1 Notes on topology by David Christian Ohnmacht.

The following is a small edited part of the introduction of my PhD thesis.
Topology concerns itself with the characterization of smooth transformations on different classes of objects.

In the following, we want to provide the basis for topology in non-interacting condensed matter systems which is applicable for band insulators, superconductors as well as for multiterminal Josephson junctions. Furthermore, we want to emphasize the topological classification of these systems, while providing some insight into the underlying mathematical field of *homotopy*. This field is extremely challenging; certain proofs related to homotopy *were* among the Millennium problems[1]. The goal of this introduction is to provide a basic understanding of the so-called ten-fold way [2–4], which can be viewed as a periodic table of non-interacting condensed matter systems. The table provides a convenient way to infer if systems can exhibit non-trivial topology just by providing the dimensionality of the system as well as its behavior under fundamental symmetries. We explicitly stay away from mathematical proofs, as we focus on the intuition to provide some insight into this field.

1.0.1 Symmetries

We have already discussed many different symmetries, in particular particle-hole and time-reversal symmetry. It will turn out that these two symmetries are corner stones of topological classification.

Generally, symmetries are transformations which leave a system invariant. Emblematic of the role of *continuous* symmetries in physics, Noether's theorem dictates that symmetries result in conserved quantities [5, 6]. Whereas this circumstance is intuitive for some symmetries; translational and rotational invariance results in the conservation of momentum and angular momentum respectively, there are other types of symmetries whose corresponding conserved quantities are not as evident.

Generally, symmetries are expressed in the Hamiltonian language, in particular by specifying how a given Hamiltonian H is modified to H' upon a symmetry operation $\mathcal{U}$. Then, the symmetry is an operator S that acts on the corresponding Fock space. By Wigner's theorem [7], symmetries can only be unitary ($S = U$) or antiunitary operators ($S = UK$), where U is a unitary matrix with $U^\dagger = U^{-1}$ and K is the complex conjugation. For a symmetry, we require it to commute with the Hamiltonian $[S, H] = 0$. The transformed Hamiltonian $H' = UHU^{-1}$ has new eigenstates given by $\Psi' = U\Psi$. Furthermore, the energy of the transformed state Ψ' has the same energy as Ψ, so a symmetry enforces degeneracy[2].

In effective non-interacting descriptions, for example in mean field theory, symmetries are expressed in terms of single-particle Hamiltonians. In these cases, com-

[1]The Poincaré conjecture was solved by G. Perelman whose works were based on works by R. Hamilton [1].

[2]This provides a nice bridge back to Noether's theorem. By considering a continuous symmetry, if we find a ground state, the transformed state is still a ground state. Thus, we can move along the ground state manifold without paying energy. Such modes are refereed to as the Goldstone modes [8].

mutation between the symmetry and the Hamiltonian is not required in contrast to the full many-body formulation. In these cases, a symmetry operation $\mathcal{U}$ is expressed by an equality [9]

$$\mathcal{U}: \; SH(k)S^{-1} = \pm H(u^{-1}k), \tag{1.1}$$

with some momentum-like d-dimensional coordinate k. S is the (anti-)unitary operator $S = U(K)$ as mentioned before whereas the transformation u specifies how the coordinate is transformed upon the symmetry transformation $\mathcal{U}$. It is evident that the symmetry operator S does not commute with the Hamiltonian anymore. Importantly, in this single-particle description, a symmetry $\mathcal{U}$ is not just an unitary operator, but also has to provide an action on coordinates. If the symmetry operation acts non-trivially on the coordinate, the symmetry is called *non-local*.

For topological classification, we are interested in non-local fundamental symmetries. Let us focus on the following three: time-reversal symmetry (TRS), particle-hole symmetry (PHS) and chiral symmetry (CS) defined by [9]

$$\text{TRS}: \quad \mathcal{T}H(k)\mathcal{T}^\dagger = U_T H^{\mathrm{T}}(k) U_T^\dagger = H(-k) \tag{1.2}$$

$$\text{PHS}: \quad \mathcal{P}H(k)\mathcal{P}^\dagger = U_P H^{*}(k) U_P^\dagger = -H(-k) \tag{1.3}$$

$$\text{CS}: \quad \mathcal{P}\mathcal{T}H(k)(\mathcal{P}\mathcal{T})^\dagger = \Pi H^\dagger(k)\Pi = -H(k), \tag{1.4}$$

with the anti-unitary operators $\mathcal{T}$ and $\mathcal{P}$, the unitary operator $\Pi = \mathcal{P}\mathcal{T}$ and the unitary matrices U_P, U_T. TRS and PHS are non-local and antiunitary whereas their composition, the CS, is local and unitary. For Hermitian Hamiltonians, it holds that $H^{\mathrm{T}} = H^*$ and $H^\dagger = H$[3]. Note that $\mathcal{T}^2 = \pm 1$ and $\mathcal{P}^2 = \pm 1$ for both TRS and PHS, which means that a Hamiltonian can be time-reversal or particle-hole symmetric in three different way: not symmetric (indexed by 0) or symmetric (index by ± 1) depending on the sign of the squared operator.

