We have talked about the difference between components of vectors and mathematical vectors. We saw that the object concerning us in Physics are the components.

In Euklidean space, this is easily handled. All n basis vectors $e_1, \dots, e_n$ ~~are~~ are fixed and one can solely consider the components.

However, in General Relativity, we do not consider Euklidean space, rather Manifolds. These are object that are only locally similar to Euklidean space. In curved manifolds, local basis vectors can generally depend on where we are, hence basis vectors may depend on coordinates. ~~which~~ This makes it non-trivial to consider the component of vectors adequately.

Let ~~us~~ us firstly start with the mathematics and then come to the important concept of the covariant Derivative. We will use the example of polar coordinates to illustrate all mathematical definitions.

[Note: This is not meant to be an introduction into Differential Geometry. The Definitions will be vague and are meant to illustrate the concepts in an easy way.]

BRUNNEN

Def. (Manifold)

An n-manifold $\mathcal{M}$ is a set of points $p \in \mathcal{M}$ which is <u>locally</u> similar to $\mathbb{R}^n$. This means that for every $p \in \mathcal{M}$ we find a set $\mathcal{U}_p$ with $p \in \mathcal{U}_p \subseteq \mathcal{M}$ such that $\mathcal{U}_p$ "is similar" to $\mathbb{R}^n$. This means that we can find (multiple) so-called <u>chart</u> maps $\underline{\varphi} : \mathcal{U}_p \to \mathbb{R}^n$ which must be homomorphism (continuous, continuous, φ^{-1} exists and continuous)

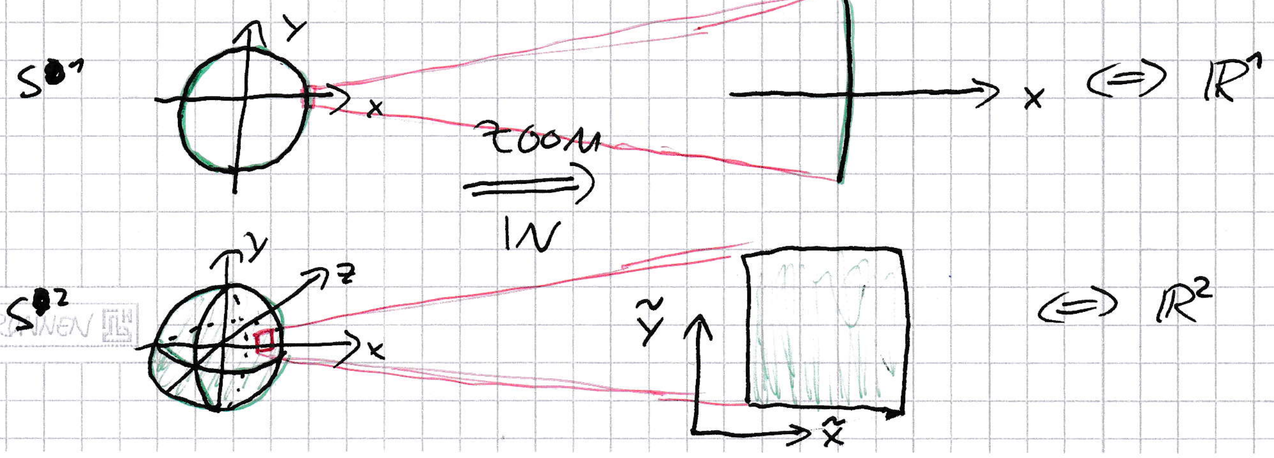
Note, that one mostly considers manifolds that have a coordinate representation and are embedded in $\mathbb{R}^m$, meaning that one can write the manifold down. An example is the S^1 sphere:

$$S^1 = \{ x^2 + y^2 = 1 \mid (x, y) \in \mathbb{R}^2 \} \subseteq \mathbb{R}^2$$

which is the unit-circle or the S^2 sphere:

$$S^2 = \{ x^2 + y^2 + z^2 = 1 \mid (x, y, z) \in \mathbb{R}^3 \} \subseteq \mathbb{R}^3$$

which is the unit-ball. Notice that S^1 is locally similar to $\mathbb{R}^1$ as it is a simple line, whereas S^2 is locally similar to $\mathbb{R}^2$, like our earth's surface is:



Optional: Topology on S^1 and the problem of ~~charts~~ charts

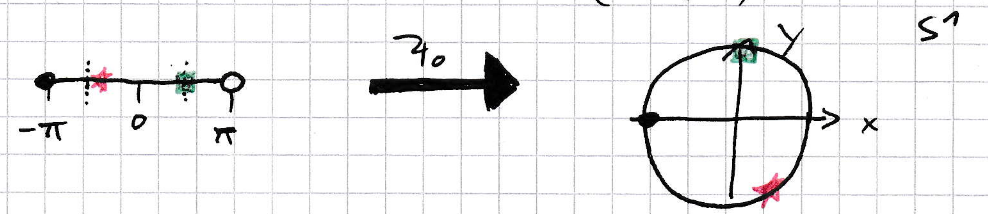
[This is not relevant for the current topic but illustrates nicely that ~~charts~~ charts are highly non-trivial].

