Multiterminal Josephson Junctions (MTJJs): topology and reflectionless modes

David Christian Ohnmacht
Wolfgang Belzig
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SFB 1432

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Multiterminal Josephson Junctions (MTJJs): topology and reflectionless modes

1) MTJJs are an excellent platform to study engineered topology

2) Refectionless scattering modes (unity transmission modes) are a source of topology in MTJJs

3) Is there a generalized topological invariant for MTJJs in the normal state properties?

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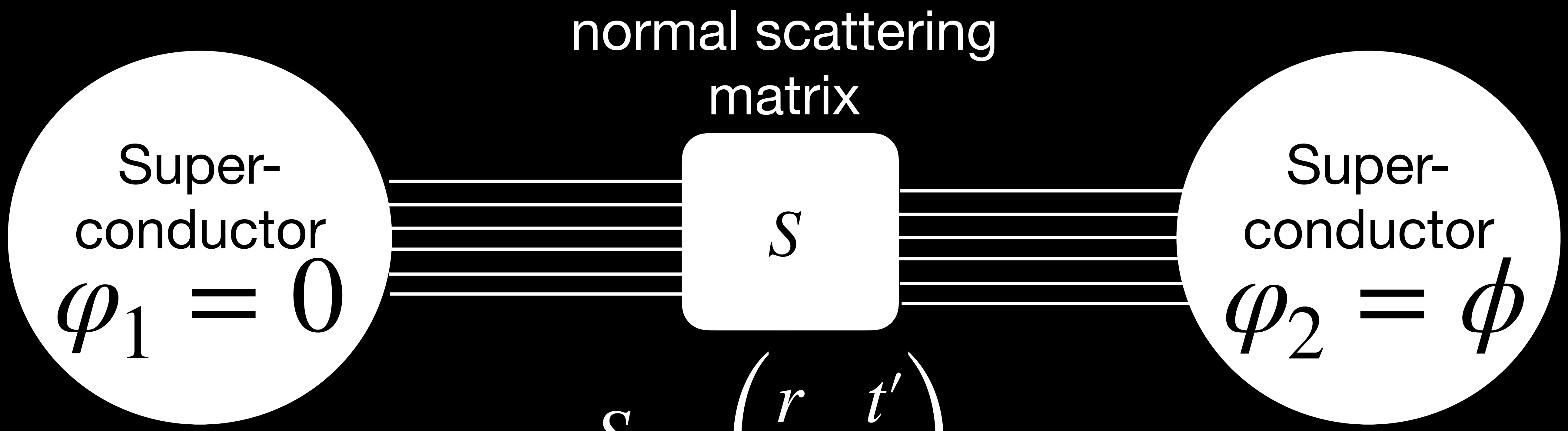
2-terminal Josephson junction (JJ): Andreev bound states



$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

$r,\ t,\ t',\ r'$ $\quad n \times n$ **matrices**

Transmission matrix with *n* transmission eigenvalues

$$T = tt^\dagger \qquad \{T_i\}_{i=1,\ldots,n}, \quad 0 \le T_i \le 1$$

Andreev Bound state spectrum (ABS):

$$E^i_{\mathrm{ABS}}(\phi) = \pm |\Delta| \sqrt{1 - T_i \sin^2(\phi/2)}$$

Supercurrent:

$$I \propto \partial_\phi E_{\mathrm{ABS}}$$

C. W. J. Beenakker and H. van Houten, Phys. Rev. Lett. 66, 3056 (1991)

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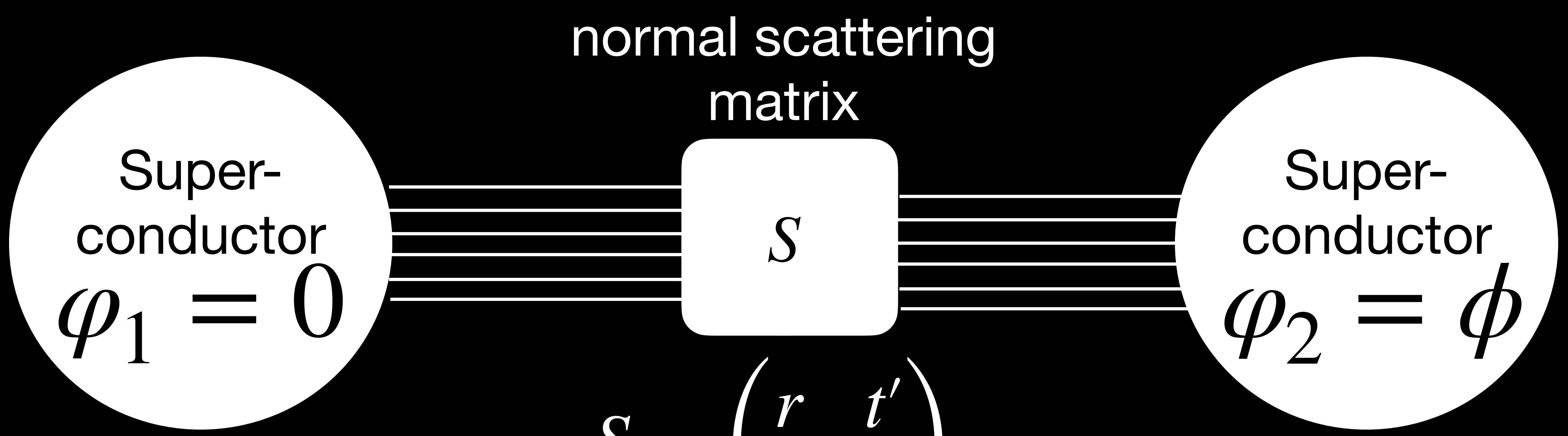

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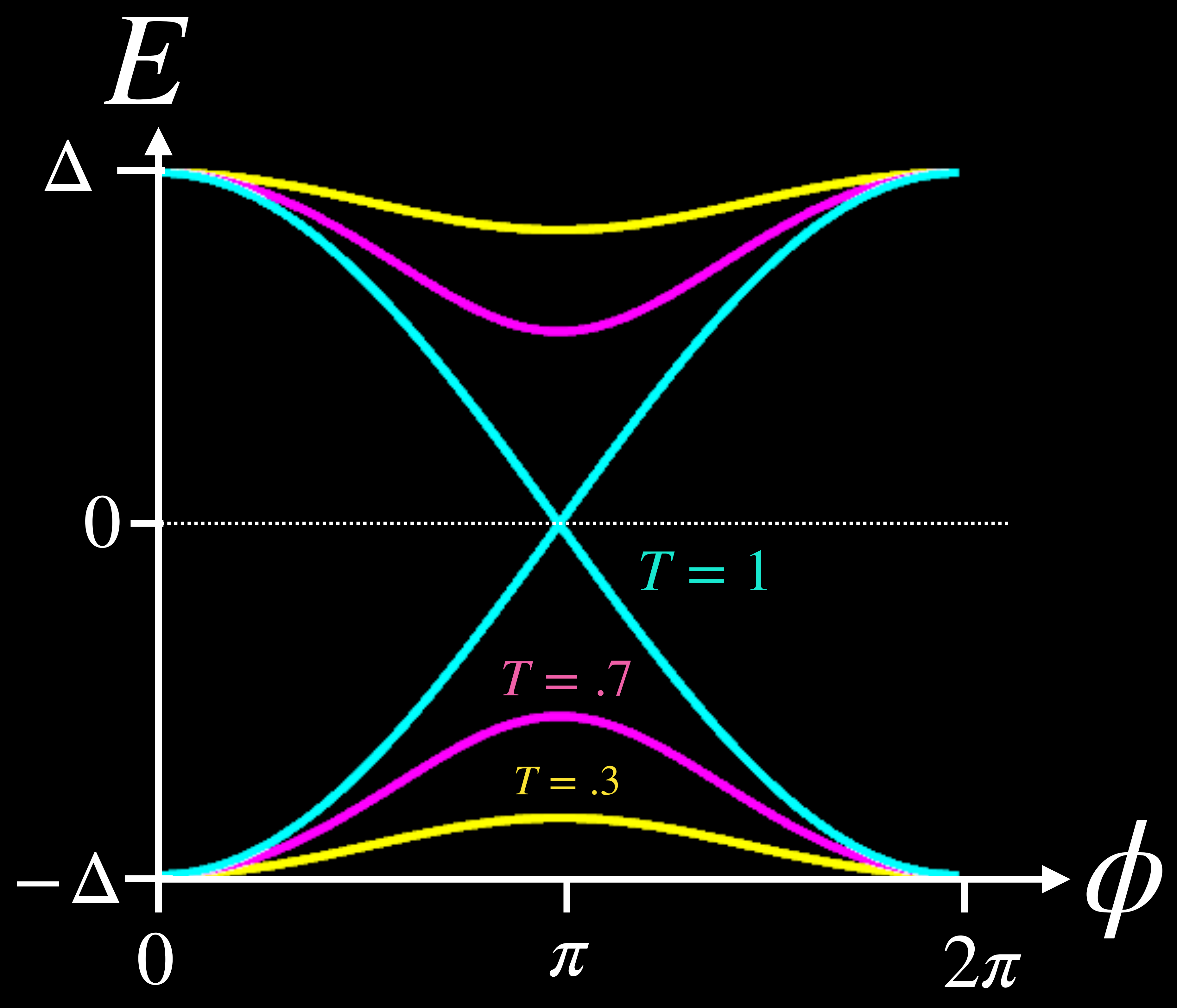

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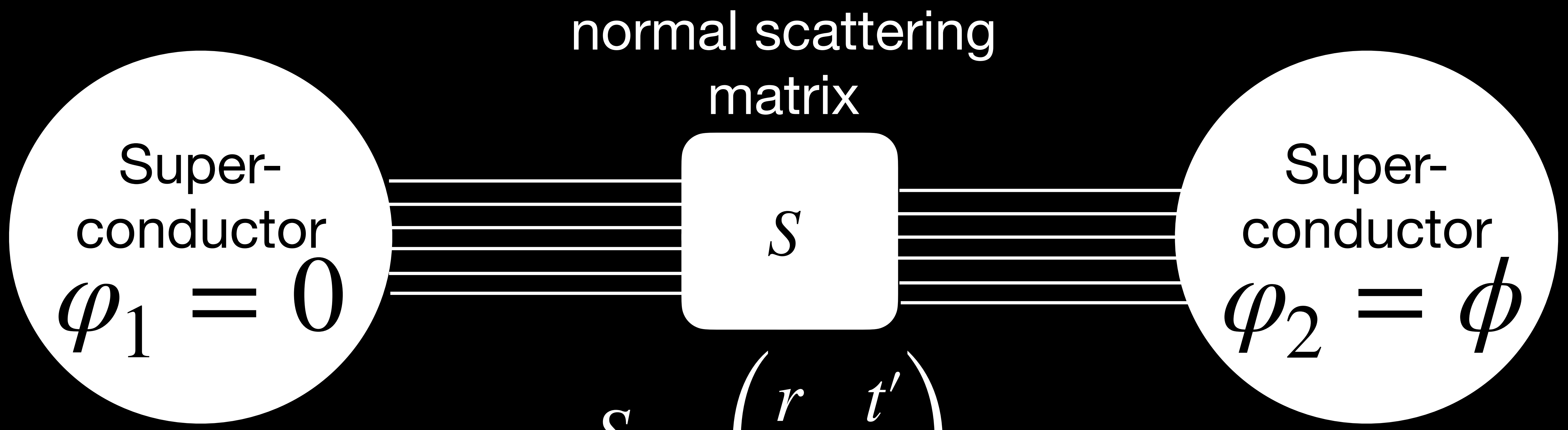

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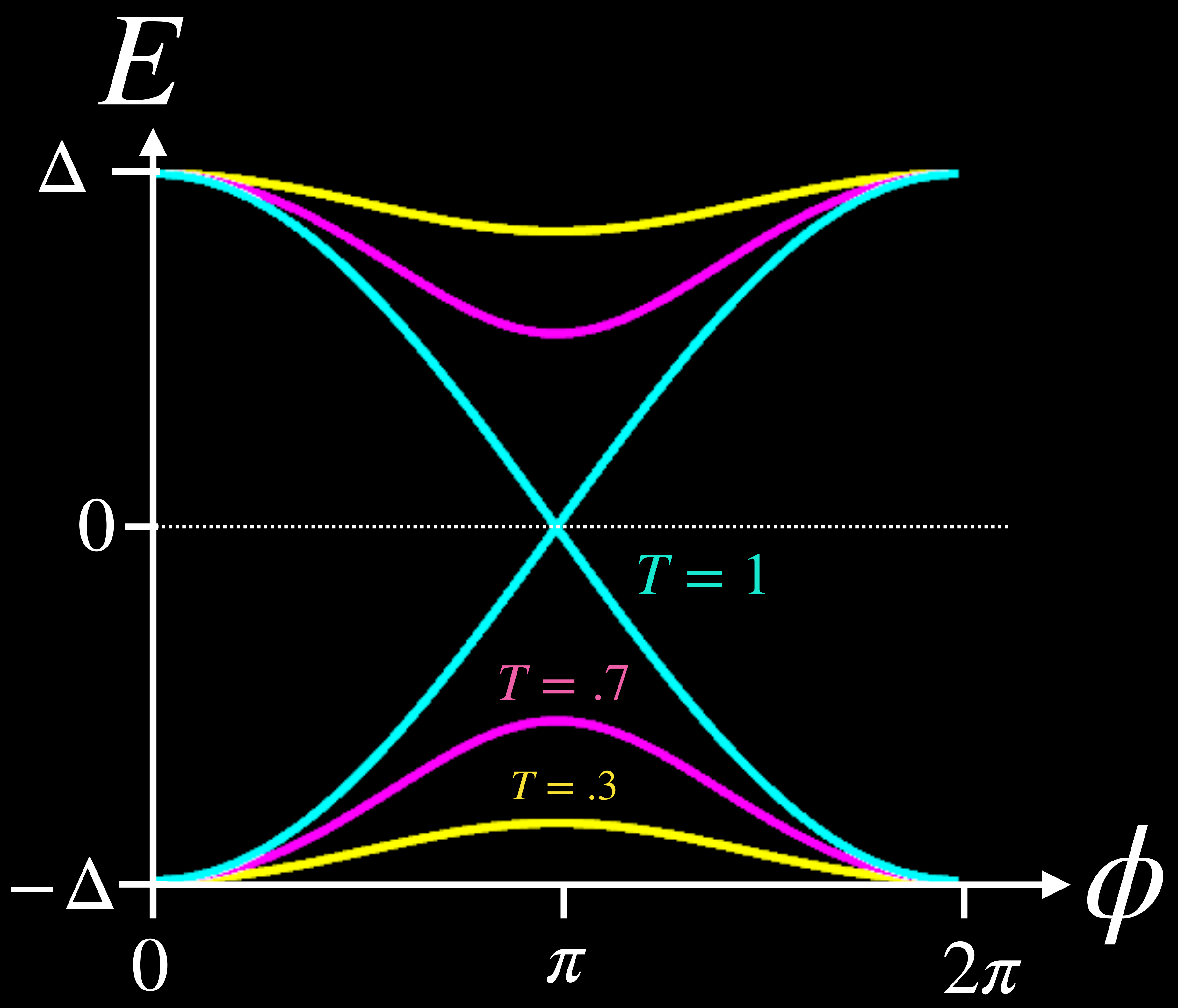


Zero energy resonance only for unity transmission

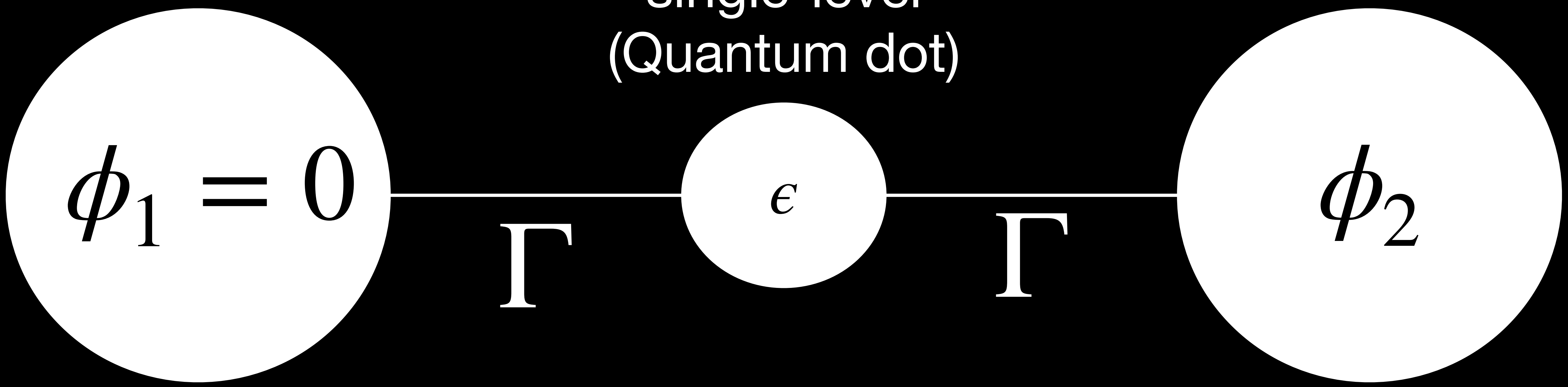
Alternatively: ABS only zero for a reflectionless scattering mode with $\det(r) = 0$

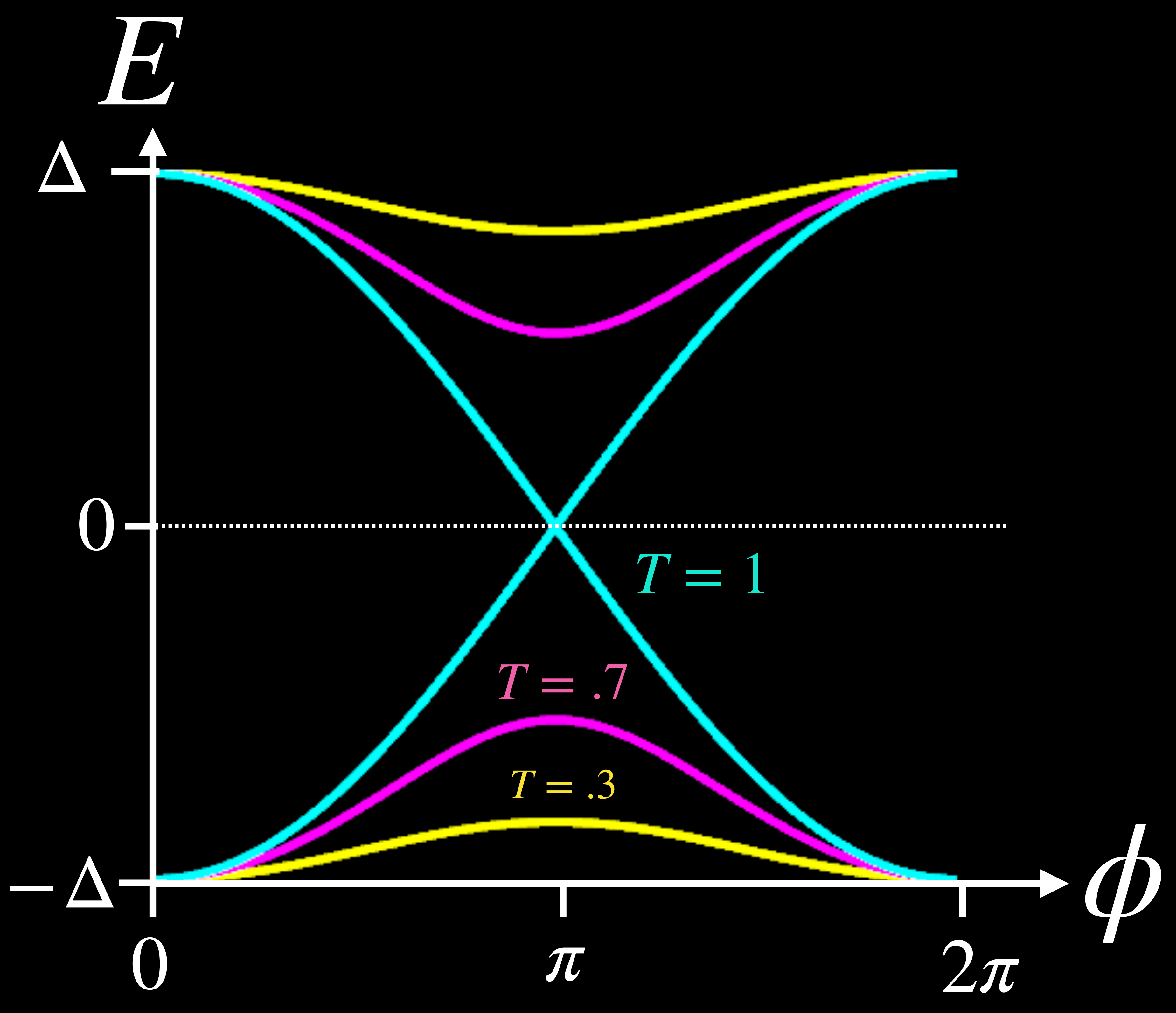
C. W. J. Beenakker and H. van Houten, Phys. Rev. Lett. 66, 3056 (1991)

2-terminal Josephson junction (JJ): Andreev bound states

Example: 1-dot 2-terminal JJ

single-level
(Quantum dot)





normal S-matrix

$$S = 1 - \begin{pmatrix} \Gamma \\ \Gamma \end{pmatrix} \frac{1}{E - H_{\mathrm{N}} + 2\imath\Gamma} \begin{pmatrix} \Gamma & \Gamma \end{pmatrix}$$

Transmission (Breit-Wigner)

$$T(E) = \frac{4\Gamma^2}{(E-\epsilon) + 4\Gamma^2}$$

equal to 1…

Reflection

$$\det(r) = 1 - \frac{2\Gamma}{2\Gamma + \imath\epsilon}$$

equal to 0…

…for $\epsilon = 0$!

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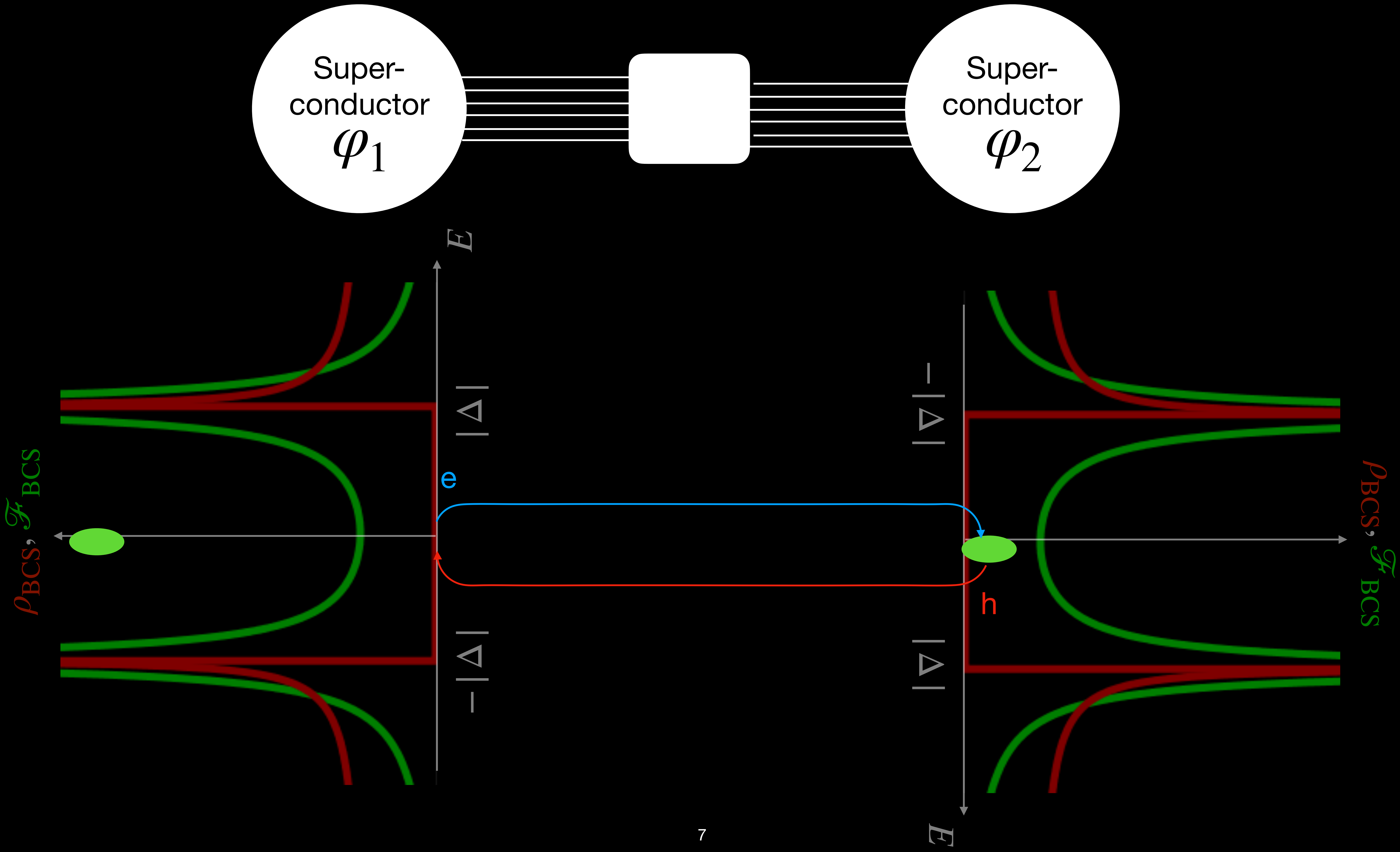

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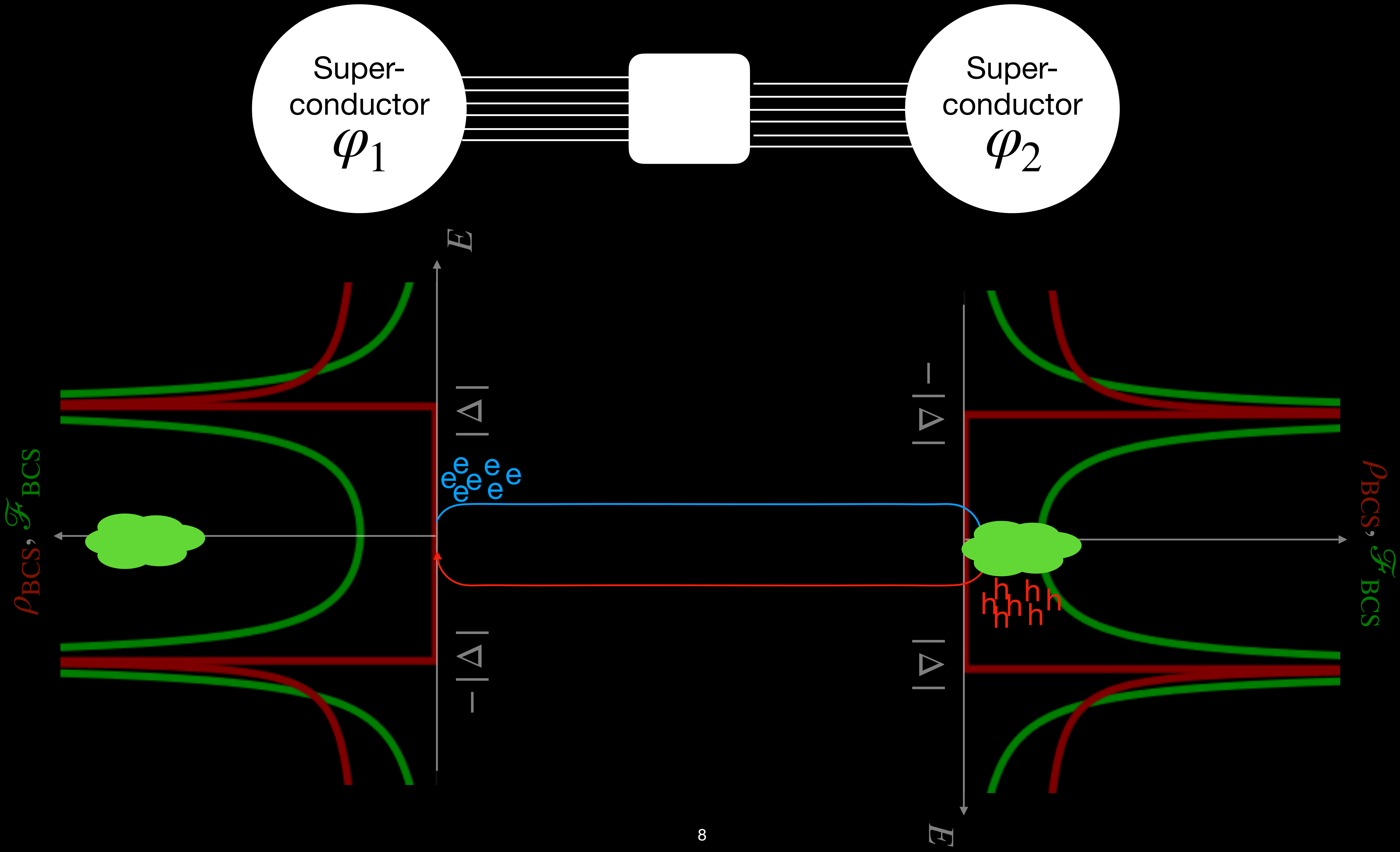
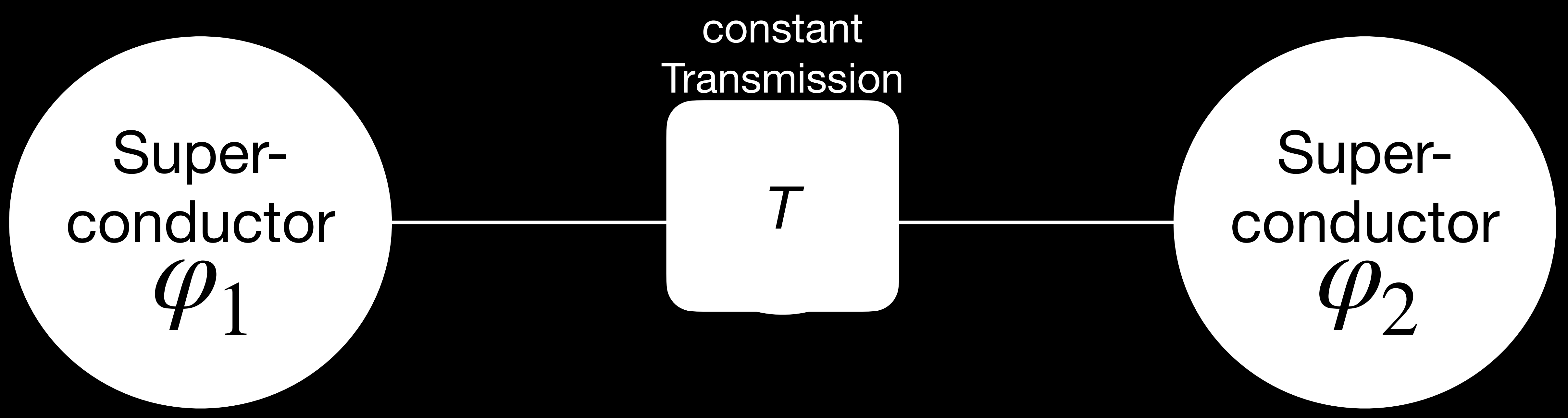


Multiterminal Josephson junctions (MTJJs)

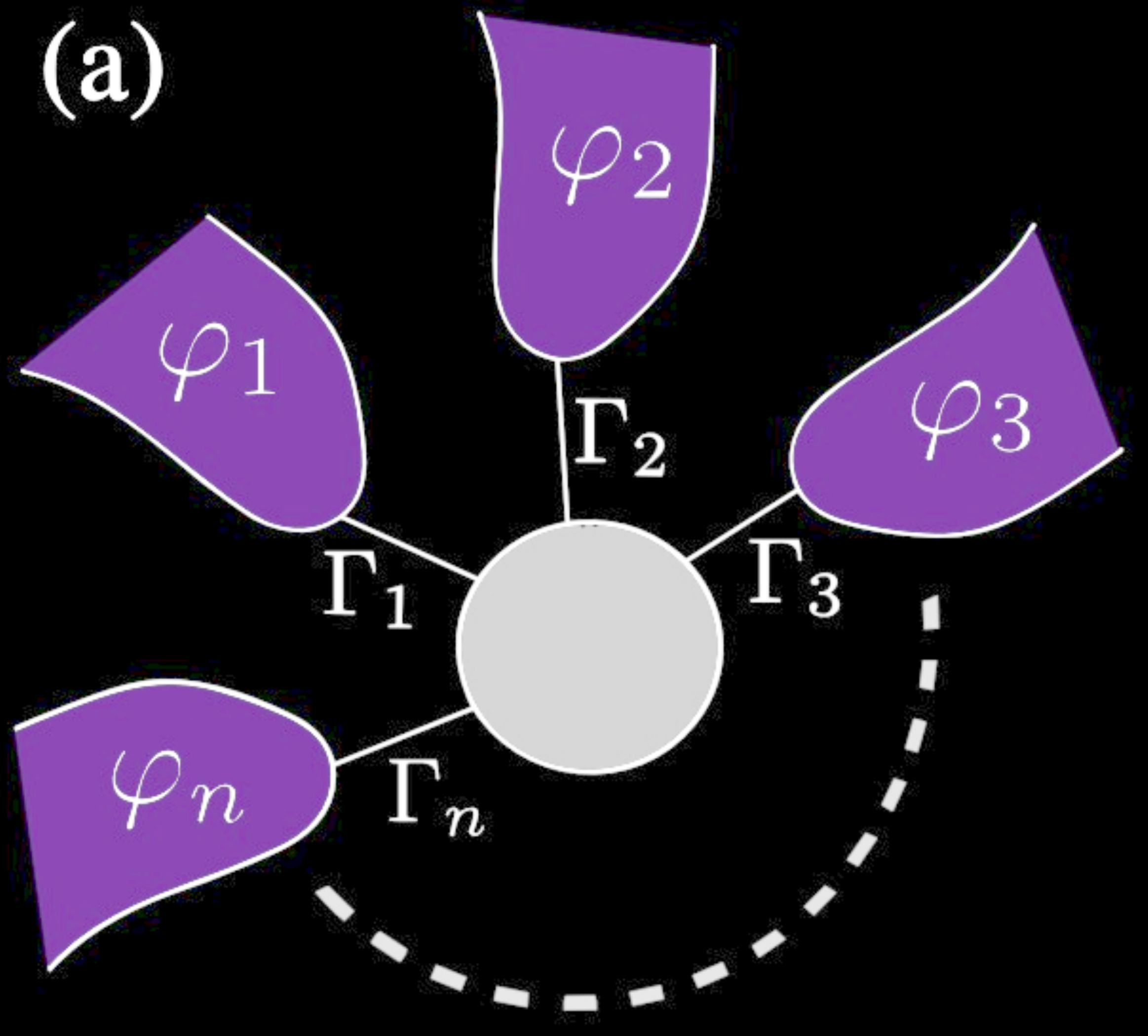
2-terminal Josephson junctions



1-to-1 correspondence between ABSs and transmission eigenvalues

$$E^i_{\text{ABS}} \leftrightarrow T_i$$

MTJJs



no 1-to-1 correspondence!

$$\cancel{E^i_{\text{ABS}} \leftrightarrow T_i}$$

Transmissions are Kramers degenerate (without TRS breaking in normal state)

Phys. Rev. B 90, 155450 (2014)

Multiterminal Josephson junctions (MTJJs)

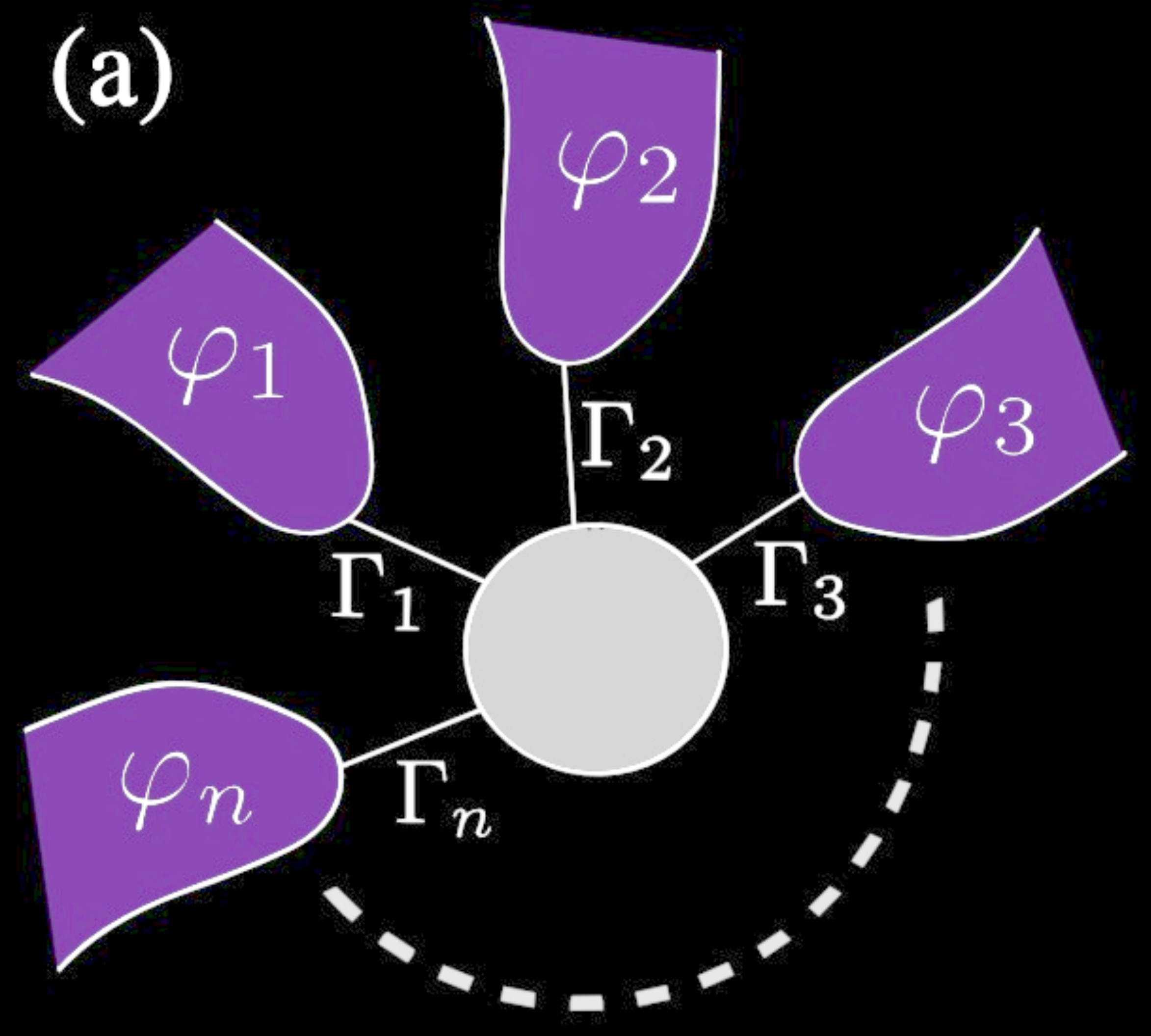
Problem concerning transmissions

Example: 4-terminal MTJJ

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}$$

6 Transmission amplitudes…

MTJJs



no 1-to-1 correspondence!

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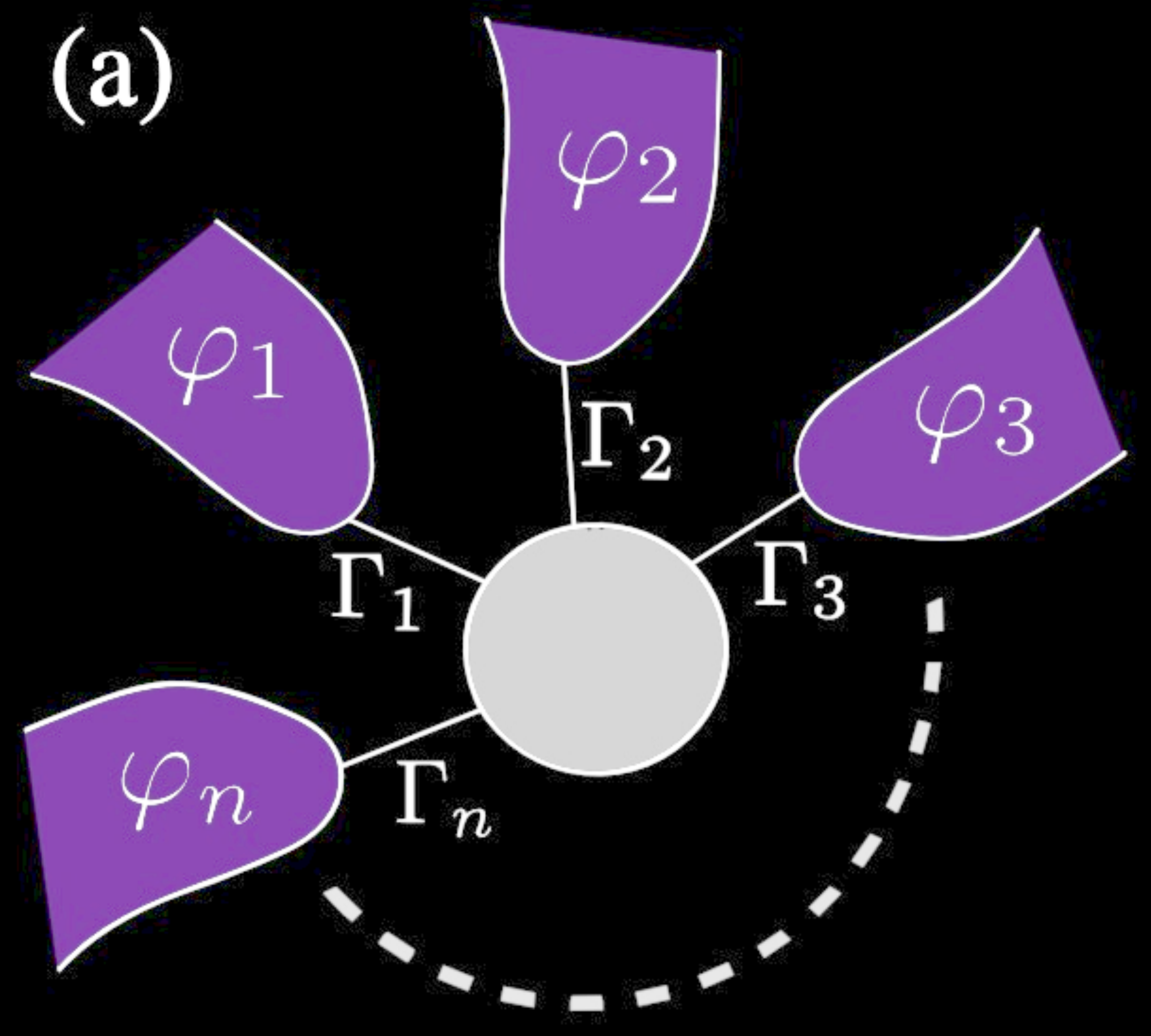
6 Transmission amplitudes…

Treat as effective 2-terminal junction?

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}$$

MTJJs



no 1-to-1 correspondence!

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Multiterminal Josephson junctions (MTJJs)

Problem concerning transmissions

Example: 4-terminal MTJJ

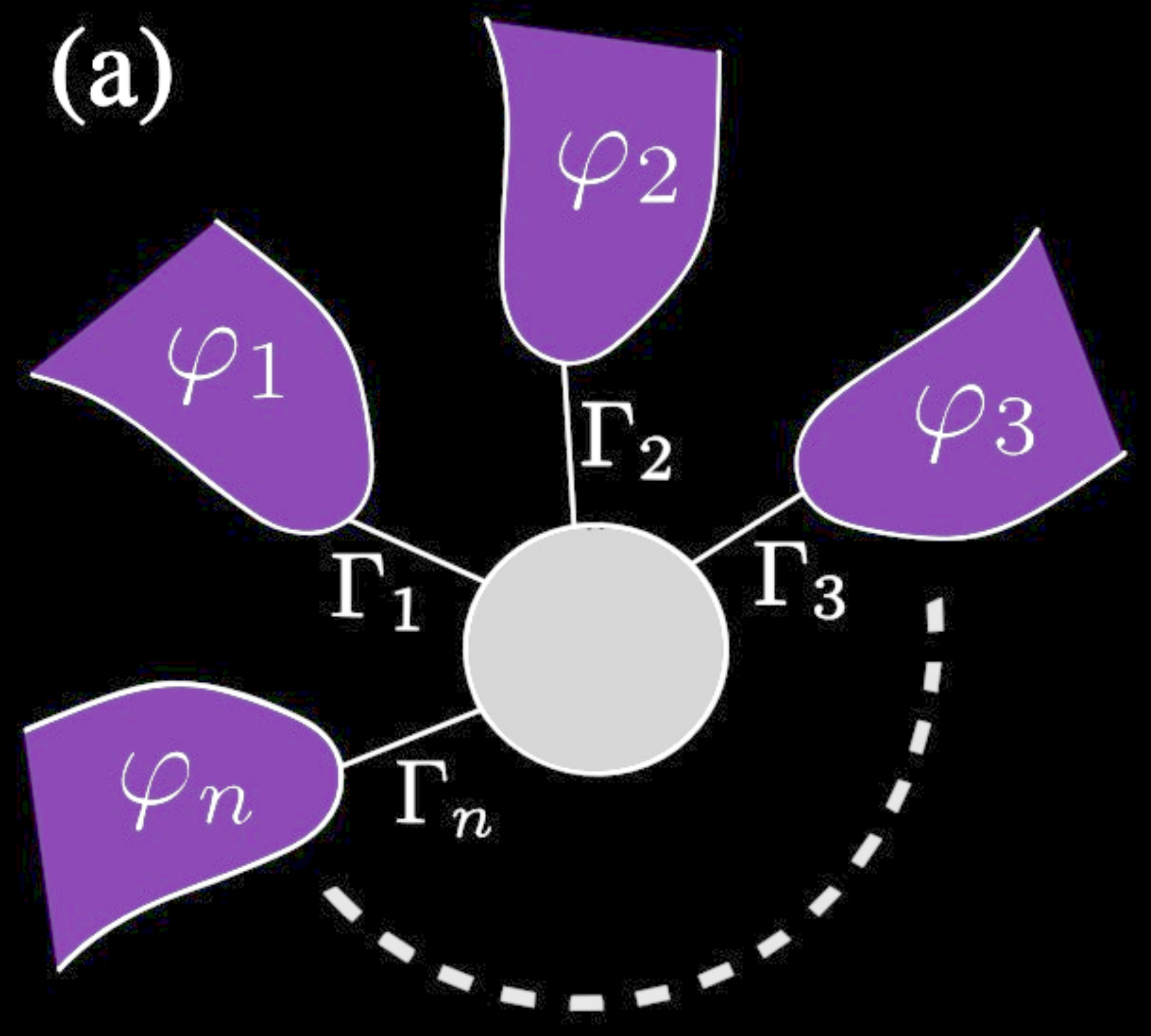
$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}$$

6 Transmission amplitudes…

Treat as effective 2-terminal junction?

$$S = \left(\begin{array}{cc|cc} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ \hline S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{array}\right) \quad \begin{array}{cc} r & t' \\ t & r' \end{array}$$

MTJJs



no 1-to-1 correspondence!

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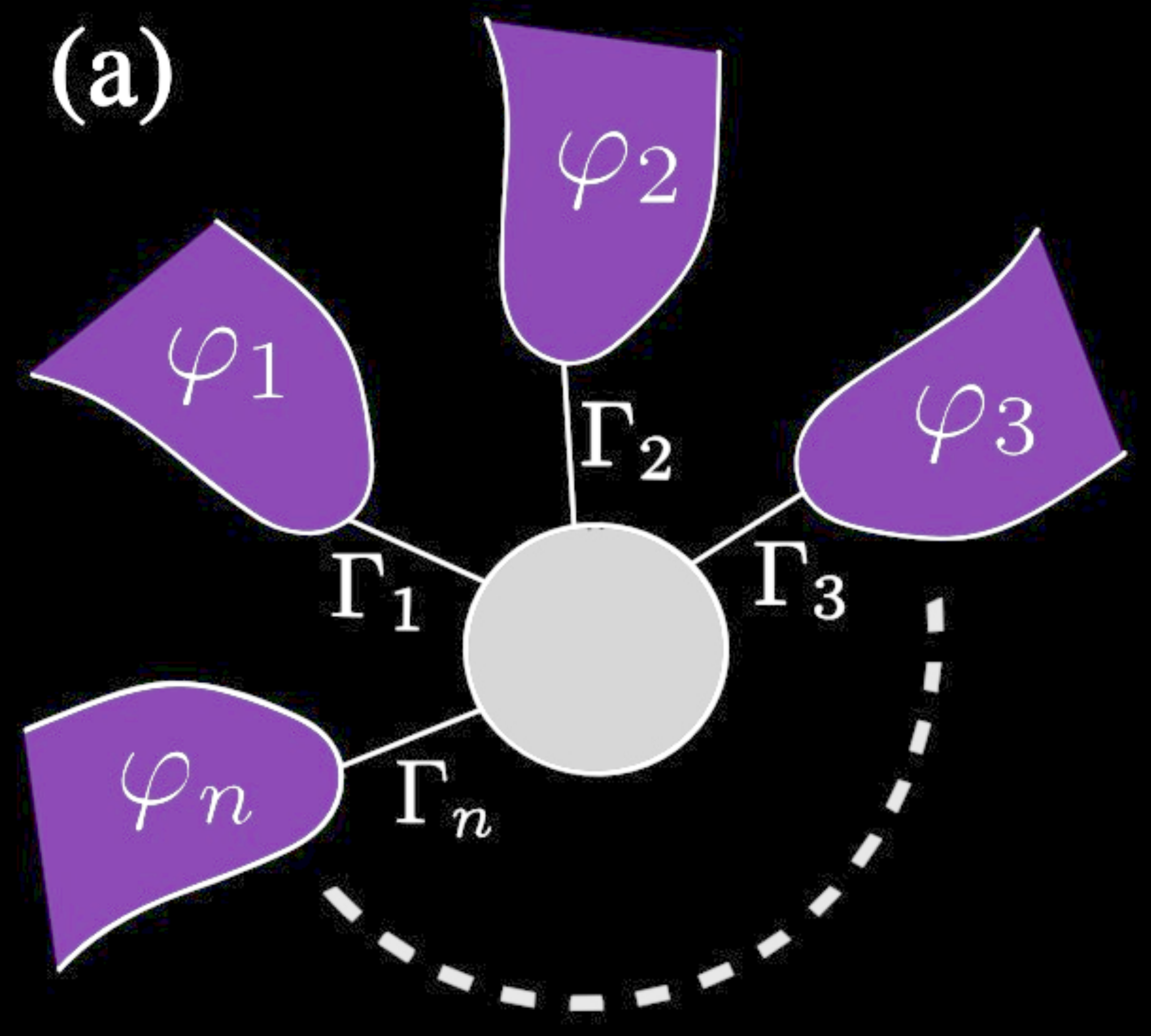
Multiterminal Josephson junctions (MTJJs)

Problem concerning transmissions

Example: 4-terminal MTJJ

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}$$ not block-diagonal…

MTJJs



Treat as effective 2-terminal junction?

$$S = \begin{pmatrix} \boxed{\begin{matrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{matrix}}_{\,r} & \boxed{\begin{matrix} S_{13} & S_{14} \\ S_{23} & S_{24} \end{matrix}}_{\,t'} \\ \boxed{\begin{matrix} S_{31} & S_{32} \\ S_{41} & S_{42} \end{matrix}}_{\,t} & \boxed{\begin{matrix} S_{33} & S_{34} \\ S_{43} & S_{44} \end{matrix}}_{\,r'} \end{pmatrix}$$

r, t', t, r'

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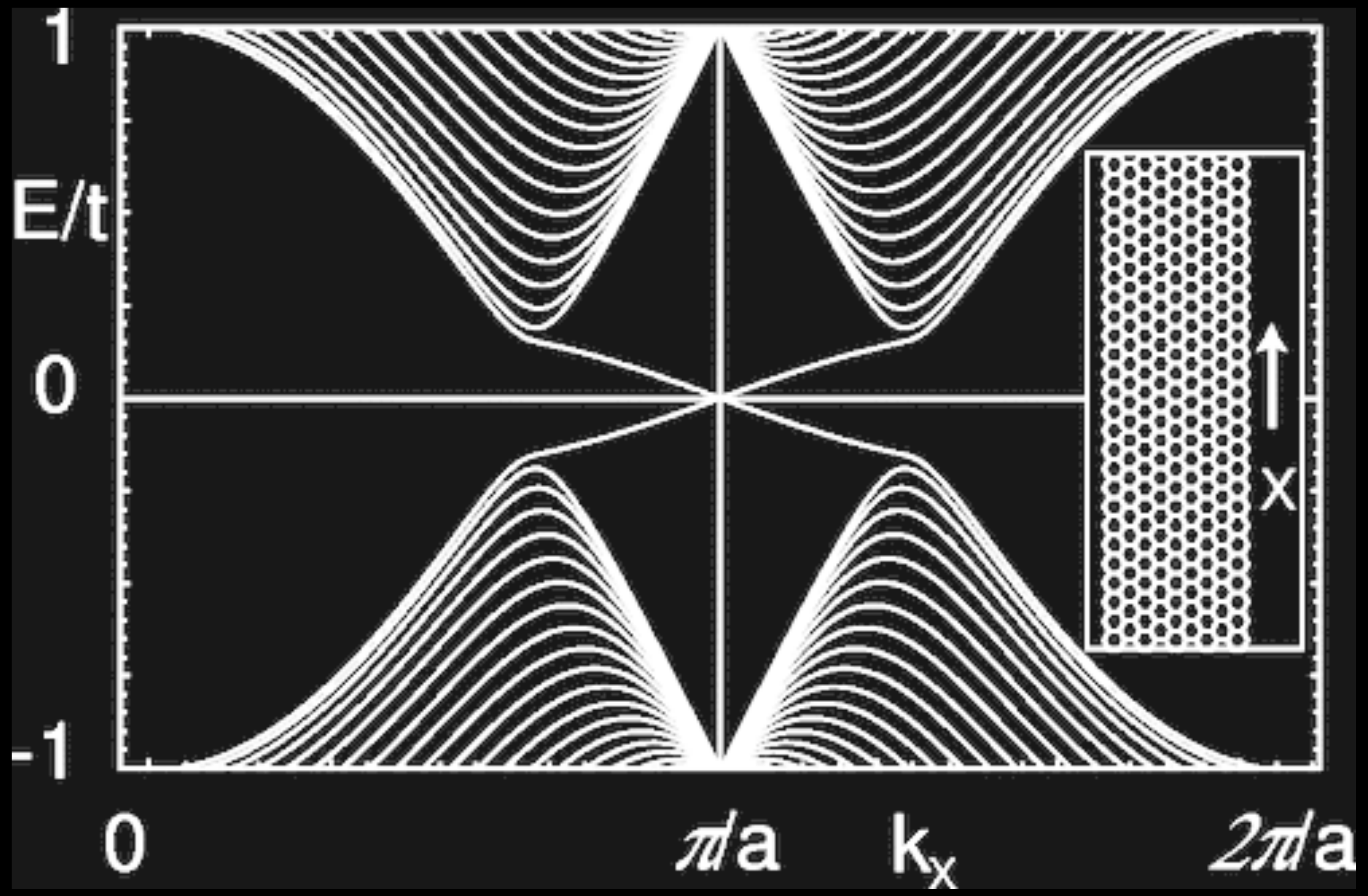
Why care about topology?

