We have handled vectors quite nicely now. We note, that the chart induced basis behaves like normal partial derivatives if the manifold is easily written down.

But the question remains on why the basis vectors are given by pseudo partial derivatives. The rather unsatisfying answer is that it really doesn't matter what the basis vectors actually are. We could've also written $\tilde{e}_r$ and $\tilde{e}_\varphi$ instead of $\left(\frac{\partial}{\partial r}\right)_p$ and $\left(\frac{\partial}{\partial \varphi}\right)_p$.

However, it remains the fact that writting the basis vectors in the second way easily translate to basis transformations. This becomes even more evident when we consider the covectors of $\left(\frac{\partial}{\partial \psi^i}\right)_p$ which will turn out to be differentials!

<u>Def.</u> (Differential)

For <u>any</u> function $f \in C^\infty(M)$, there exists ~~an~~ function $df \in (T_p M)^*$ (dualspace), s.t.:

$$(df)_p : \quad T_p M \to \mathbb{R}$$
$$v_{\gamma,p} \mapsto v_{\gamma,p}(f) = \dot{\gamma}^i \frac{\partial f}{\partial \psi^i}$$

Notice, that f-functions are <u>charts!</u> Hence, every chart with coordinates $(\psi^1, \dots, \psi^n)$ induces differentials $(d\psi^1)_p, \dots, (d\psi^n)_p$.

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7.1.

From the Definition of $(df)_p$, it becomes evident that the components of any vector $v_{\gamma,p}$ on T_pM can be <u>extracted</u> with respect to chart z by acting with the differential on it:

$$v_{\gamma,p}^{z^i} = v_{\gamma,p}(z^i) =: dz^i(v_{\gamma,p})$$

(follows from definition).

One can also easily verify that $\{(dz^i)_p\}_i$ is the dual basis of $\left(\frac{\partial}{\partial z^i}\right)_p$:

$$(dz^j)_p\left(\frac{\partial}{\partial z^i}\right)_p := \left(\frac{\partial}{\partial z^i}\right)_p z^j = \delta^j_i .$$

$\square$

Similarily to the chart induced basis, one can transform the differentials in the usual sense:

$$(dz^i)_p = \left(\frac{\partial z^i}{\partial \lambda^j}\right)_p d\lambda^j.$$

In the case of polar coordinates on S^1, it holds that:

$$(dr)_p = \left(\frac{\partial r}{\partial x}\right)_p (dx)_p + \left(\frac{\partial r}{\partial y}\right)_p (dy)_p$$

$$= x\,(dx)_p + y\,(dy)_p$$

$$(d\varphi)_p = \left(\frac{\partial \varphi}{\partial x}\right)_p (dx)_p + \left(\frac{\partial \varphi}{\partial y}\right)_p (dy)_p$$

$$= x\,(dy)_p - y\,(dx)_p .$$

Notice, that we have to write $(\ldots)_p$ on every single entry in the above equation. As long as we only use one chart of an atlas, we don't actually need this.

7.2.

But it is not obvious on how to remove this $(\dots)_p$ in a general case.
For this, we need to combine all tangent spaces T_pM into a <u>tangent bundle</u>, TM.

<u>Def.</u> (Tangent bundle)
Let M be a manifold and T_pM the tangent space at point $p \in M$. Then, we can define the tangent bundle by $TM := \bigcup_p T_pM$.

<u>Note</u>: The actual mathematical Def. of the tangent bundle requires a lot of effort an the introduction of numerous other things like fibers, bundles, etc.

This can be used to define a vector field!

<u>Def.</u> (Vector field)
A vector field X is a function that maps to every $p \in M$ a vector on the respective tangent space:

$$X : M \to TM.$$

This notion ~~are~~ is hard to handle on a theoretical level, so we will just take the following for granted, namely that every vector field X can be written in the following way for a given (global) chart $\psi : M \to \mathbb{R}^n$

$$X = \sum_{i=1}^{n} d\psi^i(X) \frac{\partial}{\partial \psi^i}$$

which takes as input a $p \in M$, s.t.

$$X_p = \sum_{i=1}^{n} (d\psi^i)_p(X)\left(\frac{\partial}{\partial \psi^i}\right)_p = \sum_{i=1}^{n} X_p^i \left(\frac{\partial}{\partial \psi^i}\right)_p \in T_pM.$$

Hence, if we want to extract the component Z_1^i out of a vector field X, we just compute $dZ_1^i(X)$.

