $E \neq 0$

$\Longrightarrow$

No symmetry: Class A

Making sense of effective dimensions

RESEARCH ARTICLE | MAY 14 2009

Periodic table for topological insulators and superconductors

Alexei Kitaev

AIP Conf. Proc. 1134, 22–30 (2009)

https://doi.org/10.1063/1.3149495

The Hamiltonian of a translationally invariant systems can be written in the momentum representation:

$$\hat{H} = \frac{i}{4}\sum_{\mathbf{p}}\sum_{j,k} A_{jk}(\mathbf{p})\hat{c}_{-\mathbf{p},j}\hat{c}_{\mathbf{p},k}, \qquad (19)$$

where j and k refer to particle flavors. The matrix $A(\mathbf{p})$ is skew-Hermitian but not real; it rather satisfies the condition $A_{jk}(\mathbf{p})^* = A_{jk}(-\mathbf{p})$. By abuse of terminology, such matrix-valued functions are called "functions from $\bar{\mathbb{R}}^d$ to real skew-symmetric matrices", where $\bar{\mathbb{R}}^d$ is the usual Euclidean space with the involution $\mathbf{p} \leftrightarrow -\mathbf{p}$ (cf. [29]).

By a *real vector bundle* over the *real space* X we mean a complex vector bundle E over X which is also a real space and such that

(i) the projection $E \to X$ is real (i.e. commutes with the involutions on E, X);

(ii) the map $E_x \to E_{\bar{x}}$ is anti-linear, i.e. the diagram

$$\begin{array}{ccc} \mathbf{C}\times E_x & \to & E_x \\ \downarrow & & \downarrow \\ \mathbf{C}\times E_{\bar{x}} & \to & E_{\bar{x}} \end{array}$$

commutes, where the vertical arrows denote the involution and $\mathbf{C}$ is given its standard real structure $(\tau(z) = \bar{z})$.

Normally, symmetries defined by

$$U_C H_i^*(k) U_C^{-1} = -H_i(-k)$$

But phase is **unusual** dimension:

$$U_C H_i^*(\phi) U_C^{-1} = -H_i(+\phi)$$

$\Rightarrow$ Phase is "just" a parameter

Solution: Effective Parameter

Effective classification parameter

$$d = d_k - d_\phi \bmod 8 = \begin{cases} 0 \ , \ d_k = d_\phi = 0 \\ 7 \ , \ d_k = 0 \ , \ d_\phi = 1 \end{cases}$$

Jeffrey C. Y. Teo and C. L. Kane Phys. Rev. B 82, 115120 (2010)
Fan Zhang and C. L. Kane Phys. Rev. B 90, 020501(R) (2014)

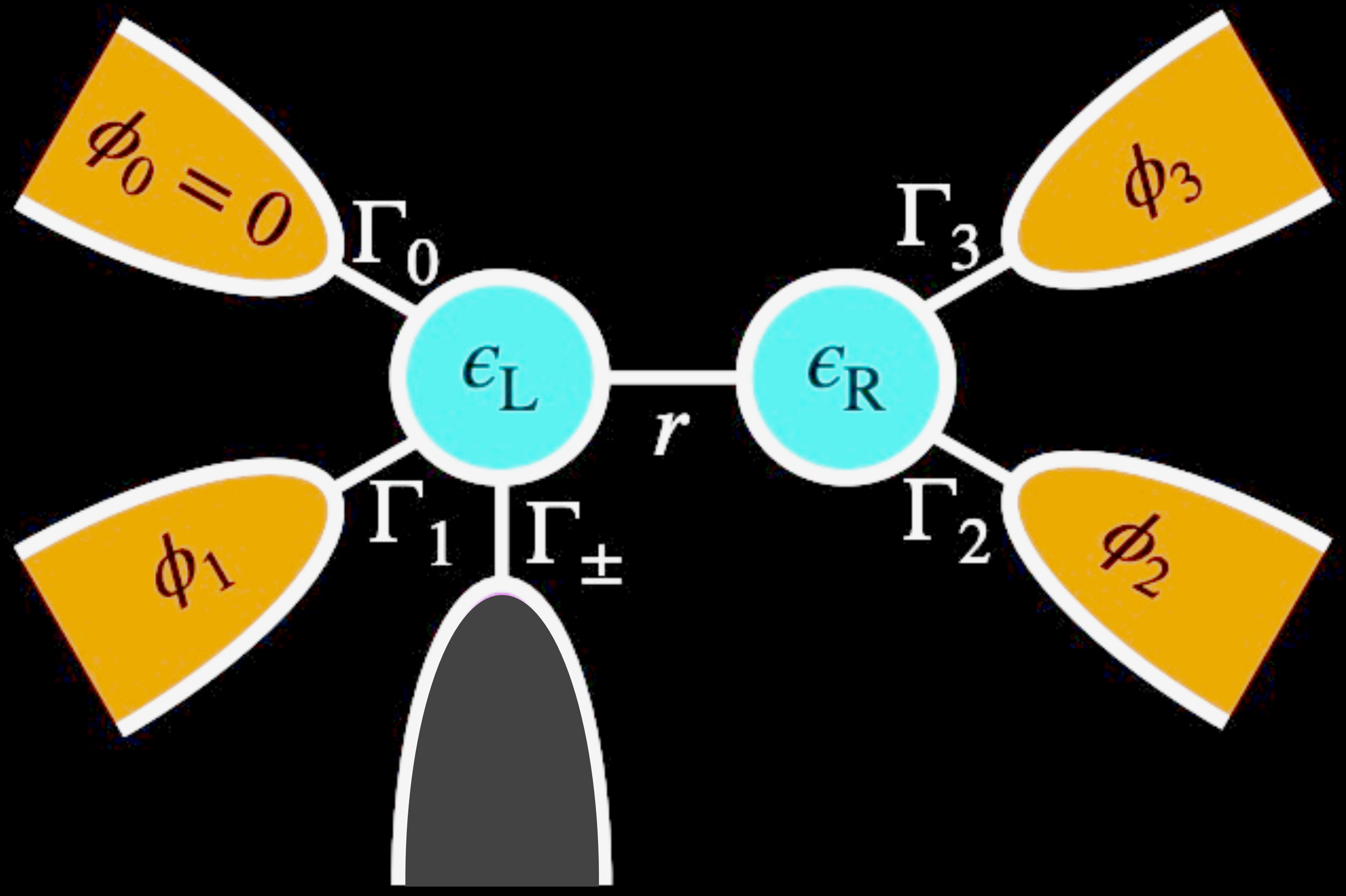
AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

winding number
NOT allowed

winding number
allowed!

Non-hermitian topology in MTJJs

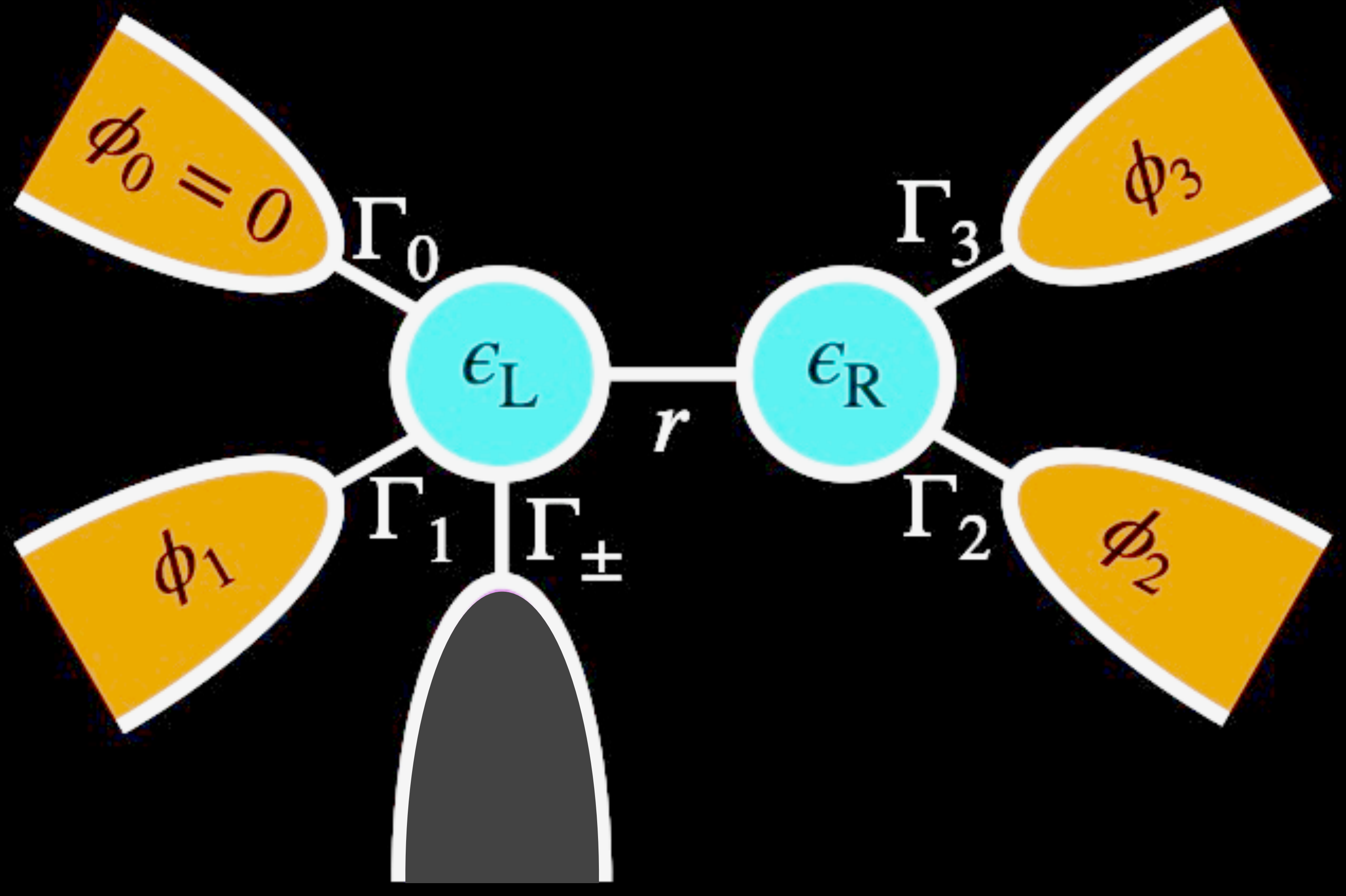
D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

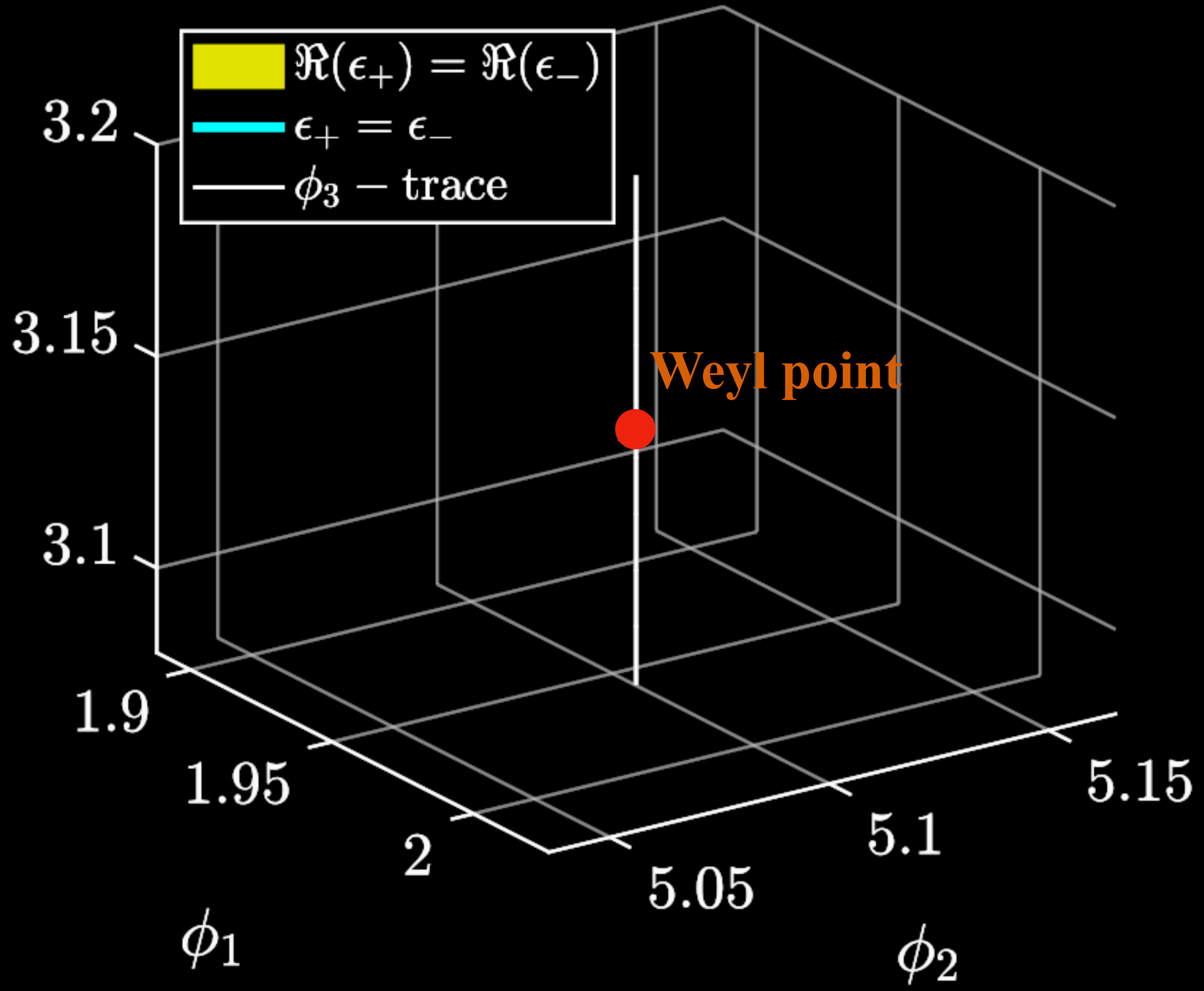


Non-hermitian topology in MTJJs

D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

Breaking spin-rotation symmetry!

