THE SKYSCRAPER

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Preface

This is a latex version of subtle or important materials I encountered while studying in Peking university. I started this project in the fall of my third undergraduate year(October 2019), noticing that I have a poor memory and consistently forget what I have already learned thus struggle to check details. So it came to me that I can compile all the proofs of theorems I cannot recall that is hard and subtle yet appearing over and over again. But finally it turns out I want to make it as comprehensive as possible. That's it.

I constantly add stuffs to this note, and I regularly put them online. You can find the newest version at https://phacademic.com/files/my-notes.pdf.

Constitution: The following are principles of the structure of this note, but the current version is far from it. they serve as ultimate goals.

- Be self-contained.
- Notations should be consistent throughout the whole note.
- Logical order is not necessary, but vicious circles are intolerable.
- Theories should be stated at the most generality. No new proof should be given for the special case, but clarify the deduction from the general case, unless it is needed in the proof of the general case, then state it as a lemma.
- When facing multiple proofs, only the most elegant and essential proof should be recorded.
- References should be traced to the original author and his specific paper.

Tips: This is hardly a *readable* book, I use it as a dictionary. It only contains materials that I'm interested in and many proofs are still missing. Hopefully I can complete them all as time goes by. The most important reason why I have to latex this note is because I need tons of hyper-references. So I believe the best way to read this book is to use a computer, and it's vital to know how to go forwards and backwards between hyper-references on your computer. For example, on a Mac, the default shortcuts is ⌘+[, ⌘+] for Preview and ⌥+←, ⌥+→ for Foxit Reader.

Acknowledgement: Sincere thanks to Yi Tian(田翊) for answering my questions when I was learning algebraic geometry and p-adic geometry. His help is fundamental.

There is already a great online book StackProject that covers considerably many of the Algebraic Geometry part of this note, but it's too long(at least for me now). I haven't finish reading it but I reordered the materials that I learned and keep track of it in my own way. I write much shorter and omitted easy proofs. The ideal to write this book is inspired by StackProject, The work of de Jong is fully respected.

And they said, Go to, let us build us a city and a tower, whose top may reach unto heaven; and let us make us a name, lest we be scattered abroad upon the face of the whole earth.

–Genesis 11:1-9

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Chapter I

Algebra

I.1 the Zermelo-Fraenkel Set Theory with Choice

Basic references are [Set Theory Jech] [Model Theory Marker].

Set theory is just one choice of foundation of mathematics. And many propositions need efforts to be rigorous. A new choice of foundation of mathematics is homotopy type theory.

1 Basic Axioms

Axiom (I.1.1.1) (Axiom of Existence). There exists a set which has no elements.

Axiom (I.1.1.2) (Axiom of Extensionality). If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

Cor. (I.1.1.3). There is at most one set which has no elements, it is called the **empty set** $\varnothing$ or 0.

Def. (I.1.1.4). Write $x \in X$ if x is an element of X.

Def. (I.1.1.5). Write $A \subset B$ iff $\forall x, x \in A \Rightarrow x \in B$.

Axiom (I.1.1.6) (Axiom Schema of Comprehension). For any set A, if $P(x)$ is a property of elements of A, there is a set B that $x \in B$ iff $x \in A$ and $P(x)$, it is denoted by $B = \{x \in A | P(x)\}$.

Axiom (I.1.1.7) (Axiom of Pair). For each A, B, there is a set C that $x \in C$ iff $x = A$ or $x = B$.

Axiom (I.1.1.8) (Axiom of Union). For any set S, there exists a set U that $x \in U$ iff $x \in A$ for some $A \in S$.

Axiom (I.1.1.9) (Axiom of Power Set). For any set S, there exists a set P that $x \in P$ iff $x \subset S$.

Def. (I.1.1.10) (Successor). The **successor** of a set x is $S(x) = x \cup \{x\}$.

Def. (I.1.1.11). A set I is called **inductive** if $0 \in I$ and if $n \in I$, then $n+1 \in I$, where $n+1 = S(n)$ the successor.

Axiom (I.1.1.12) (Axiom of infinity). An inductive set exists.

Def. (I.1.1.13) (Natural Numbers). The **set of natural numbers** $\mathbb{N}$ is defined to be $\{x \in I_0 | x \in I$ for all inductive set $I\}$, where I_0 is an inductive set. Elements of $\mathbb{N}$ are called **natural numbers**.

Cor. (I.1.1.14) (Inductive Principle). If $P(x)$ is a property that $P(0)$, and $P(n)$ implies $P(n+1)$, then $P(n)$ for each natural number n.

Proof: By definition $B = \{n \in \mathbb{N} | P(n)\}$ is an inductive set, so $\mathbb{N} \subset B$. □

Prop. (I.1.1.15). $\mathbb{N}$ is a linearly ordered set, Cf.[Set Theory Jech P43].

Cor. (I.1.1.16) (Inductive Principle Second Version). If $P(x)$ is a property that $P(0)$, and $P(k)$ holds for all $k < n$ implies $P(n)$, then $P(n)$ for each natural number n.

Proof: Use induction principle(I.1.1.14) for the property $Q(n) : P(k)$ for all $k < n$. Then $Q(n)$ implies $Q(n+1)$. □

2 Cardinal

Lemma (I.1.2.1). If $A_1 \subset B \subset A$ with $|A| = |A_1|$, then $|A| = |B|$.

Proof: Let f be a bijection from A to A_1. Define inductively $A_{n+1} = f(A_n), B_{n+1} = f(B_n)$. Then $A_{n+1} \subset B_n \subset A_n$. Let $C_n = A_n - B_n$, $C = \cup C_n$, then $f(C_n) = C_{n+1}$, so $f(C) = \cup_{i>0} C_i$.

Now define $g : A \to B = f(x)$ on C and x on $A - C$, then it is a bijection from A to B. □

Prop. (I.1.2.2) (Cantor-Schröder-Bernstein Theorem). If there is an injection from A to B and an injection from B to A, then there is a bijection from A to B. Thus the ordering of the cardinal is well-defined.

So we can define $|A| \leq |B|$ iff there is an injection from A to B. Then if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof: It $f : A \to B$, $g : B \to A$ be injection, then use the above lemma(I.1.2.1) for $g \circ f(A) \subset g(B) \subset A$. □

Def. (I.1.2.3) (Cardinal Number). A **cardinal number** is an equivalence class of sets, where equivalence and ordering is given by injections and surjections. it is used to describe the 'size' of a set.

It is by the axiom of choice that any two cardinal number can be compared.

The cardinal number of $\mathbb{N}$ is denoted by $\aleph_0$.

Countable and Uncountable

Def. (I.1.2.4). A set is called **countable** iff it has the cardinality of $\aleph_0$. It is called **finite** if it has the cardinality of n for some natural number n. It is called **uncountable** iff it is not countable or finite.

Prop. (I.1.2.5). The subset or image of an almost countable set is at most countable.

Prop. (I.1.2.6). The product of two countable set is countable. (Use diagonal enumerating).

Prop. (I.1.2.7). A countable union of almost countable subsets is almost countable.

Proof: It suffices to prove the countable case, the rest follows from(I.1.2.5). For this, choose an enumerating $a_n(k)$ for each A_n, the $\cup A_n$ is the image of $\mathbb{N}\times\mathbb{N} : (n,k)\mapsto a_n(k)$. Then it is countable by(I.1.2.6). □

Prop. (I.1.2.8). The set of finite sequences and hence the set of finite subsets of a countable set is countable.

Proof: The desired set equals $\cup_k A^k$, which is countable by(I.1.2.6) and(I.1.2.7). □

Cardinal Arithmetics

Def. (I.1.2.9). The **sum**, **multiplication** and **exponentiation** of two ordinal is the cardinality of the set $A\coprod B$, $A\times B$ or A^B respectively.

It is easily verified to be associative and commutative, just as usual operations.

Prop. (I.1.2.10). $\aleph_0\times\aleph_0=\aleph_0$ by(I.1.2.6). And $\kappa\times\kappa=\kappa$ for any infinite cardinal, if one uses the axiom of choice by(I.1.4.4).

Prop. (I.1.2.11). The image of a set X has cardinals no more than X, if axiom of choice holds.

Proof: Use axiom of choice to choose an element from each inverse image $f^{-1}(\{x\})$, then it is an injection from $f(X)$ to X. □

Prop. (I.1.2.12) (Cantor). $|P(X)|=|2^{|X|}|$, and $|X|<|P(X)|$.

Proof: The first is obvious, for the second, the function $x\to\{x\}$ is an injection of X into $P(X)$. And there are no mapping from X onto $P(X)$, because if f is one, the consider $S=\{x|x\notin f(x)\}$, then S is not in the range of f, because if $f(z)=S$, then $z\in S$ iff $z\notin S$, contradiction. □

Prop. (I.1.2.13) (Cardinality Arithmetic of $\aleph_0$). For cardinality arithmetics involving $\aleph_0$, Cf.[Set Theory Jech P98].

Prop. (I.1.2.14). if $|B|=2^{\aleph_0}$ and $|A|\leq\aleph_0$, then $|B-A|=2^{\aleph_0}$. In fact, $|B-A|=|B|$ for any $|A|<|B|$, if one uses the axiom of choice.

Proof: By(I.1.2.13), we can assume $B=\mathbb{R}\times\mathbb{R}$, then project A onto the coordinate axis, then $\pi(A)$ has cardinality$\leq\aleph_0$, so there is a $x_0\notin\pi(A)$, so $x_0\times\mathbb{R}\subset B-A$, so $|B-A|=2^{\aleph}$.

For the general case, ? □

Conjecture (I.1.2.15) (The Continuum Hypothesis). There is no cardinal κ that $\aleph_0<\kappa<2^{\aleph_0}$.

Notice $2^{\aleph_0}\geq\aleph_1$ by Cantor's theorem(I.1.2.12), and this hypothesis is equivalent to $2^{\aleph_0}=\aleph_1$.

For Infinite operation of Cardinal Arithmetics, Cf.[Set Theory Jech Chap9].

3 Ordinal

Linear Ordering

Def. (I.1.3.1) (Well-Ordering). A **linear ordering** is just a totally ordering.

A linear ordering is called a **well-ordering** if every nonempty subset has a minimal element.

Prop. (I.1.3.2) (Lexicographical Ordering). If given a family of linearly ordered set A_i indexed by a well-ordered set I, then there is a linear ordering on $\prod_I A_i$, where $(f_i) < (g_i)$ iff $(f_i) \neq (g_i)$ and for the minimal i_0(well-ordering used) that $f_{i_0} \neq g_{i_0}$, $f_{i_0} < g_{i_0}$. It is called the **lexicographical ordering**.

Def. (I.1.3.3). An ordered set X is called **dense** iff for each $a < b$, there is a x that $a < x < b$.

Prop. (I.1.3.4). Any two ordered set that is countable, dense and has no endpoints are isomorphic. In particular, any of these is isomorphic to the set of rational numbers $\mathbb{Q}$.

Proof: Cf.[Set Theory Jech P83] or [Model Theory Marker P48]. □

Cor. (I.1.3.5). Any countable linearly ordered set can be mapped isomorphically into $\mathbb{Q}$.

Proof: Cf.[Set Theory Jech P84]. □

Def. (I.1.3.6). A **initial segment** of an ordered set W is the ordered set $W[a] = \{x \in W | x < a\}$.

Lemma (I.1.3.7). If W is a well-ordered set, then any increasing function $f : W \to W$ satisfies $f(x) \geq x$.

Proof: If the set $\{x | f(x) < x\}$ is not empty, then it has a minimal element a, then $f(f(a)) < f(a)$, contradiction. □

Cor. (I.1.3.8). A well-ordered set cannot be isomorphic to an initial segment of itself, and an automorphism of a well-ordered set must be identity.

Proof: Use the above lemma(I.1.3.7), if it is isomorphic to $W[a]$, then $f(a) < a$, contradiction. For any automorphism, $f(x) \geq x, f^{-}(x) \geq x$, so $f(x) = x$. □

Prop. (I.1.3.9) (Comparison of Well-Orderings). A cut of a well-ordered set is well-ordered. And for any two well-ordered sets W_1, W_2, either they are isomorphic, or one of them is isomorphic to a initial segment of another.

Proof: The three cases are mutually exclusive by(I.1.3.8), So it suffices to show one of them holds.

Define a set $f = \{(x, y) \in W \times W | W_1[x] \cong W_2[y]\}$. (I.1.3.8) shows f is injective and monotone in both coordinates. Now we want to prove that if the domain of f is not all W_1, then it is an initial segment, and the image is all W_2, this will finish the proof.

It is clearly an initial segment $W_1[a]$ because it is well-ordered and if $h : W_1[x] \cong W_2[y]$ and $x' < x$, then $h : W_1[x'] \cong W_2[h(y)]$. If the image is not all of W_2, then similarly the image of f is an initial segment of W_2, $= W_2[b]$. But this means $W_1[a] \cong W_2[b]$, so a, b is also in the domain(image), which is a contradiction. □

Complete Linear Ordering

Def. (I.1.3.10). A **cut** of an ordered set X consists of two disjoint nonempty subsets $A \cup B = X$ that $a < b$ for any $a \in A, b \in B$.

It is called a **Dedekind cut** A doesn't has a maximal element. It is called a **gap** iff A doesn't have a maximal element and B doesn't have a minimal element.

An ordered set is called **complete** if there are no gaps.

Prop. (I.1.3.11). In a complete ordered set X, any subset T bounded from above has a supremum, and subset bounded form below has a infimum.

Proof: Consider the cut $A = \{x|x < a \text{ for some element in } T\}, B = \mathbb{R} - A$, if T is bounded above, B is not empty, so this is truly a cut, and A doesn't has a maximal element, because if $x < a \in \mathbb{R}$, then $x < \frac{x+a}{2} < a$. So by completeness of R, B has a minimal element, that is, the supremum of A exists. Similarly for the case A bounded from below. □

Prop. (I.1.3.12) (Completion of Ordering). There is an obvious ordering on the set C of all Dedekind cuts of X, and X embeds into C by $b \mapsto \{x|x < b\} \cup \{x|x \geq b\}$.

C is complete and has no endpoints, P is dense in C, which is called a **completion** of P.

Proof: Cf.[Set Theory Jech P88]. □

Prop. (I.1.3.13) (Real Numbers). $\mathbb{Q}$ has a unique completion ordering $\mathbb{R}$, called the **set of real numbers**. The set of real numbers is not countable.

Proof: $\mathbb{R}$ is a dense linear ordering without endpoints, so by(I.1.3.4) if it is countable then it is isomorphic to $\mathbb{Q}$, but this is not possible because $\mathbb{Q}$ is not complete. □

Prop. (I.1.3.14). $|P(\mathbb{N})| = |2^{\mathbb{N}}| = |\mathbb{R}|$, which is denoted by $2^{\aleph_0}$. By(I.1.3.13), $\aleph_0 < 2^{\aleph_0}$.

Proof: The first equality is by(I.1.2.12). Now by the construction of $\mathbb{R}$, it can be embedded into $P(\mathbb{N})$, so $|\mathbb{R}| \leq |P(\mathbb{Q})| = |P(\mathbb{N})|$. Conversely, $|2^{\mathbb{N}}| \leq |\mathbb{R}|$ by decimal representation, so they are equal by bernstein(I.1.2.2). □

Ordinal Number

Def. (I.1.3.15) (Ordinal Numbers). A set is called **transitive** iff each element of T is a subset of T. A set α is called a **ordinal number** iff α is transitive and well-ordered by inclusion.

Prop. (I.1.3.16). If α is an ordinal, then $S(\alpha) = \alpha \cup \{\alpha\}$ is also an ordinal, obviously. Thus any natural number is an ordinal by definition.

An ordinal is called a **successor ordinal** iff $\alpha = S(\beta)$ for some β, and a **limit ordinal** otherwise.

Lemma (I.1.3.17).

1. If α is an ordinal, then $\alpha \notin \alpha$.
2. Every element of an ordinal is an ordinal.
3. If ordinals $\alpha \subsetneq \beta$, then $\alpha \in \beta$. That is, for ordinals, $\subsetneq$ is the same as $\in$.

Proof:

1. If $\alpha \in \alpha$, then contradiction to the fact $\in$ is a ordering(I.1.3.15).
2. To show $x \in \alpha$ is transitive, it suffices to show that if $u \in v \in x$, then $u \in x$, because then v is a subset of x. But this follows from the fact $\in$ is an ordering. And because $x \subset \alpha$, the inclusion of x is the restriction of inclusion in α, so it is a well-ordering.
3. Consider $\beta - \alpha$, it has a minimal element γ. Notice $\gamma \subset \alpha$, because otherwise there is an element of $\beta - \alpha$ smaller than γ, by definition(I.1.3.15).
   Now we show $\gamma = \alpha$, then it will follow that $\alpha \in \beta$. For this, if $\delta \in \alpha$ and $\delta \notin \gamma$, then $\gamma \in \delta$ or $\gamma = \delta$. But then this implies that $\delta \in \alpha$ because α is an ordinal, contradicting the fact $\gamma \in \beta - \alpha$.

□

Prop. (I.1.3.18) (Ordinal is Well-Ordered). Define the ordering of ordinal by $\alpha < \beta$ iff $\alpha \in \beta$. The ordering of ordinals is a total ordering and is a well-ordering.

Proof: If $\alpha\beta \in \gamma$, then $\alpha \in \gamma$ because γ is transitive. If $\alpha \in \beta \in \alpha$, then $\alpha \in \alpha$, contradicting(I.1.3.17).

Given any two ordinals, $\alpha \cap \beta$ is also an ordinal by definition. If $\alpha \cap \beta = \beta$ or α, then $\alpha \subset \beta$, hence $\alpha \in \beta$ by(I.1.3.17). If $\alpha \cap \beta \subsetneq \alpha$ and $\alpha \cap \beta \subsetneq \beta$, then $\alpha \cap \beta \in \alpha \cap \beta$, contradiction.

Well-ordering: Given a set of ordinals, take $\alpha \in A$ and consider the set $\alpha \cap A$. If $\alpha \cap A = \varnothing$, then α is minimal in A, because otherwise some $\beta \in \alpha \cap A$. If $\alpha \cap A \neq \varnothing$, then it has a minimal element β in the inclusion because α is an ordinal. Then β is the minimal element of A. □

Cor. (I.1.3.19) (Supremum Ordinal). Any set of ordinals has a supremum ordinal, it is just $\cup_{\alpha\in X}\alpha$.

Proof: Firstly $\cup_{\alpha\in X}\alpha$ is transitive and it is well-ordered(for each subset $A \subset X$, choose an $\alpha \in X$ that $\alpha \cap A \neq 0$, then the minimal element of $\alpha \cap A$ is just the minimal element of A.) so it is an ordinal.

Now if $\alpha \in X$, then $\alpha \subset \cup X$, so $\alpha \leq \cup X$ by(I.1.3.17). And if $\alpha \in \gamma$ for some ordinal γ, then $\cup X \subset \gamma$. So $\cup X$ is truly the supremum. □

Cor. (I.1.3.20). For any set X of ordinals, there is an ordinal α that is not in X, just choose $S(\cup X)$.

Axiom (I.1.3.21) (Axiom Scheme of Replacement). Let $P(x, y)$ be a property that for each x there exists uniquely a y that $P(x, y)$, then for each set A there is a set B that for each $x \in A$ there is an element $y \in B$ that $P(x, y)$.

Prop. (I.1.3.22). Every well-ordered set is isomorphic to a unique ordinal.

So we can regard an ordinal as an equivalence class of isomorphic well-ordered sets.

Proof: Cf.[Set Theory Jech P111]. □

Cor. (I.1.3.23) (Cardinal as Initial Ordinal). The axiom of choice together with(I.1.3.18) asserts that every cardinal has a unique smallest ordinal, called the **initial ordinal**. So we can identify cardinal number α as an ordinal that is the initial ordinal ω_α of α. Anyway, cardinal number is fewer than ordinal numbers.

The first infinite cardinal number(or the first initial ordinal) is denoted by ω or $\aleph_0$.

Def. (I.1.3.24). The **cofinality** of an ordinal α is the smallest ordinal δ that is the order type of a cofinal subset of α.

The **cofinality** of or a poset (i.e partially ordered set) α is the is the smallest cardinality δ of a cofinal subset of α.

Prop. (I.1.3.25) (Transfinite Induction/Recursion). If a property defined for the set of ordinals satisfies:

1. $P(0)$.
2. $P(\alpha + 1)$ if $P(\alpha)$.
3. $P(\lambda)$ if $P(\beta)$ for all $\beta < \lambda$.

then P is true for all ordinals.

Transfinite recursion:

Proof: □

Ordinal Arithmetic

Cf.[Set Theory Jech Chap5.5].

Def. (I.1.3.26). We use infinite recursion to define **addition** of ordinals as

- $\beta + 0 = \beta$
- $\beta + (\alpha + 1) = (\beta + \alpha) + 1$, where $\alpha + 1$ is the successor of α.
- $\beta + \alpha = \sup\{\beta + \gamma | \gamma < \alpha\}$ for a limit ordinal α.

The **multiplication** and **exponentiation** are defined similarly.

Remark (I.1.3.27) (Cardinal and Ordinal Arithmetics). Note that the ordinal arithmetics may be smaller than the ordinal sum of the corresponding initial ordinal(I.1.3.23), because operations of initial ordinals may not be initial, the deeper reason is that the cardinal case, we can rearrange the order to get a smaller ordinal.

Prop. (I.1.3.28). The addition and multiplication of ordinals are of the order type of $\alpha \coprod \beta$ in adjunction order and $\alpha \times \beta$ in lexicographical order respectively, Cf.[Set Theory Jech P120,122]

Cantor Normal Form

Prop. (I.1.3.29) (Cantor Normal Form). Any ordinal α can be expressed uniquely as the form $\alpha = \sum_{i<n} \omega^{\beta_i}$, where $\beta_0 \geq \beta_1 \geq \ldots \beta_{n-1}$ are ordinals.

Proof: Cf.[Jech Set Theory P124]. □

Prop. (I.1.3.30) (Goodstein Sequence). The **weak Goodstein sequence** is a sequence that m_2 is any positive integer, m_{k+1} is m_k written in k-basis and replacing the base by $k+1$, and then minus 1.

The **Goodstein sequence** is a is a sequence that m_2 is any positive integer, m_{k+1} is m_k written in k-basis and even the exponents in k-basis and and replacing the base by $k+1$, and then minus 1.

Then for each Goodstein sequence and weak Goodstein sequence, it reaches 1 in a finite number of times.

Proof: Let $m_k = \sum k^{a_i} b_i$, then let the ordinal $\alpha_k = \sum \omega^{a_i} b_i$. Then it is clear that $\alpha_2 > \alpha_3 > \ldots$. But if the weak Goodstein sequence doesn't terminate, we constructed a descending sequence of ordinals that doesn't terminate, contradiction(choose a minimal element).

Similarly for Goodstein sequences, just replace every base k by ω. □

4 Alephs

Def. (I.1.4.1). For any set A, there is a least ordinal that is not equipotent to any subset of A, called the **Hartogs number** of A. This is clearly an initial ordinal.

Prop. (I.1.4.2). Hartogs number exists for any set.

Proof: By axiom schema of replacement, any well-ordered subsets of A is equipotent to an ordinal, and also by axiom schema of replacement, there is a set H that for any well-ordering of subsets of A in $P(A \times A)$, this ordered sets is equipotent to a $\alpha \in H$. Then use(I.1.3.20) to find a minimal ordinal that is not equipotent to any subset of A. In fact, this is just $h(A) = \{\alpha \in H | \alpha \text{ equipotent to some subset of A}\}$. □

Def. (I.1.4.3) (Aleph). The **alephs** for ordinal numbers are defined recursively: $\aleph_0 = \omega$, $\aleph_{\alpha+1} = h(\aleph_\alpha)$, and $\aleph_\alpha = \sup\{\aleph_\beta | \beta < \alpha\}$ for a limit ordinal α. By definition $\alpha_\alpha < \aleph_\beta$ when $\alpha < \beta$.

Then $\aleph_\alpha$ are all infinite initial ordinal numbers, and any infinite ordinal number is of the form $\aleph_\alpha$ for some ordinal α. So natural numbers together with alephs are just all the cardinal numbers.

Notice: to avoid confusion, when do arithmetic of ordinal numbers, $\aleph_\alpha$ is written as ω_α.

Proof: Use transfinite induction on α. The only nontrivial case is when α is a limit ordinal, where if $\gamma < \aleph_\alpha$ and $|\gamma| = |\aleph_\alpha|$, then there is a $\beta < \alpha$ that $\gamma \leq \aleph_\beta$ by definition, so $|\aleph_\alpha| < |\gamma| \leq |\aleph_\beta| < |\aleph_\alpha|$ as $\aleph_\beta$ is an initial ordinal.

To prove that any infinite initial ordinal is an aleph, first notice that $\alpha < \aleph_\alpha$ by a simple transfinite induction. So we may use transfinite induction on the following assertion for α: if $\Omega < \aleph_\alpha$, then there is a $\gamma < \alpha$ that $\Omega = \aleph_\gamma$. For this, $\alpha = 0$ is trivially true, if $\alpha = \beta + 1$, then $\Omega < h(\aleph_\alpha)$ implies that $|\Omega| < |\aleph_\alpha|$ by definition. Because Ω is initial, $\Omega = \aleph_\beta$ or $\Omega < \aleph_\beta$, so by induction hypothesis it is true. If α is a limit ordinal, then $\Omega < \omega_\beta$ for some $\beta < \alpha$, so also by induction hypothesis it is true. □

Aleph Arithmetics

Prop. (I.1.4.4). $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

Proof: Cf.[Set Theory Jech P134]. □

Cor. (I.1.4.5). $\aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$ for $\alpha \leq \beta$, and $n \cdot \aleph_\beta = \aleph_\beta$.

So $\aleph_\alpha + \aleph_\beta = \aleph_\beta$ for $\alpha \leq \beta$, and $n + \aleph_\beta = \aleph_\beta$.

5 The Axiom of Choice

Def. (I.1.5.1). Let S be a system of sets, a function g defined on S is called a **choice function** iff $g(X) \in X$ for each $X \in S$.

Prop. (I.1.5.2) (Zermelo). The following are equivalent:

1. (the axiom of choice)There exists a choice function for very system of sets.
2. (the well-ordering principle)Every set can be well-ordered.
3. (Zorn's lemma)If every chain in a partially ordered sets has a upper bound, then the partially ordered set has a maximal element.

Proof: $2 \to 1$: If A is well-ordered, then $P(A)$ clearly has a choice function, that is the minimal element of a set.

$1 \to 2$: Use transfinite recursion, Cf.[Set Theory Jech P137].

$2 \to 3, 3 \to 2$: Cf.[Set Theory Jech P142]. □

Axiom (I.1.5.3) (the Axiom of Choice). Any system of sets has a choice function.

Prop. (I.1.5.4). Every infinite set X has a countable subset, if the axiom of choice holds.

Proof: Choose a well-ordering of it(I.1.5.2), then it is an infinite ordinal. Then the initial segment of the first ordinal $X[\omega]$ is a countable subset. □

Prop. (I.1.5.5). For every infinite set S, there exists a unique aleph $\aleph_\alpha$ that $|S| = \aleph_\alpha$.

Proof: choose a well-ordering of S(I.1.5.2), then it is an infinite ordinal, and it has the same cardinality as an initial cardinal by(I.1.3.23), thus the result. □

Cor. (I.1.5.6). For any sets A and B, either $|A| \leq |B|$ or $|B| \leq |A|$.

Proof: Because the ordinal is totally ordered(I.1.3.18). □

6 Filters and Ultrafilters

Def. (I.1.6.1). Let S be a non-empty set, a **filter** on S is a collection F of subsets of S that:

- $S \in F$ and $\varnothing \notin F$.
- If $X, Y \in F$, then $X \cap Y \in F$.
- If $X \in F, X \subset Y$, then $Y \in F$.

The filter of all sets containing a nonempty subset A of S is called the **principle filter**. if $A = \{a\}$ for some element a.

an **ideal** on S is a collection F of subsets of S that:

- $\varnothing \in F$ and $S \notin F$.
- If $X, Y \in F$, then $X \cup Y \in F$.
- If $X \in F, X \supset Y$, then $Y \in F$.

An ideal is just the dual of a filter.

Def. (I.1.6.2) (Finite intersection property). A family of subsets of a set is said to have the **finite intersection property** if any finite collection of elements of this family is non-empty.

Lemma (I.1.6.3). let G be a collection of subsets of S that has the finite intersection property(I.1.6.2), then there is a smallest filter F that $G \subset F$. It is just the collection of subsets of S that contain some finite intersection set of elements of G.

Def. (I.1.6.4) (Ultrafilter). An **ultrafilter** is a filter F that for every subset X, $X \in F$ iff $S - X \notin F$. A **prime ideal** is an ideal that for every subset X, $X \in F$ iff $S - X \notin F$.

A ultrafilter is equivalent to a maximal filter. And it is equivalent to a $\{0,1\}$-valued finitely additive measure on S.

Proof: If F is an ultrafilter, then it is maximal, because any larger filter will have some $X, S - X$, thus has $\varnothing$, contradiction.

Conversely, if F is maximal filter but not ultra, then there is a X that $X \notin F$, $S - X \notin F$. Let $G = F \cup \{X\}$, then any finite intersection of elements of G is not empty: $X_1 \cap \ldots \cap X_n \cap X \neq 0$ otherwise $S - X \in F$. So there is a filter containing G by(I.1.6.3), contradiction. □

Prop. (I.1.6.5) (Pushforward of Filters). If $\mathcal{F}$ is a(n) (ultra)filter on X and $f : X \to Y$ is a function, then $f_*(\mathcal{F}) = \{A \subset Y | f^{-1}(A) \in \mathcal{F}\}$ is a(n) (ultra)filter on X, called the **pushforward filter** of $\mathcal{F}$.

Prop. (I.1.6.6). For an ultrafilter $\mathcal{F}$ on X , if $U_i \notin \mathcal{F}$, then $\sum_{i=1}^n U_i \notin \mathcal{F}$.

Proof: As $X - U_i \in \mathcal{F}$, there intersection are in $\mathcal{F}$, so its complement is not in $\mathcal{F}$. □

Prop. (I.1.6.7). Any filter can be extended to an ultrafilter(maximal filter), if the axiom of choice is used.

Proof: Use Zorn's lemma(I.1.5.2). It suffices to prove that a union of a chain of filters is a filter, which is trvial. □

Cor. (I.1.6.8). Non-principal ultrafilter exists on any infinite set. And in fact, any non-principal ultrafilter contains all the cofinite sets.

For any non-principle ultrafilter, it cannot contains a single pt $\{x\}$, so it contains every cofinite set.

Proof: Consider any ultrafilter containing the filter of cofinite sets of S, then it is non-principal. □

Closed unbounded and Stationary Set

Silver's Theorem

7 Combinatorial Set Theory

Prop. (I.1.7.1) (Ramsey's Theorem). For any positive natural number r, s if we color the r-subsets of a set with cardinality $\aleph_0$ into s groups, then there is a subset of cardinal $\aleph_0$ that all its r-subsets are colored the same.

Cor. (I.1.7.2). Every infinite linearly ordered set contains a subset isomorphic to $(\mathbb{N}, <)$ or $(\mathbb{N}, >)$.

Proof: Choose a well ordering of it. Then consider this new ordering and the original ordering. Then there is an infinite set that is compatible with the original ordering, or converse. Then its initial segment of order type ω_0 satisfies the requirement. □

Trees

Def. (I.1.7.3). A **tree** is a partial ordered set T that there is a minimal element r and for each x, $\{y \in T | y < x\}$ is finite and linearly ordered.

A tree is called **of finite branched** for each x, there is a finite set $\{y_1, \ldots, y_r\}$ is T that $y_i > x$ and if $z > x$, then $z \geq y_i$ for some i.

Def. (I.1.7.4) (Height). For any node x, $\{y \in T | y < x\}$ is a well-ordered set, which is isomorphic to an ordinal by(I.1.3.22), it is called the **height** of x. T_α denotes the set of all nodes of T of order α. The least α that $T_\alpha \neq \varnothing$ is called the **height** of T.

A **branch** is a maximal chain in T, its **length** is its ordinal. The length is always smaller than the height of the tree. If it equals the height of the tree, it is called **cofinal**.

Def. (I.1.7.5). A **subtree** is a subset T' of T that if $x \in T', y < x$, then $x \in T'$.

An **antichain** of a tree T is a subset $A \subset T$ that any two elements in A are incomparable.

Def. (I.1.7.6). A **path** through T is a morphism of ordering from ω to T.

Lemma (I.1.7.7) (König's Lemma). If T is an infinite finite branching tree, then there is a path through T.

Proof: Use recursion to choose for each n an element that has infinite successors. □

Def. (I.1.7.8). An **Aronszajn tree** is a tree of height κ and all its level sets are at most countable, but has no branches of length κ.

Prop. (I.1.7.9). An **Aronszajn tree** of height ω_1 exists.

Proof: Cf.[Set Theory Jech P228]. □

8 The Axiom of Foundation

9 Large Cardinals

10 Forcing Construction of Cohn

Axiom (I.1.10.1) (Martin's Axiom). If P is a partially ordered set satisfying the countable chain condition, and D is a collection of dense subsets of P with $|D| < 2^{\aleph_0}$, then there is a D-generic filter on P.

I.2 Linear Algebra

References are [Linear Algebra Hoffman], [线性代数 谢启鸿] and [Determinant 安金鹏高等代数].

In this section, we study vector spaces V over a field K, without considering the topology on K or V.

1 Basics

Def. (I.2.1.1) (Basis). If M is a vector space over a field K, then the sets S that are linearly independent over R has maximal objects by Zorn's lemma, and such a maximal object must span M, called a **basis** of M.

Prop. (I.2.1.2) (Dimension). All basis of a linear k-vector space V have the same cardinality, this cardinality is called the **dimension** $\dim_k(V)$ of V. This follows immediately from(I.3.4.1).

Prop. (I.2.1.3) (Canonical way of Writing a Basis). After so many years, I still find it confusing to write a basis and observing change of basis, so I will write it here:

A vector should always be written vertically, and so a basis should be $\overrightarrow{e} = (e_1, \ldots, e_n)$(horizontal), and a vector with basis $\overrightarrow{a}$(vertical) is in fact $\overrightarrow{e}\overrightarrow{a}$.

A change of basis should be written $\overrightarrow{e'} = ea$, with $a \in GL_n$, and then if an operator has matrix A w.r.t. the basis $\overrightarrow{e}$, it then map in the basis $\overrightarrow{e'}$ $v = \overrightarrow{e'}x = \overrightarrow{e}ax \mapsto \overrightarrow{e}Aax = \overrightarrow{e'}a^{-1}Aax$, so it has matrix $a^{-1}Aa$ w.r.t the basis $\overrightarrow{e'}$.

Prop. (I.2.1.4). Let F is a subfield of K and U is a K-vector space with a F-subspace U'. Then if every finite F-linearly independent subset of U' is K-linearly independent, then $\dim_F(U') \leq \dim_K(U)$.

Proof: This is very simple, if the converse is true, there is a F-basis u'_j of U', then some of u'_j is K-linearly dependent, contradiction. $\square$

Prop. (I.2.1.5). A, B are two $n \times n$-matrices, if $1 - AB$ is invertible, then so does $1 - BA$, and

$$(1 - BA)^{-1} = 1 + B(1 - AB)^{-1}A.$$

Proof: Immediate from(I.3.3.1) or(I.2.8.9). $\square$

2 Rank

Prop. (I.2.2.1). The row rank of a matrix A is the same as the column rank.

Proof: Let A have n rows, the column rank equals $\dim \operatorname{Im} f$, and the row rank is $n - \dim \operatorname{Ker} f$, so by the rank-nullity theorem $\dim \operatorname{Im} f + \dim \operatorname{Ker} f = n$, which is because exact sequence of vector spaces split, the conclusion follows. $\square$

Prop. (I.2.2.2) (Sylvester's Inequality). For U a $m \times n$ matrix and V a $n \times k$ matrix,

$$\operatorname{Rank}(UV) \geq \operatorname{Rank}(U) + \operatorname{Rank}(V) - n$$

Proof: This comes from $\dim \operatorname{Ker} fg \leq \dim \operatorname{Ker} f + \dim \operatorname{Ker} g$, which is because $\operatorname{Ker} fg = g^{-1}(\operatorname{Ker} f)$. $\square$

Prop. (I.2.2.3) (Finite Field General Linear Group). Over finite field $\mathbb{F}_{p^k}$, $|GL_n(\mathbb{F}_{p^k})| = (p^n-1)(p^n-p)\dots(p^n-p^{n-1})$.

Proof: This is because choose the rows are equivalent to choosing a basis for $V = \oplus_{i=1}^n \mathbb{F}_{p^k}$, and when choosing n-th row, it suffices to avoid an element in the span of the first $n-1$ rows. □

3 Dual space

Prop. (I.2.3.1). For a linear map between two spaces of the matrix form A, the adjoint map between their dual spaces is of the matrix form the transpose of A.

Prop. (I.2.3.2) (Infinite Dual space). If $\dim_K V$ is not finite, then $\dim_K V < \dim_K V^*$.

Proof: Notice $\mathrm{Hom}(\oplus_{i\in I} Ke_i, K) = \prod Ke_i^*$.

We prove first that if $|K|$ is at most countable, then $|V| = |I|$. Notice the set $S_n(I)$ of all n-element subsets of I is of the same cardinality of I(I.1.4.4). And the finite sums of K and e_i can be seen as a subset of $S_n(I)\times K^n$, so it has the same cardinality of I.

Now we can prove if $|K|$ is at most countable, then $\dim V < \dim V^*$. This is because V^* equals the functions from V to K, which is bigger than the functions from V to $\{0,1\}$, which is the power set of V, so having cardinality $2^{|V|}$ which is bigger than $|V|$, by Cantor theorem(I.1.2.12).

Now generally, K is not countable, but it has a base field F, which is countable, so we consider the F-vector space $W = \oplus_{i\in I} Fe_i$, then $\dim_F W = \dim_K V$, and $\dim_F W < \dim_F W^*$. If we can show $\dim_F W^* \le \dim_K V^*$, then we are done.

For this, first consider the natural F-linear mapping $W^*\to V^*$, which is clearly an imbedding. Now we want to use(I.2.1.4), so we check the conditions, for F-linearly independent $\varphi_1,\dots,\varphi_n$, if $\sum c_i\varphi = 0$, $c_i\in K$, then if we can find $w_k\in W$ that $\varphi_i(w_j) = \delta_{ij}$, then this is a contradiction. But this is true, by a simple argument, using the F-linearity of F. □

4 Rational Form and Jordan Form

Prop. (I.2.4.1) (Elementary and Invariant Factors). A linear operator in $L(V)$ is equivalent to a $\mathbb{K}[X]$-module structure on V, and two operators are similar iff the module structure are isomorphic.

As $K[X]$ is a PID, the elementary factors, invariant factors, cyclic and elementary decomposition theorems(I.3.4.16) can be applied to the case.

Proof: Cf.[Advanced Linear Algebra P168]. □

Cor. (I.2.4.2) (Complex Structure Form). A matrix $J\in M_n(\mathbb{R})$ that $J^2+1=0$ is similar to $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^n$.

Proof: By the elementary factor theorem, the elementary factors of J are all x^2+1, thus its Rational form is just $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^n$. □

Cor. (I.2.4.3) (Jordan Form).

- For a matrix over an alg.closed field, it is similar to a matrix of blocks $\lambda_i I + N$, $Nx_i = x_i + 1$, called the **Jordan form**.

- For a real matrix, it is similar to a matrix of blocks of the above form together with $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ on the diagonal and $I_{2\times 2}$ on the lower side.

Proof: 1: Over an alg.closed field, the elementary factors are all of the form $(x-c_i)^{m_{ij}}$. Now in the basis $v, (T-c_i)v, \dots, (T-c_i)^{m_i-1}v$, the matrix is just the Jordan form.

2: Over $\mathbb{R}$, the elementary factors are all of the form $(x-c_i)^{m_{ij}}$ and $((x-a)^2+b^2)^{m_{ij}}$. Then complexify it and consider a cyclic vector v, for $(T-(a+bi)I)$, let $v_{n+1} = (T-(a+bi)I)v_n$, and let $v_n = X_n + iY_n$, then it can be verified that T is of the Jordan form given in the basis X_i, Y_i. □

Applications

Prop. (I.2.4.4). For any two matrices $A, B \in M_{n\times n}(K)$, $(AB)^n$ and $(BA)^n$ are similar.

Proof: It suffices to show that they have the same elementary factors. Notice that for any irreducible polynomial p, if $p \neq x$, then if $p^k(AB)v = 0$, then $p^k(BA)Bv = 0$, if $p^k(BA)v = 0$, then $p^k(AB)Av = 0$. Thus there are maps $B : N(p^k(AB)) \to N(p^k(BA))$ and $A : N(p^k(BA)) \to N(p^k(AB))$. Now their composition are both injective, thus they have the same dimension.

And for $p = x$, these two both have nullity as the multiplicity of 0 in the charpoly of AB, BA(I.2.8.10), thus the same. So they have the same elementary factors, thus similar. □

5 Minimal and Characteristic Polynomial

Def. (I.2.5.1). An operator is called **nonderogatory** iff its minimal polynomial equals its characteristic polynomial, or equivalently they have the degrees.

Prop. (I.2.5.2) (Diagonalizable and Minimal Polynomial). A matrix over an alg.closed field is diagonalizable iff its minimal polynomial has no multiple factors.

Proof: If it is □

Def. (I.2.5.3). Companion matrix.

Prop. (I.2.5.4). An operator is cyclic iff it is similar to a companion matrix. A companion matrix is nonderogatory.

Prop. (I.2.5.5) (Rational Canonical Form). Every matrix is similar to splint of companion matrixes, corresponding to its elementary divisors.

Prop. (I.2.5.6) (Invariant Factor Form). Every matrix is similar to splint of companion matrixes, corresponding to its invariant factors.

Lemma (I.2.5.7). For a companion matrix A of p, $\det(xI-A) = p$.

Prop. (I.2.5.8) (Generalized Cayley-Hamilton). The characteristic polynomial of A is the product of the elementary divisors of A, thus they have the same set of irreducible factors, but may not with the same multiplicity.

Prop. (I.2.5.9). The linear functor $X \to AX - XC$ is an isomorphism iff the minimal polynomial of A and C has not common factor.

Proof: Notice if $AX = XC$, then we have $P(A)X = XP(C)$ for every polynomial P, in, particular for the minimal polynomials of A and C, thus $P(C)$ is non-invertible and A, C has a characteristic value in common. Conversely, if they have a characteristic value, then we upper triangularize A to see clearly that there is a X that $AX = XC$ (X has only the first row). □

6 Similarity(Linear map)

Prop. (I.2.6.1). If a linear map has matrix form T in a basis (X_i) and there is another basis (Y_i) that $(Y_i) = (X_i)P$, then it has matrix form PTP^{-1} in the basis (Y_i). In particular, if T can be diagonalized, with eigenvectors (X_i), then $T = (X_i)D(X_i)^{-1}$.

Prop. (I.2.6.2).
- An $n \times n$-matrix A is upper-triangulable over a field K iff its minimal polynomial is a product of linear factors.
- An $n \times n$-matrix A is diagonalizable over a field K iff its minimal polynomial is a product of linear factors with no multiple root.

Proof: 1: If it is upper-triangulable, its minimal polynomial is a product of factors because its characteristic polynomial does(I.2.5.8). Conversely, we can find an eigenvector for A, then we quotient this vector and use induction.

2: If it is diagonalizable, then the minimal polynomial is clearly polynomials. Conversely, its elementary factors are all linear factors, thus its Jordan form is just diagonal(I.2.4.3). □

Cor. (I.2.6.3) (Upper Triangulation Alg.Closed Field). If K is alg.closed, then any $n \times n$-matrix A is upper-triangulable over K. Similarly, it is lower-triangulable.

Proof: It suffices to find a flag that is stabilized by A. And for this, it suffices to find an eigenvector of A. This is clear, as the characteristic polynomial of A has a root in K. □

Prop. (I.2.6.4) (Simultaneously Triangulation). If A_i is a commutating family of upper-triangulable $n \times n$-matrices, then they are simultaneously triangulable.

Proof: As in the proof of(I.2.6.3), by induction, it suffices to show there is a common eigenvector. Now assume there are f.m. matrices in $\mathcal{F}$, we induct on the number of matrices to show there is an eigenvector. Let λ be an eigenvalue of A_1 because it is upper-triangulable, then $N(A_1 - \lambda I)$ is invariant under $\mathcal{F}$, and all the matrices are upper-triangulable on $N(A_1 - \lambda I)$, which is seen by intersecting the flag with $N(A_1 - \lambda I)$. So by induction, there is a common eigenvector for $\mathcal{F}$. □

Prop. (I.2.6.5) (Simultaneously Diagonalizable). If A_i is a commutating family of diagonalizable $n \times n$-matrices, then they are simultaneously diagonalizable.

If A_i is a commutating family of real symmetric matrixes, then they are simultaneously orthogonally diagonalizable.

Proof: We may assume there are f.m. matrices and use induction. Consider the diagonal decomposition V_i of V for A_1, then each V_i is invariant under $\mathcal{F}$. Notice then each A_i is diagonalizable on V_i, thus by induction, $\mathcal{F}$ is simultaneously diagonalizable on each V_i, then $\mathcal{F}$ is simultaneously diagonalizable.

For the second, induction on the numbers of matrices. If some matrix is cI, then clear, if some are not cI, then choose its eigenvalue decomposition, we conclude by induction hypothesis. □

Prop. (I.2.6.6) (Invariance of Field Extension). Let $A \in M_n(F)$, and E be the subfield generated by the entries of A, then the invariant factors of A are polynomials over E. In particular, two matrix are similar over the smallest field that they are defined.

Proof: Clear, because we know how an irreducible polynomial over E factors through $\overline{E}$. □

Prop. (I.2.6.7) (Computing Invariant Factors). Cf.[Hoffman Chap7].

7 Congruence(Bilinear Form)

A matrix M defines a bilinear form on V by $(x, y) \mapsto x^t M y$, so we will interchange freely between a matrix and a bilinear form on V.

Lemma (I.2.7.1). Any eigenvalue of a Hermitian(e.g., real symmetric) matrix M is real.

Proof: Consider the bilinear form defined by the matrix M, then if x is an eigenvector with eigenvalue λ, then $\lambda(x, x) = (Hx, x) = (x, Hx) = \overline{\lambda}(x, x)$, so if λ is not real, $x = 0$. □

Prop. (I.2.7.2) (Unitarily Diagonalizable). A symmetric matrix A is orthogonally diagonalizable. Similarly, a skew-symmetric matrix is orthogonally diagonalizable and an (skew)hermitian matrix is unitarily diagonalizable.

Proof: For any matrix A and any vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle.$$

Now assume that A is symmetric, and $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of A corresponding to distinct eigenvalues λ and μ, the eigenvalue is real, by(I.2.7.1). Then

$$\lambda\langle \mathbf{x}, \mathbf{y}\rangle = \langle \lambda\mathbf{x}, \mathbf{y}\rangle = \langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x}, \mu\mathbf{y}\rangle = \mu\langle \mathbf{x}, \mathbf{y}\rangle.$$

Therefore, $(\lambda - \mu)\langle \mathbf{x}, \mathbf{y}\rangle = 0$. Since $\lambda - \mu \neq 0$, then $\langle \mathbf{x}, \mathbf{y}\rangle = 0$, i.e., $\mathbf{x} \perp \mathbf{y}$.

Now we can find an orthonormal basis for each eigenspace, because for any eigenvector, we can factor A on its diagonal complement. Since the eigenspaces are mutually orthogonal, these vectors together give an orthonormal subset of V. □

Prop. (I.2.7.3) (Normal operator). More generally, a normal operator over $\mathbb{C}$ is unitary diagonalizable using resolution of identity(V.5.4.3) because the spectrum are discrete thus the point projection is orthogonal.

Prop. (I.2.7.4) (Gram-Schimidt). Any symmetric matrix over fields of characteristic$\neq 2$ is diagonalizable.

Proof: Any symmetric matrix defines a bilinear form on V. If B is not identically 0, then there is a x that $x^t Bx \neq 0$, by polarization identity. Then $W = \{Kx\}$ is non-degenerate, so we have $W \oplus W^\perp = V$ by(II.9.1.7). And by induction, we are done. □

Prop. (I.2.7.5) (Antonne-Takagi). For any complex symmetric matrix A, there is unitarily matrix U that UAU^t is a real diagonal matrix with non-negative entries.

Proof: Consider $B = A^*A$ is Hermitian and positive-semi-definite, thus there is a unitary matrix V that V^*BV is diagonal with non-negative real entries by(I.2.7.2). Now $C = V^t AV$ is complex symmetric with C^*C real diagonal. If we let $C = X + iY$, then $XY = YX$. So by(I.2.6.5), there is a real orthogonal matrix W that WXW^t and WYW^t are diagonal. Now set $U = WV^t$, which is unitary, UAU^t is complex diagonal. And easily we can modify the diagonal to be □

Prop. (I.2.7.6) (Symplectic Form). Over $\mathbb{R}$, a skew-symmetric matrix are orthogonally congruent to $\mathrm{diag}\left\{\begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}\right\}_i$.

Proof: Choose a α, β that $(a,b) \neq 0$. and choose their orthogonal complement. □

Cor. (I.2.7.7). For a matrix that $J^2+1=0$, by (I.2.4.2),there is a unique inner product s.t. J is orthogonal and then it is orthogonally congruent to $\langle\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\rangle_n$. (Use cyclic decomposition).

so this J is equivalent to a complex structure, homeomorphic to $O(n)/U(\frac{n}{2})$.

Prop. (I.2.7.8). Given a bilinear form on a field, the relation of orthogonality is symmetric iff it is symmetric or alternating, i.e. $B(x,x)=0$.

Proof: Let $w = B(x,z)y - B(x,y)z$, then $B(x,w)=0$, hence we have $B(w,x)=0$, that is

$$B(x,z)B(y,x) - B(x,y)B(z,x) = 0.$$

Let $z = x$, then $B(x,x)[B(x,y)-B(y,x)] = 0$.

If some $B(u,v) \neq B(v,u)$ and $B(w,w) \neq 0$, then $B(u,u) = B(v,v) = 0, B(w,v) = B(v,w), B(w,u) = B(u,w)$, Let $x = u$ or v se get $B(w,v) = B(v,w) = 0 = B(w,u) = B(u,w)$. Now $B(u, w+v) \neq B(w+v, u)$, hence $B(w+v, w+v) = 0 = B(w,w)$, contradiction. □

Prop. (I.2.7.9). If B is a non-degenerate bilinear form on an associative algebra V, choose a basis x_i of V, then choose a dual basis y_i, then $\sum x_i y_i$ is independent of x_i chosen.

Proof: □

For symmetric bilinear forms and more about quadratic forms, seeII.9.

8 Determinant

Def. (I.2.8.1) (Determinant). For a linear operator $T \in L(V)$, as $\dim \wedge^n V^* = 1$, the **determinant** $\det T$ is defined by $\wedge^n(T^t) = \det T \,\mathrm{id}_{\wedge^n V^*}$. That is: $L(T\alpha_1, \ldots, T\alpha_n) = \det T L(\alpha_1, \ldots, \alpha_n)$. And the determinant of a matrix is defined by the linear operator it associates in a canonical basis.

Prop. (I.2.8.2) (Properties of Determinants).

1. $\det(\mathrm{id}_V) = 1$.
2. $\det(UV) = \det U \cdot \det V$.
3. T is invertible iff $\det T$ is invertible, in which case $\det(T^{-1}) = (\det T)^{-1}$.
4. If $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V, and f_i is its dual basis, then $\det T = f_1 \wedge \ldots \wedge f_n(T\alpha_1, \ldots T\alpha_n)$.

Proof: All these are not hard. □

Prop. (I.2.8.3). $\det T = \det T^t$.

Proof: Use(I.2.8.2), if $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V, and f_i is its dual basis, then

$$\det T^t = \alpha_1 \wedge \ldots \wedge \alpha_n(T^t f_1, \ldots, T^t f_n) = f_1 \wedge \ldots \wedge f_n(T\alpha_1, \ldots T\alpha_n) = \det T$$

□

Prop. (I.2.8.4) (Expansion of Determinants). If A_i be the i-th column of A, then

$$\det A = f_1 \wedge \ldots \wedge f_n(A\varepsilon_1, \ldots, A\varepsilon_n) = f_1 \wedge \ldots \wedge f_n(A_1, \ldots, A_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) A_{\sigma(i)i}.$$

Prop. (I.2.8.5). For a matrix, the determinant satisfies the following properties:

1. adding a multiple of a column/row to another column/row, the determinant doesn't change.
2. Multiplying a row or a column with a scalar, then the determinant multiplies with this scalar.
3. Changing two rows or two columns makes the determinant multiply by -1.

Proof: All this follows from 4 of(I.2.8.2). Notice the last one follows from the first two. □

Prop. (I.2.8.6) (Laplacian Expansion Formula). Cf.[Determinant 安金鹏 P15].

Prop. (I.2.8.7). Adjunction matrix, Cf.[Determinant 安金鹏 P16].

Prop. (I.2.8.8) (Cramer's Rule). Cf.[Determinant 安金鹏 P16].

Prop. (I.2.8.9) (Sylvester's Determinant Identity). If A and B are matrices of sizes $m \times n$ and $n \times m$, then

$$\det(I_m + AB) = \det(I_n + BA)$$

Proof:

$$\begin{aligned}\begin{bmatrix} 1 & \mathrm{A} \\ \mathrm{B} & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ \mathrm{B} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-\mathrm{BA} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{A} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathrm{A} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\mathrm{AB} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathrm{B} & 1 \end{bmatrix}\end{aligned}$$

□

Cor. (I.2.8.10). Multiplying by x, we see that the characteristic polynomial of AB and BA are the same.

Remark (I.2.8.11). There is another proof in case $m = n$: It suffices to show

$$\det(I + (A + xI)(B + xI)) = \det(I + (B + xI)(A + xI)).$$

But notice $A + xI$ and $B + xI$ are invertible in $M_{n\times n}(K(X))$, thus

$$\det(I + (A + xI)(B + xI)) = \det((A + xI)((A + xI)^{-1} + (B + xI))) = \det(I + (B + xI)(A + xI)).$$

Prop. (I.2.8.12).

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B).$$

Proof: As before, consider $\det \begin{bmatrix} A + xI & B \\ C & D + xI \end{bmatrix}$, then $A + xI$ is invertible, and this equals

$$\det \begin{bmatrix} A + xI & B \\ C & D + xI - C(A + xI)^{-1}B \end{bmatrix} = \det(A + xI)\det(D + xI - C(A + xI)^{-1}B).$$

Letting $x = 0$, we get the desired result. □

Prop. (I.2.8.13) (Symplectic Group Determinant). The determinant of a symplectic matrix $\in \mathrm{Sp}(n)$ has determinant 1.

Proof: A symplectic matrix preserves the symplectic structure thus the symplectic form ω, hence preserves ω^n which is $n!$ times the volume form, so it has determinant 1 by definition(I.2.8.1). $\square$

Prop. (I.2.8.14). $GL_n(\mathbb{C})$ can be embedded into $GL_{2n}(\mathbb{R})$, with determinant $|\det|^2$. And in this way, $U(n)$ is mapped into $O(2n)$. Also, $O(n)$ embeds into $U(n)$ diagonally.

Proof:
$$X+iY\mapsto\begin{bmatrix}X & Y\\ -Y & X\end{bmatrix}\sim\begin{bmatrix}X & Y\\ iX-Y & X+iY\end{bmatrix}\sim\begin{bmatrix}X-iY & Y\\ 0 & X+iY\end{bmatrix}$$
$\square$

Prop. (I.2.8.15) (Vandermonde Matrix). The $n\times n$ Vandermonde matrix, with the k-th row $(1,x_k,\ldots,x_k^n)$, has determinant $\prod_{i<j}(x_i-x_j)$. So it is invertible when x_i are pairwisely different.

Proof: Eliminate the first row by adding columns. $\square$

Prop. (I.2.8.16) (Pfaffian). There is a polynomial Pf called **Pfaffian** s.t. $\det M=\mathrm{Pf}(M)^2$ for a skew-symmetric matrix. This is because a skew symmetric is equal to $A^t\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}^k A$ for A an orthogonal matrix (I.2.7.2), so it has determinant $(\det A)^2$ and A and depends polynomially on the entries of M.

Cor. (I.2.8.17).
$$\mathrm{Pf}(A^tMA)=\det A\cdot\mathrm{Pf}(M).$$
Because we only need to consider the sign and it is determined by letting $A=\mathrm{id}$.

9 Spectral Theory

See also 4

Def. (I.2.9.1) (Cayley-transformation). For a field k of char$k\neq 2$ and a matrix $P\in M_n(k)$ that has no eigenvalue -1, there is a **Cayley transformation** $A=\frac{1-P}{1+P}$,. $1+A$ is invertible, and $P=\frac{1-A}{1+A}$.

Then A is skew-symmetric iff P is orthogonal.

Proof: If $Av=-v$, then $v-Pv=-v-Pv$, so $2v=0$, so $v=0$. So $A+1$ is invertible.

If P is orthogonal, then $A^t=\frac{1-P^t}{1+P^t}=\frac{1-P^{-1}}{1+P^{-1}}=-A$. Conversely, if $A^t=-A$, then $P^t=\frac{1+A}{1-A}=P^{-1}$. $\square$

Prop. (I.2.9.2). If char$k\neq 2$ and P is an orthogonal matrix of odd dimension, then $\det P$ is an eigenvalue of P.

Proof: multiplying by -1, we can assume $\det P=-1$. Consider the Cayley transformation(I.2.9.1), then
$$\det P=\det(1-A)\det(1+A)^{-1}=\det(1-A)^t\det(1+A)^{-1}=1.$$
Contradiction. $\square$

10 Decompositions

Prop. (I.2.10.1). Let $G = GL(n,\mathbb{R}), K = O(n)$ or $G = GL(n,\mathbb{R})^+, K = SO(n)$, then every double coset $K\backslash G/K$ has a unique representation of diagonal matrix D with decreasing positive entries.

Proof: For the existence, given g, consider $S = g^t g = k_1^{-1} diag(\lambda_1, \ldots, \lambda_n) k_1$, where $k_1 \in SO(n)$(I.2.7.2). Then consider

$$k_2 = g k_1^{-1} diag(\lambda_1^{-1/2}, \ldots, \lambda_n^{-1/2})$$

it is orthogonal and $g = k_2 diag(\lambda_1^{1/2}, \ldots, \lambda_n^{1/2}) k_1$.

For the uniqueness, consider $g k_1^{-1} diag(\lambda_1^{-1/2}, \ldots, \lambda_n^{-1/2})$ is orthogonal, thus $k_1 S k_1^{-1}$ is diagonal with decreasing positive entries, thus uniquely defined. □

Prop. (I.2.10.2) (Polar Decomposition). $GL_n(\mathbb{R})$ can be decomposed as $P \cdot O(n)$, where P is a positive symmetric matrix and $O(n)$ the orthogonal matrix. a positive symmetric matrix can be diagonalized, so $GL_n(\mathbb{R})$ have $O(n)$ as deformation kernel.

Similarly, $Sp(2n)$ can be decomposed as $P \cdot U(n)$, because $O(2n) \cap Sp(2n) = U(n)$. And it has $U(n)$ as deformation kernel.

Prop. (I.2.10.3) (QR-Decomposition).

- Any real matrix A has the form $A = QR$ where Q is orthogonal and R is upper triangular. Moreover, if A is invertible, then there is a unique R that has positive diagonal entries.
- Any complex matrix A has the form $A = QR$ where Q is unitary and R is upper triangular, and if A is invertible, then there is a unique R that has positive diagonal entries.

Proof: Use Gram-Schimidt orthogonalization. □

Prop. (I.2.10.4) (Bruhat Decomposition).

$$GL_n[K] = BWB$$

where W is permutation matrix, B is upper triangular, and the decomposition is a disjoint union。

Proof: Cf.[群与表示 王立中] □

Prop. (I.2.10.5) (Iwasawa Decomposition).

11 Positivity

Prop. (I.2.11.1) (Positivity and Principal Minors). A matrix is symmetric positive iff it is symmetric, and all its upper principle minors has positive determinants.

Proof: Cf.[Hoffman P328]. □

Prop. (I.2.11.2) (Positive Matrices and Symmetric Matrices). The exponential defines a diffeomorphism from the space of symmetric matrices to positive definite symmetric matrices in $GL(n,\mathbb{R})$.

Proof: By(I.2.7.2) it is clearly surjective. For injectivity, consider $\exp(X) = \exp(Y)$, then at least X, Y are both similar to the same diagonal matrix $diag(d_1, \ldots, d_n)$, and then we have

$$X = \tau Y \tau^{-1}, \quad diag(D_1, \ldots, D_n) = \tau diag(D_1, \ldots, D_n)\tau^{-1}$$

where $D_i = e^{d_i}$. Consequently, $diag(D_1^k, \ldots, D_n^k) = \tau diag(D_1^k, \ldots, D_n^k)\tau^{-1}$, and we can choose c_i that $\sum c_i D_j^i = d_j$ for any j, because the Vandermonde matrix is nonsingular. Hence $diag(d_1, \ldots, d_n) = \tau diag(d_1, \ldots, d_n)\tau^{-1}$, and $X = Y$. □

Prop. (I.2.11.3) (Farkas' Lemma). For a matrix A, and a vector b, exactly one of the following equation has a solution:

$$\begin{cases} AX = b, X \geq 0 \\ Y^t A \leq 0, Y^t b > 0 \end{cases}$$

Proof: First notice if both have a solution, then $0 \geq Y^t AX > 0$, contradiction. The rest follows form the Hahn-Banach separation theorem. □

Cor. (I.2.11.4) (Gordan's Theorem). exactly one of the following has a solution:

$$\begin{cases} AX > 0 \\ Y^t A = 0, Y \geq 0, Y \neq 0 \end{cases}$$

Proof: If both have a solution, then $0 = Y^t AX > 0$, contradiction. If the first has no solution, then $A'x = e, z \geq 0$, where $A' = [A, -A, -I]$ has no solution, by Farkas' lemma, there is a solution of $Y^t A' \leq 0$ and $Y^t b = 0$. Which shows that $Y^t A = 0$ and $Y \neq 0$. □

Cor. (I.2.11.5). For any subspace in $\mathbb{R}^m$, either it has an intersection with the open first quadrant, or its orthogonal complement has an intersection with the closed first quadrant minus 0. (Regard it has the image of a AX).

12 Miscellaneous

I.3 Abstract Algebra(Lang)

References are [Algebra Lang], [Advanced Linear Algebra] and [Finite Groups Issac].

This section differs from Commutative Algebra because it contains basic, and maybe non-commutative properties.

1 Group Theory

Def. (I.3.1.1). A **simple group** is a group that has no normal subgroups.

Prop. (I.3.1.2) (Fundamental Isomorphisms). For a normal subgroups U, H of G,

- $G/HU \cong (G/U)/(H/H \cap U)$.
- $UH/U \cong U/U \cap H$.

Proof: □

Cor. (I.3.1.3). if H_1, H_2 are subgroups of a group G that has finite indexes, then $H_1 \cap H_2$ also has finite index in G.

Proof: By fundamental isomorphism(I.3.1.2), $H_1/H_1 \cap H_2 \cong H_1H_2/H_2 \subset G/H_2$, so $H_1 \cap H_2$ has finite index in H_1, so by transitivity of indexes, $H_1 \cap H_2$ has finite index in G. □

Def. (I.3.1.4) (Index of Subgroup). The **index** of a group H in a group G is defined to be the number of the left coset G/H, if it is finite. Now if H has finite index in G, then $|G/H| = |H\backslash G|$.

Proof: Because for any system of representative a_i for the left coset G/H, a_i^{-1} is a representative for the right coset $H\backslash G$, and vise versa. □

Prop. (I.3.1.5). If a finite group G has an automorphism α that $\alpha^2 = \mathrm{id}$ and α has no fixed point other than e, then G is an Abelian group of odd order.

Proof: G is clearly of odd order. Consider the map $g \mapsto \alpha(g)g^{-1}$, then it is injective, hence it is also surjective, and consider $\alpha(\alpha(g)g^{-1}) = g\alpha(g)^{-1} = (\alpha(g)g^{-1})^{-1}$, thus $\alpha(h) = h^{-1}$ for all $h \in G$, thus clearly G is Abelian. □

Prop. (I.3.1.6). If H is a subgroup of a finite group G, then $G \neq \cup g^{-1}Hg$.

Proof: There are at most $|G/H|$ different summands in the right hand side, so it doesn't have enough elements. □

Prop. (I.3.1.7) (Classifying F.g. Abelian Groups). As $\mathbb{Z}$ is PID, the theorem follows immediately from(I.3.4.15).

Prop. (I.3.1.8). If G is a f.g. group and H is a group of finite index in G, then H is f.g.

Proof: Suppose G has generators g_i, we may add their inverses to it, and let $Ht_1, \ldots, Ht_m$ are all the right cosets with $t_1 = 1$, then there are h_{ij} that $t_ig_j = h_{ij}t_{k_{ij}}$, then we claim H is generated by h_{ij}.

For this, consider any $h = \prod g_{i_r}$, then $g_1 = h_{1i_r}t_{k_{1i_r}}$, and we can do this from left to right. Now $h = \prod h_{i_sj_s}t_o$, then t_o must by 1, and we are done. □

Prop. (I.3.1.9).

Automorphism Group

Prop. (I.3.1.10). Any group of order>2 have at least 2 automorphisms.

Proof: Assume the contrary, consider its inner automorphism, then it is Abelian, and then multiplying by p for p large prime is not identity, Then $|G||p-1$ for such G. Now it is clear $|G|=2$, because otherwise we can choose $p \equiv 2 \mod |G|$. □

Def. (I.3.1.11). A group is called **complete** if all automorphisms of G are inner.

Prop. (I.3.1.12). S_n is the automorphism group of A_n for $n=5$ or $n \geq 7$.

Proof: □

Prop. (I.3.1.13). If G is a non-Abelian simple group, then $\mathrm{Aut}(G)$ is a complete group.

Prop. (I.3.1.14). S_n are complete groups except for S_6.

Proof: S_n is the automorphism group of A_n for $n=5$ or $n \geq 7$ by(I.3.1.12), thus it is complete by(I.3.1.13). □

Prop. (I.3.1.15) (Wielandt Theorem). If G is a finite subgroup, then the sequence

$$G < \mathrm{Aut}(G) < \mathrm{Aut}(\mathrm{Aut}(G)) < \ldots$$

must terminate in finite steps.

Proof: □

Free Groups

Prop. (I.3.1.16) (Nielsen-Schreier). A subgroup of a free group is a free group. Moreover, a subgroup of index m in a free group on n generators is a free group on $1+m(n-1)$ generators.

Proof: A free group is the fundamental group of a wedge sum of circles, and a cover of it is a connected 1-graph. Now the graph has a maximal tree and module the tree gets us a wedge sum of circle. The second statement follows by two ways of counting Euler number χ. □

Sylow Theory

Prop. (I.3.1.17) (Class Equation). For a finite group G, if $G_x = C((x))$, then

$$|G| = |C(G)| + \sum |G|/|G_x|$$

where the summation is over non-trivial conjugate classes of G.

Proof: Consider the left action of G on itself, and calculate elements. □

Cor. (I.3.1.18) (p-Group has non-trivial Center). if G is a p-group, then G has a non-trivial center.

Cor. (I.3.1.19). If $p||G|$, then G has an element of order p.

Proof: Follows from Sylow theory and any p-group has a non-trivial center. □

Lemma (I.3.1.20). For any p-group G acting on a finite set X, $|X| \equiv X^G \mod p$.(trivial).

Prop. (I.3.1.21) (Sylow Theorem). For a finite group of order $|G| = p^k m$.

- There is a Sylow p-group.
- For a Sylow p-subgroup, any p-subgroup is contained in a conjugate of P. In particular, any two Sylow p-subgroups are conjugate.
- the number of Sylow p-groups n_p satisfies: $n_p|m, n_p \equiv 1 \mod p$.

Proof: 1: Use induction, let $Z = C(G)$, if $p||Z|$, then Z contains a cyclic group of order p. Choose a p-Sylow subgroup of G/C, then its inverse image in G is a p-Sylow subgroup. If Z is prime to p, consider the conjugate action of G on $G - Z$, then some conjugacy class has order prime to p, by(I.3.1.20), then the stablizer H of this class satisfies $[G : H]$ is prime to p. Thus H contains a p-Sylow subgroup by induction.

2: If Q is a p-subgroup, then Q acts on G/P by left translation, so it has a fixed element by(I.3.1.20), $QxP = xP$ for some x, thus $Q \subset xPx^{-1}$.

3: $n_p|m$ by considering the conjugate action of P on the set of conjugates of P, then as in the proof of item2, P is the only fixed element, so $n_p \equiv 1 \mod p$ by(I.3.1.20). □

Lemma (I.3.1.22). If G has a sylow subgroup H that $|G/H|!$ is not divisible by $|G|$, then G is not simple.

Proof: Consider the conjugate action of G on the conjugacy classes of H, then it is a group homomorphism of G into a subgroup of $S_{|G/H|}$, but the hypothesis shows that it is not injective, thus the kernel is non-trivial normal. □

Prop. (I.3.1.23) (Frattini Argument). If G is a finite subgroup, N is normal in G and P is a Sylow subgroup of N, then $NN_G(P) = G$.

Proof: For any element $g \in G$, consider $g^{-1}Pg \subset N$ is a Sylow subgroup of N, thus by Sylow theorem(I.3.1.21), there is a $n \in N$ that $g^{-1}Pg = n^{-1}Pn$, thus $gn^{-1} \in N_G(P)$, thus $g \in NN_G(P)$. □

Split Extension

Prop. (I.3.1.24) (Cyclic Central Extension Split). If there is an exact sequence $0 \to Z \to G \to C \to 0$ where $Z \subset C(G)$ and C is cyclic, then G is Abelian.

Proof: This is because we can choose an inverse image of a generator of C. □

Prop. (I.3.1.25) (Schur-Zassenhaus). An exact sequence of finite groups $0 \to A \to E \to G \to 0$ must split when $|A|$ and $|G|$ are relatively prime.

Proof: □

Prop. (I.3.1.26). Let $\alpha, \beta : G \to \mathrm{Aut}(H)$ be two actions of G on H, then theirs semiproduct sequences

$$1 \to H \to G \ltimes H \to G \to 1$$

are isomorphic iff α, β are equivalent modulo $\mathrm{Inn}(H)$.

Subnormality

Def. (I.3.1.27) (Normal Series). A **normal series** of a group G is a descending chain of groups:

$$G = G_0 > G_1 > \ldots > G_r = \{e\}$$

that G_{k+1} is normal in G_k. It is called a **composite series** iff each G_k/G_{k+1} is simple.

Def. (I.3.1.28) (Central Series). A **central series** of a group G is an ascending chain of groups:

$$\{e\} = Z_0 < G_1 < \ldots < G_r = G$$

that Z_{k+1}/Z_k is in the center of G/Z_k.

Prop. (I.3.1.29). A group is

- solvable iff it has a normal series that G_i/G_{i+1} is Abelian.
- defined to be **supersolvable** iff it has a normal series that G_i/G_{i+1} is cyclic.
- nilpotent iff it has an upper central series.

Proof: □

Lemma (I.3.1.30) (p-Group is Nilpotent). Any p-group is nilpotent.

Proof: Using induction by(I.3.1.18), we see it has a central series, thus nilpotent(I.3.1.29). □

Prop. (I.3.1.31) (Nilpotent Finite Groups). If G is a finite group, then the following are equivalent:

- G is nilpotent.
- $N_G(H) > H$ for every proper subgroup $H < G$.
- Every maximal subgroup of G is normal.
- Every Sylow subgroup of G is normal.
- G is a direct product of its non-trivial Sylow subgroups.

Proof: $1 \to 2$: Choose a central series Z_n, let $Z_n \subset H$ and $Z_{n+1} \not\subset H$, then $[Z_{n+1}, H] \subset [Z_{n+1}, G] \subset Z_n \subset H$, thus $Z_{n+1} \subset N_G(H)$.

$2 \to 3, 4 \to 5$: trivial.

$3 \to 4$ For any p-Sylow subgroup G, if $N_G(P)$ is proper subgroup, then it is contained in some maximal subgroup M, and M is normal, thus by Frattini argument(I.3.1.23), $G = N_G(P)M = M$, contradiction.

$5 \to 1$: By lemma(I.3.1.30). □

Prop. (I.3.1.32) (Jordan-Horder). For a finite group G, any two of its composite series has the same length, then the quotient groups G_k/G_{k+1} are in bijection with each other as sets.

Proof: Cf.[代数学引论 P89]. □

Prop. (I.3.1.33) (Minimal Normal Subgroup). The minimal normal subgroup N of a finite group G is a direct product of simple groups L^n.

Proof: Let N_1 be a maximal normal subgroup of N, then N/N_1 is simple, and let N_i be the conjugates of N_1 in G, then they are all maximal normal subgroup of N. the simple groups N/N_i are mutually isomorphic, and $\cap N_i = 1$ by the minimality of N.

Now we use induction to prove $N/N_1 \cap \ldots \cap N_i$ is isomorphic to a product of N/N_1, which will finish the proof.

Now assume $N_1 \cap \ldots \cap N_{i-1} \not\subset N_i$, then $(N_1 \cap \ldots \cap N_{i-1})N_i = N$, and notice

$$N/N_1 \cap \ldots \cap N_i \cong N_1 \cap \ldots \cap N_{i-1}/N_1 \cap \ldots \cap N_i \times N_i/N_1 \cap \ldots \cap N_i \cong N/N_i \times N/N_1 \cap \ldots \cap N_{i-1}.$$

□

Prop. (I.3.1.34). If a finite group $|G| = \prod p_i$, where p_i are different primes that $\prod p_i$ and $\prod(p_i - 1)$ are coprime, then G is cyclic.

Proof: We prove all the Sylow groups are normal. Choose the maximal Sylow group A_n, then it is normal by Sylow theorem, and other Sylow groups act by conjugation is trivial(consider the center(I.3.1.18), then the center of the quotient, and so on), hence A_n is in the center. Now consider the quotient, by induction it is cyclic, hence this is a central extension of a cyclic group, hence G is Abelian(I.3.1.24), so cyclic. □

Prop. (I.3.1.35). If G is a finite group and p is the minimal prime number of $|G|$, then all subgroups N of G of index p is normal.

Proof: Consider the left action of G on G/H, then the kernel is $\cap a^{-1}Ha$, which is the maximal normal subgroup contained in H. Now this is group homomorphism of G into S_p, thus it has kernel at lest $|G|/p$, so the kernel equals H, showing H is normal. □

Prop. (I.3.1.36) (Burnside's Theorem). If p, q are primes, then any finite groups of order $p^a q^b$ is solvable.

Proof: Cf.[Serre Linear representations of finite groups, P65]. □

Prop. (I.3.1.37) (Thompson). A finite group is not solvable iff there exist non-trivial elements x, y, z of coprime orders a, b, c that $xy = z$.

Prop. (I.3.1.38) (Feit-Thompson). All finite groups that has odd order is solvable.

Proof: □

Commutators

Def. (I.3.1.39) (Notation).
- $[a, b] = a^{-1}b^{-1}ab$.
- $x^y = y^{-1}xy$.

Prop. (I.3.1.40) (Commutator relations). .

Def. (I.3.1.41). A **metabelian** group is a group G that G' is Abelian.

Prop. (I.3.1.42). If $G = AB$ where A, B are Abelian, then $[G, G] = [A, B]$ and G is metabelian.

Proof: The first one is easy to verify, the second because if we let $b^{a_1} = a_2b_2$, $a^{b_1} = b_3a_3$, then

$$[a, b]^{a_1b_1} = [a, b^{a_1}]^{b_1} = [a, b_2]^{b_1} = [a^{b_1}, b_2] = [a_3, b_2]$$

and similarly, $[a, b]^{b_1a_1} = [a_3, b_2]$, so we have $[a, b]$ commutes with $[b_1^{-1}, a_1^{-1}]$, which shows $[A, B]$ is Abelian. □

Prop. (I.3.1.43). If G is a metabelian finite group, then the transfer of $Ver : G \to G'$ is trivial map.

Transfer

Permutation Groups

Lemma (I.3.1.44). If $n \geq 3$, then any proper normal subgroup of A_n has index divisible by 3.

Proof: Otherwise consider $n = |G/H|$, then every p-power is in H. But then an element c of order 3 is in H, because $c = c^{3k+1} = (c^{-1})^{3k+2}$ for any k. But A_n is generated by 3-Cycles. □

Lemma (I.3.1.45). A_5 is simple.

Proof: By(I.3.1.44), any proper normal subgroup H has order dividing 20. H cannot contain a 5-cycle, because a 5-cycle has □

Prop. (I.3.1.46). A_n is simple for $n \geq 5$.

Proof: Cf.[代数学引论 P66]. □

Classification of Small Groups

Prop. (I.3.1.47) (Classification).

1. A group G of prime order p or order p^2 is Abelian.
2. A group G of order p^aq^b that it has q^b p-Sylow subgroups, then its q-Sylow subgroup is normal thus it is not simple.
3. A non-Abelian group G of order 6 is isomorphic to S_3.
4. Any non-Abelian group of order 8 is isomorphic to D_4 or quadratic numbers Q.
5. A group of order smaller than 60 is solvable.
6. Any simple group G of order 60 is isomorphic to A_5.
7. A group of order 148 is not simple, by(I.3.1.22) applied to the 37-Sylow subgroup.
8. A group of order 150 is not simple, by(I.3.1.22) applied to the 5-Sylow subgroup.

Proof:

1. Because G has non-trivial center Z by(I.3.1.18), if $Z = G$, then it is Abelian, otherwise the $|Z| = p$, and the quotient G/Z is cyclic, thus G is Abelian by(I.3.1.24).
2. Calculating elements.
3. Consider its normal 3-Sylow group, then the quotient is cyclic thus G is semi-product which must by S_3 when non-Abelian.

4.

5.

6. Consider G has 6 5-Sylow groups, thus there are 24 elements of order 5.
   G has 4 or 10 3-Sylow subgroups, if it have 4 3-Sylow subgroups, then the normalizer contains a 5-Sylow subgroup, so we have a subgroup of order 15, which must by $\mathbb{Z}/15$, so it contains a normal 5-Sylow subgroup, which shows there are at most $60/15 = 4$ 5-Sylow subgroups, contradiction.
   So we have 10 3-Sylow subgroups, which shows there are at most 15 elements of order 2 or 4. So we have 3 or 5 2-Sylow subgroups. If it is 3, then we can do the same as that for 3-Sylow to construct a 20-order group and reach contradiction.
   So now it have 5 2-Sylow subgroups, and then we consider the conjugate action on Sylow subgroups, which is transitive, so it has trivial kernel, and $G \hookrightarrow S_5$. Now $G = [G, G] \subset [S_5, S_5] = A_5$.

$\square$

Prop. (I.3.1.48). There is a group that is group that $a^3 = 1$ for any $a \in G$, but is not Abelian. It is the uni-upper-triangular matrices in $M_3(\mathbb{F}_3)$.

2 Polynomials

Prop. (I.3.2.1) (Descartes's Rule of Sign). Let $p(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ be a real polynomial with nonzero a_i, where $A_0 < B_1 < \cdots < b_n$, then the number of positive roots of $p(x)$ is the number of changing signs of $\{a_n\}$ minus $2k$.

Proof: Lemma: when $a_0a_n > 0$, the number of positive roots are even and when $a_0a_n < 0$, it is odd. This is seen by consider $p(0)$ and $p(\infty)$.

Then we consider the derivative p' and use induction. Denote the number of changing sign by $v(p)$ and the number of positive roots by $z(p)$, then if $z_0a_1 > 0$, then $v(p) = v(p')$ and $z(p) \equiv z(p') \mod 2$. Then we have $z(p) \equiv v(p) \mod 2$ and middle value theorem shows that $z(p') \leq z(p) - 1$, hence by induction and parity argument, we have $v(p) \geq z(p)$.

If $a_0a_1 < 0$, then the same method shows that $v(p) = v(p') + 1 \geq z(p') + 1 \geq z(p')$ and the have the same parity by the lemma. $\square$

Prop. (I.3.2.2) (Lagrange Interpolation). if K is a field, a_i are $n+1$ elements of K, b_i are $n+1$ elements of K, then there is a unique polynomial f of degree no greater than n that $f(a_i) = b_i$.

Proof: The polynomial in search is

$$f(x) = \sum \prod_{j \neq i} b_i \frac{x - a_j}{a_i - a_j}.$$

It is a polynomial of degree smaller than $n + 1$, and it satisfies the hypothesis. And clearly there is at most one such polynomial, otherwise their difference has $n + 1$ zeros. $\square$

Cor. (I.3.2.3). If $f(x) = a_nx^n + \ldots + a_0$, then for any $n + 1$ different integers $a_0, \ldots, a_n$, there exists some $|f(a_i)| \geq \frac{n!}{2^n}|a_n|$.

Proof: Use Lagrange interpolation and consider the leading coefficient. $\square$

Prop. (I.3.2.4). If a degree n polynomial p satisfies $p(n) = 2^n$ for $n = 0, 1, \dots, n$, then $p(n+1) = 2^{n+1} - 1$.

Proof: The polynomial in search is

$$p(x) = \sum_{k=0}^{n} C_x^k.$$

□

Prop. (I.3.2.5) (Newton Identities). For n indeterminants x_i, denote $s_k = \sum x_i^k, \sigma_k$, then there are **Newton Identities**:

$$\begin{cases} s_k - \sigma_1 s_{k-1} + \dots + (-1)^n \sigma_n s_{k-n} = 0 & k \geq n \\ s_k - \sigma_1 s_{k-1} + \dots + (-1)^k k\sigma_k = 0. & k \leq n \end{cases}$$

Proof: The case of $k \geq n$ is simple. Now if $k \leq n$, notice that the equation is true for $n = k$. Then we prove by induction on n. If n is already proven, then the term is 0 if one of them is 0, but this implies that this equation is divisible by $\prod_{i=1}^{n+1} x_i$, but is has degree $k \leq n$, so it must by 0. □

Prop. (I.3.2.6) (Combinatorial Nullstellensatz). If F is a field and $f \in F[X_1, \dots, X_n]$ is a polynomial. Let $S_1, \dots S_n$ be nonempty finite subsets of F and $g_i = \prod_{s \in S_i}(x_i - s)$, then if f vanishes at the common zeros of g_i, then there are polynomials $h_i \in F[X_1, \dots, X_n]$ that $\deg h_i \leq \deg F - \deg g_i$ and $g = \sum h_i g_i$.

Proof: The proof is very simple, just replace terms of f by lower degree terms, using equation of g_i, then we get a polynomial that has degree in x_i smaller than $|S_i|$ and vanish on $S_1 \times \dots \times S_n$, so it must be 0, as easily checked. □

Cor. (I.3.2.7) (Combinatorial Nullstellensatz). If F is a field and $f \in F[X_1, \dots, X_n]$ is a polynomial of degree n, if $\prod X_i^{t_i}$ is a highest degree term of f and S_i are arbitrary subsets of F that $|S_i| > t_i$, then there are some $s_i \in S_i$ that $f(s_1, \dots, s_n) \neq 0$.

Proof: May assume $|S_i| = t_i + 1$. By combinatorial Nullstellensatz(I.3.2.6), if no such s_i exist, then there are h_i that $f = \sum h_i \prod_{s \in S_i}(x_i - s)$, but the term $\prod X_i^{t_i}$ needs to appear, so it must by some term of h_i times $x_i^{t_i+1}$, which is a contradiction. □

Remark (I.3.2.8). For many combinatorial applications of Combinatorial Nullstellensatz, Cf.[Combinatorial Nullstellensatz].

Irreducibility

Prop. (I.3.2.9). If $f = a_n x^n + \dots + a_1 x + p \in \mathbb{Z}[X]$ satisfies p is a prime and $\sum |a_i| < p$, then f is irreducible in p.

Proof: The ideal is that all of its roots has norm bigger than 1, because otherwise $p = |\sum a_k x^k| \leq \sum |a_k| < p$, contradiction. So if now $f = gh$, then g, h all have roots with norm greater than 1, in particular it has constant coefficients norm greater than 1, which is a contradiction because p is a prime. □

Prop. (I.3.2.10). The cyclotomic polynomial $\Psi_n(x)$ is irreducible over $\mathbb{Z}$.

Proof: It suffices to show that for any irreducible factor $f | \Psi_n(x)$, if ξ is a root of f and $(p, n) = 1$, then ξ^p is also a root of f. Cf.[Lang]. □

Diophantine Equations

Prop. (I.3.2.11) (Fermat's Equation in Function Case). If $n \geq 2$, then the only non-trivial solution to the equation in $\mathbb{C}[t]$ of
$$X^n + Y^n = Z^n$$
is $n = 2, X = (a^2 - b^2)/2, Y = ab, Z = (a^2 + b^2)/2$.

Proof: The $n = 2$ case is easy. If $n > 2$, we show there are no non-trivial solutions: Differentiate it to get:
$$X^{n-1}X' + Y^{n-1}Y' = Z^{n-1}Z',$$
And cancelling X^{n-1}, we get
$$Y^{n-1}(X'Y - Y'X) = Z^{n-1}(X'Z - Z'X).$$
Now $X'Z - Z'X \neq 0$ because X, Z are not linearly equivalent, and Y, Z is coprime, so $Y^{n-1} | X'Z - Z'X$. But then if we assume $\dim Y \geq \dim X$, then $(n-1)\dim Y \leq 2\dim Y - 1$, which means $n < 2$. $\square$

Resultant

Def. (I.3.2.12). Over a commutative ring R, the **resultant** $res(A, B)$ of two polynomials A, B of degree d, e respectively is the determinant of the map $W_e \times W_d \to W_{d+e}$ that $(X, Y) \mapsto AX + BY$, where W_t is the free module of polynomials of degree $< t$.

Prop. (I.3.2.13). The resultant can be seen as the determinant of the matrix with values the coefficient of A or B in different places, multiplying X^*s with different degree and add to the last row, we can get $A \cdot X^*$s and $B \cdot X^*$s, so: $res(A, B) = AC + BD$ for some C, D.

Now if $R \subset S$ and A, B has common roots in S, then $\mathrm{res}(A, B) = 0$.

Cor. (I.3.2.14). Resultant is stable under Euclidean division, so it can be seen as a suitable division remainder of the two polynomial.

Prop. (I.3.2.15). When $R \subset L$ a field and A, B decompose into linear factors in L, let t_i be roots of A and u_j be roots of B, then
$$\mathrm{res}(A, B) = v_0^d w_0^e \prod_{i=1}^{d} \prod_{j=0}^{e} (t_i - u_j)$$

Proof: See the resultant as polynomials of the roots of A and B, then we proved that if they has the same root, then $res = 0$, so it is divisible by $(t_i - u_j)$ for all i, j. Then notice the RHS is homogenous of degree d in u_j and homogenous of degree e in t_i, so does res. So they are equal. $\square$

Invariant Theory

Prop. (I.3.2.16) (Elementary Symmetric Polynomial). For n indeterminants x_i, define the **elementary polynomials** $\sigma_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k}$. Then any symmetric polynomial is a polynomial of the fundamental symmetric polynomials.

Proof: Set the first coordinate to 0, then the rest is a polynomial of the fundamental symmetric polynomials by induction, $f(0, x) = h(\sigma_1, \ldots, \sigma_n)$, then consider $f - h$, with h the same expression but x_n is included, we get it is a symmetric polynomial, and it is divisible by x_1, thus also divisible by $\prod x_i$, thus divide it by $\prod x_i$ and use induction, we get f is a polynomial of elementary symmetric polynomials. □

Prop. (I.3.2.17). Any polynomial on the entries of matrixes $M_n(k)$ that is invariant under conjugation is generated by coefficients of $\det(\lambda I + X)$ and can also be generated by $\mathrm{tr}(X^k)$.

Proof: We notice that the matrixes having disjoint eigenvalues is dense in $M_n(k)$, thus the restriction of the polynomial on these matrixes is a symmetric polynomial(I.3.2.16) thus identical to a polynomial described above. Hence they are equal. □

Prop. (I.3.2.18). For any polynomial on the entries of matrixes $M_n(k)$ that $f(BA) = f(A)$ for $B \in O(n)$, there is a polynomial F that $f(A) = F(A^*A)$. Cf.[Heat Equation and the Index Theorem Atiyah P323].

Prop. (I.3.2.19) (Weyl). Any linear map f from $(\mathbb{R}^m)^{\otimes n}$ to R that is $O(m)$-equivariant is a linear combinations of maps of the form:

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \langle v_{i_1}, v_{i_2} \rangle \langle v_{i_3}, v_{i_4} \rangle \cdots \langle v_{n-1}, v_n \rangle.$$

Where $i_1, \ldots, i_n$ is a permutation of $1, 2, \cdots, n$ when n is even and when n is odd, f must be 0.

Proof: Cf,[Heat Equation and the Index Theorem]. □

3 Ring Theory

Prop. (I.3.3.1). If $1 - ab$ is invertible in a ring, then so does $1 - ba$, and

$$(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a.$$

Proof: Direct Calculation. □

Bezout Domain

Def. (I.3.3.2). A **Bezout domain** is an integral domain that any sum of two principal domains is also principal.

Prop. (I.3.3.3). The localization of a Bezout domain is Bezout. A local ring is Bezout iff it is a valuation ring(by(V.3.2.10)).

Prop. (I.3.3.4). Any finite submodule of a free module over a Bezout ring is finite free, Cf.[StackProject 0ASU].

Prop. (I.3.3.5). Finite locally free R-module over a Bezout domain is free.

Prop. (I.3.3.6). A module over a Bezout domain is flat iff it is torsion-free.

Proof:

□

Prop. (I.3.3.7). finitely presented projective R-module is free.

UFD

Def. (I.3.3.8). An element x in a domain R is called **irreducible** iff for $x = yz$, y or z is a unit.

An element x in a domain R is called a **prime** iff (x) is a prime ideal.

A domain R is called a **UFD** iff each element $x \in R$ has a factorization into irreducible, unique up to units.

Prop. (I.3.3.9). if R is Noetherian domain, then each element has a decomposition into irreducible.

Proof: Trace the decomposition inductively, if it doesn't stop, then it contradicts with Noetherian hypothesis. □

Prop. (I.3.3.10). A domain R is a UFD iff each element x factors into irreducibles, and each irreducible element is a prime.

Proof: If R is a UFD, then if x is irreducible, if $ab \in (x)$, $ab = xc$, then x is one irreducible in the decomposition of a and b, by UFD, so (x) is a prime ideal.

Conversely, if there are two decompositions $\prod a_i = \prod b_j$, then some $b_j \in (a_i)$ by primeness of (a_i), so $b_j = a_i u$, so u must by units, so by induction, this two decompositions are the same. □

Prop. (I.3.3.11). A Noetherian domain is UFD iff all minimal primes are principal.

Proof: If it is UFD, then for a prime ideal $\mathfrak{p}$ of height 1, choose an element $x \in \mathfrak{p}$ and a decomposition $x = \prod a_i$, then as $\mathfrak{p}$ is a prime, some $a_i \in \mathfrak{p}$, but (a_i) is a prime, so $\mathfrak{p} = (a_i)$.

Conversely, by(I.3.3.9) and(I.3.3.10), it suffices to prove every irreducible is a prime. Let $\mathfrak{p}$ be a minimal prime over (x), then $\mathfrak{p}$ has height 1 by(I.5.6.9), so $\mathfrak{p} = (y)$, so $x = yz$ and z is a unit as x is irreducible, so x is a prime. □

Prop. (I.3.3.12). A polynomial ring over a UFD is a UFD.

Proof: □

Prop. (I.3.3.13). Let k be a field, then the power series $k[[X_1, \ldots, X_n]]$ is a UFD.

Proof: Cf.[Algebra Lang P209]. □

Prop. (I.3.3.14) (Gauss Lemma).

Prop. (I.3.3.15) (Chinese Remainder Theorem).

Prop. (I.3.3.16) (PID Structures). In a PID,

- An element t is irreducible iff (t) is maximal.
- A PID is UFD hence Noetherian.
- An element t is irreducible iff it is a prime.
-

Proof: 1:

2: By(I.3.3.11).

3: By item2 and(I.3.3.9).

□

Skew-Field

Def. (I.3.3.17). A **skew field**(or division algebra) is a unital ring that every non-zero element is invertible (but may not be commutative).

Def. (I.3.3.18) (Wedderburn). A finite division algebra D is a field.

Proof: Use the class equation for the invertible elements of D, if it is not isomorphic, consider the center $Z(D)$ of D, let $|Z(D)| = z$, then it is a field, and any other centralizer can be seen as a vector space over $Z(D)$, let it of dimension k, then $z^n - 1 = z - 1 + \sum \frac{z^n-1}{z^{k_i}-1}$. But then let Ψ_n be the cyclotomic character of degree n, then $\Psi_n(z)$ divides $z-1$. But this is not true, as it is bigger. □

Prop. (I.3.3.19). If D is a f.d. division algebra over $\mathbb{R}$, then it is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Proof: Cf.[Advanced Linear Algebra P466]. □

Prop. (I.3.3.20). A skew field A of at most countable dimension over $\mathbb{C}$ is isomorphic to $\mathbb{C}$.

Proof: Notice it suffices to find an eigenvalue of ϕ for each $\varphi \in A$, but otherwise $\{(\varphi - a)^{-1}\}$ is a uncountable set of linearly elements of A, so some $\sum a_i(\varphi - c_i)^{-1} = 0$, spanning the expression, we see $\prod_k(\varphi - \mu_k) = 0$, so some $\varphi - \mu_k = 0$, contradiction. □

Prop. (I.3.3.21) (Hualuogeng Equation). In a skew field K, if $a, b \neq 0$ and $ab \neq 1$, then

$$a - (a^{-1} + (b^{-1} - a)^{-1})^{-1} = aba$$

Proof: suffices to show

$$1 = (1 - ab)a(a^{-1} + (b^{-1} - a)^{-1}) = (1 - ab)(1 + a(b^{-1} - a)^{-1}).$$

But this is equal to

$$(1 - ab)(1 - (b^{-1} - a)(b^{-1} - a)^{-1} + b^{-1}(b^{-1} - a)^{-1}) = (1 - ab)(1 - ab)^{-1} = 1.$$

□

4 Module Theory

Prop. (I.3.4.1) (Rank of Free Modules). If M is a free module over a unital ring R, then any basis of M is of the same cardinality, called the **rank** of M, and any spanning subset of M has greater cardinalities.

Proof: Cf.[Advanced Linear Algebra P127]. □

Prop. (I.3.4.2). If M is a free module over an integral domain R, then any linearly independent set has cardinality smaller that $\mathrm{rk}(M)$.

Proof: Let Q be the quotient field of R. The thing is, if S is a set linearly independent over R, then S is linearly independent over Q, thus it is smaller than the cardinality of the basis of $M \otimes_R Q$ over Q by(I.2.1.1), which is the same as $\mathrm{rk}(M)$. □

Prop. (I.3.4.3). For an endomorphism T of a R module M, if we denote p the minimal integer that $R(T^p) = R(T^{p+1})$ and q the minimal integer that $N(T^q) = N(T^{q+1})$. Then the morphisms are stable afterward. Then if there is a m, n that $R(T^m) \oplus N(T^n) = X$ for a R-module endomorphism $T \in \mathrm{End}(M)$, then $p, q < \infty$ and they are equal. Moreover, if we know $p, q < \infty$, then we have $R(T^p) \oplus N(T^q) = M$.

Proof: We notice that

$$T^i : N(T^{i+j})/N(T^i) \to R(T^i) \cap N(T^j), \quad T^i : M/(R(T^j) + N(T^i)) \to R(T^i)/R(T^{i+j})$$

are isomorphisms. Thus $R(T^m) \oplus N(T^n) = X$ shows $q \le m$ and $p \le n$, thus we have $R(T^p) \oplus N(T^q) = M$, which implies $p \ge q$ and $q \ge p$. thus the result. The rest also follows easily from these isomorphisms. □

Prop. (I.3.4.4). For a f.g. module over a Noetherian ring, if an endomorphism is surjective, then it is injective.

Proof: The kernel $\mathrm{Ker}(\varphi^i)$ stablize, thus there is a $\mathrm{Ker}(\varphi^i) = \mathrm{Ker}(\varphi^{2i}) \to \mathrm{Ker}(\varphi^i)$ that is also surjective, and it is also zero, thus φ is injective. □

Prop. (I.3.4.5) (Induced & Coinduced). Given a ring homomorphism $S \to$R.

- $f^*M = M_S$, the restriction.
- $f_!M = M \otimes_S R$ is the induced module, it is left adjoint to restriction.
- $f_*M = \mathrm{Hom}_S(R, M)$ is the coinduced module, it is right adjoint to restriction. (It is a R-mod by $s(f)(t) = f(ts)$.)

Prop. (I.3.4.6) (Nakayama). If M is a finite A-module, and $I \subset A$ is an ideal that $IM = M$, then there is a $a \in 1 + I$ that $aM = 0$.

In particular, if $I \subset \mathrm{rad}(A)$, then a is a unit(I.5.7.1), so $M = 0$.

Proof: Because $M = IM$, choose a set of generators $\{x_i\}$ of M, then $x_i = \sum a_{ij}x_j$, where $a_{ij} \in I$. Then if the matrix $M = (\delta_{ij} - a_{ij})$, then $Mx_i = 0$. So taking the adjoint matrix, then $det(M)x_i = 0$. Notice $det(M)$ is a morphism. But the determinant must be element like $1 + k, k \in I$, so we are done. □

Cor. (I.3.4.7). If M is finite and $M = \mathrm{rad}(A)M + N$, then $M = N$.

Cor. (I.3.4.8). If a finite R-module M satisfies $M \otimes_R k(p) = 0$, then there is a $f \notin p$ that $M_f = 0$.

Proof: Because $M_p = 0$, and the support of M is closed(finiteness used). □

Prop. (I.3.4.9) (Jordan-Horder).

Prop. (I.3.4.10). If A is an algebra of countable dimension over $\mathbb{C}$ with unit, and $\alpha \in A$ is not nilpotent, then there exists a simple A-module M that $a|_M \neq 0$.

Proof: First we claim that there is some $\lambda \neq 0 \in \mathbb{C}$ that $a - \lambda$ is not invertible in A, this is nearly the same as the proof of(I.3.3.20), noticing that a is not nilpotent. Now we can take $M = A/(a - \lambda)A$, then $a1 = \lambda \neq 0$. □

Torsion-Free Module

Def. (I.3.4.11). Let R be a domain, an R-module M is called **torsion-free** iff there are no elements $0 \neq x \in R, 0 \neq f \in M$ that $xf = 0$.

Prop. (I.3.4.12). If $0 \to M_1 \to M_2 \to M_3$ and M_1, M_3 are torsion-free, then M_2 is torsion-free. Torsion-free is a stalk-wise property(I.5.1.27).

Proof: Trivial. □

Prop. (I.3.4.13). Let M be a finite R-module, then M is torsion-free if it is a submodule of a finite free module.

Proof: One direction is trivial, for the other, if M is torsion-free, then $M \subset M \otimes_R K$, and $M \otimes_R K$ is a finite K-vector space, with basis e_i. Now let x_i be a basis of M, let $x_i = \sum a_{ij}/b_{ij} e_j$, then let $e = \prod_{ij} b_{ij}$, then $M \subset Re_1/b \oplus \ldots \oplus Re_n/b$. □

Prop. (I.3.4.14). If M, N are R-modules that M is torsion-free, then $\mathrm{Hom}(N, M)$ is torsion free.

Proof: Choose a surjection $\bigoplus_I R \to N \to 0$, then $\mathrm{Hom}(N, M) \hookrightarrow \prod_I M$ is torsion free. □

Modules over PID

Prop. (I.3.4.15) (Classification of Modules over PID).

1) Submodule of a free module over a PID is free of smaller rank. Thus a projective module over a PID is free
2) Finite torsion-free module over a PID is free.
3) Finite module over a PID has a primary decomposition $M = \bigoplus_i R/(q_i)$,where (q_i) is primary ideals.

So projective $\iff$ free $\iff$ torsion-free(when f.g.).

Proof: 1: Choose a well ordering on the basis of F, let F_i is the submodule generated by $e_j, j \leq i$. Then $\pi_i(P \cap F_i) \subset R$ is a module of the form (a_i), thus choose $u_i \in P$ that $p_i(u_i) = a_i$. Then u_i is a basis for P: they are linearly independent, because for any finite linear combination that are 0, the maximal coordinate are 0. It also spans P, because we can choose an element in $P - \{u_i\}$ whose maximal nonzero coordinate α is minimal among them, by well-orderedness. But we can subtract a multiple of u_α, thus producing a smaller element, contradiction.

2: If it is finite torsion-free, then it is a submodule of a finite free module(I.3.4.13), so it is free by item1.

3: Follows immediately from(I.5.3.22). □

Prop. (I.3.4.16) (Primary Cyclic Decomposition). There is a primary cyclic decomposition theorem for a torsion module M over a PID R. Thus the multisets of elementary divisors of M is a complete set of invariants for M.

Proof: Cf.[Advanced Linear Algebra P153]. □

Cor. (I.3.4.17) (Invariant Factor Decomposition). By reordering the cyclic decomposition, we can get the **invariant factor decomposition** of M, there are scalars $d_m | d_{m-1} | \ldots | d_1$ that are called the **invariant factors** of M.

Proof: Cf.[Advanced Linear Algebra P157]. □

5 Field Theory

Field Extensions

Def. (I.3.5.1). A family L of extensions are called **distinguished** iff it is closed under base change and $k \subset F \subset E \in L$ iff $k \subset F \in L$ and $F \subset E \in L$.

Prop. (I.3.5.2). The family of finite extensions form a distinguished class.
The family of algebraic extensions form a distinguished class.
The family of f.g. extensions form a distinguished class.

Proof: Finite case is trivial. For the alg. extensions, for $k \subset F \subset E$, for any $\alpha \in E$, α satisfies an polynomial function with f.m coefficients in F, the coefficients form a subfield F_0 of F which is finite over k, so $k \subset F_0 \subset F_0(\alpha)$ is a finite tower, so it is finite, hence algebraic. The base change is easy to check.

For f.g. extensions, it suffice to check composition: □

Prop. (I.3.5.3). For an alg.extension $k \subset E$, any injective field map $E \to E$ over k is an automorphism. (This is because it induce a permutation of any α with its conjugates in E, so it is surjective).

Lemma (I.3.5.4). Let $f \in k[X]$ be a polynomial of degree≥ 1, then there is a field K that f has a root in K. Hence for any finite set of polynomials, there is a field K that all of them have roots in K.

Proof: Cf.[Algebra Lang P231]. □

Lemma (I.3.5.5). For any field k, there exists uniquely an alg.closed field K containing k.

Proof: Firstly, we construct a field that every polynomial in $k[X]$ of degree≥ 1 has a root. Consider the polynomial ring $k[X_f]$, where there is a indeterminant X_f for each $f \in k[X]$ of deg≥ 1. Then the ideal generated by $f(X_f)$ is not a unit ideal, which can be seen by constructing a finite field extension that f_i all have a root in it(I.3.5.4).

So if $\mathfrak{m}$ is a maximal ideal containing all $f(X_f)$, then the quotient field is a field that all f have a root(X_f).

So now if we construct inductively like this, and consider their union, then it is clearly a field and any polynomial of degree≥ 1 have a root in it. □

Prop. (I.3.5.6) (Algebraic Closure Exists). For any field K. There exists uniquely an alg.closed field K/k that is algebraic over K, up to isomorphism over k.

Proof: Let E be a field that is alg.closed and contains k by(I.3.5.5). Let k^a be the union of subextensions that are algebraic over k. k^a is a field, by(I.3.5.2), and k^a is alg.closed, because if $f(X)$ is a polynomial of degree≥ 1 in $k^a[X]$, then it has a root $\alpha \in E$, and α is algebraic over k^a, so $\alpha \in k^a$. □

Prop. (I.3.5.7). If E/F is an algebraic field extension, then for any R that $E \subset R \subset F$, R is a field.

Proof: If $\alpha \in R$, then α is algebraic over E, so there is a relation $\alpha^n + \ldots + a_0 = 0$, so $\alpha^{-1} = -a_0^{-1}(a_1 + \ldots \alpha^{n-1}) \in R$. □

Separable, Normal & Galois Extensions

Def. (I.3.5.8). An extension K/k is called **normal extension** iff it satisfied the following equivalent conditions:

- Any embedding of K into k^{alg} induce an automorphism on K.
- K is the splitting field of a family of polynomials in $k[X]$.
- Every irreducible polynomial in $k[X]$ that has a root in K splits in K.

Proof: Cf.[Algebra Lang P237]. □

Prop. (I.3.5.9). Normal extension are stable under base change and forms a lattice, this is immediate from the first definition of(I.3.5.8).

Def. (I.3.5.10). If we define the **separable degree** $[E:k]_s$ of an extension E/k as the number of embedding into a fixed alg.closure, then it commutes with composition and when E/k is finite, $[E:k]_s \leq [E:k]$.

Def. (I.3.5.11). A finite extension is called **separable** iff $[E:k]_s = [E:k]$, an algebraic number α over k is called **separable** iff $k(\alpha)/k$ is separable. A polynomial $f \in k[X]$ is called **separable** iff it has no multiple roots in k^{alg}.

Def. (I.3.5.12). An extension E/k is called **separable** iff it satisfies the following equivalent conditions:

- every f.g. subfield is separable over k,(this is compatible because subfield of a finite separable extension is separable, by the compatibility of separable degree).
- Every element of E is separable.
- It is generated by a family of separable elements.

Proof: If E/k is separable and $k \subset k(\alpha) \subset E$, then by(I.3.5.11), $k(\alpha)$ is separable. And if it is generated by a family of separable elements $\{\alpha_i\}$, then any f.g. subfield can are f.g. by elements $\{\alpha_i\}$. Now it is a tower of separable extensions, hence separable by the compatibility of separable degree. □

Prop. (I.3.5.13). Separable extensions form a distinguished class.

Proof: Cf.[Algebra Lang P241]. □

Prop. (I.3.5.14) (Primitive element Theorem). A finite extension E/k is primitive iff there are only f.m. middle fields. And if E/k is separable, this is satisfied.

Proof: If k is finite, this is simple. Assume k infinite, for any two elements α, β, consider $k(\alpha + c_i\beta)$, if there is only finitely many middle fields, there exists two that is equal, so $k(\alpha, \beta) = k(\gamma)$. Proceeding inductively, E is primitive.

Conversely, if $k(\alpha) = E$, every middle field corresponds to a divisor of the irreducible polynomial of α. This map is injective, because for any g_F, degree of α over F is the same over the degree over the coefficient field of g_F, so it must be equal to F.

If E/k is separable, Let

$$P(X) = \prod_{i \neq j} (\sigma_i\alpha + X\sigma_i\beta - \sigma_j\alpha - X\sigma_j\beta)$$

for different embedding σ_i, σ_j of $E(\alpha, \beta)$ into k^{alg}. Then it is not identically zero, thus there exists c that $\sigma_i(\alpha + c\beta)$ is all distinct, thus generate $K(\alpha, \beta)$. □

Cor. (I.3.5.15). If L/K is a finite Galois extension, then there is an isomorphism:

$$L \otimes_K L \cong L \times L \times \ldots \times L : (a, b) \mapsto (ab, a\sigma_1(b), \ldots, a\sigma_{n-1}b)$$

where σ_i are Galois elements.

Proof: Choose a primitive element x and its minimal polynomial $f(x)$, then $L \cong K[X]/(f)$, and $L \otimes_K L \cong L[X]/(f)$, but f decomposes completely in $L[X]$, thus by Chinese remainder theorem(I.3.3.15), the given map is an isomorphism of rings. □

Prop. (I.3.5.16). Automorphisms of a field L are linearly independent over L.

Proof: If there are some $\sum \alpha_i \sigma(b) = 0$ for all b, then if $\alpha_i \neq 0, \alpha_j \neq 0$, choose a t that $\sigma_i(t) \neq \sigma_j(t)$, then we can subtract and make the non-zero terms one less, so by induction this is false. □

Inseparable Extensions

Prop. (I.3.5.17). Any irreducible polynomial of fields of characteristic 0 is separable and if char$= p$, then all roots have the same multiplicity and thus $[k(\alpha) : k] = p^n [k(\alpha) : k]$ for some n.

Proof: All roots have the same multiplicity because there are Galois actions. If the multiplicity is not 1, the derivative f' is zero, otherwise f is not irreducible. Then $f(X) = g(X^p)$. We can choose $f(X) = h(X^{p^n})$ with h separable, then $[k(\alpha) : k(\alpha^{p^n})] = p^n$, thus the result. □

Def. (I.3.5.18). The **inseparable degree** $[E : k]_i$ is defined as the quotient $[E : k]/[E : k]_s$. An algebraic element α is called **purely inseparable** over k iff there is a n that $\alpha^{p^n} \in k$.