If the manifold can be written down, like $\mathbb{R}^2$, one can write a vector field by:

$$A(x,y) = dx(A)_p \left(\frac{\partial}{\partial x}\right)_p + dy(A)_p \left(\frac{\partial}{\partial y}\right)_p$$

$$= A^x(x,y) \left.\frac{\partial}{\partial x}\right)_{(x,y)} + A^y(x,y) \left.\frac{\partial}{\partial y}\right)_{(x,y)}$$

Hence, we arrive at our standard understanding of vector fields, namely the components of a vectorfield A, $\underline{A}$ are given by:

$$\underline{A} = (A^x(x,y), A^y(x,y)).$$

However, what happens if we consider S^1 in its natural chart coordinate?
Then, a vectorfield is given by:

$$A(\varphi) = d\varphi(A)_p \left(\frac{\partial}{\partial \varphi}\right)_p$$

$$= A^\varphi(\varphi) \left.\frac{\partial}{\partial \varphi}\right)_\varphi ,$$

where φ is the parameter corresponding to the one-dimensional coordinate.

Notice that $\left(\frac{\partial}{\partial \varphi}\right)_\varphi$ depends on the coordinate φ. This is very normal in curved (curvilinear) coordinates. This makes it problematic to perform spatial derivatives on our vectorfields.

7.4.

Example (Polar coordinates in standard framework)

Consider the polar coordinate parametrization in $\mathbb{R}^2$:

$$(*)\begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \quad \text{for} \quad r \in (0,\infty) \text{ and } \varphi \in (0, 2\pi).$$

We have a good grasp on the basis vectors in $\mathbb{R}^2$, namely $\underline{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

But what are $\underline{e}_r$ and $\underline{e}_\varphi$? These vectors are given by the coordinate transformation $(*)$:

$$\underline{e}_i(\underline{r}) = \frac{\frac{\partial \underline{r}}{\partial q_i}}{\left|\frac{\partial \underline{r}}{\partial q_i}\right|} \quad \text{where} \quad \underline{r} = (x(r,\varphi), y(r,\varphi)) \text{ and } q = (r,\varphi)$$

It holds that:

$$\underline{e}_r = \cos\varphi \, \underline{e}_x + \sin\varphi \, \underline{e}_y$$
$$\underline{e}_\varphi = -\sin\varphi \, \underline{e}_x + \cos\varphi \, \underline{e}_y.$$

(this can be easily verified)

Consider the coordinate vector: $\underline{r} = x \underline{e}_x + y \underline{e}_y$ [redacted] which can also be written by: $\underline{r} = r \, \underline{e}_r$, as $\underline{e}_r$ depends on φ and thus rotates with the position vector $\underline{r} = (r, \varphi)$.

But what about vector fields?

$$\underline{A} = A^x \underline{e}_x + A^y \underline{e}_y \overset{!}{=} \tilde{A}^r \underline{e}_r + \tilde{A}^\varphi \underline{e}_\varphi$$

$$= \tilde{A}^r (\cos\varphi \, \underline{e}_x + \sin\varphi \, \underline{e}_y) + \tilde{A}^\varphi (-\sin\varphi \, \underline{e}_x + \cos\varphi \, \underline{e}_y)$$

$$= (\tilde{A}^r \cos\varphi - \tilde{A}^\varphi \sin\varphi) \underline{e}_x + (\tilde{A}^r \sin\varphi + \tilde{A}^\varphi \cos\varphi) \underline{e}_y$$

$$\Rightarrow \begin{aligned} A^x &= \tilde{A}^r \cos\varphi - \tilde{A}^\varphi \sin\varphi \\ A^y &= \tilde{A}^r \sin\varphi + \tilde{A}^\varphi \cos\varphi. \end{aligned} \quad \Bigg\| \, (A)$$

7.5.

Notice that A^x and A^y have almost the same transformation properties as dx and dy

$$\Rightarrow \quad dx = \frac{\partial x}{\partial \varphi} d\varphi + \frac{\partial x}{\partial r} dr$$

$$= \cos\varphi \, dr - r \sin\varphi \, d\varphi$$

$$dy = \sin\varphi \, dr + r \cos\varphi \, d\varphi, \qquad (*)$$

where the factor of r in front of $d\varphi$ prevents the whole equality. This factor of r will follow us from here on out.

From $(*)$ and $(*)$, by assuming that $A^x = dx(A)$; $dy(A) = A^y$; $dr(A) = A^r$; $d\varphi(A) = A^\varphi$, we conclude that:

$$\tilde{A}^\varphi = r A^\varphi = r \, d\varphi(A); \quad \tilde{A}^r = A^r = dr(A)$$

which translates to:

$$\underline{e}_\varphi = \frac{1}{r} \frac{\partial}{\partial \varphi}; \quad \underline{e}_r = \frac{\partial}{\partial r}$$

This circumstance will solve itself once we study the covariant derivative in detail in the next few pages.

It is however strange that the "partial derivative" and unit vector are equal for $\underline{e}_r$ and $\frac{\partial}{\partial r}$ but not for $\underline{e}_\varphi$ and $\frac{\partial}{\partial \varphi}$. This leaves the question on whether $\tilde{A}^\varphi$ or A^φ is the physical component of A.