To further highlight the importance of these fundamental symmetries, note that we have already utilized these symmetries to discuss relevant physical aspects. Namely, the BdG Hamiltonian of a superconductor is by construction particle-hole symmetric.

1.0.2 Role of symmetries for classification

The ten fold way allows for the topological classifications of non-interacting band insulators and superconductors [3]. From fundamental symmetries alone, we can infer if a system is allowed to have non-trivial topology. In the following, we follow the main reasoning in Ref. [3] and provide examples on why one can anticipate fundamental symmetries to have a drastic influence on the topology.

Topology boils down to the question whether a collection of states can be *continuously* transformed into another collection of states. However, requiring just this transformation condition, every set of states is continuously transformable into any other set of states. Why? To answer this question, take any two Hermitian Hamiltonians H and H' of the same basis. A continuous transformation can be seen as changing the parameters of this Hamiltonian by the continuous transformation $f : \alpha \to (1-\alpha)H + \alpha H'$ for $\alpha \in [0, 1]$. It is clear that every Hermitian Hamiltonian

[3] which is not the case for non-Hermitian Hamiltonians, which will become relevant later.

can be continuously transformed into any other Hermitian Hamiltonian if their dimensionality is the same. We have to invoke another restriction. Thus, we demand that a *spectral gap* must always be present upon the transformation [4]. With this restriction, it becomes evident that not every Hamiltonian can be transformed into any other any longer. But how do we quantify the difference between two Hamiltonians then? This role is handled by so-called *topological invariants*, which are quantities which remain invariant as long as the system is in the gapped state.

Let us take the example of a zero-dimensional (in spatial sense) system without any additional symmetries. Upon diagonalizing the respective Hamiltonian, we obtain n_+ unoccupied and n_- occupied states upon setting the value of the chemical potential. We notice that as long as the number of occupied states is unchanged, we can continuously transform states into each other because we do not close the gap. We infer that the number of occupied states n_- is a topological invariant [10, 11]. When there is a gapless state, and thus a zero crossing, the number of occupied states may change, and we may have a topological phase transition when n_- changes. The topological number n_- can be any integer, which is why we use the symbol $\mathbb{Z}$ as a classification symbol of the topological class of this system [10, 11].

As another example, we consider a spin-1/2 system which is time reversal symmetric. Due to Kramer's theorem [12], each state in the system is doubly degenerate. Thus, taking the topological invariant from the previous example, namely the number of occupied states n_-, it can only change in even steps, meaning that it is an even integer, and thus characterized by $2\mathbb{Z}$ [10, 11].

However, the topological index is not always simply the number of occupied states. Namely, the topological index can vary in its physical meaning drastically. As an example, consider a Hamiltonian with particle-hole symmetry (in zero dimension). It is evident that the number of occupied states n_- is not a good measure for changes of topological properties in this system, because the number of occupied states always remains the same (there are always the same number of states above and below the chemical potential). One could expect the system to not exhibit non-trivial topology, but this is not the case. It turns out that *parity* is the topological invariant, however, computing it is not as straightforward [11].

We have seen examples where these symmetries determine how the topological invariants manifest themselves. In other cases, symmetries might forbid the system from even having non-trivial topology. For these topologically trivial systems, every Hamiltonian can be continuously transformed into another. However, this does not necessarily mean that we cannot have zero crossings. There are zero crossings which are not accompanied by a topological phase transition.

We have only discussed the three fundamental symmetries TRS, PHS and CS. What about other symmetries? Firstly, we are not interested in local symmetries which commute with the single-particle Hamiltonian. Namely, if we find such a symmetry, we can block-diagonalize the Hamiltonian in the shared eigenbasis and analyze each block separately. Then, the topological index of the whole system is just the sum of topological indices in each block. But what about other non-local symmetries like reflections, rotations, etc.? These have drastic influences on the topology and there are numerous studies on the topological classification of real systems with respect to their point groups [13, 14]. However, right now we are interested in the most fundamental aspects of condensed matter systems and point group symmetries are easily broken by local impurities and other perturbations [15].

1.0.3 Ten-fold way

The original idea for the classification of noninteracting systems was proposed by A. Altland and M. R. Zirnbauer [2]. By considering the effect of certain symmetries, they investigated the influences on the Hamiltonians and their topological properties. Later, A. Kitaev explained the classes by K-theory [4]. The tenfold way states that every noninteracting system can be classified using its symmetry properties under TRS, PHS and CS, in addition to specifying the effective dimension of the coordinate of the system (mostly momentum). The classification table is shown in Tab. 1.

Class	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	$\delta=0$	$\delta=1$	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	+1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI/G_0	0	0	+1	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI/G_1	+1	+1	+1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D/G_2	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII/G_3	+1	+1	-1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII/G_4	0	0	-1	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII/G_5	+1	-1	-1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C/G_6	0	-1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI/G_7	+1	-1	+1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Table 1: Tenfold way topological classification of non-interacting fermionic systems and superconductors [9]. Each class is defined by its properties under PHS ($\mathcal{P}$), TRS ($\mathcal{T}$) and CS ($\mathcal{C}$), denoted by ± 1 and 0. The topological invariants 0, $\mathbb{Z}$, $2\mathbb{Z}$ or $\mathbb{Z}_2$ are indicated for each effective dimension from $\delta = 0$ to $\delta = 7$.