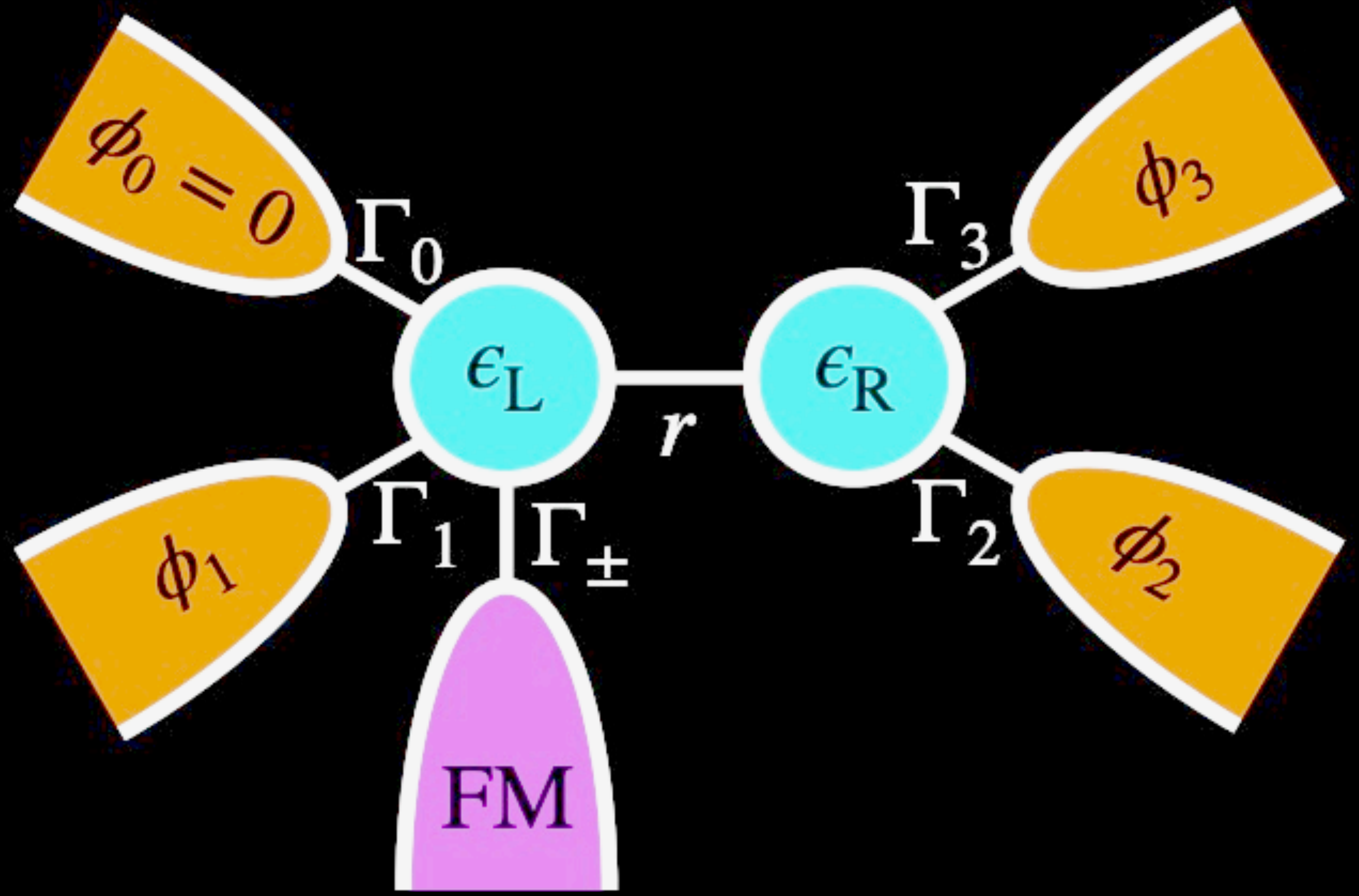


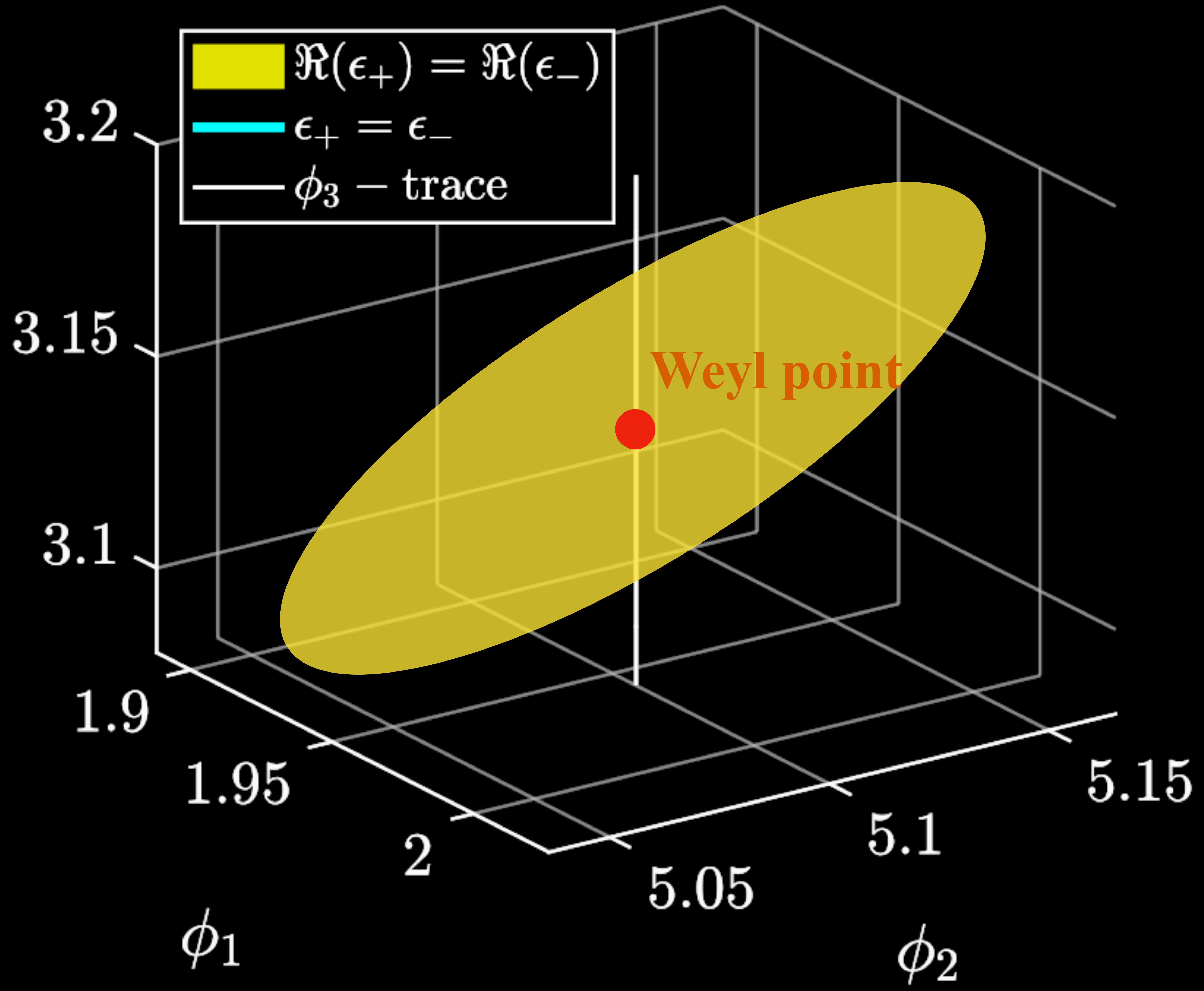


Non-hermitian topology in MTJJs

D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

Breaking spin-rotation symmetry!