Topological Phases of Matter



C. L. Kane and E. J. Mele Phys. Rev. Lett. 95, 226801 (2005)
König, M. et al. Science 318, 766–770 (2007)

Fundamental Insights into Symmetry and Topology

Class	T	C	S	1	2	3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	0	0	0
BDI	1	1	1	$\mathbb{Z}$	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Altland and M. R. Zirnbauer, Phys. Rev. B 55, 1142 (1997)
Andreas P. Schnyder, et. al., Phys. Rev. B 78, 195125 (2008)

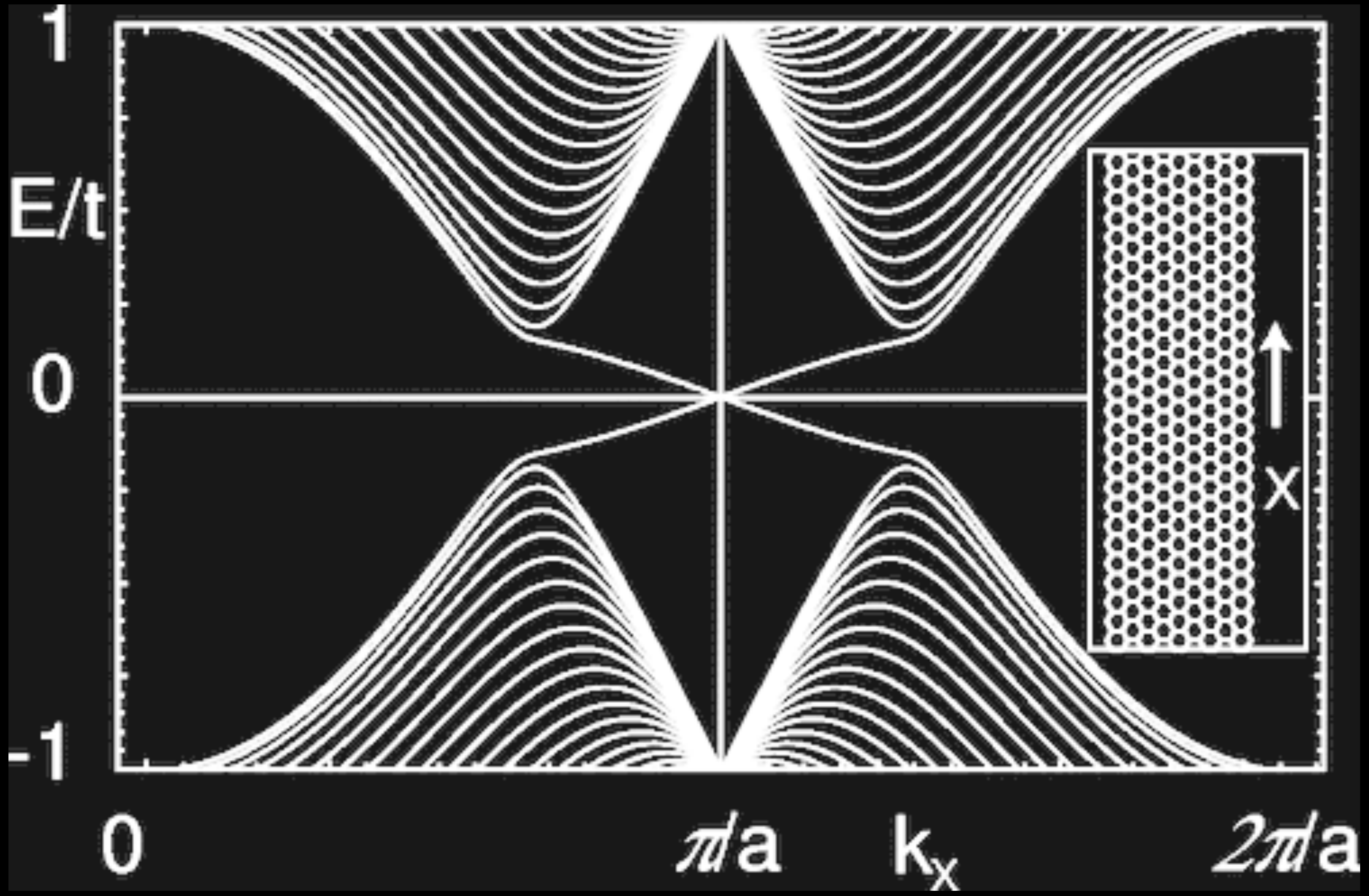
Stability Against Perturbations

Applications in Quantum Computing and Electronics

...

Why care about topology?

Topological Phases of Matter



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Stability Against Perturbations

Applications in Quantum Computing and Electronics

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How to define topology (in MTJJs)?

Effective Hamiltonian with eigenenergies and eigenvectors

$$H_{\text{eff}} = \overrightarrow{d} \cdot \overrightarrow{\sigma} / \;\; H_{\text{eff}} = -G_{\text{C}}^{-1}(E=0)$$

$$E_i / |v_i\rangle$$

Define the Chern number of a band E_i like
for topological insulators

$$C_i^{jk} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} B_i^{jk}(\overrightarrow{k})dk_j dk_k \qquad C_i^{jk} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} B_i^{jk}(\overrightarrow{\phi})d\phi_j d\phi_k$$

Berry curvature

$$B_i^{jk} = -2\mathrm{Im}\langle \partial_{k_j} v_i | \partial_{k_k} v_i \rangle \qquad B_i^{jk} = -2\mathrm{Im}\langle \partial_{\phi_j} v_i | \partial_{\phi_k} v_i \rangle$$

Example

Example: Qi-Wu-Zhang Modell

$$H_{\text{eff}} = \vec{d} \cdot \vec{\sigma} \qquad \vec{d} = \begin{pmatrix} \sin k_x \\ \sin k_y \\ m + \cos k_x + \cos k_y \end{pmatrix} \qquad \vec{e} = \vec{d} / |\vec{d}|$$

Topological phase transition for

$E_\pm = |\vec{d}| = 0$:

Weyl/Dirac point

Chern number as function of m

$$C = \begin{cases} 1, & 2 > m > 0 \\ -1, & 0 > m > -2 \\ 0, & \text{else} \end{cases}$$

Example

Phys. Rev. B 74, 085308 (2006)

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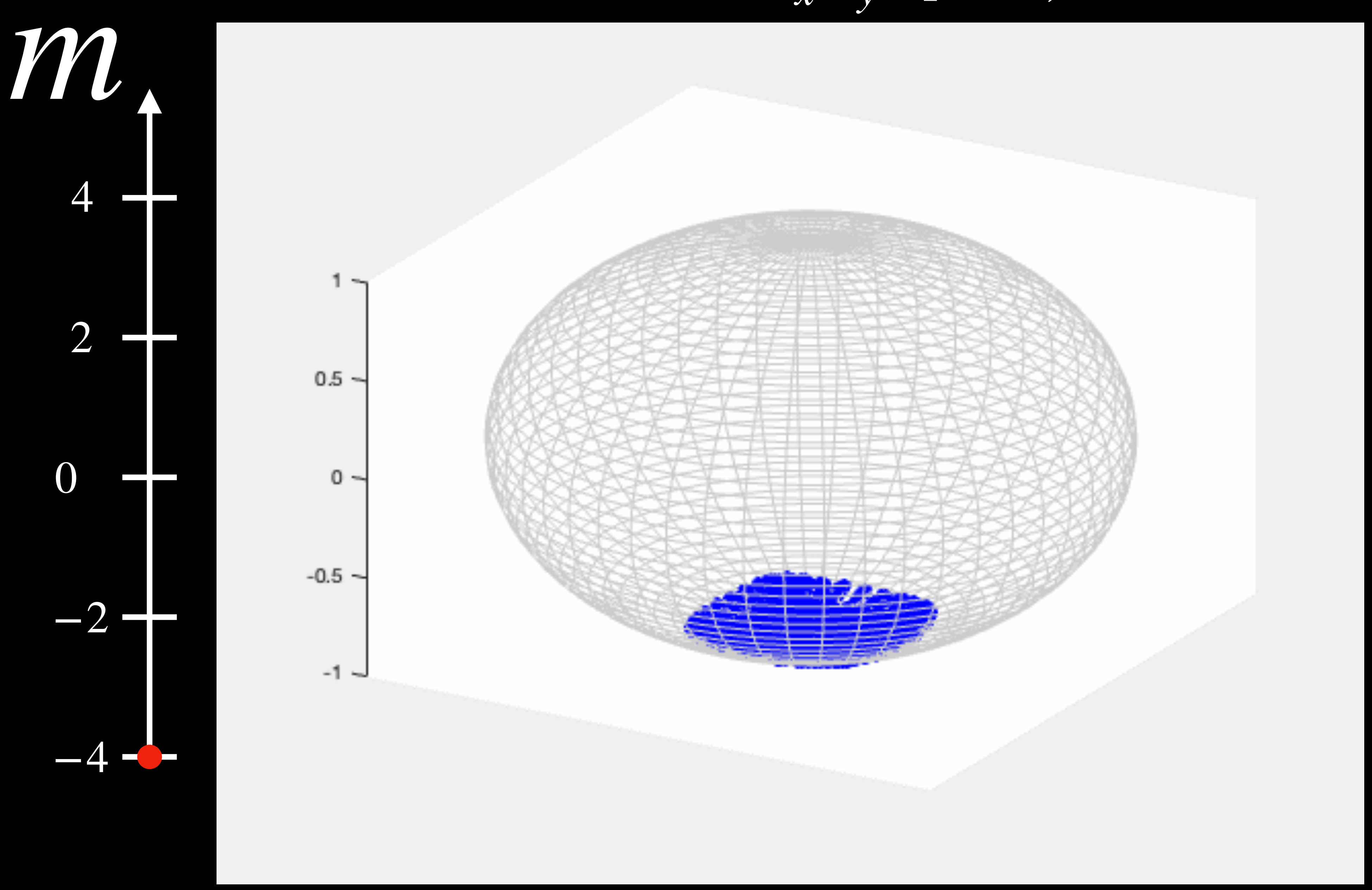
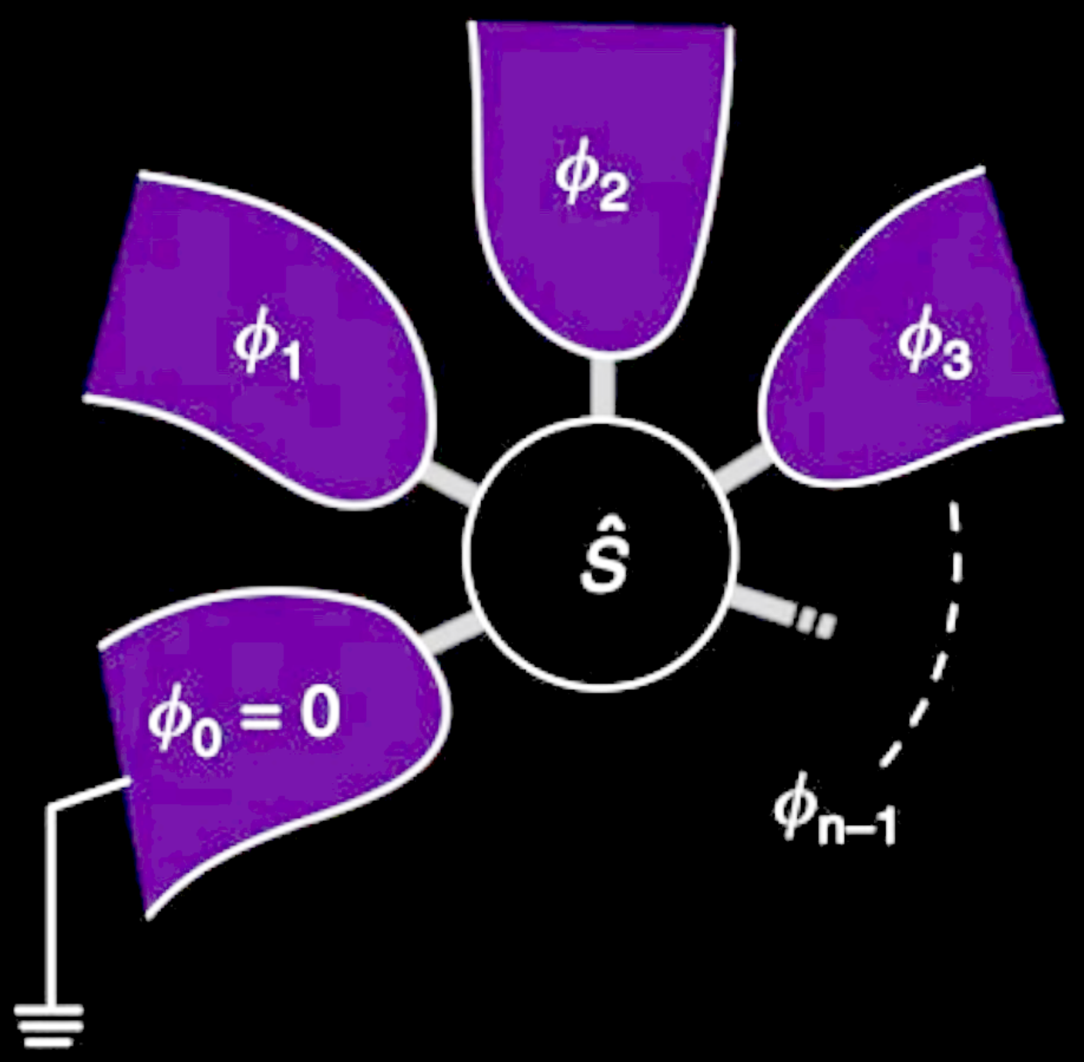
$\vec{e}(k_x, k_y)|_{k_x, k_y \in [0, 2\pi)}$



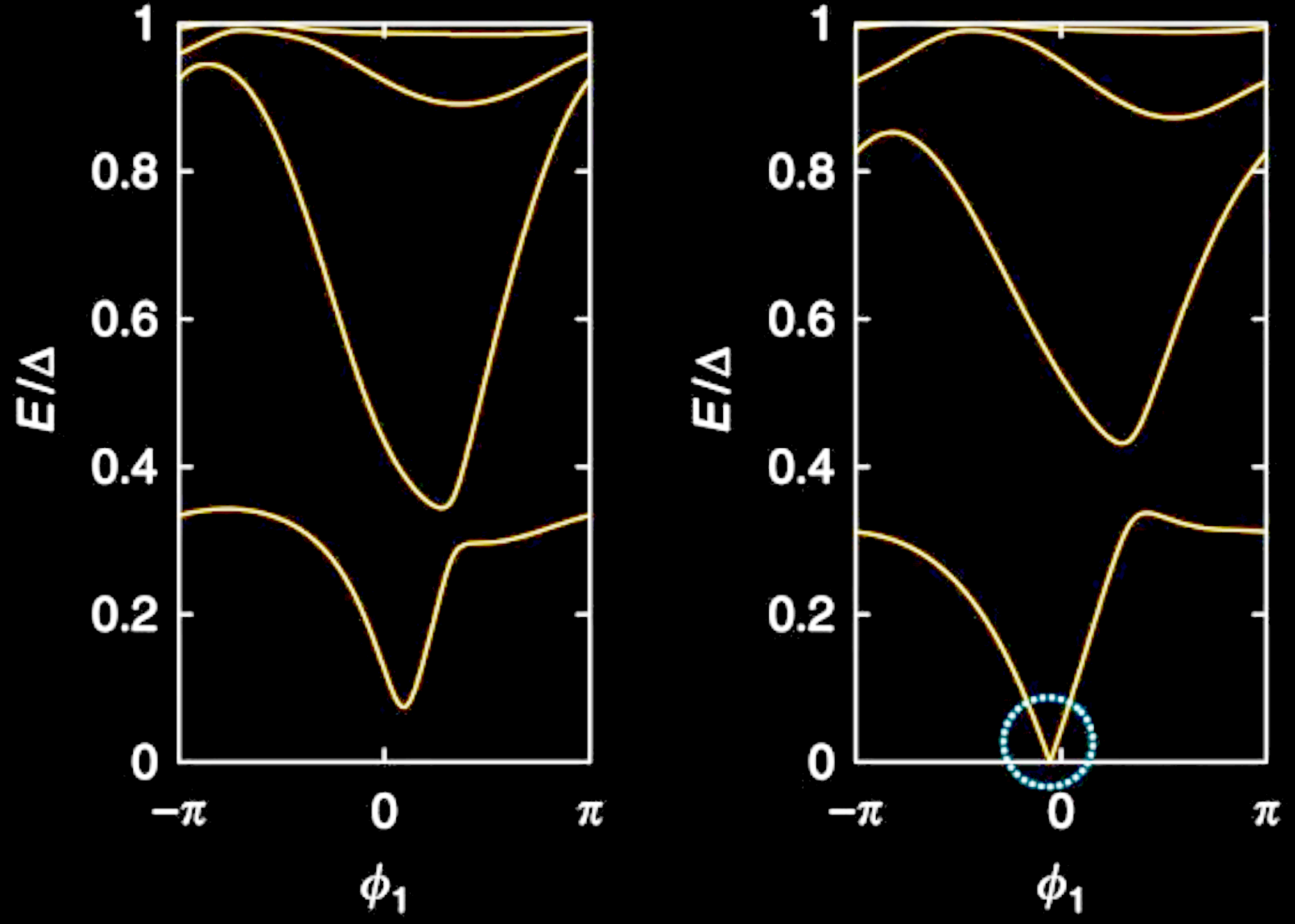
Topology in Multiterminal Josephson junctions (MTJJs)

MTJJ



Andreev bound states

Weyl Node



Superconducting phases function as pseudo momenta

R.-P. Riwar, et. al., Nature Commun. 7, 1 (2016)
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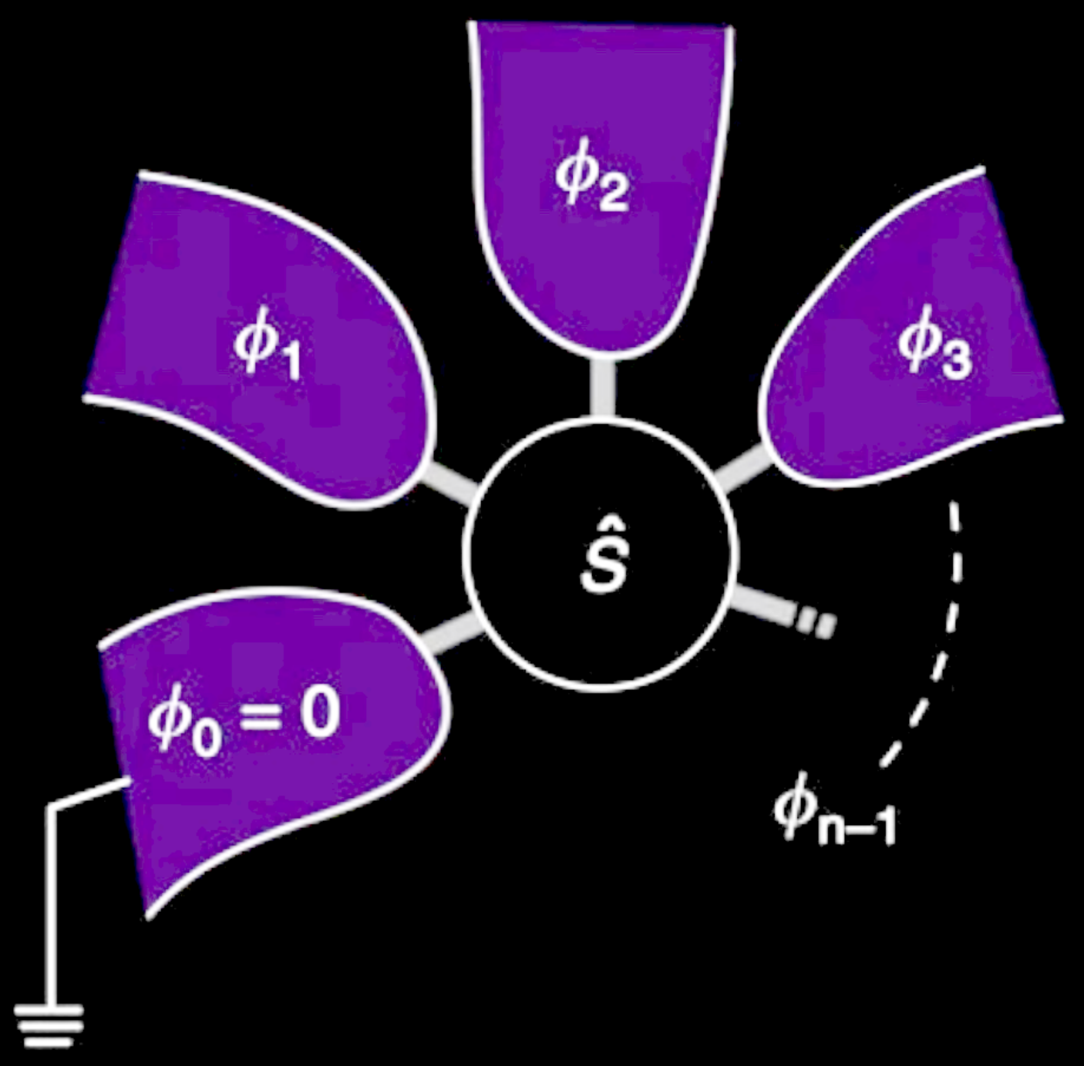

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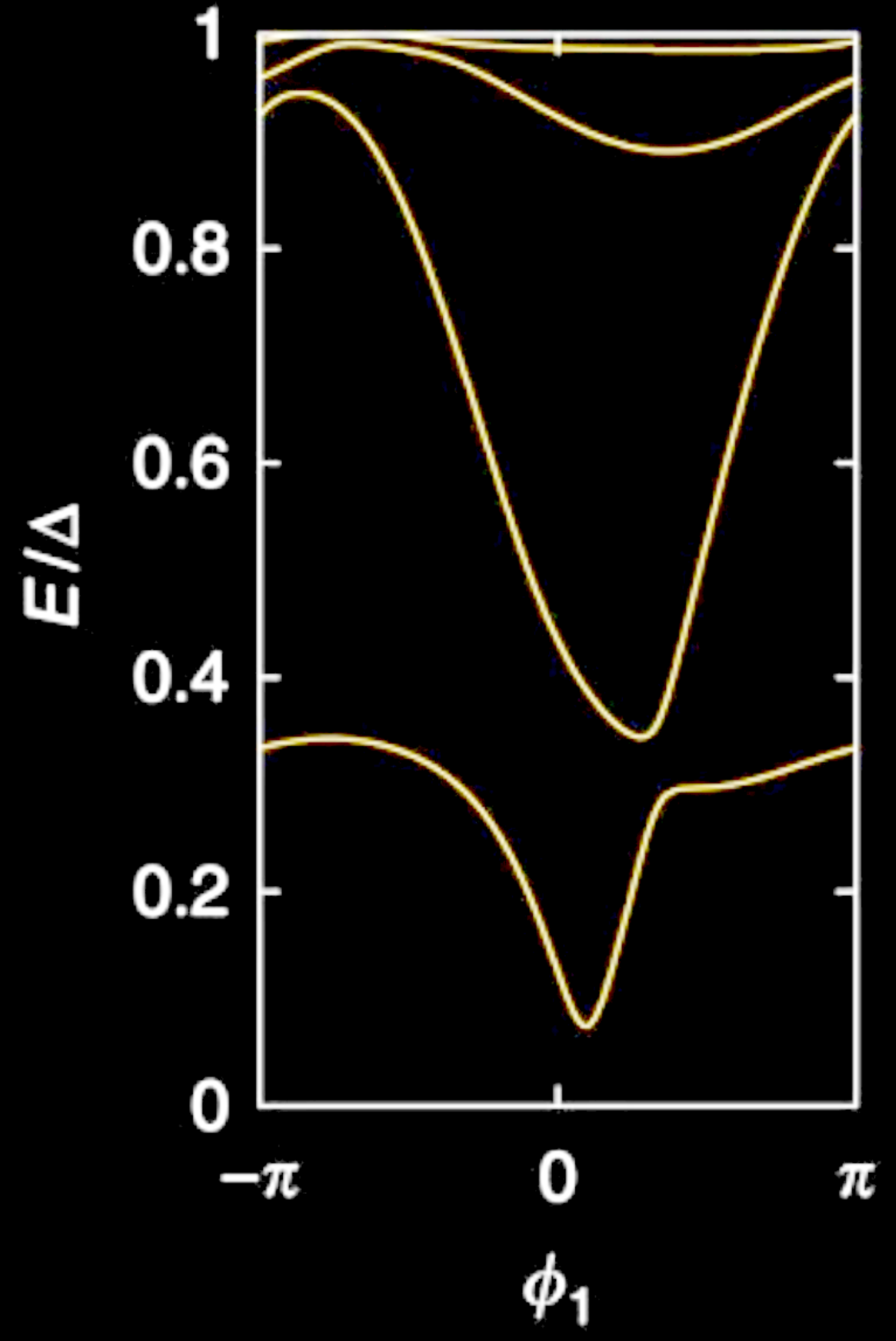

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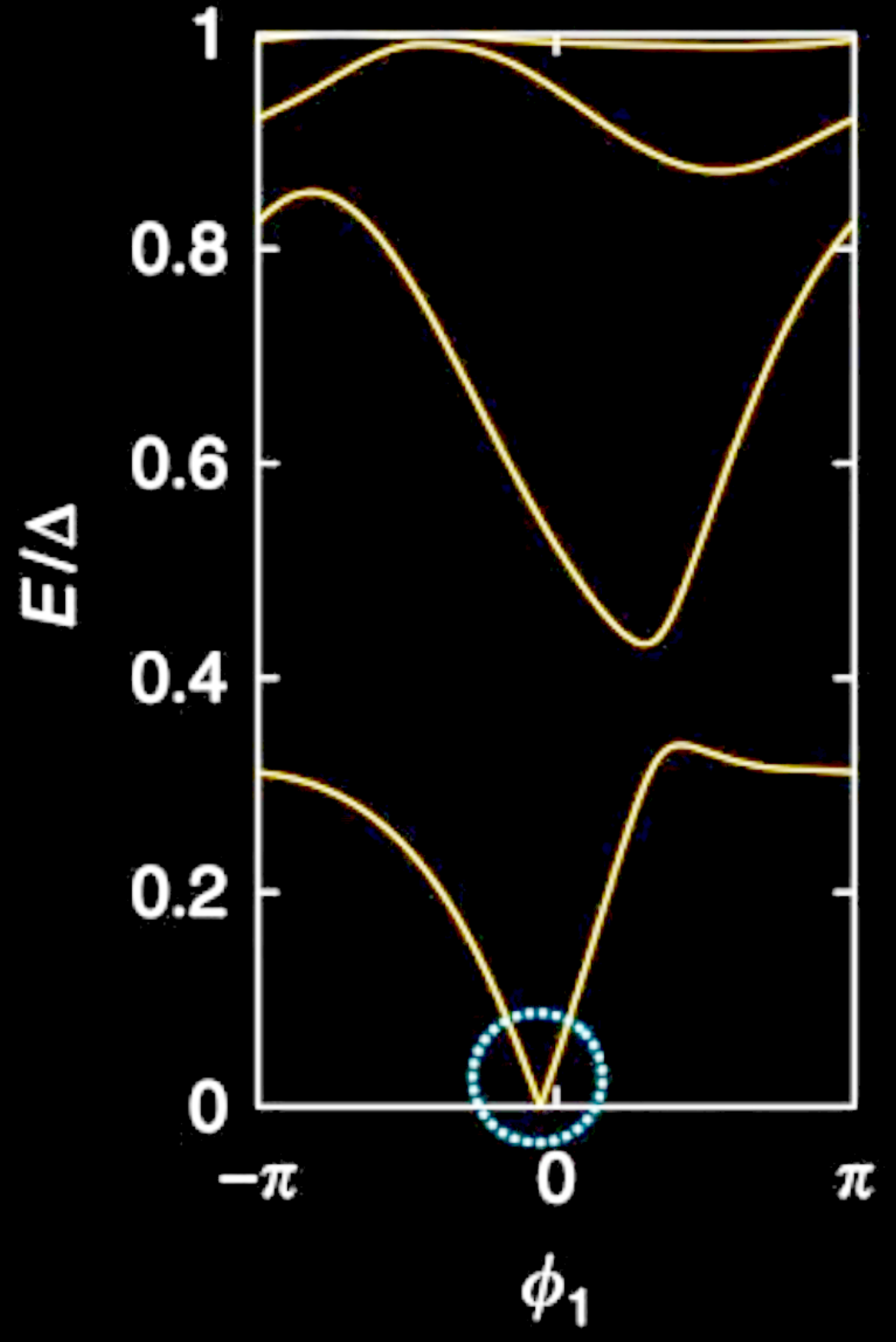
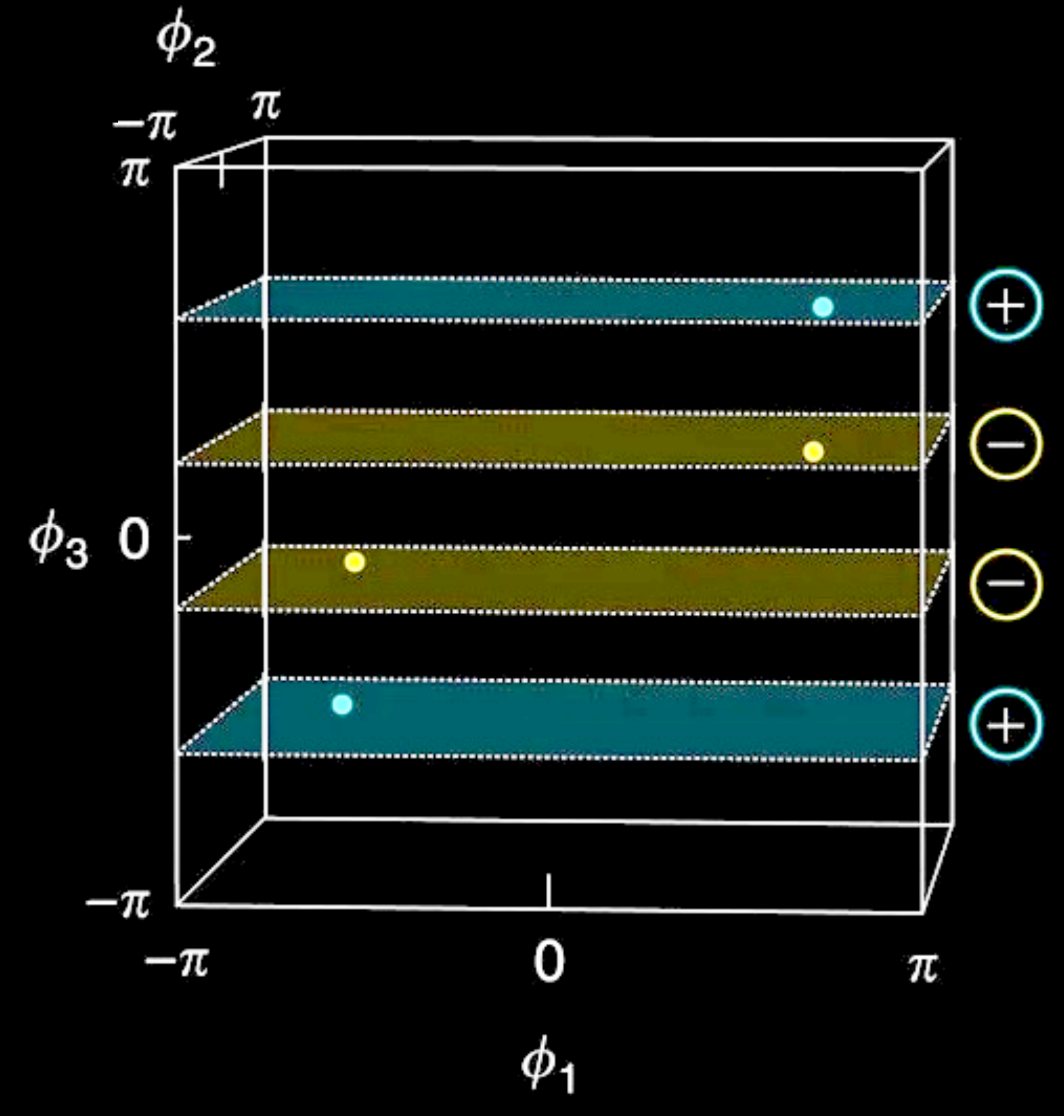
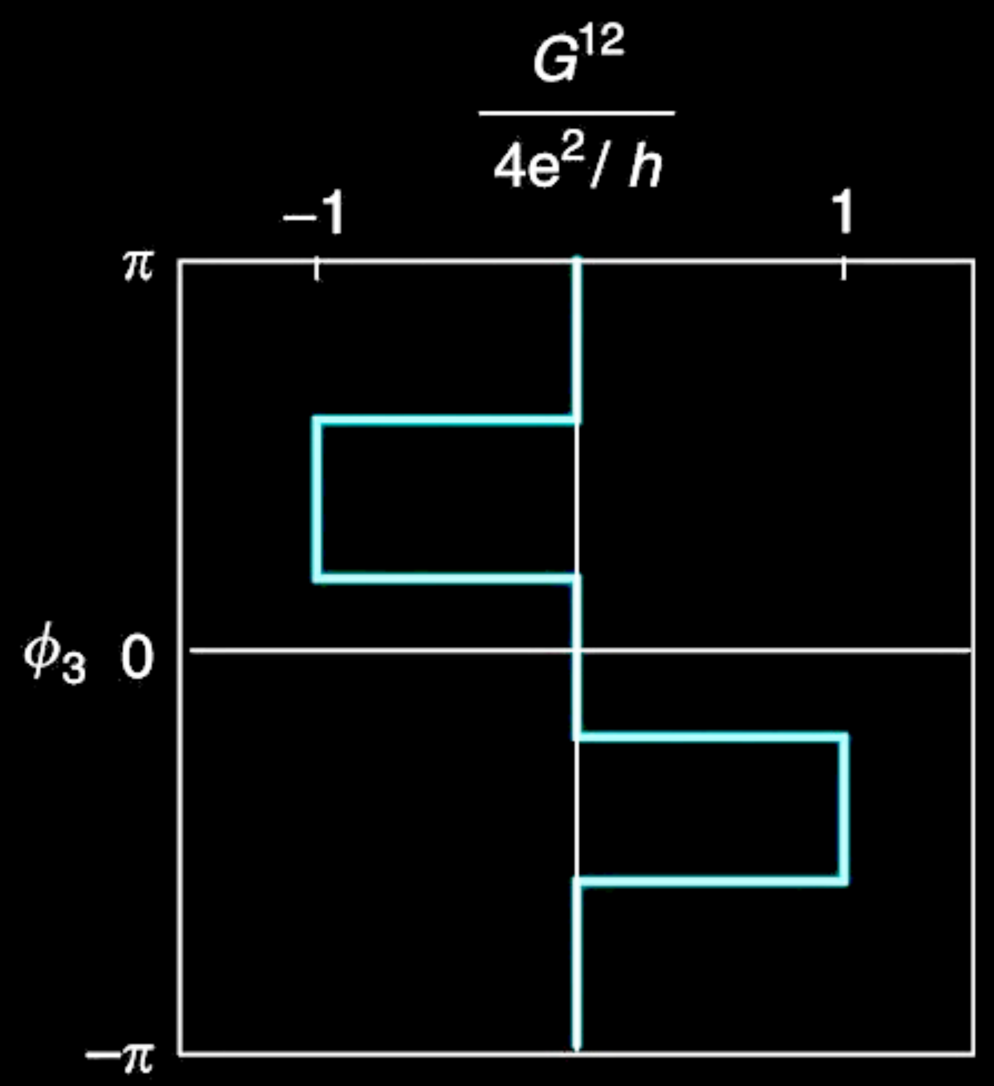


Weyl Nodes in phase space



Transconductance (Chern number)



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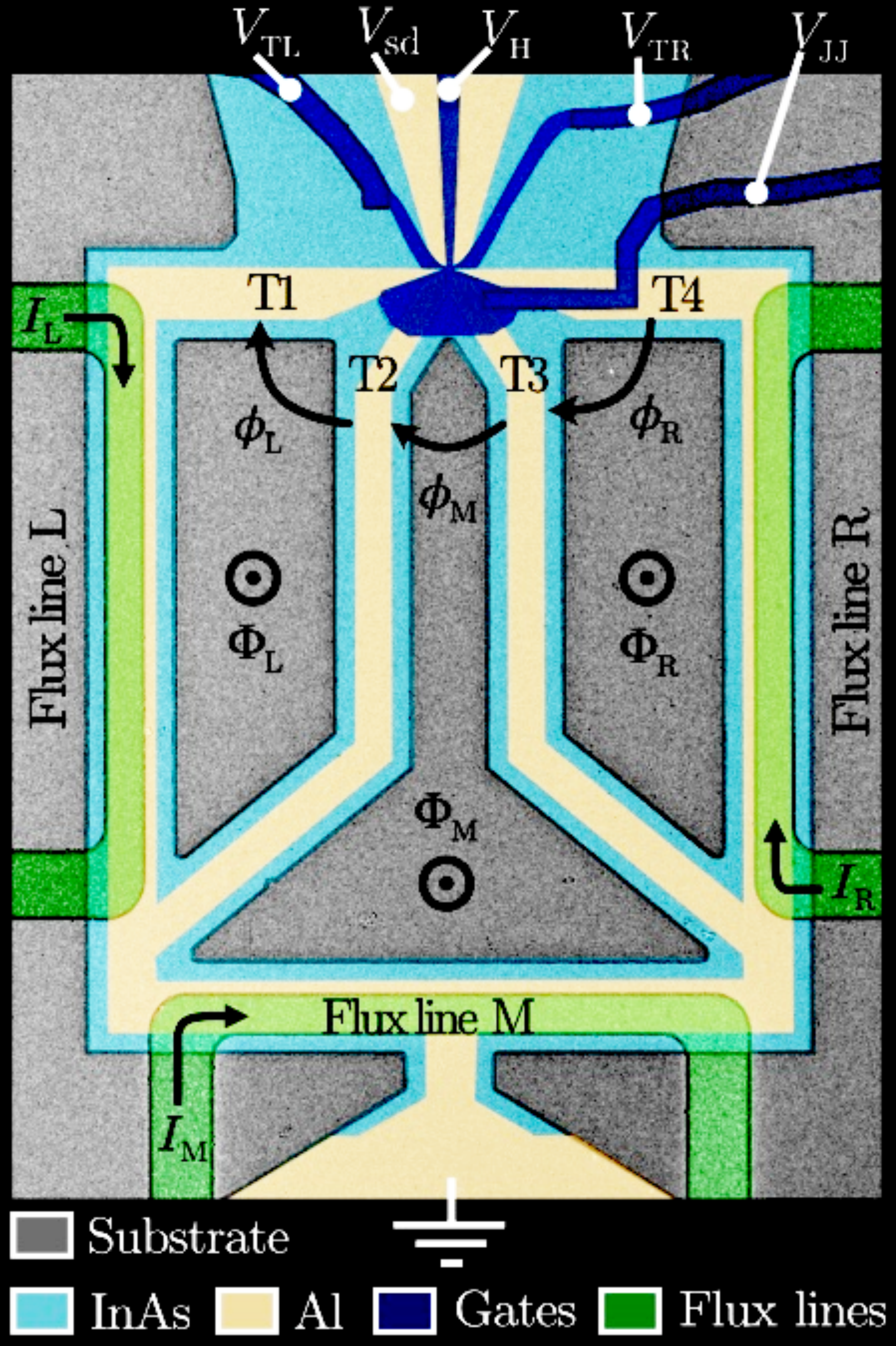
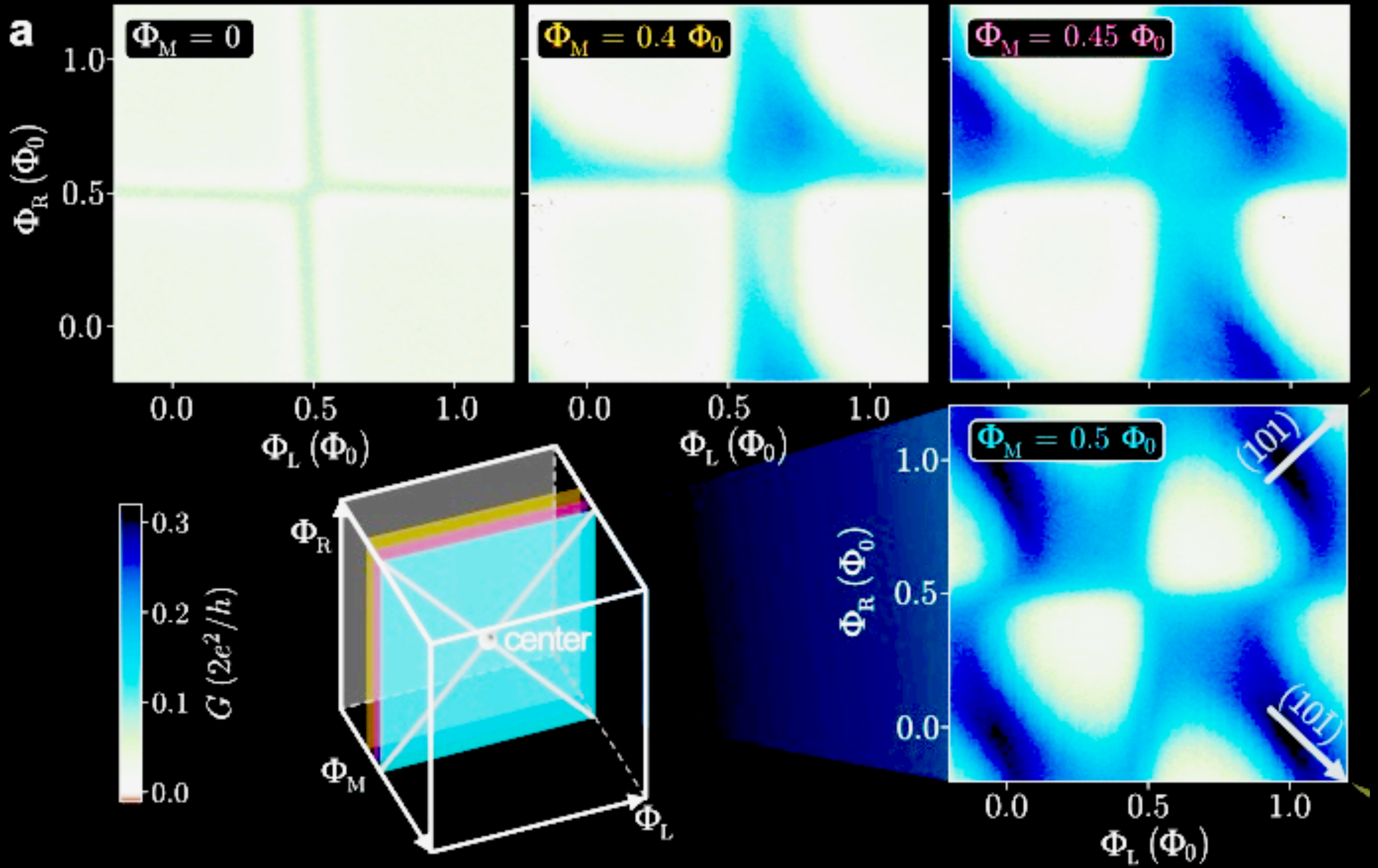
Experiments on MTJJs: Topology?

Topography of
4-terminal MTJJ



Conductance measurement at $E = 0$



T. Antonelli, et. al., arXiv:2501.07982 (2025).

Experiments on MTJJs: Topology?