First, let us analyze the table without providing that much context into the separate classes. There are two so-called complex classes (A and AIII) and eight real classes which are represented by the conventional notation as well as by an integer $s = 0, 1, 2, ..., 7$ with the symbol G_s. Why are there ten classes? For each class, its properties under TRS, PHS and CS are stated. Namely, as established in Sec. 1.0.1, a system can behave under TRS or PHS in three different ways (index by ± 1) or not (index by 0) as seen in Tab. 1. As these two symmetries have three different ways to act on a system, we have nine symmetry classes to classify how TRS and PHS act on a Hamiltonian. We note that the system always obeys CS when it obeys TRS and PHS. Furthermore, when the system obeys only one of TRS or PHS, it does not obey CS. However, there is an additional case where the system does not obey TRS and PHS, but it does obey CS (class AIII), resulting in ten classes. Whether the system may host non-trivial topology is indicated by the respective entry for each dimension $d = 0, ..., 7$ of the ten classes. We observe four different indices, namely $\mathbb{Z}$, $2\mathbb{Z}$, $\mathbb{Z}_2$ and 0. $\mathbb{Z}$ describes an integer valued topological invariant as we have seen in the previous example (number of occupied states). $2\mathbb{Z}$ refers to even numbered invariants, for example due to Kramer's degeneracy as shown before. $\mathbb{Z}_2$ refers to a system being non-topological or topological, so there are only two different phases. A system with entry 0 is *always* topologically trivial.

However, getting an intuition towards this table is difficult. A lot of results are based on mathematical arguments, and the physics is often not transparent from the entries in the table alone. Just the index $\mathbb{Z}$ does not communicate what its physical meaning actually is. For example, $\mathbb{Z}$ can also stand for a *winding number* or *Chern number*, which is not evident from the table. Furthermore, there are certain periodicities in the table. Notice that when we increase the dimensionality of the system and the index s by one, we find the same topological index, so $G_{s+1}(\delta+1)$ and $G_s(\delta)$ are similar, which we mathematically write as $G_{s+1}(\delta+1) \cong G_s(\delta)$, meaning the objects are isomorph[4]. Furthermore, note that the dimensionality extends up to $\delta = 7$, as $\delta = 8$ is actually equivalent to $d = 0$, which is referred to as Bott periodicity [17][5]. In the following we want to go further into detail to provide meaning to this table, which requires the notion of *homotopy*.

1.0.4 Homotopy

Homotopy concerns itself with the question if (continuous) functions can be continuously transformed into one another. In mathematics, these functions can map from any base space to another target space. The most important homotopy is the one of functions from an n-sphere S^n to another m-sphere S^m, which is called homotopy group $\pi_n(S^m)$ [18, 19]. But what does this mean? Let us consider the easiest example, namely $\pi_n(S^0)$ which is the Homotopy group of functions from an n-sphere to the zero sphere, which is set of two disjointed points $\{\pm 1\}$. Every function $f : S^n \to \{\pm 1\}$ must map to one of the two points to be continuous, meaning that the image of every function is just a point. Homotopy concerns itself with functions that can be continuously transformed into one another and we conclude that every possible function's image is just a point meaning that every possible map is equivalent. Homotopy counts the number of so-called *equivalence classes*, sets of homotopically equivalent functions. Thus, there exists only one equivalence class of possible homotopic mappings in this case, making the corresponding homotopy group trivial, $\pi_n(S^0) = 0$ [19, 20]. Now, let us focus on the homotopy groups $\pi_1(S^n)$ for $n > 0$, which are called fundamental groups. The question is whether functions $f : S^1 \to S^n$ can be continuously transformed into each other. We can rephrase this aspect in a more intuitive way by considering the previous example. In that case, every function's image is just a point. Therefore, we ask the question whether a one-dimensional loop on the manifold S^n can be contracted to a single point in the respective target space. Let us illustrate this by the most intuitive example, namely $\pi_1(S^2)$. In Fig. 1.1 we show an example of a loop on a sphere which is continuously contracted to a single point. In fact, every loop on a sphere's surface can be contracted in such a way. The homotopy group is therefore trivial with $\pi_1(S^2) = 0$ [19, 20].

[4] We will show later that a K-group homomorphism can be established by dimensionality reduction and extension. One can also show that these have inverses. For details see Ref. [16].

[5] By considering the aforementioned similarity between $G_{s+1}(\delta+1) \cong G_s(\delta)$ of the eight real classes, we infer that the modulo 8 periodicity of the dimension is inherited by the total number of real classes.