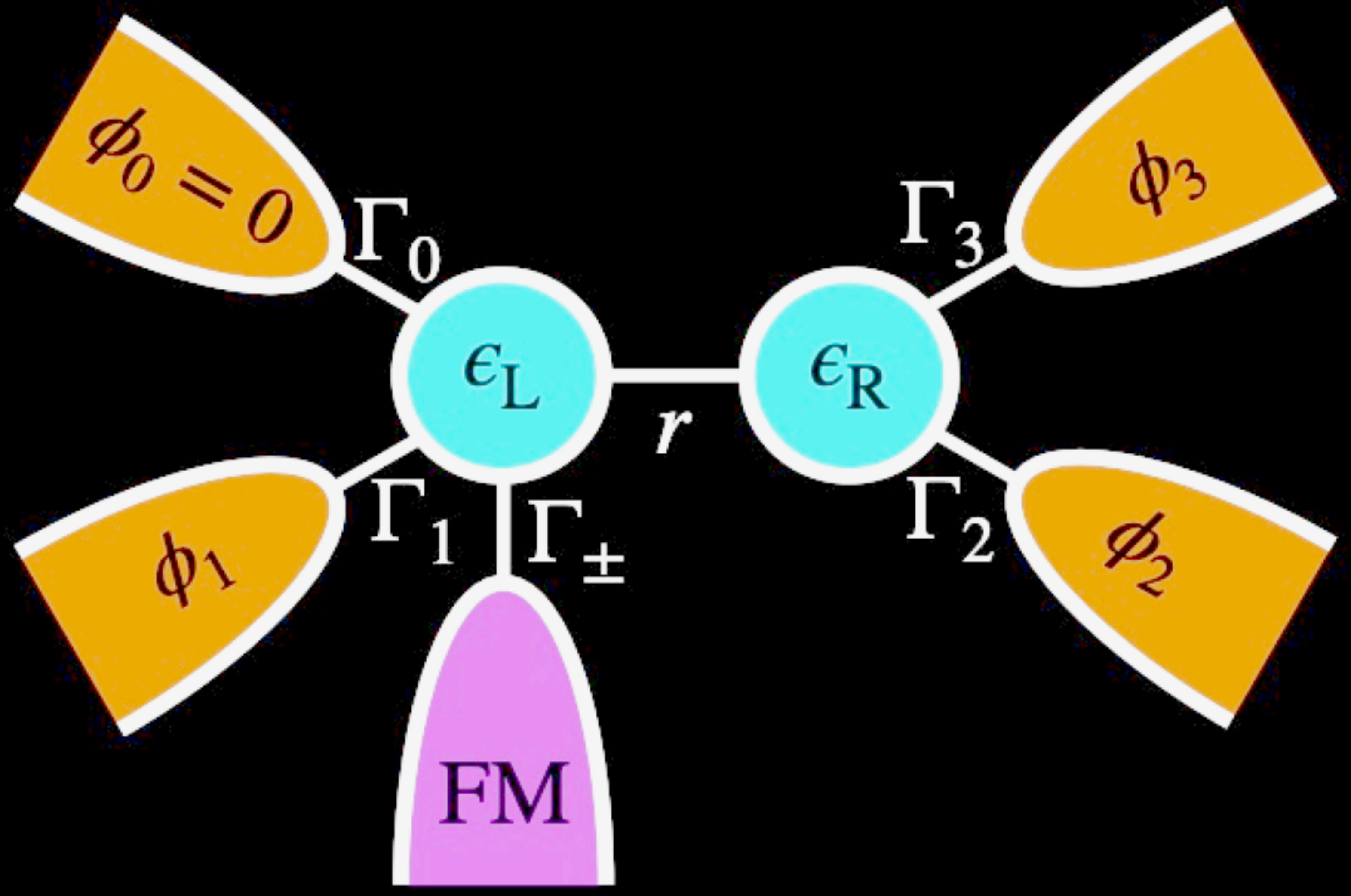




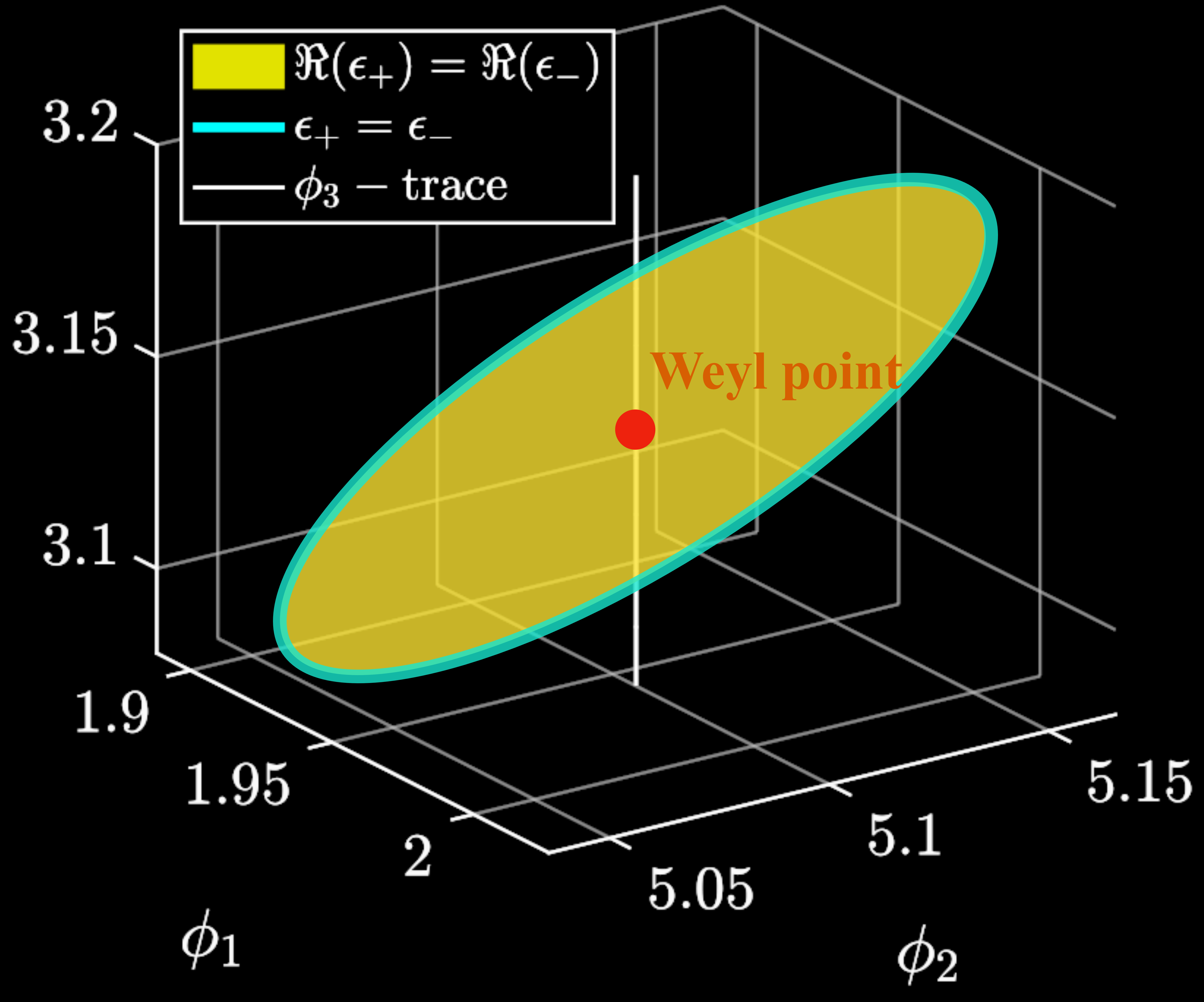
Non-hermitian topology in MTJJs

D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

Breaking spin-rotation symmetry!



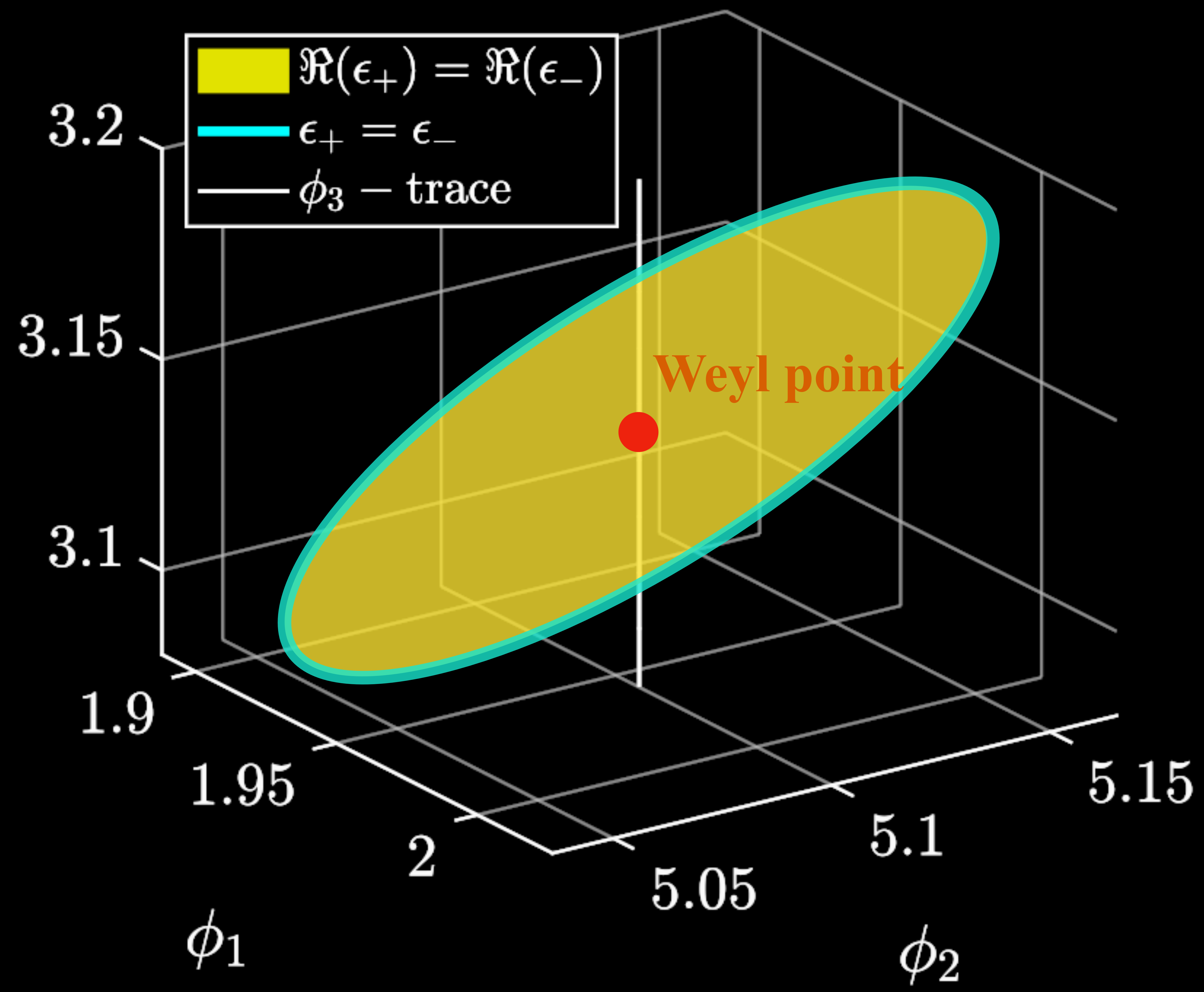
Exceptional Ring



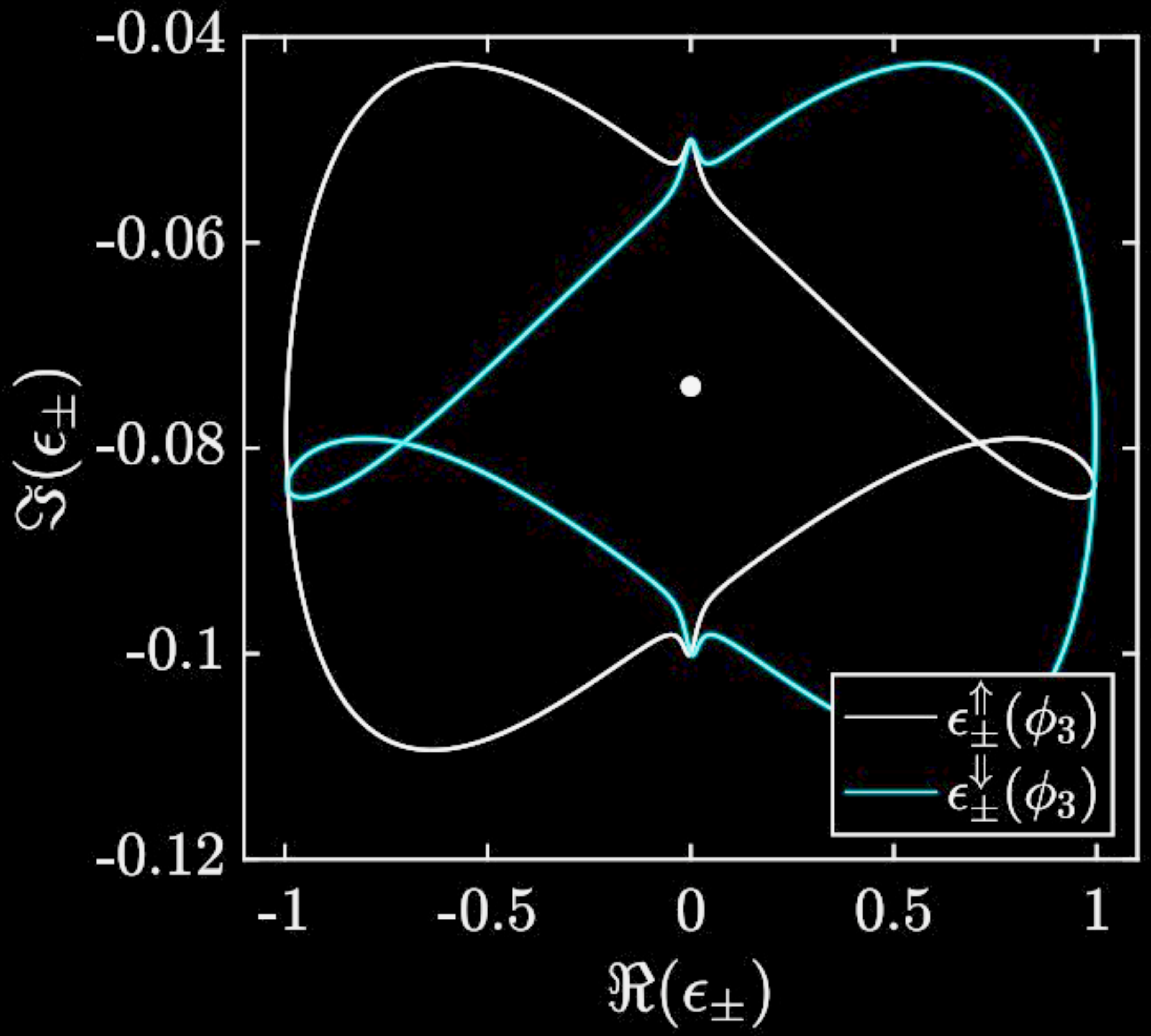
Non-hermitian topology in MTJJs

D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

Exceptional Ring



Point Gap



Effective classification parameter

Jeffrey C. Y. Teo and C. L. Kane Phys. Rev. B 82, 115120 (2010), Fan Zhang and C. L. Kane Phys. Rev. B 90, 020501(R) (2014)

Point gap in system with 1 superconducting phase $d_\phi = 1$ ($d_k = 0$)

$$d = d_k - d_\phi \mod 8 = 7$$

Effective classification parameter

Jeffrey C. Y. Teo and C. L. Kane Phys. Rev. B 82, 115120 (2010), Fan Zhang and C. L. Kane Phys. Rev. B 90, 020501(R) (2014)

Point gap in system with 1 superconducting phase $d_\phi = 1$ ($d_k = 0$)

$$d = d_k - d_\phi \mod 8 = 7$$

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$\mathrm{C}^\dagger$	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_r}$	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	$\mathrm{L_i}$	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0

Effective classification parameter

Jeffrey C. Y. Teo and C. L. Kane Phys. Rev. B 82, 115120 (2010), Fan Zhang and C. L. Kane Phys. Rev. B 90, 020501(R) (2014)

Point gap in system with 1 superconducting phase $d_\phi = 1$ ($d_k = 0$)

$$d = d_k - d_\phi \bmod 8 = 7$$

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$\mathrm{C}^\dagger$	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_r}$	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	$\mathrm{L_i}$	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0

winding number
NOT allowed

Effective classification parameter

Jeffrey C. Y. Teo and C. L. Kane Phys. Rev. B 82, 115120 (2010), Fan Zhang and C. L. Kane Phys. Rev. B 90, 020501(R) (2014)

Point gap in system with 1 superconducting phase $d_\phi = 1$ ($d_k = 0$)

$$d = d_k - d_\phi \bmod 8 = 7$$

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$C^\dagger$	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	L_r	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	L_i	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0

winding number
NOT allowed

Break spin-rotation symmetry!