Def. (I.3.5.19). An extension is called **purely inseparable** if it satisfies the following equivalent conditions:

- $[E : k]_s = 1$.
- Every element α of E is purely inseparable over k.
- For every $\alpha \in E$, the irreducible equation of α over k is of type $X^{p^n} - a$.
- It is generated by a family of purely inseparable elements.

Proof: Cf.[ALgebra Lang P249]. □

Prop. (I.3.5.20). Purely inseparably extensions form a distinguished class.

Proof: Cf.[Algebra Lang P250]. □

Prop. (I.3.5.21). If E/k is algebraic and E_0 be the maximal separable extension contained in E, then E/E_0 is purely inseparable. And if E/k is normal, then E_0/k is normal, too.

Proof:

By the proof of(I.3.5.17), any α has a p^n that α^{p^n} that α^{p^n} is separable, hence it is purely inseparable over E_0 by(I.3.5.19). E_0/k is normal because any σ maps E to itself, and E_0 to $\sigma(E_0) \in E$ separable, hence $\sigma(E_0) \subset E_0$. □

Def. (I.3.5.22). A field is called **perfect** iff there are no purely inseparable extensions of it. From the third definition of(I.3.5.19), this is equivalent to $x \to x^p$ is an automorphism of K, where p is the characteristic of K.

Prop. (I.3.5.23). For any field k of char p, there is a unique purely inseparable field extension k^{perf}/k that k^{perf} is perfect, called the **perfect closure** of k. It is generated by adding all the p^n-th roots to k.

6 Transcendental extension

Prop. (I.3.6.1) (Transcendental Basis). Let K be an extension of a field k, a **transcendental base** is an algebraically independent set that any element is algebraic over it. Then the number of elements in any algebraically independent set $\leq$ the number of elements in any transcendental base. In particular, given any algebraically independent set $S \subset T$ a set over which K is algebraic, S can be extended to a transcendental base.

Proof: Let $X = \{x_1, \dots, x_m\}$ transcendental base of minimal number, $S = \{w_1, \dots, w_n\}$ an algebraically independent set. If $n > m$, we proceed by changing one element a time using induction and prove that K is algebraic over $\{w_1, \dots, w_r, x_{r+1}, \dots, x_m\}$, contradiction.

Because w_{r+1} is algebraic over $\{w_1, \dots, w_r, x_{r+1}, \dots, x_n\}$, we have a minimal polynomial

$$f = \sum g_j(w_{r+1}, w_1, \dots, w_r, x_{r+2}, \dots, x_m)x_{r+1}^j$$

s.t. $f(w_{r+1}, w_1, \dots, x_m) = 0$ (after possibly renumbering x_i, this x must exists because S is itself algebraically independent). So x_r is algebraic over $\{w_1, \dots, w_{r+1}, x_{r+2}, \dots, x_m\}$, hence K is independent over it, too. □

Prop. (I.3.6.2). If K is of finite transcendental degree over k, then $|K| = |k|$.

Proof: We find a purely transcendental L/k that K/L is algebraic, then the element of L are all polynomials of finite indeterminants of elements of k, so $|L| = |k|$ by(I.1.4.5), and similarly $|K| = |L|$. □

Prop. (I.3.6.3) (Luroth Theorem). The automorphism group of $K(x)$ is $PGL_2(K)$.

Proof: Consider $\theta = \sigma(x) = \frac{f(x)}{g(x)}$, then x is algebraic over $K(\theta) : \theta g(x) - f(x) = 0$. Now x is transcendental over K, thus θ is transcendental over K as well. Now the minimal polynomial of x over $K(\theta)$ is just $\theta g(x) - f(x)$, because it is irreducible, as it is linear over θ. But $K(x) = K(\theta)$, thus the polynomial must have degree 1, so $f(x), g(x)$ is of degree 1. Now the rest is clear. □

7 Galois Theory

Prop. (I.3.7.1) (Artin). If G is a monoid and K is a field, any distinct characters of G in K are linearly independent over K.

Proof: Consider the minimal length of linear combination that is 0, then we substitute a suitable z in it, then we can can cancel a character, contradicting the minimality. □

Cor. (I.3.7.2). If α_i are different elements in K and there are element a_i that $\sum a_i\alpha_i^v = 0$ for every $v \geq 0$, then $a_i = 0$ for all n. (Seen as characters from $\mathbb{Z}_{\geq 0} \to K$).

Prop. (I.3.7.3) (Artin Algebraic Independence). Let K be an infinite field and σ_i be distinct elements of a finite group of automorphisms of K, then σ_i are alg.indepent over K.

Proof: Cf.[Algebra Lang P311]. □

Prop. (I.3.7.4) ((Artin)Galois Main Theorem). Let G be a finite group of automorphisms of K. Then K/K^G is Galois of Galois group G.

Proof: For every element x, set $\{\sigma_1 x, \dots, \sigma_r x\}$ be distinct conjugates, then $f(X) = \prod_i^r = (X - \sigma_i x)$ shows that K is separable and normal over K^G. And primitive element theorem shows that $[K : K^G] \leq |G|$, so it must equals G. □

Prop. (I.3.7.5) (Infinite Galois Theorem). The middle fields correspond to the closed subgp of $G(L/K)$.

Proof: The highlight is that $G(L/L^H) = H$ for a closed subgp H of $G(L/K)$. If σ fixes L^H but is not in H, because for every finite field M, $H \cdot G(L/M)$ corresponds to $M/(M \cap L^H)$, so $\sigma G(L/M) \cap H \neq \varnothing$. So σ is in the closure of H thus in H. □

Prop. (I.3.7.6) (Normal Basis Theorem). For a finite Galois extension, normal basis exists.

Proof: Finite case: The Galois group is cyclic, and the linear independent of characters shows that the minimal polynomial of σ is n-dimensional thus equals $X^n - 1$. Regard L as a $K[X]$ module thus by (I.3.4.15) is a direct sum of modules of the form $K[X]/(f(x))$, $f(x)|X^n - 1$ and the minimal polynomial for the action of X is $X^n - 1$. So it must be isomorphic to $K(X)/(X^n - 1)$.

Infinite Case: Let

$$f([X_\sigma]) = \det(t_{\sigma_i,\sigma_j}), \qquad t_{\sigma,\tau} = X_{\sigma^{-1}\tau}$$

We see $f \neq 0$ by substituting 1 for X_{id} and 0 otherwise. So it won't vanish for all x if we substitute $X_\sigma = \sigma(x)$ because $[\sigma(x)]$ is pairwisely different. Thus there exists w s.t.

$$\det(\sigma^{-1}\tau(w)) \neq 0.$$

Now if

$$\sum a_\tau \tau(w) = 0, \qquad a_\sigma \in K,$$

act by σ for all σ, we get $[\sigma^{-1}\tau(w)][a_\sigma] = 0$, thus $[a_\sigma] = 0$. □

Prop. (I.3.7.7) (Kummer Theory). Let K be a field containing the n-th roots of unity, a **Kummer extension** L/K of order n is one that the Galois group is Abelian and of exponent n. There exists an inclusion preserving isomorphism between the lattice of Kummer extensions L of K and the lattice of subgroups of L containing K^n:

$$L \mapsto \Delta = (L^\times)^n \cap K^\times, \qquad \Delta \mapsto K(\sqrt[n]{\Delta}).$$

And $\Delta/(K^\times)^n$ is isomorphic to $\mathrm{Hom}_{cont}(G_{L|K}, \mathbb{Q}/\mathbb{Z})$.

Proof: Notice the composite of two Kummer extension is an extension, so we consider the maximal Kummer extension L, then $K^* \subset (L^*)^n$, because otherwise we can add a $\sqrt[n]{a}$, this is another Kummer extension.

We use the exact sequence $1 \to \mu_n \to L^* \xrightarrow{n} (L^*)^n \to 0$, then the profinite cohomology exact sequence says

$$1 \to K^* \to (L^*)^n \cap K^* \xrightarrow{\delta} H^1_{cts}(G, \mu_n) \to H^1_{cts}(G, L^*) = 1$$

And G acts trivially on $\mu_n \subset K^*$, then $H^1_{cts}(G, \mu_n) = \mathrm{Hom}_{cts}(G_{L|K}, \mathbb{Q}/\mathbb{Z})$. δ maps $a \mapsto \chi_a(\sigma) = \sigma(\sqrt[n]{a})/\sqrt[n]{a} \in \mu_n$.

Thus if we let L be the maximal Kummer extension, then by Galois theory, Kummer extensions of K corresponds to closed subgroups of G, subgroups of $\mathrm{Hom}_{cts}(G, \mathbb{Q}/\mathbb{Z})$ correspond to subgroups of $K^*/(K^*)^n$. These two correspond by the intersection of the kernel of all them, because they correspond for finite subgroup and open subgroup. And closed subgroups are intersection of open subgroups, and any open subgroup containing it must contain an open subgroup of chosen form, by compactness. Thus they correspond. □

Prop. (I.3.7.8). $\mathrm{Gal}(F_{p^n}/F_p)=\mathbb{Z}/n\mathbb{Z}$. and is generated by the Frobenius.

Brauer Groups

Prop. (I.3.7.9). The **Brauer group** $\mathrm{Br}(K)$ is defined as the profinite cohomology $H^2(G(K_s/K),K_s^*)$. For a Galois extension L/K, $\mathrm{Br}(L/K)$ is defined as $H^2(G(L/K),L^*)$. Then by(II.3.3.2) we have

$$\operatorname{colim}\mathrm{Br}(L/K)=\mathrm{Br}(K).$$

Cf.[Neukirch Cohomology of Number Fields Chap6.3].

Prop. (I.3.7.10). The **Brauer group** $\mathrm{Br}(K)$ is defined as the profinite cohomology $H^2(G(K_s/K),K_s^*)$. For a Galois extension L/K, $\mathrm{Br}(L/K)$ is defined as $H^2(G(L/K),L^*)$. Then by(II.3.3.2) we have

$$\operatorname{colim}\mathrm{Br}(L/K)=\mathrm{Br}(K).$$

And by Hochschild-Serre spectral sequence and by Hilbert's multiplicative theorem90:$H^1(H,K_s^*)=0$, we have the low term:

$$0\to\mathrm{Br}(L/K)\xrightarrow{inf}\mathrm{Br}(K)\xrightarrow{res}\mathrm{Br}(L)^G\to H^3(G(L/K),L^*)\to H^3(K,K_s^*).$$

So $\mathrm{Br}(L/K)$ is the kernel of $\mathrm{Br}(K)\to\mathrm{Br}(L)$.

Cor. (I.3.7.11). $\mathrm{Br}(\mathbb{F}_q)$=0 for finite fields, because the finite Galois extension are cyclic and unramified.

In fact, the Brauer group has semisimple algebraic interpretations. Cf.[Milne].

8 Ordered Fields & Real Fields

Ordered Fields

Cf.[Lang Algebra Chap 11].

Prop. (I.3.8.1). An ordered field has char0.
A square>0 in an ordered field.

Real Fields

Def. (I.3.8.2). A field K is called **real** if -1 is not a sum of squares in K. A field K is called **real closed** iff it is real, and any alg.extension that is real must be itself. An ordered field is clearly a real field, the converse is in fact true, by(I.3.8.11).

Prop. (I.3.8.3). If K is real, $a\in K$, a and $-a$ cannot both be sum of squares. $-a$ is not a sum of squares in K, then $K(\sqrt{a})$ is real. Hence either $K(\sqrt{a})$ or $K(\sqrt{-a})$ is real.

Proof: If a is a square, then $K(a)=K$ is real. If a is not square, then if $K(a)$ is not real, then

$$-1=\sum(b_i+c_i\sqrt{a})^2=\sum(b_i^2+ac_i^2+2b_ic_i\sqrt{a})$$

Since $\sqrt{a}\notin K$, $-1=\sum(b_i^2+ac_i^2)$, so

$$-a=\frac{1+\sum b_i^2}{\sum c_i^2}$$

This implies that $-a$ is a sum of squares. But if a is a sum of squares. □

Prop. (I.3.8.4). If the minimal polynomial f of an α algebraic over a real field K is of odd degree, then $K(\alpha)$ is real.

Proof: If $K(a)$ is not real, then $-1 = \sum g_i(X)^2 + h(X)f(X)$, where g_i has degree smaller than n. This can happen if $h(X)$ has degree odd and $\leq n-2$. Then if β is a root of h, then $K(\beta)$ is also not real. So the proof is finished if we use induction. □

Def. (I.3.8.5) (Real Closure Exists). For any real field K, there exists a **real closure** K^a of K. That is, it is real closed and algebraic over K.

Proof: This is an easy consequence of Zorn's lemma. □

Cor. (I.3.8.6). There exists a unique ordering on a real closed field R. The elements> 0 are just the squares in R. Now every real closed field is assumed to have this ordering tacitly. In particular, any real closed field has char0, so does any real field.

Proof: The set of finite sum of squares in R is closed under addition and multiplication, and all of them are squares, by(I.3.8.3) and maximality of R. Also by(I.3.8.3) either a is a square or $-a$ is a square, but not simultaneously. So it is truly an order on R. □

Prop. (I.3.8.7) (Fundamental Theorem of Algebra). For a field R, R is real closed iff $R \neq R[i]$ and $\overline{R} = R[i]$.

Proof: One direction is trivial, the other follows from the lemma below(I.3.8.8), it satisfies the condition by(I.3.8.3) and maximality. □

Lemma (I.3.8.8). If R is a real field that: for all $a \in R$, $\sqrt{a} \in R$ or $\sqrt{-a} \in R$, and any polynomial of odd degree has a root in R, then $K = \overline{R}$ is alg.closed.

Proof: For any order of R, the first condition in fact says that any $a > 0$ in R is a square. Now $\frac{a+\sqrt{a^2+b^2}}{2}$ is non-negative, so there is a $c^2 = \frac{a+\sqrt{a^2+b^2}}{2}$, that is $(c + \frac{b}{2c}i)^2 = a + b_i$, so K has all squares.

As R is of char0(I.3.8.6)(I.3.8.1), so it suffices to show any Galois extension L/K is trivial. Let $G = G(L/R)$, and H be its 2-Sylow subgroup, then $G = H$ by condition. Now if $G_1 = G(L/K)$, then G_1 is nontrivial, because otherwise there is a subgroup of index 2, then its fixed field is a square extension of K, which is impossible by what we have proved. So $G = G_1$, that is $L = K$. □

Cor. (I.3.8.9). $\mathbb{C} = \mathbb{R}[i]$ is alg.closed.

Prop. (I.3.8.10) (Intermediate Property). An ordered field is real closed iff it has the intermediate property.

Proof: If R is real closed, as $R[i]$ is alg.closed(I.3.8.7), f can be decomposed into factors of degree 1 or 2. For a factor $X^2 + \alpha X + \beta$, $4\beta > \alpha$, otherwise it has a root hence not irreducible. So the change of sign is because of a linear factor, the rest is easy.

Conversely, if it has the intermediate property, then for $a > 0$, consider $p(X) = X^2 - a$, then $p(0) < 0, p(a+1) > 0$, so p has a root, that is, a is a square. For a polynomial of odd degree, for M large enough, $f(M) > 0, f(-M) < 0$, so f has a root. So by(I.3.8.8) R is real closed. □

Real Fields and Order

Prop. (I.3.8.11) (Real Field and Order). If R is real, then it is orderable, in fact, if $-a$ is not a sum of squares in F, then there is an ordering that $a > 0$. So a real field is equivalent to an ordered field.

Proof: By(I.3.8.3), $F(\sqrt{a})$ is real, so it has a real closure(I.3.8.5) and has the induced order(I.3.8.6), and $a > 0$ because it is a square(I.3.8.1). □

Prop. (I.3.8.12). For any ordered field F, there is a unique real closure R of F that every positive element of F is a square in R.

Proof: The existence is by adding all the square roots of elements> 0 to F, the resulting field is real because of(I.3.8.3) and the fact a union of real fields is real.

The uniqueness: because an ordered field is of char0(I.3.8.1), so the primitive element theorem(I.3.5.14) applies that each finite subextension of R_0 is of the form $F(\alpha)$, where α is a root of a irreducible separable polynomial f. Then the roots of f are different so can be ordered $\alpha_1 < \ldots < \alpha_n$. Similarly, f has the same number of different roots in R_1 $\beta_1 < \ldots < \beta_n$ by the lemma(I.3.8.14) below, so there is a map $h : \alpha_i \to \beta_i$, and it is the unique map that a ordered map from $F(\alpha)$ to R_1 extending id on F can be. it is this uniqueness that makes us able to use Zorn's lemma to show that there is a maximal ordered map, must be a map from R_0 to R_1, which is an isomorphism, by primitive element theorem again. □

Prop. (I.3.8.13) (Sturm's Algorithm). Cf.[Model Theory Marker P327].

Lemma (I.3.8.14). If F is an ordered field and R_0, R_1 be two real closure of F that is compatible with the ordering, then any irreducible polynomial has the same number of roots in R_0 and R_1.

Proof: Cf.[Model Theory Marker P328]. □

I.4 Representation Theory

1 Semisimple Algebras

Basic references are [StackProject Chap11] and [Algebra Lang Chap17].

Def. (I.4.1.1). An R-module E is called **simple** iff it has no submodules other than 0 and E.

Prop. (I.4.1.2) (Shur's lemma). For a simple module E, $\mathrm{End}_R(E)$ is a division ring, this is because the kernel and image are all 0 or E.

Def. (I.4.1.3) (Semisimple Module). A module E is called **semisimple** iff it satisfies the following equivalent conditions:

- It is a sum of simple modules.
- It is a direct sum of simple modules.
- Any submodule F of E has a complement in E.

Proof: $3 \to 2 \to 1$ is immediate, it suffices to show $1 \to 3$: for any submodule $N \subset M$, consider all the simple modules that intersect N trivially, denote their sum by V, I claim $N \oplus V = M$, otherwise, let S be a simple submodule that contained in $N + V$, then $S \cap (N + V) = 0$, so $N \cap (S + V) = 0$, contradicting the maximality. □

Cor. (I.4.1.4). Any submodule and quotient module of a semisimple module is semisimple. This is because any submodule is a direct sum of simple modules contained in it, and it has a complement.

Def. (I.4.1.5) (Semisimple Ring). A ring is called **semisimple** iff it is a semisimple module over itself, it is called **simple** iff it is simple module over itself.

2 Linear Representation of Finite Groups

Basic references are [Serre Linear Representations of Finite Groups]. [群表示论 notes 薛航]. The representations in this subsection is assumed to be of char0.

Results for Arbitrary Algberas

Prop. (I.4.2.1) (Schur's lemma). If π is an at most countable dimensional irreducible $\mathbb{C}$-representation of an algebra A, then $\mathrm{End}(V) \cong \mathbb{C}$. In particular this holds for $\dim A$ countable.

Proof: First the dimension of $\dim_{\mathbb{C}} \mathrm{End}(V)$ is at most countable, because V is acyclic by irreducibly, so $\dim_{\mathbb{C}} \mathrm{End}(V) \leq \dim_{\mathbb{C}} V$. And $\mathrm{End}(V)$ is a skew field, by irreducibility. So the result follows from(I.3.3.20). □

Results for Arbitrary Groups

Prop. (I.4.2.2). Any f.g.(i.e. f.g. over $F[G]$) representation of a group has an irreducible quotient.

Proof: Use Zorn's lemma for the set of proper G-subspaces of U, the combine of a chain of proper G-space is proper, because it is f.g., so it has a maximal proper G-space, so the quotient is irreducible. □

Prop. (I.4.2.3). By(I.3.4.5), the induced and coinduced representation is that of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} -$ and $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], -)$. If $[G : H]$ is finite, then induced is the same as coinduced.

Proof: Choose a left coset representation of H, then check $x \otimes a \to f : hx^{-1} \mapsto ha$ is an isomorphism Cf.[Weibel P172]. □

Prop. (I.4.2.4) (Clifford's Theorem). If $\rho : G \to GL(V)$ is a semisimple representation and H is a normal subgroup of G, then $\rho|_H$ is also semisimple.

Proof: Use definition(I.4.1.3), we reduce to the case ρ is simple. Now an H-subrepresentation is simple iff it has no proper H-subrepresentations, so clearly G maps a simple H-subrepresentation to another simple H-subrepresentation. So if W is the sum of all simple H-subrepresentations, then G preserves W, which shows $W = V$, and V is H-semisimple by(I.4.1.3). □

Finite Case

Prop. (I.4.2.5). If G is a finite p-group and A is a nonzero p-torsion G-module, then $A^G \neq 0$.

Proof: We may consider A generated by a single element. Because A is p-torsion, $|A| = p^n$ for some n. Now consider the orbit, then if the orbit is not a single element, then its order is divisible by p, so $|A^G|$ is divisible by p. But 0 is fixed, so $A^G \neq 0$. □

Prop. (I.4.2.6) (Maschke's Theorem). If F is a field of char p and G is a finite group of order prime to p, then for any representation U of $F[G]$ and a submodule V, there exists a complement of V in U.

Proof: Choose an arbitrary projection π of U to V, and let $\rho(v) = 1/|G| \sum g^{-1}\pi(g(v))$, then it can be checked ρ commutes with G-actions, thus its kernel is also a G-modules, and it is identity on V, so $U = V \oplus k \operatorname{Ker} \rho$. □

Prop. (I.4.2.7) (Brauer-Nesbitt). For a finite group G, if two finite dimensional semisimple representations over a field has the same char poly for every element g of G, then they are isomorphic.

Proof: Just use the irreducible representations are orthogonal and that they have the same and for char p, we can use divide by p and the char poly becomes p-th power and we can do this forever, contradiction. □

Important representations of small finite groups

Prop. (I.4.2.8). There is a 2-dimensional representation of the quadratic group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$:

$$i \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad j \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Prop. (I.4.2.9). There is a representation of S_n on the $n-1$-dimensional hypersurface $\sum x_i = 0$.

Modularity Problems

Prop. (I.4.2.10). For a finite group G, all representations of G has characters in $\mathbb{Q}$ iff it all reps has characters in G, iff every two element generating the same subgroup of G is conjugate.

Proof: Cf.[Serre P103]. □

Cor. (I.4.2.11). representations of S_n all has characteristic in $\mathbb{Z}$.

3 Modular Representations

Prop. (I.4.3.1). The only irreducible representation of a p-group over a a field of char p is the trivial representation.

Proof: For any $v \in V$, consider the additive subgroup generated by $g(s)v$, then it is a finite group of prime power order. Then(I.3.1.20) shows it has a element other than 0 fixed by all G, thus it is not irreducible unless trivial representation. □

4 Topological Groups

Def. (I.4.4.1). Let G be a topological group and V a TVS, then a **representation** of G on V is a group action that $G \times V \to V$ is continuous.

Locally Compact Groups

Cf.V.6.

5 Locally Profinite Groups

Def. (I.4.5.1) (Test Functions and Distributions). The set of **test functions** on a locally profinite space X is the set of locally constant functions with compact supports. Test functions form an $\mathbb{C}$-algebra S without topology. The space S^* of **distribution** on X is a functional on the space of test functions on X.

There structure sheaf $C^\infty(G)$ on a locally profinite space X is defined to be the sheaf of locally constant functions on X.

Prop. (I.4.5.2). For X locally profinite and $U \in X$ open, $Z = X - U$, there is an exact sequence:

$$0 \to S(U) \to S(X) \to S(Z) \to 0,$$

Thus also an exact sequence

$$0 \to S^*(Z) \to S^*(X) \to S^*(U) \to 0,$$

Proof: The first is the exact sequence(III.5.5.19) applied to the constant sheaf(structure sheaf of X). The second is immediate from the first. □

Prop. (I.4.5.3) (Smooth Representations). A representation of a locally profinite group G on a complex vector space is called **smooth** iff it is continuous w.r.t the discrete topology.

The category $\mathcal{M}(G)$ of smooth representations is a full Abelian subcategory of the category of continuous G-modules, and there is a right adjoint to the forgetful functor:

$$V^\infty = \bigcup_{H \subset G \text{ compact open}} V^K$$

So it preserves injectives and $\mathcal{M}(G)$ has enough injectives.

Def. (I.4.5.4) (Equivariant Sheaf). Let G be a locally profinite group acting on a locally profinite space X, let $p : G \otimes X \to X$ be the projection and $a : G \times X \to X$ be the action, then a **equivariant sheaf** on X is a pair $(\mathcal{F}, \rho)$, where $\mathcal{F}$ is a sheaf on X and ρ is an isomorphism of sheaves $p^*(\mathcal{F}) \cong a^*(\mathcal{F})$ that:

- ρ is identity on $e \otimes X$.
- $p_{23}^*\rho \circ (\mathrm{id}_G \times a)^*\rho = (m \times \mathrm{id}_X)^*\rho$ on $G \times G \times X$.

Prop. (I.4.5.5). If X is a pt with the trivial G-action, then the equivariant sheaves on X is equivalent to a representation of G on X.

Proof: For any equivariant sheaf on X, the pullback are just locally constant functions of G with value in V. Then ρ on each stalk g defines an action of g on V. Compatibility with the G-action shows that this is a group action. And consider the stalk at e, because ρ is id at e, for each v, there is an open nbhd U that $\rho = \mathrm{id}$ on U, thus it is smooth. The converse is obvious. □

Hecke Algebra

Def. (I.4.5.6) (Idempotented Algebra). An **idempotented algebra** $\mathcal{H}$ is an algebra that for any subsets of elements f_i, there is an idempotented e that $ef_ie = f_i$.

An $\mathcal{H}$-module V is called non-degenerate iff $\mathcal{H}V = V$. The category of all the non-degenerate $\mathcal{H}$-modules are denoted by $\mathcal{M}(\mathcal{H})$.

Prop. (I.4.5.7). If $\mathcal{H}$ is an idempotented algebra, then $\mathcal{M}(H)$ has enough projectives.

Proof: For any idempotent $e \in \mathcal{H}$, consider the module $\mathcal{H}e$, then it is projective, because $\mathrm{Hom}_{\mathcal{H}}(\mathcal{H}e, X) = eX$, thus it is clearly exact. Now any $m \in V$ has an idempotent $e \in \mathcal{H}$ that $ev = v$(use definition). Thus by taking the direct sum, we are done. □

Def. (I.4.5.8) (Hecke Algebra). The space of all compactly supported distributions on a locally profinite G forms an algebra under convolution, and G acts on this algebra by left translation. A compactly supported distribution is called **locally constant** if it is invariant under the action of an open subgroup of G.

The algebra $\mathcal{H}(G)$ of locally constant compactly supported distribution on a locally profinite group G is called the **Hecke algebra** of G.

Notice we are doing distribution on arbitrary locally compact group, but this hasn't been written yet?.

Prop. (I.4.5.9). If Γ is a compact subgroup of G, then the normalized Haar measure e_Γ is a distribution on G with compact support, so does ε_g, the delta distribution at g . And if Γ is open compact, then $e_\Gamma \in \mathcal{H}(G)$. Moreover,

$$e_{\Gamma'} * e_\Gamma = e_\Gamma, \quad e_\Gamma * \varepsilon_g = \varepsilon_g * e_\Gamma = e_\Gamma,$$

for $g \in \Gamma$, $\Gamma' \subset \Gamma$.

Prop. (I.4.5.10). Let h be a K-invariant distribution with compact support, where K is open compact in G, then $h = \sum a_i(e_K * \varepsilon_{g_i})$ is a finite sum. (Easy).

Cor. (I.4.5.11). Multiplication by Haar measure gives an isomorphism from $S(G) \cong \mathcal{H}(G)$. And any $h \in \mathcal{H}(G)$ is also locally constant with right action of G, because Haar measure on a compact space is right invariant(over $\cap g_i^{-1}Kg_i$).

Prop. (I.4.5.12) (Main Theorem of Hecke Algebra). For a smooth representation (π, V) of G, for any v, $\pi(g)v$ can be regarded as a locally constant function with value in V, thus an element of $C^\infty(G) \otimes V$. Thus for $h \in \mathcal{H}(G)$, we can define $h(v) = h(\pi(g)v)$, thus given a $\mathcal{H}(G)$-module V. Then:

- $\mathcal{H}(G)$ is an idempotented algebra.
- The $\mathcal{H}(G)$ module V is non-degenerate. And this gives an equivalence of categories $\mathcal{M}(G) \cong \mathcal{M}(\mathcal{H}(G))$.

Proof: 1: This is obvious from(I.4.5.11) when choosing e_K for K large enough.

2: the module is non-degenerate because for any v, it is invariant by some open compact K, thus $e_K v = v$.

Conversely, for a non-degenerate module M of $\mathcal{H}(G)$, we consider any element of V is of the form $\sum h_i v_i$, with $h_i \in \mathcal{H}(G)$, so we can construct a G-action on V as $\sum h_i v_i \mapsto m\sum(\varepsilon_g * h_i)v_i$. This is then a representation of G on V. This representation is smooth because each v is stabilized by some e_K: because $v = \sum h_i v_i$, let e_K be such that $e_K h_i = h_i$(this is done by(I.4.5.10)(I.4.5.9)), then $e_K v = v$, thus v is fixed by $g \in K$, by(I.4.5.9). $\square$

Cor. (I.4.5.13). $(-)^\infty$ is exact, this is because $(-)^K$ do: it is left exact clearly, and it is right exact because it is the image of e_K. Then use filtered colimits.

Semisimple Representation

Prop. (I.4.5.14). If G is profinite, then Any smooth representation is semisimple(I.4.1.3), because for any element v, G_v is compact open thus of finite index in G, then the orbit of v is finite and some open normal subgroup H of G_v fixes all the orbits of v, so it is a finite representation of the finite group G/H, hence by Mackey's theorem(I.4.2.6), it is a finite sum of irreducible representations.

So if G is locally profinite, then any G-representation is K-semisimple for compact open subgroup K of G.

Admissible Representations

Def. (I.4.5.15) (Contragradient Representation). For a smooth representation V of G, the **contragradient representation** $\widetilde{V}$ is the smooth part of V^*, $= (V^*)^\infty$.

Prop. (I.4.5.16).
- For any compact open subset K of G, $V^K = (\widetilde{V})^{K*}$.
- $\mathrm{Hom}_G(V, \widetilde{W}) = \mathrm{Hom}(W, \widetilde{V})$.
- $V \hookrightarrow \widetilde{\widetilde{V}}$.

Proof: 1: Using(I.4.5.12), because $(\widetilde{V})^K = V^{*K}$. There is a homomorphism $V^{*K} \to V^{K*}$, it is injective, because if $f(v) = 0$ for each $v \in V^K$, then $f(w) = f(e_K v) = 0$. It is also injective, because for each $f \in V^{K*}$, the inverse image is $g(w) = g(e_K w)$.

2: $\mathrm{Hom}(V, \widetilde{W}) = \mathrm{Hom}(V, W^*) = \mathrm{Hom}(V \otimes W, \mathbb{C})$.

3: by the proof of item1,

$$\widetilde{\widetilde{V}} = \cup_K((\cup_K V^{*K})^{*K}) = \cup_K((\cup_K V^{*K})^{K*}) = \cup_K(V^{*K*}) = \cup_K(V^{K**}).$$

So the filtered colimits of the injections $V^K \to (V^K)^{**}$ gives an injection $\cup_K(V^{K**})$. $\square$

Cor. (I.4.5.17) (Contragradient-Functor-Exact). The contragradient functor $V \mapsto \widetilde{V}$ is exact. Because $(-)*$, $(-)^\infty$ are all exact(I.4.5.13).

Cor. (I.4.5.18). If P is projective in $\mathcal{M}(G)$, then $\widetilde{P}$ is injective in $M(G)$.

Proof: $\mathrm{Hom}(X, \widetilde{P}) = \mathrm{Hom}(P, \widetilde{X})$, and notice that the contragradient functor is exact(I.4.5.17). $\square$

Def. (I.4.5.19) (Admissible Representation). A representation is called **admissible** iff for any compact open subgroup K of G, V^K is a finite G-module.

A representation is admissible iff $V \cong \widetilde{\widetilde{V}}$. In particular, the contragradient of an admissible representation is admissible.

Proof: If V^K is finite for each K, then by the proof of item3 of(I.4.5.16), $V \cong \widetilde{\widetilde{V}}$. Conversely, if $V \cong \widetilde{\widetilde{V}}$, then $V^K \cong V^{K**}$, thus V must be finite, by(I.2.3.2). $\square$

Prop. (I.4.5.20). Every smooth irreducible representation is admissible. In fact, this is true for general connected reductive group.

Proof: $\square$

Cor. (I.4.5.21). $\mathcal{M}(G)$ has enough injectives.

Proof: As $\mathcal{M}(G)$ has enough projectives(I.4.5.7)(I.4.5.12), there is a surjection $P \to \widetilde{X}$, thus an injection $\widetilde{\widetilde{X}} \hookrightarrow \widetilde{P}$(I.4.5.17). Now $X \hookrightarrow \widetilde{\widetilde{X}}$ by(I.4.5.16). $\square$

Jacquet Functor

Def. (I.4.5.22) (Jacquet Functor). For any group G, The **Jacquet functor** J is tensoring $\mathbb{C}$ on the category of G-modules, where $\mathbb{C}$ is the trivial representation. It is equal to the space $V \mapsto V_G = V/(\pi(g)v - v)$, by the split exact sequence $0 \to I_G \to \mathbb{C}[G] \to \mathbb{C} \to 0$.

Prop. (I.4.5.23). J is right exact, and if G is σ-compact, then it is exact.

Proof: Direct limit of exact functors are exact, thus the σ-compact case follows from the compact case. In the compact case, $V^G = e_G V = V_G$(check). $\square$

Irreducible Representations

Prop. (I.4.5.24) (Shur's Lemma). If G is a σ-compact locally profinite group, then any irreducible representation V is of at most countable dimension, thus $\mathrm{End}(V) = \mathbb{C}$ by(I.3.3.20).

Proof: If it is of at most finite dimension because if ξ generate V, then notice its stablizer is compact open, and G is σ-compact, so V is at mots countable. $\square$

Prop. (I.4.5.25) (Separation Lemma). If G is a σ-compact locally profinite group, then for any $0 \neq h \in \mathcal{H}(G)$, there is an irreducible representation ρ that $\rho(h) \neq 0$.

Proof: $\square$

Compact Representations

Def. (I.4.5.26). A smooth representation of a locally profinite group G is called **compact** iff for every $\xi \in V$ and every open compact subgroup $K \subset G$, the function $D_{\xi,K} : g \mapsto \pi(e_K)\pi(g^{-1})\xi$ has compact support.

Prop. (I.4.5.27). If $\xi \in V, \widetilde{\xi} \in \widetilde{V}$, then the function $m_{\widetilde{\xi},\xi}(g) = \widetilde{\xi}(\pi(g^{-1})\xi)$ is called the **matrix coefficients**. Then V is compact iff every matrix coefficient is compactly supported.

Proof: Cf.[Bernstein P22]. □

6 $(\mathfrak{g}, K)$-Modules

Real Reductive Groups

Def. (I.4.6.1) (Admissible Representation). Let G be a connected real reductive group(which is relevant, thus the complex representations of G and $G(\mathbb{R})$ are the same(IV.7.5.6)), Let $K \subset G(\mathbb{R})$ be a maximal compact subgroup. Consider $V^\infty, V^\rho, V^{K-fin}$ as in(V.6.4.7)(IV.7.6.5).

A representation V of G is called **admissible** if for any ρ, V^ρ is of f.d.

Prop. (I.4.6.2). For any ρ, $V^\infty \cap V^\rho$ is dense in V^ρ.

Cor. (I.4.6.3). If V is admissible, then $V^{K-fin} \subset V^\infty$ by(IV.7.6.11).

Prop. (I.4.6.4). If V is admissible, then V^{K-fin} is a $(\mathfrak{g}, K)$-module(I.4.6.8), and the map $V \mapsto V^{K-fin}$ induces a functor

$$Rep(G)_{adm} \to (\mathfrak{g}, K) - mod_{adm}.$$

And we call two admissible representations V_1, V_2 of G **infinitesimal equivalent** iff they are isomorphic after this functor.

Proof: By(I.4.6.3), $\mathfrak{g}$ acts on V^{K-fin}, and we check

$$T_k T_\eta T_{k^{-1}} = T_{Ad_k \eta}$$

which is by definition, and the second condition in(IV.7.6.11) is also obvious. □

Lemma (I.4.6.5). Let V be an admissible representation of G, and $v \in V^{K-fin}$, then for any $\eta \in V^*$, the function $g \mapsto \eta(g(v))$ is real analytic.

Proof: Cf.[Gaitsgory P37]. □

Prop. (I.4.6.6) ($Rep(G)$ and $(\mathfrak{g}, K)$-Modules).

- If V_1, V_2 be two admissible representations of G, if $S : V_1 \to V_2$ is a continuous map of TVS. Assume $S(V_1^{K-fin}) \subset V_2^{K-fin}$ and induces a $(\mathfrak{g}, K)$-module map, then the initial S is a map of G-representations.
- If $V \in Rep(G)_{adm}, M = V^{K-fin}$, then the functors

  $$(V_1 \subset V) \mapsto (V_1)^{K-fin} \subset M; \quad (M_1 \subset M) \mapsto \overline{M}_1 \subset V$$

  induces mutually inverse bijections between closed G-subrepresentations of V and $(\mathfrak{g}, K)$-submodules of M.

Proof: 1: It suffices to show for $v_1 \in V^{K-fin}$, $T_gS(v_1)ST_g(v_1)$. So by Hahn-Banach it suffices to show for any $\eta \in V_2^*$,

$$\eta(T_gS(v_1)) = \eta(ST_g(v_1)).$$

Both sides are analytic in g by(I.4.6.5), so it suffices to show all their derivatives at 1 are equal, and use the fact $\pi_0(K) \to \pi_0(G)$ is surjective(IV.7.5.9). And the derivatives equal because S commutes with $\mathfrak{g}$-action.

2: Firstly $\overline{M}_1$ is a G-representation: because $\overline{M}_1 = ((M_1)^\perp)^\perp$ by Hahn-Banach. so it suffices to show for $v_1 \in M_1, \eta(g(v_1)) = 0$ for any $\eta \in (M_1)^\perp$. Then this uses analyticity(I.4.6.5) as above and the fact M_1 is a $(\mathfrak{g}, K)$-subrepresentation.

For the bijection, notice V_1^{K-fin} is dense in V_1 by(V.6.4.12). Conversely, for a submodule M_1, it suffices to show the image of $T_{\xi_\rho \mu_{Haar}}(\overline{M}_1) \subset M_1^\rho$ by(V.6.4.10). However $T_{\xi_\rho \mu_{Haar}}(M_1) \in M_1^\rho$ by(V.6.4.10), so this is true by continuity. □

Cor. (I.4.6.7) (Irreducibility of $(\mathfrak{g}, K)$-Modules). An admissible G-representation V is irreducible iff V^{K-fin} is irreducible as $(\mathfrak{g}, K)$-modules.

$(\mathfrak{g}, K)$-Modules

Def. (I.4.6.8) ($(\mathfrak{g}, K)$-Modules). If G is a Lie group and $K \subset G$ be a maximal compact Lie subgroup(IV.7.5.9), then K acts on $\mathfrak{g}$. Then a $(\mathfrak{g}, K)$**-Module** is a $\mathbb{C}$-vector space that has a K-finite action and a $\mathfrak{g}$ action that satisfies:

- for $k \in K, \eta \in \mathfrak{g}$, $T_kT_\eta T_{k^{-1}} = T_{Ad_k(\eta)}$.
- The action of $\mathfrak{k}$ on V induced by K agrees with the restriction of the action of $\mathfrak{g}$.

A $(\mathfrak{g}, K)$-module is called **admissible** iff every V^ρ part is of f.d. as K-representation.

Def. (I.4.6.9). If M is an admissible $(\mathfrak{g}, K)$-module, then we can define its **algebraic dual** as

$$(M^*)^{alg} = \oplus_\rho (M^\rho)^*,$$

which is the K-finite part of the usual dual of M.

$(\mathfrak{g}, K)$-Modules and $\mathfrak{g}$-Modules

Def. (I.4.6.10). Let K_0 be the unital component of K, then there are forgetful functors

$$(\mathfrak{g}, K) - mod \to (\mathfrak{g}, K_0) - mod \to \mathfrak{g} - mod.$$

Prop. (I.4.6.11). The functor $(\mathfrak{g}, K_0) - mod \to \mathfrak{g} - mod$ is fully faithful, and its essential image is stable under taking submodules.

Proof: Cf.[Gaitsgory P39]. □

Cor. (I.4.6.12). If $M \in (\mathfrak{g}, K) - mod$ is irreducible as a $\mathfrak{g}$-module, then it is irreducible.

Prop. (I.4.6.13). The functor $(\mathfrak{g}, K) - mod \to \mathfrak{g} - mod$ sends f.g. objects to f.g. objects.

Proof: By(I.4.6.11), it suffices to consider the functor $(\mathfrak{g}, K) - mod \to (\mathfrak{g}, K_0) - mod$. Let M be f.g. $(\mathfrak{g}, K)$-module, and $\cup M_i = M$ be a chain of $(\mathfrak{g}, K_0)$-submodules. Pick $k \in K$ for each element of $\pi_0(K)$(f.m.), then each $M_i' = \sum_k k(M_i)$ is a $(\mathfrak{g}, K)$-submodule, thus $M_i' = M$ for some i. Now we can choose j large that $k(M_i) \in M_j$ for any k, then $M_j = M$(because we may choose M_i be f.g. $(\mathfrak{g}, K_0)$-modules?. □

Cor. (I.4.6.14). The category $(\mathfrak{g}, K) - mod$ is Noetherian.

Proof: If $M \in (\mathfrak{g}, K) - mod$ is f.g. and $M_1 \subset M$, then M is f.g. as $\mathfrak{g}$-module, then M_1 is f.g. as $\mathfrak{g}$-module by(I.10.7.10). So clearly it is also f.g. as a $(\mathfrak{g}, K)$-module. □

Prop. (I.4.6.15). For an irreducible $(\mathfrak{g}, K)$-module M, the underlying $\mathfrak{g}$-module is a direct sum of f.m. irreducibles.

Proof: By(I.4.6.11), it suffices to prove M is a direct sum of f.m. irreducible $(\mathfrak{g}, K_0)$-modules. M is f.g. as a $(\mathfrak{g}, K_0)$-modules by the proof of(I.4.6.13), so it has a maximal submodule M' that $N = M/M'$ is irreducible. Pick $k \in K$ for each component of K, consider

$$M'' = \cap_k k(M')$$

which is a proper $(\mathfrak{g}, K)$-submodule of M, so it is 0, Hence the map

$$M \to \oplus(N)^k$$

is injective, where N^k is N twisted by conjugate action of k, so it is a submodule of a semisimple-module, thus semisimple. □

Properties of $(\mathfrak{g}, K)$-Modules

Cor. (I.4.6.16) (Schur's Lemma). Schur's lemma holds for irreducible $(\mathfrak{g}, K)$-modules.

Proof: It suffice to show any endomorphism S of an irreducible $(\mathfrak{g}, K)$-module M has an eigenvalue. But S preserves M^ρ for any ρ, and M^ρ is of f.d, thus it has an eigenvalue over $\mathbb{C}$. □

Cor. (I.4.6.17) (Irreducible Unitary Representation Determined by Finite Part). If V_1, V_2 are two irreducible unitary representations of G that are infinitesimal equivalent, then they are isomorphic.

Proof: Firstly they are admissible by(I.4.6.28), so we can talk about their corresponding $(\mathfrak{g}, K)$-modules M_i, then V_i are the Hilbert space completion of M_i by(I.4.6.6).

Now M_i has Hermitian forms, so $M_i \cong (M^*)^{alg}$, and if $S : M_1 \cong M_2$, then S^*S is an automorphism of M_1, thus by(I.4.6.7)(I.4.6.16) it is a scalar map, so after a scalar change, we may assume S preserves Hermitian structure thus induces an isomorphism of vector spaces $V_1 \cong V_2$, so by(I.4.6.6) it is an isomorphism of G-representations. □

Prop. (I.4.6.18). Any irreducible $(\mathfrak{g}, K)$-module has a Banach space structure.

Action of $Z(U(\mathfrak{g}))$

Prop. (I.4.6.19). Let M be an admissible $(\mathfrak{g}, K)$-module, then

$$M \cong \oplus_{\chi \in \mathrm{Spec}(Z(\mathfrak{g}))} M_\chi$$

s.t. $Z(\mathfrak{g})$ acts on each M_χ with a generalized character χ.

Now let $(\mathfrak{g}, K) - mod_\chi$ be the full subcategory of $(\mathfrak{g}, K)$-modules on which $Z(\mathfrak{g})$ acts with a generalized character χ.? Cf.[Gaitsgory P42].

Proof: $Z(\mathfrak{g})$ commutes with G thus K action, so it preserves each M^ρ, which are of f.d.. □

Prop. (I.4.6.20). The category $(\mathfrak{g}, K) - mod_\chi$ has only f.m. isomorphism classes of irreducible objects.

Proof: Cf.[Gaitsgory]. □

Prop. (I.4.6.21). If M is a f.g. $(\mathfrak{g}, K)$-module, then for any ρ of K, M^ρ is f.g. over $Z(\mathfrak{g})$.

Proof: Cf.[Gaitsgory P43]. □

Prop. (I.4.6.22). For $M \in (\mathfrak{g}, K) - mod_\chi$, the following are equivalent:

- M is f.g..
- M is of finite length.
- M is admissible.

Proof: $2 \to 1$ is trivial, $1 \to 3$ is by(I.4.6.21).

For $3 \to 2$: Use(I.4.6.20), there are only f.m. irreducible classes ρ_α, let $\rho = \oplus_\alpha \rho_\alpha$, then if there is a chain of length n, then there are at least n linearly independent morphisms in $\mathrm{Hom}_K(\rho, M)$. Thus n is bounded, because $\dim_K \mathrm{Hom}_K(\rho, M)$ is finite because M is admissible. □

Cor. (I.4.6.23). The category $(\mathfrak{g}, K) - mod_\chi$ is Artinian(I.8.2.17).

Cor. (I.4.6.24). Every irreducible $(\mathfrak{g}, K)$-module is admissible.

Proof: Firstly irreducible module are in $(\mathfrak{g}, K) - mod_\chi$ for some χ, and then use the proposition and(I.4.6.15). □

Cor. (I.4.6.25) (Harish-Chandra Modules). For a $(\mathfrak{g}, K)$-module, the following conditions are equivalent:

- M is f.g. and admissible.
- M is f.g. and its support over $\mathrm{Spec}(Z(\mathfrak{g}))$ is finite.
- M is admissible and its support over $\mathrm{Spec}(Z(\mathfrak{g}))$ is finite.
- M is of finite length.

Then such modules are called a **Harish-Chandra module**.

Proof: ? □

Unitary Irreducible Representation is Admissible

Prop. (I.4.6.26). If V is an irreducible unitary representation of G, then the image of the induced action of $Meas_c(G)$ is dense in $\mathrm{End}(V)$ in the strong topology(V.3.5.4).

Proof: This follows immediately from the von Neumann theorem(V.5.3.13) and Schur's lemma(V.6.2.4): if we denote the algebra generated by $Meas_c(G)$ by A, then

$$\overline{A} = (A^c)^c = (\mathbb{C})^c = \mathrm{End}(V).$$

□

Prop. (I.4.6.27). If V Is a representation of G that the image of the induced action of $Meas_c(G)$ is dense in $\mathrm{End}(V)$ in the strong topology, then

$$\dim(V^\rho) \leq \dim(\rho)^2.$$

Proof: Follows directly from the following two lemmas????. □

Lemma (I.4.6.28). For any $\rho \in \mathrm{Irrep}(K)$, let $A_\rho = \xi_\rho \cdot Meas_c(G) \cdot \xi_\rho$, this is an algebra that acts on V^ρ by(V.6.4.10). Then there exists a family of f.d. representations π of A_ρ that:

- Each π is of dimension$\leq n = \dim(\rho)^2$.
- For every element $a \in A_\rho$, there exists a π that $\pi(a)$ is non-trivial.

Proof: Consider the set of all irreducible f.d. representations of G, and π^ρ their ρ-isotopic parts. Then these are representations of A_ρ, and for any $\varphi \in Meas_c(G)$, there is a π that $T_\varphi \neq \mathrm{Id}$?, and each π^ρ has dimension$\leq \dim(\rho)^2$?. Cf.[Gaitsgory P46]. □

Lemma (I.4.6.29). If A is an associative algebra equipped with a family of f.d. modules satisfying conditions in??, then if V is a representation of A that the image of A is dense in $\mathrm{End}(V)$ in the strong topology, then $\dim V \leq n$.

Proof: For an associated algebra A, consider the minimal integer r that the property $P(r)$:

$$\sum_{\sigma \in \Sigma_r} \mathrm{sgn}(\sigma) a_{\sigma(1)} \cdots a_{\sigma(r)} = 0$$

for any $a_1, \ldots, a_r$, then Amitsur-Levitski showed that for $A = GL(n, \mathbb{C})$, $r = 2n$??.

Now the condition of A in?? shows that P_{2n} is true for A. If $\dim V \geq n+1$, then the image of A satisfies $P(2n)$, so also $\mathrm{End}(V)$ satisfies $P(2n)$ because A is dense in $\mathrm{End}(V)$. But V contains a subgroup $GL(n+1, \mathbb{C})$, so it cannot satisfy $P(2n)$ by??, contradiction. □

Cor. (I.4.6.30). Let V be an irreducible unitary representation of G, then for any $\rho \in Irrep(K)$,

$$\dim(V^\rho) \leq \dim(\rho)^2.$$

In particular, every unitary irreducible representation of G is admissible.

Proof: Directly from Lemmas(I.4.6.26) and(I.4.6.27) above, as the action of A_ρ on V^ρ is also have dense image in the strong topology. □

Prop. (I.4.6.31). If M is an irreducible $(\mathfrak{g}, K)$-module equipped with an invariant inner product $((km_1, km_2) = (m_1, m_2), (\xi m_1, m_2) + (m_1, \xi m_2) = 0)$, then the Hilbert space completion of M carries a unique unitary G-representation s.t. $V^{K-fin} = M$ as $(\mathfrak{g}, K)$-modules.

Proof: By(I.4.6.30), the Hermitian form can be extended continuously to the Banach space completion of M, It suffices to prove the extended Hermitian form is continuous, because them we can choose its completion w.r.t. $(-,-)$.

For the invariance, consider $f(g) = (gm_1, m_2) - (m_1, g^{-1}m_2)$, then notice $(a, -)$ are continuous functional on V, thus by(I.4.6.5) and similar analytic method as in(I.4.6.6) using the invariance of inner product. □

Lemma (I.4.6.32). Situation as in(I.4.6.29), M has a Banach norm that $(m, m) \leq ||m||^2$.

Proof: (I.4.6.18) shows M does have a Banach norm. Then let $M \cong V^{K-fin}$ and $M^{*alg} \cong (V^*)^{K-fin}$. However the Hermitian form induces $M \cong M^{*alg}$, thus we can form

$$M \xrightarrow{\Delta} M \oplus M \to V \oplus V^*$$

let V' be the closure of the image of M, then it is a G-representation by(I.4.6.6), and then

$$(m, m) \leq ||i_1(m)||||i_2(m)|| \leq (||i_1(m)|| + ||i_2(m)||)^2.$$

$\square$

Cor. (I.4.6.33). The above proposition(I.4.6.29) is true for M admissible.

7 l-adic Representations

G_K denotes the separable Galois group of K.

Def. (I.4.7.1). A **Galois representation** is a continuous representation of the Galois group G_K.

Prop. (I.4.7.2). The most common representation is the cyclotomic representation $G_k \to \mathbb{Z}_l^*$, where $(l, p) = 1$. And for any representation of G_k over a module V over a $\mathbb{Z}_l$-algebra R, we can associate the representation $V(n)$ which is the twist with the cyclotomic representation.

Prop. (I.4.7.3) (Continuous Action of Compact Group Stable Lattice). Let Γ be a compact group and let $\rho : \Gamma \to GL_n(\overline{\mathbb{Q}_l})$ be a continuous homomorphism, then there exists a finite extension $L/\mathbb{Q}_p$ that $\rho(\Gamma) \subset GL_n(L)$, and up to conjugation, it is $\rho(\Gamma) \subset GL_n(\mathcal{O}_L)$.

Proof: Notice $\rho(\Gamma)$ is compact and Hausdorff, so by Baire category theorem, now that $GL_n(L)$ is closed in $GL_n(\overline{\mathbb{Q}_l})$ for all L/Q_p finite, and all this extensions are countable by primitive element theorem, so there is an L that $\rho(\Gamma) \cap GL_n(L)$ contains an open subset of $\rho(\Gamma)$, so it is an open subgroup, thus of finite index, hence by adding all the coset representations into L, we get an L' finite.

For the second assertion, it suffices to find an $\mathcal{O}_L$-lattice that is stable under Γ-action. Notice $\rho(\Gamma) \cap GL_n(\mathcal{O}_{L'})$ is open in $\rho(\Gamma)$, thus of finite index, so taking the coset representation, it is a lattice that is stable under Γ. $\square$

Prop. (I.4.7.4). If $\rho : \Gamma \to GL_n(k) = GL(V)$ is a representation, then it has a filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$ where V_{i+1}/V_i is irreducible, then there is a semisimplification of ρ, which is $\rho^{ss} = \oplus V_{i+1}/V_i$.

Prop. (I.4.7.5). If $k = L/\mathbb{Q}_p$ finite, we may take a Γ-stable $\mathcal{O}_L$-lattice Γ, then the **residual representation** $\overline{\rho}_L$ is defined by $\Gamma \to GL(\Lambda/\pi\Lambda)$. Then the simisimplification of $\overline{\rho}_L$ is independent of Λ chosen.?

Prop. (I.4.7.6) (Brauer-Nesbitt). If two representations satisfies they have the same char polynomial or that char$k = 0$ or char$k > n$ and trace is the same, then their semisimplification are the same.

Proof: The proof is not hard, use the Artin-Wedderburn theorem, and the fact the representation may not by semisimple. $\square$

8 p-adic Groups

References are [Representations of p-adic Groups Bernstein].

I.5 Commutative Algebra(Atiyah)

Basic References are [Commutative Algebra Atiyah] and [Commutative Ring Theory Matsumura], [StackProject Chap10], [Commutative Algebra with a View Towards Algebraic Geometry]. also referenced [Weibel Homologocal Algebra Ch4].

1 Basics

Prop. (I.5.1.1) (Prime Avoidance). If finitely many primes cover an ideal, then one of them cover it.

Proof: Assume otherwise, use induction. For two primes, use $x+y$, for r primes, choose $x \notin p_i, i<r$, then $x \in p_r$, and choose $y \in JI_1 \ldots I_{r-1}$ and $y \notin y_r$, then $x+y$ suffice. □

Prop. (I.5.1.2). Any non-zero commutative ring has a maximal ideal.

Proof: Use Zorn's lemma, the union of a chain of ideals is an ideal. □

Prop. (I.5.1.3). If $r(I)$ and $r(J)$ are coprime, then I, J are coprime.

Proof: As $a+b=1$, and $a^m \in I, b^n \in J$, $1=(a+b)^{m+n} \in I+J$. □

Localization

Prop. (I.5.1.4) (Localization is exact). S^{-1} is an exact functor from $R-\mathrm{mod}$ to $R-\mathrm{mod}$. Because it is a filtered colimit,(I.8.2.24).

Cor. (I.5.1.5). $(R/I)_{\bar{P}} \cong R_P/IR_P$, in particular, $k(R/P) \cong R_P/PR_P$.

Def. (I.5.1.6). A map between two local rings are called **local ring map** iff it maps non-invertible elements to non-invertible elements, equivalently, $f^{-1}(\mathfrak{m}_S)=\mathfrak{m}_R$.

Prop. (I.5.1.7). If $F_i, i \in I$ is a collection of fields, then the prime ideals in the ring $\prod F_i$ is in bijection with the ultrafilters on I, where the ultrafilter $\mathcal{F}$ corresponds to the ideal $p_{\mathcal{F}} = \{(a_i) | \text{the set of coordinates that} a_i=0 \text{ is in } \mathcal{F}\}$. And in the same way, ideals of $\prod F_i$ corresponds to the filters on I.

Proof: Clearly $p_{\mathcal{F}}$ is an ideal, and if $\mathcal{F}$ is an ultrafilter, let $Z(a)$ be the coordinates that a is zero on, and notice $Z(ab)=Z(a)\cup Z(b)$, then $ab \in p$ iff $a \in p$ or $b \in p$, by(I.1.6.6), so it is a prime ideal.

Conversely, notice that any two a, b with $Z(a)=Z(b)$ differs by a unit, so $\mathcal{F}_p=\{Z(a)|z \in p\}$ is easily checked to be a filter. And if p is a prime, then for any $A \subset I$, let $a, b \in \prod F_i$ be that $Z(a)=A, Z(b)=I-A$, then $ab=0 \in p$, so $a \in p$ or $b \in p$. □

Noetherian

Prop. (I.5.1.8). Quotient ring, f.g. module, f.g. ring, localization and power series of a Noetherian ring A are Noetherian, hence graded algebra of a A by an ideal I is Noetherian.

Proof: Only need to prove $A[X]$ and $A[[X]]$, localization and others are quotients of these. For an ascending chain of ideal I_j of $A[X]$, we consider the coefficients ideal $I_{i,j}$ of X^i of I_j, then there are only f.m. different $I_{i,j}$s, so we have I_j stabilize as well.

Similarly for $A[[X]]$, we prove any ideal I is f.g. Consider the lowest terms coefficient ideal at degree i, then it is ascending and stabilize, then a set of generators as a whole generate I. □

Remark (I.5.1.9). The subring of a Noetherian ring is NOT necessarily Noetherian, by the example of $k[X_1, \ldots, X_n, \ldots] \subset k(X_1, \ldots, X_n, \ldots)$.

Prop. (I.5.1.10). When A is Noetherian and is quipped with I-adic topology, then I is f.g., and there is surjective ring map $A[[X]] \to A^*$ the completion, mapping to the generators of I, hence the completion is Noetherian. (It is surjective can be seen by the Cauchy sequence construction of completion).

Prop. (I.5.1.11). If $R \to R'$ is ring map of f.t., then if $R \to S$ and S is Noetherian, then $S \otimes_R R'$ is Noetherian, because $S \times_R R'$ is of f.t. over S, and use(I.5.1.8).

Cor. (I.5.1.12). If S is a Noetherian k-algebra over a field k, then for any f.g. field extension K/k, $S \otimes_k K$ is Noetherian. (Because there is a f.g. algebra B over k that K is the localization of B, and use(I.5.1.8)).

Prop. (I.5.1.13). If R is Noetherian and M is a f.g. R-module, then there is a filtration $\{M_i\}$ of M that the quotients are all isomorphic to $R_{\mathfrak{p}_i}$ where $\mathfrak{p}_i$ are primes.

Proof: M is generated by x_i, so $(x_1) \cong R/I_i$, and so we modulo x_i, then the result follows by induction. So we may assume $M = R/I$. We use Noetherian condition to choose a maximal element J that is a counterexample, then J is not a prime, so there are $a, b \notin J$ that $ab \in J$. Then we have a filtration $0 \subset aR/(J \cap aR) \subset R/J$. Notice $R/(J + bR) \to aR/(J \cap aR) \to 0$, and the second quotient is $R/(J + aR)$, so they all can be factorized. □

Prop. (I.5.1.14) (Cohen). If every prime ideals of a commutative unital ring R is f.g., then R is Noetherian.

Proof: Suppose P is not Noetherian. Firstly the set of non-finitely generated ideals has the chain property: if I_i is a chain of non-f.g. ideals of R, then $I = \cup_{i \in \Phi} I_i$ is non-f.g., otherwise there are $(f_i) = I$, but $f_i \in I_h$ for some h, thus I_h is f.g.. Next we use the Zorn's lemma to find a maximal non-f.g. ideal I, and show that I is a prime ideal:

$I \neq R$ because $R = (1)$, so if $a, b \in R - I$ that $ab \in I$, then $I + (a)$ and $I + (b)$ is f.g. by $p_i + r_i a$, by maximality, and let $K = (P : a)$, then $I \subset I + (b) \subset K$, thus K is f.g., so does aK.

Now I claim $I = (p_1) + \ldots + (p_n) + aK$: one direction is clear, and if $r \in I \subset I + (a)$, then $r = \sum c_i(p_i + r_i a)$, thus $(\sum c_i r_i)a = r - \sum c_i p_i \in I$, thus $\sum c_i r_i \in K$, thus $r = \sum c_i p_i + (\sum c_i r_i)a \in (p_1) + \ldots + (p_n) + aK$.

So now I is f.g., contradiction, which shows I is a prime, but this contradicts the hypothesis. □

Length

Def. (I.5.1.15). The **length** of a R-module M is the supremum of lengths of chains of submodules of M. It is checked to be an additive function on R-modules.

Prop. (I.5.1.16). If $\mathrm{length}_A(M) < \infty$, then any maximal chain of submodules has the same length.

Proof: Let $l(M)$ be the minimal length of a maximal chain, then if $M \subsetneq N$, then firstly $l(M) < l(N)$, because a maximal chain of M restricts to a maximal chain of N, and if the length is the same, then each term is in M, so $N \subset N$, contradiction. Now any chain has length $l(M)$, because if there is a chain M_i, then $l(M_0) < l(M_1) < \ldots < l(M)$. □

Prop. (I.5.1.17) (Order of Vanishing). If R is a semi-local Noetherian domain of dimension 1 and a, b are not zero-divisors, then $f(a) = \mathrm{length}(R/(a))$ satisfies $f(a) + f(b) = f(ab) < \infty$.

So this $R \subset K$ and has fraction field K, then f extends to an additive function on K^*, denoted by $\mathrm{ord}_R(f)$.

Proof: It is finite by[StackProject 00PF]. It is additive because length is additive(I.5.1.15) and $0 \to R/(a) \to R(ab) \to R(b) \to 0$. □

Artinian Ring

Prop. (I.5.1.18) (Artinian Ring). A ring A is called **Artinian** iff it satisfies the following equivalent definitions:

1. descending chain condition holds for A.
2. A is Noetherian of dimension 0.
3. $\mathrm{length}_A A < \infty$.
4. A is a finite product of local rings with descending conditions.
5. A is Noetherian, and has f.m. maximal ideals, and its Jacobson radical is locally nilpotent.

Proof: $1 \to 5$: Consider $\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \ldots \supset$, then it is a descending chain, by Chinese remainder theorem. So it has f.m. maximal ideals.

Consider the Jacobson radical I, $I^n = I^{n+1}$ for some n, let $J = \mathrm{Ann}(I^n)$, it suffices to show $J = A$. If not, choose a minimal J' that contains J but not J(exists by Artinian property), then $J' = J + Ax$, and $IJ' \subset J$ by Nakayama, so $xI^{n+1} \subset JI^n = 0$, so $x \in J$, contradiction.

$5 \to 2$: By lemma(I.5.1.19) below.

$2 \to 5$: A has f.m. minimal ideals, and they are all maximal, so $\mathrm{Spec}\, A$ is discrete, and the radical is nilpotent.

$5 \to 3$: By lemma(I.5.1.19) below, R is a product of its local rings, and the local rings are all Noetherian and have nilpotent maximal ideals, so by Nakayama, they have finite length. So also R has finite length.

$3 \to 1$: It has finite length so is Artinian.

$5 \to 4$: By lemma(I.5.1.19) below, A is a product of its localization, and its localizations also satisfies 5, and by $5 \to 3 \to 1$, they both have descending conditions.

$4 \to 5$: An Artinian ring is noetherian and its Jacobson radical is locally nilpotent by $1 \to 5$, then so does their product. □

Lemma (I.5.1.19). If a ring A has f.m. minimal ideals and the Jacobson radical is locally nilpotent, then it is the product of its localizations at maximal primes and $\dim A = 0$.

Proof: Since the set I of locally nilpotents element is the intersection of all primes, so any prime contains a maximal primes, so $\dim A = 0$. Now $A/I = \oplus A/\mathfrak{m}_i$ by Chinese remainder theorem, so $\mathrm{Spec}\, A/I$ is discrete with n pts, so by(I.5.7.2) there are n idempotents e_i that $e_i \equiv \delta_{ij} \mod \mathfrak{m}_j$, $\sum e_i = 1$. Thus $R = \prod Re_i$. And Re_i is just the localization at a maximal prime. □

Prop. (I.5.1.20). For an Artinian local ring A, the following are equivalent:

1. A is a PID.
2. the maximal ideal $\mathfrak{m}$ is maximal.

3. $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq 1$.

Proof: It suffices to prove $3 \to 1$: If $\mathfrak{m} = \mathfrak{m}^2$, then $\mathfrak{m} = 0$ by Nakayama, so A is a filed. If $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$, then $\mathfrak{m}$ is principle by Nakayama. And $\mathfrak{m}$ is nilpotent by(I.5.1.18), so for any ideal a there is a minimal n that $a \subset \mathfrak{m}^n$. Now choose $y \in a - \mathfrak{m}^{n+1}$, then $y = ux^r$, and $u \notin (x)$, so u is a unit, thus $x^r \in a$, meaning $a = \mathfrak{m}^n$ hence principal. □

Tensor Product, Limits and Colimits

Def. (I.5.1.21) (Tensor Product). For R-modules A, B, their **tensor product** is an R-module defined by universal properties: Any $A \times B \to A \times_R B$ is bilinear, and any bilinear map $A \times B \to C$ factors uniquely through $A \times_R B$.

Prop. (I.5.1.22). By definition, $\mathrm{Hom}(M \otimes_R N, P) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, P))$, so tensoring is right adjoint.

Prop. (I.5.1.23). If A is a right R-module and B is a left R-module, then $A \times_R B$ is a left A-module as well as a right B-module. If A, B, R are

Prop. (I.5.1.24). If X is f.g module over a Noetherian ring, then $\mathrm{Hom}(X, -)$ commutes with direct sums and $X \otimes -$ commutes with direct products.

Proof: Write X as a cokernel of free modules. □

Prop. (I.5.1.25). For a ring map $R \to S$, let q be a prime in S, $p = q \cap R$, then $(M \otimes_R S)_q = M_p \otimes_{R_p} S_q$ for any R-module M.

Proof: $(M \otimes_R)_q = M \otimes_R S_q = M \otimes_R R_p \otimes_{R_p} S_q = M_p \otimes_{R_p} S_q$. □

Local Properties

Def. (I.5.1.26). A property P of rings or modules over a ring is called **local property** iff X has P iff X_{f_i} all has P for a covering $(f_1, \ldots, f_n) = 1$.

A property of morphisms of rings is called **local on the target** iff $R \to S$ has P iff $R_{f_i} \to S_{f_i}$ has P for a covering $(f_1, \ldots, f_n) = 1$ in R.

Prop. (I.5.1.27) (Stalkwise Properties). For a ring R,

1. Trivial is stalkwise for modules over R, it is even stalkwise. Hence so does injectivity and surjectivity because localization is exact.
2. Torsion-free is stalkwise for modules over R integral.
3. Flatness for modules over R.
4. Flatness for rings over R.
5. Formal unramifiedness for rings over R, both on the target and source.
6. normal for rings.

Proof:

1. It suffice to prove an element is trivial on every localization then it is 0. For this, consider the annihilator $\mathrm{Ann}(x)$, it is not contained in any maximal ideal so it contains 1.

2. if $xf=0$ but $f\neq 0$, then $x\in Ann(f)\neq (1)$, so $Ann(f)\subset \mathfrak{m}$ maximal, so f is torsion in $M_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$. Conversely, if f is torsion in $R_{\mathfrak{m}}$, then it is clearly torsion over R.
3. We use the definition(I.7.1.1). Notice $(IM)_{\mathfrak{p}}=I_{\mathfrak{p}}M_{\mathfrak{q}}$ and every ideal of $R_{\mathfrak{p}}$ is of the form $I_{\mathfrak{p}}$. Then use the fact injective is stalkwise(I.5.1.27).
4. We use the definition(I.7.1.1). Notice $(I\otimes_R S)_{\mathfrak{q}}=I_{\mathfrak{p}}\otimes_{R_{\mathfrak{p}}}S_{\mathfrak{q}}$ for all primes $\mathfrak{q}$ of S and $\mathfrak{p}=\mathfrak{q}\cap R$. And every ideal of $R_{\mathfrak{p}}$ is of the form $I_{\mathfrak{p}}$. Then use the fact injective is stalkwise(I.5.1.27)
5. Because formally unramified is equivalent to $\Omega_{R/S}=0$(I.7.5.1), so we get the result by functorial properties of $\Omega_{S/R}$(VI.2.1.4) and triviality is stalkwise(I.5.1.27).
6. By definition.

□

Prop. (I.5.1.28) (Local Properties). For a fixed ring R,

1. Finite is a local property for modules over R.
2. F.p. is a local property for modules over R.
3. Noetherian is a local property for rings.
4. F.g. is a local property for rings over R, both on the source and target.
5. F.p. is a local property for rings over R, both on the source and target.
6. Smoothness for rings over R, both on the target and source.

Proof: Cf.[StackProject 00EO].

1.
2.
3.
4. If S_{g_i} is finite type over R, choose $\sum h_i g_i=1$. Let S_{g_i} be generated by $y_{ij}/g_i^{n_{ij}}$, then we claim S equals the subalgebra generated by generated by g_i,h_i,y_{ij}. For this, notice g_i generate the unit ideal in S', and $S'_{g_i}\to S_{g_i}$ is surjective by definition, so $S'\to S$ is surjective because surjection is stalkwise(I.5.1.27). It is local on the target because it is local on source and stable under composition and $R\to R_f$ is f.g..
5. By f.t. is local property(I.5.1.28), we know S is f.g. over R. The rest Cf.[StackProject 00EP]. It is local on the target by what we have already proved.
6. (I.7.4.15).

□

2 Spectra

Prop. (I.5.2.1) (Chevalley). The Spec map of a f.p. ring map maps constructible sets to constructible sets.

Proof: Cf.[StackProject 00FE]. □

Prop. (I.5.2.2). For $R\subset S$, all the minimal primes of R are in the image of the Spec map of a minimal prime of S.

Proof: Localize w.r.t. to the minimal prime $\mathfrak{p}$, then it is a local ring with only one prime. And $S_{\mathfrak{p}}$ is nonzero because localization is exact, so it has a maximal ideal $\mathfrak{q}$. Now we choose a minimal prime of S contained in $\mathfrak{q}$, then it is also mapped to $\mathfrak{p}$. $\square$

Idempotents

Prop. (I.5.2.3) (Clopen Subsets). The clopen subsets of $\operatorname{Spec} A$ corresponds to idempotents in A.

Proof: This is all equivalent to the fact that there exists $e+f=1, ef=0$:

If $A=U\coprod V$, then both U,V are closed hence qc, so $\operatorname{Spec} A=\cup V(f_i)\coprod \cup V(g_j)$, then f_ig_j is nilpotent by(I.5.7.1). Denote $I=(f_i), J=(g_j)$, then $(IJ)^N=0$ and $I+J=A$, there are $1=x+y$, $x\in I^N, y\in J^N$.

For uniqueness, if $e_1\neq e_2$, then $0\neq e_1-e_2=e_1(e_2+f_2)-e_2(e_1+f_1)=e_1f_2-e_2f_1$, so may assume $e_1f_2\neq 0$, and it is not nilpotent, so there is a $e_1f_2\subset\mathfrak{p}$, which is a contradiction. $\square$

Cor. (I.5.2.4). A local ring has no non-trivial idempotents, and then an idempotent is defined by the maximal ideals that it vanishes.

Cor. (I.5.2.5). If I is an ideal of R that $I=I^2$, and I is f.g., then $V(I)$ is open and closed in $\operatorname{Spec} R$, and $V(I)=R_e$ for some idempotent e.

Proof: By Nakayama, there is a a $f=1-e$ with $e\in I$ that $fI=0$. So $e-e^2=0$ and $f^2=f$. $V(I)=D(f)=D(e)$. $\square$

Going-up and down

Prop. (I.5.2.6). Integral injective ring extension satisfies going-up(I.5.4.5). Flat ring map satisfies going-down(I.7.1.23).

Prop. (I.5.2.7). Going-up and Going-down are stable under composition, trivially.

Prop. (I.5.2.8). If the image of the Spec map of a ring map is closed under specialization, then this image is closed.

Proof: Let it be $R\to S$, let I be the kernel, then the image is contained in $V(I)$, so we may replace R be R/I, then $R\subset S$, so by(I.5.3.10), the image contains all the minimal primes of R, then it is all of R, thus closed. $\square$

Cor. (I.5.2.9) (Going-up and Spec Closed). Going-up is equivalent to Spec closed.

Proof: If going-up holds, composing with a quotient map, it suffices to prove the image is closed, and this is by(I.5.2.8). Conversely, a closed map satisfies going-up, by(IV.1.13.4). $\square$

Prop. (I.5.2.10). If Spec map is open, then going-down holds.

Universal Homeomorphism

Cf.[StackProject 10.45] and [StackProject 28.44].

Prop. (I.5.2.11). If $\varphi : R \to S$ is a ring map and p is a prime number that satisfies:

- S is generated over R by elements x that there is n that $x^{p^n} \in \varphi(R)$ and $p^n x \in \varphi(x)$.
- $\mathrm{Ker}(\varphi)$ is locally nilpotent.

then $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is a homeomorphism, and any base change of φ satisfies the above conditions, so it is a universal homeomorphism.

In particular, this applies to any base change of a field extension k'/k that is purely inseparable, because it is f.f. hence injective.

Proof: Cf.[StackProject 0BRA]. □

3 Support and Associated Primes

Def. (I.5.3.1). The **support** $\mathrm{Supp}(M)$ of a module M is the set of all p that $M_p \neq 0$. When M is f.g., $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$.

The **associated primes** $\mathrm{Ass}(M)$ of a A-module M is the set of $p = \mathrm{Ann}(m)$. I is called **unmixed** if primes in $\mathrm{Ass}(A/I)$ don't contain each other, and of the same height.

Prop. (I.5.3.2). The support of a nonzero module is not empty, because triviality is stalkwise by(I.5.1.27).

Prop. (I.5.3.3). If $0 \to N \to M \to Q \to 0$, then we have $\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(Q)$, this is because localization is exact.

Prop. (I.5.3.4). For E, F f.g over a ring A, $\mathrm{Supp}(E \otimes F) = \mathrm{Supp}(E) \cap \mathrm{Supp}(F)$, this is because on a local ring A_p, $E \neq 0, F \neq 0 \to E \times F \neq 0$, which can be seen by passing to the residue field and use Nakayama.

Prop. (I.5.3.5). Let A be Noetherian and I be an ideal, then $I^n M = 0$ for some n iff $\mathrm{Supp}(M) \subset V(I)$.

Proof: If $I^n M = 0$, then if $I \not\subseteq P$, then $M_P = 0$. Conversely, we have a filtration of M, and by(I.5.3.3) we have all the P_i include I, so I^n annihilate M. □

Prop. (I.5.3.6) (Associated Primes and Exact Sequence). Note that $P \in \mathrm{Ass}(M)$ iff M contains a submodule isomorphic to A/P. So for an exact sequence $0 \to M_1 \to M \to M_2$, $\mathrm{Ass}(M) \subset \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2)$ and $\mathrm{Ass}(M_1) \subset \mathrm{Ass}(M)$. Hence for a f.g module over a Noetherian ring, $\mathrm{Ass}(M)$ is finite by(I.5.1.13).

Prop. (I.5.3.7) (Associated Primes and Support). $\mathrm{Ass} M \subset \mathrm{Supp}\, M$, and when R is Noetherian, their minimal elements are the same.

In particular, $Ass(M)$ is not empty by(I.5.3.2), and $\mathrm{Ass}(A/I)$ contains all the minimal primes over I.

Proof: If $\mathfrak{p} = \mathrm{Ann}(m)$, then m is nonzero in $M_\mathfrak{p}$, so $M_\mathfrak{p}$ is nonzero, i.e. $\mathfrak{p} \in \mathrm{Supp}(M)$.

For the second assertion, we first prove for M finite, and then write any module as sum of finite submodules, and use the fact Supp and ass are all unions of those of the submodules. Cf.[StackProject 02CE]. □

Prop. (I.5.3.8) (Associated Primes and Zero-divisors). When R is Noetherian and M a R-module, the union of the associated primes of M is the set of zero-divisors in M.

Proof: Elements in associated points are zero-divisors obviously, and conversely, if $xm = 0$, then $x \in \mathrm{Ann}(m)$ and $\mathrm{Ann}(m)$ has an associated point $\mathfrak{q}$ by(I.5.3.7). Now x must be in $\mathfrak{q}$ and $\mathfrak{q}$ is also an associated point of M by(I.5.3.7). □

Cor. (I.5.3.9). Use the prime avoidance(I.5.1.1), we can prove if R is Noetherian and M is a finite R-module, then $I \subset \mathfrak{p}$ for some $\mathfrak{p} \in Ass(M)$ iff I consists of zero-divisors.

Prop. (I.5.3.10) (Associated Primes and Maps). For a ring map $\varphi : R \to S$ and a S-module M, then $\mathrm{Spec}(\varphi)(Ass_S(M)) \subset Ass_R(M)$, and equal if S is Noetherian.

Proof: We prove it is equal. If $\mathfrak{p} = \mathrm{Ann}_R(m)$, then we let $I = \mathrm{Ann}_S(m)$, then $R/\mathfrak{p} \subset S/I \subset M$, so by(I.5.3.10), there is a minimal prime of S over I that are mapped to $\mathfrak{p}$, now this prime is in $Ass(S/I)$ by(I.5.3.7) and also in $Ass_S(M)$ by(I.5.3.6). □

Prop. (I.5.3.11) (Associated Primes and Localization). Let A is Noetherian and φ : $\mathrm{Spec}\, S^{-1}A \to \mathrm{Spec}\, A$, if M is a A-module, then

$$\mathrm{Ass}_A(S^{-1}M) = \varphi(\mathrm{Ass}_{S^{-1}A}(S^{-1}M)) = \mathrm{Ass}_A(M) \cap \{p | p \cap S = \varnothing\}.$$

Proof: The first equality is by(I.5.3.10). For the second, if $\mathrm{Ann}_A(x) = \mathfrak{p}$ and $\mathfrak{p} \cap S = \varnothing$, then $\mathrm{Ann}_{S^{-1}A}(x/1) = S^{-1}(\mathfrak{p})$. Conversely, if $\mathrm{Ann}_{S^{-1}A}(x/s) = S^{-1}\mathfrak{p}$, then $\mathfrak{p} \cap S = \varnothing$, and $\mathrm{Ann}_A(x) \subset S^{-1}\mathfrak{p} \cap A = \mathfrak{p}$. □

Prop. (I.5.3.12). If R is Noetherian and M is a R-module, then $M \to \prod_{\mathfrak{p} \subset Ass(M)} M_\mathfrak{p}$ is injective.

Proof: Notice $(x) \subset M$, if (x) is nonzero, then there is an associated prime $\mathfrak{p}$ of (x)(I.5.3.7), then it is an associated prime of M, and then $(x)_\mathfrak{p} \subset M_\mathfrak{p}$ is not zero, contradiction. □

Prop. (I.5.3.13). An associated point that is not minimal among them is called a **embedded point**. An embedded point correspond to a nilpotent element, because $px = 0$ is contained in every minimal element but p is not, so x is contained in every minimal prime ideal.

Cor. (I.5.3.14) (Reduced Ring No Embedded Primes). A reduced ring has no embedded primes, because it has no nilpotent elements. Hence all its associated primes are just the minimal primes.

Primary Decomposition

Def. (I.5.3.15). For R Noetherian, a R-module M is called **coprimary** iff it has only one associated primes. A submodule N of M is called p**-primary** iff $Ass(M/N) = \{p\}$. A ring is called p**-primary** iff (0) is p-primary.

Notice coprimary is equivalent to the following: if $a \in A$ is a zero divisor for M, then for each $x \in M$, there is a n that $a^n x = 0$, i.e. **locally nilpotent**. And for ideals in a Noetherian ring, this is equivalent to $r(I)$ is a prime.

Proof: If M is p-primary, if $x \in M$ is nonzero, then $Ass(Rx) = \{p\}$, so p is the unique minimal element of $\mathrm{Supp}(Rx) = V(\mathrm{Ann}(x))$ by(I.5.3.7). So p is the radical of $\mathrm{Ann}(x)$, i.e. $a^n x = 0$ for some n(I.5.7.1).

Conversely, we know the ideal p of locally nilpotent elements equals the union of the associated primes(I.5.3.8), so if $q \in AssM = \mathrm{Ann}(x)$, then by definition, $p \subset q$. So $p = q$, and thus $AssM = \{p\}$. □

Lemma (I.5.3.16). A primary ring has no nontrivial idempotent element, because e and $1-e$ will all belong to the same minimal ideal p.

Lemma (I.5.3.17). The intersection of p-primary submodules are p-primary. (Because there is a injection $M/Q_1 \cap Q_2 \to M/Q_1 \oplus M/Q_2$).

Lemma (I.5.3.18) (Associated Prime and Primary Decomposition). If $N = \cap Q_i$ is an irredundant primary decomposition and if Q_i belongs to p_i, then we have $Ass(M/N) = \{p_1, \ldots, p_r\}$.

Proof: There is a injection $M/N \to M/Q_1 \oplus \ldots M/Q_r$ which shows $Ass(M/N) \subset \{p_1, \ldots, p_r\}$. And for the inverse, notice $Q_2 \cap \ldots \cap Q_r/N$ is a submodule of M/Q_1, which shows $Ass(Q_2 \cap \ldots \cap Q_r/N) = \{p_1\}$ by(I.5.3.7). □

Prop. (I.5.3.19). If N is a p-primary submodule of a R-module M, and p' is a prime ideal, then

- $N_{p'} = M_{p'}$ if $p \not\subseteq p'$.
- $N = M \cap N_{p'}$ if $p \subset p'$.

Proof: $M_{p'}/N_{p'} = (M/N)_{p'}$, and $Ass((M/N)_{p'}) = Ass(M/N) \cap \{\text{primes contained in } p'\} = \varnothing$ by(I.5.3.11). So $M_{p'} = M_{p'}$ by(I.5.3.7).

For the second, notice it suffices to show $M/N \to M_{p'}/N_{p'}$ is injective. But this is because $A - p'$ contains no nonzero-divisor, by(I.5.3.8). □

Cor. (I.5.3.20) (Second Uniqueness of Primary Decomposition). For an irredundant primary decomposition $N = \cap Q_i$, if Q_1 corresponds to p_1 and p_1 is minimal in $Ass(M/N)$, then $Q_1 = M \cap N_{p_1}$. In particular, the minimal prime part of a irredundant primary decomposition is uniquely determined.

Proof: By the above proposition, there are elements u_i of Q_i, i$\neq 1$ that are mapped to units in M_{p_1}, so $Q_1 \cdot u_2 u_3 \ldots u_r$ is mapped onto the image of $Q_1 \to M_{p_1}$. Then $Q_1 = M \cap (Q_1)_{p_1} = M \cap N_{p_1}$.
□

Prop. (I.5.3.21). If R is Noetherian and M is a R-module, there are p-primary submodules $Q(p)$ for each $p \in \mathrm{Ass}M$ that $(0) = \bigcap_{p \in \mathrm{Ass}M} Q(p)$.

Proof: For a $p \in AssM$, we seek $Q(p)$ to be the maximal submodule N that $p \notin AssN$. This has a maximal ideal because of Zorn and the fact $Ass(\cup N_\lambda) = \cup Ass(N_\lambda)$. Then We have $\mathrm{Ass}(M/Q(p)) = \{p\}$, otherwise there is another p', then there is a $Q'/Q(p) \cong A/p'$. Now Q' is bigger than $Q(p)$. Finally, $(0) = \bigcap_{p \in \mathrm{Ass}M} Q(p)$ because it has no associated primes. □

Cor. (I.5.3.22) (Primary Decomposition). If M is f.g. over a Noetherian ring R, then any submodule has a primary decomposition. (Notice M has only f.m. associated primes).

Def. (I.5.3.23) (Symbolic Power). For a prime ideal in a Noetherian ring, The n-th **symbolic power** $p^{(n)}$ of p is defined to be the p-primary component of p^n, who has only one minimal prime(hence one associated prime). The symbolic power is giving by $p^n A_p \cap A$ by(I.5.3.20).

4 Integral Extension

Def. (I.5.4.1) (Totally Integrally Closed). For two rings $A \to B$, $f \in B$ is called **almost integral**(or totally integral when almost mathematics is performed:)) over A if $f^{\mathbb{N}}$ lies in a f.g. A-module of B. It is clear that the elements of totally integral elements of B is a subring. And A is called **totally integrally closed** in B iff any $f \in B$ totally integral over A is in A.

Prop. (I.5.4.2). For a ring map $\varphi : A \to B$, an element x is integral over A iff x is contained in a finite A-module in B. In particular, the elements of B that are integral over A is a ring containing $\varphi(A)$.

Proof: If x is integral, then $\varphi(A)[x]$ is finite. If $\varphi(A)[x]$ is finite, then there is a set of generators of polynomials in x. Then for m large, $x^m = \sum a_i f_i(x)$, so x is integral over A. □

Prop. (I.5.4.3). For $A \subset B$, if B is integral over A, then A is a field iff B is a field.

Proof: If A is a field, $y^{-1} = -a_n^{-1}(y^{n-1} + \ldots + a_{n-1}) \in B$. If B is a field, $x^{-1} = -(b_1 + b_2 x + \ldots + b_m x^{m-1}) \in A$. □

Cor. (I.5.4.4). If B is integral over A, then a prime p of B is maximal iff $p \cap A$ is maximal.

Proof: Look at the integral extension $A/p \cap A \to B/p$. □

Prop. (I.5.4.5). Let $A \to B$ integral. Then:

1. There is no inclusion relation between prime ideals of B lying over a fixed prime ideal of A.
2. The Spec map is surjective.
3. The going-up holds.

Proof:

1. If $p \cap A = p' \cap A = q$, Localize at q, then p, p' are both maximal ideals of B_q, they cannot contain each other.
2. For any prime q of A, since $A_q \neq 0$, $B_q \neq 0$ (it contains 1), so it has a maximal ideal(I.5.1.2).
3. for any $q_1 \subset q_2$ Localize at q_1 and use 2.

□

Cor. (I.5.4.6). In particular, The Spec map of an integral ring map is closed, by(I.5.2.9) and(I.5.4.5).

Prop. (I.5.4.7). Let A a subring of $B, A \to B$ integral Noetherian. Then:

1. $\dim(A) = \dim(B)$
2. $\mathrm{ht(P)} = \mathrm{ht(P \cap A)}$
3. If going up holds, then $\mathrm{ht(J)} = \mathrm{ht(J \cap A)}$ for any ideal J.

Proof: 1:By the preceding lemma, there is no inclusion relation between prime over a fixed prime, so $\dim(B) \leq \dim(A)$. On the other hands, going-up holds, so $\dim(B) \geq \dim(A)$.

2:Follows from (I.5.6.5)(1) since $\mathrm{ht(P/(P \cap A)B)} = 0$ by the preceding lemma.

3:by 2 and surjectiveness of Spec for integral extension. □

5 Graded Ring & Completion

Cf.[Matsumura Ch11].

Def. (I.5.5.1). A **graded ring** is a ring $A = \oplus_{n=0}^{\infty} A_n$ that $A_m A_n \subset A_{m+n}$. A **graded module** over a graded ring A is a module $M = \oplus_{n=0}^{\infty} M_n$ that $A_m M_n \subset M_{m+n}$.

Def. (I.5.5.2). Let A be a ring and $\mathfrak{a}$ be an ideal of A, an **$\mathfrak{a}$-filtration** of M is a descending sequence $M = M_0 \supset M_1 \supset \ldots$ that $\mathfrak{a}M_n \subset M_{n+1}$. It is called a **stable filtration** iff there is an N that $\mathfrak{a}M_n = M_{n+1}$ for $n \geq N$.

For an ideal $\mathfrak{a} \subset A$, there can be associated a graded ring $A^* = \oplus \mathfrak{a}^n$, and an $\mathfrak{a}$-filtration M can be associated a graded module over $A^* : M^* = \oplus M_n$. When A is Noetherian, then so is A^*, because it is a quotient of a polynomial ring over A(I.5.1.8).

Lemma (I.5.5.3). If A is a Noetherian ring and M is a f.g. A-module that has a $\mathfrak{a}$-filtration M_n, then M^* is f.g. over A^* iff M_n is a stable filtration.

Proof: As every M_n is finite over A, if it is stable, then M^* is generated over A^* by all the generator of $M_n, n \leq N$, so it is f.g.. Conversely, if it is f.g., then it is clear that M_n is a stable filtration. □

Prop. (I.5.5.4) (Artin-Rees). For A Noetherian and I an ideal, let $N \subset M$ be <u>finite</u> A-modules, then if M_n is a stable filtration of M, then $M_n \cap N$ is a stable filtration of N.

In particular, let $M_n = I^n M$, then $I^n M \cap N = I^{n-r}(I^r M \cap N)$, hence the I-adic topology on M induce the I-adic topology on N.

Proof: This is immediate from the lemma above, as N^* is an A^*-submodule of M^*, and A^* is Noetherian(I.5.5.2). □

Cor. (I.5.5.5) (Intersection Theorem). Notation as above, let $N = \cap^{\infty} I^n M$, then the I-adic topology on N is trivial, by Artin-Rees, $IN = N$. So if $I \subset \mathrm{rad}(A)$, Nakayama tells us $N = 0$. This can be used to use induction to prove some theorem.

Cor. (I.5.5.6) (Krull). For A Noetherian, if $I \subset \mathrm{rad}(A)$ or A is a domain, then $\cap^{\infty} I^n = 0$.

Prop. (I.5.5.7). Notice for any ring A and a non-zero-divisor f, if $I = \cap_n f^n A$, then $fI = I$, needless of the Noetherian property.

Proof: If $x \in I$, $x = fy$, because $x \in f^n A$, $fy = f^n t$ for some t, so $y = f^{n-1}t$, so $f \in I$. Thus $I = fI$. □

Def. (I.5.5.8) (Hilbert-Serre). Let A be an Artinian ring and $B = A[X_i]$. For a f.g. graded B-module $\oplus M_n$, we have $l(M_n)$ is a polynomial of n for n big, called the **Hilbert Polynomial**. Its degree is $\mathrm{Supp}\, M$.

Proof: The case when A is a field follows from(III.5.4.17). Cf.[Hartshorne P51]. □

Prop. (I.5.5.9) (Hilbert Polynomial and Dimension). For a Noetherian local ring A, the Hilbert polynomial of a f.g. module M w.r.t $\mathfrak{m}$ has degree $\dim M$. And $\dim M$ is the smallest integer r s.t. there exists $x_1, \ldots, x_r$ that $l(M/x_1 M + \ldots, x_r M) < \infty$.

Proof: Cf.[Mat P76]. □

Completion

Prop. (I.5.5.10). Let the topology on a A-module be defined by countable filtration of submodules, then iff M is complete, then M/N is complete in the quotient topology.

Proof: Write $x_{i+1} - x_i = y_i + z_i$ with $y_n \in M_n$ and $z_n \in N$, then the image of the limit of $\sum y_i$ is the limit of $\overline{x_i}$. □

Prop. (I.5.5.11). For a local ring map of two power series map, it is an isomorphism iff its Jacobian is invertible.

Proof: □

Def. (I.5.5.12). The **completion** of a topological A-module is a functor $\varphi : M \to M'$ that are left adjoint to the forgetful functor from the category of complete Hausdorff A-modules. It is defined as composition of the Hausdorffization functor followed by $\lim M/M_n$ with the topology like that of profinite groups.

The completion is right exact. For left exactness, notice the limit process is exact, but the Hasudorffization can go astray.

Prop. (I.5.5.13). The completion of a submodule $N \subset M$ is the closure of $\varphi(N)$ (By direct construction). The completion of M/N is M^*/N^* because it is right exact.

Cor. (I.5.5.14). If N is open in M then $M/N \cong M^*/N^*$ because M/N is discrete hence complete Hausdorff.

Prop. (I.5.5.15). When N is finite, $0 \to N^* \to M^* \to (M/N)^* \to 0$ is exact, because the Hasfdorffization of N embeds in that of M by Artin-Rees.

Prop. (I.5.5.16). When A is Noetherian and M is finite A-module, then the natural map $M \otimes_A A^* \to M^*$ is an isomorphism (use M is finite presentation and tensor & completion is right exact), and five lemma.

Cor. (I.5.5.17). When A is Noetherian, A^*/A is flat (because flatness is check for finite module), and when A is complete Hausdorff, any finite module M is complete Hausdorff and hence any its submodule is complete thus closed in it. Hence the the completion of a submodule $N \subset M$ is $\varphi(N)A^*$ in $M^* = MA^*$. In fact this implies complete Hausdorff adic-ring is Zariski.

Prop. (I.5.5.18). Let A be a ring with a non-zero-divisor t, then any limits of t-adically complete ring is t-adically complete.

Proof: Check the definition directly. □

Prop. (I.5.5.19) (t-adically Complete). If A is a ring with a nonzero-divisor t and there is an exact sequence $0 \to M \to N \to Q \to 0$ of A-modules, if M, N is t-torsion-free and $tQ = 0$, then M is t-adically complete iff N is adically complete.

Proof: Consider multiplying by t^n, use snake lemma and conditions, we can get an exact sequence $0 \to Q \to M/t^n \to N/t^n \to Q \to 0$, now take inverse image, notice the $Q \to Q$ term is multiplying by t, so its inverse image is 0 and we get an exact sequence $0 \to \widehat{M} \to \widehat{N} \to Q \to 0$ by the left exactness of inverse image and right exactness of completion. Thus use snake lemma again, it is clear that $M \cong \widehat{M}$ iff $N \cong \widehat{N}$. □

Prop. (I.5.5.20). A Noetherian I-adic ring is called **Zariski ring** if it satisfies the following equivalent conditions:

- Every finite module is Hausdorff in the I-adic topology.
- Every submodule in a finite module is closed in the I-adic topology.
- Every ideal is closed.
- $I \subset \mathrm{rad}A$.
- A^*/A is f.f.

Hence every complete Hausdorff ring is Zariski.

Proof: $1 \to 2$:apply it to the submodule M/N.

$3 \to 4$:If $I \not\subseteq m$, then $I^n + m = A$, thus $\overline{M} = A$, contradiction.

$4 \to 1$: by intersection theorem(I.5.5.5).

$4 \to 5$: for any maximal ideal m, $I \subset m$ so it is open, thus $A^*/mA^* = A/m \neq 0$ by(I.5.5.14) thus f.f. by(I.7.1.11).

$5 \to 1$:by(I.7.1.12), for any m maximal, there is a maximal ideal m' lying over m, so $IA^* \subset m^*$ by(I.5.5.17), thus $I \subset m$, hence $I \subset \mathrm{rad}A$. □

Cor. (I.5.5.21). In a Zariski ring A, maximal ideals are open, thus $A/m \cong A^*/mA^*$ by(I.5.5.14), thus $\mathrm{Spec}\, A^* \to \mathrm{Spec}\, A$ is bijection on closed pt.

Prop. (I.5.5.22) (Cohen Structure Theorem). If A is a complete local ring containing a field k that the residue field is separably generated over k, then there is a field K containing k that is a Cohen ring, i.e. complete local ring with a prime number as a uniformizer, that has the same residue field as A.

Proof: □

Prop. (I.5.5.23) (Beauville-Laszlo). Let A be a Noetherian ring and $f \in A$, let $\widehat{A}$ be the f-adic completion, then there is a pullback diagram of categories:

$$\begin{array}{ccc} A - Mod & \longrightarrow & A[\frac{1}{f}] - Mod \\ \downarrow & & \downarrow \\ \widehat{A} - Mod & \longrightarrow & \widehat{A}[\frac{1}{f}] - Mod \end{array}$$

Proof: Cf.[StackProject Ch15.81]. □

6 Dimension

Def. (I.5.6.1). For a A-module M, $\dim(M)$ is defined as $\dim(A/\mathrm{Ann}(M))$.

The **height** of an ideal I in A is defined as the height of the minimal prime ideal over I.

Prop. (I.5.6.2). For a Noetherian ring A, $\dim A = \sup \dim A_p$, because X is Noetherian hence has f.m. minimal primes.

Def. (I.5.6.3). A ring is called **universally catenary** if all its f.g. algebra is catenary, i.e. the dimension behave well.

Dedekind domain, e.g. field is universally catenary, so f.g. domain over fields is catenary.

Prop. (I.5.6.4). If A is a Noetherian local ring with maximal ideal m, then $\dim A \leq \dim_k m/m^2$. Cf.[Matsumura P78].

Prop. (I.5.6.5) (Dimension Extension Formula). Let $A \to B$ Noetherian, let $p = P \cap A$,then:

- ht(P) $\leq$ ht(p) + ht(P/pB),in other words $\dim(B_P) \leq \dim(A_p) + \dim(B_P \otimes k(p))$. Where $k(p) = A_p/pA_p$ and $B \otimes k(p) = B_p/pB_p$.
- equality holds if going-down holds. For example, if it is flat.
- if Spec map is surjective and going-down holds, then we have i)$\dim B \geq \dim A$, and ii)ht(I) = ht(IB) for ideal I of A.
- if going-up holds, then $\dim B \geq \dim A$. e.g. B integral over A Cf.(I.5.4.7)

Proof: Cf.[Commutative Algebra Matsumura (13.B)]. □

Lemma (I.5.6.6). For a Noetherian ring A, $\dim A[X] = \dim A + 1$.

Proof: Cf.[Matsumura P83]. □

Prop. (I.5.6.7). If $R \to S$ is a map of Noetherian rings, if the going down property holds, then $\dim S_q = \dim R_p + \dim S_q/pS_q$ for a prime q of S over p.

Proof: Cf.[StackProject 00ON]. □

Prop. (I.5.6.8) (Noetherian Normalization Theorem). If A is a f.g. algebra over a field. then there are r alg. independent elements y_i that A is integral over $k[y_i]$. Also $\dim A = \text{tr.deg}\, A$ in case A is integral because integral extension of integral Noetherian ring has the same dimension(I.5.4.7).

Proof: Cf.[Commutative Algebra Matsumura P91]. □

Cor. (I.5.6.9) (Krull's Height Theorem). In a Noetherian domain, the height of an ideal generated by n elements is at most n.

Proof: Cf.[StackProject 0BBZ]. □

Prop. (I.5.6.10) (Completion). For a local ring A, $\dim A = \dim \hat{A}$.

Proof: □

7 Jacobson and Nilradical

Def. (I.5.7.1). The **Jacobson radical** $J = \mathrm{rad}(R)$ is the intersection of all maximal primes of R. $J = \{r \in R | 1 + rs \text{ is a unit } \forall \text{s} \in \text{R}\}$.

The **nilradical** is the intersection of all primes. It consists of nilpotent elements.

Proof: Jacobson: One way is trivial and for the other if r is not in a maximal ideal $\mathfrak{m}$, then $(r) + \mathfrak{m} = (1)$, so contradiction.

Nilpotent: Every nilpotent element is contained in every prime, and if a is not nilpotent, then A_a is nonzero, hence there is a maximal ideal, i.e. there is a prime of A not containing a. □

Lemma (I.5.7.2). If I is a locally nilpotent ideal, then $R \to R/I$ induces a bijection on idempotents.

Proof: Because $R \to R/I$ induces a bijection on the spectra, and clopen subsets of the spectrum corresponds to the idempotents(I.5.2.3). □

Jacobson Ring

Def. (I.5.7.3). A commutative ring is called **Jacobson** if every prime ideal is an intersection of maximal ideals. In particular, the Jacobson radical equals the nilradical. This is equivalent to every radical ideal is an intersection of maximal primes.

Prop. (I.5.7.4). R is Jacobson iff $\mathrm{Spec}\, R$ is Jacobson space(IV.1.13.16). In particular, the closed pts are dense in any closed subsets (Hilbert's Nullstellensatz satisfied).

Proof: We need to show that a locally closed subset contains a closed pt, we assume this set is of the form $V(I) \cap D(f)$, I is radical, then $f \notin I$, then by the condition, there is a $I \subset \mathfrak{m}$ that $f \notin \mathfrak{m}$, thus the result.

Conversely, for a radical ideal, let $J = \cap_{I \subset \mathfrak{m}} \mathfrak{m}$, then J is radical and $V(J)$ is the closure of $V(I) \cap X_0$, $V(I) = V(J)$, and because they are both radical, $I = J$. □

Cor. (I.5.7.5). Being Jacobson is a local property, and quotient of Jacobson ring is Jacobson, and maximal ideals of R_f are maximal in R. (Immediate from(I.5.7.4)(IV.1.13.17) and(IV.1.13.18)).

Lemma (I.5.7.6) (Hilbert's Nullstellensatz).
1. For any maximal ideal $\mathfrak{m}$ of $k[X_1, \ldots, X_n]$, the field extension $k(\mathfrak{m})/k$ is a finite field extension.
2. $R = k[X_1, \ldots, X_n]$ is Jacobson.

The same is true for any f.g. algebra over a field k.

Proof: It suffices to prove the polynomial case, because any f.g. algebra is a quotient of such. Use induction on n, consider $\mathfrak{p} = k[X_n] \cap \mathfrak{m}$. If $\mathfrak{p} \neq (0)$, then it is maximal, because $\dim k[X_n] = 1$(I.5.6.6), and the quotient field k' is finite over k because $\mathfrak{p}$ is generated by an irreducible polynomial. Then $k(m)$ is in fact a quotient of $k'[X_1, \ldots, X_{n-1}]$, so it is finite over k' hence finite over k.

If $\mathfrak{p} = 0$, then consider $k[X_n] \subset k[X_1, \ldots, X_n]/\mathfrak{m}$. It is of finite presentation, so by Chevalley(I.5.2.1), the image of the Spec map is constructible, and it contains (0), which is generic, so it contains a standard open $D(f)$. But $D(f)$ is infinite, which contradicts the fact that $\mathrm{Spec}\, k[X_1, \ldots, X_n]/\mathfrak{m}$ has only one point.

For the second assertion, if I is a radical ideal, and $f \notin I$, choose a maximal ideal m' in the ring R_f/IR_f, let $m = m' \cap R$, then $I \subset m$ and $f \notin m$. If we show m is maximal, then the proof is finished. It is clear that $k \subset R/m \subset k(m')$, and $k(m')/k$ is finite, so R/m is a field by(I.3.5.7). □

Lemma (I.5.7.7). If R is a ring and $R \subset K$ where K is field, and K is f.g. over R, then there is a f that R_f is a field and K/R_f is a finite field extension.

Proof: Cf.[StackProject 00FY]. □

Lemma (I.5.7.8). If R is a Jacobson ring and $R \subset K$ where K is a field, and K is f.g. over R, then R is a field and K/R is a finite field extension.

Proof: By(I.5.7.7), there is a f that R_f is a field and K/R_f is a finite field extension, hence (0) is a maximal ideal for R_f, but then by(I.5.7.5), (0) is also a maximal ideal of R. So R is a field and $R = R_f$. □

Prop. (I.5.7.9) (Generalized Nullstellensatz). If R is Jacobson and S is a finitely generated R-algebra, then:

- The maximal ideal of S intersect with R a maximal ideal, and the quotient ring extension is finite, (in particular algebraic).
- S is Jacobson.

In particular, a f.g. algebra over a ring of dimension 0, (e.g. Artinian ring or field) is Jacobson.

Proof: If $\mathfrak{m}$ is maximal in S, then $R/\mathfrak{m}\cap R \to S/\mathfrak{m}$ satisfies the condition of(I.5.7.8), by(I.5.7.5), so the last two assertions are proved.

Now we show S is Jacobson. Cf.[StackProject 00GB]. □

Zariski Pairs

Def. (I.5.7.10). A pair (A, I) is called a **Zariski pair** iff I is contained in the Jacobson radical of A.

Prop. (I.5.7.11). If (A, I) is a Zariski pair, then the map $A \to A/I$ induces a bijection between the idempotents.

Proof: idempotents are determined by the maximal ideals that it vanishes(I.5.2.4), and $A \to A/I$ induces a bijection on the maximal ideals. □

8 Dedekind Domain

Def. (I.5.8.1). A **Dedekind domain** is an integrally closed Noetherian domain of dimension 1. A PID is a Dedekind domain by(I.3.3.10)(I.3.3.16).

Prop. (I.5.8.2) (Equivalent Definitions of Dedekind Domain). For a domain R, the following are equivalent:

1. R is a Dedekind domain.
2. R is Noetherian and each $R_\mathfrak{m}$ is DVR for maximal ideal $\mathfrak{m}$.
3. each ideal of R can be written as a product of prime ideals uniquely.

Proof: $1 \iff 2$ as normal is a stalkwise and(I.6.5.11).

$3 \to 2$: If 3 is true, then $\mathfrak{p} \neq \mathfrak{p}^2$ for each prime $\mathfrak{p}$, so choose $x \in \mathfrak{p} - \mathfrak{p}^2$, then for each $y \in \mathfrak{p}$, $(x, y) = \prod p_i$, then exactly one p_i(may assume p_1) is contained in p, so $(x, y)R_p = p_1 R_p$. Now in fact $(x)R_p = p_1 R_p$, because $(x, y^2)R_p$ is also a prime, so $y = ax + by^2$ in R_p, $(1 - by)y = ax \in (x)R_p$ is a prime, so $y \in (x)R_p$. This is for all $y \in p$, so $(x)R_p = p_1 R_p$.

Now if $(x) = \mathfrak{p}_1 \cdots p_r$, then $p_1 R_p = pR_p$, so $p_1 = p$, and p is f.g. by the lemma(I.5.8.3) below. $\mathfrak{p}$ is arbitrary, so R is Noetherian and $R_\mathfrak{m}$ is DVR for $\mathfrak{m}$ maximal, by(I.6.5.11).

$1 \to 3$: if 1 is true, then any ideal is a unique intersection of primary ideals, and primary ideals are their radical are different, so they are coprime(I.5.1.3), so this is in fact a unique decomposition into products of primary ideals. And any primary ideal is a power of its radical, because this is the case after localization. □

Lemma (I.5.8.3). I, J be ideals in a ring A and $IJ = (f)$ where f is a non-zero-divisor, then I, J are f.g. and finitely locally free of rank 1 as A-modules.

Proof: The second assertion implies the first, by(I.6.1.4). $f = \sum x_i y_i$, and $x_i y_i = a_i f$, so $\sum a_i = 1$ as f is non-zero-divisor. Now we show I_{a_i} as J_{a_i} is free of rank 1. Now after localization, $f = xy$, so x, y are non zero divisors. Now if $x' \in I$, then $x'y = af = axy$ for some a, so $x' = ax$. □

Prop. (I.5.8.4) (Extension of Dedekind Domain). If A is a Noetherian domain of dimension 1 with fraction field K and L/K is a finite field extension, then the integral closure B of A in L is a Dedekind domain, and $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is surjective, and have finite fibers and induces finite residue field extension.

Proof: Cf.[StackProject 09IG]. $\square$

Cor. (I.5.8.5). The integral closure of a Dedekind domain in a finite extension fields of its quotient fields is again a Dedekind domain.

In particular, the ring of integers in a number field is a Dedekind domain.3

Cor. (I.5.8.6). The ring of integers in an algebraic number field K is a Dedekind domain.

Prop. (I.5.8.7) (Flat Module over a Dedekind Domain). If A is a Dedekind domain, then an A-module is flat iff it is torsion-free.

Proof: Because flatness and torsion-freeness is stalkwise(I.5.1.27), so it suffices to prove for its localization, which is DVR(I.5.8.2), so the result follows from(I.7.1.8). $\square$

Fractional Ideals

Def. (I.5.8.8). For A an integral domain and K its quotient field, then an A-module M in K is called a **fractional ideal** if $xM \subset A$ for some $x \neq 0$.

Every f.g. submodule in K is a fractional ideal, and if A is Noetherian, then the converse is true, because it is of the form $x^{-1}\mathfrak{a}$.

Prop. (I.5.8.9). An A-submodule M of K is called an **invertible ideal** if there is a submodule N that $MN = A$. It follows that M, N are f.g., because there are $\sum x_i y_i = 1$, so M is generated by x_i and N is generated by y_i.

Prop. (I.5.8.10). Invertibility is a stalkwise property.

Proof: Notice $(A : M)_{\mathfrak{p}} = (A_{\mathfrak{p}} : M_{\mathfrak{p}})$, and M is invertible iff $M(A : M) = A$. Then use the fact isomorphism is stalkwise(I.5.1.27). $\square$

Prop. (I.5.8.11). A local domain is a DVR iff every non-zero fractional ideal of A is invertible.

Proof: If is a DVR, let $\mathfrak{m} = (x)$, for any fractional ideal $M <$ let $yM \subset A = (x^r)$, then $M = (x^{r-s})$, where $v(y) = s$. Conversely, if every non-zero fractional ideal of A is invertible, then they are all f.g.(I.5.8.9), so A is Noetherian. Now it suffices to prove that every ideal of A is a power of $\mathfrak{m}$, by(I.6.5.11). If this is not true, choose a maximal element $\mathfrak{a}$ in the set of ideals that is not a power of $\mathfrak{m}$(by Noetherian), then $\mathfrak{m}^{-1}\mathfrak{a} \subset \mathfrak{m}^{-1}\mathfrak{m} = A$, and $\mathfrak{m}^{-1}\mathfrak{a} \supset \mathfrak{a}$, but it is not $\mathfrak{a}$, so $\mathfrak{m}^{-1}\mathfrak{a} = \mathfrak{m}^k$ for some k, so $\mathfrak{a} = \mathfrak{m}^{k+1}$, contradiciton. $\square$

Cor. (I.5.8.12). An integral domain is a Dedekind domain iff every non-zero fractional ideal is invertible.