Notice that we can parametrize S^1 by a simple number $\varphi \in (-\pi, \pi)$ by

$$S^1 = \{x^2+y^2=1 \mid x,y \in \mathbb{R}\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \middle| \varphi \in [-\pi, \pi) \right\}$$

which suggests that S^1 is similar to the interval $(-\pi, \pi)$. However, this is not the case. For this consider that $[-\pi, \pi) \subseteq \mathbb{R}$ is a manifold. Consider the candidate for a ~~chart~~ chart from $[-\pi, \pi)$ to S^1 by:

$$\psi_0 : (-\pi, \pi) \to S^1$$
$$\psi_0(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$



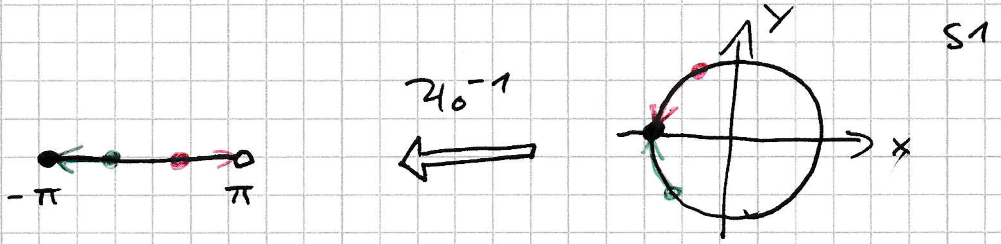
Notice that ψ_0 is continuous. In addition, ψ_0 is bijective and thus ~~have~~ has an inverse function ψ_0^{-1} (every chart must have an inverse)

$$\psi_0^{-1} : S^1 \to (-\pi, \pi)$$
$$\psi_0^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \arctan\left(\frac{y}{x}\right),$$

Note: $\arctan\left(\frac{y}{x}\right)$ is the phase.

which is not continuous. To see this, consider the point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ which can be approached in S^1 from the top and bottom of the circle.

However, approaching it from the top results in a phase of π whereas the point $\binom{-1}{0}$ is obviously mapped to $-\pi$.



Meaning that $(-\pi, \pi)$ is not similar to S^1. The reason for this is the different topoloy of the two systems. To really quantify what "similar" means, we need the notion of Homeomorphisms.

Note: Topology in this case refers to the fact that $p_1 = -\pi$ and $p_2 \Rightarrow \pi$ are not connected on $(-\pi, \pi)$

These are functions which are continous, bijective but their inverse is also continous. The latter is exactly what we were missing before.

By just removing the point $\binom{x}{y} = \binom{-1}{0}$ out of S^1, we can contruct a homeomorphism:

$$\psi_0' : (-\pi, \pi) \to S^1 / \{\tbinom{-1}{0}\},$$

thus ψ_0' is a chart.

However, as we want to understand S^1 as a whole, this feels unsatisfyong.

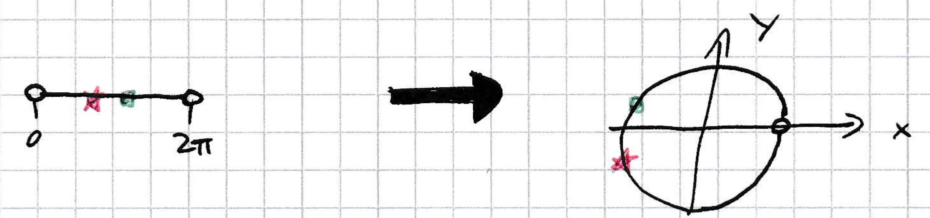
The problem is that we cannot parametrize S^1 globally with $\mathbb{R}$. ~~However To solve this~~

But this is not a problem. In the definition of a manifold, we never required that a manifold should be globally similar / homeomorph to $\mathbb{R}^n$, only locally.

We are only missing the point $\binom{x}{y} = \binom{-1}{0}$ out of S^1. However, as the circle is symmetric, we could have cut out any other point too. Hence, let us consider another homeomorphism:

$$\psi'_\pi : (0, 2\pi) \to S^1 \setminus \binom{1}{0}$$

$$\psi'_\pi(\varphi) = \begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix}$$



Notice that ψ'_0 and ψ'_π cover S^1 fully. Such a collection of charts is called <u>Atlas</u>, and it an important object always belonging to a manifold.

<u>Def.</u> (Altas)

A set $\mathcal{A}$ of chart such that the domains $\mathcal{U}$ of the charts $\psi : \mathcal{U}_p \to \mathbb{R}^n$ with $(\psi, \mathcal{U}_p) \in \mathcal{A}$ cover $\mathcal{M}$, is called <u>Atlas</u>.

$$\bigcup \{\mathcal{U} : (\mathcal{U}, x) \in \mathcal{A}\} = \mathcal{M}.$$