Topography of
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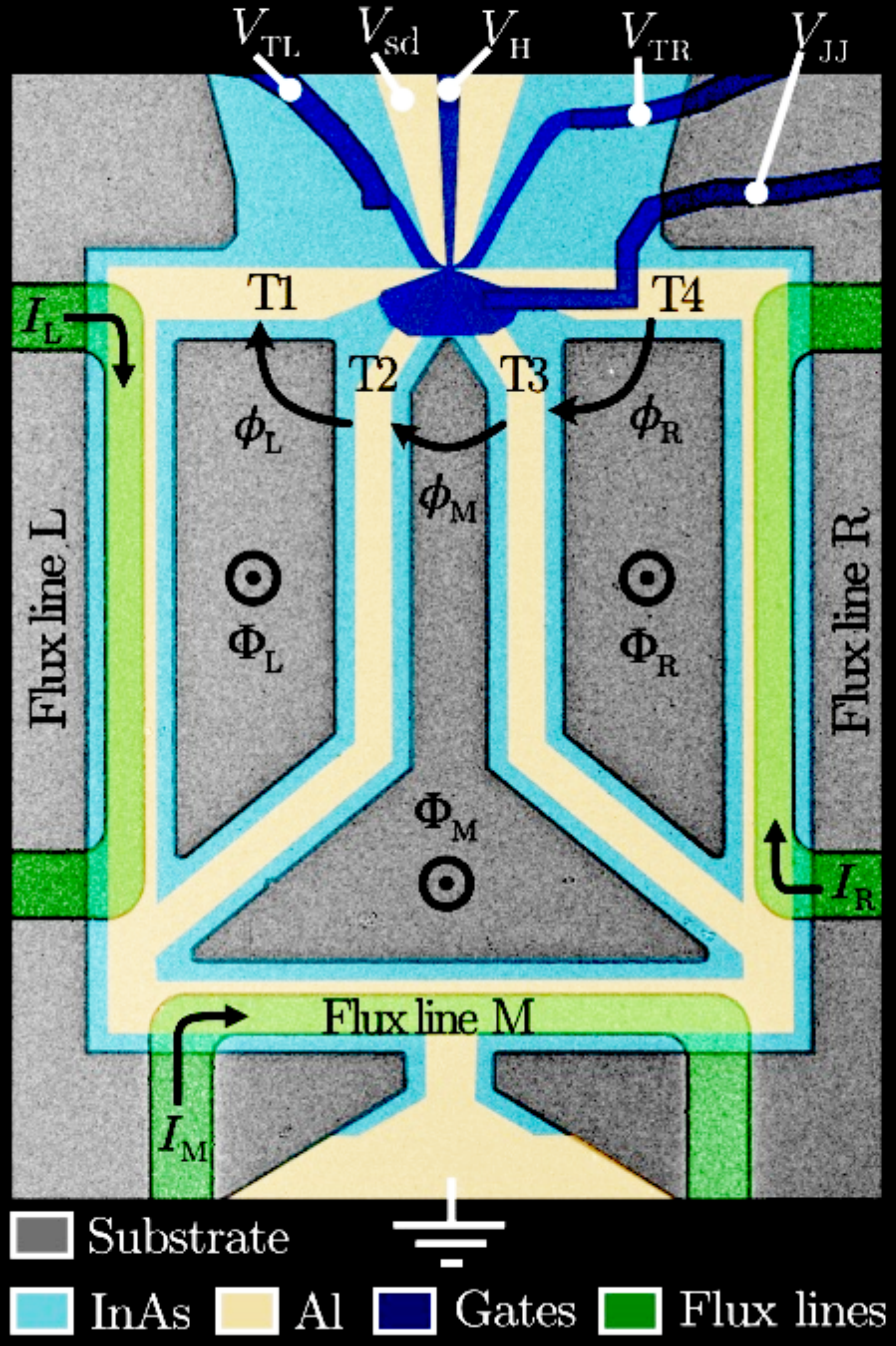
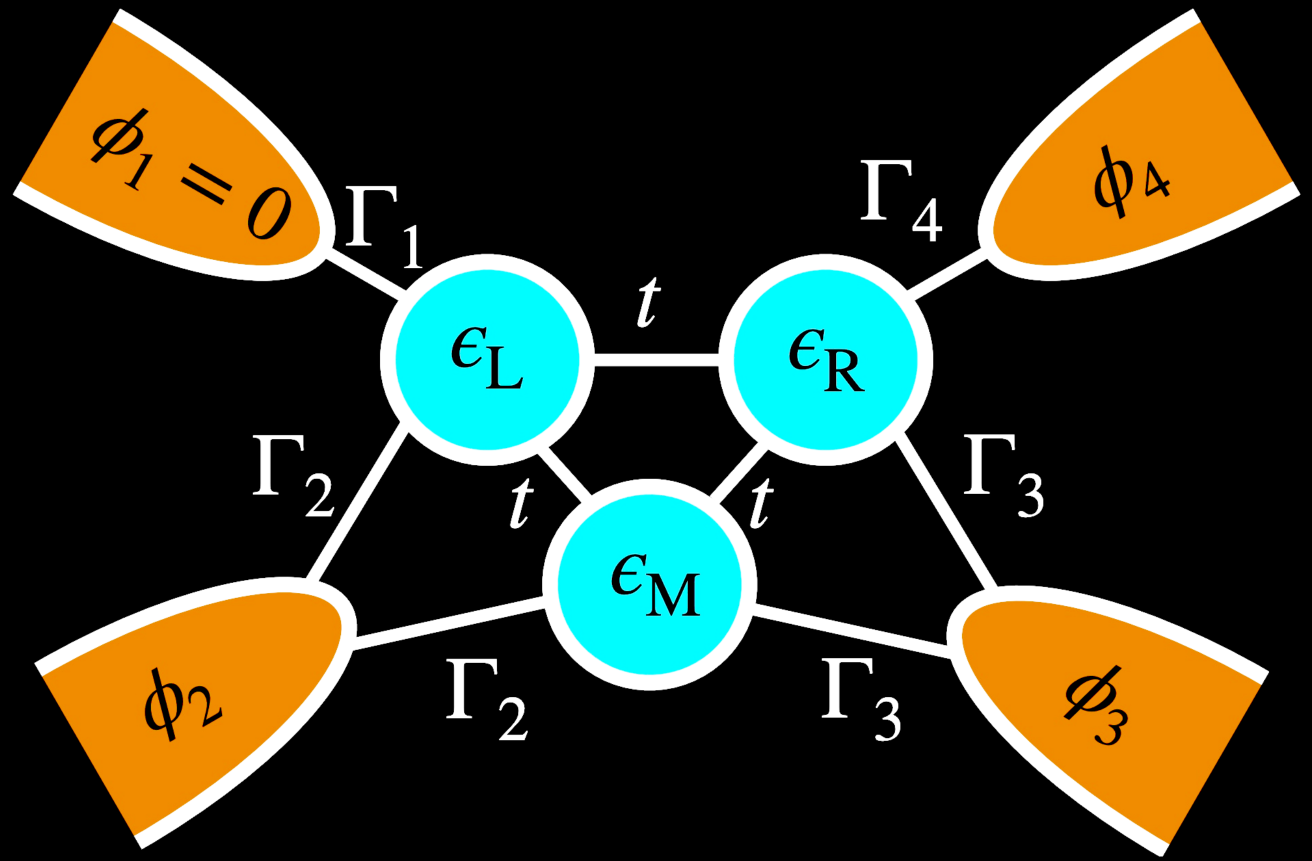


Three state Andreev molecule model



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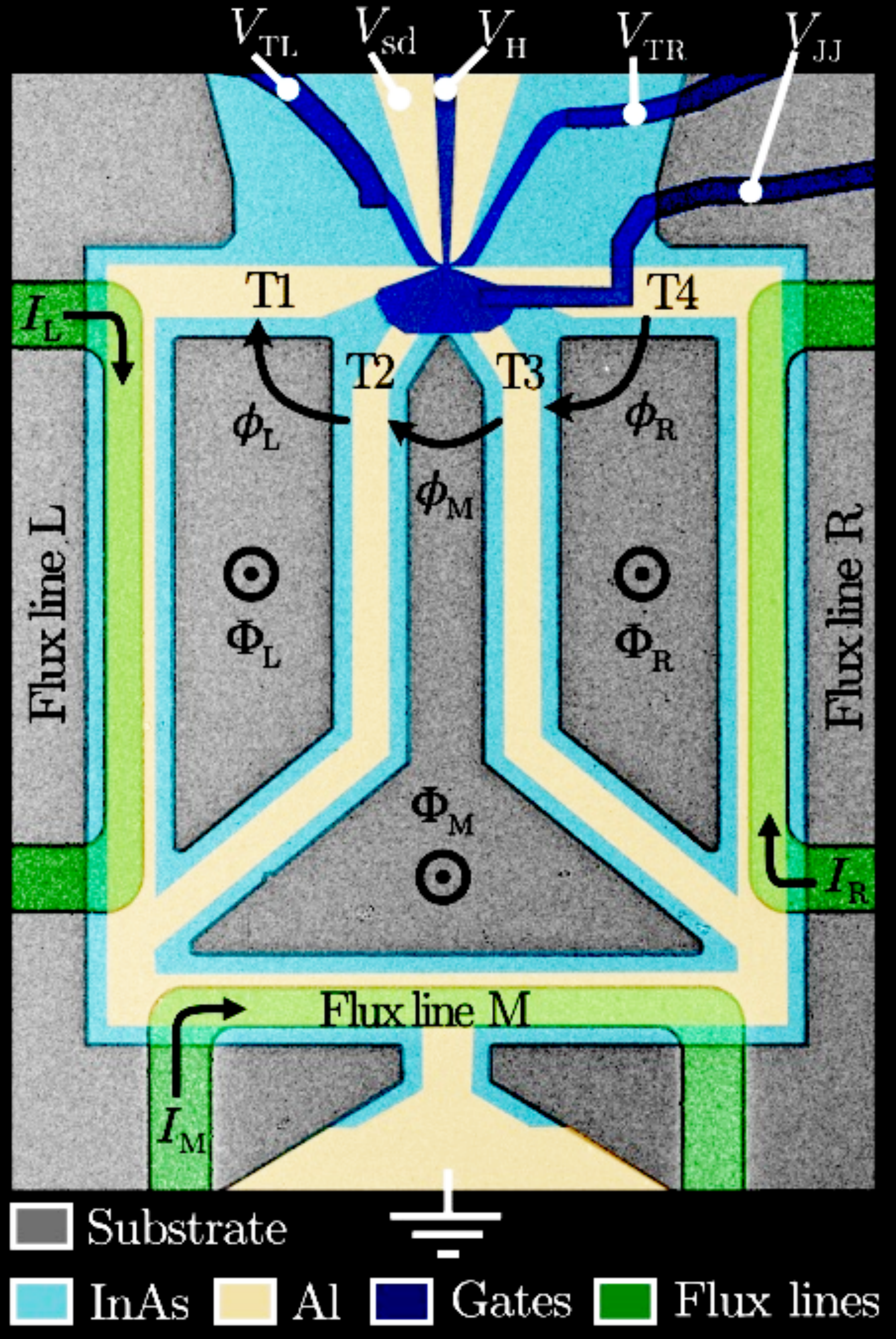
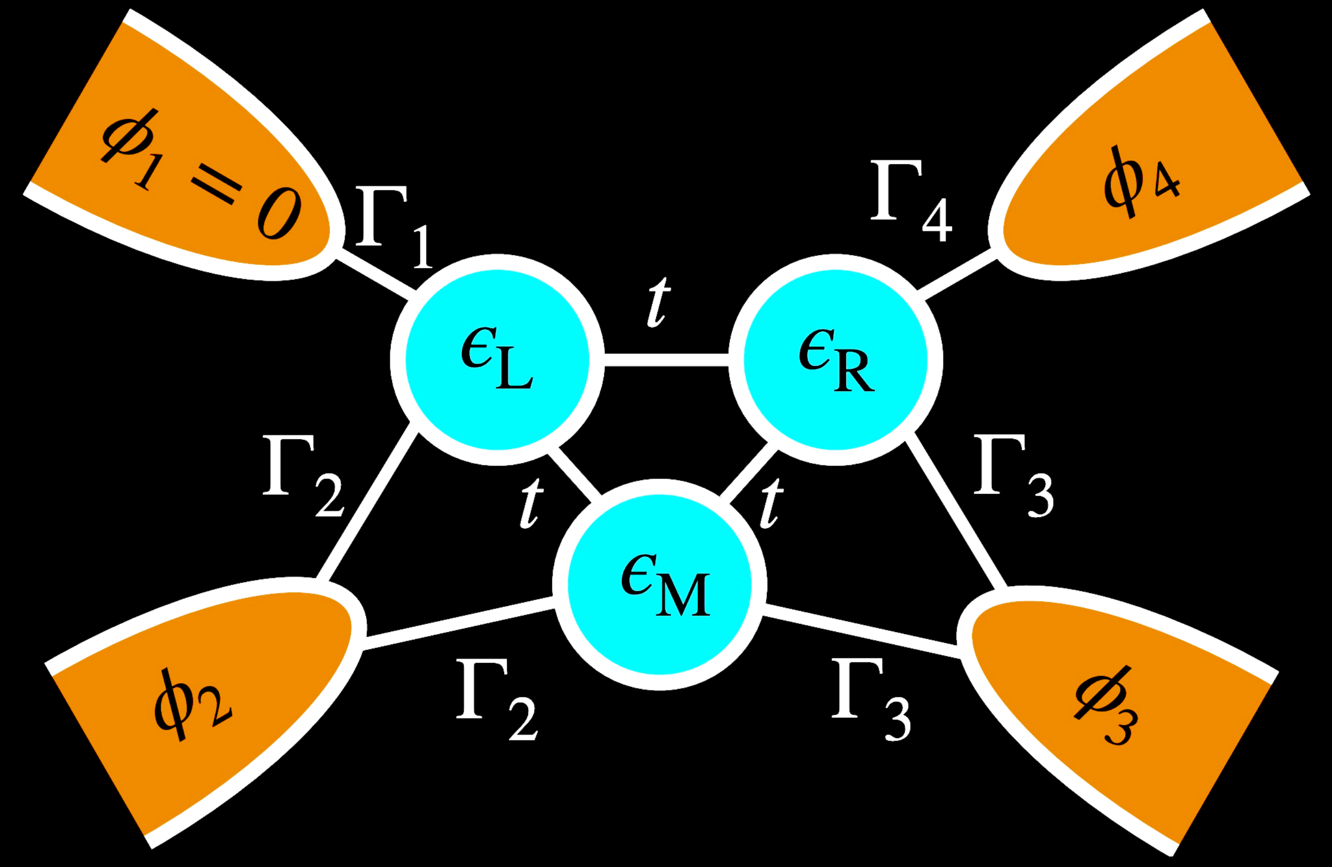
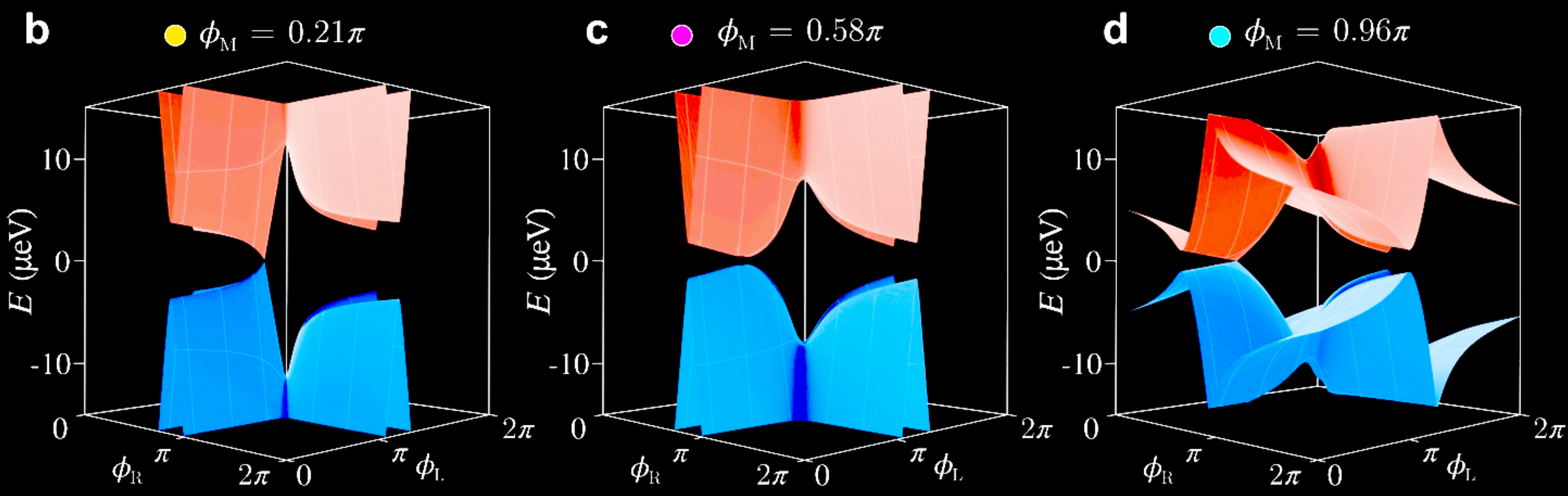


Three state Andreev molecule model



Model predicts Weyl points and non-trivial topology



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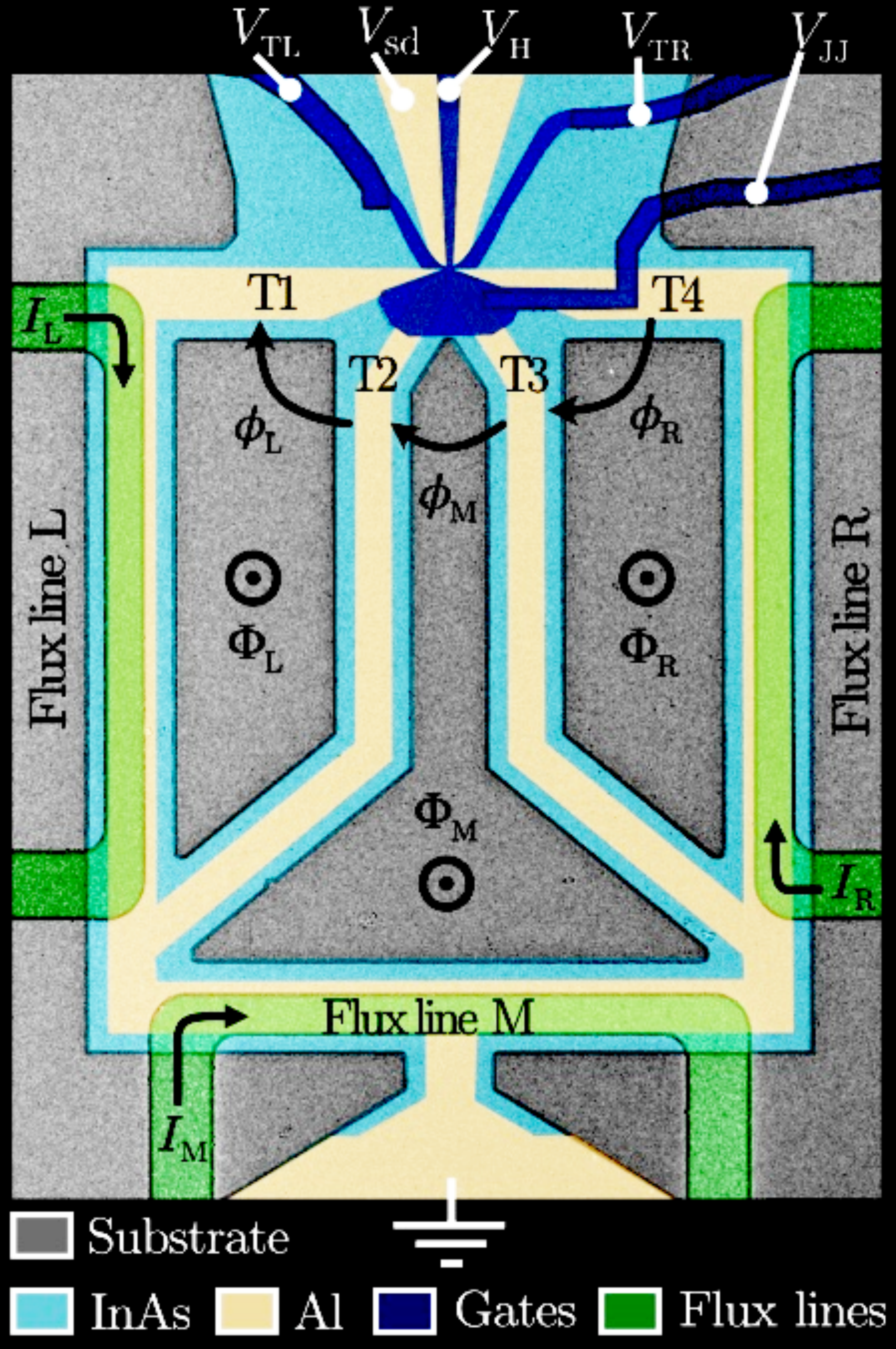
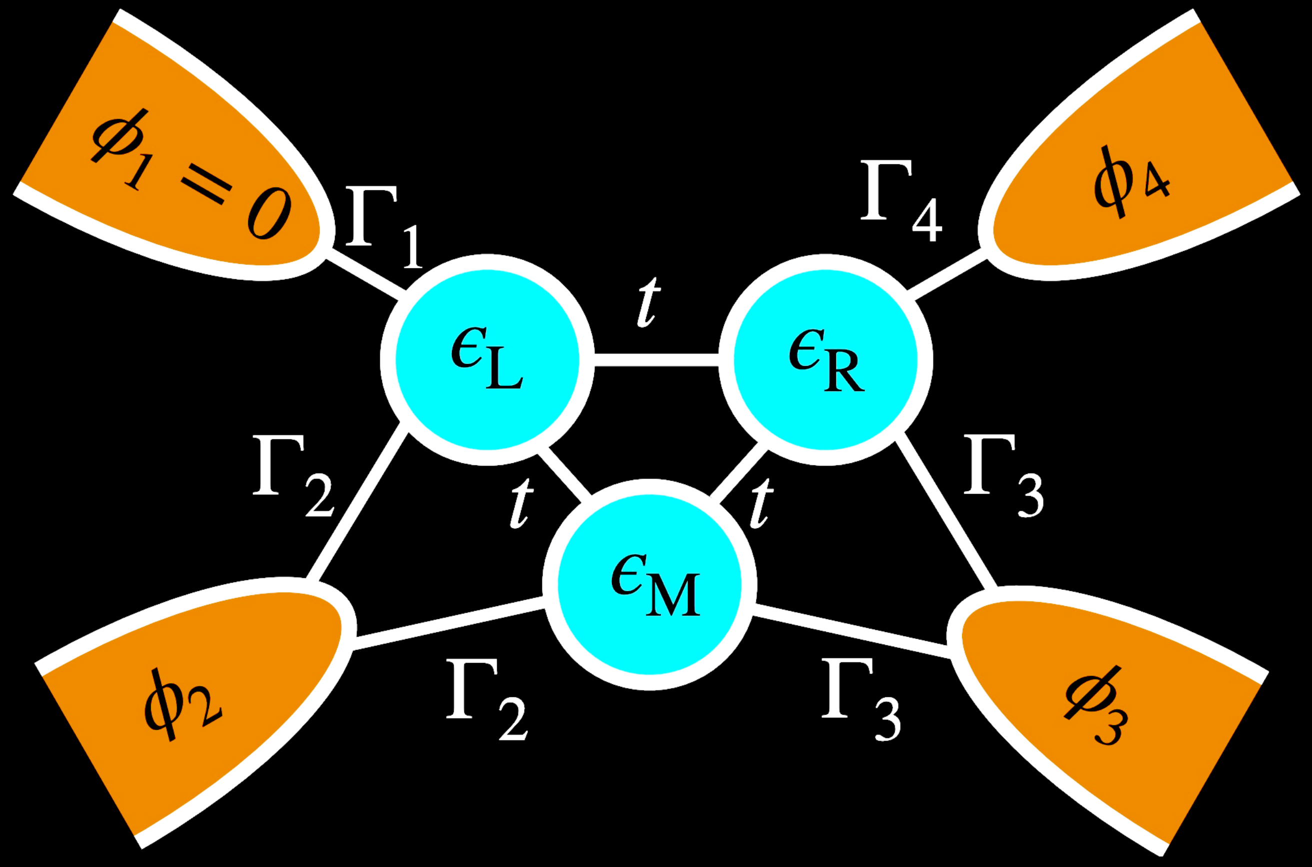
Experiments on MTJJs: Topology?

Topography of 4-terminal MTJJ



Three state Andreev molecule model



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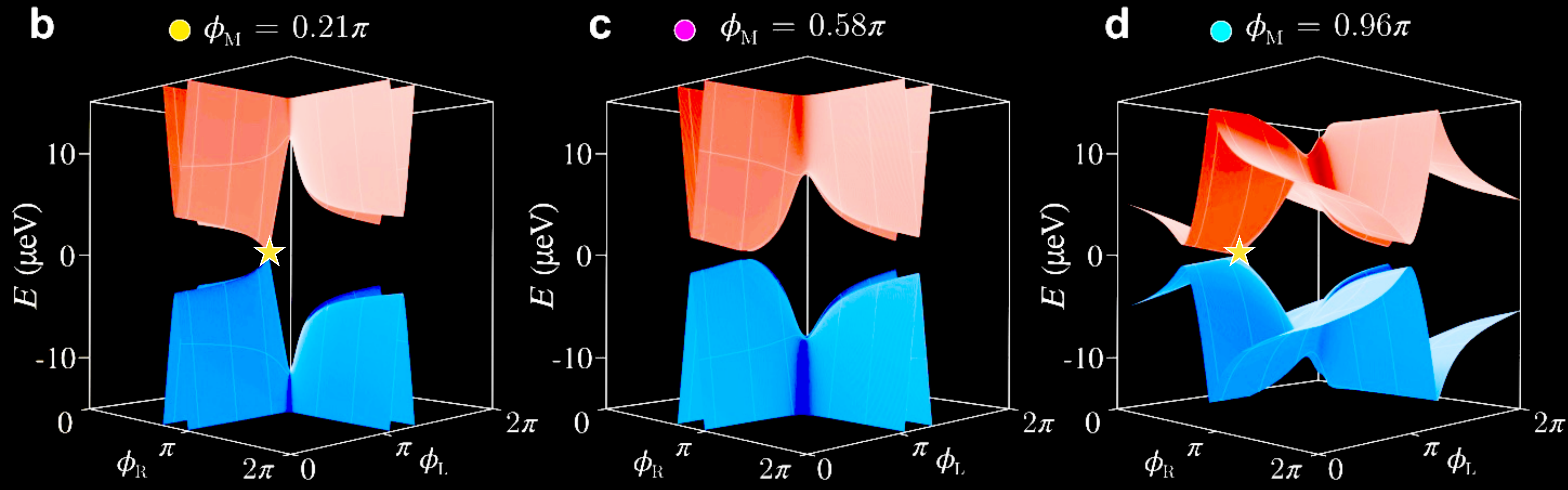

★ Weyl points

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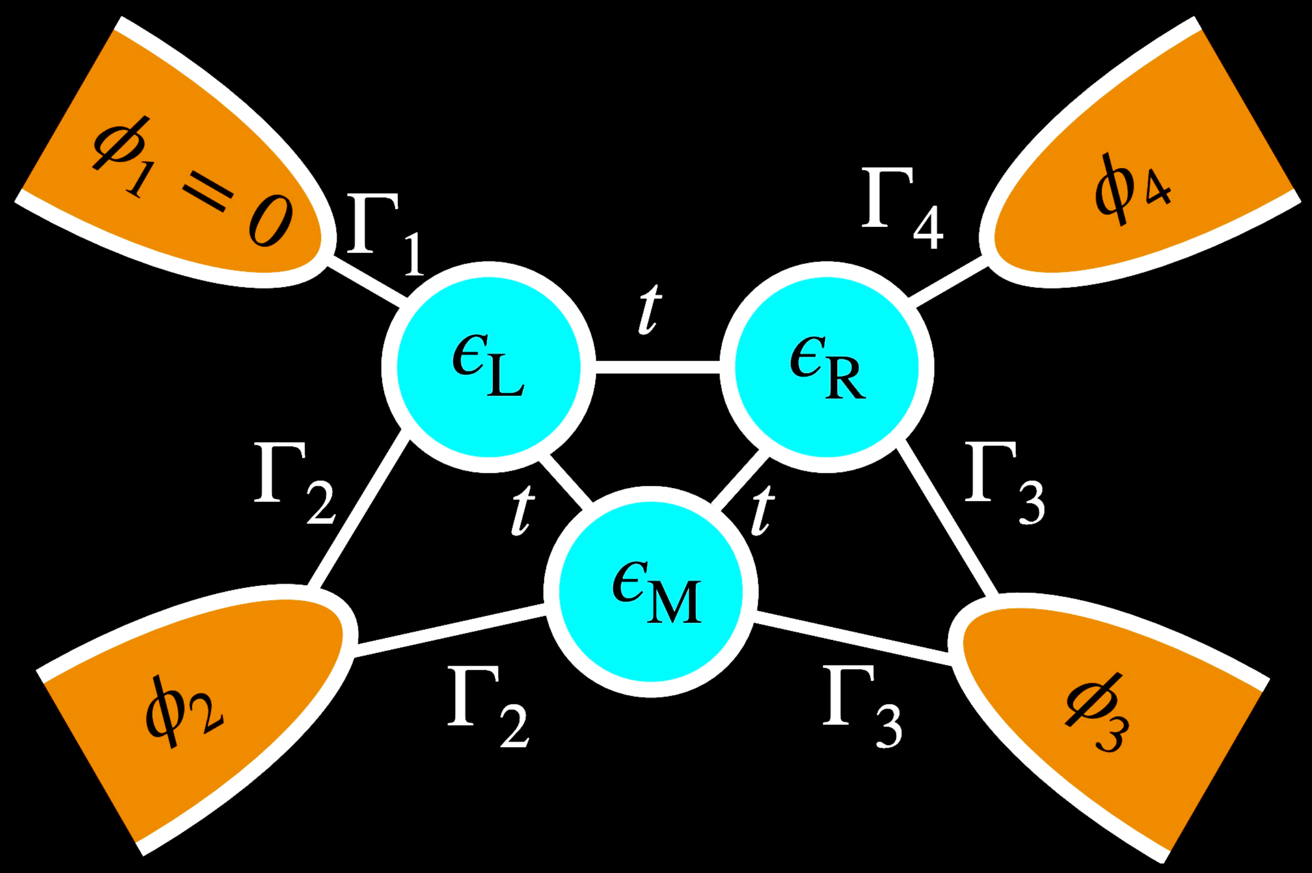
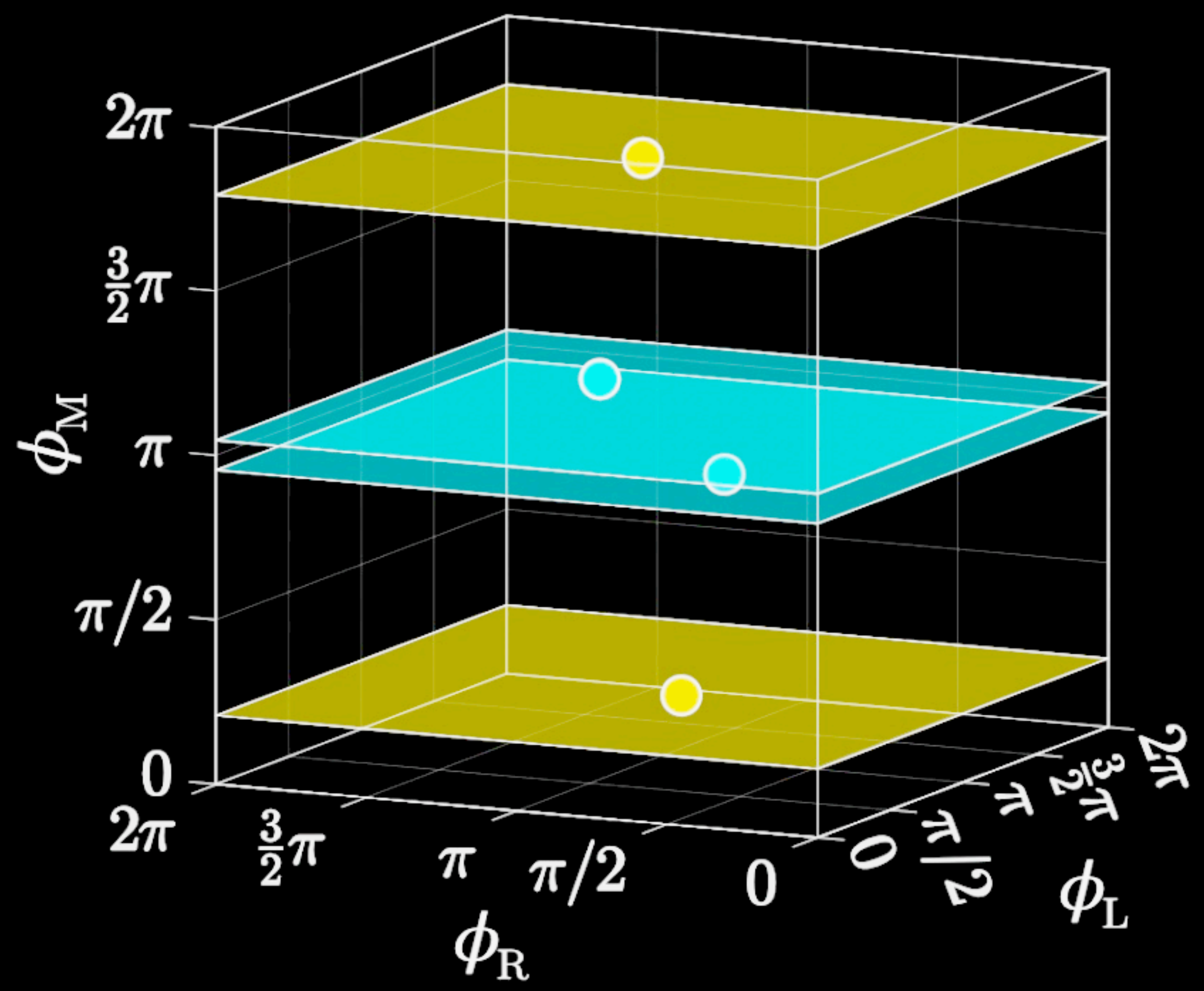
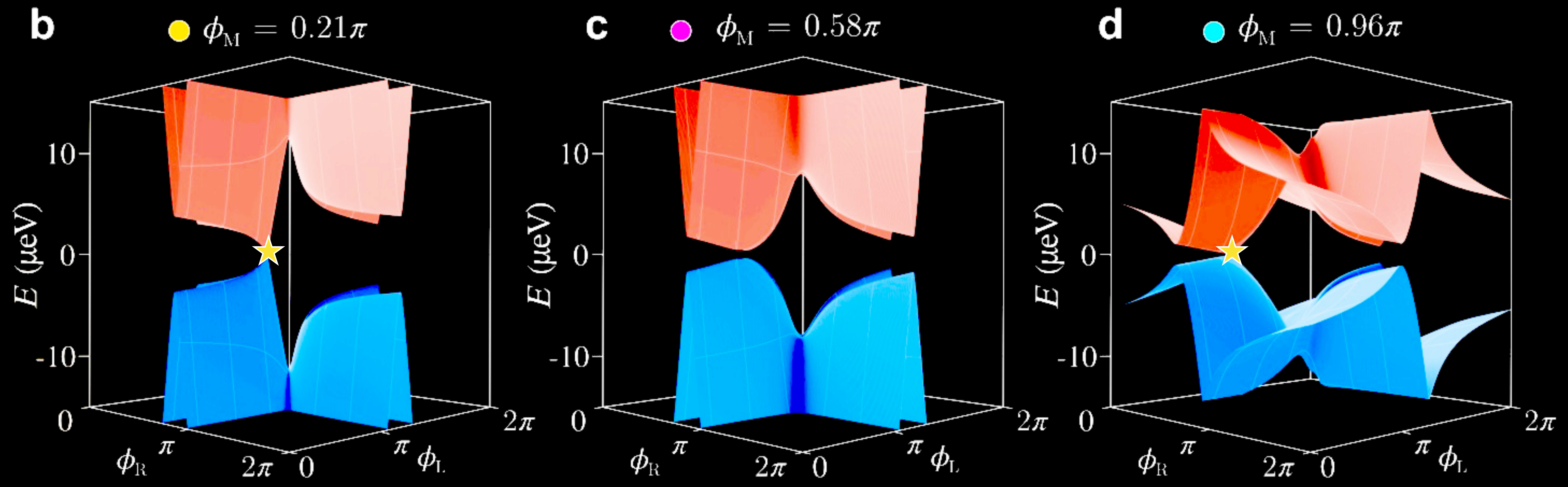
Three state Andreev molecule model



Position of Weyl points

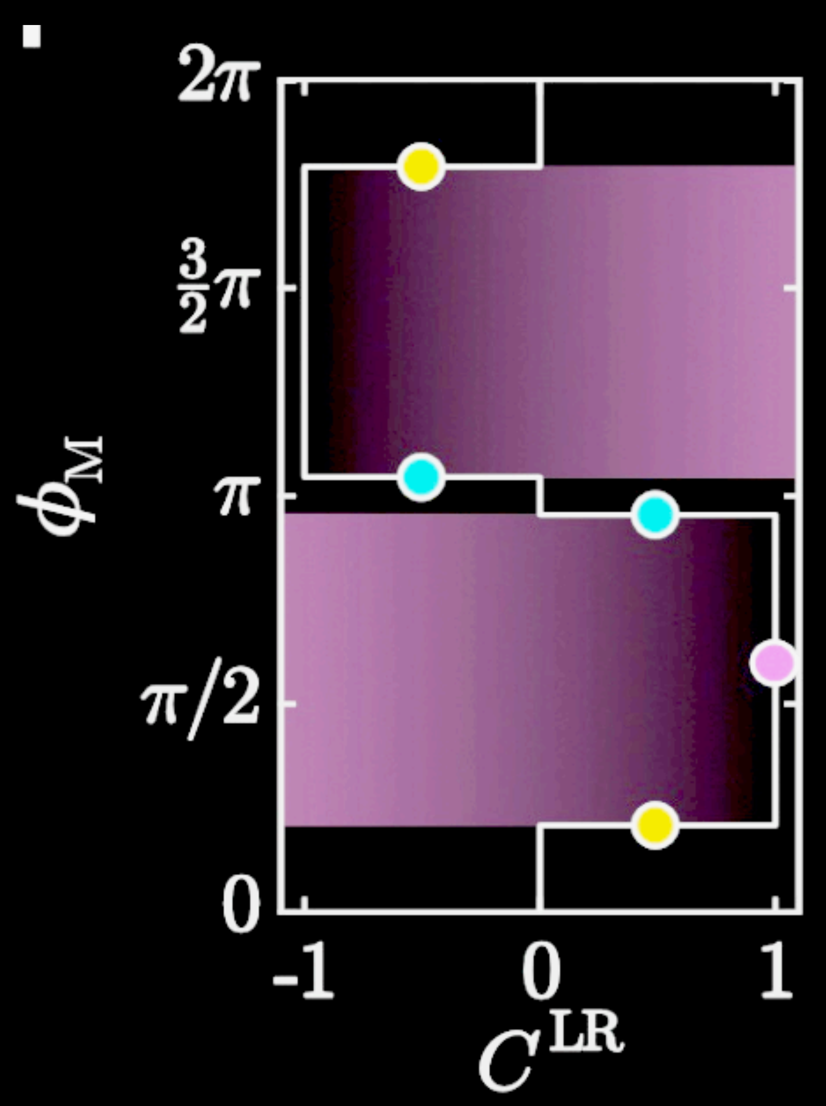


Model predicts Weyl points and non-trivial topology



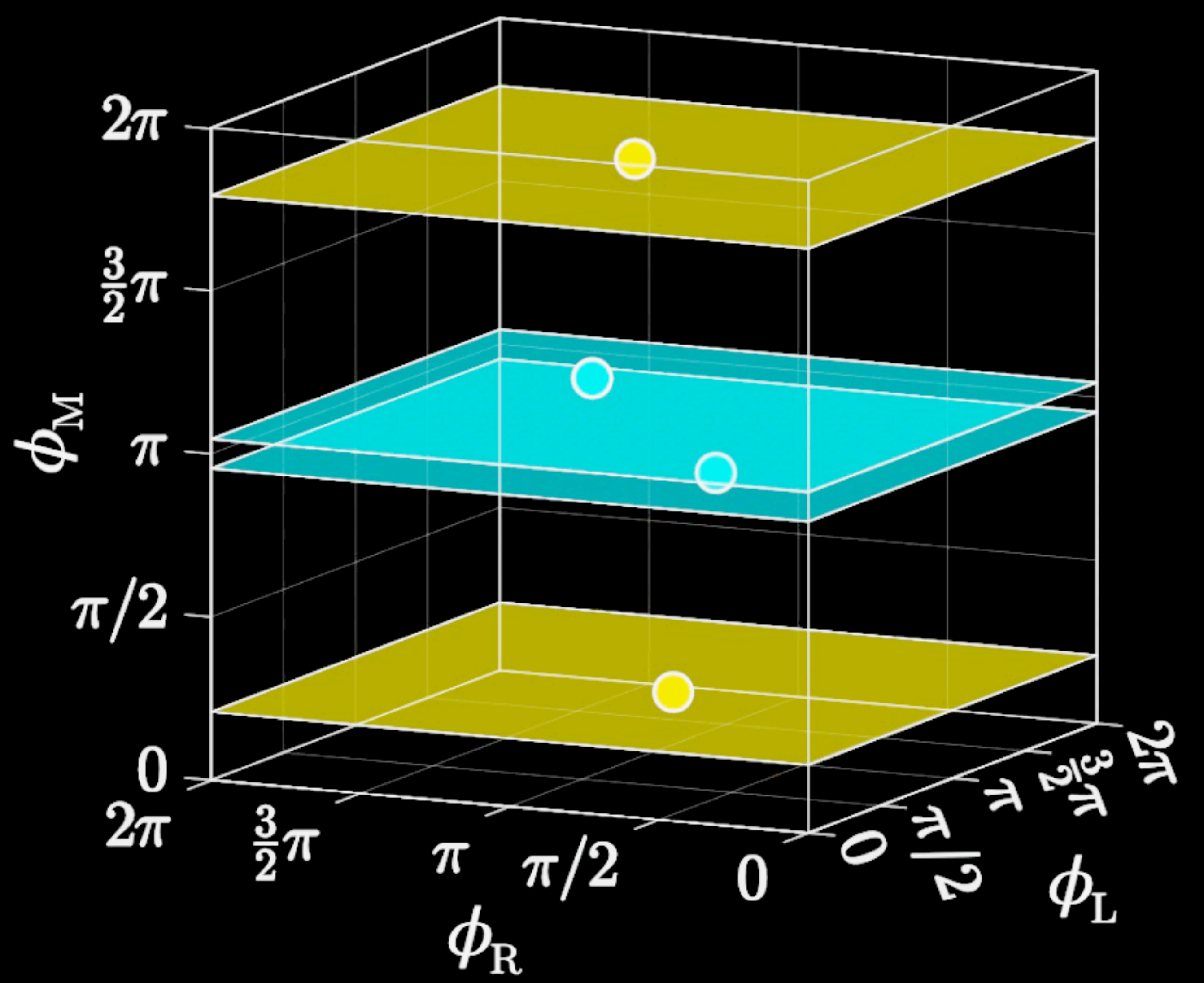
T. Antonelli, et. al., arXiv:2408.01281 (2024).

Phase transition



Experiments on MTJJs: Topology?

Position of Weyl points



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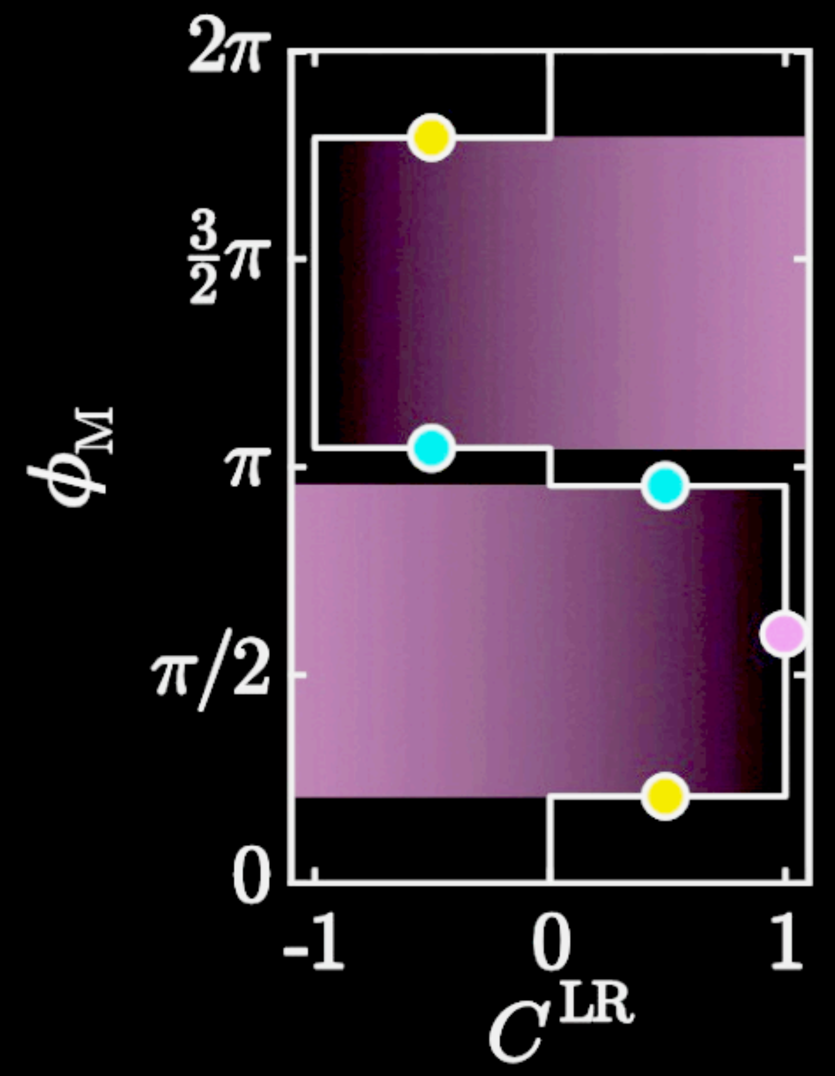
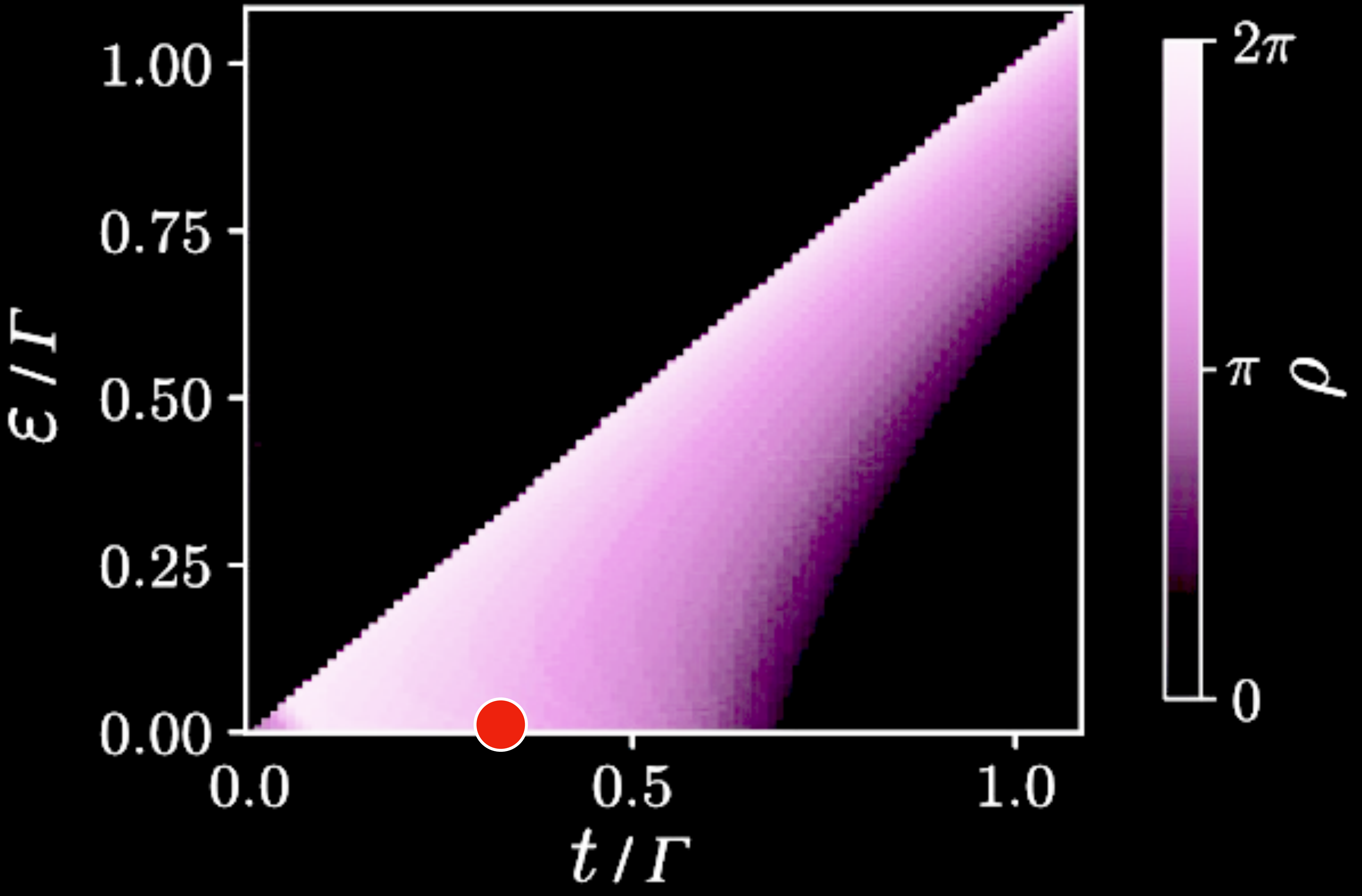


Stability in parameter space
"Transmission" vs Hybridization

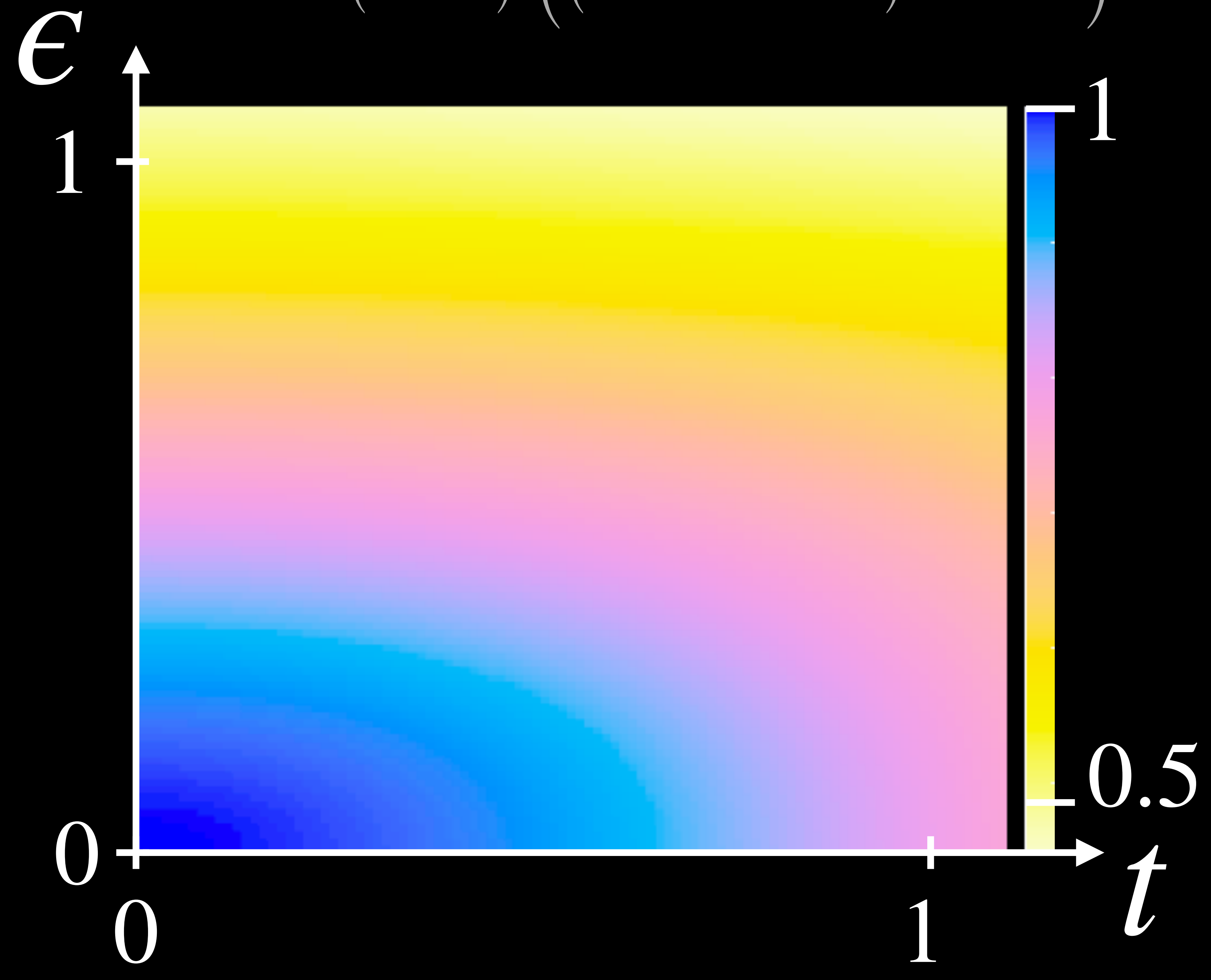


T. Antonelli, et. al., arXiv:2408.01281 (2024).

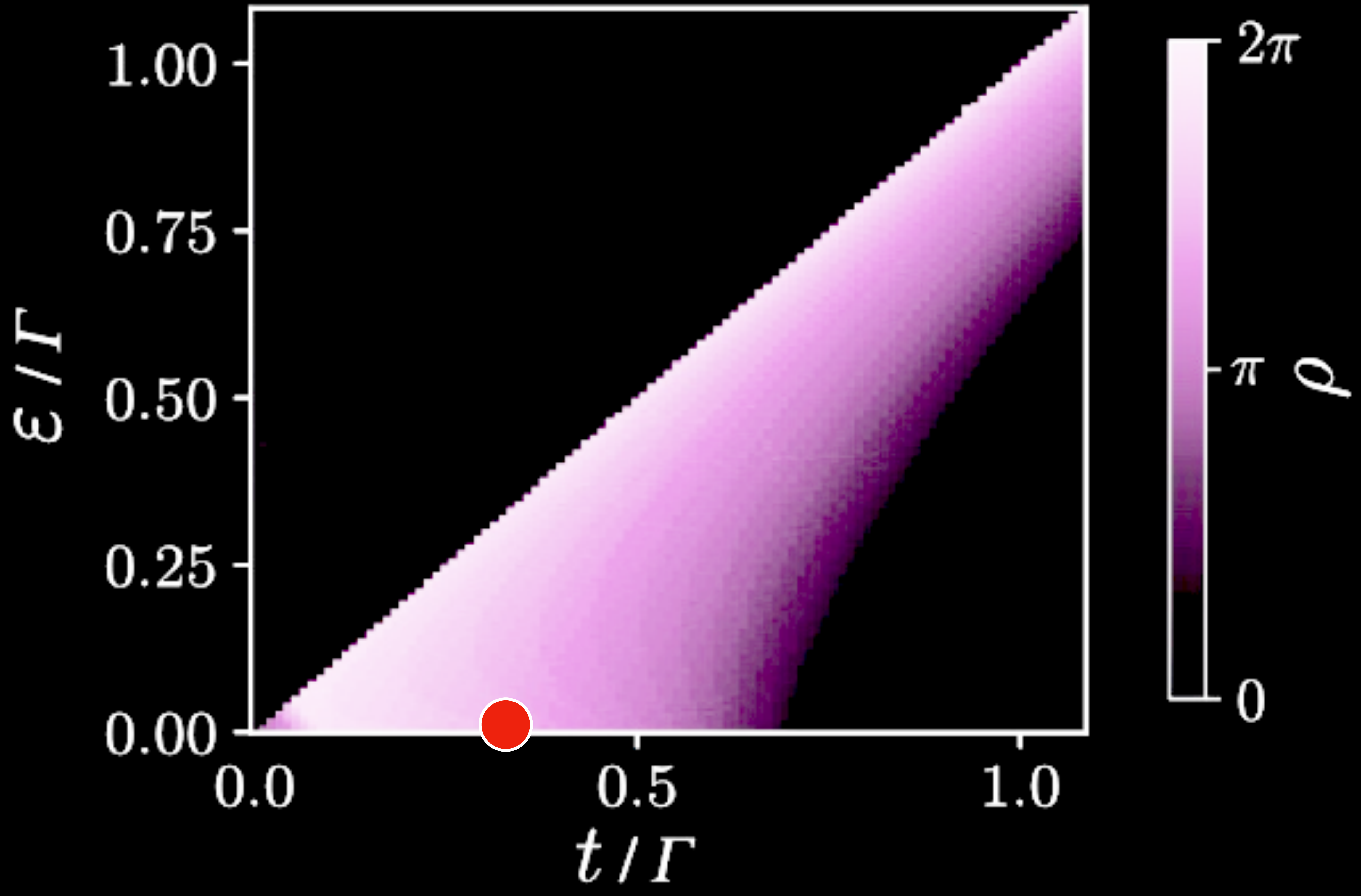
Experiments on MTJJs: Topology?