Proof: Immediate from the proposition and(I.5.8.2)(I.5.8.10). $\square$

I.6 Commutative Algebra(Matsumura)

1 Projective

References are [Projective Modules].

Prop. (I.6.1.1). Localization and tensor product preserves projective because they are left adjoints.

And when tensoring f.f. map, then the converse is also true(I.7.2.1).

Prop. (I.6.1.2). A module over a ring is projective iff it is a direct summand of a free module, in particular, it is flat. Moreover, there is a free module Q that $P \oplus Q = F$ free.

Proof: For the second assertion, we can choose an arbitrary Q that $P\oplus Q$ free, and see $\bigoplus_{\mathbb{N}}(P\oplus Q)$ is free. □

Prop. (I.6.1.3) (Projective over Local Ring). A projective module over a local ring or a PID is free.

Proof: Local ring case: We only prove for P finite, the infinite case is proved in [Kaplansky. Projective modules]. Choose a set of generators of minimal number n, then $R^n = P \oplus N$, pass to the quotient field, we have $k^n = P/mP \oplus N/mN$. P/mP has rank m by Nakayama, thus $N/mN = 0$, thus $N = 0$ by Nakayama.

PID case: directly from(I.3.4.15). □

Prop. (I.6.1.4) (Finite Projective,Locally Free,Flat,F.P.). Let M be a R-module, the following are equivalent:

1. M is finite projective.
2. M is f.p. and flat.
3. M is f.p. and all its localizations at (maximal)primes are free.
4. M is finite locally free.
5. M is finite and locally free.
6. M is finite and all its localizations at primes are free and the function $p \to \dim_{k(p)} M \otimes_R k(p)$ is a locally constant function on $\operatorname{Spec} R$.

Proof: $1 \to 2$: $M \otimes K = R^m$ for some K and m, so K is finite and $M = R^m/K$ is f.p. And M is flat because it is a summand of R^n(I.7.1.3).

$2 \to 4$: For any prime p, choose a basis for the $k(p)$-vector space $M \otimes k(p)$, then by Nakayama, their inverse image generate M_g for some $g \notin p$(I.3.4.8), and the kernel K of this generation is finite because M_g is f.p. And $K \otimes k(p) = 0$ by the flatness of M_g. Then by Nakayama again there is a $g' \notin p$ that $M_{gg'} = 0$(I.3.4.8).

$4 \to 3$: Because f.p. is local(I.5.1.28).

$3 \to 2$: Because flatness is trivial.

$4 \to 5$: Because finite is local(I.5.1.28).

$5 \to 4, 4 \to 6$: Trivial.

$6 \to 4$: Cf.[StackProject 00NX]

$2 + 3 + 4 + 5 + 6 \to 1$: Cf.[StackProject 00NX].

Consider the stalk, it is all free by(I.6.1.1) and(I.6.1.3), thus by(III.2.3.10), it is locally free. □

Cor. (I.6.1.5) (Partially Stalkwise). If P is fo f.p., then finite projectiveness is a stalkwise property for P.

Cor. (I.6.1.6) (Projective and Flat). A finite module over a Noetherian ring is projective iff it is flat.

Cor. (I.6.1.7). If M is finite projective, then the canonical map $\mathrm{Hom}(M,N)\otimes L \to \mathrm{Hom}(M,N\otimes L)$ is an isomorphism.

Proof: By proposition above M is f.p. and finite locally free, so by(I.6.7.6) and tensor commutes with localization, we can check locally, where M is finite free so the isomorphism is obvious. □

Duality of Projective Modules

Prop. (I.6.1.8) (Basis Criterion of Projectiveness). An A-module P is projective iff there are elements x_i in P and f_i in P^* that for any x, $f_i(x)=0$ a.e. i, and $\sum f_i(x)x_i = x$. Moreover, P is finite projective iff there are f.m. of them.

Proof: If P is projective, as a summand of a free module, then we can choose the coordinates of the inclusion map as f_i, and choose the image of the quotient map of the coordinate as x_i. The converse is verbatim. □

Cor. (I.6.1.9). If P is projective, then $P\to P^{**}$ is injective, and if P is finite projective, then it is an isomorphism.

Proof: If $f(x)=0$ for all $f\in P^*$, then the proposition says $x=0$. And if P is finite projective, it can be seen x_i, f_i forms a "basis" of P^*(finiteness used), so f_i generate P^*, and similarly x_i generate P^*, so $P\to P^{**}$ is surjective. □

Cor. (I.6.1.10). If P is projective over R, then $P^*\neq 0$.

Cor. (I.6.1.11). In the meanwhile of the proof, we already get: if P is finite projective, then P^* is finite projective, by(I.6.1.8).

Cor. (I.6.1.12). If P is finite projective, the the map $P\otimes M\to \mathrm{Hom}(P^*,M)$ is an isomorphism.

Proof: In(I.6.1.7), let $N=R$ and let $M=P^*$, then use the fact $P\cong P^{**}$. □

Prop. (I.6.1.13). Any finite projective module over $K[X_1,\ldots,X_k]$ is free. (Highly nontrivial).

Proof: □

Prop. (I.6.1.14). $\prod^{\mathbb{N}}\mathbb{Z}$ is not free thus not projective over $\mathbb{Z}$(I.6.1.3). And

$$\mathrm{Hom}(\prod^{\mathbb{N}}\mathbb{Z},\mathbb{Z}) = \bigoplus^{\mathbb{N}}\mathbb{Z}.$$

Proof: Cf.https://wildtopology.wordpress.com/2014/07/02/the-baer-specker-group/.
□

2 Injective

Prop. (I.6.2.1) (Baer's Criterion). A right $R-$module I is injective iff for every right ideal J of R, every map $J \to I$ can be extended to a map $R \to I$. (Directly from (I.8.2.20)).

Cor. (I.6.2.2). A module over a PID is injective iff it is divisible.

Cor. (I.6.2.3). A is injective iff $\mathrm{Ext}^1(R/I, A) = 0$ for every ideal I of R.

Prop. (I.6.2.4). The category of R-mod has enough injectives by(I.8.2.24), and it has enough projectives trivially.

Prop. (I.6.2.5). If I is an injective A-module, then for any ideal α of A, $\Gamma_\alpha(I) = \{m|\alpha^n m = 0\}$ for some n is injective.

Proof: Use Baer criterion, for any ideal b of A, it is f.g. so there is a n that $\phi(\alpha^n b) = 0$, and Artin-Rees tells us that $\phi(\alpha^N \cap b) = 0$ for some N. So we have an extension of ϕ over $b/b \cap \alpha^N$ to $A/\alpha^N \to I$, and this obviously factor through $\Gamma_\alpha(I)$, so it is done. □

Prop. (I.6.2.6). For an injective module A-module I, $I \to I_f$ is surjective.

Proof: we have the sheaf of modules $\widetilde{I}$ is flabby(III.5.1.6), thus the map to the stalk is surjective. □

Pontryagin Duality

Basic references are [Weibel Homological Algebra].

Def. (I.6.2.7). The **Pontryagin dual** $M^\vee$ of a left R-module M is the right R-module $\mathrm{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$, where $(fr)(b) = f(rb)$.

It is easily verified that if $A \neq 0$, then $A^\vee \neq 0$, and $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, thus the Pontryagin dual is faithfully exact.

Prop. (I.6.2.8). M is flat R-module iff $M^\vee$ is an injective right R-module (Because $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is exact).

3 Homological Dimension

Def. (I.6.3.1). For a R-mod A, the **projective dimension** $\mathrm{pd}(A)$ is the minimal length of a projective resolution of A. The **injective dimension** $\mathrm{id}(A)$ is the minimal length of a injective resolution of A. The **flat dimension** $\mathrm{fd}(A)$ is the minimal length of a flat resolution of A.

Prop. (I.6.3.2). If R is Noetherian, then $fd(A) = pd(A)$ for every f.g. module A.

Proof: Use(I.6.3.3), we see that if we choose a syzygy and look at the n-th term, then it is f.p and flat, so we have it is projective by(I.7.1.9). □

Lemma (I.6.3.3) (pd). If $\mathrm{Ext}^{d+1}(A, B) = 0$ for every B, then for every resolution

$$0 \to M \to P_{d-1} \to \ldots, P_1 \to P_0 \to A \to 0$$

where P_k is projective, then M is projective. Hence we have $pd(A) \leq d$. (Use dimension shifting, the following two are the same).

Lemma (I.6.3.4) (id). If $\mathrm{Ext}^{d+1}(A,B)=0$ for every A, then for every resolution

$$0 \to B \to P_0 \to \dots, P_{n-1} \to M \to 0$$

where P_k is injectives, then M is injective. Hence we have $id(B) \leq d$

Lemma (I.6.3.5) (fd). If $\mathrm{Tor}_{d+1}(A,B)=0$ for every B, then for every resolution

$$0 \to M \to F_{d-1} \to \dots, F_1 \to F_0 \to A \to 0$$

where F_k is flat, then M is flat. Hence we have $fd(A) \leq d$

Prop. (I.6.3.6) (Global Dimension Theorem). The following are the same for any ring R and called the **left global dimension** of R:

1. $\sup\{id(B)\}$
2. $\sup\{pd(A)\}$
3. $\sup\{pd(R/I)\}$
4. $\sup\{d : \mathrm{Ext}^d_R(A,B) \neq 0$ for some module A,B.

Proof: This follows from(I.6.3.3),(I.6.3.4) and (I.6.2.3). □

Prop. (I.6.3.7). A $\mathbb{Z}$ has global dimension 1 because injective is equivalent to divisible, and this shows that a quotient of an injective is injective.

Prop. (I.6.3.8) (Tor Dimension Theorem). The following are the same for any ring R and called the **Tor dimension** of R:

1. $\sup\{fd(A)\}$ for A a left module.
2. $\sup\{fd(B)\}$ for B a right module.
3. $\sup\{pd(R/I)\}$ for I a left ideal.
4. $\sup\{pd(R/J)\}$ for J a right ideal.
5. $\sup\{d : \mathrm{Tor}^R_d(A,B) \neq 0$ for some module A,B.

Proof: This follows from(I.6.3.5) applied to R and R^{op} and also(I.7.1.1). □

Prop. (I.6.3.9) (Change of Rings). Let $S \to R$ be a ring map, let A be a R-mod, then we have $pd_S(A) \leq pd_R(A) + pd_S(R)$.

Proof: Use the Cartan-Eilenberg resolution and the total complex has length $pd_R(A) + pd_S(R)$. □

4 Depth & Cohen-Macaulay Ring

Prop. (I.6.4.1) (Rees). For a f.g. module M and $IM \neq M$,

$$\mathrm{depth}_I(M) = \min\{i | \mathrm{Ext}^i_A(A/I,M) \neq 0\} = \min\{i | \mathrm{Ext}^i_A(N,M) \neq 0\}$$

where $\mathrm{depth}_I(M)$ is the length of the maximal M-regular sequence in I, N is a finite A-module with $\mathrm{Supp}(N) \subset V(I)$.

Proof: If No elements of I are M-regular, then $i \subset \cup\mathrm{Ass}(M)$ thus in one of them, so $\mathrm{Hom}_{A_p}(k, M_p) \neq 0$, and we have $N_p/PN_p = N \otimes_A k_p$ nonzero by Nakayama, thus $\mathrm{Hom}_k(N \otimes_A k_p, k_p) \neq 0$, thus $\mathrm{Hom}_{A_p}(N_p, M_p) = (\mathrm{Hom}_A(N, M))_p \neq 0$, so $\mathrm{Ext}^0_A(N, M) \neq 0$. Other dimensions follows by induction, consider the cokernel of $M \xrightarrow{a_1} M$.

Conversely, use induction, then we have an injection $\mathrm{Ext}^i_A(N, M) \xrightarrow{a_1} \mathrm{Ext}^i_A(N, M)$ for $i < n$. And the condition shows that $I \subset \sqrt{\mathrm{Ann}(M)}$, so $a_1^r N = 0$, thus the result. □

Cor. (I.6.4.2). Two maximal regular sequence in a f.g. module have the same length.

Cor. (I.6.4.3). For a module M over a Noetherian ring A, we know $\Gamma_I(M) = \{m | I^n m = 0 \text{ for some } n\}$, and H^n_I is its right derived functor, then we have $\mathrm{depth}_I(M) \geq n \iff H^i_I(M) = 0$ for $i < n$. (Because derived functor commutes with colimits, consider $N = A/I^k$).

Lemma (I.6.4.4) (Ischebeck). For a Noetherian local ring A, if M, N are finite modules, then we have $\mathrm{Ext}^i_A(N, M) = 0$ for $i < \mathrm{depth}(M) - \dim N$. Cf.[Matsumura P104].

Prop. (I.6.4.5). Let A be a local ring and M is finite A-module, then $\mathrm{depth}(M) \leq \dim A/P \leq \dim M$ for every $P \in \mathrm{Ass}(M)$. (Because $\mathrm{Hom}(A/P, M) \neq 0$.)

Prop. (I.6.4.6) (Auslander-Buchsbaum Formula). For a local ring R, if M is a finitely generated R-mod, if $pd(M) < \infty$, then we have $\mathrm{depth}(R) = \mathrm{depth}(M) + \mathrm{pd}(M)$. Cf.[Weibel P109].

Prop. (I.6.4.7). For a A-module M, if $x_1, \ldots, x_n$ is an M-regular sequence in A, then the Koszul complex has higher homology vanish and $H_0 = M/\sum x_i M$.

Proof: Cf.[Hartshorne P135]. □

Cohen-Macaulay

Def. (I.6.4.8). For A Noetherian local, a f.g. A-module M is called **Cohen-Macaulay** if $\mathrm{depth}(M) = \dim M$. In view of(I.6.4.5), this is equivalence to $\mathrm{depth}(M) = \dim A/P$ for all $P \in \mathrm{Ass}(M)$.

A localization of a C.M local ring is C.M, so we call a ring **C.M.** if all its localization at primes are C.M.

Prop. (I.6.4.9). A ring R is called **Gorenstein** iff $id_R R < \infty$. A Gorenstein local ring is C.M. In this case, $\mathrm{depth}(R) = id_R R = \dim R$, and $\mathrm{Ext}^q_R(R/m, R) \neq 0 \iff q = \dim R$. Cf.[Weibel P107].

Prop. (I.6.4.10). A ring is C.M. iff for all ideals, the associated primes of A/I all have the same height as I, i.e. unmixed.

Prop. (I.6.4.11). If a local ring is C.M. and $I = (x_1, \ldots, x_r)$ is a regular sequence, then there is an isomorphism $(A/I)[t_1, \ldots, t_r] \to \mathrm{gr}_t A = \oplus I^n/I^{n+1}$. In particular, I/I^2 is a free A/I module.

Prop. (I.6.4.12). Let A is a Noetherian local ring and M a f.g. module, if a set of elements $(x_1, \ldots, x_r)$ forms a regular sequence for M, then $\dim M/(x_1, \ldots, x_r) = \dim M - r$. The converse is also true when A is C.M. If this is the case, then $A/(x_1, \ldots, x_r)$ is also C.M.

Proof: By(I.5.5.9), we have $<$, for the converse, $\mathrm{Supp}(M/fM) = \mathrm{Supp}(M) \cap \mathrm{Supp}(A/fA) = \mathrm{Supp}(M) \cap V(f)$, and when f is M-regular, $V(f)$ doesn't contain any $\mathrm{Ass}(M)$ thus no minimal elements of $\mathrm{Supp}(M)$, so $\dim(M/fM) < \dim M$, thus we have $>$.

When A is C.M.: □

5 Normal Ring & Regular Local Ring

Normal Ring

Def. (I.6.5.1). A domain is called **normal** iff it is integrally closed in its fraction field. This property is local, so we can generally call a ring **normal** iff all its stalks are integrally closed domain.

The **normalization** of an integral domain is the alg.closure of it in its quotient field. It commutes with localization.

Proof: The localization of a normal domain is normal, and the converse follows from $A = \cap A_{\mathfrak{m}}$ where $\mathfrak{m}$ are maximal. □

Def. (I.6.5.2). A domain is called **completely normal** iff all almost normal elements are in A, i.e. $\{u|\exists a, au^n \in A\ \forall n\} \in A$. For Noetherian ring, completely normal is equivalent to normal.

Proof: Cf.[StackProject 00GX]. □

Prop. (I.6.5.3). A UFD is normal integral.

Proof: If x is integral over R, check the indices of each prime ideal of x is non-negative. □

Prop. (I.6.5.4). A is a normal domain, then so does $A[X]$. If A is Noetherian normal domain, then so does $A[[X]]$.

Proof: Cf.[StackProject 030A, 0BI0]. □

Prop. (I.6.5.5). Direct limits of normal rings are normal.

Proof: Let $\mathfrak{p}$ be an ideal of $R = \varinjlim R_i$, $\mathfrak{p}_i = \mathfrak{p} \cap R_i$, then $R_{\mathfrak{p}} = \varinjlim (R_i)_{\mathfrak{p}_i}$, so it suffices to prove for normal domains, the rest is easy. □

Prop. (I.6.5.6). Principal ideals in a Noetherian normal domain is unmixed and $A = \cap_{\text{ht } p=1} A_p$.

Proof: Cf.[Matsumura P124]. □

Prop. (I.6.5.7) (Hironaka). Let A be a local Noetherian domain that is a localization of an algebra of f.t. over a field k. Let $t \in A$ that

- tA has only on eminimal associated prime ideal p.
- t generate the maximal ideal of A_p.
- A/p is normal.

Then $p = tA$ and A is normal.

Proof: Cf.[Hartshorne P264]. □

Prop. (I.6.5.8) (Quadratic Normal Ring). If K is a field of char$\neq 2$ and $f \in K[X_1, \ldots, X_n]$ is a polynomial with no square factors, then $K[X_1, \ldots, X_n, Z]/(Z^2 - f)$ is integrally closed.

Proof: It is an integral domain because $K[X_1, \ldots, X_n, Z]$ is UFD thus it suffices to show $Z^2 - f$ is irreducible, and this is easy. Now we see the quotient field is just $K(X_1, \ldots, X_n)[Z]/(Z^2 - f)$. It is Galois extension of degree 2, for any element $g + hz$, its minimal polynomial is $X^2 - 2gX + (g^- h^f)$, thus it is integral over $K[X_1, \ldots, X_n, Z]$ iff g, h are both polynomials(char$\neq 2$ and f has no square factors used). □

Regular Ring

Def. (I.6.5.9). A Noetherian local ring is called **regular** iff it $\mathrm{rank}_k\ \mathfrak{m}/\mathfrak{m}^2 = \dim A$. This is equivalent to $\mathrm{gr}\ A \cong k[X_1, \ldots, X_d]$ by(I.5.5.9).

Localization of a regular local ring at primes are regular local. Hence we can call a Noetherian ring **regular** iff all its localization at primes are regular local.

Proof: Cf.[Matsumura P139]. □

Prop. (I.6.5.10). If A is regular, then $A[X_1, \ldots, X_n]$ is regular, and $A[[X_1, \ldots, X_n]]$ is regular.

Proof: Cf.[Matsumura P176]. □

Prop. (I.6.5.11) (Local in Dimension 1 Case). For a Noetherian local domain of dimension 1 with maximal ideal $\mathfrak{m}$ and residue field k, the following are equivalent:

1. A is a DVR.
2. A is normal.
3. $\mathfrak{m}$ is a principal ideal.
4. A is regular.
5. Every nonzero ideal is a power of $\mathfrak{m}$.
6. There exists $x \in A$ that every nonzero ideal is of the form (x^k).

Proof: $1 \to 2$: Valuation ring is integrally closed, by(V.3.2.6).

$2 \to 3$: As the radical of any ideal a is $\mathfrak{m}$, and A is Noetherian, so there is an n that $\mathfrak{m}^n \subset a$ and $\mathfrak{m}^{n-1} \not\subseteq a$. Then choose $b \in a - \mathfrak{m}^{n-1}$, $x = a/b \in K$, then $x^{-1} \notin A$, then it is not integral over A. So $x^{-1}\mathfrak{m} \not\subseteq \mathfrak{m}$, but $x^{-1}\mathfrak{m} \subset A$, so it equals A, which means $\mathfrak{m} = (x)$.

$3 \to 4$: Clear.

$4 \to 5$: For any ideal a, its radical is $\mathfrak{m}$ and A is Noetherian, so $\mathfrak{m}^n \subset a$. Now $A/\mathfrak{m}^n$ is Artinian by(I.5.1.18), so by(I.5.1.20) a is a power of $\mathfrak{m}$.

$5 \to 6$: And $x \in \mathfrak{m} - \mathfrak{m}^2$ will do.

$6 \to 1$: Define $v(a) = k$ if $(a) = (x^k)$. □

Prop. (I.6.5.12) (Auslander-Buchsbaum). A regular local ring is UFD. A priori it is an normal domain.

Proof: Cf.[Matsumura P142],[Weibel P106]. □

Prop. (I.6.5.13). A regular local ring Gorenstein hence C.M.

Proof: □

Prop. (I.6.5.14). If a quotient of a Noetherian local ring by a non-zerodivisor is regular, then it is itself regular.

Prop. (I.6.5.15) (Serre). A Noetherian local ring A is regular iff the global dimension of A is finite.

Proof: Cf.[Mat P139]. □

Prop. (I.6.5.16). For A a regular local ring and M a f.g. A-module,

$$pd(M) + \text{depth } M = \dim A.$$

Cf.[Hartshorne P237].

Cor. (I.6.5.17). For a f.g. module M over a regular local ring A, $pd(M) \le n$ iff $\mathrm{Ext}^i(M, A) = 0$ for all $i > n$.

Proof: This is because we can use dimension shifting to show $\mathrm{Ext}^i(M, N) = 0$ for all N f.g., then(I.6.3.3) says that $pd(M) \le n$. □

Serre Conditions R_k & S_k

Def. (I.6.5.18). A ring is called R_k iff for all prime p of height $\le k$, A_p is regular.
A ring is called S_k iff $\mathrm{depth}(A_p) \ge \min(k, \mathrm{ht}(p))$ for all prime p.
A module M is called S_k iff $\mathrm{depth}(M_p) \ge \min(k, \dim \operatorname{Supp} M_p)$ for all prime p.

Prop. (I.6.5.19).

- M is S_1 iff M has no associated embedded primes. Cf.[StackProject 031Q].
- A Noetherian ring is reduced iff it is R_0 and S_1. Cf.[StackProject 031R].
- (Serre Criterion)A Noetherian ring is normal iff it is R_1 and S_2. Cf.[StackProject 031S].
- A ring is C.M. iff it is $S_{\mathbb{N}}$.

Proof: □

Cor. (I.6.5.20) (Regular and Normal). A regular ring is normal, and normal ring is regular in codimension 1.

Proof: By(I.6.5.19), it suffices to prove that a regular ring satisfies R_1 and S_2. A regular ring is C.M.(I.6.5.13) so it is S_2 by(I.6.5.19), it is R_1 by(I.6.5.9) □

Cor. (I.6.5.21). A Noetherian local ring of dim 1 is normal iff it is regular. i.e. integrally closed iff maximal ideal principal.

Local Complete Intersection

Basic references are [StackProject 10.133, 23.8].

Def. (I.6.5.22). A f.g. k-algebra S is called a **complete intersection** if $S = k[X_1, \ldots, X_n]/(f_1, \ldots, f_c)$ with $\dim S = n - c$. It is called a **local complete intersection** if it is locally a complete intersection. Notice by Krull's theorem(I.5.6.9), this is equivalent to it is equidimensional of dimension $n - c$.
Relative complete intersection definition, Cf.[StackProject 00SP].
To be well-defined, we in fact need the following lemma(I.6.5.23).

Lemma (I.6.5.23). For a f.g. k-algebra, (Local)Complete intersection is stable under localization.

Proof: Cf.[StackProject 00SA]. □

Prop. (I.6.5.24). For a f.g. k-algebra S and a field extension $k \to K$, S is a local complete intersection iff $S \otimes_k K$ is a local complete intersection.

6 Geometric Properties

Def. (I.6.6.1).

- A k-algebra S is called **geometrically reduced/integral/connected**... over a field k iff for any field extension k'/k, $S_{k'}$ is reduced/integral/connected....
- A Noetherian k-algebra S is called **geometrically regular** iff for any f.g. field extension K/k, S_K is regular(Notice $A \otimes_k k'$ is Noetherian(I.5.1.12), so this makes sense).

Prop. (I.6.6.2) (Geometrically reduced). If S is a k-algebra, the following are equivalent.

1. $S \otimes_k R$ is reduced for every reduced k-algebra R.
2. S is geometrically reduced.
3. residue fields of S at maximal points are reduced.
4. $S \otimes_k \overline{k}$ is reduced.
5. $S \otimes_k k^{per}$ is reduced.
6. $S \otimes_k k'$ is reduced for any finite purely inseparable field extension k'/k.
7. $S \otimes_k k^{1/p}$ is reduced.

Proof: Cf.[StackProject 030V] and [Gortz 135]. □

Prop. (I.6.6.3).

Prop. (I.6.6.4). It suffices to check geometrically regular for k'/k finite purely inseparable.

Proof: Cf.[StackProject 0381]. □

7 Finitely Presented

Finite Presented Module

Def. (I.6.7.1). A module is called **finitely presented** iff it is like R^m/R^n.

Finite presentation is stable under base change because tensoring is right exact.

Prop. (I.6.7.2). For a surjective module map $F \to M$, if F is f.g. and M is f.p., then the kernel is f.g.

Proof: use the diagram
$$\begin{tikzcd} & R^m \arrow[r] \arrow[d, "\alpha"] & R^n \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Ker} \arrow[r] & F \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$
. and snake lemma, then image and cokernel of α are all finite, then Ker is finite. □

Prop. (I.6.7.3). For $0 \to M_1 \to M_2 \to M_3 \to 0$, if M_1, M_3 are f.p., then so does M_2. This is because we can find a compose a diagram of $R^* \to M_*$, and look at the kernel.

Prop. (I.6.7.4). If $R \to S$ is a f.g. ring map and a S-module M is f.p. over R, then it is f.p. over S.

Proof: Let $S = R[x_1, \ldots, x_n]$, and $M = R[y_1, \ldots, y_m]/(\sum a_{ij} y_j), 1 \leq i \leq t$, then as M is a S-module, we let $x_i y_j = \sum a_{ijk} y_k$, and forms a quotient $S^{mn+t} \to S^m \to N \to 0$, where S^{mn+t} corresponds to the relations $\sum a_{ij} y_j$ and $x_i y_j - \sum a_{ijk} y_k$. Then there is a surjective A-module map $N \to M$, and we check it is injective: if $z = \sum b_j y_j$ are mapped to 0, where $b_j \in S$, then we can transform z into the shape $\sum c_j y_j$, where $c_j \in R$ by relations $x_i y_j - \sum a_{ijk} y_k$. Thus it is zero by definition. □

Prop. (I.6.7.5). Any module is a direct limit of f.p. modules. This can be seen by considering all f.g. submodules and f.m relations between them.

Prop. (I.6.7.6) (FP and Localization). For M f.p., $S^{-1} \operatorname{Hom}_R(M, N) = \operatorname{Hom}_{S^{-1}R} \operatorname{Hom}(S^{-1}M, S^{-1}N)$ for any R-module N. (Use the presentation and Hom is left exact).

Finitely Presented Ring Map

Def. (I.6.7.7). A ring map is called **of finite presentation** iff it is a quotient of a free algebra by a free algebra.

Prop. (I.6.7.8). Finite presentation is stable under composition(choose a presentation form to see) and base change because tensoring is right exact.

It is local on the source and target by(I.5.1.28).

Prop. (I.6.7.9). If $g \circ f : R \to S' \to S$ is of finite presentation and f is of finite type, then g is of finite presentation.

Proof: Let $S' = R[y_1, \ldots, y_a]$ and $S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$, then let $h_i(X) \cong y_i$ in S, then $S = S'[X_1, \ldots, X_n]/(f_1, \ldots, f_m, h_i - y_i)$. □

Prop. (I.6.7.10). For S f.p. over R, then the kernel of any surjective ring map $R[X_1, \ldots, R_n] \xrightarrow{\alpha} S$ is f.g..

Proof: Let $S = R[Y_1, \ldots, Y_m]/(f_1, \ldots, f_k)$, then if $\alpha(X_i) \cong g_i(Y)$, then $\alpha : R[X_1, \ldots, R_n] \to R[X_1, \ldots, X_m, Y_1, \ldots, Y_m]/(f_1, \ldots, f_k, X_i - g_i)$. And the Y_i are in the image, thus we let Y_i are mapped onto by $h_j(X)$, then $\operatorname{Ker} \alpha = (f_i(h_j(X)), X_i - g_i(X))$. □

Prop. (I.6.7.11). If S is f.p. over R that S has a presentation $S = R[X_1, \ldots, X_n]/I$ that I/I^2 is free over S, then S has a presentation $R[X_1, \ldots, X_m]/(f_1, \ldots, f_c)$ that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is freely generated by $f_1, \ldots, f_c$.

Proof: Cf.[StackProject 07CF]. □

I.7 Commutative Algebra(StackProject)

1 Flatness

Prop. (I.7.1.1). Flatness need only be checked for finite modules, and it is equivalent to $\mathrm{Tor}_1(M, A/I) = 0$ for any f.g. ideal I(i.e $I \otimes M \to M$ is injective). This is all because tensor product commutes with colimit.

Cor. (I.7.1.2). If $0 \to M' \to M \to M'' \to 0$, then M' and M'' flat implies M is flat.

Prop. (I.7.1.3). If M is flat then $\mathrm{Tor}_i^A(M, N) = 0$ for all $i > 0$, because we have: if $0 \to M_1 \to M_2 \to M_3 \to 0$ M_2, M_3 flat, then M_1 is flat (Use 9 entry sequence and the fact that Tor is symmetric(I.9.6.5)). So $\mathrm{Tor}_{n+1}(M_3, N) = \mathrm{Tor}_n(M_1, N) = 0$ by induction.

And a direct summand of a flat module is flat. Thus we have the class of flat modules is adapted $-\otimes N$ for all N (because free is flat).

Prop. (I.7.1.4) (Flatness and Base Change).

- (Faithfully)Flatness is stable under base change.
- If $R \to S$ is f.f., then M is flat iff its base change is flat.
- Flatness is stable under filtered limit because filtered limit commutes with tensoring and is exact. $S^{-1}A$ are A-flat because localization is exact.
- If $R \to S$, and a S-module is R-flat and S-f.f., then $R \to S$ is flat.

Proof: Use definition and tensor trick. □

Prop. (I.7.1.5) (Equational Criterion of Flatness). For a R module M, a relation $\sum f_i x_i = 0$ of elements of M are called **trivial** iff $x_i = \sum a_{ij} y_j$ and $0 = \sum f_i a_{ij}$ for any j. Then M is flat iff all relations of elements of M is trivial.

Proof: Cf.[StackProject 00HK]. □

Prop. (I.7.1.6) (Gororov-Lazard). Any flat $A-$module is isomorphic to a direct colimit of free modules of finite type.

Proof: □

Prop. (I.7.1.7). A finite module M over a local ring A is flat iff it is free. In particular, finite flat modules over a field are all flat.

Proof: Let $A/\mathfrak{m} = k$, choose a k-basis x_i of $M/\mathfrak{m}M$, then they generate M by Nakayama. It suffices to prove that x_i are independent over R. For this, use equational criterion of flatness(I.7.1.5), we prove that if x_i is independent over k, then they are independent over A. Use induction, if $x \neq 0$ in $M/\mathfrak{M}$, if $fx = 0$ for some $f \in A$, then $x = \sum a_j y_j$ that $f a_j = 0$, but then some a_j is a unit, so $f = 0$.

If $\sum f_i x_i = 0$, then by hypothesis, $f_i \in \mathfrak{m}$, and there are y_j that $x_i = \sum a_{ij} y_j$, $\sum f_i a_{ij} = 0$. As $x_n \notin \mathfrak{m}M$, there is a $a_{nj} \notin \mathfrak{m}$, so $f_n = \sum(-a_{ij}/a_{nj}) f_i$. then $\sum_{i \neq n} f_i (x_i - a_{ij}/a_{nj} x_n) = 0$, but $x_i - a_{ij}/a_{nj} x_n$ is also independent over k, so by induction, $f_i = 0$, also does f_n, so we are done. □

Prop. (I.7.1.8) (Flat over Valuation Ring Locally Free). A module over a valuation ring is flat iff it is torsion free, because valuation ring is Bezout(V.3.2.10) and use(I.3.3.6).

Prop. (I.7.1.9) (Finite Flat and Projective). Finitely presented flat module is equivalent to finite projective. (Immediate from(I.6.1.4)).

Prop. (I.7.1.10). if M is a flat R-module, then $IM \cap JM = (I \cap J)M$ for ideals of A.

Proof: Tensoring the exact sequence $0 \to I \cap J \to I \oplus J \to J \cup J \to 0$ with M. □

Prop. (I.7.1.11) (Faithfully Flat). The following are equivalent:

- M is f.f.
- M is flat and for any $N \neq 0$, $N \otimes M \neq 0$.
- M is flat and for any (maximal)prime ideal $\mathfrak{m}$ of A, $k_{\mathfrak{m}} \otimes_R M \neq M$. (When $\mathfrak{m}$ is maximal, this says $\mathfrak{m}M \neq M$).

Proof: $3 \to 2$: any nonzero module has a submodule A/I, and thus $(A/I)M = M/IM \neq 0$.

$2 \to 1$:first show S is a complex if $S \otimes M$ is exact, then $H^*(S) \otimes M = H^*(S \otimes M)$ by flatness, thus $H^*(S) = 0$. □

Flat ring extension

Prop. (I.7.1.12). The following are equivalent:

- $A \to B$ is f.f.
- It is flat and $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.
- It is flat and Spec map contains all the closed pts.

This follows form(I.7.1.11) as we see that $\mathfrak{p}$ is in the image of Spec map iff $k_{\mathfrak{p}} \otimes_R S \neq S$.

Cor. (I.7.1.13). Integral flat injective of rings is f.f..

Cor. (I.7.1.14). Flat local ring map of local rings is f.f..

Cor. (I.7.1.15). Direct limits of f.f. rings over R is f.f.

Proof: If is flat by(I.7.1.4), and for a maximal ideal $\mathfrak{m}$ of R, $S_i/\mathfrak{m}S_i$ is non-zero, hence there direct limit is non-zero because 1 is contained. So $\mathfrak{m}$ is in the image, hence it is f.f. by(I.7.1.12). □

Prop. (I.7.1.16). if rings $A \subset B \subset C$ and $C/A, C/B$ is flat, then B/A is flat.

Proof: Cf.[GAGA Serre P26]. □

Prop. (I.7.1.17). If B is flat over A, then

$$\operatorname{Tor}_i^A(M,N) \otimes B = \operatorname{Tor}^B(M_{(B)}, N_{(B)}), \quad \operatorname{Ext}_i^A(M,N) \otimes B = \operatorname{Ext}^B(M_{(B)}, N_{(B)}).$$

Prop. (I.7.1.18). If $R \to S$ is (faithfully)flat ring map and M is a (faithdully)flat S-module, then M is a (faithfully)flat R-module. In particular, (faithfully) flatness is stable under composition.

Also (faithfully)Flatness is stable under base change(I.7.1.4) and local on the target and source by(I.7.1.22).

Prop. (I.7.1.19) (Faithfully Flat Injective). A f.f. ring map $R \to S$ is universally injective. In particular, tensoring with R/I, we get $R \cap IS = I$ for an ideal I of R.

Proof: Because $R \to S$ is f.f., we only need to show that $N \otimes_R S \to N \otimes_R S \otimes_R S$ is injective for any N, but this is true because it has a left inverse. □

Prop. (I.7.1.20). A flat ring map maps a non-zero-divisor to a non-zero-divisor, because if we consider the principal ideal generated by it, then(I.7.1.1) shows the ideal in M is also injective, so it is not a zero-divisor.

Prop. (I.7.1.21). If A is Noetherian and I is an ideal, the the I-adic completion $\hat{A}/A$ is flat by(I.5.5.17).

Prop. (I.7.1.22) (Flatness is Local). Flatness is stalkwise both on the target and source, thus flatness is local both on the target and the source**??**

Cor. (I.7.1.23) (Going-down). Going-down holds for flat ring map.

Proof: The ring map $R_{\mathfrak{p}'} \to S_{\mathfrak{q}'}$ is flat by(I.7.1.22), thus it is f.f. by(I.7.1.14). Then(I.7.1.12) says $\mathfrak{p} \subset \mathfrak{p}'$ is in the image. □

Prop. (I.7.1.24). The Spec map of a ring map $R \to S$ of f.p. that satisfies going-down(e.g. flat), is open.

Proof: $S \to S_f$ satisfies going-down and is of f.p, so we see that $R \to S_f$ satisfies going down. It suffice to prove the image of this map is open. By Chevalley, the image is constructible, and it is stable under specialization. So it is closed by(IV.1.14.6). □

Prop. (I.7.1.25). The Spec map f of a f.f. ring map is submersive.

Proof: For a T that $f^{-1}(T)$ is closed, we see that $f^{-1}(T) \to T$ satisfies going-down because f does, so its complement is closed under specialization, so it is closed by(I.5.2.8) it is closed. So T is open. □

2 Faithfully Flat Descent

Prop. (I.7.2.1) (Faithfully Flat Descent). List of properties that descent through faithfully flat morphism.

1. Projectiveness for modules over a ring.
2. Finiteness for modules over a ring.
3. F.p. for modules over a ring.
4. Flatness for modules over a ring.
5. (Formal)Smoothness for ring maps.
6. Noetherian for rings over a ring.
7. Reducedness for rings over a ring.
8. Normal for rings over a ring.
9. Regular for rings over a ring.

Proof:

1. Cf.[StackProject 05A9].

2. Cf.[StackProject 03C4].
3. Cf.[StackProject 03C4].
4. Cf.[StackProject 03C4].
5. Use criterion(I.7.4.3), we see by flatness that the sequence $I/I^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$ commutes with flat base change, and when it is f.f., then use(VI.2.1.4) and descent for projectiveness(I.7.2.1) that $\Omega_{S/R}$ is projective, so it is a split exact sequence. The smooth case follows from definition(I.7.4.14) as f.p. can descend.
6. Because for $S \to S'$ faithfully flat and a chain of ideals I_k in S, $I_k S' = I_k \otimes_S S'$, and $I_k S'$ is stable if S' is Noetherian, so also I_k is stable because it is faithfully flat.
7. Trivial as $S \to S'$ is f.f. hence injective(I.7.1.19).
8. Cf.[StackProject 033G].
9. Cf.[StackProject 07NG].

□

3 Syntomic

Def. (I.7.3.1). A ring map is called **syntomic** iff it is of f.p., flat and the fibers are all local complete intersection rings.

4 Smooth

Formally Smoothness

Def. (I.7.4.1). A ring map $R \to S$ is called **formally smooth** if for every R-ring A and an ideal I of A that $I^2 = 0$, a map $S \to A/I$ can extends to a map $S \to A$.

Formal smooth is stable under base change and composition, by universal arguments. A polynomial algebra is formally smooth.

Prop. (I.7.4.2). Giving a presentation $S = P/J$ where P is formally smooth(e.g. polynomial algebra), S is formally smooth iff there is a map $S \to P/J^2$ that is right converse to the obvious projection.

Proof: One way is from the definition of formally smooth applied to P/J^2 and J. Conversely, for any A and I, we notice the map $P \to S \to A/I$ can be lifted to $P \to A$, and J is mapped to I, so J^2 is mapped to 0, so we have a map $P/J^2 \to A$. Then $S \to P/J^2 \to A$ is the lifting. □

Cor. (I.7.4.3). If $P \to S$ is a presentation of S/R by polynomial algebra with kernel I, then S/R is formal smooth iff

$$0 \to I/I^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

is split exact as in(VI.2.1.6).(by(VI.2.1.6)).

Now we consider the relation of Formal Smoothness and Cotangent Complexes.

Lemma (I.7.4.4) (Formally Smooth Replacement 1). If $A \to B$ is a ring map that has two surjective presentations $C \to B, D \to B$ with kernels I, J. If there is a map $C \to D$ commutating these two presentation, D formally smooth, and $C \to D$ is surjective or C is formally smooth, then their corresponding naive cotangent complexes are quasi isomorphic.

Proof: Cf.[Foundations of Perfectoid Geometry P123]. □

Prop. (I.7.4.5) (Formally Smooth Replacement 2). formally-smooth-re2snt-naive-cotantgent-complex If $A \to B$ is a ring map that has two formally smooth presentation $C \to B, D \to B$ with kernels I, J. then their corresponding naive cotangent complexes are quasi isomorphic.

Proof: It suffices to prove they are both quasi isomorphic to the canonical cotangent complex. For this, we first consider the diagram $\begin{array}{ccc} D & \longrightarrow & C \\ \downarrow & & \downarrow \\ A[B] & \longrightarrow & B \end{array}$, where $D = A[S]$ and $S = C \coprod A[B]$ as sets. The two map $D \to A[B]$ and $D \to A[B]$ can be chosen because $C \to B$ is surjective. So the results follows from(I.7.4.4). □

Cor. (I.7.4.6) (Equivalence Definition). S/R is formally smooth iff $NL_{S/R}$ is quasi-isomorphic to a projective S-module at degree 0.

Proof: If S/R is formally smooth, then choose a presentation will suffice by(I.7.4.3). The converse is also true by projectiveness and(I.7.4.3). □

Cor. (I.7.4.7). If C/B is formally smooth, then the Jacobi-Zariski sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

as in(VI.2.1.5) is split exact, by(VI.2.2.5). In particular, any derivation of B to a C-module can be extended to a derivation C to a C-module.

Cor. (I.7.4.8). If $A \to B \to C$ with $A \to C$ formally smooth and $B \to C$ surjective with kernel I, then there is an split sequence

$$0 \to I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

by(VI.2.2.5).

Standard Smooth Algebra

Def. (I.7.4.9). A **standard smooth algebra** over R is a algebra $S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_c)$, where $c \le n$ and $J(\frac{f_1,\ldots,f_c}{X_1,\ldots,X_c})$ is invertible in S.

Prop. (I.7.4.10) (Standard Smooth Localization). If $R \to S$ is standard smooth, then $R \to S_g$ is standard smooth, and $R_f \to S_f$ is standard smooth(because stable under base change(I.7.4.11)).

Proof: For localization at $g \in S$, let h be an inverse image of g in $R[X_1, \ldots, X_n]$, then $S_g = R[X_1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_c, X_{n+1}h - 1)$, and it is standard smooth. □

Prop. (I.7.4.11). Standard smoothness is stable under base change and composition.

Proof: For base change, notice the Jacobi matrix is the base change of the Jacobi matrix, so it is also invertible. For composition, write out the presentation, the determinant is the product of the presentation. □

Prop. (I.7.4.12). A standard smooth algebra is a relative global complete intersection.

Proof: Cf.[StackProject 00T7]. □

Prop. (I.7.4.13) (Jacobian Criterion). For a f.p. ring $S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_c)$, S/R is standard smooth in a nbhd of q iff the Jacobian matrix has rank c at q, i.e. $J(\frac{f_1,\ldots,f_c}{X_1,\ldots,X_c})$ is not in q for some permutation of $X_1, \ldots, X_n$.

Proof: If $h = J(\frac{f_1,\ldots,f_c}{X_1,\ldots,X_c}) \notin q$, let $S_h = R[X1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_c, X_{n+1}g - 1)$ is a standard smooth algebra. Conversely, $S_h = R[X1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_c, X_{n+1}g - 1)$ is standard smooth for some $g \notin q$, so $J(\frac{f_1,\ldots,f_c}{X_1,\ldots,X_c}) \notin q$ by calculation. □

Smooth

Def. (I.7.4.14) (Smooth Ring Map). A ring map $R \to S$ is called **smooth** if it satisfies the following equivalent conditions:

- It is of f.p. and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module placed at degree 0. In other words,

$$0 \to I/I^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

  is exact and $\Omega_{S/R}$ is finite S-projective. By(VI.2.2.3), we only need to prove for a single presentation of S.

- It is locally standard smooth.
- It is formally smooth and of f.p..

Proof: $1 \to 3$: by(I.7.4.6). $3 \to 1$: By(I.7.4.6), $\Omega_{S/R}$ is f.p. and projective, so it is finite projective.

At this point we already know that the first definition is stable under base change and composition, because f.p. and formally smoothness both do(I.7.4.1)(I.6.7.8).

And also the first definition is local on source because f.p. does(I.5.1.28) and NL commutes with localization(VI.2.2.8) so we can use the local properties of triviality(I.5.1.27) and finite projectiveness(I.6.1.4).

Now it is also local on the source because it is stable under base change and composition and $R \to R_{f_i}$ does by locality on the source.

$2 \to 1$:Now the property are all local on source. It suffices to prove a standard smooth map is smooth: its naive cotangent complex is $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \to S[dX_1, \ldots, dX_n]$, and it is a split injection by linear algebra, and $\Omega_{S/R} = S[dX_{c+1}, \ldots, dX_n]$, so it is a smooth ring map.

$1 \to 2$:We need to prove, assuming the first definition, it is locally standard smooth. For this, Cf.[StackProject 00TA]. □

Cor. (I.7.4.15). Smoothness is stable under composition and base change. Smoothness is local on the source and target(In particular, $R \to R_f$ is smooth). (Already proved in the proof of(I.7.4.14)).

Cor. (I.7.4.16). A smooth map is syntomic, hence flat.

Proof: Cf.[StackProject 00TA]. □

Prop. (I.7.4.17) (Noetherian Descent). A smooth ring map $R \to S$ is a base change of smooth ring map over a ring f.g. over $\mathbb{Z}$.

Proof: Use the equivalence definition(I.7.4.2), we know that there is a map

$$S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_c) \to R[X_1, \ldots, X_n]/(f_1, \ldots, f_c)^2,$$

which if we write $\sigma(X_i) = h_i$, then must satisfy

$$f_i(h_1, \ldots, h_n) = \sum a_{ijk} f_j f_k.$$

Then we consider the subalgebra generated by f_i, h_i, a_{ijk}, then by the same reason, they form a smooth algebra over $\mathbb{Z}$, and its tensor with R gives out S. □

Cor. (I.7.4.18). The lifting property in the definition of formally smooth is true for any I that is locally nilpotent, when S/R is moreover smooth.

Proof: By the proposition, we can retract to a f.g. algebra S_0 over $\mathbb{Z}$, then the image of S_0 is a f.g. algebra A_0. Then A_0 is Noetherian and so I_0 is nilpotent, then we can use finite induction to find lifting to $S_0 \to A_0$ over $\mathbb{Z}$. □

Prop. (I.7.4.19) (Stalkwise). If $R \to S$ is f.p., then it is smooth iff it is S_q/R_p is smooth for every (maximal)prime q of S and p under it.

Proof: Because of f.p., we only need to check triviality of $H_1(NL)$ and finite projectivity of $\Omega_{S/R}$(fp used). But both triviality and fintie projectivity is stalkwise(I.5.1.27). (Notice $R \to R_p$ is smooth). □

Cor. (I.7.4.20) (Fiberwise). For a ring map $R \to S$ and q is a prime of S over p. Then S/R is smooth at q iff S/R is of f.p. and S_q/R_p is flat and $S \otimes k(p)/k(p)$ is smooth.

Proof: One way is because smooth is flat, f.p. and stable under base change. Conversely, Cf.[StackProject 00TF]. □

Cor. (I.7.4.21) (Smooth Points and Flat Base Change). If $R \to S$ is of f.p. and $R \to R'$ is flat. Then the set of primes in $S' = S \otimes_R R'$ that has a nbhd that is smooth over R' is the inverse image of set of primes in S that has a nbhd that is smooth over R.

Proof: One direction is because smooth is stable under base change. Conversely, the local ring map is f.f., so $H_1(NL_{S'/R',q}) = H_1((NL_{S/R} \otimes_S S')_q) = H_1(NL_{S/R,p} \otimes_{S_p} S'_q)$. Then the result follows as S'_q/S_p is f.f. and triviality and finite projective descents for f.f. map(I.7.2.1). □

Prop. (I.7.4.22) (Strong Lifting Property). For a smooth ring map, the lifting property is true for $A \to A/I$, where I is locally nilpotent.

Proof: By(I.7.4.17), $R \to S$ is a base change of a smooth ring map $R_0 \to S_0$ where R_0 is f.g. over $\mathbb{Z}$. Now if S_0 is generated by $x_1, \ldots x_n$ and $a_1, a_n \in A$ maps to the image of $x_1, \ldots, x_n$ in A/I, then consider the subring A_0 generated by R_0 and a_i, and let $I_0 = A_0 \cap I$, then it suffices to prove this case followed by base change. But now A_0 is f.g. over $\mathbb{Z}$, so it is Noetherian, and then I is nilpotent, thus we have a desired lifting. □

Smooth over Fields

Prop. (I.7.4.23). A smooth k-algebra is a local complete intersection.

Proof: Immediate from(I.7.4.16). ☐

Lemma (I.7.4.24). Let S be f.g. over a alg.closed field k and $\mathfrak{m}$ a maximal ideal, then the following are equivalent:

- S_m is regular.
- $\dim_k \Omega_{S/k} \otimes k \leq \dim S_{\mathfrak{m}}$
- $\dim_k \Omega_{S/k} \otimes k = \dim S_{\mathfrak{m}}$
- S/k is smooth in a nbhd of $\mathfrak{m}$.

Proof: Cf.[StackProject 00TS]. ☐

Prop. (I.7.4.25) (Smooth Differential Criterion). For a ring S f.g. over a field, S is smooth in a nbhd of x corresponding to q iff $\dim_{k(q)} \Omega_{S/k} \otimes k(q) \leq \dim_x(X)$.

And in this case, equality hold, and S_q is regular.

Proof: Cf.[StackProject 00TT]. ☐

Prop. (I.7.4.26) (Smooth and Regular). Let S be f.g. over a field k, if $k(q)/k$ is separable(e.g. char 0) for q a prime of S, then S is smooth in a nbhd of p iff S_q is regular.

Proof: Cf.[StackProject 00TV]. ☐

Prop. (I.7.4.27). An injective morphism of domains is smooth at (0) iff the quotient field map is separable.

Proof: Cf.[StackProject 07ND]. ☐

5 Unramified

Formally Unramified

Def. (I.7.5.1). A ring map $R \to S$ is called **formally unramified** if for every R-ring A and an ideal I of A that $I^2 = 0$, a map $S \to A/I$ has at most one extension to a map $S \to A$.

Formally unramified is equivalent to $\Omega_{S/R} = 0$. So it is stable under composition by Jacobi-Zariski sequence(VI.2.1.5).

Proof: Let $J = \mathrm{Ker}(S \otimes_R S \to S)$, let $A_{univ} = S \otimes_R S/J^2$, then $J/J^2 \cong \Omega_{S/R}$(VI.2.1.2), so we have two natural map from S to A_{univ}, they differ by the universal differential $S \to \Omega_{S/R}$. If S/R is unramified, then $ds = 0$ for all $s \in S$, so $\Omega_{S/R} = 0$.

Conversely, if there is a A and A/J that there are two liftings τ_1, τ_2, then we let $A_{univ} \to A$ defined by $s_1 \otimes s_2 \to \tau_1(s_1)\tau_2(s_2)$, this is well-defined, and because $A_{univ} \cong S$, this map descends to S, so $\tau_1(s_1 s_2) = \tau_2(s_1 s_2)$. ☐

Prop. (I.7.5.2) (Formally Unramified Stalkwise). Formally unramified is stalkwise both on the source and target(I.5.1.27).

Prop. (I.7.5.3). Colimits of formally unramified rings over R is formally unramified. (Trivial as one renders on the diagram in the definition of formally unramified).

Unramified Map

Def. (I.7.5.4). A ring map is called **unramified** iff it is formally unramified and f.g..

A ring map is called G-**unramified** iff it is formally unramified and of f.p.. In particular, an étale map is G-unramified.

These two notions are stable under composition and base change. These two notions are local on the source and target. $R \to R_f$ is G-unramified.(I.7.5.1)(I.5.1.27)

Prop. (I.7.5.5). $R \to R/I$ is unramified, and if I is f.g., then it is G-unramified. (Trivial).

Prop. (I.7.5.6) (Stalkwise and Fiberwise). If $R \to S$ is of f.t(f.p.), then it is unramified(G-unramified) at a prime q of S iff $(\Omega_{S/R})_q = 0$ iff $\Omega_{S/R} \otimes_S k(q) = 0$ iff $(\Omega_{S\otimes k(p)/k(p)})_q = 0$ iff $\Omega_{S\otimes k(p)/k(p)} \otimes k(q) = 0$.

Proof: By Nakayama, two pair of them are equivalent, and if $\Omega_{S/R,q} = 0$, then $\Omega_{S/R,g} = 0$ for some $g \notin q$(because support of finite module is open), so $R \to S_g$ is (G-)unramified. And notice in fact $\Omega_{S/R} \otimes_S k(q) = \Omega_{S\otimes k(p)/k(p)} \otimes_{k(p)} k(q)$. □

Prop. (I.7.5.7) (Equivalent Definition of Unramifiedness). A f.g. ring map $R \to S$ is unramified at a prime q of S over p iff $pS_q = qS_q$ and $k(q)/k(p)$ is finite separable.

Proof: Suppose $R \to S_g$ is unramified, then $S \otimes k(p)$ is unramified over $k(p)$, hence by(I.7.4.25), it is also smooth, so it is étale, and(I.7.6.9) gives the result.

For the converse, Cf[StackProject 02FM]. □

Prop. (I.7.5.8). A ring map is unramified iff it is locally a quotient of a standard étale map.

Proof: Cf.[StackProject 0395]. □

Prop. (I.7.5.9). Any G-unramified map is a base change of a G-unramified map over a ring R_0 f.g. over $\mathbb{Z}$. And similarly any unramified map is a quotient of a base change of a G-unramified map over a ring R_0 f.g. over $\mathbb{Z}$.

Proof: Let $S = R[X_1, \ldots, X_n]/(g_1, \ldots, g_c)$, then we have $dX_i = \sum a_{ij}dg_j + a_{ijk}g_j dX_k$, so we let R_0 be generated by g_i, a_{ij}, a_{ijk}, so $S_0 = R_0[X_1, \ldots, X_n]/(g_1, \ldots, g_c)$ is G-unramified. □

Prop. (I.7.5.10) (Unramifiedness and Idempotent). If $R \to S$ is of f.t., then it is unramified iff $S \times_R S \to S$ is isomorphic to $S \otimes_R S \to (S \otimes_R S)_e$ for some diagonal idempotent $e \in S \otimes_R S$ that $e\,\mathrm{Ker}(\mu) = 0$, i.e. $S \otimes_R S \cong S \times S'$.

Proof: If it is G-unramified, the kernel I satisfies $I/I^2 = 0$, and I is f.g.(by $x_i \otimes 1 - 1 \otimes x_i$) so we can use(I.5.2.5).

Conversely, the existence of the diagonal idempotent e implies that $I = I^2$. □

6 Étale

Formally Étale

Def. (I.7.6.1). A ring map $R \to S$ is called **formally étale** iff it is formally smooth and formally unramified.

Prop. (I.7.6.2). Colimits of formally étale rings over R is formally étale. (The lifting are compatible because of uniqueness).

Prop. (I.7.6.3). $R \to S^{-1}R$ is formally étale.

Proof: It suffice to prove that if $\varphi(s)$ is invertible modulo I, then $\varphi(s)$ is invertible, but this is true because I is nilpotent. □

Étale Map

Def. (I.7.6.4). A ring map $R \to S$ is called **étale** if it is of f.p. and the naive cotangent complex is exact, i.e. $I/I^2 \cong \Omega_{P/R} \otimes_P S$.

In particular, étale is equivalent to smooth+formally unramified($\Omega_{R/S} = 0$).

Cor. (I.7.6.5) (Properties of Étale).

1. Étale map is stable under base change and composition.
2. Étale map is local on the source and target. In particular, $R \to R_f$ is étale.
3. If $R \to S$ is of f.p. and $R \to R'$ is flat. Then the set of primes in $S' = S \otimes_R R'$ that has a nbhd that is étale over R' is the inverse image of set of primes in S that has a nbhd that is étale over R. (The same as(I.7.4.21)).
4. Étale map is syntomic, hence flat.
5. Any Étale map is a base change of an étale map over a ring R_0 f.g. over $\mathbb{Z}$. (Cf.[StackProject 00U2]).

Prop. (I.7.6.6) (Jacobson Criterion). Any étale map is equivalent to a standard smooth ring map $S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$ that $J(\frac{f_1,\ldots,f_n}{X_1,\ldots,X_n})$ is invertible in S.

Proof: $I/I^2 \cong \Omega_{P/R} \otimes_P S$, so I/I^2 is free, so by(I.6.7.11), there is a presentation of S that $f_1, \ldots, f_c$ freely generate I/I^2, then obviously $c = n$ and $J(\frac{f_1,\ldots,f_c}{X_1,\ldots,X_n})$ is invertible in S, i.e. S is standard smooth. □

Cor. (I.7.6.7) (Example of Étale Maps). The ring

$$S = R[X_1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_n, X_{n+1}\det(\frac{f_1, \ldots, f_n}{X_1, \ldots, X_n}) - 1)$$

is étale over R.

Prop. (I.7.6.8) (Étale over Fields). An algebra over a field k is étale iff it is a finite product of finite separable extensions of k.

Proof: If k'/k is finite separable, then $k' = k(\alpha)$ for some α by primitive element theorem, thus $k' = k[X]/(f)$ that f' is invertible in k', thus it is étale by(I.7.6.6).

Conversely, Cf.[StackProject 00U3]. □

Cor. (I.7.6.9). If $R \to S$ is étale at a nbhd of a prime q of S over p, then $pS_q = qS_q$, and $k(q)/k(p)$ is finite separable.

Proof: We can replace S by S_g so S_g/R is étale. Then $S \otimes k(p)/k(p)$ is étale, that is S_p/pS_p is a finite product of finite separable fields, so $S_q/pS_q = (S_p/pS_p)_q =$ some separable closed field. □

Lemma (I.7.6.10). If $R \to S$ is an étale map and q is a prime of S over p, then S/R is étale in a nbhd of q if

- $R \to S$ is of f.p.
- $R_p \to S_q$ is flat.
- $pS_q = qS_q$.
- $k(q)/k(p)$ is a finite separable field extension.

Proof: Cf.[StackProject 00U6]. □

Prop. (I.7.6.11) (Equivalent Definition of Étale). A ring map $R \to S$ is **étale** iff it is flat, of f.p. and $\Omega_{S/R}$ vanishes.

Proof: One way is by definition, and the converse is by(I.7.6.10) and(I.7.5.7). □

Prop. (I.7.6.12). A ring map of f.p. is formally étale iff it is étale. (Because in this case, formally smooth is equivalent to smooth(I.7.4.14).)

Prop. (I.7.6.13). If S/R and S'/R are étale, then any R-algebra map $S \to S'$ is étale.

Proof: $S \to S'$ is of f.p. by(I.6.7.9), the rest Cf.[StackProject 00U7]. □

Prop. (I.7.6.14) (Étale Algebra seen explicitly as Finite Projective Modules). Étale algebras are finite projective, by(I.6.1.4). And we can see this clearly as follows: There is an diagonal idempotent as it is unramified(I.7.5.10), If $e = \sum a_i \otimes b_i$, then we can realize S as a direct command of R^n through maps

$$S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$$

where $\alpha(f) = (\mathrm{tr}_{S/R}(fa_i))$, and $\beta((g_i)) = \sum g_i b_i$.

Proof: We check that $\beta \circ \alpha = \mathrm{id}$: Notice first that $\mathrm{tr}_{i_2}(e) = \mathrm{tr}_{S/S}(1) = 1$, following from the decomposition above, so $\sum \mathrm{tr}_{S/R}(a_i)b_i = 1$, thus shows that $\beta\alpha(1) = 1$.

Now for general f, using the formula $(f \otimes 1)e = (1 \otimes f)e$, we get $\sum \mathrm{tr}(fa_i)b_i = \sum \mathrm{tr}(a_i)b_i f = f$. □

Prop. (I.7.6.15). If R is a ring and I is an ideal, then any étale ring map $R/I \to \overline{S}$ comes from an étale ring map $R \to S$.

Proof: Use(I.7.6.6), an étale map is of the form $\overline{S} = R/I[X_1, \ldots, X_n]/(\overline{f}_1, \ldots, \overline{f}_n)$ that $\delta = J(\frac{f_1,\ldots,f_n}{X_1,\ldots,X_n})$ is invertible in S, then we take $S = R[X_1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_n, X_{n+1}\delta - 1)$, then it is étale by(I.7.6.7) and maps to $\overline{S}$. □

Standard Étale

Def. (I.7.6.16). A ring map $R \to R' = R[X]_g/(f)$ is called **standard étale** iff f is monic and the derivative f' is invertible in R'.

Standard étale is stable under base change and principal localization, but not stable under composition.

Prop. (I.7.6.17) (Étale and Standard Étale). A ring map is étale iff it is locally standard étale.

Proof: For a standard étale algebra $R[X]_g/(f) = R[X,Y]/(f, gY-1)$ which is standard smooth and $\Omega_{R'/R} = 0$(VI.2.1.8), so it is étale. To prove if it is locally standard étale then it is étale, Cf.[StackProject 00UE]. □

Prop. (I.7.6.18). Giving any ring R and a prime p, if there is a finite separable extension $L/k(p)$, then there is a standard étale map $R \to R'$ that for some q', $k(q') \cong L$ over k.

Proof: $L = k(p)[\alpha]$ by primitive element theorem, so the minimal polynomial of α is separable, and if we change α to $c\alpha$ for some $c \in k(p)$, we can assume f can be lifted to a $f \in R[X]$. Now $f'(\alpha)$ is invertible in L, so there is a map from $R[X]_{f'}/(f)$ to L, whose kernel gives the desired prime q. □

Étale over Fields

Prop. (I.7.6.19) (Étale and Unramified over Fields). A f.g. algebra is étale over field k iff it is G-unramified over it, by(I.7.4.25).

7 Separability

Basic reference is [Weibel Chap P309] and [StackProject 10.41].

Def. (I.7.7.1). A f.d simisimple algebra R over a field k is called **separable** iff for every field extension l/k, $R \otimes_k l$ is semisimple.

Prop. (I.7.7.2).

8 Japanese & Nagata Rings

Def. (I.7.8.1). Let R be a domain with quotient field K, then R is called N**-1** iff the integral closure of R in K is a finite R-module.

R is called N-2 or **Japanese** iff for any finite field extension L/K, its integral closure in L is a finite R-module.

A ring R is called **universally Japanese** if for any finite type domain S/R, S is Japanese.

A ring R is called **Nagata** if it is Noetherian and for any prime p, R/p is Japanese.

Prop. (I.7.8.2). A f.g. algebra A over a field is Nagata.

Proof: Cf.[Hartshorne P20]. □

Cor. (I.7.8.3). The normalization of a f.g. integral domain over a field is f.g. over A.

9 Separably Generated Field Extension

Basic reference is [Matsumura Ch10] and [StackProject 10.41, 10.43].

Def. (I.7.9.1). A field extension K/k is called **separably generated** iff it K is a separable algebraic extension of a purely transcendental field L/k.

A field extension K/k is called **separable** iff all f.g. subextensions are separably generated.

An algebra A/k is called **separable** iff $A \otimes_k k'$ is reduce for any k'/k algebraic.

Prop. (I.7.9.2). If K/k is a separable field extension and S is a reduced k-algebra, then $S \otimes_k K$ is reduced.

Proof: Cf.[StackProject 030U]. □

Prop. (I.7.9.3) (Separable and Geo.Reduced). Let K/k be a field extension of char p, then K/k is separable iff $K \otimes_k k^{1/p}$ is reduced, iff K/k is geometrically reduced.

Proof: Cf.[StackProject 030W]. □

Cor. (I.7.9.4). A separably generated field extension is separable, Cf.[StackProject 030X].

10 Henselian Local Ring

Basic References are [StackProject Chap10.148].

Def. (I.7.10.1). A local ring $(R, \mathfrak{m}, k)$ is called **Henselian** iff for every $f \in R[X]$ and $a_0 \in k$ that $\overline{f}(a_0) = 0$ and $\overline{f}'(a_0) \neq 0$, then there is a root α of f lifting a_0. It is called **strict Henselian** if moreover its residue field is separably closed.

Henselian Pairs

Def. (I.7.10.2). A **Henselian pair** is a pair (A, I) that is Zariski and for any f, g in $A[T]$ monic and $\overline{f} = \overline{g}\overline{h} \in A/I[T]$ that is coprime and monic, there is a factorization $f = gh$ lifting the decomposition.

Prop. (I.7.10.3). Filtered limits of Henselian pairs is Henselian, this is clear from the definition(I.7.10.2).

Lemma (I.7.10.4). If A is a ring with ideal I, if $\overline{f} = \overline{g}\overline{h}$ be a factorization of a polynomial $f \in A[T]$ in $A/I[T]$, then there is an étale ring map $A \to A'$ that $A/IA \cong A'/IA'$, and a factorization $f = g'h' \in A'[T]$ lifting the factorization.

Proof: Cf.[StackProject 0ALH] □

Prop. (I.7.10.5). If I is locally nilpotent, then $A_{ét} \cong (A/I)_{ét}$, and (A, I) is Henselian.

Proof: First if $A \to S$ is étale, then clearly $A/I \to S/IS$ is étale by Jacobson Criterion, and the map is essentially surjective by(I.7.6.15). And any map $B/IB \to B'/IB'$ can be lifted to $B \to B$ because étale is smooth and use(I.7.4.22). And the lifting is unique, otherwise if f, g are two lifting, because étale is unramified, so if we choose an idempotent e generating the kernel of $B \otimes_A \to B \to B$(I.7.5.10), then $f \otimes g(e) \in IB'$, which is locally nilpotent, thus $f \otimes g(e) = 0$, thus $f = g$.

For then Henselian, I is clearly contained in the Jacobson radical, and for the decomposition, by(I.7.10.4) these is an étale map $A \to A'$ that $A/IA \cong A'/IA'$ that lifts the factorization, but $A = A'$, by what we have seen above. □

Cor. (I.7.10.6) (Complete Pair is Henselian). If (A, I) is a pair that A is I-adically complete, then (A, I) is Henselian.

Proof: I is in the Jacobson radical because $1 + I$ consists of units, and by(I.7.10.5) and(I.7.10.4) we can lift the decomposition to A/I^n inductively. As $A = \lim A/I^n$, we are done. □

Prop. (I.7.10.7) (Equivalent Definitions of Henselian Pair). The following are all equivalent to (A, I) being Henselian:

- Given any étale ring map $A \to A'$, then any $A' \to A/I$ lifts to an A-algebra map $A' \to A$.
- For any finite/integral A-algebra B, the map $B \to B/IB$ induces a bijection on idempotents.
- (Gabber) (A, I) is Zariski and every monic polynomial $f(T) \in A[T]$ of the form $T^n(T-1) + a_nT^n + \ldots + a_1T + a_0$ with $a_i \in I$ has a root $\alpha \in 1 + I$.

Moreover, root in item3 is unique.

Proof: Cf.[StackProject 09XI]. □

Cor. (I.7.10.8). if (A, I) is Henselian and $A \to B$ is integral, then (B, IB) is also Henselian.

Prop. (I.7.10.9). A Zariski pair (R, I) is Henselian iff the pair $(\mathbb{Z} \oplus I, I)$ is Henselian. In particular, the property of being Henselian only depends on the non-unital ring I.

Proof: Cf.[Almost Ring Theory, 5.1.9]. □

11 Local Algebra

Prop. (I.7.11.1). If A is a Noetherian local integral domain with residue field k and quotient field K, if M is a f.g. A-module that $\dim_k M \otimes_A k = \dim_K M \otimes_A K = r$, then M is free of rank r.

In other words, if the rank of M at the generic point and closed pt of B are the same, then M is free.

Proof: First M is generated by r elements by Nakayama and the kernel R of $A^r \to M$ vanishes when tensoring K, thus vanish because it is torsion-free. □

12 Dualizing Module

I.8 Homological Algebra

1 Category

Exactness

Prop. (I.8.1.1). In an Abelian category, the functor $X \mapsto \mathrm{Hom}(X,Y)$ and $X \mapsto \mathrm{Hom}(Y,X)$ is both left exact. Note that left and right is seen on the image.

Adjointness

Def. (I.8.1.2). Two functors $f : \mathcal{C} \to \mathcal{D}$, $g : \mathcal{D} \to \mathcal{C}$ are called **left/right adjoint** iff there is natural isomorphism of functors:

$$\mathcal{C}^{op} \times \mathcal{D} \to Set : \mathrm{Hom}(fX,Y) \cong \mathrm{Hom}(X,gY)$$

Prop. (I.8.1.3). If f,g are adjoints, then there are natural maps $u : X \to gfX$, and $v : fgY \to Y$, called **unit/counit maps**. They satisfies $fX \to fgfX \to fX$ is id, and $gY \to gfgY \to fY$ is id.

Conversely, if f,g satisfies this two identity, then they are adjoint, by

$$\mathrm{Hom}(fX,Y) \to \mathrm{Hom}(gfX,gY) \to \mathrm{Hom}(X,gY) \to \mathrm{Hom}(fX,fgY) \to \mathrm{Hom}(fX,Y).$$

Proof:

□

Prop. (I.8.1.4). A right adjoint functor is left exact and it preserves injectives if its left adjoint is exact.

A left adjoint functor is right exact and it preserves projectives if its right adjoint is exact.

Prop. (I.8.1.5). Any presheaf on a small category is a colimit of representable sheaves h_X. (Consider all $h_X \to \mathcal{F}$ and take colimit, prove it is isomorphism).

Prop. (I.8.1.6). The sheaf Γ functor is right adjoint to the constant sheaf functor over arbitrary site.

Prop. (I.8.1.7). The inclusion functor is right adjoint to the shifification functor over arbitrary site.

Prop. (I.8.1.8). The forgetful functor is right adjoint to the Shifification functor, and shifification is exact, so it preserves injectives.

Prop. (I.8.1.9). The stalk functor is left adjoint to the skyscraper sheaf operator.

Prop. (I.8.1.10). The valuation at k'th coordinate is left adjoint to the functor $k_*(A)(i) = \prod_{\mathrm{Hom}\, i,k} A$ and is exact. So k_* preserves injectives.

Kan Extension

Prop. (I.8.1.11) (Yoneda Lemma). $h_X : Y \mapsto \mathrm{Hom}(Y, X)$ is a presheaf, and $\mathrm{Hom}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$ for any presheaf $\mathcal{F}$.

So $X \to h_X$ is a fully faithful embedding from $\mathcal{C}$ to $\hat{\mathcal{C}} = \mathrm{Func}(\mathcal{C}^\circ, \mathrm{Set})$. In particular, if a $X \to Y$ induces isomorphism $\mathrm{Hom}(W, X) \to \mathrm{Hom}(W, Y)$ for every W, then $X \cong Y$.

So we can regard $\mathcal{C}$ as a fully faithful subcategory of $\widehat{\mathcal{C}}$.

Proof: The map $\mathrm{Hom}(h_X, \mathcal{F}) \to \mathcal{F}(X)$ maps a u to $u(X)(\mathrm{id}_X)$. And the inverse map is defined to be $x \in \mathcal{F}(X) \mapsto (s \in \mathrm{Hom}(Y, X) \mapsto s^*(x) \in \mathcal{F}(Y)) \in \mathrm{Hom}(h_X, \mathcal{F})$. □

Cor. (I.8.1.12). A **universal object** for a presheaf $\mathcal{F}$ is a pair (X, ζ) that $\zeta \in F(X)$ with the property that for any U and a $\sigma \in \mathcal{F}()$, there is a unique arrow $U \to X$ that $Ff(\zeta) = \sigma$.

In fact, a universal object is equivalent to an isomorphism $h_X \cong \mathcal{F}$, and this makes it easy to check whether a presheaf $\mathcal{F}$ is representable.

Prop. (I.8.1.13). A presheaf of sets in C, i.e. $C^{op} \to \mathcal{S}et$ is a colimit of presentable sheaves of C. More precisely, there is an isomorphism

$$\mathcal{F} \cong \varinjlim_{h_X \to \mathcal{F}} h_X.$$

From this we see that any morphism $\widehat{\mathcal{C}} \to D$ is determined by its restriction on $\mathcal{C}$.

Proof: For any presheaf $\mathcal{G}$, there is a morphism $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\varinjlim_{h_X \to \mathcal{F}} h_X, \mathcal{G})$, i.e. a set of sections $f_s \in \mathcal{G}(X)$ for every $h_X \xrightarrow{s} \mathcal{F}$, that if $t \circ u = s$, then $u^*(f_t) = f_s$. Conversely, by Yoneda lemma, this just says that there is a morphism of presheaves $\mathcal{F} \to \mathcal{G} : F(X) \to G(X) : s \mapsto f_s$. □

Cor. (I.8.1.14) (Kan Extension). For a cocomplete category $\mathcal{D}$, there is a natural bijection between functor $\widehat{\mathcal{C}} \to \mathcal{D}$ that commutes with colimits and functors $\mathcal{C} \to \mathcal{D}$ by Yoneda embedding.

Proof: For this, we only have to notice the functor $\mathcal{D} \to \widehat{\mathcal{C}} : D \to \mathrm{Hom}(FX, D)$ is right adjoint to $F : \widehat{\mathcal{C}} \to \mathcal{D}$ when F is defined by colimit as in(I.8.1.13). □

Cor. (I.8.1.15). Any contravariant functor $F : \widehat{\mathcal{C}} \to \mathcal{S}et$ that take colimits to limits, F is representable. (Just use G in the last proof, F is representable by $G(\mathrm{pt})$.

Prop. (I.8.1.16) (Ends and Coends). Cf.[MacLane].

Prop. (I.8.1.17) (Category Equivalence). A Functor $\mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it's fully faithful and essentially surjective.

Proof: There exist an object $G(X) \in \mathcal{C}$ and an isomorphism $\xi_X : FG(X) \to X$ for every $X \in \mathcal{D}$. Because F is fully faithful, there exists a unique morphism $G(f) : G(X) \to G(Y)$ such that $F(G(f)) = \xi_Y^{-1} \circ f \circ \xi_X$ for every morphism $f : X \to Y$ in $\mathcal{D}$. Thus we obtain a functor $G : \mathcal{C} \to \mathcal{D}$ as well as a natural isomorphism $\xi : F \circ G \to \mathrm{Id}_{\mathcal{D}}$. Moreover, the isomorphism $\xi_{F(Z)} : FGF(Z) \to F(Z)$ decides an isomorphism $\eta_Z : GF(Z) \to Z$ for every $Z \in \mathcal{C}$.This yields a natural isomorphism $\eta : G \circ F \to \mathrm{Id}_{\mathcal{C}}$. □

Morita Equivalence

Basic References are [Morita Equivalence] and [Fuller Rings and Categories of Modules].

Def. (I.8.1.18). Two ring R, S are called **Morita equivalent** if the category of mod-R is equivalent to the category of mod-S.

Prop. (I.8.1.19). For an Abelian category $\mathcal{A}$ satisfying AB3 (i.e arbitrary sum exists), An object P of $\mathcal{A}$ is a **progenerator** if the functor $h' : X \mapsto \mathrm{Hom}_{\mathcal{A}}(P, X)$ is exact and and strict: $h'(X) = 0 \to X = 0$. Then h' determines an equivalence from $\mathcal{A}$ to mod-R, where $R = \mathrm{Hom}_{\mathcal{A}}(P, P)$.

Similarly, if $\mathcal{A}$ is an Abelian Noetherian category and P is a progenerator, then R is Noetherian and $\mathcal{A}$ is equivalent to the category of finitely generated R-categories.

Proof: Essentially surjective: construct using direct limit and cokernel.

Notice that $h'(X) \cong h'(X') \to X \cong X'$ by strictness and A4 axiom. So let $X = \mathrm{Coker}(P^{\oplus I}, P^{\oplus J})$,

$$\begin{aligned}\mathrm{Hom}(h'(X), h'(Y)) &= \mathrm{Hom}(\mathrm{Coker}(h'(P^{\oplus J}), h'(P^{\oplus I}), h'(Y)) \\ &= \mathrm{Ker}(\mathrm{Hom}(h'(P^{\oplus J}), h'(Y)) \to \mathrm{Hom}(h'(P^{\oplus I}), h'(Y))) \\ &= \mathrm{Ker}(h'(Y^{\Pi I}) \to h'(Y^{\Pi J})) \\ &= \mathrm{Hom}(X, Y)\end{aligned}$$

□

Prop. (I.8.1.20). In the case when A is the category mod-R, P is a generator $\iff$ $h' : X \mapsto \mathrm{Hom}_R(P, X)$ is faithful $\iff$ every M is a quotient of direct sums of P. And a **progenerator** is a f.g. projective generator.

Prop. (I.8.1.21). Let P be a (A, B)-bimodule, iff P is a progenerator as a right B module, then it is a progenerator as a left A module.

Prop. (I.8.1.22). Let P be a progenrator as a

Prop. (I.8.1.23) (Morita). The following are equivalent:

- categories A-mod and B-mod are equivalent.
- categories mod-A and mod-B are equivalent.
- There exist a finitely generated progenerator P of mod-A that $B \cong \mathrm{End}_A P$.

Proof: $2 \to 3$:A is a progenerator in mod-A, thus when $A \sim B$, F : mod-$A \to$ mod-B, $A \cong \mathrm{End}_A A = \mathrm{End}_B F(A)$, and $F(A)$ is a left A module as well as a progenerator of B. Thus there is a (A, B)-bimodule P that $A \cong \mathrm{End}_B P$, and a (B, A)-bimodule Q that $B \cong \mathrm{End}_A Q$. □

Prop. (I.8.1.24). There can be defined another Morita invariance that $R \sim S$ iff there are (R, S)-bimodule P and (S, R)-bimodule Q that $P \otimes_S Q \cong R$ as a (R, R)-bimodule and $Q \otimes_R P \cong S$ as a (S, S)-bimodule. This will immediately generate equivalence between R-mod and S-mod as well as equivalence between mod-R and mod-S by tensoring. And P and Q are projective modules respectively, because equivalence is a kind of adjoint.

Prop. (I.8.1.25) (Properties Preserved under Morita Invariance). Cf.[Rings and Categories of Modules P54].

Fiber Product

Prop. (I.8.1.26). For a category C, the following are equivalent:

- It has arbitrary limits.
- it has arbitrary products and equalizer.
- it has arbitrary products and fibered products.

Proof: $1 \to 2, 1 \to 3$ is trivial. $3 \to 2$: The equalizer for $f, g : X \to Y$ can be constructed as the base change of $Y \to Y \otimes Y$ along $(f, g) : X \to Y \times Y$. $2 \to 1$: for any diagram $F : I \to C$, the fibered pullback can be constructed as the equalizer of two morphisms:

$$s, t : \prod_{i \in \mathrm{Ob}(I)} F(i) \to \prod_{f: j \to k \in \mathrm{Mor}(I)} F(k)$$

where $\pi_{(f:j\to k)} s = \pi_k$, and $\pi_{(f:j\to k)} t = (Ff)\pi_j$. □

Prop. (I.8.1.27) (Diagonal Base Change). The diagonal commutes with base change:

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\Delta} & (X \times_Y Z) \times_Z (X \times_Y Z) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad\Delta\quad} & X \times_Y X \end{array}$$

Proof: □

Prop. (I.8.1.28). $(X \times_E Y) \times_S (Z \times_F W) = (X \times_S Z) \times_{E \times_S F} (Y \times_S W)$.

Proof: □

Prop. (I.8.1.29). For $X \to T$ and $Y \to T$ and $T \to S$, $X \times_T Y = T \times_{T \times_S T} (X \times_S Y)$. In particular, $X \times_T Y \to X \times_S Y$ is a base change of $T \to T \times_S T$.

Proof: □

Prop. (I.8.1.30). The diagonal map $X \to X \times_Y X$ is an isomorphism iff $X \to Y$ is monomorphism. (Because this is equivalent to $\mathrm{pr}_1 = \mathrm{pr}_2$).

Localization

Def. (I.8.1.31) (Localizing System). A class of morphisms S in a category is called **left(resp. right) localizing** if:

- S is closed under composition and has all the identities.
- for every $s \in S$ and f with the same source, there is a $t \in S$ and g, s.t. $f \circ t = s \circ g$(resp. $t \circ f = g \circ s$).
- the existence of a $t \in S$ s.t. $ft = gt$ implies(resp. is implied by) the existence of a $s \in S$ s.t. $sf = sg$.

It is called **localizing** if it is both left localizing and right localizing.

Def. (I.8.1.32) (Saturation).

For localizing of a category, Cf.[StackProject Chap4.26].

Prop. (I.8.1.33). If S is localizing in a category $\mathcal{C}$, then the morphisms in $\mathcal{C}$ that is mapped to an isomorphism in $S^{-1}\mathcal{C}$ is exactly the saturation of S.

Proof: Cf.[StackProject 05Q9]. □

Prop. (I.8.1.34) (Localization Category).

- If $\mathcal{C}$ is a category and S is a left localizing class, then the rule $X \mapsto X, (f : X \to Y) \mapsto \mathrm{id}^{-1} f$ is a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.
- Any $s \in S$ is mapped via Q to an isomorphism, and for any other category $\mathcal{D}$ and $F : \mathcal{C} \to \mathcal{G}$ that satisfies s is mapped to isomorphisms, it factors through $S^{-1}\mathcal{C}$ uniquely.
- The functor Q preserves finite colimits.

Proof: Cf.[StackProject 04VG], [StackProject 05Q2]. □

Group Objects

Def. (I.8.1.35) (Group Object). In a category C with finite products and a final object e, a **group object** is an object G that represents a functor from C to *Grp*. And a homomorphism of group objects is a natural transformation as a functor from C to *Grp*.

This is in fact equivalent to a morphism $m_G : G \times G \to G$ and $i_G : G \to G$, $e_G : e \to G$ that satisfies the desired commuting diagrams.

Def. (I.8.1.36). A **(left)action** of a group object on an object X is equivalent to a morphism $G \times X \to X$ that satisfies the desired commuting diagrams.

Prop. (I.8.1.37). The group objects in the category of groups is abelian groups.

Proof: By Eckmann-Hilton argument(I.8.1.39), the category multiplication is the same as the group multiplication, so the unit is obviously the sam unit, thus the inverse. So the commutativity of m with inverse implies that it is abelian. □

Prop. (I.8.1.38). One should notice that the group object structure in any category $(m, id, i, X$ definition) is equivalent to a group structure on $\mathrm{Hom}(Y, X)$ that are preserve under composition with morphisms.

Others

Prop. (I.8.1.39) (Eckmann-Hilton argument). If $\circ$ and $\otimes$ are two unital binary operator that commutes:$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$, then they are equal and in fact commutative and associative. Cf.[Wiki].

2 Abelian Category

Def. (I.8.2.1) (Axioms for Abelian Category).

- **A1**:$\mathrm{Hom}(X, Y)$ is an Abelian group.
- **A2**:There exists a zero object.

- **A3**:There exists a canonical sum&product with projections, and the sum induce the Abelian structure of $\mathrm{Hom}(X, Y)$.

(Satisfying this three is called an **additive category**.)

- **A4**:Coimage equals image.

Def. (I.8.2.2). A functor between additive categories $F : \mathcal{C} \to \mathcal{D}$ is called **additive** if $F : \mathrm{Mor}(X, Y) \to \mathrm{Mor}(FX, FY)$ is a homomorphism of Abelian groups for all $X, Y \in \mathcal{C}$.

Remark (I.8.2.3). WARNING: An additive category that epimorphism+monomorphism is isomorphism need not be an Abelian category. Cf.[https://mathoverflow.net/questions/41722/is-every-balanced-pre-abelian-category-abelian] for a counter-example.

Prop. (I.8.2.4). The $\mathrm{Hom}(X, -)$ operator is left exact in Abelian category by definition.

Prop. (I.8.2.5). Axiom A3 asserts the good existence of product and sum of objects as we wanted, and it can be used to prove that monomorphism and epimorphism are stable under pushout and pullback.

But this uses $A4$ strongly, Cf.[MacLane Categories for working mathematicians P203]. (For epimorphism, first prove $0 \to X \times_U Y \to X \times Y \to U \to 0$ is exact when $X \to U$ is epi).

Prop. (I.8.2.6). equalizer and finite product derives finite limit, thus finite limits and finite colimits exists in Abelian categories.

Prop. (I.8.2.7) (Mitchell's embedding theorem). If $\mathcal{A}$ is a small category, then there exists a unital ring R, not necessary commutative and a fully faithful and exact functor $\mathcal{A} \to R-$mod that preserves kernel and cokernel. WARNING: it may not preserve sum and product, let alone limits and colimits.

Proof: □

Prop. (I.8.2.8). If $\mathcal{C}, \mathcal{A}$ are categories and $\mathcal{A}$ is Abelian, then $\mathcal{H}om(\mathcal{C}, \mathcal{A})$ is an Abelian category. In particular, $\mathrm{Ch}(\mathcal{A})$ is Abelian.

Localization

Prop. (I.8.2.9) (Localization Category). If $\mathcal{C}$ is a preadditive category and S is a left or right localizing system of $\mathcal{C}$, then there exists a natural additive structure on $\mathcal{S}^{-1}\mathcal{C}$ and a localizing functor $\mathcal{C} \to S^{-1}\mathcal{C}$ that is additive.

Proof: Cf.[StackProject 05QD]. □

Lemma (I.8.2.10).] If $\mathcal{C}$ is additive and S is localizing, let X be an element of $\mathcal{C}$, then: $Q(X) = 0$ iff there is a morphism $0 : X \to Y$ that is an element of S iff there is a morphism $0 : Z \to X$ that is an element of S.

Proof: If such $0 : X \to Y \in S$, then it maps to isomorphisms in $S^{-1}(\mathcal{C})$ by(I.8.1.34), so $Q(X) = 0$ If $Q(X) = 0$, then the morphism $0 \to X$ is mapped to an isomorphism, so by(I.8.1.33), there are g, h that $fg = hf = 0$, so $Z \to 0 \to X \in S$. Dually for the other direction. □

Prop. (I.8.2.11). If $\mathcal{A}$ is Abelian and S is localizing, then $S^{-1}\mathcal{A}$ is an Abelian category and $\mathcal{A} \to S^{-1}\mathcal{A}$ is exact.

Proof: Cf.[StackProject 05QG]. □

Serre Subcategory

Def. (I.8.2.12). A **Serre subcategory** of an Abelian category is a non-empty full subcategory $\mathcal{C}$ that if

$$A \to B \to C$$

is exact and $A, C \in Ob(\mathcal{C})$, then $B \in Ob(C)$.

A **weak Serre subcategory** of an Abelian category is a non-empty full subcategory $\mathcal{C}$ that if

$$A \to B \to C \to D \to E$$

is exact and $A, B, D, E \in \mathcal{C}$, then $C \in \mathcal{C}$.

Prop. (I.8.2.13).

- A Serre category is equivalent to a full subcategory $\mathcal{A}$ that contains 0, all the subobjects and quotient objects of $\mathcal{A}$, and extensions of objects of $\mathcal{A}$ are in $\mathcal{A}$. In this case, the functor $i : \mathcal{A} \to \mathcal{C}$ is exact.
- A weak Serre category is equivalent to a full subcategory $\mathcal{A}$ that contains 0, and all the kernels, cokernels of objects in $\mathcal{A}$, and all the extensions of objects in $\mathcal{A}$. In this case, the functor $i : \mathcal{A} \to \mathcal{C}$ is exact.

Proof: One direction of these two are trivial, it suffices to prove the converse. For the first, $0 \to \operatorname{Im} A \to B \to \operatorname{Im} B \to 0$, so $B \in \mathcal{C}$. For the second, $0 \to \operatorname{Coker}(A \to B) \to C \to \operatorname{Ker}(D \to E) \to 0$, so $C \in \mathcal{C}$. □

Prop. (I.8.2.14) (Quotients by Serre Subcategory). For an exact functor between Abelian categories, the objects that mapped to 0 forms a Serre subcategory. And any Serre subcategory is the kernel of an essentially surjective map. i.e. the quotient Abelian category $\mathcal{C}/\mathcal{A}$ can be constructed, and satisfies universal properties.

Proof: The full subcategory of $\operatorname{Ker}(F)$ is clearly a Serre subcategory by checking the definition. Conversely, consider $S =$all the morphisms that has kernel and cokernel in $\mathcal{C}$, first we prove it is a localizing system(I.8.1.31).

The long exact seuqence(I.8.3.10) shows that if $f, g \in S$, then $gf \in S$. For other verifications, Cf.[StackProject 02MS].

Next we construct $\mathcal{C}/\mathcal{A}$ as $S^{-1}\mathcal{C}$. Consider which objects are mapped to 0 in $\mathcal{C}/\mathcal{A}$, use(I.8.2.10) and consider the kernel and cokernel, it is easy to see that $\operatorname{Ker}(Q) = \mathcal{C}$. If another category $\mathcal{D}$ and $F : \mathcal{C} \to \mathcal{D}$ satisfies $\mathcal{C}$ is mapped to0, then it is clear that elements in S is mapped to isomorphism, so it factors through $\mathcal{C}/\mathcal{A}$ by universal property(I.8.1.34). □

Prop. (I.8.2.15). For a Serre subcategory $\mathcal{B}$ of an Abelian category $\mathcal{A}$, the set of all complexes that has cohomology group in $\mathcal{B}$ is a strictly full triangulated subcategory of $\mathcal{D}(\mathcal{A})$.

Proof: Cf.[StackProject 06UQ]. □

Others

Def. (I.8.2.16) (Essential Morphism). In an Abelian category, an injection $A \to B$ is called **essential** iff every non-zero subobject of B intersects A. A surjection is called **essential** iff every proper subobject of A is not mapped to B.

Def. (I.8.2.17). In a Grothendieck Abelian category $\mathcal{A}$, an object M is called **finitely generated** if for every ascending chain

$$M_1 \subset M_2 \subset \ldots \subset M$$

with $\cup_i M_i = M$, we have $M_i = M$ for some i.

$\mathcal{A}$ is called **Noetherian** iff a subobject of a f.g. object is f.g.. $\mathcal{A}$ is called **Artinian** iff every f.g. object has finite length?.

Grothendieck Abelian Category

Prop. (I.8.2.18) (Axioms for Grothendieck Abelian Category).

- **AB3**: It is an Abelian category and arbitrary direct sums exists. (Thus colimits over small categories exists.)
- **AB5**: Filtered colimits over small categories are exact. This is equivalent to { for any family of subobjects $\{A_i\}$ of A to B indexed by inclusion can induce a morphism $\sum A_i \to B$ (internal sum)}?
- **GEN**: It has a generator, that is, an object U s.t. for any proper subobject $N \subsetneq M$, there is a map $U \to M$ that doesn't factor through N.

Prop. (I.8.2.19). The presheaf category $\mathcal{A}^C$ is a Grothendieck Abelian category if $\mathcal{A}$ is Grothendieck Abelian.

Proof: For the presheaf, the only problem is the existence of generator, for that, just construct a family of presheaves and sum them. Take $Z_X = i_{f_X}(U)$, where U is the generator of $\mathcal{A}$ and $f = \mathrm{pt} \to \mathcal{A}^C : \mathrm{pt} \to U$. Then $F(X) = \mathrm{Hom}(Z_X, F)$ by adjointness(III.1.2.5). So they are a family of generators. □

Prop. (I.8.2.20) (Injectives). In a Grothendieck Abelian category with generator U, an object is injective iff it is extendable over subobjects of U. (AB5 assures we can extend by Zorn's lemma. Then use GEN, Cf.[StackProject 079G]). If it is a family of objects, it suffice to extend over each one of them.

Prop. (I.8.2.21). Grothendieck Abelian category has a functorial injective embedding.

Proof: Cf.[StackProject 079H]. □

Prop. (I.8.2.22). A contravariant functor from a Grothendieck category to $\mathcal{S}ets$ is representable iff it takes colimits to limits.

Proof: $M \oplus M \to M$ with induce a map $F(M) \times F(M) \to F(M)$ thus $F(M)$ is a semigroup, and the inverse of id_Min $\mathrm{Hom}(M, M)$ maps to a $F(M) \to F(M)$ which is the inverse, Thus in fact F is a left adjoint functor to Ab.

Let U be a generator, $A = \sum_{s \in F(U)} U$, let $s_{univ} = (s) \in F(A) = \prod_{s \in F(U)} F(U)$. let A' be the largest objects that s_{univ} restrics to 0 in A', let $\overline{s}_{univ}$ be in $F(A/A')$ that maps to s_{univ} in $F(A)$ (because F is left exact). Then we claim $(A/A', \overline{s}_{univ})$ represents F. Cf.[StackProject 07D7]. □

Cor. (I.8.2.23). Grothendieck Category satisfies AB3*. (because $F = \prod_i \mathrm{Hom}(-, M_i)$ commutes with colimits).

Examples of Grothendieck Category

Prop. (I.8.2.24). The category of R-modules is a Grothendieck Abelian category with generator R because in R-mod category, taking filtered colimits is exact. (Diagram chasing).

Prop. (I.8.2.25). The category of Abelian presheaves and Abelian sheaves on a site is a Grothendieck category.

Proof: For the presheaf, C.f(I.8.2.19). For the sheaf, it follows from (I.8.2.27). □

Remark (I.8.2.26). The category of Abelian sheaves doesn't satisfy AB4*, i.e. not every limit of epimorphisms is epimorphism.

Proof: Consider the constant sheaf $\oplus B(\frac{p}{q}, \frac{1}{n})$ on $[0, 1]$. □

Prop. (I.8.2.27). The category of sheaf of $\mathcal{O}_X$-modules on a ringed site is a Grothendieck Abelian category. Moreover, injectives are flabby.

Proof: It is obviously an Abelian category and have limits as presheaves, and for a family of generators, take $j_!\mathcal{O}_U$ as the representative for $\Gamma(U, -)$, which is the sheaf associated to the sheaf Z_U in the proof of(I.8.2.19). Use $j_!\mathcal{O}_U$, we can see injectives are flabby, (because $j_!\mathcal{O}_U \to j_!\mathcal{O}_V$ is a monomorphism for $V \subset U$). □

Prop. (I.8.2.28). The category of Qco sheaves on a scheme is Grothendieck category, and there is a **coherentor** left adjoint to the forgetful functor.

Proof: Qco: First by(III.2.3.4), Qco is an Abelian category, and on affine open set, the colimit is an Qco sheaf, thus the limit exists in Qco and equals that of limits in the category of sheaves, thus filtered colimits is exact because $\mathcal{O}_X$-Mod is Grothendieck(I.8.2.27). The generator exists, Cf.[StackProject 077P].

The coherentor exists by the fact that $h_{\mathcal{F}}$ commutes with colimits and by the property of Grothendieck category(I.8.2.22). □

Lemma (I.8.2.29) (Gabber). Let X be a scheme, then there exists a cardinal κ that every Qco sheaf is a colimit of its κ-generated Qco subsheaves.

Proof: Cf.[StackProject 077N]. □

3 Cohomology of Complexes

Remark (I.8.3.1). Remember the translation operator $K[n]$ makes the complex lower n dimensions.

Def. (I.8.3.2). A **universal** δ **functor** between Abelian categories is one that any natural transformation from T^0 to another δ-functor will generate a δ-map. A **effaceable** δ functor is one that for any $n > 0$ and any object A, there is an injection $A \to B$ that $T^n(A) \to T^n(B) = 0$.

Prop. (I.8.3.3) (Grothendieck). A δ-functor is universal if it is effaceable.

Proof: We construct by induction on n. choose a $0 \to A \to B \to C \to 0$ such that $T^{n+1}(A) \to T^{n+1}(B) = 0$ then there is an isomorphism $T^{n+1}(A) \cong \mathrm{Coker}(T^n(B) \to T^n(C))$, and so we can construct the map on T^{n+1} induces by

$$\mathrm{Coker}(T^n(B) \to T^n(C)) \to \mathrm{Coker}(G^n(B) \to G^n(C)) \to G^{n+1}(A).$$

This can be verified to be a δ map. □

Def. (I.8.3.4) (Cone & Cylinder). The **mapping cone** of $f : K^\bullet \to L^\bullet$ is the complex $C(f)^\bullet$ that:

$$C(f) = K[1]^\bullet \oplus L^\bullet, \quad d(k^{i+1}, l^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L l^i)$$

The **mapping cylinder** of $f : K^\bullet \to L^\bullet$ is the complex $\mathrm{Cyl}(f)$ that:

$$\mathrm{Cyl}(f) = K^\bullet \oplus K[1]^\bullet \oplus L^\bullet, \quad d(k^i, k^{i+1}, l^i) = (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L l^i)$$

It is a shame I haven't see clearly the similarity of this with the topological cone and cylinder, should study it further.

Prop. (I.8.3.5) (Distinguished Triangle of $K^*(\mathcal{A})$). For any morphism $K^\bullet \to L^\bullet$, there exists a termwise-spliting exact sequence of Complexes commuting in $K(\mathcal{A})$.

$$\begin{tikzcd}
 & K^\bullet \arrow[r] \arrow[d, equal] & L^\bullet \arrow[d, "\alpha"] & & \\
0 \arrow[r] & K^\bullet \arrow[r] & \mathrm{Cyl}(f) \arrow[r] \arrow[d, "\beta"] & C(f) \arrow[r] \arrow[d, equal] & 0 & \\
 & 0 \arrow[r] & L^\bullet \arrow[r] & C(f) \arrow[r] & K^\bullet[1] \arrow[r] & 0
\end{tikzcd}$$

where $\beta\alpha = \mathrm{id}$ and $\alpha\beta \sim \mathrm{id}$. And $K^\bullet \to L^\bullet \to C(f) \to K^\bullet[1]$ is called a distinguished triangle. Any exact triple of complexes in $\mathrm{Kom}(\mathcal{A})$ is quasi-isomorphic to a distinguished triangle. In fact, we can define the distinguished triangle in $K(\mathcal{A})$ as that induced by a split exact sequence, Cf.[StackProject 014L].

Notice all this can imitate the similar parallel construction in the topology category.

Proof: Cf.[Gelfand P157] □

Cor. (I.8.3.6). A distinguished triangle will induce a long exact sequence, for this, just need to verify that the δ-homomorphism coincide with the morphism that $C(f) \to K^\bullet[1]$ induces.

Cor. (I.8.3.7). A morphism $f : K \to L$ is quasi-iso iff $C(f)$ is acyclic. It is homotopic to 0 iff f can be extended to a morphism $C(f) \to L$.

Prop. (I.8.3.8) (Five lemma). In an Abelian category, if there is a diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d, "s"] & * \arrow[r] \arrow[d, "g"] & * \arrow[r] \arrow[d, "f"] & * \arrow[r] \arrow[d, "h"] & * \arrow[d, "i"] \\
* \arrow[r] & * \arrow[r] & * \arrow[r] & * \arrow[r] & *
\end{tikzcd}$$

Where the rows are exact and g, h are isomorphisms. If i is injective, then f is surjective; if s is surjective, then f is injective.

Proof: Rotate the diagram counterclockwise 90°. Then use the two different filtration both converge(I.9.5.7). □

Prop. (I.8.3.9) (Snake lemma). In an Abelian category, if there is a diagram

$$\begin{tikzcd}
 & * \arrow[r, "i"] \arrow[d, "f"] & * \arrow[r] \arrow[d, "g"] & * \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & * \arrow[r] & * \arrow[r, "s"] & * &
\end{tikzcd}$$

where the rows are exact, then there is a long exact sequence

$$\operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h \to \operatorname{Coker} f \to \operatorname{Coker} g \to \operatorname{Coker} h$$

And if i is injective, then the first one is injective; if s is surjective, then the last one is surjective.

Proof: Rotate the diagram counterclockwise 90°. Then use the two different filtration both converge(I.9.5.7). □

Cor. (I.8.3.10). In an Abelian category, if $f : A \to B, g : B \to C$, then there is a long exact sequence:

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} gf \to \operatorname{Ker} g \to \operatorname{Coker} f \to \operatorname{Coker} gf \to \operatorname{Coker} g \to 0.$$

Proof: Use snake lemma(as modules), there is a diagrams:

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "gf"] & B \arrow[r] \arrow[d, "g"] & \operatorname{Coker} f \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C \arrow[r, equal] & C \arrow[r] & 0 & \end{tikzcd}$$

So by Snake lemma,

$$\operatorname{Ker} gf \to \operatorname{Ker} g \to \operatorname{Coker} f \to \operatorname{Coker} gf \to \operatorname{Coker} g \to 0.$$

As Abelian category is dual, we can do this dually to get:

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} gf \to \operatorname{Ker} g \to \operatorname{Coker} f \to \operatorname{Coker} gf.$$

They splint together to get the desired long exact sequence. □

Prop. (I.8.3.11). For a 3×3 diagram of complexes, the connection homomorphism satisfies an anti-commutative diagram:

$$\begin{tikzcd} H^{q-1}(Z'') \arrow[r, "\delta"] \arrow[d, "\delta"] & H^q(X'') \arrow[d, "-\delta"] \\ H^q(Z) \arrow[r, "\delta"] & H^{q+1}(X) \end{tikzcd}$$

by(I.9.1.4) as the category $K(\mathcal{A})$ is triangulated.

Prop. (I.8.3.12) (Universal Coefficient Theorem). Should be somewhere in [Weibel].

Def. (I.8.3.13) (Herbrand Quotient). For a complex of R-modules cyclic of order 2, we define the **additive Herbrand quotient** as $\operatorname{length}_R(H^0) - \operatorname{length}_R(H^1)$, when both are definable and the **multiplicative Herbrand quotient** as $|H^0|/|H^1|$ when they are both finite.

Prop. (I.8.3.14). For an exact sequence $0 \to M \to N \to K \to$ of complexes of cyclic order 2, we have $h(N) = h(M)+h(K)$ and $h^*(N) = h^*(M)h^*(K)$ in the sense that if two of them are definable, then so is the third. This is an easy consequence of long exact sequence.

Prop. (I.8.3.15). If each term of this complex has finite length, then $h(M) = 0$. If each term is finite, then $h^*(M) = 0$. This is an consequence of isomorphism theorem. So we have, if a a morphism of complexes has kernel and cokernel finite, then it induce an isomorphism on h or h^*.

4 Injectives & Projectives

Def. (I.8.4.1). An **injective object** in a Abelian category is a I s.t. $\mathrm{Hom}(-, I)$ is an exact functor, equivalently, maps to I can be extended along injections.

A **projective object** in a Abelian category is a I s.t. $\mathrm{Hom}(I, -)$ is an exact functor, equivalently, maps to I can be pulled back along surjections.

Prop. (I.8.4.2). Product of injective elements are injective, coproducts of projective elements are projective.

Prop. (I.8.4.3). In an Abelian category, the direct summand of a projective object is projective. (The summand has definition in an Abelian category).

Prop. (I.8.4.4). If a functor f between Abelian categories is left adjoint to an exact functor, then it preserves injectives. Dually for projectives.

Prop. (I.8.4.5). If A is an Abelian category, the chain complex category $Ch(A)$ is abelian by(I.8.2.8). A chain complex P is projective iff it is a split exact complex of projective objects. The same is true by dual argument for injectives.

Proof: If K is projective, use the surjection $C(\mathrm{id}_K) \to K[1]$, there is a homotopy between id_K and 0. Thus we have $x = dhx + hdx$. And if $dhx = hdy$, then $dhdy = 0$, thus $dy = 0$, so $K = dhK \oplus hdK$ and thus $K[n] = B_n \oplus B_{n+1}$. Thus K is a direct product of $0 \to B \to B \to 0$. And this one is projective if B is projective. □

Injective Resolutions

Prop. (I.8.4.6) (Horseshoe Lemma). For a exact sequence $0 \to X_1 \to X \to X_2 \to 0$ and a injective resolution of X_1 and X_2, there is a injective resolution of X commuting with them. (Choose them one-by-one, in fact, $I_n = I_n^1 \oplus I_n^2$ using the injectivity of I_n^1. Snake lemma told us that the cokernel is an exact sequence, use that to define the next one.

Prop. (I.8.4.7). For two lifting of morphisms $X_1 \to Y_1$ and $X_2 \to Y_2$, there is a lifting of the morphism $X \to Y$ compatible with that. Cf.[Weibel P2.4.6].

Prop. (I.8.4.8) (Cartan-Eilenberg Resolution). If $\mathcal{I}_{\mathcal{B}}$ is sufficiently large, for any K in $K^+(\mathcal{B})$ there is a functorial Cartan-Eilenberg resolution, that is, It induces simultaneous injective resolutions of K^n, Z^n, B^n and H^n. Moreover, the resolution for $B^i \to Z^i \to H^i$ and $Z^i \to K^i \to B^{i+1}$ splits.

This is achieved by the functoriality of resolutions, it is natural and induces a functor from $K^+(\mathcal{B})$ to $K^{++}(\mathcal{I}_{\mathcal{B}})$. Cf., [Gelfand P210].[Weibel P146].

For a CE resolution, the spectral sequence can be applied, one side gets us: $K \to \mathrm{Tot}(L)$ is a quasi-isomorphism, i.e. $\underline{\mathrm{Tot}(L) \text{ is a injective resolution of } K}$. so $RG(K) = G(\mathrm{Tot}L)$ in $D(C)$

Prop. (I.8.4.9) (Functorial Injective Resolution). If $\mathcal{A}$ has an functorial injective embedding, then $K^+(\mathcal{A})$ has a functorial injective resolution (just construct one row by one row and use spectral sequence to show it is a quasi-isomorphism). This resolution functor induces a functor from $K^+(\mathcal{A})$ to $K^+(\mathcal{I})$. In particular, this applies to Grothendieck category(I.8.2.21).

5 K-injective

Prop. (I.8.5.1) (K-injective). For an Abelian category, a complex $I^\bullet$ in $K(\mathcal{A})$ is called a **K-injective** object iff it satisfies the following equivalent conditions:

- $\mathrm{Hom}_{K(\mathcal{A})}(S^\bullet, I^\bullet) = 0$ for any acyclic $S^\bullet$ in $K(\mathcal{A})$.
- $\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \cong \mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$ for quasi-iso $M^\bullet \to N^\bullet$.
- $\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, I^\bullet) \cong \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, I^\bullet)$ for every $X^\bullet$.
  In particular, a quasi-iso between two K-injective objects is a homotopy equivalence.

Proof: $1 \to 2$ is by(I.8.3.6); $2 \to 3$ use(I.9.2.2), for $3 \to 1$, notice an acyclic is quasi-iso to 0. ☐

Prop. (I.8.5.2). Objects in $K^+(\mathcal{I})$ are all K -injectives. thus the injective resolution is unique in K^+. Dually $K^-(\mathcal{P})$ are all K-projectives.

Proof: Use the first definition of K-injectives. Use induction, we construct the first homotopy, because $I^\bullet$ is bounded below, we see the map h^n factors through $\mathrm{Coker}\, d^{n-1} = \mathrm{Im}\, d^n$ because $S^\bullet$ is acyclic, so by injectivity, it can be extended to $S^{n+1} \to I^n$. ☐

Prop. (I.8.5.3). If a functor f between Abelian categories is left adjoint to an exact functor, then it preserves K-injectives (use definition1).

Prop. (I.8.5.4) (Functorial K-injective Resolution). If $\mathcal{A}$ is a Grothendieck category, then $K(\mathcal{A})$ has a functorial K-injective resolution $M^\bullet \to I^\bullet$, moreover, $I^\bullet$ consists of injective objects.

Proof: Cf.[StackProject 079P]. ☐

6 Ring Category Case

Prop. (I.8.6.1). If A is Noetherian and $C^\bullet$ is a complex of A-modules bounded above that every cohomology group H^i is a finite A-module, then there is a complex $L^\bullet$ of finite free A-modules, that $g : L^\bullet \to C^\bullet$ is a quasi-isomorphism.

Moreover, if C^i are all flat A-modules, then $L^\bullet \otimes_A M \to C^\bullet \otimes_A M$ is quasi-isomorphism for every M.

Proof: $C^\bullet$ is bounded above so we choose $L^n = 0$, and use induction to construct L^n that $H^i(L) \to H^i(C)$ is isomorphism for $i > n+1$ and surjection for $i = n+1$. For this, choose a generator $x_1, \ldots, x_r$ of $H^n(C)$ in $Z^n(C)$, and let $y_{r+1}, \ldots, y_s$ be a generator of $g^{-1}(B^{n+1}(C))$(Noetherian used), and let $g(y_i) = dx_i$ for $x_i \in C^n$.