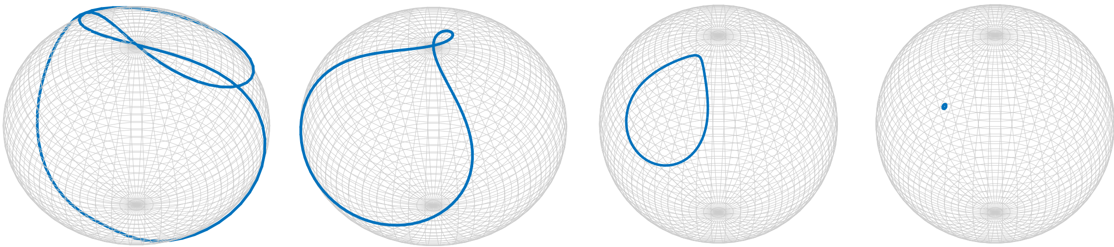

Figure 1.1: Demonstration of homotopy group $\pi_1(S^2)$, the fundamental group of the sphere. Illustration of a loop on a 2-sphere that can be continuously contracted to a single point (from left to right). This is possible for any continous mapping $f : S^1 \to S^2$.

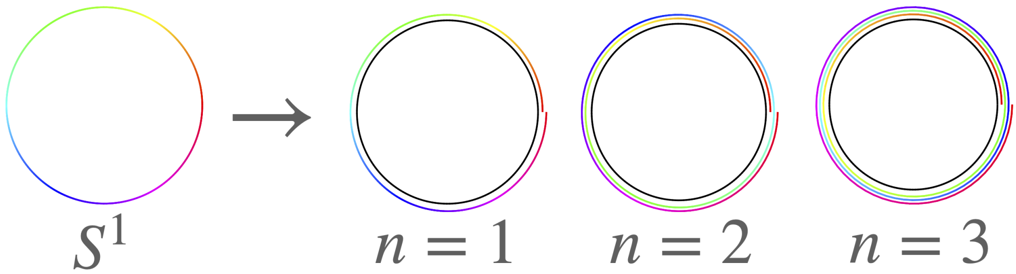

Figure 1.2: Illustration of fundamental group of S^1. The target space is color-coded to illustrate the different mappings on the right hand side. For $n = 1, 2, 3$ the base space is wrapped around the target space S^1 a total of n times.

In contrast, consider the homotopy group $\pi_1(S^1)$. We can parametrize the 1-sphere by $S^1 = \{(\cos(\phi), \sin(\phi))|\phi \in [0, 2\pi)\}$ and consider the mapping of the circle onto another circle by $f : S^1 \to S^1$, $f(\phi) = \phi$, meaning that the same point on the base space circle is mapped onto the same point on the target space circle as shown in Fig. 1.2 for $n = 1$. It is clear that the circle in the target space cannot be continuously transformed into a single point as we would have to cut the loop. Thus, we have constructed an example of a function which is not homotopic to a single point mapping ($f(\phi) = 0$) and is consequently part of a different equivalence class. Alternatively, we can wrap around the circle n number of times by $f(\phi) = n\phi$ which can neither be transformed into a point nor into a circle of $m \neq n$ number of loops, as seen in Fig. 1.2. Thus, we have identified our topological invariant, namely the integer winding number and obtain the homotopy group $\pi_1(S^1) = \mathbb{Z}$ [19, 20]. It becomes increasingly more complicated to envision higher homotopy groups like $\pi_1(S^3)$ as these higher dimensional spheres cannot be easily visualized. As a general rule of thumb, the higher the dimensionality of the n-sphere, the easier it is to contract a loop to a single point, namely $\pi_1(S^n) = 0$ for $n > 1$ [19, 20]. The intuition is the following: for a higher dimensional n-sphere, we have an additional dimension which can be use to unwind the loops continuously.

Developing intuition beyond these previously mentioned homotopy groups becomes increasingly more difficult. In particular, let us consider $\pi_2(S^n)$ for $n > 0$. It holds that $\pi_2(S^1) = 0$ [19, 20], meaning that a sphere mapping onto the unit circle is contractable to a point. However, there is no easy visual analog for this. Of major importance is the fact that $\pi_2(S^2) = \mathbb{Z}$ [19, 20]. This is the same homotopy

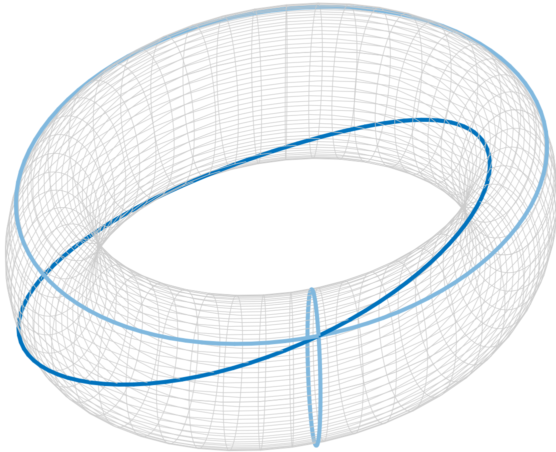

Figure 1.3: Example of loops on a two-torus, which cannot be contracted to a single point continuously. The dark blue loop corresponds to a combined winding around both edges of the torus. The lighter loops wind around one edge of the torus respectively. All loops belong to different equivalence classes.

group responsible for the existence of topologically protected *Skyrmions* [21] in 2D magnetic materials [22]. Furthermore, $\pi_2(S^n) = 0$ for $n > 2$ [19, 20], where for a large enough dimension of the sphere, the homotopy group becomes trivial.

