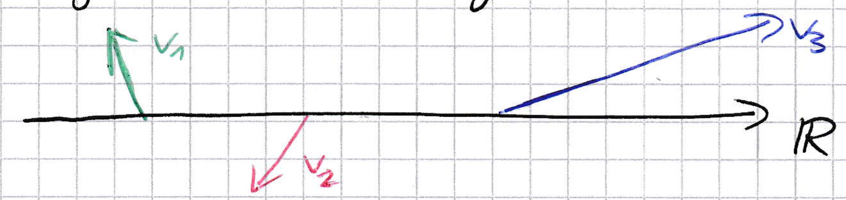
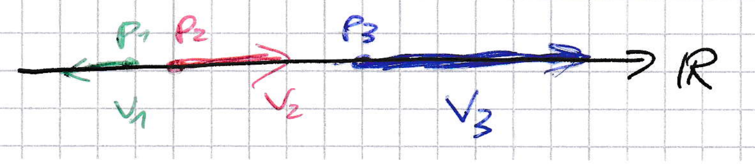
We saw that one needs two charts to fully cover S^1 by $\mathbb{R}$. Now, we want to understand more what vectors are actually allowed to exist in which ~~space~~ type of manifold.

This problem may sound unintuitive at first glance but it is really important. To emphasize the problem, consider the manifold $\mathbb{R}$, and ask us the question: "What vectors can be build in $\mathbb{R}$?" On first thought, one might say every vector in $\mathbb{R}^2$ as one could imagine the following vectors:



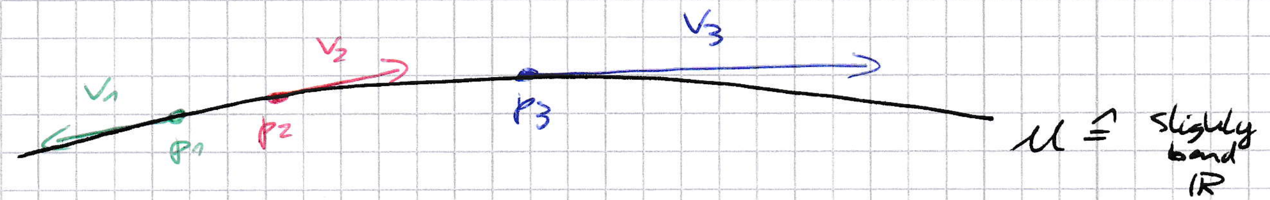
But, these vectors <u>point outside of $\mathbb{R}$</u> and can thus not be described in $\mathbb{R}$ alone. Of course, if we were to embed $\mathbb{R}$ in $\mathbb{R}^2$, it would make sense to allow for them, but differential geometry tries to describe n-manifolds exactly <u>not</u> be embedding them into a higher dimensional spaces but to map them to $\mathbb{R}^n$. With this premise, it becomes rather obvious, that the allowed vectors mustn't point outside of $\mathbb{R}$, rather allow for ~~the~~ following type of vectors, namely all $v \in \mathbb{R}$.



<u>Note</u>: In this case, it doesn't matter on which point we are as $\mathbb{R}$ is the same on each point.

This generalizes straightforward to $\mathbb{R}^n$, for which every vector $v \in \mathbb{R}^n$ is allowed.

However, what are the allowed vectors for S^1? The vectors necessarily have to point outwards of the manifold because of the curvy nature of S^1. To ~~understand~~ solve this problem, consider again the vectors in $\mathbb{R}$ and tilt ~~the~~ $\mathbb{R}$ slightly. Then the vectors v_1, v_2 and v_3 point outward of $\mathcal{M}$ but are still <u>tangent</u> to $\mathcal{M}$ on the same points p_1, p_2 and p_3!



Hence, we allow for all <u>tangents</u> of $\mathcal{M}$ to be suitable vectors. But how do we define tangents on arbitrary manifolds? We have to define the tangent space $T_p\mathcal{M}$ on every point $p \in \mathcal{M}$, because $\mathcal{M}$ ~~is~~ can be locally different on each p.