Now let L^n be freely generated by $e_1, \ldots, e_s$ and $de_i = 0$ for $i \le r$ and $de_i = y_i$ for $i > r$, and let $g : L^n \to C^n$ be $ge_i = x_i$. Then it can be verified to be a quasi-isomorphism.

If C^i are all flat, we check isomorphism for all f.g. modules M, because $\otimes$ and cohomology all commutes with direct limits. Use induction, for n large, both are 0, and if we write $0 \to R \to F \to M \to 0$, for F finite free, then there is a commutative diagram of long exact sequences, and for F, H^i are obviously isomorphism, so I can use five lemma. ☐

I.9 Derived Category

Basic references are [Gelfand Homological Algebra], should consult [StackProject Ch13].

1 Triangulated Cateogory

Def. (I.9.1.1). A **triangulated category** is an additive category with a T: additive automorphism and an isomorphism class of distinguished triangles satisfying the following axiams:

TR1) $X \xrightarrow{id} X \to 0 \to X[1]$ is distinguished. Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle.

TR2) A morphism $X \to Y$ will generate a helix and a triangle is distinguished iff the helix it generate is all distinguished.

TR3) Any two consecutive morphisms of two distinguished class can be extended to a morphism of distinguished class.

TR4) Any diagram of the type "upper cap" can be completed to a octahedron diagram.

Def. (I.9.1.2). A functor from a triangulated category to an Abelian category is called **(co)homological** iff it maps a distinguished triangle to an exact sequence.

Conversely, A functor from an Abelian category to a triangulated category is called **δ-functor** iff it functorially maps an exact sequence to a distinguished triangle.

A functor between two triangulated category is called **exact** iff it preserves $-[1]$ and maps distinguished triangle to a distinguished triangle.

Prop. (I.9.1.3). For a distinguished category, $\mathrm{Hom}(-, C)$ and $\mathrm{Hom}(C, -)$ is (co)homological. In particular, composition of consecutive maps in a distinguished triangle is 0, (Easily from TR1 and TR3).

Thus the extension of TR3 of two isomorphisms is an isomorphism (by 5-lemma, $\mathrm{Hom}(C, X) \to \mathrm{Hom}(C, X')$ is an isomorphism, then use Yoneda). Hence the completion in TR2 is unique by TR3.

Prop. (I.9.1.4). In a triangulated category $\mathcal{D}$, any commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

can be extended to a diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X[1] & \longrightarrow & Y[1] & \longrightarrow & Z[1] & \longrightarrow & X[2] \end{array}$$

where the lower right is anti-commutative. Cf.[StackProject 05R0].

Prop. (I.9.1.5) ($K^*(\mathcal{A})$ is Triangulated). For Abelian category A, the categories $K^*(A)$ with distinguished triangles(I.8.3.6) is triangulated, and they are all subcategories of $K(\mathcal{A})$. This is hard to verify, but it solves every problem. Cf[Gelfand P246][StackProject 014S]. And an additive functor will induce exact functor between K^* because distinguished is split.

Localization of Triangulated Category

2 Derived Cateogry

Def. (I.9.2.1). The **derived category** $D(\mathcal{A})$ of an Abelian category $\mathcal{A}$ represents the universal property that any functor to a category $\mathcal{A} \to \mathcal{C}$ s.t. quasi-isomorphisms is mapped to isomorphisms uniquely factors through $D(\mathcal{A})$.

It can be defined as the localization of quasi-isomorphisms, but the class of quasi-isomorphisms is not localizing. But one can prove the quasi-isomorphisms in $K(\mathcal{A})$ is localizing and the localization by quasi-isomorphisms of $K(\mathcal{A})$ is equivalent to $D(\mathcal{A})$. Cf.[Gelfand P159]

Notice that equivalent roofs induce the same map on homology, so the cohomology functor can be regarded defined on $D(\mathcal{A})$.

$$\mathcal{A} \to K(\mathcal{A}) \to K(\mathcal{A})[S^{-1}] = D(\mathcal{A}) \xrightarrow{H^*} \mathcal{A}.$$

Prop. (I.9.2.2). The category $D^*(\mathcal{A})$ are the localized category of $K^*(\mathcal{A})$ at the class of quasi-isomorphisms respectively. The isomorphisms in $D^*(\mathcal{A})$ is of the form $t \circ s^{-1}$. (look at the homology map they induced).

Prop. (I.9.2.3). If $\mathcal{B}$ is a full subcategory that $S \cap \mathcal{B}$ is a localizing category of $\mathcal{B}$ and any $s \in S$ can be 'denominated' in one given side (any one is OK) into $\mathcal{B}$, then $\mathcal{B}[S \cap \mathcal{B}^{-1}]$ is a full subcategory of $\mathcal{C}[S^{-1}]$. The proof is easy, use left roof or right roof.

Prop. (I.9.2.4). K is a triangulated category and a localizing class S compatible with T, i.e. $s \in S \iff T(s) \in S$ and the extension in $TR3$ of f, g in S is in S. Then the localizing category $K[S^{-1}]$ is triangulated.

Cor. (I.9.2.5) (Derived Category is Triangulated). $D(\mathcal{A})$ is a triangulated category. The distinguished triangle is just the obvious one, and for a distinguished triangle, the long exact sequence exists,(I.8.3.5). In other words, H^0 is a cohomological functor for $D(\mathcal{A})$.

Derived Colimit

Def. (I.9.2.6). A **derived colimt** for a complex K_n in a triangulated category $\mathcal{D}$ is a K that

$$\oplus K_n \xrightarrow{(1-d)} \oplus K_n \to K \to \oplus K_n[1]$$

This exists when $\oplus K_n$ exists by TR1 and then it is unique by TR3. And the derived colimit is natural.

Dually for the definition of **derived limit**.

3 Acyclic Elements and Derived Functors

Def. (I.9.3.1). For a left exact F, a class R of elements is called **adapted to** F if it is sufficiently large and F maps acyclic objects in $Kom^+(\mathcal{R})$ to acyclic objects.

Injectives are F-acyclic for all left exact F because id : $I^\bullet \to I^\bullet$ is homotopic to 0, Cf(I.8.5.2).

Prop. (I.9.3.2). When RF exists, an object X is called F-acyclic iff $R^iF(X) = 0$ for all $i > 0$. Then: there is an adapted class of F iff the class of F-acyclic objects Z is sufficiently large.

If this is the case, then adapted class of F are exactly sufficiently large subclass of Z, and Z contains all injectives, Cf.[Gelfand P195].

Prop. (I.9.3.3) (Acyclic Criterion). If a class T of elements in an Abelian category of enough injectives is:

- sufficiently large.
- If $A \oplus A' \in T$ implies $A \in T$. (This implies all injectives are in T).
- Cokernel of elements of T is in T and $0 \to F(A) \to F(A') \to F(\mathrm{Coker}) \to 0$ is exact. (To use induction).

Then T is adapted to F.

Prop. (I.9.3.4). For a class of objects $\mathcal{R}$ in $\mathcal{A}$ stable under finite direct sum and are adapted to a left exact functor F, i.e. $Kom^+(\mathcal{R})$ is F-acyclic and every object in $\mathcal{A}$ is a subobject of an object from $\mathcal{R}$. Just need to verify the condition of(I.9.2.3). Similarly for the opposite category.

And in this case $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ is equivalent to $D^+(\mathcal{A})$.

Proof: The hard part is to prove every complex in $K^+(\mathcal{A})$ is quasi-isomorphic to a complex in $K^+(\mathcal{R})$, for this, use direct construction. Cf.[Gelfand P187]. □

Prop. (I.9.3.5). By(I.8.5.2), $K^+(\mathcal{I})$ is a saturated subcategory of $D+(\mathcal{A})$. And if $\mathcal{A}$ has enough injectives, this is an equivalence of category. (We only need to verify that the localization of $K^+(\mathcal{I})$ is itself, using the last proposition). In particular, this applies to Grothendieck categories. Cf.[Gelfand P179].

Prop. (I.9.3.6). By(I.8.5.2), if $\mathcal{A}$ contains sufficiently many injectives, then injective objects are adapted to any left exact functor F. (Because id on acyclic injective complexes is homotopic to 0 by the lemma).

Def. (I.9.3.7) (Derived functor). The **right derived functor** $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ for an additive functor F between Abelian categories is defined by the following universal property:

RF is exact and there is a natural tranformation

$$\varepsilon_F : Q_{\mathcal{B}}\ K^+F \to RF\ Q_{\mathcal{A}}.$$

and any other exact $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and a similar transformation must factor through ε_F uniquely. Thus this RF is unique up to natural isomorphism.

If a left exact functor F between Abelian categories has an adapted class, then by preceding proposition, $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ is equivalent to $D^+(\mathcal{A})$, then the derived functor exists, it is just F^+ on $K^+(\mathcal{R})$, Cf.[Gelfand P188].

Notice there is a more general derived functor that use inductive limits in $\hat{\mathcal{A}}$ that it maps $D^*(\mathcal{A})$ to $\mathrm{Ind}(D^*(\mathcal{B}))$, and if it has image in the subcategory of representable objects, then it coincide with RF. Similarly for right exact functor F. (This is easy to check)Cf.[Gelfand P198].

Yet there is another way to just look at the derived functors, it is the hypercohomology of the Cartan-Eilenberg resolution of the complex(I.9.3.11).

Prop. (I.9.3.8). $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Abelian categories with enough injectives in $\mathcal{A}, \mathcal{B}$ and $F : \mathcal{A} \to \mathcal{B}, G : \mathcal{B} \to \mathcal{C}$ are left exact functors. If $F(\mathcal{I}) \subset R_{\mathcal{B}}$ for $\mathcal{I}$ injective, then $R(G \circ F) \to RG \circ RF$ is an isomorphism . (Because the definition of RF is just F on $K^+(\mathcal{I})$).

Prop. (I.9.3.9). The derived functors form a universal δ-functor (when it exists).

Proof: It is a δ functor by(I.9.2.5), it is universal by(I.8.3.3). □

Prop. (I.9.3.10). Derived functor commutes with filtered colimits, when B is an Grothendieck category, this is by $AB5$.

Prop. (I.9.3.11) (Hypercohomology). we can define the **hypercohomology** of a left exact functor as $H^n(\mathrm{Tot}^{\Pi} F)$ if $\mathcal{B}$ satisfies AB3*.

Dually we can define the **hyperhomology** if $\mathcal{A}$ satisfies AB3* and AB4* and $\mathcal{B}$ satisfies AB3.

For complexes in $K^+(\mathcal{A})$, there is no restriction and everything is smooth.

When the Abelian category $\mathcal{A}$ satisfies AB3* and AB4*, i.e. the direct product is exact, then Tot^{Π} of the Cartan-Eilenberg resolution of any complex is a quasi-isomorphism to it by the dual of(I.9.5.10). (Take horizontal filtration, AB4* assures it collapse).

4 (Co)Homological Dimension

Prop. (I.9.4.1). If $\mathcal{A}$ has enough projectives, then the projective dimension of an object X is the length of projective resolutions. (Use resolution and long sequence).

Prop. (I.9.4.2) (Hilbert Theorem). For an Abelian category $\mathcal{A}$, the category $\mathcal{A}[T]$ is an Abelian category. If $\mathcal{A}$ has enough projectives and have infinite direct sum, then $\mathrm{dhp}_{\mathcal{A}[T]}(X, t) \le \mathrm{dhp}_{\mathcal{A}}(X) + 1$ and equality with $t = 0$.

Cor. (I.9.4.3). The Categories Ab and $K[X]$-mod have homological dimension 1. $K[X_i, \ldots, X_k]$ has homological dimension k.

5 Spectral Sequence

Reference for this section is [Weibel Ch5]. All the definition below is dual for homology and cohomology, just rotate the diagram 180 degree.

We work in an Abelian category.

Def. (I.9.5.1). A convergent **Spectral Sequence** is a three-dimensional arrange of entries $E_r^{p,q}$ that:

1. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ that $d_r d_r = 0$.
2. $H^{p,q}(E_r^{p,q}) \cong E_{r+1}^{p,q}$. And $E_r^{p,q}$ has a direct limit $E_\infty^{p,q}$.
3. There is a complex E^n and a decreasing bounded filtration $F^p E^n$ on each E^n and $E_\infty^{p,q} \cong F^p E^{p+q} / F^{p+1} E^{p+q}$.

For a morphism of spectral sequences, if it defines an isomorphism for some r, then by five-lemma, it defines isomorphisms afterward, so it defines an isomorphism on E_∞^{pq}.

Def. (I.9.5.2). We say a (co)homology filtration is bounded below $F_{n_s}E_n = 0$ for some n_s, bounded above $F_{n_s}E_n = E_n$ for some n_s. It is exhaustive iff $\cap F_i E_n = E_n$. The spectral sequence is called regular iff $d^r_{pq} = 0$ for sufficiently large r.

Def. (I.9.5.3) (Spectral Sequence of a Filtered Complex). For a complex $K^\bullet$ and a filtration F^pK^n on K^n, we have a natural spectral sequence

$$E_1^{pq} = H^{p+q}(F^pE^{p+q}/F^{p+1}E^{p+q}), \quad E^n = H^n(K^\bullet), \quad F^pE^n = H^n(F^pK^\bullet).$$

For a morphism of filtered complexes that are isomorphism for some r, induction on the exact sequence $0 \to F^pE^n \to F^{p+1}E^n \to E_\infty^{p,n-p}$ and use five-lemma shows it induces isomorphism on H^*E.

Prop. (I.9.5.4) (Comparison Theorem). For a morphism between two convergent spectral sequences, if it is an isomorphism for some r, then it induce isomorphism on the infinite homology, because there are exact sequence

$$0 \to F^{p+1}H^n \to F^pH^n \to E_\infty^{p,n-p} \to 0$$

we can use five lemma and induction.

Prop. (I.9.5.5) (Classical Convergence). For homology, if the filtration is bounded below and exhaustive for all n, we have a convergence to E_n. Cf[Gelfand P203] for cohomological case and [Weibel P133] for homological case.

Prop. (I.9.5.6) (Complete convergence). For homology, if the filtration is complete, exhaustive, bounded above, and the spectral sequence is regular, then the spectral sequence converges to E_n.

There are two examples, the stupid filtration and the canonical filtration, the canonical filtration is natural and factors through $D(\mathcal{A})$.

Prop. (I.9.5.7) (Spectral Sequence of a Double Complex). A double complex has two natural filtration of the total complex, they defines two spectral sequence, one has

$$E_{2,x}^{p,q} = H_x^p(H_y^{\bullet,q}(L^{\bullet,\bullet}))$$

and the other has

$$E_{2,y}^{p,q} = H_y^p(H_x^{q,\bullet}(L^{\bullet\bullet})).$$

Cf.[Gelfand P209]. In fact under reflection, there is only one spectral sequence. For the horizontal filtration, the differential goes vertical first, for the vertical filtration, the differential goes horizontal first. The differential goes one way, the convergence goes reversely.

If both the filtration is finite and bounded, in particular if E is in the first quadrant, then they both converges to $H^n(E)$, this will generate important consequences.

Cor. (I.9.5.8). If a double complex in the first quadrant has its all column acyclic (3rd-quadrant pointing), then the total complex is acyclic. Thus a morphism of double complex inducing quasi-isomorphism on each column induces a quasi-isomorphism on the total complex.

If a double complex has $H_p(C_{*,q}) = 0, \forall p > 0, q$, then

$$H_n(\mathrm{Tot}C_{*,*}) = H_n(\mathrm{Coker}(C_{1,*} \to C_{0,*})$$

Prop. (I.9.5.9) (Horizontal Filtration). For a second-quadrant free homology double complex, the filtration is bounded below and exhaustive for $\mathrm{Tot}^{\oplus}$, so the classical convergence applies.

For a fourth-quadrant free homology double complex, the filtration is complete and exhaustive and regular ? for Tot^{Π}, so the complete spectral sequence applies. Cf.[Weibel P142].

Prop. (I.9.5.10) (Vertical Filtration). For a fourth-quadrant free homology double complex, the filtration is bounded below and exhaustive for $\mathrm{Tot}^{\oplus}$, so the classical convergence applies.

For a second-quadrant free homology double complex, the filtration is complete and exhaustive and regular ? for Tot^{Π}, so the complete spectral sequence applies. Cf.[Weibel P142].

Cor. (I.9.5.11) (Grothendieck Spectral Sequence). If $\mathcal{A},\mathcal{B},\mathcal{C}$ be Abelian categories and $\mathcal{A},\mathcal{B}$ has enough injectives, and $F:\mathcal{A}\to\mathcal{B}, G:\mathcal{B}\to\mathcal{C}$ are left exact functors. If $\mathcal{R}_{\mathcal{A}}$ is adapted to F, $\mathcal{R}_{\mathcal{B}}$ is adapted to G and $F(I_{\mathcal{A}})\subset\mathcal{R}_{\mathcal{B}}$, then for any $X\in K^+(\mathcal{A})$, there is a spectral sequence with $E_2^{p,q}=R^pG(R^qF(X))$ (to upper left) that converges to $E^n=R^n(G\circ F)(X)$. And this spectral sequence is functorial in X.

In particular, this applies to F is a right adjoint and its left adjoint is exact, then we may choose $\mathcal{R}_{\mathcal{A}}=\mathcal{I}_A$ and $\mathcal{R}_{\mathcal{A}}A=\mathcal{I}_{\mathcal{A}}$.

Proof: Let $K=F(I_X)=RF(X)$, and choose the CE resolution of K(I.8.4.8), because the resolutions for $B^i\to Z^i\to H^i$ and $Z^i\to K^i\to B^{i+1}$ split and G is additive, we have

$$H_x^{q,\bullet}(G(L^{\bullet,\bullet}))=G(H_x^{q,\bullet}(L^{\bullet,\bullet}))=RG(H^q(K))$$

So

$$E_{2,y}^{p,q}=H_y^p(H_x^{q,\bullet}(L^{\bullet\bullet}))=R^pG(H^q(K))=R^pG(R^qF(X))$$

and

$$E^{\bullet}=RG(\mathrm{Tot}(L))=G(\mathrm{Tot}(L))=RG(K)=RG\circ RF(X)=R(G\circ F)(X)(I.9.3.8).$$

□

Cor. (I.9.5.12). The low degree parts read:

$$0\to R^1G(F(A))\to R^1(G\circ F)(A)\to G(R^1F(A))\to R^2(G(F(A))\to R^2(G\circ F)(A).$$

(Check definition). More generally, if $R^pG(R^qF(A))=0, 0<q<n$, then

$$R^mG(F(A))\cong R^m(G\circ F)(A)\quad m<n$$

And

$$0\to R^nG(F(A))\to R^n(G\circ F)(A)\to G(R^nF(A))\to R^{n+1}(G(F(A))\to R^{n+1}(G\circ F)(A).$$

The Grothendieck spectral sequence is tremendously important.

Cor. (I.9.5.13). For chain complex K in $K^+(\mathcal{A})$ and a left exact functor F, the CE resolution will generate two spectral sequences: $E_{2,x}^{p,q}=H_x^p(R^qF(A_\bullet))$ and the other has $E_{2,y}^{p,q}=R^pF(H^q(A))$ that converges to the hypercohomology $\mathbb{R}^{p+q}F(K)$. Dually for derived homology.

6 Tor Hom and Ext

Prop. (I.9.6.1) (Ext). $\mathcal{A}$ is categorically equivalent to the subcategory of $D(\mathcal{A})$ that has only H^0 nonzero. If we define $\mathrm{Ext}^i_{\mathcal{A}}(X,Y)$ as $\mathrm{Hom}_{D(\mathcal{A})}(X[0],Y[i])$, then it is equivalent to the i-term extension of Y by X, and it is an abelian group. We have a

$$\mathrm{Ext}^i_{\mathcal{A}}(X,Y)\times\mathrm{Ext}^i_{\mathcal{A}}(Y,Z)\to\mathrm{Ext}^{i+j}_{\mathcal{A}}(X,Z)$$

by composition or equivalently the conjunction of extensions.

Proof: Cf.[Gelfand P167] □

Cor. (I.9.6.2). The definition of $\mathrm{Ext}^n(X,Y)$ is equivalent to the usual definition as the derived functor of $\mathrm{Hom}(X,-)$. Because by(I.8.5.2) when we use a projective resolution or an injective resolution, then it is equivalent to hom in $K(\mathcal{A})$(I.8.5.1), which is exactly the homology group of the Hom.

Prop. (I.9.6.3). In an Abelian category with enough injectives, the extension $\mathrm{Ext}^1(N,M)$ is equivalent with the Abelian group of extensions with Baer sum as addition.

Proof: We choose a projective resolution $0\to K\to P\to N\to 0$, so $\mathrm{Hom}(K,M)\to\mathrm{Ext}^1(N,M)$ is surjective, so choose a lifting and the pushout $0\to M\to L\to N\to 0$ with be the corresponding extension, Now the Baer sum is easy to define and verify. □

Prop. (I.9.6.4). In an Abelian category with enough injectives, we have the $\mathrm{Ext}^i(-,G)$ forms a long exact sequence, by injective resolution.

Ring Category Case

Prop. (I.9.6.5) (Balancing Tor). In the category of rings, $\mathrm{Tor}_n(A,B)=\mathrm{Tor}_n(B,A)$. This can be seen using spectral sequence of the double complex of flat resolutions of A and B. Similarly, we have two definitions of $\mathrm{Ext}^i(M,N)$ are compatible.

Prop. (I.9.6.6) (Base Change). For a ring extension $R\to S$, using projective resolution and spectral sequence, there is a first quadrant homology spectral sequence:

$$E^2_{pq}=\mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A,S),B)\Rightarrow\mathrm{Tor}^R_{p+q}(A,B).$$

Similarly, for Ext,

$$E_2^{pq}=\mathrm{Ext}^p_S(A,\mathrm{Ext}^q_R(S,B))\Rightarrow\mathrm{Ext}^{p+q}_R(A,B).$$

Prop. (I.9.6.7) (Universal Coefficient Theorem). Let P be a free R-module so $d(P_n)$ are all flat, then $Z(P_n)$ are also flat and

$$0\to d(P_{n+1})\to Z_n\to H_n\to 0$$

is a free resolution. we have an exact sequence:

$$0\to H_n(P)\otimes_R M\to H_n(P\otimes_R M)\to\mathrm{Tor}^R_1(H_{n-1}(P),M).$$

$$0\to\mathrm{Ext}^1_R(H_{n-1}(P),M)\to H^n(\mathrm{Hom}_R(P,M))\to Hom(H_n(P),M)\to 0$$

and these exact sequences non-canonically split because Z_n is a direct summand of P_n, thus $Z_n\otimes M$ is a direct summand of $P_n\otimes M$ and a fortiori $Z_n(P_n\otimes M)$. so $H_n(P)\otimes M$ is a direct summand of $H_n(P\otimes_R M)$.

Rtensor

Lemma (I.9.6.8). if P is a complex of R-modules and if $\alpha, \beta : L \to M$ are homotopy equivalent maps, then the map $Tot(L \otimes P) \to Tot(M \otimes P)$ they induce are also homotopy equivalent.

So $Tot(- \otimes_R L)$ is a functor $K(R) \to K(R)$, and moreover, the tensor product of complexes in R-mod is an exact functor of triangulated categories $K(R)$.

Proof: Easily constructed. For the second, notice the distinguished triangles in $K(R)$ are termwise-split exact sequences(I.8.3.5). □

Def. (I.9.6.9) (K-flat). A complex $K^\bullet$ in an Abelian category is called **K-flat** if for any acyclic complex $M^\bullet$, the total complex $\mathrm{Tot}(M^\bullet \otimes K^\bullet)$ is acyclic. This is equivalent to tensoring with $K^\bullet$ maps quasiiso to quasiiso, because quasiiso is equivalent to the cone is acyclic and tensoring is exact.

Prop. (I.9.6.10). If K, K' are R-K-flat complexes,

- $K \otimes K'$ are K-flat.
- $K \otimes_R R'$ are R'-K-flat.
- If (K_1, K_2, K_3) is a distinguished triangle in $K(\mathrm{Mod}_R)$, if two of them is K-flat, then the third is also K-flat.
- If P is a bounded above complex of flat R-modules, then P is K-flat.
- Any filtered colimits of K-flat complexes are K-flat.

Proof: $1, 2$: trivial.
3: use(I.9.6.8), and the long exact sequence.
4: Cf.[StackProject 064K].
5: because Tot and tensor all commutes with filtered colimits.

□

Prop. (I.9.6.11) (K-Flat Resolutions). Any complex of R-modules has a K-flat resolution, moreover, each term is a flat module.

Proof: Cf.[StackProject 06Y4]. □

Lemma (I.9.6.12). Let $P \to Q$ be a quasi-iso of K-flat complexes, then for any complex L, $Tor(L \otimes P) \to Tor(L \otimes Q)$ is a quasi-iso.

Proof: Choose a K-flat approximation(I.9.6.11) K of L, then notice $Tor(L \otimes P) \cong Tor(K \otimes P) \cong Tor(K \otimes Q) \cong Tor(L \otimes Q)$ by definition of K-flatness. □

Prop. (I.9.6.13) (Derived Tensor Product). In the Abelian category of R-modules, we can define a **derived tensor product**

$$\otimes^L : D(\mathcal{A}) \times D(\mathcal{A}) \to D(\mathcal{A})$$

that is, for complexes $F^\bullet$ and $G^\bullet$, there are K-flat resolutions P and Q of F, by(I.9.6.11), thus we can define $F \otimes^L G = P \otimes Q$ as the total complex of the double complex, this is independent of the resolution chosen up to quasi-isomorphism, so descends to a functor on $D(\mathcal{A})$, by(I.9.6.12) and(I.9.6.9). In fact, only one resolution will suffice.

Cor. (I.9.6.14). For complexes K, L, M,

$$K \otimes^L L \cong L \otimes^L K, \quad (K \otimes^L L) \otimes^L M \cong K \otimes^L (L \otimes^L M).$$

Derived Change of Rings

Def. (I.9.6.15). Let $R \to A$ be a ring map, then for N a complex of A-modules, we can regard $-\otimes_R^L N$ as a functor from $D(R) \to D(A)$, because the Abelian category structure is compatible. In particular, if $N = A$, then this is called the **derived change of rings**.

Remark (I.9.6.16) (WARNING). If M, N are two A-modules, then we can define $M_R \otimes_R^L N$ and $M \otimes_R^L N_R$, but there are no reason for them to be isomorphic.

Prop. (I.9.6.17). Let $A \to B \to C$ be ring maps, $M \in K(A)$, $N \in K(B)$ and $K \in K(C)$, then

$$(M \otimes_A^L N) \otimes_B^L K = M \otimes_A^L (N \otimes_B^K) = (M \otimes_A^L C) \otimes_C^L (N \otimes_B^L K)$$

and

$$(M \otimes_A^L K) \otimes_B^L K \cong (M \otimes_A^L K) \otimes_C^L (N \otimes_B^L C)$$

Proof: For the first equation, see K-flat resolutions, noticing(I.9.6.10). Similarly for the second equality. The last isomorphism follows from the above, Cf.[StackProject 08YU]. □

RHom

Def. (I.9.6.18) (Hom Complex). For two complex P, Q, there is a **Hom complex** $\mathrm{Hom}^\bullet(P, Q)$ as

$$\mathrm{Hom}^n(P, Q) = \prod \mathrm{Hom}(P_i, Q_{n+i}),$$

with the differential giving by $d_n(\{f_k\})_i = \{df_i + (-1)^n f_{i+1} d\}$ and suitable signatures.

It is clear that $H^n(\mathrm{Hom}^n(P, Q)) = \mathrm{Hom}_{K(\mathcal{A})}(P, Q[n])$.

Prop. (I.9.6.19). There is a canonical isomorphism:

$$\mathrm{Hom}^\bullet(K, \mathrm{Hom}^\bullet(L, M)) = \mathrm{Hom}^\bullet(Tot(K \otimes_R L), M).$$

Proof: Cf.[StackProject 0A5Y]. □

Cor. (I.9.6.20). In the category $D(R)$, If K is K-flat and I is K-injective, then $\mathrm{Hom}^\bullet(K, I)$ is K-injective.

Proof: Use definitions(I.9.6.9)(I.8.5.1). □

Prop. (I.9.6.21). Given complexes K, M, L of R-complexes, there are canonical functorial morphisms:

- $Tot(\mathrm{Hom}^\bullet(L, M) \otimes_R \mathrm{Hom}^\bullet(K, L)) \to \mathrm{Hom}^\bullet(K, M)$,
- $Tot(\mathrm{Hom}^\bullet(L, M) \otimes_R K) \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K, L), M)$,
- $Tot(K \otimes_R \mathrm{Hom}^\bullet(M, L)) \to \mathrm{Hom}^\bullet(M, Tot(K \otimes_R L))$,
- $K \to \mathrm{Hom}^\bullet(L, Tot(K \otimes_R L))$.

Proof: 1: [StackProject 0A8I].
2:[StackProject 0A60].
3:[StackProject 0BYM].
4:[StackProject 0A62]. □

Prop. (I.9.6.22). In a Grothendieck Abelian category $\mathcal{A}$, we can define a **derived Hom**

$$R\operatorname{Hom} : (D(\mathcal{A}))^{op} \times D(\mathcal{A}) \to D(\mathcal{A})$$

that is, for complexes X and Y, there is a K-injective resolution I of Y by(I.8.5.4). Thus we define $R\operatorname{Hom}^n(X,Y) = R\operatorname{Hom}(X,I)$.

This does descend to D because first it is independent of X chosen because of the second definition of(I.8.5.1) and homotopy induce a homotopy in the the double complex.

Also, two K-injective resolutions are quasi-isomorphic and quasi-isomorphisms induce isomorphism on E_1 of the spectral sequence associated to the double complex (used injectiveness) thus on the homology of total complex by comparison.

Cor. (I.9.6.23) (Derived Ext). For any $X^\bullet \in K(\mathcal{A}), Y^\bullet \in K^+(\mathcal{A})$, we define

$$\operatorname{Ext}^n_{\mathcal{A}}(X,Y) = H^i(R\operatorname{Hom}(X,Y)),$$

this is seen to be equal to $\operatorname{Hom}_{K(\mathcal{A})}(X^\bullet, Q^\bullet[n]) = \operatorname{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet[n])$ by definition of K-injective(I.8.5.1). Dually for $P^\bullet$.

Prop. (I.9.6.24). If R is a ring and $K, L, M \in D(R)$, then

$$R\operatorname{Hom}_R(K, R\operatorname{Hom}(L,M)) = R\operatorname{Hom}_R(K \otimes^L_R L, M).$$

Proof: Choose a K-flat resolution K of L and a K-injective resolution I of M, then $R\operatorname{Hom}^\bullet(K,I) = \operatorname{Hom}^\bullet(K,I)$ is K-injective(I.9.6.20), and this isomorphism is just(I.9.6.19). □

Cor. (I.9.6.25). By(I.9.6.23), taking H^0, we get:

$$\operatorname{Hom}_{D(R)}(K, R\operatorname{Hom}(L,M)) = \operatorname{Hom}_{D(R)}(K \otimes^L_R L, M).$$

Cor. (I.9.6.26). For a ring map $R \to S$ and any $L \subset D(R), M \subset D(S)$, there is an isomorphism

$$R\operatorname{Hom}_R(L,M) \cong R\operatorname{Hom}_S(L \otimes^L_R S, M)$$

Proof: Choose a K-flat resolution of L and a K-injective resolution of M, then this is just the usual adjunction of Tor and Hom. □

Inverse Limit

Prop. (I.9.6.27). The derived functor of lim from $K^+(\mathcal{A}) \to \mathcal{A}$ is $\operatorname{Coker}(a_i) \to (a_i - a_{i+1})$ for $\mathcal{A}$ Abelian, has enough injectives and satisfies $AB4^*$ (R-mod). $\lim^1$ vanishes for a complex that satisfies Mittag-Leffler conditions.

Proof: If A satisfies the M-L condition, the essential image $\{B_i\}$ is surjective so acyclic and $\{A_i/B_i\}$ is acyclic because the inverse image can be defined as a finite sum. So the long exact sequence gives A is acyclic.

The δ-functor is defined by the snake lemma and $AB4^*$ and we have to prove it is effaceable. For this, we use(I.8.1.10) to see that $E = \prod_k k_* A_k$ exists in $\mathcal{A}^C$ is injective and $A \to E$ is an injection. In this case E is a product of towers $\cdots \to A_k \to A_k \to 0 \to 0 \to \cdots$, hence surjective by $AB4^*$ so is acyclic. □

For applications, Cf.[Weibel P82].

7 T-Structures

8 Derived Category of Rings

[StackProject Chap15] contains many beautiful results working on the derived category of rings, and is used heavily on Scholze's Thesis, should be noted.

I.10 Lie Algebra

Basic references are [Lie Algebras of Finite and Affine Type, Carter], [Complex Semisimple Lie Algebras Serre] and [Lie Algebras and Lie Groups Serre]. Notice that [Lie Algebras Humphreys] is not good.

Notice all construction is done over a field of char0 or specifically $\mathbb{C}$.

1 Basics

Def. (I.10.1.1). A **Lie algebra** L is an non-associative algebra over a field k with a bilinear **Lie bracket** operation that satisfies:

$$[x,x]=0, \quad [x[yz]]=[[xy]z]+[y[xz]](\text{Jacobi Identities}).$$

It is easily deduced that $[xy]=-[yx]$.

We denote $\mathrm{ad}x(y)=[xy]$, then $\mathrm{ad}x$ are all derivatives of L.

An element $x\in\mathfrak{g}$ is called **nilpotent** or **semisimple** if $\mathrm{ad}x$ is nilpotent or semisimple.

Prop. (I.10.1.2). For any associative algebra over k, it is naturally a Lie algebra by defining $[xy]=xy-yx$.

Def. (I.10.1.3). A submodule $I\subset L$ is called an **ideal** iff $[\mathfrak{g},\mathfrak{a}]\subset\mathfrak{a}$. If I is an ideal of L, then L/I can be made into a Lie algebra by defining $[I+x,I+y]=I+[xy]$.

Def. (I.10.1.4). The **center** of a Lie algebra $\mathfrak{g}$ is the elements a that $\mathrm{ad}a=0$. It is an ideal.

Def. (I.10.1.5). A Lie algebra $\mathfrak{g}$ is called **simple** if it is not 1-dimensional and it has no nontrivial ideal.

Def. (I.10.1.6). A **representation** of a Lie algebra L over a vector space V is a Lie homomorphism $\mathfrak{g}\to\mathfrak{gl}(V)$.

Def. (I.10.1.7). A bilinear form B on $\mathfrak{g}$ is called **invariant** if $B([x,y],z)+B(x,[y,z])=0$.

The **Killing form** on $\mathfrak{g}$ is an invariant symmetric bilinear form defined by $B(x,y)=\mathrm{tr}(\mathrm{ad}x\circ\mathrm{ad}y)$. It can be proved that any invariant symmetric bilinear form on $\mathfrak{g}$ is a multiple of the Killing form.

Nilpotent and Solvable Lie Algebras

Def. (I.10.1.8) (Nilpotent and Solvable Lie Algebras). Let $\mathfrak{g}$ be a Lie algebra, the **lower central series** of $\mathfrak{g}$ is the descending sequence of ideals of $\mathfrak{g}$ defined inductively by $C^1\mathfrak{g}=\mathfrak{g}$ and $C^n\mathfrak{g}=[\mathfrak{g},C^{n-1}\mathfrak{g}]$.

Let $\mathfrak{g}$ be a Lie algebra, the **derived series** of $\mathfrak{g}$ is the descending sequence of ideals of $\mathfrak{g}$ defined inductively by $D^1\mathfrak{g}=\mathfrak{g}$ and $D^n\mathfrak{g}=[D^{n-1}\mathfrak{g},D^{n-1}\mathfrak{g}]$.

A Lie algebra is called **nilpotent** if there is an n that $C^n\mathfrak{g}=0$. This is equivalent to $\mathrm{ad}x_1\mathrm{ad}x_2\ldots\mathrm{ad}x_n=0$ for any n element $x-1,\ldots,x_n$. It is called **solvable** if $D^n=0$ for some n. It is clear that $D^n\subset C^n$, so nilptent Lie algebra is solvable.

Prop. (I.10.1.9). The lower central series satisfies: $[C^n\mathfrak{g},C^m\mathfrak{g}]\subset C^{m+n}\mathfrak{g}$.

Proof:

□

Prop. (I.10.1.10). Subalgebras, quotient algebras and extension algebras of solvable algebras are solvable.

Proof: □

Cor. (I.10.1.11). If $\mathfrak{a},\mathfrak{b}$ are solvable ideals of a Lie algebra $\mathfrak{g}$, then the ideal $\mathfrak{a}+\mathfrak{b}$ is also solvable, because $0\to\mathfrak{a}\to\mathfrak{a}+\mathfrak{b}\to\mathfrak{b}/\mathfrak{a}\cap\mathfrak{b}$.

Thus there is a maximal solvable ideal $\mathfrak{r}\subset\mathfrak{g}$, called the **radical** $\mathrm{Rad}(\mathfrak{g})$.

Def. (I.10.1.12). A Lie algebra L is called **semisimple** if $\mathrm{Rad}\ L=0$. Notice $L/\mathrm{Rad}(L)$ is semisimple, by(I.10.1.11).

Prop. (I.10.1.13) (Engel). If all elements of L are ad-nilpotent(i.e. $\mathrm{ad}x=0$), then L is nilpotent. Equivalently, elements of L has a common 0-eigenvector.

Proof: only need to show that If a subalgebra of $GL(n)$ consists of nilpotent elements, then there is a common 0-eigenvector. Use Induction, choose a maximal subalgebra of L, then it must be of codimension 1,$L=K+Fz$. Thus the 0-eigenvector for K is a nonzero subspace, and a 0-eigenvector for z will suffice. □

Prop. (I.10.1.14) (Lie's prop). Let $\mathfrak{g}\subseteq\mathfrak{gl}(V)$ be a solvable lie algebra over an alg.closed field of char0, then $\mathfrak{g}$ is upper triangulable. Equivalently, there exists a vector $v\in V$ which is a common eigenvector for all $X\in\mathfrak{g}$, and moreover equivalently, any irreducible representation of $\mathfrak{g}$ is 1-dimensional.

Proof: Idea is to prove by induction on dimension of $\mathfrak{g}$.

Produce a codimension 1 ideal $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g}$ be generated (as a vector space) by $\mathfrak{h}$ and Y.Being a subalgebra of solvable algebra $\mathfrak{g}$, $\mathfrak{h}$ is itself a solvable lie algebra. Apply induction step on $\mathfrak{h}$ and choose $v\in V$ such that v is an eigenvector for all $X\in\mathfrak{h}$.

The idea is to consider set W all common eigenvectors of elements of $\mathfrak{h}$ and produce an eigenvector of Y from this W. Let

$$W=\{v\in V|X(v)=\lambda(X)v\ \forall\ X\in\mathfrak{h}\text{ for a fixed }\lambda(X)\in\mathbb{C}\}.$$

Suppose W is an invariant subspace of Y, we then have restriction map $Y:W\to W$. As we are in complex vector space (algebraically closed) there exists an eigenvector for Y in W say w_0. Thus, w_0 is common eigenvector for all elements of $\mathfrak{g}$.

It remains to show that W is an invariant subspace of Y i.e., $Y(w)\in W$ for all $w\in W$ i.e., given $X\in\mathfrak{h}$, we need to have $X(Y(w))=\lambda(X)Y(w)$.

Let $w\in W$, we have

$$\begin{aligned}X(Y(w))&=Y(X(w))+[X,Y](w)\\&=Y(\lambda(X)w)+\lambda([X,Y])\,w\\&=\lambda(X)Y(w)+\lambda([X,Y])\,w\end{aligned}$$

This is almost the same as what we want but with an extra term $\lambda([X,Y])\,w$. Suppose we prove $\lambda([X,Y])=0$ for all $X\in\mathfrak{h}$ then we are done.

Then considers subspace U spanned by elements $\{w,Y(w),Y^2(w),\cdots\}$ and then says that U is invariant subspace of each element of $\mathfrak{h}$ and (assuming n is the smallest integer such that $Y^{n+1}w$ is in the subsapce generated by $\{w, Y(w), \cdots, Y^n(w)\}$) representation of an element Z of $\mathfrak{h}$ with the basis $\{w, Y(w), \cdots, Y^n(w)\}$ is an upper triangular matrix with $\lambda(Z)$ in the diagonal. So, $tr(Z) = n\lambda(Z)$.

So, $tr([X,Y]) = n\lambda([X,Y])$. As $[X,Y] = XY - YX$, we have $\mathrm{tr}([X,Y]) = \mathrm{tr}(XY) - \mathrm{tr}(YX) = 0$. Thus, $\lambda([X,Y]) = 0$ and we are done. $\square$

Prop. (I.10.1.15) (Cartan's Criteria for Solvability). If $\mathfrak{g}$ is a Lie algebra $\subset \mathfrak{gl}_n$, then

$$\mathfrak{g} \text{ is solvable} \iff Tr(xy) = 0,\ \forall x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}].$$

Note that a Lie algebra is solvable if the adjoint representation is solvable because the kernel is abelian.

Proof: Cf.[Humphreys P20], [Serre P42]. $\square$

2 Semisimple Lie Algebra

Prop. (I.10.2.1) (Levi Decomposition). Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{r}$ its radical, then there is a subalgebra $\mathfrak{s}$ that is a complement of $\mathfrak{r}$. So $\mathfrak{g}$ is a semidirect product $\mathfrak{r} \rtimes \mathfrak{s}$.

Proof: Cf.[Serre P49]. $\square$

Prop. (I.10.2.2) (Cartan-Killing Criteria for Semisimplicity). A lie algebra is semisimple $\iff$ the Killing form is non-degenerate.

Proof: Cf.[Humphreys P22]. Show that the kernel of the Killing form is a solvable ideal and that $\mathrm{ad}x \cdot \mathrm{ad}y$ is nilpotent for x in an abelian ideal. $\square$

Cor. (I.10.2.3). Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then the orthogonal space $\mathfrak{a}'$ w.r.t the Killing form is a complement for $\mathfrak{a}$ in $\mathfrak{g}$, and $\mathfrak{g} \cong \mathfrak{a} \times \mathfrak{a}'$.

Proof: $\square$

Cor. (I.10.2.4). A Lie algebra is semisimple iff it is isomorphic to a product of simple algebras.

Proof: Cf.[Carter P43]. $\square$

Cor. (I.10.2.5). If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Prop. (I.10.2.6) (Examples of Semisimple Lie Algebras).

- The subalgebra $\mathfrak{sl}(V)$ of all elements of $\mathrm{End}(V)$ of trace 0 is semisimple.

Prop. (I.10.2.7) (Weyl). Finite representation of a semisimple lie algebra is completely reducible.

Proof: Cf.[Humphreys P28]. $\square$

Prop. (I.10.2.8). If L is semisimple, then every derivative of L is inner.

Proof: Cf.[Humphreys P23]. $\square$

Prop. (I.10.2.9) (Abstract Jordan Decomposition). If $\mathfrak{g}$ is a semisimple algebra, then any $x \in \mathfrak{g}$ is of the form $x = s + n$, where s is semisimple and n is nilpotent, and $[s, n] = 0$. Moreover, any y that commutes with x also commutes with s and n.

Prop. (I.10.2.10). Let L be a semisimple lie algebra and $\phi : L \to GL(v)$ be a representation. If $x = s + n$ is the abstract Jordan decomposition of x, then $\phi(x) = \phi(s) + \phi(n)$ is the usual Jordan decomposition of $\phi(x)$.

Proof: In fact, we only need to prove that if L is a semisimple algebra $\subset \mathfrak{gl}(V)$, then L contains the semisimple and nilpotent element of all its element. Because the image of L is semisimple and the usual Jordan decomposition must be its abstract decomposition. The last assertion is due to the fact that if z is semisimple(nilpotent), then $\mathrm{ad}_{\mathfrak{gl}_n} z$ is semisimple(nilpotent), thus so do $\mathrm{ad}_L z$.

Cf.[Humphreys P27] for the following proof. □

Complex Semisimple Lie Algebra

Following [Serre].

Simple Lie Algebras

Def. (I.10.2.11) (A_1). A_1 is also called $\mathfrak{sl}_2(\mathbb{C})$. It has a basis f, h, e with

$$[he] = 2e, \quad [hf] = -2f, \quad [ef] = h.$$

In matrix form,

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

It can also be realized by

$$H = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad R = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \quad L = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}.$$

These two representation differ by a conjugation by the Cayley transformation $C = -\frac{1+i}{2}\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix}$.

Proof: □

3 Singular element

Introduction

Singular element in $\mathfrak{g}$ is a linear space and is defined by some homogenous ideal in $S(\mathfrak{g})$.

The paper [Singular element] of Kostant tells in fact it is defined by some r-homogenous functions M^r in $S(\mathfrak{g})$,and further describes the properties of this ideal such as the G-module decomposition and as span of determinant minors.

Preliminary

Let complex simple Lie algebra $\mathfrak{g} = \mathrm{Lie}\ G$,$n = l + 2r$.The non-degenerate Killing form $\mathcal{B} \triangleq (x, y)$ on $\mathfrak{g}$ generate a nonsingular pair on $S(\mathfrak{g})$ and $\wedge(g)$ by

$$(x_1 \cdots x_k, y_1 \cdots y_k) = \sum_{\sigma \in \Sigma_k} (x_1, y_{\sigma(1)}) \cdots (x_k, y_{\sigma(k)})$$

$$(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots y_k) = \sum_{\sigma \in \Sigma_k} sg(\sigma)(x_1, y_{\sigma(1)}) \cdots (x_k, y_{\sigma(k)})$$

So $\mathfrak{g} \longleftrightarrow \mathfrak{g}'$,$S(\mathfrak{g}) \longleftrightarrow S(\mathfrak{g}') \longleftrightarrow$ polynomial functions on $\mathfrak{g}$; and $S(\mathfrak{g})$ and $\wedge(g)$ are $\mathfrak{g}$ thus G modules extending the adjoint representation.

recall that δ and ∂ are called $\mathcal{B}$-dual if $(\delta x, y) = (x, \partial y)$.Set antiderivation $-d$ $\mathcal{B}$-dual to the operator

$$\partial(x_1 \wedge \cdots \wedge x_p) = \sum_{i<j} (-1)^{i+j+1}[x_i, x_j] \wedge x_1 \wedge \cdots \hat{x}_i \wedge \cdots \hat{x}_j \wedge \cdots \wedge x_p$$

on $\wedge(\mathfrak{g})$ and antiderivation $\iota(u)$ $\mathcal{B}$-dual to the operator $\epsilon(u)v = u \wedge v$ on $\wedge(\mathfrak{g})$.

Element v of $S(\mathfrak{g})$ are called **invarient** iff $gv = v, \forall g \in G$ and element u of $S(\mathfrak{g})$ are called **harmonic** iff $(u, v) = 0,\ \forall v$ invariant and no constant term.

Denote by J, H respectively the graded subspace of invariant and harmonic elements, then:

Prop. (I.10.3.1) (Separation of Variables (in K2)).

$$S(\mathfrak{g}) \cong J \otimes H$$

<u>the Ideal of Sing $\mathfrak{g}$</u>

In the projection

$$\tau : \ T(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$$

PBW theorem asserts that $S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ is an isomorphism.Denote:

$$\Gamma = \tau|_{S(\mathfrak{g})}^{-1} \circ \tau$$

Γ is a G-map (as a consequence of the next prop).

Denote by $\Gamma_{2r,2}$ the subgroup of permutation that preserves the set of unordered pairs {(1,2),(3,4),...,(2r-1,2r)} and let Π_r be a left coset representative of $\Gamma_{2r,2}$ in Γ_{2r} that $sg(\Pi_r) = 1$

In [Amitsur-Levitski],Kostant proved:

Prop. (I.10.3.2) (in K4).

$$\Gamma(\wedge^{2k}(\mathfrak{g})) = R^k \in S^k(\mathfrak{g})$$

$$\Gamma(x_1 \wedge \cdots \wedge x_k) \longrightarrow \sum_{v \in \Pi_k} [x_{v(1)}, x_{v(2)}] \cdots [x_{v(2k-1)}, x_{v(2k)}]$$

Prop. (I.10.3.3) (in K4).

$$M = R^r \in H^r$$

so it consists of harmonic functions.

Let $w \in \wedge^2\mathfrak{g}$ of rank k standardized as $v_1 \wedge v_2 + \cdots + v_{2k-1} \wedge v_{2k}$.Let

$$\mathrm{Rad}w = \{y \in \mathfrak{g} | \iota(y)w = 0\} = \{y \in \mathfrak{g} | (w, \epsilon(y)z) = 0, \forall z\}$$

then w of rank $2k \iff w^k \neq 0 \ \&\ w^{k+1} = 0 \iff \dim \mathrm{Rad}w = n - 2k$.

Lemma (I.10.3.4).

$$\iota(y)dx = [y, x]$$

Thus

$$\mathrm{Rad}dx = \mathfrak{g}^x, \ \mathrm{Sing}\mathfrak{g} = \{x \in \mathfrak{g} | (dx)^r = 0\}.$$

Proof: $(\iota(y)dx, z) = (dx, y \wedge z) = (x, -[y, z]) = ([y, x], z)$ □

So in order to find the module M,it's the best to find the dual of

$$\gamma : S(\mathfrak{g}) \longrightarrow \wedge^{\text{even}}\mathfrak{g} : x \longrightarrow -dx$$

Luckily:

Prop. (I.10.3.5) (in K4). γ is $\mathcal{B}$ dual to Γ,in particular,

$$(\Gamma(\zeta), x) = \frac{(-1)^r}{r!}(\zeta, (dx)^r) \ (\forall \zeta \in \wedge^{2r}\mathfrak{g} \text{ and } x \in \mathfrak{g})$$

So:

$$f(x) = 0, \forall f \in M \iff x \in \mathrm{Sing}\mathfrak{g}$$

Cor. (I.10.3.6). Let $\mathfrak{a}$ be a CSA of $\mathfrak{g}$,$\Delta_+(\mathfrak{a})$ be the positive roots, then

$$f|\mathfrak{a} = C_f \cdot \prod_{\beta \in \Delta_+(\mathfrak{a})} \beta \qquad (\forall f \in M)$$

Proof: This is because that an element in a CSA is singular iff it commutes with an element outside this CSA, and taking root decomposition, this is equivalent to annihilated by a root, and by counting degree, the cor follows. □

By props of [K1] a **regular nilpotent** element e is **uniquely** in a nilpotent radical $\mathfrak{n}$ of a Borel subalgebra and that $\mathfrak{g}^e \cap [\mathfrak{n}, \mathfrak{n}] = (\mathrm{Sing}\mathfrak{g}) \cap \mathfrak{g}^e$.So there is a linear function ξ on $\mathfrak{g}^e$ that $\mathrm{Ker}\,\xi = (\mathrm{Sing}\mathfrak{g}) \cap \mathfrak{g}^e$.Thus:

Cor. (I.10.3.7).

$$f|\mathfrak{g}^e = C_f \cdot \xi^r \qquad (\forall f \in M)$$

Proof: By counting degree, the same reason as before. □

Now we think of a natural question:Can singular elements be defined be functions of even lower degree? The answer is NO.

Prop. (I.10.3.8). Assume $0 \neq f$ homogenous vanishes on $\mathrm{Sing}\mathfrak{g}$,then $\deg f \geq r$

Proof: By the last cor, if f has degre less than r then f vanish on any CSA, but semisimple regular element, thus CSAs are Zariski dense in $\mathfrak{g}$ (this is because semisimple element are defined by a polynomial),so $f = 0$. □

Thus we have established that $\mathrm{Sing}\mathfrak{g}$ is an algebraic set defined by a set of harmonic r-homogenous functions on $\mathfrak{g}$ and not by functions of degree lower than r.

Next we offer a different formation of M.

M **as minors of determinants**

for a $\mathcal{B}$ dual basis y_i, w_j,define a derivation

$$d_W(f \otimes u) = \sum_i^n \partial_{y_i} f \otimes \epsilon(w_i)u \qquad \text{on } S(\mathfrak{g}) \otimes \wedge\mathfrak{g}.$$

Here $\partial_{\sum a_i x_i}$ is defined as $\sum a_i \frac{\partial}{\partial x_1}$ for a standard basis x_i of $\mathfrak{g}$. It's easy to verify that d_W is well defined and is a G-map(Take a different basis Aw_i and Bz_i,then $AB^t = I$,substitute into the formula of d_W,it doesn't change).

Chevalley Thm tells us J is a polynomial ring $\mathbb{C}[p_1, \ldots, p_l]$, where p_i are homogenous polynomials of fixed degree d_i and $\sum_{j=i}^{l}(d_j - 1) = r$.So:

$$d_W p_1(x) \wedge \cdots \wedge d_W p_l(x) = \sum_{1 \le i_1 < \ldots < i_l \le n} \phi(y_{i_1}, \ldots, y_{i_l})(x) w_{i_1} \wedge \cdots \wedge w_{i_l}$$

Where $\phi(y_{i_1}, \ldots, y_{i_l}) = \det \partial_{y_i} p_j$ is homogenous of degree r.(counting degree).

To see this, notice that $f \otimes u$ acts as a function from $\mathfrak{g}$ to $\wedge\mathfrak{g}$: $f \otimes u(x) = f(x)u$.So:

$$d_W p_j(x) = \sum_{i=1}^n \partial_{z_i} p_j(x) w_i.$$

Prop. (I.10.3.9). for any CSA $\mathfrak{h}$ of $\mathfrak{g}$ and a basis $\{v_i\}$ of $\mathfrak{h}$,$\forall x \in \mathfrak{h}$,

$$d_W p_1(x) \wedge \cdots \wedge d_W p_l(x) = \kappa \cdot \prod_{\phi \in \Delta_+} \phi(y) v_1 \wedge \ldots \wedge v_l$$

Lemma (I.10.3.10) (in K2).

$$\{d_W p_1(x), \cdots, d_W p_l(x)\} \text{ linearly independant } \iff x \in \mathrm{Reg}\mathfrak{g}$$

Proof: Notice that $d_W p_j$ is a $\mathfrak{g}$-map,ad$y \cdot d_W p_j = d_W p_j([y, x])$ so $d_W p_j(x)$ commutes with $\mathfrak{g}^x$; so $\in \mathfrak{g}^x$.Then the lm tells us when y is regular, $d_W p_j(y)$ forms a basis of $\mathfrak{g}^y$.Considering in $\mathfrak{g}^y, x$ is regular iff $\prod_{\phi \in \Delta_+} \phi(x) \neq 0$,the prop follows. ☐

Next we give a explicit expression for γ_r.

It can be verified (taking a z_i basis) that

$$dx = \frac{1}{2} \sum_{i=1}^n w_i \wedge [z_i, x]$$

Now $x \in \mathfrak{h}$,

$$dx = \sum_{\phi \in \Delta_+} \phi(x) e_\phi \wedge f_{-\phi}$$

(just take the basis w_i and z_i as a standard basis of $\mathfrak{g}$ consisting of $\{h_i, \ldots, h_l, e_\phi, f_\phi\}$)

So

$$\gamma_r(x^r) = r!(-1)^r \prod_{\phi \in \Delta_+} \phi(x) e_\phi \wedge f_{-\phi}$$

Let $\mu = i^r v_1 \wedge \cdots \wedge v_l \wedge \prod_{\phi \in \Delta_+} e_\phi \wedge f_{-\phi}$ then $(\mu, \mu) = 1$.

Denote for $v \in \wedge\mathfrak{g}$

$$v^* = \iota(v)\mu$$

then

$$(v_1 \wedge \cdots \wedge v_l)^* = i^r \prod_{\phi\in\Delta_+} e_\phi \wedge f_{-\phi} = C_o\gamma_r(x^r)$$

(notice that $\iota(u)\iota(v) = \iota(v \wedge u)$ and use lm 1)

Prop. (I.10.3.11).

$$(d_W p_1(x) \wedge \cdots \wedge d_W p_l(x))^* = \kappa_o\gamma_r(\frac{x^r}{r!}) \neq 0$$

Proof: For $y \in \mathfrak{h}$ regular, this follows from previous calculations, and notice both side are G-maps, and semisimple regular elements are Zariski open, conclusion follows. □

Lemma (I.10.3.12). $(s,t) = (s^*, t^*)$,so that $-^*$ is a $\mathcal{B}$-isomorphism.

Prop. (I.10.3.13). Let $\{w_1, \ldots, w_{2r}\}$ be linearly independent and $\{u_1, \ldots, u_l\}$ be a basis of $\{w_1, \ldots, w_{2r}\}^\perp$,then

$$\Gamma(w_1 \wedge \cdots \wedge w_{2r}) = \kappa_1 \det \partial_{u_i} p_j \neq 0$$

Thus,M is the span of all the minors $\det \partial_{u_i} p_j$.

Proof: By the preceding props,

$$\begin{aligned}
\det \partial_{u_i} p_j &= \phi(u_1, \ldots, u_l)(x) \\
&= (d_W p_1(x) \wedge \cdots \wedge d_W p_l(x), u_1 \wedge \cdots \wedge u_l) \\
&= ((d_W p_1(x) \wedge \cdots \wedge d_W p_l(x))^*, (u_1 \wedge \cdots \wedge u_l)^*) \\
&= \kappa_o\kappa_2(\phi_r(\frac{x^r}{r!}), w_1 \wedge \cdots \wedge w_{2r}) \\
&= \kappa^{-1}\Gamma(w_1 \wedge \cdots \wedge w_{2r})(x).
\end{aligned}$$

□

G-module structure of M

Now we show the G-module structure of M.
Let θ be the derivative that $\theta(x)(y) = [x,y]$ on $\mathfrak{g}$.

$$\mathrm{Cas} = \sum_{i=1}^n \theta(z_i)\theta(w_i).$$

It's in fact just the action of the Casimir element in center of $U(\mathfrak{g})$.Let m_l and M_l be the maximal eigenvalue and eigenspace of Cas.

For a commutative Lie subalgebra $\mathfrak{c}$ of rank l ,denote by $[\mathfrak{c}]$ the line it defines on $\wedge^l\mathfrak{g}$.The span of these $[\mathfrak{c}]$ is denoted A_l.Notice that $[g^y] \subset A_l$ for a regular y,and A_l is a G-submodule.

Prop. (I.10.3.14) (in K3).

$$A_l = M_l; \qquad m_l = l.$$

An ideal in a Borel subalgebra of $\mathfrak{g}$ is necessarily spanned by root vectors and a prop of [KW] says any ideal of dim l is (denoted by $\mathcal{I}$) in fact abelian.

A prop in [K3] asserts that for two different ideals Φ_1, Φ_2,sum of their weight vectors $\langle\Phi\rangle$ is distinct.

So $G[\Phi_i]$ is an irreducible G-module V_Φ with highest weight $\langle\Phi\rangle$ and V_Φ are inequivalent G-modules(because an irreducible representation have only one highest vector).

Prop. (I.10.3.15) (in K3).

$$M_l = \oplus_{\Phi\in\mathcal{I}} V_\Phi.$$

Now denote M_{2r} image of M_l under the isomorphism $u \longrightarrow u^*$, then

Prop. (I.10.3.16) (in KW). M_l is the span of $G \cdot [\mathfrak{g}^x]$ for x regular.

but by precious prop,

$$[\mathfrak{g}^x] = \mathbb{C}d_W p_1(x) \wedge \ldots \wedge p_l(x).$$

thus M_{2r} is the span of $G \cdot (\gamma_r(\frac{x^r}{r!})), x$ regular.

Prop. (I.10.3.17) (Final).

$$\Gamma|_{M_{2r}} : M_{2r} \longrightarrow M$$

is an isomorphism and

$$M \cong M_{2r} \cong M_l = A_l$$

as G-module.

So M is a multiplicity one module with $|\mathcal{I}|$ irreducible components.

Proof: Notice that

$$\Gamma(\zeta)(x) = (\zeta, \gamma_r(\frac{x^r}{r!}))$$

and M_{2r} is the span of $G \cdot (\gamma_r(\frac{x^r}{r!}))$,the first part follows, and the rest is a recapitulation of previous props. $\square$

4 Amitsur-Levitski

Prop. (I.10.4.1) (Classical Amitsur-Levitski). Classical Amitsur-Levitski proposition asserting that

$$[[x_1, x_2, \dots, x_n]] = \sum_{\sigma} sg(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)} = 0 \qquad \forall x_i \in \mathfrak{gl}_n$$

called n-fold **standard identity**.
It's clear if π satisfies the m-identity, it satisfies the $m+1$-identity(By taking a summation on a fixed first element $\sigma(1)$).So Kostant seeks in this paper for the minimum m-identity satisfied by a representation.

He finds for any reductive lie algebra $\mathfrak{g}$ and representation π the smallest k that the image of $\mathfrak{g}$ satisfies the k-fold standard identity.

And he finds a computable formula regarding to the highest weight of the irreducible representation.

Notice that when $\mathfrak{g}$ is abelian, $k = 2$, so the standard identity in a way describes the degree of failure of commutativity of the resulting associative algebra.

Preliminary

Notice that in this paper, Kostant considers **reductive** lie groups, which is an abelian lie group plus a semisimple group.But in the range of this paper, the abelian part makes no contribution in the alternative part because they commutes with all elements.So We well just assume a **semisimple** lie algebra in order to get a non-degenerate Killing form.

As mentioned before:

Prop. (I.10.4.2).

$$\Gamma(\wedge^{2k}(\mathfrak{g})) = R^k \in S^k(\mathfrak{g})$$

$$\Gamma(x_1 \wedge \cdots \wedge x_k) = \sum_{v \in \Pi_k} [x_{v(1)}, x_{v(2)}] \cdots [x_{v(2k-1)}, x_{v(2k)}]$$

Proof: The proof is in fact simple, just notice that for every $v \in \Pi_k$ a representative of the subgroup $\Sigma_{2k,2}$ permuting the unordered pairs $\{(1,2),(3,4),\dots,(2k-1,2k)\}$,the element in $v\Sigma_{2k,2}$ in fact combine in pairs to $[x_{v(2i-1)}, x_{v(2i)}]$ and together the $k!$ permutation of them compose a $[x_{v(1)}, x_{v(2)}] \cdots [x_{v(2k-1)}, x_{v(2k)}]$. $\square$

Later he finds the dual of Γ,that is:

Prop. (I.10.4.3). γ is $\mathcal{B}$ dual to Γ,

$$(\Gamma(\zeta), y_1 \cdots y_r) = (-1)^r (\zeta, dy_1 \wedge \cdots \wedge dy_r) \qquad \forall \zeta \in \wedge^{2r}\mathfrak{g} \text{ and } y_i \in \mathfrak{g}.$$

In particular,

$$\Gamma(\zeta)(x) = \frac{(-1)^r}{r!}(\zeta, (dx)^r) \qquad \forall \zeta \in \wedge^{2r}\mathfrak{g} \text{ and } x \in \mathfrak{g}.$$

For the proof just notice that $(-dw, x_i \wedge x_j) = (w_i, [x_i, x_j])$ and

$$(x_1 \otimes \cdots \otimes x_r, y_1 \otimes \cdots \otimes y_r) = \sum_{\sigma} (x_1, y_{\sigma(1)}) \cdots (x_r, y_{\sigma(r)})$$

So dim of R^k equals the dim of image of γ_r,that is, spanned by $(dx)^k$(because they are dual).

We say that a representation of $\mathfrak{g}$ satisfies m-fold standard identity if the alternating sum of any m elements of image of $\mathfrak{g}$ is 0.Obviously, this is equivalent to:

$$\tau(R^k(\mathfrak{g})) \subset \operatorname{Ker} \pi_V$$

Now let $o(\mathfrak{g})$ be the maximum rank of $dw, w \in \mathfrak{g}$,then by the discussion of the first paper, when $\mathfrak{g}$ is semisimple,$o(\mathfrak{g}) = r$.So the $2r$-identity is satisfied by any representation of $\mathfrak{g}$.

Furthermore a prop of [Harish-Chandra] assert for any nonzero element $u \in U(\mathfrak{g})$,there is a representation such that $\pi(U) \neq 0$.So this is a sharp bound for general representations.

But one might naturally ask:Can we find the specific bound for a particular representation of a specific $\mathfrak{g}$?The answer is YES.

Prop. (I.10.4.4). γ vanishes on the ideal $J'_+ \cdot S(\mathfrak{g})$.

Proof: The proof comes from the observation π is a G-map and by **Cartan-Koszul** theory, invariant elements in $\wedge\mathfrak{g}$ are naturally isomorphic to the cohomology of $\mathfrak{g}$ and $\gamma(w) = -dw$ is clearly exact,Thus $\gamma(w_1 w_2 \dots w_i) = (-1)^i dw_1 \wedge \cdots \wedge dw_i$ is exact too. So $\gamma(w) = 0$.

□

Cor. (I.10.4.5).

$$M = R^r \in H^r$$

so it consists of harmonic functions.

Proof: $(u, \Gamma(y)) = (\gamma(u), y) = 0\ \forall u$ invariant, by Thm; so R^k =Image Γ is harmonic. □

Generalized Amitsur-Levitski

Let $E^k \subset U(\mathfrak{g})$ be spanned be y^k, y nilpotent in $\mathfrak{g}$,and Z the center.

in [K5] Kostant proved that the PBW isomorphism

$$\delta : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$$

induces $\delta(J) = Z$,$\delta(H) = E$.

so $\tau : T(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ induces

$$\tau(A^{2k}(\mathfrak{g})) = \delta(R^k(\mathfrak{g})) \subset E^k.$$

Define $\epsilon(\pi)$ the minimum integer k that $\pi(y)^k = 0, \forall y$ nilpotent in $\mathfrak{g}$.Then clearly:

Prop. (I.10.4.6) (Generalized Amitsur-Levitski). π satisfies the $2\epsilon(\pi)$-fold standard identity.

Lemma (I.10.4.7).

- Let π be the natural representation of $\mathfrak{gl}_n$ on $\mathbb{C}^n$ then $\epsilon(\pi) = n$.
- If n even and π the natural representation of skew-symmtric matrixs on $\mathbb{C}^n$ then $\epsilon(\pi) = n-1$.

From this one derives the **classical Amitsur-Levitski prop** that $GL_n(\mathbb{C})$ satisfies the n-fold standard identity.

Proof: The first part comes from the **abstract Jordan decomposition** which assures x is nilpotent in $\mathfrak{gl}_n$ if $\pi(x)$ is nilpotent.

the second part of the lamma comes from the **Lacobson-Morosov** Thm that any nilpotent element of $\mathfrak{g}$ is contained in a $\mathfrak{sl}_2$-triple.then we only need to show that W is reducible considered as this $\mathfrak{sl}_2$-triple-module.

But then an irreducible representation of $\mathfrak{sl}_2$ preserves a non-degenerate bilinear form it must be odd dimensional cause a non-degenerate bilinear form is equivalent to a $\mathfrak{g}$-map from V to V^*.

And there can be constructed an anti-symmetric form defined on the $\mathfrak{sl}_2$-representation on $Sym_{2k}[x, y]$, so there can't exist symmetric $\mathfrak{g}$-invariant form.So this representation must be reducible. □

Prop. (I.10.4.8). For a construction of the anti-symmetric form, notice

$$\pi(g)f(x_1, x_2) = f(g_{11}x_1 + g_{12}x_2, g_{21}x_1 + g_{22}x_2).$$

Set

$$v_k = \binom{m}{k} x_1^{m-k} x_2^k.$$

and

$$\Omega(v_k, v_{m-k}) = (-1)^k \binom{m}{k}. \quad \Omega(v_k, v_p) = 0, \qquad k + p \neq m.$$

One verifies:

$$\Omega(g \cdot u, g \cdot v) = (\det g)^m \Omega(u, v) \qquad \forall g \in GL(2, \mathcal{C})$$

So when $m = n - 1$ is odd, this is a symplectic form preserved by $\mathfrak{sl}_2$.

A computable Formula

Finally, Kostant gave a computable formula for determining $\epsilon(\pi)$.Clearly we just need to consider irreducible representation.

Let π_λ be the irreducible representation of highest weight λ,then the dual representation $\pi_{\lambda'}$ has highest weight the negative of the lowest weight of π_λ,that is, $-w_o(\gamma)$.

But then $\lambda + \lambda'$ is a sum of simple positive roots.$\lambda + \lambda' = \sum_{i=1}^n m_i\alpha_i$.Put $\epsilon(\lambda) = 1 + \sum_{i=1}^n m_i$.then:

Prop. (I.10.4.9).

$$\epsilon(\pi_\lambda) = \epsilon(\lambda)$$

Proof: Just choose a $\mathfrak{sl}_2$-triple $\{e, x, f\}$ with $\alpha(x) = 2$ $\forall\alpha$ simple root.Then $\lambda(x)$ and $-\lambda'(x)$ are respectively the maximal and minimal eigenvalues of $\pi(x)$.

$$(\lambda + \lambda')(x) = 2(\epsilon(\lambda) - 1).$$

Thus f has nilpotent degree $\epsilon(\lambda)$.And any nilpotent element action increases the eigenvalue of a eigenvector of x by at lest 2, the prop follows. □

Further Work (cf. Procesi 7)

Another different proof of the Amitsur-Levitski prop is given by Kostant using technics related to **trace identities**.It turns out that method sheds more light.Later is studied the polynomial of matrices invariant under the conjugation action. Artin conjectured that all the invariants is polynomials of the Trace polynomial $Tr(A_1A_2\cdots A_n)$(Proved)

And further, the relations among these invariant all turned out to be consequences of the prop of Hamilton-Cayley.All this is made into the **Invariant Theory**.

Prop. (I.10.4.10) (Interesting results). 1)If an algebra over a field of characteristic 0 satisfies the identity $X^n = 0$,then it satisfies all the identities of $n \times n$ matrices.
2)The space of multilinear identities of degree m of $n \times n$ matrices can be described completely in terms of Young diagrams.

Remark (I.10.4.11) (Bibliography).

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5 Reductive Lie Algebra

Prop. (I.10.5.1). A lie algebra is called reductive if $\mathrm{Rad}(L) = Z(L)$.

1. If L is reductive, then L is completely reducible ad L-module.
2. $L = [LL] \oplus Z(L)$.
3. If $L \subset GL(V)$ acting irreducibly on V, then L is reductive with $\dim \mathrm{Rad}(L) \leq 1$. In particular, If $L \in SL(V)$ and $\mathrm{char} F \neq 0$, it must be semisimple. This can be used to prove that all classical algebras are semisimple. And the diagonal matrix will be toral and finding a set of simple roots will suffice to prove that every calssical lie algebra is simple.
4. If L is a completely reducible ad L-module, then L is reductive.
5. If L is reductive, then all finite dimensional representations of L in which $Z(L)$ is represented by semisimple endomorphism are completely reducible.
6. If $[LL]$ is semisimple, then L is reductive.

Proof: (1): Because $L/Z(L)$ is a semisimple lie algebra and $Z(L)$ is mapped to the kernel.

(2): Let $L = M \oplus Z(L)$ as a ad-L module, then $[LL] \subset [MM] \subset M$, but $[LL]$ maps onto $L/Z(L)$ because a semisimple is a sum of simple algebra. So $[LL] = M$.

(3): Cf.[Humphreys P102].

(4): In this way L decompose into $Z(L)$ and simple algebras, so it is reductive.

(5): First simultaneously diagonalize $Z(L)$, then the subspace corresponding to different characters are stable under L. Then decompose w.r.t. $[LL]$ with get the result. (6): Note that the element in $\mathrm{Rad}(L)$ will all be central. $\square$

Prop. (I.10.5.2). Let L be a simple lie algebra, then any two symmetric associative bilinear forms on L is proportional. Because any of this form corresponds to a L-morphism from L to L^*. In particular, when $L \subset \mathfrak{gl}_n$, the usual trace is proportional to the Killing form.

6 Real Lie Algebra

Prop. (I.10.6.1) (Passage from Real to Complex). If $\mathfrak{g}_0$ is a Lie algebra over $\mathbb{R}$ and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ its complexification, then $\mathfrak{g}_0$ is Abelian/nilpotent/solvable/semisimple iff $\mathfrak{g}$ does.

Def. (I.10.6.2). A **compact real form** is a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ s.t. $\mathfrak{g}$ is the complexification of $\mathfrak{l}$ and $\mathfrak{l}$ is the lie algebra of a compact simply-connected Lie group.

Prop. (I.10.6.3). A real Lie algebra is compact iff there exists a inner product s.t.

$$([X, Y], Z) + (X, [Y, Z]) = 0,$$

iff the Killing form is negative definite.

Proof: One direction is easy, just use the average method to find a G-invariant inner product and then take derivative. For the other direction, the identity shows that a complement of an ideal is an ideal so $\mathfrak{g}$ is decomposed into simple lie groups and reduce to the case that $\mathfrak{g}$ is simple. The ideal is to show that $\mathfrak{g} \cong \mathrm{ad}(\mathfrak{g})$ is the whole outer derivative group $\partial(\mathfrak{g})$(the following lm). So $\mathfrak{g}$ equals to the identity component of $\mathrm{Aut}(\mathfrak{g})$ which is a closed subgroup thus closed but it is also a subgroup of the compact group $O(\mathfrak{g})$ thus it is compact. $\square$

Lemma (I.10.6.4). If a real semisimple Lie algebra X has a invariant inner product, then every outer derivative is inner.(In fact, this is true by Cartan Criterion for semisimplicity (I.10.2.8).

Proof: since $\mathrm{ad}(X)$ is skew-symmetric, it's diagonalizable and its eigenvalue is pure imaginary, so the Killing form of X is negative definite. Now choose the complement $\mathfrak{a}$ of $\mathrm{ad}(X)$ in $\partial(X)$, then $\mathfrak{a} \cap X = 0$. Thus for $D \in \mathfrak{a}$, $\mathrm{ad}(D(g)) = [D, \mathrm{ad}(g)] = 0$ for all g in X, so $D = 0$, thus $\mathrm{ad}(X) = \partial(X)$. $\square$

Prop. (I.10.6.5). -

1. The complexification of the Lie algebra of a connected compact Lie group is reductive.
2. A complex Lie algebra is semisimple iff it is isomorphic to the complexification of the Lie algebra of a simply-connected compact Lie group. i.e. every complex semisimple Lie algebra has a compact real form.

Proof: 1: Because a connected compact Lie group is completely reducible so the does the Lie algebra and so does the complexification. So it is reductive by (I.10.5.1)4.

2: Cf.[Varadarajan Lie Groups Lie algebras and Their Representations]. The ideal is to find a real form whose corresponding simply-connected group is compact. □

Prop. (I.10.6.6). If $\mathfrak{g}$ is the Lie algebra of a matrix Lie group G, then:

1. every Cartan subalgebra comes from a maximal commutative subalgebra of a compact real form and any two Cartan subalgebras are conjugate under the Ad-action of G.
2. any two compact real form is conjugate under the Ad-action of G.
3. any two maximal commutative subalgebra of a compact real form is conjugate under the Ad-action of the corresponding compact compact subgroup.

Prop. (I.10.6.7). A real Lie algebra is semisimple iff its complexification is semisimple. Cf.[Varadarajan].

Cor. (I.10.6.8). The real Lie algebra of a compact simply-connected group is semisimple.

Note: For the classification of real semisimple Lie algebras, Cf.[李群讲义项武义 §6]

Prop. (I.10.6.9). If a complex representation of a Lie group admits an invariant bilinear form, then it is non-degenerate and unique. In fact, this is equivalent to a G-map from V to V^*. Thus there is unique invariant inner product in a compact real form by the preceding proposition.

7 Universal Constructions

Def. (I.10.7.1) (Universal Enveloping Algebra). The **universal enveloping algebra** of a Lie algebra $\mathfrak{g}$ is defined to be

$$U(g) = T(\mathfrak{g})/J, \quad T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

which is a graded algebra $T(\mathfrak{g})$ quotients the ideal $J = (\{x \otimes y - y \otimes x - [xy]\})$.

There is a natural linear map $\sigma : \mathfrak{g} \to U(\mathfrak{g})$.

Prop. (I.10.7.2). The universal enveloping algebra $U : \mathfrak{g} \mapsto U(\mathfrak{g})$ defines a functor $LieAlg \to AssAlg$ that is left adjoint to the canonical functor $LieAlg \to LieAlg$.

Proof: For any associative algebra A and a morphism $\mathfrak{g} \to [A]$, there is a easily seen to be a morphism $U(\mathfrak{g}) \to A$, and it is unique. □

Prop. (I.10.7.3) (Poincaré-Birkhoff-Witt). If L is a Lie algebra with basis $\{x_i\}, i \in I$ and $<$ is an order on I, then $U(\mathfrak{g})$ has a basis consisting of elements $\{x_{i_1}^{r_1} \cdots x_{i_k}^{r_k}\}$ where $i_1 < i_2 < \ldots < i_r$.

Proof: Cf.[Carter, P174]. □

Cor. (I.10.7.4). The map $\mathfrak{g} \to U(\mathfrak{g})$ is injective.

Cor. (I.10.7.5). $U(\mathfrak{g})$ has no zero-divisors.

Proof: We can use the identities $x \otimes y - y \otimes x = [xy]$ to make any element in their right representations under the PBW prop(I.10.7.3), so it is clear two nonzero elements cannot product to be 0. □

Def. (I.10.7.6) (Free Lie Algebra). Let X be a set, then we define the **free Lie algebra** $FL(X)$ to be the intersection of Lie subalgebras in $[F(X)]$ containing $\sigma(X)$, where $F(X)$ is the free algebra generated by X.

Prop. (I.10.7.7). The free algebra $FL : X \mapsto FL(X)$ defines a functor $Set \to LieAlg$ that is left adjoint to the forgetful functor.

Proof: We need to show that for any Lie algebra L and a map fo sets $\theta : X \to L$, there is a unique φ completing the upper left triangular diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\theta"] & FL(X) \arrow[r] \arrow[dl, "\varphi"] \arrow[dr, "\overline{\varphi}"] & F(X) \arrow[d, "\overline{\theta}"] \\ L \arrow[rr, "\sigma"] & & U(L) \end{tikzcd}$$

Notice $\overline{\varphi}^{-1}(\sigma(L))$ is a Lie algebra containing X thus containing $FL(X)$, so it induces a φ.

And for the uniqueness, if there are two φ_1, φ_2, then the element that they coincide is a Lie algebra containing X, thus containing $FL(X)$, so $\varphi_1 = \varphi_2$. □

Cor. (I.10.7.8). $U(FL(X)) \cong X$ for any set X.

Proof: Because $U \circ FL$ and F are both left adjoint to the forgetful functor $AssAlg \to Set$. □

Prop. (I.10.7.9). Let L be a Lie algebra, if we let $S(L) = T(L)/(x \otimes y - y \otimes x)$ be the universal symmetric algebra of L, then it is a graded algebra. There is a filtered structure on $U(L)$ given by $U_i = \{\text{subalgebra generated by} a_1 a_2 \ldots a_j, j \leq i\}$, then the associated graded algebra of $U(L)$ is isomorphic to $S(L)$.

Proof:

□

Prop. (I.10.7.10). $U(\mathfrak{g})$ is Noetherian.

Proof: □

<u>**Center of $U(\mathfrak{g})$**</u>

Prop. (I.10.7.11) (Chevalley). The center of the universal enveloping algebra is isomorphic to the polynomial ring over $\mathbb{C}$ of l elements, where L is a semisimple lie algebra of rank l. In particular, The center for $\mathfrak{sl}_2$ is the algebra generated by the Casimir element $1/2h^2 + ef + fe$.

Proof: Because there is a commutative diagram of isomorphisms of algebras:

$$\begin{tikzcd} S(L)^G \arrow[r, "\alpha"] \arrow[d, "\eta"] & P(L)^G \arrow[d, "\phi"] \\ S(H)^W \arrow[r, "\beta"] & P(H)^W \end{tikzcd}$$

Where P is the polynomial ring$\cong S(L^*)$, the horizontal is Killing isomorphisms and vertical is the restriction maps. Cf.[Carter prop 13.32]**?**.

The twisted Harish-Chandra map gives an isomorphism of algebras $Z(L) \to S(H)^W$ (It just maps $z \in Z(L)$ to its pure H part and transform every indeterminants h_i to $h_i - 1$). e.g. $z = h^2 + 2h + 1 + 4fe \in Z(\mathfrak{sl}_2)$ is maped to h^2 in $S(H)$. And $P(H)^W$ is isomorphic to a polynomial ring in l generators over $\mathbb{C}$. □

Def. (I.10.7.12) (Casimir Element). If L is semisimple Lie algebra, by(I.10.2.2) the Killing form is non-degenerate, thus we choose a basis x_i of L and a dual basis y_i, then $c = \sum x_i y_i$ is independent of x_i chosen by(I.2.7.9), and is called the **Casimir element** of $U(L)$.

Prop. (I.10.7.13). The Casimir element lies in the center of $U(L)$.

Proof: □

Prop. (I.10.7.14) (Quillen's lm). If K is an alg.closed field of char 0 that $\mathfrak{g}$ is a f.d. Lie algebra over K. If $U = U(\mathfrak{g})$ is its universal enveloping algebra, then for any irreducible U-module M, $\mathrm{End}_U(M) = K$.

Miscellaneous

Prop. (I.10.7.15) (Grading on $U(\mathfrak{sl}_2(\mathbb{C}))$). Let H, R, L be a basis of $\mathfrak{sl}_2(\mathbb{C})$(I.10.2.11), if we define a grading as $\deg R = 1, \deg H = 0, \deg L = -1$, then this is descends to a grading on $U(\mathfrak{g})$, and the degree 0 part is the ring $R = \mathbb{C}[\Delta, H]$. Also, there is a decomposition:

$$U(\mathfrak{g}) = \bigoplus_{i \geq 0} L^i R \oplus \bigoplus_{i > 0} R^i R.$$

8 $\mathfrak{g}$-Modules

Prop. (I.10.8.1) (Schur's Lemma). Let $\mathfrak{g}$ be a finite Lie algebra, M be an irreducible $\mathfrak{g}$-module, then $\dim M$ is countable. In particular, Shur's lemma holds by(I.4.2.1).

Proof: It is of countable dimensional because $\dim U(\mathfrak{g})$ is countable. □

representations of $\mathfrak{sl}_2(\mathbb{C})$

Prop. (I.10.8.2) (Irreducible Representations of $\mathfrak{sl}_2(\mathbb{C})$). Cf.[Bump P206-208], [Complex Semisimple Lie Algebras, Serre]. Please use the fact that the Casimir element Δ acts by a scalar on an irreducible representation.

9 Lie Algebra Cohomology

Prop. (I.10.9.1) (Chevalley-Eilenberg resolution).

I.11 Almost Ring Theory

References are [Almost Ring Theory Gabber/Ramero] and [Almost Ring Theory Foundations Gabber/Ramero].

Def. (I.11.0.1). The setup of most mathematics is an flat ideal $I \subset R$, that $I^2 = I$. This implies that $I \otimes I \cong I^2 = I$.

Denote $i : R \to R/I$. Then there is a map $i_* : M \mapsto M_R$, which has a left adjoint $i^* : N \mapsto N \otimes_R R/I$, and a right adjoint $i^! : N \mapsto \mathrm{Hom}(R/I, N)$.

1 Homological Theory

Almost Modules

Prop. (I.11.1.1) (Examples).

- If K is a perfectoid field, $R = K^0, I = K^{00}$, then I is flat over R, because if π is a pseudo-uniformizer of K(II.7.2.6), then $I = (\pi^{\frac{1}{p^\infty}})$, which is a colimit of free modules thus flat, and $I^2 = I$ clearly.
- Let R be a ring and f is an arbitrary element with compatible p^n-th roots, let $I = (f^{\frac{1}{p^\infty}})$, then $I^2 = I$. To show I is flat, consider:
$$M_0 \xrightarrow{f^{1-\frac{1}{p}}} M_1 \xrightarrow{f^{\frac{1}{p}-\frac{1}{p^2}}} M_2 \to \ldots \to M_n \xrightarrow{\frac{1}{p^n}-\frac{1}{p^{n+1}}} M_{n+1} \to \ldots$$
where $M_i \cong R$, and $M = \mathrm{colim}\, M_i$, then M is flat, and there is a map $M \to I : 1 \in M_n \to f^{\frac{1}{p^n}}$, then this map is surjective, and it is injective: if α maps to 0, then $\alpha f^{\frac{1}{p^n}} = 0$, so $\alpha^{p^m} f = 0$ for all $m \geq n$, and by perfectness of R, $\alpha f^{\frac{1}{p^m}} = 0$, so in particular, $\alpha = 0 \in M_{n+1}$.

Prop. (I.11.1.2) (The Category of Almost R-Modules in disguise). Let $\mathcal{A} \subset \mathrm{Mod}_R$ be the category of all R-modules M that the action $I \otimes M \to M$ is an isomorphism(By $I \otimes I = I$ this is equivalent to $M = I \otimes N$ for some N) then:

- The inclusion $j_! : \mathcal{A} \to \mathrm{Mod}_R$ is exact, i.e. the cokernel, kernel of objects in $\mathcal{A}$ are also in $\mathcal{A}$.
- $j_!$ has a right adjoint $j^* : M \mapsto I \otimes M$, and the unit map $N \to j^* j_! N$ is an isomorphism on $\mathcal{A}$.
- j^* has its right adjoint $j_*(M) = \mathrm{Hom}(I, M)$, and the counit $j^* j_* M \to M$ is an isomorphism on $\mathcal{A}$.

Proof: 1: an easy consequence of five-lemma.

2: We need to show for $N \in \mathcal{A}$, $\mathrm{Hom}(N, I \otimes M) \cong \mathrm{Hom}(N, M)$. Notice there is a distinguished triangle
$$I \otimes M \to M \to M \otimes R/I,$$
as $- \otimes_R^L M$ is a derived functor and I is flat. So it suffices to show
$$R\,\mathrm{Hom}_R(N, M \otimes_R^L R/I) = 0 = R\,\mathrm{Hom}_{R/I}(N \otimes_R^L R/I, M \otimes_R^L R/I).$$
And in fact $N \otimes_R^L R/I = 0$, because $N \otimes_R^L R/I = N \otimes_R^L I \otimes_R^L R/I$, and $I \otimes_R^L R/I = I \otimes_R R/I = I/I^2 = 0$ by flatness and hypothesis.

$N \cong j^* j_! N$ is an easy consequence of $I \otimes I = I$.

3: The adjointness is just Tor-Hom-adjunction, and for the isomorphism $I \otimes \mathrm{Hom}(I, M) \cong M$, as I is flat, it suffices to prove the stronger result that $I \otimes^L_R R\mathrm{Hom}(I, M) = M[0]$. As there is an exact triangle

$$R\mathrm{Hom}(R/I, M) \to M \to R\mathrm{Hom}(I, M),$$

so it suffices to show $I \otimes^L R\mathrm{Hom}(R/I, M) = 0$, because $I \otimes^L M = M$. But this is because $I \otimes^L R\mathrm{Hom}(R/I, M) = I \otimes^L R/I \otimes^L R\mathrm{Hom}(R/I, M)$, and $I \otimes^L R/I = 0$ as before. □

Prop. (I.11.1.3) (Category of Almost R-modules).

- The image of the functor $i_* : \mathrm{Mod}_{R/I} \to \mathrm{Mod}_R$ is a Serre subcategory of Mod_R, so the quotient $\mathrm{Mod}^a_R = \mathrm{Mod}_R/\mathrm{Mod}_{R/I}$ exists by(I.8.2.14),
- The quotient $q : \mathrm{Mod}_R \to \mathrm{Mod}^a_R$ admits fully faithful left and right adjoints. In particular, q preserves all limits and colimits.
- The image of i is a 'tensor ideal' of Mod_R, so the quotient Mod^a_R inherits a natural symmetric monoidal $\otimes$-product structure.
- There is a functor alHom : $(\mathrm{Mod}^a_R)^{op} \times \mathrm{Mod}^a_R \to \mathrm{Mod}^a_R : (X, Y) \to \mathrm{alHom}(X, Y)$ that $\mathrm{alHom}(X, -)$ is right adjoint to $- \otimes X$:

$$\mathrm{Hom}(Z \otimes X, Y) \cong \mathrm{Hom}(Z, \mathrm{alHom}(X, Y)).$$

Proof: 1: the image of i_* is just the category of modules killed by I, if M is killed by I, then subobjects and quotients of M is killed by I, and if M is an extension of two elements killed by I, then $IM = I^2M = 0$.

2: In fact we show that the category $\mathcal{A}$ in(I.11.1.2) and the functor j^* is just equivalent to Mod^a_R: First: $j^*(\mathrm{Mod}_{R/I}) = 0$, because $I \otimes M = I \otimes_R R/I \otimes_{R/I} M = I/I^2 \otimes_{R/I} M = 0$ as $I = I^2$, and j^* is exact because I is flat.

And for any R-module M, consider $I \otimes M \to M$, it has kernels and cokernels, then tensoring I, it becomes $I \otimes M \to I \otimes M$(I.11.0.1). as I is flat, the kernel and cokernels are killed by I, so for any functor q to another category that kills $\mathrm{Mod}_{R/I}$, $q(M) = q(I \otimes M) = qj_!j^*(M)$, so q factors through M, uniquely, as j^* is surjective.

Now the left/right adjoints exist by(I.11.1.2).

3: if $IM = 0$, then $IM \otimes N = 0$, so the tensor products pass to the quotient, and j^* is a map between symmetric monoidal categories.

4: alHom is defined by $\mathrm{alHom}(j^*M, j^*N) = j^*(\mathrm{Hom}(M, N)) = \mathrm{Hom}(M, N)^a$. This is well defined, because if $IM = 0$ or $IN = 1$, then $I\,\mathrm{Hom}(M, N) = 0$. □

Cor. (I.11.1.4).

- $i^*j_! = 0$.
- $i^!j_* = 0$.
- $j^*i_* = 0$, and the kernel of j^* is just $i_*(\mathrm{Mod}_{R/I})$.

Proof: 1: $R/I \otimes I \otimes M = 0$, because $I \otimes R/I = 0$.

2: $\mathrm{Hom}(R/I, \mathrm{Hom}(I, M)) = 0$, because $I \otimes R/I = 0$.

3: This is by 2 of the proposition(I.11.1.3). □

Remark (I.11.1.5). The construction above can be summarized as the following diagram:

$$\mathrm{Mod}_{R/I}\ \begin{matrix}\xleftarrow{\ i^*\ }\\ \xrightarrow{\ i_*\ }\\ \xleftarrow{\ i^!\ }\end{matrix}\ \mathrm{Mod}_R\ \begin{matrix}\xleftarrow{\ j_!\ }\\ \xrightarrow{\ j^*\ }\\ \xleftarrow{\ j_*\ }\end{matrix}\ \mathrm{Mod}_R^a$$

with four adjoint pair and three vanishing. This should be seen as an analogy of the case of topology: X is a space and $i : U \to X$ is open in X, and $j : Z \to X$ is closed, $Z = X - U$, then the defined sheaf operations are the same as written above.

However, one should not consider Mod_R^a as the sheaf of modules on the open subscheme $\mathrm{Spec}\, R_f$ for some pseudo uniformizer, because the map $M \to M \otimes R_f$ factors through Mod_R^a as it vanishes on $\mathrm{Mod}_{R/I}$, but it is not $\mathrm{Mod}_{R/I}^a$. For example, if k is a perfect field, and consider $R = k[t^{\frac{1}{p^\infty}}]$, then the module $M = R/(t)$ is also killed by $\otimes R_f$, but it is not killed by I.

Then one may consider it is the category of Qco sheaves on $D(I)$, but first this is not an affine scheme, and second this is false, anyway. And we should imagine an non-existent open subscheme $\overline{U}$ bigger than U, as it contains any affine opens of U.

Remark (I.11.1.6). Notice j^* is both left exact and right exact, so it preserves both arbitrary limits and colimits, so almostification nearly loses anything. In particular, the category Mod_R^a has all colimits and limits.

Def. (I.11.1.7) (Almost Commutative Algebras). As Mod_R^a has a symmetric monoidal structure, it is possible to define the category of **almost commutative algebras** as the category of commutative unitary monids in Mod_R^a, denoted by $CAlg(\mathrm{Mod}_R^a)$. Notice that its unit object is $I = R^a$.

There is an obvious map

$$(-)^a : \mathrm{Alg}(\mathrm{Mod}_R) \to \mathrm{Alg}(\mathrm{Mod}_R^a),$$

and yet another functor

$$(-)_* : \mathrm{Alg}(\mathrm{Mod}_R^a) \to \mathrm{Alg}(\mathrm{Mod}_R),$$

because $M \to M_*$ is lax symmetric monoidal, i.e. there are natural map $M_* \otimes N_* \to (M \otimes N)_*$. This is a right adjoint of $(-)^a$, as j^* and j_* is adjoint.

Finally there is a functor

$$(-)_{!!} : \mathrm{Alg}(\mathrm{Mod}_R^a) \to \mathrm{Alg}(\mathrm{Mod}_R),$$

whose construction is a little complicated, first notice the functor $(-)_!$ preserves multiplication but it has no units, so in order to give it a unit, consider the module pushout: $(A_! \oplus V)/I$, which has a natural multiplicative structure that can be made into a R-module, and $(-)_{!!}$ is left adjoint to $(-)^a$,Cf.[Almost Ring Theory P22].

Prop. (I.11.1.8). $(-)_{!!}$ preserves faithfully flatness.

Proof: Cf.[Almost Ring theory P52]**?** □

Def. (I.11.1.9). For an almost commutative algebra A, a **left module** is an almost module $M \subset \mathrm{Mod}_R^a$ that has a left action $A \otimes M \to M$ that has natural commutative diagrams as one expects. And for any R-algebra A, there are natural maps $\mathrm{Mod}_A \to \mathrm{Mod}_{A^a}^a$.

<u>**Almost Homological Algebra**</u>

2 Almost Commutative Algebra

Def. (I.11.2.1) (Almost Notations). Given a R-module M, an element $f \in M$ is called **almost zero** if $I \cdot f = 0$, and M is called **almost zero** if all $f \in M$ is almost zero.

Denote

$$M^a = j^* M \in \mathrm{Mod}_R^a, \quad M_* = j_* M^a = \mathrm{Hom}(I, M), \quad M_! = j_! M^a = I \otimes M.$$

Then there are morphisms $M_! \to M \to M_*$, which becomes isomorphisms after almostification.

Prop. (I.11.2.2). If $I = (f^{\frac{1}{p^\infty}})$, then $M_* = \{x \in M[f^{-1}] | f^{\frac{1}{p^n}} x \in A\}$ for all n.

Prop. (I.11.2.3). If $M \to N$ is almost surjective maps of K^0-algebras that $M/I \to N/I$ is surjective, then $M \to N$ is surjective.

Proof: As I is flat over K^0, if $M \to N \to Q \to 0$ is the cokernel, tensoring A/I, as $M/I \to N/I$ is surjective, $Q/IQ = 0$, but Q is almost zero thus $IQ = 0$, so $Q = 0$. $\square$

Def. (I.11.2.4) (Almost Properties). Something is called **almost XXX** if it is XXX when passed to the category of almost R-modules. For example,

- elements of M_* are called **almost elements** of M.
- M is called **almost flat** iff $M^a \otimes -$ is exact on Mod_R^a, which is equivalent to $Tor_{>0}^R(M, N)$ is almost zero for all N.
- M is called **almost projective** iff $\mathrm{alHom}(M, -)$ is exact on Mod_R^a, which is equivalent to $\mathrm{Ext}_R^{>0}(M, N)$ is almost zero for all N.
  Notice this is not equivalent to projective in Mod_R^a, because R is almost projective, but $\mathrm{Hom}_{R^a}(R^a, M^a) = \mathrm{Hom}(I, M)$ is not exact as I is not projective: $\mathrm{Ext}^1_{R^a}(R^a, R^a) = \mathrm{Ext}^1_R(I, R) = \mathrm{Ext}^2(k, R)$, which is not 0 if R is the valuation ring of a non-spherically complete perfectoid field K, like $\widehat{\overline{\mathbb{Q}_p}}$**?**.
- M is called **almost finitely generated/almost finitely presented** if for any $\varepsilon \in I$, there is a f.g./f.p. $M_\varepsilon \to M$ with N_ε generators that the kernel and cokernel are killed by ε. It is called **uniformly almost finitely generated** iff N_ε is independent of ε.
  Notice this definition doesn't depends on M chosen**?**
- If S is of char$p$, it is called **almost perfect** iff S_* is perfect.

Prop. (I.11.2.5) (Enough Almost Injectives). The category mod_R^a has enough injectives. In fact j^*, j_* both preserves injectives, because they has exact left adjoints, so I is injective R-module iff I^a is injective R^a-module, and J is injective R^a-module iff J_* is injective R-module. So to construct an injective resolution in R^a, pass to R-modules using either $(-)_*$ or $(-)_!$ and find an injective resolution, then almostificate it.

Prop. (I.11.2.6) (Derived Functors of $(-)_*$). Notice that $\mathrm{Hom}_{R^a}(M^a, N^a) = \mathrm{Hom}_R(I \otimes M, N)$ by adjointness, so using(I.11.2.5),

$$\mathrm{Ext}^k_{R^a}(M^a, N^a) = \mathrm{Ext}^k_R(M_*, N) = \mathrm{Ext}^k_R(M, R\,\mathrm{Hom}(I, N)),$$

then as $M_* = \mathrm{Hom}(I, M)$, the derived functor of $(-)_*$ is just $\mathrm{Ext}^k_R(I, M) = \mathrm{Ext}^k_{R^a}(R^a, M^a)$.

Notice that $\mathrm{Ext}^k_{R^a}(M, N)$ are all almost zero, as $j^a j_* = \mathrm{id}$, and use trivial Grothendieck spectral sequence.

Prop. (I.11.2.7) ((Example)A Quadratic Extension of a Perfectoid Field). If $K = \widehat{Q_p[p^{\frac{1}{p^\infty}}]}$ and $L = K(\sqrt{p})$ with $p \neq 2$, then L^0 is a uniformly almost f.p. projective K^0-module.

Proof: It suffices to find for each n a K^0-module R_n of rank 2 that $R_n \to L^0$ is injective with cokernel annihilated by $p^{\frac{1}{p^n}}$. For this, consider $R_n = K^0 \oplus K^0 p^{\frac{1}{2p^n}}$, then $L^0 = \widehat{\operatorname{colim}_n R_n}$.

Notice that the cokernel of $R_n \to R_{n+1}$ is killed by $p^{\frac{1}{p^n}}$, because

$$p^{\frac{1}{p^n}} \cdot p^{\frac{1}{2p^{n+1}}} = p^{\frac{(p+1)/2}{p^{n+1}}} \cdot p^{\frac{1}{2p^n}} \subset R_n.$$

So by killing one by one, the cokernel of $R_n \to \operatorname{colim}_n R_n$ is killed by any p-power with power larger than $\sum \frac{1}{p^n}$, in particular by $p^{\frac{1}{p^{n-1}}}$. So $\operatorname{colim}_n R_n$ is an extension of R_0 by a cokernel killed by p, so it is also p-adically complete, and $L^0 = \operatorname{colim}_n R_n$. Now Consider $0 \to R_n \to \operatorname{colim}_n R_n \to \operatorname{Coker}$, then $\operatorname{Ext}^n(\operatorname{colim}_n R_n, N) = \operatorname{Ext}^n(\operatorname{Coker}, N)$ is killed by $p^{\frac{1}{p^{n-1}}}$ for all n, so it is killed by I, thus $\operatorname{colim}_n R_n$ is almost projective. $\square$

Completions and Closures

Prop. (I.11.2.8) (prc and Completion). If A is a ring with a nonzero-divisor f that $A \subset A[f^{-1}]$ is p-root closed(prc), then:

- $\widehat{A} \subset \widehat{A}[f^{-1}]$ is p-root closed.
- If f admits a compatible p-power roots, then $A_* \subset A_*[f^{-1}]$ is p-root closed(where almost mathematics is performed w.r.t $(f^{\frac{1}{p^\infty}})$.

Proof: We first replace A with its maximal separated quotient $A/(\cap_n f^n A = I)$: f is still non-zero-divisor, because if $fg \in I$, then $fg \in f^n A$ for all n, so $g \in f^{n-1}A$ as f is non-zero-divisor. And it is p-root closed, because if $a^p \in A/I[f^{-1}]$, then $a^p = b + f^{-c}d$ for c integer and $d \in I$. Notice $I = fI = f^c I$ by(I.5.5.7), so $f^{-c}d \in I$ as well, so $a \in A$.

Now A is f-separated, in particular, $A \hookrightarrow \widehat{A}$.

1: If $g \in \widehat{A}[f^{-1}]$ and $g^p \in \widehat{A}$, then $f^N g \in \widehat{A}$ for some N and choose a $m \geq N(p-1)$, then by the density, $g = g_0 + f^m g_1$ for some $g_0 \in A[f^{-1}]$, $g_1 \in \widehat{A}$. Notice $f^N g_0 \in \widehat{A}$, now

$$g^p = g_0^p + p g_0^{p-1} f^m g_1 + \ldots + (f^m g_1)^p,$$

By definition of m, all terms except g_0^p are in $\widehat{A}$, so $g_0^p \in A$, so $g_0 \in A$, and $g \in \widehat{A}$.

2: Use the convention(I.11.2.2), if $g \in A_*[f^{-1}]$ that $g^p \in A_*$, then $f^{\frac{1}{p^n}} g^p \in A$ for all n, so $(f^{\frac{1}{p^{n+1}}} g)^p \in A$, thus $f^{\frac{1}{p^{n+1}}} g \in A$, hence $g \in A_*$. $\square$

Prop. (I.11.2.9) (ic and Completion). Let A be a ring with a non-zero-divisor f, if $A \subset A[f^{-1}]$ is integrally closed, then:

- $\widehat{A} \subset \widehat{A}[f^{-1}]$ is integrally closed.
- If f admits a compatible p-power roots, then $A_* \subset A_*[f^{-1}]$ is integrally closed(where almost mathematics is performed w.r.t $(f^{\frac{1}{p^\infty}})$.

Proof: We first replace A with its maximal separated quotient $A/(\cap_n f^n A = I)$: f is still non-zero-divisor and I is f-divisible as in the proof of(I.11.2.8). And it is integrally closed, because if g

satisfies a monic polynomial $h(X) \in A/I[f^{-1}][X]$, then choose a lifting, $h(g) \in I[f^{-1}] = I \subset A$, so g is integral over A thus $g \in A$, and $g \in A/I$. Now A is f-separated and $A \hookrightarrow \widehat{A}$.

1: If $g \in \widehat{A}[f^{-1}]$ satisfies a polynomial $H \in \widehat{A}[X]$, then $g = f^{-c}h$ for $h \in \widehat{A}$, and then h satisfies a polynomial $H(f^c x)$, and choose an approximation of coefficients of $H(x)$ and h_0 of h mod f^{cn}, then it is clear that $H(f^c h_0) \in f^{cn}\widehat{A} \cap A = f^{cn}A$, so when dividing back, $g_0 = f^{-c}h_0$ is integral over A thus $g_0 \in A$, thus $h_0 \in f^c A$, and $h \equiv h_0 \mod f^{cn}$, thus $h \in f^c\widehat{A}$, and $g \in A$.

2: Use the convention(I.11.2.2), if $g \in A_*[f^{-1}]$ is integral over A_*, then there are polynomial H that $H(g) = 0$, now if $\varepsilon = f^{\frac{1}{p^k}}$ consider another polynomial $H(x/\varepsilon)$, then its coefficients are all in A, thus εg is integral over A thus $\varepsilon g \in A$, and then $g \in A_*$. □

Prop. (I.11.2.10) (tic and Completion). Let A be a ring with a non-zero-divisor f that admits a compatible system of p-power roots $f^{\frac{1}{p^n}}$ for all $n > 0$, and A is totally integrally closed(tic) in $A[f^{-1}]$, then $\widehat{A}$ is totally integrally closed in $\widehat{A}[f^{-1}]$ and $A = A^*$.

Proof: 1: Notice totally integrally closed is p-root closed, so $\widehat{A} \subset \widehat{A}[f^{-1}]$ is p-root closed. Now if $f^k g^{\mathbb{N}} \subset \widehat{A}$ for some k, then by prc, $f^{\frac{k}{p^n}} g \in \widehat{A}$ for all n, thus g in an almost zero element in $\widehat{A}[f^{-1}]/\widehat{A} \cong A[f^{-1}]/A$, and then g is totally integrally closed over A, because for any n, let $n < p^k$, then $f^{\frac{1}{p^k}} g \in A$, thus $f^{\frac{n}{p^k}} g \in A$, and $fg^n \in A$.

2: Because $f^{\frac{1}{p^k}} A_* \subset A$ by convention(I.11.2.2), clearly A_* is totally integrally closed in A, thus $A_* \subset A$. □

Almost Étale Map

Def. (I.11.2.11). A map $A \to B$ of R^a-algebras is called **almost étale** iff:

- B is almost f.p. projective over A.
- (Unramifiedness(I.7.5.10))There exists a diagonal idempotent $e \in (B \otimes_A B)_*$. i.e. $e^2 = e$ and $\mu_*(e) = 1$, and $\mathrm{Ker}(\mu)_* \cdot e = 0$, where $\mu : B \otimes_A B \to B$ is the multiplication map.

Prop. (I.11.2.12) (Example of Almost Étale Maps). Let $K = \widehat{Q_p[p^{\frac{1}{p^\infty}}]}$ and $L = K(\sqrt{p})$ with $p \neq 2$, then L^0/K^0 is uniformly almost f.p projective K^0-module, by(I.11.2.7). We show it is finite étale: flatness is clear, as L^0/K^0 is torsion-free and K^0 is a valuation ring and use(II.7.3.3).

For unramifiedness, notice that

$$L \otimes_K L \cong L \times L : (a, b) \mapsto (ab, a\sigma(b)).$$

by(I.3.5.15), the diagonal idempotent e is given by

$$e = \frac{1}{2p^{\frac{1}{2p^n}} \otimes 1}(1 \otimes p^{\frac{1}{2p^n}} + p^{\frac{1}{2p^n}} \otimes 1)$$

for any $n \geq 0$, then we see $p^{\frac{1}{p^n}} e \in L^0 \otimes_{K^0} KL^0$ for all n, thus $e \in (L^0 \otimes_{K^0} L^0)_*$.

Lemma (I.11.2.13) (Lemma for Almost Purity in Characteristic p). If $\eta : R \to S$ is an integral map of perfect rings. If $\eta[t^{-1}]$ is finite étale for some $t \in R$, then η is almost finite étale w.r.t the ideal $I = (t^{\frac{1}{p^\infty}})$.

Proof: Firstly, we may assume R, S are both t-torsion-free, because the t-torsion part $R[t^\infty]$ and $S[t^\infty]$ is almost zero: if $t^c\alpha = 0$, then $t^c\alpha^{p^n} = 0$, so $t^{\frac{c}{p^n}}\alpha = 0$. So we reduce to $R/R[t^\infty] \to S/S[t^\infty]$, which doesn't change anything.

Now we reduce to the case that R, S are integrally closed in $R[t^{-1}]$ and $S[t^{-1}]$: it suffices to show that $R_{int} \subset R_*$, thus they are almost isomorphic. For this, an element $f \in R_{int}$ satisfies $f^{\mathbb{N}}t^k \in R$ for some k, so by perfectness, $ft^{\frac{k}{p^n}} \in R$ for all n, so $f \in R_*$.

Now check unramifiedness: let $e \in (S \otimes_R S)[t^{-1}]$ be a diagonal idempotent, then $t^c e \in S \otimes_R S$ for some c, now $e^2 = e$, so easily $e \in (S \otimes_R S)_*$.

Now check almost finite projective: for $m > 0$, represent $t^{\frac{1}{p^m}}e = \sum a_i \otimes b_i \in S \otimes_R S$, then use the map $S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$ as in(I.7.6.14), then $\beta\alpha = t^{\frac{1}{p^m}}$ on S, as S is t-torsion free, $R^n \to S$ is injective with $t^{\frac{1}{p^m}}$-torsion cokernel, for any m. So S is almost finite projective. □

Prop. (I.11.2.14) (Almost Purity in Characteristic p). If R is a perfect ring of char$p$, then using the almost mathematics w.r.t. $I = (t^{\frac{1}{p^\infty}})$, $S \to S_*[t^{-1}]$ gives an isomorphism of categories: $R_{af\acute{e}t} \cong R[t^{-1}]_{f\acute{e}t}$.

Proof: As in the proof of(I.11.2.13), we may assume R is t-torsion-free. Notice that any integral extension of $R[t^{-1}]$ comes from an integral extension of R(choose the integral closure), so the lemma above(I.11.2.13) tells us the functor is essentially surjective.

Now we construct an inverse functor, $S_*[t^{-1}]$ maps to T^a, where T is the integral closure of R in $S_*[t^{-1}]$. By lemma(I.11.2.15) below, S is almost perfect. So S_* is t-torsion-free, as if $t^c f = 0$, then $t^{\frac{c}{p^n}} f = 0$ for all n, so $f = 0 \in (S_*)_* = S_*$. So now $S_* \subset S_*[t^{-1}]$. Clearly T is also perfect and t-torsion-free. So $R \to T$ is an integral extension that is identified with $R[t^{-1}] \to S_*[t^{-1}]$ after inversion of t.