All introductions of scientific talks concerning non-trivial topology start with the donut shaped 2-torus $\mathbb{T}^2$. It is said that it has a different topology to the 2-sphere S^2 because "the torus has a hole". But what does this concretely mean? Let us consider the fundamental group of the torus $\pi_1(\mathbb{T}^2)$. For this, notice that the torus can be written as a direct product of two 1-spheres, namely $\mathbb{T} = S^1 \times S^1$. Furthermore, it holds that [23]

$$\pi_1(\mathbb{T}^2) = \pi_1(S^1 \times S^1) \cong \pi_1(S_1) \times \pi_1(S_1) = \mathbb{Z}^2. \tag{1.5}$$

In particular, circles in the target space can wind around the two separate edges of the torus, which cannot be contracted to a point respectively, as portrayed in Fig. 1.3. Thus, the topological index $\mathbb{Z}^2$ counts the numbers of windings around each orientation of the torus. We observe that this hole in the torus is key for its fundamental group to be non-trivial. Also note that we cannot continuously transform a 2-sphere surface into a torus surface because of the hole in it. We conclude that target spaces with different fundamental groups cannot be continuously transformed into one another.

But how does this apply to physics? In a physical context, we are interested in how states evolve as functions of a momentum-like variable k which is a d dimensional quantity. In the case of band insulators that are subject to Bloch's theorem, k lives in a d dimensional torus $\mathbb{T}^d$. The question is what the target space G is. This is a very difficult question, because we impose the existence of a spectral gap. In principle, we would expect the meaningful physical quantity to be the homotopy of functions from the d-torus to this target space G. However, by reducing the base space from a torus to a sphere, we allow for the breaking of crystalline symmetries, which makes the classification robust against disorder [10]. With this, we can make use of homotopy groups $\pi_d(G)$ of target space G to classify real physical systems.

However, in the topological classification, we are interested not only in the topology of certain Hamiltonians in isolation, but in their topology with respect to additional underlying structure. This is usually captured by effectively considering an infinite number of states, which then leads to the classifying spaces [4].

1.0.5 Examples of homotopy groups

Let us start with a $N \times N$ Hamiltonian $H(k)$ without any additional symmetry with a d dimensional momentum k, which corresponds to class A of Tab. 1. We want to classify this Hamiltonan while preserving the constraint that no gap should be opened. To accommodate this, we have to diagonalize our Hamiltonian with a unitary transformation $U(k)$

$$U^\dagger(k)H(k)U(k) = \mathrm{diag}(\epsilon_1^-, ..., \epsilon_{n_-}^-, \epsilon_{n_-+1}^+, ..., \epsilon_N^+), \tag{1.6}$$

where energies ϵ_i^- are occupied and energies ϵ_j^+ are unoccupied with the number of occupied states n_- and the number of unoccupied states $n_+ = N - n_-$. While demanding that n_- is not changing, we can adiabatically transform the eigenvalues by a process called *spectral flattening* such that $\epsilon_i^- = -1$ and $\epsilon_j^+ = +1$ [9]. With this, we can define the so-called flatband Hamiltonian [9]

$$F(k) = U(k)\mathrm{diag}(\mathbb{1}_{n_+}, -\mathbb{1}_{n_-})U^\dagger(k), \tag{1.7}$$

with the same unitary matrix that was used to diagonalize H. Note that we therefore did not change the eigenstates of the system. We see that characterizing the flatband Hamiltonian, which is similar (in a homotopy sense) to the total Hamiltonian, requires characterizing a unitary matrix $U(k)$. Naively, one would expect the target space to then be $U(N)$. However, the ground state of the flatband Hamiltonian has a degeneracy, which results in an additional $U(n_-) \times U(n_+)$ gauge symmetry. The resulting target space is the *symmetric space* [24]

$$C_0 = \frac{U(n_- + n_+)}{U(n_-) \times U(n_+)}, \tag{1.8}$$

and it holds that $\pi_2(C_0) = \mathbb{Z}$, with the corresponding topological invariant, the so-called *Chern number* [9]. However, it is not at all evident why $\pi_2(C_0) = \mathbb{Z}$. Let us focus on the simplest example, a 2×2 Hamiltonian with no additional symmetries. Then, the target space becomes $C_0 = U(2)/(U(1) \times U(1)) = \mathbb{CP}^1$, where $\mathbb{CP}^1$ is the complex projection space [9][6]. To give meaning to this projection space, we note that it is equivalent to the the set of all 1-dimensional linear subspaces of $\mathbb{C}^2$. From the Hopf fibration, it holds that $\mathbb{CP}^1 \cong S^2$ [26]. Hence, the relevant topology is obtained by the homotopy group $\pi_d(S^2)$, namely $\pi_2(S^2) = \mathbb{Z}$, so the topology boils down to mappings from 2-spheres to 2-spheres[7].