Transmission from terminal 1 to 2

$$S_{12}(E=0) = \frac{4\left(\left(-\epsilon^2+v^2+4\right)^2+16\,\epsilon^2\right)}{\left(\epsilon^2+4\right)\left(\left(-\epsilon^2+3\,v^2+4\right)^2+16\,\epsilon^2\right)}$$



Stability in parameter space
"Transmission" vs Hybridization



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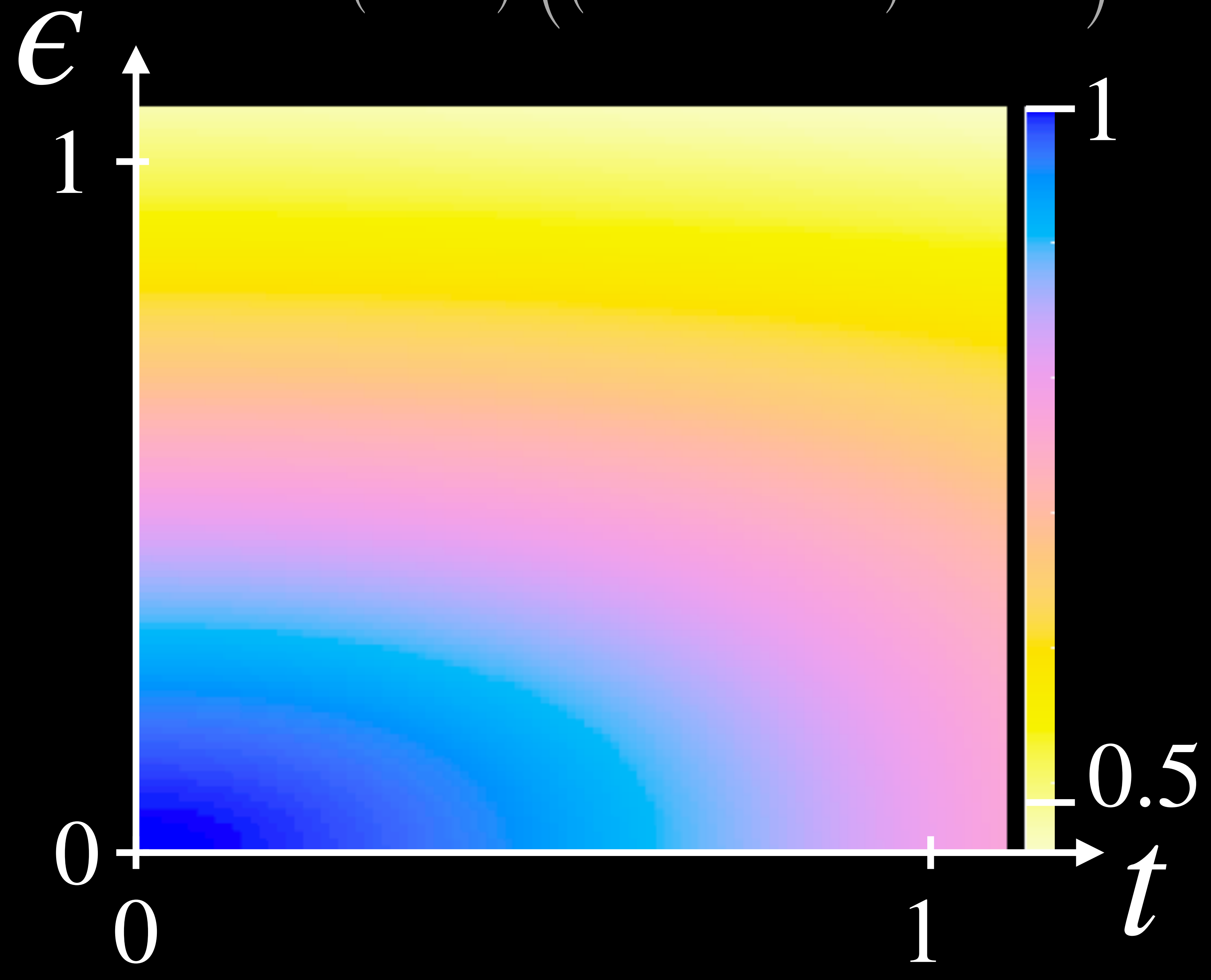

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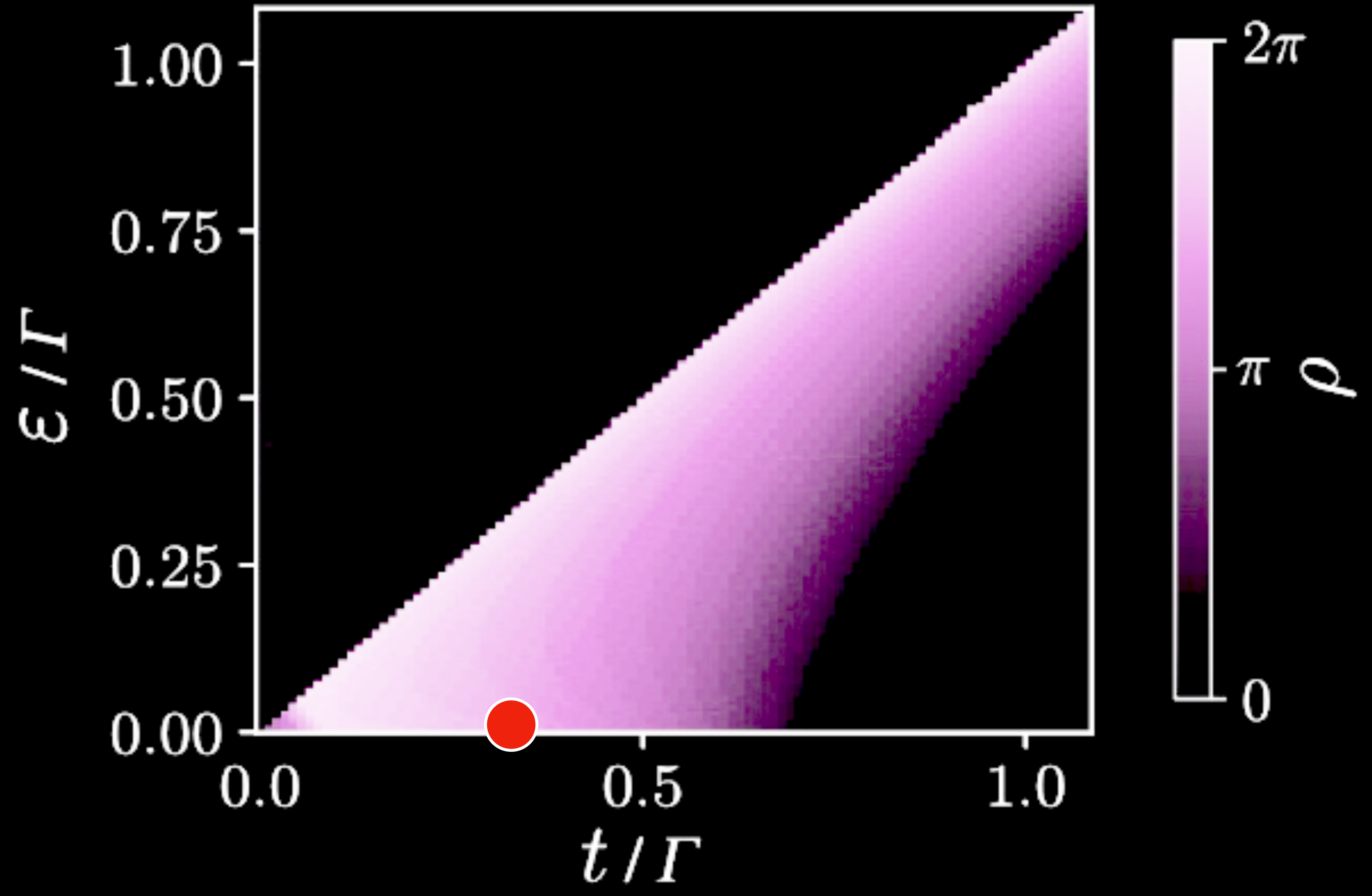


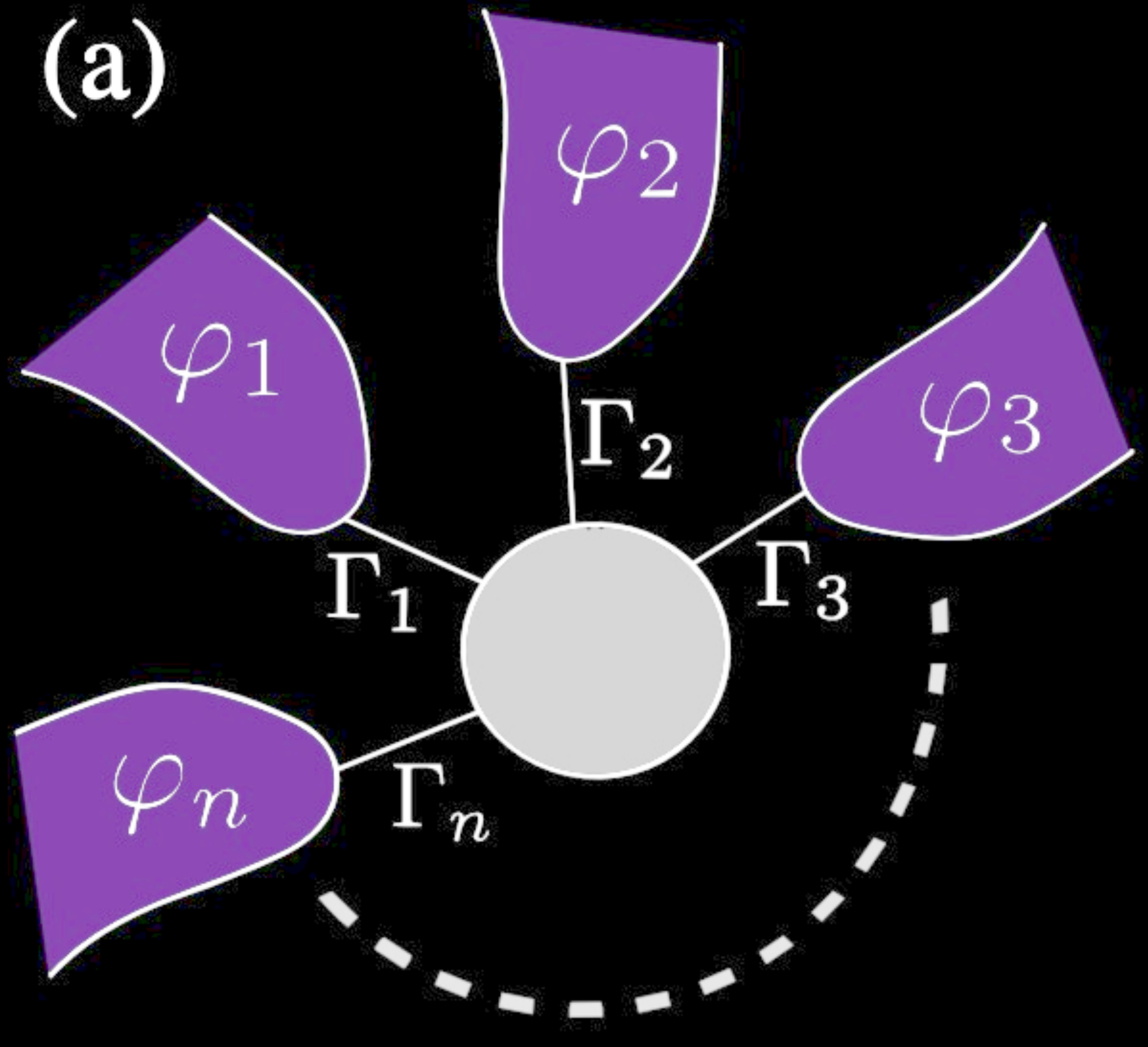
Is there any insight theory can provide for experiment?

T. Antonelli, et. al., arXiv:2408.01281 (2024).

What's the simplest model?

1-dot model?

Phys. Rev. Lett. 132, 126505 (2024)



Effective Hamiltonian

$$H_{\text{eff}} = \begin{pmatrix} \epsilon & \sum_i \Gamma_i e^{\imath\varphi_i} \\ \sum_i \Gamma_i e^{-\imath\varphi_i} & -\epsilon \end{pmatrix}$$

What's the simplest model?

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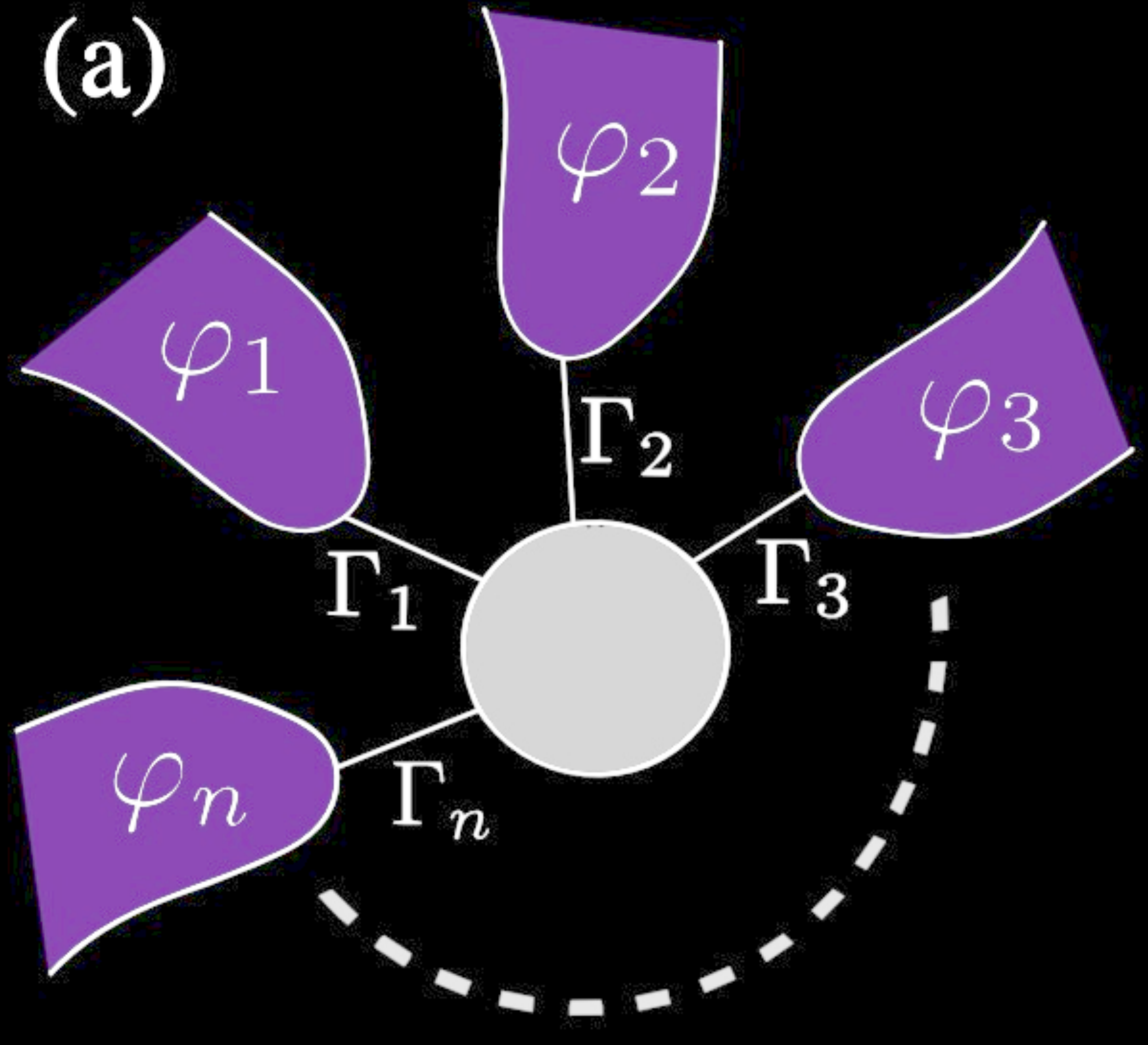
Phys. Rev. Lett. 132, 126505 (2024)



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Problem: off-diagonal can be set real with gauge trafo

$$\chi \equiv |\vec{\chi}| = \sqrt{1 - 4\sum_{j>l=1}^{n} \gamma_j \gamma_l \sin^2\left(\frac{\phi_j - \phi_l}{2}\right)}$$

$$\gamma_i = \Gamma / \left(\sum_i \Gamma_i\right)$$

What's the simplest model?

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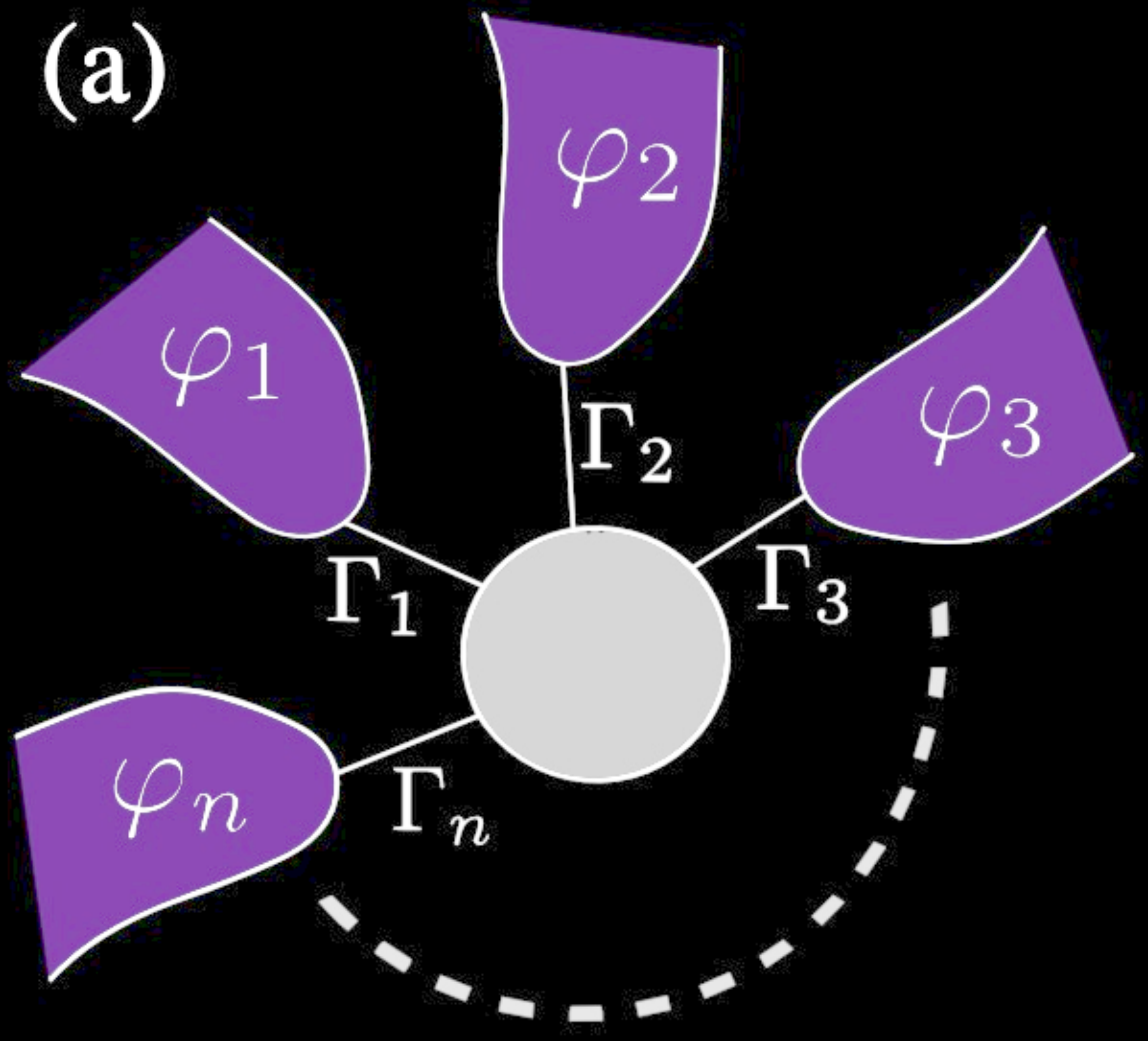
Phys. Rev. Lett. 132, 126505 (2024)



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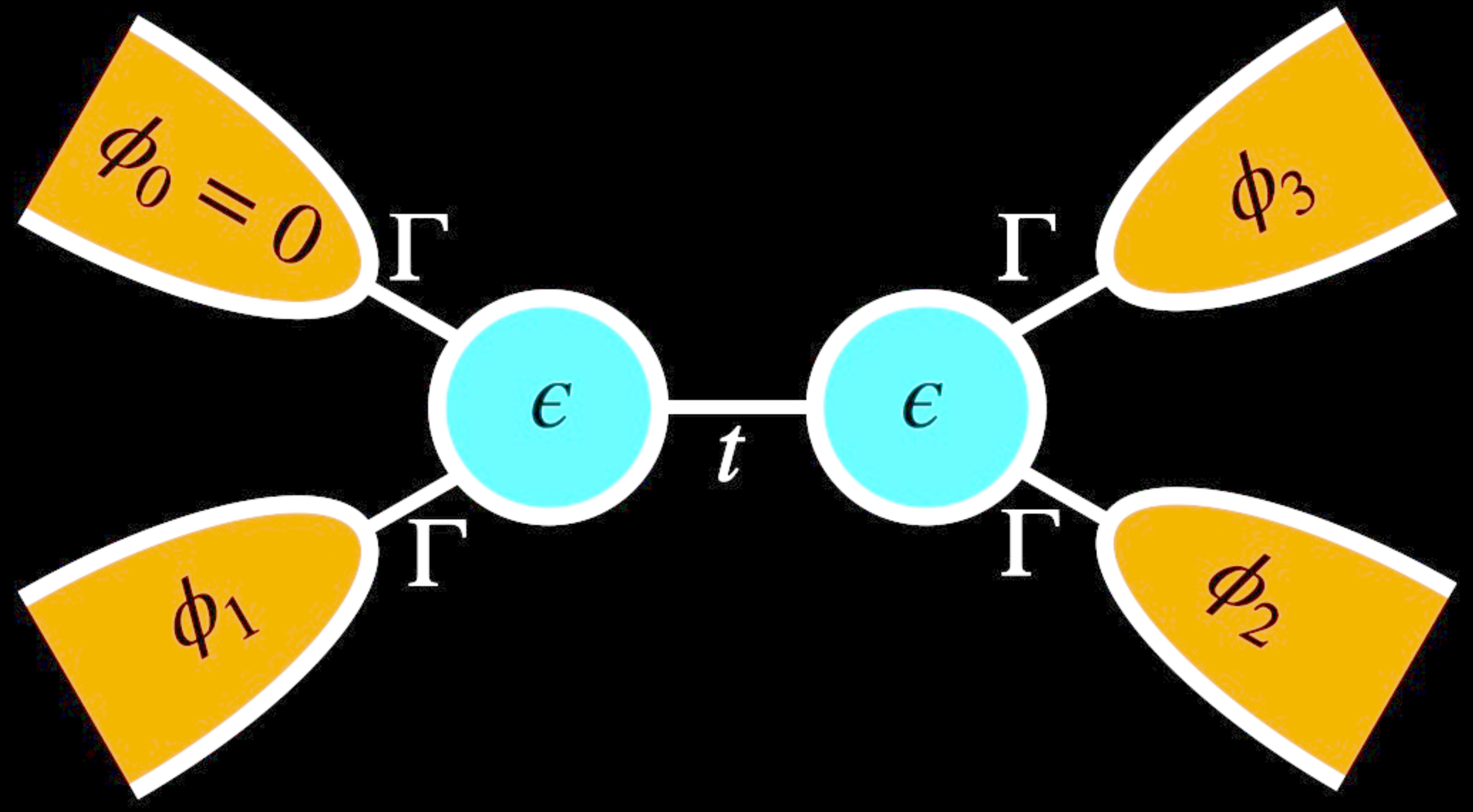
2-dot model!

SciPost Phys. 15, 214 (2023)



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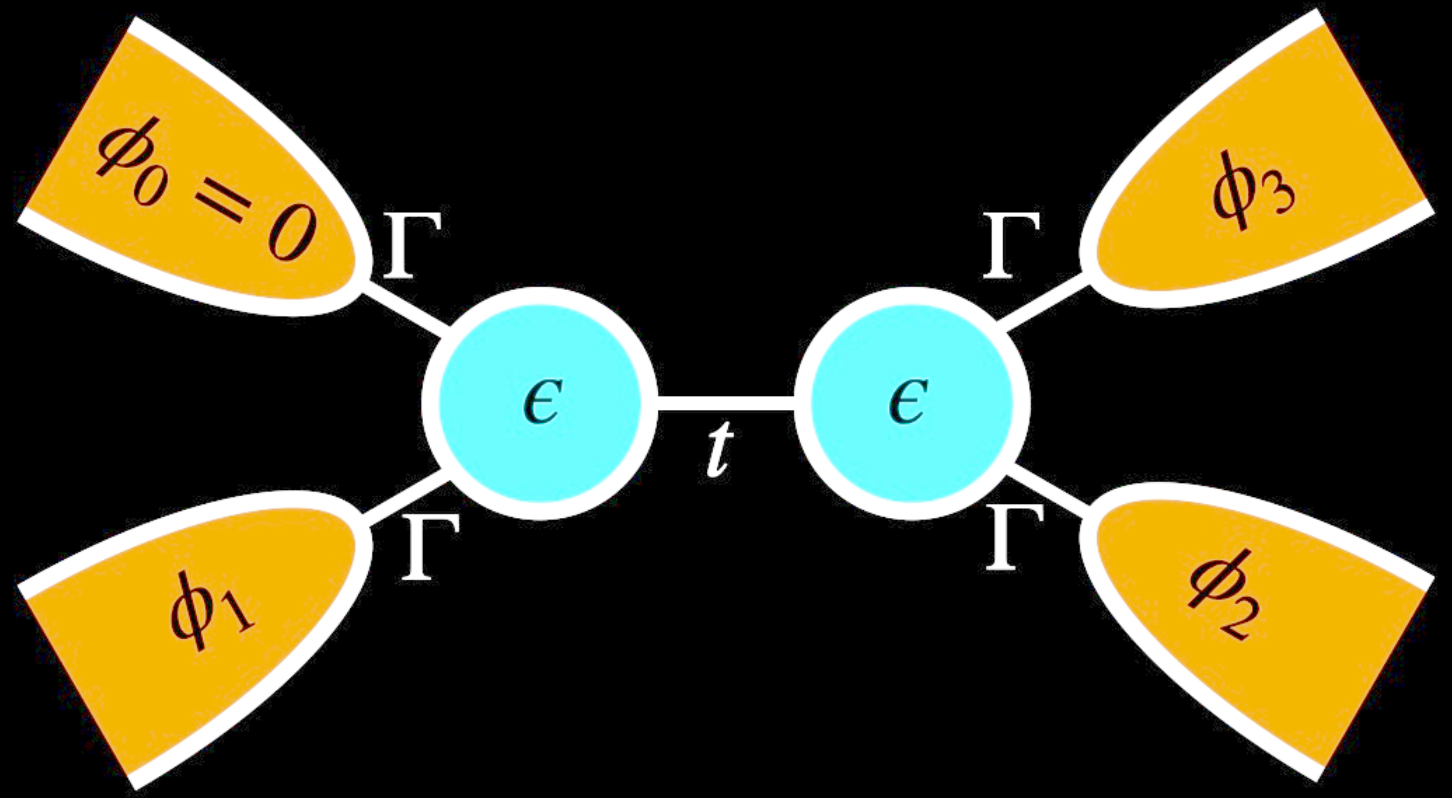
2-dot modell is minimal modell

MTJJs considered here

Effective quantum dot models

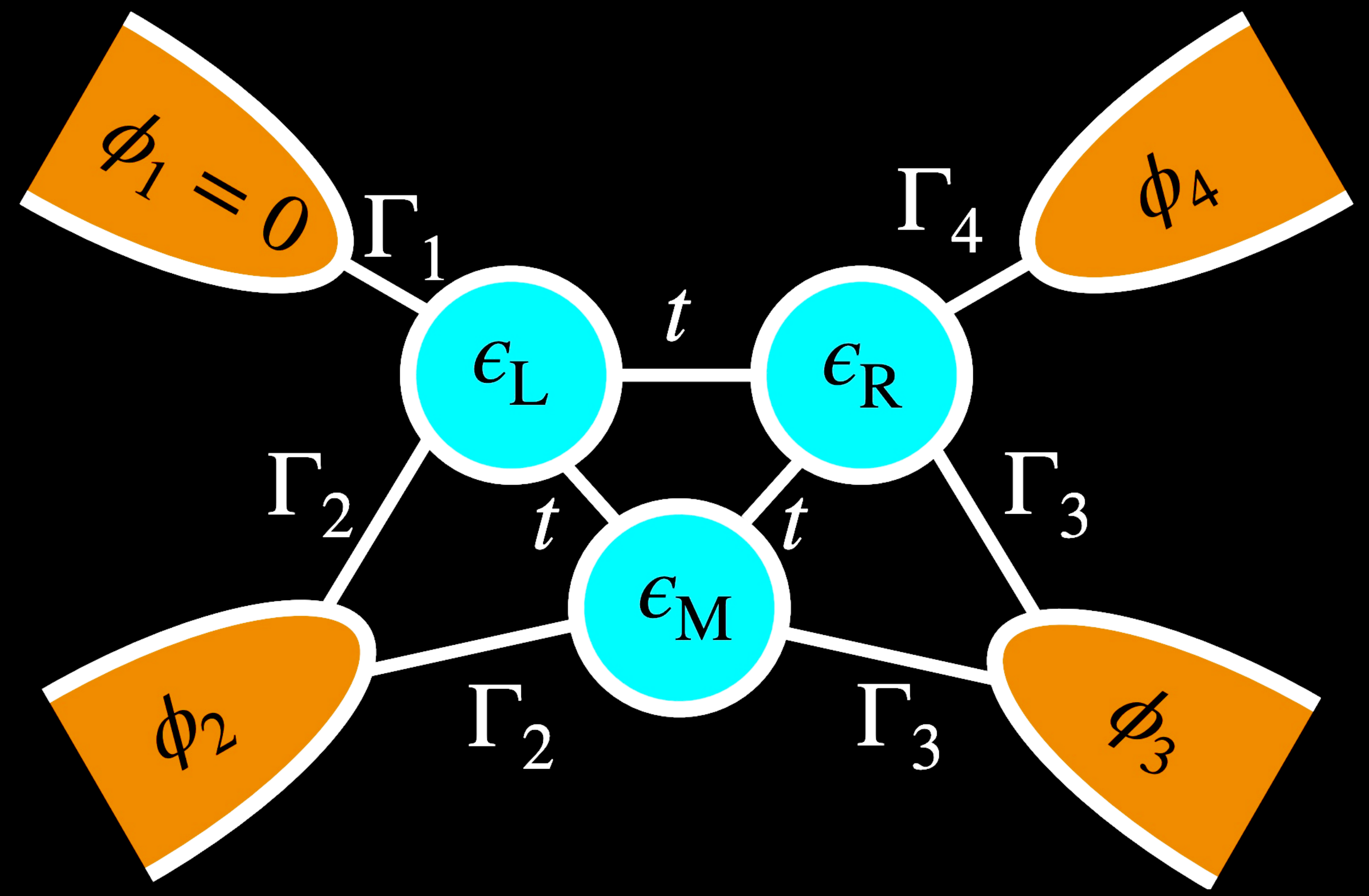
2-dot modell



Effective Hamiltonian

$$H_{\mathrm{eff}} = \begin{pmatrix} H_{\mathrm{N}} & \Sigma \\ \Sigma^* & -H_{\mathrm{N}} \end{pmatrix}$$

3-dot modell



$$H_{\mathrm{N}} = \begin{pmatrix} \epsilon & t \\ t & \epsilon \end{pmatrix}, \ \sigma(H_{\mathrm{N}}) = \{\epsilon \pm t\}$$

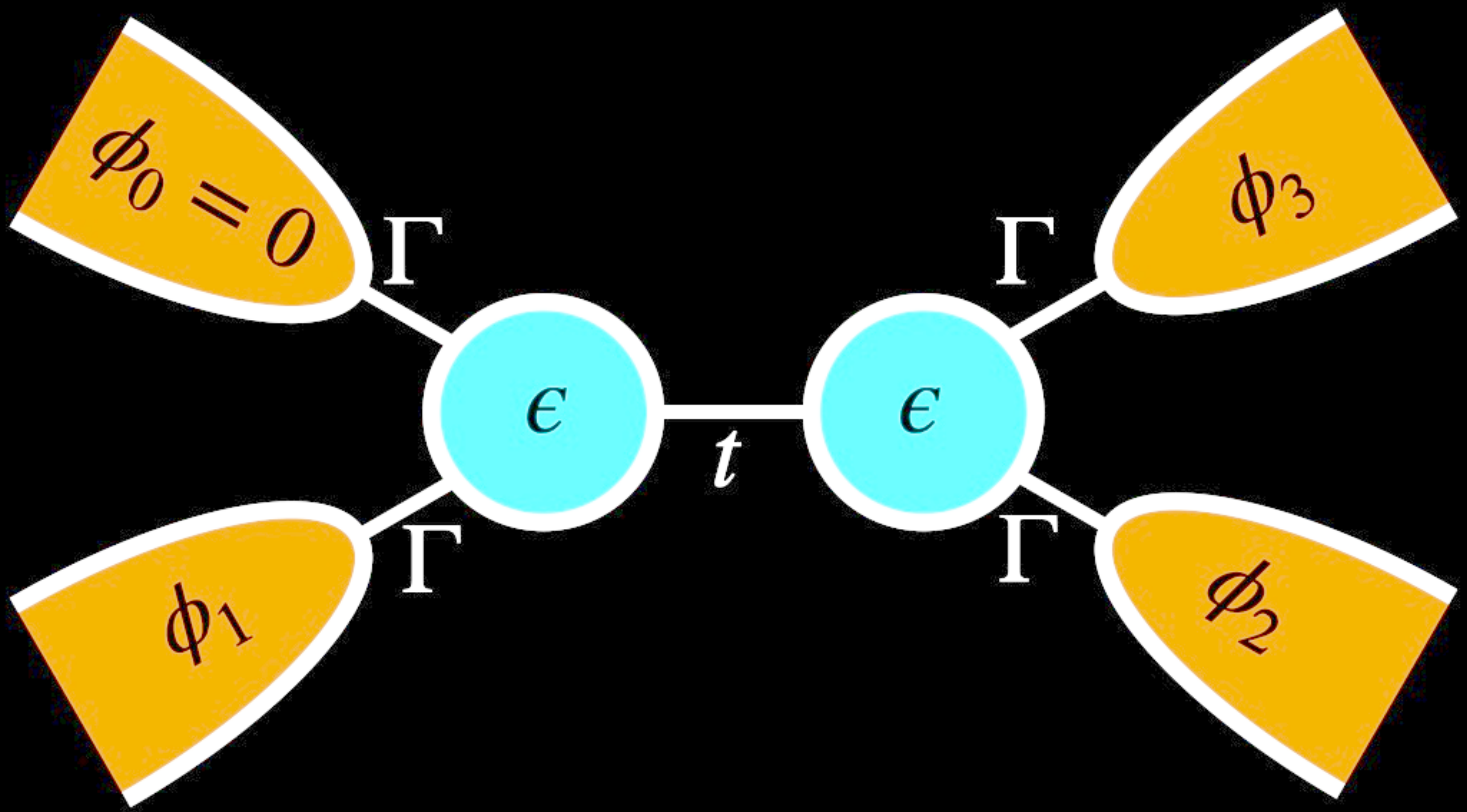
$$\Sigma = \begin{pmatrix} \Gamma + \Gamma e^{\imath\phi_1} & 0 \\ 0 & \Gamma e^{\imath\phi_2} + \Gamma e^{\imath\phi_3} \end{pmatrix}$$

$$H_{\mathrm{N}} = \begin{pmatrix} \epsilon & t & t \\ t & \epsilon & t \\ t & t & \epsilon \end{pmatrix}, \ \sigma(H_{\mathrm{N}}) = \{\epsilon + 2t, \ \epsilon - t, \ \epsilon - t\}$$

$$\Sigma = \begin{pmatrix} \Gamma_1 + \Gamma_2 e^{\imath\phi_2} & 0 & 0 \\ 0 & \Gamma_2 e^{\imath\phi_2} + \Gamma_3 e^{\imath\phi_3} & 0 \\ 0 & 0 & \Gamma_3 e^{\imath\phi_3} + \Gamma_4 e^{\imath\phi_4} \end{pmatrix}$$

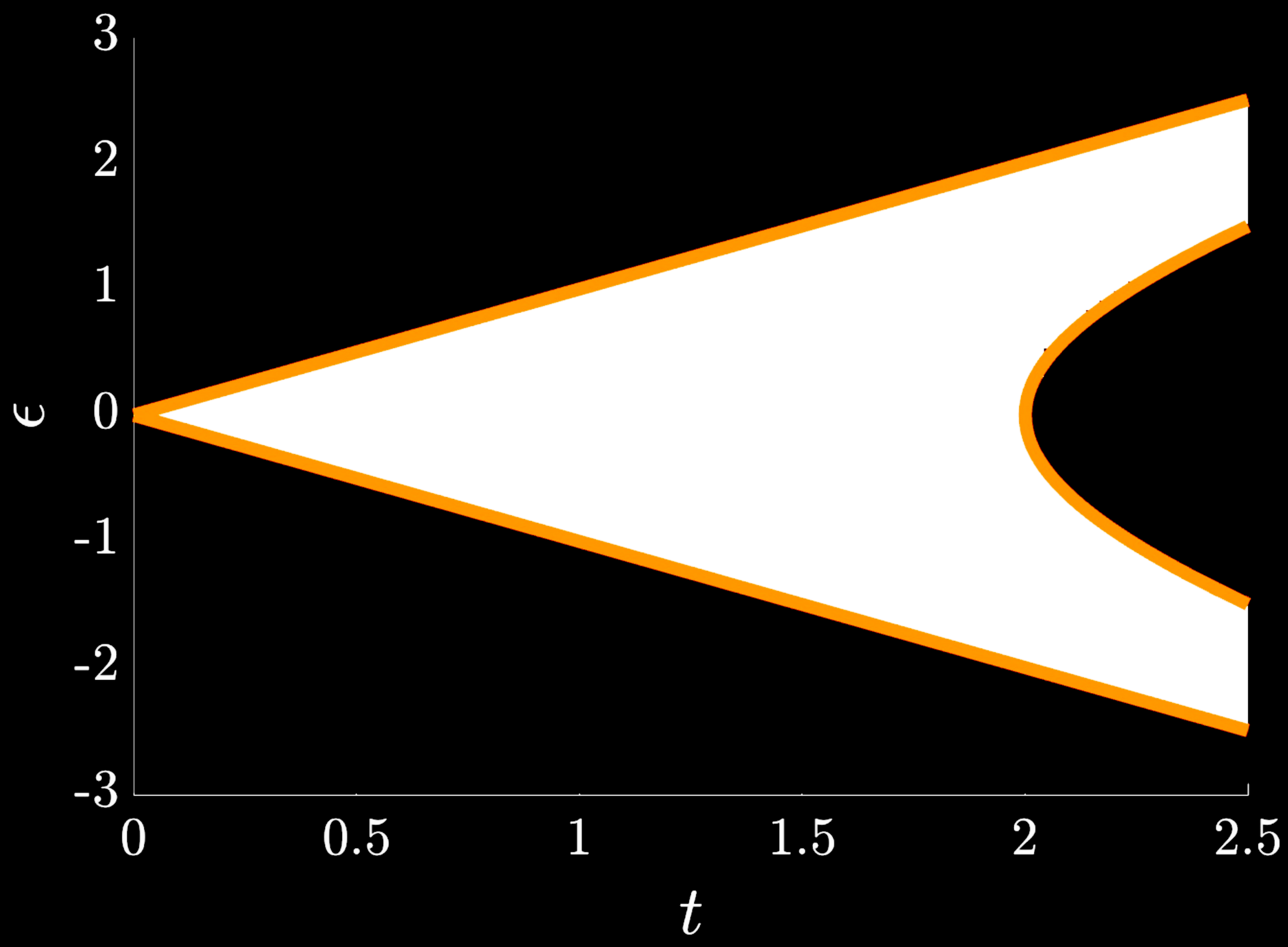
What's the source of topology in MTJJs?

4-terminal MTJJ with two dots



SciPost Phys. 15, 214 (2023)

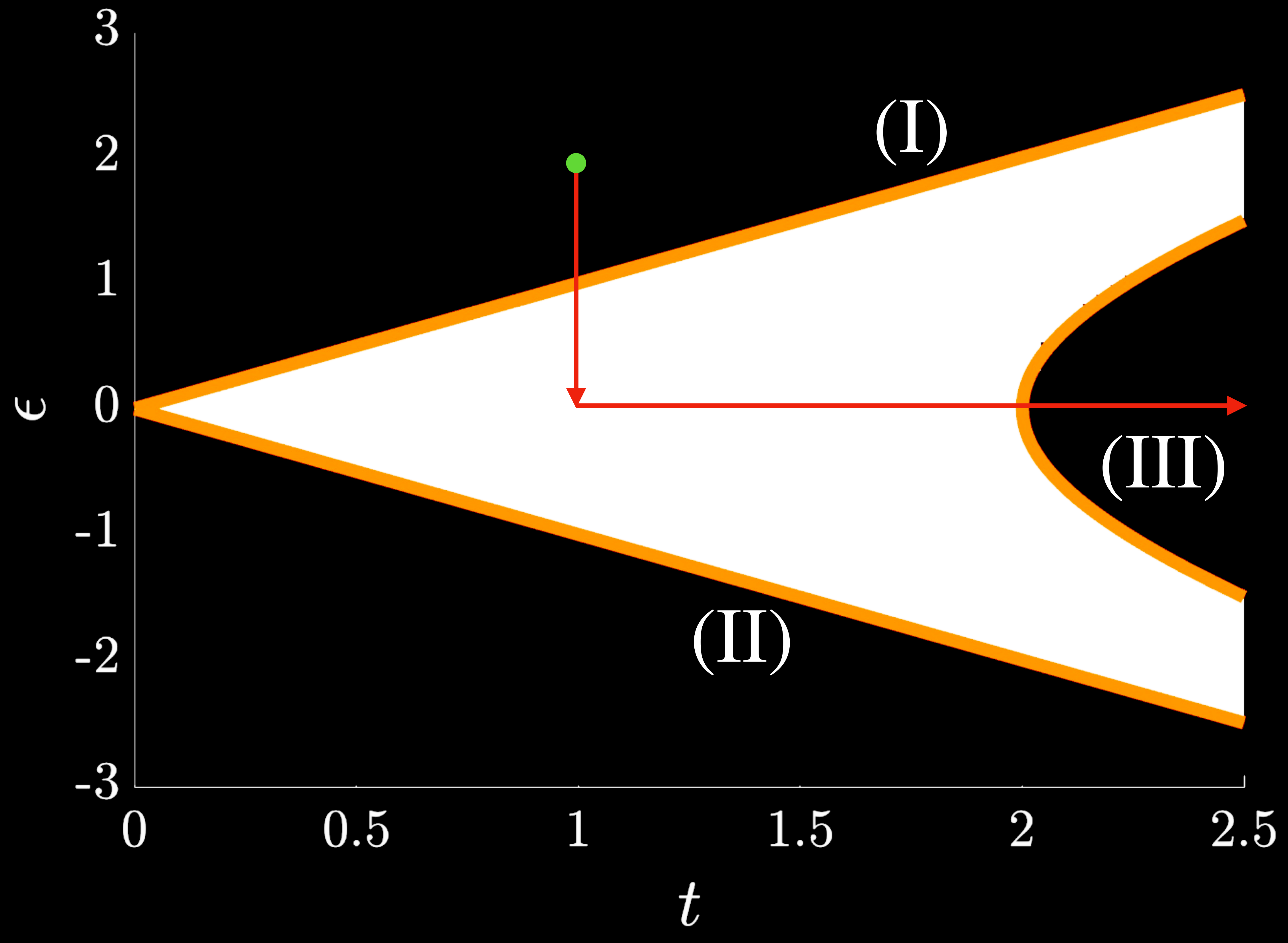
Topological phase diagram



non-trivial Chern number in superconducting phase space $\left\{ \vec{\phi} \right\}$

What determines boundaries?

Topological phase diagram



non-trivial Chern number

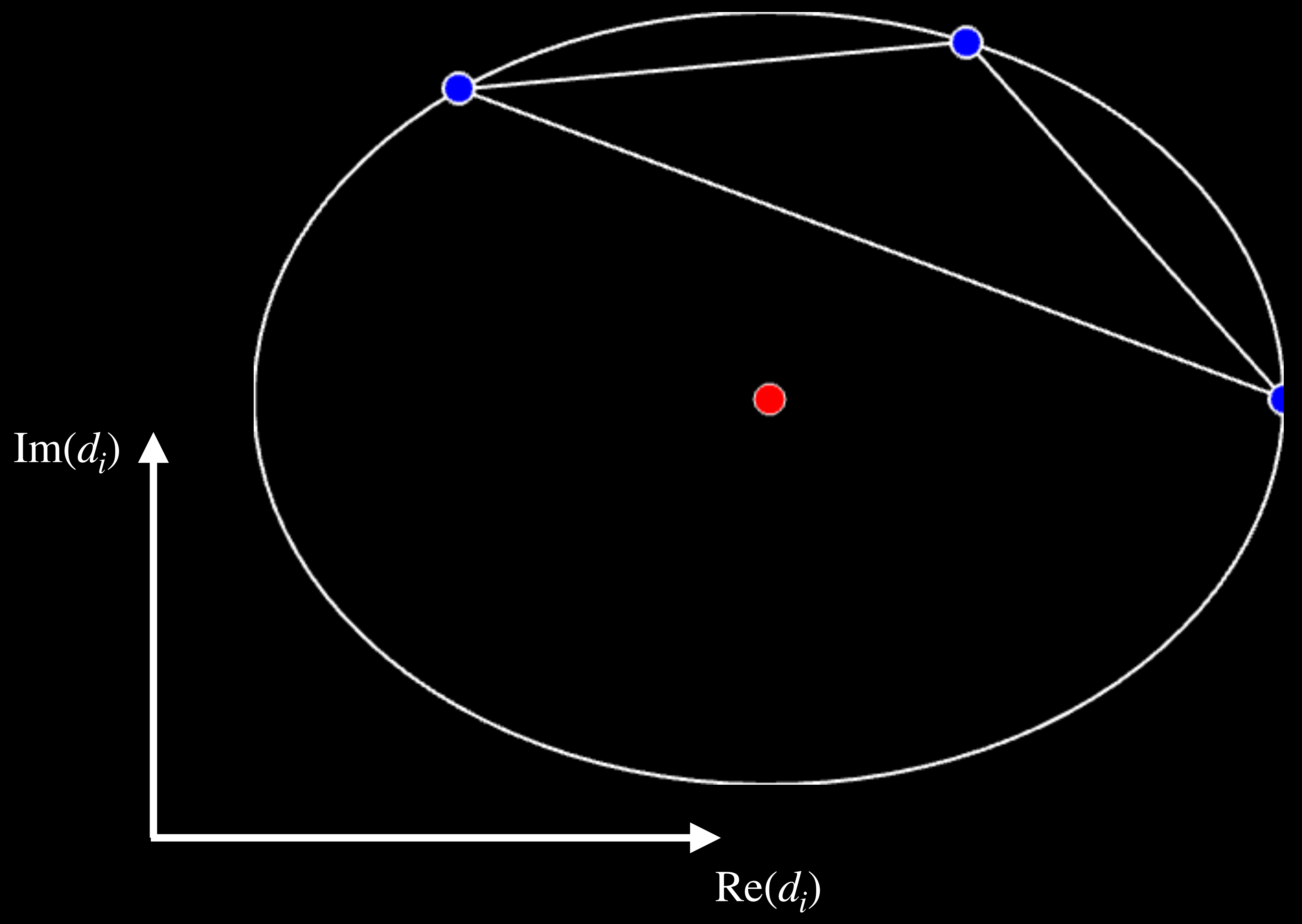
in superconducting phase space $\left\{ \vec{\phi} \right\}$

Consider eigenvalues of **normal** metal scattering matrix $S(E=0)$!

$$D_\mathrm{S} = \mathrm{diag}\left(1, 1, d_+, d_-\right) = \mathrm{diag}\left(1, 1, \frac{\epsilon_+^*}{\epsilon_+}, \frac{\epsilon_-^*}{\epsilon_-}\right)$$

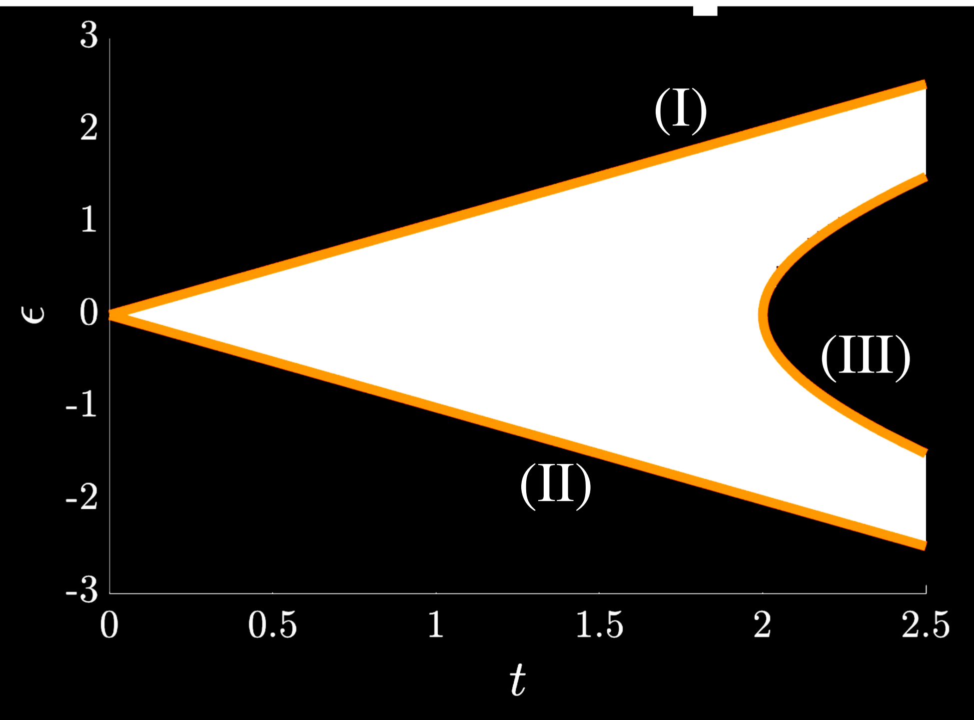
$$\epsilon_\pm = \epsilon \pm t + 2\imath\Gamma$$

$$\sigma(H_\mathrm{N}) = \{\epsilon \pm t\}$$



When do we obtain a topological boundary?

Eigenvalues of **normal** metal scattering matrix $S(E=0)$



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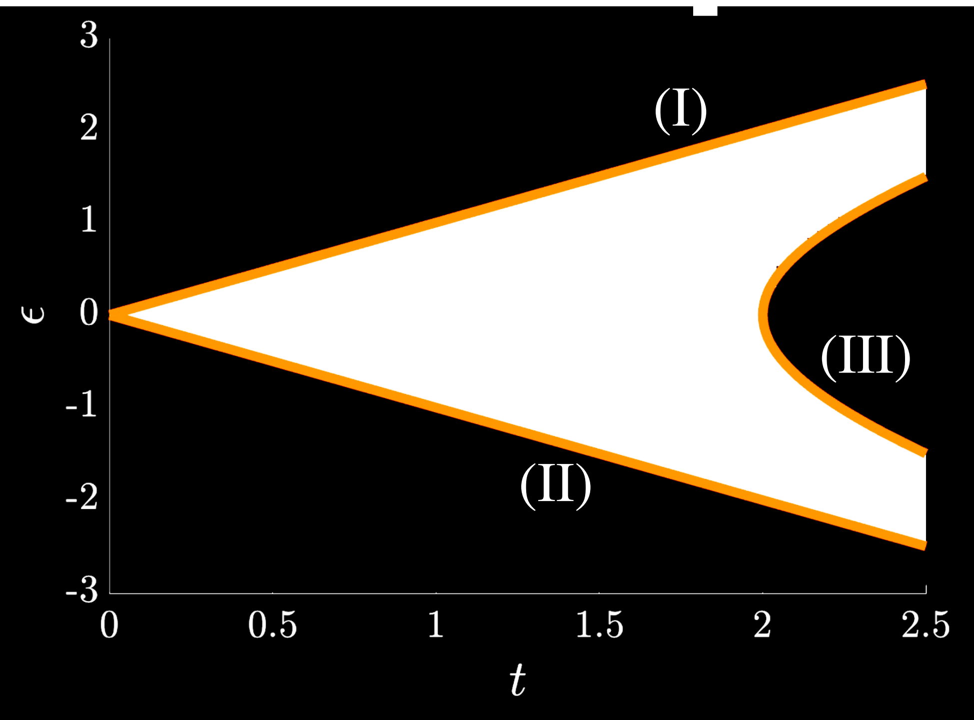

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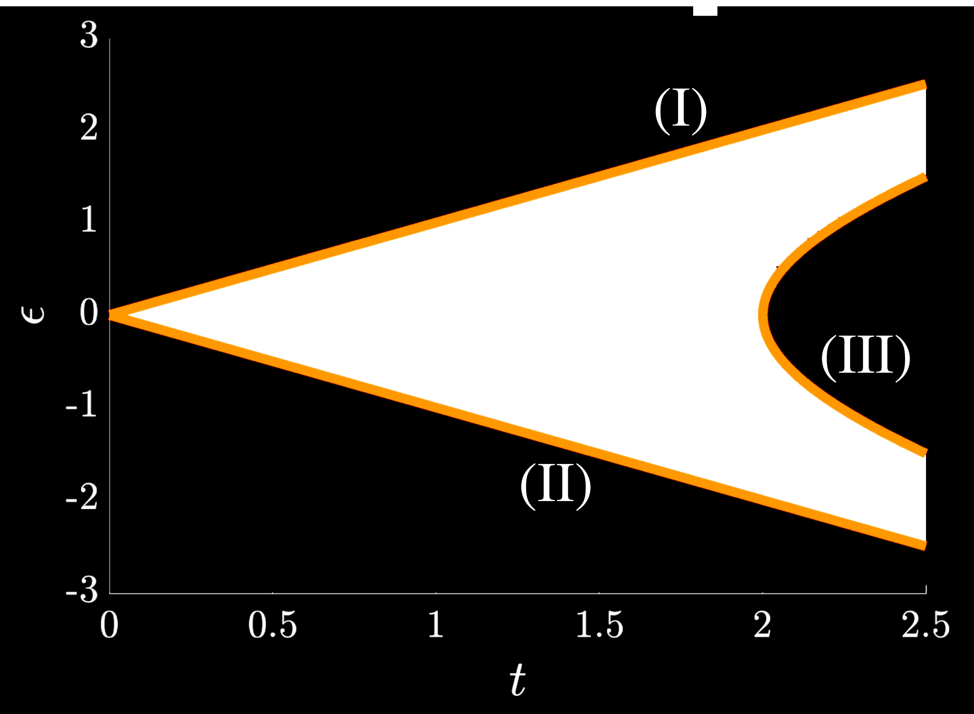
lie oppositely to each other

(I) $d_- = -1$

(II) $d_+ = -1$

(III) $d_+ = -d_-$

When do we obtain a topological boundary?