To show that $T^a = S$, it suffices to show $T_* = S_*$. for $f \in T$, $f^{\mathbb{N}}$ spans a finite module of $T[t^{-1}] = S_*[t^{-1}]$, so $t^c f^{\mathbb{N}} \subset S_*$, then by perfectness, $f \in (S_*)_* = S_*$, so $T_* \in S_*$. Conversely, if $g \in S_*$, then $tg^{\mathbb{N}}$ lies in a f.g. R-module of $S_*[t^{-1}]$, by almost f.g.. So $t^c g^{\mathbb{N}} \subset T$, and then by perfectness $g \in T_*$. □

Lemma (I.11.2.15). Almost finite étale map of rings of char$p$ is almost relatively perfect.

Proof: Cf.[Bhatt notes on Perfectoid Spaces P28]. □

Chapter II

Number Theory & Arithmetic Geometry

II.1 p-adic Analysis

This section should only contain theorems that are only applicable to non-Archimedean valuations. Theorems that are applicable to both Archimedean and non-Archimedean valuations should be put intoV.3.

As far as I know, all properties proved in Functional Analysis independent of complex analysis is applicable to the non-Archimedean case, and in fact, the goal of this section is to build an analytic theory parallel to complex analysis.

References are [Non-Archimedean Analysis Part A].

1 (Non-Archimedean)Valuation Theory

Normed Rings

Def. (II.1.1.1). A **semi-normed group** is a group with a non-Archimedean valuation, it is called a **normed group** iff the valuation has kernel 0, which is equivalent to Hausdorff.

A normed group is totally connected, because open balls at 0 are subgroups hence closed.

Def. (II.1.1.2) (Normed Ring). A **(semi-)normed ring** is a (semi-)normed additive group that

- $|1| = 1$. or the valuation is trivial.
- $|ab| \leq |a||b|$.

A **valued ring** is a normed ring with $|ab| = |a||b|$. It is called **degenerate** if all non-zero valuation value≥ 1.

Prop. (II.1.1.3). A valuation on a ring is non-Archimedean iff $\{|n|\}$ is bounded. Thus any valuation on a field with finite characteristic is non-Archimedian.

Prop. (II.1.1.4). In a normed ring, every triangle is an acute isosceles triangle. (This is because the biggest is smaller than the maximal of the other two, thus the biggest two are equal). Hence we have, for a circle $B(O, r)$, any interior point P is a center of circle, because $OP < r$.

Def. (II.1.1.5). A normed ring R is call a B**-ring** if elements of valuation 1 is invertible, it is called **bald** if there is a ε that no elements has valuation in $(1-\varepsilon, 1)$.

Prop. (II.1.1.6). If K is a normed field with valuation ring R, the smallest subring containing a zero sequence $a_0, a_1, \ldots$ is bald.

Proof: Cf.[Formal and Rigid Geometry P25]. □

Def. (II.1.1.7). An element a in a normed ring A is called **topologically nilpotent** iff $\lim a^n = 0$. The set of all topological nilpotent elements in A are denoted by $\check{A}$ or A^0.

Prop. (II.1.1.8). $\check{A}$ is a subgroup of A^+, which is multiplicatively closed. And $\check{A}$ is Clopen in A. In particular, $\check{A}$ is complete if A is complete.

Proof: Cf.[Non-Archimedean analysis P27]. □

Prop. (II.1.1.9) (Nakayama's Lemma). If A is complete normed ring and M is a A-module, if there are f.m. elements x_i of M that $M = N + \sum x_i M$, then $M = N$.

Proof: The proof is verbatim as the proof of the usual Nakayama lemma. □

Normed Modules

Def. (II.1.1.10) (Normed Module). A module M over a normed ring A is called **normed module** iff it is a normed additive group and $|ax| \leq |a||x|$ for $a \in A, x \in M$. If A is valued and the equality always holds, we call it **faithfully normed** or **valued module**.

If A is a valued field, any normed module is valued.

Prop. (II.1.1.11) (Normed Algebra). A normed algebra is an A algebra B with $A \to B$ bounded of norm 1.

Prop. (II.1.1.12). For two valued module over A, if A is non-degenerate, a morphisms is bounded iff it is continuous. This is because we can multiply by elements of A to reduce to a nbhd of 0.

This applies to the case when A contains a field where the valuation is non-trivial, because we can use(II.1.1.10).

Def. (II.1.1.13) (Completed Tensor Product). For two normed modules over a normed ring R, there is a complete normed R-module $M \hat{\otimes} N$ called the **completed tensor product**, satisfying the following universal properties: $M \times N \to M \hat{\otimes} N$ is bounded by 1, and for any complete normed R-module T and a R-map $M \times N \to T$ bounded by a, then it factor through a R-map $M \hat{\otimes} N \to T$ bounded by a.

It satisfies many universal properties as you can imagine.

Proof: Cf.[Formal and Rigid Geometry P238]. □

Cor. (II.1.1.14). By(II.1.1.12), when A is non-degenerate, then the amalgamated sum is just the fibered pushout when restricted to the category of complete valued module over A with continuous maps as morphisms, because it satisfies the universal property.

Prop. (II.1.1.15) (Amalgamated Sum). For two normed R-algebras there is an operation of **amalgamated sum** which satisfies universal properties similar to(II.1.1.13). In fact, it is just the completed tensor product when seen as modules.

Proof: Cf.[Formal and Rigid Geometry P242]. □

Weakly Cartesian Space

Def. (II.1.1.16). A normed K-vector space over a valued field K is called **weakly Cartesian** iff**?**

Prop. (II.1.1.17). If K is a complete valued field, then each normed K-vector space V is weakly Cartesian.

Proof: Cf.[Non-Archimedean Analysis P92]. □

Extensions of Norms and Valuations

Def. (II.1.1.18). If L/K is a finite extension of valued field of degree n, then v extends uniquely to $w(\alpha) = \frac{1}{n}v(N_{L/K}(\alpha)$, now we define the **ramification degree** as $(w(L^*) : v(K^*))$, and the **inertia degree** as the degree of the residue field extension.

Completeness

Prop. (II.1.1.19) (Cauchy Sequence of Non-Archimedean field). For a sequence $\sum a_i$ in a non-Archimedean field, it is a Cauchy sequence iff $\lim |a_i| = 0$.

In particular, convergent sequence are all absolutely convergent and for a Cauchy sequence not converging to 0, the valuations of the terms stabilize.

Proof: One way is easy, the other way, notice $|\sum_{v=i}^{j} a_i| \leq \max_{i,i+1,\ldots j} |a_v| < \varepsilon$. □

Prop. (II.1.1.20) (Completion of a Field). The completion of a non-Archimedean field is preferred to choose the definition of Cauchy sequence, so we see by(II.1.1.19) that $v(\hat{K}) = v(K)$.

Prop. (II.1.1.21). For a complete field K and any finite vector space L, L has only one norm up to equivalence and it is complete.

Proof: Cf.[Formal and Rigid Geometry P230]. □

Prop. (II.1.1.22). A complete valuation on a field can extend uniquely to a valuation on its alg.closure. And in the finite case, it is $|\alpha| = |N(\alpha)|^{\frac{1}{d}}$. This is an immediate consequence of(II.1.1.30) and(II.1.1.27), and $|\alpha| \leq 1$ iff it is integral over valuation ring R of K.

Prop. (II.1.1.23). Any infinite separable algebraic extension of a complete field is never complete.

Proof: We use Krasner's lemma(II.1.1.31). By Ostrowski theorem(V.3.2.19), we can assume it is non-Achimedean, otherwise it cannot by infinite dimensional. Choose an infinite linearly independent basis of decreasing value rapidly enough, then we can see the field generated by the limit contains all the partial sums, contradiction. □

Prop. (II.1.1.24). If K is alg.closed valued field, then its completion is also alg.closed.

Proof: Let $L = (\hat{K})^{alg}$, then we can extend to a valuation on L, now let f be a monic polynomial with coefficients in $\hat{K}$, we show its root $\alpha \in L$ can be approximated by elements in K, now let g monic in $K[X]$ be an approximation of f that $|g(\alpha)| \leq \varepsilon^n$, then there is a root β of g that $|\alpha - \beta| < \varepsilon$, and $\beta \in K$ by alg.closedness. □

Prop. (II.1.1.25). If F is a complete valued field, then F^{sep} is dense in F^{alg}.

Proof: Assume F is non-Archimedean, then for $y \in F^{alg}$, there is a n that $y^{p^n} = \alpha \in F^{sep}$. We may assume $|\alpha| \leq 1$, then let π be an element that $|\pi| < 1$, then if y_i is a root of the separable polynomial $Y^{p^n} - \pi^i Y - \alpha = 0$, then $(y - y_i)^{p^n} = \pi^i y_i$. So $|y - y_i| \to 0$. □

Henselian Value Field

Def. (II.1.1.26). A valued field K is called **Henselian** iff the valuation ring is a Henselian local ring(I.7.10.1).

Prop. (II.1.1.27). K is Henselian iff the valuation of K has a unique extension to any finite extension L/K.

Proof: Cf.[Algebraic Number Theory Neukirch P144]. □

Cor. (II.1.1.28). Thus for a normal extension, x and $\sigma(x)$ has the same valuation. Hence any polynomial in $K[X]$ has a decomposition into polynomials where all their roots has the same valuation.

Prop. (II.1.1.29) (Hensel's Lemma Generalized). Let K be a complete valued non-Archimedean field and $\mathcal{O}_K$ be the valuation ring. If $P, Q, R \in \mathcal{O}_K[X]$ and $0 \le \lambda < 1$ that $\deg P = m+n, \deg Q = n, \deg R = m$, and

$$\deg(P - QR) \le m+n-1, \quad |P - QR|_G \le \lambda |\operatorname{res}(Q,R)|^2$$

Where $|-|_G$ is the induced Gauss norm on $K[X]$. Then there exist polynomials U, V that

$$|U|_G, |V|_G \le \lambda |\operatorname{res}(Q,R)|^2, \deg U \le n-1, \deg V \le m-1$$

and $P = (Q+U)(R+V)$.

Proof: If $\rho = |\operatorname{res}(Q,R)| = 0$, then $P = QR$. Otherwise, the map $\theta_{Q,R}: W_m \oplus W_n \to W_{m+n}$ is invertible(I.3.2.12). Then we let $\varphi(U,V) = \theta_{Q,R}^{-1}(P - QR - UV)$, then If $U, V \in B(0, \lambda\rho)$, then $|\varphi(U,V)|_G \le \lambda\rho$. And it can be proved φ is a contraction map from $B(0,\lambda\rho)^2$ to itself with contraction factor λ, so it has a fixed point (U,V) by(IV.1.7.6). So $QU + RV = P - QR - UV$. □

Cor. (II.1.1.30) (Hensel's Lemma). Let K be a complete valued non-Archimedean field and A be the valuation ring. If $P(X) \in A[X]$ and α_0 is an element of A s.t. $|P(\alpha_0)/P'(\alpha_0)^2| = \varepsilon < 1$, then there exists a $\alpha \in A$ that $P(\alpha) = 0$ and $|\alpha - \alpha_0| \le |P(\alpha_0)/P'(\alpha_0)|$.

The usual form is when $|P'(\alpha_0)| = 1$, in which case we can pass to the residue field. Equivalently, the valuation ring of a complete non-Archimedean field is a Henselian local ring.

Proof: Let $\lambda = |P(\alpha_0)/P'(\alpha_0)|$ and $\operatorname{res} = |P'(\alpha_0)$. Notice If $P(X) = Q(X)(X - \alpha_0) + P(\alpha)$, then $\operatorname{res}(Q(X), X - \alpha_0) = Q(\alpha_0) = P'(\alpha_0)$(I.3.2.14). □

Prop. (II.1.1.31) (Krasner's Lemma). For a Henselian non-Archimedean field K, the if $\alpha, \beta \in \overline{K}$ that $|\alpha - \beta| < |\alpha - \sigma(\alpha)|$ for all σ, then $K(\alpha,\beta)/K(\beta)$ is purely inseparable. So when α is separable over K, $K(\alpha) \in K(\beta)$.

Proof: It suffice to prove that for all field morphism $\tau: K(\alpha,\beta) \to \overline{K}$ fixing $K(\beta)$, $\tau(\alpha) = \alpha$. This is because $|\tau(\alpha) - \beta| = |\alpha - \beta| < |\alpha - \sigma(\alpha)|$, thus $|\tau(\alpha) - \alpha| \le \max\{|\tau(\alpha) - \beta|, |\beta - \alpha|\} < |\alpha - \sigma(\alpha)|$. □

Cor. (II.1.1.32). If f is a separable irreducible polynomial and α is a root, then for g closed enough to f, there is a root β of g that $K(\beta) = K(\alpha)$. (Immediate consequence of(V.3.2.20)).

Cor. (II.1.1.33). Let K be a non-Archimedean valued field with completion $\hat{K}$, then any finite separable extension $\mathcal{L}/\hat{K}$ is of the form $L\hat{K}$. (Because of Primitive element theorem).

Cor. (II.1.1.34). If a Henselian field is dense in its alg.closure, then it is alg.closed.

Prop. (II.1.1.35) (Kaplansky-Schilling). A field which is Henselian w.r.t two inequivalent valuation is separably closed, and separably closed field is Henselian w.r.t any valuation.

Proof: □

2 K-Banach Algebra

In this section, denote K a complete non-Archimedean field(of rank 1), $K^0 = \{x \in K||x| < 1\}$, and $|t| < 1$ a uniformizer.

Normed K-modules

Prop. (II.1.2.1). If K is complete, then each normed K-module is weakly-Cartesian, Cf.[Non-Archimedean Analysis P92].

Cor. (II.1.2.2). If K is complete, any two valuation on a finite K-vector space are equivalent.

Proof: Cf.[Non-Archimedean Analysis P93]. □

Prop. (II.1.2.3). If V is a normed Q_p vector space and $V_0 = \{x \in V||x| \leq 1\}$, then $\widehat{V} \cong \widehat{(V_0)}_p[p^{-1}]$.

K-Banach Space

Def. (II.1.2.4). In the non-Archimedean case, a **K-Banach algebra** is defined as usual, but $|a+b| \leq \max\{|a|, |b|\}$.

Def. (II.1.2.5) (Uniform Banach Space). For a complete non-Archimedean field K and a Banach algebra R, define R^0 to be the ring of **power bounded elements**. Then it is a subring, and it is open, as it contains the closed ball $B(0,1]$.

R is called **uniform Banach space** if R^0 is itself bounded in R. Notice a uniform Banach space must be reduced, because a nilpotent element is clearly power bounded, and any scalar multiple of it is nilpotent.

Lemma (II.1.2.6). Fix a uniformizer t in a non-Archimedean complete field K, if $|K^*|$ is discrete, then if A is a t-adically complete and t-torsion-free K^0-algebra, let $R = A[t^{-1}]$, then the norm

$$|f| = \inf\{|t|^n | f \in t^n A\},$$

then this makes R into a K-Banach space that the t-adic topology of A is the same as the metric topology of A, so $A \subset R_{\leq 1} \subset R^0$.

Notice if $|K^*|$ is not discrete but there is a pseudo-uniformizer t that has a compatible system of p^n-th roots, if A is a t-adically complete and t-torsion-free K^0-algebra, let $R = A[t^{-1}]$, then the norm

$$|f| = \inf\{|t|^{\frac{n}{p^k}} | f \in t^{\frac{n}{p^k}} A\},$$

then this makes R into a K-Banach space that the t-adic topology of A is the same as the metric topology of A, so $A \subset R_{\leq 1} \subset R^0$, and in this case $R^0 = A_* = \mathrm{Hom}((t^{\frac{1}{p^\infty}}), A)$(I.11.2.2).

Prop. (II.1.2.7) (Uniform K-Banach Space and K^0-Algebra). Fix a pseudo uniformizer t in a non-Archimedean complete field K, the following category are equivalent:

- The category $\mathcal{C}$ of uniform Banach K-algebras R.
- The category $\mathcal{D}_{tic}$ of t-adically complete and t-torsionfree K^0-algebras A with A totally integrally closed(I.5.4.1) in $A[t^{-1}]$.

Proof: The functor $F : \mathcal{C} \to \mathcal{D}_{tic}$: if R is uniform Banach space, then $F(R) = R^0$: R^0 is open subring by(II.1.2.5), and $R^0 \in B(0,r]$ for some $r > 0$ by uniformity. As R is K-Banach, $\cap t^n B(0,r] = 0$, so R^0 is t-adically separated, and also it is complete. If $f^{\mathbb{N}} \in t^{-k}R^0 \subset t^{-k}B(0,r]$, then clearly f is power bounded thus $f \in R^0$, so R^0 is totally integrally closed in R. $R \to R^0$ is preserved by continuous mappings, so F is truly a functor.

Conversely, lemma above(II.1.2.6) shows $R = A[t^{-1}]$ is a K-Banach algebra, this is a functor $G : \mathcal{D}_{tic} \to \mathcal{C}$, and $A \subset R^0$. We show $A = R^0$, as this is equivalent to $FG \cong \mathrm{id}$: as the t-adic topology and metric topology are the same(II.1.2.6), if $t^c f^{\mathbb{N}} \subset A$ for some c, thus f is totally integral over A, thus $f \in A$ by tic.

Finally, we need to show $GF \cong \mathrm{id}$, which in fact that the given Banach algebra norm on R is equivalent to the norm $|\cdot|'$ given in(II.1.2.6) w.r.t R^0. $R'_{<1} \subset R^0 \subset R_{\leq c}$ by uniformity, and conversely, $R_{\leq 1} \subset R^0 \subset R'_{\leq 1}$, thus this two norms are equivalent. □

Prop. (II.1.2.8). Let $\varphi : A \to B$ be a k-homomorphism between k-Banach algebras that there is a family $\mathfrak{B}$ of ideals of B that for each $b \in \mathfrak{B}$:

- B is closed and $\varphi^{-1}(b)$ is closed in A.
- $\dim_k B/b < \infty$.
- $\cap_{b\in\mathfrak{B}} b = (0)$.

Then φ is continuous.

Proof: Consider the map $A/\varphi^{-1}(b) \to B/b$ with the residue norms, Cf.[non-Archimedean analysis P167]. □

Cor. (II.1.2.9). Let $\varphi : A \to B$ be a k-homomorphism between Noetherian k-Banach algebras that there is a family $\mathfrak{B}$ of ideals of B that for each $b \in \mathfrak{B}$, $\dim_k B/b < \infty$ and $\cap_{b\in\mathfrak{B}} b = (0)$, then φ is continuous. (Because the closedness condition is automatic by(II.1.2.12)).

Cor. (II.1.2.10). All complete k-algebra norms on a Noetherian k-algebra B satisfying the condition of(II.1.2.9) are equivalent.

Modules over K-Banach Spaces

Prop. (II.1.2.11). If M is a normed module over a k-Banach algebra A, if the completion of M is a finite A-module, then M is complete.

Proof: There are morphism $\pi : A^n \to \hat{M}$ that are surjective continuous, so by open mapping theorem(V.3.4.4), this map is open, so $\sum \check{A} x_i = \pi(A^n)$ is a nbhd of 0 in $\hat{M}$, because $\check{A}$ is open(II.1.1.8) and then $\hat{M} = M + \sum \check{A} x_i$, because M is dense in $\check{M}$, then we are done by(II.1.1.9). □

Cor. (II.1.2.12) (Noetherian and Submodule Closed). For a complete normed module over a k-Banach algebra A, M is Noetherian iff all submodules of M are closed. In particular, A is Noetherian iff all ideals of A are closed.

Proof: If M is Noetherian, then the completion of any submodule is finite over A, so it is complete hence closed by(II.1.2.11). Conversely, if any ideal of M is closed, then for a chain of ideals of $M : \cup M_i = M'$, M' is complete hence Baire space by(IV.1.8.2), so some M_i must contain a nbhd of M', because it is an ideal, but then $M_i = M'$. □

3 p-adic Analysis

Basic References are [p-adic Analysis Robert].

Prop. (II.1.3.1). For $b \in \mathbb{Z}_p$, we can define a power series in $\mathbb{Z}_p[[T]]$ as the limit of $(1+a)^{b_n}$ for $b_n \to b$ in $\mathbb{Z}_p$. So for $a \in \mathbb{C}_p$ with $v(a) > 0$, there can be defined an element $(1+a)^b \in \mathbb{C}_p$, and we have $(1+a)^b = \sum C_b^k a^k$.

Prop. (II.1.3.2). $\overline{\mathbb{Q}_p}$ is isomorphic to $\mathbb{C}$ as a field, uncanonically.

Cor. (II.1.3.3). The p-adic valuation on $\mathbb{Q}$ can be extended to $\mathbb{R}$ uncanonically.

Holomorphic functions

Def. (II.1.3.4). For a p-adic field L, denote by $\mathcal{L}_L$ the set of Laurent series with coefficients in $\mathcal{L}$, then the set of valuations that a Laurent series converges $Conv(f)$ is an interval of $[-\infty, +\infty]$. Let $\mathcal{A}(I)$ denote the set of elements in L of valuation in I.

If f is bounded at r_1, r_2, then it is convergent on (r_1, r_2).

Def. (II.1.3.5). Denote
$\mathcal{L}_L[r_1, r_2] = \{f | f \text{ is convergent on } [r_1, r_2]\}$.
$\mathcal{L}_L(r_1, r_2] = \{f | f \text{ is convergent on } (r_1, r_2]\}$.
$\mathcal{L}_L]r_1, r_2] = \{f | f \text{ is convergent on } (r_1, r_2] \text{ and bounded at } r_1\}$.
$\mathcal{B}_L(I)$ is the subset of bounded functions. These are all rings under addition and multiplication. And if we define $v^{(r)}(f)$ as the minimum of $v(a_n) + nr$, then it is a valuation on these rings.

Proof: Cf.[Foundations of Theory of (φ, Γ)-modules over the Robba Ring P31]. □

Def. (II.1.3.6). If we set for $\mathcal{L}_L]r_1, r_2]$ the the valuation $v^{[r_1, r_2]}(f) = \min\{v^{(r_1)}(f), v^{(r_2)}(f)\}$, then this is a valuation on it.

Prop. (II.1.3.7). $\mathcal{L}_L(\{r\})$ is complete under valuation $v^{(r)}$. Similarly the valuation $v^{[r_1, r_2]}(f)$ makes $\mathcal{L}_L[r_1, r_2]$ a Banach space unless $r_1 = r_2 = \infty$.

Proof: We let $r = 0$. For a Cauchy sequence of Laurent series, we see that each coefficient is a Cauchy sequence, hence converge to some element in L, so it converge term-wise to a Laurent series f, so it converse to f in $v^{(r)}$. □

Cor. (II.1.3.8). We consider $\mathcal{L}_L(0, r]$, then it has a countable sequence of norms $v^{1/n, r}$, which makes it a locally convex space, and the last proposition shows that these valuations are complete, and a Cauchy sequence must converge to the term-wise limit, so $\mathcal{L}_L(0, r]$ is a complete Fréchet space in the Fréchet topology.

Cor. (II.1.3.9). The same method shows that $\mathcal{L}_L(I)$ is a Fréchet space for any interval I.

Def. (II.1.3.10) (Robba Ring and Overconvergent Elements). We define $\mathcal{E}$ as the Laurent sequences that are bounded at 0 and $\varinjlim_{n \to -\infty} v(a_n) = \infty$, and we define the **overconvergent elements** $\mathcal{E}^\dagger$ and **Robba ring** $\mathcal{R}$ as

$$\mathcal{E}^\dagger = \bigcup_{r>0} \mathcal{L}_L]0, r], \quad \mathcal{R} = \bigcup_{r>0} \mathcal{L}_L(0, r], \quad \mathcal{E}^\dagger \subset \mathcal{R}$$

and equip them with the final topology w.r.t. the Fréchet topologies on $\mathcal{L}_L(0, r]$. And denote by $\mathcal{E}^+ = \mathcal{E}^\dagger \cap L[[T]]$ and $\mathcal{R}^+ = \mathcal{R} \cap L[[T]]$.

For more properties of Robba ring, See [Foundations of Theory of (φ, Γ)-modules over the Robba Ring Chap4].

Def. (II.1.3.11) (Newton Polygon). For a non-Archimedean valued field K and a polynomial or power series $P(X) = a_0 + a_1X + \cdots + a_dX^d \in K[X]$, we denote by **Newton polygon** as the lower convex hull of the set of points $(0, v(a_0)), (1, v(a_1)), \ldots, (d, v(a_d))$.

Prop. (II.1.3.12). Foe a non-Archimedean field K the number of roots of P in $\overline{K}$ with valuation λ equals the horizontal width of the segment of Newton polynomial of P of slope $-\lambda$.

Proof: We may assume P is monic, then its coefficients are elementary polynomials of roots of P. And the conclusion follows as K is non-Archimedean. □

For Newton polynomial of power series, see[Berger Galois Representations Chap3] and Reference [Zeros of Power Series over complete Valued Field Lazard].

Prop. (II.1.3.13). If $I =]0, +\infty]$ and $f(X) \in \mathcal{H}(I)$, then the number of zeros of $f(X)$ in $\mathcal{A}(I)$ equals the length of the segment of $NP(f)$ whose slope is $-s$, and these roots gives a $P_s(X) \in K[X]$ that $f(X) = P_s(X)G(X)$, $G(X) \in \mathcal{H}(I)$.

Proof: Cf.[Zeros of Power Series over complete Valued Field Lazard]. □

Cor. (II.1.3.14). If $f(X) \in \mathcal{H}(I)$, then $f(X) \in \mathcal{B}_L(I)$ iff it has f.m. zeros in $\mathcal{A}(I)$.

Proof: Let $r = \inf I$ and $s = \sup I$. First notice that $f \in \mathcal{L}_L(I)$ is in $\mathcal{B}(I)$ iff $v(a_n) + nr$ is bounded from below as $n \to +\infty$ and $v(a_n) + ns$ is bounded below as $n \to -\infty$. And from the graph of $NP(f)$, this is equivalent to f has f.m. zeros in $\mathcal{A}(I)$. □

Prop. (II.1.3.15). $\mathcal{H}(I)$ is a Bezout domain.

II.2 Algebraic Number Theory

References are [Algebraic Number Theory Neukirch], should also include notes of Pete.L.Clark. [Neukirch Chap2.8, 2.9, 3.1, 3.2] should be added quickly.

1 Ramification Theory

Prop. (II.2.1.1). If a prime $\mathfrak{p}$ splits completely in two separable extension LM of K, then it also splits completely in the composite LM.

Proof: We use the language of valuation. The extension of a valuation v of K corresponds to the set of equivalent classes of algebra map from L to $\overline{K_v}$ module conjugacy over K_v. So We only need to show that two different maps of LM are not conjugate over K_v. But the restrict of them to L or M is different, thus not conjugate over K_v by the assumption. □

Cor. (II.2.1.2). A prime splits completely in a separable extension L if it splits completely in the Galois closure N of L.

Proof: This is because the Galois closure is the composite of the conjugates of L.

But it also can be proven directly : Set $H = \mathrm{Gal}(N/L)$,$\mathcal{P}$ a prime of N over $\mathfrak{p}$, then

$$H\backslash G/G_{\mathcal{P}} \longrightarrow \{\text{Primes of } L \text{ over } \mathfrak{p}\}, \quad H\sigma G_{\mathcal{P}} \mapsto \sigma\mathcal{P} \cap L$$

is a bijection. So it splits completely in $L \iff G_{\mathcal{P}}$ is trivial $\iff$ it splits completely in N by counting numbers. □

Prop. (II.2.1.3). A prime p splits in $\mathbb{Z}[\xi_n]$ iff $p \equiv 1 (\mathrm{mod}\ n)$.

Proof: First, if it splits, then $f = 1$, Because the ring of integers is $\mathbb{Z}[\xi_n]$, so $X^n - 1$ splits in $\mathbb{F}_p$(II.2.3.3), thus $p \equiv 1 (\mathrm{mod}\ n)$. And if $p \equiv 1 (\mathrm{mod}\ n)$, it is unramified and $X^n - 1$ splits in $\mathbb{F}_p$, so $f = 1$. □

Unramified Extension

Def. (II.2.1.4). For K a Henselian non-Archimedean valued field, L/K a finite extension is called **unramified** iff the residue field extension λ/k is separable and $[L : K] = [\lambda : k]$. Any algebraic extension is called **unramified** iff any finite extension is unramified.

This is compatible because unramified extensions form a distinguished class. So we can talk about the **maximal unramified extension** T of K.

Proof: It is faithfully transitive because the field extension degree is transitive, and for base change, as[the residue field is separable, we let $\lambda = k[\overline{\alpha}]$, and choose a lift $\alpha \in \mathcal{O}_L$, the minipoly of α is $f(X) \in \mathcal{O}_K[X]$. Then we have

$$[\lambda : k] \leq \deg \overline{f} = \deg f = [K(\alpha) : K] \leq [L : K] = [\lambda : k]$$

So $L = K(\alpha)$ and $\overline{f}$ is the minipoly of $\overline{\alpha}$. Then $L' = K'(\alpha)$, and let $g(X)$ be the minipoly of α over K', then $\overline{g}$ is a factor of $\overline{f}$ so separable, hence irreducible by Hensel's lemma. Noe:

$$[\lambda' : k'] \leq [L' : K'] = \deg g = \deg \overline{g} = [k'(\alpha) : k'] \leq [\lambda' : k].$$

So $[\lambda' : k'] = [L' : K']$. □

Prop. (II.2.1.5). The residue field of the maximal unramified extension T/K is $\overline{k}$, and the value group is the same as K.

Proof: The first assertion is because for any separable polynomial, it has a lift which is irreducible has a root lifting $\overline{\alpha}$, contradicting the maximality. For the second, look at finite subextensions, then it results from the fundamental inequality(V.3.2.21). □

Tamely Ramified Extension

Def. (II.2.1.6). For K a Henselian non-Archimedean valued field, L/K a finite extension is called **tamely ramified** iff the residue field extension is separable and $([L:T],p)=1$, where T is the maximal unramified subextension.

Prop. (II.2.1.7). A finite extension L/K is tamely unramified iff the extension is generated by radicals: $L=T(\sqrt[m]{a_i})$, where $a_i\in L$, (WARNING: make sure if $a_i\in K$ or not?).

Proof: Cf.[Algebraic Number Theory Neukirch P155]. □

Prop. (II.2.1.8). Tamely unramified extensions form a distinguished class, so we can talk about the maximal tamely unramified extensions.

Proof: Cf.[Algebraic Number Theory Neukirch P156]. □

Prop. (II.2.1.9). The value field of tamely ramified extensions. Cf.[Neukirch P157].

Totally Ramified Extension

Def. (II.2.1.10) (Eisenstein Polynomial). An **Eisenstein** polynomial is a polynomial that.

Ramification Groups

Prop. (II.2.1.11). For an extension valuation $w|v$, the **decomposition group** is $G_w(L/K)=\{\sigma\in G(L/K)|w\circ\sigma=w\}$. The **decomposition group** Z_w is the fixed field of G_w.

When w is non-Archimedean, we further define:

The **inertia group** is $I_w(L/K)=\{\sigma\in G_w(L/K)|\sigma(x)\equiv x \mod \mathfrak{P}\}$. The **inertia field** T_w is the fixed field of I_w.

The **ramification group** is $R_w(L/K)=\{\sigma\in G_w(L/K)|\sigma(x)/x\equiv 1 \mod \mathfrak{P}\}$. The **ramification field** V_w is the fixed field of R_w.

Prop. (II.2.1.12). For a local field, the ramification degree e equals the order of inertia group $|I_{L/K}|$.

Prop. (II.2.1.13). When w is non-Archimedean, the residue field extension λ/k is normal and there is an exact sequence

$$1\to I_w\to G_w\to G(\lambda/k)\to 1.$$

Proof: Cf.[Neukirch P172]. □

Prop. (II.2.1.14). T_w/Z_w is the maximal unramified subextension of L/Z_w.

Proof: Cf.[Neukirch P173]. □

Prop. (II.2.1.15). V_w/Z_w is the maximal tamely ramified subextension of L/Z_w.

Proof: Cf.[Neukirch P175]. □

Higher Ramification Groups

Def. (II.2.1.16). For L/K be a finite Galois extension of CDVR, we define the **s-th ramification group** $G_s(L/K) = \{\sigma \in G | v_L(\sigma(x) - x) \geq s + 1 \text{ for all } a \in \mathcal{O}_L\}$.

Then we have $G = G_{-1} \supset G_0 \supset G_1 \subset \ldots$. And G_0 is the inertia group.

When K has finite quotient field, then G_1 is the ramification group(one way is trivial, for the other, we use Teichmüller representatives, then R_w preserves all them, and $\sigma(x) - x \equiv 0 \mod \mathfrak{P}^2$ is true for π, so it is true for all). In this case, we have

$$G_s(L/K) = \{\sigma \in G_0 | \frac{\sigma(\pi_L)}{\pi_L} \in U_L^s\}, \text{ for } s \geq 0.$$

So there are injective morphism $G_s/G_{s+1} \to U_L^s/U_L^{s+1} : \sigma \mapsto \sigma(\pi_L)/\pi_L$ for $s \geq 0$.(This is independent of π_L chosen because units are mapped mod U_L^{s+1}).

Prop. (II.2.1.17). For local fields L/K, if σ is in the inertia group, then

$$v_L(\frac{\sigma(x)}{x} - 1) \geq v_L(\frac{\sigma(\pi_L)}{\pi_L} - 1) + \delta_{v_L(x),0}$$

for any $x \in \mathcal{O}_L$ and a uniformizer π_L. Equality holds when $v_L(x) = 1$.

Proof: if L has residue field $\mathbb{F}_q$, then any element of $\mathcal{L}$ can be written as $\sum \xi_n \pi_L^n$, where ξ_n are all $q-1$-th roots of unity. And because σ is inertia group, all $q-1$-th roots of unity are preserved, so $\sigma(\xi_n \pi_L^n) - \xi_n \pi_L^n = \xi_n \pi_L (\frac{\sigma(\pi_L)}{\pi_L} - 1)(\sigma(\pi_L)^{n-1} + \sigma(\pi_L)^{n-2}\pi_L + \ldots + \pi_L^{n-1})$ has valuation$\geq$ $v(\frac{\sigma(\pi_L)}{\pi_L} - 1) + n$. Thus the result. □

Prop. (II.2.1.18). For a finite extension of CDVR, if the residue field extension λ/k is separable, then there exists a $x \in \mathcal{O}_L$ that $\mathcal{O}_K[x] = \mathcal{O}_L$.

Proof: If $\overline{x}$ is an element of λ that generate λ over k, by primitive element theorem, then let $\overline{f}$ be the minipoly of $\overline{x}$, then let f, x be lifting of them, then $f(x)$ is a uniformizer, otherwise, we now $f'(x)$ has valuation 0, so $f(x + \pi_L)$ is a uniformizer. Now we see that $x^i f(x)^j$ is a basis of $\mathcal{O}_L$ over $\mathcal{O}_K$, □

Now in the sequel, we assume that the residue field extension is separable.

Lemma (II.2.1.19). We define $i_{L/K}(\sigma) = v_L(\sigma x - x)$, where x is the generator of $\mathcal{O}_L/\mathcal{O}_K$.

If $L/L'/K$ are Galois extensions that e is the ramification index of L/L'. Then

$$i_{L'/K}(\sigma') = \frac{1}{e} \sum_{\sigma|_{L'} = \sigma'} i_{L/K}(\sigma).$$

Proof: Cf.[Neukirch Algebraic Number Theory P178]. □

Def. (II.2.1.20) (Upper Numbering). We define the **Herbrand function** $\varphi_{L/K}(u) = \int_0^u \frac{\mathrm{dx}}{(G_0 : G_x)}$. It maps $\{x \geq 1\}$ to itself and is strictly increasing.

If $m \leq s < m+1$, then it is just $\varphi_{L/K}(s) = \frac{1}{g_0}(g_1 + g_2 + \ldots + g_m + (s - m)g_{m+1})$, where $g_i = |G_i|$. By a double counting, it is

$$\varphi_{L/K}(s) = \frac{1}{g_0} \sum_{\sigma \in G} \min\{i_{L/K}(\sigma), s + 1\} - 1.$$

The derivative of $\varphi_{L/K}$ is $\varphi'_{L/K}(s)=\dfrac{|G_s|}{g_0}$.

Let $\psi_{L/K}$ be the inverse function of We define $G^t=G_{\psi_{L/K}(t)}$, this is called the **upper numbering**.

Lemma (II.2.1.21). For $L/L'/K$ Galois extensions, one has $G_s(L/K)H/H=G_t(L'/K)$, where $t=\varphi_{L/L'}(s)$. Equivalently, $G_s/H_s=(G/H)_{\varphi_{L/L'}(s)}$.

Proof: For $\sigma'\in G(L'/K)$, we choose a inverse image $\sigma\in G(L/K)$ of maximal $i_{L/K}(\sigma)$, then $i_{L'/K}(\sigma')-1=\varphi_{L/L'}(i_{L/K}(\sigma)-1)$. To prove this, let $i_{L/K}(\sigma)=m$, then we see $i_{L/K}(\sigma\tau)=\min\{i_{L/K}(\tau),m\}$, so by(II.2.1.19), $i_{L'/K}(\sigma')=\frac{1}{e}\sum_{\tau\in H}\min\{i_{L/K}(\tau),m\}$. And $e=|H_0|$ by(II.2.1.12). So the assertion follows from(II.2.1.20).

Now σ' is in the image of G_s is equivalent to $i_{L/K}(\sigma)-1\geq s\iff\varphi_{L/L'}(i_{L/K}(\sigma)-1)\geq\varphi_{L/L'}(s)$, which by what proved is equivalent to $\sigma'\in G_t(L'/K)$. □

Cor. (II.2.1.22). For $L/L'/K$ Galois extensions, $\varphi_{L/K}=\varphi_{L'/K}\circ\varphi_{L/L'}$, hence similar formula holds for ψ.

Proof: By the proposition and multiplicity of ramification index e, we get

$$\frac{1}{e_{L/K}}|G_s|=\frac{1}{e_{L'/K}}|(G/H)_t|\frac{1}{e_{L/L'}}|H_s|.$$

where $t=\varphi_{L/L'}(s)$, which is equivalent to the derivative $\varphi'_{L/K}(s)=\varphi'_{L'/K}(t)\varphi'_{L/L'}(s)=(\varphi_{L'/K}\circ\varphi_{L/L'})'(s)$, and they are equal at 0, so the conclusion follows. □

Prop. (II.2.1.23) (Herbrand's Theorem). For $L/L'/K$ Galois extensions, $G^t(L'/K)$ is the image of $G^t(L/K)$ under the quotient.

Proof: Let $r=\varphi_{L'/K}(t)$, by the above lemme and corollary,

$$G^tH/H=G_{\varphi_{L/K}(t)}H/H=G'_{\varphi_{L/L'}(\psi_{L/K}(t))}=G'_{\varphi_{L/L'}(\psi_{L/L'}(r))}=G_r(L'/K)=G^t(L'/K)$$

□

Prop. (II.2.1.24) (Hasse-Arf). For an Abelian extension of CDVRs L/K that the residue field extension is separable, the jump in the upper numbering of higher ramification group G^v must happen at integers. (Note: The proof in the case where K is a local field is much easier by Lubin-Tate group, See(II.4.1.33).

Proof: The theorem is just saying that if $G_s\neq G_{s+1}$ for s integer, then $\varphi_{L/K}(s)$ is an integer.

This follows follows from the following lemma, because if G is not totally ramified, then we can change it to the Galois field of G_0, this didn't change anything by the definition of(II.2.1.20), and the fact $\varphi(0)=0$. And when $G^v\neq G^{v+}$, then we consider splitting G/G^{v+} into product of cyclic groups, thus there is one cyclic group H that the projection of G^v into H is not trivial. Now H is a Galois group of some L'/K, and Herbrand's theorem shows that $H^v\neq H^{v+}$, hence v is an integer by the following lemma. □

Lemma (II.2.1.25). For a cyclic totally ramified extension of CDVRs L/K that the residue field extension is separable, if μ is the maximal integer that $G_\mu\neq 1$, then $\varphi_{L/K}(G_\mu)$ is an integer.

Proof: Cf.[Serre Local Fields P94]. ☐

Remark (II.2.1.26). An example: If $F_n = \mathbb{Q}_p(\zeta_{p^n})$, then

$$G(F_n/\mathbb{Q}_p)_s = G(F_n/F_t) \quad \text{for } p^t - 1 \leq s < p^{t+1} - 2.$$

(This is because $\zeta_{p^n} - 1$ is a uniformizer of F_n). Thus $G(F_n/\mathbb{Q}_p)^i = G(F_n/F_i)$.

Different and Discriminant

Def. (II.2.1.27). Let L/K be a finite separable field extension with separable residue field extensions, and $\mathcal{O}_K$ is a Dedekind domain with integral closure $\mathcal{O}_L$ in L, there is a **trace form** on L: $(x, y) \to \mathrm{tr}(xy)$.

We define the **dual module** for a fractional ideal I as $\check{I} = \{x \in L | \mathrm{tr}(xI) \in \mathcal{O}_K\}$. This is truly a fractional ideal because if $\alpha_i \in \mathcal{O}_L$ is a basis of L/K, and let $d = \det(\mathrm{tr}(\alpha_i\alpha_j))$, then for any $a \in I \cap \mathcal{O}_L$, $ad\check{I} \in \mathcal{O}_L$, because if $x = \sum x_i\alpha_i \in \check{I}$, then $\sum ax_i \,\mathrm{tr}(\alpha_i\alpha_j) = \mathrm{tr}(xa\alpha_j) \in \mathcal{O}_K$, so solve the equation shows $dax_i \in \mathcal{O}_K$.

The **different** of K/L is defined to be $\mathcal{D}_{L/K} = \check{\mathcal{O}}_L^{-1}$.

Prop. (II.2.1.28). Different is compatible with composition, localization and completion.

Proof: Cf.[Neukirch Algebraic Number Theory P195]. ☐

Prop. (II.2.1.29). If $\mathcal{O}_L = \mathcal{O}_K[\alpha]$, then $\mathcal{D}_{L/K} = (f'(\alpha))$, where f is the minipoly of α.

Proof: Let $f = a_0 + a_1X + \ldots + a_nX^n$ and $f(X)/(x - \alpha) = b_0 + b_1X + \ldots + b_{n-1}X^{n-1}$, and denote the roots of f be α_i, then

$$\sum \frac{f(X)}{X - \alpha_i} \frac{\alpha_i^r}{f'(\alpha_i)} = X^r$$

for all r by Lagrange interpolation(I.3.2.2). This is equivalent to

$$\mathrm{tr}(\frac{\alpha_i^r b_j}{f'(\alpha_i)}) = \delta_{ij}$$

So $\mathcal{D}_{L/K} = f'(\alpha)^{-1}(b_0\mathcal{O}_K + b_1\mathcal{O}_K + \ldots + b_{n-1}\mathcal{O}_K)$. Now the result follows if $(b_0\mathcal{O}_K + b_1\mathcal{O}_K + \ldots + b_{n-1}\mathcal{O}_K) = \mathcal{O}_L$, which is easy to see if we write b_i as polynomials of α. ☐

Cor. (II.2.1.30). If L/K is finite extension of local fields, then

$$v_L(\mathcal{D}_{L/K}) = \sum_{\sigma \in G, \sigma \neq 1} i_{L/K}(\sigma) = \int_{-1}^{\infty} (|G(L/K)_t| - 1)dt.$$

Notation as in(II.2.1.19).

Prop. (II.2.1.31). If L/K is a finite extension and if I is an ideal of $\mathcal{O}_L$, then $v_K(\mathrm{tr}_{L/K}(I)) = \lfloor v_K(I \cdot \mathcal{D}_{L/K}) \rfloor$.

Proof: By definition, $\mathrm{tr}_{L/K}(x\mathcal{O}_L) \subset \mathcal{O}_K$ iff $x \in \mathcal{D}_{L/K}^{-1}$, thus $\mathrm{tr}_{L/K}(I) \subset J$ iff $I \subset \mathcal{D}_{L/K}^{-1}J$, i.e. $\mathrm{tr}_{L/K}(I)$ is the smallest ideal J of $\mathcal{O}_K$ that contains $I \cdot \mathcal{D}_{L/K}$, thus the result. ☐

2 Local Fields

Def. (II.2.2.1) (Local Fields). A **local field** is a field that is complete w.r.t. a discrete valuation and has a finite residue field.

For a local field, the normalized exponential valuation is denoted by v_p, and the normalized absolute valuation is defined by $|x|_\mathfrak{p} = q^{-v_p(x)}$, where $q = |k|$, where k is the residue field.

Prop. (II.2.2.2). For a local field, $\mathcal{O}_K$ and K are locally compact.

Proof: $\mathcal{O}_K = \lim \mathcal{O}/\mathfrak{p}^n$, because $\mathcal{O}_K$ is a complete DVR, so it is profinite, hence closed and compact. K is locally compact because for any a, $a + \mathcal{O}_K$ is compact. □

Prop. (II.2.2.3) (Local Fields). The local fields are precisely the finite extensions of the field $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$.

Proof: Cf.[Neukirch Algebraic Number Theory P135]. □

The Group Structure of Local Fields

Prop. (II.2.2.4). For $m > 0$, there is an isomorphism $(-)^m : U^n \cong U^{n+v(m)}$ when n is sufficiently large.

Proof: Let $m = u\pi^{v(m)}$. For surjectivity, we need to find x, that $1 + a\pi^{n+v(m)} = -(1 + x\pi^n)^m$. i.e.

$$-a + ux + \pi^{n-v(m)} f(x) = 0.$$

This has a solution x by Hensel's lemma. □

Cor. (II.2.2.5). $(K^*)^m$ is an open subgroup of K^*, and $\bigcap_m (K^*)^m = 1$. (Because if $a \in \bigcap_m (K^*)^m = 1$, then a is a unit, thus $a \in \bigcap_m (U)^m = 1$, thus $a \in U^n$ for every n thus $a = 1$.

Prop. (II.2.2.6). $[K^* : (K^*)^m]$ can be calculated, Cf.[Neukirch CFT P81].

Proof: □

Prop. (II.2.2.7) (p-adic Logarithm). For a p-adic umber field K, there is a unique **p-adic logarithm** function $\log : K^* \to K$ that $\log(p) = 0$, and for $1 + x \in U^1$, it is defined to be

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots.$$

Moreover, for $n > \frac{e}{p-1}$, where e is the ramification index of K, there is a map $\exp : \mathfrak{p}^n \to U^n$: which is an inverse to log on U^n, so $U^n \cong \mathfrak{p}^n$.

Proof: This is easy by a Newton polygon analysis, the slope of the Nowton polygon is $\frac{1}{p^{n-1}(p-1)}$, which converges to 0, so this is definable for all x that $v_p(x) > 0$.

For the exp, $v_p(n!) = \frac{n-c(n)}{p-1}$ by(IX.2.2.1), so its Nowton polygon is a single line with slope $\frac{1}{p-1}$, so it is definable for $v_p(x) \geq \frac{1}{p-1}$, which is equivalent to $x \in U^{\lceil \frac{e}{p-1} \rceil}$. That exp and log are converse to each other is just a formal calculus. □

Remark (II.2.2.8). In fact, this map can be extended to a function from $\mathbb{C}_p^*$ to $\mathbb{C}_p$.

Cor. (II.2.2.9). For a local field K, $\mathcal{O}_K^*$ thus also K^* are locally compact.

Proof: For n large, $U^n \cong \mathfrak{p}^n$ is compact. □

Prop. (II.2.2.10) (Multiplicative Group Structure). For a local field K,

- If char $K = 0$, then $\mathcal{O}_K^* \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}_p^d$, where $d = [K : \mathbb{Q}_p]$.
- If char $K = p$, then $\mathcal{O}_K^* \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}_p^{\mathbb{N}}$.

Proof: Cf.[Neukirch P140]. □

Prop. (II.2.2.11). Any automorphism of $\mathbb{R}$ is identity, and any automorphism of a p-adic number field is identity.

Proof: It suffices to show that an automorphism is continuous. For $\mathbb{R}$, this is because $a > 0 \iff a = b^2 \iff \sigma(a) = \sigma(b)^2 \iff \sigma(a) > 0$, and $\mathbb{Q}$ is dense in $\mathbb{R}$.

For a local field, we prove that $\sigma(\mathcal{O}_K^*) \subset \mathcal{O}_K^*$. $\mathcal{O}_K^*$ is characterized by the property that $\{n | y^n = x\}$ are infinite. This is because $x^p = a$ has a root for $a \in \mathcal{O}_K^*$ for p large prime, by Henselian lemma. □

Ramification of Cyclotomic Fields

Prop. (II.2.2.12) (Unramified case). For K a finite extension of $\mathbb{Q}_p$ of residue field $\mathbb{F}_q$, we consider $L = K(\zeta_n)/K$, where $(n, p) = 1$. Then it is unramified of degree f where f is the minimal number that $q^f \equiv 1 \mod n$. And $\mathcal{O}_L = \mathcal{O}_K[\zeta_n]$.

Proof: ζ is a root of $\Phi_n | X^n - 1$, which is separable in k, so Φ and $\overline{\Phi}$ are both irreducible of the same degree by Hensel's lemma, so it is unramified, and λ is the minimal extension of $\mathbb{F}_q$ that contains the n-th roots and are generated by it, thus the result by the theory of finite fields.

For the last assertion, notice it is unramified so $\mathcal{O}_L = \mathcal{O}_K[\zeta_n] + p\mathcal{O}_L$ hence the result follows from Nakayama. □

Cor. (II.2.2.13). The maximal unramified extension of K is generated by adjoining all n-th roots where $(n, p) = 1$. This is because there is an inclusion relation and their residue field $\overline{\mathbb{F}}_p$ is already generated by roots of unity.

Prop. (II.2.2.14) (Totally Ramified case). Consider $\mathbb{Q}_p$(other local fields behave different), we have the $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is totally ramified of degree $\varphi(p^n)$ the Galois group is $(\mathbb{Z}/p^n\mathbb{Z})^*$. The ring of valuation of $\mathbb{Q}(\zeta_{p^n})$ is $\mathbb{Z}_p[\zeta_{p^n}]$ and $1 - \zeta$ is a uniformizer.

Proof: ζ is a root of the polynomial $\Phi = X^{p^{n-1}(p-1)} + X^{p^{n-2}(p-1)} + \ldots + 1 = 0$, which equals $\frac{X^{p^n}-1}{X^{p^{n-1}}-1} \equiv (X-1)^{p^{n-1}(p-1)} \mod p$ and $\Phi(1) = p$, so $\Phi(X+1)$ is a Eisenstein polynomial, hence irreducible. So $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is totally ramified of degree $p^{n-1}(p-1)$ and $N(1 - zeta) = \prod(1 - \sigma(\zeta)) = \Phi(1) = p$, so it is a uniformizer. The ring of integer is generated by a uniformizer by(V.3.2.21) as the extension is totally ramified. □

Prop. (II.2.2.15). The cyclotomic field $\mathbb{Q}[\zeta_n]$ has integral basis.

Proof: Cf.[Algebraic Number Theory Milne P98]. □

Prop. (II.2.2.16) (Infinite Cyclotomic Field). For a p-adic number field K, let $K_n = K(\zeta_{p^n})$ and $K_\infty = \cup K_n$ and $F = \mathbb{Q}_p$. Let χ be the cyclotomic character, then $\chi(G_K)$ is an open subgroup of $\mathbb{Z}_p^*$, thus contains a U_n for some n. Thus there is an isomorphism of groups: $\chi^{-1}(U_n)\cap G_K/\chi^{-1}(U_n+1)\cap G_K \cong U_n/U_{n+1}$ which has order p, for n large.

So K_{n+1}/K_n is totally ramified of degree p, because $K_n = K\cdot F_n$, and its value group extension is of degree p, too.

And $|\{K_n : F_n\}|$ is decreasing and eventually equals to $[K_\infty : F_\infty]$. This is because its order equals $\chi^{-1}(U_n)/\chi^{-1}(U_n)\cap G_K \cong \chi^{-1}(U_n)G_K/G_K$, which is eventually $\mathrm{Ker}(\chi)G_K/G_K$, because $U_n \subset \chi(G_K)$.

Cor. (II.2.2.17). For n large, if x_i is a set of basis of $\mathcal{O}_{K_n}$ over $\mathcal{O}_{F_n}$, then they form a basis of K_N over F_N for all $N \geq n$. This is because it generate K_N over F_N and $[K_N : F_N] = [K_n : F_n]$.

Prop. (II.2.2.18). $p^n v_p(\mathcal{D}_{K_n/F_n})$ is bounded and eventually constant. In particular $v_p(\mathcal{D}_{K_n/F_n})$ converges to 0.

Proof: Cf.[Galois representation Berger P20]. □

Cor. (II.2.2.19). If L/K is a finite extension, then $\mathrm{tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty}) = \mathfrak{m}_{K_\infty}$.

Proof: By(II.2.1.31) and the fact $G(L_\infty/K_\infty) \cong G(L_n/K_n)$ for n large by(II.2.2.16), we have $\mathrm{tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_n}) = \mathfrak{m}_{K_n}^{c_n}$, where $c_n = \lfloor v_{K_n}(\mathfrak{m}_{L_n}\mathcal{D}_{L_n/K_n})\rfloor$. By the above proposition, c_n is bounded by a c. But if $x \in \mathfrak{m}_{K_\infty}$, $x \in \mathfrak{m}_{K_n}^c$ for n large, so $x \in \mathrm{tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty})$. □

Lemma (II.2.2.20). For any $\delta > 0$, when n is large, if $x \in \mathcal{O}_{K_{n+1}}$ and $g \in G(K_{n+1}/K_n)$, $v_p(g(x)-x) \geq \frac{1}{p-1} - \delta$. In particular, $v(N_{K_{n+1}/K_n}(x) - x^p) \geq \frac{1}{p-1} - \delta$.

Proof: Choose a basis e_i of $\mathcal{O}_{K_n}/\mathcal{O}_{F_n}$, then e_i^* is a basis for $\mathcal{D}_{K_n/F_n}$, and if $x_i = \mathrm{tr}_{K_{n+1}/F_{n+1}}(xe_i)$, then $x_i \in \mathcal{O}_{F_{n+1}}$ and $x = \sum x_i e_i$, by(II.2.2.17), and we have by(II.2.1.26), $v(g(x_i) - x_i) \geq 1/(p-1)$, so when n is large, by(II.2.2.18), $v(x_i) \geq -\delta$, so the require is satisfied. □

Prop. (II.2.2.21). if $\delta > 0$ and I is the ideal of elements of valuation$\geq 1/(p-1) - \delta$, then for n large, there is a map $x \mapsto x^p : \mathcal{O}_{K_{n+1}}/I \cap \mathcal{O}_{K_{n+1}} \to \mathcal{O}_{K_n}/I \cap \mathcal{O}_{K_n}$, and it is surjective.

Proof: For n large, choose a uniformizer π_{n+1} of K_{n+1}, then $\pi_n = N_{K_{n+1}/K_n}(\pi_{n+1})$ is the uniformizer of K_n because it is totally ramified(II.2.2.16), so any element $x \in \mathcal{O}_{K_{n+1}}$ can be written as $\sum \pi_{n+1}^i[x_i]$, where $x_i \in k_{K_{n+1}} = k_\infty$. Then $x^p \equiv \sum \pi_{n+1}^{pi}[x_i]^p \equiv \sum \pi_n^i[x_i^p] \mod I$ by the above proposition. And the surjection is verbatim. □

Def. (II.2.2.22) (Tate's Normalized Trace). The function $R_n(x) = p^{-k}\,\mathrm{tr}_{F_{n+k}/F_n}(x)$ is compatible with k and defines a F_n-linear projection from F_∞ to F_n, and it commutes with G_F action, called the **Tate's normalized trace**.

it's easily verified that $R_n(\mathcal{O}_{F_{n+k}}) \subset \mathcal{O}_{F_n}$, thus $R_n(\pi_n^j\mathcal{O}_{F_{n+k}}) \subset \pi_n^j\mathcal{O}_{F_n}$. So we have $v(R_n(x)) > v(x) - v(\pi_n)$. So R_n extends by continuity to a map $R_n : \hat{F}_\infty \to F_n$. If $x \in F_\infty$, then $R_n(x) = x$ for n large, thus $R_n(x) \to x$ for any $x \in \hat{F}_\infty$.

Now for a finite extension $K/\mathbb{Q}_p$, for n large, if e_i is a set of basis of $\mathcal{O}_{K_n}/\mathcal{O}_{\mathcal{F}_n}$, then for any $x \in \mathcal{O}_{K_N}$, $x = \sum x_i e_i^*$, where $x_i = \mathrm{tr}_{K_\infty/F_\infty}(xe_i) \in \mathcal{O}_{F_N}$, as in the proof of(II.2.2.20). So know we define $R_n(x) = \sum R_n(x_i)e_i^*$. Notice this is defined only for n large, and is independent of x_i chosen, and by the following lemma, it is continuous and extends to a K_n-linear projection from $\hat{K}_\infty$ to K_n.

Lemma (II.2.2.23). for any $\delta > 0$, when n is large, $v(R_n(x)) \geq v(x) - \delta$.

Proof: We have $v(x_i) > v(x) - v(\pi_N)$ by F_N-linearity, and $v(R_n(x_i)) > v(x_i) - v(\pi_n)$ as in(II.2.2.22), and $v(e_i^*) \geq -\delta$ when n is large, by(II.2.2.18). Thus the result. □

Prop. (II.2.2.24). There is a decomposition of $\hat{K}_\infty = X_n \oplus X_n$, where $X_n = \operatorname{Ker} R_n$. If $\delta > 0$, then for n large, $\alpha \in \mathbb{Z}_p^*$ and γ_n that $\chi(\gamma_n)$ is a topological generator Γ_{F_n}, $1 - \alpha\gamma_n : X_n \to X_n$ (because γ commutes with R_n) is invertible and $v_p((1-\alpha\gamma_n)^{-1}x) \geq v_p(x) - 1/(p-1) - \delta$, unless $\alpha = -1$ and $p = 2$, in which case it is only invertible on X_{n+1}.

Proof: As usual, x_i is a basis of $\mathcal{O}_{K_n/F_n}$, then $x = \sum x_i e_i^*$, $x_i = \operatorname{tr}_{K_\infty/F_\infty}(xe_i) \in \hat{F}_\infty$, and $R_n(x) = 0$. Then $(1-\alpha\gamma_n)$ acts on x_i, so it reduce to the case $K = \mathbb{Q}_p$, if one notices(II.2.2.23) and(II.2.2.18).

Injectivity: If $\alpha = 1$, this is Ax-Sen-Tate. In other situations, $(1-\alpha\gamma_n)(R_{n+k}(x)) = 0$ for all $k \geq 0$, so $R_{n+k}(x) = \alpha^{p^k}\gamma_n^{p^k}(R_{n+k}(X)) = \alpha^{p^k} R_{n+k}(X)$, so $R_{n+k}(x) = 0$, hence $x = 0$.

Surjectivity: Cf.[Galois representation Berger P23]. □

Miscellaneous

Prop. (II.2.2.25). $\sqrt{p}$ is contained in $\mathbb{Q}(\zeta_p)$, In fact, $(\sum_0^{p-1} \zeta_p^{a^2})^2 = p$.

Proof: □

3 Global Fields

Def. (II.2.3.1). A **global field** is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p((t))$, without a valuation.

Prop. (II.2.3.2). $\mathrm{G}(\mathbb{Q}[\mu_n]/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$.

Proof: We choose a prime p prime to n and show that μ_n^p is conjugate to μ_n.

Let $X^n - 1 = f(X)h(X)$ with f(X) minimal polynomial of μ_n.If $f(\mu_n^p) \neq 0$, then $h(\mu_n^p) = 0$,thus $h(X^p) = f(X)g(X)$.So module p, $X^n - 1$ has a multi root, which is impossible. □

Prop. (II.2.3.3). The ring of integers in a cyclotomic field is generated by the roots of identity.

Proof: First consider the case n a prime power. Because $d(1, \zeta, \cdots, \zeta^{d-1}) = \pm l^s$, $l^s\mathcal{O} \subset \mathbb{Z}[\zeta] \subset \mathcal{O}$. Because p totally splits, $\mathcal{O} = \mathbb{Z}[\zeta] + \pi\mathcal{O}$, thus $\mathcal{O} = \mathbb{Z}[\zeta] + \pi^t\mathcal{O}$. Choose $t = s\phi(n)$ yields $\mathbb{Z}[\zeta] = \mathcal{O}$.

Then for different p, the fields are disjoint and the discriminant are pairwisely coprime, thus by (2.11) in Neukirch, the products of the integral basis form an integral basis. □

Prop. (II.2.3.4) (Unit Theorem). If S is a finite set of primes containing all the infinite primes, the group K^S of elements of K^* that has only prime divisors in S, is a f.g. group of rank $|S| - 1$.

Prop. (II.2.3.5) (Class Number). The **ideal class group** is defined as the group of ideals in K quotients J_K the principal ideals, it has finite order, class the **class number** of K.

Prop. (II.2.3.6) (Hermite's theorem). There exists only f.m. number fields with bounded discriminant.

Proof: Cf.[Neukirch Algebraic Number Theory P206]. □

Prop. (II.2.3.7) (Minkowski's theorem). The discriminant of a number field different from $\mathbb{Q}$ is not ± 1.

Proof: Cf.[Neukirch Algebraic Number Theory P207]. □

Cor. (II.2.3.8). The field $\mathbb{Q}$ doesn't contain any unramified extensions.

Prop. (II.2.3.9) (Strong Approximation Theorem).

4 Adele and Idele

Restricted Direct Product

Def. (II.2.4.1) (Restricted Direct Product). Let $\{\mathfrak{p}\}$ be a set of indices and given a family of LCA gps $G_\mathfrak{p}$, and for a.e. $\mathfrak{p}$ an open compact subgroup $H_\mathfrak{p} \subset G_\mathfrak{p}$. Then the **restricted direct product** is defined to be

$$G = \prod{}'(G_\mathfrak{p}, H_\mathfrak{p}) = \varinjlim_{S \in \{\mathfrak{p}\}, |S| < +\infty} \prod_{\mathfrak{p} \in S} G_\mathfrak{p} \times \prod_{\mathfrak{p} \notin S} H_\mathfrak{p}$$

given the colimit space topology. And we denote $G_\mathfrak{p} \times \prod_{\mathfrak{p} \notin S} H_\mathfrak{p} = G_S, \prod_{\mathfrak{p} \notin S} H_\mathfrak{p} = G^S$.

This topology is stronger than the product topology of $\prod_\mathfrak{p} G_\mathfrak{p}$. It has an open basis $N = \prod_\mathfrak{p} N_\mathfrak{p}$, where $N_\mathfrak{p}$ is open in $G_\mathfrak{p}$ and $N_\mathfrak{p} = H_\mathfrak{p}$ for a.e. $\mathfrak{p}$. It is locally compact because every G_S does.

Prop. (II.2.4.2). Every compact subset N of G is contained in a $\prod_\mathfrak{p} N_\mathfrak{p}$, where $N_\mathfrak{p}$ is compact and $N_\mathfrak{p} = H_\mathfrak{p}$ for a.e. $\mathfrak{p}$.

Proof: This is because G_S is an open covering of G, and the union of f.m. G_{S_i} is also of the form G_S. So N is contained in some G_S, thus its projection in the S-coordinates is compact. □

Prop. (II.2.4.3) (Quasi-Characters on G). Quasi-characters on G are all of the form $\otimes_\mathfrak{p} c_\mathfrak{p}$, where $c_\mathfrak{p}$ is trivial on $H_\mathfrak{p}$ for a.e. $\mathfrak{p}$.

Proof: Let c be a quasi-character, choose a nbhd of $1 \in U \subset \mathbb{C}$ that contains no subgroup, then $c^{-1}(U)$ has contains an open basis $\prod_{\mathfrak{p} \in S} N_\mathfrak{p} \times G^S$, where $N_\mathfrak{p}$ are open nbhds of 1, so $c(G^S) = 1$. Thus $c(\mathfrak{a}) = \prod_\mathfrak{p} c_\mathfrak{p}(a_\mathfrak{p})$ is true for any $\mathfrak{a} \in G$.

Conversely, clearly $\otimes_\mathfrak{p} c_\mathfrak{p}$ is a quasi-character on G, it is continuous. □

Prop. (II.2.4.4) (Dual of G). In each $\widehat{G_\mathfrak{p}}$, by(V.6.3.6) $H_\mathfrak{p}$ are compact, so $\widehat{H_\mathfrak{p}} = \widehat{G_\mathfrak{p}}/H_\mathfrak{p}^\perp$ are discrete, so $H_\mathfrak{p}^\perp$ is open; $H_\mathfrak{p}$ are open, so $H_\mathfrak{p}^\perp = \widehat{G_\mathfrak{p}/H_\mathfrak{p}}$ are compact. So we can define the space $\prod'(\widehat{G_\mathfrak{p}}, H_\mathfrak{p}^\perp)$.

Then the dual group $\widehat{G} \cong \prod'(\widehat{G_\mathfrak{p}}, H_\mathfrak{p}^\perp)$ as a topological group.

Proof: (II.2.4.3) shows that this is an algebraic isomorphism, so it suffices to prove this is a topological homeomorphism(V.6.3.5):

For any compact $B \in G_1 = \prod_{\mathfrak{p} \in S} N_\mathfrak{p} \times \prod_{\mathfrak{p} \notin S} H_\mathfrak{p}$, for any $\varepsilon > 0$, if $c \in \prod_{\mathfrak{p} \in S} N'_\mathfrak{p} \times \prod_{\mathfrak{p} \notin S} H_\mathfrak{p}^\perp$, where $N'_\mathfrak{p} = \{c_\mathfrak{p} || c_\mathfrak{p}(N_\mathfrak{p} - 1)| < \varepsilon/|S|\}$, then $|c(B) - 1| < \varepsilon$.

Conversely, if ε is small enough, then if $c(\prod_{\mathfrak{p} \in S} N_\mathfrak{p} \times_{\mathfrak{p} \notin S} H_\mathfrak{p}) - 1| < \varepsilon$, then $c \in \prod_{\mathfrak{p} \in S} N'_\mathfrak{p} \times_{\mathfrak{p} \notin S} H_\mathfrak{p}^\perp$, where $N'_\mathfrak{p} = \{c_\mathfrak{p} || c_\mathfrak{p}(N_\mathfrak{p} - 1)| < \varepsilon\}$ □

Prop. (II.2.4.5) (Restricted Product Measure). Let measures $d\alpha_{\mathfrak{p}}$ be given on $G_{\mathfrak{p}}$ that $\alpha_{\mathfrak{p}}(H_{\mathfrak{p}})=1$ for a.e. $\mathfrak{p}$, define a Haar measure on G as follows:

On G_S, $d\alpha_S=\prod_{\mathfrak{p}\in S}d\alpha_{\mathfrak{p}}\cdot d\alpha^S$, where α^S is the product measure on G^S.

Then these can define a functional a positive left-invariant functional I that $|I(f)|\leq ||f||$ for any f that depends only on f.m. coordinates $\mathfrak{p}\in S$. Then Stone-Weierstrass theorem shows these functions are dense in $C(G)$, thus I can be uniquely extended to a functional on $C(G)$, and this defines a Haar measure on G by Riesz representation(V.1.1.5), denoted by $d\alpha=\prod'_{\mathfrak{p}}d\alpha_{\mathfrak{p}}$, called the **restricted product measure**.

Adele and Idele

Def. (II.2.4.6) (Notations). We fix some notation:

- K is a global field.
- S is a finite set of primes.
- The **Idele** of F is defined to be $\prod'(K_{\mathfrak{p}}^*,\mathcal{O}_{\mathfrak{p}}^*)$.
- The **ideal class group** $C_K=I_K/K^*$.
- The group $I_K^S=\prod_{\mathfrak{p}\in S}K_{\mathfrak{p}}^*\times\prod_{p\notin S}U_{\mathfrak{p}}$ is called the **group of S-ideles** of K.
- $K^S=K^*\cap I_K^S$ is the set of **S-units** of K.

Prop. (II.2.4.7). For a field extension L/K, $I_K\subset I_L$, and $I_L^G=I_K$, this is be the diagonal inclusion to all the primes above a given prime, and the action is by $(\sigma\mathfrak{a})_{\mathfrak{P}}=\sigma\mathfrak{a}_{\sigma^{-1}\mathfrak{P}}$. This induces an inclusion $C_K\subset C_L$ and $C_L^G=C_K$. The last assertion uses long exact sequence and $H^1(G,L^*)=0$.

Prop. (II.2.4.8). I_K is locally compact in the restricted product topology, and K^* is a discrete subgroup of I_K, thus C_K is also Hausdorff locally compact.

Proof: Cf.[Neukirch P157]. □

Prop. (II.2.4.9). There is an absolute valuation on I_K and it vanish on K^*, thus induce a valuation on C_K. Then he kernel C_K^0 is compact and $C_K=C_K^0\times\mathbb{R}_+^*$.

Proof: Cf.[Neukirch P159]. □

Prop. (II.2.4.10). We let I_K^S be the group of Ideles that has unit as components at all primes except S. Then we have a canonical isomorphism

$$I_K/J_K^{S_\infty}\cong J_K,\quad I_K/I_K^{S_\infty}\cdot K^*\cong J_K/P_K.$$

The proof is easy, just cut out the infinite prime part of $\mathfrak{a}$.

Prop. (II.2.4.11). If S is sufficiently large(containing a S_0) then $I_K=I_K^S\cdot K^*$ hence $C_K=I_K^S\cdot K^*/K^*$.

Proof: The ideal class group is finite, hence we can find a finite set of representative for it. Only finite set of primes are involved in it, thus we let S contain all these primes and infinite primes, then for any $\mathfrak{a}$, $\prod_{\mathfrak{p}\nmid\infty}a_{\mathfrak{p}}=A_i\cdot(x)$, and $A_i\in I_K^S$, hence $\mathfrak{a}\in I_K^S\cdot K^*$. □

II.3 Profinite Cohomology

Basic Reference is Neukirch's Wonderful book [Neukirch Class Field Theory 2015] and the giant book [Neukirch Cohomology of Number Fields]. More should be added to the discussion of CFT.

1 Group Cohomology

We usually consider finite group G, at least it should be discrete.

Def. (II.3.1.1). The **group cohomology** $H^n(G,A)$ is the derived functor of the left exact functor $H^0(G,A)=A^G=\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},A)$, so $H^n(G,A)=\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},A)$.

The **group homology** $H_n(G,A)$ is the derived functor of the right exact functor $H_0(G,A)=A_G=\mathbb{Z}\otimes_{\mathbb{Z}[G]}A$, so $H_n(G,A)=\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},A)$。

A^H is left exact from G-mod to G/H-mod because it is right adjoint to the inclusion functor: $\mathrm{Hom}_G(X,A)=\mathrm{Hom}_{G/H}(X,A^H)$ and it preserves injectives **??**. Dually for A_H.

Prop. (II.3.1.2) (Serre-Hochschild Spectral Sequence). By Grothendieck Spectral sequence, the relation $A^G=(A^H)^{G/H}$ gives us a spectral sequence E that

$$E_2^{p,q}=H^p(G/H,H^q(H,A))\Longrightarrow E^n=H^n(G,A).$$

The lower parts give us:

$$0\to H^1(G/H,A^H)\xrightarrow{inf}H^1(G,A)\xrightarrow{res}H^1(H,A)^{G/H}\xrightarrow{transgression}H^2(G/H,A^H)\xrightarrow{inf}H^2(G,A).$$

dually for homology group.

Moreover if $H^k(H,A)=0$ for $k=1,\ldots,n-1$, then the rows are blank, thus the above lower part can change to dimension n.

Cor. (II.3.1.3) (Hopf). If $G=F/R$, F is free, then use the homology spectral sequence, $H_2(G,\mathbb{Z})\cong\frac{R\cap[F,F]}{[F,R]}$. Cf.[Weibel P198].

Prop. (II.3.1.4). For $G=\mathbb{Z}$, we have a free resolution $0\to Z[t,t^{-1}]\xrightarrow{1-t}Z[t,t^{-1}]\to Z\to 0$. In particular, thus $H_n(\mathbb{Z},\mathbb{Z})=\mathbb{Z}$ iff $n=0,1$ and vanish otherwise.

Prop. (II.3.1.5) (Tate Cohomology). Neukirch Constructed a standard resolution of the $\mathbb{Z}[G]$-module $\mathbb{Z}$, Cf.[Neukirch CFT P13]:

$$\cdots\longleftarrow X_{-2}\longleftarrow X_{-1}\xleftarrow{\mu\circ\varepsilon}X_0\longleftarrow X_1\longleftarrow\cdots$$

that $X_q=X_{-q-1}$ are $\mathbb{Z}$-module generated by q-cells $(\sigma_1,\ldots,\sigma_q)$, $X_0=X_{-1}=\mathbb{Z}[G]$.

It then can be verified that for G finite, Hom from this resolution gives out the Tate cohomology

$$H^n_T(G,A)=\begin{cases}H^n(G,A) & n\geq 1\\ A^G/N_GA & n=0\\ {}_{N_G}A/I_GA & n=-1\\ H_{-1-n}(G,A) & n\leq -2\end{cases}$$

and H^n_T is a long exact sequence.

In particular, the Hom complex looks like:

$$\cdots \to A_{-2} \xrightarrow{\partial_{-1}} A_{-1} \xrightarrow{\partial_0} A_0 \xrightarrow{\partial_1} A_1 \xrightarrow{\partial_2} A_2 \to \cdots$$

where $A_{-1} = A_0 = A$ and $\partial_0 x = N_G x$, $(\partial_1 x)(\sigma) = \sigma x - x$,
$\partial_2(x)(\sigma_1, \sigma_2) = \sigma_1 x(\sigma_2) - x(\sigma_1\sigma_2) + x(\sigma_1)$.

From now on, consider only Tate cohomology.

Prop. (II.3.1.6).

$$H^{-2}(G, \mathbb{Z}) = G^{ab}, \quad H^{-1}(G, \mathbb{Z}) = 0, \quad H^0(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z}, \quad H^1(G, \mathbb{Z}) = 0, \quad H^2(G, \mathbb{Z}) = \chi(G).$$

Proof: H^0 is trivial and $H^1(G, \mathbb{Z}) = H^0(G, \mathbb{Q}/\mathbb{Z}) = 0$, $H^2(G, \mathbb{Z}) = H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. $H^{-1}(G, \mathbb{Z}) = {}_{N_G}\mathbb{Z}/I_G A = 0$.

For $H^{-2}(G, \mathbb{Z})$, use the dimension shifting $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$, $= H^{-1}(G, I_G) = I_G/I_G^2$. And $G^{ab} \cong I_G/I_G^2$ by $\sigma \mapsto \sigma - 1$. □

Prop. (II.3.1.7). $H^n(\mathbb{Z}/n\mathbb{Z}, A) = A^G/NA$ for n even and $H^n(\mathbb{Z}/n\mathbb{Z}, A) = {}_N A/(\sigma - 1)A$ for n odd.

Prop. (II.3.1.8). For a finite group G, $|G| \cdot H^n(G, A) = 0$ for any G-module A. (True for H^0 and use dimension shifting). In particular, a divisible G-module A has trivial cohomology).

Prop. (II.3.1.9).

Operations

Prop. (II.3.1.10) (Dimension Shifting). There are fundamental split exact sequence $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \to \mathbb{Z}[G] \to J_G \to 0$, thus $A_G = A/I_G A$. This can be used to tensor with A and define natural dimension shifting of cohomology δ.

Def. (II.3.1.11). The **inflation** is defined for $p \geq 0$ by composing with $G \to G/H$.

The **restriction** is the map $H^q(G, A) \to H^q(H, A)$ that is id when $q = 0$ and commutes with δ.

The **corestriciton** is the map $H^q(H, A) \to H^q(G, A)$ that maps a to $N_{G/H} a$ when $q = 0$ and commutes with δ.

Prop. (II.3.1.12). $cor \circ \mathrm{res} = [G : H]$ for a subgroup H. (check at degree 0 and use dimension shifting).

Prop. (II.3.1.13). For an isomorphism (σ^*, σ) of a group and its cochain map in the sense that $\sigma^*(g)(\sigma(a)) = g(a)$, we have an isomorphism of Conjugation acts trivially on the group cohomology, because it does on H^0 because $H^0 = A^G$ fixed by G, and it commutes with dimension shifting. (Warning, if you count directly $a(\sigma\tau\sigma^{-1}) - \sigma a(\tau)$, you won't get 0, but a 1-coboundary).

Prop. (II.3.1.14) (Cup Product). The cup product is defined by $C^p(X, A) \times C^q(X, B) \to C^{p+q}(X, A \otimes B)$:

$$(a \smile b)(\sigma_1, \ldots, \sigma_{p+q}) = a(\sigma_1, \ldots, \sigma_p) \otimes \sigma_1 \ldots \sigma_p b(\sigma_{p+1}, \ldots, \sigma_{p+q}).$$

It satisfies $\partial(a \smile b) = \partial(a) \smile b + (-1)^p a \smile \partial(b)$, thus defines a:

$$\smile : H^p(G, A) \times H^q(G, B) \to H^{p+q}(G, A \otimes B)$$

for $p, q \geq 0$. And in negative dimension this is also definable but not computable, Cf.[Neukirch Cohomology of Number Fields P42] or [Neukirch Class Field Theory 2015 P45].

- $a \smile b = a \otimes b$ for $a \in H^0(G, A), b \in H^0(G, B)$.
- $\delta(a \smile b) = \delta a \smile b, \delta(a \smile b) = (-1)^p(a \smile \delta b)$ for $a \in H^p(G, A)$.
- $\smile$ is associative and skew-symmetric (follows from dimension shifting and the last one.

Prop. (II.3.1.15) (Duality and Cup Product). Let $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ and $0 \to B' \xrightarrow{u} B \xrightarrow{v} B'' \to 0$ be exact and there is a pairing $\varphi : A \times B \to C$ that $\varphi(A' \times A') = 0$ hence induce a compatible pairing on $A' \times B''$ and $A'' \times B'$, then we have

$$\delta(\alpha) \smile \beta + (-1)^p \alpha \smile \delta(\beta) = 0$$

for $\alpha \in H^p(G, A'')$ and $\beta \in H^q(G, B'')$.

Proof: Use the definition of δ, let a, b be the preimage of α, β in A and B, and $ia' = \partial a, ub' = \partial b$, then $\delta(\alpha) \smile \beta + (-1)^p \alpha \smile \delta(\beta) = a' \smile vb + (-1)^p ja \smile b' = \partial a \smile b + (-1)^p a \smile \partial b = \partial(a \smile b)$ is a boundary. $\square$

Prop. (II.3.1.16).

$$\mathrm{res}(a \smile \beta) = \mathrm{res}(a) \smile \mathrm{res}(b), \quad \mathrm{cor}(\mathrm{res} a \smile b) = a \smile \mathrm{cor} b$$

Cf.[Neukirch CFT P48].

Prop. (II.3.1.17). Let $\sigma \in G^{ab} = H^{-2}(G, \mathbb{Z})$ and $a_1 \in H^1(G, A), a_2 \in H^2(G, A)$, then

$$a_1 \smile \sigma = a_1(\sigma), \quad a_2(\sigma) = \sum_\tau a_2(\tau, \sigma).$$

Cf.[Neukirch CFT P50,P51].

Prop. (II.3.1.18). For cyclic group, the Tate cohomology is 2-cyclic.

Proof: There is an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \to \mathbb{Z} \to 0$, and this defines an isomorphism $\delta^2 : H^0(G, \mathbb{Z}) \cong H^2(G, \mathbb{Z})$. And this is also true for any A when tensored with it. The isomorphism is $a \mapsto \delta^2 a = \delta^2(1) \smile a$. $\square$

Prop. (II.3.1.19) (Duality). The cup product induces an isomorphism $H^i(G, A^\vee) \cong (H^{-i-1}(G, A))^\vee$, i.e, $H^n(G, A^\vee)$ and $H_n(G, A)$ are dual to each other when $n > 0$, where $A^\vee = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

Proof: We only need to verify $A^{*G}/N_G A^* \cong ({}_{N_G}A/I_G A)^*$ and use dimension shifting. Should use the injectivity of $\mathbb{Q}/\mathbb{Z}$ and the compatibility of cup product with dual. $\square$

Cor. (II.3.1.20). When A is $\mathbb{Z}$-free, the cup product also induce an isomorphism $H^i(G, \mathrm{Hom}(A, \mathbb{Z})) \cong H^{-i}(G, A)^\vee$.

Prop. (II.3.1.21) (Theorem of Cohomological Triviality). For a G-module A, if there is a q s.t. $H^q(g, A) = H^{q+1}(g, A) = 0$ for all subgroups of G, then $H^p(g, A) = 0$ for any p and subgroup g. Cf.[Neukirch CFT P57].

Prop. (II.3.1.22) (Tate's Theorem). Assume A is a G-module that $H^1(G,A)=0$ and $H^2(g,A)$ is cyclic of order $|g|$ for any subgroup g of G, then for a generator a of $H^2(G,A)$, there is an isomorphism

$$a\smile: H^q(G,\mathbb{Z})\cong H^{q+2}(G,A).$$

Cf.[Neikirch CFT P79].

Cor. (II.3.1.23). In particular, by dimension shifting, if A is a G-module that $H^1(G,A)=0$ and $H^2(g,A)$ is cyclic of order $|g|$ for any subgroup g of G this gives an isomorphism:

$$a\smile: H^q(G,\mathbb{Z})\cong H^{q+2}(G,A).$$

for a generator a of $H^2(G,A)$, because cup product commutes with dimension shifting.

Miscellaneous

Prop. (II.3.1.24) ((Schreier)H^2 and Extensions). For a G-module A, there is a correspondence of equivalence classes of extension of G over A that are compatible with the G action and $H^2(G,A)$.

Proof: Cf.[Weibel P183]. In fact there are also interpretations of $H^3(G,A)$ as $0\to A\to N\to E\to G$ under some equivalences. □

Prop. (II.3.1.25). When G is a cyclic group and A is a G-module, let $f=\sigma-1$, $g=1+\sigma+\ldots+\sigma^{n-1}$, then we can form a cyclic complex of order 2 and compute the Herbrand quotient(I.8.3.13). In this case, $g_{f,g}$ is just $|H^0(G,A)|/|H^{-1}(G,A)|$. And by(I.8.3.15), if a G-morphism $A\to B$ has finite kernel and cokernel, then they have the same Herbrand quotient.

2 Profinite Groups

Basic references are [Neukirch Cohomology of Number Fields], [Serre Galois Cohomology] [Profinite Groups Zalesskii]and [Shatz Profinite Groups, Arithmetic and Geometry].

Def. (II.3.2.1). A **profinite group** is defined as an inverse limit of finite discrete groups.

Lemma (II.3.2.2). For a compact totally disconnected group G, any nbhd U of e contains a normal open subgroup.

Proof: U contains a precompact nbhd of e, then by(IV.1.11.7), U contains an open subgroup V, so by(IV.1.11.4), there is a nbhd V' of e that $xV'x^{-1}\subset V$ for all $x\in G$, this says $\cap x^{-1}Vx$ is open, so it is an open normal subgroup. □

Prop. (II.3.2.3) (Profinite Compact and Totally Disconnected). A profinite group is the same thing as a totally disconnected, compact Hausdorff topological group. In particular, $G\cong\varprojlim G/N$ for all open normal subgroups of G.

Proof: One way is because $\lim G_i$ is a closed subgroup of $\prod G_i$ which by Tychonoff's theorem is compact.

Conversely, by(II.3.2.2), G has a basis of e consisting of normal open subgroups, and by(IV.1.11.6), the intersection of open normal subgroups is $\{e\}$. For any open normal subgroup N of G, G/N is compact discrete hence finite, the map $G\to\varprojlim G/N$ is continuous and has dense image, but G is compact and the right is Hausdorff, so the image is closed, hence it is surjective. It is injective because $\cap N=\{e\}$. Hence $G\cong\varprojlim G/N$. □

Cor. (II.3.2.4). A closed subgroup of a profinite group is profinite, and a quotient group is profinite.

A direct product of profinite groups are profinite, and so the inverse limit profinite groups are profinite, as it is a closed subgroup of a direct product.

Proof: The closed subgroup is totally disconnected by(IV.1.1.8).

To show the quotient group is totally disconnected, by(IV.1.11.6), it suffice to prove H is intersection of compact open nbhds in G/H. If $x \notin H$, then there is an open subgroup U disjoint from xH by(IV.1.11.5), so it is closed hence compact. So UH is a compact nbhd of H in G/H that doesn't contains xH, hence the result. □

Cor. (II.3.2.5). A closed subgroup of a profinite group is a intersection of open normal subgroups of G containing it, as G/H is profinite and as in the proof of(II.3.2.3), H is the intersection of open normal subgroups of G/H.

Prop. (II.3.2.6). The category of profinite Abelian groups is Pontryagin dual to the category of torsion abelian group. (not that hard to verify).

Pro-p-Groups

Def. (II.3.2.7). To consider indexes of closed subgroups of a profinite group, the notion of surnatural numbers are needed. A **surnatural number** is a formal product $\prod_p p^{n_p}$, $n_p \in \mathbb{N} \cup \{0, \infty\}$.

For a closed subgroup H of a profinite group G, $[G : H]$ is defined to be the least common multiple of $[G/U : H/H \cap U]$ where U goes over all open normal subgroups of G. This also equals the least common multiple of $[G : V]$ for V open containing H(because for any such V, there is an open normal subgroup U that $HU \subset V$(IV.1.11.4)).

Prop. (II.3.2.8). The index is compatible with composition and quotient: $[G : K] = [G : H][H : K]$ and $[G : H] = [G/K : H/K]$ for K closed normal in G.

$[G : H]$ is finite iff H is open. For a decreasing family of closed subgroups H_i of G, $[G : \cap H_i]$ equals the least common multiple of $[G : H_i]$.

Proof: $[G/U : K/K \cap U] = [G/U : H/H \cap U][H/H \cap U : K/K \cap U] | [G : H][H : K]$, giving one way of inequality. For the converse, Cf.[Etale Cohomology Fulei P150]. The quotient case is trivial.

If $[G : H]$ is finite, then For the final assertion, notice for a open subgroup V, $G - V$ is compact, so $\cap H_i \subset V$ iff $\cap H_i \subset V$ for some i. □

Def. (II.3.2.9). A profinite group G is called a **pro-p-group** iff $[G : 1]$ is a power of p. This is equivalent to G is an inverse limit of finite p-groups($G = \lim G/N$).

Given a profinite group, a closed subgroup H is called **Sylow p-subgroup** of G if H is pro-p and $[G : H]$ is prime to p.

Prop. (II.3.2.10). Any pro-p subgroup H of G is contained in a Sylow p-subgroup of G, and any two Sylow p-subgroups are conjugate. And a surjective morphism of profinite groups maps a pro-p group to a pro-p group.

Proof: For any open normal subgroup U of G, let I_U be the sets of all Sylow groups of G/U containing $H/H \cap U$, then the map $G/VG/U$ maps I_V to I_U, and I_U is finite nonempty by Sylow theory. So the inverse limit of I_U is nonempty, and let (P_U) be such an element, and $P = \varprojlim_U P_U$, then P is a pro-p subgroup of G, and $[G : P]$ equals the least common multiple of $[G/U : P_U]$, which is prime to p, so it is a Sylow p-group. Similarly, for two Sylow-p subgroup, we consider A_U the set of all $x \in G/U$ that $x^{-1}(PU/U)x = P'U/U$, then there is a inverse element x, and $x^{-1}Px = P'$.

If $G' = G/N$, then $[G/N : PN/N] = [G : PN] | [G : P]$ is prime to p, and $[PN/N : 1] = [P : P \cap N] | [P : 1]$ is a power of p, so PN/N is Sylow-p in G'. $\square$

Prop. (II.3.2.11). For a pro-p group G, any nonzero simple p-torsion G-module is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with trivial G-action.

Proof: The action of G on A factors through a finite quotient group which is a p-group, by**??**, $A^G \neq 0$, so $A = A^G$, then A must be $\mathbb{Z}/p\mathbb{Z}$. $\square$

3 Cohomology of Profinite Groups

Prop. (II.3.3.1) (Abelian Sheaves on T_G). If G is profinite, the category of Abelian sheaves on the canonical topology T_G of G-sets is equivalent to the the category of G-modules, by Yoneda functor. The inverse map is $F \mapsto \varinjlim F(G/H)$.

Proof: The task is to prove $F \cong h_{\varinjlim F(G/H)}$. Cf.[Tamme P29].

The inverse of the Yoneda functor is the functor $F \mapsto F(G)$ as a left G-set where $gs = F(\cdot g)s$. The task is to show that $F \cong h_{F(G)}$. For this, for any U we consider the covering $\{G \xrightarrow{\varphi_u} U$ where $\varphi_U(g) = gu$. Sheaf condition says

$$F(U) \to \prod_{u \in U} F(G) \rightrightarrows F(G \times_U G)$$

is exact, in other words, $F(U) \cong \mathrm{Hom}_G(U, F(G))$. $\square$

Prop. (II.3.3.2) (Profinite Cohomology). The profinite cohomology is the derived functor of $A \to A^G$ in the Abelian category C_G(It has enough injectives by(I.4.5.3)). And

$$H^*(G, A) \cong H^*(C(G, A)) \cong \varinjlim H^*(G/U, A^U)$$

where $C(G, A)$ is the set of continuous cochain complex of morphisms from G to A. Moreover, for the same reason, when $G = \varprojlim G_i$, and $A = \varinjlim A_i$, then

$$H^*(G, A) \cong \varinjlim H^*(G_i, A_i).$$

Proof: The second is an isomorphism because $C^n(G, A) = \mathrm{colim}\, C^n(G/U, A^U)$ and direct limit is exact.

For the first, the H^0 obviously coincide, so it suffice to prove $H^*(C(G, A))$ form a universal δ-functor. It is effaceable because I^U is injective G/U-module.

For the last one, we need to check $C^n(G, A) = \varinjlim C^n(G_i, A_i)$. Notice G has the profinite topology, thus must factor through some G_i, and the right through some A_i because the image of a morphism from G^n to A has finite image. Thus the result follows. $\square$

Prop. (II.3.3.3). *cor* $\circ$ res $= [G : H]$ for a subgroup H is also true for profinite cohomology(II.3.1.12), if H is an open subgroup of G. This is because of(II.3.3.2).

Prop. (II.3.3.4). If H is a closed subgroup of a profinite group G that $[G : H]$ is relatively prime to p, then for any G-module A and i, the restriction map $H^i(G, A) \to H^i(H, A)$ is injective on the p-primary part of $H^i(G, A)$.

Proof: $H^i(H,A) = \varinjlim_U H^i(U,A)$ for open subgroups U containing H, by(II.3.3.2), and $H^i(G,A) \to H^i(U,A)$ is injective on the p-primary part by(II.3.3.3), so it is injective. □

Lemma (II.3.3.5) (Shapiro).

$$H_*(G, \mathrm{ind}_H^G(A)) \cong H_*(H,A), \quad H^*(G, \mathrm{Coind}_H^G(A)) \cong H^*(H,A)$$

because (co)induced is adjoint to exact functors, so it preserves injectives(projectives) and it is exact because $\mathbb{Z}[G]$ is free $\mathbb{Z}[H]$-module.

And in the finite case, this is also true for Tate cohomology using dimension shifting.

Prop. (II.3.3.6) (Serre-Hochschild Spectral sequence). Same as the finite case(II.3.1.2) also applies to profinite cohomology with H closed normal in G.

Cohomological Dimensions

Def. (II.3.3.7). The **p-cohomological dimension** $cd_p(G)$ of a profinite group G is defined as the smallest integer n that the p-primary part of $H^i(G,A)$ vanish for any torsion G-module A. The **strict p-cohomological dimension** $scd_p(G)$ of a profinite group G is defined as the smallest integer n that the p-primary part of $H^i(G,A)$ vanish for any G-module A.

The **cohomological dimension** $cd(G)$ is defined to be $\sup_p(cd_p(G))$. The **strict cohomological dimension** $scd(G)$ is defined to be $\sup_p(scd_p(G))$.

Prop. (II.3.3.8). For a profinite group G, the following are equivalent:

- $cd_p(G) \leq n$.
- $H^i(G,A) = 0$ for any $i > n$ and any p-torsion G-module A.
- $H^{n+1}(G,A)$ for any simple p-torsion G-module A.

And if G is pro-p, then it suffice to check $H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$

Proof: For any torsion G-module A, $A = \oplus_p A(p)$, so $H^i(G, A(p))$ is the P-primary part of $H^i(G,A)$, so $1,2$ are equivalent. For $3 \to 1$: use the fact cohomology commutes with colimits(II.3.3.2), reduce to the case of A finite, and then use use the quotient tower.

The last assertion is by(II.3.2.11). □

Prop. (II.3.3.9). $cd_p(G) \leq scd_p(G) \leq cd_p(G) + 1$.

Proof: Let $A_p = \mathrm{Ker}(p : A \to A)$. There are exact sequences $0 \to A_p \to A \xrightarrow{p} pA \to 0$ and $0 \to pA \to A \to A/pA \to 0$. A_p and A/pA are p-torsion G-modules, so if $i > cd_p(G) + 1$, then $H^i(G, A_p)$ and $H^{i-1}(G, A/pA)$ vanish. so $H^i(G,A) \xrightarrow{p} H^i(G < pA)$ and $H^i(G, pA) \to H^i(G,A)$ are injections, so their composition $H^i(G,A) \xrightarrow{p} H^i(G,A)$ is injective, showing $(H^i(G,A))_p = 0$, so $scd_p(G) \leq cd_p(G) + 1$. □

Prop. (II.3.3.10). For a closed subgroup H of a profinite group G, $cd_p(H) \leq cd_p(G)$ and $scd_p(H) \leq scd_p(G)$, and if $[G : H]$ is relatively prime to p, then equality holds.

Proof: The first is because of Shapiro's lemma(II.3.3.5). For the equality, use(II.3.3.4). □

Cor. (II.3.3.11). $cd_p(G) = cd_p(G_p) = cd(G_p)$, $scd_p(G) = scd_p(G_p) = scd(G_p)$.

Prop. (II.3.3.12). If H is a closed normal subgroup of G, then $cd_p(G) \leq cd_p(H) + cd_p(G/H)$, by Hochschild-Serre spectral sequence.

Prop. (II.3.3.13). If K is a field of char p, then $cd_p(G(K_s/K)) = 0$.

If $H^2(G(K_s/L), K_s^*) = 0$ for all L/K separable, then $cd(G(K_s/K)) \leq 1$. In particular $H^i(G(K_s/K), K_s^*) = 0$ for $i \geq 1$.

Proof: Let G_p be the Sylow p-subgroup of $G(K_s/K)$ and $M = K_s^{G_p}$. There is an exact sequence $0 \to \mu_p \to K_s \xrightarrow{x^p - x} K_s \to 0$, and combined with the fact that $H^i(G_p, K_s) = H^i(G(K_s/M), K_s) = 0$ for $i \geq 1$(II.3.4.1), so $H^i(G_p, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq 2$. Thus by(II.3.3.8) and(II.3.3.11), $cd_p(G(K_s/K)) \leq 1$.

For the second assertion, similarly, for $l \neq p$, consider the kernel of x^l, μ_l of l-th roots of unity in K_s, and $H^2(G_l, \mu_l(K_s)) = \varinjlim_L H^2(G(K_s/L), \mu_l(K_s)) = 0$, so $cd_l(G(K_s/K)) \leq 1$. Then $cd(G(K_s/K)) \leq 1$, and $scd(G(K_s/K)) \leq 2$, so $H^i(G(K_s/K), K_s^*) = 0$ for $i \geq 1$. □

Prop. (II.3.3.14). For L/K field extension, $cd_p(G(L_s/L)) \leq cd_p(G(K_s/K)) + tr.deg(L/K)$.

Proof: Cf.[Etale Cohomology Fulei P169]. □

Cor. (II.3.3.15). If k is separably closed and K be a function field over k, then $cd(G(K_s/K)) \leq 1$.

And if K is of char $p > 0$, $H^2(G(K_s/K), K_s^*)$ is a p-torsion group.

Proof: Th first one is clear, for the second, for any $l \neq p$, use the exact sequence $\mu_l(K_s) \to K_s^* \xrightarrow{x \to x^l} K_s^* \to 0$, then $H^2(G(K_s/K), \mu_l(K_s)) = 0$, and $H^2(G(K_s/K), K_s^*) \xrightarrow{l} H^2(G(K_s/K), K_s^*)$ is injective. l is arbitrary, so $H^2(G(K_s/K), K_s^*)$ must be a p-torsion group. □

4 Galois Cohomology

References are [Neukirch Chap6]. Should include [Galois Cohomology Serre].

This subsection is not included in the following subsection about Galois/Profinite Cohomology because the G-groups may not be Abelian and it may not be endowed with the discrete topology.

Prop. (II.3.4.1) (Hilbert's Additive Satz 90). For L/K a Galois extension, $H^n(G(L/K), L) = 0$ for $n > 0$, where L is equipped with the discrete topology.

Proof: Form the normal basis theorem**??**, for finite Galois extension L/K, L is an induced module over K, thus $H^*(G, L) = H_*(G, L) = 0$ for $* \neq 0$ and $H_T^*(G, L) = 0$ by(II.3.3.5).

Hence the same is true, for arbitrary Galois extension, when L is equipped with the discrete topology, the same as in the proof of(II.3.4.7). □

Prop. (II.3.4.2) (Hilbert's Multiplicative Satz 90). $H^1(G_{L/K}, L^*) = 0$ for Galois extension L/K, where L is equipped with the discrete topology,(follows from(II.3.4.7)).

Non-Abelian Cohomology

Def. (II.3.4.3) (Non-Abelian Cohomology). Let G, M be topological groups, with a continuous action of G on M, then we define $H^0(G, M) = M^G$.

We define $Z^1(G, M)$=continuous maps $x : G \to M$ that

$$\sigma_1(x(\sigma_2)(x(\sigma_1\sigma_2))^{-1}x(\sigma_1) = 1, \quad \text{i.e.} \quad x(gh) = x(g)g(x(h))$$

If $x \in Z^1(G,M)$, then $x_m : \sigma \to m^{-1}x(\sigma)\sigma(m) \in Z^1(G,M)$ too. This defines an equivalence relation on $Z^1(G,M)$, the equivalence classes are called $H^1(G,M)$. This is compatible with the commutative case.

Prop. (II.3.4.4). Restriction map and inflation map is definable for H^0 and H^1, and $H^1(H,M)$ is a G/H-set where G acts on $H^1(H,M)$ by $g(c)(h) = g(c(g^{-1}hg))$.

Prop. (II.3.4.5). There is an exact sequence of pointed sets:

$$0 \to H^1(G/H, M^H) \xrightarrow{inf} H^1(G,M) \xrightarrow{res} H^1(H,M)^{G/H}.$$

Proof: First $\mathrm{res}(H^1(G,M)) \subset H^1(H,M)^{G/H}$ because $g(c)(h) = c(g)^{-1}c(h)h(c(g))$ is checked so $g(c)$ is cohomologous to c.

$\mathrm{res} \circ inf = 0$ is easy, if $\mathrm{res}(c) = 0$, then c is trivial on H, hence $c(gh) = c(g)$ and $h(c(g)) = c(hg) = c(g \cdot g^{-1}hg) = c(g)$, so c is inflated from $H^1(G/H, M^H)$.

For the injectivity of inf. If $c(\overline{g}) = g^{-1}g(a)$, then $a \in M^H$, so it is a coboundary in $H^1(G/H, M^H)$. □

Prop. (II.3.4.6). Let $1 \to A \to B \to C \to 1$ be an exact sequence of G-groups, then there is a long exact sequence of pointed sets

$$1 \to A^G \to B^G \to C^G \xrightarrow{\delta} H^1(G,A) \to H^1(G,B) \to H^1(G,C) \xrightarrow{\Delta} H^2(G,A)$$

the last term is defined only when A is in the center of G.

Where δ is defined as follows: for $c \in C^G$, let b be an inverse image of c in B, then $a_\sigma = b^{-1}\sigma(b) \in A$, and it defines a cocycle in $H^1(G,A)$, different choice differ by a coboundary, so it is well-defined.

Δ is defines as: for c_σ a cocycle in $H^1(G,C)$, choose b_s inverse images of c_s, then $a_{\sigma,\tau} = b_\sigma\sigma(b_\tau)b_{\sigma\tau}^{-1}$ is a cocycle in $H^2(G,A)$.

Proof: The verification of well-definedness of Δ is checked at [Serre Local Fields P124].

For the exactness at C^G, the definition of δ shows that $\delta(c) = 1$ iff there is an inverse image b that $b^{-1}\sigma(b) = 1$ for all σ.

For the exactness at $H^1(G,A)$, $a_\sigma = b^{-1}\sigma(b)$ if a_σ is in the image of δ. Conversely, the image of b in C is in C^G, so it is in the image of δ.

For the exactness at $H^1(G,B)$, one way is clear, and for the other, if $\pi(b_\sigma) = c^{-1}\sigma(c)$, then if t is an inverse image of c, then $tb_\sigma\sigma(t)^{-1}$ is a cocycle in A cohomologous to b_σ.

For the exactness at $H^1(G,C)$, one way is clear, and if b_s is an inverse image of b_s and $a_{\sigma,\tau} = b_\sigma\sigma(b_\tau)b_{\sigma\tau}^{-1}$ is a coboundary, then it is $a_\sigma\sigma(a_\tau)a_{\sigma\tau}^{-1}$, so we change b to $a_\sigma^{-1}b_\sigma$, as A is in the center of B, this lifts c to a cocycle in B. □

Prop. (II.3.4.7). For L/K a Galois extension, $H^1(G(L/K), GL_n(L)) = 1$, where L is equipped with the discrete topology.

Proof: We prove any cocycle is a coboundary, for this, notice any cocycle factor through a finite quotient, and the images of it is contained in a finite extension of K, hence it reduce to the case of L/K finite.

For some $a \in H^1(G, GL_n(L))$, for a vector $x \in L^n$, let $P(x) = \sum a(\sigma)\sigma(x)$, then $\{P(x)\}$ generate L^n, because if f is a linear functional that vanish on it, then

$$0 = f(P(\lambda x)) = \sum f(a(\sigma)\sigma x)\sigma\lambda.$$

But automorphisms are linearly independent over L, hence $f(a(\sigma)\sigma(x)) = 0$ for all σ, so $f = 0$ as $a(\sigma) \in GL_n(L)$

Now let $\{P(x_i)\}$ generate L^n, then let T be the matrix with x_i as rows, then $P = \sum a(\sigma)\sigma(T)$ is invertible. Now $a(\sigma) = P \cdot \sigma(P)^{-1}$ is a cocycle. □

Cor. (II.3.4.8). $H^1(G(L/K), SL_n(L)) = 1$. This is seen from the exact sequence $1 \to SL_n(L) \to GL_n(L) \to L^* \to 1$.

Continuous Cochain Complex

In this subsubsection cohomology of G-modules with non-discrete topology is studied. References are [Cohomology of Number Fields Neukirch Chap 2.7].

Prop. (II.3.4.9). $H^*_{cts}(G, -)$ forms a long exact sequence for any $0 \to A \to B \to C \to 0$ of continuous G-modules.

Prop. (II.3.4.10). If A is a compact G-module which is an inverse limit of finite discrete G-modules A_n, then if $H^i(G, A_n)$ is finite for all n, then

$$H^{i+1}_{cts}(G, A) = \varprojlim_n H^{i+1}(G, A_n).$$

Proof: Cf.[Cohomology of Number Fields Neukirch P142]. □

Prop. (II.3.4.11). Let π be a topologically nilpotent element of A which is complete in the π-adic topology and π is not a zero-divisor, let $R = A/\pi A$ equipped with discrete topology. Let G be a group which acts continuously on A and fix π, then if $H^1(G, R)$ is trivial, then $H^1(G, A)$ is trivial, and if moreover $H^1(G, GL_n(R))$ is trivial, then $H^1(G, GL_n(A))$ is trivial.

Proof: Cf.[Galois Representations Berger P15]. □

Prop. (II.3.4.12) (Cyclic Case). if G is a topological cyclic group $\overline{\langle g \rangle}$, then the map $H^1(G, M) \to M/(1-g)$ is well-defined and injective. And when M is profinite, p-adically complete, then the map is also surjective.

Proof: The surjection: there is only one choice: $c(g^i) = (1 + g + \ldots + g^{i-1})(m)$. And we need to verify that it is continuous. The case of p-adic can be deduced from profinite case, because $c(\gamma) \in p^{-k}M$ for some k, and $p^{-k}M$ is then profinite. For any finite quotient N of M, there is a k that $kM = 0$, and a n that $g^n = \mathrm{id}$ on N, so $c(g^{rkn}) = 0$ on N, which shows c is continuous. □

Interpretation of H^1

Prop. (II.3.4.13) (Semilinear Actions). For a topological group G and a G-ring R, now giving a free R module X of degree d with a G semilinear action of G(i.e. $\sigma(rm) = \sigma(r)\sigma(m)$), then the matrix of the G action defines a $[X] \in H^1(G, GL_d(R))$ and change of basis defines exactly the equivalence relations on $H^1(G, GL_d(R))$(II.3.4.3).

So there is a bijection of $H^1(G, GL_d(R))$ with the set of isomorphisms classes of semilinear representations of G on free R-modules of rank d.

Def. (II.3.4.14). A G-set X is a discrete set with a continuous G-action on X. Let A be a G-group, an **A-torsor** is a right A-action that is simply transitive and semi-linear in G.

Prop. (II.3.4.15) (H^1 and Torsors). We have a canonical bijection of pointed sets: $H^1(G, A) \cong TORS(A)$.

Proof: Let X be an A-torsor, choose $x \in X$, then $\sigma(x) = xa_\sigma$ for $a_\sigma \in A$. Now that $\sigma \mapsto a(\sigma)$ is checked to be a cocycle, and change of x changes to $\sigma \mapsto b^{-1}a_\sigma\sigma(b)$. Conversely, for an $a \in H^1(G, A)$, we let $X = A$ be a right A-module, and let $\sigma'(x) = a_\sigma\sigma(x)$, i.e. regarding coming from $x = 1$, then this is a inverse map. $\square$

Prop. (II.3.4.16) (Extension of Rings).

Prop. (II.3.4.17). There is an isomorphism of pointed sets $H^1(G, O(\varphi_L)) \cong E_\varphi(L/K)$. Cf.[Neukirch Cohomology of Number Fields P346].

Prop. (II.3.4.18). There is an isomorphism of pointed sets $H^1(G, PSL_n(L)) \cong BS_n(L/K)$, where $BS_n(L/K)$ is the isomorphism classes of Brauer-Severi varieties of dimension $n-1$ that splits in L. Cf.[Neukirch Cohomology of Number Fields P348].

5 Iwasawa Modules

II.4 Cohomology of Number Fields

1 Class Field Theory

Abstract Class Field Theory

Def. (II.4.1.1). A formation consists of a profinite group G regarded as a Galois group $G(K)$ and a G-module A. It is called a **field formation** iff for any normal extension L/K, $G(L/K, A^L) = 0$.

For a field extension, by(II.3.1.2), inf is an injection on H^2. We denote $H^2(K)$ as the profinite cohomology group $H^2(G, A) = \mathrm{Br}(K)$. Inflation should be thought of as inclusions.

It is called a **class formation** if moreover for every normal extension L/K, there is an canonical isomorphism

$$\mathrm{inv}_{L/K} : H^2(L/K) \to \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$$

that is compatible with inflation and restriction in the sense that:

- If $N/L/K$ with N/K and L/K normal, then $\mathrm{inv}_{L/K} = \mathrm{inv}_{N/K}|_{H^2(L/K)}$ via inflation.
- If $N/L/K$ with N/L and N/K normal, then $\mathrm{inv}_{N/L} \circ \mathrm{res}_L = [L:K] \cdot \mathrm{inv}_{N/K}$.

The element of H^2 that is mapped to $\frac{1}{[L:K]} + \mathbb{Z}$ is called the **fundamental class** $u_{L/K}$.

Prop. (II.4.1.2). inv also commutes with cor and conjugation:

$$\mathrm{inv}_{N/K}(\mathrm{cor}_K c) = \mathrm{inv}_{N/L} c, \quad \mathrm{inv}_{\sigma N/\sigma K}(\sigma^* c) = \mathrm{inv}(c).$$

The first is because inv commutes with res thus res is surjective, thus there is a c' that $c = \mathrm{res} c'$. Because of cor res$=[L:k]$, we have $\mathrm{cor}_K(c) = c'^{[L:K]}$. Thus $\mathrm{inv}_{N/K}(\mathrm{cor}_K c) = [L:K]\mathrm{inv}_{N/K}(c') = \mathrm{inv}_{N/L}(\mathrm{res}_L c') = \mathrm{inv}_{N/L}(c)$.

For the conjugation, Cf.[Neukirch CFT P69].