[6] Also referred to as Grassmannian $Gr(1, \mathbb{C}^2)$ [25].

[7] Note that this example does not relate directly to the classification given by the ten-fold way. Namely $\pi_1(C_0) = 0$ and $\pi_3(C_0) = 0$ from class A in the tenfold way whereas $\pi_1(S^2) = 0$ and $\pi_3(S^2) = \mathbb{Z}$. For the ten-fold way classification, the limit of large matrices is considered, namely

However, what is the actual physics behind this? Note that for the example of a 2×2 Hamiltonian, we obtain a single ground state $|\Psi\rangle \in \mathbb{C}^2$ of the occupied state. This state is defined up to a normalization factor, meaning that the eigenstate spans the 1-dimensional linear subspace $\mathrm{span}(|\Psi\rangle) \in \mathbb{CP}^1$, which is an element of the exact target space described before. We conclude that the relevant target space is the space of *all* available possible ground states.

Let us discuss a different example, namely a $2N \times 2N$ Hamiltonian $H(k)$ with chiral symmetry in class AIII in Tab. 1. In the eigenbasis of the chiral symmetry operator, the flatband Hamiltonian and the chiral symmetry read [9]

$$F(k) = \begin{pmatrix} 0 & U(k) \\ U^\dagger(k) & 0 \end{pmatrix} \quad , \quad U_C = \mathrm{diag}(1_N, -1_N), \tag{1.9}$$

where $U(k)$ is a unitary. We conclude that the flatband Hamiltonian is characterized solely by a unitary matrix $U(k)$ without any residual gauge symmetry in contrast to before. Thus, in this case, we are interested in the homotopy group $\pi_d(U(N))$ for this symmetry class. This is a much more manageable target space. To get a grasp on the homotopy, we note that $U(N)$ is isomorph to $S^1 \times SU(N)$ [28]. The intuition behind this similarity is given by the fact that one can transform any unitary matrix into a special unitary matrix by the transformation $U \to U/\det(U)$, which roughly translates to the idea that any matrix in $SU(N)$ with a phase factor in $U(1)$ (this is the determinant) gives a unitary matrix $U(N)$. Then, we obtain

$$\pi_d(U(N)) = \pi_d(S^1) \times \pi_d(SU(N)). \tag{1.10}$$

We established that $\pi_1(S^1) = \mathbb{Z}$ while $\pi_j(S^1) = 0$ for $j > 1$. Furthermore, $\pi_{1/2}(SU(N)) = 0$ [29, 30]. Let us consider a specific example, namely $U(2)$. Then, the homotopy group is given by $\pi_d(S^1) \times \pi_d(SU(2))$. By using that $SU(2) \cong S^3$ [30], we find [29, 30]

$$\pi_1(U(2)) = \underbrace{\pi_1(S^1)}_{=\mathbb{Z}} \times \underbrace{\pi_1(SU(2))}_{=0} = \mathbb{Z}, \tag{1.11}$$

$$\pi_2(U(2)) = \underbrace{\pi_2(S^1)}_{=0} \times \underbrace{\pi_2(SU(2))}_{=0} = 0, \tag{1.12}$$

$$\pi_3(U(2)) = \underbrace{\pi_3(S^1)}_{=0} \times \underbrace{\pi_3(SU(2))}_{=\mathbb{Z}} = \mathbb{Z}, \tag{1.13}$$

in accordance with the entries in the tenfold way table in Tab. 1 for class AIII.

1.0.6 Berry curvature and Chern number

In 1980, K. v. Klitzing and coworkers discovered that the Hall voltage in two-dimensional electron gases has fixed values at particular well-defined integer values insensitive to the geometry of the device [31]; the experimental discovery of the integer quantum Hall effect. In particular the Hall conductance exhibits plateaus

$N \to \infty$ to grant stability against stronger forms of perturbation, which includes adding states [4]. Only in this limit, we talk about *strong* topological invariants. Thus, we find that in a system with less states, namely only two, we can have non-trivial topology of systems in three dimensions, but expect the invariant to be a so-called *weak* topological invariant [27].

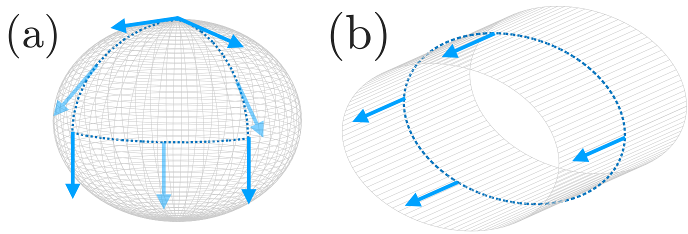

Figure 1.4: Parallel transport of a tangent vector on the surface of a sphere and cylinder. (a) On a sphere, the initial tangent vector on the north pole changes its orientation when undergoing a closed looped motion. (b) On a cylinder, the orientation can never change along closed loops, because the target space has trivial curvature.