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meaning that

(I) $\mathrm{Re}(\epsilon_-) = \epsilon - t = 0$

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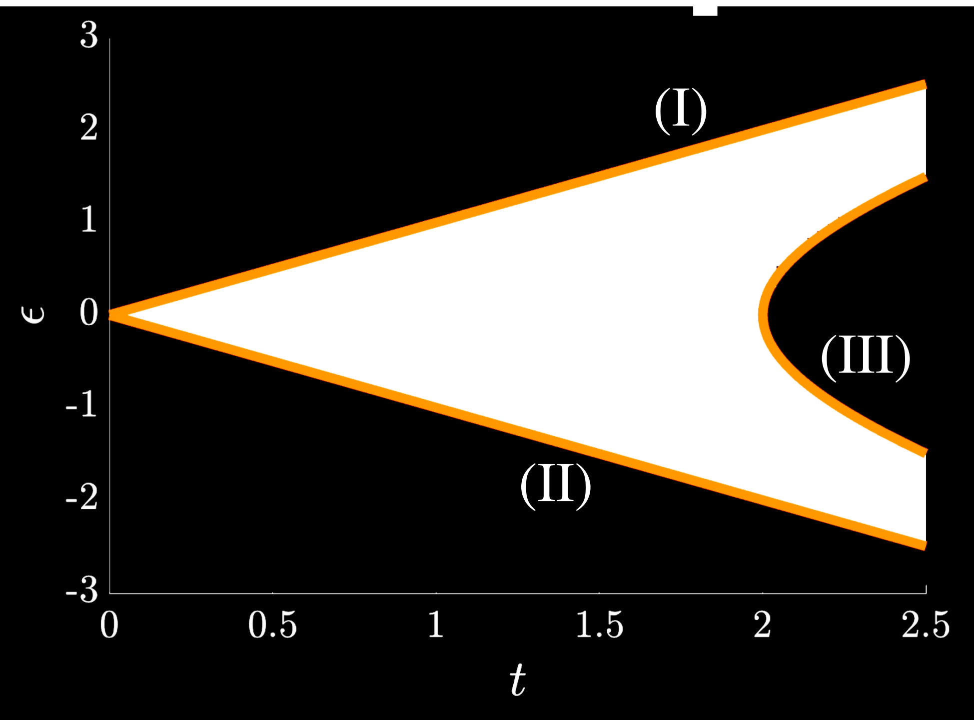
(III) $\mathrm{Re}(\epsilon_+\epsilon_-^*) = \epsilon^2 - t^2 + 4\Gamma^2 = 0$

<u>When do we obtain a topological boundary?</u>

Eigenvalues of **normal** metal scattering matrix $S(E=0)$



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Consider effective Hamiltonian

$$H_{\mathrm{eff}} = \begin{pmatrix} H_{\mathrm{N}} & \Sigma \\ \Sigma^* & -H_{\mathrm{N}} \end{pmatrix}$$

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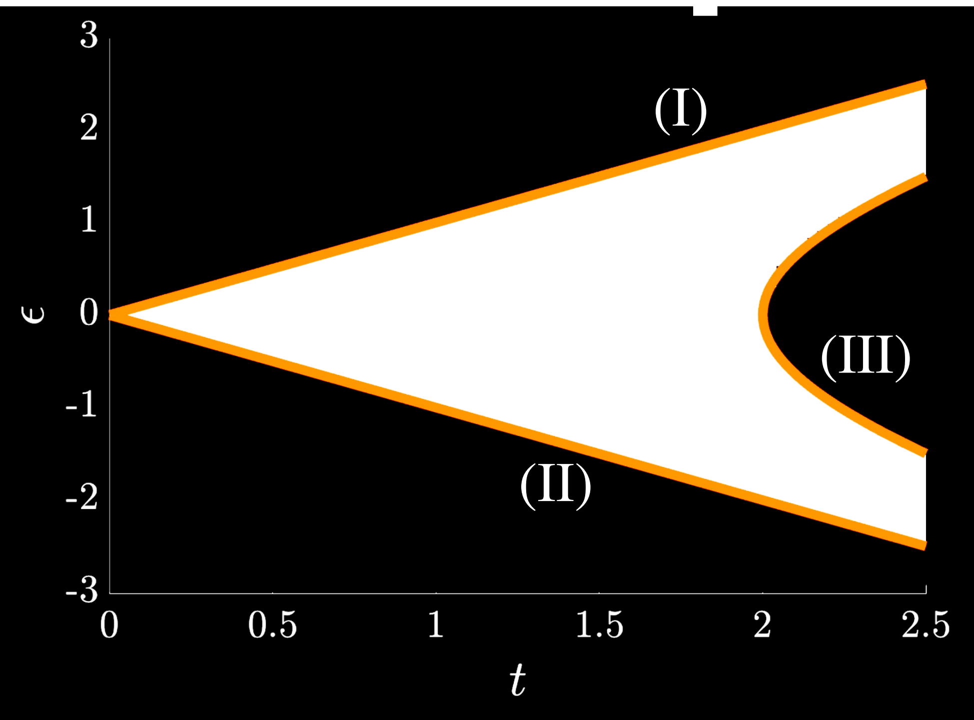

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$$H_{\mathrm{eff}} = \begin{pmatrix} H_N & \Sigma \\ \Sigma^* & -H_N \end{pmatrix}$$

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(I)(II) $\vec{\phi} = (0, \pi, 0, \pi)$

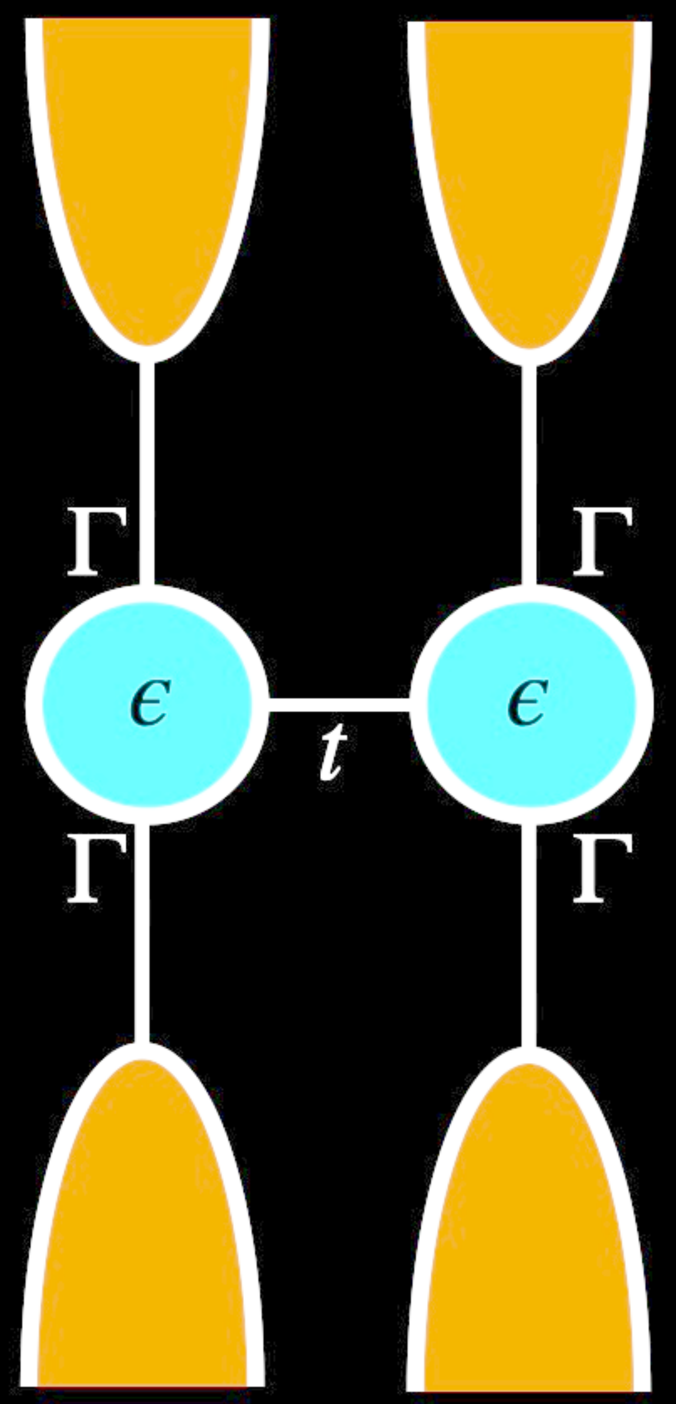
$$\Sigma = 0 \Leftrightarrow \det H_{\mathrm{eff}} = 0 \Leftrightarrow \epsilon \pm t = 0$$

(III) $\vec{\phi} = (0, 0, \pi, \pi)$

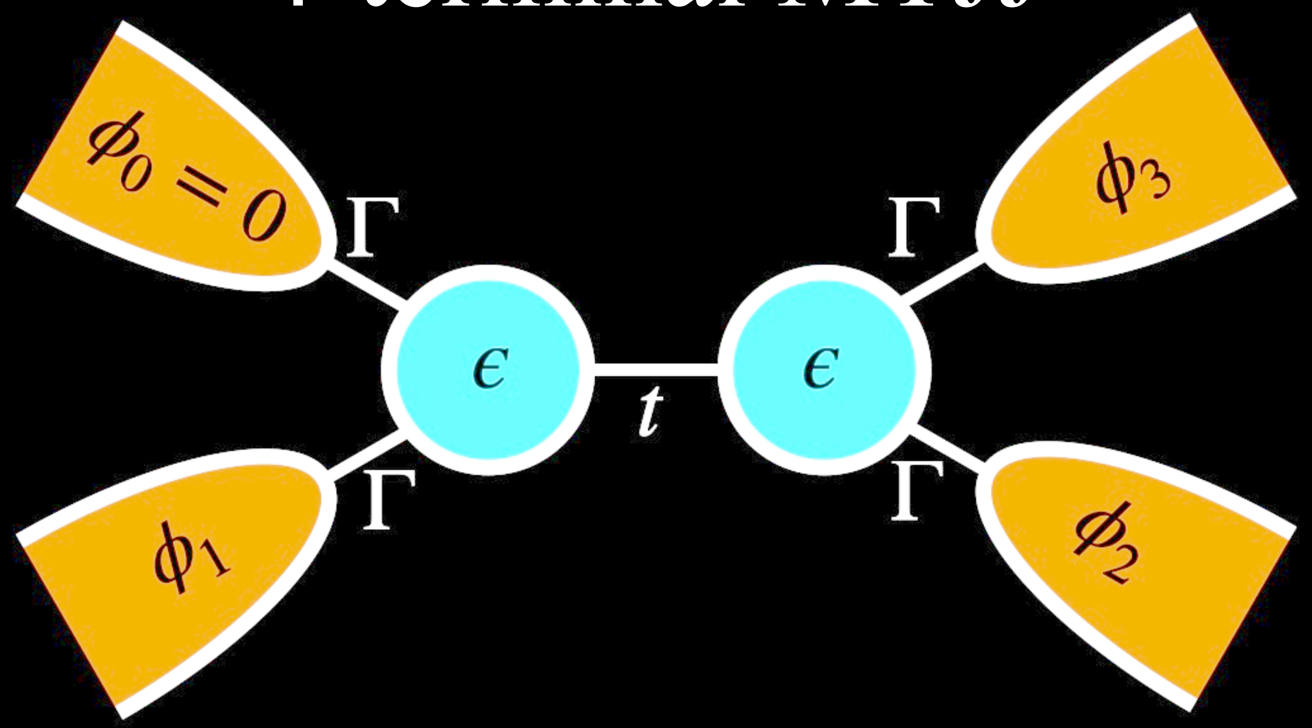
$$\mathrm{tr}(\Sigma) = 0 \Leftrightarrow \det H_{\mathrm{eff}} = 0 \Leftrightarrow \epsilon^2 - t^2 + 4\Gamma^2 = 0$$

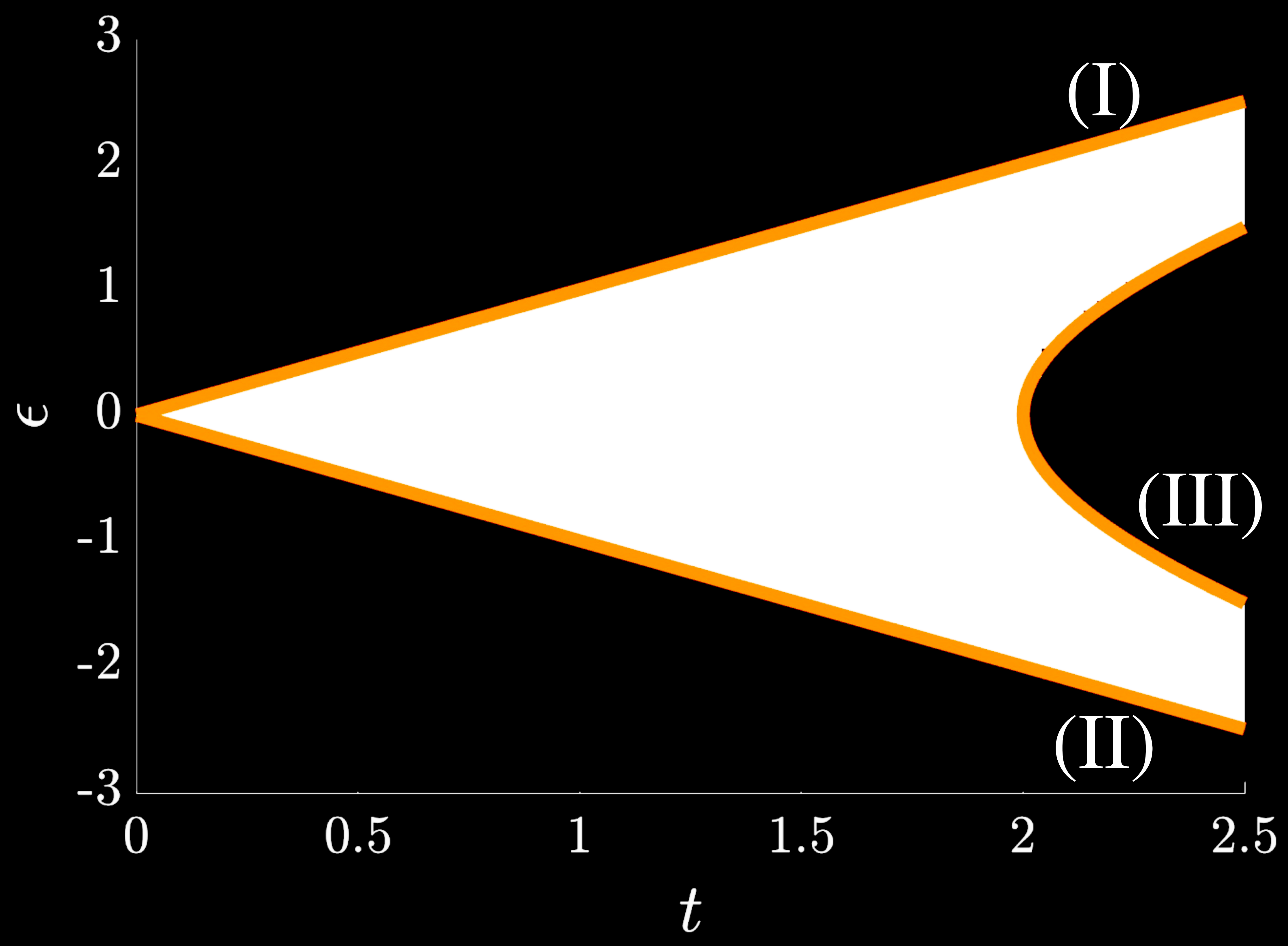
Reflectionless modes as source of Weyl Nodes

effective 2-terminal MTJJ

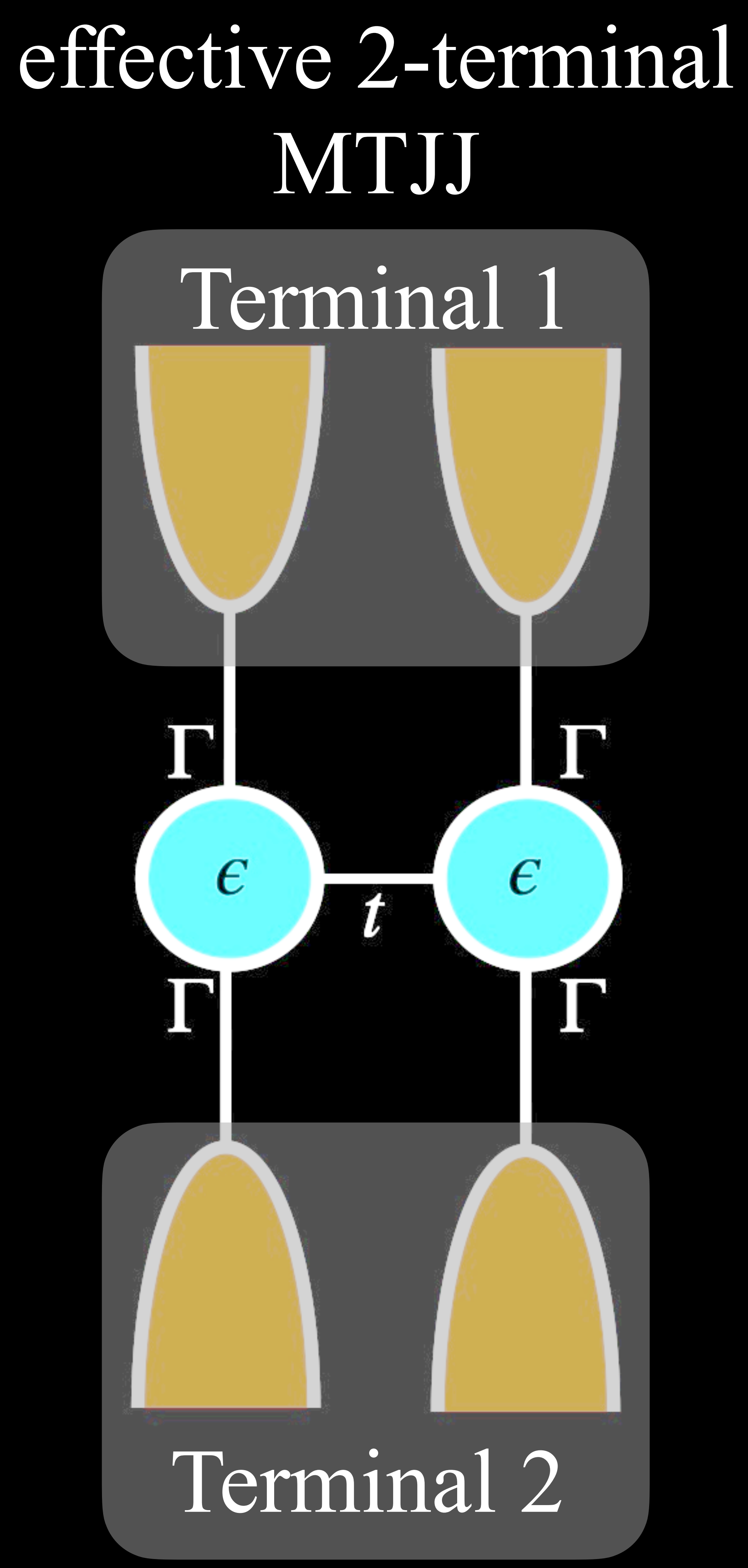


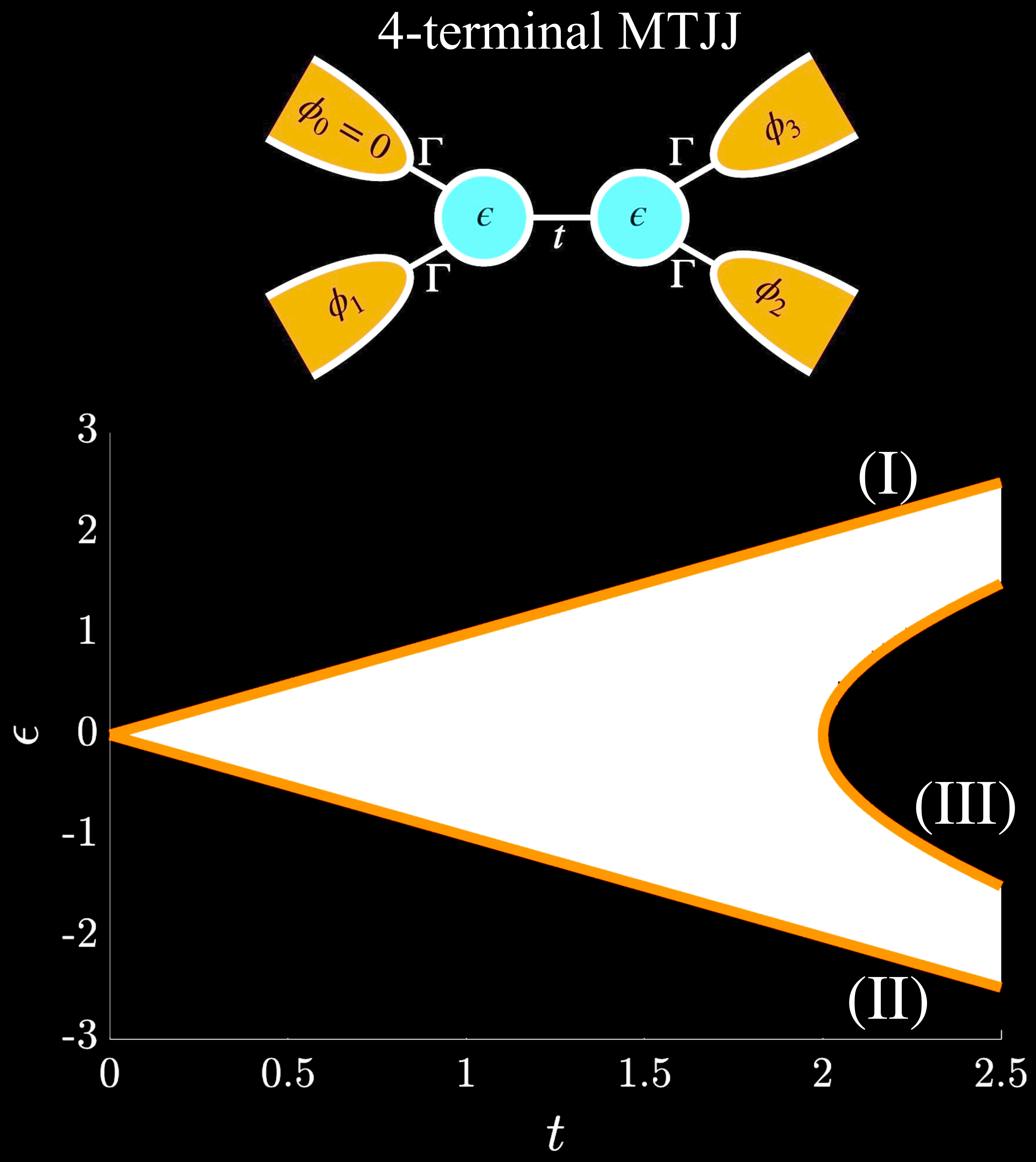
4-terminal MTJJ





Reflectionless modes as source of Weyl Nodes





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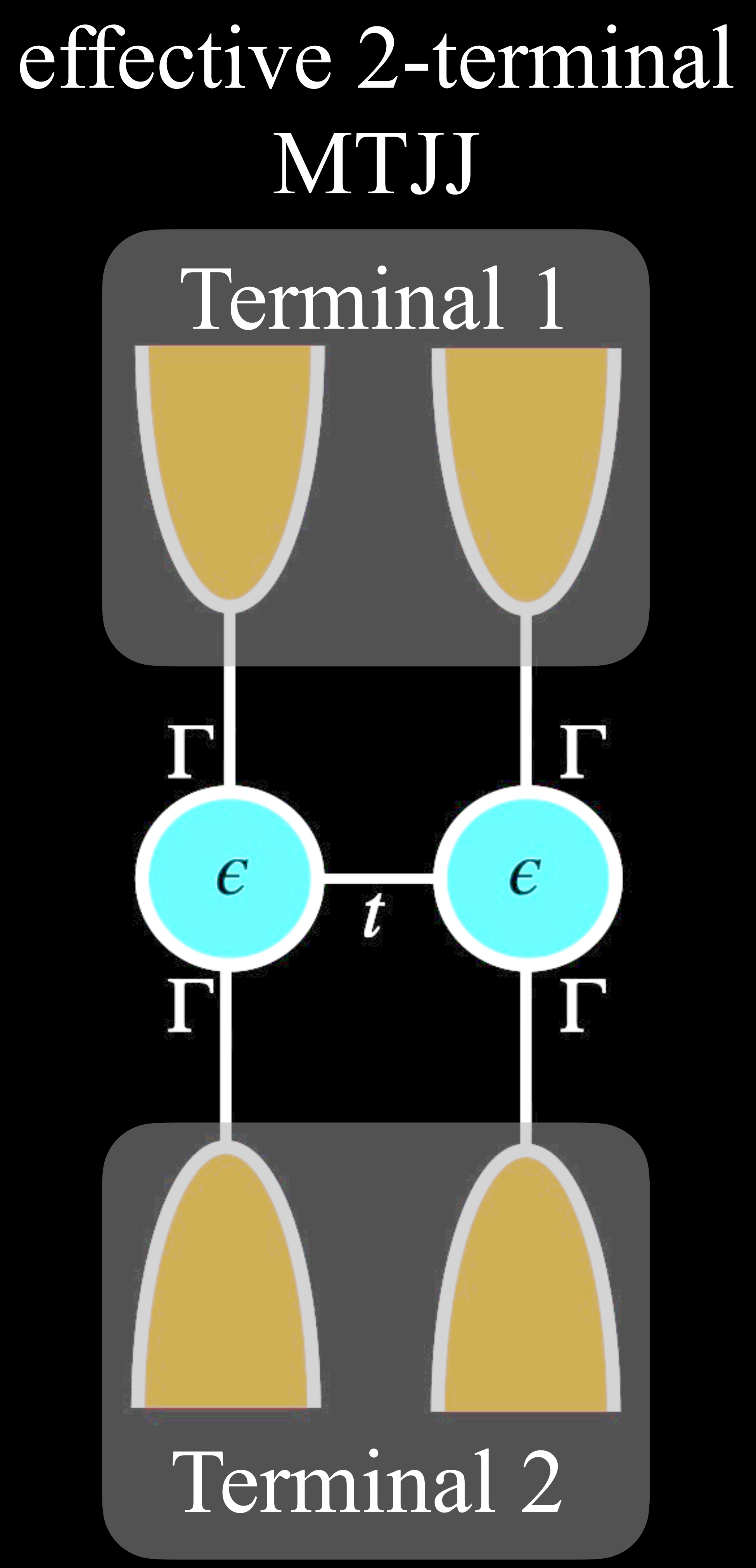
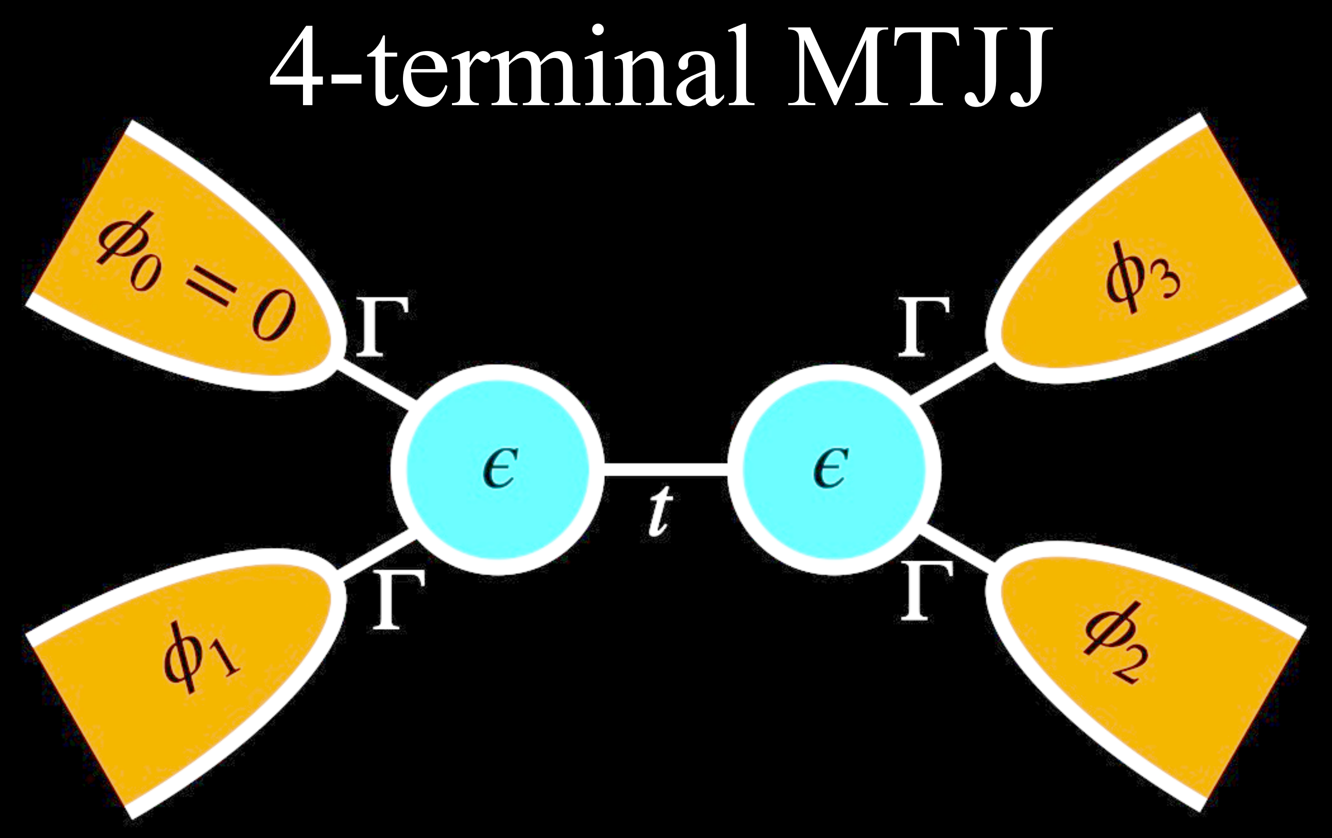
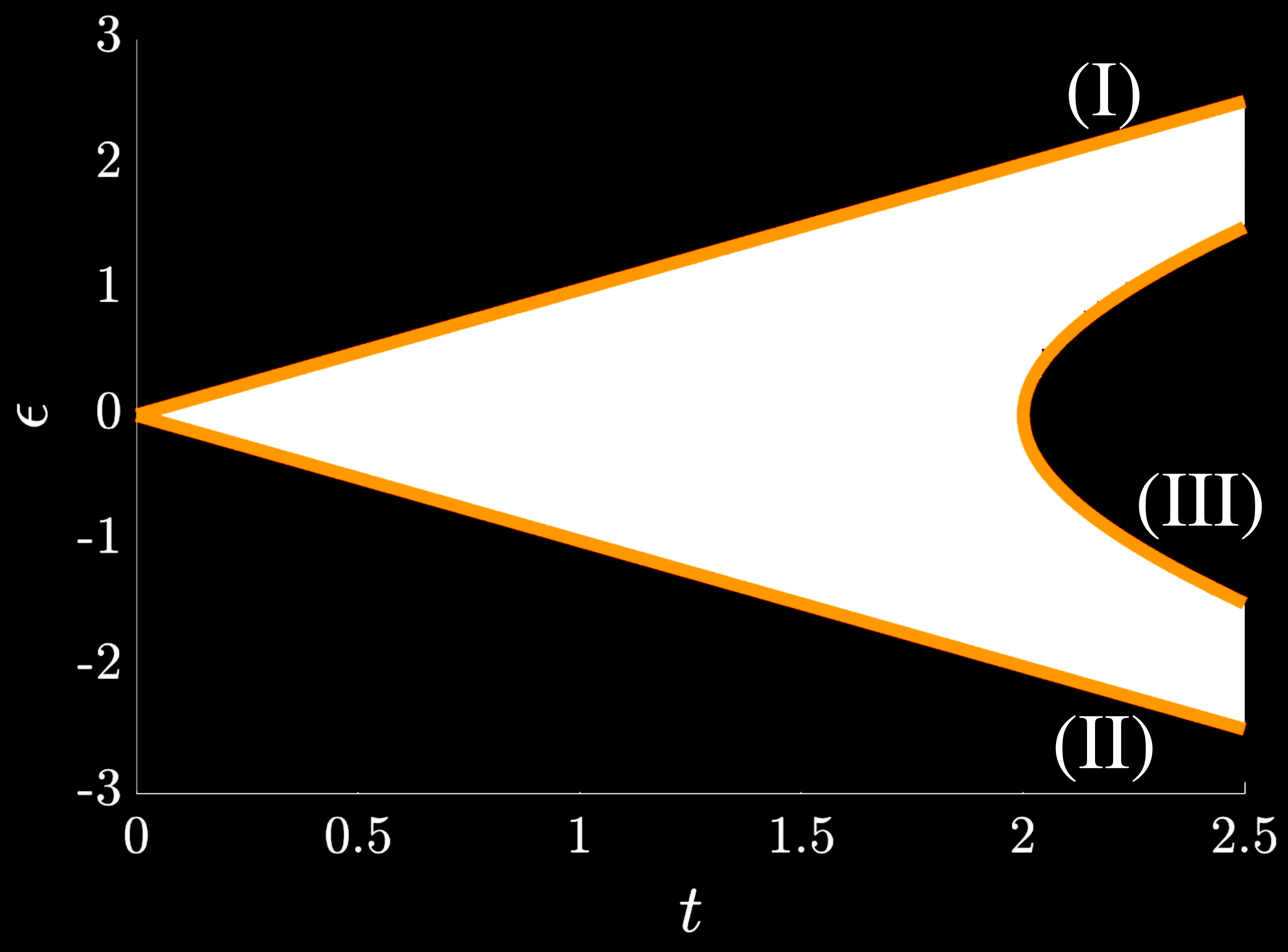


Normal state
Scattering matrix

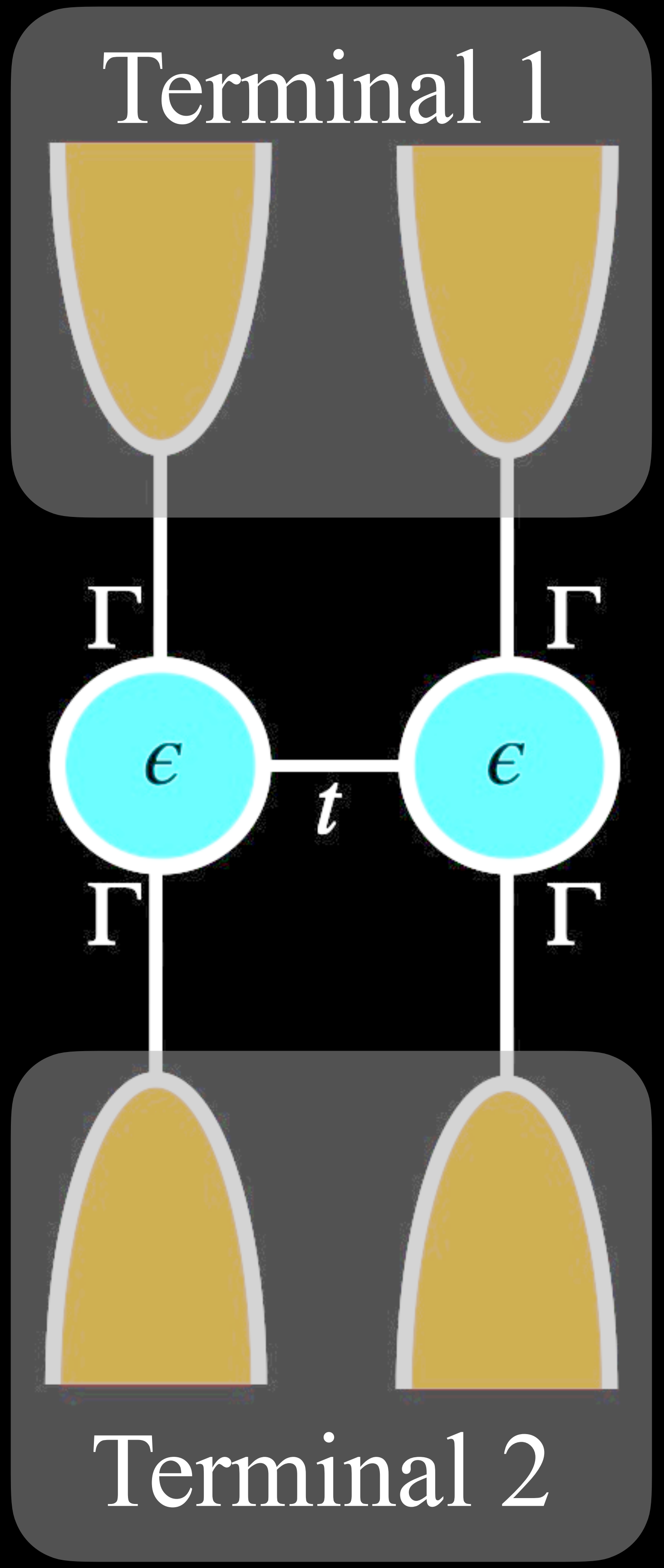
$$S_{\mathrm{N}} = \begin{pmatrix} r_{2\times2} & t_{2\times2} \\ t'_{2\times2} & r'_{2\times2} \end{pmatrix}$$





Reflectionless modes as source of Weyl Nodes

effective 2-terminal MTJJ



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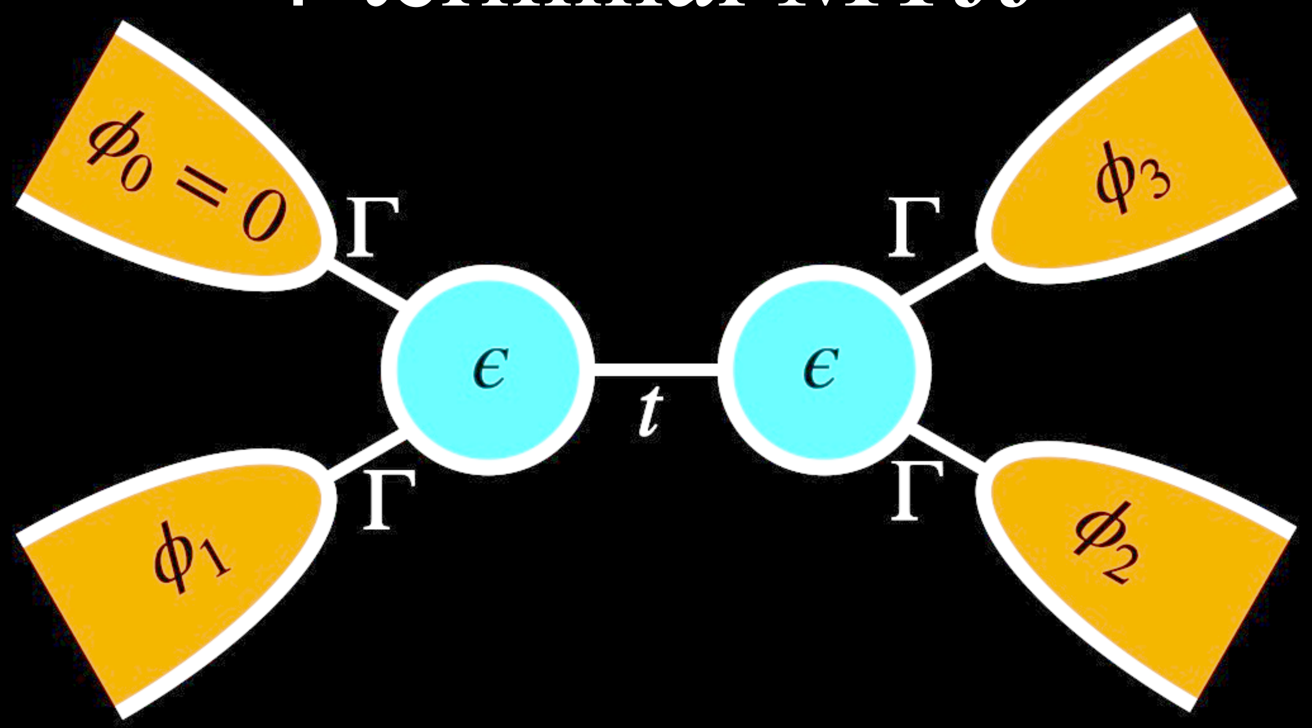
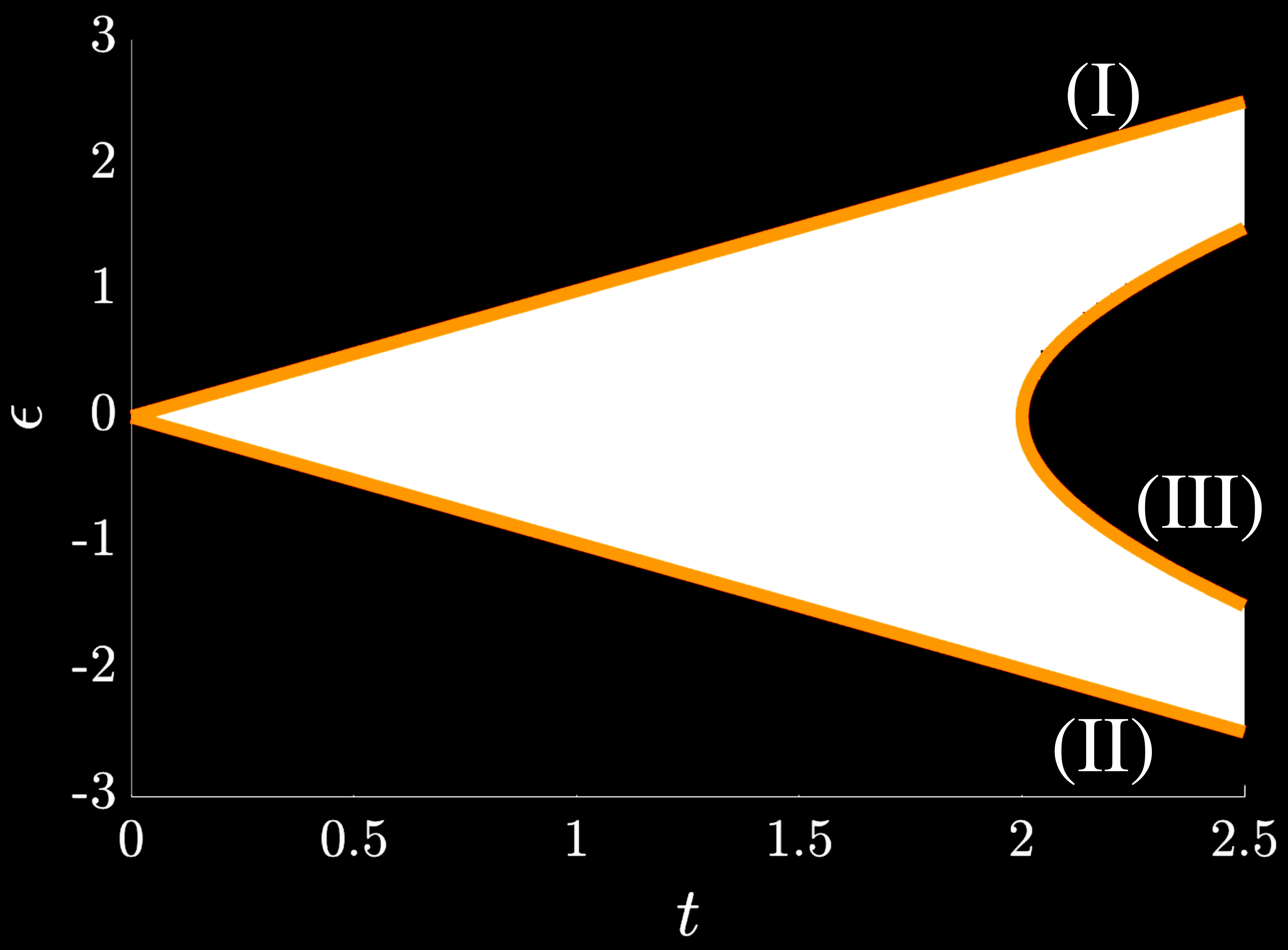
Diagonalized refl. matrix

$$D_{r_{2\times2}} = \begin{pmatrix} \frac{E-(\epsilon-t)}{E-(\epsilon-t-2\imath\Gamma)} & 0 \\ 0 & \frac{E-(\epsilon+t)}{E-(\epsilon+t-2\imath\Gamma)} \end{pmatrix}$$

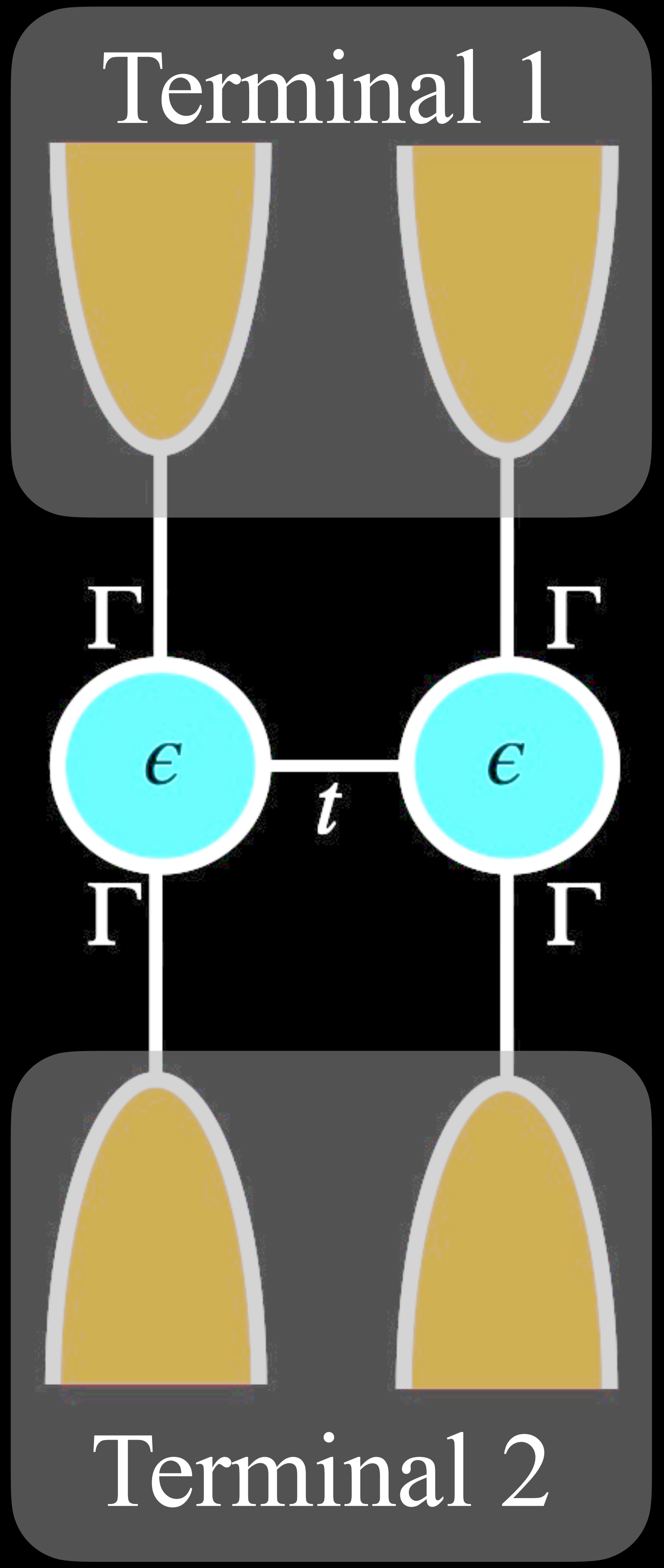
4-terminal MTJJ





Reflectionless modes as source of Weyl Nodes

effective 2-terminal
MTJJ



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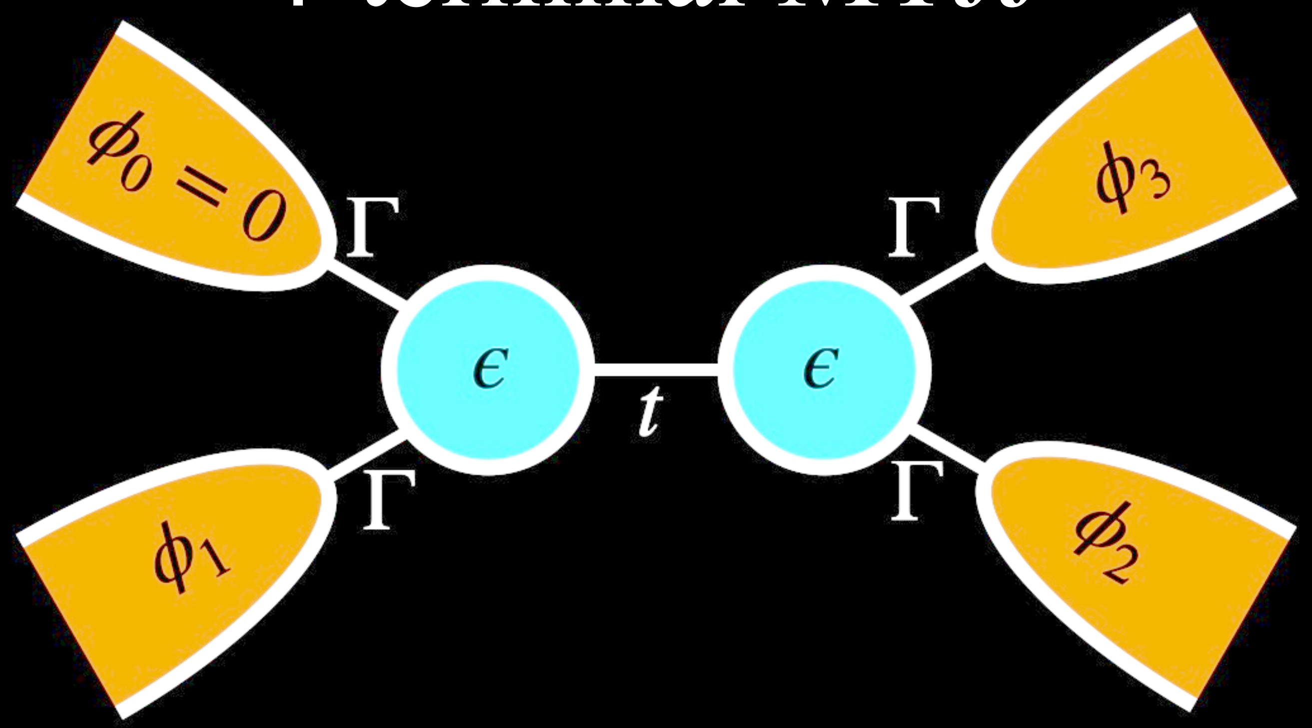
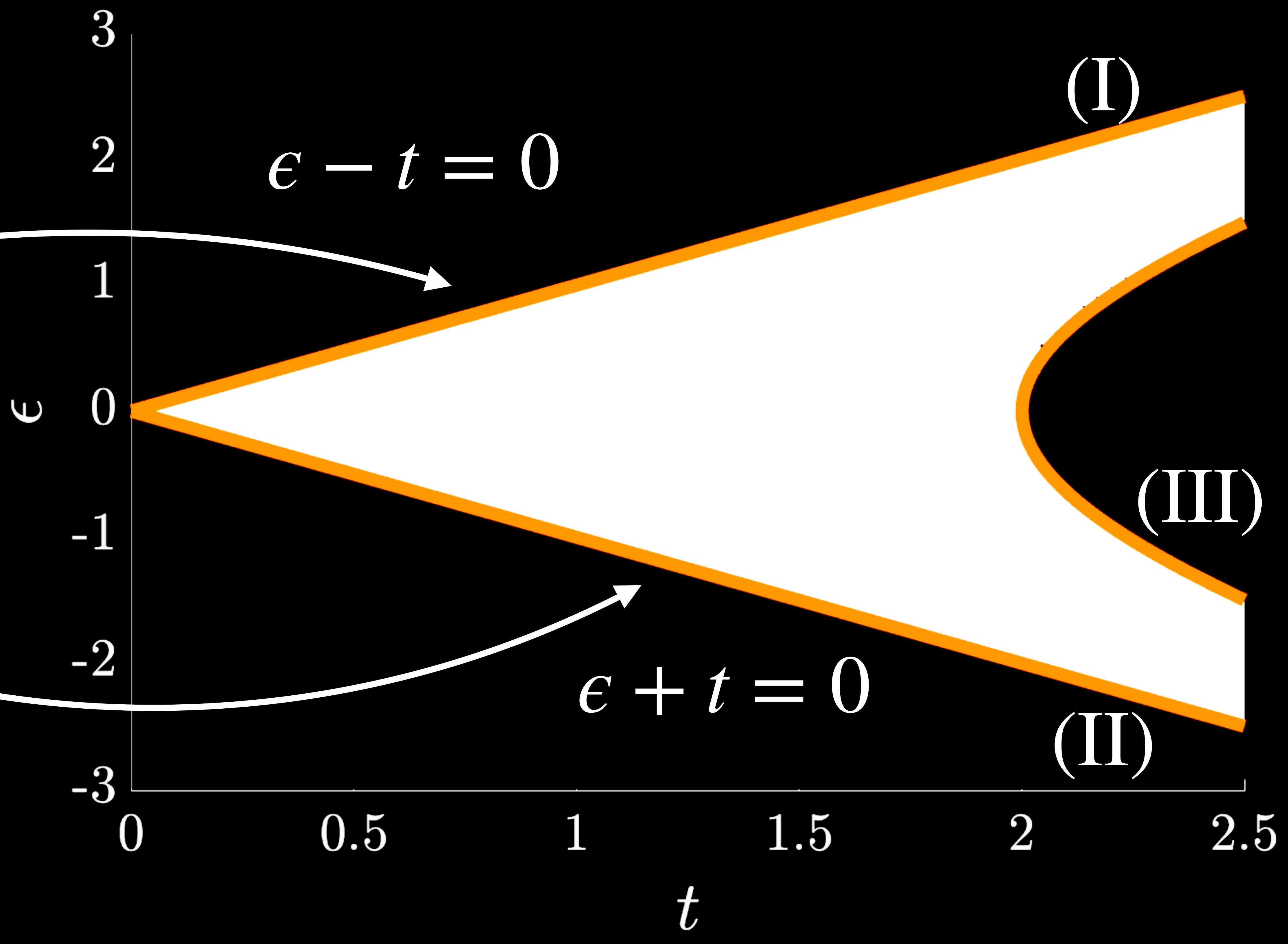
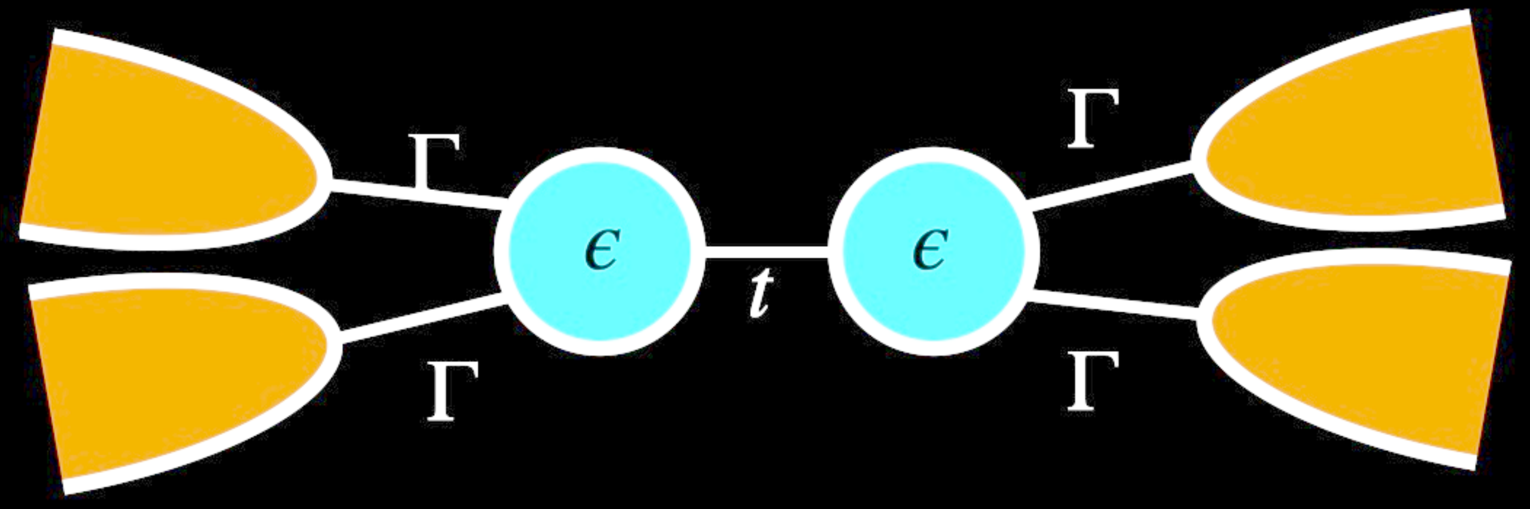
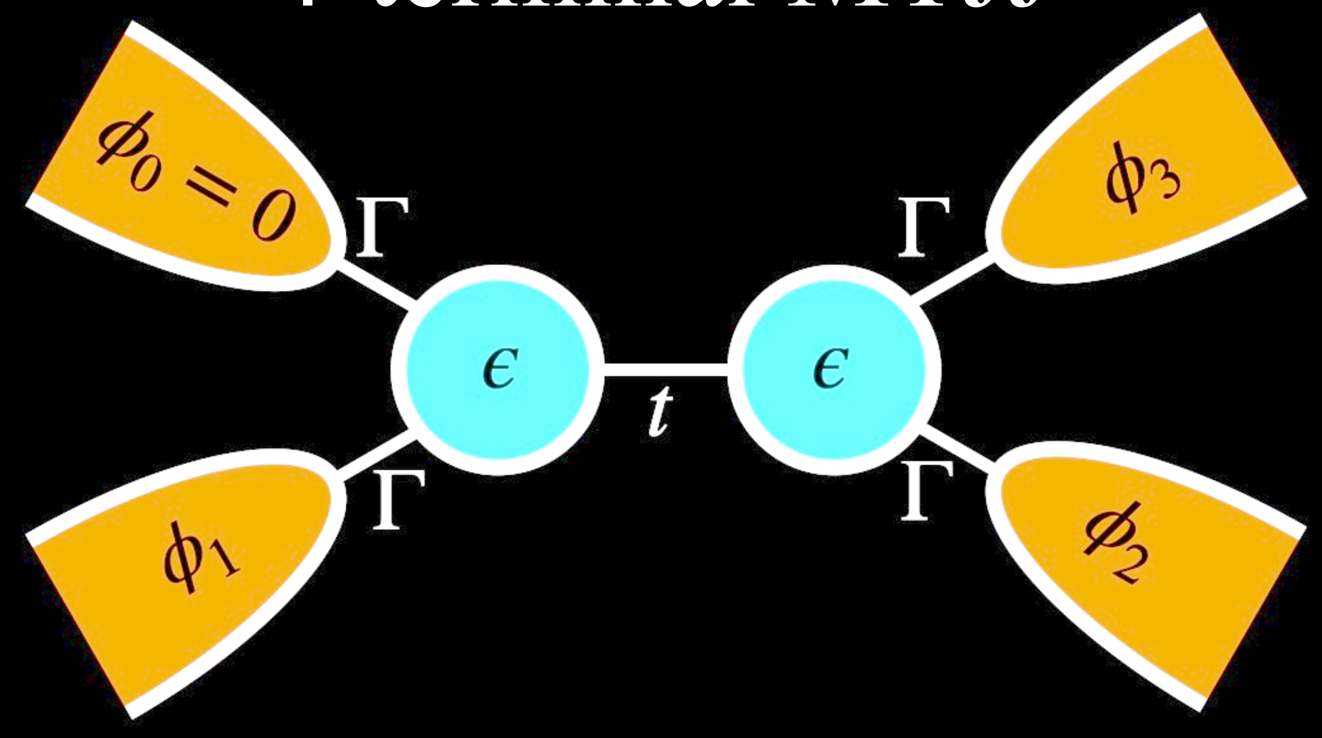
4-terminal MTJJ

