We have seen how to compute the derivatives of components of vector field A w.r.t. coordinates of a chart, namely $\frac{\partial A^x}{\partial y}$.

Now, let us take the same vector field $A^r = 0$ and $A^\varphi = \Omega$ and compute derivatives w.r.t r and φ. Namely consider:

$$\partial_\varphi A^{r|\varphi} = \frac{\partial A^{r|\varphi}}{\partial \varphi} = 0 \quad ; \quad \overset{\partial_r A^{\varphi|r}}{\frac{\partial A^{\varphi|r}}{\partial r}} = 0 \quad \text{as } A^\varphi, A^r = \text{const.},$$

$$\Rightarrow \partial_\varphi A = \partial_r A = 0 \,?$$

So we conclude that both components don't change with φ. However, let us take a closer look and consider the embedding of A in $\mathbb{R}^2$ with the basis vectors $\underline{e}_\varphi, \underline{e}_r \in \mathbb{R}^2$:

$$A(r,\varphi) = r\Omega\, \underline{e}_\varphi.$$

So, compute: $\frac{\partial A}{\partial \varphi} = r\Omega \frac{\partial \underline{e}_\varphi}{\partial \varphi} = -r\Omega\, \underline{e}_r \neq 0$

$$\frac{\partial A}{\partial r} = \Omega\, \underline{e}_\varphi \neq 0$$

which suggest that $\left(\frac{\partial A}{\partial r}\right)^\varphi = \Omega \overset{\partial_r(A^\varphi)}{\times}$ and $\left(\frac{\partial A}{\partial \varphi}\right)^r = -r\Omega \neq \partial_\varphi(A^r)$ if we were to translate the embedding formalism back into diff. geo. formalism.

Why is it not the same? The answer lies in the curviness of the coordinates r and φ. Whereas before, where we computed derivatives w.r.t x and y, we were in Euklidean space where basis vectors are constant.

Mo. 9.1.

However, when we perform partial derivatives w.r.t. r and φ of the <u>components</u>, we neglect the change of basis vectors!

In the way of writing $\underline{e}_\varphi$ and $\underline{e}_r$, we can actually see why the two ways of calculating this derivative are not equal as we can't compute $\left(\frac{\partial}{\partial \varphi}\right)\left(\frac{\partial}{\partial r}\right)$ in mathematical diff. geometry.

So how do we resolve this? The answer lies in the covariant derivative and the Christoffel symbols!

<u>Def.</u> (Covariant derivative)
Let A be a vector field. Then, the derivative of A along the direction $\frac{\partial}{\partial \psi^k}$ for a chart $\psi : M \to \mathbb{R}^n$ that includes the change of basis vector is given by:

$$A_{;k} = \nabla_{\frac{\partial}{\partial \psi^k}} A = \left(A^i_{,k} + \Gamma^i_{jk} A^j\right) \frac{\partial}{\partial \psi^i},$$

where $A^i_{,k} = \partial_k A^i$, the partial derivative of the i-th component of A, $A^i = d\psi^i(A)$.

In particular: $(A_{;k})^i = A^i_{,k} + \Gamma^i_{jk} A^j$.

The symbol Γ^i_{jk} is called <u>Christoffel symbols</u>.

It encodes how the basis vectors change with change of coordinates.

9.2.

One can define the Christoffel-symbols self-consistently by choosing $X = \frac{\partial}{\partial y_m}$ as the vector field:

$$\Rightarrow \left(\frac{\partial}{\partial y_m}\right)_{;k} = \Big[\underbrace{\partial_k \delta^i_{\ m}}_{=0} + \Gamma^i_{\ \gamma k}\, \delta^\gamma_{\ m}\Big] \frac{\partial}{\partial y^i}$$

$$= \Gamma^i_{\ mk} \frac{\partial}{\partial y^i}$$

Now, we see what the relationship between the embedding and the diff. geo. language is:

Namely observe that:

$$\boxed{\begin{aligned} \frac{\partial (r \underline{e}_\varphi)}{\partial \varphi} &= -r\, \underline{e}_r \quad \Longleftrightarrow \quad \frac{\partial}{\partial \varphi}_{;\varphi} = -r \frac{\partial}{\partial r} \\ \text{and} \quad \frac{\partial (r\, \underline{e}_\varphi)}{\partial r} &= \underline{e}_\varphi \quad \Longleftrightarrow \quad \frac{\partial}{\partial \varphi}_{;r} = \frac{1}{r} \frac{\partial}{\partial \varphi} \end{aligned}}$$

which results in $\Gamma^r_{\ \varphi\varphi} = -r$

and $\Gamma^\varphi_{\ \varphi r} = \frac{1}{r}$.