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Effective classification parameter

Jeffrey C. Y. Teo and C. L. Kane Phys. Rev. B 82, 115120 (2010), Fan Zhang and C. L. Kane Phys. Rev. B 90, 020501(R) (2014)

Point gap in system with 1 superconducting phase $d_\phi = 1$ $(d_k = 0)$

$$d = d_k - d_\phi \bmod 8 = 7$$

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$C^\dagger$	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	L_r	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	L_i	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0

winding number NOT allowed

Break spin-rotation symmetry!

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

winding number allowed!

Conclusion II

1) MTJJs are an excellent platform to study engineered (non-hermitian) topology

Non-hermitian topology in MTJJs
D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)

Topology in three state Andreev Molecule
T. Antonelli, et. al., arXiv:2501.07982 (2025)

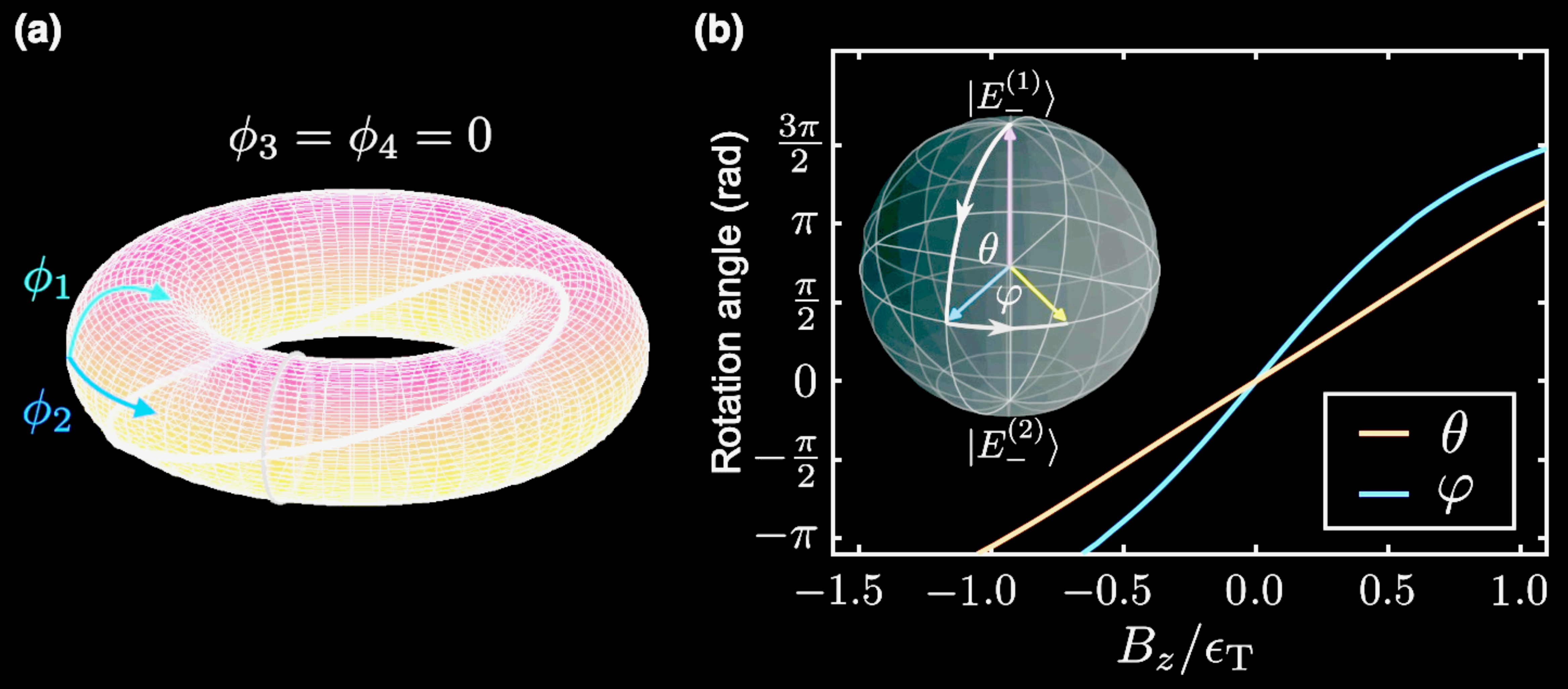
2) Refectionless scattering modes are a source of topology in MTJJs

Reflectionless Modes lead to Weyl nodes in MTJJs
D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

david.ohnmacht@uni-konstanz.de

Outlook

Higher order topology
(2nd Chern number)



H. Weisbrich, et. al., PRX Quantum 2, 010310 (2021)

Fundamental bounds?

Y. Onishi and L. Fu, Phys. Rev. X 14, 011052 (2024)

Andreev spin qubits

M. Hays,et. al., Science 373, 430 (2021).

Holonomic quantum
computation

V. Boogers, et. al., Phys. Rev. B 105, 235437 (2022)

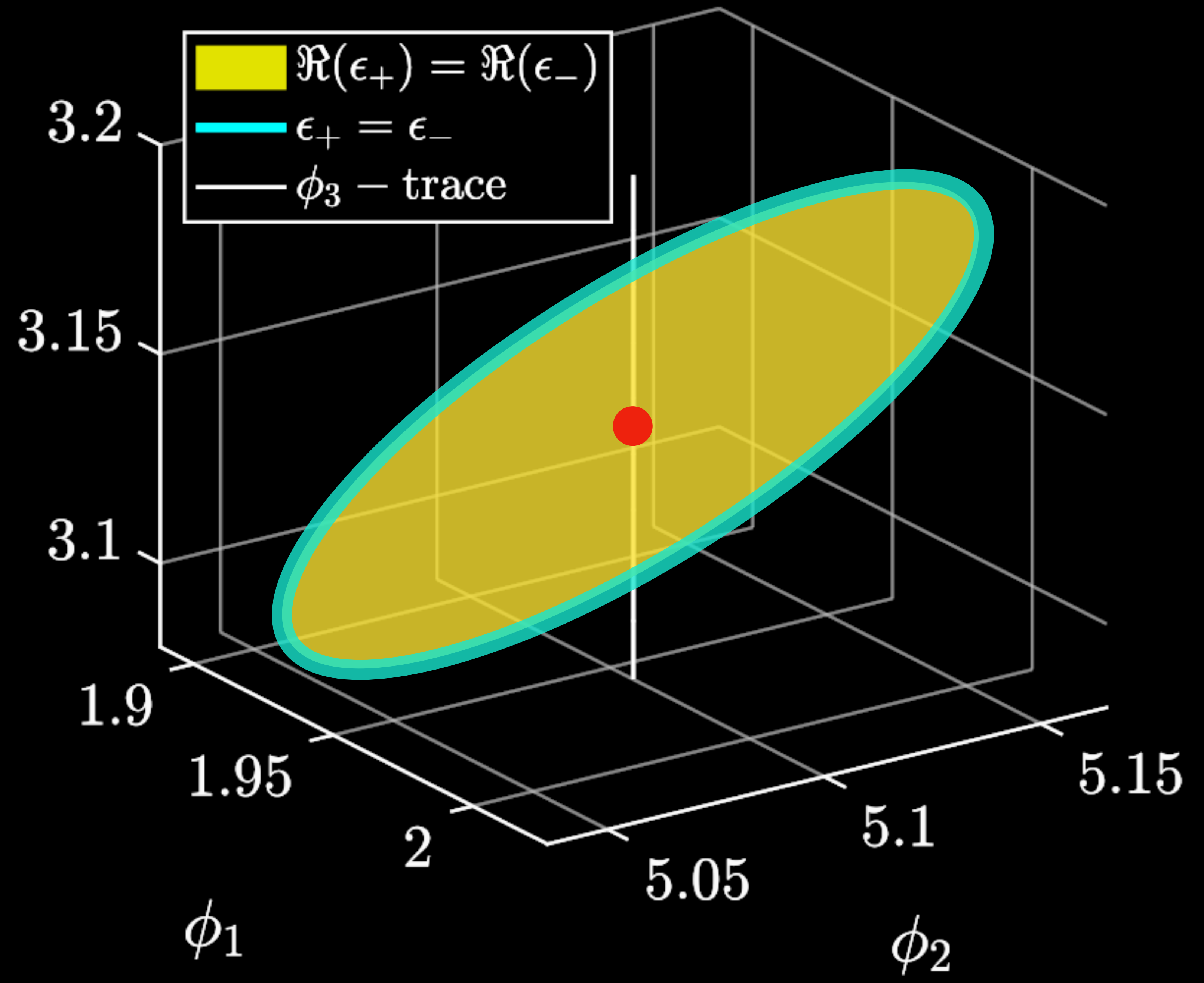
Edge States?

Thank you!

david.ohnmacht@uni-konstanz.de

Non-hermitian topology in MTJJs

D. C. Ohnmacht, et. al., Phys. Rev. Lett. 134, 156601 (2025)



Reflectionless modes as a source of Weyl nodes in multiterminal Josephson junctions

D. C. Ohnmacht, et. al., arXiv:2503.10874 (2025)

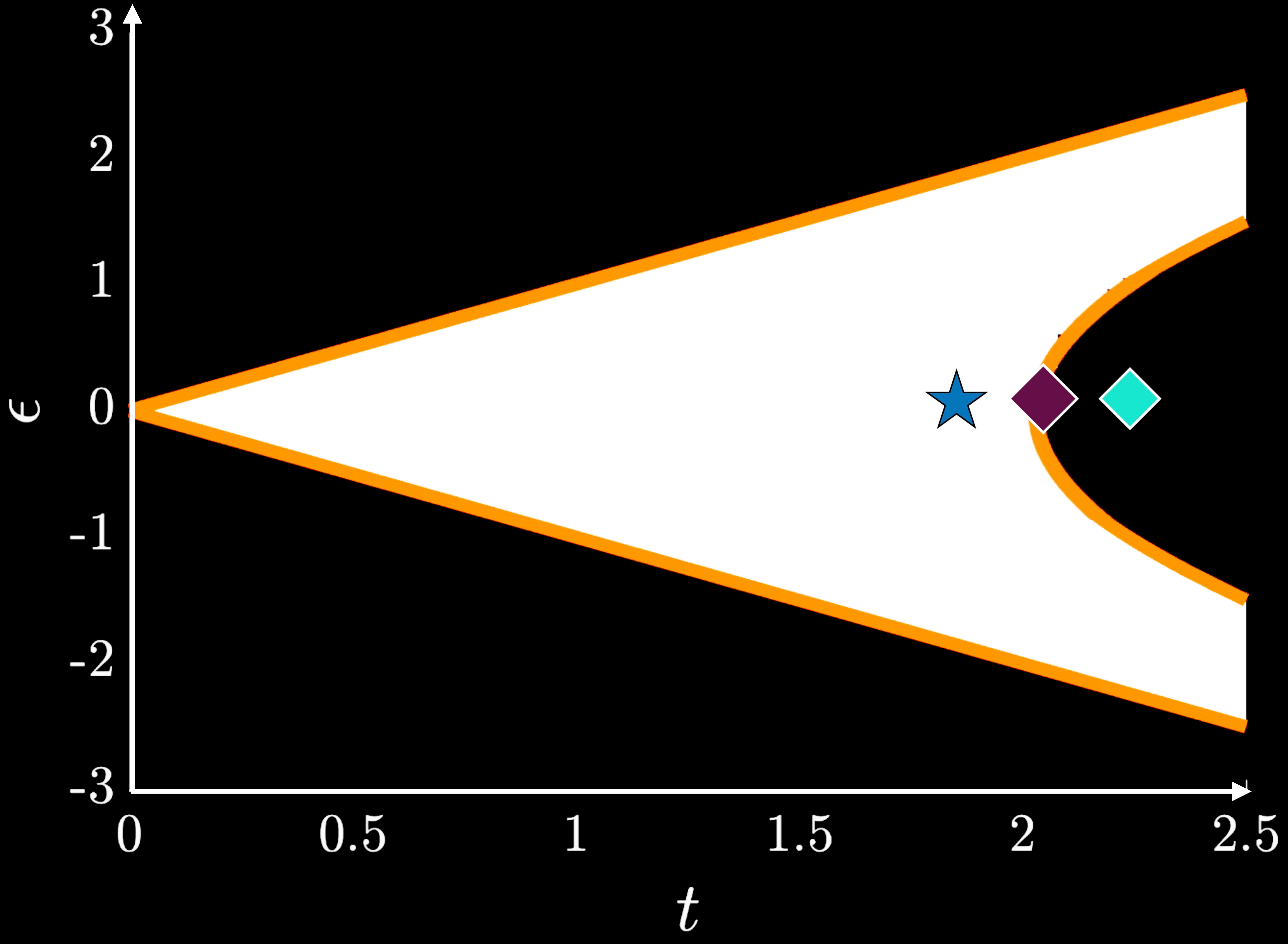
Exploring the energy spectrum of a four-terminal Josephson junction: Towards topological Andreev band structures