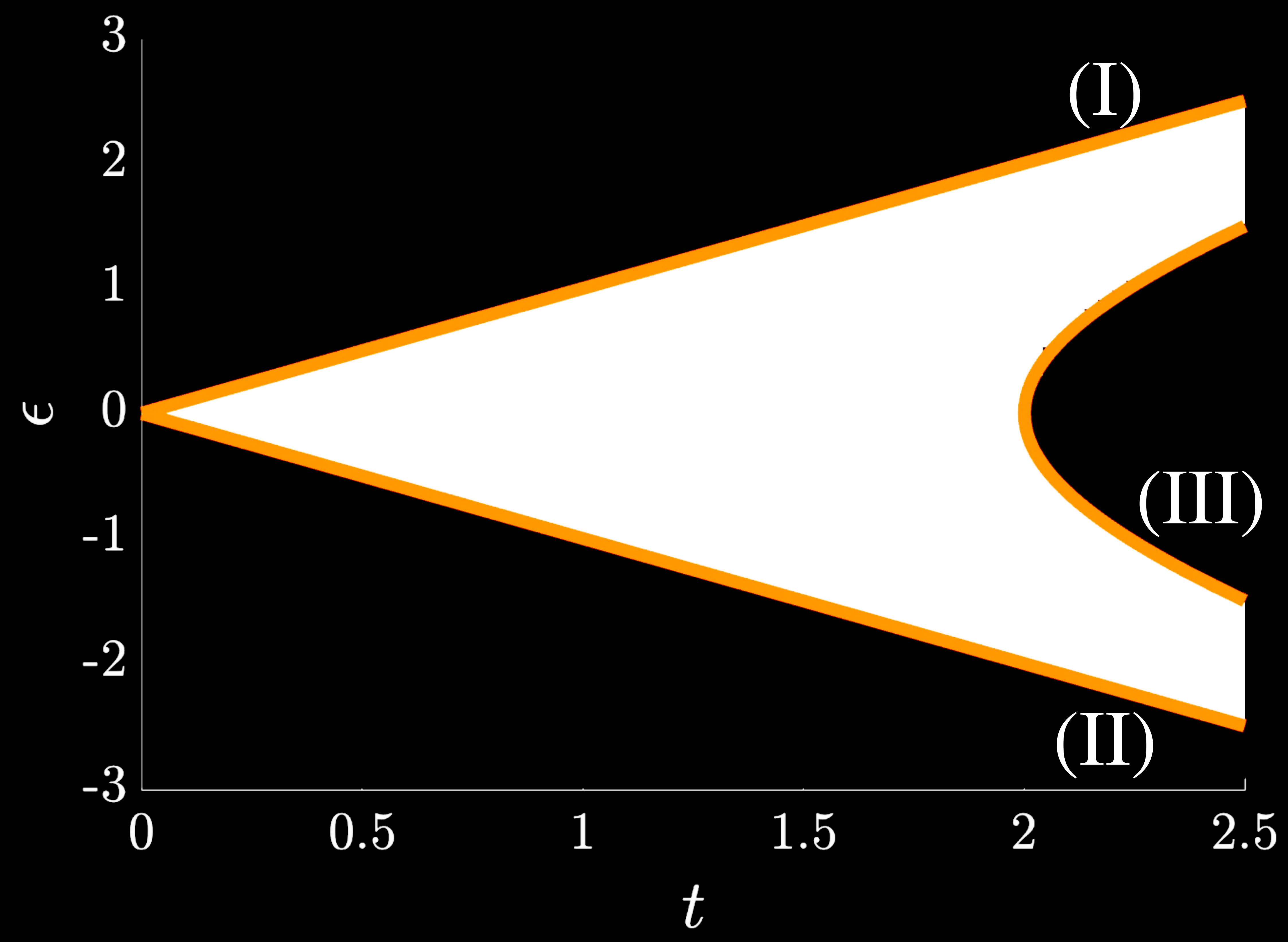




Reflectionless modes as source of Weyl Nodes







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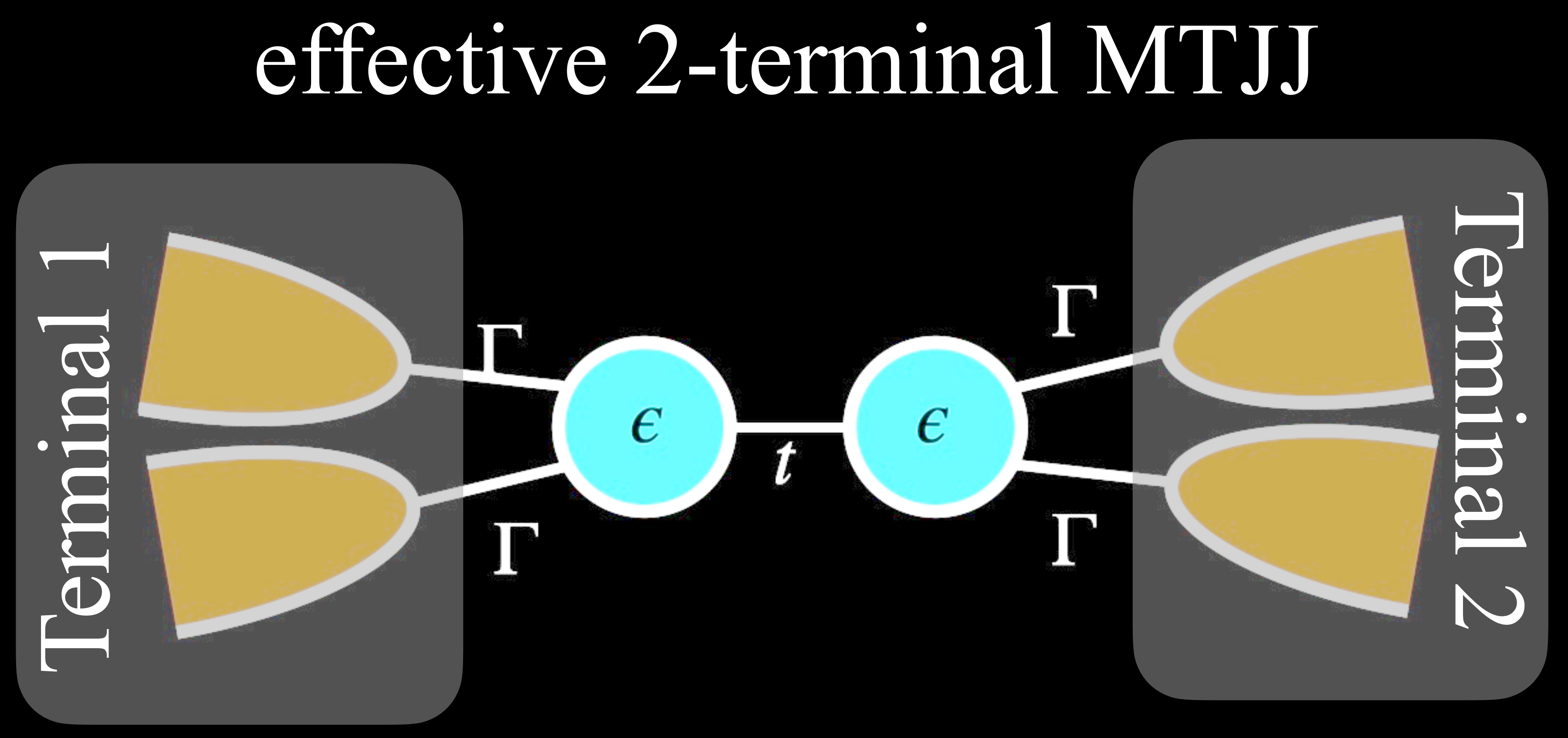
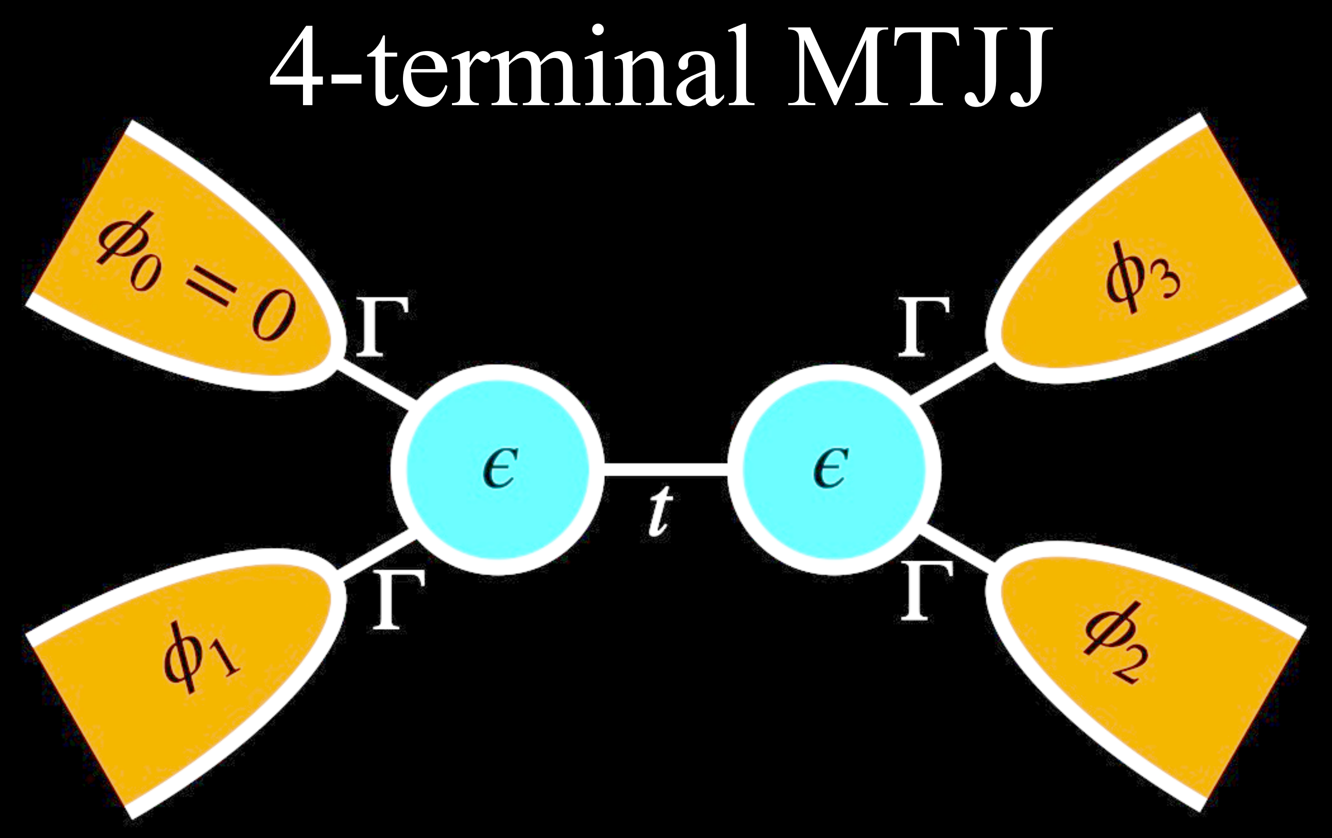
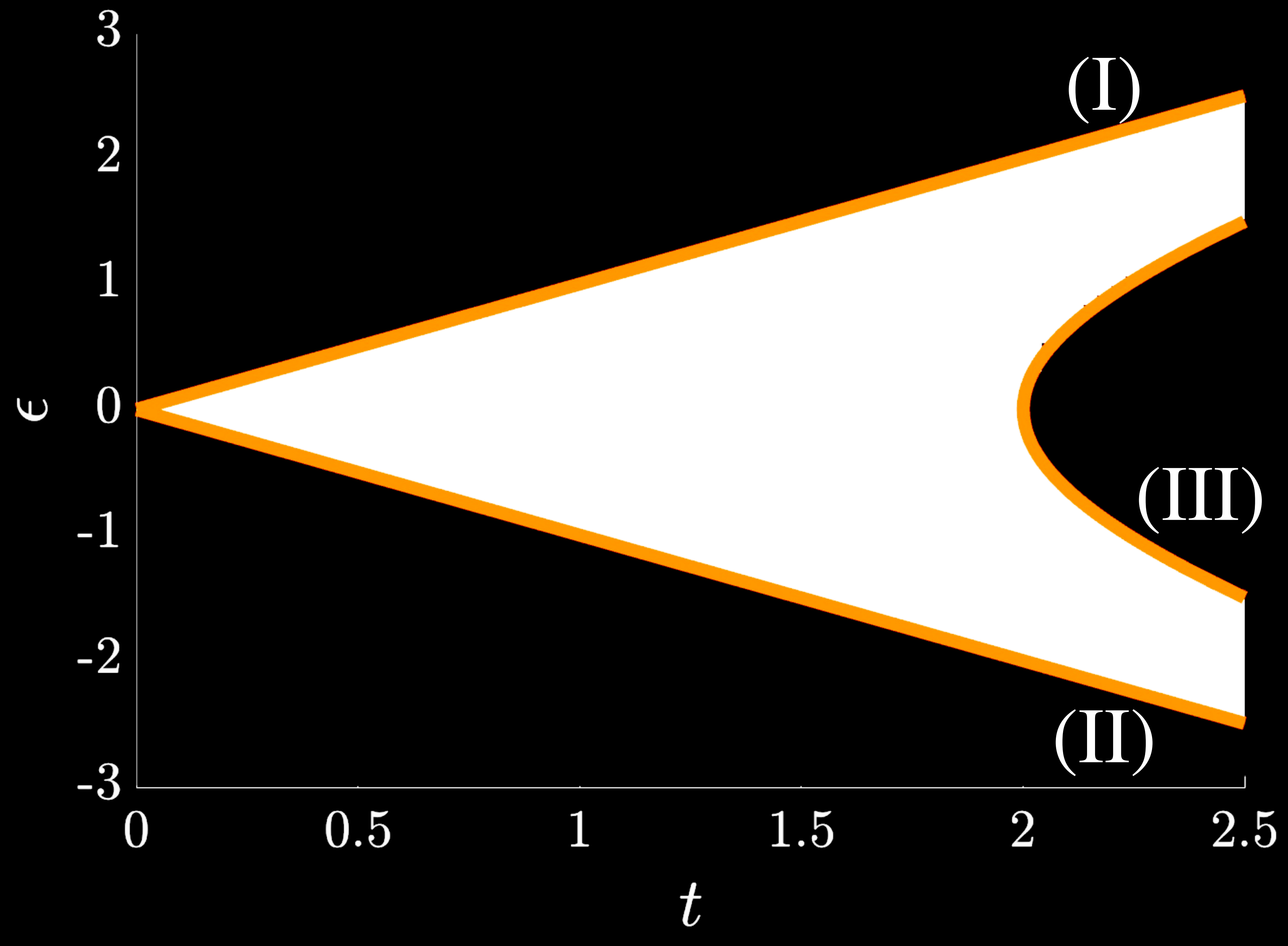
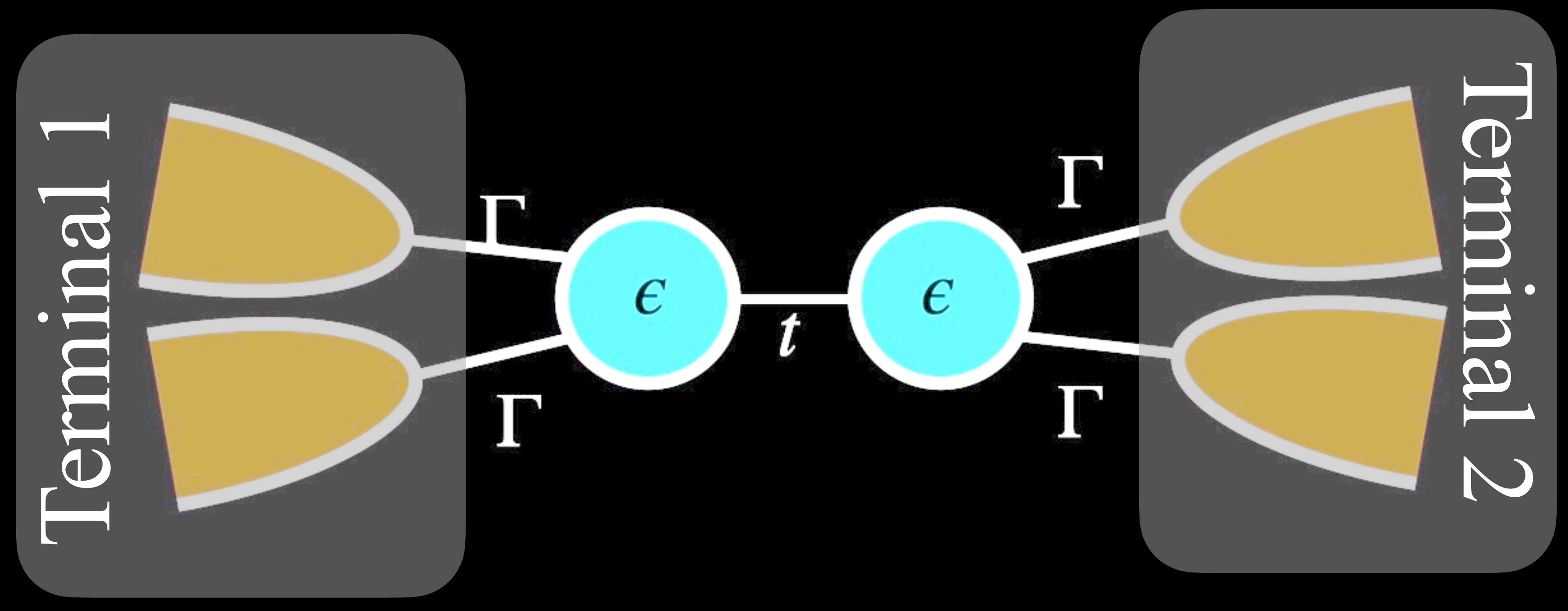






Reflectionless modes as source of Weyl Nodes

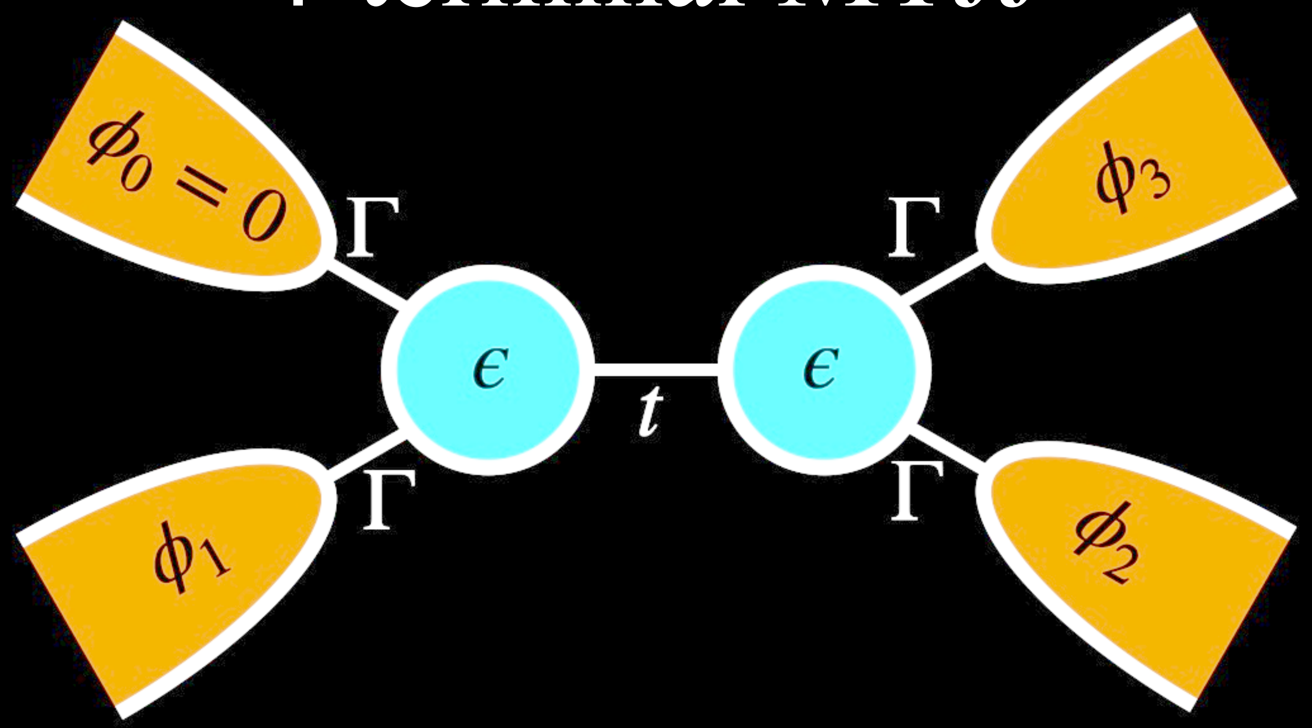
effective 2-terminal MTJJ

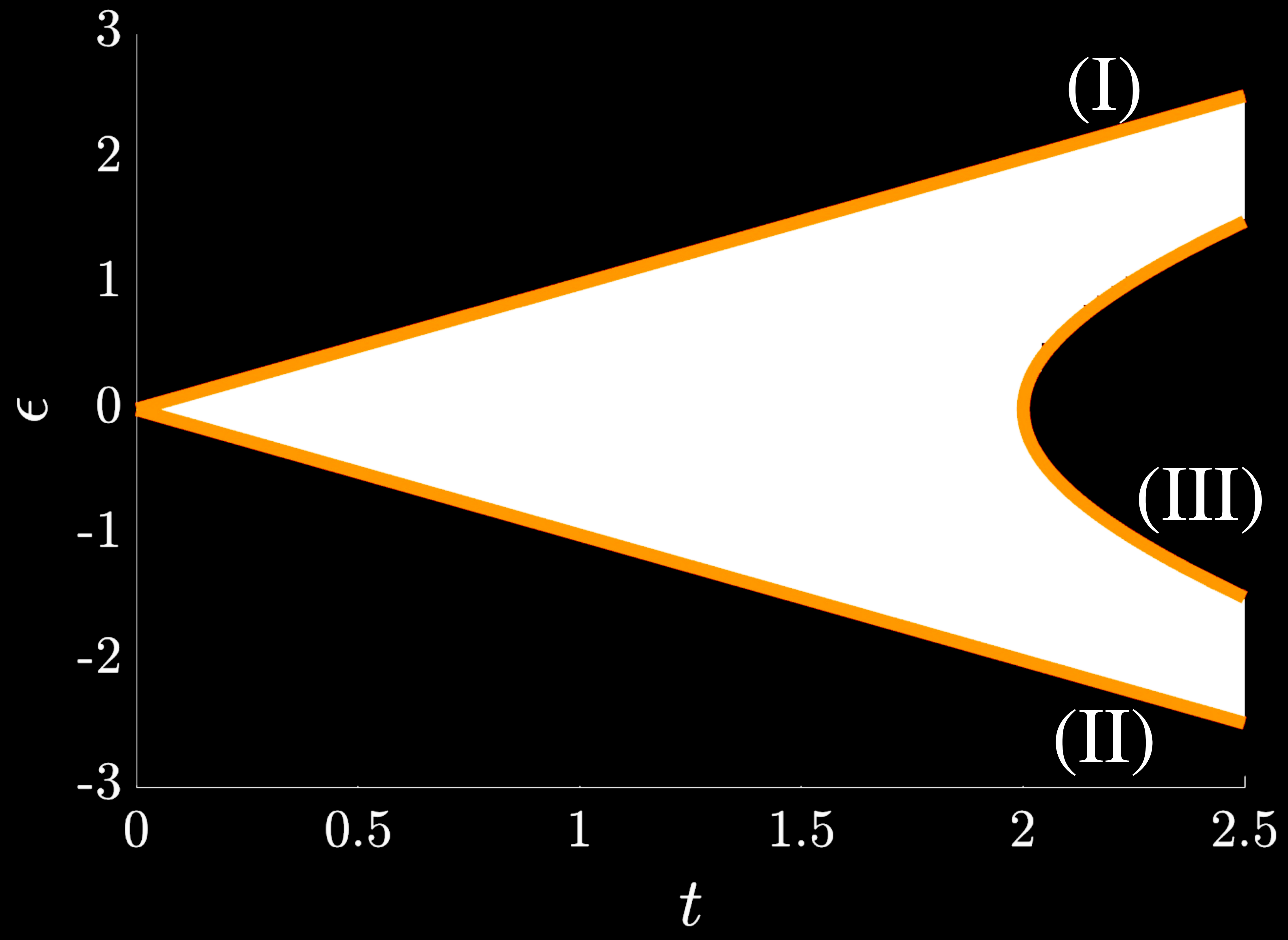


Scattering matrix

$$S_{\mathrm{N}} = \begin{pmatrix} r_{2\times2} & t_{2\times2} \\ t'_{2\times2} & r'_{2\times2} \end{pmatrix}$$

4-terminal MTJJ





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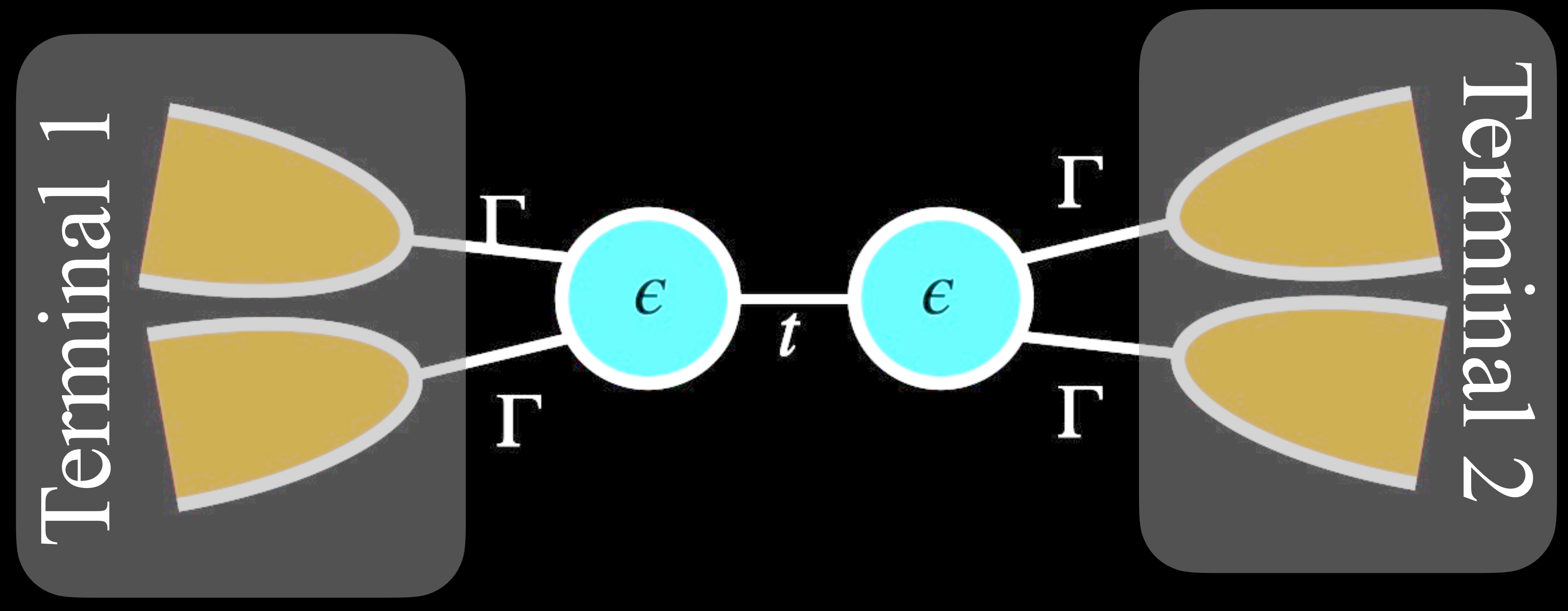

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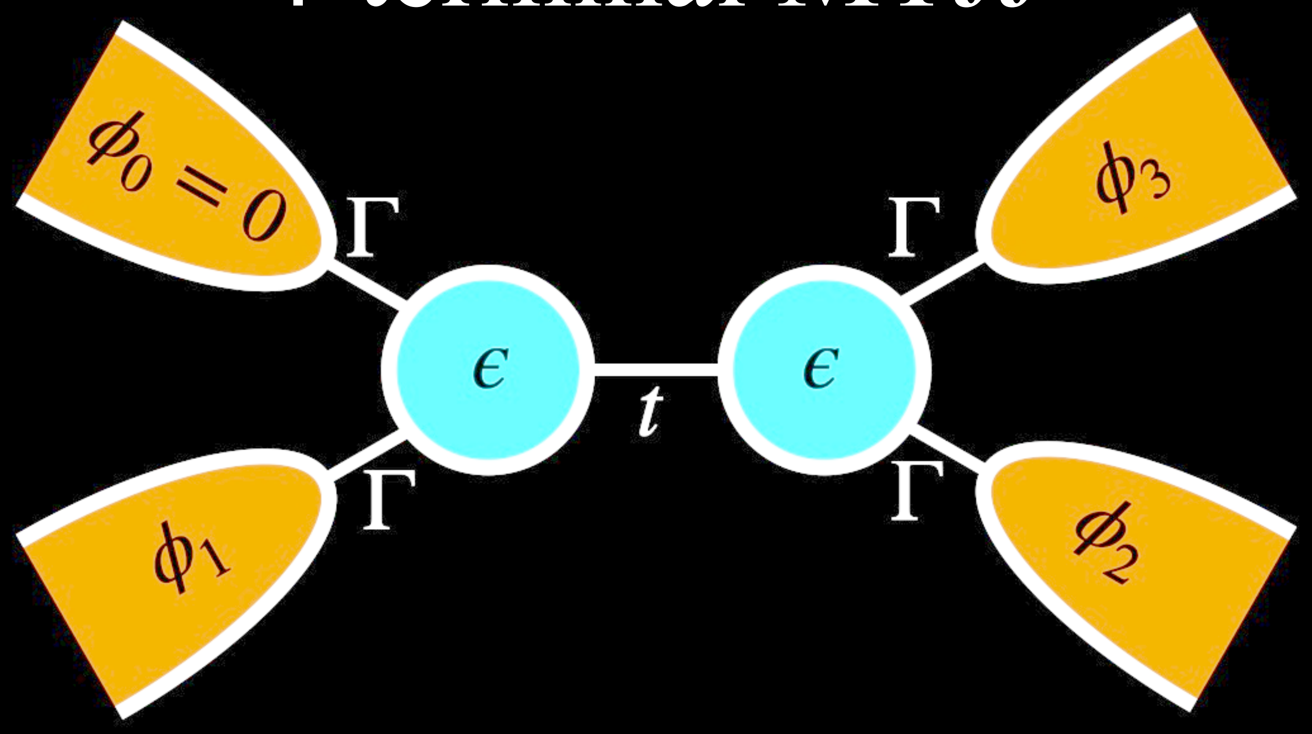

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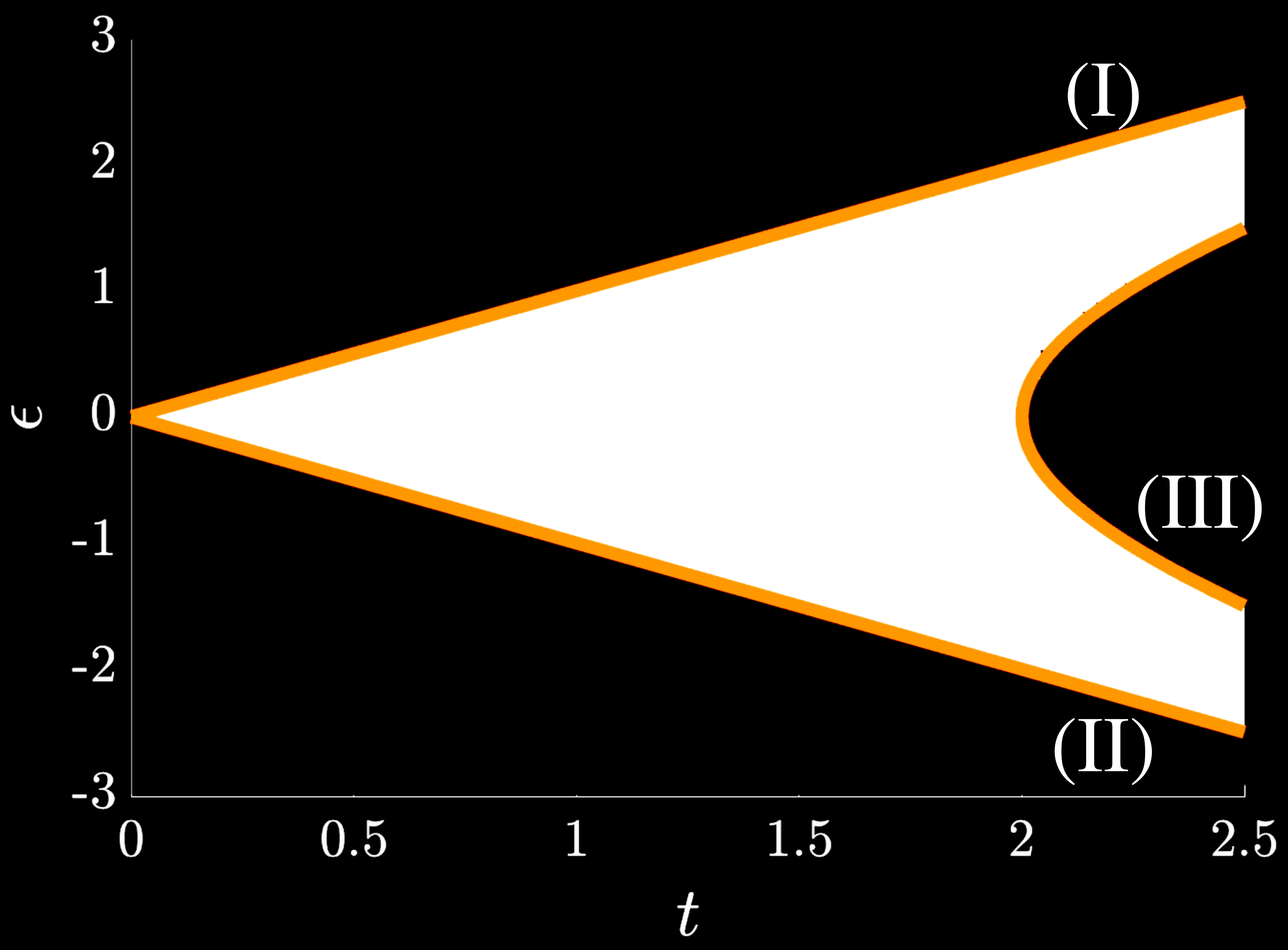
Diagonalized refl. matrix

$$D_{r_{2\times2}} = \begin{pmatrix} \left[E(E-e) - \left(t^2 - \epsilon^2 - 4\Gamma^2\right)\right]/D(E) & 0 \\ 0 & 1 \end{pmatrix}$$



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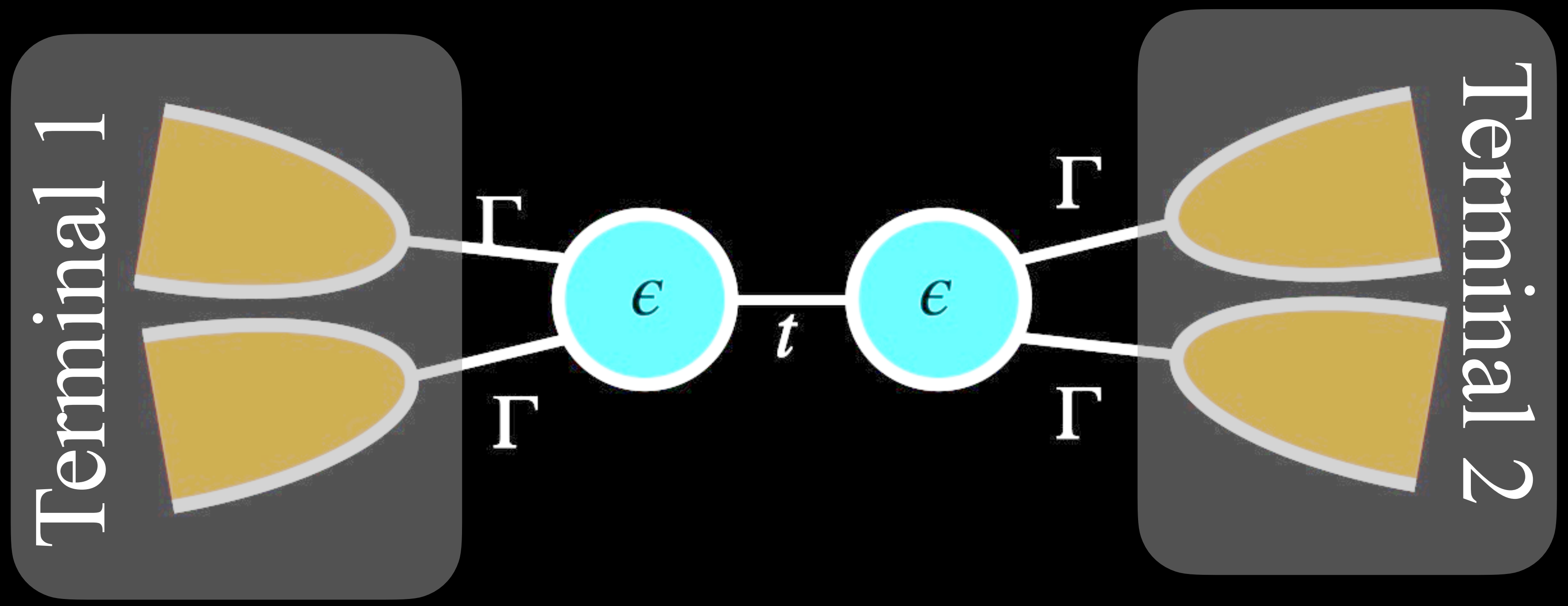

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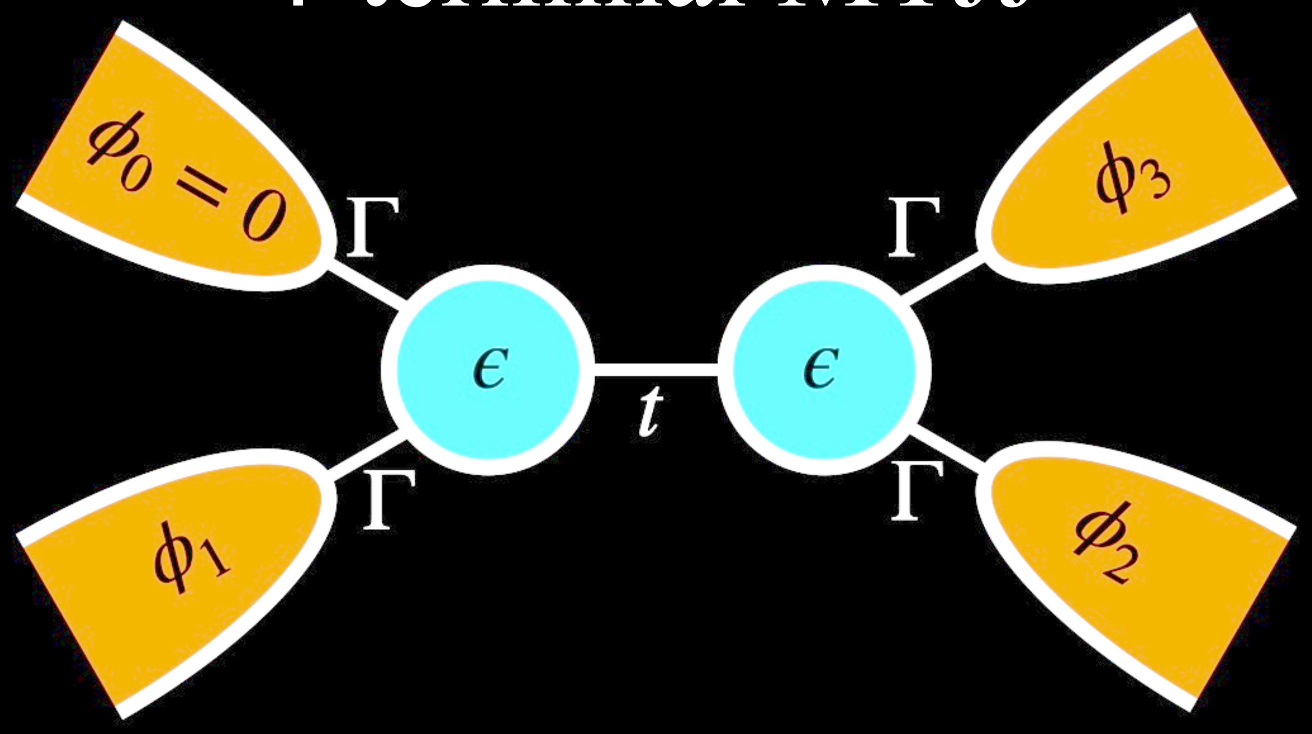

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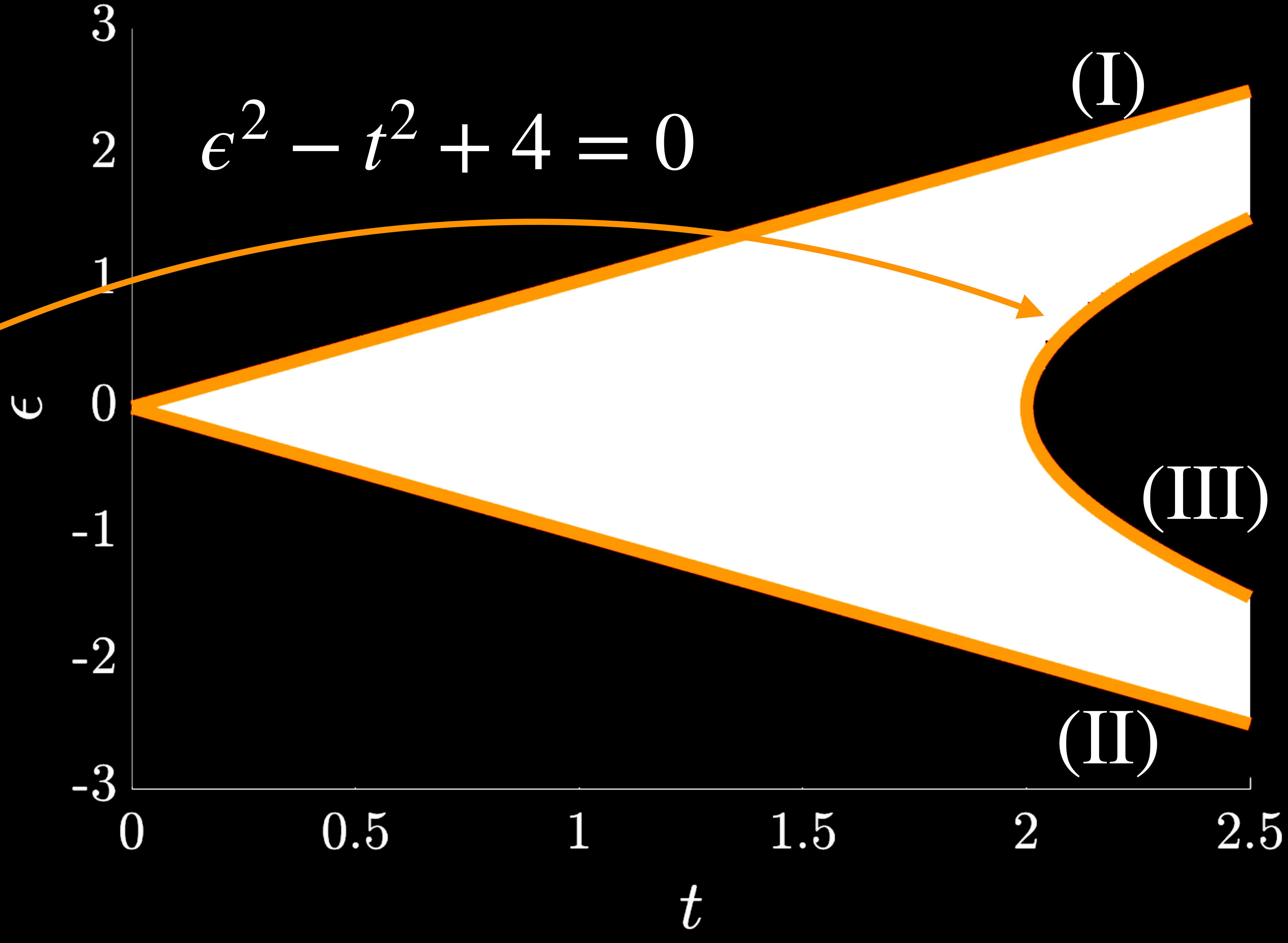

normal state S-matrix ⇔ effective Hamiltonian?