$\sigma_{xy} = r_{\mathrm{H}}^{-1} = ne^2/h$ for integer values ($n = 1, 2, ...$). A few years later, a similar effect was found at fractional values [32], coined fractional quantum Hall effect [33]. The results where explained in two different ways. First, R. B. Laughlin provided a gauge argument where the plateaus have to manifest themselves by adiabatic transfer along states [34]. D. J. Thouless and coworkers made a direct connection from the Hall conductivity to a topological invariant [35], namely the Chern number [36]. It was a hallmark discovery in that it illustrated a connection between topological properties and quantized observables in condensed matter systems. The quantum Hall effect therefore accounts for three Nobel Prizes of Physics, 1985 for the initial experimental discovery, 1998 for the fractional quantum Hall effect and 2016 for the relationship to topology.

As was often alluded to in these discussions about topology, the role of geometry is crucial. Recall that a torus and a sphere have different fundamental groups because of their different *geometrical* properties. How can this be quantified? The bridge between geometry and topology is given by the Gauss-Bonnet theorem [37]: Given a two-dimensional manifold $\mathcal{M}$ *without boundary*, the integral over the Gaussian curvature $\mathcal{F}$ over this manifold is given by

$$\chi(\mathcal{M}) = 2(1-g) = \frac{1}{2\pi}\int_{\mathcal{M}} dA\ \mathcal{F}, \tag{1.14}$$

with the area integral dA and the *genus* g and Euler-characteristic $\chi(\mathcal{M})$ of the manifold. The genus is related to the hole of the torus. Namely, $g = 1$ for a torus and $g = 0$ for a sphere. How can we convince ourselves of this formula? Namely, for a sphere of radius R, the Gaussian curvature is given by $\mathcal{F} = 1/R^2$ while the area integral gives $4\pi R^2$. Then on the right hand side we obtain 2 resulting in $g = 0$ and $\chi(\mathcal{M}) = 2$.

The Euler characteristic describes the *curvature* of the underlying space. Namely, consider a vector in the tangent space of the sphere's surface being transported along the surface in a closed loop like portrayed in Fig. 1.4(a). Parallel transport means that we never change the orientation of the initial vector abruptly while it is being transported [38]. It is seen that upon a looped motion, the orientation of the vector changes in comparison to the initial configuration. In the view of this example, it becomes evident that the surface of a torus, which is equivalent to a cylinder with periodic boundary conditions, does not have curvature because a loop motion never tilts a vector away from its starting position, like illustrated for one loop in Fig. 1.4(b). In the case of condensed matter systems, the manifold is given by the Brillouin zone, spanned by a d dimensional momentum-like variable k. Given a Hamiltonian H, which can be represented as a $N \times N$ matrix, upon diagonalizing this Hamiltonian with the unitary matrix U, the tangent space comprises these eigenstates for each respective k. By specifying the chemical potential, we have n occupied states. The single-particle Hamiltonian Hilbertspace is the total N-dimensional complex projective space [9]. The occupied states span an n dimensional subspace of this projective space, which functions as a vector bundle. Whereas Gauss-Bonnet's theorem was expressed for 2-dimensional manifolds with tangent spaces isomorphic to $\mathbb{R}^2$, the target space in the classes of the tenfold way are more complicated.

Namely, given a $N \times N$ Hamiltonian H, the Gaussian curvature is generalized to a curvature form, the so-called Berry curvature F [9]

$$F = F_{ij}^{ab}(k) dk_i \wedge dk_j = \left(\partial_i A_j^{ab}(k) - \partial_j A_i^{ab}(k) + [A_i(k), A_j(k)]^{ab}\right) dk_i \wedge dk_j \tag{1.15}$$

with its respective Berry connection $A = A_i^{ab}(k) dk_i$ specified for all occupied states $a, b = 1, ..., n$ with

$$A_i^{ab}(k) = \sum_{j=1}^{N} U_{aj}^\dagger(k) \partial_i U_{jb}(k), \tag{1.16}$$

with the components of the unitary which was used to diagonalize the Hamiltonian U_{ij}. In two dimensions, Chern generalized the Gauss-Bonnet theorem to [9, 36]

$$C^{(1)} = \imath \frac{1}{4\pi} \int_{\mathcal{M}} \mathrm{tr} F = \frac{\imath}{2\pi} \int_{\mathcal{M}} d^2k \ \mathrm{tr}\ F_{12}(k), \tag{1.17}$$

with the (1st) Chern number $C^{(1)}$. Notably, this Chern number only takes integer values. The above formula might not be recognizable at first glance. If there is no degeneracy in the ground state, we can express the unitary as $U = (v_1, ..., v_N)$ where v_j are the eigenvectors of the Hamiltonian. Then, it holds that

$$\sum_{a=1}^{n} A_i^{aa}(k) = \sum_{a=1}^{n} \sum_{i=1}^{N} U_{ai}(k)^\dagger \partial_\alpha U_{ia}(k) = \sum_{a=1}^{n} \sum_{i=1}^{N} v_a^{(i)*} \partial_\alpha v_a^{(i)} = \sum_{a=1}^{n} v_a^\dagger \partial_\alpha v_a, \tag{1.18}$$