T. Antonelli, et. al., arXiv:2501.07982 (2025)

Topological charge of refl. less modes

$$\vec{V} = \left(\frac{\mathrm{Re}\,\det(r(z))}{|\det(r(z))|}, \frac{\mathrm{Im}\,\det(r(z))}{|\det(r(z))|} \right)$$

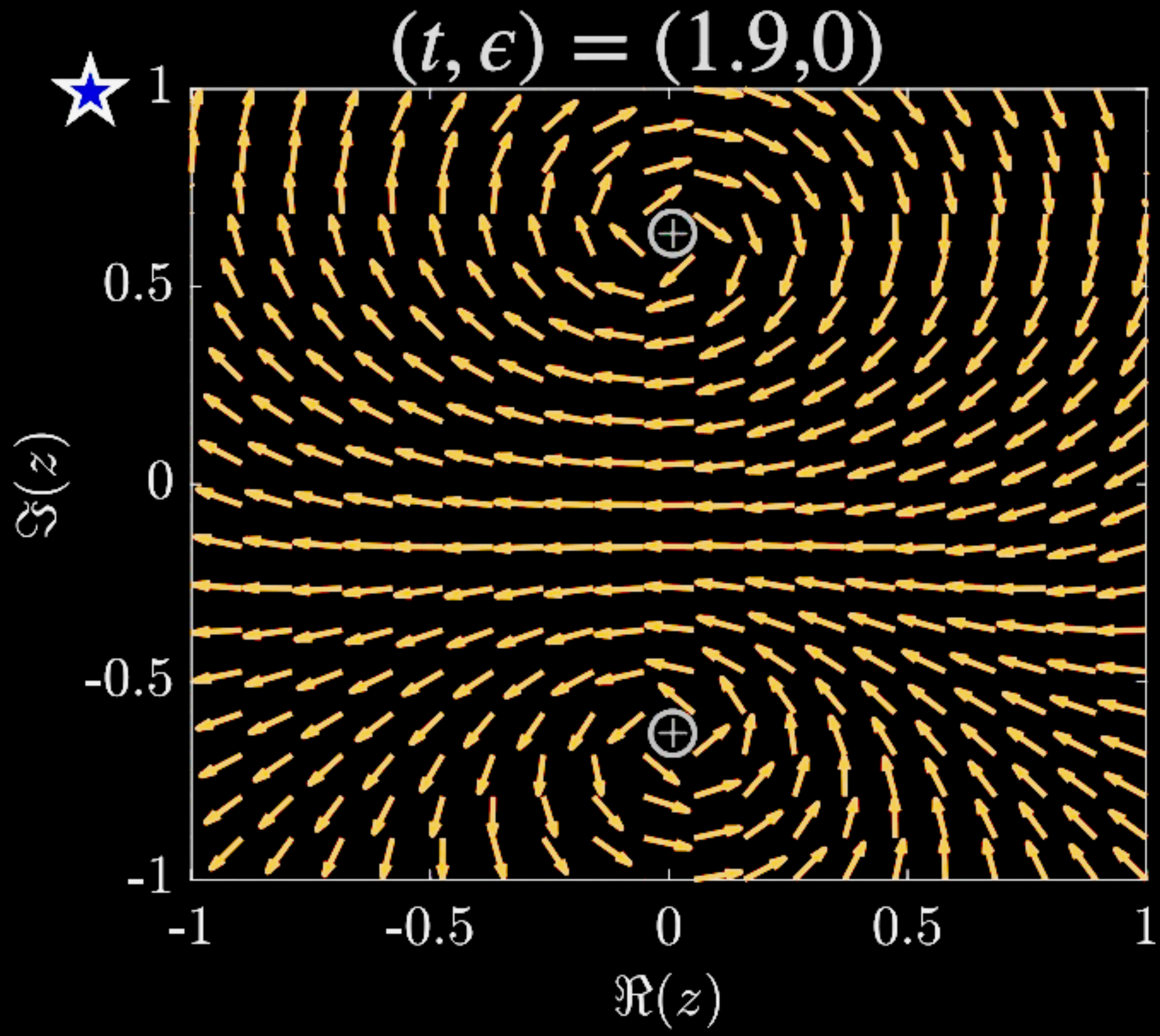
complex energy
$z \in \mathbb{C}$



W. R. Sweeney, et. al., Phys. Rev. A 102, 063511 (2020)

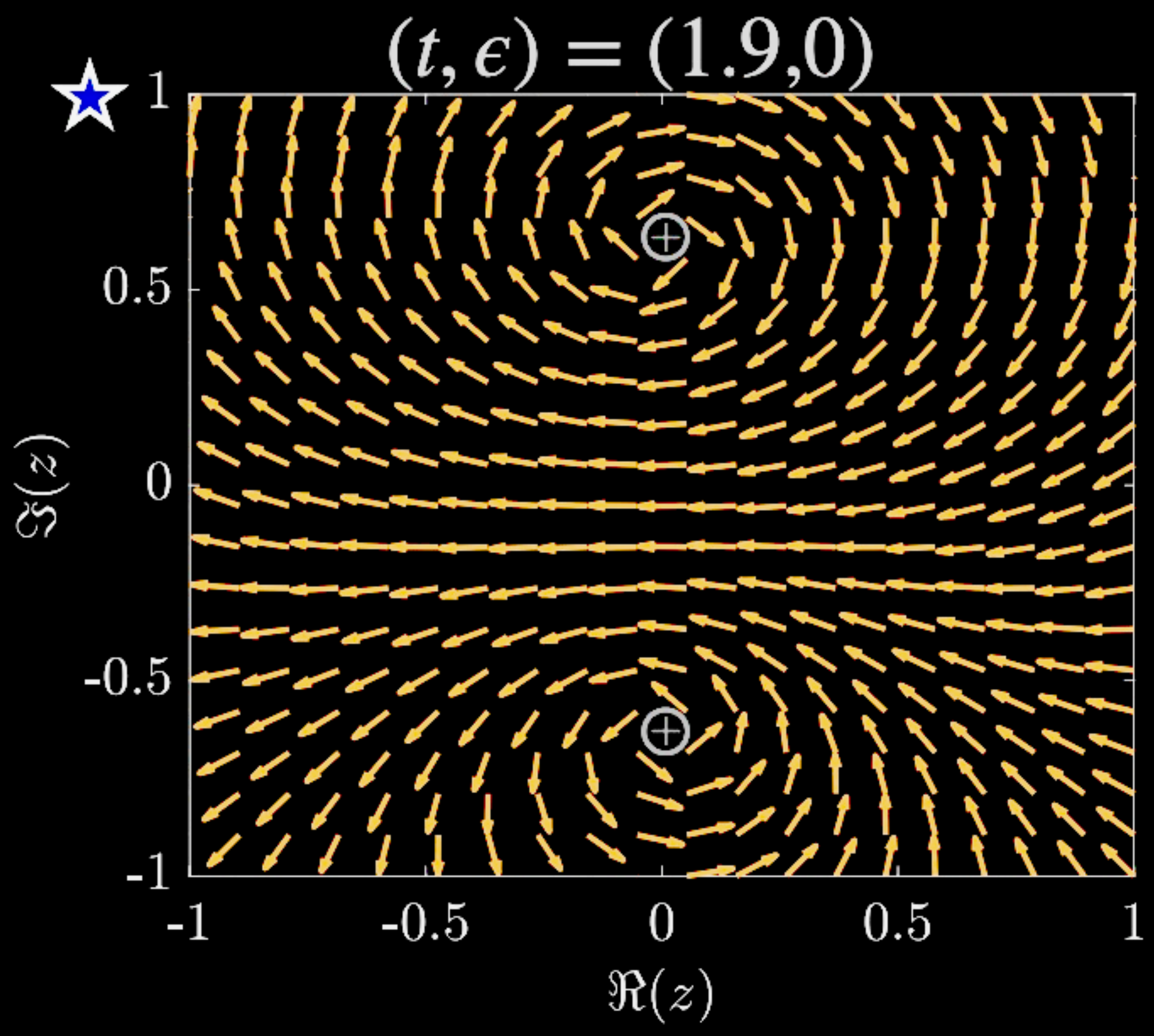
Topological charge of refl. less modes

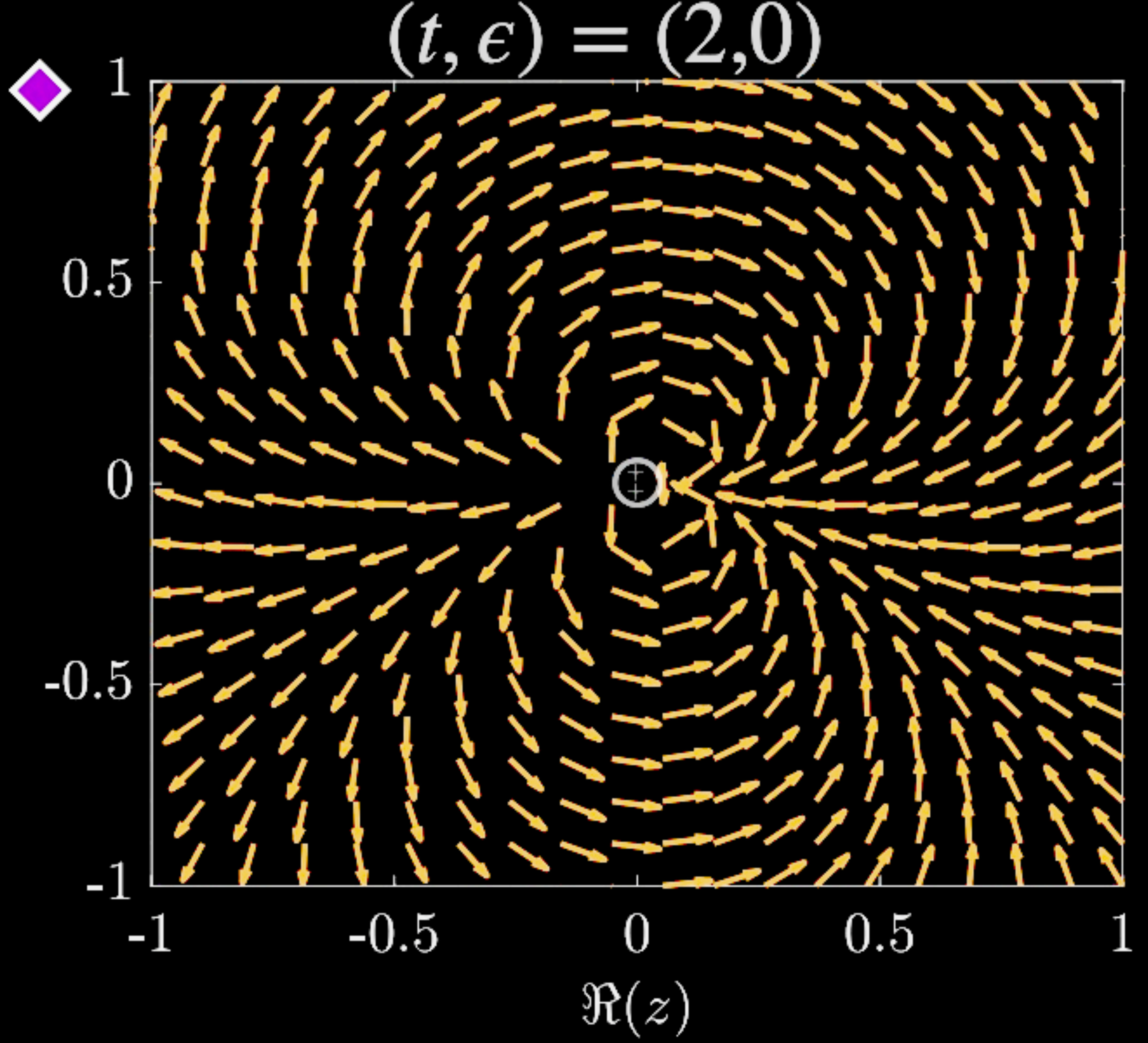
$$\vec{V} = \left(\frac{\mathrm{Re}\,\det(r(z))}{|\det(r(z))|}, \frac{\mathrm{Im}\,\det(r(z))}{|\det(r(z))|} \right)$$