Cor. (II.4.1.3). From this we easily get that

$$u_{L/K} = (u_{N/K})^{[N:L]}, \quad \mathrm{res}_L(u_{N/K}) = u_{L/K}$$

$$\mathrm{cor}_K(u_{N/L}) = (u_{N/K})^{[L:K]}, \quad \sigma^*(u_{N/K}) = u_{\sigma N/\sigma K}.$$

Prop. (II.4.1.4) (Main Theorem). Tate's theorem(II.3.1.22) tells us for a class formation, for L/K normal extension, there is an isomorphism

$$u_{L/K} \smile : H^q(G_{L/K}, \mathbb{Z}) \cong H^{q+2}(L/K).$$

Especially, for $q = -2$, there is a canonical isomorphism $G_{L/K}^{ab} \cong A_K/N_{L/K}A_L$ that its inverse is called **reciprocity isomorphism** and $A_K \to G_{L/K}^{ab}$ is called **norm residue symbol**$(-, L/K)$. This norm residue symbol also induce a **universal residue symbol** $(-, K)$ on the limit G_K^{ab}, i.e. the maximal Abelian extension of K.

Lemma (II.4.1.5). Let L/K be a normal extension, $a \in A_K$ and $\chi \in \chi(G_{L/K}^{ab}) = H^1(G_{L/K}, \mathbb{Q}/\mathbb{Z})$ is a character, then

$$\chi((a, L/K)) = \mathrm{inv}_{L/K}(a \smile \delta\chi) \in \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}.$$

Proof: Cf.[Neukirch CFT P71]. □

Prop. (II.4.1.6) (Properties of Inv). There are commutative diagrams:

$$\begin{tikzcd} A_K \arrow[r] \arrow[d, "\mathrm{id}"] & G^{ab}_{N/K} \arrow[d, "\pi"] \\ A_K \arrow[r] & G^{ab}_{L/K} \end{tikzcd} \qquad \begin{tikzcd} A_K \arrow[r] \arrow[d, shift left, "i"] & G^{ab}_{N/K} \arrow[d, shift left, "\mathrm{Ver}"] \\ A_L \arrow[r] \arrow[u, shift left, "N_{L/K}"] & G^{ab}_{N/L} \arrow[u, shift left, "k"] \end{tikzcd} \qquad \begin{tikzcd} A_K \arrow[r] \arrow[d, "\sigma"] & G^{ab}_{N/K} \arrow[d, "\sigma^*"] \\ A_{\sigma K} \arrow[r] & G^{ab}_{\sigma L/\sigma K} \end{tikzcd}$$

Where Ver is the transfer map defined in**??**.

Proof: Cf.[Neukirch CFT P72]. □

Prop. (II.4.1.7). For a finite normal extension L/K, $N_{L/K}A_L = N_{L^{ab}/K}A_{L^{ab}}$. This is because the quotient both correspond to $G^{ab}_{L/K}$. So class field theory doesn't tell about non-Abelian extension.

Prop. (II.4.1.8) (Norm Group and Abelian Extension). The map $L \mapsto I_L = N_{L/K}A_L$ defines a inclusion reversing isomorphism between the lattice of Abelian extension L of K and the lattice of norm groups of A_K, i.e.:

$$I_{L_1L_2} = I_{L_1} \cap I_{L_2}, \quad I_{L_1 \cap L_2} = I_{L_1} \cdot I_{L_2}.$$

And any group that contains a norm group is a norm group.

Proof: By the first commutative diagram of inv, if $(a, L_i/K) = 0$, then $(a, L_1L_2/K)$ is trivial on $G_{L_i/K}$, thus trivial on $G_{L_1L_2/K}$, thus $a \in I_{L_1L_2}$. so $I_{L_1} \cap I_{L_2} \subset I_{L_1L_2}$, the other side is easy. the second is because $|I_{L_1\cap L_2}/I_{L_1}| = |G_{L_1/L_1\cap L_2}| = |G_{L_1L_2/L_2}| = |I_{L_1}I_{L_2}/I_{L_1}|$. Also we deduce $I_{L_1} \subset I_{L_2} \iff L_2 \subset L_1$, thus by canonical isomorphism, groups containing $N_{L/K}A_L$ are one-to-one correspondence with middle fields of L/K by counting numbers. □

Remark (II.4.1.9). This shows the philosophy of CFT, i.e. the property of Abelian extensions of a field is can be read from its multiplicative group structure. Of course, determining and characterizing these norm groups requires some work.

Local Class Field Theory

The strategy is to first establish CFT for unramified extensions, then show that unramified extensions already cover $H^2(\overline{K}/K)$.

Lemma (II.4.1.10). Let L/K be an unramified extension, then $H^q(G_{L/K}, U_L) = 0$ for all q.

Proof: Cf.[Neukirch P83]. □

Prop. (II.4.1.11). The unramified extensions of K forms a class formation.

Proof: We first define the inv map: use the exact sequence $1 \to U_L \to L^* \xrightarrow{v_L} \mathbb{Z} \to 0$, using the lemma(II.4.1.10), we have

$$H^2(G_{L/K}, L^*) \xrightarrow{v_K} H^2(G_{L/K}, \mathbb{Z}) \xrightarrow{\delta^{-1}} H^1(G_{L/K}, \mathbb{Q}/\mathbb{Z}) = \chi(G_{L/K}).$$

And there is an isomorphism $\chi(G/K) \xrightarrow{\varphi} \frac{1}{[L;K]}\mathbb{Z}/\mathbb{Z}$, where φ is the Frobenius which generate $G_{L/K}$, and $\varphi(\chi) = \chi(\varphi)$.

To verify this is a class formation, we should verify(II.4.1.1), Cf.[Neukirch P85]. □

Prop. (II.4.1.12). If L/K is unramified, then $(a, L/K) = \varphi_{L/K}^{v_K(a)}$, Cf.[Neukirch CFT P86]. The same holds for L replaces by T, in which case

$$1 \to U_K \to K^* \to G_{T/K} \to 0$$

is exact. Cf[Neukirch P88].

Proof: We use(II.4.1.5), then $\chi(a, L/K) = \mathrm{inv}_{L/K}(\overline{a} \smile \delta\chi) = \varphi \circ \delta^{-1} \circ v_K(\overline{a} \smile \delta\chi) = \varphi(\delta^{-1}(v_K(a)\delta\chi) = \varphi(v_K(a)\chi) = v_K(a)\chi(\varphi_{L/K}) = \chi(\varphi_{L/K}^{v_K(a)})$, for any χ. The second assertion follows from the last prop(II.4.1.13). □

Cor. (II.4.1.13). The norm group of an unramified extension of degree f is

$$U_K \times \{\pi^{fn}\}(n = 0, 1, \ldots).$$

(This follows from the proposition as the degree f is the order of the Frobenius map).

Now we pass to ramified extensions.

Lemma (II.4.1.14). If L/K is normal, then $|H^2(L/K)|\Big|[L : K]$.

Proof: Cf.[Neukirch CFT P89]. Should use the fact that $G_{L/K}$ is solvable and Herbrand quotient. □

Lemma (II.4.1.15). If L/K is a normal extension and L'/K is another unramified extension of the same degree, then $H^2(L/K) = H^2(L'/K) \subset Br(K)$.

Proof: In view of(II.4.1.14) and(II.4.1.11), we only need to prove $H^2(L'/K) \subset H^2(L/K)$. For this, we let $N = LL'$, then there is an exact sequence(II.3.1.2)

$$1 \to H^2(L/K) \to H^2(N/K) \xrightarrow{res_L} H^2(N/L)$$

then we only need to prove $res_L(c) = 0$, and this follows from $\mathrm{inv}_{N/L}(res_L c) = 0$. This will follow, if we have

$$\mathrm{inv}_{N/L}(res_L c) = [L : K] \cdot \mathrm{inv}_{L'/K} c.$$

This follows from the lemma below(II.4.1.16). □

Lemma (II.4.1.16). For two subextensions L/K, L'/K in M/L normal with L'/K unramified extension, $N = LL'$, for $c \in H^2(L'/K)$,

$$\mathrm{inv}_{N/L}(\mathrm{res}_L c) = [L : K] \cdot \mathrm{inv}_{L'/K} c.$$

Proof: Cf[Neukirch CFT P90]. □

Prop. (II.4.1.17). (G_K, K^*) forms a class formation.

Proof: This almost follows from that of unramified extensions(II.4.1.11). We verify axioms(II.4.1.1) that inf is natural and commutes with res. It is natural because it is natural on unramified extensions, it commutes with res because we can assume $c \in H^2(L'/K)$ unramified and use(II.4.1.16). □

Cor. (II.4.1.18) (Main Theorem of Local Class Field Theory). Let L/K be a normal extension, then the homomorphism

$$u_{L/K}\smile: H^q(G_{L/K},\mathbb{Z})\cong H^{q+2}(L/K)$$

is an isomorphism.

Cor. (II.4.1.19). $H^3(L/K)=1, H^4(L/K)=\chi(G_{L/K})$, by(II.3.1.6).

Cor. (II.4.1.20). For a $\mathfrak{p}$-adic number field K, $Br(K)\cong\mathbb{Q}/\mathbb{Z}$.

Prop. (II.4.1.21). By(II.4.1.6), there is commutative diagrams

$$\begin{tikzcd} K^* \arrow[r] \arrow[d, "\mathrm{id}"] & G^{ab}_{N/K} \arrow[d, "\pi"] \\ K^* \arrow[r] & G^{ab}_{L/K} \end{tikzcd}\quad \begin{tikzcd} K^* \arrow[r] \arrow[d, shift left, "i"] & G^{ab}_{N/K} \arrow[d, shift left, "\mathrm{Ver}"] \\ L^* \arrow[r] \arrow[u, shift left, "N_{L/K}"] & G^{ab}_{N/L} \arrow[u, shift left, "k"] \end{tikzcd}\quad \begin{tikzcd} K^* \arrow[r] \arrow[d, "\sigma"] & G^{ab}_{N/K} \arrow[d, "\sigma^*"] \\ \sigma K^* \arrow[r] & G^{ab}_{\sigma L/\sigma K} \end{tikzcd}$$

Prop. (II.4.1.22). For an Abelian extension L/K, the higher principal units U_K^n are mapped under the higher ramification groups of $G_{L/K}$ under the upper numbering. ?

Prop. (II.4.1.23). Getting things together, we get a universal norm residue map

$$K^*\xrightarrow{(-,K)} G_K^{ab}$$

It is injective because $(K^*)^n$ are all norm groups by(II.4.1.25), so the kernel is there intersection with is 1 by(II.2.2.5). It image is called the **Weil group**.

Now we want to characterize the norm groups of K^*.

Prop. (II.4.1.24) (Norm Group and Abelian Extension). The map $L\mapsto I_L=N_{L/K}A_L$ defines a inclusion reversing isomorphism between the lattice of Abelian extension L of K and the lattice of norm groups of A_K, i.e.:

$$I_{L_1L_2}=I_{L_1}\cap I_{L_2},\quad I_{L_1\cap L_2}=I_{L_1}\cdot I_{L_2}.$$

And any group that contains a norm group is a norm group. This follows from(II.4.1.8) and (II.4.1.17).

Prop. (II.4.1.25). The norm groups are precisely the open(closed) subgroups of finite index in K^*. In fact finite index are itself open because it contains $(K^*)^n$ which is open.

Proof: One part follows from(II.4.1.24) and the fact that $(K^*)^m$ is open(II.2.2.5). For the converse, we only need to prove $(K^*)^m$ is a norm group. This uses Kummer theory and Cf.[Nuekirch CFT P96]. □

Prop. (II.4.1.26) (Norm Groups of Local Fields). The norm groups of K^* are exactly the groups containing $U_K^n\times(\pi^f)$ for some n,f.

Proof: $U_K^n\times(\pi^f)$ is a norm group because it is closed of finite index. Conversely, any norm group contains some U_K^n because it is open and contains some (π^f) because it is of finite index. □

Lubin-Tate Formal Group

This is a continuation of3.

Prop. (II.4.1.27). There is an isomorphism of $\mathcal{O}$-modules $\Lambda_{f,n} \cong \mathcal{O}/\pi^n\mathcal{O}$, Cf.[Neukirch CFT P101]. Thus the automorphism of $\Lambda_{f,n}$ is all of the form u_f for units, isomorphic to U_K/U_K^n.

So we can define a **Tate module** $TG = \varprojlim \mathrm{Ker}[\pi_K^n]$, it is a free $\mathcal{O}_K$-module of rank 1.

Def. (II.4.1.28). As TG is a free $\mathcal{O}_G$-module of dimension 1, and G_K acts on TG, there can be attached a **Lubin-Tate character** $\chi_K : G_K \to \mathcal{O}_K^*$ by $g(\alpha) = [\chi_K(g)](\alpha)$, this depends on π_K, but its restriction on I_K doesn't depend on π_K, and is just the local CFT isomorphism composed with $x \to x^{-1}$.

Proof: $[\chi_K(g)]$ is, by definition, the morphism that is id on K^{ur} and g on L_π. So it equals g on all K^{ab} iff g is id on K^{ur}, that is, $g \in I_K$. So if $g \in I_K$, by local CFT, $(\chi(g))^{-1}$ corresponds to g, uniquely. □

Prop. (II.4.1.29). $G_{\pi,n} \cong \mathcal{O}_K^*/U_K^n$, thus we have $G_\pi \cong \mathcal{O}_K^*$. $L_{\pi,n}/K$ is Abelian totally ramified of degree $p^{n-1}(p-1)$ generated by a Eisenstein polynomial with constant coefficient π so π is in the norm group.

Proof: For this, first note Galois action induce an isomorphism on $\Lambda_{f,n}$, thus correspond to an element of U_K/U_K^n by(II.4.1.27), this is an injection because $\Lambda_{f,n}$ generate $L_{\pi,n}$. Then we use the canonical polynomial $f(Z) = \pi Z + Z^q$, $f^n = f^{n-1}\varphi(n)$, where $\varphi(n)$ is a Eisenstein polynomial, thus $L_{\pi,n}/K$ is totally ramifies with $|G_{\pi,n}| = q^{n-1}(q-1) = |U_K/U_K^n|$, thus the result. □

Prop. (II.4.1.30) (Explicit Local Norm Residue Symbol). Now we can write the universal residue symbol little bit more explicitly. For $a = u\pi^m$, (a, K) acts by φ^m on T and generated by the action $(u^{-1})_f$ on $\Lambda_{f,n}$ on $L_{\pi,n}$. Cf.[Neukirch CFT P106].

Thus the norm group of $L_{\pi,n}$ is just U^n by(II.4.1.29).

Cor. (II.4.1.31). The norm groups of the totally ramified Abelian extension is precisely the groups that contains some $U_K^n \times (\pi)$ for some uniformizer π. And every totally ramified Abelian extension L/K is contained in some $L_{\pi,n}$.

Proof: For any totally ramified extension, choose a uniformizer, then its norm is a uniformizer π of K. And $N_{L/K}$ is open (as it contains $(K^*)^m$**??**.) Thus it contains some U^n. The rest follows from local CFT(II.4.1.24). □

Cor. (II.4.1.32) (Maximal Abelian Extension of Local Fields). Let $L_\pi = \cup L_{\pi,n} = K(\Lambda_f)$, where $\Lambda_f = \cup\Lambda_{f,n}$, then $T \cdot L_\pi$ is the maximal extension of Abelian extension of K. Hence $G_K^{ab} = G_{T,K} \times G_\pi$. This follows immediately from(II.4.1.26).

Cor. (II.4.1.33) (Hasse-Arf). We can prove Hasse-Arf(II.2.1.24) in the case where K is a local field. This is because we already know the maximal Abelian extension, and $G(K^{ab}/T) \cong G(L_\pi/K) \cong \mathbb{Z}_p$ for which we know the Galois action well(II.4.1.27)(II.4.1.29), so $i(\sigma) = v(\sigma(\alpha_n) - \alpha_n) = v([\sigma - 1](\alpha))$, which jumps at U_K^n(the same pattern as $K = \mathbb{Q}_p$(II.2.1.26), thus the result.

Remark (II.4.1.34). There is a concrete example. When $K = \mathbb{Q}_p$, we can choose $f(Z) = (1 + Z)^p - 1$, thus $L_{\pi,n}$ is just $\mathbb{Q}_p(\xi_{p^n})$. And we have $r_f = (1 + Z)^r - 1$, thus we have

$$(a, \mathbb{Q}_p(\xi_{p^n})/\mathbb{Q}_p)\zeta = \zeta^r$$

where $a = up^m$, and $r \equiv u^{-1} \bmod p^n$.

Global Class Field Theory

For Basic Notations regarding Idele and Adele, See(II.2.4.6).

The **ideal class group** $C_K = I_K/K^*$ is a the main object of global class field theory. We will denote $H^q(G_{L/K}, C_L)$ by $H^q(L/K)$. $H^2(G_{L/K}, I_L)$ is the secondary object.

Prop. (II.4.1.35). Let $\mathfrak{P}$ be a prime of L lying over $\mathfrak{p}$, then $H^q(G, I_L^{\mathfrak{p}}) \cong H^q(G_{\mathfrak{P}}, L_{\mathfrak{P}}^*)$. If $\mathfrak{p}$ is a finite unramified prime of L, then $H^q(G, U_L^{\mathfrak{p}}) = 1$ for all q.

Proof: Notice $I_L^{\mathfrak{p}} = \prod_{\sigma \in G/G_{\mathfrak{P}}} L_{\sigma\mathfrak{P}}^* = \prod_{\sigma \in G/G_{\mathfrak{P}}} \sigma L_{\mathfrak{P}}^*$ is an induced module, so by(II.3.3.5), we have $H^q(G_{\mathfrak{P}}, L_{\mathfrak{P}}^*)$, and similarly for $U_{\mathfrak{p}}$, which vanish by(II.4.1.10). □

Cor. (II.4.1.36).

$$H^q(G, I_L^S) = \bigoplus_{p \in S} H^q(G_{\mathfrak{P}/\mathfrak{p}}, L_{\mathfrak{P}}^*), \quad H^q(G, I_L) = \bigoplus_{p} H^q(G_{\mathfrak{P}}, L_{\mathfrak{P}}^*).$$

And the isomorphism is natural, by restriction to components.

Proof: For this, just notice $I_L = \cup_S I_L^S$, then use the last proposition, notice group cohomology commutes with colimits(II.3.3.2). □

Cor. (II.4.1.37). $H^1(G, I_L) = H^3(G, I_L) = 0$, by(II.4.1.19).

Cor. (II.4.1.38). An idele $\mathfrak{a} \in I_K$ is the norm of an idele $\mathfrak{b}$ in I_L if each component $\mathfrak{a}_{\mathfrak{p}}$ is the norm of an element $b_{\mathfrak{P}} \in L_{\mathfrak{P}}^*$.

Prop. (II.4.1.39). The decomposition commutes with inf, res and cor. Cf.[Neukirch CFT P125].

The strategy is to first establish CFT for cyclic extensions, then show they cover all $H^2(\overline{K}/K)$.

Lemma (II.4.1.40). For a cyclic extension L/K of order p, C_L is a Herbrand module with Herbrand quotient $h(C_L) = p$.

Proof:

□

Prop. (II.4.1.41) (First Fundamental Inequality). $(C_K : N_G C_L) \geq p$

Prop. (II.4.1.42). If K contains p-th roots of unity and L/K is a cyclic extension of order p, then $(C_K : N_G C_L) \leq p$.

Cor. (II.4.1.43) (Second Fundamental Inequality). If L/K is a cyclic extension of order p, then $(C_K : N_G C_L) = p$.

Cor. (II.4.1.44) (Hass Norm Theorem). For a cyclic extension L/K, an element $x \in K^*$ is a norm iff it is locally a norm everywhere.

Proof: Use the long exact sequence for $1 \to L^* \to I_L \to C_L \to 1$, we see that $H^0(G, L^*) \to H^0(G, I_L)$ is an injection, which is

$$0 \to K^*/N_{L/K}L^* \to \bigoplus_{p} K_{\mathfrak{p}}^*/N_{L_{\mathfrak{P}/K_{\mathfrak{p}}}} L_{\mathfrak{P}}^*.$$

In fact, by(II.4.1.37), we say that this is equivalent to $H^1(G_{L/K}, C_L) = 1$, which is equivalent to second fundamental inequality. □

Prop. (II.4.1.45). For L/K normal extension, $|H^2(G, C_L)|\Big|[L:K]$.

Proof: Cf.[Neukirch P137]. $\square$

Prop. (II.4.1.46). Let K be a finite algebraic number field, then

$$Br(K) = \bigcup_{L/K\text{ cyclic}} H^2(G_{L/K}, L^*), \quad H^2(G_{\overline{K}/K}, I_{\overline{K}}) = \bigcup_{L/K\text{ cyclic}} H^2(G_{L/K}, I_L).$$

Proof: Cf,[Neukirch P127]. $\square$

Next we construct the Invariant map, first for $H^2(G_{L/K}, I_L)$, then for $H^2(G_{L/K}, C_K)$.

Def. (II.4.1.47). We define for $c = (c_{\mathfrak{p}}) \in H^2(G_{L/K}, I_L)$ by

$$\mathrm{inv}_{L/K} c = \sum_{\mathfrak{p}} \mathrm{inv}_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} c_{\mathfrak{p}}.$$

For an Abelian extension L/K, we define for $\mathfrak{a} \in I_K$:

$$(\mathfrak{a}, L/K) = \prod_{\mathfrak{p}} (a_{\mathfrak{p}}, L_{\mathfrak{P}}/K_{\mathfrak{p}}) \in G_{L/K}.$$

Prop. (II.4.1.48). If $c \in H^2(G_{L/K}, L^*)$, then $\mathrm{inv}_{L/K} c = 0$. Cf.[Neukirch P141].

Cor. (II.4.1.49). Now we can define the inv map for C_K when . By the exact sequence $1 \to L^* \to I_L \to C_K \to 1$, we have

$$1 \to H^2(G_{G/K}, L^*) \to H^2(G_{L/K}, I_L) \to H^2(G_{L/K}, C_L) \to H^3(G_{L/K}, L^*)$$

The last one is 1 if L/K is cyclic, thus tby this proposition, inv is defined for $H^2(G_{L/K}, C_L)$.

Prop. (II.4.1.50) (Hasse's Main Theorem). For every finite algebraic number field K, there is a canonical exact sequence

$$1 \to Br(K) \to \bigoplus_{\mathfrak{p}} Br(K_{\mathfrak{p}}) \xrightarrow{\mathrm{inv}_K} \mathbb{Q}/\mathbb{Z} \to 0$$

Proof: Cf.[Neukirch P146]. $\square$

Prop. (II.4.1.51). If L/K is normal and L'/K is cyclic and they have the same degree, then $H^2(L'/K) = H^2(L/K) \subset H^2(\overline{K}/K)$.

Cor. (II.4.1.52). $H^2(\overline{K}/K) = \cup_{L/K\text{ cyclic}} H^2(L/K)$, thus the homomorphism $H^2(G_K, I_{\overline{K}}) \to H^2(\overline{K}/K)$ is surjective by(II.4.1.49).

Prop. (II.4.1.53). The inv map is defined for $H^2(\overline{K}/K)$, and $inv_{L/K} : H^2(L/K) \to \frac{1}{[L;K]}\mathbb{Z}/\mathbb{Z}$ is an isomorphism for every normal extension L/K.

Prop. (II.4.1.54) (Main Theorem). The formation $(G_K, C_{\overline{K}})$ is a class formation with the inv map.

Cor. (II.4.1.55) (Artin's Reciprocity Law). The cup product with the fundamental class in $H^2(L/K)$ defines an isomorphism **reciprocity map**

$$G_{L/K}^{ab} \cong H^{-2}(G_{L/K}, \mathbb{Z}) \to H^0(L/K) = C_K/N_{L/K}C_L.$$

And the reverse map is called the **norm residue symbol**

$$1 \to N_{L/K}C_L \to C_K \xrightarrow{(-,L/K)} G_{L/K}^{ab} \to 1$$

Remark (II.4.1.56). WARNING: we have already defined a norm residue map in(II.4.1.47), they are compatible with that derived from CFT mechanism. i.e. local global correspondence and vanish on K^*.

Proof: Cf.[Neukirch CFT P154]. □

Prop. (II.4.1.57). Properties of Norm Residue symbol.

Cor. (II.4.1.58). By(II.4.1.8), the map $L \mapsto I_L = N_{L/K}C_L$ defines a inclusion reversing isomorphism between the lattice of Abelian extension L of K and the lattice of norm groups of C_K, i.e.:

$$I_{L_1L_2} = I_{L_1} \cap I_{L_2}, \quad I_{L_1 \cap L_2} = I_{L_1} \cdot I_{L_2}.$$

And any group that contains a norm group is a norm group.

Prop. (II.4.1.59). Let L/K be an Abelian extension, then $(\mathfrak{a}, L/K) = \prod_{\mathfrak{p}}(a_{\mathfrak{p}}, L_{\mathfrak{P}}/K_{\mathfrak{p}})$

Proof: Cf.[Neukirch P154]. □

Prop. (II.4.1.60) (Existence Theorem). The norm groups of C_K are precisely the (open)closed subgroups of finite index.

Proof: Cf.[Neukirch P162]. □

Now we want to further characterize the norm groups of C_K in an arithmetic way.

Def. (II.4.1.61) (Notations). A **modulus** $\mathfrak{m}$ is a $\prod_{\mathfrak{p}} \mathfrak{p}_{\mathfrak{p}}^n$ that $n_{\mathfrak{p}} = 0$ a.e.. $\{\mathfrak{m}\}$ is the set of primes in $\mathfrak{m}$.

$I_K^{\mathfrak{m}} = \{\mathfrak{a} \in I_K | \mathfrak{a} \equiv 1 \bmod \mathfrak{m}\}$.

The **congruence subgroup mod** $\mathfrak{m}$ $C_K^{\mathfrak{m}} = I^{\mathfrak{m}} \cdot K^*/K^* \subset C_K$.

$C_K^{\mathfrak{m}}$ is a norm group by(II.4.1.63), the Abelian class field L/K associated with $C_K^{\mathfrak{m}}$ is called the **ray class field mod** $\mathfrak{m}$, so its Galois group is isomorphic to $C_K/C_K^{\mathfrak{m}}$.

Prop. (II.4.1.62). For a field K, if S is a finite set of primes that contains all the infinite primes and all the primes lying above the primes dividing n, and $I_K = I_K^S \cdot K^*$, then $C_K^n \cdot U_K^S$ is a norm group. If K contains the n-th roots of unity, then it corresponds to the Kummer extension $T = K(\sqrt[n]{K^S}/K)$.

Prop. (II.4.1.63). The norm groups $\mathcal{N}_{L/K}$ of C_K is precisely the groups containing some congruence subgroup $C_K^{\mathfrak{m}}$. Such $\mathfrak{m}$ are called a **modulus of definition for** L/K.

Proof: Cf.[Neukirch P164]. □

Prop. (II.4.1.64). Getting things together, we get a **universal norm residue symbol** $C_K \xrightarrow{(-,K)} G_K^{ab}$, and its kernel is $D_K = \cap_L N_{L/K} C_L$.

Then we have $D_K = \cap C_K^n$ and it is the connected component of $1 \in C_K$ and $C_K/D_K \to G_K^{ab}$ is an isomorphism.

Proof: Cf.[Neukirch P167]. and [Class Field Theory Artin Tate Chap9]. □

Prop. (II.4.1.65). When $K = \mathbb{Q}$ and $\mathfrak{m} = m \cdot p_\infty$, then the ray class field mod $\mathfrak{m}$ is $\mathbb{Q}(\zeta_m)$.

Proof: Cf.[Neukirch P165]. □

Cor. (II.4.1.66) (Kronecker Theorem). Every Abelian extension of $\mathbb{Q}$ is a subfield of $Q(\zeta_m)$ for some cyclotomic field.

Remark (II.4.1.67). The ray class field mod 1 is important, it is the **Hibert class field** of K, its Galois group is isomorphic to $C_K/C_K^1 \cong I_K/I_K^{S_\infty} \cdot K^* \cong J_K/P_K$ by(II.2.4.10). Its degree is equal to the ideal class number h of K.

Next we investigate the relation of CFT with the decomposition of primes in extension fields.

Prop. (II.4.1.68). If L/K is an Abelian extension, then $N_{L/K} C_L \cap K_\mathfrak{p}^* = N_{L_\mathfrak{P}/K_\mathfrak{p}} L_\mathfrak{P}^*$.

Proof: For the non-trivial part, notice if $\mathfrak{a} \in N_\mathfrak{P} L_\mathfrak{P}^*$ is a norm times a $a \in K^*$, then it is a norm at all primes except $\mathfrak{p}$, thus it is also norm at $\mathfrak{p}$ by the multiplicative definition of the inv map(II.4.1.47). □

Cor. (II.4.1.69). Let L/K be Abelian and $\mathcal{N} = N_{L/K} C_L$ be the norm group, then $\mathfrak{p}$ is unramified in L iff $U_\mathfrak{p} \subset \mathcal{N}$ and $\mathfrak{p}$ splits completely in L iff $K_\mathfrak{p}^* \subset \mathcal{N}_{L/K}$.

Cor. (II.4.1.70) (Conductor). We can define the **conductor** $\mathfrak{f}$ of L/K as the gcd of all $\mathfrak{m}$ that $C_K^\mathfrak{m} \in \mathcal{N}_{L/K}$. Then all primes not in $\mathfrak{f}$ are unramified and in particular, all primes not in $\mathfrak{m}$ are unramified in $C^\mathfrak{m}$.

Prop. (II.4.1.71) (Ramification and Norm Group). Let L/K is an Abelian extension of degree n and $\mathfrak{p}$ is an unramified prime ideal of K and π is a uniformizer, then if f is the smallest number that $(\ldots, 1, \pi^f, 1, \ldots) \in N_{L/K} C_L$, then $\mathfrak{p}$ factors in the extension L into $r = n/f$ distinct primes of degree f.

Proof: The degree the extension of $\mathfrak{p}$ is just the order of the Frobenius automorphism of $G_{\mathfrak{P}/\mathfrak{p}}$, which is just the order in $G_{L/K} \cong C_K/N_{L/K} C_L$. The Frobenius of $\mathfrak{p}$ correspond exactly to $(\ldots, 1, \pi, 1, \ldots)$ by(II.4.1.12), so the result follows. □

Prop. (II.4.1.72). The Hilbert class field is the maximal unramified extension of K.

Prop. (II.4.1.73) (Principal Ideal Theorem). In the Hilbert class field over K, every ideal $\mathfrak{a}$ of K becomes a principal ideal.

Proof: Cf.[Neukirch P171]. It should use a Finite Group theory theorem(I.3.1.41) □

Next we interpret the conclusions of GCFT in the language of ideals.

Def. (II.4.1.74) (Notations). $J^{\mathfrak{m}}$ is the group of all ideals relatively prime to $\mathfrak{m}$.

The **ray mod** $\mathfrak{m}$ $P^{\mathfrak{m}}$ is the group of all principal ideals (a) that $a \equiv 1 \bmod \mathfrak{m}$.

All subgroups of $J^{\mathfrak{m}}/P^{\mathfrak{m}}$ are called **ideal groups defined mod** $\mathfrak{m}$.

If L/K is an Abelian extension with a modulus of definition $\mathfrak{m}$, then $H^{\mathfrak{m}} = N_{L/K}J_L^{\mathfrak{m}} \cdot P^{\mathfrak{m}}$ is called the **ideal group defined mod** $\mathfrak{m}$.

Def. (II.4.1.75). We have a homomorphism $J^{\mathfrak{m}} \to G_{L/K}$ called the **Artin symbol** $(\frac{L/K}{})$. On primes $\mathfrak{p}$, it maps a prime $\mathfrak{p}$ which is unramified by(II.4.1.70) to its local Frobenius automorphism in $G_{\mathfrak{P}/\mathfrak{p}} \subset G_{L/K}$, which doesn't depend on $\mathfrak{P}$ because it is Abelian.

Lemma (II.4.1.76). When $\mathfrak{m}$ is a modulus of definition, the restriction to finite part defines isomorphism $C/C^{\mathfrak{m}} \cong J^{\mathfrak{m}}/P^{\mathfrak{m}}$ and $N_{L/K}C_L/C^{\mathfrak{m}} \cong H^{\mathfrak{m}}/P^{\mathfrak{m}}$.

Proof: Cf.[Neukirch CFT P176]. □

Prop. (II.4.1.77) (classical Artin Reciprocity Law). If L/K is an Abelian extension and $\mathfrak{m}$ is a modulus of definition, then there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_{L/K}C_L & \longrightarrow & C_K & \xrightarrow{(-,L/K)} & G_{L/K} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H^{\mathfrak{m}}/P^{\mathfrak{m}} & \longrightarrow & J^{\mathfrak{m}}/P^{\mathfrak{m}} & \xrightarrow{(\frac{L/K}{})} & G_{L/K} & \longrightarrow & 1 \end{array}$$

Thus the second row is exact by(II.4.1.55), and $G_{L/K} \cong J^{\mathfrak{m}}/H^{\mathfrak{m}}$.

Prop. (II.4.1.78) (Ramification and Ideal Group). Let L/K is an Abelian extension of degree n with a modulus of definition $\mathfrak{m}$(e.g. the conductor) and $\mathfrak{p}$ doesn't divid $\mathfrak{m}$. then if f is the smallest number that $\mathfrak{p}^f \in H^{\mathfrak{m}}$, then $\mathfrak{p}$ factors in the extension L into $r = n/f$ distinct primes of degree f.

Proof: The degree the extension of $\mathfrak{p}$ is just the order of the Frobenius automorphism of $G_{\mathfrak{P}/\mathfrak{p}}$, which is just the order in $G_{L/K} \cong J^{\mathfrak{m}}/H^{\mathfrak{m}}$. The Frobenius of $\mathfrak{p}$ correspond exactly to $\mathfrak{p}$ by(II.4.1.12), so the result follows. □

2 Cohomology of Local Fields

Def. (II.4.2.1) (Notations). For an algebraic extension K/Q_p, we let G_K be $G(\overline{\mathbb{Q}_p}/K)$.

For a finite extension $K/\mathbb{Q}_p$, K_∞ is defined to be K adding all the p^n-th roots of unities.

H_K is defined to be $G(\overline{\mathbb{Q}_p}/K_\infty)$, $\Gamma_K = G_K/H_K$.

The **cyclotomic character** χ is defined to be the multiplicative map $\chi : G_K \to \mathbb{Z}_p^*$ that $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ for every $\sigma \in G_K$ and ζ a p^n-th root of unity. The kernel of χ is H_K, and it identifies $\Gamma_{\mathbb{Q}_p}$ as $\mathbb{Z}_p^*$ and Γ_K as an open subgroup of $\mathbb{Z}_p^*$.

Prop. (II.4.2.2). The profinite group $\mathbb{Q}_p^{\mathrm{tame}}$ is $\widehat{\mathbb{Z}} \ltimes \Delta_p$. Which is the profinite group generated by the relationship $\sigma\tau\sigma^{-1} = \tau^p$, where σ is a lift of Frobenius. Which means that it is the limit of finite quotients of the group $\langle \sigma\tau\sigma^{-1} = \tau^p \rangle$.

Proof: Cf.[Local Fields Clark]. □

3 Cohomology of Global Fields

4 Iwasawa Theory

II.5 Diophantine Geometry

1 Artin's Conjecture

Basic references are [Serre Galois Cohomology Chap2.4.5].

Def. (II.5.1.1). A field K is called C_1 or for any homogenous polynomial $F(X_1,\ldots,X_n)$ of degree$d$ with coefficient in K that $d^k<n$ has a non-zero solution in K^n.

C_0 fields are just alg.closed fields, C_1 fields are also called **quasi-algebraically closed**.

Prop. (II.5.1.2). Any field L algebraic over a quasi-alg.closed field is quasi-alg.closed.

Proof: For a homogenous polynomial $F(X_1,\ldots,X_n)$, its coefficient lies in a finite extension of K contained in L, so we may assume L/K is finite. Then choose a basis $\{e_1,\ldots,e_m\}$ of L over K, then consider the function $f(x_{11},\ldots,x_{1m},\ldots,x_{n1},\ldots,x_{nm})=N_{L/K}(F(x_11e_1+\ldots+x_{1m}e_m,\ldots,x_{n1}e_1+\ldots+x_{nm}e_m)$, which is a homogenous polynomial of degree nm with coefficient in K, because it has values all in K. So it has a nonzero solution in K^{nm} by(III.7.2.4), Krull's height theorem and k is alg.closed. □

Prop. (II.5.1.3) (Chevalley-Warning). Any finite field $\mathbb{F}_q$ is quasi-algebraically closed. In fact, for any system of polynomials f_i, if $\sum_{i=1}^r\deg f_i<d$, then the number of solutions to this equation on $\mathbb{F}_q$ is divisible by p.

Proof: The number of solutions to this system is equivalent to

$$\sum_{x\in\mathbb{F}_q^n}\prod(1-f_i^{q-1}(x))$$

modulo p.

But notice that if $i<q-1$, then $\sum_{x\in\mathbb{F}_q}x^i=0$ in $\mathbb{F}_q$ by(IX.2.2.5), but as the degree of the highest term of $\sum_{x\in\mathbb{F}_q^n}\prod(1-f_i^{q-1}(x))$ modulo p is smaller than $n(q-1)$, some x_i has power smaller than $q-1$, thus when summed over $\mathbb{F}_q$, it vanishes. □

Prop. (II.5.1.4) (Tsen). Algebraic function fields of dimension 1 over an alg.closed field K is quasi-alg.closed.

Proof: By(II.5.1.2), it suffice to consider the case $K=k(t)$ purely transcendental. for a polynomial F with coefficient in $k(t)$, we can assume it has coefficient in $k[t]$, then let δ be their maximal degree. If substituted with $X_i=\sum_{i=0}^N a_ijt^j$, the function becomes a system of $\delta+dN+1$ homogenous equation with $n(N+1)$ unknowns a_{ij}, since $d<n$, $\delta+dN+1<n(N+1)$ for N large. In this case, □

Prop. (II.5.1.5). If K is quasi-alg.closed, then $H^2(G(K_s/K),K_s^*)=0$.

Proof: Cf.[Etale Cohomology Fulei 5.7.15]. □

Cor. (II.5.1.6). By this and(II.5.1.2), the condition of(II.3.3.13) are satisfied. So if K is quasi-alg.closed, then $cd(G(K_s/K))\leq 1$ and $H^i(G(K_s/K),K_s^*)=0$ for $i\geq 1$.

Prop. (II.5.1.7) (Ax-Kochen). For any d, there is a N_d that if $p>N_d$, then any homogenous polynomial $f(X_1,\ldots,X_n)$ of degree d with coefficient in $\mathbb{Q}_p$ that $d^k<n$ has a non-zero solution in $\mathbb{Q}_p^n$. The proof uses Model theory.

II.6 Classical Rigid Analytic Geometry

Basic references are [Formal and Rigid Geometry Siegfried Bosch] and [Non-Archimedean Analysis Remmert], but this treatment is already classical and there are other approaches, such as given by Berkovich, or given by Huber, and used in Scholze's work, which is most natural because it behaves well w.r.t. the formal model.

1 Affinoid Algebras

Tate Algebras

Def. (II.6.1.1). For a complete non-Archimedean field K with residue field k, we define the **Tate algebra** $T_n = K\langle x_1, \ldots, x_n\rangle$ to be the subalgebra of $K[[x_1, \ldots, x_n]]$ consists of elements $\sum_v a_v x^v$ that $\varinjlim_{|v|\to\infty} |a_v| = 0$. It is endowed with the norm $|f| = \max |a_v|$.

The norm satisfies $|fg| = |f||g|$ and $|f + g| \leq |f| + |g|$.

There is a **reduction map** from T_n to $k[x_1, \ldots, x_n]$, it is surjective.

Proof: T_n is an algebra because the values of coefficients of f is bounded. $|fg| \leq |f||g|$ is easy, to show $|fg| \geq |f||g|$, we assume $|f| = |g| = 1$, then their reduction in $K[x_1, \ldots, x_n]$ is non-zero, thus $\overline{fg}$ is non-zero, which shows $|fg| \geq 1$. □

Prop. (II.6.1.2) (Maximum Principle). A formal power series f converges in $B^n(\overline{K})$ iff it is in T_n.

And when it is in T_n, $|f(x)|$ attains a maximum$= |f|$ in $B^n(\overline{K})$.

Proof: If it converges at $(1, \ldots, 1)$, then $\varinjlim_{|v|\to\infty} |a_v| = 0$ by(II.1.1.19). Conversely, for any point in $B^n(\overline{K})$, it can be considered in a finite extension field of K, thus complete, hence we can apply(II.1.1.19) again.

For the second assertion, we assume $|f| = 1$, then consider its reduction to $k[x_1, \ldots, x_n]$, then there is a $\overline{x}$ in the alg.closure of k that $\overline{f}(\overline{x}) \neq 0$. Now $\overline{k}$ can be seen as the residue field of $\overline{K}$. Then the lifting of $\overline{x}$ to a $x \in \overline{K}$ has valuation 1 and $|f(x)| = 1$. □

Prop. (II.6.1.3). T_n is a Banach algebra(Easy).

Cor. (II.6.1.4). An element f of norm 1 of T_n is invertible in T_n iff its reduction in $k[x_1, \ldots, x_n]$ is a unit. Elements of other norms can be reduced to the case of norm 1.

Proof: One direction is trivial, the other is because $|f - f(0)| < 1$, hence $f = f(0)(1 + g)$, this is invertible by power expansion as T_n is complete. □

Def. (II.6.1.5). A restricted power series $g = \sum g_v X_n^v \in T_n$ with coefficients in T_{n-1} is called X_n**-distinguished of order** s iff g_s is a unit in T_{n-1}, $|g_s| = |g|$ and $|g_s| > |g_v|$ for all $v > s$.

Lemma (II.6.1.6). For any f.m. elements $f_i \in T_n$, there is a continuous automorphism of T_n that maps $T_n \to T_n, T_i \to T_i + T_n^{\alpha_i}$ that maps f_i to X_n-distinguished elements.

Proof: Cf.[Rigid and Formal Geometry P16]. □

Prop. (II.6.1.7) (Weierstrass Division). If $g \in T_n$ is X_n-distinguished of order s, for any $f \in T_n$, there is a unique form $f = qg + r$, where $q \in T_n$ and $r \in T_{n-1}[X_n]$ of degree $r < s$. Moreover, $|f| = \max\{|q||g|, |r|\}$.

Proof: Cf.[Rigid and Formal Geometry P17]. □

Cor. (II.6.1.8) (Weierstrass Preparation). If $g \in T_n$ is X_n-distinguished of order s, then there exists uniquely a $r \in T_{n-1}[X_n]$ of degree s and $g = re$, where e is a unit in T_n.

Proof: By(II.6.1.7) applied to $X_n^s = qg + r$ with $|r| \leq 1$. Then $\omega = X_n^s - r$ is X_n is the desired polynomial, it suffice to show q is a unit. Let g be normalized that $|g| = 1$, then $|q| = 1$, by reduction to polynomials, we see $\widetilde{\omega} = \widetilde{q}\widetilde{g}$, and $\widetilde{\omega}, \widetilde{g}$ are both polynomials of degree s, so $\widetilde{q} \in k^*$, so q is a unit by(II.6.1.4).

Uniqueness: if $g = e\omega$, then $X_n^s = e^{-1}g + (X_n^s - \omega)$, so uniqueness follows from that of Weierstrass division. □

Prop. (II.6.1.9) (Noether Normalization). For any proper ideal $\mathfrak{a}$ of T_n, There is a d and a finite injection $T_d \to T_n/\mathfrak{a}$.

Proof: We may assume $\alpha \neq 0$, thus choose a $g \in \alpha \neq 0$, then using(II.6.1.6), we may assume g is X_n-distinguished. Now the Weierstrass division theorem(II.6.1.7) says that $T_{n-1} \to T_n/(g)$ is finite. Hense $T_{n-1} \to T_n/(g) \to T_n/\mathfrak{a}$ is finite. Now we can use induction to find a $T_d \to T_{n-1}/T_{n-1} \cap \mathfrak{a}$ finite, thus also $T_d \to T_n/\mathfrak{a}$ is finite. □

Cor. (II.6.1.10) (Residue Field of Tate Algebra). The residue field of a maximal ideal of T_n is a finite extension field of K, because $T_n/\mathfrak{m}$ has dimension 0, thus $K \to T_n/\mathfrak{m}$ finite injective.

Proof: Because finite injection $T_d \to T_n/\mathfrak{m}$ shows T_n is a field(I.5.4.3), thus we must have $d = 0$. □

Cor. (II.6.1.11). The map from $B^n(\overline{K})$ to the set of maximal ideals of T_n are surjective.

Proof: Evaluation map defines a map $T_n \to K(x_1, \ldots, x_n)$ that is surjective, thus the kernel is a maximal ideal. Conversely, for any maximal ideal $\mathfrak{m} \subset T_n$, $K' = T_n/\mathfrak{m}$ is finite over K, so we may assume $K' \subset \overline{K}$.

We show that this map $\varphi : T_n \to \overline{K}$ is contractive, otherwise there is a $|a| = 1, |\alpha = \varphi(a)| > 1$. Consider the minimal polynomial p of $|\alpha|$, all the conjugates of α has the same valuation as K, as K is Henselian, thus p has ascending Newton polygon, thus by(II.6.1.4) it is invertible in T_n. But $\varphi(p(a)) = 0$, contradiction.

So $|\varphi(x)| \leq |x|$, then it is continuous, and $(\varphi(T_1), \ldots, \varphi(T_n)) \subset B^n(K^n)$, so we are done. □

Cor. (II.6.1.12) (Main Theorem). T_n is Noetherian, UFD, Jacobson of Krull dimension n.

Proof: Noetherian: Use induction, as in the proof of(II.6.1.9), $T_{n-1} \to T_n/(g)$ is finite for some $g \subset \mathfrak{a}$, then also $T_n/\mathfrak{a}$ is finite over T_{n-1}, thus Noetherian as a T_{n-1} module, thus Noetherian as a ring.

UFD: Cf.[Rigid and Formal Geometry P20].

Jacobson: We need to show that any prime ideal $\mathfrak{p}$ is an intersection of maximal ideals. The case of $\mathfrak{p}$ is by(II.6.1.2). For $\mathfrak{p} \neq 0$, by Noetherian normalization(II.6.1.9), there is a $T_d \subset T_n/\mathfrak{p}$ finite. Then use induction and generalized Nullstellensatz(I.5.7.9), $T_n/\mathfrak{p}$ is Jacobson, thus $\mathfrak{p} = \mathrm{rad}(T_n/\mathfrak{p})$.

Dimension n: Cf.[Formal and Rigid Geometry P22]. □

Prop. (II.6.1.13). For an ideal $\mathfrak{a} \in T_n$, there are $a_1, \ldots, a_r$ which generate $\mathfrak{a}$ that $|a_i| = 1$, and any elements in f has a representation of the form $\sum f_i a_i$ with $|f_i| \leq |f|$.

The same assertion holds for submodules of T_n^k.

Proof: Cf.[Formal and Rigid Geometry P27,29]. □

Cor. (II.6.1.14). Each ideal of T_n is closed hence complete in T_n. This follows immediately from(II.6.1.12) and(II.1.2.12).

Cor. (II.6.1.15). For any ideal $\mathfrak{a}$ of T_n, the distance from an element to $\mathfrak{a}$ attains minimum.

Proof: Cf.[Formal and Rigid Geometry P28]. □

Affinoid Algebras

Def. (II.6.1.16) (Affinoid Tate Algebra). A normed algebras of the form $A = T_n/\mathfrak{a}$ are called **affinoid algebras**, so it is Noetherian and Jacobson by(II.6.1.12). An affinoid algebra has a natural semi-norm by $|f|_{sup} = \sup |f|_{\mathfrak{m}}$ in $A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ of A by(II.6.1.10).

Proof: We need to show the sup is finite, for this, notice $|f| = |g|$ for some g in the residue norm(II.6.1.17), so for any maximal ideal $\mathfrak{m}$ of A, the inverse is a maximal ideal $\mathfrak{n}$ in T_n by finiteness, thus $|f|_{\mathfrak{m}} = |g|_{\mathfrak{n}} \leq |g|_{sup} = |g| = |f|$, so $|f|_{\mathrm{sup}} \leq |f|$.

For the second-last equality, notice on T_n, $|\cdot|_{sup}$ and $|\cdot|$ equal, by(II.6.1.2) and(II.6.1.11). □

Def. (II.6.1.17) (Residue Norm). For a Tate algebra $A = T_n/\mathfrak{a}$, there is a natural residue norm on it. This is a complete K-algebra norm on A, and $T_n \to A$ is continuous and open. For any $f \in A$, the residue norm is attained at an element of T_n.

Any residue norm is bigger than the sup-norm, by the proof of(II.6.1.16).

Proof: It is a K-algebra norm is easily verified, should notice $|f| = 0$ iff $f = 0$, because $\mathfrak{a}$ is closed(II.6.1.14). The last assertion follows from(II.6.1.15). □

Remark (II.6.1.18). The sup norm may not even be a norm, if $\mathfrak{a}$ is not radical, but the fact that sup norm is smaller than any residue norm, together with(II.6.1.22), is enough for use.

Prop. (II.6.1.19). For $T_d \to A$ a finite injection, assume A is a torsion-free T_d-module, then for any $f \in A$, there is a unique minimal monic polynomial P of f over T_d.

In this case, $|f|_{sup} = \sup |a_i|_{sup}^{1/i}$ where a_i are coefficients of P.

Proof: Because A is torsion-free, we reduce to the quotient field of T_n, then f has a minimal monic polynomial, and T_n is UFD, hence Gauss lemma shows that this polynomial has coefficients in T_d. Hence $T_n[f] = T_n[X]/(p)$.

For the second, notice first for finite extension the Spec map is surjective, thus we may assume $A = T_n[f] = T_n[X]/(p)$, and for a maximal ideal $\mathfrak{m}$ of T_n, let $T_n/\mathfrak{m} = k$, then $A/(\mathfrak{m}) = k[X]/(\overline{p})$, then maximal ideals of $A/(\mathfrak{m})$ corresponds to roots α_i of $\overline{p}$ in $\overline{k}$, so $\sup_{\mathfrak{n} \text{ over } \mathfrak{m}} |f|_{\mathfrak{n}} = \sup |\alpha_i| = \max |a_i|_{\mathfrak{m}}^{1/i}$, so the result follows. □

Cor. (II.6.1.20). $|f|_{sup} \in \sqrt[N]{|K|}$ for some N and all $f \in A$, because the minimal polynomial has coefficients in T_n, and sup norm and Gauss norm coincide on T_n by the proof of(II.6.1.16).

Cor. (II.6.1.21) (Maximum Principle). $|f|_{sup} = |f|_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$.

Proof: Since A is Noetherian(II.6.1.12), it has f.m. minimal primes, hence $|f|_{sup} = |f|_{sup}$ in A/p_i for some minimal prime of A. Hence we reduce to the case of(II.6.1.19), hence the conclusion follows from(II.6.1.2) and the proof of(II.6.1.19). □

Prop. (II.6.1.22) (Residue Norms Equivalent). Any morphism from a Noetherian k-algebra to an affinoid algebras A is continuous w.r.t any residue norms. In particular, any k-Banach algebra topology on A coincides with the k-affinoid topology on A, and all residue norms on an affinoid algebra are equivalent.

Moreover, for any morphism of k-affinoid algebras $B \to A$, the norm on A can be replaced by an equivalent one that makes A into a normed B-algebra.

Proof: Use(II.1.2.9), it suffices to show the condition holds, for $\mathfrak{B} = \{\mathfrak{m}^v\}$ where $\mathfrak{m}$ are maximal ideals of A: The residue field is finite by(II.6.1.10), their intersections is (0) because if $f \in \cap_{\mathfrak{m}} \cap_n \mathfrak{m}^n$, Krull's theorem(I.5.5.6)(use localization) says for each maximal ideal $\mathfrak{m}$ there is a $m \in \mathfrak{m}$ that $(1-m)f = 0$, so $Ann(f) = (1)$, so $f = 0$.

For the second assertion, see [non-Archimedean Analysis P229]. □

Cor. (II.6.1.23). The notion of power-boundedness and topological nilpotence is independent of residue norm chosen.

Cor. (II.6.1.24). For an affinoid algebra A, the restricted power series in A:

$$A\langle X_i\rangle = \{\sum a_v X^v | \varinjlim_{|v|\to\infty} a_v = 0\}$$

is an affinoid algebra, this is independent of the residue norm chosen.

Lemma (II.6.1.25). The image A is dense in $A\langle X\rangle/(X-f)$(in the residue norm, and thus in all other norms, by(II.6.1.22)(II.6.1.17)), this is because a restricted power series can be truncated by a finite part and a part with small norm, and the finite part is in the image of A.

Def. (II.6.1.26) (Affinoid Generator). For a morphism of affinoid algebras $A \to A'$, a set of elements h_i in A' is called a set of **affinoid generator** iff there is a surjection

$$A\langle X_1, \ldots, X_n\rangle \to A', \quad X_i \mapsto h_i$$

Of course h_i is power-bounded, by the residue norm given.

Lemma (II.6.1.27). If $\pi' : A\langle X_1, \ldots, X_n\rangle \to A' : X_i \to h_i'$ is a surjective morphism of affinoid algebras that $A\langle X_1, \ldots, X_n\rangle$ is endowed with the Gauss norm and A' is endowed with the residue norm, then any set of elements $h = (h_1, \ldots, h_n)$ that $|h_i - h_i'| < 1$ is a set of affinoid generators.

Proof: By non-Archimedean property, $|h_i| \leq 1$ thus also $|h_i'| \leq 1$, and Let $\varepsilon = \max\{|h_i - h|\} < 1$. The strategy is simple, if for each g in A', we can find a f that $|f| = |g|, |\pi(f) - g| \leq \varepsilon|g|$, then by iteration, there is a f that $\pi(f) = g$. But by(II.6.1.17) and(II.6.1.15), if we choose a f that $\pi'(f) = g$ and $|f| = |g|$, then

$$|\pi(f) - g| = |\sum a_v h^v - \sum a_v h'^v| = |\sum a_v(h^v - h'^v)| \leq \varepsilon|f| = \varepsilon|g|.$$

□

Def. (II.6.1.28) (Distinguished Element). For an affinoid algebra A and an element $x \in \operatorname{Sp} A$(II.6.2.1), a element $f \in A\langle X_1, \ldots, X_n\rangle$ is called **X_n-distinguished of order s at x** iff it is distinguished in $A/\mathfrak{m}_x\langle X_1, \ldots, X_n\rangle$ is distinguished of order s in the sense of(II.6.1.5)(notice $A/\mathfrak{m}_x$ is a complete valued field by(II.6.1.10)).

Prop. (II.6.1.29) (Fibererd-Pushouts). When R, A_1, A_2 are all affinoid algebras, the amalgamated sum is also an affinoid algebra. In other words, the category of affinoid algebras admits amalgamated sums(fibered pushouts by(II.1.1.14)).

Proof: Cf.[Formal and Rigid Geometry P245]. □

Prop. (II.6.1.30). $T_n\hat{\otimes}T_m \cong T_{m+n}$. $K'\hat{\otimes}T_{n,K} = T_{n,K'}$.

Prop. (II.6.1.31). For affinoid algebras R, A_1, A_2 and ideals $\mathfrak{a}_1 \subset A_1, \mathfrak{a}_2 \subset A_2$, there is an isomorphism:

$$(A_1\hat{\otimes}_R A_2)/(\mathfrak{a}_1, \mathfrak{a}_2) \cong (A_1/\mathfrak{a}_1)\hat{\otimes}_R(A_2/\mathfrak{a}_2)$$

Proof: Cf.[Rigid and Formal Geometry P248]. □

Prop. (II.6.1.32) (Finite Extension of Affinoid Algebras). If B is an affinoid K-algebra and $\varphi: B \to A$ is a finite ring map, then A can be provided a topology to make it an affinoid K-algebra, and φ is continuous.

Proof: We can associate to A a Banach algebra topology induced from $B^n \to A \to 0$ that is continuous. Now it is an affinoid K-algebra: we may assume $B = T_n$, then $A = \sum T_n a_i$, and we may assume $|a_i| < 1$ then clearly there is a continuous extension $T_n\langle X_i\rangle \to A$ extending this map, so A is affinoid. □

2 Affinoid K-Spaces

Def. (II.6.2.1) (Affinoid K-Space). an Affinoid algebra A can be viewed as the function ring on the space $\mathrm{Sp}\,A$ of maximal ideals of A with the usual Zariski topology called the **affinoid K-space associated to A**. A morphism of affinoid algebras induce a map on their $\mathrm{Sp}\,A$. This is because residue fields of maximal ideals are finite over K. So we *define* the category of affinoid K-spaces as the opposite category of affinoid K-algebras.

Cor. (II.6.2.2). The category of affinoid spaces admits fiber products, because of(II.6.1.29).

Prop. (II.6.2.3). By the properties of a Jacobson space(IV.1.13.19)(IV.1.13.16), the affinoid K-space has good properties w.r.t. closed, open hence irreducible compared to $\mathrm{Spec}\,A$ in Zariski topology. In particular, it is a Noetherian space.

Def. (II.6.2.4). The affinoid K-space has another topology, called the **canonical topology**, generated by $X(f, \varepsilon) = \{x | f(x) \leq \varepsilon\}$ as a subbasis. And this topology is in fact generated by $X(f) = X(f, 1)$ as a subbasis.

Proof: For the last assertion, notice $f(x)$ assume value in $|\overline{K}|$, which is dense in $\mathbb{R}_+$, so we can assume $\varepsilon \in |\overline{K}|$(by approximation from below), hence $\varepsilon^n = |c|$, where $c \in K$, so $X(f, \varepsilon) = X(f^n, c) = X(c^{-1}f^n)$. □

Prop. (II.6.2.5). $\{x | f(x) = \varepsilon\}$ is open in $\mathrm{Sp}\,A$.

Proof: We let $f(x) = \varepsilon$ and $k = A/\mathfrak{m}_x$, let the minipoly of f in $A/\mathfrak{m}_x$ be P of degree n, and let $g = P(f)$, then $g(x) = 0$, and if $|g(y)| < \varepsilon^n$, then $|f(y)| = \varepsilon$, otherwise $|f(y) - \alpha_i| \geq |\alpha_i| = \varepsilon$ for every root α_i of P, hence $|P(f(y))| \geq \varepsilon^n$, contradiction. □

Cor. (II.6.2.6). By the proof, we have, $X(f_1,\ldots,f_r)$, $f_i\in\mathfrak{m}_x$ forms a basis of x in $\operatorname{Sp}A$.(Replace every $X(f_i)$ by $\{y||f_i(y)|=\varepsilon\}$, then by some $X(g_i)$ for $g_i\in\mathfrak{m}_x$.

Def. (II.6.2.7) (Affinoid Subdomain). For an affinoid K-space X, a subset U is called a **affinoid subdomain** of X if there is an closest affinoid space map $X'\to X$ with image in U, i.e. any other these maps factor through it. The definition is weird but the situation is clarified by the following proposition.

Prop. (II.6.2.8). For an affinoid subdomain $i:X'\to X$,
- i is injective and $\operatorname{Im} i=U$.
- i^* induce an isomorphism $A/\mathfrak{m}_{i(x)}^k\cong A'/\mathfrak{m}_x^k$.
- $\mathfrak{m}_x=\mathfrak{m}_{i(x)}A'$.

Proof: Consider a point $y\in U$, there is a commutative diagram
$$\begin{tikzcd} A \arrow[r, "i^*"] \arrow[d] & A' \arrow[d] \arrow[dl, "\alpha"] \\ A/\mathfrak{m}_y^n \arrow[r, "\sigma"] & A'/\mathfrak{m}_y^n A' \end{tikzcd}$$
. Then there is a map $\alpha:A'\to A/\mathfrak{m}_y^n$ that makes the upper diagram commutative by universal property of subdomain, and the lower triangle is commutative by universal properties again. Then we see σ is surjective and notice the kernel of the projection is $\mathfrak{m}_yA'$ is in the kernel of α, thus σ is injective.

Now the case $n=1$ shows $\mathfrak{m}_yA'$ is maximal, hence i is surjective and the inverse image is just one point. □

Prop. (II.6.2.9) (Special Subdomains). There are three special affinoid subdomain of X: **Weierstrass domain** $X(f_1,\ldots,f_r)$, **Laurent domain** $X(f_1,\ldots,f_r,g_1^{-1},\ldots,g_s^{-1})$, **rational domain** $X(\frac{f_1}{f_0},\ldots,\frac{f_r}{f_0})=\{x||f_i(x)|\le|f_0(x)|\}$ for $(f_0,\ldots,f_r)=(1)$. They are all open by(II.6.2.5).

Proof: The Weierstrass domain corresponds to $A\to A\langle X_1,\ldots,X_r\rangle/(X_i-f_i)$.

The Laurent domain corresponds to $A\to A\langle X_1,\ldots,X_{r+s}\rangle/(X_i-f_i,1-X_{r+j}g_j)$.

The rational domain corresponds to $A\to A\langle X_1,\ldots,X_r\rangle/(f_i-f_0X_i)$.

They are affinoid subdomains is in fact, easily checked. □

Lemma (II.6.2.10). Weierstrass domain are Laurent, and Laurent domain are rational, this is because intersection of rational domains are rational.

Any rational domain is a Weierstrass domain of a Laurent domain.

Proof: Notice a Laurent subdomain is a finite intersections of $X(\frac{f}{1})$ and $X(\frac{1}{g})$, so it is rational.

For a rational domain U, f_0 is a unit in $\mathcal{O}(U)$, hence its inverse has a bounded value, then $|cf_0|>1$ for some $c\in K^*$. Hence U is Weirerstrass in $X((cf_0)^{-1})$. □

Cor. (II.6.2.11) (Pullback & Composition of Affinoid Subdomain). The pullback(hence intersections) of affinoid subdomains is affinoid subdomain and it is just the set-theoretic inverse image, and specialness are preserved.

The affinoid subdomain of an affinoid subdomain is affinoid subdomain, and Weierstrassness and rationalness are preserved(while Laurentness not).

Proof: Pullback: fiber product exist in the category of affinoid K-spaces, then the universal property is checked. The set-theoretic property follows from(II.6.2.8).

Speciality: Clear.

Transitivity: Clear by universal property.

For the speciality, if $V = X(f_i)$, $U = V(g_j)$ is Weierstrass, then because by(II.6.1.25) A is dense in $A\langle f_i \rangle$, we can replace g_j by elements from A, by adding elements of small sup-norm, because valuation is non-Archimedean. Then $U = X(f_i, g_j)$. For the rational subdomain $V = X(\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0})$, use(II.6.2.10), it suffices to prove for $U = V(g)$ or $U = V(g^{-1})$. For this, notice the image of $A[f_0^{-1}]$ is dense in $A\langle \frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0} \rangle$, by(II.6.1.25), so as before, we change g that it $g_0 f_0^n g \in A$ for some n. Now

$$V(g) = V \cap \{x \in X \| g_0(x)| \leq |f_0^n(x)|\}, \quad V(g^{-1}) = V \cap \{x \in X \| g_0(x)| \geq |f_0^n(x)|\}.$$

But now f_0^n is a unit in $A\langle \frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0} \rangle$, so $|f(x)|_{\sup} \geq |c|$ for some $c \in K^*$, so

$$V(g) = V \cap X(\frac{g_0}{f_0^n}, \frac{c}{f_0^n}), \quad V(g^{-1}) = V \cap X(\frac{f_0^n}{g_0}, \frac{c}{g_0}).$$

is rational in X. $\square$

Cor. (II.6.2.12). For a special subdomain U of X, the canonical topology induces the canonical topology of U, by the transitivity property of affinoid subdomains and(II.6.2.10). In fact, by(II.6.2.14), any affinoid subdomain is open and the topology coincides.

Prop. (II.6.2.13). Let $\varphi : Y = \operatorname{Sp} B \to X = \operatorname{Sp} A$ be a morphism, if x is a point of X that $A/\mathfrak{m}_x \to B/\mathfrak{m}_x B$ is a surjection, then there is an affinoid nbhd U of x that φ restricts to a closed immersion on $\varphi^{-1}(U)$. If $A/\mathfrak{m}_x^n \cong B/\mathfrak{m}^n$ for all n, then there is an affinoid nbhd U of x that φ restricts to an isomorphism $\varphi^{-1}(U) \cong U$.

Proof: Cf.[Rigid and Formal Geometry P57]. $\square$

Cor. (II.6.2.14). Every affinoid subdomain of X is open and has the restriction topology of X(canonical topology), because it satisfies the second condition of(II.6.2.13), by(II.6.2.8).

Lemma (II.6.2.15). If $f \subset A\langle X_1, \ldots, X_n \rangle$ is X_n-distinguished of order $\leq s$ for each element of $\operatorname{Sp} A$, then the set of elements that f is X_n-distinguished of exact order s is a rational subdomain of A.

Proof: Let $f = \sum f_v X_n^v$, let the constant coefficient of f_v be a_v, then the set is in fact $U = \{x \in \operatorname{Sp} A \| a_v(x)| \leq |a_s(x)|\}$. This is because, if f is distinguished of order s_x at x, then $a_{s_x} \neq 0$ because f_{s_x} is a unit, and $|a_v|_x \leq |f_v| \leq |f_{s_x}| = |a_{s_x}|_x$ for $v \leq s_x$ and strict inequality holds for $v > s_x$. In particular, $a_0, \ldots, a_s$ cannot have a common zero, so it is truly a rational subdomain. $\square$

Prop. (II.6.2.16). If $f \subset A\langle X_1, \ldots, X_n \rangle$ is X_n-distinguished of order s for each element of $\operatorname{Sp} A$, then the map

$$A\langle X_1, \ldots, X_{n-1} \rangle \to A\langle X_1, \ldots, X_n \rangle/(f)$$

is finite.

Proof: Cf.[Rigid And Formal Geometry P79]. $\square$

Presheaf of Affinoid Functions

Def. (II.6.2.17). The **weak Grothendieck category**(affine topology) on an affinoid space X has coverings defined by the finite cover by affinoid subdomains, called **affinoid covering**.

The **strong Grothendieck category**(fpqc topology) on an affinoid space X is defined by: objects are unions of affinoid subdomains $U = \cup U_i$ that for any morphism from an affinoid space $\varphi : Z \to U \subset X$, the pullback covering $\cup\varphi^{-1}(U_i)$ has a finite subcover by affinoid subdomains. A covering is defined by the same finiteness property.

The strong Grothendieck topology satisfies completeness conditions G_0, G_1, G_2 defined in(III.1.1.4), as easily verified.

The weak Grothendieck topology is a temporary notion, it will be obsolete after Tate's acyclicity theorem is proved. Admissible opens and admissible covers are notions w.r.t. the strong Grothendieck topology.

Proof: The weak Grothendieck category is a Grothendieck category by(II.6.2.11). The strong Grothendieck category is a Grothendieck category because: the finiteness condition lifts along base change, and also for base change, because we can first choose a finite subcover, then choose a finite subcover of the base change covering of that finite covering. □

Def. (II.6.2.18). For n functions $f_1, \ldots, f_n$ without common zeros, the rational subdomains $U_i = X(\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i})$ is an affinoid covering, called the **rational covering**. For n functions $f_1, \ldots, f_n$, there is a **Laurent covering** $X(\prod f_i^{\varepsilon_i}), \varepsilon_i = \pm 1$.

Prop. (II.6.2.19). Morphisms of affinoid spaces are continuous in weak Grothendieck topology by(II.6.2.11). It is also continuous in the strong Grothendieck topology, as one can check the finiteness conditions.

Prop. (II.6.2.20). Let X be an affinoid K-space, for any $f \in \mathcal{O}_X(X)$, consider the following sets:

$$U_1 = \{x||f(x)| < 1\}, \quad U_2 = \{x||f(x)| > 1\}, \quad U_3 = \{x||f(x)| > 0\}.$$

Then any finite union of sets of the form is admissible, and any finite cover by finite union of sets of the form is an admissible covering.

Proof: We first show that U_1 is admissible open, the others are similar. Let ε_n be an ascending sequence of elements in $\sqrt{|K^*|}$ converging to 1, then $U_1 = \cup_n X(\varepsilon_n^{-1} f)$ is a union of open subsets because $\varepsilon_n \in \sqrt{|K^*|}$. Now for any affinoid space Z mapping into U_1, $|\varphi^*(f)(z)|_{\sup} < 1$ for all $z \in Z$, thus by maximal principle(II.6.1.21), $|f|_{\sup} < 1$, thus the cover $U_1 = \cup_n X(\varepsilon_n^{-1} f)$ can be refined by a finite cover, thus it is admissible open.

For the admissibility of covering, the proof is similar, but use the following lemma(II.6.2.21). □

Lemma (II.6.2.21). For any affinoid K-algebra A, if f_i, g_j, h_k are system of functions on A that: for every $x \in A$, either $|f_i(x)| < 1, |g_j > 1$ or $h_k(x) > 0$, then we can replace $>, <$ by $\geq, \leq$ and elements in $\sqrt{|K^*|}$ that the same condition is true.

Proof: Cf.[Rigid and Formal Geometry P97]. □

Cor. (II.6.2.22). The strong Grothendieck category is finer than the Zariski category, because any standard affine open set is of the form U_3 and also Zariski covering is open covering because $\mathrm{Sp}(A)$ is Noetherian(II.6.1.16).

Def. (II.6.2.23) (Presheaf of Affinoid Functions). There is a **presheaf of affinoid functions** defined on the weak Grothendieck category because of the universal property of the affinoid subdomains, and **stalks** are defined routinely.

Then the stalk $\mathcal{O}_{X,x}$ are local ring with maximal ideal $\mathfrak{m}_x\mathcal{O}_{X,x}$. Hense let $X = \operatorname{Sp} A$, the stalk map factor thorough $A \to A_{\mathfrak{m}_x} \hookrightarrow \mathcal{O}_{x,X}$, and

$$A/\mathfrak{m}_x^n \cong A_{\mathfrak{m}_x}/\mathfrak{m}_x^n A_{\mathfrak{m}_x} \cong \mathcal{O}_{X,x}/\mathfrak{m}_x^n\mathcal{O}_{X,x}$$

so it induces isomorphisms between their $\mathfrak{m}_x$-adic completions.

Proof: By(II.6.2.8), there is an isomorphism $K' = \mathcal{O}_X(X)/\mathfrak{m}_x \cong \mathcal{O}_X(U)/\mathfrak{m}_x\mathcal{O}(U)$. Take the converse and pass to direct colimit(it is exact), $\mathcal{O}_{X,x}/\mathfrak{m}_x O_{X,x} \cong K'$. This map will be regarded as evaluation at x. The kernel $\mathfrak{m}_x\mathcal{O}_{X,x}$ is a maximal ideal. There are no other maximal ideals in $\mathcal{O}_{X,x}$ because if f in not in the kernel, then $f(x) \neq 0$, and multiply by an element in K^*, it can be made $|f(x)| \geq 1$, and then $U(f^{-1})$ is an affinoid subdomain containing x that f is invertible in it.

For the second assertion, for an affinoid subdomain $\operatorname{Sp} A'$, there are maps

$$A/\mathfrak{m}^n \to A'/\mathfrak{m}^n \to \mathcal{O}_{X,x}/\mathfrak{m}^n\mathcal{O}_{X,x}.$$

We first show these are isomorphisms: the first map is an isomorphism by(II.6.2.8), then take direct colimit, the composition map is also isomorphism.

$A/\mathfrak{m}_x^n \cong A_{\mathfrak{m}_x}/\mathfrak{m}_x^n A_{\mathfrak{m}_x}$ is classical.

$A_{\mathfrak{m}_x} \hookrightarrow \mathcal{O}_{x,X}$ is injective because by Krull's intersection theorem(I.5.5.6), $A_{\mathfrak{m}_x} \hookrightarrow \mathcal{O}_{x,X} \to \widehat{\mathcal{O}_{X,x}} \cong \widehat{A_{\mathfrak{m}}}$ is injective. □

Cor. (II.6.2.24). $f \in A = \mathcal{O}_X(X)$ vanish iff it vanish at every stalk, this is because $A \to \prod_{\mathfrak{m}} A_{\mathfrak{m}} \to \prod \mathcal{O}_{X,x}$ is injective.

Cor. (II.6.2.25). Giving a covering of affinoid subdomain of an affinoid space $X_i \to X$, then $\mathcal{O}_X(X) \to \prod \mathcal{O}_{X_i}(X_i)$ is an injection. (This is because the kernel vanishes at each stalk.)

Cor. (II.6.2.26). For a subdomain of an affinoid space X, the corresponding ring map is flat.

Proof: Cf.[Formal and Rigid Geometry P68]. □

Prop. (II.6.2.27). The stalk $\mathcal{O}_{X,x}$ is Noetherian, in particular it is $\mathfrak{m}$-adically separated by Krull's intersection theorem(I.5.5.6).

Proof: First it is $\mathfrak{m}$-adically separated, because by(II.6.2.23), for a $f \in \cap\mathfrak{m}^n\mathcal{O}_{X,x}$, we can choose an affinoid subdomain $\operatorname{Sp} A$ that $f \in A$(II.6.2.8), then $f \in \mathfrak{m}^n A$, so by Krull's intersection theorem(I.5.5.6), we have $f = 0$ in $A_{\mathfrak{m}}$.

In the same way, any f.g. ideal $\mathfrak{a}$ of $\mathcal{O}_{X,x}$ is $\mathfrak{m}$-adically closed, this is because it is generated by an ideal in the affinoid algebra of a nbhd, and then $\mathcal{O}_{x,X}/\mathfrak{a}$ is separated as the stalk of an affinoid algebra $A'/\mathfrak{a}'$.

Now pass a chain of f.g. ideals to their completion, then that chain is stationary because $\hat{\mathcal{O}}_{X,x} = \hat{A}_{\mathfrak{m}_x}$ is Noetherian(I.5.1.10). And now this chain is also stationary because ideals are closed in $\mathfrak{m}$-adic topology. □

Locally Closed Immersions

Def. (II.6.2.28) (Immersions). A morphism of affinoid spaces is called a **closed immersion** iff the corresponding ring map is surjective. It is called a **locally closed immsetion** iff it is injective and the stalk map are all surjective. It is called an **open immersion** iff it is injective and the corresponding stalk maps are isomorphism. All these notions are stable under compositions.

An affinoid subdomain is an open immersion by(II.6.2.8)(II.6.2.23) and(II.6.2.27).

Lemma (II.6.2.29). Base change by affinoid subdomain of closed/locally closed/open immersions are of the same type.

Proof: This is obvious for locally close and open, because affinoid subdomains are open(II.6.2.14), for the closed immersion, use(II.6.1.31). □

Prop. (II.6.2.30). A closed immersion of affinoid spaces is equivalent to a locally closed immersion that the corresponding ring map is finite.

Proof: Cf.[Rigid and Formal Geometry P70]. A closed immersion $X' \to X$ is a locally closed immersion because the canonical topology of $\mathrm{Sp}\, A$ restricts to the canonical topology on $\mathrm{Sp}\, A/\mathfrak{a}$(II.6.2.4), then use(II.6.1.31), and the fact direct limit is exact. □

Prop. (II.6.2.31) (Clopen Immersion). The image of an open and closed immersion is Zariski closed and open. In particular, it is a Weierstrass subdomain.

Proof: Cf.[Rigid and Formal Geometry P71]. □

Def. (II.6.2.32). A **Runge immersion** is a closed immersion followed by an open immersion of Weierstrass subdomain. Runge immersion is table under base change of affinoid subdomains by(II.6.2.29)

Prop. (II.6.2.33) (Equivalent Definition of Runge Immersions). For a morphism $\sigma : A \to A'$, $\mathrm{Sp}\, A' \to \mathrm{Sp}\, A$ is a Runge immersion iff $\underline{\sigma(A) \text{ is dense in } A'}$ iff $\sigma(A)$ contains a set of affinoid generator of A' over A.

Proof: For a Runge immersion, $\sigma(A)$ is dense in A', because this is true for Weierstrass subdomain and closed immersion.

If $\sigma(A)$ is dense in A', then by(II.6.1.27), we can modify a set of affinoid generators by a set of affinoid generators in $\sigma(A)$.

If h_i is a set of affinoid generators in $\sigma(A)$, then $A \to A\langle h_i\rangle \to A'$ is a Runge immersion. □

Cor. (II.6.2.34). Runge immersion is stable under composition.

Prop. (II.6.2.35). An open and Runge immersion is an immersion of Weierstrass subdomain.

Proof: By localizing on this Weierstrass subdomain, and notice Weierstrass subdomain is stable under composition(II.6.2.11), we reduce to clopen immersion case, and result follows by(II.6.2.31). □

Lemma (II.6.2.36) (Extension of Runge Immersion). For a morphism of affinoid spaces $X' \to X = \mathrm{Sp}\, A$, if $f_1, \ldots, f_n, g$ generate A, for $\varepsilon \in \sqrt{\{|K^*|\}}$, denote $X_\varepsilon = \{x| |f_i(x)| \leq \varepsilon |g|\}$, this is a rational subdomain. The inverse image of X_ε is X'_ε, then if $X'_{\varepsilon_0} \to X_{\varepsilon_0}$ is a Runge immersion for some ε_0, then there is a $\varepsilon > \varepsilon_0$ that $X'_\varepsilon \to X_\varepsilon$ is also a Runge immersion.

Proof: Cf.[Rigid and Formal Geometry P73]. □

Prop. (II.6.2.37) (Gerritzen-Grauert). For a locally closed immersion $\varphi: X' \to X$, there is a finite cover of X of rational subdomains X_i that $\varphi^{-1}(X_i) \to X_i$ are Runge immersions.

Proof: Cf.[Formal and Rigid Geometry P79]. □

Cor. (II.6.2.38) (Gerritzen-Grauert). Any affinoid subdomain is equivalent to a finite union of rational subdomains.

Proof: An affinoid subdomain is an open immersion by(II.6.2.28), so $\varphi^{-1}(X_i) \to X_i$ is open and Runge, so it is Weierstrass by(II.6.2.35). In particular, $X \cap X_i$ is rational in X by transitivity, thus the result. □

Tate's Acyclicity

Lemma (II.6.2.39) (Reduction of Weak Grothendieck Topology).

- Every affinoid covering has a refinement of rational covering.
- For every rational covering, there is a Laurent covering $\{V_i\}$ that restriction on each V_i is rational covering generated by units.
- Every rational covering generated by units has a refinement of Laurent covering.

Proof: 1: By(II.6.2.38), we can assume the covering consists of rational subdomains $U_i = X(\frac{f_{i1},\ldots,f_{ii_k}}{f_{i0}})$, then consider the elements $f_{v_1\ldots v_n} = \prod_{i=1}^n f_{iv_i}$, where at least some $v_i = 0$. Denote the set of these elements by I.

Firstly, these elements has no common zero on X, thus generating a rational covering of X: for any $x \in U_i$, f_{i0} doesn't vanish at x, thus the product $\prod_{j\neq i} f_{jv_j}$ vanishes for all choices of v_j, but this is impossible because for each j, $(\{f_{ik}\}_k) = (1)$.

Secondly, this is a refinement of U_i: We show $X_{f_{v_1\ldots v_n}} \subset U_k$ where $v_k = 0$. For this, consider $x \in X_{f_{v_1\ldots v_n}}$, then $x \in U_j$ for some j. If $j = k$, we are done, otherwise,

$$|f_{v_1\ldots\mu_k\ldots v_n}(x)| \leq |f_{v_1\ldots 0\ldots\mu_k\ldots v_n}(x)| \leq |f_{v_1\ldots v_n}(x)|.$$

Where the last inequality is because $(v_1, \ldots, 0, \ldots, \mu_k, \ldots, v_n)$ has a 0, thus $f_{v_1\ldots 0\ldots\mu_k\ldots v_n} \in I$.

2: For a rational covering, f_i is invertible in the ring of $U = X(\frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i})$, thus it has a inverse that attains maximum value on U(II.6.1.21). Hence there is a $c \in K^*$ that $|c|^{-1} < \inf(\max\{|f_i(x)|\})$.

I claim the Laurent covering w.r.t. the elements $cf_0, \ldots, cf_n$ satisfies the requirement. Because for example, on $V = X((cf_0)\cdots(cf_s)(cf_{s+1})^{-1}\cdots(cf_n)^{-1})$, $|f_i(x)| < |f_j(x)|$ for $i \leq s < j$, so the covering restricted to V is just the rational covering generated by $f_{s+1}, \ldots, f_n$, and they are all invertible in $\mathcal{O}(V)$.

3: In fact the Laurent covering generated by the element $f_i f_j^{-1}$ for $i < j$ is a refinement of the rational covering generated by $f_1, \ldots, f_n$, because in any one of this Laurent subdomains V, for any two i, j, either $|f_i(x)| < |f_j(x)|$ or $|f_j(x)| < |f_i(x)|$ for all $x \in V$, so there is a maximal one f_s, then $V \subset X(\frac{f_0}{f_s}, \ldots, \frac{f_n}{f_s})$. □

Prop. (II.6.2.40) (Tate's Acyclicity Theorem). The presheaf of affinoid functions on an affinoid space $X = \mathrm{Sp}\, A$ is a sheaf w.r.t the weak Grothendieck category. In fact, for any A module M, the presheaf $\widetilde{M} = M \otimes_A \mathcal{O}_X$ is a sheaf w.r.t. the weak Grothendieck topology, called the **quasi-coherent** sheaf on X.

Moreover, for any finite cover of affinoid subdomains, the Čech cohomology group $\check{H}^q(\mathrm{Sp}\, A, \widetilde{M})$ vanish for $q \neq 0$.

Proof: It suffices to prove the last assertion. First reduce to the case of Laurent covering by(II.6.2.39) and(III.5.2.6)(III.5.2.7). Noticing the base change invariance of the specialities of affinoid subdomains(II.6.2.11). Even more, by(III.5.2.8) and a induction process, it suffices to prove for the simple Laurent covering $X(f), X(f^{-1})$.

It suffices to prove for the sheaf of affinoid functions $\mathcal{O}_X$, because for any Qco sheaf $\widetilde{M}$, choose a free resolution of M, then use dimension shifting, notice the covering is finite(the flatness of the algebra map(II.6.2.26) is used to deduce the long exact sequence).

For the sheaf $\mathcal{O}_X$, the main tool is the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & (X-f)A\langle X\rangle \times (1-fY)A\langle Y\rangle \arrow[r, "\delta''"] \arrow[d] & (X-f)A\langle X, X^{-1}\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\varepsilon'"] \arrow[d, equal] & A\langle X\rangle \times A\langle Y\rangle \arrow[r, "\delta'"] \arrow[d] & A\langle X, X^{-1}\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\varepsilon"] & A\langle f\rangle \times A\langle f^{-1}\rangle \arrow[r, "\delta"] & A\langle f, f^{-1}\rangle \arrow[r] & 0
\end{tikzcd}$$

where δ' is given by $(h_1(X), h_2(Y)) \mapsto h_1(X) - h_2(X^{-1})$, and δ'' is induced by δ'. The columns are all exact, and the first row and the second row are exact. ε is injective by(II.6.2.25). Then the last row is also exact, by spectral sequence. □

Prop. (II.6.2.41) (Strong/Weak Topos The same). If X is an affinoid K-space, the category of sheaves w.r.t the strong Grothendieck topology is equivalent to the category of sheaves w.r.t. the weak Grothendieck topology by pushforward and pullback of sheaves by(III.1.2.13) because the strong and weak Grothendieck category satisfies the conditions.

In particular, this applies to the case $\mathcal{O}_X$ by(II.6.2.40), the resulting sheaf is called the **sheaf of rigid analytic functions** on X, also denoted by $\mathcal{O}_X$.

3 Rigid Analytic Spaces

Def. (II.6.3.1). A ***G*-ringed *K*-space** is a pair $(X, \mathcal{O}_X)$ where X is a G-topological space and $\mathcal{O}_X$ is a sheaf of K-algebras. It is called **local *G*-ringed *K*-space** if the stalks are all local rings. Their morphisms are defined routinely.

Prop. (II.6.3.2). An affinoid K-space with the sheaf of analytic functions $(X, \mathcal{O}_X)$(II.6.2.41) is an example of local G-ringed K-space(II.6.2.23). And a morphism induce a local G-ringed morphism. And all morphisms come from these.

Moreover, an affinoid K-space is a complete G-ringed K-space(i.e. rigid)(III.1.1.4).

Proof: It is a G-space by(II.6.2.40)(II.6.2.41), morphisms by(II.6.2.19), notice the $\mathfrak{m}_x$ generate the maximal ideal of $\mathcal{O}_{X,x}$(II.6.2.23), so the morphism is local.

To show all morphisms are like these, we need to show a morphism $\sigma : A \to B$ gives at most one $\mathrm{Sp}\, B \to \mathrm{Sp}\, A$: the morphism is local, so it maps $\mathfrak{m}_{\varphi(x)}$ to $\mathfrak{m}_x$, and from the commutative diagram

$$\begin{tikzcd} A/\mathfrak{m}_{\varphi(x)} \arrow[r] \arrow[d, "\cong"] & B/\mathfrak{m}_x \arrow[d, "\cong"] \\ \mathcal{O}_{\varphi(x),\mathrm{Sp}\, A}/\mathfrak{m}_{\varphi(x)}\mathcal{O}_{\varphi(x),\mathrm{Sp}\, A} \arrow[r] & \mathcal{O}_{x,\mathrm{Sp}\, B}/\mathfrak{m}_x\mathcal{O}_{x,\mathrm{Sp}\, B} \end{tikzcd}$$

(II.6.2.23) shows $\mathfrak{m}_{\varphi(x)}$ is mapped into $\mathfrak{m}_x$, so $\mathfrak{m}_{\varphi(x)} = (\sigma^*)^{-1}\mathfrak{m}_x$, which shows φ is unique set-theoretically, and on the level of structure sheaf, the uniqueness of $\mathcal{O}_{\mathrm{Sp}\, A}(V) \to \mathcal{O}_{\mathrm{Sp}\, B}(\varphi^{-1}(V))$ is unique by the definition of affinoid subdomain(II.6.2.7). □

Def. (II.6.3.3). The category of **rigid (analytic) space** is a full subcategory of local G-ringed K-spaces that it is complete G_0, G_1, G_2, and it has an admissible covering $\{X_i \to X\}$ that $(X_i, \mathcal{O}_X|_{X_i})$ are affinoid K-spaces.

It follows easily that an admissible open subset of a rigid space is again rigid.

Prop. (II.6.3.4). Glueing rigid analytic spaces is legitimate, so does glueing morphisms on the source.

Proof: First glue the set, then use(III.1.1.6) to glue G-topology, finally the glue of structure sheaf is similar to(III.1.6.10). □

Cor. (II.6.3.5) (Spectrum Adjointness). If X is rigid and Y is affinoid, then $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_X)$. This follows from(II.6.3.2) and glue(II.6.3.4).

Prop. (II.6.3.6). Fiber products exist in the category of rigid analytic space. This is fiber product of affinoid spaces are affinoid so we can glue them by universal property, the same as(III.2.7.1).

Prop. (II.6.3.7). An affinoid space is connected in the weak Grothendieck topology iff it is connected in the strong Grothendieck topology iff it is connected in the Zariski topology.

Proof: Firstly the weak and strong are equal because any strong covering of X has a refinement of weak covering, and a weak covering is a strong covering. So it suffices to prove the equivalence of the last two.

One direction is trivial, for the other direction, use Tate's acyclicity, if $X_1, X_2 \to \mathrm{Sp}\, A$, $X_1 \cap X_2 = 0$, then $A = \mathcal{O}_X(X_1) \times \mathcal{O}_X(X_2)$, so $\mathrm{Spec}\, A$ is not connected, neither do $\mathrm{Sp}\, A$. □

Prop. (II.6.3.8). We can define the connected components of X as the equivalence classes of elements that can be reached using connected admissible open subsets of X. Then the connected components are admissible and forms an admissible cover of X.

Proof: Notice that there exists a finite covering consisting of connected Zariski subsets, by(II.6.3.7) and the fact $\mathrm{Sp}\, A$ has f.m. connected components because $\mathrm{Spec}\, A$ does as A is Noetherian(II.6.1.12) and(II.6.2.3).

Thus we are done, because by(II.6.2.22), a Zariski covering is admissible, and clearly the connected components of X are just this Zariski covering. □

Rigid GAGA

Lemma (II.6.3.9). Let Z be an affine scheme algebraic over K, and Y a rigid K-space, then the set of morphisms of local G-ringed spaces $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ corresponds to K-algebra morphisms from $\mathcal{O}_Z(Z)$ to $\mathcal{O}_Y(Y)$.

Proof: Cf.[Formal and Rigid Geometry P111]. □

Def. (II.6.3.10) (Rigid Analytification). There is a partial functor X^{rig} from the category of schemes X locally algebraic over a valued field K to the category of rigid K-spaces that are right adjoint to the forgetful functor from the category of rigid K-spaces to local ringed K-space, called the **GAGA functor**.

The existence of this functor is proven in(II.6.3.14).

Def. (II.6.3.11) (Analytification of Affine Schemes). Let $T_n(r)$ be the elements $\sum a_v \zeta^v$ in T_n that $\lim a_v r^{|v|} = 0$. Then choose a $c \in K, |c| > 1$, define $T_n^{(i)} = T_n(|c|^i)$. Then $T_n^{(i)} = K\langle c^{-i}X_1, \ldots, c^{-i}X_n\rangle$, so clearly $\mathrm{Sp}(T_n^{(i)})$ is an affinoid subdomain of $\mathrm{Sp}(T_n^{(i+1)})$ by(II.6.2.9). Thus there is a chain of inclusions of affinoid subdomains:

$$B^n = \mathrm{Sp}(T_n^{(0)}) \hookrightarrow \mathrm{Sp}(T_n^{(1)}) \hookrightarrow \mathrm{Sp}(T_n^{(2)}) \hookrightarrow \ldots$$

Then we can use(II.6.3.4) to glue them together as $\mathbb{A}_K^{n,rig}$.

Prop. (II.6.3.12). The maximal spectrum $\mathrm{Max}(K[X_i]) = \cup_n \mathrm{Spa}(T_n^{(i)})$ as sets.

Proof: It suffices to show the following two.

- For any maximal ideal $\mathfrak{m} \subset T_n$, $\mathfrak{m}' = \mathfrak{m} \cap K[X_i]$ is maximal.
- For any maximal ideal $\mathfrak{m}' \subset K[X_i]$, there is some N that $\mathfrak{m}' T_n^{(i)}$ is maximal in $T_n^{(i)}$ for all $i > N$.

For 1: Consider the $K \subset K[X_i]/\mathfrak{m}' \subset T_n/\mathfrak{m}$, $T_n/\mathfrak{m}$ is a finite extension of K by(II.6.1.10), then so does $K[X_i]/\mathfrak{m}'$, by(I.5.4.3). To prove $\mathfrak{m}' = \mathfrak{m} \cap K[X_i]$, consider the following diagram:

$$\begin{array}{ccc} K[X_i]/\mathfrak{m}' & \longrightarrow & T_n/\mathfrak{m}' T_n \\ \| & & \downarrow \\ K[X_i]/\mathfrak{m}' & \longrightarrow & T_n/\mathfrak{m} \end{array}$$

As $K[X_i]/\mathfrak{m}'$ is finite over K, it is complete, but $K[X_i]$ is dense in T_n, thus the horizontal maps are surjective. But then the lower horizontal is isomorphism, then the upper horizontal is also isomorphism, and then the vertical map is isomorphism, thus we are done.

For 2, $K[X_i]/\mathfrak{m}'$ is a finite extension of K, thus has a unique valuation, let N be large that $|\overline{X}_i| \leq |c|^N$, then for $i > N$, the quotient map factors uniquely as $K[X_i] \to T_n^{(i)} \to K/\mathfrak{m}'$. Then the kernel $\mathfrak{m}$ of $T_n^{(i)}$ is a maximal ideal(same reason as before) that satisfies $\mathfrak{m} \cap K[X_i] = \mathfrak{m}'$. Then we finish by item1. □

Cor. (II.6.3.13) (Analytification for Affine Schemes). Similarly, for an affine scheme $Z = \mathrm{Spec}\, K[X_i]/\mathfrak{a}$ of f.t over K, we construct its analytification Z^{rig} as the glue of the inclusions:

$$\mathrm{Sp}(T_n^{(0)}/\mathfrak{a}) \hookrightarrow \mathrm{Sp}(T_n^{(1)}/\mathfrak{a}) \hookrightarrow \ldots$$

Then Z^{rig} is the analytification of $K[X_i]/\mathfrak{a}$.

And we see from the proof of(II.6.3.12) the maximal spectrum $\mathrm{Max}(K[X_i]/\mathfrak{a}) = \cup_n \mathrm{Spa}(T_n^{(i)}/\mathfrak{a})$ as sets.

Proof: The canonical map $K[X_i]/\mathfrak{a} \to T_n^{(i)}/\mathfrak{a}$ glue together to be a morphism $\mathcal{O}_Z(Z) \to O_{Z^{rig}}(Z^{rig})$, which by(II.6.3.9) corresponds to a map $Z^{rig} \to Z$ of local ringed spaces.

Now any other morphism $Y \to Z$ from a rigid K-space Y to Z, choose an affinoid K-space covering Y_i of Y, then the map $Y_i \to Z$ corresponds by(II.6.3.9) to a morphism $\sigma : K[X_i]/\mathfrak{a} \to \mathcal{O}_{Y_i}(Y_i)$, thus if we choose i large enough that $|\sigma(X_i)| \leq |c|^i$, then σ can be extended uniquely to

$$K[X_i]/\mathfrak{a} \to T_n^{(i)}/\mathfrak{a} \xrightarrow{\sigma} \mathcal{O}_{Y_i}(Y_i),$$

By the universality of affinoid subdomains. This σ corresponds to a morphism $Y_i \to \mathrm{Sp}(T_n^{(i)}) \to Z^{rig}$, and these clearly glue together to give a morphism $Y \to Z^{rig}$, thus proving the universal property. □

Prop. (II.6.3.14) (General Analytification). For any locally algebra scheme X over K, choose an affine covering Z_i, consider the analytification of Z_i by(II.6.3.13), then $Z_i \cap Z_j$ obviously has the inverse image as the rigid analytification by universal property, thus unique, so the analytifications of Z_i can be glued to an analytification of X.

Moreover, the underlying set of X^{rig} is identified with the closed pts of the scheme X, because this is the case of Z_i(II.6.3.11).

Prop. (II.6.3.15). Rigid analytification preserves fiber products.

Proof: This follows from the construction of fibered product of schemes(III.2.7.1), so we only need to prove the affine case. For this, Cf.[U. Köpf, Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen. Schriftenr, Satz 1.8]. □

Prop. (II.6.3.16) (Stalks). For a point $z \in Z^{rig}$, the completion of $\mathcal{O}_{Z^{rig},z}$ and $\mathcal{O}_{Z,z}$ are the same.

Proof: Cf.[U. Köpf, Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen. Schriftenr, Satz 2.1]. □

4 Coherent Sheaves on Rigid Spaces

Prop. (II.6.4.1). For an affinoid K-space X, there is a Qco module construction $M \to M \otimes_A \mathcal{O}_X$ as in(II.6.2.40) in the weak Grothendieck topology, and it extends uniquely to a sheaf w.r.t. the strong Grothendieck topology by(II.6.2.41), also denoted by $M \otimes_A \mathcal{O}_X$. This is a faithfully exact, fully faithful functor between Abelian categories from *Ab* to $\mathcal{O}_X$-modules, and it preserves tensor product and direct sums.

Proof: Because $\Gamma(X, M \otimes_A \mathcal{O}_X) = M$ and obviously fully faithful, this map is fully faithful, and it is exact because the restriction map of an affinoid subdomain is flat(II.6.2.26), and shifification is exact. □

Def. (II.6.4.2) (Coherent Sheaves). For an $\mathcal{O}_X$-module $\mathcal{F}$ on a rigid space X, **finite type, of finite presentation, coherence** are defined w.r.t the strong topology as X is a ringed site. All these notions are stable under passing to an admissible open subspaces.

Proof: For the passing of coherence to admissible open subspaces, use the fact that restriction maps are flat(II.6.2.26). □

Cor. (II.6.4.3). Notice $\mathcal{O}_X^n = A^n \otimes_A \mathcal{O}_X$, by(II.6.4.1) and the fact A is Noetherian, passing to a refinement covering, $\mathcal{F}$ is coherent iff there is an admissible affinoid covering $\mathfrak{U} : X_i \to X$ that $F|_{X_i}$ is associated to a finite $\mathcal{O}_{X_i}$-module. In this case, $\mathcal{F}$ is said to be **$\mathfrak{U}$-coherent**. Thus the coherent sheaves form a weak Serre subcategory of $\mathcal{O}_X$-modules.

In particular, $\mathcal{O}_X$ is coherent.

Prop. (II.6.4.4). If $\mathcal{F}, \mathcal{G}$ are all $\mathfrak{U}$-coherent modules, then:

- $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{F} \oplus \mathcal{G}$ are $\mathfrak{U}$-coherent,
- if $\mathcal{F} \to \mathcal{G}$ is a $\mathcal{O}_X$-module morphism, then the kernel and image are all $\mathfrak{U}$-coherent.
- If $\mathcal{I}$ is a $\mathfrak{U}$-coherent sheaf of ideal of $\mathcal{O}_X$, then $\mathcal{I}\mathcal{F}$ is $\mathfrak{U}$-coherent.

Proof: The first and the second are consequences of(II.6.4.1), noticing A_i is Noetherian. The third is an image of $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$. □

Lemma (II.6.4.5). If $\mathcal{F}$ is $\mathfrak{U}$ coherent for a simple Laurent covering $\mathfrak{U}$, then $H^1(\mathfrak{U}, \mathcal{F}) = 0$.

Proof: The goal is to show any element in $\mathcal{F}(U_1 \cap U_2)$ can be represented by $u_1 + u_2$, where $u_i \in \mathcal{F}(U_i)$. Let $U_1 = \mathrm{Sp}\, A\langle f\rangle, U_2 = \mathrm{Sp}\, A\langle f^{-1}\rangle, U_1 \cap U_2 = \mathrm{Sp}\, A\langle f, f^{-1}\rangle$. Now $A\langle f\rangle = A\langle X\rangle/(X - f), A\langle f^{-}\rangle = A\langle Y\rangle/(Yf-1), A\langle f, f^{-1}\rangle = A\langle X, Y\rangle/(X - f, Yf - 1)$, and we endow them with the residue norm.

Now we want to give norms to $M_1 = \mathcal{F}(U_1), M_2 = \mathcal{F}(U_2), M_{12} = \mathcal{F}(U_1 \cap U_2)$. M_i are finite $\mathcal{O}_X(U_i)$-modules, so there are elements $v_i, w_j, i \leq m, j \leq n$ that generate M_1, M_2 respectively. So there are attached morphisms

$$(A\langle f\rangle)^m \to M_1, \quad (A\langle f^{-1}\rangle)^n \to M_2, \quad (A\langle f, f^{-1}\rangle)^m \to M_{12}$$

And endow them with the residue norm, which is complete..

Notice that to prove the assertion, it suffice to show for each $\varepsilon > 0$, there is an α that for each $u \in_{12}$, there are u_1 and u_2 in M_i respectively that $|u_i| < \alpha|u|$ and $|u - u_1 - u_2| < \varepsilon|u|$, because then we can use iteration and completeness to get the result.

Giving $\beta > 1$, any $g \in A\langle f, f^{-1}\rangle$ can be lifted to an element $\sum c_{ij} X^i Y^j$ that $|c_{ij}| \leq \beta|g|$. Then by regrouping terms that $i \geq j$ or $i < j$, there are two element $g^+ \in A\langle f\rangle$ and $g^- \in A\langle f^{-1}\rangle$ that $g^+ + g^-$ restricts to g on $U_1 \cap U_2$, and $|g^+|, |g^-| \leq \beta|g|$.

Now that $\mathcal{F}$ is coherent, so v_i and w_j both generate M_{12} separately. Then there are equations $v_i = \sum c_{ij} w_j$ and $w_i = \sum d_{ij} v_j$, where $c_{ij}, d_{ij} \in A\langle f, f^{-1}\rangle$. The image of $A\langle f\rangle$ is dense in $A\langle f, f^{-1}\rangle$(II.6.1.25), so there are elements $c'_{ij} \in A\langle f^{-1}\rangle$ s.t. $\max_{ijl} |c_{ij} - c'_{ij}||d_{jl}| < \beta^{-2}\varepsilon$.

Now I claim the above approximation process is true for $\alpha = \beta^2 \max(|c'_{ij}| + 1)$. For this, notice for any $u = \sum a_i v_i$ with $a_i \in A\langle f, f^{-1}\rangle$, which we may assume $|a_i| \leq \beta|u|$ by the definition of the norm on M_{12}, then $a_i = a_i^+ + a_i^-$, that $|a_i^*| \leq \beta|a_i|$. Consider the following element

$$u^+ = \sum a_i^+ v_i \in M_1, \quad u^- = \sum a_i^- \sum c'_{ij} w_j \in M_2$$

Then it is easily verified that $|u^*| < \alpha|u|$, and

$$u - u^- - u^+ = \sum\sum a_i^-(c_{ij} - c'_{ij}) w_j = \sum\sum\sum a_i^-(c_{ij} - c'_{ij}) d_{jl} v_l.$$

which has norm smaller than $\max |a_i^-(c_{ij} - c'_{ij}) d_{jl}| \leq \beta^2|u| \cdot \beta^{-2}\varepsilon = \varepsilon|u|$, finishing the proof. □

Prop. (II.6.4.6) (Kiehl). An $\mathcal{O}_X$-module $\mathcal{F}$ on an affinoid K-space $\operatorname{Sp} A$ is coherent iff it is associated to a finite A-module.

Proof: The converse is obvious, for the other direction, by(II.6.2.39), it suffices to prove for $\mathfrak{U}$ a Laurent covering, and further, it suffices to prove for the simplest Laurent covering $X(f), X(f^{-1}) \to X$ because: $U(f,g)\cup U(f,g^{-1})\cup U(f^{-1},g)\cup U(f^{-1},g^{-1}) = (U(f,g)\cup U(f,g^{-1}))\cup(U(f^{-1},g)\cup U(f^{-1},g^{-1}))$.

Thus the above lemma shows that $H^1(\mathfrak{U},\mathcal{F})=0$. Now I prove that for any finite affinoid covering $\mathfrak{U}=\cup \operatorname{Sp} A_i$, if $H^1(\mathfrak{U},\mathcal{F})=0$ for any coherent sheaf $\mathcal{F}$, then any $\mathfrak{U}$-coherent sheaf $\mathcal{F}$ is associated to a finite A-module, this will finish the proof.

Consider any maximal ideal $\mathfrak{m}_x$ of A, $\mathfrak{m}_x\otimes_A \mathcal{O}_X$ is a coherent sheaf as $\mathfrak{m}_x$ is finite because A is Noetherian, so there is a short exact sequence

$$0\to \mathfrak{m}_x\mathcal{F}\to\mathcal{F}\to\mathcal{F}/\mathfrak{m}_x\mathcal{F}\to 0$$

of $\mathfrak{U}$-coherent sheaves, because $A/\mathfrak{m}_x$ is a field, thus flat.

Now for any affinoid space U' in U_i for some i, the section of this exact sequence is exact, because the ring morphism associated to an affinoid subdomain is flat(II.6.2.26). In particular, this can be applied to any intersections of U_i, in particular the Čech complex of these sheaves. Then the long exact sequence and the fact $H^1(\mathfrak{U},\mathfrak{m}_x\mathcal{F})=0$ shows

$$0\to \mathfrak{m}_x\mathcal{F}(X)\to\mathcal{F}(X)\to\mathcal{F}/\mathfrak{m}_x\mathcal{F}(X)\to 0$$

Next we want to show $\mathcal{F}/\mathfrak{m}_x\mathcal{F}(X)\to\mathcal{F}/\mathfrak{m}_x\mathcal{F}(U_i)$ is isomorphism for any $x\in U_i$. To prove this, first for any affinoid subspace $U'=\operatorname{Sp} B$ contained in some U_j, let $U'\cap U_i=\operatorname{Sp} B_i$, $\mathcal{F}|_{U'}=M'\otimes_A\mathcal{O}_{U'}$, we show $\mathcal{F}/\mathfrak{m}_x\mathcal{F}(U')\cong\mathcal{F}/\mathfrak{m}_x\mathcal{F}(U'\cap U_i)$, this is equivalent to

$$M'/\mathfrak{m}_xM'\to M'/\mathfrak{m}_xM'\otimes_B B_i = M'/\mathfrak{m}_xM'\otimes_{B/\mathfrak{m}_xB} B_j/\mathfrak{m}_xB_j$$

is an isomorphism. But $B/\mathfrak{m}_xB\cong B_j/\mathfrak{m}_xB_j$: This is true when $x\in U'$ by(II.6.2.8), and they are both trivial ring if $x\notin U'$. Then look at the morphism of Čech complex induced by $\mathcal{F}/\mathfrak{m}_x\mathcal{F}\to\mathcal{F}/\mathfrak{m}_x\mathcal{F}|_{U_i}$, then it is an isomorphism, by what we just proved, so its H^0 is also isomorphism, which is $\mathcal{F}/\mathfrak{m}_x\mathcal{F}(X)\cong\mathcal{F}/\mathfrak{m}_x\mathcal{F}(U_i)$.

Finally, by the commutative diagram $\begin{array}{ccc}\mathcal{F}(X) & \longrightarrow & \mathcal{F}/\mathfrak{m}_x\mathcal{F}(X)\\ \downarrow & & \downarrow\\ \mathcal{F}(U_i) & \longrightarrow & \mathcal{F}/\mathfrak{m}_x\mathcal{F}(U_i)\end{array}$, the right vertical arrow is isomorphism, so if denote $\mathcal{F}(U_i)$ by M_i, then $\mathcal{F}(X)$ generate $M_i/\mathfrak{m}_xM_i$ for every x, then consider $L=M_i/\mathcal{F}(X)$, then $\mathfrak{m}_xL=L$ for every x, then by Nakayama, for each maximal ideal $\mathfrak{m}$, there is a $m\in\mathfrak{m}$ that $(1+m)L=0$, so $Ann(L)=(1)$, so $L=0$, i.e. $\mathcal{F}(X)$ generate M_i for each i.

Now choose f_i in $\mathcal{F}(X)$ that generate M_i simultaneously, then the map $\mathcal{O}_X^n\to\mathcal{F}$ is a surjection of $\mathfrak{U}$-coherent sheaves, its kernel $\mathcal{G}$ is also coherent by(II.6.4.4), now all the above argument works for $\mathcal{G}$, so there is a surjection $\mathcal{O}_X^m\to\mathcal{G}$, so $\mathcal{O}_X^m\to\mathcal{O}_X^n\to\mathcal{F}\to 0$, so $\mathcal{F}$ is associated to the cokernel of the map $A^m\to A^n$. □

Cor. (II.6.4.7). Coherence for a $\mathcal{O}_X$-module on a rigid K-space is affinoid local on the target.

Cohomology on Rigid Analytic Spaces

Lemma (II.6.4.8). The category of $\mathcal{O}_X$-modules on a rigid K-space is a Grothendieck category by(I.8.2.27).

Def. (II.6.4.9) (Derived Cohomologies). Consider the right derived functor for Γ and more general f_*, these are left exact by(III.1.2.5). Then $R^p f_*\mathcal{F} = (f_*\mathcal{H}^p(\mathcal{F}))^\sharp$ by Grothendieck spectral sequence.

The Cech-to-Derived spectral sequence(III.5.2.10) is applied: $H^p(\{U_i \to U\}, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(U,F)$, $\check{H}^p(U,\mathcal{H}^q(F)) \Rightarrow H^{p+q}(U,F)$ and $\check{H}^1(U,F) \cong H^1(U,F)$.

In particular, if $H^q(U_{i_0i_1\ldots i_r},\mathcal{F}) = 0, q>0$, then $H^p(\{U_i \to U\},\mathcal{F}) = H^p(U,\mathcal{F})$(III.5.2.12). And it is enough to have $\check{H}^q(U_{i_0i_1\ldots i_r},\mathcal{F}) = 0, q>0$ by(III.5.2.14).

Cor. (II.6.4.10). A Qco sheaf on an affinoid space has vanishing higher sheaf cohomology by Tate's acyclicity(II.6.2.40) and(III.5.2.14).

Properties of Rigid K-Spaces

This subsubsection is strongly suggested to read after reading the parallel part of schemes.

Def. (II.6.4.11). A morphism is called a **closed immersion** if there is an admissible affinoid covering that it restricts to a closed immersion of affinoid spaces(It is compatible with definition(II.6.2.28) before by(II.6.4.15)). It is called an **open immersion** iff it is injective and the corresponding stalk maps are isomorphisms. The **(quasi-)separatedness, quasi-compactness, finiteness** are defined similarly as for schemes.

Lemma (II.6.4.12) (Nike's Trick). In a rigid analytic K-space X and $\operatorname{Sp} A$, $\operatorname{Sp} B$ be affinoid subspaces, then there is an admissible affinoid covering of $\operatorname{Sp} A \cap \operatorname{Sp} B$.