Computing the desired basis-vector-compatible derivative of A w.r.t. φ and r now yields (in diff. geo. language)

$$\nabla_{\frac{\partial}{\partial \varphi}} A = \Big(\underbrace{\partial_\varphi A^i}_{=0 \text{ as } A = \text{const.}} + \Gamma^i_{\ \gamma\varphi} A^\gamma\Big) \frac{\partial}{\partial y^i}$$

$$= \left(\Gamma^i_{\ \varphi\varphi}\, \Omega\right) \frac{\partial}{\partial y^i}$$

$$= \Omega\, \Gamma^r_{\ \varphi\varphi} \frac{\partial}{\partial r} = -r\Omega \frac{\partial}{\partial r}$$

as $\underbrace{\Gamma^\varphi_{\ \varphi\varphi} = 0}$ as $\frac{\partial (r\, \underline{e}_\varphi)}{\partial \varphi} = -r\, \underline{e}_r + \underbrace{0}\cdot \underline{e}_\varphi$

and $\nabla_{\frac{\partial}{\partial r}} A = \left(\underbrace{\partial_r A^i}_{=0} + \Gamma^i_{\varphi r} A^\varphi \right) \frac{\partial}{\partial z^i}$

$$= \Gamma^\varphi_{\varphi r}\, \Omega \frac{\partial}{\partial \varphi} = \frac{1}{r} \Omega \frac{\partial}{\partial \varphi}$$

as $\Gamma^r_{\varphi r} = 0$.

We can actually use this relation to compute the covariant derivative

$$\left[\nabla_{\frac{\partial}{\partial x}} A \right]^y$$

in contrast to before where we considered only $\frac{\partial A^y}{\partial x}$.

Observe that $\frac{\partial}{\partial x} = \underbrace{\frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x} \frac{\partial}{\partial \varphi} = \frac{x}{\sqrt{\;}} \frac{\partial}{\partial r} - \frac{y}{\sqrt{\;}^2} \frac{\partial}{\partial \varphi}}$

and $dy = \underbrace{\frac{\partial y}{\partial \varphi} d\varphi + \frac{\partial y}{\partial r} dr = r\cos\varphi \, d\varphi + \sin\varphi \, dr}$

to write:

$$\left[\nabla_{\frac{\partial}{\partial r}} A \right]^y = dy \left(\nabla_{\frac{\partial}{\partial x}} (A) \right)$$

$$= \underbrace{(\qquad)}\, \nabla_{\underbrace{\quad}} (A)$$

$$= \Bigg[\frac{r\cos\varphi\, x}{\sqrt{\;}} d\varphi\, \nabla_{\frac{\partial}{\partial r}}$$

$$- \frac{r\cos\varphi}{\sqrt{\;}^2} y \, d\varphi\, \nabla_{\frac{\partial}{\partial \varphi}}$$

$$+ \frac{\sin\varphi\, x}{\sqrt{\;}} dr\, \nabla_{\frac{\partial}{\partial r}}$$

$$- \frac{\sin\varphi\, y}{\sqrt{\;}^2} dr\, \nabla_{\frac{\partial}{\partial \varphi}} \Bigg] (A)$$

$$= r\cos^2\varphi \, A^\varphi{}_{;r} - r\underbrace{\cos\varphi\sin\varphi}_{\sqrt{}} A^\varphi{}_{;\varphi} + \sin\varphi\cos\varphi \, A^r{}_{;r} - \underbrace{\sin^2\varphi}_{\sqrt{}} A^r{}_{;r}$$

$$= \begin{cases} A^\varphi{}_{,r} + \Gamma^\varphi_{\gamma r} A^\gamma \\ \underbrace{A^\varphi{}_{,\varphi}}_{=0} + \Gamma^\varphi_{\gamma\varphi} A^\gamma \\ \underbrace{A^r{}_{,r}}_{=0} + \Gamma^r_{\gamma r} A^\gamma \\ \underbrace{A^r{}_{,\varphi}}_{=0} + \Gamma^r_{\gamma\varphi} A^\gamma \end{cases}$$

$$= \begin{cases} \Gamma^\varphi_{\varphi r} A^\varphi \\ \Gamma^\varphi_{r\varphi} A^r \\ \Gamma^r_{\varphi r} A^\varphi \\ \Gamma^r_{\varphi\varphi} A^\varphi \end{cases} = \begin{cases} \frac{1}{r}\Omega \\ 0 \\ 0 \\ -r\Omega \end{cases}$$

$$= \cos^2\Omega + \sin^2\varphi\,\Omega = \Omega.$$

$\square$

Notice that before in 8.5, we found that

$$(\partial_i A)^\gamma = \partial_i (A^\gamma).$$

We see that by translating $\partial_i A \mapsto D_i A$, that:

$$(D_i A)^\gamma = \partial_i A^\gamma = A^\gamma{}_{,i},$$

so the Christoffel symbols are zero in Euklidean coordinates.