Topological charge of refl. less modes

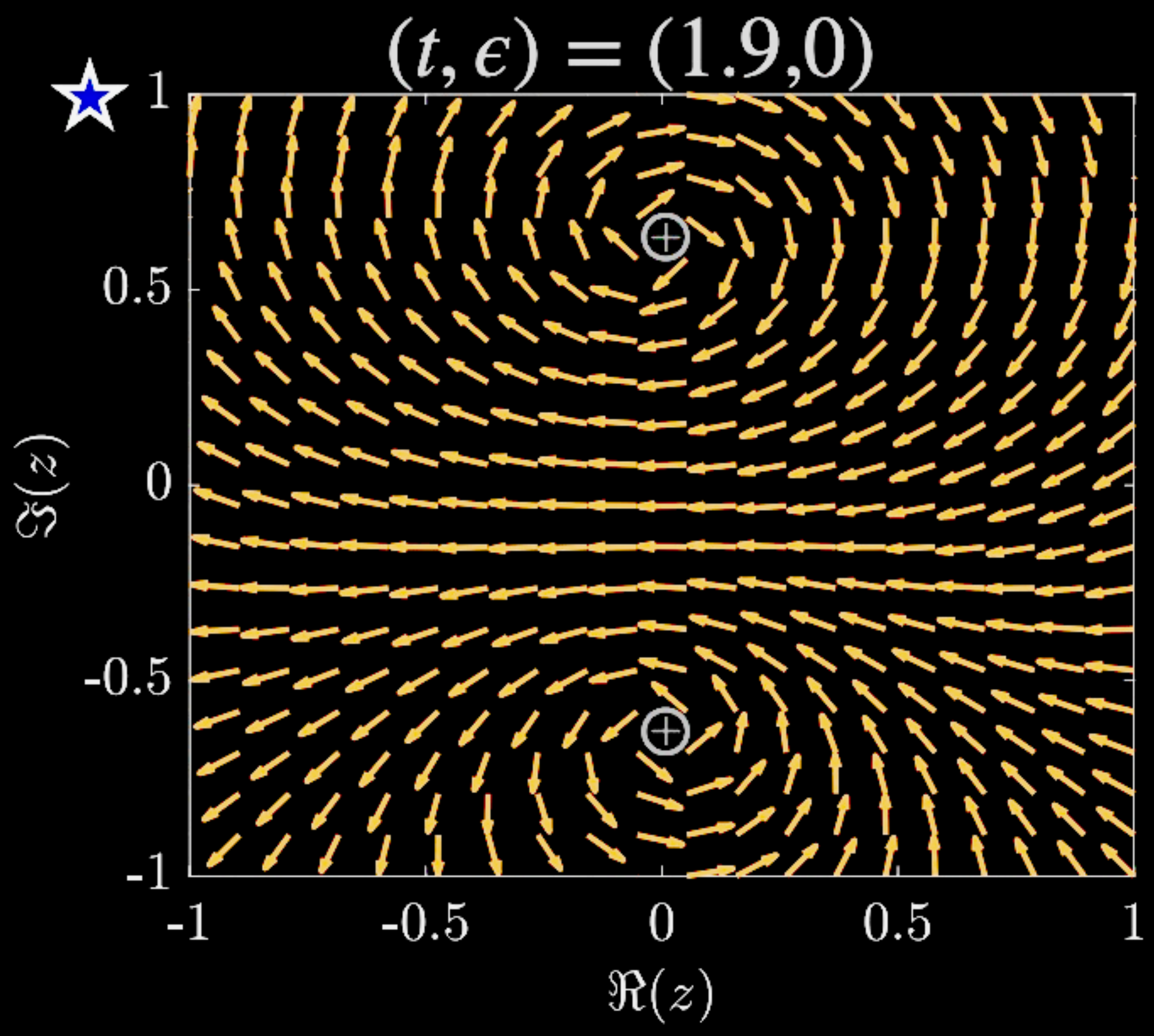
$$\vec{V} = \left(\frac{\mathrm{Re}\,\det(r(z))}{|\det(r(z))|}, \frac{\mathrm{Im}\,\det(r(z))}{|\det(r(z))|} \right)$$

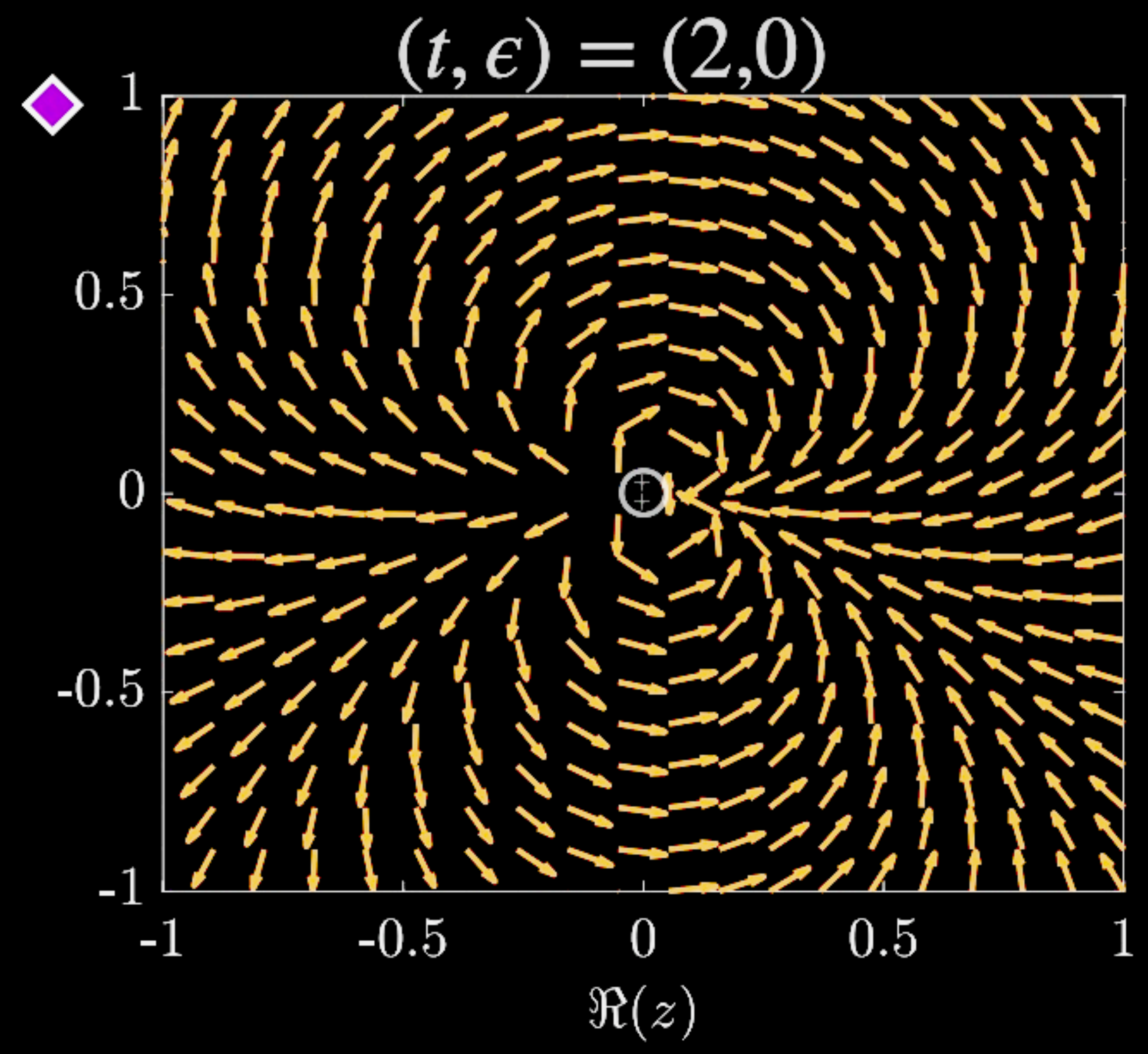


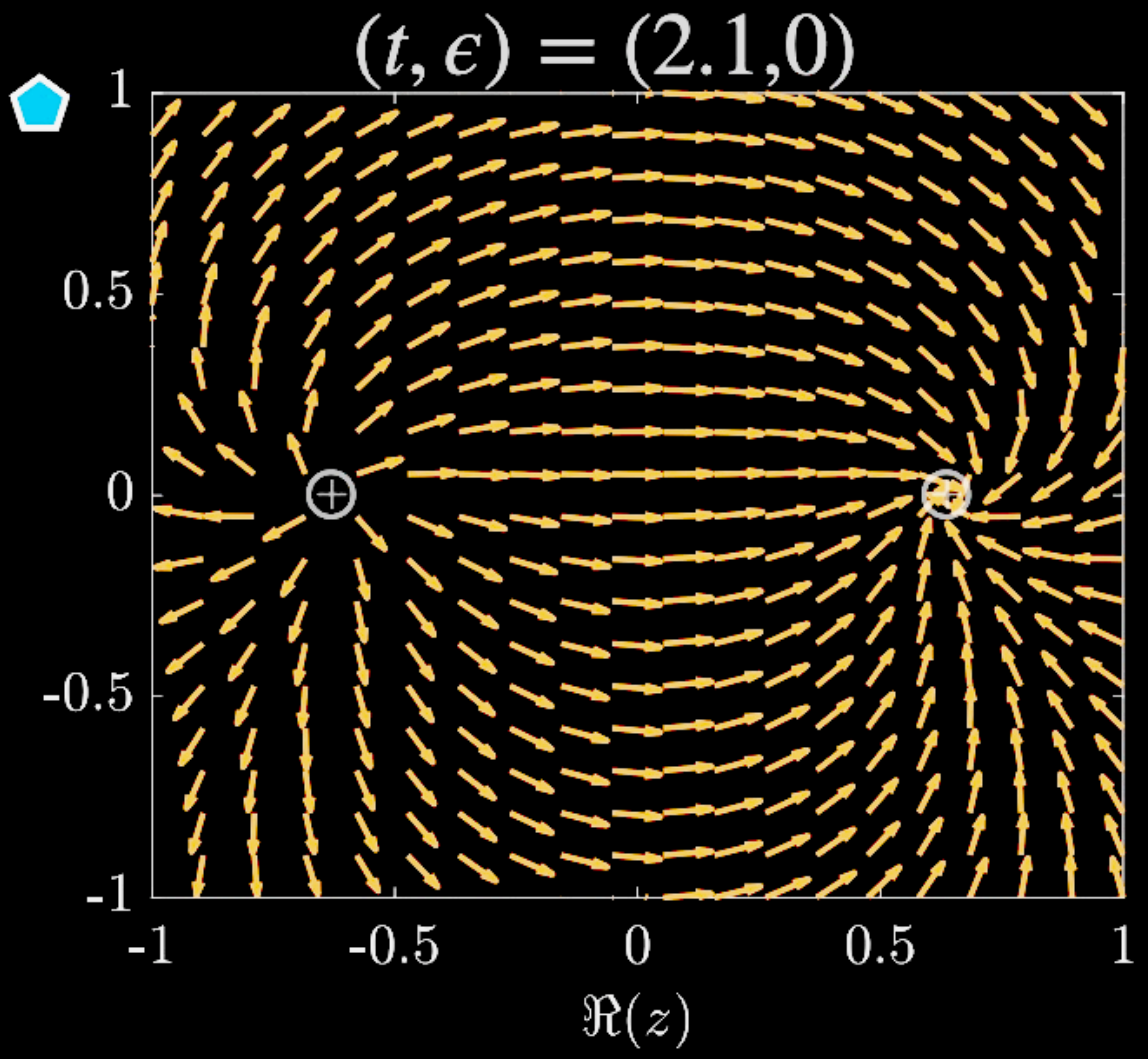


Topological charge of refl. less modes

$$\vec{V} = \left(\frac{\mathrm{Re}\,\det(r(z))}{|\det(r(z))|}, \frac{\mathrm{Im}\,\det(r(z))}{|\det(r(z))|} \right)$$







Topological charge of refl. less modes

$$\vec{V} = \left(\frac{\mathrm{Re}\,\det(r(z))}{|\det(r(z))|}, \frac{\mathrm{Im}\,\det(r(z))}{|\det(r(z))|} \right)$$

One can assign top. number, but
what about other boundaries?
→ No generalized number…

normal state S-matrix ⇔ effective Hamiltonian?