Proof: This is analogous to the scheme case(III.3.1.1), but the proof is different: X has an admissible covering, this restricts to an admissible covering of $\operatorname{Sp} A \cap \operatorname{Sp} B$, and any admissible covering can be refined by an affinoid admissible covering. □

Prop. (II.6.4.13) (Affinoid Communication Theorem). A property P of affinoid open subsets of X is called **affiniod local** if: $\operatorname{Sp} A$ has $P \Rightarrow$ all affinoid subdomains of $\operatorname{Sp} A$ has P, and any admissible affinoid cover of $\operatorname{Sp} A$ has $P \Rightarrow \operatorname{Sp} A$ has P. Notice a stalk-wise property is obviously affine-local.

Now if we call X has $\widetilde{P}$ if there is an admissible affinoid covering $A_i \to X$ that A_i has P. Then the following are equivalent:

- all open affinoid subsets of X has P.
- all open subspace of X has $\widetilde{P}$.
- X has a cover of open subspaces that has $\widetilde{P}$.
- X has $\widetilde{P}$.

Proof: The proof is the same as the scheme case(III.3.1.2). □

Prop. (II.6.4.14). Separated morphism is quasi-separated because closed immersion is affinoid hence quasi-compact(II.6.2.3).

Prop. (II.6.4.15) (Finite Morphism). For a morphism $\varphi : X \to Y$ of rigid K-spaces

- It is finite iff the inverse image of any affinoid space is affinoid, and $\varphi_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module. In particular, finiteness is local on the target because coherence do.
- It is closed immersion iff it is finite and $\mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ is surjective, this shows the definition of closed immersion is compatible with before.

Proof: Coherence is affinoid local on the target by Kiehl's theorem, so it suffices to prove the inverse image of any affinoid space is affinoid for a finite morphism: Consider any affinoid subdomain $U \subset X$ with inverse image $\varphi^{-1}(U)$, by Kiehl's theorem, $B = \mathcal{O}_X(f^{-1}(U))$ is finite over $A = \mathcal{O}_Y(U)$, thus can be given an affinoid K-algebra structure(II.6.1.32). Now

$$\varphi^{-1}(U) \xrightarrow{\chi} \operatorname{Sp} B \xrightarrow{\rho'} \operatorname{Sp} A$$

χ is locally an isomorphism, as ρ is finite, so χ is an isomorphism.

The second assertion is because locally $\mathcal{O}_Y, \varphi_*\mathcal{O}_X$ are both Qco so surjectivity is equivalent to the global section is surjective(II.6.4.1). □

Prop. (II.6.4.16). Closed/Open immersion, quasi-compactness, (quasi-)separatedness are all local on the target, and stable under base change.

Proof: Closed immersion is local on the target because finiteness do and surjectiveness of $\mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ is checked locally. Open immersion is local on the target because stalk and injectivity are all checked locally.

Then Closed/Open immersion are stable under base change because the affinoid case is true(II.6.2.29).

Quasi-compact is easily seen local on the target and stable under base change.

(Quasi-)Separateness is local on the target because closed immersion and quasi-compact do.

(Quasi-)Separateness is stable under base change because closed immersion and quasi-compact do, because diagonal commutes with base change(I.8.1.27). □

Prop. (II.6.4.17). Morphisms between affinoid K-spaces are separated. Moreover, because of localness, any finite morphism is separated.

Proof: The diagonal is $\operatorname{Sp} A \to \operatorname{Sp} A\hat{\otimes}_B A$, whose ring map is surjective. □

Prop. (II.6.4.18). By(I.8.1.29), for $X \to S$ and $Y \to S$, the map $X = X \times_Y Y \to X \times_S Y$ is an immersion. It is closed immersion if $Y \to S$ is separated, and it is qc if $Y \to S$ is quasi-separated.

Cor. (II.6.4.19). If $s : S \to X$ is a section of $f : X \to S$, the above proposition applies to this case, because $S = S \times_X X \to S \times_S X = X$.

Prop. (II.6.4.20). A morphism is quasi-separated iff there is an admissible affinoid covering W_i that, for any two affinoid open U, V that are mapped to an affinoid open, their intersection is quasi-compact. This is because quasi-compact is local on the target.

A morphism is separated iff there is an admissible affinoid covering W_i that, for any two affinoid open U, V that are mapped to an affinoid open, their intersection is affinoid open and $\mathcal{O}(U)\hat{\otimes}_{\mathcal{O}(W_i)}\mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is surjective. This is because closed immersion is local on the target(II.6.4.16).

Cor. (II.6.4.21). If $g \circ f$ is (quasi-)separated, then so is f.

Cor. (II.6.4.22). If X is (quasi-)separated, then $X \to Y$ is (quasi-)separated.

Proper Mapping Theorem

Def. (II.6.4.23). For a rigid space X over affinoid space Y, if $U \subset U' \subset X$ be affinoid subspaces, U is called **relatively compact** in U' iff there is a set of affinoid generators f_i of $\mathcal{O}_X(U')$ over $\mathcal{O}_Y(Y)$ that $|f_i(x)| < 1$ on U. This is denoted by $U \Subset_Y U'$.

Prop. (II.6.4.24). If X_1, X_2 are affinoid spaces over an affinoid space Y, and U_i are affinoid space of X_i, then

- if $U_1 \Subset_Y X_1$, then $U_1 \times_Y X_2 \Subset_{X_2} X_1 \times_Y X_2$.
- if $U_i \Subset_Y X_i$, then $U_1 \times_Y U_2 \Subset_Y X_1 \times_Y X_2$.
- If $U_i \Subset_Y X_i$, and X_i are affinoid subspaces of a rigid space separable over Y, then $U_1 \cap U_2 \Subset_Y X_1 \cap X_2$.
- If $U_1 \Subset_Y X_1$, and $i : T \to X_1$ is a closed immersion, then $i^{-1}(U_1) \Subset_Y i^{-1}(X_1)$.

The proof is easy. For the last one, should notice $|f(x)| = |f(i(x))|$, because it is closed immersion, so the residue field is the same.

Def. (II.6.4.25) (Proper Morphism). A **proper** morphism $\varphi : X \to Y$ of rigid K-spaces is a separated morphism that there is an admissible affinoid covering Y_i of Y that there are two admissible affinoid coverings X_{ij}, X'_{ij} of $\varphi^{-1}(Y_i)$ that $X_{ij} \Subset_{Y_i} X'_{ij}$ for any i, j.

Prop. (II.6.4.26). Properness is stable under base change and composition

Proof: The base change follows directly from(II.6.4.24).
For the composition, Cf.[Formal and Rigid Geometry P131](difficult). □

Prop. (II.6.4.27). Properness is local on the target.

Proof: This is because separatedness is local on the target(II.6.4.16) and the second condition of properness is itself local. □

Prop. (II.6.4.28). If $g \circ f : X \to Y \to Z$ is proper and g is separated, then f is proper.

Proof: By(II.6.4.18), $\tau : X \to X \times_Z Y$ is closed immersion, and f is separated by(II.6.4.21). Now proper is local, so we may assume Z is affinoid, so there are two admissible covering X_i, X'_i of X that $X_i \Subset_Z X'_i$, and choose an admissible affinoid covering $Y_i \to Y$, then $X_j \times_Z Y_i, X'_j \times_Z Y_i$ are admissible coverings of Y_i that is $X_j \times_Z Y_i \Subset_{Y_i} X'_j \times_Z Y_i$. And it can be pulled back to an affinoid admissible coverings of $f^{-1}(Y_i)$ that $\tau^{-1}(X_j \times_Z Y_i) \Subset_{Y_i} \tau^{-1}(X'_j \times_Z Y_i)$, because τ is closed immersion. So $X \to Y$ is proper. □

Prop. (II.6.4.29). Finite morphism is proper, in particular, closed immersion is proper.

Proof: Finite morphism is separated by(II.6.4.17), and locally, assume both space are affinoid, $X = \operatorname{Sp} B \to \operatorname{Sp} A = Y$, then B is a finite A-module, in priori a f,g, A-algebra, so there is a set of generators f_i of B over A that (by multiplying a constant in K^*) $|f_i|_{sup} < 1$(II.6.1.21), so $X \Subset_Y X$, hence it is proper. □

Prop. (II.6.4.30) (Proper and Analytification). For a morphism between schemes locally of f.t. over K, it is proper iff its rigid analytification is proper.

Proof: Cf.[U. Köpf, Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen. Schriftenr. Math. Inst. Univ. Münster, 2. Serie, Heft 7 (1974) Satz 2.16]. □

For the following: Cf.[Formal and Rigid Geometry P132].

Prop. (II.6.4.31) (Proper Mapping theorem Kiehl). The higher direct images of a proper map of rigid analytic spaces takes coherent sheaves to coherent sheaves.

Prop. (II.6.4.32). For a scheme X locally of f.t. over K, an $\mathcal{O}_X$-module $\mathcal{F}$ on X gives rise to an $\mathcal{O}_{X^{rig}}$-module on X^{rig}, and it is coherent iff $\mathcal{F}$ is coherent.

Prop. (II.6.4.33). For a proper scheme over K, $H^q(X,\mathcal{F}) \cong H^q(X^{rig},\mathcal{F}^{rig})$ for $\mathcal{F}$ coherent.

Prop. (II.6.4.34). When X is proper, coherent sheaves on X^{rig} corresponds to coherent sheaves on X. This gives an analog of Chow's theorem when applied to $X = \mathbb{P}^n_K$ and $\mathcal{F}'$ is a sheaf of ideal in $\mathcal{O}_{X^{rig}}$.

5 Formal Geometry

II.7 Adic Space and Perfectoid Space

Basic References are [Adic Spaces Morel], [Lecture notes for a class on Perfectoid Spaces Bhatt] and [Perfectoid Spaces Scholze].

1 Perfection and Tilting

Remark (II.7.1.1). The basis setting is a non-Archimedean complete valued field K, and denote $K^0=\{x\in K||x|\le 1\}$, and $K^{00}=\{x\in K||x|<1\}$, which are rings because K is non-Archimedean. $k=K^0/K^{00}$ is the residue field of K. A non-zero element of K^{00} is called a **pseudo-uniformizer**.

K^{00} is just the set of topological nilpotent elements of K.

Prop. (II.7.1.2). The valuation can in fact be constructed from K^0 as $|x|=\sup\{\frac{n}{k}|x^k\in t^nK^0\}$ by(V.3.1.3), as it is a rank 1 valuation.

Def. (II.7.1.3). A ring of characteristic p is called **perfect** iff the Frobenius $\mathrm{Frob}/\varphi:P\to P$ is an isomorphism. It is called **semi-perfect** iff Frob is surjective.

Def. (II.7.1.4) (Perfection and Tilting). If R is of char p, we define $R_{perf}=\varinjlim_\varphi R$ and $R^{perf}=\varprojlim_\varphi R$.

For any ring R, let $R^\flat=(R/p)^{perf}$, endowed with the profinite topology.

Prop. (II.7.1.5) (Universal Property). The $(\cdot)_{perf}$ and $(\cdot)^{perf}$ are respectively the left and right adjoint to the forgetful functor from the category of perfect rings to the category of rings of characteristic p.

Proof: First both R_{perf} and R^{perf} are perfect: for R_{perf}, every element in R_{perf} is represented by an element $a_n\in R_n$, and this element is equivalent to $a_n^p\in R_{n+1}$, so its p-th root is $a_n\in R_{n+1}$. For R^{perf}, an element $(\ldots,x_n,\ldots,x_0)$ has p-th root $(\ldots,x_{n+1},\ldots,x_1)$.

Second it is easily checked to be a functor because Frob is natural. The universal property is easy. □

Prop. (II.7.1.6) (Perfection Kills Nilextensions). If $f:R\to S$ is a map of rings of characteristic p that is surjective with nilpotent kernel, then $R_{perf}\to S_{perf}$ and $R^{perf}\to S^{perf}$ are both isomorphisms.

Proof: $-_{perf}$ map is clearly surjective, and it is injective because if a maps to 0, then $\mathrm{Frob}^k(a)\in\mathrm{Ker}\,f$ for some k, so it is nilpotent, so $\mathrm{Frob}^{k+n}(a)=0$.

$-^{perf}$ is clearly injective, and it is surjective because: suppose $\mathrm{Ker}\,f^n=0$, then for a $(s_n)\in S$, let t_m be the inverse image of s_{mn}, for each m, and let $x=(x_n),x_{mn-k}=\mathrm{Frob}^kt_m$, then $(x)\in R^{perf}$ and x maps to s. □

Prop. (II.7.1.7). If R is a f.g. algebra over an alg.closed field k of char$p$, then $R^\flat\cong k^{\pi_0(\mathrm{Spec}\,R)}$.

Proof: It suffice to prove for $\mathrm{Spec}\,R$ connected and reduced, because by(II.7.1.6). We first prove the case $\mathrm{Spec}\,R$ is irreducible, i.e. R is integral:

In this case, choose a closed point x, then there is a map $R\to\widehat{R_x}$, where $\widehat{R_x}$ is the $\mathfrak{m}_x$-adic completion. By Krull(I.5.5.6) and the fact R is integral, this map is injective, so it suffices to show that $0=(\widehat{R_x})^{perf}=\varinjlim(R_x/\mathfrak{m}_x^n)^{perf}$. But $(-)^{perf}$ is a right adjoint so commutes with colimits and $\widehat{R_x}=\varinjlim R_x/\mathfrak{m}_x^n$. But $(R_x/\mathfrak{m}_x^n)^{perf}=(R_x/\mathfrak{m}_x)^{perf}=k^{perf}=k$, by(II.7.1.6) again.

If R is not irreducible, **?** □

Prop. (II.7.1.8) (Examples of Perfectoid Field).

- $\mathbb{F}_p[t]_{perf} = \mathbb{F}_p[t^{\frac{1}{p^\infty}}]$, $\mathbb{F}_p[t]^\flat = \mathbb{F}_p[t]^{perf} = \mathbb{F}_p$.
- $(\mathbb{Z}_p)^\flat = \mathbb{F}_p$.
- If R is a perfect ring of char$p$ and $f \in R$ is a non-zerodivisor, then $(R/f)^{perf}$ is the f-adic completion of R. In particular, $(\mathbb{F}_p[t^{\frac{1}{p^\infty}}]/(t))^{perf} \cong \widehat{\mathbb{F}_p[t^{\frac{1}{p^\infty}}]}$.
- $(\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]})^{perf} \cong \widehat{\mathbb{F}_p[t^{\frac{1}{p^\infty}}]} \cong \widehat{\mathbb{F}_p[t]_{perf}} \cong (\mathbb{F}_p[t]_{perf}/(t))^{perf}$.

Proof: The first two are trivial, for the third, notice $\widehat{R_f} = \varprojlim_n R/f^n = \varprojlim_n R/f^{p^n}$, and there are commutative diagram

$$\begin{array}{ccc} R/f & \xrightarrow{\varphi} & R/f \\ \downarrow{\scriptstyle \varphi^{k+1}} & & \downarrow{\scriptstyle \varphi^{k}} \\ R/f^{p^{k+1}} & \xrightarrow{i} & R/f^{p^k} \end{array}$$

, so $\varprojlim_n R/f^{p^n} \cong (R/f)^{perf}$.

For the fourth, only the first equivalence needs proving, the others are consequences of the first three items. Then notice

$$(\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]})^\flat \cong \varprojlim (\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}/p^k)^{perf} \cong (\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}/p)^{perf} \cong (\mathbb{F}_p[t^{\frac{1}{p^\infty}}]/t)^{perf} \cong \widehat{\mathbb{F}_p[t^{\frac{1}{p^\infty}}]}$$

The last isomorphism by item3. □

Prop. (II.7.1.9). If R is a p-adically complete ring (char0), then the map $R \to R/p$ induces an homeomorphism of monoids:

$$\varprojlim_{x \to x^p} R \cong \lim_\varphi R/p = R^\flat$$

Proof: Injectivity: if $(a_n), (b_n) \in \lim_{x \to x^p} R$ satisfies $a_n \equiv b_n \mod p$ for all n, then applying power lifting(IX.2.2.4), $a_n \cong b_n \mod p^{n+k}$ for all k, so $a_n = b_n$.

Surjectivity: for $(\overline{a_n}) \in R^\flat$, choose arbitrary lifting a_n, then $a_{n+k+1}^p \equiv a_{n+k} \mod p$ for all $n+k$, so $k \mapsto a_{n+k}^{p^k}$ is a Cauchy sequence by power lifting(IX.2.2.4) again, thus converging to some point b_n. then it's easily checked that $b_{n+1}^p = (\lim a_{n+1+k}^{p^k})^p = \lim a_{n+1+k}^{p^{k+1}} = b_n$. so (b_n) maps to $(\overline{a_n})$.

For the topology: it is clearly continuous, and for the reverse, if $(a_i), (b_i)$ satisfies that $a_i \equiv b_i \mod p$ for $i < k$, then the image in $\varprojlim_{x \to x^p} R$ satisfies $x_i \equiv y_i \mod p^{k-i}$ for $i < k$, thus it is open. □

Cor. (II.7.1.10) (Sharp Map). From this proposition, we get a **sharp map**: $\sharp : R^\flat \to R$, and its image is just the elements that has a compatible system of p^k-th roots $x^{\frac{1}{p^k}}$. These elements are also called **perfect**.

Cor. (II.7.1.11). By(II.7.1.6), we in fact have $\varprojlim_{x \to x^p} R \cong \lim_\varphi R/\pi = R^\flat$ for any $|p| \le |\pi| < 1$.

Cor. (II.7.1.12) (Addition in $R^\flat$). From the isomorphism(II.7.1.9) above, we can read what the addition looks like in the presentation $\varprojlim_{x \to x^p} R$: if $(f_n), (g_n)$ are two elements, then their addition is given by (h_n), where $h_n = \lim_k (f_{n+k} + g_{n+k})^{p^k}$.

Prop. (II.7.1.13) (Tilting as a Valuation Ring). If R is a domain or a valuation ring, then the same is true for $R^\flat$. In the valuation case, the valuation of $R^\flat$ can in fact be chosen to be $|\cdot| \circ \sharp$, so in particular, the rank of $R^\flat$ is no more than the rank of R.

Proof: Use the isomorphism $\varprojlim_{x\to x^p} R \cong \lim_{\varphi} R/p = R^\flat$(II.7.1.9).

For the domain case, if $(a_n)(b_n) = 0$, then $a_n b_n = 0$, so $a_0 = 0$ or $b_0 = 0$, so $(a_n) = 0$ or $(b_n) = 0$. Similarly, if R is a valuation ring, then $R^\flat$ is firstly a domain, and it suffices to prove that for any $(a_n), (b_n) \in R^\flat$, the quotient of one by another is in $R^\flat$, by(V.3.2.3). For this, because R is valuation ring, we may assume $a_0/b_0 \in R$, so a_n/b_n is also in R, because their power do, and R is normal(V.3.2.6), thus $(a_n)/(b_n) \in R^\flat$.

For the valuation given, notice in the above proof, $|(a_n)| \leq |(b_n)|$ iff $|a_0| \leq |b_0|$, so the valuation are equivalent to $|\cdot| \circ \sharp$ by(V.3.2.16), so it can be chosen to be so. □

2 Perfectoid Fields

Def. (II.7.2.1) (Perfectoid Field). A **perfectoid field** is a non-Archimedean complete field with residue field of char$p$ s.t.:

- The value group $|K^*| \subset R^*$ is not discrete.
- K^0/p is semi-perfect.

Prop. (II.7.2.2) (Examples of Perfectoid Fields).

- $K = \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$. Its valuation ring K^0 is $\widehat{\mathbb{Z}_p(p^{\frac{1}{p^\infty}})}$, and $K^0/p \cong \widehat{\mathbb{F}_p(t^{\frac{1}{p^\infty}})}/(t)$, which is clearly semi-perfect. And its value group is $\mathbb{Z}[p^{-1}]$.
- $K = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$, its value group is $\mathbb{Q}$, and K is alg.closed, so K^0 is clearly perfect.
- if K is a non-Archimedean field of char$p$, then K is a perfectoid field iff K is perfect: if K is perfect, then it is clearly perfectoid, and the semi-perfectness of K^0 implies its perfectness, so also K is perfect(multiply by a p-power of an element in K^{00}).
- If K is a perfectoid field and $|p| \leq |\pi| < 1$ is a pseudo-uniformizer, then K/π is perfect hence perfectoid.

Prop. (II.7.2.3). If K is a perfectoid field, then

- $|K^*|$ is p-divisible.
- $(K^{00})^2 = K^{00}$, and K^{00} is flat.
- K^0 is not Noetherian.

Proof: $1:$ First if $|p| < |x| \leq 1$, we show $|x|$ is p-divisible: there is a $y, z \in K^0$ that $y^p = x + pz$, so $|y|^p = |x|$. Now because $|K^*|$ is not discrete, so there is a $|x| \notin |p|^{\mathbb{Z}}$, by rescaling, we may assume $|p| < |x| \leq 1$, thus $p = xy$ for some y, and $|p| < |y| \leq 1$, too. So $|p|$ is also divisible by p, so it is clear now $|K^*|$ is divisible by p.

$2:$ for $f \in K^{00}$, by perfectoidness, $f = g^p + ph$ for some $g \in K^{00}$ and $h \in K^0$, 1 shows $p \in (K^{00})^2$(as $p = ua^p$ with $|u| = 1, a \in K^{00}$), so $f \in (K^{00})^2$. The flatness of K^{00} is because any torsion-free module over a valuation ring is flat(I.7.1.8).

$2 \to 3$ by Nakayama's lemma, because otherwise $K^{00} = 0$. □

Cor. (II.7.2.4). The proof of 1 also shows that $|K^*|$ is generated by $|x|$ that $|p| < |x| < 1$.

Lemma (II.7.2.5). If $C^\flat$ is a perfectoid space of char$p$, then $1 + \mathfrak{m}_{C^\flat}$ is a $\mathbb{Q}_p$-algebra.

Proof: Both φ and exponentiation of $\mathbb{Z}_p^*$ is definable, so $p^n t \cdot (1+x) = (\varphi^n(1+x))^k$. □

Tilting

Prop. (II.7.2.6). Fix a **pseudo-uniformizer** $|p| \le |\pi| < 1$, we let $K^{0\flat} = \lim_\varphi K^0/\pi$, then by(II.7.1.11), the group together with the topology doesn't depends on π chosen.

Remark (II.7.2.7). There are diagrams:

$$\begin{tikzcd} \lim_{x\to x^p} K^0 \arrow[r] \arrow[d, "\cong"] & K^0 \arrow[d] \\ K^{0\flat} = \lim_\varphi K^0/p \arrow[r] \arrow[ur, "\sharp"] \arrow[d, "\cong"] & K^0/p \\ \lim_\varphi K^0/\pi \arrow[r] & K^0/\pi \arrow[from=u, crossing over] \end{tikzcd}.$$

Prop. (II.7.2.8) (Tilting of K^0). Let $K^{0\flat}$ be given as in(II.7.2.6), then there is an element $t \in K^{0\flat}$ that $|t^\sharp| = |\pi|$, and t maps into (π) and gives an isomorphism $K^{0\flat}/t \cong K^0/\pi$.

Moreover, the t-adic topology on $K^{0\flat}$ is complete, and coincides with the topology of $K^{0\flat}$ given as in(II.7.1.4).

Proof: There are canonical surjective map $K^{0\flat} \to K^0/p \to K^0/\pi$, and by p-divisibility of the value group(II.7.2.3), there is a $f \in K^0$ that $|f|^p = |\pi|$, so in particular $|f| > |\pi|$, thus $f \neq 0 \in K^0/\pi$. and choose a $g \in K^{0\flat}$ lifting f, then $g^\sharp \equiv f \mod \pi$, see diagram(II.7.2.7). so $|g^\sharp| = |f|$ as $|f| > |\pi|$. Now let $t = g^p$, then $|t^\sharp| = |f|^p = |\pi|$.

Now clearly t maps into (π), and if g maps to 0 in K^0/π, then by the diagram again, $g^\sharp \in (\pi)$, and $(t^\sharp) = (\pi)$, so $g^\sharp = at^\sharp$ for some $a \in K^0$. so $t|g$ in $K^{0\flat}$, as by(II.7.1.13), $R^{0\flat}$ is a valuation ring in the valuation $|\cdot| \circ \sharp$.

For the last assertion, just use the commutative diagrams:
$$\begin{tikzcd} K^{0\flat}/(t^{p^n}) \arrow[r] \arrow[d] & K^{0\flat}/(t^{p^{n-1}}) \arrow[d] \\ K^0/(\pi) \arrow[r, "\varphi"] & K^0/(\pi) \end{tikzcd}$$
, where the vertical are isomorphisms, and compute their inverse limits. □

Cor. (II.7.2.9) (Tilting of Perfectoid Field).

- $K^{0\flat}$ is a valuation ring of rank 1, with the field of fraction $K^\flat = K^{0\flat}[t^{-1}]$ which is a perfectoid.
- Its maximal ideal is $(t^{\frac{1}{p^\infty}})$, and of Krull dimension 1.
- The value group and residue field of K and $K^\flat$ is canonical isomorphic.

Proof: $K^{0\flat}$ has rank no more than K^0 which is 1(II.7.2.1), and it is non-trivial because $|t| = |\pi|$, so the rank is 1, and it is perfect by definition(II.7.1.5), so K is perfectoid by(II.7.2.2).

For the maximal ideal, the maximal ideal of $K^{0\flat}/t$ is its nilradical, as it is a valuation ring of rank 1(V.3.1.3), which is clearly $(t^{\frac{1}{p^\infty}})$. For the dimension, by(V.3.2.9), the Krull dimension equal the rank, which is 1.

For the residue field, use the isomorphism(II.7.2.8), $K^{0\flat}/t = K^0/\pi$ and the second item just proved, and for the value group, the same lemma(II.7.2.8) gives any $|p| \le |\pi| < 1$ are in the value group of $K^\flat$, and $|K^*|$ is generated by these values by(II.7.2.4). □

Prop. (II.7.2.10) (Tilting Continuous Valuations). If K is perfectoid, for any continuous valuation on K of any rank, the function $|\cdot|^\flat = |\cdot| \circ \sharp$ is a continuous valuation on $K^{0\flat}$, and all continuous valuation of $K^{0\flat}$ comes from this way.

Proof: Clearly $|-|^\flat$ is multiplicative and has trivial kernel, and it is non-Archimedean: for $f=(f_n), g=(g_n)\in K^\flat$, $f+g=(\lim_k(f_{n+k}+g_{n+k})^{p^k})$ by(II.7.1.12). So

$$|f+g|^\flat=|(f+g)^\sharp|=|\lim_k(f_k+g_k)^{p^k}|=\lim_k|f_k+g_k|^{p^k}\leq\lim_n\max\{|f_n|,|g_n|\}^{p^n}=\max\{|f_0|,|g_0|\}.$$

so it is non-Archimedean. It is also continuous because $\sharp$ is continuous.

Conversely, we notice a continuous valuation on a rank 1 valuation field corresponds to valuation rings in the residue field K^0/K^{00}, so by(II.7.2.9), we get a bijection on the continuous valuations. □

Prop. (II.7.2.11) (Almost Purity in Dimension 0). If K is a perfectoid field, L/K is a finite field ext, with the natural topology, then:

- L is perfectoid.
- $[L^\flat:K^\flat]=[L:K]$.
- The map $L\to L^\flat$ defines an isomorphism $K_{fet}\cong K^\flat_{fet}$.

Proof: This is a special case of almost purity theorem(II.7.5.1). □

Lemma (II.7.2.12) (Kedlaya). If $K^\flat$ is alg.closed, then K is alg.closed.

Proof: Let $P(X)=X^d+a_{d-1}X^{d-1}+\ldots+a_0\in K^0[X]$ be a irreducible monic polynomial, then its Newton polygon is a line, and we may assume $|a_0|=1$, as $K^{0\flat}$ is alg.closed, so $|K^{0*}|=|K^{0\flat*}|$(II.7.2.9) is a Q=vector space.

Next we choose a $Q(X)\in K^{0\flat}[X]$ that $Q(X)\equiv P[X] \mod t$, as $K^{0\flat}/t\cong K^0/\pi$(II.7.2.8). Now we consider $P(x+y^\sharp)$, then $P(y^\sharp)$ is divisible by π, so its Newton polygon is now of positive slope, so $c^{-d}P(cx+y^\sharp)\in K^0[X]$ again, where $c^d=|P(y^\sharp)|\leq|\pi|$. Then notice by iteration this argument, we get a sequence of $y_n^\sharp$, and then $y_1^\sharp+c_1y_2^\sharp+c_1c_2y_3^\sharp+\ldots+c_1\ldots c_ny_{n+1}^\sharp$ that converges to a root of $P(X)$. □

Prop. (II.7.2.13) (Examples of Tilting).

- If $K=\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$, then $K^0=\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$, thus $K^\flat=\mathbb{F}_p\widehat{((t))_{perf}}$(II.7.1.8). And if $L=K(\sqrt{p})$, then similarly $L^0=\widehat{\mathbb{Z}_p[p^{\frac{1}{2p^\infty}}]}$, and $L^\flat=K^\flat(\sqrt{t})$.
- If $K=\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, then $K^0=\widehat{\mathbb{Z}_p[\mu_{p^\infty}]}$, and Notice there is a map $\mathbb{Z}_p[\varepsilon^{\frac{1}{p^\infty}}]\to\mathbb{Z}_p[\mu_{p^\infty}]$ with kernel $(1+\varepsilon^{\frac{1}{p}}+\ldots+\varepsilon^{\frac{p-1}{p}})$, so

$$K^0/p=\mathbb{F}_p[\varepsilon^{\frac{1}{p^\infty}}]/(\varepsilon^{\frac{1}{p}}-1)^{p-1}\cong\mathbb{F}_p[t^{\frac{1}{p^\infty}}]/(t^{p-1})$$

  with the substitution $t=\varepsilon^{\frac{1}{p}}-1$. Then by(II.7.1.8), $K^{0\flat}=\widehat{\mathbb{F}_p[t^{\frac{1}{p^\infty}}]}$, and $K^\flat=\mathbb{F}_p\widehat{((t))_{perf}}$.

Remark (II.7.2.14). Notice that $K=\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ and $K=\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$ has the same tilting, so the tilting functor is not faithful. this is due to the fact that $\mathbb{Q}_p$ is not perfectoid. This will not happen over a perfectoid base field.

3 Almost Mathematics

Prop. (II.7.3.1) (Almost Elements). If K is a perfectoid field, $R = K^0$ and $I = K^{00}$, M is an R-module, then

- If M is torsion-free, then $M_* = \{m \in M \otimes_{K^0} K | Im \in M\} = \{m \in M \otimes_{K^0} K | t^{\frac{1}{p^n}} m \in M\}$,by(II.7.2.9) and(I.11.2.2).
- $I_* = R_* = R$. More generally, for an ideal $J \subset R$, let $c = \sup\{|x||x \in J\}$, then $J_* = \{a \in K, |a| \le c\}$.

Prop. (II.7.3.2). Let K be a perfectoid field with a pseudo-uniformizer π. If $\alpha : M \to N$ is an almost surjective map of K^0-algebras that M is π-adically separated and N is π-torsion-free that $\alpha \mod \pi$ is an almost isomorphism, then α is an almost isomorphism.

Proof: We may replace N with the image of α as to assume α is surjective. Now if $L = \operatorname{Ker} \alpha$, then the π-torsion-freeness of N shows L/π is the kernel of $(\alpha \mod \pi)$, and L/π is almost zero, thus L is almost π-divisible, but it is also π-separated, thus it is almost zero(using $t^{\sum_i \frac{1}{p^{a_i}}} m \in \cap_n t^n L$). □

Prop. (II.7.3.3) (Almostification and Completeness). Let K be a perfectoid field with a pseudo uniformizer t and $R = K^0, I = K^{00}$, let $M \in \mathrm{Mod}^a_R$, then:

- M is almost flat iff M_* is R-flat iff $M_!$ is R-flat.
- Assume M is almost flat, then M is t-adically complete iff M_* does.
- Assume M is almost flat, then for each $f \in K^0$, $fM_* \cong (fM)_*$, and $M_*/fM_* \subset (M/fM)_*$. And for any $\varepsilon \in I$, the image of $(M/f\varepsilon M)_*$ and M_*/fM_* in $(M/fM)_*$ are identical.

Proof: 1: R is a valuation ring, so M_* is R-flat iff $M_*[t]$ is flat by(I.7.1.8), as t is a pseudo uniformizer. As $(-)_*$ is left exact, $M_*[t] = (M[t])_*$, so if M is almost flat, then $M[t] = 0$ as t is nonzero-divisor, so M_* is R-flat. The converse is true as $M = (M_*)^a$, and the tensor is compatible.

For $(-)_!$, this follows from the observation that $(-)_!$ and $(-)^a$ are both exact and commute with tensor products, and notice $M_! \otimes N = (M \otimes N^a)_!$.

2: As $(-)^a$ commutes with all limits and colimits, if M_* is t-adically complete then so does $M = (M_*)^a$. Conversely, if M is R-flat and t-adically complete, then $M_!, M_*$ are also R-flat, and consider the commutative diagram:

$$\begin{array}{ccc} M_! & \xrightarrow{a} & \lim(M/t^n M)_! = \lim M_!/t^n M_! = \widehat{M_!} \\ \downarrow d & & \downarrow b \\ M_* & \xrightarrow{c} & \lim(M/t^n M)_* \end{array}$$

then d is almost isomorphism by(I.11.1.2) and so does b because $(-)^a$ commutes with all limits, and c is an isomorphism as $(-)_*$ commutes with limits and M is t-adically complete. So a is also almost isomorphism.

Now notice $M_!$ is flat hence t-torsion-free, so the kernel of a, d must be 0, with almost zero cokernels. Now(I.5.5.19) shows first $M_!$ is complete and next M_* is complete.

3: Notice $(fM_*)^a = fM$ as $(-)^a$ is exact, so

$$(fM)_* = \mathrm{Hom}(I, fM_*) = \{y \in M_*[t^{-1}] | Iy \subset fM_*\} = f\{y \in M_*[t^{-1}] | Iy \subset M_*\} = fM_*$$

and $M_*/fM_* \subset (M/fM)_*$ follows from the left exactness of $(-)_*$.

For the last assertion, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d] & M \arrow[r] \arrow[d] & M/fM \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M/\varepsilon M \arrow[r, "f"] & M/f\varepsilon M \arrow[r] & M/fM \arrow[r] & 0
\end{tikzcd}$$

and apply $(-)_* = \mathrm{Hom}_{R^a}(R^a, -)$ and use(I.11.2.6), then

$$\begin{tikzcd}
0 \arrow[r] & M_*/fM_* \arrow[r, "a"] \arrow[d] & (M/fM)_* \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{R^a}(R^a, M)[f] \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & (M/f\varepsilon M)_* \arrow[r, "b"] & (M/fM)_* \arrow[r] & \mathrm{Ext}^1_{R^a}(R^a, M/\varepsilon M)
\end{tikzcd}$$

To show a, b has the same image, it suffices to show that c is injective. For this, it suffices to show $\mathrm{Ext}^1_{R^a}(R^a, M) \to \mathrm{Ext}^1_{R^a}(R^a, M/\varepsilon M)$ is injective. Consider the exact sequence $0 \to M \xrightarrow{\varepsilon} M \to M/\varepsilon M \to 0$, it suffices to show that $\varepsilon\,\mathrm{Ext}^1_{R^a}(R^a, M) = 0$, and this is obvious as $\varepsilon \in I$(I.11.2.6). □

Prop. (II.7.3.4) (General Completeness and Almostification). More generally, if $J = (f_1, \ldots, f_r) \subset R$ is a f.g. ideal, then an R^a-module M is J-adically complete iff M_* does.

Proof: Cf.[Perfectoid Spaces Bhatt P32]. □

Banach Space

Prop. (II.7.3.5) (Uniform Banach K-Algebra). If K is a non-Archimedean perfectoid(perfect) field with a pseudo uniformizer t, then the following categories are equivalent:

- The category of uniform Banach K-algebras.
- The category $\mathcal{D}_{tic}$ of t-adically complete and t-torsionfree K^0-algebras A with A totally integrally closed in $A[t^{-1}]$(I.5.4.1).
- The category $\mathcal{D}_{ic}$ of t-adically complete and t-torsionfree K^0-algebras that A is integrally closed in $A[t^{-1}]$ and $A = A_*$.
- The category $\mathcal{D}_{prc}$ of t-adically complete and t-torsionfree K^0-algebras that A is p-root closed in $A[t^{-1}]$ and $A = A_*$.

Proof: The last three are equivalent, because if $A \in \mathcal{D}_{tic}$, then $A = A_*$ by(I.11.2.10), as K is perfect by(II.7.2.2). So $\mathcal{D}_{tic} \subset \mathcal{D}_{ic} \subset \mathcal{D}_{prc}$, so it suffices to show that $\mathcal{D}_{prc} \subset \mathcal{D}_{tic}$. Now for any f that $f^{\mathbb{N}} \subset t^{-k}A$, then $t^k f^{p^n} \subset A$, and A is p-root closed, so $t^{\frac{k}{p^n}} f \subset A$ for all n, so $f \in A_*$(I.11.2.3), but $A_* = A$.

The equivalence of $1, 2$ is general, by(II.1.2.7). □

Prop. (II.7.3.6). If K is a perfectoid field, then the category of uniform Banach spaces has all colimits and limits.

Proof: Cf.[Bhatt P38]. □

4 Perfectoid Algebras

Def. (II.7.4.1) (Perfectoid Algebra). For K a perfectoid field with tilt $K^\flat$, let $t \in K^\flat$ be a pseudo-uniformizer with $\pi = t^\sharp$ satisfying $|p| \le |\pi| < 1$, so it has a compatible collection of p^n-th roots $(t^{\frac{1}{p^n}})^\sharp$. Now:

- A **perfectoid algebra** over K is a uniform Banach K-algebra R that R^0/π is semi-perfect.
- A **perfectoid algebra** over K^{0a} is a K^{0a}-algebra A that is t-adically complete and flat over K^{0a}(or A_* over K^0, by(II.7.3.3)), and $K^{0a}/\pi \to A/\pi$ is relative perfect, i.e. the Frobenius induces an isomorphism $A/\pi^{\frac{1}{p}} \cong A/\pi$.
- A **perfectoid algebra** over K^{0a}/π is a K^{0a}/π-algebra A that is flat over K^{0a}/π(or A_* over K^0/π, by(II.7.3.3)), and the map $K^{0a}/\pi \to A$ is relatively perfect, i.e. the Frobenius induces an isomorphism $A/\pi^{\frac{1}{p}} \cong A$.

Remark (II.7.4.2). notice the definition regarding the relative perfectness doesn't depends on π chosen, by the power lifting theorem(IX.2.2.4).

Prop. (II.7.4.3) (Faithfully flatness of Perfectoids). Nonzero flat K^{0a}/π-algebras are faithfully flat, so does t-adically complete flat K^{0a}-algebras. In particular, $\mathrm{Perf}_{K^{0a}/\pi}$ and $\mathrm{Perf}_{K^{0a}}$ are all faithfully flat modules.

Proof: If $K^{0a}/\pi \to A$ is not faithfully flat, then there is an ideal $J \subset K^{0a}/\pi$ that $K^0/J \neq 0$ but $A/J = 0$. Now this implies $J \subsetneq I$, so there is a $\varpi \in I - J$ hence $J \subset (\varpi)$. Hence $A/\varpi = 0$ as well. Now there are exact sequences $0 \to K^{0a}/\varpi^n \xrightarrow{\varpi} K^{0a}/\varpi^{n+1} \to K^{0a}/\varpi \to 0$, so tensoring with A and induct, we get $K^{0a}/\varpi^n \otimes A = 0$, but $|\varpi^n| < |\pi|$ for some n, so $A = 0$.

The other case is similar, now $A/\varpi = 0$, so use(II.7.3.3), $A_*/\varpi \subset (A/\varpi)_* = 0$, but A_* is also t-adically complete, so $A_* = 0$, and $A = (A_*)^a = 0$. □

Prop. (II.7.4.4) (Examples of Perfectoid Algebras).

- If K has char$p$, then a K-Banach algebra is perfectoid iff it is uniform and perfect. Likewise, a π-adically complete and π-torsion free K^{0a}-algebra is perfectoid iff it is perfect.
- Let $A = K^0\widehat{[x_1^{\frac{1}{p^\infty}}, \ldots, x_n^{\frac{1}{p^\infty}}]}$, then $A^a \in \mathrm{Perf}_{K^{0a}}$, and $R = A[\pi^{-1}] \in \mathrm{Perf}_K$ in the Banach metric as in(II.1.2.7).

Proof: 1: a perfectoid algebra of char$p$ is perfect, because by semi-perfectness, $x = x_1^p + \pi z_1 = x_1^p + \pi x_2^p + \pi^2 z_2 = \ldots$, so $x = (x_1 + \pi^{\frac{1}{p}} x_2 + \pi^{\frac{2}{p}} x_3 + \ldots)^p$. In fact, uniformity is automatically implied by perfectness, by(II.7.6.11). The case of K^{0a}-algebra is similar.

2: A^a is K^{0a}-flat because A_* does, because it is a colimit of completions of polynomial algebras over I and I is flat over K^0(II.7.2.3). and R is perfectoid by(II.1.2.7) because A is totally integrally closed in R, because $K^0[x_1^{\frac{1}{p^\infty}}, \ldots, x_n^{\frac{1}{p^\infty}}]$ does(trivially), and use(I.11.2.10). □

Tilting Equivalence

Prop. (II.7.4.5) (Tilting Equivalence). There are canonical isomorphisms of categories:

$$\mathrm{Perf}_K \cong \mathrm{Perf}_{K^{0a}} \cong \mathrm{Perf}_{K^{0a}/\pi},$$

where the first map is by $R \mapsto R^{0a}$ and $A \to A_*[t^{-1}]$ just as in(II.1.2.7). The second map is reduction by π.

In particular, using tilting(II.7.2.8), there are canonical isomorphisms of categories:

$$\mathrm{Perf}_K \cong \mathrm{Perf}_{K^{0a}} \cong \mathrm{Perf}_{K^{0a}/\pi} = \mathrm{Perf}_{K^{\flat 0a}/t} \cong \mathrm{Perf}_{K^{\flat 0a}} \cong \mathrm{Perf}_{K^\flat}.$$

Now if $R \in \mathrm{Perf}_K$ corresponds to $S \in \mathrm{Perf}_{K^\flat}$, then we call $S = R^\flat$ the **tilting** of R and $R = S^\sharp$ the **untilting** of S.

Proof: $[\mathrm{Perf}_K \cong \mathrm{Perf}_{K^{0a}}]$

Firstly, if $R \in \mathrm{Perf}_K$, then $A = R^{0a} \in \mathrm{Perf}_{K^{0a}}$: $R^0/\pi^{\frac{1}{p}} \to R^0/\pi$ is surjective by definition, for injectivity, if $x^p/\pi \in R^0$, then $x/\pi^{\frac{1}{p}}$ is also power bounded, thus in R^0. And by(II.1.2.7), A is π-adically complete and π-torsion-free, hence R-flat by(I.7.1.8).

Next we show if $A \in \mathrm{Perf}_{K^{0a}}$, then A_* is π-adically complete, t-torsion free and p-root closed in $A[\pi^{-1}]$, hence is an left inverse to the mapping $R \to R^{0a}$, by(II.7.3.5). It is complete by(II.7.3.3)2,3.

For p-root closedness, by(II.7.3.3), $A_*/\pi^{\frac{1}{p}} \subset (A/\pi^{\frac{1}{p}})_* \hookrightarrow (A/\pi)_*$ by Frobenius, and then so does $A_*/\pi^{\frac{1}{p}} \to A_*/\pi$. Now if $x \in A_*[\pi^{-1}]$ satisfies $x^p \in A_*$, then $y = \pi^{\frac{k}{p}} x \in A_*$ for some k, and we want to lower k by 1 inductively, thus showing $x \in A_*$: As $y^p \in \pi A_*$, $y \in \pi^{\frac{1}{p}} A_*$ by what we have proved, thus $\pi^{\frac{k-1}{p}} x \subset A_*$.

For surjectivity of Frob: $A_*/\pi \to A_*/\pi$, notice first it is almost surjective, because $(A_* \to A_*/\pi)^a = A \to (A_*/\pi)^a \subset (A/\pi)^a_* = A/\pi$ is surjective by hypothesis, then by(I.11.2.3), it suffices to show that Frob is surjective on A/IA. For some $x \in A^*$, choose $0 < 1 < c$, almost surjectivity shows that $\pi^c x \equiv y^p \mod \pi A_*$, so $(y/p^{\frac{c}{p}})^p \in A_*$, thus $y \in p^{\frac{c}{p}} A_*$, thus $x \equiv (y/p^{\frac{c}{p}})^p \mod \pi^{1-c} A_* \subset IA$, so we are done.

Finally, this is also a right inverse, because we know that $A_* \cong R^0$ by(II.7.3.5), thus $A \cong R^{0a}$ in Mod^a_R. □

Proof: $[\mathrm{Perf}_{K^{0a}} \cong \mathrm{Perf}_{K^{0a}/\pi}]$

Firstly the reduction is a perfectoid K^{0a}/π-algebra: it is flat because flatness is stable under base change, and the rest are trivial. To construct a converse is a problem of deformation theory, we need to lift from K^{0a}/π-algebra via K^{0a}/π^n-algebras to a K^{0a}-algebra, suppose each lifting is unique up to isomorphism and the lift A_n is flat over K^{0a}/π^n, then we can form their inverse limit, which is flat, because it is π-torsion-free: if $\pi(x_n) = 0$, then by $0 \to \pi^n K^{0a}/\pi^{n+1} \to K^{0a}/\pi^{n+1} \xrightarrow{\pi} K^{0a}/\pi^n \to 0$ and the flatness of A_{n+1}, $x_{n+1} \in \pi^n A_{n+1}$, thus $x_n = 0$, and $x = 0$.

Now $0 \neq A \in \mathrm{Perf}_{K^{0a}/\pi}$, then A is faithfully flat by(II.7.4.3), then by(I.11.1.8), $A_{!!}$ is faithfully flat, and $(-)_{!!}$ is preserves all colimits and also Frobenius, so $A_{!!}$ is relatively perfect. Then we use the above argument, and(VI.2.4.1) to show that there is a $\widetilde{A} \in \mathcal{C}$ which is π-adically complete and K^0-flat, then $\widetilde{A} = (\widetilde{A_{!!}})^a$ is also p-adically complete and K^{0a}-flat, by(II.7.3.3).

And we check $\widetilde{A}/\pi = (\widetilde{A_{!!}/\pi})^a = (A_{!!})^a = A$ as $(-)_{!!}$ commutes with colimits, and conversely, if $A \in \mathrm{Perf}_{K^{0a}}$, we need to show $A = \widetilde{A/\pi}$, notice by hypothesis, $A_{!!}$ is faithfully flat K^0-algebra that is relatively flat over K^0/π, now it is also complete, because $A_{!!} \to A_*$ is an injection(because $(-)^a$ is exact) and almost isomorphism, so the cokernel is π-torsion, and A_* is complete, so does $A_{!!}$, by(I.5.5.19). Now $A_{!!}/\pi = (A_{!!}/\pi)$ as $(-)_{!!}$ commutes with colimits, so $A_{!!}$ is just the lift, and $(\widetilde{A_{!!}/\pi})^a = A^a_{!!} \cong A$. □

Cor. (II.7.4.6) (Tilting via Fountain's Functors). The tilt $R^\flat$ is just the Foutain's tilting, i.e. $R^\flat = R^{0\flat}[t^{-1}]$, and $R^\flat = \lim_{x \mapsto x^p} R$, $R^{0\flat} = (R^\flat)^0$.

Proof: Consider the diagram

$$\begin{tikzcd} K^{\flat 0}/t^{p^n} \arrow[r, "\varphi^{-n}", "\cong"'] \arrow[dr] & K^{\flat 0}/t \cong K^0/\pi \arrow[r] \arrow[d, "\varphi^n"] & R^0/t \arrow[d, "\varphi^n"] \\ & K^{\flat 0}/t \cong K^0/\pi \arrow[r] & R^0/t \end{tikzcd}$$

Then the upper row is just the unique flat and relative perfect lifting along $K^{\flat 0}/t^{p^n} \to K^{\flat 0}/t$. Taking inverse limit, we get the structure map $K^{\flat 0} \to R^{\flat 0}$, so after almostification, this is just the lifting we are looking for, because it is unique. So $(R^{0\flat})^a = (R^{\flat 0})^a$, and $R^\flat = R^{0\flat}[t^{-1}]$ unwinding the tilting equivalence.

For $R^\flat$, notice there is a map

$$R^\flat \cong (\lim_{x\mapsto x^p} R^0)[t^{-1}] \to \lim_{x\mapsto x^p}(R^0[\pi^{-1}]) \cong \lim_{x\mapsto x^p} R$$

Now injectivity is clear as t is non-zero-divisor, and if $(f_n) \in \lim_{x\mapsto x^p} R$, then $\pi^c f_n \in R^0$ for some c, then $\pi^{\frac{c}{p^n}} f_n \in R^0$ because R^0 is p-root closed(II.7.3.5), so $t^c(f_n) \subset R^{0\flat}$.

For the last assertion, it is true if $R^{0\flat}$ is totally integrally closed in $R^\flat$, by(II.7.3.5). For this, if $t^c f^{\mathbb{N}} \subset R^{0\flat}$, then $\pi^c(f^\sharp)^{\mathbb{N}} \subset R^0$, thus $f^\sharp \in R^0$. And by p-root closedness, p^n-th roots of $f^\sharp$ are all in R^0, so $f = (f_n) \in \lim_{x\mapsto x^p} R$ is in $R^{0\flat}$. □

Prop. (II.7.4.7) (Fountain's Functor θ). Using(VI.2.4.4), given a perfectoid field K, the canonical map $\overline{\theta} : K^{0\flat} \to K^{0\flat}/t \cong K^0/\pi$ lifts via inverse limit to a map θ:

$$\begin{tikzcd} A_{inf}(K^{0\flat}) = W(K^{0\flat}) \arrow[r, "\theta"] \arrow[d] & K^0 \arrow[d] \\ K^{0\flat} \arrow[r, "\overline{\theta}"] & K^{0\flat}/t = K^0/\pi \end{tikzcd}.$$

Then $\operatorname{Ker}\theta$ is generated by a non-zero-divisor, in fact, if $\operatorname{char}K = 0$, the generator can be chosen to be any element that maps to a generator of $\operatorname{Ker}\overline{\theta}$ and if $\operatorname{char}K = p$ this diagram is trivial. In particular, the diagram is a pushout.

Proof: Consider an element $t \in K^{0\flat}$ that $t^\sharp = \pi u$ for a unit u, then choose $\xi = \pi u - [t]$, where $[t]$ is the Techmuller lift, then $\xi \subset \operatorname{Ker}\theta$, and it generate the kernel after modulo π, so it generate the kernel by completeness of $A_{inf}(K^{0\flat})$, and Nakayama. □

Cor. (II.7.4.8) (Untilting via A_{inf}). In the above diagram, for any perfect $K^{0\flat}$-algebra A, by deformation theory(or in fact Witt theory) there is a unique lifting $W(A)$ lifting it to $A_{inf}(K^{0\flat})$. And then pushout $W(A) \otimes_{A_{inf}(K^{0\flat})} K^0$ is just the lifting of A/π, because the diagram above is pushout. This is in fact the method of [Kedlaya-Liu] used to prove the tilting-equivalence without the use of almost mathematics and deformation theory.

Cor. (II.7.4.9) (Limits and Colimits). Any of the categories in(II.7.4.5) has arbitrary limits and colimits.

Proof: We construct for $\mathrm{Perf}_{K^{0\flat a}}$: The limits is just the limits of topological rings, as the properties of t-adically complete, t-torsion free and perfect is preserved by limits(I.5.5.18). For the colimit, just use the t-adic completion of the left perfection of the colimits in the category of $K^{0\flat a}$-algebras, its t-torsion is almost zero because of perfectness, thus it is almost flat(II.7.3.3). □

Remark (II.7.4.10). Note also for further reference that in the category $\mathrm{Perf}_{K^{0a}}$, a filtered colimits is just the π-adically completion of the filtered limits as rings, because perfectness and flatness is preserved(I.7.1.4).

Prop. (II.7.4.11) ((Un)Tilting Preserves Fields). A perfectoid K-algebra R is a perfectoid field iff its tilt $R^\flat$ is a perfectoid field.

Proof: It is proven that if R is a perfectoid field, then $R^\flat$ is a perfectoid field. Conversely, R is a perfectoid field if the spectral norm given by $||x|| = \inf\{|t|^{-1}|t \in R^*, tx \in R^0\}$ is the Banach valuation of R and R is a field.

For the multiplicativeness of $||-||_R$, notice that $R^\flat$ is a perfectoid field, so its non-Archimedean valuation coincides with the spectral norm of $||-||_{R^\flat}$, and this equals $||-||_R\circ\sharp$, because $R^{0\flat} = R^{\flat 0}$, an element $f \in R^{\flat 0}$ iff $f^\sharp \in R^0$. Now the norm extends that of K and commutes with scalar multiplication, so for any f, g, we may assume $f, g \in R^0 - 0\pi^{\frac{1}{p}}R^0$, now choose $a, b \in R^\flat$ that $a^\sharp - f, b^\sharp - g \in \pi R^0$, this can be done because $R^{\flat 0} = R^{0\flat} \to R^0/\pi$ is surjective, then $a, b, ab \notin tR^{\flat 0}$ because $R^{0\flat} = R^{\flat 0}$. Then clearly $||f||_R = ||a||_{R^\flat}, ||g|| = ||b||_{R^\flat}, ||fg||_R = ||ab||_{R^\flat}$, so it is multiplicative by the multiplicativeness of $R^\flat$.

To show R is a field, consider and $f \in R - \pi^{\frac{1}{p}}R$, choose $a \in R^\flat$ that $f = a^\sharp + \pi g$, then as $R^\flat$ is a field, there is a b that $ab = 1$. Now $||\pi||_R < ||\pi^{\frac{1}{p}}||_R \subset ||f||_R = ||a||_{R^\flat} \leq 1$, so we get $||\pi b^\sharp g|| < 1$, then

$$f^{-1} = \frac{1}{a^\sharp + \pi g} = \frac{b^\sharp}{1 + \pi b^\sharp g} = b^\sharp(\sum(-\pi b^\sharp g)^k)$$

can be constructed in R. □

5 Almost Purity Theorem

Prop. (II.7.5.1) (Almost Purity Theorem). For a perfectoid K-algebra R and its tilt S,

- Almost purity in characteristic p: (take $(-)_*$ and)Inverting t gives an equivalence $S^0_{af\acute{e}t} \cong S_{f\acute{e}t}$.
- Almost purity in characteristic 0: Inverting π gives an equivalence $R^0_{af\acute{e}t} \cong R_{f\acute{e}t}$.
- Tilting and untilting functors induce equivalences $R^0_{af\acute{e}t} \cong S^0_{af\acute{e}t}$.

In particular, there are equivalences

$$S_{f\acute{e}t} \xleftarrow{a} S^{0a}_{af\acute{e}t} \xrightarrow{b} (S^{0a}/t)_{af\acute{e}t} \cong (R^{0a}/\pi)_{af\acute{e}t} \xleftarrow{c} R^{0a}_{af\acute{e}t} \xrightarrow{d} R_{f\acute{e}t}$$

Proof: The map a is already given in(I.11.2.14) by passing the power bounded-elements(equivalently, S_*) and inverting t. And it is an isomorphism.

The equivalence of b and c follows from[Almost Ring theory, Thm5.3.27].

The functor d is given by $A \to A_*[t^{-1}]$. Firstly, A is a perfectoid K^{0a}-algebra. This is because it is almost finite projective thus almost flat, and $R^{0a}/\pi \to A/\pi$ is weakly relative perfect by(I.11.2.15), so does $K^{0a}/\pi \to A/\pi$ because relative perfect is stable under composition. And it is finite projective thus almost direct summand of a finite free module.

So now the tilting equivalence(II.7.4.5) shows that $A_*[t^{-1}] \in R_{f\acute{e}t}$: it is finite etale because the A_* is finite projective by the right adjointness of $(-)_*$, and unramified is defined in terms of A_*. The converse of d is supposed to be the functor that extract from A_* from $A_*[t^{-1}]$ the total integral closure A_{tic} of R^0, which is functorial. We already know that A_* is totally integrally closed in $A_*[t^{-1}]$ by(II.7.4.5), so $A_{tic} \subset A_*$. Conversely, as A is almostly finitely generated over R^0, for $f \in A_*$, $\pi f^{\mathbb{N}}$ lies in a f.g. R^0-submodule of A_*, so $f^{\mathbb{N}}$ is totally integral over R^0, so $A_* = A_{tic}$.

It's left to show that d is essentially surjective, but this uses perfectoid spaces. For now, we only check that this is true for R being a perfectoid field(of char 0).For this, we show directly that the untilting functor $\sharp : K^\flat_{f\acute{e}t} \to K_{f\acute{e}t}$ is essentially surjective. Now $\sharp$ is an equivalence of categories $\mathrm{Perf}_{K^\flat} \to \mathrm{Perf}_K$, and it preserves degree, at least for field extensions, so it preserves Galois extensions. Now that finite étale algebra over fields are just disjoint of finite separable extensions(I.7.6.8), so it suffices to show that any finite extension of K is contained in some $L^\sharp$.

Consider $M = \widehat{\overline{K^\flat}}$, it is alg.closed of char$p$ so clearly a perfectoid field, and by(II.7.2.12) $M^\sharp$ is alg.closed. $M^\sharp$ is just the colimit in the category of uniform Banach K-algebras, so its valuation ring is just the completion of the valuation ring of $L^\sharp$ for $L/K^\sharp$ finite Galois. Then if $N = \cup L^\sharp$, then N is dense in $M^\sharp$, and N/K is clearly algebraic and in particular Hensel. So $N \subset \overline{N} \subset M^\sharp$ is dense, so by Krasner's lemma(II.1.1.34), $N = \overline{N}$. Now $N = \cup L^\sharp$ is an alg.closure of K, so every finite extension of K is contained in some $L^\sharp$.

The proof of the general case of the essentially surjectivity of d is continued at(II.7.7.28). □

6 Adic Space

Tate Ring

Def. (II.7.6.1) (Tate Ring). A topological ring is called **Tate** if there exists an open subring A_0 that the induced topology on A_0 is t-adic for some $t \in A_0$ which becomes a unit in A. Such a A_0 is called a **ring of definition**, and t is called a **pseudo uniformizer**. Morphisms of Tate rings are just a continuous morphisms of topological rings.

Def. (II.7.6.2) (Huber Ring). A topological ring is called **Huber** if there exists an open subring A_0 that the induced topology on A_0 is I-adic for some f.g. ideal I of A.

Prop. (II.7.6.3) (Examples of Tate Rings). If K is a complete non-Archimedean field and R is a K-Banach algebra, then R is Tate with a ring of definition by $(R_{\leq 1}, t)$, where t is a pseudo-uniformizer of K.

Prop. (II.7.6.4). For a Tate ring, $A = A^0[t^{-1}]$.

Proof: For any $f \in A$, by continuity of multiplication by f, $t^n f \in A^0$ for n large, so $f \in A^0[t^{-1}]$. □

Prop. (II.7.6.5) (Boundedness and Rings of Definition). A subset S is called **bounded** iff $S \in t^{-k}A_0$ for some ring of definition (A_0, t). The notion of boundedness is independent of the ring of definition. A subset S is bounded $S \subset t^{-n}A_0$ for some A_0.

A ring of definition is equivalent to an open bounded subset of A.

Proof: Firstly, if we define S is bounded iff $S \subset t^{-n}A_0$ for a fixed A_0, then any other ring of definition (A_1, t_1) is bounded: $t_1^l A_1 \subset A_0$, thus $A_1 \subset t_1^{-l}A_0$ for some l. Now $t_1^l A_0$ is open thus $t_1^l A_0 \supset t_0^k A_0$ for some k, $t_1^{-l}A_0 \subset t_0^{-k}A_0$, thus A_1 is bounded w.r.t A_0.

Now similarly, if $S \subset t_1^{-k}A_1$, then $S \subset t_0^{-\alpha_{10}(k)}A_0$, thus boundedness is independent of the ring of definition.

Now a ring of definition is open and bounded, conversely, choose a ring of definition (A_0, t), if A is open and bounded, then $t^n A_0 \subset A \subset t^{-n}A_0$, so it is clearly a ring of definition in the t-adic topology. □

Prop. (II.7.6.6) (Power-Bounded Elements). The subset A^0 of power-bounded elements in A is a subring, and it is the filtered colimit of all the ring of definition in A. It is integrally closed in A. And we call A **uniform** iff A^0 is bounded.

Proof: By(II.7.6.5), an element f is power-bounded iff $t^c f^{\mathbb{N}} \subset A_0$ for a ring of definition (A_0, t). It is then clear it is a subring. Now by the arbitrariness of ring of definition, all rings of definitions are contained in A^0, conversely, if (A_0, t) is a ring of definition, then the subring $A_0[f]$ is open and bounded, so it is a ring of definition(II.7.6.5). So now A^0 is the union of all rings of definitions. Moreover, the rings of definitions are filtered.

To see A^0 is integrally closed, notice that if f is integral over A^0, then it is integral over some ring of definition A_0, so f is power-bounded thus $f \in A^0$. □

Prop. (II.7.6.7) (Topological Nilpotent Elements). The subset A^{00} of elements that is topologically nilpotent is an ideal of A^0, and it is the radical of the ideal generated by any pseudo-uniformizer t.

A topological nilpotent unit is equivalent to a pseudo-uniformizer.

Proof: For $f \in A^{00}$ and $g \in A^0$, $t^c g^{\mathbb{N}} \subset A_0$ for a ring of definition A_0, thus fg is topologically nilpotent. So A^{00} is an ideal.

For any pseudo-uniformizer t and $f \in A^{00}$, $f^n \subset tA^0$ for some n, thus $A^{00} \subset \mathrm{rad}((t))$, and if $g^m = f \in A^{00}$, then clearly g is itself topological nilpotent.

for the last assertion, for $t \in A^{00}$ choose any ring of definition (A_0, f), then $t^n \in A_0$ for some n as A_0 is open. So $A_0[t]$ is open and bounded thus a ring of definition(II.7.6.5), and it is clear that the t-adic topology coincides with the f-adic topology on A_0. □

Prop. (II.7.6.8). If K is a complete non-Archimedean field, then any Banach K-algebra R is a complete Tate ring, and if K, R are perfectoids, then $R^{00} = K^{00}R^0$.

Proof: In the perfectoid case, first $K^{00}R^0 \subset R^{00}$, and for any topological nilpotent α, $\alpha^n \subset tR^{00}$ for a pseudo-uniformizer t. Thus R^{00} and $K^{00}R^0$ has the same radical, it suffices to show $K^{00}R^0$ is radical, but the quotient $R^0/K^{00}R^{00}$ is a perfect K^0/K^{00}-algebra by perfectoidness, thus it must be radical. □

Prop. (II.7.6.9) (Completion of Tate Rings). For any Tate ring A with a ring of definition (A_0, t), A is complete iff A_0 is t-adically complete. Now for any Tate ring A, we can form $\widehat{A}_0 = \lim A_0/t^n A_0$, then let $\widehat{A} = \widehat{A}_0[t^{-1}]$, then this is the completion of A, with $\widehat{A}_0$ as a ring of definition.

Prop. (II.7.6.10) (Morphisms and Ring of Definition). For a map of Tate rings $f : A \to B$, we can modify the pseudo-uniformizer of B to be the image of the pseudo-uniformizer of A, and also we can modify the rings of definitions in either A or B to make sure that A_0 is mapped into B_0.

Proof: For the pseudo-uniformizer, notice that the image of a topological nilpotent element is topological nilpotent. And choose arbitrary rings of definition, notice that $A' \cap f^{-1}(B_0)$ is open and bounded, thus a ring of definition(II.7.6.5), and reversely, since $t^n A_0 \in A$ for some n, so if we let B' be the subring of B generated by A_0, then $f(t)^n B' \subset B_0$, thus it is open and bounded, thus is a ring of defition. □

Prop. (II.7.6.11) (Complete Perfect Tate ring is Uniform, André). If A is a complete Tate ring of char$p$ that is perfect, then A is uniform.

Proof: Let (A_0, t) be a ring of definition, let $A_n = A_0^{\frac{1}{p^n}}$, then $A_\infty = \operatorname{colim} A_n = (A_0)_{perf}$. We check $t^{\frac{1}{p^n}} A^0 \subset A_\infty \subset t^{-1} A_0$, which shows A^0 is bounded.

If $f \in A^0$, then $t^a f^{\mathbb{N}} \subset A \subset A_\infty$, and A_∞ is perfect, so $t^{\frac{a}{p^n}} f \in A_\infty$ for all n. Notice the Frobenius is a continuous bijection of Banach spaces, so it is open by Banach theorem(V.3.4.4), so $A_0^p \supset t^{mp} A_0$, thus $t^m A_1 \subset A_0$, and then $t^{\frac{m}{p^n}} A_{n+1} \subset A_n$. So $t^{\sum_n m/p^n} A_n \subset A_1$. So $t^c A_\infty \subset A_0$, for c large. □

Affinoid Tate Ring

Def. (II.7.6.12) (Affinoid Tate Ring). For a Tate ring A, a **ring of integral elements** is an open and integrally closed subring of A^0(for example A^0 itself(II.7.6.6)). A **affinoid Tate ring** is a pair (A, A^+) that A is a Tate ring and T^+ is a ring of integral elements. A morphism of affinoid Tate ring should preserve the ring of integers.

Prop. (II.7.6.13). $A^{00} \subset A^+$ as an ideal for any A^+. In particular, A^+ contains any pseudo-uniformizer, and the set of rings of integral elements is in bijection with integrally closed subrings of A/A^{00}.

Also, A^+ is a filtered colimits of rings of definitions.

Proof: $t \in A^{00}$ is topologically nilpotent hence $t^n \subset A^+$ as it is open, and then $t \in A^+$ as it is integrally closed. It is an ideal because it is an ideal of A^0(II.7.6.7).

For the last assertion, notice that A^0 is the filtered colimits of rings of definitions(II.7.6.6), and the intersection of a ring of definition with A^+ is also a ring of definition, because it is open and bounded(II.7.6.5), the result follows. □

Def. (II.7.6.14) (Zariski, Henselian, Complete). An affinoid Tate ring (A, A^+) is called

- **complete** iff A is complete.
- **Henselian** iff (A^+, A^{00}) is Henselian.
- **Zariski** iff (A^+, A^{00}) is Zariski.

Prop. (II.7.6.15). An affinoid Tate ring (A, A^+) with a ring of definition (A_0, t) that $A_0 \subset A^+$ is

- Zariski iff t is in the Jacobson radical of A_0.
- Henselian iff the pair (A_0, tA_0) is Henselian.
- Complete then it is Henselian.
- Henselian then it is Zariski.

Proof: 1: We prove that if $t \in \operatorname{rad}(A_0)$, the for any other $B_0 \supset A_0$, $t \in \operatorname{rad}(B_0)$. If this is true, then as A^+ is a filtered colimits of rings of definitions(because A^0 does), it is clear that t lies in the maximal ideal (check $1 + at$ is unit). For this, if $\mathfrak{m} \subset B_0$ is maximal and $t \notin \mathfrak{m}$, choose n that $t^n B_0 \subset A_0$, and an element $b \in B_0$ that maps to t^{-n-1} modulo $\mathfrak{m}$, then $a = t^n b \in A_0$ is mapped to t^{-1}. Thus the composition $A_0 \to B_0 \to B_0/\mathfrak{m}$ is surjective: $\bar{b}$ is the image of $a^n(t^n b) \in A$. So t is not in a maximal ideal of A_0, contradiction.

Conversely, Cf.[Bhatt Perfectoid Space P57].

2: Cf.[Bhatt Perfectoid Spaces P57].

3: A is complete then A_0 is complete, hence (A_0, tA_0) is Henselian by(I.7.10.6), so it is Henselian by item2. 4: Trivial. □

Prop. (II.7.6.16) (Completion, Henselization, Zariski Localization). There are left adjoint to the forgetful functors from the category of Complete/Henselian/Zariski pairs to the category of pairs, called the **Completion/Henselization/Zariski Localization** of pairs. And there are natural maps

$$(A, A^+) \to (A, A^+)_{Zar} \to (A, A^+)_{Hens} \to (\widehat{A}, \widehat{A}^+)$$

Proof: We only prove for the completion: $\widehat{A}_0$ is a ring of definition in $\widehat{A}$(II.7.6.9), if we define $\widehat{A}^+$ to be the integral closure of $\widehat{A}^+ \otimes_{A_0} \widehat{A}_0$, then it is an open and integrally closed subring of $\widehat{A}$. Now it is also contained in $\widehat{A}^0$, because $A^0 \otimes_{A_0} \widehat{A}_0$ is contained $\widehat{A}^0$, and $\widehat{A}^+$ is integrally closed over it. Then we can use the fact $\widehat{A}^0$ is integrally closed(II.7.6.6), so $\widehat{A}^+ \subset \widehat{A}^0$. □

Valuation Spectrum

Def. (II.7.6.17) (Riemann-Zariski Space). Let K be a field and A be a subring, the Riemann-Zariski space $RZ(K, A)$ is defined to be the set of all valuation subrings of K containing A that has the topology generated by

$$U(x_1, \ldots, x_n) = \{P \in RZ(K, A) | x_1, \ldots, x_n \in P\}.$$

$RZ(K, 0)$ is also denoted by $RZ(K)$.

Prop. (II.7.6.18). Clearly the specialization relations of $RZ(K, A)$ is identical to inclusion relations.

Prop. (II.7.6.19). $RZ(K, A)$ is quasi-compact.

Proof: Cf.[Adic Space, Morel P18].? □

Def. (II.7.6.20) (Valuation Spectrum). Let A be a ring, the **valuation spectrum** $\mathrm{Spv}(A)$ is the set of equivalent classes of valuations on A, topologized by the open subsets

$$\mathrm{Spv}(A)(\frac{f}{g}) = \{x \in \mathrm{Spa}(A) || f(x)| \leq |g(x)| \neq 0\}.$$

Def. (II.7.6.21). There is a kernel map $\mathrm{Spv}(A) \to \mathrm{Spec}\, A$ sending a valuation to its kernel(support). Then this map is continuous, and the fiber of this map over $\mathfrak{p}$ is just isomorphic to the Riemann-Zariski space $RZ(k(\mathfrak{p}))$.

Proof: For $\mathfrak{p} \in \mathrm{Spec}\, A$, $(\mathfrak{p}, k(\mathfrak{p})) \in \mathrm{Ker}^{-1}(\mathfrak{p})$, so the kernel map is surjective, and $\mathrm{Ker}(D(f)) = \mathrm{Spv}(A)(\frac{f}{f})$, so it is continuous.

Now $\mathrm{Ker}^{-1}(\mathfrak{p}) = RZ(k(\mathfrak{p}))$ as set, and the open sets $\mathrm{Spv}(\frac{f_1, \ldots, f_n}{g})(g \notin \mathfrak{p})$ corresponds to $U((f_1 + \mathfrak{p})(g + \mathfrak{p})^{-1}, \ldots, (f_n + \mathfrak{p})(g + \mathfrak{p})^{-1})$, so they are homeomorphic as topological spaces. □

Lemma (II.7.6.22) (Valuation Spectrum is Spectral). The valuation spectrum of any ring is spectral, with sub-basis generated by $\mathrm{Spv}(\frac{f}{g})$. And Spv induces a map from the category of rings to the category of spectral spaces.

Moreover, the map $\mathrm{Ker} : \mathrm{Spv}(A) \to \mathrm{Spec}\, A$ is spectral, as the kernel of $D(f)$ is $U(\frac{f}{f})$.

Proof: Cf.[Adic Space, Morel P31].? □

Affinoid Adic Space

Def. (II.7.6.23) (Adic Spectrum). Let (A, A^+) be a affinoid Tate ring, the **adic spectrum** $\mathrm{Spa}(A, A^+)$ is a defined as the set of the set of equivalence classes of valuations $x : A \to \Gamma \cup \{0\}$ s.t. $x(A^+) \leq 1$ and is continuous w.r.t the order topology on $\Gamma \cup \{0\}$.

Given $f, g \in A$, let
$$\mathrm{Spa}(A, A^+)(\frac{f}{g}) = \{x||f(x)| \leq |g(x)| \neq 0\}$$
and we endow $\mathrm{Spa}(A, A^+)$ with the topology generated by all these $\mathrm{Spa}(A, A^+)(\frac{f}{g})$.

Prop. (II.7.6.24). The adic spectrum construction defines a contravariant functor from the category of affinoid Tate rings to the category of topological spaces. And for any ring of integers A^+, $\mathrm{Spa}(A, A^0) \hookrightarrow \mathrm{Spa}(A, A^+)$ is an immersion of spaces.

Def. (II.7.6.25) (Kernel map). Taking kernels of valuations gives a map $\mathrm{Ker} : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}\, A$. This map is continuous, as the inverse image of $D(f)$ is $\mathrm{Spa}(A, A^+)(\frac{f}{f})$. We call a subset a **Zariski open subset** of $\mathrm{Spa}(A, A^+)$ iff it is open in the initial topology along Ker.

Prop. (II.7.6.26) (Microbial Valuation Ring Characterization of Spa). For an affinoid adic space (A, A^+), there is a natural bijection between $\mathrm{Spa}(A, A^+)$ and the set S of maps $\varphi : A^+ \to V$ where V is a microbial valuation ring that φ preserves pseudo-uniformizers, under the equivalence relation that if $A^+ \to V \to W$ that W/V is faithfully flat that preserves pseudo-uniformizers, then $A^+ \to V \sim A^+ \to W$.

Moreover, by(V.3.1.7), we can assume a point always corresponds to a map $A \to V$ that V is complete.

Proof: For any valuation $x \in \mathrm{Spa}(A, A^+)$, consider $x(A^+) \subset R_x \subset k(\mathfrak{p}_x)$, and $x(t)$ is a pseudo-uniformizer of R_x.

Conversely, if $x \in S$, then $|\operatorname{Im}(x)|$ is clearly a totally ordered Abelian group, thus defines a valuation on $A = A^+[t^{-1}]$, and $\operatorname{Im}(x) \subset V$ is an injection of valuation rings, so it is faithfully flat that preserves pseudo-uniformizers. If $V \to W$ is faithfully flat and preserves pseudo-uniformizers, then it is injective and continuous, so they define the same valuation group and the same kernel, thus the same element in $\mathrm{Spa}(A, A^+)$. □

Remark (II.7.6.27). Notice that we can interprete points of $\mathrm{Spa}(R, R^+)$ as continuous mapping to a complete affinoid field such that the quotient field of R in K is dense.

Remark (II.7.6.28). Visualization of adic points, Cf.[Bhatt Perfectoid Spaces P64].

Prop. (II.7.6.29).

- The canonical map $(A, A^+) \to (\widehat{A}, \widehat{A}^+)$ induces an isomorphism on Spa that preserves rational subsets.
- $\mathrm{Spa}(A, A^+)$ vanishes iff its completion $\widehat{A}$ vanishes.
- (Adic Nullstellensatz)$A^+ = \{f \in A | x(f) \leq 1, \forall x \in \mathrm{Spa}(A, A^+)\}$.
- If $x \to y$ is a specialization, then $\mathfrak{p}_x = \mathfrak{p}_y$, and R_x is a localization of R_y. Conversely, if $\mathfrak{p}$ is a prime of R_y, then the valuation group $R_{y,\mathfrak{p}}$ corresponds to an $x \in \mathrm{Spa}(A, A^+)$ specializing to y.
- Assume (A, A^+) is Zariski, then $f \in A$ is a unit iff $x(f) \neq 0$ for all $x \in \mathrm{Spa}(A, A^+)$.

Proof: 1: Use the valuation ring characterization(II.7.6.26), now a point of x determined a map $\varphi : A^+ \to V$ for some complete microbial valuation V that $\varphi(t)$ is topologically nilpotent, thus φ extends to a continuous map $(A, A^+) \to (\mathrm{Frac}(V), V)$(with the $\varphi(t)$-adic topology). Now this extends to a map under completion, thus determines a point of $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)$, so the Spa map is surjective. And injectivity follows from as A is dense in $\widehat{A}$.

For the homeomorphism, just notice that if $f_i - f_i', g - g' \in t^{N+1}\widehat{A}$, then

$$\mathrm{Spa}(\frac{f_1, \ldots, f_n}{g}) = \mathrm{Spa}(\frac{f_1', \ldots, f_n'}{g'}).$$

where $f_n = t^N$. So now A is dense in $\widehat{A}$, if we choose $f_i, g \in A$, then this rational subset is clearly induced from A.

2: Cf.[Bhatt Perfectoid Spaces P65].

3: ?

4: For the kernel: if $f \in \mathfrak{p}_x$, then $x \notin \mathrm{Spa}(\frac{f}{f})$, thus $y \notin \mathrm{Spa}(\frac{f}{f})$, thus $f \in \mathfrak{p}_y$. Conversely, if $f \in \mathfrak{p}_y$, then $y \in \mathrm{Spa}(\frac{f}{t^n})$ for all n, hence so does x, thus $f \in \mathfrak{p}_x$.

This rest from the valuation ring characterization of(II.7.6.26) and the fact that any inclusion of two valuation rings in a field is a localization is generated by localization(V.3.2.12). Notice that if $R_y \subset R_x$, then any open subset containing y contains x, so x specializes to y.

5: One direction is trivial, for the other, we must show f is a unit in every residue field of A. As (A, A^+) is Zariski, t is in the Jacobson radical of A^+. In particular, every point of $\mathrm{Spec}\, A \subset \mathrm{Spec}\, A^+$ specializes to some point of $\mathrm{Spec}(A^+/(t))$. Now it suffices to prove for each $\mathfrak{p} \subset A^+/(t)$, there is a $x \in \mathrm{Spa}(A, A^+)$ that has support $\mathfrak{p}$. But this is easy, as it has a valuation ring(by localizing A^+ at a maximal ideal) and by t-adic completion, we get a point in $\mathrm{Spa}(A, A^+)$ by(V.3.1.7). □

Cor. (II.7.6.30) (Generalizations in Spa). The above proposition shows that the generalization relations of Spa are easily determined, for an element y, all generalizations of y are in bijection with $\mathrm{Spec}(R_y/(t))$ as a poset, thus totally ordered, and each y has a unique generic point as generalization, because it is microbial.

Moreover, $\mathrm{Spa}(A, A^0)$ is closed under generalization in $\mathrm{Spa}(A, A^+)$, and they have the same set of generic points.

Proof: The last assertion is because the generalizations of a point y is just valuation rings containing R_y, and R_y contains $A^0/\mathfrak{p}_y$, so does its generalizations. And for any generic point $x \in \mathrm{Spa}(A, A^+)$, A^0 is mapped to the valuation ring R_x, because it is a rank 1 valuation, so if $t^k f^{\mathbb{N}} \subset R_x$, then $f \in R_x$ because otherwise we have $|t| < |f^{-n}|$ for n large. □

Spectrality of Affinoid Adic Spaces

Def. (II.7.6.31) (Rational Subsets). A **rational subset** of $\mathrm{Spa}(A, A^+)$ is defined to be

$$\mathrm{Spa}(A, A^+)(\frac{f_1, \ldots, f_n}{g}) = \{x \in \mathrm{Spa}(A, A^+) | x(f_i) \leq x(g)\},$$

where $(f_i) = (1)$.

Prop. (II.7.6.32) (Modifying Rational Subsets). By the help of a pseudo-uniformizer t, we can modify f_i and g as they are in A^+, moreover, because $1 + \sum a_i f_i$, for N large, we can assume

$t^N = \sum a_i f_i$ that $a_i, f_i \in A^+$. Then:

$$\mathrm{Spa}(A, A^+)(\frac{f_1, \ldots, f_n}{g}) = \mathrm{Spa}(A, A^+)(\frac{f_1, \ldots, f_n, t^N}{g}).$$

That is, we doesn't require $(f_i) = 1$, but require $f_i, g \in A^+$, and $f_n = t^N$.

Prop. (II.7.6.33). Rational subsets are stable under intersection(Easy).

Prop. (II.7.6.34). The rational subsets forms a subbasis for the topology of $\mathrm{Spa}(A, A^+)$. This is because of the formula

$$\mathrm{Spa}(\frac{f}{g}) = \cup_n \mathrm{Spa}(\frac{f, t^n}{g}).$$

But in generally $\mathrm{Spa}(\frac{f}{g})$ is not quasi-compact, in particular, $\mathrm{Spv}(A) \to \mathrm{Spa}(A, A^+)$ is not quasi-compact(proper).

Prop. (II.7.6.35) (Adic Spectrum is Spectral). The adic spectrum $\mathrm{Spa}(A, A^+)$ is a spectral space, and a basis given by quasi-compact opens are given by rational subsets. And the Spa functor is naturally a functor from the category of affinoid Tate rings to the category of spectral spaces.

Proof: Cf.[Bhatt Perfectoid Spaces P71].? □

Remark (II.7.6.36). In spite of the proposition that adic spaces are spectral, it is very different from classical algebraic geometry. For example, the generalizations of a point y is totally ordered(localization of the valuation ring), but this nearly never happen for an affine variety.

Cor. (II.7.6.37) (Detecting Nilpotence Locally). If (A, A^+) is an affinoid Tate ring and $f \in A$, then $f \in A^{00}$ iff $|f(x)|^n \to 0$ for all x.

Proof: If f is topological nilpotent, then $f^N \in tA^+$ for some n, so $|f(x)|^{nN} \le |t(x)|^n \to 0$ because x is continuous. Conversely, if $|f(x)|^n \to 0$ for all x, then $X = \cup_n X(\frac{f^n}{t})$. But X is quasi-compact, so $|f(x)|^n \le |t(x)|$ for all x, for some n. So by(II.7.6.29) $f^n \in tA^+$. Now A^+ is a filtered colimits of rings of definitions(II.7.6.13), so $f^n \in tA_0$ for some tA_0, which shows that $f \in A^{00}$. □

Prop. (II.7.6.38) (Direct Limits of Uniform Affinoids). The direct limits exists in the category of uniform affinoid Tate rings. and $A^+ = \mathrm{colim}\, A_i^+$.

Moreover, $\mathrm{Spa}(A, A^+) \cong \lim \mathrm{Spa}(A_i, A_i^+)$, and each rational subset of $\mathrm{Spa}(A, A^+)$ is pulled back from some rational subset of $\mathrm{Spa}(A_i, A_i^+)$.

The same conclusion also hold in the category of complete uniform affinoid Tate rings(For the homeomorphism, (II.7.6.29) is used).

Proof: Suppose the colimit index has a minimal element i_0, let t be a pseudo-uniformizer, then each A_i^+ is a ring of definition with pseudo-uniformizer t. Now we set $A = \mathrm{colim}\, A_i$ with ring of definitions $A^+ = \mathrm{colim}\, A_i^+$, then A^+ is integrally closed in A, thus (A, A^+) is truly a uniform affinoid Tate ring. Now we check it is the colimit: For any compatible map $(A_i, A_i^+) \to (B, B^+)$, there is a map $f : (A, A^+) \to (B, B^+)$ as abstract rings. We check it is continuous: we may assume B^+ is the ring of definition, then $t^n A^+ \subset f^{-1}(t^n B^+)$, thus it is continuous.

For the adic spectrum, now a point $x \in \mathrm{Spa}(A, A^+)$ is determined by the map of uniform affinoid Tate rings $(A, A^+) \to (k(\mathfrak{p}), R_x)$, and by the universal property, it is defined by a compatible set of maps $(A_i, A_i^+) \to (k(\mathfrak{p}), R_x)$. Now it is easy to see the desired bijection of topological spaces, as the elements defining rational subsets are pullbacks from some A_i. □

Prop. (II.7.6.39) (Perfection of Adic Spectrum). Let (A, A^+) be an affinoid Tate ring of char$p$, then

- The Frobenius map induces a homeomorphism on the adic spectrum of (A, A^+).
- If (A, A^+) is uniform, then there is a perfection functor, which is left adjoint to the forgetful functor from the category of perfect uniform affinoid Tate rings to the category of affinoid Tate rings. And it is just (A_{perf}, A^+_{perf}).
- The natural map $(A, A^+) \to (A_{perf}, A^+_{perf})$ induces a homeomorphism on the adic spectrum.

Proof: 1: The Frobenius pulls a valuation a multiple of itself, thus equals itself.

2: Clearly A^+_{perf} is integrally closed in A^+ and is in A^0_{perf}. It suffices to show (A_{perf}, A^+_{perf}) is uniform, but this is because $A^0_{perf} = (A_{perf})^0 \subset (t^{-n}A_0)_{perf} = t^{-n}(A_0)_{perf}$.

3: The Spa map is checked to be continuous and injective, for the converse, using the microbial valuation ring characterization, a point x of $\mathrm{Spa}(A, A^+)$ corresponds to a map A^+ to a complete field $k^+(x)$, thus a map $A^+_{perf} \to k^+(x)_{perf}$, now $k^+(x) \to k^+(x)_{perf}$ is faithfully flat that preserves pseudo-uniformizers, thus it is a point y that maps to y. □

Def. (II.7.6.40) (Specialization Map). The **specialization map**

$$\mathrm{Sp} : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A^+/A^{00})$$

that maps a point x to the inverse image of the maximal ideal of R_x along the valuation map $A^+ \to R_x$ it corresponds. It clearly lies in $\mathrm{Spec}(A^+/A^{00})$ as any pseudo-uniformizer is mapped to a pseudo-uniformizer in R_x thus in the maximal ideal.

This map is continuous and spectral, unlike that the kernel map(II.7.6.34): the inverse image of a $D(f)$ for $f \in A^+$ is the set of points $x \in \mathrm{Spa}(A, A^+)$ that $x(f)$ is a unit, i.e. $|x(f)| = 1$. As $|x(f)| \leq 1$ for all $f \in A^+$, this set is just $\mathrm{Spa}(A, A^+)(\frac{1}{f})$, so specialization map sp is both continuous and spectral.

Prop. (II.7.6.41) (Maximal Hausdorff Quoteint). Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid Tate space, the if $\overline{X}$ is the quotient of X by the equivalence relation generated by specialization, then $\overline{X}$ is the Hausdorffization of X, i.e. $\overline{X}$ is Hausdorff.

Proof: To show $\overline{X}$ is Hausdorff, if $x, y \in X$ is not mapped to the same point in $\overline{X}$, then by(II.7.6.30), we may assume x, y is generic in X, and $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$. Now we must find two disjoint open subsets of x, y that is stable under specialization. Cf.[Bhatt Perfectoid Spaces P75]. □

Structure Presheaf and Adic Spaces

Lemma (II.7.6.42) (Functions on Rational Subsets). If $X = \mathrm{Spa}(A, A^+)$ is an affinoid Tate ring, and U is a rational subset, then there is a unique complete affinoid Tate ring $(\mathcal{O}_X(U), \mathcal{O}^+_X(U))$ over (A, A^+) that the the Spa map

$$\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}^+_X(U)) \to \mathrm{Spa}(A, A^+)$$

is universal in all the complete affinoid Tate algebras that has image in U.

And in this, this Spa map is a homeomorphism identifying the rational subsets contained in T to rational subsets of $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}^+_X(U))$.

Proof: Choose a ring of definition (A_0, t), and $U = \mathrm{Spa}(A, A^+)(\frac{f_1,\ldots,f_n}{g})$ for $f_i, g \in A_0$, and $f_n = t$(II.7.6.32), and let $B = A[g^{-1}]$ and $B_0 = A_0[\frac{f_i}{g}]$. Then $B = B_0[t^{-1}]$(notice that $A_0[t^{-1}] = A$). So B is a Tate A-algebra with ring of definition (B_0, t). Now if B^+ is the integral closure of the subring of B generated by $A^+[\frac{f_i}{g}]$, then (B, B^+) is an affinoid Tate ring. Set $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ to be its completion.

By construction $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ maps into U, because $x(g) \neq 0$ because g is a unit, and $|x(f_i)| \leq |x(g)|$ as $f_i/g \in B^+$.

Now check universal property: if $\mathrm{Spa}(C, C^+)$ maps into U, then g is a unit in C by(II.7.6.29), and then $f_i/g \in C^+$ by(II.7.6.29) again. Now C^0 is the filtered colimit of all rings of definition, so there is a ring of definition C_0 that contains A_0 and all f_i/g((II.7.6.10) is used). Then this gives a map of affinoid Tate rings that maps B_0 into C_0, and when passed to completion, induces a map $\mathcal{O}_X(U) \to C$ of Tate algebras. Now also B^+ is mapped into C^+ because C^+ is integrally closed, so we are done.

For the last assertion, by(II.7.6.29), we only have to prove $\mathrm{Spa}(B, B^+) \to U$ is a homeomorphism preserving rational subsets, for this, the injectivity is clear as B is a localization of A. And the surjectivity follows immediately from the valuation ring characterization and universal property. Continuity is also clear.

For the openness, for any rational subset $X(\frac{f_1,\ldots,f}{g})$ of $X = \mathrm{Spa}(B, B^+)$, because $B = A[g^{-1}]$, g is unit in B, we can assume that $f_i, g \in A_0$. Now we show $U \cap \mathrm{Spa}(A, A^+)(\frac{f_1,\ldots,f}{g})$ is rational, for this, it suffices to add a t^N to f_i, and this is possible, as $X(\frac{f_1,\ldots,f}{g})$ is quasi-compact by(II.7.6.35). (This is in fact similar to the proof that continuous bijection from compact to Hausdorff is homeomorphism). □

Remark (II.7.6.43). The proof goes through with complete replaced by Zariski or Henselian, because we only use item5 of(II.7.6.29), which is true for all Zariski pairs.

And by looking at the construction, if a rational subset U has a representation $X(\frac{f_1,\ldots,f_n}{g})$, then $f_i \in \mathcal{O}_X^+(U)$, and g is invertible in $\mathcal{O}_X(U)$.

Def. (II.7.6.44) (Stalks). For an affinoid Tate space, the **stalks** is defined as in the case of schemes, i.e. the colimit of the function ring of rational subsets containing x, without topology, and similarly for the **integral stalk**. notice that the function rings are defined by universal property w.r.t to complete affinoid Tate algebras, so the stalks only depend on the completion of (A, A^+).

Prop. (II.7.6.45) (Valuations on the Stalks). Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid Tate ring, and $x \in X$ is a point, then

- There is a valuation x on $\mathcal{O}_{X,x}$ extending that on A, and $\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x} | |f(x)| \leq 1\}$.
- $\mathcal{O}_{X,x}$ is local with maximal ideal $\mathfrak{m}_x = \mathrm{Ker}\, x$, and $\mathcal{O}_{X,x}^+$ is local with maximal ideal $\{f \in \mathcal{O}_{X,x} | |f(x)| < 1\}$.
- If $k(x)$ is the residue field of $\mathcal{O}_{X,x}$ and $k(x)^+$ be the image of $\mathcal{O}_{X,x}^+$ in $k(x)$, then $k^+(x)$ is naturally a valuation ring, and (k, k^+) is an affinoid field over (A, A^+), and there is a map $R_x \to k^+(x)$ that becomes an isomorphism after t-adic completion.
- The ring $\mathcal{O}_{X,x}^+$ is t-adically Henselian, and $\mathcal{O}_{X,x}^+ \to k^+(x)$ induces an isomorphism after t-adic completion.
- The pairs $(\mathcal{O}_{X,x}^+, \mathfrak{m}_x)$ and $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ are Henselian.

Proof: 1: Consider the t-adic completion of the valuation ring R_x corresponding to x, then $(\widehat{k}(\mathfrak{p}_x),\widehat{R}_x)$ is an affinoid Tate ring over (A,A^+) that is mapped to x(and its generalizations), thus by universal property, there are unique maps from every rational subsets containing x to $(\widehat{k}_x,\widehat{R}_x)$, thus inducing a map $(\mathcal{O}_{X,x},\mathcal{O}_{X,x}^+)\to(\widehat{k}_x,\widehat{R}_x)$, which induces the desired valuation. And also we have $\mathcal{O}_{X,x}^+\subset\{f\in\mathcal{O}_{X,x}||f(x)|\leq 1\}$, for the converse, if $|f(x)|\leq 1$, then $U(\frac{f,1}{1})$ is rational subsets in U containing x, so by(II.7.6.43), $f\in\mathcal{O}_X(V)^+$, thus $f\in\mathcal{O}_{X,x}^+$.

2: for g not in $\mathfrak{m}_x$, $|g(x)|>|t(x)|^n$ for some n, so g is invertible in $U(\frac{t^n}{g})$ by(II.7.6.43), hence invertible in $\mathcal{O}_{X,x}$. Similarly for $\mathcal{O}_{X,x}^+$, as g is invertible in $U(\frac{1}{g})$.

3: This is clear from the construction of the valuation on $\mathcal{O}_{X,x}$ in item1.

4: As filtered colimits of Henselian pair is Henselian(I.7.10.3) and the function ring is complete, the stalk is Henselian. As for the completion, notice $\mathfrak{m}_x\subset\mathcal{O}_{X,x}^+$ and is t-divisible, thus $\mathcal{O}_{X,x}^+$ has the same t-adic completion as $k^+(x)$.

5: We first prove $(\mathcal{O}_{X,x}^+,t)$ is Henselian, for this, it suffices to prove $(\mathcal{O}_X^+(U),t)$ is Henselian, by(I.7.10.3). And $\mathcal{O}_X^+(U)$ is a filtered colimits of rings of definitions(II.7.6.13) and they are t-adically complete hence Henselian, so we are done by(I.7.10.3) again. Then so does $(\mathcal{O}_{X,x},\mathfrak{m}_x)$ because the property of being Henselian only depends on I(I.7.10.9). □

Cor. (II.7.6.46). By the construction of the valuation on the stalk, we have an inclusion of rings $k(\mathfrak{p}_x)\subset k(x)\subset\widehat{k}(x)$ that has the same completions, where the first is induced by the compatible map $\mathcal{O}_X(X)\to\mathcal{O}_X(U)$.

Def. (II.7.6.47) (Huber's Presheaf). Now by the universal property of function ring, we have a map between them induced by inclusion of rational subsets, so we can define the **structure presheaf** to be

$$\mathcal{O}_X(W)=\lim_{U\subset W\text{ rational}}\mathcal{O}_X(U),$$

and similarly for the **integral structure sheaf** $\mathcal{O}_X^+$.

Then there is a valuation of a point on $\mathcal{O}_X(W)$ by passing to the stalk, and

$$\mathcal{O}_X^+(W)=\{f\in\mathcal{O}_X(W)||f(x)|\leq 1,\forall x\in W\}.$$

because this is true for all rational subsets by adic nullstellensatz(II.7.6.29).

An affinoid Tate ring is called **sheafy** iff the structure sheaf $\mathcal{O}_X$ on $X=\mathrm{Spa}(A,A^+)$ is a sheaf. In this case, $\mathcal{O}_X^+$ is also a sheaf by the above formula.

Def. (II.7.6.48). The **Huber categrory** $\mathcal{V}$ is the category of triples $(X,\mathcal{O}_X,v_x)$ that $(X,\mathcal{O}_X)$ is local ringed space, and $\mathcal{O}_X$ has the structure of sheaf of complete topological rings, and v_x are continuous valuations on the stalk $\mathcal{O}_{X,x}$.

Def. (II.7.6.49). The category of **adic spaces** is the full subcategory of Huber category that is locally the adic spectrum of a sheafy affinoid Tate ring.

Prop. (II.7.6.50) (Spectrum Adjointness). For any affinoid adic space $X=(R,R^+)$ that $\mathcal{O}_X$ is sheaf, and Y is an arbitrary adic space, then there is a natural isomorphism

$$\mathrm{Hom}(Y,X)\cong\mathrm{Hom}((R,R^+),(\mathcal{O}_Y(Y),\mathcal{O}_Y^+(Y))).$$

Proof: Cf.[Generalizations of Formal And Rigid Varieties Huber Prop2.1(2)]. □

7 Perfectoid Spaces

Perfectoid Affinoid Algebra

Def. (II.7.7.1) (Perfectoid Affinoid K-algebras). Fix a perfectoid field K and write $\mathfrak{m} \subset K^0$ and $\mathfrak{m}^\flat \subset K^{\flat 0}$ for the maximal ideals, and choose a pseudo-uniformizer t that $|p| \le |t^\sharp| < 1$, $\pi = t^\sharp$. Then an **affinoid K-algebra** (R, R^+) is just an affinoid Tate ring over (K, K^0). It is called a **perfectoid affinoid K-algebra** iff R is a perfectoid algebra.

Prop. (II.7.7.2) (Tilting Equivalence). The categories of perfectoid affinoid algebras over K and $K^\flat$ are equivalent, where (R, R^+) is identified with $(R^\flat, R^{\flat +})$ iff $R^\flat$ is the tilting of R and

$$\begin{tikzcd} R^+/\mathfrak{m}R^0 \arrow[r, "\cong"] \arrow[d, hook] & R^{\flat +}/\mathfrak{m}^\flat R^{\flat 0} \arrow[d, hook] \\ R^0/\mathfrak{m}R^0 \arrow[r, "\cong"] & R^{\flat 0}/\mathfrak{m}^\flat R^{\flat 0} \end{tikzcd}.$$

Moreover, R^+/π is semi-perfect, and $R^{+\flat} \cong R^{\flat +}$ as a subring of $R^{0\flat} \cong R^{\flat 0}$.

Proof: The case $R^+ = R^0$ is already known by tilting equivalence(II.7.4.5) and(II.7.4.6).

By(II.7.6.8) and(II.7.6.13), $\mathfrak{m}R^0 = R^{00} \subset R^+ \subset R^0$, thus $R^+ \to R^0$ is an almost isomorphism and R^+ is determined by its image $\overline{R^+} \subset R^0/\mathfrak{m}R^0$, which is integrally closed if R^+ does, so the identification is clear.

For the semi-perfectness: as $R^+/\mathfrak{m}R^0$ is integrally closed, it is perfect. Now $R^+ \to R^0$ is an almost isomorphism, so Frob on R^+/π is almost surjective because it does on R^0/π by definition, and now we know Frob is surjective on R^+/π by(I.11.2.3).

To show $R^{+\flat} \cong R^{+\flat}$, we show there is a Cartesian diagram

$$\begin{tikzcd} R^{+\flat} \arrow[r] \arrow[d, hook] & R^{\flat +}/\mathfrak{m}^\flat R^{\flat 0} \arrow[d, hook] \\ R^\flat \arrow[r] & R^\flat/\mathfrak{m}^\flat R^{\flat 0} \end{tikzcd}$$

, but this is the Cartesian diagram

$$\begin{tikzcd} R^+/\pi \arrow[r] \arrow[d, hook] & R^+/\mathfrak{m}R^0 \cong R^{\flat +}/\mathfrak{m}^\flat R^{\flat 0} \arrow[d, hook] \\ R^0/\pi \arrow[r] & R^0/\mathfrak{m}^\flat R^0 \end{tikzcd}$$

applied the functor $(-)^{perf}$, which preserves limits(II.7.1.4). (Notice that $R^+/\mathfrak{m}R^0 \cong R^{\flat +}/\mathfrak{m}^\flat R^{\flat 0}$ is already perfect). $\square$

Cor. (II.7.7.3). Notice that the proof also shows that $R^+ \to R^0$ is an almost isomorphism, thus if R is a perfectoid K-algebra, then R^+ is automatically a perfectoid K^{0a}-algebra by(II.7.4.1).

Cor. (II.7.7.4) (Perfectoid Affinoid Field). A perfectoid affinoid K-algebra (R, R^+) is called an **affinoid field** iff R is a perfectoid field and R^+ is an open valuation ring.

Notice this is equivalent to $R^+/\mathfrak{m}R^0$ is a valuation ring in $R^0/\mathfrak{m}R^0$. In particular, combining with(II.7.4.11), affinoid perfectoid fields are preserved under tilting and untilting.

Prop. (II.7.7.5) (Filtered Colimits of Perfectoid Affinoid K-Algebras). The category of perfectoid affinoid K-algebras has filtered colimits, and it is just the colimits in the category of complete uniform affinoid Tate rings(II.7.6.38). In particular, the filtered colimits of (A_i, A_i^+) is $(\operatorname{colim}_i A_i, \operatorname{colim} A_i^+)$.

Proof: The colimit is perfectoid because the filtered colimits is exact. $\square$

Affinoid Perfectoid Spaces and Tilting

Prop. (II.7.7.6) (Tilting Rational Subsets). For perfectoid affinoid K-algebra (R, R^+) over a perfectoid field K,

- The $\sharp$ map induces an isomorphism $X = \mathrm{Spa}(R, R^+) \cong X^\flat = \mathrm{Spa}(R^\flat, R^{\flat 0})$ that identifies rational open subsets.
- For a rational subset U with tilting $U^\flat$, the complete affinoid Tate algebra $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is perfectoid over R with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.

Proof: This follows from(II.7.7.10). $\square$

Lemma (II.7.7.7) (Huber's Presheaf in Char p). Assume $\mathrm{char}K = p$ and $U = X(\frac{f_1,\ldots,f_n}{g})$ is a rational subset that $f_i, g \in R^+$ and $f_n = \pi^N$, then:

- Consider the subring $R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}]$, its π-adic completion $(R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle)^a$ is a perfectoid K^{0a}-algebra.
- The map $R^+[X_i^{\frac{1}{p^\infty}}] \to R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}]$ has kernel containing and almost equal to $I = (g^{\frac{1}{p^m}} X_i^{\frac{1}{p^m}} - f_i^{\frac{1}{p^m}})$.
- $\mathcal{O}_X(U)$ is a perfectoid K-algebra and $\mathcal{O}_X(U)^{0a} \cong (R^+\langle(\frac{f_i}{g})^{\frac{1}{p^m}}\rangle)^a$.

Proof: 1: The ring $R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}]$ is perfect and π-torsion-free, and R^+ is semi-perfect, thus its completion is clearly a perfectoid K^{0a}-algebra by(II.7.4.1).

2: Clearly $I \subset \mathrm{Ker}$ and notice $R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}][\pi^{-1}] = R[g^{-1}]$ as $f_n = \pi^N$, so $I[\pi^{-1}] = \mathrm{Ker}[\pi^{-1}]$. Now consider the mapping

$$P_0 = R^+[X_i^{\frac{1}{p^\infty}}]/I \to R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}]$$

Now this map is an isomorphism ofter inverting π, so the kernel is π^∞-torsion. But we have $I = I^{[p]}$ because R^+ is semi-perfect, so P_0 is perfect, so the kernel must be almost zero.

3: Consider the inclusion $R^+[\frac{f_i}{g}] \hookrightarrow R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}]$, we show the cokernel is killed by π^{nN}: as $f_n = \pi^N$,

$$\pi^{nN}\prod_{i=1}^n (\frac{f_i}{g})^{\frac{1}{p^{a_i}}} = \prod_{i=1}^n (f_i^{\frac{1}{p^{a_i}}} g^{1-\frac{1}{p^{a_i}}})\frac{f_n}{g} \in R^+[\frac{f_i}{g}].$$

So these two ring has the same π-adic completion, the first one is just $\mathcal{O}_X(U)$ by the construction(II.7.6.42), so $\mathcal{O}_X(U)$ is perfectoid K-algebra, and the isomorphism is by tilting equivalence $\mathrm{Perf}_K \cong \mathrm{Perf}_{K^{0a}}$(II.7.4.5). $\square$

Lemma (II.7.7.8) (Huber's Presheaf in Char 0). Let $U = X(\frac{f_1,\ldots,f_n}{g})$ is a rational subset that f_i, g are perfect elements in R^+, $f_i = a_i^\sharp, g = b^\sharp$, and $f_n = \pi^N$, so f_i, g have compatible p^n-th roots, then let $U^\flat = X^\flat(\frac{f_1,\ldots,f_n}{g})$ be the tilting of U, U is the inverse image of $U^\flat$ along the map $X \to X^\flat$. Then the conclusion of(II.7.7.7) is also true, and moreover, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ tilts to $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.

Proof: 1, 2 of(II.7.7.7): Notation as before, there is a map $P_0 = R^+[X_i^{\frac{1}{p^\infty}}]/I \to R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}]$, and an inclusion $R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}] \to \mathcal{O}_X^+(U)$. Now write (S, S^+) for the untilt of the perfectoid $(R^\flat, R^{\flat+})$-algebra $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$, then by the tilting process(II.7.7.2), $\mathrm{Spa}(S, S^+)$ maps into U, so by the universal property, there is a map

$$\mu : (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (S, S^+).$$

Consider the composition

$$P_0 \xrightarrow{a_0} R^+[(\frac{f_i}{g})^{\frac{1}{p^\infty}}] \xrightarrow{d_0} S^+,$$

we prove their completion gives the same K^{0a}-algebras(notice S^+ is already complete): a_0 is surjective, thus so does its completion, the map $d_0 \circ a_0$ is almost isomorphism modulo π by(II.7.7.7) item2 and tilting equivalence, so does its completion. Now(II.7.3.2) tells us the completion of $d_0 \circ a_0$ is almost isomorphism, so does a and d as a is surjective.

By the way, we know that $R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle[\pi^{-1}]$ is the untilt of $\mathcal{O}_{X^\flat}^+(U^\flat)$.

3 of(II.7.7.7) is proved as before.

For the tilting, by the above, we already know $R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}$ tilts to the perfectoid $K^{\flat 0a}$-algebra $\mathcal{O}_{X^\flat}(U^\flat)^{0a}$, and by item3 $\mathcal{O}_X(U)$ tilts to the perfectoid $K^\flat$-algebra $\mathcal{O}_{X^\flat}(U^\flat)$. Now the question is the tilt of $\mathcal{O}_X^+(U)$, notice as in the proof of item1, there is a natural map

$$(R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle[\pi^{-1}], R^+\langle(\frac{f_i}{g})^{\frac{1}{p^\infty}}\rangle) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U)),$$

whose tilting gives by university of Huber's presheaf a map

$$\xi : (\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))^\flat.$$

These two map μ, ξ are inverse to each other, showing that the tilting of $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$. □

Lemma (II.7.7.9) (Approximation Lemma). Assume $R = K\langle T_0^{\frac{1}{p^\infty}}, \ldots, T_0^{\frac{1}{p^\infty}}\rangle$, $f \in R^0$ is homogenous of degree $d \in \mathbb{N}[p^{-1}]$, then for any $c > 0, \varepsilon > 0$, there exists some $g_{c,\varepsilon} \in R^{\flat 0}$ homogeneous of degree d that

$$|(f - g^\sharp)(x)| \leq |\pi|^{1-\varepsilon} \max\{|f(x)|, |\pi|^c\}.$$

In particular, if $\varepsilon < 1$, then

$$\max\{|f(x)|, |\pi|^c\} = \max\{|g_{c,\varepsilon}^\sharp(x)|, |\pi|^c\}.$$

Proof: Cf.[Sholze Perfectoid Spaces, Lemma6.5]**?**. □

Prop. (II.7.7.10). For an arbitrary perfectoid K-algebra R,

- The same conclusion of(II.7.7.9) holds.
- For $f, g \in R$, there exist $a, b \in R^\flat$ that $X(\frac{f,\pi^c}{g}) = X(\frac{a^\sharp,\pi^c}{b^\sharp})$. In particular, any rational subsets U of X comes from $X^\flat$, thus(II.7.7.8) applies for U.
- For any $x \in X$, the non-Archimedean field $\widehat{k(x)}$ is perfectoid.
- $X \to X^\flat$ is a homeomorphism preserving rational subsets.

Proof: 1: Using the tilting equivalence, we can write $f = g_0^\sharp + \pi g_1^\sharp + \ldots + \pi^c g_c^\sharp + f_{c+1}\pi^{c+1}$, let $f_0 = g_0^\sharp + \pi g_1^\sharp + \ldots + \pi^c g_c^\sharp$. Then notice that the solution $g_{c,\varepsilon}$ for f_0 is suitable for f as well. So now if we consider the mapping

$$\mu : K\langle T_0^{\frac{1}{p^\infty}}, \ldots, T_n^{\frac{1}{p^\infty}}\rangle \to R : T_i \to g_i^\sharp,$$

together with its tilting

$$\mu^\flat : K^\flat\langle T_0^{\frac{1}{p^\infty}}, \ldots, T_n^{\frac{1}{p^\infty}}\rangle \to R^\flat.$$

Then for the approximation $g_{c,\varepsilon}^\sharp$ for $f' = \sum \pi^i T_i$, $\mu^\flat(g)$ is what we are searching for.

2: Using 1, we find $a, b \in R^\flat$ that $|(g - b^\sharp)(x)| < \max\{|g(x)|, \pi^c\}$ and $\max\{|f(x)|, |\pi|^c\} = \max\{|a^\sharp(x)|, |\pi|^c\}$ for all $x \in \mathrm{Spa}(R, R^+)$. Then it is routine to check that $X(\frac{f,\pi^c}{g}) = X(\frac{a^\sharp,\pi^c}{b^\sharp})$.