<u>Def.</u> (Tangent space)
Let $p \in \mathcal{M}$ be a point on a manifold and let $\gamma : \mathbb{R} \to \mathcal{M}$ be a curve such that $\gamma(\lambda_0) = p$ for some $\lambda_0 \in \mathbb{R}$. So p lies on the <u>path</u> γ. We define the vector $\boxed{V_{\gamma,p}}$ of the curve γ at point $p \in \mathcal{M}$ as a map that takes as input a function $f: \mathcal{M} \to \mathbb{R}$; $f \in C^\infty(\mathcal{M})$ (well-behaved) and outputs a number in $\mathbb{R}$:

Note: well-behaved usually means that the function is continuous and ~~its~~ all its derivatives are continuous too.

5.2. $$\boxed{V_{\gamma,p}} : C^\infty(\mathcal{M}) \to \mathbb{R}$$

$$V_{\gamma,p} : f \longmapsto V_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0)$$
$$= f(\gamma(t))'(\lambda_0),$$

where $(\circ)'$ denotes the derivative. Notice that $f \circ \gamma$ is a function from $\mathbb{R}$ to $\mathbb{R}$, hence a derivative is defined. However, it is not straight forward to perform the chain rule:

$$f(\gamma(t))' = \frac{\partial f}{\partial x^i} \dot{\gamma}^i(t),$$

as it is not clear what $\frac{\partial f}{\partial x^i}$ means as f is a function from $\mathcal{M}$ to $\mathbb{R}$ and we don't have coordinates in $\mathcal{M}$. Same for $\dot{\gamma}^i(t)$ where one can neither ~~can't in fact easily~~ perform ~~the~~ the derivative nor is clear what the i-th component of a function in $\mathcal{M}$ is. However, both of these problems have the same origin, namely we have no coordinate system in $\mathcal{M}$.

We will solve this problem later. First, consider the example curve γ in S^1:

$$\gamma : (0, 2\pi) \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \in S^1$$

which goes through the point $p = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at ~~[illegible]~~ $(\lambda_0) = \varphi_0 = 0$

The problem is that we still have this unknown function $f \in C^\infty(\mathcal{M})$, so we can't compute $V_{\gamma,p}$ until we specify f. (Reminder: $f : \mathcal{M} \to \mathbb{R}$)

We have already encountered such functions, namely **charts**, or rather parts of a chart.

As a quick reminder, a chart maps a point (and its vicinity) to $\mathbb{R}^n$. So, $\psi : M \to \mathbb{R}^n$.

Now, as ψ outputs something in $\mathbb{R}^n$, we can write:

$$\psi(p) = \begin{pmatrix} \psi^1(p) \\ \psi^2(p) \\ \vdots \\ \psi^n(p) \end{pmatrix},$$

So $\psi(p)$ can be written as a vector in $\mathbb{R}^n$.

where $\psi^i : M \to \mathbb{R}$, so $\psi^i \in C^\infty(M)$. So let's take these ψ^i as functions f.

In the case of S^1, take

$$\psi_0 \begin{pmatrix} x \\ y \end{pmatrix} = \arctan\left(\frac{y}{x}\right)$$

which is a chart in the specified range of γ. In particular:

$$\begin{aligned}
V_{\gamma,p}(\psi_0) &= (\psi^0 \circ \gamma)'(\varphi_0) \\
&= (\psi \circ (\gamma(\varphi)))'(\varphi_0) \\
&= \psi\begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}'(0) \\
&= \arctan\left(\frac{\sin(\varphi)}{\cos(\varphi)}\right)'(0) \\
&= \varphi'(0) \\
&= \frac{\partial \varphi}{\partial \varphi}(0) = 1,
\end{aligned}$$

So the allowed vectors on the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are 1, for the specified γ.

That sounds strange. Why don't we obtain a multicomponent vector?

The answer lies in the fact that we considered S^1 in $\mathbb{R}$. We can't obtain a two-component vector this way.