and therefore

$$\begin{aligned} \mathrm{tr} F_{12} = \sum_{a=1}^{n} \partial_1 [v_a^\dagger (\partial_2 v_a)] - \partial_2 [v_a^\dagger (\partial_1 v_a)] &= \sum_{a=1}^{n} (\partial_1 v_a^\dagger)(\partial_2 v_a) - (\partial_2 v_a^\dagger)(\partial_1 v_a) \\ &= \sum_{a=1}^{n} -2\mathrm{Im}\{(\partial_1 v_a^\dagger)(\partial_2 v_a)\}, \end{aligned} \tag{1.19}$$

reminiscent of the more standard way to portray the Berry curvature. We conclude that in non-degenerate settings, we need to compute the occupied eigenvectors of the Hamiltonian as functions of the momentum in order to compute the Chern number and thus determine if the system exhibits non-trivial topology.

References

[1] H.-D. Cao and X.-P. Zhu, Asian Journal of Mathematics **10**, 165 (2006).

[2] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).

[3] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).

[4] A. Kitaev, AIP Conf. Proc. **1134**, 22 (2009).

[5] E. Noether, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse **1918**, 235 (1918).

[6] E. Noether, Transport Theory and Statistical Physics **1**, 186 (1971).

[7] E. Wigner, *Group Theory: And Its Application to the Quantum Mechanics of Atomic Spectra* (Elsevier, 2012).

[8] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[9] S. Huber and T. Neupert, Lecture Notes: Topological Condensed Matter Physics, `https://cmt-qo.phys.ethz.ch/education/spring-semester-2021.html` (2021), accessed: 6 November 2025.

[10] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New Journal of Physics **12**, 065010 (2010).

[11] A. Akhmerov, J. Sau, B. van Heck, S. Rubbert, R. Skolasiński, B. Nijholt, I. Muhammad, and T. Örn Rosdahl, Online course on topology in condensed matter, `https://topocondmat.org/index.html`, accessed: 6 November 2025.

[12] B. A. Bernevig, *Topological Insulators and Topological Superconductors* (Princeton University Press, 2013).

[13] L. Fu, Phys. Rev. Lett. **106**, 106802 (2011).

[14] K. Shiozaki and M. Sato, Phys. Rev. B **90**, 165114 (2014).

[15] C. Chamon, M. O. Goerbig, R. Moessner, and L. F. Cugliandolo, *Topological Aspects of Condensed Matter Physics: École de Physique Des Houches, Session CIII, August 2014* (Oxford University Press, 2017).

[16] J. C. Y. Teo and C. L. Kane, Phys. Rev. B **82**, 115120 (2010).

[17] R. Bott, Annals of Mathematics **70**, 313 (1959), 1970106 .

[18] M. M. Cohen, *A Course in Simple-Homotopy Theory* (Springer Science & Business Media, 2012).

[19] T. U. F. Program, Homotopy Type Theory: Univalent Foundations of Mathematics (2013), arXiv:1308.0729 [math] .

[20] H. Toda, *Composition Methods in Homotopy Groups of Spheres* (Princeton University Press, 1962).

[21] T. H. R. Skyrme, Nuclear Physics **31**, 556 (1962).

[22] A. Belavin and A. Polyakov, Metastable states of two-dimensional isotropic ferromagnets, Sov. Phys. JETP Lett. **22**, 245 (1975), translated from *Pis'ma Zh. Eksp. Teor. Fiz.* **22**, No. 10, 503-506 (1975).

[23] A. Hatcher, *Algebraic Topology* (Cambridge University Press, 2002).

[24] E. Cartan, Bulletin de la Société Mathématique de France **54**, 214 (1926).

[25] Wikipedia contributors, Grassmannian — Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/wiki/Grassmannian` (2025), accessed: 6 November 2025.

[26] H. Hopf, in *Selecta Heinz Hopf: Herausgegeben zu seinem 70. Geburtstag von der Eidgenössischen Technischen Hochschule Zürich*, edited by H. Hopf (Springer, Berlin, Heidelberg, 1964) pp. 38–63.

[27] R. Kennedy and C. Guggenheim, Phys. Rev. B **91**, 245148 (2015).

[28] N. Lambert and C. Papageorgakis, J. High Energ. Phys. **2010** (4), 104, arXiv:1001.4779 [hep-th] .

[29] M. Mimura and H. Toda, *Topology of Lie Groups, I and II* (American Mathematical Soc., 1991).

[30] C. Mathematical Society of Japan and K. Itô, *Encyclopedic Dictionary of Mathematics (2nd Ed.)* (MIT Press, Cambridge, MA, USA, 1993).

[31] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[32] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).

[33] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[34] R. B. Laughlin, Phys. Rev. B **23**, 5632 (1981).

[35] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).

[36] S.-s. Chern, Annals of Mathematics **47**, 85 (1946), 1969037 .

[37] L. W. Tu, *Differential Geometry: Connections, Curvature, and Characteristic Classes* (Springer, 2017).

[38] J. W. Zwanziger, M. Koenig, and A. Pines, Annual Review of Physical Chemistry **41**, 601 (1990).