Phys. Rev. A 102, 063511 (2020)

Determinant of Reflection matrix with incoming and outgoing self-energy

$$\det(r) = \frac{\det(E - H_{\mathrm{N}} - \imath(\Sigma_{\mathrm{in}} - \Sigma_{\mathrm{out}}))}{\det(E - H_{\mathrm{N}} + \imath\Sigma(\overrightarrow{\phi} = 0))}$$

normal state S-matrix ⇔ effective Hamiltonian?

Phys. Rev. A 102, 063511 (2020)

Determinant of Reflection matrix with incoming and outgoing self-energy

$$\det(r) = \frac{\det(E - H_{\mathrm{N}} - \imath(\Sigma_{\mathrm{in}} - \Sigma_{\mathrm{out}}))}{\det(E - H_{\mathrm{N}} + \imath\Sigma(\overrightarrow{\phi} = 0))}$$

$$\text{For } r_{12} = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix}: \Sigma_{\mathrm{int/out}} = \sigma_0 \mp \sigma_z$$

$$\text{(III)} \ \det(r_{12}(E=0)) \propto \det(H_{\mathrm{N}} + 2\imath\sigma_z) \propto (\epsilon^2 - t^2 + 4)$$

normal state S-matrix ⇔ effective Hamiltonian?

Phys. Rev. A 102, 063511 (2020)

Determinant of Reflection matrix with incoming and outgoing self-energy

$$\det(r) = \frac{\det(E - H_\mathrm{N} - \imath(\Sigma_\mathrm{in} - \Sigma_\mathrm{out}))}{\det(E - H_\mathrm{N} + \imath\Sigma(\overrightarrow{\phi} = 0))}$$

$$\text{For } r_{12} = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix}: \Sigma_\mathrm{int/out} = \sigma_0 \mp \sigma_z$$

$$\text{(III)} \det(r_{12}(E=0)) \propto \det(H_\mathrm{N} + 2\imath\sigma_z) \propto (\epsilon^2 - t^2 + 4)$$

Beenaker determinant formula

$$\det(\{S_\mathrm{N}(E=0), e^{\imath\overrightarrow{\phi}}\}) = 0$$

$$S_\mathrm{N} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \quad \text{with} \quad \det(r(E=0)) = 0$$

normal state S-matrix ⇔ effective Hamiltonian?

Phys. Rev. A 102, 063511 (2020)

Determinant of Reflection matrix with incoming and outgoing self-energy

$$\det(r) = \frac{\det(E - H_{\mathrm{N}} - \imath(\Sigma_{\mathrm{in}} - \Sigma_{\mathrm{out}}))}{\det(E - H_{\mathrm{N}} + \imath\Sigma(\overrightarrow{\phi} = 0))}$$

For $r_{12} = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix}$: $\Sigma_{\mathrm{int/out}} = \sigma_0 \mp \sigma_z$

(III) $\det(r_{12}(E=0)) \propto \det(H_{\mathrm{N}} + 2\imath\sigma_z)$
$\propto (\epsilon^2 - t^2 + 4)$

Beenaker determinant formula

$$\det(\{S_{\mathrm{N}}(E=0), e^{\imath\overrightarrow{\phi}}\}) = 0$$

$$S_{\mathrm{N}} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \quad \text{with} \quad \det(r(E=0)) = 0$$

Choose: $e^{\imath\overrightarrow{\phi}} = \mathrm{diag}(1_{n\times n}, -1_{n\times n})$

$$\det(\{S_{\mathrm{N}}(E=0), e^{\imath\overrightarrow{\phi}}\}) = \det(r(E=0))\det(-r'(E=0))$$

normal state S-matrix ⇔ effective Hamiltonian?

Phys. Rev. A 102, 063511 (2020)

Determinant of Reflection matrix with incoming and outgoing self-energy

$$\det(r) = \frac{\det(E - H_\mathrm{N} - \imath(\Sigma_\mathrm{in} - \Sigma_\mathrm{out}))}{\det(E - H_\mathrm{N} + \imath\Sigma(\overrightarrow{\phi} = 0))}$$

$$\text{For } r_{12} = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix}: \Sigma_\mathrm{int/out} = \sigma_0 \mp \sigma_z$$

$$\text{(III)}\ \det(r_{12}(E=0)) \propto \det(H_\mathrm{N} + 2\imath\sigma_z)$$
$$\propto (\epsilon^2 - t^2 + 4)$$

Beenaker determinant formula

$$\det(\{S_\mathrm{N}(E=0), e^{\imath\overrightarrow{\phi}}\}) = 0$$

$$S_\mathrm{N} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \quad \text{with} \quad \det(r(E=0)) = 0$$

$$\text{Choose: } e^{\imath\overrightarrow{\phi}} = \mathrm{diag}(1_{n\times n}, -1_{n\times n})$$

$$\det(\{S_\mathrm{N}(E=0), e^{\imath\overrightarrow{\phi}}\}) = \det(r(E=0))\det(-r'(E=0))$$

Effective Hamiltonian

$$H_\mathrm{eff} = \begin{pmatrix} H_\mathrm{N} & \Sigma \\ \Sigma^* & -H_\mathrm{N} \end{pmatrix} \qquad \Sigma = 2\imath\sigma_z$$

$$\det(H_\mathrm{eff}) = \det(H_\mathrm{N} + 2\imath\sigma_z)\det(-H_\mathrm{N} + 2\imath\sigma_z)$$

normal state S-matrix ⇔ effective Hamiltonian?

Phys. Rev. A 102, 063511 (2020)

Determinant of Reflection matrix with incoming and outgoing self-energy

$$\det(r) = \frac{\det(E - H_\mathrm{N} - \imath(\Sigma_\mathrm{in} - \Sigma_\mathrm{out}))}{\det(E - H_\mathrm{N} + \imath\Sigma(\vec{\phi} = 0))}$$

$$\text{For } r_{12} = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix} : \Sigma_\mathrm{int/out} = \sigma_0 \mp \sigma_z$$

$$\text{(III)}\ \det(r_{12}(E=0)) \propto \det(H_\mathrm{N} + 2\imath\sigma_z) \propto (\epsilon^2 - t^2 + 4)$$

Beenaker determinant formula

$$\det(\{S_\mathrm{N}(E=0), e^{\imath\vec{\phi}}\}) = 0$$

$$S_\mathrm{N} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \quad \text{with} \quad \det(r(E=0)) = 0$$

$$\text{Choose: } e^{\imath\vec{\phi}} = \mathrm{diag}(1_{n\times n}, -1_{n\times n})$$

$$\det(\{S_\mathrm{N}(E=0), e^{\imath\vec{\phi}}\}) = \det(r(E=0))\det(-r'(E=0))$$

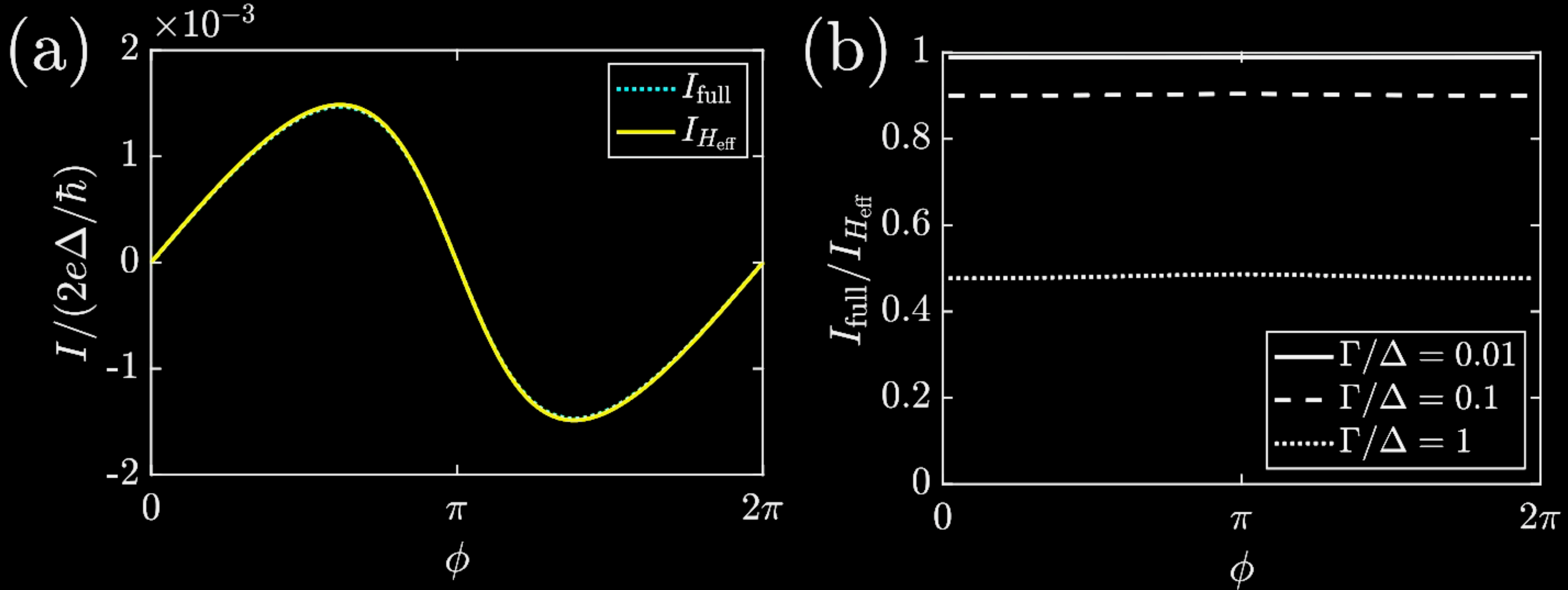
Effective Hamiltonian

$$H_\mathrm{eff} = \begin{pmatrix} H_\mathrm{N} & \Sigma \\ \Sigma^* & -H_\mathrm{N} \end{pmatrix} \qquad \Sigma = 2\imath\sigma_z$$

$$\det(H_\mathrm{eff}) = \det(H_\mathrm{N} + 2\imath\sigma_z)\det(-H_\mathrm{N} + 2\imath\sigma_z)$$

The superconducting phases play the same role as scattering with incoming and outgoing self-energies!

Validity of non-hermitian Hamiltonian



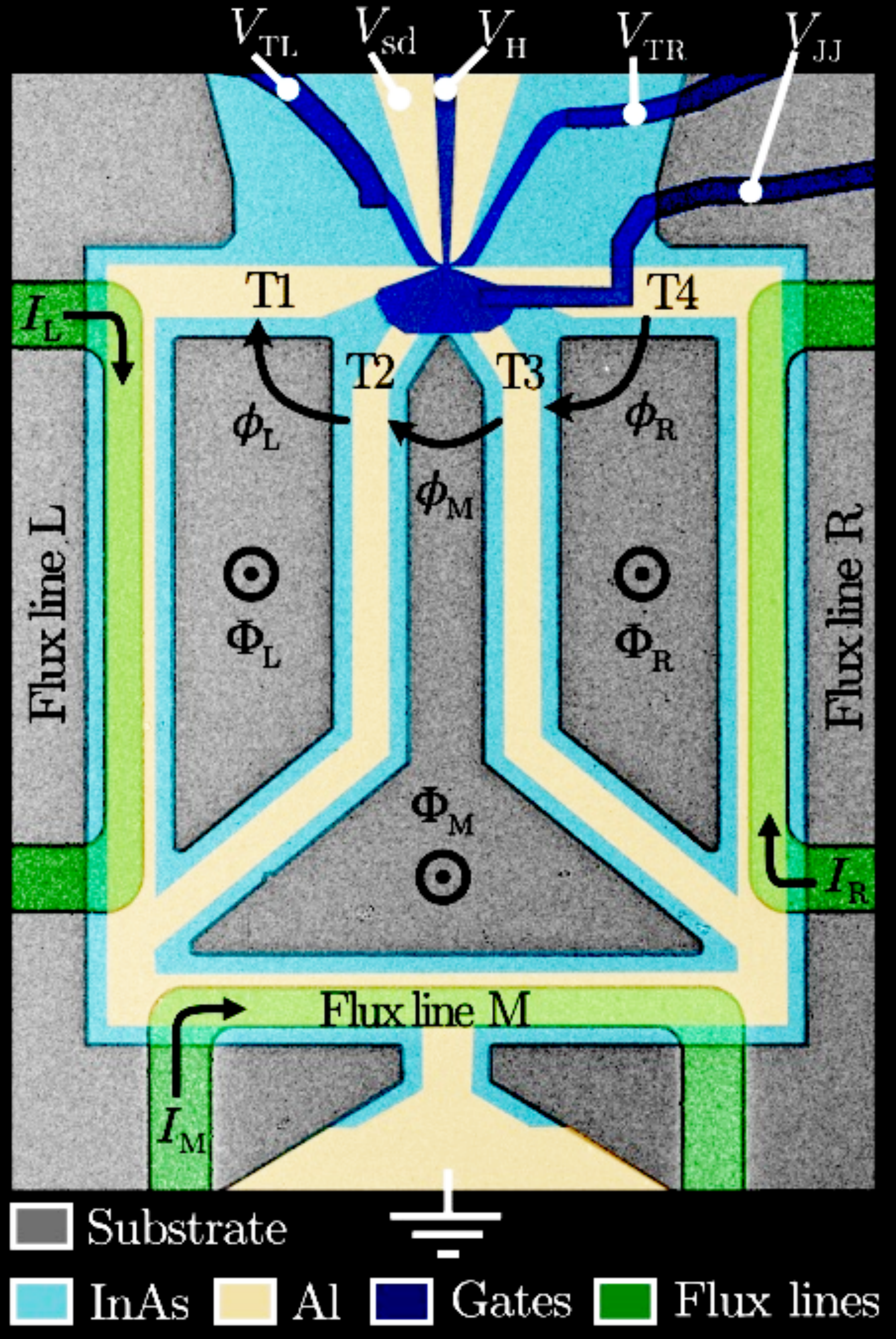
$$I = -\frac{4e\Delta}{h}\frac{d}{d\phi}\mathrm{Im}\sum_i \epsilon_i \ln(\epsilon_i)$$