Phys. Rev. A 102, 063511 (2020)

Determinant of Reflection matrix with incoming and outgoing self-energy

$$\det(r) = \frac{\det(E - H_\mathrm{N} - \imath(\Sigma_\mathrm{in} - \Sigma_\mathrm{out}))}{\det(E - H_\mathrm{N} + \imath\Sigma(\overrightarrow{\phi} = 0))}$$

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(III) $\det(r_{12}(E = 0)) \propto \det(H_{\mathrm{N}} + 2\imath\sigma_z)$

$\propto (\epsilon^2 - t^2 + 4)$

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Phys. Rev. A 102, 063511 (2020)

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Effective Hamiltonian

$$H_\mathrm{eff} = \begin{pmatrix} H_\mathrm{N} & \Sigma \\ \Sigma^* & -H_\mathrm{N} \end{pmatrix} \qquad \Sigma = 2\imath\sigma_z$$

$$\det(H_\mathrm{eff}) = \det(H_\mathrm{N} + 2\imath\sigma_z)\det(-H_\mathrm{N} + 2\imath\sigma_z)$$

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Phys. Rev. A 102, 063511 (2020)

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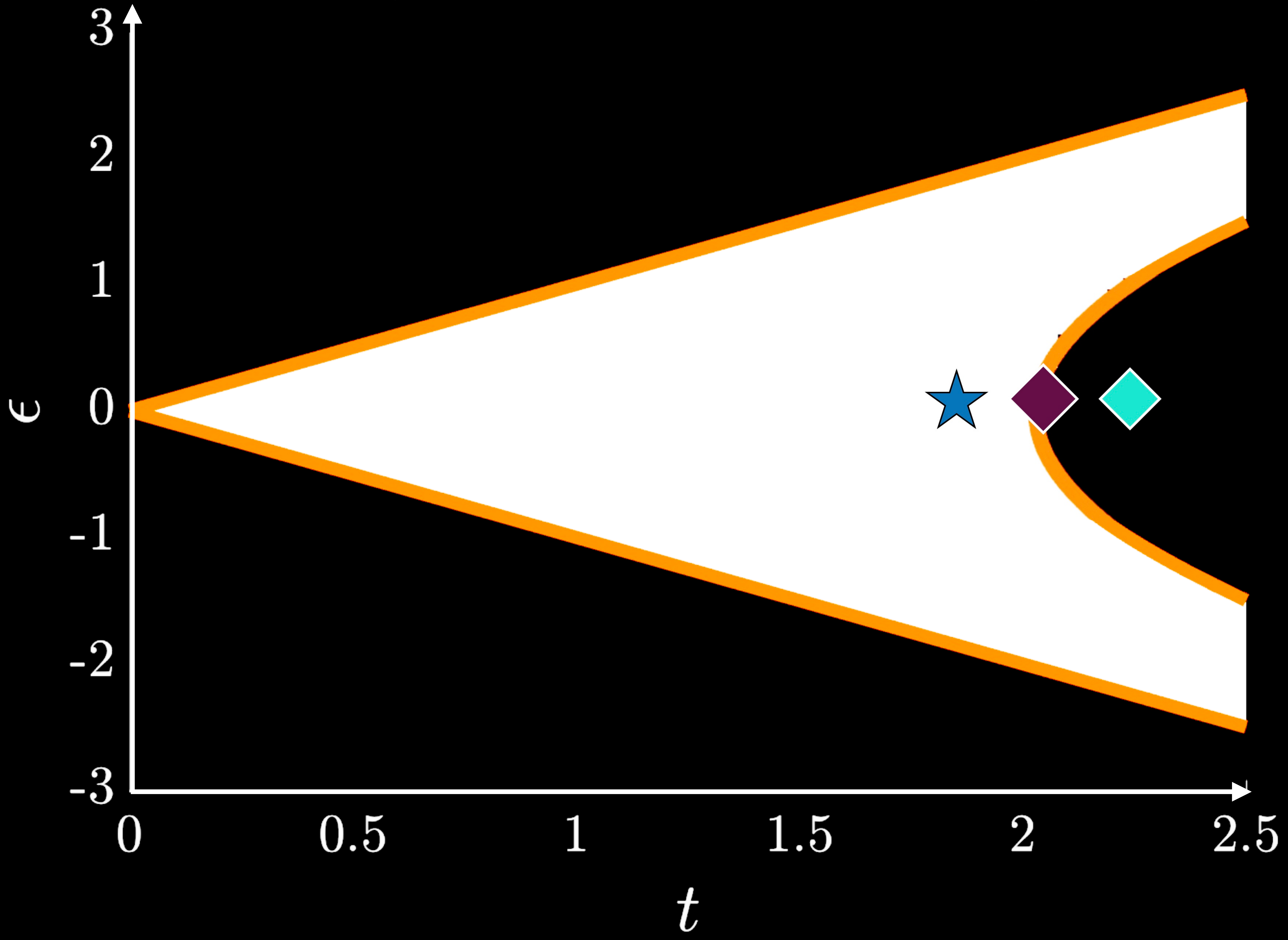
The superconducting phases play the same role as scattering with incoming and outgoing self-energies!

Topological charge of refl. less modes

$$\vec{V} = \left(\frac{\mathrm{Re}\,\det(r(z))}{|\det(r(z))|}, \frac{\mathrm{Im}\,\det(r(z))}{|\det(r(z))|} \right)$$

complex energy
$z \in \mathbb{C}$



W. R. Sweeney, et. al., Phys. Rev. A 102, 063511 (2020)

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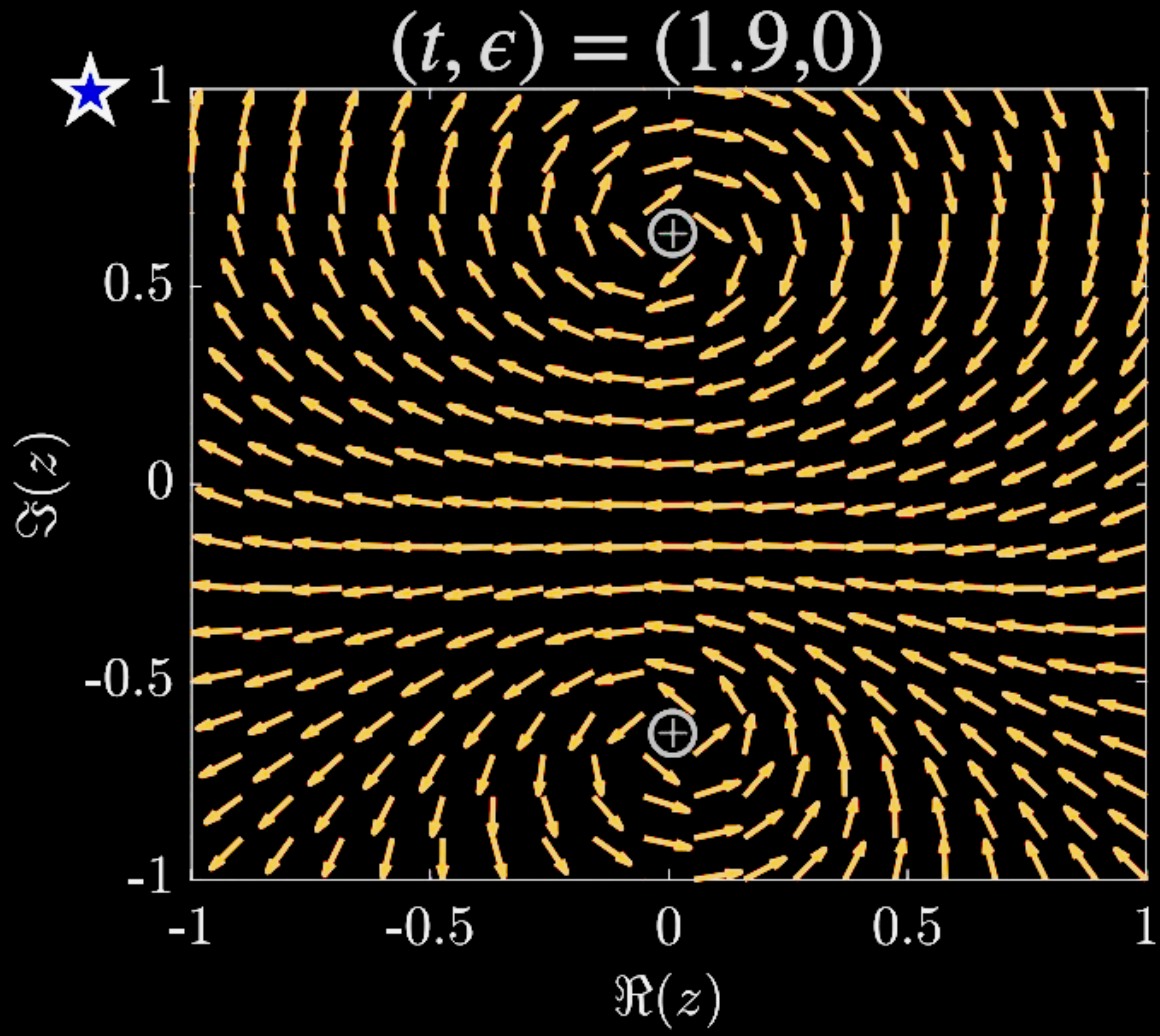

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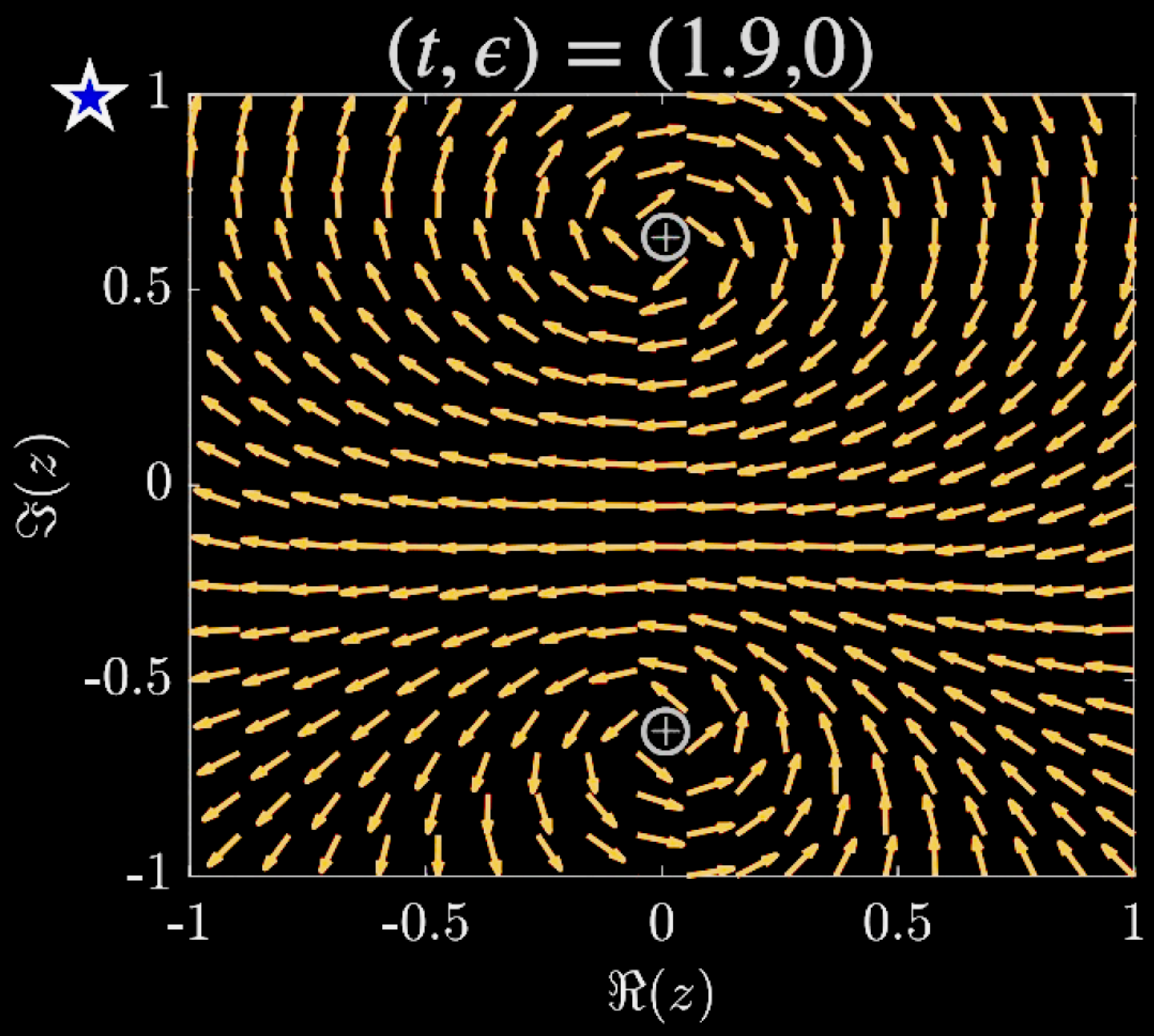

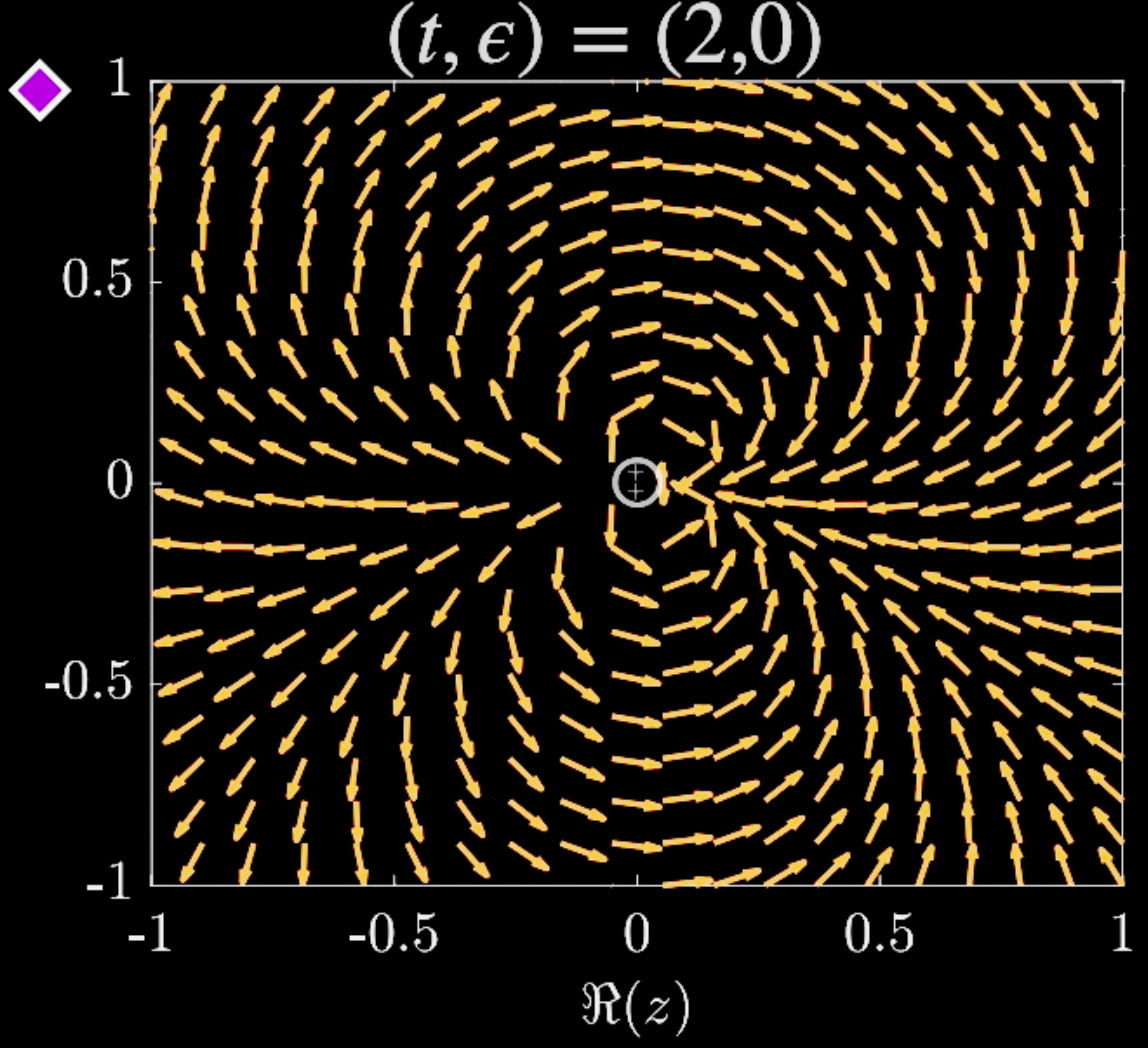

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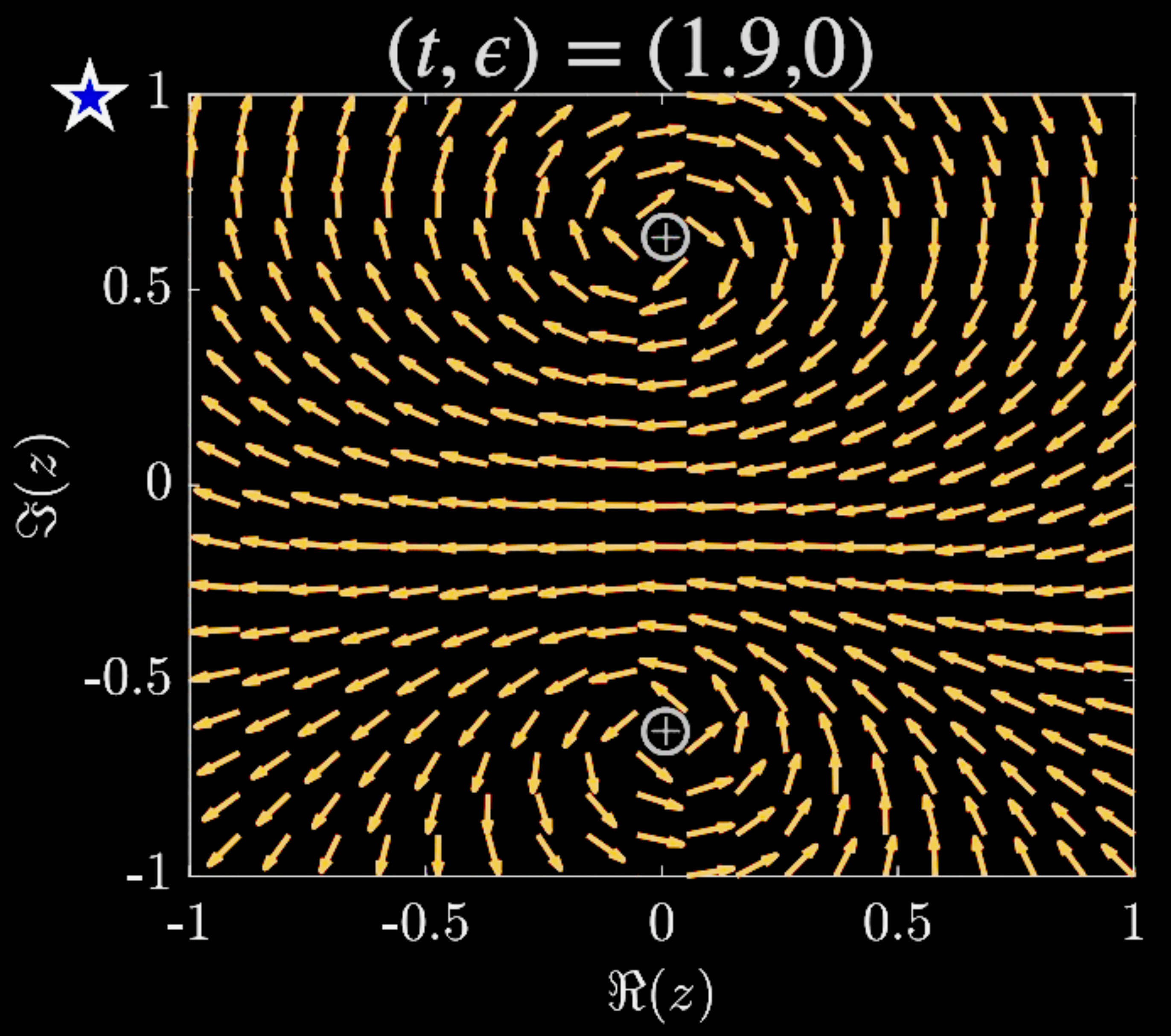

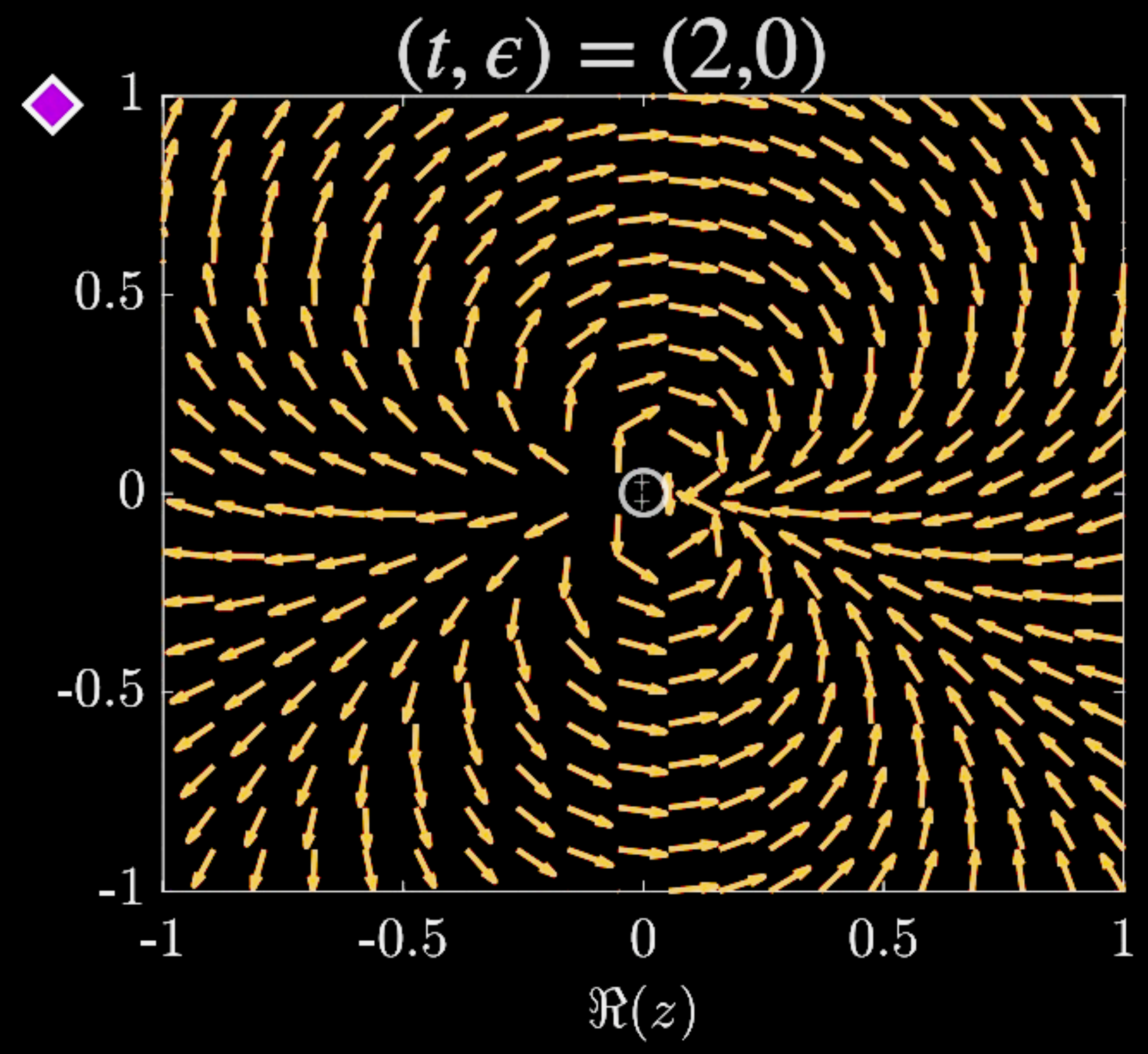

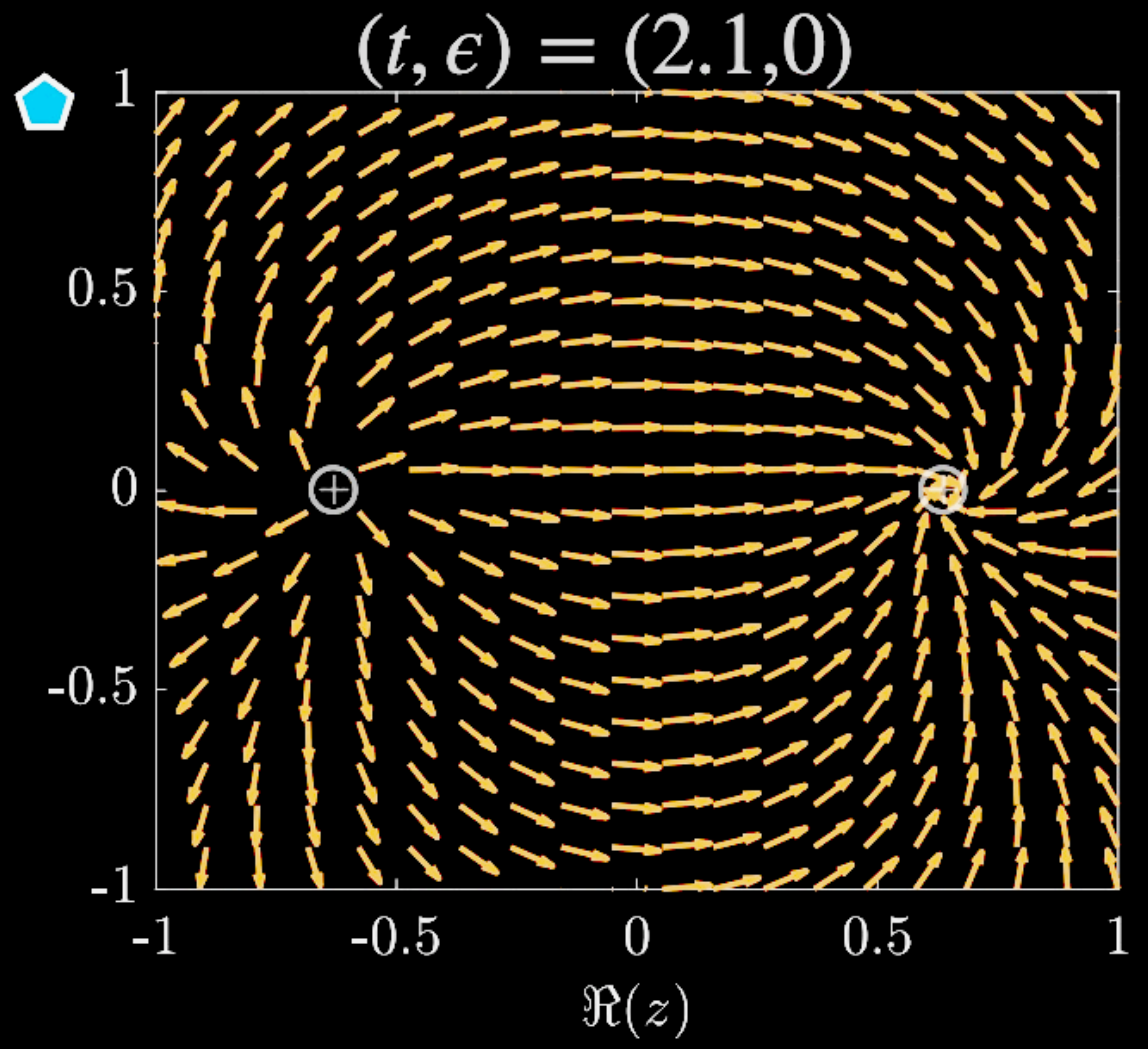

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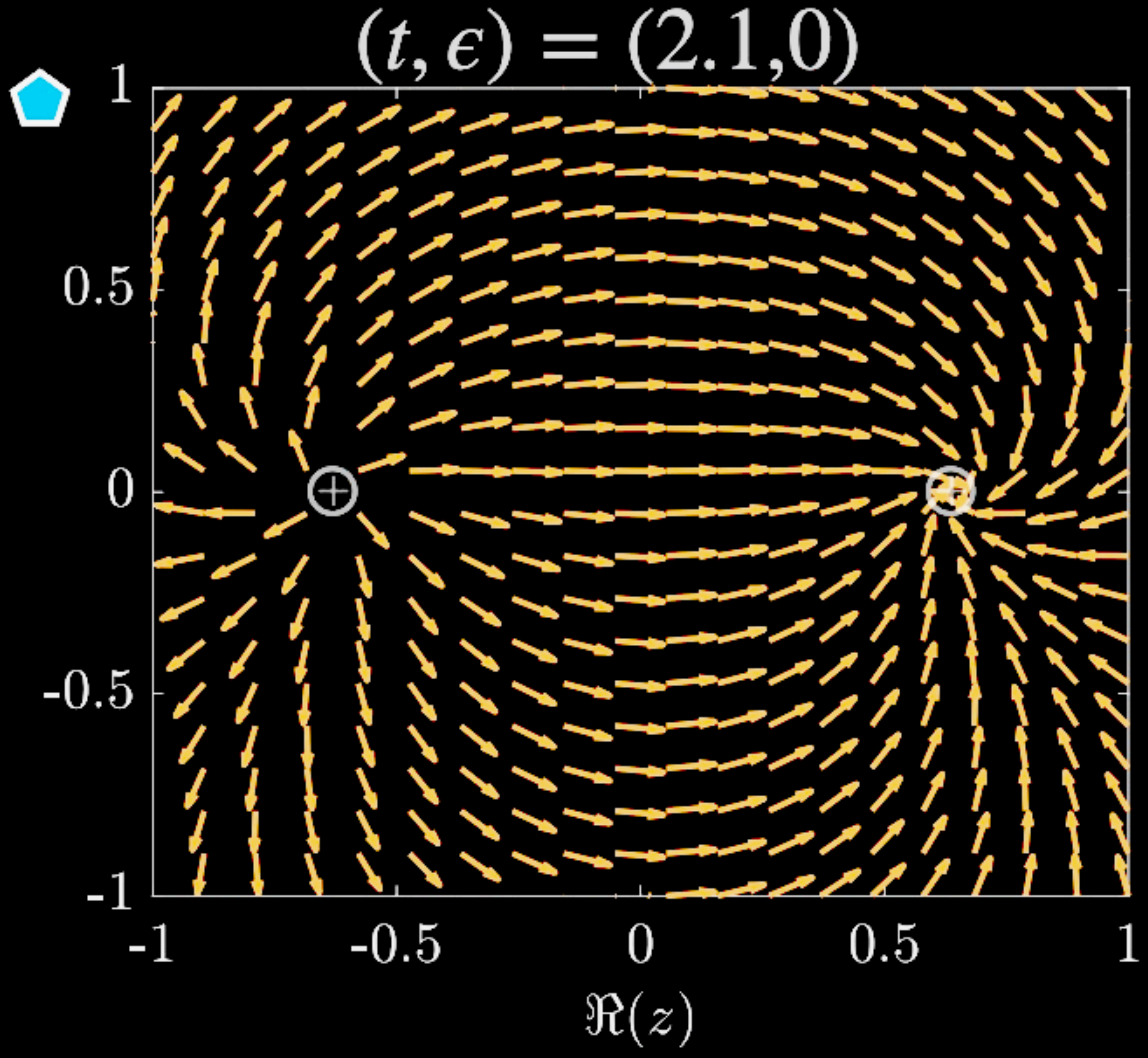
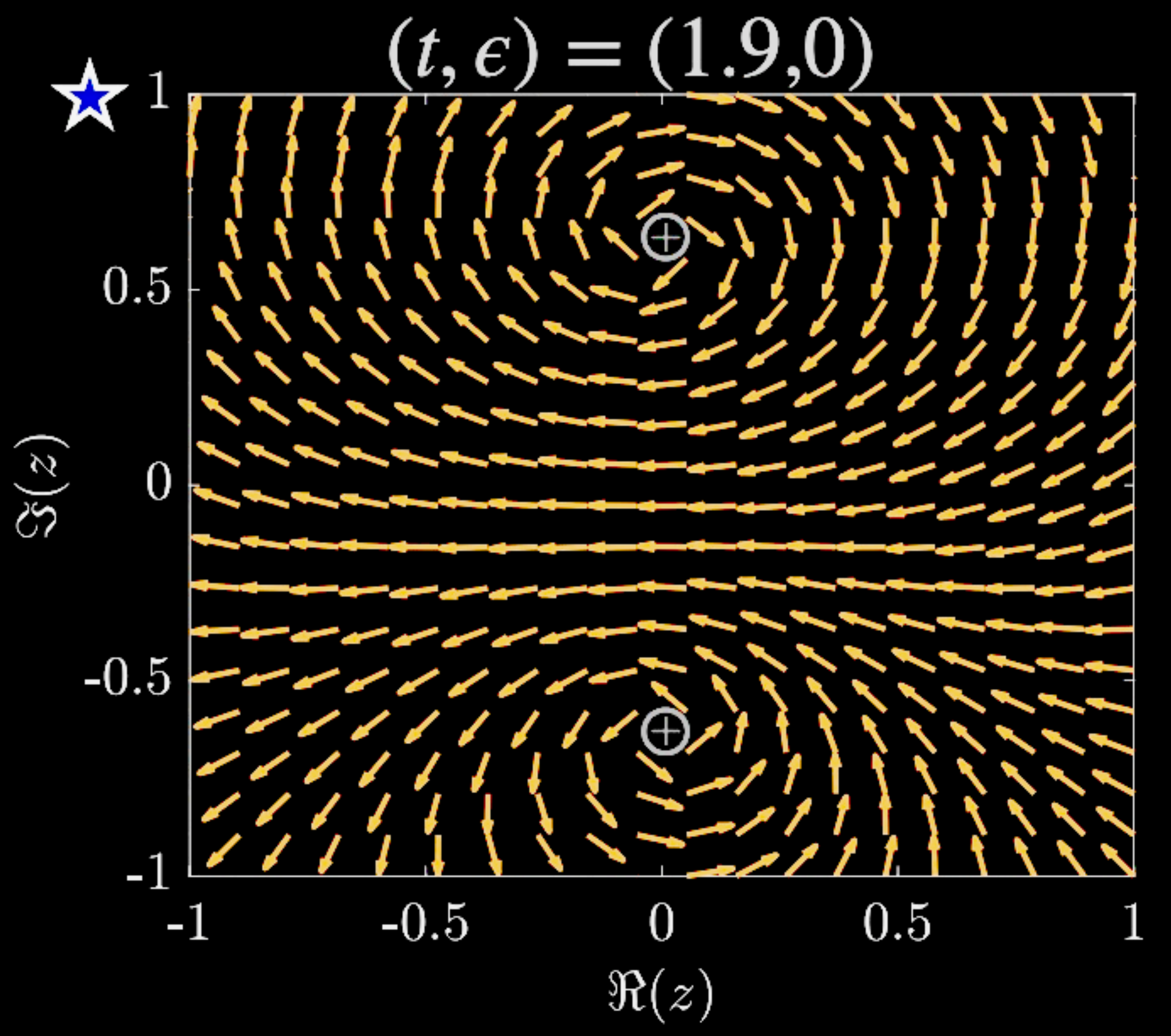
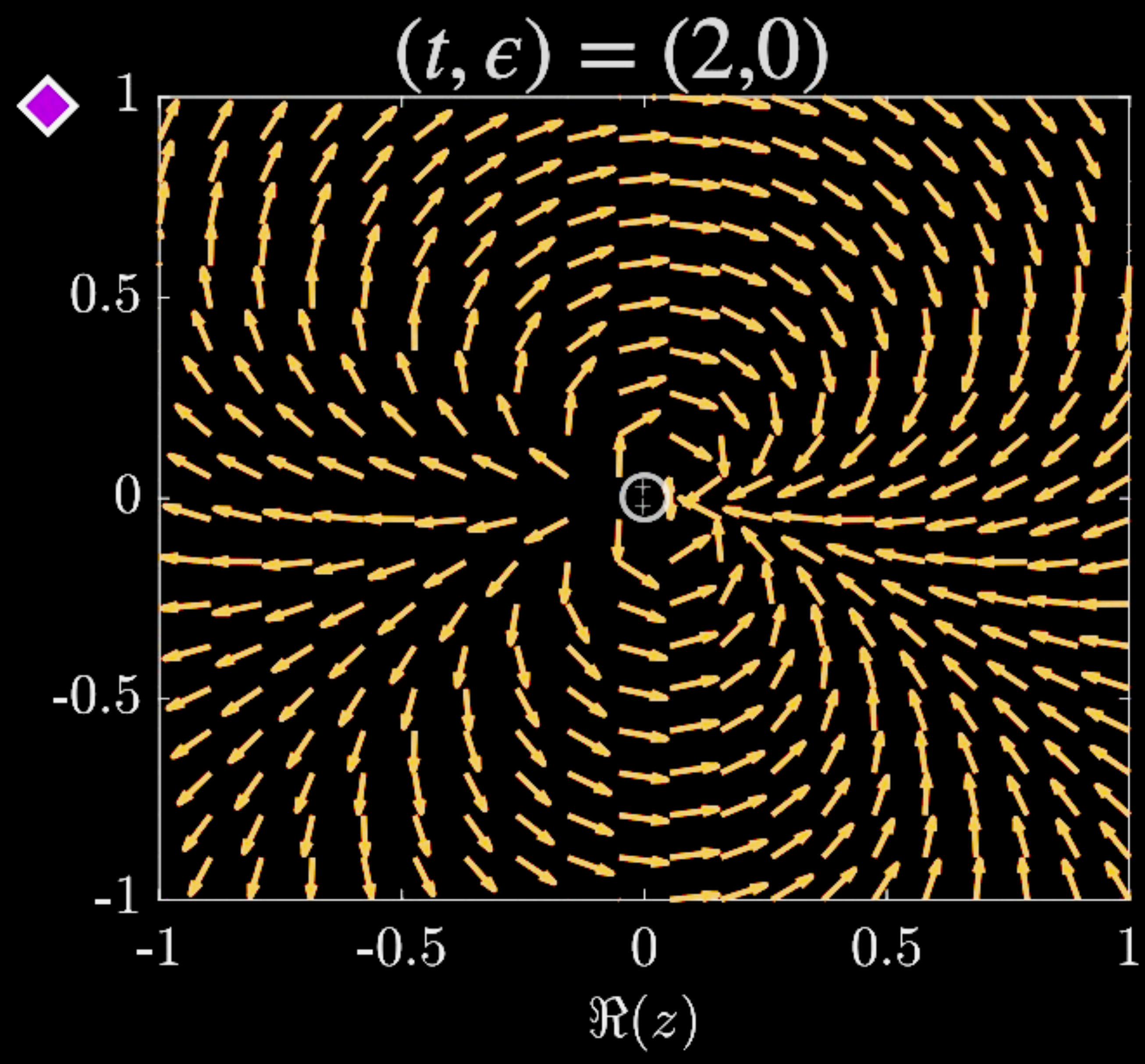
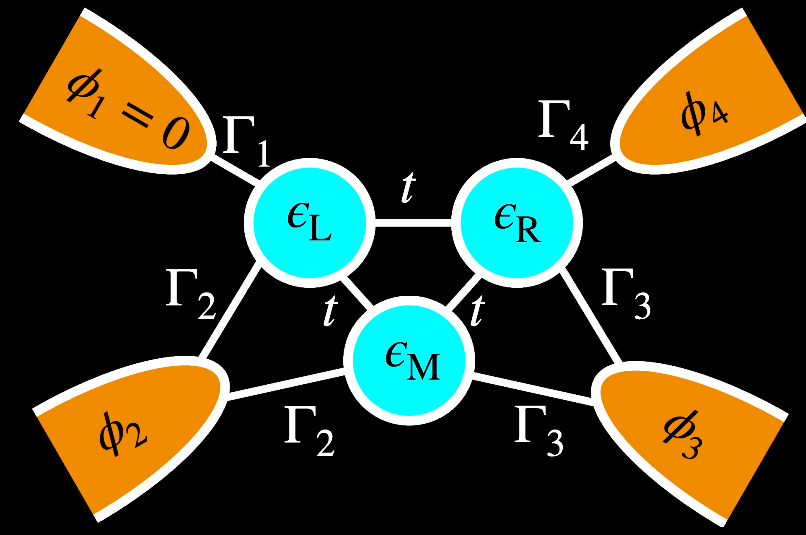




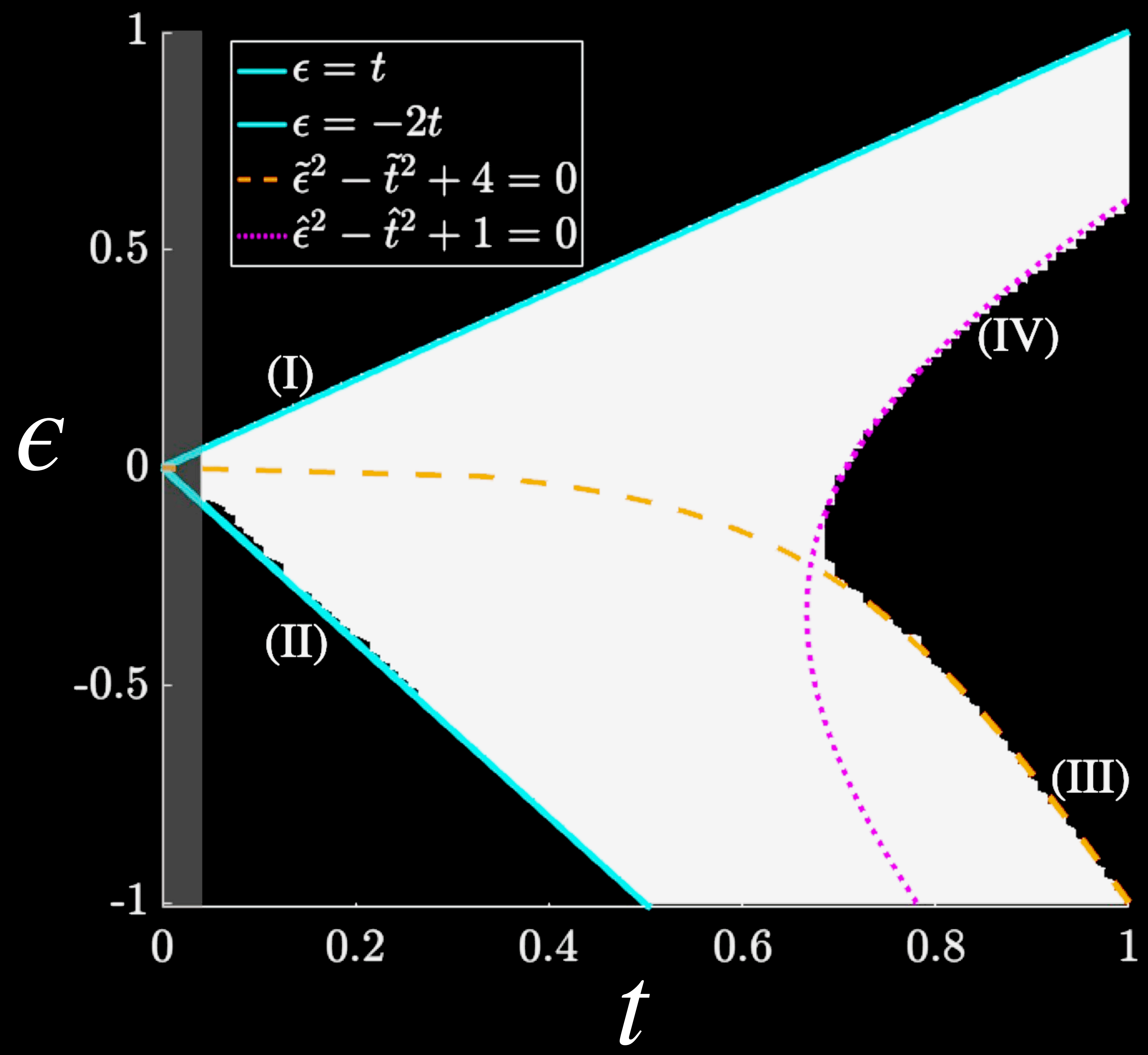


One can assign top. number, but
what about other boundaries?
→ No generalized number…

3-dot system



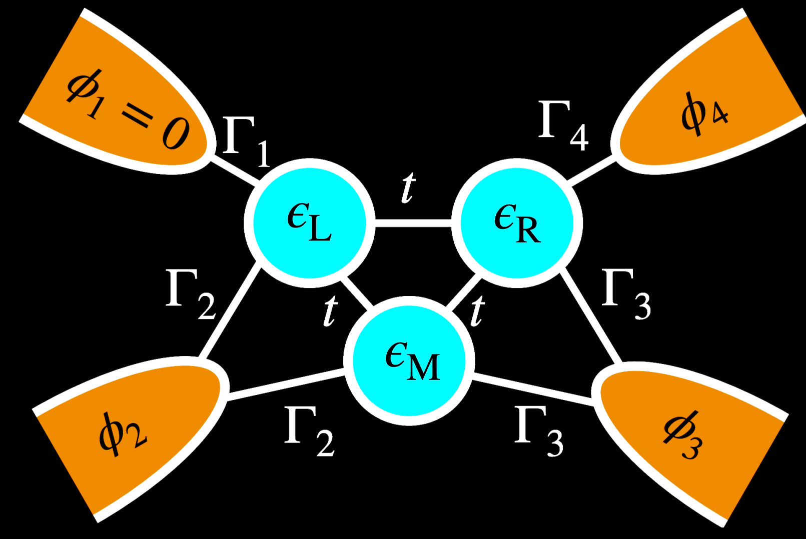
Phase diagram



non-trivial Chern number

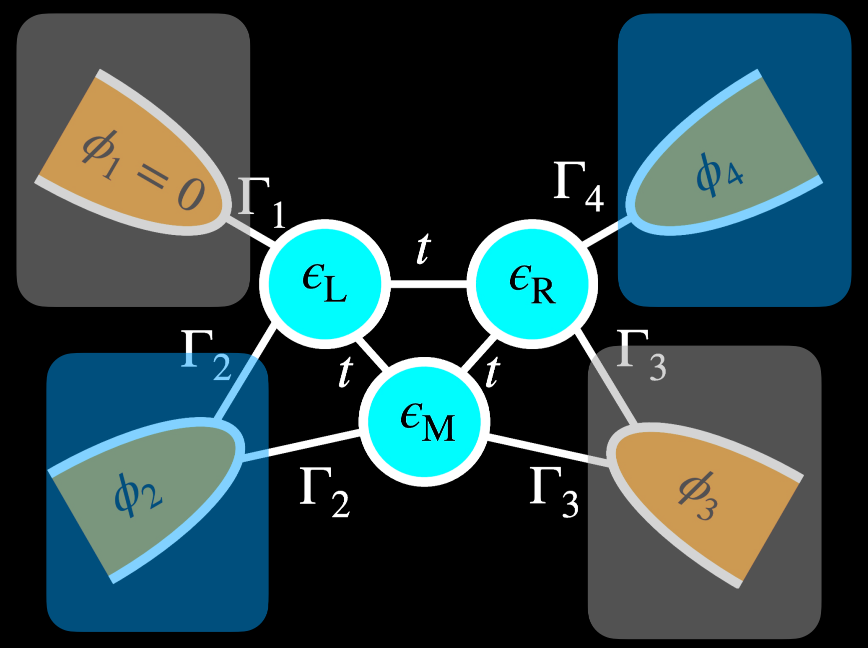
in superconducting phase space $\left\{ \vec{\phi} \right\}$

3-dot system



(I),(II): $\vec{\phi} = (0, \pi, 0, \pi)$

$\Sigma = 0$



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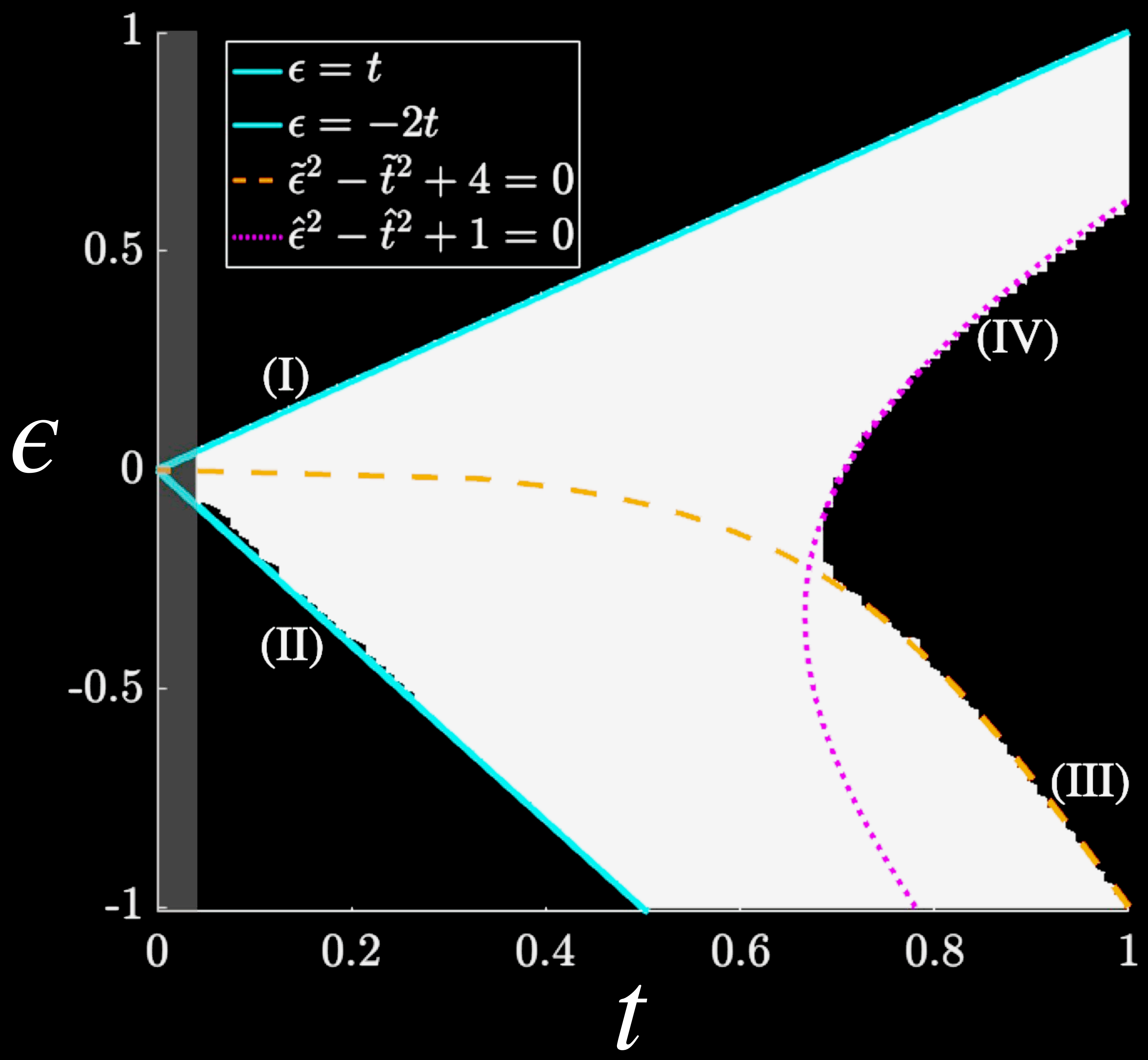

non-trivial Chern number
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(III): $\vec{\phi} = (0,0,\pi,\pi)$

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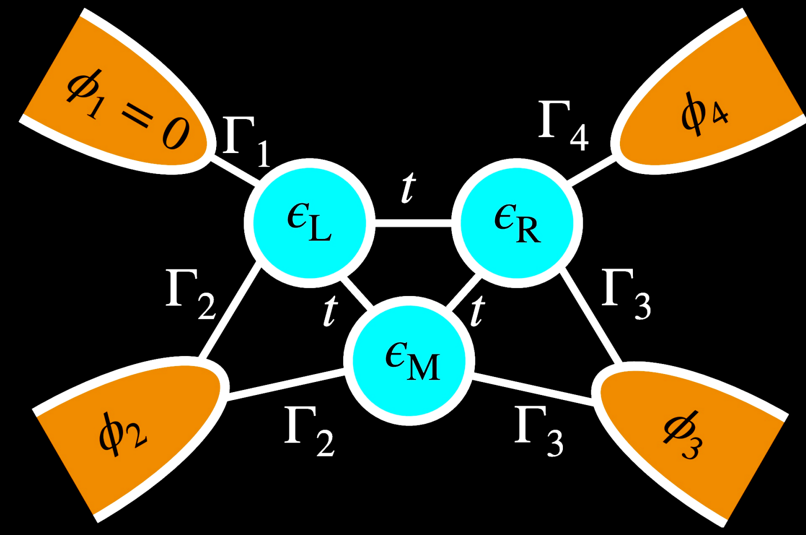

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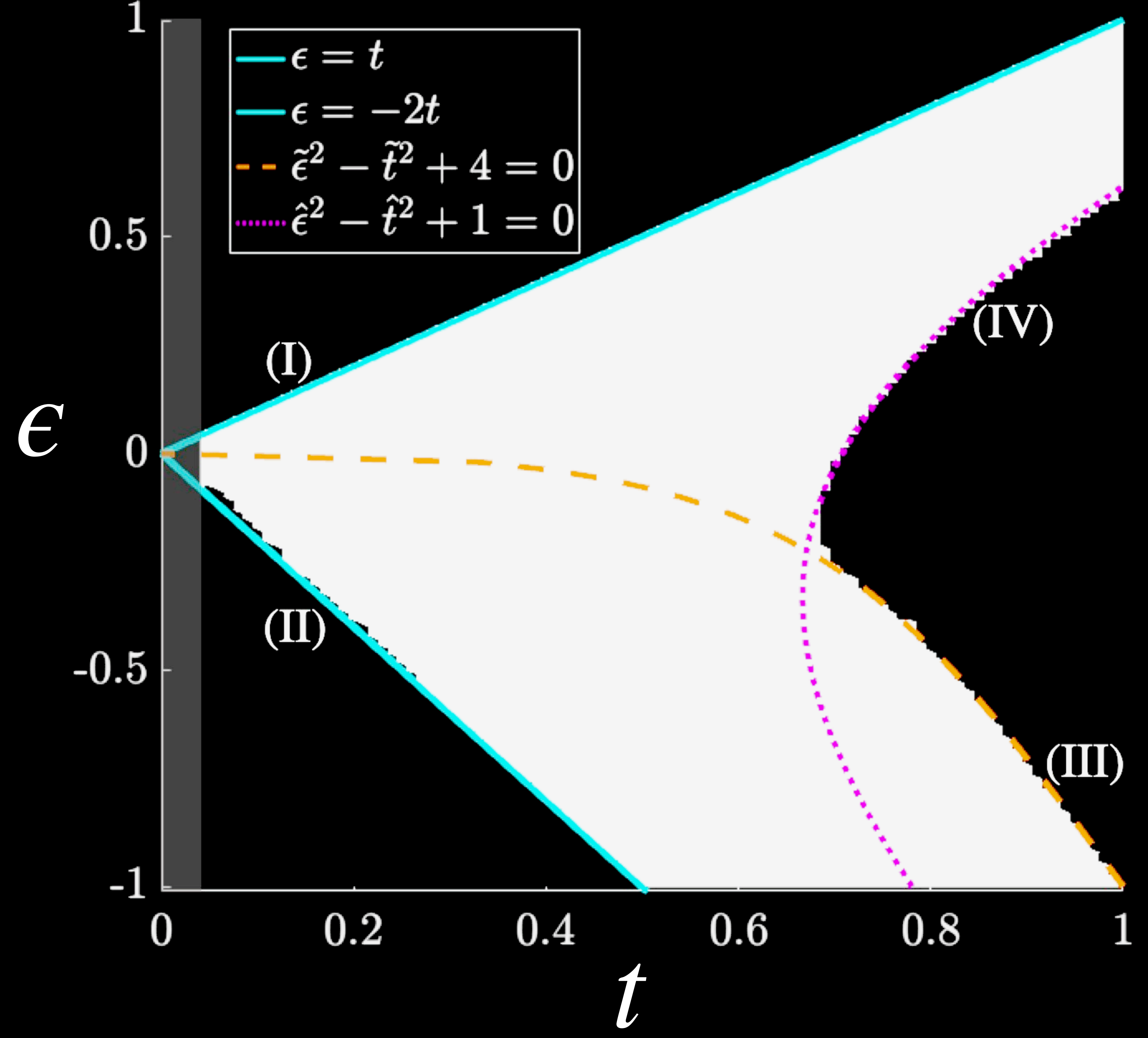