Notice that we can also treat S^1 as the usual unit circle embedded in $\mathbb{R}^2$. Then, we can use the identity chart:

$$\begin{pmatrix} \psi^1_{\mathbb{R}^2}\begin{pmatrix}x\\y\end{pmatrix} \\ \psi^2_{\mathbb{R}^2}\begin{pmatrix}x\\y\end{pmatrix}\end{pmatrix} = \psi_{\mathbb{R}^2}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2,$$

hence $\psi^1_{\mathbb{R}^2} \in C^\infty(M)$ as well as $\psi^2_{\mathbb{R}^2} \in C^\infty(M)$

Compute the velocity $v_{\gamma,p}$ with the same p and γ but for these charts!

$$\Rightarrow v_{\gamma,p}(\psi^1_{\mathbb{R}^2}) = \left(\psi^1_{\mathbb{R}^2}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\right)'(0)$$

$$= \frac{\partial \cos\varphi}{\partial\varphi}(0)$$

$$= -\sin\varphi(0) = 0$$

$$\Rightarrow v_{\gamma,p}(\psi^2_{\mathbb{R}^2}) = \left(\psi^2_{\mathbb{R}^2}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\right)'(0)$$

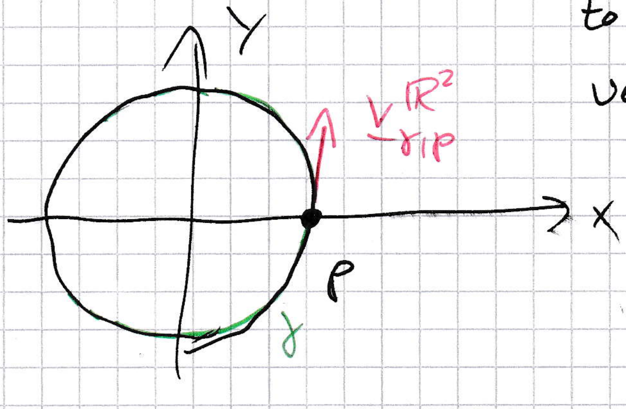
$$= \frac{\partial \sin\varphi}{\partial\varphi}(0)$$

$$= \cos\varphi(0) = 1.$$

We can combine the two velocities into a two-component vector:

$$\underline{v}^{\mathbb{R}^2}_{\gamma,p} = \begin{pmatrix} v_{\gamma,p}(\psi^1_{\mathbb{R}^2}) \\ v_{\gamma,p}(\psi^2_{\mathbb{R}^2})\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

which is exactly the usual velocity vector that one computes. So the notion of $v_{\gamma,p}$ generalizes to our well known velocity when treating circular motion as an embedding in $\mathbb{R}^2$!



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That is good and all, but how do we jusity that we seem to get two diffent results. For the chart φ_0, one obtains

$$V_{\gamma,p}(\varphi_0) = 1$$

and for the charts $\varphi^1_{\mathbb{R}^2}$ and $\varphi^2_{\mathbb{R}^2}$, we obtain

$$\underline{V}_{\gamma,p}(\varphi_{\mathbb{R}^2}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The answer lies in components of vectors vs. vectors. $V_{\gamma,p}$ is a vector in the tangent space. It remains the same for any coordinate transformation. As soon as we feed charts to $V_{\gamma,p}$, we choose a basis. The (*) we calculated are components of $V_{\gamma,p}$!

(*) numbers $V_{\gamma,p}(\varphi_0)$, $V_{\gamma,p}(\varphi^1_{\mathbb{R}^2})$ and $V_{\gamma,p}(\varphi^2_{\mathbb{R}^2})$

Hence we seem to have solved our problem of coordinates in M. The charts determine a coordinate system in $\mathbb{R}^n$ which changes the basis vectors and thus the components one calculates. Hence, it also becomes apparent why there is a function f appearing in the definition of $V_{\gamma,p}$. We can extract the components of $V_{\gamma,p}$ in the basis φ by feeding φ to $V_{\gamma,p}$!