$$\Gamma \ll \Delta$$

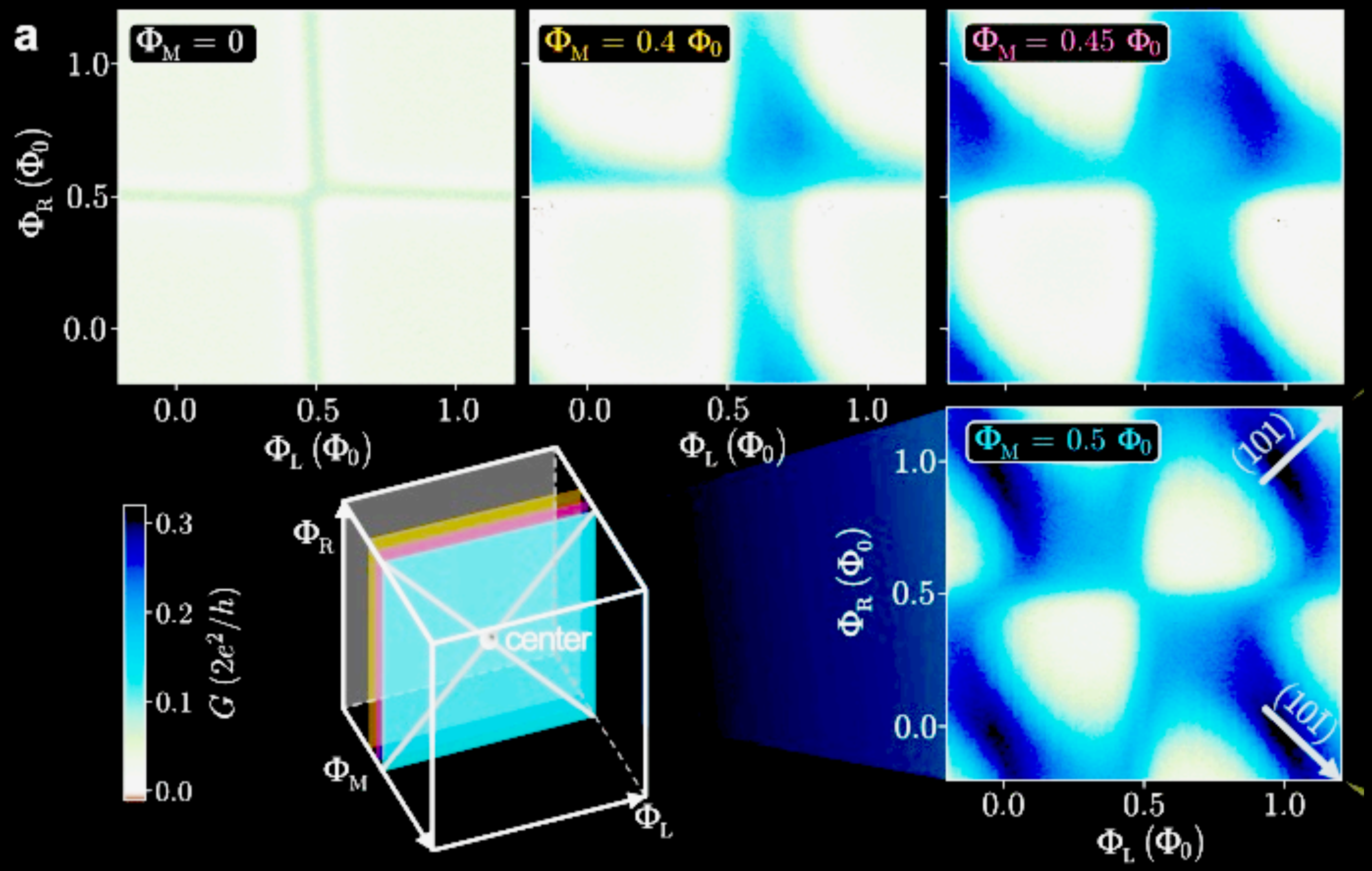
Experiments on MTJJs: Topology?

S-2DEG Hybrid

Topography of
4-terminal MTJJ

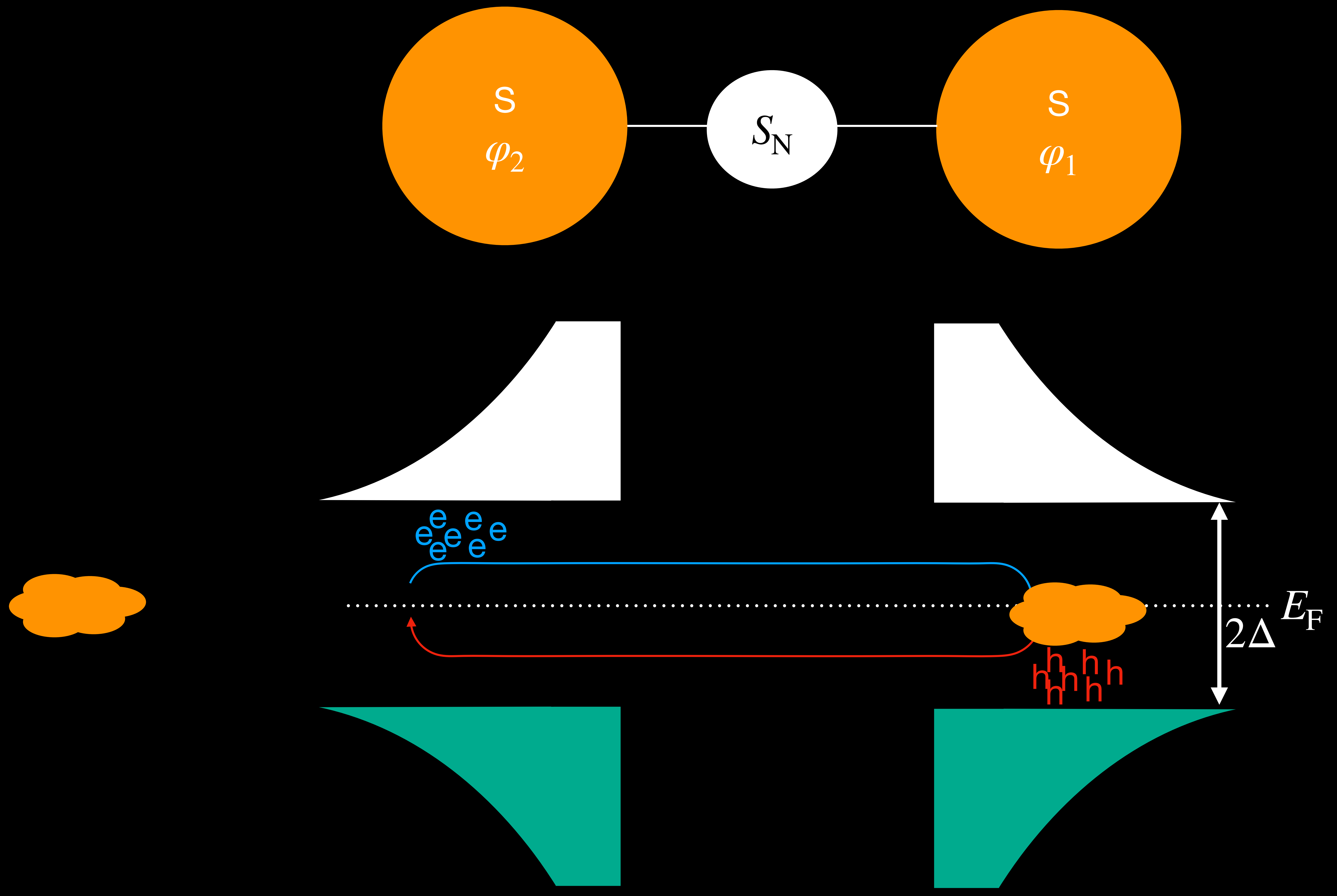


Conductance measurement at $E = 0$



T. Antonelli, et. al., arXiv:2408.01281 (2024).

Supercurrent



☑ **Supercurrent in equilibrium!**

$$
, -\frac{\sqrt{2}\,\Gamma_1\Gamma_2^2\sqrt{|\Gamma_1|^2+|\Gamma_2|^2+2r^2+\sqrt{|\Gamma_1|^4+|\Gamma_2|^4+4r^2|\Gamma_1|^2+4r^2|\Gamma_2|^2-2|\Gamma_1|^2|\Gamma_2|^2-\frac{4\Gamma_2 r^2|\Gamma_1|^2}{\Gamma_1}-\frac{4\Gamma_1 r^2|\Gamma_2|^2}{\Gamma_2}}}\left(\frac{|\Gamma_2|^2\sqrt{|\Gamma_1|^4+|\Gamma_2|^4+4r^2|\Gamma_1|^2+4r^2|\Gamma_2|^2-2|\Gamma_1|^2|\Gamma_2|^2-\frac{4\Gamma_2 r^2|\Gamma_1|^2}{\Gamma_1}-\frac{4\Gamma_1 r^2|\Gamma_2|^2}{\Gamma_2}}}{2\Gamma_2}-\frac{|\Gamma_2|^4}{2\Gamma_2}+\frac{|\Gamma_1|^2|\Gamma_2|^2}{2\Gamma_2}+\frac{r^2|\Gamma_1|^2}{\Gamma_1}-\frac{r^2|\Gamma_2|^2}{\Gamma_2}\right)}{2\left(\Gamma_2 r^2+\Gamma_1|\Gamma_2|^2\right)\left(\Gamma_1|\Gamma_2|^2-\Gamma_2|\Gamma_1|^2\right)}
$$

$$
\frac{\sqrt{2}\,\Gamma_1\Gamma_2^2\sqrt{|\Gamma_1|^2+|\Gamma_2|^2+2r^2-\sqrt{|\Gamma_1|^4+|\Gamma_2|^4+4r^2|\Gamma_1|^2+4r^2|\Gamma_2|^2-2|\Gamma_1|^2|\Gamma_2|^2-\frac{4\Gamma_2 r^2|\Gamma_1|^2}{\Gamma_1}-\frac{4\Gamma_1 r^2|\Gamma_2|^2}{\Gamma_2}}}\left(\frac{|\Gamma_2|^4}{2\Gamma_2}+\frac{|\Gamma_2|^2\sqrt{|\Gamma_1|^4+|\Gamma_2|^4+4r^2|\Gamma_1|^2+4r^2|\Gamma_2|^2-2|\Gamma_1|^2|\Gamma_2|^2-\frac{4\Gamma_2 r^2|\Gamma_1|^2}{\Gamma_1}-\frac{4\Gamma_1 r^2|\Gamma_2|^2}{\Gamma_2}}}{2\Gamma_2}-\frac{|\Gamma_1|^2|\Gamma_2|^2}{2\Gamma_2}-\frac{r^2|\Gamma_1|^2}{\Gamma_1}+\frac{r^2|\Gamma_2|^2}{\Gamma_2}\right)}{2r\left(\Gamma_2 r^2+\Gamma_1|\Gamma_2|^2\right)\left(\Gamma_1|\Gamma_2|^2-\Gamma_2|\Gamma_1|^2\right)}
$$

$$1$$

$$
' -4\,e^{-\chi_1 1i+\chi_2 1i}\left(f_2+e^{\chi_1 1i-\chi_2 1i}-f_2\,e^{\chi_1 1i-\chi_2 1i}\right)\left(1+f_1\,e^{\chi_1 1i-\chi_2 1i}-f_1\right)'
$$