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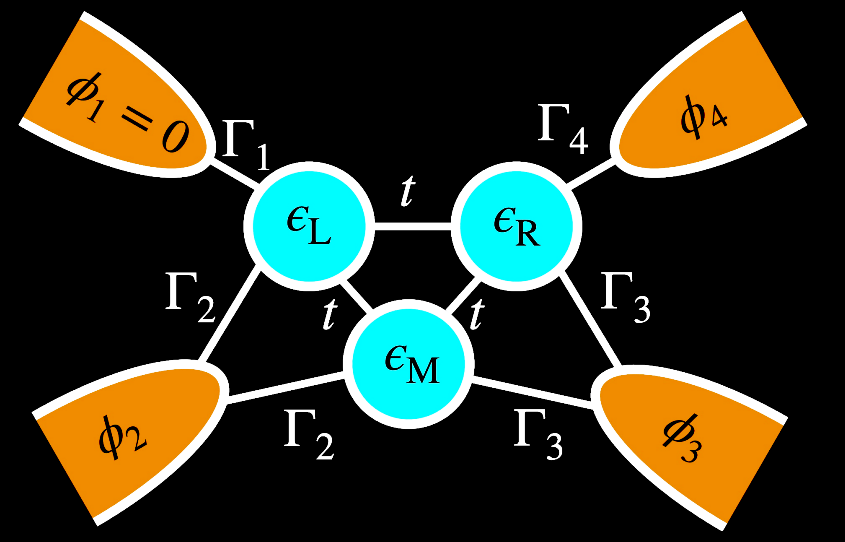

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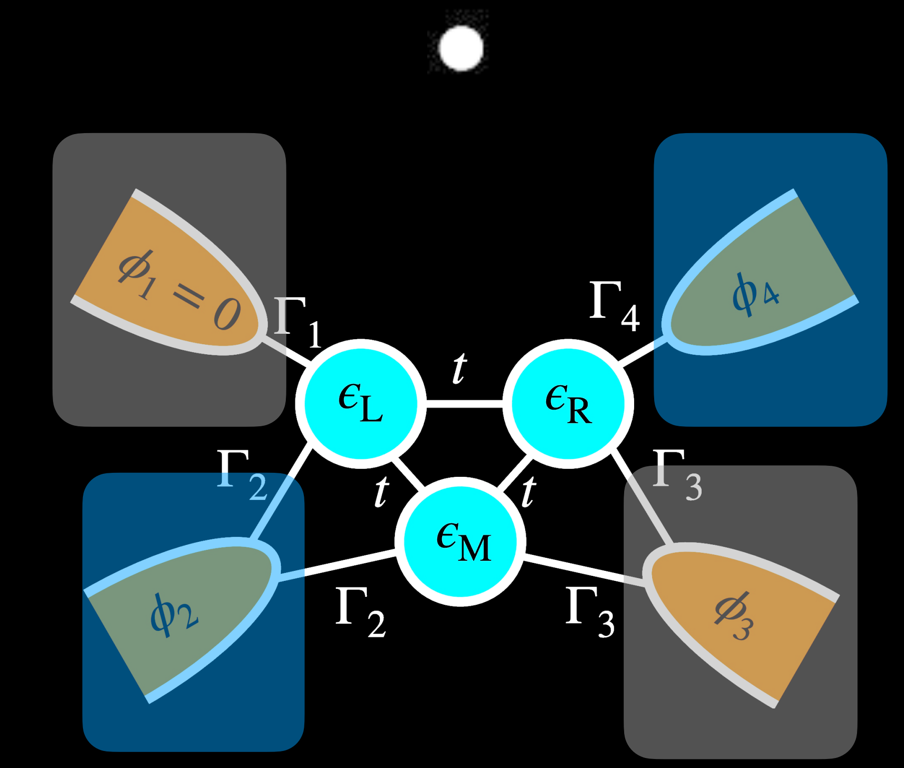

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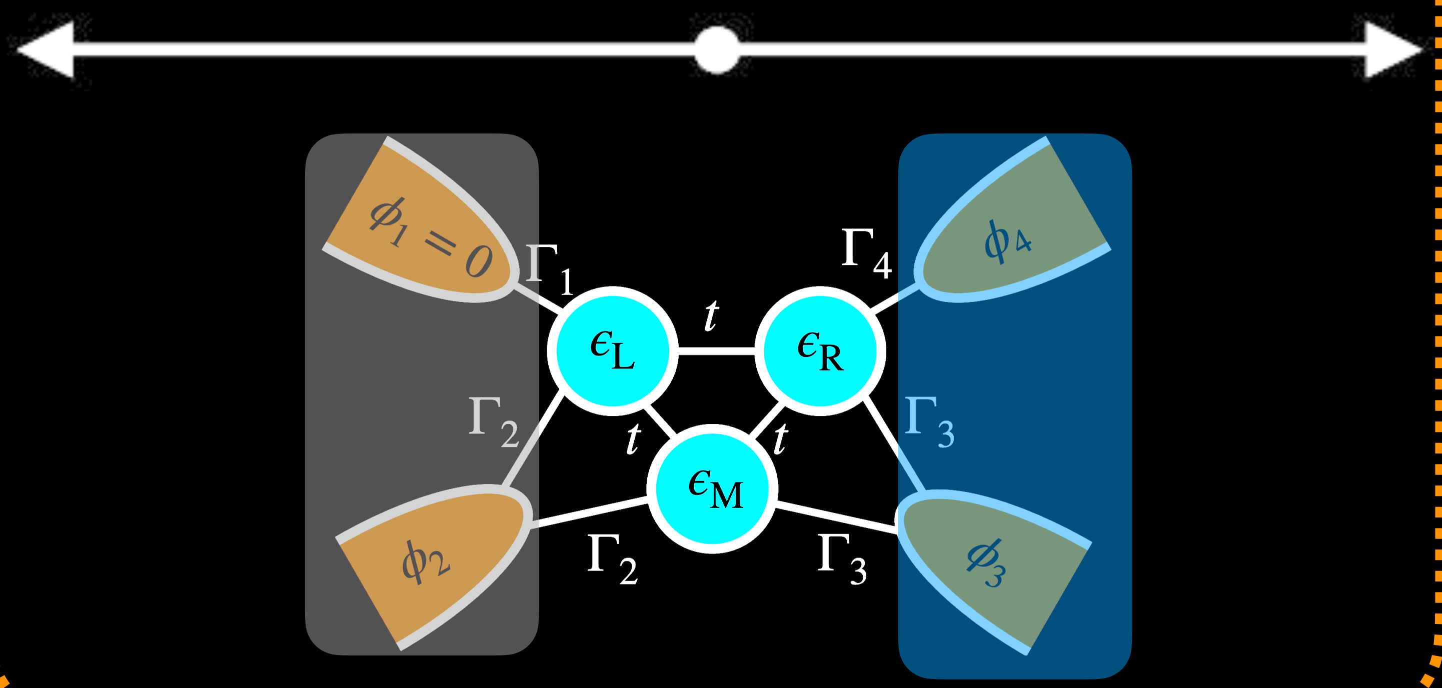

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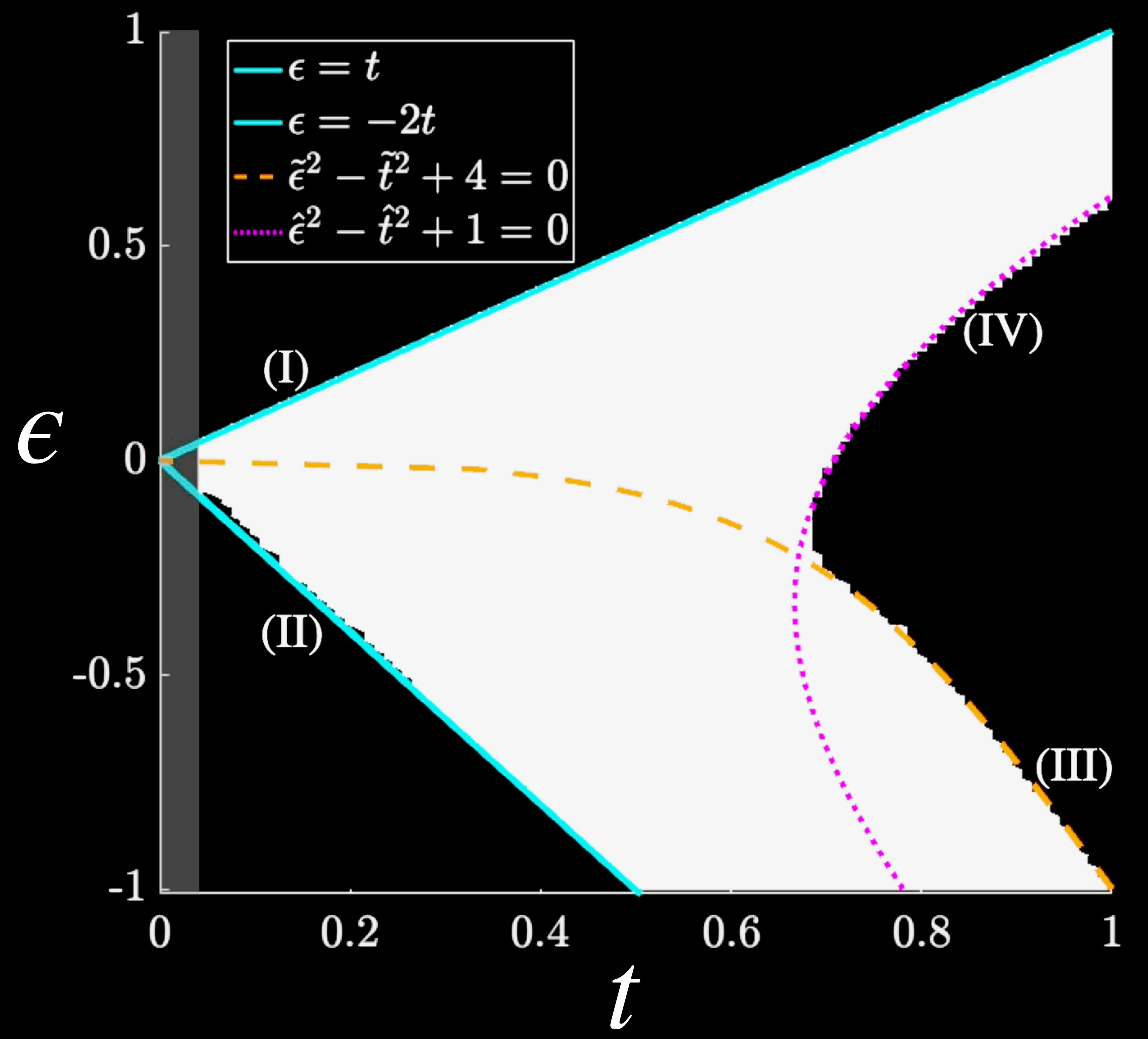

non-trivial Chern number in superconducting phase space $\left\{\overrightarrow{\phi}\right\}$

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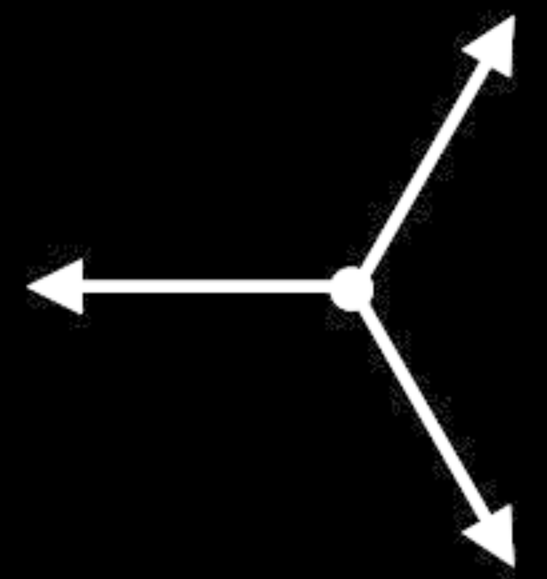

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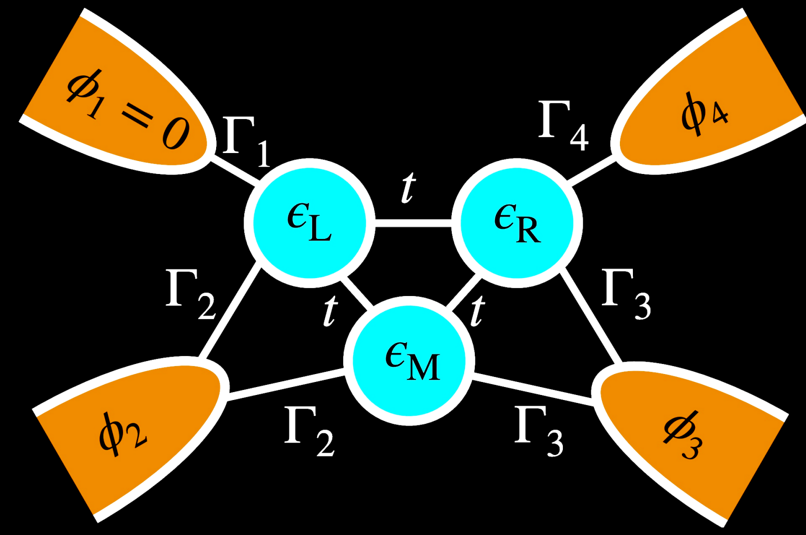

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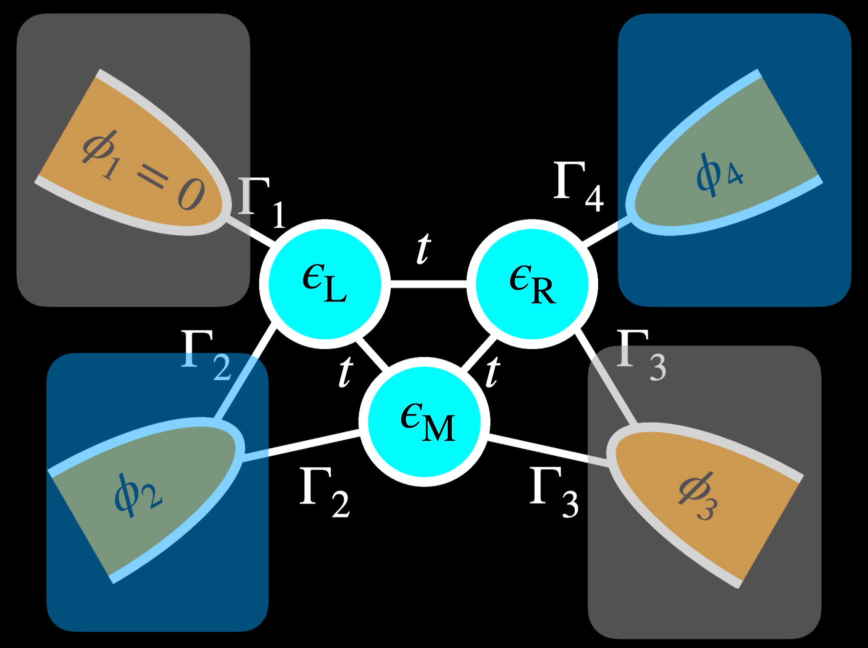

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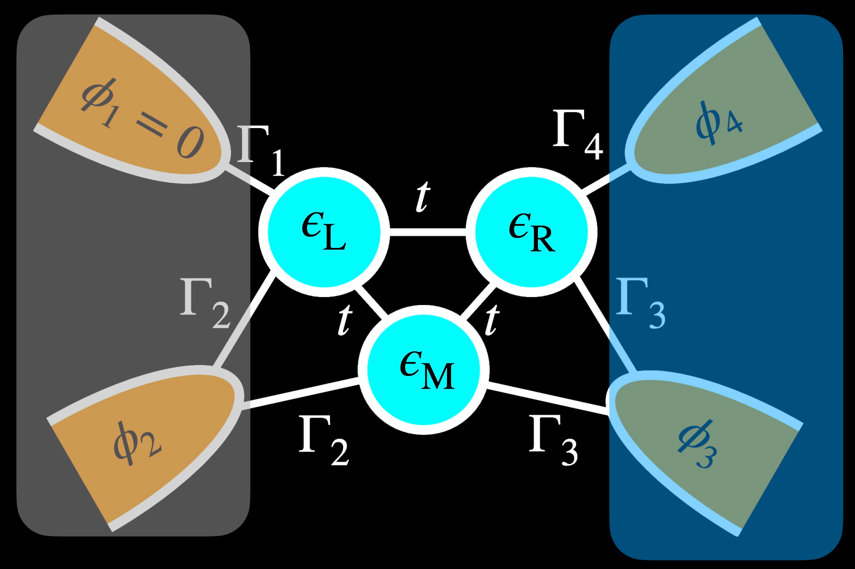

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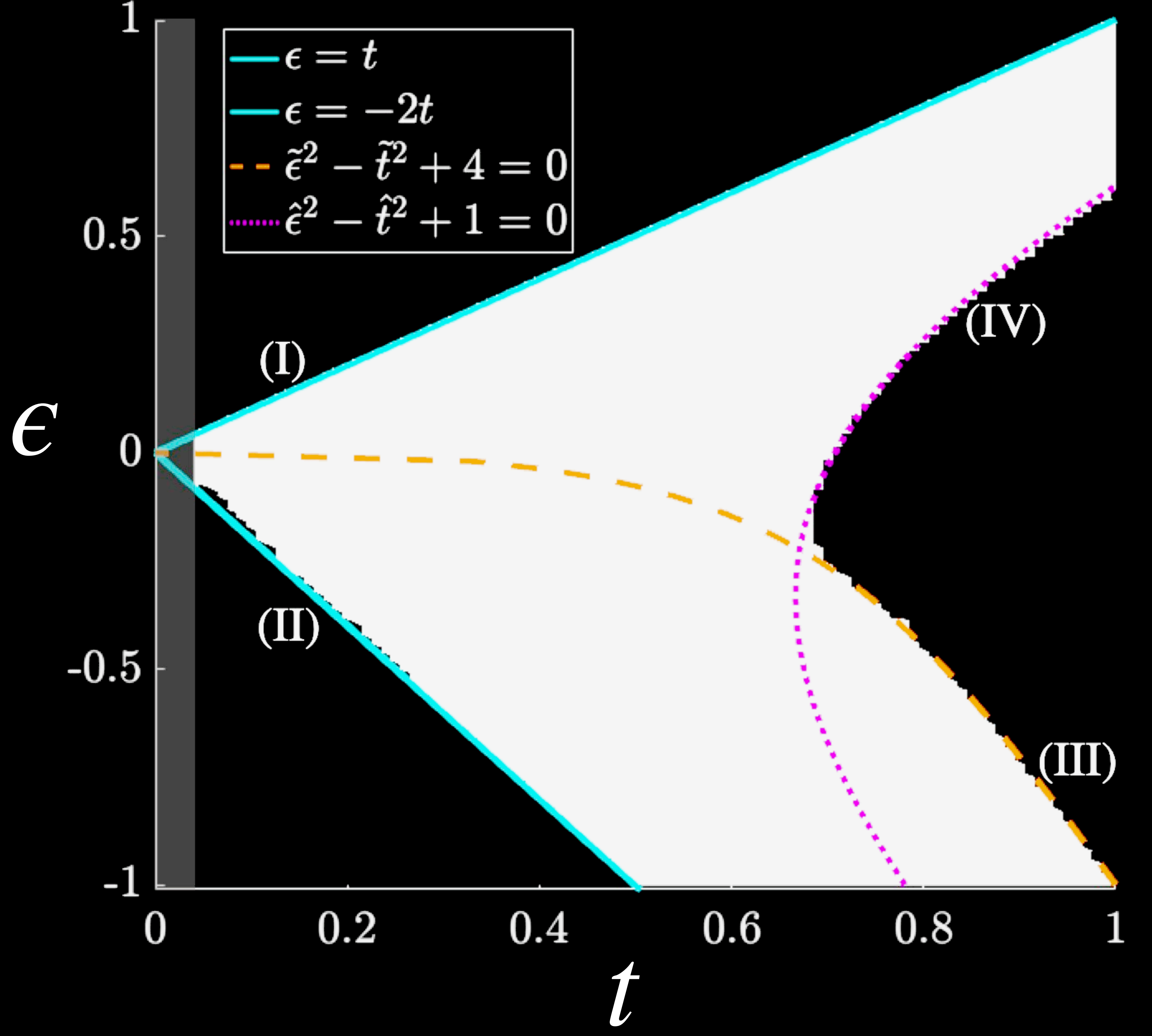

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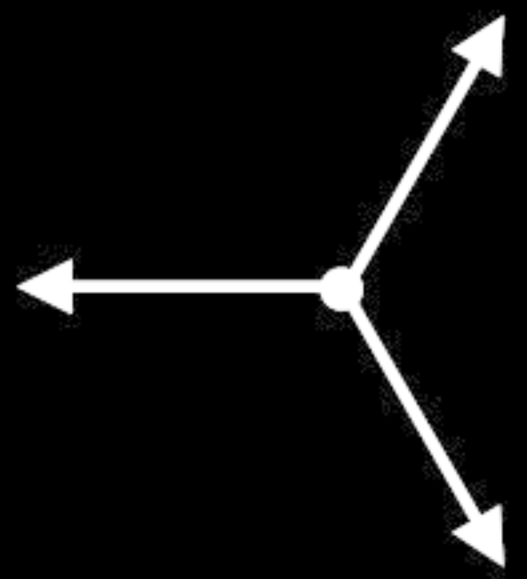
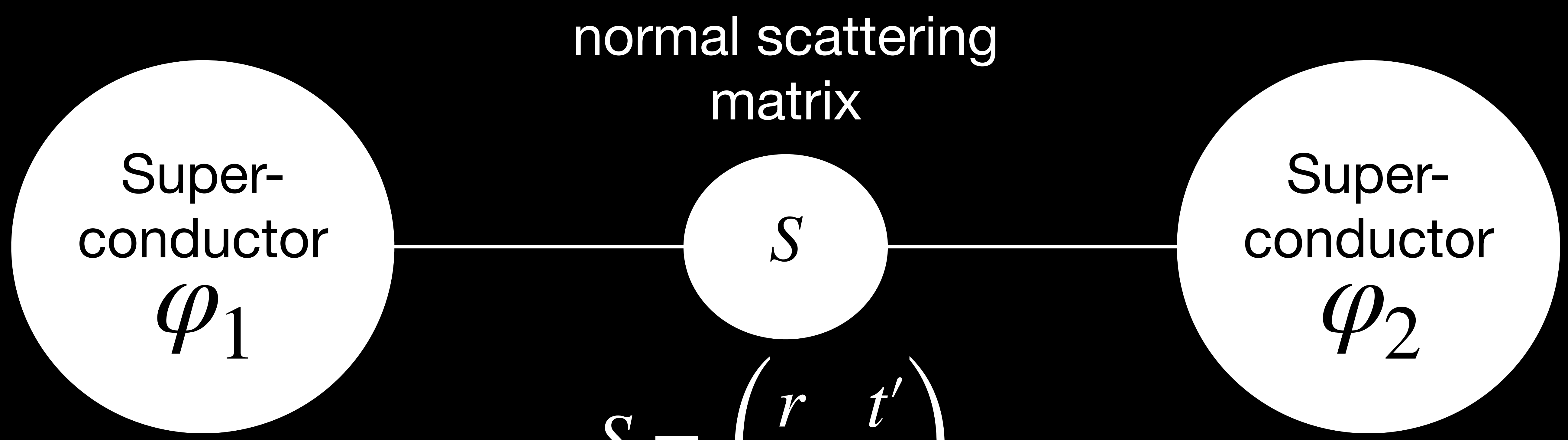


Not every phase boundary is reflectionless mode of normal metal S-matrix…

Topology in 2-terminal Josephson junctions

Phys. Rev. B 83, 155429 (2011)



$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

Classification depending on symmetry

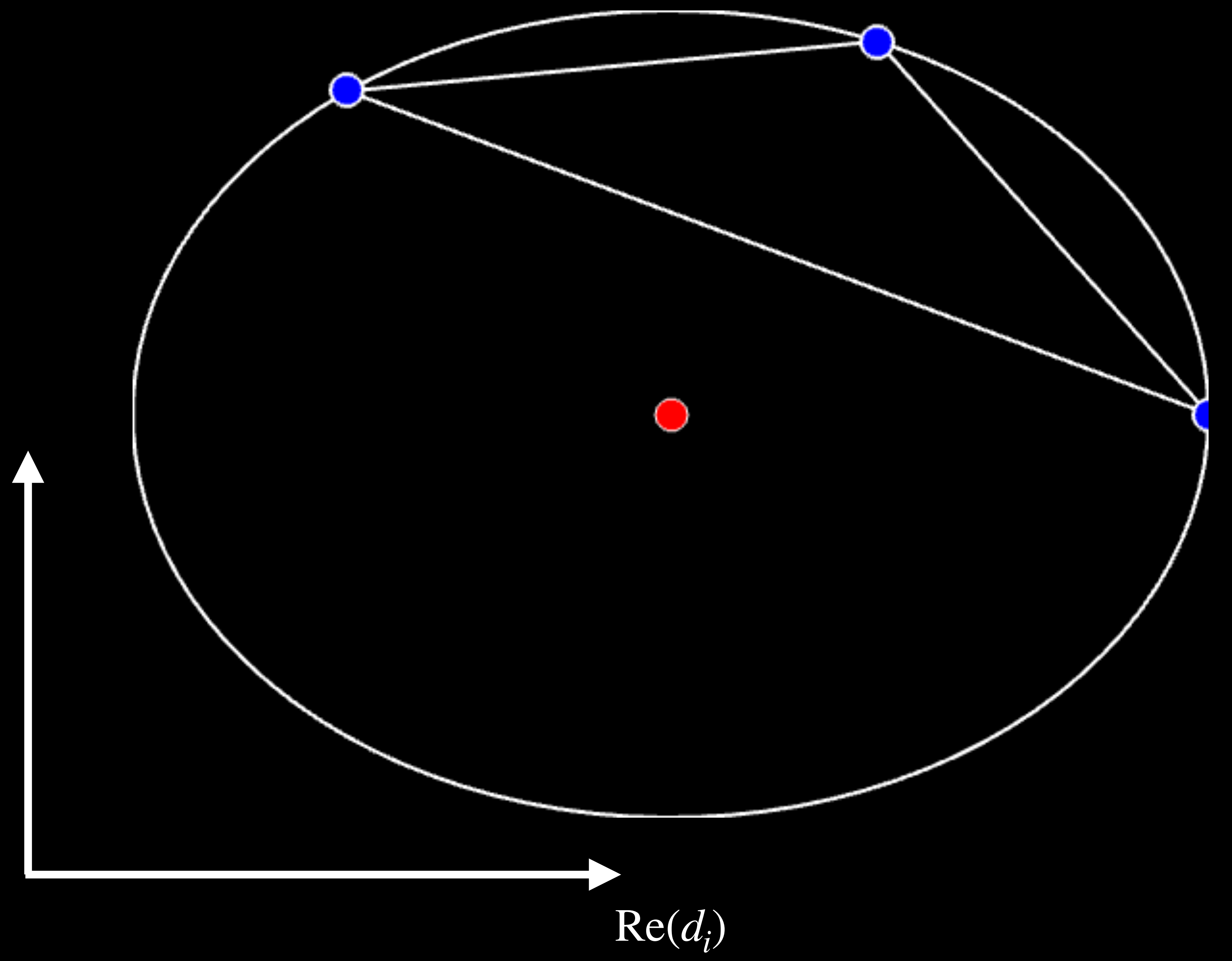
	D	DIII	BDI	AIII	CII
Topological phase	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
Particle-hole symmetry	$S = S^*$	$S = S^*$	$S = S^*$	×	$S = \Sigma_y S^* \Sigma_y$
Time-reversal symmetry	×	$S = -S^{\mathrm{T}}$	$S = S^{\mathrm{T}}$	×	$S = \Sigma_y S^{\mathrm{T}} \Sigma_y$
Spin-rotation symmetry	×	×	√	√ or ×	×
Chiral symmetry	×	$S^2 = -1$	$S^2 = 1$	$S^2 = 1$	$S^2 = 1$
Reflection matrix	$r = r^*$	$r = r^* = -r^{\mathrm{T}}$	$r = r^* = r^{\mathrm{T}}$	$r = r^\dagger$	$r = r^\dagger = \Sigma_y r^{\mathrm{T}} \Sigma_y$
Topological quantum number	$\mathrm{sign}\,\mathrm{Det}\,r$	$\mathrm{sign}\,\mathrm{Pf}\,ir$	$\nu(r)$	$\nu(r)$	$\frac{1}{2}\nu(r)$

Eigenvalues of the scattering matrix

Consider eigenvalues of **normal** metal scattering matrix $S(E=0)$!

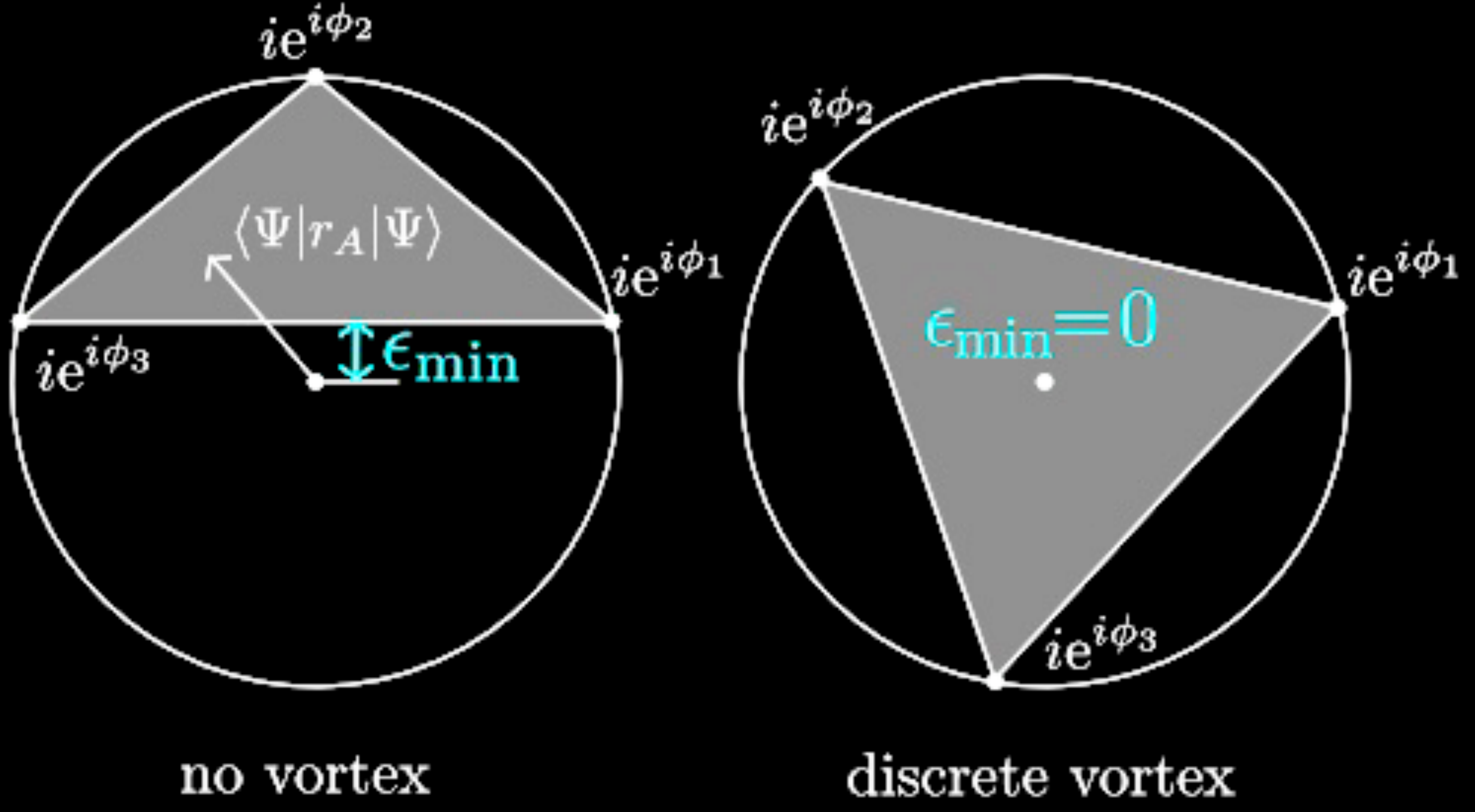
$$D_\mathrm{S} = \mathrm{diag}\left(1,1,d_+,d_-\right) = \mathrm{diag}\left(1,1,\frac{\epsilon_+^*}{\epsilon_+},\frac{\epsilon_-^*}{\epsilon_-}\right)$$

$$\epsilon_\pm = \epsilon \pm t + 2\imath\Gamma$$



Consider superconducting phases

Phys. Rev. B 90, 155450 (2014)



Relevant operator works for both!

$$A = \frac{1}{2}\{S, e^{\imath\vec{\phi}}\}$$

Conclusion

1) MTJJs are an excellent platform to study engineered topology

Topology in three state Andreev Molecule

arXiv:2501.07982 (2025)

2) Refectionless scattering modes are a source of topology in MTJJs

Reflectionless Modes can lead to topological phase boundaries in MTJJs

arXiv:2503.10874 (2025)

david.ohnmacht@uni-konstanz.de

Open questions

1) Intuitive invariant for arbitrary MTJJ?

2) Underlying mechanism between reflection zero and Chern number

3) Deeper meaning of S-matrix eigenvalue configuration?

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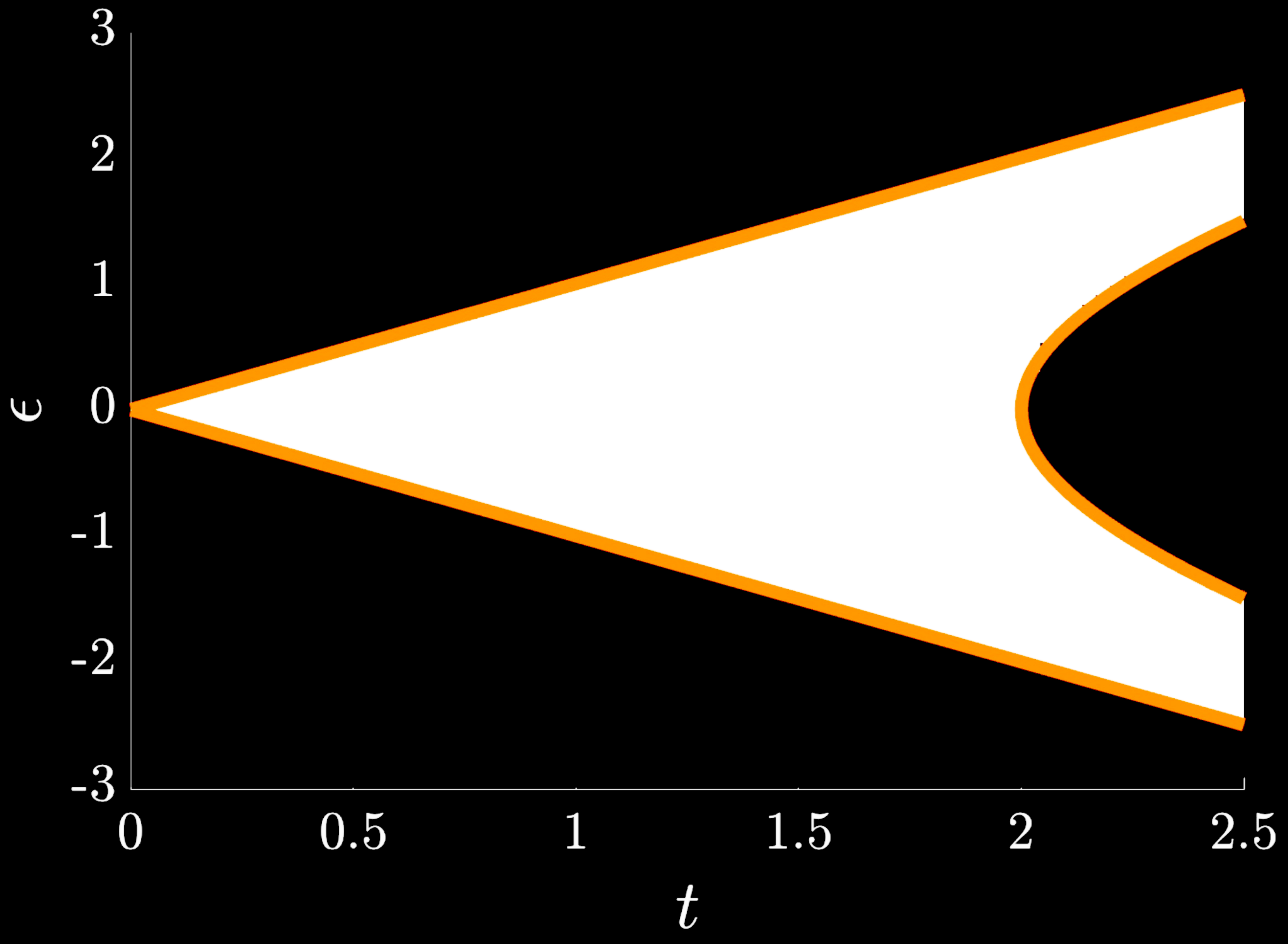
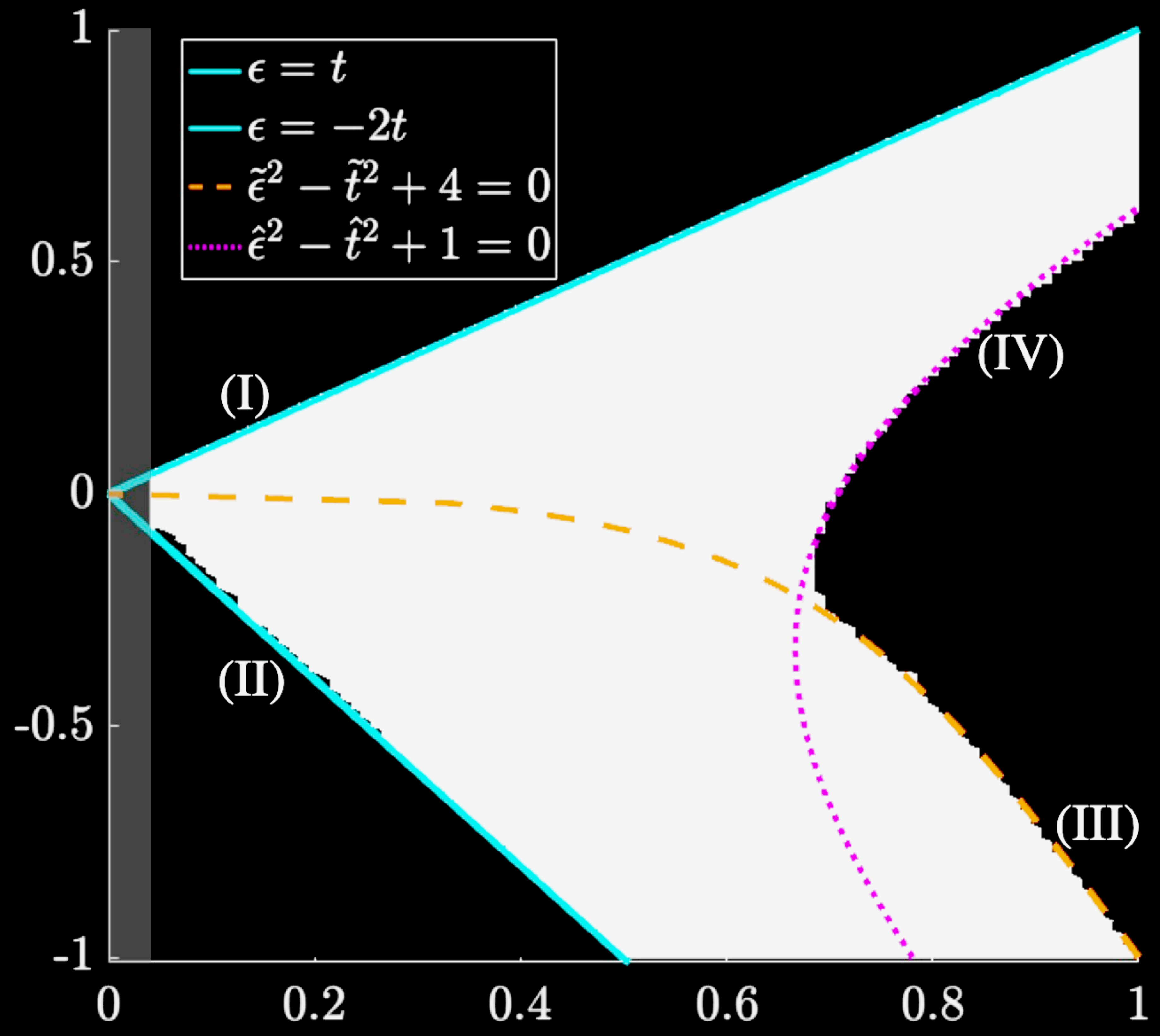
Reflectionless modes as a source of Weyl nodes in multiterminal Josephson junctions

arXiv:2503.10874 (2025)

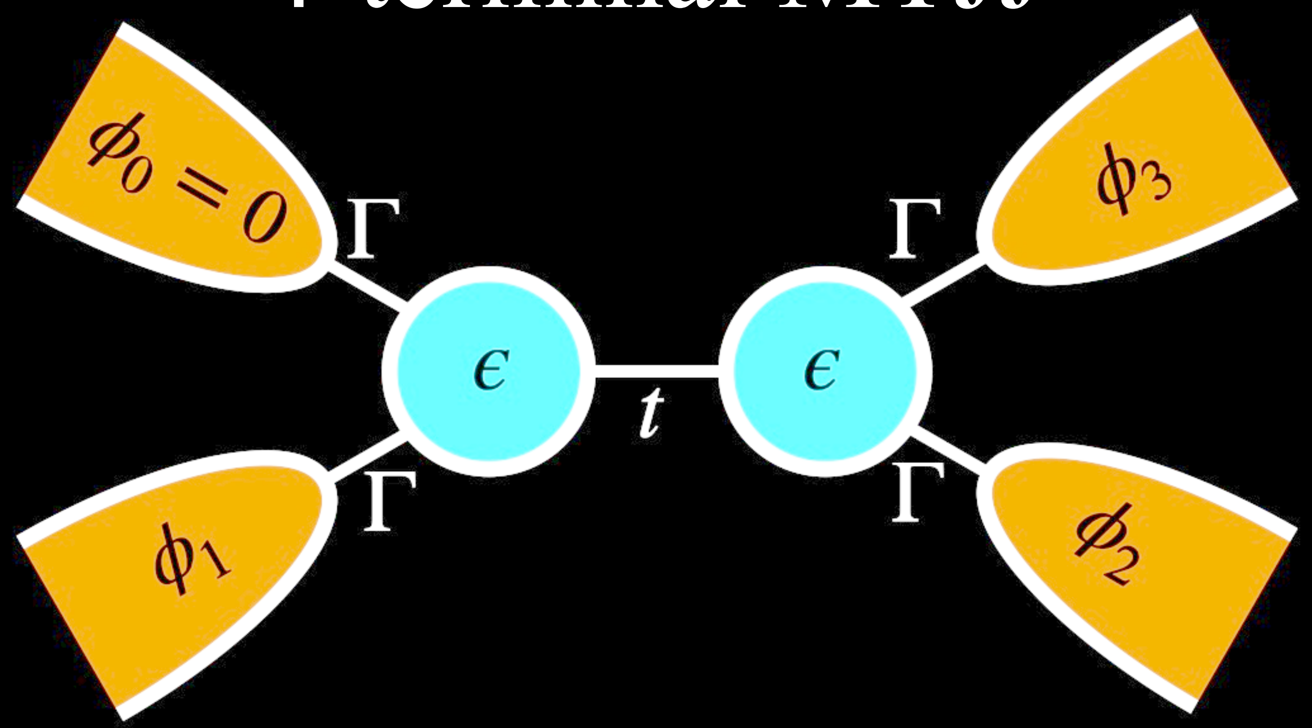
Thank you!

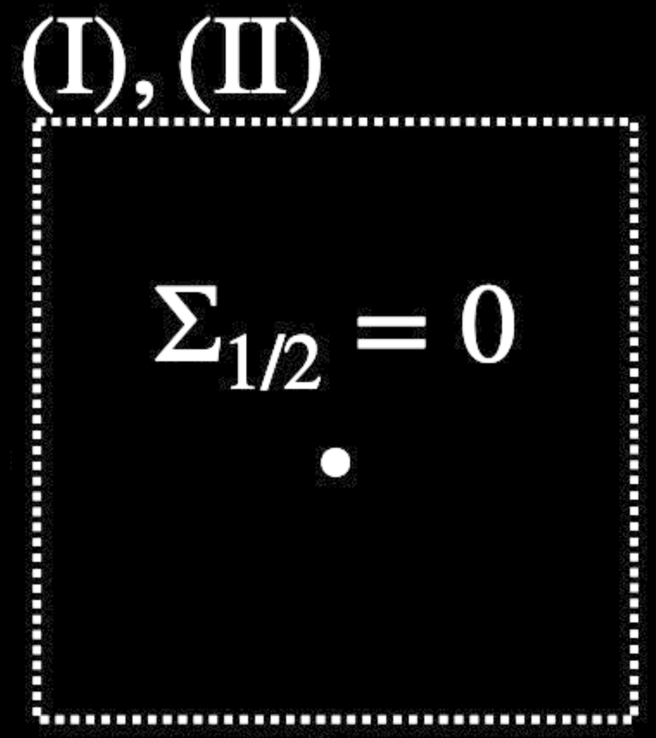


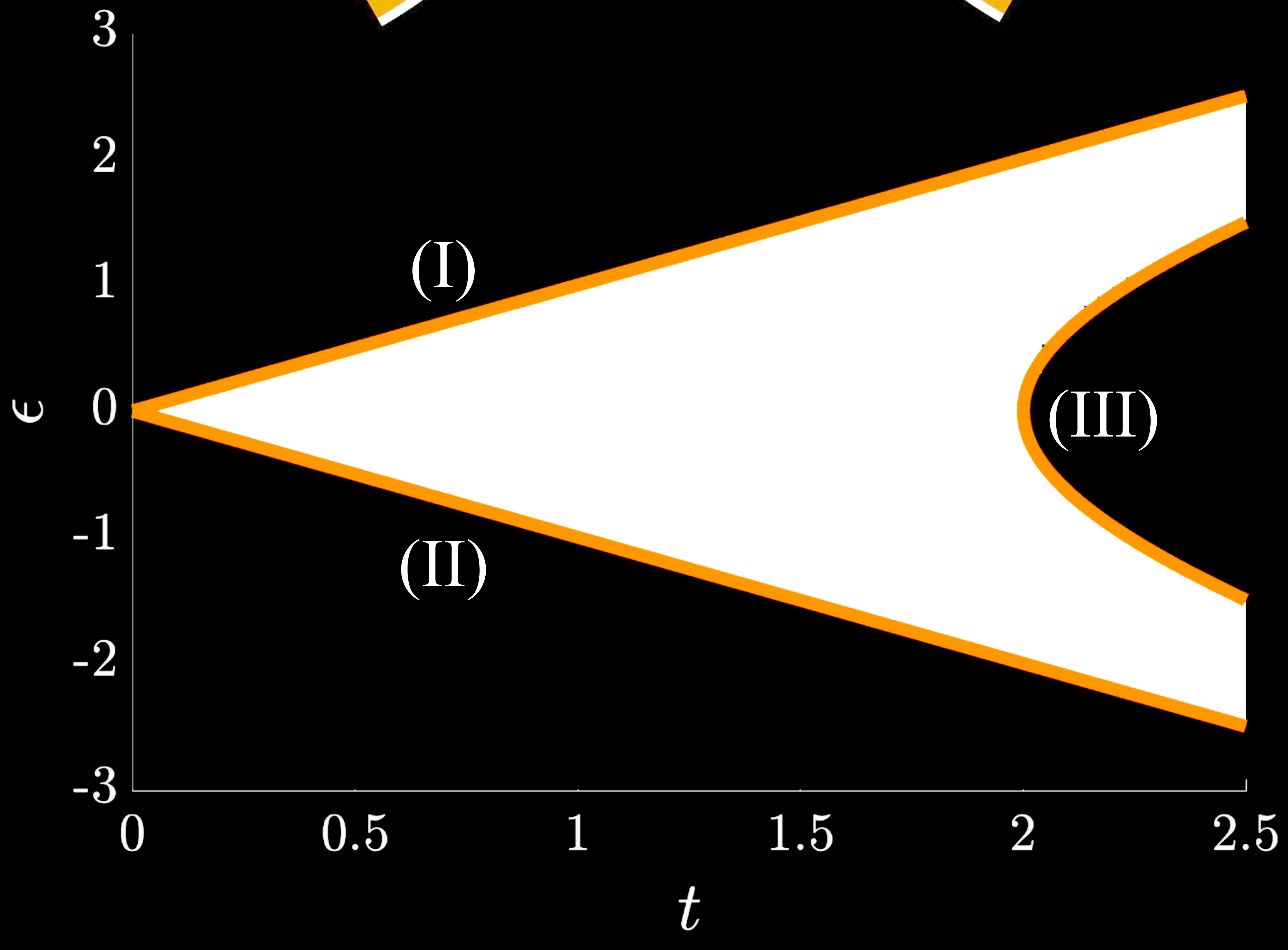


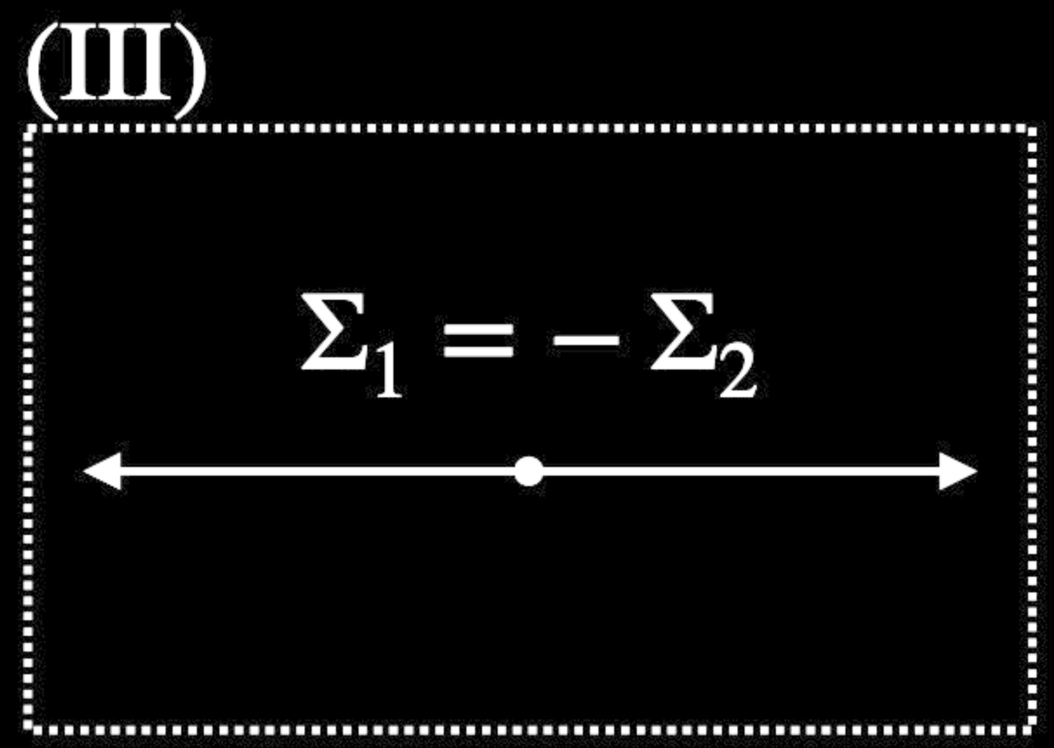
2-dot Summary

4-terminal MTJJ









Effective classification parameter

Jeffrey C. Y. Teo and C. L. Kane Phys. Rev. B 82, 115120 (2010), Fan Zhang and C. L. Kane Phys. Rev. B 90, 020501(R) (2014)

Point gap in system with 1 superconducting phase $d_\phi = 1$ ($d_k = 0$)

$$d = d_k - d_\phi \mod 8 = 7$$

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$\mathrm{C}^\dagger$	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_r}$	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	$\mathrm{L_i}$	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0

winding number NOT allowed

Break spin-rotation symmetry!

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

winding number allowed!