3: By(II.7.6.45), $\widehat{k^+(x)}$ equals the completion of the colimit colim $\mathcal{O}_X^+(U)$ over rational subsets U containing x. As these are all perfectoid K^{0a}-algebras(II.7.7.3), and completion of the filtered limits of perfectoid K^{0a}-algebras is perfectoid(II.7.4.10), we know that $\widehat{k^+(x)}$ is perfectoid over K^{0a}, thus inverting π shows $k(x)$ is also perfectoid over K.

4: this is injective because X is T_0 and a rational subset is the untilt of a ratinal subset of $X^\flat$ by item2. For surjectivity, a point of $X^\flat$ determines a continuous map $(R^\flat, R^{\flat+}) \to (\widehat{k(x)}, \widehat{k^+(x)})$, thus by untilting(II.7.7.2) corresponds to a map $(R, R^+) \to (L, L^+)$, then (L, L^+) is a perfectoid field, by(II.7.7.4), so it sorresponds to a point $y \in X$. Then it is clear that y maps to x, because the diagram $\begin{array}{ccc} R^\flat & \xrightarrow{\sharp} & R \\ \downarrow & & \downarrow \\ \widehat{k(x)} & \xrightarrow{\sharp} & L \end{array}$ is commutative. □

Tate's Acyclicity

Def. (II.7.7.11) (p-Finite Tate Ring). Denote $L = \widehat{\mathbb{F}_p[[t]]_{perf}}[t^{-1}]$, an $\mathbb{F}_p[t]$ algebra A^+ is called **algebraically admissible** if it is f.p., reduced, t-torsion-free, and integrally closed in $A^+[t^{-1}]$. A perfectoid affinoid L-algebra (R, R^+) is called **p-finite** if it is the completion of the perfection(II.7.6.39) of a uniform Tate ring of the form $(A^+[t^{-1}], A^+)$, where A^+ is algebraically admissible.

Lemma (II.7.7.12) (Tate's Acyclicity for Classical Affinoid Algebra). If A^+ is an algebraically admissible $\mathbb{F}_p[t]$-algebra, then $(A^+[t^{-1}], A^+)$ is a uniform affinoid Tate algebra(because it is finite), and:

- For any rational subset $U \subset X$, the structure presheaf $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is also uniform, and it is a perfection of an algebraically admissible $\mathbb{F}_p[t]$-algebra, so $\mathcal{O}_X^+(U) = \mathcal{O}_X(U)^0$.
- For any covering $\mathfrak{U} : U_i \to X$ of rational subsets, the Čech cohomology groups $H^i(\mathfrak{U}, \mathcal{O}_X^+)$ are all killed by t^N for N large.
- (A, A^+) is sheafy, with $H^i(X, \mathcal{O}_X^+)$ being t^∞-torsion for all i.

Proof: ? □

Lemma (II.7.7.13) (Tate's Acyclicity for p-Finite Perfectoid Algebras). Let (R, R^+) be a p-finite perfectoid L-algebra that comes from the completion of perfection of (A, A^+), then:

- The map $X = \mathrm{Spa}(R, R^+) \to Y = \mathrm{Spa}(R, R^+)$ is a homeomorphism.

- For rational subset $V \subset Y$ with preimage $U \subset X$, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is the completion of the perfection of $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$.
- For any covering $\mathfrak{U} : U_i \to X$ of rational subsets, the Čech cohomology groups $H^i(\mathfrak{U}, \mathcal{O}_X^0)$ are all almost zero.
- (R, R^+) is sheafy, with $H^i(X, \mathcal{O}_X^+)$ almost zero for all $i > 0$.

Proof: 1: This is because the adic spectrum is insensitive for perfection(II.7.6.39) and completion(II.7.6.29).

2: This is by the universal property, as these two are both the universal elements for the complete and affinoid adic spaces mapping to X that factors through U.

3: The complex calculating $H^i(\mathfrak{U}, \mathcal{O}_X^0)$ is just the complex calculating $H^i(\mathfrak{U}, \mathcal{O}_Y^+)$ under completion of perfection((II.7.6.38) used). So(II.7.7.12) and(II.7.7.14)(applied to every element) shows that the perfection makes the complex almost acyclic, and this is preserved under completion as $(-)^a$ is exact.

4: The complex calculating $H^i(\mathfrak{U}, \mathcal{O}_X)$ is just the complex calculating $H^i(\mathfrak{U}, \mathcal{O}_X^+)$ inverting t, thus they are all 0 as localization is exact. For the second, it is because of item3 and the fact $\mathcal{O}_X^+$ is almost isomorphic to $\mathcal{O}_X^0$(II.7.7.3). □

Lemma (II.7.7.14). Let A be a ring with an element t that admits compatible p^n-th roots, then for an A-module M that $t^N M = 0$, consider the Frobenius pushforward $M \to F_* M$, then the colimit $\mathrm{colim}_{F_*} F_*^n M$ is naturally a module over A_{perf}, and it is annihilated by $t^{\frac{1}{p^n}}$ for all n.

Proof: The A_{perf} structure is natural, and notice $F_*^k M$ is annihilated by $t^{\frac{N}{p^k}}$, thus naturally the colimit is annihilated by $t^{\frac{1}{p^n}}$ for all n. □

Prop. (II.7.7.15) (Noetherian Approximation in Char$p$). If K is a perfectoid field of char$p$ with pseudo-uniformizer t, then K is an extension of $L = \widehat{\mathbb{F}_p[[t]]_{perf}}[t^{-1}]$, and If A is an K^0-perfectoid algebra that is integrally closed in $A[t^{-1}]$, then:

- A is a completion of a filtered colimit $\widehat{\mathrm{colim}_i B_i}$ that B_i are p-finite, that induces an homeomorphism
$$\mathrm{Spa}(A[t^{-1}], A) \cong \lim_i \mathrm{Spa}(B_i[t^{-1}], B_i)$$
that each rational subset of $\mathrm{Spa}(A[t^{-1}], A)$ comes from a rational subset of some $\mathrm{Spa}(B_i[t^{-1}], B_i)$.
- If $U_i \subset \mathrm{Spa}(B_i[t^{-1}], B_i)$ is a compatible system of rational subsets that corresponds to $U \subset \mathrm{Spa}(A, A^+) = X$, then
$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \cong \mathrm{colim}_j(\widehat{\mathcal{O}_j(U_j)}, \mathcal{O}_j^+(U_j)).$$

Proof: 1: $A = \mathrm{colim}_i A_i$, where A_i are all the f.p. $\mathbb{F}_p[t]$-algebras in A. Then each A_i is reduced(as A is complete and integrally closed in $A[t^{-1}]$) and t-torsion-free, and we can assume they are integrally closed in $A_i[t^{-1}]$ because A does, by passing to their integral closure.

Then applying the $(-)_{perf}$ functor gives $\mathrm{colim}_i (A_i)_{perf} = A$, as A is perfect, and applying the completion gives
$$\widehat{\mathrm{colim}_i \widehat{A_i}} = A,$$
as A is already complete, so we are done.

2: This is immediate from 1 and(II.7.6.38).
3: This is because by universal property for Huber presheaves, there are pushouts diagrams

$$\begin{tikzcd}
(B_i[t^{-1}],B_i) \arrow[r] \arrow[d] & (B_j[t^{-1}],B_j) \arrow[r] \arrow[d] & \ldots \arrow[r] & (A[t^{-1}],A) \arrow[d] \\
(\mathcal{O}_{X_i}(U_i),\mathcal{O}^+_{X_i}(U_i)) \arrow[r] & (\mathcal{O}_{X_j}(U_j),\mathcal{O}^+_{X_j}(U_j)) \arrow[r] & \ldots \arrow[r] & (\mathcal{O}_X(U),\mathcal{O}^+_X(U))
\end{tikzcd}$$

So the conclusion follows as colimits commutes with colimits. □

Prop. (II.7.7.16) (Tate's Acyclicity for Perfectoids). Fix a perfectoid field K and a perfectoid affinoid K-algebra (R,R^+) with adic spectrum $X=\mathrm{Spa}(R,R^+)$, then

- (R,R^+) is sheafy, i.e. $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves.
- $\mathcal{O}_X^+(X)=R^+$, and $H^i(X,\mathcal{O}_X^+)$ is almost zero for $i>0$.
- $\mathcal{O}_X(X)=R$, and $H^i(X,\mathcal{O}_X)=0$ all $i>0$.

Proof: As in the proof of(II.7.7.13), it suffices to prove that the complex computing $H^i(\mathfrak{U},\mathfrak{O}^0_X)$ is almost exact. For this, notice each term is π-adically complete and flat by(II.7.7.6), so it suffices to prove it is almost exact modulo π(II.7.3.2). Then by the tilting equivalence, it suffices to prove for $X^\flat$. So we may assume at first that K is of char$p$. Then we may replace K by $L=\widehat{\mathbb{F}_p[[t]]_{perf}}[t^{-1}]$.

But then Noetherian approximation(II.7.7.15) shows that the rational subrings are completion of filtered colimits of p-finite K-algebras((II.7.6.38) used), and then we reduced to the p-finite case, as in the proof of(II.7.7.13). □

Cor. (II.7.7.17) (Perfectoid Space). Now for any perfectoid affinoid K-algebra, we associated to it an affinoid adic space $\mathrm{Spa}(R,R^+)$, called an **affinoid perfectoid space**.

Tate's acyclicity(II.7.7.16) shows that the adic spectrum of a perfectoid affinoid K-algebra is sheafy, so we can defined the category of **perfectoid spaces** is defined to be the full subcategory of adic spaces that is locally isomorphic to an affinoid perfectoid space.

Prop. (II.7.7.18) (Tilting Equivalence for Perfectoid Spaces). Fix a perfectoid field K, then for any perfectoid space X/K, there is a unique perfectoid space $X^\flat/K^\flat$ that satisfies: $X(R,R^+)\cong X^\flat(R^\flat,R^{\flat+})$ functorially, called the **tilt** of X. Moreover, this $X^\flat$ satisfies naturally $|X|\cong|X^\flat|$.

When X is an affinoid perfectoid space, this tilting coincides with that of(II.7.7.6).

And this tilting induces an equivalence between the category of perfectoid spaces over K and $K^\flat$.

Proof: Firstly, the universal property truly determines the tilt $X^\flat$ uniquely: if there are two tilts X_1,X_2, as they are locally affinoid perfectoid like $\mathrm{Spa}(R^\flat,R^{\flat+})$ by(II.7.7.6), the immersion map $\mathrm{Spa}(R^\flat,R^{\flat+})\to X_1$ determines via the functorial isomorphism a morphism $\mathrm{Spa}(R^\flat,R^{\flat+})\to X_2$. Now X_1 has a sheaf structure, so these morphisms glue to give a morphism $X_1\to X_2$. The same argument shows conversely there is a morphism $X_2\to X_1$, and they are clearly converse to each other, so $X_1\cong X_2$.

The construction of X is just the glueing of the tilting of the affinoid perfectoid spaces, as the tilting defined in(II.7.7.6) is a functor. The universal property is verified by just checking on the affinoid perfectoid spaces, as we can glue using the sheaf property. For the affinoid case, we should use(II.7.6.50). The last assertion is by(II.7.7.6). □

Prop. (II.7.7.19) (Fiber Products of Perfectoid Spaces). The category of perfectoid spaces over K admits fiber products.

Proof: Perfectoid spaces are constructed by glueing, thus we just need to show that the perfectoid $K^\flat$-algebras has fiber pushouts. For this, if $X = (A, A^+), Y = (B, B^+), Z = (C, C^+)$, define $X \otimes_Y Z = (D, D^+)$, where D is the completion of $A \otimes_B C$, and D^+ is the completion of the integral closure of $A^+ \otimes_{B^+} C^+$ in D. Then D is a perfectoid affinoid K-algebra, and it is truly the filtered colimits, as in(II.7.6.16). □

Final Proof of Almost Purity Theorem

Def. (II.7.7.20) (Finite Étale Map of Adic Spaces). A map $(A, A^+) \to (B, B^+)$ is called **finite étale** if $A \to B$ is finite étale, and B^+ is the integral closure of A^+ in B.

A map $f : X \to Y$ of adic spaces is called **finite étale** if there is a cover of Y by affinoids $V \subset Y$ that $U = f^{-1}(V)$ are all affinoids, and the map $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is finite étale. Write $Y_{fét}$ for the category of all such maps.

Def. (II.7.7.21). For convenience, in case of perfectoid affinoid K-algebras, we call $(A, A^+) \to (B, B^+)$ **strongly finite étale** if it is finite étale and B^{+a} is almost finite étale over A^{+a}. And similarly define the strongly finite étale maps of perfectoid affinoid K-spaces. Write $Y_{sfét}$ for the category of all such maps.

Finally we will prove that if (A, A^+) is perfectoid, then any finite étale map $(A, A^+) \to (B, B^+)$ is strongly finite étale.

Prop. (II.7.7.22). We have an equivalence of categories $X_{sfét} \cong X^\flat_{sfét}$. For this, use(II.7.7.18) and the proven part of(II.7.5.1) and notice that

$$A^{+a}_{afét} = A^{0a}_{afét} \cong A^{\flat 0a}_{afét} = A^{\flat +a}_{afét},$$

and the integral closure clearly corresponds.(It ignores the problem that $R^{0a}_{fét} \to R_{fét}$ hasn't been proven essentially surjective).

Prop. (II.7.7.23) (Gabber-Ramero). If A is a finite K^0-algebra that is π-adically Henselian, then

$$A[\pi^{-1}]_{fét} \cong \widehat{A}[\pi^{-1}]_{fét}.$$

Proof: Cf.[Almost Ring Theory P5.4.53]. □

Cor. (II.7.7.24) (Finite Étale Covers and Direct Limits of Complete Uniform Rings). Let (A_i, A_i^+) be a filtered system of complete uniform affinoid K-algebras, and (A, A^+) be their colimit in the category of complete uniform affinoid Tate rings, then $2-\mathrm{colim}_i\, A_{i,fét} \cong A_{fét}$ as categories.

Proof: By(II.7.6.38), A^+ is the π-adic completion of the algebraic colimit B^+ of A_i^+, and $A = A^+[t^{-1}]$. Each A_i is complete and π-torsion-free, thus the colimit is Henselian and torsion-free(I.7.10.3)(I.7.10.6). Then the proposition(II.7.7.23) shows that $B^+[\pi^{-1}]_{fét} \cong A_{fét}$. Now the theorem follows from(II.7.6.38). □

Prop. (II.7.7.25) (Strongly Finite Étale Maps Form a Stack). If $f : X \to Y$ is a strongly finite étale map that $Y = \mathrm{Spa}(A, A^+)$ is an affinoid perfectoid, then X is also affinoid perfectoid, and the structure map $(\mathcal{O}_X(X), \mathcal{O}_X^+(X)) \to (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ is strongly finite étale.

Proof: By(II.7.7.22), it suffices to prove in char$p$. Then we can replace K by $L = \widehat{\mathbb{F}_p[[t]]_{perf}}[t^{-1}]$. Then by Noetherian approximation(II.7.7.15), we can assume that Y is a limit of p-finite affinoids $\mathrm{Spa}(B_i, B_i^+)$. As both rational subsets and finite étale algebras pass through filtered colimit, and adic spectrum is quasi-compact(II.7.6.35), we can assume that a finite étale cover of Y arises through base change of some $\mathrm{Spa}(B_i, B_i^+)$. So it suffices to prove the proposition in case of Y p-finite. Then Y is a completion of perfection of some algebraically admissible ring over $\mathbb{F}_p[t]$. Then by the above argument again, we can assume that Y is algebraically admissible.

Now a classical theorem(Cf.[Étale cohomology of rigid analytic varieties and adic spaces, Huber 1.6.6(2)]) shows that the finite étale cover of Y is global finite étale $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ in this case. Notice the strongness is not needed because we are working in char p, where almost purity theorem is already proven. □

Cor. (II.7.7.26). For an affinoid perfectoid space $Y = \mathrm{Spa}(R, R^+)$, the functor $X \mapsto \mathcal{O}_X^+(X)$ defined an equivalence of categories $Y_{sf\acute{e}t} \cong R^{+a}_{af\acute{e}t}$, and the functor $X \mapsto \mathcal{O}_X(X)$ gives a fully faithful functor $Y_{sf\acute{e}t} \to R_{f\acute{e}t}$.

Lemma (II.7.7.27). If $X = \mathrm{Spa}(A, A^+)$ is a perfectoid affinoid space, then there is a fully faithful functor $M \mapsto M \otimes_A \mathcal{O}_X$ from the category of sheaves on X. This is because M is flat and X is sheafy.

Prop. (II.7.7.28) (Proof of Almost Purity Theorem). Fix a perfectoid affinoid K-algebra (R, R^+), if $S \in R_{f\acute{e}t}$, then the integral closure of S^+ in R^+ lies in $R^{+a}_{af\acute{e}t}$, and this gives an inverse to the morphism d in(II.7.5.1), thus finishing the proof of almost purity theorem.

Proof: Continuing the proof of(II.7.5.1), it suffices to show that $d : R^{+a}_{af\acute{e}t} \to R_{f\acute{e}t}$ is essentially surjective, because $S^+ \to \overline{S} \subset R^+$ is the only possible inverse, by the almost purity theorem in char$p$ and tracing the tilting equivalence(II.7.4.6). Given(II.7.7.26), it suffices to prove that for $X = \mathrm{Spa}(R, R^+)$, the prestacks $X_{sf\acute{e}t} \cong X_{f\acute{e}t}$, where $X_{sf\acute{e}t}(U) = \mathcal{O}_X^{+a}(U)_{sf\acute{e}t}$, and $X_{f\acute{e}t}(U) = \mathcal{O}_X(U)_{f\acute{e}t}$.

We use(III.12.1.2), firstly $X_{sf\acute{e}t}$ is a stack, by(II.7.7.25), and for each U, $X_{sf\acute{e}t}(U) \to X_{f\acute{e}t}(U)$ is fully faithful by almost purity theorem(II.7.5.1). $X_{f\acute{e}t}$ is separated by(II.7.7.27), because the structure section of an element $S \in X_{f\acute{e}t}$ is determined its value on the stalk.

Its left to prove that their stalks are equal, for this, use the formula

$$\mathrm{colim}_{x\in U}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (\widehat{k(x)}, \widehat{k^+(x)})$$

in the category of complete uniform affinoid K-algebras(they are all perfectoids(II.7.7.6) thus uniform), by definition. So we get by(II.7.7.24):

$$\mathrm{colim}_{x\in U}\, \mathcal{O}_X(U)_{f\acute{e}t} \cong \widehat{k(x)}_{f\acute{e}t},$$

and by(II.7.7.24) together with almost purity theorem:

$$\mathrm{colim}_{x\in U}\, \mathcal{O}_X^+(U)_{af\acute{e}t} \cong \mathrm{colim}_{x^\flat\in U^\flat}\, \mathcal{O}_X^{\flat+}(U^\flat)_{af\acute{e}t} \cong \widehat{k(x^\flat)}_{af\acute{e}t} \cong \widehat{k^+(x)}_{af\acute{e}t}.$$

Now we have already proved the almost purity over fields, thus $\widehat{k^+(x)}_{f\acute{e}t} \cong \widehat{k(x)}_{f\acute{e}t}$, so their stalks are the same. □

II.8 Fargues-Fontaine Curve

Basic references are [FF curves Lurie], [FF Curve Johannes], [The Fargues-Fontaine Curve and Diamonds Mathew Morrow], [Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de Hodge p-adique]

1 Fontaine's Ring

Fontaine's Ring A_{inf}

Def. (II.8.1.1) (Fontaine's Ring A_{inf}). Let $C^\flat$ be a perfectoid field of char$p$, Fontaine's ring A_{inf} is defined to be the Witt vectors $W(\mathcal{O}_C^\flat)$.

All the char0 untilts of C is denoted by Y. and $Y_{[a,b]}$ denotes those untilts that $a \le |p|_K \le b$.

Prop. (II.8.1.2). By(II.7.4.7), if K is perfectoid field of char0 with tilt $C^\flat$, there is a diagram

$$\begin{array}{ccc} A_{\text{inf}} & \xrightarrow{\theta} & \mathcal{O}_K \\ \downarrow & & \downarrow \\ \mathcal{O}_C^\flat & \xrightarrow{\overline{\theta}} & \mathcal{O}_K/\pi \end{array}.$$

And the kernel $\operatorname{Ker}\theta$ is generated by $\xi = \pi u - [t]$, where $[t]$ is the Techmuller lift, because it generate after modulo p, and use Nakayama.

Def. (II.8.1.3) (Distinguished Elements). We stick to the case $\pi = p$, and call an element $\xi \in A_{\text{inf}}$ **distinguished** iff it is of the form $[t] - pu$, where $|t|_{C^\flat} < 1$ and u is invertible in A_{inf}. Equivalently, ξ is of the form

$$[c_0] + [c_1]p + [c_2]p^2 + \dots$$

where $|c_0|_{C^\flat} < 1$ and $|c_1|_{C^\flat} = 1$.

Lemma (II.8.1.4). If two distinguished elements divides each other, then they differ by a unit.

Proof: Use the Teichmuller expansion. □

Lemma (II.8.1.5). If R is a commutative ring, $x, y \in R$, if x is not a zero-divisor in R and R is x-adically complete Hausdorff, and y is not a zero-divisor in R/x and R/x is y-adically complete, then the same is true with x, y interchanged.

Proof: ? □

Prop. (II.8.1.6) (Untilts and Distinguished Elements). (II.8.1.2) shows that for any untilt K of $C^\flat$, the kernel is generated by a distinguished element. Conversely, for any distinguished element ξ, $A_{\text{inf}}/(\xi)$ can be identified with the valuation ting $\mathcal{O}_K$ of a perfectoid field K. and

$$\mathcal{O}_C^\flat = A_{\text{inf}}/p \to A_{\text{inf}}/(\xi, p) \cong \mathcal{O}_K/p$$

exhibits K as an untilt of $C^\flat$.

Proof: May assume $\xi = [t] - up$ and $t \neq 0$. Consider the mapping $\theta : A_{\text{inf}} \to A_{\text{inf}}/(\xi) = \mathcal{O}_K$, and denote $\theta([x])$ by $x^\sharp$.

Firstly, we can apply lemma(II.8.1.5) to ξ and p to conclude that A_{inf} is ξ-complete and ξ-torsion-free, and $\mathcal{O}_K$ is p-adically complete and p-torsion-free.

Now for any $y \in \mathcal{O}_K$ is p-adically complete, there is a $x \in \mathcal{O}_C^\flat$ that $(y) = (x^\sharp)$: multiplying p-power, we can assume y is not divisible by p, and there is a x that $y \equiv x^\sharp \mod p$, thus x is not divisible by t. Now $t = xx'$ for some $x' \in \mathfrak{m}_C^\flat$, thus $y = x^\sharp + t^\sharp w = x^\sharp(1 + x'w)$, and $1 + x'w$ is invertible in $\mathcal{O}_K$.

Next we prove $\mathcal{O}_K$ is an integral domain: It suffices to show any $y \neq 0 \in \mathcal{O}_K$ is not a zero-divisor. We can assume $y = x^\sharp$, by what just proved, and then x divides t^n for some n, so it suffices to consider $y = t^{n\sharp} = p^n$, and p^n is not a zero-divisor by what just proved.

Now we can endow $\mathcal{O}_K$ with the valuation $|y| = |x|_{C^\flat}$ for $y = x^\sharp u$, and extend it to the quotient field K. Then this is a Non-Archimedean valuation and the residue field has char p because $|p| < 1$, and K has char0, because $p \neq 0$ in K. And it is p-adically complete.

Finally, $\mathcal{O}_K/p\mathcal{O}_K \cong A_{\text{inf}}/(\xi, p) = \mathcal{O}_{C^\flat}/\pi$, so the Frobenius is surjective, thus $K = K(\mathcal{O}_K)$ is a perfectoid field. ☐

Cor. (II.8.1.7). The correspondence $\xi \mapsto Quot(A_{\text{inf}}/(\xi))$ induces a bijection

$$\{\text{Distinguished elements}\}/\text{units} \cong \{\text{Untilts of } C^\flat\}/\text{isomorphisms}.$$

Prop. (II.8.1.8) (A_{inf} as Holomorphic Function in p). Any element in A_{inf} can be written uniquely as a unique Teichmuller representation $[c_0] + [c_1]p + [c_2]p^2 + \ldots +$. Now we can regard these elements as holomorphic functions on $B(0,1)$, and any untilts K of $\mathcal{O}_{C^\flat}$ can be regarded as points, where A_{inf} take value $c_0^\sharp + c_1^\sharp p + \ldots \in \mathcal{O}_K$ at the point K.

This map can in fact be extended to $A_{\text{inf}}[\frac{1}{[t]}, \frac{1}{p}]$ that

$$A_{\text{inf}} \hookrightarrow A_{\text{inf}}[\frac{1}{[t]}] \hookrightarrow A_{\text{inf}}[\frac{1}{[t]}, \frac{1}{p}] \to K.$$

called the valuation map.

Ring B

Def. (II.8.1.9) (Fontaine's Ring B). If compared to the complex case, the elements of A_{inf} are just elements $\sum a_n z^n$ that $|a_n| \leq 1$, this are not all the holomorphic functions on $B(0,1)$, which is $\sum a_n z^n$ that $\limsup |a_n| \leq 1$. This leads to a enlargement of A_{inf}:

For $0 < a \leq b < 1$ in the value group of $C^\flat$, $|\pi_a| = a, |\pi_b| = b$, define

$$B_{[a,b]} = \widehat{A_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]}[p^{-1}],$$

this is definable at any untilts K that $a \leq |p|_K \leq b$.

Then $B_{[a,b]}$ is an algebra over $A_{\text{inf}}[\frac{1}{[t]}, \frac{1}{p}]$, and define $B = \varprojlim B_{[a,b]}$.

Prop. (II.8.1.10) (Gauss Norm). Any element f in $A_{\text{inf}}[\frac{1}{[t]}, \frac{1}{p}]$ is of the form $\sum_{n>>-\infty}[c_n]p^n$, where $\{|c_n|\}$ is bounded. So we can define the valuation $|f|_\rho = \sup\{|c_n|\rho^n\}$, it is realizable by some term $|a_n|\rho^n$. Notice that for an until $y = (K, \iota)$, if $\rho = |p|_K$, then $|f(y)| \leq |f|_\rho$.

Then this is a non-Archimedean valuation on $A_{\text{inf}}[\frac{1}{[t]}, \frac{1}{p}]$.

Proof: Firstly $|f+g|_\rho \le \max\{|f|_\rho, |g|_\rho\}$ for every ρ that is generic for $f+g$ and in the value group of $C^\flat$: In this case,

$$|f+g|_\rho = |(f+g)(y)| \le \max\{|f(y)|, |g(y)|\} \le \max\{|f|_\rho, |g|_\rho\}$$

for some point y by(II.8.1.10), then by continuity and(II.8.1.11), this is true for any ρ.

The same method shows that $|f|_\rho |g|_\rho = |fg|_\rho$. □

Lemma (II.8.1.11) (Generic Norms). ρ is called **generic** for f iff the valuation is realized exactly once. Notice if ρ is generic for f and in the value group of $C^\flat$, then $|f|_\rho = |f(y)|$ for some y(Choose $K = A_{\inf}/([c]-p)$ where $|c|_{C^\flat} = \rho$).

For any f, the numbers ρ that ρ is not generic for f is discrete in ρ.

Proof: Consider the Newton polygon of f, then only the slopes of the Newton polygon are not generic. □

Lemma (II.8.1.12). If y is a point that $|p|_K = \rho$, then $|f(y)| \le |f|_\rho$, and equality holds if either ρ is generic or f is invertible.

Prop. (II.8.1.13) (Valuation Map). For $0 < a \le b < 1$ in the value group of $C^\flat$, $|\pi_a| = a, |\pi_b| = b$,

$$A_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}] = \{f \in A_{\inf}[\frac{1}{[t]}, \frac{1}{p}] | |f|_a \le 1, |f|_b \le 1\} = V_0,$$

Thus the ring $B_{[a,b]}$ is identified with the completion of $A_{\inf}[\frac{1}{[t]}, \frac{1}{p}]$ w.r.t the valuation $|\cdot|_a + |\cdot|_b$(II.1.2.3). In particular, for any point y that $a \le |p|_K \le b$, the valuation map(II.8.1.8) can be extended to a map

$$B_{[a,b]} \to K.$$

Proof: Notice V_0 is a subring by(II.8.1.10), so clearly $A_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}] \subset V_0$.

For the reverse containment, notice that $\{|c_n|\}$ is bounded, so there is an m that $\pi_b^m c_n \in C^\flat$ for any n. Now

$$f = \sum_{n<m} [c_n]p^n + (\sum_{n\ge 0} [c_{n+m}\pi_b^m]p^n)(\frac{p}{[\pi_b]})^m,$$

so it suffices to prove the case f has finite presentation. Now $c_n\pi_a^n, c_n\pi_b^n \in \mathcal{O}_C^\flat$, thus $[c_n]p^n = [c_n\pi_a^n](\frac{\pi_n}{p})^{-n} = [c_n\pi_b^n](\frac{p}{\pi_b})^n \in A_{inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$, where $n \ge 0$ or $n \le 0$. Thus the inverse containment is true. □

Prop. (II.8.1.14) (Topology of B). For $0 < a \le c \le b < 1$, $|f|_c \le \max\{|f|_a, |f|_b\}$(trivial), thus the Fontaine's ring B can be realized as the completion of all the norms, and endowed with the topology of p-adic Fréchet space.

Prop. (II.8.1.15) (Teichmuller Expansion). An infinite sum $f = \sum [a_n]p^n$ converges in B iff it converges in any norm $|\cdot|_\rho$ for $0 < \rho < 1$, which is equivalent to

$$\limsup_{n>0} |c_n|_{C^\flat}^{1/n} \le 1, \quad \lim_{n\to\infty} |c_{-n}|_{C^\flat}^{1/n} = 0.$$

This is analogous to the complex case(V.2.5.3). However, for now, we don't know iff every element of B is of this form, and whether the representation is unique?.

Prop. (II.8.1.16) (Frobenius Action). Notice the Frobenius action of $C^\flat$ extends to a Frobenius action on the Witt victor A_{inf}, and it satisfies

$$|\varphi(f)|_{\rho^p} = (|f|_\rho)^p,$$

Thus induces an isomorphism $B_{[a,b]} \cong B_{[a^p,b^p]}$. Passing to the limit, we get an automorphism of B, denoted also by φ.

Fargues-Fontaine Curve

Def. (II.8.1.17) (Fargues-Fontaine Curve). The sum $\oplus_n B^{\varphi=p^n}$ is a graded ring. In fact, it is non-negatively graded(II.8.2.29), and we define the **Fargues-Fontaine curve** as the scheme

$$\mathrm{Proj}(\oplus_{n\geq 0} B^{\varphi=p^n}).$$

Def. (II.8.1.18) (Formal Logarithm). For $x \in 1+\mathfrak{m}_{C^\flat}$, $[x]-1 = [x-1]+\sum_{n>0}[c_n]p^n$, thus $|[x]-1|_\rho \geq |x-1| > 0$, thus the formal logarithm

$$\log([x]) = \sum_{k>0} \frac{(-1)^{k+1}}{k}([x]-1)^k$$

converges for every Gauss norm $|\cdot|_\rho$, thus converges to some element in B. And clearly $\varphi(\log([x])) = p\log([x])$, thus $\log([x]) \in B^{\varphi=p}$. And $\log([xy]) = \log([x])\log([y])$.

Prop. (II.8.1.19) (Artin-Hasse Exponential). There is another way of constructing elements in $B^{\varphi=p}$, which is

$$T : a \in \mathfrak{m}_{C^\flat} \mapsto \sum_n \frac{[a^{p^n}]}{p^n}.$$

We want to relate this one to the formal logarithm:

There is a bijection of sets $\mathfrak{m}_{C^\flat} \cong 1+\mathfrak{m}^b_{C^\flat}$ that $\log([E(a)]) = T(a)$, which is defined by the **Artin-Hasse exponential**

$$E(x) = \prod_{(d,p)=1} (\frac{1}{1-x^d})^{\frac{\mu(d)}{d}}.$$

Proof: Firstly, it has coefficients in $\mathbb{Z}_{(p)}$, because $(1-x^d)^{\frac{1}{d}} = \sum(-1)^k C^k_{\frac{1}{n}} x^{kd}$ has coefficient in $\mathbb{Z}_{(p)}$. And $[1-x] = \lim_k(1-[x^{p^{-k}}])^{p^k}$, so

$$\log([\prod_{(d,p)=1}(\frac{1}{1-x^d})^{\frac{\mu(d)}{d}}]) = \sum_{(d,p)=1}\frac{\mu(d)}{d}\log(\frac{1}{[1-d]}) = \sum_{(d,p)=1}\mu(d)\sum_{\alpha\in p^{-n}\mathbb{Z}}\frac{[x^{d\alpha}]}{d\alpha}$$

Notice the right hand side stablizes for any term $[x^\beta]$, and if $\beta \neq \frac{1}{p^k}$, it will vanish, thus for $x \in \mathfrak{m}_{C^\flat}$, it converges, and the sum equals $\sum_n \frac{[x^{p^n}]}{p^n}$. □

Cor. (II.8.1.20). The set of elements of the form $\sum_n \frac{[a^{p^n}]}{p^n}$ is closed under addition.

the Field B_{dR}^+

Prop. (II.8.1.21) (Untilts with Roots of Unitry). Let $\mathbb{Q}_p^{cyc}=\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, and $\varepsilon=(1,\mu_p,\mu_{p^2},\ldots)$ be a compatible p^n-th roots of unity that is an element of $(\mathbb{Q}_p^{cyc})^\flat$. Then $\varepsilon-1$ is a pseudo-uniformizer of $(\mathbb{Q}_p^{cyc})^\flat$, For any untilts K of $\mathbb{C}^\flat$ and an embedding of $\mathbb{Q}_p^{cyc}$ in K, the tilting maps $\varepsilon-1$ to a pseudo-uniformizer of $\mathbb{C}^\flat$. This induces a bijection:

$$\{\text{Untilts } (K,\iota) \text{ of } C^\flat \text{ with an embedding } \mathbb{Q}_p^{cyc}\hookrightarrow K\}\cong\{x\in\mathbb{C}^\flat|0<|x-1|<1\}.$$

Proof: In fact, the left hand side is equivalent to K has a compatible p^n-th roots of unity, and we want to prove that for any x in the right hand side, there is a unique untilts K that $(x^{\frac{1}{p^k}})^\sharp$ is a compatible primitive roots of unity, and this is equivalent to $(x^{\frac{1}{p}})^\sharp$ satisfies $1+x+\ldots+x^{p-1}=0$, and further equivalent to $\theta:A_{inf}\to\mathcal{O}_K$ annhilalates $1+[x^{\frac{1}{p}}]+\ldots+[x^{\frac{p-1}{p}}]$.

It suffices to show $\xi=1+[x^{\frac{1}{p}}]+\ldots+[x^{\frac{p-1}{p}}]$ is distinguished(II.8.1.7). Let $\xi=\sum[c_n]p^n$, consider reducing to the residue field: $W(\mathcal{O}_{C^\flat})\to W(\mathcal{O}_{C^\flat}/\mathfrak{m}_{C^\flat})$, then $\overline{x}=1$, and $\overline{\xi}=p$, thus $|c_0|<1,|c_1-1|<1$, so it is distinguished(II.8.1.3). ☐

Cor. (II.8.1.22). Considering different p^n-th roots of unities, there is a correspondence:

$$\{\text{Char 0 Untilts } (K,\iota) \text{ of } C^\flat \text{ with a compatible } p^n-\text{th roots of unity}\}\cong\{x\in\mathbb{C}^\flat|0<|x-1|<1\}/\mathbb{Z}_p^*.$$

where $\mathbb{Z}_p^*$ acts by exponentiation(II.7.2.5).

Furthermore, there is a correspondence:

$$\{\text{Char 0 Untilts } (K,\iota) \text{ of } C^\flat \text{ with a compatible } p^n-\text{th roots of unity}\}/\varphi_{C^\flat}^{\mathbb{Z}}\cong\{x\in\mathbb{C}^\flat|0<|x-1|<1\}/\mathbb{Q}_p^*.$$

where the inverse is given by $x\mapsto$ the vanishing locus of $\log([x])\in B$.

Proof: The only thing needed to be proven is the inverse is given by $N(\log([x]))$. Notice for any untilts K, $|(x^{p^n})\sharp-1|<|p|_K^{1/(p-1)}$ for n large, then $\log((x^{p^n})\sharp)=0$ iff $(x^{p^n})\sharp=1$ by Newton polygon. Now $x^\sharp\neq1$ because $x\neq1$ and $\sharp$ is injective. Hence composing φ^n for some unique n, we can assume $x^\sharp=1, x^{\frac{1}{p}}\neq1$, thus it corresponds an untilt as in(II.8.1.21). ☐

Def. (II.8.1.23) (B_{dR}^+). For an untilts K of $C^\flat$ that corresponds to a distinguished element ξ(II.8.1.7), p is not a zero-divisor in $A_{\mathrm{inf}}/(\xi^n)$, as in the proof pf(II.8.1.6), so we can define

$$B_{dR}^+=\varprojlim_n A_{\mathrm{inf}}/(\xi^n)[p^{-1}]$$

Prop. (II.8.1.24). B_{dR}^+ is a complete discrete valuation ring with ξ a uniformizer, and the residue field is isomorphic to K. Hence we can define B_{dR} as the quotient field of B_{dR}^+.

Proof: Firstly ξ is not a zero divisor in B_{dR}^+, because if $\xi x=0, x=(x_n)$, then for any $n>0$, and some k that $p^kx_n\in A_{\mathrm{inf}}/(\xi^n)$, so p^kx_n is annihilated by ξ in $A_{\mathrm{inf}}/(\xi^n)$, thus $p^kx_n=\xi^{n-1}y_n$ for some y_n, because ξ is a non-zero-divisor in A_{inf}(II.8.1.6). So $p^{n-1}x^{n-1}=0\in A_{\mathrm{inf}}/(\xi^n)$, thus $x_{n-1}=0$, because p is non-zero-divisor in $A_{\mathrm{inf}}/(\xi^n)$(II.8.1.6).

Next there is a map $B_{dR}^+/(\xi^m)\to A_{\mathrm{inf}}/(\xi^m)[p^{-1}]$. I claim this is an isomorphism: it is clearly a surjection, and if $x=(x_n)$ is mapped to 0, then for each $n\geq m$, we choose $p^{k(n)}x_n=0\in A_{\mathrm{inf}}/(\xi^n)$, then $p^{k(n)}x_n = \xi^m y_n$ for a unique $y_n \in A_{\inf}/(\xi^{n-m})$. So $x = \xi^m \cdot (\frac{y_n}{p^{k(n)}}) \in \xi^m B_{dR}$.(Notice the uniqueness of y_n shows $(\frac{y_n}{p^{k(n)}})$ is an element in B_{dR}^+).

Then it follows $B_{dR}^+ \cong \varprojlim_m B_{dR}^+/(\xi^m)$, which shows that B_{dR}^+ is ξ-adically complete, and $m=1$ shows the residue field is equal to K. ☐

Remark (II.8.1.25). remark if $\xi = [t] - pu$, then $A_{inf}/(\xi^n)[p^{-1}] = A_{inf}/(\xi^n)[[t]^{-1}]$, so if K is of char p, then B_{dR}^+ is just $W(C^\flat)$.

When K is of char0, then B_{dR} contains a field isomorphic to K by(II.10.1.1), thus B_{dR}^+ is (non-canonically) isomorphic to $K[[t]]$, and should be thought as the completed local ring at the point $y = (K, \iota)$.

Prop. (II.8.1.26) (The Stalk Map). Notice $A_{\inf} = \varprojlim_n A_{\inf}/(\xi^n)$ by(II.8.1.6), thus there is a natural map $A_{\inf} \to B_{dR}^+$, whose composition with $B_{dR}^+ \to B_{dR}^+/\xi \cong K$ maps $p, [t]$ to $p, t^\sharp$, which shows they are invertible in B_{dR}^+, so there is a map

$$e : A_{\inf}[\frac{1}{p}, \frac{1}{[t]}] \to B_{dR}^+.$$

In case $a \le |p|_K \le b$, this can be further extended to a map $e : B_{[a,b]} \to B_{dR}^+$(The stalk map).

Proof: It suffices to prove for $a = |p|_K = b$ because the topology is stronger. In this case, choose $t = p^\flat \in C^\flat$, then $|t|_{C^\flat} = |p|_K$, thus $\overline{e}_n$ determined a map

$$A_{\inf}[\frac{[t]}{p}, \frac{p}{[t]}] \to B_{dR}^+/(\xi^n) \cong (A_{\inf}/\xi^n)[p^{-1}],$$

It suffices to prove the image is contained in $p^{-k}(A_{\inf}/\xi^n)$ for some $k = k(n)$, because then $\overline{e}_n$ is p-adically continuous, and extends to map of $B_{[a]} = (A_{\inf}[\frac{[t]}{p}, \frac{p}{[t]}])^\wedge_p \to (A_{\inf}/(\xi^n))[p^{-1}]$, which is compatible w.r.t n, thus gives a map $B_{[a]} \to B_{dR}^+$.

For this, consider $f = \overline{e}_n(\frac{[t]}{p}), g = \overline{e}_n(\frac{p}{[t]})$, then their reduction under $B_{dR}^+/(\xi^n) \to B_{dR}^+/\xi \cong K$ is in $\mathcal{O}_K \cong A_{\inf}/(\xi)$, thus

$$f = f_1 + \frac{\xi}{p^c} f_2, \quad g = g_1 + \frac{\xi}{p^c} g_2$$

for $f_1, f_2, g_1, g_2 \in A_{\inf}/(\xi^n)$ for some c. Then any

$$f^m = (f_1 + \frac{\xi}{p^c} f_2)^m = \sum_{i=0}^{n-1} C_m^i f_1^{m-i} (\frac{\xi}{p^c} f_2)^i \in p^{-nc}(A_{\inf}/(\xi^n)).$$

Thus $\overline{e}_n(A_{\inf}[\frac{[t]}{p}, \frac{p}{[t]}]) \in p^{-nc}(A_{\inf}/(\xi^n))$. ☐

Cor. (II.8.1.27). The stalk map $e : B_{[a,b]} \to B_{dR}^+$ composed with the map $B_{dR}^+/(\xi) \cong K$ are in fact equivalent to the valuation map(II.8.1.13).

2 Divisors

Valuation Function

Def. (II.8.2.1) (Exponential Valuation). For any positive real number s, define a valuation on $A_{\inf}[\frac{1}{p}, \frac{1}{[t]}]$ by the formula $v_s(f) = -\log |f|_{\exp(-s)}$, then it is a valuation by(II.8.1.10).

If f has a Teichmuller expansion $\sum_{n>>-\infty}[c_n]p^n$, then

$$|f|_\rho = \sup\{|c_n|_{C^\flat}\rho^n\}, \quad v_s(f) = \inf\{v(c_n) + ns\}.$$

Prop. (II.8.2.2). For any $f \neq 0 \in A_{\inf}[\frac{1}{p}, \frac{1}{[t]}]$, $s \mapsto v_s(f)$ is a concave function in s which is piecewise linear with integral slopes.

Proof: Consider the Newton Polygon. □

Lemma (II.8.2.3). If $s > 0$ and f_n is a Cauchy sequence in $A_{\inf}[\frac{1}{p}, \frac{1}{[t]}]$ for the norm $|\cdot|_{\exp(-s)}$ and doesn't converge to 0, then the sequences

$$v_s(f_n), \quad \partial_- v_s(f_n), \quad \partial_+ v_s(f_n)$$

stablize.

Proof: Easy, Cf.[ff Curve Lurie P44]. □

Prop. (II.8.2.4). If $0 < a \leq b < 1$, and f_n is a Cauchy sequence in $A_{\inf}[\frac{1}{p}, \frac{1}{[t]}]$ and doesn't converge to 0 for either the norm $|\cdot|_a$ or $|\cdot|_b$, then the sequence of functions $s \mapsto v_s(f)$ stablizes on $[-\log(b), -\log(a)]$.

Proof: Assume f_n doesn't converge to 0 for the form $|\cdot|_b$, then by(II.8.2.3), the sequences $v_s(f_n), \partial_+ v_s(f_n)$ converges, thus $v_s(f_n)$ is bounded uniformly, thus $v_s(f)$ is bounded.

Then choose N large that $|f - f_m|_\rho$ very small for any $m > N$ and $a \leq \rho \leq b$, then $v_s(f) = v_s(f_m)$ for any $a \leq s \leq b$, thus it stablizes. □

Cor. (II.8.2.5). Let f be a non-zero element in B, then the construction $s \mapsto v_s(f)$ is a concave function in s with piecewise linear function with integral slopes. This is analogous to the Hadamard three circle theorem(V.2.6.10).

Proof: This is true for $f \in B_{[a,b]}$, because any f is a limit of a sequence f_n in both the norm $|\cdot|_a$ and $|\cdot|_b$, so by the proposition, there for n large, $v_s(f) = v_s(f_n)$ on $[-\log(b), -\log(a)]$, thus the conclusion is true by(II.8.2.2). And for $f \in B$, for any interval $[a, b]$ we can do the same, thus the conclusion is true on each interval, thus it is true. □

Metric Structures on Y

Def. (II.8.2.6) (Metric on $\overline{Y}$). Let $\overline{Y} = Y \cup \{0\}$ be the isomorphism classes of untilts of $C^\flat$, where 0 corresponds to $C^\flat$ itself.

(II.8.1.7) show $\overline{Y}$ corresponds to distinguished elements in $A_{\inf}$ up to units. So for any $x, y \in \overline{Y}$, we let $d(x, y) = |\xi_x(y)|_{K_y} \leq 1$. Then this is a metric, and it is non-Archimedean.

Proof: Firstly, if $d(x, y) = 0$, then ξ_x divides ξ_y, which is equivalent to $(\xi_x) = (\xi_y)$, by(II.8.1.4).

Secondly, for any x, y, since $C^\flat$ is alg.closed, we can assume $\xi_x(y) = c^\sharp$ for some $c \in C^\flat$. Notice $\xi(y) = t^\sharp + pu(y)$ is in $\mathfrak{m}_K$, thus $c \in \mathfrak{m}_{C^\flat}$. So $\xi_x - [c]$ is also a distinguished element and vanishes at y, so we may assume that $\xi_y = \xi_x - [c]$ by(II.8.1.4) again. Then

$$d(y, x) = |\xi_y(x)|_{K_x} = |c^\sharp|_{K_x} = |c|_{C^\flat} = |c^\sharp|_{K^y} = d(x, y).$$

Finally it is non-Archimedean because any valuation field K is non-Archimedean. □

Prop. (II.8.2.7) ($\overline{Y}$ is Complete). $\overline{Y}$ is complete w.r.t this metric.

Proof: Given a Cauchy sequence of points y_n in $\overline{Y}$, as in the proof of(II.8.2.6), we can assume that $\xi_{y_n} = \xi_{y_{n-1}} + [c_n]$ for some $c_n \in \mathfrak{m}_{C^\flat}$, and $|c_n|_{C^\flat} = d(y_{n-1}, y_n)$. Now $A_{\inf}$ is $[t]$-adically complete for a uniformizer $t \in C^\flat$, thus $\sum[c_n]$ is definable in $A_{\inf}$, and $\xi = \xi_0 + \sum[c_n]$ is also distinguished, and corresponds to a point y which y_n clearly converges to. □

Divisors

Lemma (II.8.2.8). $B_{[a,b]}$ is an integral domain.

Proof: By(II.8.2.5), the valuation function $v_s(f)$ and $v_s(g)$ are bounded, thus it is clear that $v_s(fg)$ is also finite, so $fg \neq 0$. □

Prop. (II.8.2.9) (Divisors). Assume $C^\flat$ is alg.closed, then for any $f \in B_{[a,b]}$ and $y = (K, \iota) \in Y_{[a,b]}$, we define the **order of vanishing** $\mathrm{ord}_K(f) \in \mathbb{Z} \cup \{\infty\}$ as the valuation of $e_K(f) \in B^+_{dR}(K)$, then

- if $f \neq 0 \in B_{[a,b]}$, then $\mathrm{ord}_K(f) < \infty$ for each $K \in Y_{[a,b]}$, and there are only f.m. K that $\mathrm{ord}_K(f) \neq 0$. In particular, $B_{[a,b]}$ is an integral domain.
- if $x, y \neq 0 \in B_{[a,b]}$, then x divides y iff $\mathrm{ord}_K(x) \geq \mathrm{ord}_K(y)$ for each $K \in Y_{[a,b]}$.

Thus for each $f \in B_{[a,b]}$, we can define the **divisor** of K as the formal sum $\sum_{K \in Y_{[a,b]}} \mathrm{ord}_K(f) K$, and this is also definable for $f \in B$, but it may be an infinite but locally finite sum.

Proof: Firstly, by(II.8.2.13) and(II.8.2.14), if $\mathrm{div}(f) \cap Y_{[a,b]} \neq 0$, then there is a distinguished element ξ that $f = \xi f_1$. And we can iterate this, and eventually end up with $f = \xi_1 \ldots \xi_n f_n$ that $\mathrm{div}(f_n) \cap Y_{[a,b]} = 0$, by(II.8.2.15), so by(II.8.2.16), f_n is invertible in f, so $\mathrm{div}(f)$ is finite. And if $\mathrm{div}(g) \geq \mathrm{div}(f)$, then g also divides $\xi_1 \ldots \xi_n$, so g divides f. □

Remark (II.8.2.10). Notice by(II.8.1.27), for a $f \in B_{[a,b]}$, $\mathrm{ord}_K(f) > 0$ iff $f(y) = 0 \in K$.

Cor. (II.8.2.11) (B is Integral Domain). B is an integral domain, and if $C^\flat$ is alg.closed, then x is divisible by y if and only if $\mathrm{div}(x) \geq \mathrm{div}(y)$.

Cor. (II.8.2.12). B is integrally closed.

Proof: B is an integral domain by(II.8.2.11), it is integrally closed because if f/g is integral over B, then there image in $B^+_{dR}(y)$ is integral over $B^+_{dR}(y)$ for all $y \in Y$, thus in $B^+_{dR}(y)$ because it is a valuation ring, and then f is divisible by g by(II.8.2.11). □

Prop. (II.8.2.13) (Examples of Divisors).

- For a distinguished element ξ, if $\xi = up$, then ξ is invertible in B, thus $\mathrm{div}(\xi) = 0$. Otherwise ξ defines a char0 untilts K of $C^\flat$, and ξ is a uniformizer of $B^+_{dR}(K)$, and it doesn't divides other distinguished elements(II.8.1.4), thus $\mathrm{div}(\xi) = K$.
- $\mathrm{div}(\log([x])) = \sum_{n \in Z} \varphi^n(K)$.

Proof: 2: $\log([x])$ vanishes at a single φ-orbits of Y, and one of them is given by the distinguished element $\xi = 1 + [x^{1/p}] + \ldots + [x^{p-1/p}] = \frac{[x]-1}{[x^{1/p}]-1}$. Notice $[x^{1/p}] - 1$ is mapped to an invertible element in K, thus it is invertible in $B^+_{dR}(K)$, so $[x] - 1$ is associated to ξ, and notice

$$\log([x]) = \sum_{k>0} \frac{(-1)^{k+1}}{k}([x] - 1)^k \equiv [x] - 1 \mod ([x] - 1)^2,$$

so $\mathrm{ord}_K(\log([x])) = 1$, and because $\varphi(\log([x])) = p\log([x])$, $\mathrm{ord}_{\varphi^n(K)}(\log([x])) = 1$ for any n, so we are done. □

Lemma (II.8.2.14). Let $C^\flat$ be alg.closed. If ξ is a distinguished element of $A_{\inf}$ vanishes at a point $y \in Y_{[a,b]}$ and $g \in B_{[a,b]}$ also vanishes at y, then g is divisible by ξ in $B_{[a,b]}$.

Proof: If $g \in A_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$, then this is easy by(II.8.2.10) and $A_{\inf}/(\xi) = \mathcal{O}_K$(II.8.1.6).

Now generally g is a limit of $g_n \in A_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$, so $g(y)$ is the limit of $g_n(y) \in K$. Now $g(y) = 0$, so $\lim_n g_n(y) = 0$. Now K is alg.closed by(II.7.2.12), so we can let $g_n(y) = c_n^\sharp$, so c_i converges to 0 in $C^\flat$. So $\{[c_n]\}$ converges to 0 in norm $|\cdot|_a$ and $|\cdot|_b$, so we can replace g_n by $g_n - [c_n]$ and assume $g_n(y) = 0$.

Now the first part shows $g_n = \xi h_i$, and now h_i is a Cauchy sequence for both $|\cdot|_a$ and $|\cdot|_b$, so converges to some h, and then $g = \xi h$. □

Lemma (II.8.2.15). Given this lemma(II.8.2.14), we have a strategy of proving(II.8.2.9), that is, decomposing f, g into distinguished elements, but we need to show this decomposition is finite. And this is true:

If $f \neq 0 \in B_{[a,b]}$, denote $\beta = -\log(b), \alpha = -\log(a)$ and let $N = \partial_- v_\beta(f) - \partial_+ v_\alpha(f) \geq 0$, then f cannot be divisible by a product of $\xi_1, \ldots, \xi_{N+1}$ of $N+1$ distinguished elements.

Proof: By(II.8.1.3), if ξ is distinguished, then $v_s(\xi) = \max\{s, v(v_0)\}$. Now $v(v_0) = v(t^\sharp) = v(|p|_K)$ in $\mathcal{O}_K = A_{\inf}/([t] - up)$, so if K corresponding to ξ belongs to $Y_{[a,b]}$, then $v(v_0) \in [\beta, \alpha]$, so $\partial_- v_\beta(\xi) = 1, \partial_+ v_\alpha(\xi) = 0$.

So if $f = \xi_1, \ldots, \xi_{N+1} u$, then $N(f) \geq \sum N(\xi_i) \geq N+1$. □

Lemma (II.8.2.16) (Valuation Funtion And Invertibility). Let $C^\flat$ be alg.closed and $f \neq 0 \in B_{[a.b]}$, then the following are equivalent:

- f is invertible.
- $\partial_- v_\beta(f) = \partial_+ v_\alpha(f)$.
- $\mathrm{div}(f) \cap Y_{[a,b]} = \varnothing$.

Proof: $2 \to 3$: by(II.8.2.15).

$1 \to 2$: Because $N(f) + N(f^{-1}) = 0$, and $N(f) \geq 0, N(f^{-1}) \geq 0$, so $N(f) = 0$.

$2 \to 1$: Assume first that $f = \sum_{n>>-\infty}[c_n]p^n \in A_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$, then the hypothesis just says that $s \to v_s(f)$ is linear in a small nbhd of $[\beta, \alpha]$, that is, there is a n_0 that $v(c_n) + ns > v(c_{n_0}) + n_0 s$ for all $n \neq n_0$ and $s \in [\beta, \alpha]$.

Now we can normalize f that $n_0 = 0$ and $c_0 = 1$, so $|f - 1|_\rho < 1$ for all $\rho \in [\beta, \alpha]$, so $f - 1$ is topologically nilpotent in $B_{[a,b]}$, and thus f is invertible.

Generally, f is a limit of a sequence $f_n \in A_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$, and by(II.8.2.4) we can assume the hypothesis holds for all f_n. Then f_n is invertible, and it is easily shown that f_n^{-1} is a Cauchy sequence in $B_{[a,b]}$, so converges to some f^{-1}.

$3 \to 2$: Firstly, if $\partial_- v_\beta(f) > \partial_+ v_\alpha(f)$, then we must have $\partial_- v_\rho(f) > \partial_+ v_\rho(f)$ for some s, so wlog, we can assume $a = b = s$, and we need to show f vanishes at some point in $Y_{\exp(s)}$. Now combining with(II.8.2.14) and(II.8.2.15), this is equivalent to another statement that any element $y \in B_{[\rho,\rho]}$ has a decomposition $y = g\xi_1 \ldots \xi_n$ where ξ_k corresponds to points in Y_ρ and g is invertible in B_ρ. The proof is finished at(II.8.2.26). □

Primitive Elements and the Proof of $3 \to 2$ of The Lemma on Valuation Function and Invertibility

Def. (II.8.2.17). Let $C^\flat$ be alg.closed, an element in $B_{[\rho,\rho]}$ is called **good** iff it has a decomposition as in the proof of $3 \to 2$ of(II.8.2.16).

Prop. (II.8.2.18) (Approximating Zero). If f is a good element having n-zeros on Y_ρ, and $g \in B_\rho$ that $|f-g|_\rho < |f|_\rho$, then for any zero y of g on Y_ρ, there exists a zero y' of f on Y_ρ that $d(y',y) < \rho(\frac{|f-g|_\rho}{|g|_\rho})^{1/n}$.

Proof: $|f-g|_\rho \geq |(f-g)(y)|_K = |f(y)|_K$. Now $f = g\xi_1 \dots \xi_n$, and ξ corresponds to y_i, then

$$|f(y)|_K = |g(y)|_K |\xi_1(y)|_K \dots |\xi_n(y)|_K = \frac{|f|_\rho}{\prod_i |\xi_i|_\rho} \prod_i d(y_i,y) = |f| \prod_i \frac{d(y_i,y)}{\rho}.$$

(Notice $|g|_\rho = |g(y)|_K$ because g is invertible(II.8.1.12)) and $d(y_i,y) \leq \rho$. So at least one ξ satisfies the desired inequality. □

Cor. (II.8.2.19). If $f \in B_\rho$ is given by a Cauchy sequence of good elements, and $\partial_- v_\rho(f) > \partial_+ v_\rho(f)$, then f has a root on Y_ρ.

Proof: By(II.8.2.3), passing to a subsequence, we may assume

$$v_s(f) = v_s(f_n), \quad \partial_- v_s(f) = \partial_- v_s(f_n), \quad \partial_+ v_s(f) = \partial_+ v_s(f_n), \quad |f_{n+1} - f_n|_\rho < |f|_\rho.$$

Let $n = \partial_- v_s(f) - \partial_+ v_s(f) > 0$, then each f_i has exactly n roots on Y_ρ, and applying(II.8.2.18), we can find successively roots y_n of f_n that $d(y_{n+1}, y_n) \leq \rho(\frac{|f_{n+1}-f_n|_\rho}{|f|_\rho})^{1/n}$, so the sequence $\{y_n\}$ is Cauchy and converges to some point $y \in \overline{Y}$, so

$$|f_i(y)|_K \leq |f_i|_\rho \frac{d(y_i,y)}{\rho} = |f|_\rho \frac{d(y_i,y)}{\rho} \to 0.$$

so $f(y) = 0$. □

Def. (II.8.2.20) (Primitive Elements). An element $f = \sum_{n\geq 0}[c_n]p^n \in A_{\text{inf}}$ is called **primitive of degree** d if $c_0 \neq 0, |c_d| = 1$ for some smallest element d.

Clearly an element is distinguished of degree 1 iff it is distinguished and corresponds to an untilts of $X^\flat$ of char0.

Prop. (II.8.2.21).

- Any element $f \in A_{\text{inf}}$ of finite Teichmuller expansion can be written uniquely as a $f = p^m[c]g$, where $c \in C^\flat$ and g is primitive.
- For an element $f \in A_{\text{inf}}[\frac{1}{p}, \frac{1}{[t]}]$, f can be written as $p^m[c]g$ iff $v_s(f)$ consists of f.m. line segments iff $\sup\{|c_n|\}$ is achieved by some n.
- If $f = gh$ in A_{inf} is primitive, then g,h are also primitive, and $\deg(f) = \deg(g) + \deg(h)$.

Prop. (II.8.2.22). Let $f = \sum[c_n]p^n \in A_{\text{inf}}$ be primitive of degree $d > 0$, and let $\lambda \in (0,1)$ be the number that $s = -\log(\lambda)$ is the minimal number that $v_s(f)$ is non-differentiable at, i.e. s is -1 times the slope of the line segment on the left of $v(c_d)$. Then f has a zero on Y_λ.

Proof: By(II.8.2.23) there is a $y \in Y_\lambda$ that $|f(y_1)| \le \lambda^{d+1}$, and then(II.8.2.24) shows we can find successively y_n that

$$d(y_n, y_{n+1}) \le \lambda^{1+\frac{d}{n}}, \quad |f(\lambda_n)| \le \lambda^{d+m}.$$

So y_n is a Cauchy sequence thus converges to some y, and then $f(y) = 0$. □

Lemma (II.8.2.23) (Lemma for Approaching a Zero). If $C^\flat$ is alg.closed and $f \in A_{\text{inf}}$ is primitive of degree $d > 0$, and let λ as in(II.8.2.22), then there is a point $y \in Y_\lambda$ that $|f(y)|_{K_y} \le \lambda^{d+1}$.

Proof: Let $f = \sum [c_n] p^n$, we may assume $c_d = 1$, and let $F = x^d + c_{d-1}x^{d-1} + \ldots + c_0$, then the largest valuation of the roots of F on $C^\flat$ is λ, by Newton polygon. Let r be such a root, then c_i is divisible by r^{d-i}, and let $\xi = p - [r]$ be a distinguished element of A_{inf} and corresponds to an untilt K, then $|p|_K = \lambda$, and

$$p^{-d} f(y) = \sum_{n \ge 0} c_n^\sharp p^{n-d} \equiv \sum_{i=0}^{d} (\frac{c_i}{r^{d-i}})^\sharp \mod p = (r^{-d} F(r)) \mod p = 0$$

thus $f(y)$ is divisible by p^{d+1}, which is equivalent to $|f(y)|_K \le \lambda^{d+1}$. □

Lemma (II.8.2.24) (Lemma for Approaching a Zero). Situation as in(II.8.2.23), if $y \in Y_\lambda$ and $|f(y)| = \lambda^d \cdot \alpha$, then there is a y' that $d(y, y') \le \lambda \cdot \alpha^{1/d}$ that $|f(y')| \le \lambda^{d+1}\alpha$.

Proof: Since A_{inf} is ξ-complete and every element of $A_{\text{inf}}/\xi \cong \mathcal{O}_K$ belongs to the image of $\sharp : \mathcal{O}_C^\flat \to \mathcal{O}_K$, thus by induction, we can write $f = \sum_{n \ge 0} [c_n] \xi^n$. Because f is primitive of degree d, we may assume $c_d = 1$, and $|c_0|_{C^\flat} = |f(y)|_K = \lambda^d \alpha$.

Let $F(x) = c_0 + c_1 x + \ldots + c_{d-1}x^{d-1} + x^d$, because $C^\flat$ is alg.closed, let r be a root of minimal absolute value, then $|r|_{C^\flat}^m |c_m|_{C^\flat} \le \lambda^n \alpha$, in particular $|r|_{C^\flat} \le \lambda \alpha^{1/n}$. So let $\xi' = \xi - [r]$, then ξ is also distinguished, and $d(y', y) = |r|_{C^\flat} \le \lambda \alpha^{1/n}$(II.8.2.6), and $d(0, y') = \lambda$, and $\xi(y') = r^\sharp$.

Now

$$\frac{(f(y'))^\sharp}{c_0^\sharp} = \sum_{n \ge 0} \frac{c_n^\sharp}{c_0^\sharp} \xi(y')^n = \sum_{n \ge 0} (\frac{c_n r^n}{c_0})^\sharp \equiv \sum_{i=0}^{n} (\frac{c_n r^n}{c_0})^\sharp \mod p = (\frac{F(r)}{c_0})^\sharp = 0,$$

So $|f(y')|_{K'} \le |c_0^\sharp|_{K'} |p|_{K'} = |c_0|_{C^\flat} \lambda = \lambda^{d+1}\alpha$. □

Cor. (II.8.2.25) (Primitive Elements Decompose as Distinguished Elements). If $f \in A_{\text{inf}}$ is a primitive element of degree $d > 0$, then f admits a factorization as products of distinguished elements ξ corresponding to points in Y.

Proof: Use induction on d. If $d = 1$, then f is distinguished by(II.8.2.20), and if $d > 1$, then by(II.8.2.22), $f = \xi g$, so g is primitive of degree $d - 1$ by(II.8.2.21), so induction is finished. □

Prop. (II.8.2.26) (Finite Teichmuller Expansion is Good). Any element of finite Teichmuller expansion is good.

In particular, because any element of B_ρ can be approximated by elements in $A_{\text{inf}}[\frac{1}{p}, \frac{1}{[t]}]$, and such element can be approximated by elements of finite Teichmuller expansion, by(II.8.2.19), we finishes the proof of $3 \to 2$ of(II.8.2.16).

Proof: If f has finite Teichmuller expansion, then $f = p^m [c] g$, where g is primitive of degree d. If $d = 0$, then g is invertible in A_{inf}, thus f is invertible in B_ρ. Otherwise, we can use(II.8.2.25) to factorize g into distinguished elements, and the elements that corresponds to points outside Y_ρ is invertible in B_ρ because $v_s(\xi) = \max\{s, v(v_0)\}$ and $2 \to 1$ of(II.8.2.16), so f is good. □

Bounded Meromorphic Functions

Prop. (II.8.2.27). If $f \in B$, then $f \in A_{\inf}$ iff $|f|_\rho \leq 1$ for any $0<\rho<1$.

Easily we can get characterization of f being in $A_{\inf}[\frac{1}{p}], A_{\inf}[\frac{1}{[t]}]$ or $A_{\inf}[\frac{1}{p}, \frac{1}{[t]}]$.

Proof: One direction is trivial, for the other, by(II.8.2.28), we can find successively f_n that $f = \sum_{i<n}[c_i]p^i + f_n$, and $|f_n|_\rho \leq \rho^n$ for all $0<\rho<1$. So f_n converges to 0 in any norm ρ, thus it converges to 0 in B, and $f = \sum_{n\geq 0}[c_n]p^n \in A_{\inf}$. □

Lemma (II.8.2.28). If $f \in B$ satisfies $|f|_\rho \leq \rho^m$ for all $0<\rho<1$, then there is a $c \in \mathcal{O}_{C^\flat}$ that $f = [c]p^m + g$ that $|g|_\rho \leq \rho^{m+1}$ for all $0<\rho<1$.

Proof: Replace f by $\frac{f}{p^m}$, we may assume $m=0$. Choose a sequence f_i in $A_{\inf}[\frac{1}{p}, \frac{1}{[t]}]$ converging to f in B, where $f_i = \sum_{n>>-\infty}[c_{n,i}]p^n$.

Firstly we want to truncate f_i with the positive part f_i^+. Notice for each ρ and any $0<\varepsilon<1$, because $\lim |f_i - f|_{\varepsilon\rho} = 0$, thus for i large, $|f_i|_{\varepsilon\rho} = |f|_{\varepsilon\rho} \leq 1$, thus $|c_{-n,i}|_{C^\flat}\rho^{-n} \leq \varepsilon^n < \varepsilon \leq \varepsilon$, so $|f_i - f_i^+|_\rho < \varepsilon$ for i large, so $\lim f_i^+ = f$ also in B.

Secondly, $|c_{0,i} - c_{0,j}|_{C^\flat} \leq |f_i - f_j|_\rho$ for each ρ, thus $c_{0,i}$ is Cauchy in $C^\flat$ thus converges to some $c \in C^\flat$, and when i is large, $|c_{0,i}|_{C^\flat} \leq |f_i|_\rho = |f|_\rho \leq 1$, so $c \in \mathcal{O}_{C^\flat}$. Now let $g_i = \sum_{n>0}[c_{n,i}]p^n$, then g_i is also Cauchy in B for any norm $|\rho|$ and converges to some g, and $f = g + [c]$.

It's left to check $|g|_\rho \leq \rho$: each $v_s(g_i)$ has positive slopes, then so does $v_s(g)$ because by(II.8.2.3), $v_s(g_i)$ stablizes to $v_s(g)$ uniformly on compact intervals. So if $v_s(g) < s-\varepsilon$ for some s, then $v_\varepsilon(g) \leq v_s(g) - (s-\varepsilon) < 0$, but this cannot happen because $v_\varepsilon(g) \leq \max\{v_\varepsilon(f), -\log|c|_{C^\flat}\} \geq 0$. □

Eigenspaces of Frobenius

Prop. (II.8.2.29).

- The vector space $B^{\varphi=p^n}$ vanish for $n<0$.
- The canonical map $\mathbb{Q}_p \to B^{\varphi=\mathrm{id}}$ is an isomorphism.

Proof: 1: Consider $v_{ps}(\varphi(f)) = pv_s(f)$(II.8.1.16), $v_s(p^n f) = ns + v_s(f)$, so if $\varphi(f) = p^n f$, then

$$pv_{s/p}(f) = v_s(\varphi(f)) = v_s(p^n f) = ns + v_s(f).$$

Let $h(s) = \partial_+ v_s(f)$, then $h(s/p) = n + h(s)$, but h must by non-increasing(II.8.2.2), so $n \geq 0$.

2: Firstly we prove $B^{\varphi=\mathrm{id}}$ is a field: by(II.8.2.11), it suffices to show that $\mathrm{div}(f) = 0$ for $f \neq 0 \in B^{\varphi=\mathrm{id}}$. If $\mathrm{div}(f) \neq 0$, because f is fixed by φ, so $\mathrm{div}(f) \geq \sum_{n\in\mathbb{Z}}\varphi^n(y)$ for some y, and $\sum_{n\in\mathbb{Z}}\varphi^n(y) = \mathrm{div}(\log([\varepsilon])$ for some $\varepsilon \in 1 + \mathfrak{m}_{C^\flat}$ because K is alg.closed and by (II.8.1.22). So by(II.8.2.11) again $f = g\log([\varepsilon])$, and $g \in B^{\varphi=p^{-1}}$ by(II.8.1.18), then $g=0$ by item1.

3: From(II.8.2.30) and(II.8.2.27), $f \in A_{\inf}[\frac{1}{p}]$, thus $f = \sum_{n>>\infty}[c_n]p^n$, so $\varphi(f) = f$ shows $c_n^p = c_n$, which is equivalent to $c_n \in \mathbb{F}_p$. So $f \in W(\mathbb{F}_p)[\frac{1}{p}] = \mathbb{Q}_p$. □

Lemma (II.8.2.30). If $f \neq 0 \in B^{\varphi=\mathrm{id}}$, then there is an integer n that $|f|_\rho = \rho^n$.

Proof: Notice $|f|_\rho^p = |\varphi(f)|_{\rho^p} = |f|_{\rho^p}$, so $v_{ps}(f) = pv_s(f)$, differentiation shows that $\partial_- v_{ps}(f) = \partial_- v_s(f)$. This is for all $s<0$, and $\partial_- v_s(f)$ is non-decreasing, thus it is constant, and $v_{ps}(f) = pv_s(f)$ shows $v_s(f) = ns$ for some integer n. □

Cor. (II.8.2.31). For $n \geq 0$, any element $f \in B^{\varphi=p^n}$ factors uniquely up to action of $\mathbb{Q}_p^*$ as $\lambda\log([\varepsilon_1])\ldots\log([\varepsilon_n])$ where $\lambda \in B^{\varphi=\mathrm{id}}$, $0<|\varepsilon_i - 1|<1$.

Proof: The existence is by(II.8.2.26)

For the uniqueness: it suffices to prove $\log([\varepsilon])$ is a prime element in $\oplus_{n\geq 0}B^{\varphi=p^n}$. For this, notice for any $f\in B^{\varphi=p^n}$, $\mathrm{div}(f)$ is fixed by φ, and $\mathrm{div}(\log([\varepsilon]))$ is a single orbit of φ, thus by(II.8.2.11), if $\log([\varepsilon])$ divides fg, then $\log([\varepsilon])$ divides f or g. □

Applications

Cor. (II.8.2.32). If $C^\flat$ is alg.closed, then every untilts K of $C^\flat$ belongs to the vanishing locus of $\log([x])$ for some $x\in C^\flat$ that $0<|x-1|<1$, and the map

$$\psi : 1+\mathfrak{m}_{C^\flat}\to K : y\mapsto \log(y^\sharp)$$

is surjective with kernel generated by x (as a $\mathbb{Q}_p$-subspace of $1+\mathfrak{m}_{C^\flat}$).

Proof: By(II.7.2.12), any untilts of $C^\flat$ is alg.closed, thus it has a compatible p^n-th roots of unity. So it belongs to some locus of $\log([x])$ by(II.8.1.22). Now if $|z|<|p|_K^{1/(p-1)}$, then $z=\log(\exp(z))$, and $\exp = y^\sharp$ for some y because K is alg.closed. So ψ contains sufficiently small elements, but it is a map of $\mathbb{Q}_p$-vector spaces, thus it is surjective. For the kernel, if $\log(y^\sharp)=0$, then $\log([y])$ vanish on K, thus by(II.8.1.22), y,x is in the same $\mathbb{Q}_p$-vector space. □

Cor. (II.8.2.33). If $C^\flat$ is alg.closed, then the map

$$1+\mathfrak{m}_{C^\flat}\xrightarrow{\log([x])} B^{\varphi=p}$$

is an isomorphism.

Proof: Firstly any untilts of $C^\flat$ is alg.closed by(II.7.2.12). It is injective because of the correspondence(II.8.1.22), and for the surjectivity, for each $f\in B^{\varphi=\mathrm{id}}$, if $f=0$, then $f=\log([1])$, and if $f\neq 0$, then notice $\mathrm{div}(f)\neq\varnothing$, because in this case f is invertible in B by(II.8.2.11), thus $f^{-1}\in B^{\varphi=p^{-1}}$, so $f^{-1}=0$ by(II.8.2.29), contradiction.

Now if $\mathrm{ord}_K(f)\geq 1$, then $\mathrm{ord}_{\varphi^n(K)}(f)\geq 1$ for any $n\in\mathbb{Z}$ since $\varphi(f)=pf$. Consider $\mathrm{div}(\log([x]))=\sum_{n\in\mathbb{Z}}\varphi^n(K)$(II.8.2.13), then f is divisible by $\log([x])$ by(II.8.2.11), $f=\log([x])g$, then $g\in B^{\varphi=\mathrm{id}}$, then $g\in\mathbb{Q}_p^*$ by(II.8.2.29), thus $f=\log([x^g])$. □

Cor. (II.8.2.34) (Filtration on B_{dR}). By(II.8.2.13) and(II.8.2.32), we see that for any untilt K of $C^\flat$, there is a unique up to $\mathbb{Q}_p$-constant ε that $t=\log([\varepsilon])$ is the uniformizer of $B_{dR}(y)$. In fact, this ε can be to be $\varepsilon=(1,\xi_p,\ldots,x_{p^n},\ldots)$, where ξ_{p^n} is a compatible roots of unity in the alg.closed field K.

Now we prefer to use the filtration $Fil^n=t^{-n}B_{dR}^+$ on B_{dR} because it is $G_{\mathbb{Q}_p}$ invariant, as ε does.

Prop. (II.8.2.35). Let $C^\flat$ be alg.closed, then any point x of the Fargues-Fontaine curve X_{FF} that is not the generic point corresponds to the prime $x_K=(\log([\varepsilon]))$(II.8.2.31) where $K\in\mathrm{div}(\log([\varepsilon]))$(II.8.2.32). And the residue field of x_K can be identified to K.

Proof: By(II.8.2.31), we can cover X_{FF} by affine schemes of the form $\mathrm{Spec}(R_f=B[f^{-1}]^{\varphi=\mathrm{id}})$ for $f\in B^{\varphi=p}$, now for any prime $\mathfrak{p}\subset R_f$, let $\frac{g}{f^n}\in\mathfrak{p}$, then $g=\lambda\log([\varepsilon_1])\ldots\log([\varepsilon_n])$, thus some $\frac{\log([\varepsilon])}{f}\in\mathfrak{p}$. Let K be a point that $\log([\varepsilon])$ vanish(II.8.1.22), then we claim $(\log([\varepsilon])/f)$ is maximal.

In fact, we may assume f doesn't vanish on K, otherwise $\log([\varepsilon])/f$ is a unit, then there is a map $\rho: B[f^{-1}]^{\varphi=1}\subset B[f^{-1}]\to K$, and this map is surjective with kernel $(\log([\varepsilon])/f)$: it is surjective even on $f^{-1}B^{\varphi=p}$ by(II.8.2.32), and if $\log([\varepsilon_1])/f$ is mapped to 0, then $\log([\varepsilon_1])$ differs from $\log([\varepsilon])$ by some Q_p^* by(II.8.1.22). □

Cor. (II.8.2.36). If $C^\flat$ is alg.closed, there is a bijection of sets:

$$Y/\varphi_{C^\flat}^{\mathbb{Z}} \cong \{\text{Closed points of } X_{FF}\}.$$

by(II.8.1.22).

Cor. (II.8.2.37). X_{FF} is a Dedekind scheme(III.3.2.7).

Proof: Let $\mathrm{Spec}(R_f = B[f^{-1}]^{\varphi=\mathrm{id}})$, two elements $f = \log([\varepsilon]), g = \log([\mu])$ can cover it. The proof of(II.8.2.35) shows that every prime ideals of R_f is maximal principal, in particular f.g, thus by(I.5.1.14), it is Noetherian. And it has Krull dimension 1 and it is regular because all of its maximal ideals are principal, hence normal(I.6.5.20). So X_{FF} is a Dedekind scheme. □

3 Harder-Narasimhan Formalism

Def. (II.8.3.1) (Harder-Narasimhan Formalism). A **Harder-Narasimhan formalism** consists of

- A category $\mathcal{C}$ with the notion of short exact sequences.
- A function $\deg : Ob(\mathcal{C}) \to \mathbb{Z}$ that is additive w.r.t short exact sequences.
- An exact faithful functor **generic fiber functor** to an Abelian category $\mathcal{C} \to \mathcal{A}$ that induces for each object $F : \mathcal{E} \in \mathcal{C}$ a bijection

  $$\{\text{strict objects of } \mathcal{C}\} \cong \{\text{subobjects of } F(\mathcal{E})\}$$

  where a **strict subobject** is an object that can be prolonged to an exact sequence.
- An additive function rank: $\mathcal{A} \to \mathbb{N}$ on $\mathcal{A}$ that $\mathrm{rank}(\mathcal{L}) = 0 \iff \mathcal{L} = 0$, and its composition with F is also called rank.
- If $u : \mathcal{E} \to \mathcal{E}'$ is a morphism in $\mathcal{C}$ that $F(u)$ is an isomorphism, then $\deg(\mathcal{E}) \leq \deg(\mathcal{E}')$ with equality iff u is an isomorphism.

Cor. (II.8.3.2).

- We are free to choose the "kernel" for u that $F(u)$ is surjection.
- The subobjects of subobjects are subobjects, by axiom3.

Prop. (II.8.3.3) (HN-Filtration on the Category of Filtered Vector Spaces). If L/K is a field extension, there is a category $VectFil_{L/K}$ consisting of (V, Fil) where V is a K-vector space and Fil is a finite filtration on $V \otimes_K L$. It is an exact category by declaring exact sequences be those induce exact sequences on the gradeds.

The generic fiber functor is $VectFil_{L/K} \to Vect_K : (V, Fil) \mapsto V$, and rank is as usual, the degree is defined to be $\deg((V, Fil)) = \sum i \dim_K gr^i(V \otimes_K L)$. This is a HN-filtration.

Proof: The axioms can be directly checked, notice a filtration W_n of a filtration V_n is a strict object iff $W_k = W_n \cap V_k$. □

Prop. (II.8.3.4) (HN-Filtration on Isocrystals). Let k be perfect and $K = W(k)[\frac{1}{p}]$, then the category of $\varphi - Mod_K$(Isocrystals) is an exact category, with the generic fiber functor the forgetting functor, and the rank defined as usual.

The degree $\deg(D)$ is defined by considering the determinant φ-module $K_0 e$, if $\varphi(e) = de$, then define $\deg(D) = v_p(d)$. This is a HN-filtration, by Dieudonné-Manin Classification(II.10.4.10).

Def. (II.8.3.5) (Slope). In a HN-formalism, the **slope** is defined to be $\mathrm{slope}(E) = \frac{\deg(E)}{\mathrm{rank}(E)}$.

$\mathcal{E}$ is called **semistable of slope** λ iff $\mathrm{slope}(\mathcal{E}) = \lambda$, and $\mathrm{slope}(\mathcal{E}') \le \lambda$ for any nonzero strict subobject $\mathcal{E}' \subset \mathcal{E}$.

Prop. (II.8.3.6). If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be a short exact sequence in $\mathcal{C}$, then:

- If two of them have the same slope, then so does the third.
- If two of them have different slope, then we know the ordering of these slopes.

Proof: Just notice that the degree and rank are all additive functions. □

Prop. (II.8.3.7). If $\mathcal{E}$ is semistable of slope λ, then for any morphism $u : \mathcal{E} \to \mathcal{E}''$ that $F(u)$ is an isomorphism, $\mathrm{slope}(\mathcal{E}'') \ge \lambda$.

Proof: Take the kernel of $F(u)$ in $\mathcal{A}$, which corresponds to a strict object $\mathcal{E}'$ of $\mathcal{E}$, and $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is exact, so we can use(II.8.3.6). □

Cor. (II.8.3.8). If $\mathcal{E}, \mathcal{F}$ are semistable of slopes $\lambda > \mu$, then $\mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{F}) = 0$.

Prop. (II.8.3.9) (Semistable Vector Bundles Form a Weak Serre Subcategory). If $f : \mathcal{E} \to \mathcal{F}$ be a map of vector bundles of the same slope λ, then $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are all semistable vector bundles of slope λ, and if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is exact and $\mathcal{E}', \mathcal{E}''$ are semistable of slope λ, then so does $\mathcal{E}$.

Proof: Use $F(f)$ to find the "coimage" A and the "image" B of f, then there is a map from $F(A)$ to $F(B)$ which is an isomorphism, but they have the same degree and rank, thus $A \cong B$ by the last axiom. And the image must has slope λ. Then $\mathrm{Ker}(f), \mathrm{Coker}(f)$ all can be defined, and they have the same slope λ by(II.8.3.6).

$\mathrm{Ker}(f)$ is semistable because strict subobjects of $\mathrm{Ker}(f)$ are also strict subobjects of $\mathcal{E}$(II.8.3.2). And for $\mathrm{Coker}(f)$, if it is not semistable, choose $\overline{\mathcal{F}'} \subset \mathrm{Coker}(f)$ that has slope$> \lambda$, let $\mathcal{F}'$ be the inverse image, then $0 \to \mathrm{Im}(f) \to \mathcal{F}' \to \overline{\mathcal{F}'} \to 0$, then by(II.8.3.6) $\mathrm{slope}(\mathcal{F}') > \lambda$, contradicting the semi-stability of $\mathcal{F}$.

For the extension, $\mathrm{slope}(\mathcal{E}) = \lambda$ by(II.8.3.6), and for a strict subobject $\mathcal{F}$ of $\mathcal{E}$, then we can find F', F'' be strict objects of $\mathcal{E}'$, $\mathcal{E}''$ respectively that there is an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$?, which shows $\mathrm{slope}(\mathcal{F}) \le \lambda$, so $\mathcal{E}$ is semistable. □

Def. (II.8.3.10) (Harder-Narasimhan Filtration). Let $\mathcal{E} \in \mathcal{C}$, a chain of objects $0 \subset \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \ldots \subsetneq \mathcal{E}_m = \mathcal{E}$ is called a **Harder-Narasimhan filtration** iff each quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable of slope λ_i and $\lambda_1 > \lambda_2 > \ldots > \lambda_m$.

Prop. (II.8.3.11). Every object $\mathcal{E} \in \mathcal{C}$ has a unique functorial Harder-Narasimhan filtration.

Proof: For uniqueness: if there are two filtrations, it suffices to show that $\mathcal{E}_1' = \mathcal{E}_1$, because notice by(II.8.3.2) $\mathcal{E}_i$ is a strict subobjects of $\mathcal{E}_j$ for any $i < j$, so we finish by induction on the length of the filtration and considering $\mathcal{E}/\mathcal{E}_1$.

For this, firstly $\lambda_1 = \lambda_1'$, suppose the contrary and $\lambda_1 > \lambda_1'$, then $\lambda_1 > \lambda_i'$ for each i, so $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i'/\mathcal{E}_{i-1}') = 0$ for each i by(II.8.3.8), so by induction $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}) = 0$, contradiction.

Next by the same reason as in the proof above, $\mathcal{E}_1 \hookrightarrow \mathcal{E}$ has image in $\mathcal{E}_1'$, and the reverse is true for $\mathcal{E}_1'$, so $\mathcal{E}_1 \cong \mathcal{E}_1'$ in $\mathcal{E}$.

For existence: Use induction on $\mathrm{rank}(\mathcal{E})$. If $\mathcal{E}$ is semistable, then we finish. Otherwise, there is a strict subobject $\mathcal{F}$ and $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ that $\mathrm{slope}(\mathcal{F}) > \mathrm{slope}(\mathcal{E})$, so $\mathrm{rank}(\mathcal{F}), \mathrm{rank}(\mathcal{G}) <$

rank($\mathcal{E}$). Now by induction $\mathcal{F}$ and $\mathcal{G}$ has HN-filtration, thus by argument as above, we see that $\mathcal{E}$ cannot have strict subobject with slope bigger than slopes appearing in the HN-filtration of $\mathcal{F},\mathcal{G}$. So if we choose a strict subobject of $\mathcal{E}_1$ of maximal rank among the strict subobjects of maximal slope, we claim the subobjects of $\mathcal{E}/\mathcal{E}_1$ all have slopes smaller than slope($\mathcal{E}_1$): if some slope($\mathcal{G}$) $\geq$ slope($\mathcal{E}_1$), consider its inverse image $\mathcal{G}'$, then $0\to\mathcal{E}_1\to\mathcal{G}'\to\mathcal{G}\to 0$, thus slope($\mathcal{G}'$) $\geq$ slope($\mathcal{E}_1$) and has bigger rank, contradiction. So we can use induction on $\mathcal{E}/\mathcal{E}_1$. □

4 Line Bundles and Filtrations

Def. (II.8.4.1). By(II.8.2.31), the graded algebra $\oplus_{n\geq 0}B^{\varphi=p^n}$ is generated over $\mathbb{Q}_p$ by $B^{\varphi=p}$, so we can define the Serre twisting sheaf $\mathcal{O}(1)$ on X_{FF}, which is a line bundle, and on an open affine scheme $U=X-\{x\}$, where x corresponds to $\log([\varepsilon])$, $\mathcal{O}(1)(U)=(B[\frac{1}{\log([\varepsilon])}])^{\varphi=p}$. Similarly we can define $\mathcal{O}(m)$, and $\mathcal{O}(m)=\mathcal{O}(1)^m$.

Lemma (II.8.4.2). There is an isomorphism $\mathrm{Div}(X)\to\mathrm{Pic}(X)$ that maps each x to the inverse of its ideal sheaf(III.6.1.16)(I.6.5.11). And there is also a degree map $\mathrm{Div}(X)\to\mathbb{Z}$. Then:

$$\begin{tikzcd} \mathrm{Div}(X) \arrow[r, "\deg"] \arrow[dr] & \mathbb{Z} \arrow[d, "\rho"] \\ & \mathrm{Pic}(X) \end{tikzcd}$$

commutes.

Proof: It suffices to show that any $\mathcal{O}(x)$ is isomorphic to $\mathcal{O}(1)$. As $\log([\varepsilon])$ is a global section of $\mathcal{O}(1)$ that vanishes of order 1 at x, it induces an isomorphism $\mathcal{O}(1)\cong\mathcal{O}(x)$ by(III.6.1.16). □

Lemma (II.8.4.3) (Cohomology of Line Bundles). For any integer m, $B^{\varphi=p^m}\to H^0(X,\mathcal{O}(m))$ is an isomorphism and $H^i(X,\mathcal{O}(m))=0$ for $i>0, m>0$.

Proof: This is trivial using Cech cohomology, as $\oplus_{n\geq 0}B^{\varphi=p^n}$ is PID, so X is separated. □

Prop. (II.8.4.4). The construction induces an isomorphism $\rho:\mathbb{Z}\cong\mathrm{Pic}(X):m\mapsto\mathcal{O}(m)$.

Proof: By lemma(II.8.4.2), ρ is surjective because $\mathrm{Div}(X)\to\mathrm{Pic}(X)$ does, and it is injection because if $\mathcal{O}(m)\cong\mathcal{O}(n)$, then tensoring $\mathcal{O}(-m)$, we can assume $\mathcal{O}\cong\mathcal{O}(-k)$, but they have different global sections by lemma(II.8.4.3) and(II.8.2.29)(II.8.2.31). □

Harder-Narasimhan Filtration of Vector Bundles

Prop. (II.8.4.5) (Harder Narasimhan Formalism for Bun_X). For a vector bundle L on X, we can define $\deg(L)=n$ iff $L\cong\mathcal{O}(n)$(II.8.4.2), and for a vector bundle E, define $\deg(E)=\deg(\wedge(E))$. And define the generic rank on the category of coherent sheaves on X. Then this is a Harder-Narasimhan formalism on $\mathcal{C}=Bun_X$ with $\mathcal{A}=Vect_{K(X)}$.

Proof: Only the last axiom needs proof, but if $\mathcal{E}'\subsetneq\mathcal{E}$, notice $\wedge\mathcal{E}'\subsetneq\wedge\mathcal{E}$(The stalks are PID), so by taking their top exterior power product, we reduce to the case of line bundles.

But $\mathcal{O}(m)$ cannot map into $\mathcal{O}(n)$ if $m>n$ and must by isomorphism if $m=n$, by tensoring $\mathcal{O}(-m)$ and looking at global sections(II.8.4.3), so the assertion is true. □

Cor. (II.8.4.6). Every vector bundle $\mathcal{E}$ on X has a unique functorial Harder-Narasimhan filtration, by(II.8.3.11).

5 Base Change of Fields

Prop. (II.8.5.1) (Base Change). Let $C^\flat$ be alg.closed. For any finite extension E of $\mathbb{Q}_p$ of degree n, $\operatorname{Spec} E \to \operatorname{Spec}\mathbb{Q}_p$ is finite étale, and finite locally free of degree n, so does $X_E = X\otimes_{\mathbb{Q}_p} E \to X$(III.4.7.8).

In particular, X_E is also a Dedekind scheme?. For any closed point x of X corresponding to an untilt K of $C^\flat$, which is alg.closed, the fiber of X_E over x is identical to the spectrum of $E\otimes_{\mathbb{Q}_p} K \cong K^n$ as K is alg.closed.

In this situation and use(II.8.2.36), we see that the closed points of X_E are in bijection with isomorphism classes of (K,ι,u) module φ-actions, where (K,ι) is an untilt of $C^\flat$, and $u: E\to K$ is an embedding of E into K over $\mathbb{Q}_p$, isomorphism classes of these triples are denoted by Y_E.

Prop. (II.8.5.2). By(II.8.5.1) and flat base change(III.5.4.31), we know $H^0(X_E,\mathcal{O}_{X_E}) = E$, in particular X_E is connected.

Lemma (II.8.5.3). If E is unramified of degree n over $\mathbb{Q}_p$, then $E\cong W(\mathbb{F}_{p^n})[\frac{1}{p}]$. In particular,

$$\operatorname{Hom}_{\mathbb{Q}_p}(E,K)\cong \operatorname{Hom}_{\mathbb{Z}_p}(W(\mathbb{F}_{p^n}),\mathcal{O}_K)\cong \operatorname{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^n},\mathcal{O}_K/p)\cong \mathcal{O}_{C^\flat}/[t])\cong \operatorname{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^n},C)$$

where the last isomorphism is by Henselian lemma.

Therefore, $Y_E\cong Y\otimes \operatorname{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^n},C)$, and

$$\text{Closed points of } Y_E\cong Y_E/\varphi^{\mathbb{Z}}\cong (Y\otimes \operatorname{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^n},C))/\varphi^{\mathbb{Z}}\cong Y/\varphi^{n\mathbb{Z}}.$$

Prop. (II.8.5.4). If E is unramified of degree n over $\mathbb{Q}_p$ and $U\neq X$ is an affine open defined by a homogenous element t, then

$$U_E = \operatorname{Spec}((B[t^{-1}]\otimes_{\mathbb{Q}_p} E)^{\varphi=\mathrm{id}}) = \operatorname{Spec}(B[t^{-1}]^{\varphi^n=1}).$$

where φ acts trivially on E.

Proof: Each $u\in \operatorname{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^n},C^\flat)$ induces a map $W(\mathbb{F}_p)\to W(\mathcal{O}_{C^\flat}) = A_{\mathrm{inf}}\to B$, which extends to a map $\overline{u}: E\to B[t^{-1}]$. and induces a map $q_u: B[t^{-1}]\otimes_{\mathbb{Q}_p} E\to B[t^{-1}]$. Now

$$B[t^{-1}]\otimes_{\mathbb{Q}_p} E\to \prod_{\operatorname{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^n},C)} B[t^{-1}]$$

is an isomorphism, which is just because $x^{p^n}-x$ splits in $B[t^{-1}]$.

And under this isomorphism, the action of φ is

$$\varphi((f_0,\ldots,f_{n-1}) = (\varphi(f_{n-1}),\varphi(f_0),\ldots,\varphi(f_{n-2})),$$

so proposition is clear. $\square$

Cor. (II.8.5.5). Fix now a finite extension $E/\mathbb{Q}_p$ with uniformizer π that has ramification degree e and inertia degree d, and E_0 is the maximal unramified subextension, then there are maps $E_0\to B$ by(II.8.5.4), fix forever one of them p_u, this induces a map

$$B[\frac{1}{t}]\otimes_{\mathbb{Q}_p} E\to B[\frac{1}{t}]\otimes_{E_0} E$$

and this induces an isomorphism

$$(B[\frac{1}{t}]\otimes_{\mathbb{Q}_p} E)^{\varphi=\mathrm{id}} = (B[\frac{1}{t}]\otimes_{\mathbb{Q}_p} E_0)^{\varphi=\mathrm{id}}\otimes_{E_0} E = (B[\frac{1}{t}]\otimes_{E_0} E)^{\varphi^d=\mathrm{id}}$$

Def. (II.8.5.6) (Y_E^0). Define $Y_E^0 \subset Y_E$ = triples (K, ι, u), where (K, ι) is an untilt of $C^\flat$, and $u : E \to K$ is an embedding that $u|_{E_0}$ is identical to $e_K \circ p_u : E_0 \to B \to K$. Notice Y_E^0 is not stable under the Frobenius, but it is stable under φ^d, and induces an isomorphism

$$Y_E^0/\varphi^{d\mathbb{Z}} \cong Y_E/\varphi^{\mathbb{Z}}.$$

Prop. (II.8.5.7). Notice for an element y of Y_E^0, the map $u : E \to K_y = B_{dR}^+(y)/\xi$ extends uniquely to a map $E \to B_{dR}^+(y)$ that is compatible with $e_K \circ p_u : E_0 \to B_{dR}^+(y)$, because E is separable over E_0. i.e.

$$\begin{tikzcd} E_0 \arrow[r, "e_K \circ p_u"] \arrow[d] & B_{dR}^+(y) \arrow[d] \\ E \arrow[r, "u"] \arrow[ur, "\widetilde{u}"] & K \end{tikzcd}$$

Then this defines a map $B \otimes_{E_0} E \to B_{dR}^+(y)$, also called the stalk map.

Prop. (II.8.5.8). For any finite extension $E/\mathbb{Q}_p$, the degree map $\deg : \mathrm{Pic}(X_E) \cong \mathbb{Z}$ is an isomorphism.

Proof: It suffices to show that $\mathcal{O}_{X_E}(x) \cong \mathcal{O}_{X_E}(x')$ for each pair of closed points x, x' of X_E.

We attempt to construct a line bundle $\mathcal{O}_{X_E}(1)$ on X_E that $\mathcal{O}_{X_E}(1)(U_E) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d = \pi}$, because $\mathcal{O}_{X_E}(U_E) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d = 1}$.

We show simultaneously that $\mathcal{O}_{X_E}(1)$ is a line bundle and it is isomorphic to $\mathcal{O}_{X_E}(x)$ for any closed point $x \in X_E$: For any $x \in X_E$ corresponding to a φ^d-orbit of Y_E^0, let f be the element constructed by lemma(II.8.5.9) below, we show that for any affine open $U = D(t)$, multiplying by $f : \mathcal{O}_{X_E}(x)(U_E) \to \mathcal{O}_{X_E}(1)(U_E)(\star)$ is an isomorphism:

Notice $B \otimes_{E_0} E$ is free over B, let $N(f) \in B$ be its norm, the norm is local, so for each $y \in Y$, $N(f)_y = \prod_{\overline{y}} f_{\overline{y}}$, where $\overline{y} \in Y_E$ are over y, so it vanishes with order 1 in a φ^d-orbit of Y(order 1 because f only vanishes at $\overline{y}$ in the orbit corresponding to x), and then $N(f)\varphi(N(f))\ldots\varphi^{d-1}(N(f))$ vanishes at a single φ-orbit of Y with order 1, thus equals $u \log([\varepsilon])$ for some $\varepsilon \in \mathfrak{m}_{C^\flat}$, by(II.8.2.13)(II.8.2.9). In particular, y divides $\log([\varepsilon])$.

Now if $x \notin U_E$, then $\log([\varepsilon])$ divides t, so f divides t, thus f is invertible in $B[\frac{1}{t}] \otimes_{E_0} E$, thus $(\star)$ is an isomorphism.

Otherwise if $x \in U_E$, then choose some x' not in U_E, then the same argument shows that f' is invertible in $B[\frac{1}{t}] \otimes_{E_0} E$, so $f/f' \in (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi = \mathrm{id}}$ that vanishes with a single zero at x, so multiplying by f'/f defines an isomorphism $\mathcal{O}_{X_E}(U_E) \cong \mathcal{O}_{X_E}(x)(U_E)$, so it suffices to show the composition

$$\mathcal{O}_{X_E}(U_E) \xrightarrow{f'/f} \mathcal{O}_{X_E}(x)(U_E) \xrightarrow{f} \mathcal{O}_{X_E}(1)(U_E)$$

is an isomorphism, but this reduces to the first case. □

Lemma (II.8.5.9) (Uniformizer Existence). If x be a closed point of X_E corresponding to an orbit of φ in Y_E thus an orbit S of φ^d in Y_E^0, then there is an element $f \in (B \otimes_{E_0} E)^{\varphi^d = \pi}$ that $\mathrm{ord}_{\overline{y}}(f) = 1$ if $\overline{y} \in S$, and 0 otherwise.

Proof: The map defined (II.8.5.18) composed with the Teichmuller section(II.8.5.16) in fact has image in $(B \otimes_{E_0} E)^{\varphi^n = \pi}$ because $[\pi] = \pi t + t^{p^n} = \varphi^n$ on $\mathcal{O}_{C^\flat}$, and it is an isomorphism of $\mathcal{O}_E$

modules. Now there are commutative diagrams:

$$\begin{tikzcd} G_{LT}(\mathcal{O}_{C^\flat}) \arrow[r, "\sigma"] & G_{LT}(A_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \arrow[r, "\log_G"] \arrow[d] & B \otimes_{E_0} E \arrow[d] \\ & G_{LT}(\mathcal{O}_K) \arrow[r, "\log_G"] & K \end{tikzcd}$$

The map $G_{LT}(\mathcal{O}_{C^\flat}) \to G_{LT}(\mathcal{O}_K)$ has kernel $\mathcal{O}_E u$ for some u, thus we can let $f = \log_G(\sigma(u))$, then the image of $f \in K$ is 0, which means f has a zero at the point $y \in Y_E^0$. And(II.8.5.13) shows the the zeros of f is just the φ^d-orbit containing y. □

Lubin-Tate Formal Groups and the Proof of the Lemma of Uniformizer

Prop. (II.8.5.10). The ring $B \otimes_{E_0} E$ is an integral domain.

Proof: Cf.[Lurie P95]. In fact this is the ramified Witt vector, which is by the same reason as before an integral domain, Cf.[FF Curve Johannes]. □

Cor. (II.8.5.11). If $f \neq 0 \in B \otimes_{E_0} E$, then $N_{E/E_0}(f) \neq 0 \in B$, in particular, the vanishing locus of f is finite.

Cor. (II.8.5.12). If $f, g \in B \otimes_{E_0} E$, then f is divisible by g iff for each $\overline{y} \in Y_E^0$, $\mathrm{ord}_{\overline{y}}(f) \geq \mathrm{ord}_{\overline{y}}(g)$.

Proof: If $\mathrm{ord}_{\overline{y}}(f) \geq \mathrm{ord}_{\overline{y}}(g)$, suppose $N_{E/E_0}(g) = gh$, then multiplying by h, we can assume $g \in B$. Now f is written uniquely as $f_0 + f_1\pi + \ldots + f_{e-1}\pi^{e-1}$ where $f_k \in B$, thus it suffices to show f_i is divisible by g, which is equivalent to $\mathrm{ord}_y(f) \geq \mathrm{ord}_y(g)$ for each $y \in Y$, by(II.8.2.11). Now if $\mathrm{ord}_y(g) = n$, the hypothesis shows f vanishes in

$$\prod_{\overline{y} \to y} B_{dR}^+(y)/\xi^n = (B_{dR}^+(y)/\xi^n) \otimes_{E_0} E = B_{dR}^+(y)/\xi^n + \pi B_{dR}^+(y)/\xi^n + \ldots + \pi^{e-1} B_{dR}^+(y)/\xi^n$$

thus $\mathrm{ord}_y(f) \geq n = \mathrm{ord}_y(g)$. □

Cor. (II.8.5.13). If $f \in (B \otimes_{E_0} E)^{\varphi^n = \pi}$, then the vanishing locus of f is a single φ^d-orbit, and all zeros are simple.

Proof: Set $N_{E/E_0}(f) = f'$ and $N_{E/E_0}(\pi) = \pi'$, then f belongs to $B^{\varphi^d = \pi'}$, and its divisor is just the image of divisor of f in Y_E^0. So it suffices to show that f' vanishes on a single $\varphi^{d\mathbb{Z}}$-orbit.

Now for $0 < \rho < 1$,

$$\rho^{p^d} |f'|_{\rho^{p^d}} = |\pi' f'|_{\rho^{p^d}} = |f'^{\varphi^d}|_{\rho^{p^d}} = |f'|_\rho^{p^d},$$

thus

$$p^d s + v_{p^d s}(f') = p^d v_s(f')$$

for each $s > 0$, differentiating, we get

$$1 + \partial_- v_s(f') = \partial_- v_s(f')$$

Now the divisor of f' is $\varphi^{d\mathbb{Z}}$-invariant, and it has exactly one zero on any annulus $(\rho^n, \rho]$(II.8.2.15), thus its divisor is a single φ^d-orbit. □

Def. (II.8.5.14) (Universal Lubin-Tate Formal Group). Recall that if E is a finite extension of Q_p with uniformizer π, for a $\mathcal{O}_E$-algebra A complete w.r.t π, $G_{LT}(A)$ is the Lubin-Tate formal group, with elements the topological nilpotent elements of A3.

Now we define the **universal cover of Lubin-Tate formal group** $\widetilde{GT}$ as the functor

$$A \mapsto \lim\{\cdots \xrightarrow{[\pi]} G_{LT}(A) \xrightarrow{[\pi]} G_{LT}(A)\}.$$

Prop. (II.8.5.15).

- Notice for K an alg.closed extension of E, $G_{LT}(\mathcal{O}_K)$ is in bijection with $\mathfrak{m}_K$, and the kernel of $[\pi^n]$ on $G_{LT}(\mathcal{O}_K)$ has order $\mathcal{O}_E/\pi^n$, thus the kernel of $\widetilde{G}_{LT}(\mathcal{O}_K) \to G_{LT}(\mathcal{O}_K)$ is a 1-dimensional $\mathcal{O}_E$-module.
- If π vanishes on A and A is perfect, then $[\pi] = \pi t + t^q = t^q$ on A, so it is just the Frobenius, and $\widetilde{G}_{LT}(A) \to G_{LT}(A)$ is a bijection.
- $\widetilde{G}_{LT}(A) \to \widetilde{G}_{LT}(A/I)$ is an isomorphism for $\pi \in I$ and A is I-adic.
- $\widetilde{G}_{LT}(A) \to \widetilde{G}_{LT}(A/I)$ is an isomorphism for any ideal I that $I + (\pi) \neq (1)$, because both of them is isomorphic to $\widetilde{G}_{LT}(A/(I + (\pi))$.

Proof: For 3, it suffices to prove that $\widetilde{G}_{LT}(A/I^{n+1}) \to \widetilde{G}_{LT}(A/I^n)$ for $n \geq 1$. Notice $F(u, v) \equiv u + v \mod I^{2n}$, so we have an exact sequence

$$0 \to I^n/I^{n+1} \to G_{LT}(A/I^{n+1}) \to G_{LT}(A/I^n) \to 0.$$

In particular the kernel is annihilated by π, so there is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\pi"] & G_{LT}(A/I^{n+1}) \arrow[r, "\pi"] \arrow[d] & G_{LT}(A/I^{n+1}) \arrow[d] \\
\cdots \arrow[r, "\pi"] \arrow[ur] & G_{LT}(A/I^{n+1}) \arrow[r, "\pi"] \arrow[ur] & G_{LT}(A/I^{n+1})
\end{tikzcd}$$

which show that $\widetilde{G}_{LT}(A/I^{n+1}) \cong \widetilde{G}_{LT}(A/I^n)$. $\square$

Cor. (II.8.5.16) (Teichmuller Section). Consider the $\mathcal{O}_E$-algebra $A_{\text{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E$. Because there are isomorphism $\mathcal{O}_{E_0}/p \cong \mathcal{O}_E/\pi$, we have an isomorphism

$$C^\flat \cong A_{\text{inf}}/p \cong (A_{\text{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E)/\pi$$

Now(II.8.5.15) shows the diagram

$$\begin{tikzcd}
\widetilde{G}_{LT}(A_{\text{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \arrow[r, "\cong"] \arrow[d] & \widetilde{G}_{LT}(\mathcal{O}_{C^\flat}) \arrow[d, "\cong"] \\
G_{LT}(A_{\text{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \arrow[r] & G_{LT}(\mathcal{O}_{C^\flat})
\end{tikzcd}$$

So the lower horizontal map is surjective, and it even has a canonical section σ, called the **Teichmuller section**.

Cor. (II.8.5.17). Given a point of Y_E^0 which corresponds to an untilt of $C^\flat$ together with a E_0-map $E \to K$, then this gives a commutative diagram

$$\begin{array}{ccc} \widetilde{G}_{LT}(A_{\text{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) & \xrightarrow{\cong} & \widetilde{G}_{LT}(\mathcal{O}_K) \\ \downarrow & & \downarrow \\ G_{LT}(A_{\text{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) & \longrightarrow & G_{LT}(\mathcal{O}_K) \end{array}$$

where the right vertical arrow is surjective with kernel free of rank 1 over $\mathcal{O}_E$. So this together with(II.8.5.16) shows there is a surjection $G_{LT}(\mathcal{O}_{\mathcal{O}_{C^\flat}}) \to G_{LT}(\mathcal{O}_K)$ with kernel a rank-1 $\mathcal{O}_E$-module.

Prop. (II.8.5.18). There is a canonical $\mathcal{O}_E$-module map

$$G_{LT}(A_{\text{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \xrightarrow{\log_G} B \otimes_{E_0} E.$$

and it is equivariant w.r.t φ.

Proof: $G_{LT}(A_{\text{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E)$ are in bijection with the maximal ideal of $A_{\text{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E$, and $\log_G(x)$ is of the form $x + \frac{c_2}{2}x^2 + \ldots + \frac{c_n}{n}x^n + \ldots$, with $c_n \in \mathcal{O}_E$.

Now for $x \in G_{LT}(A_{\text{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E)$, we show that $\log_G(x)$ converges in $B \otimes_{E_0} E = B + \pi B + \ldots + \pi^{e-1}B$: Let $c_n x^n = \sum a_{n,i}\pi^i$, then we need to show $a_{n,i}/n$ converges to 0 for each of the norm $|\cdot|_\rho$. And this is because if $x = x_0 + \pi y_0$, then for $n \geq em$, $|x|_\rho \leq \max\{|x_0|_\rho^{em}, \rho^m$, which decays exponential in n, and $|\frac{1}{n}|_\rho$ decays linearly in n. □

Prop. (II.8.5.19). The map $\log_G(\sigma(\cdot)) : G_{LT}(\mathcal{O}_{C^\flat}) \to (B \otimes_{E_0} E)^{\varphi^n = \pi}$ as in(II.8.5.9) is an isomorphism.

Proof: For surjectivity, as any $f \in (B \otimes_{E_0} E)^{\varphi^n = \pi}$ vanishes at a single $\varphi^{d\mathbb{Z}}$-orbit, then by(II.8.5.9) we can find a $\log_G(u)$ that vanishes at the same locus, so $f = \log(u)\lambda$ where λ is a unit in $B \otimes_{E_0} E$(II.8.5.12), so

$$\lambda \in (B \otimes_{E_0} E)^{\varphi^n = \text{id}} = (B \otimes_{\mathbb{Q}_p} E)^{\varphi = \text{id}}(II.8.5.5) = B^{\varphi = \text{id}} \otimes_{\mathbb{Q}_p} E = E.$$

For injectivity, we proved in(II.8.5.9) that each $\log_G(\sigma(u))$ only vanishes at a single φ^d-orbit in Y_E^0, so it cannot by 0, which vanishes at all points. □

Cor. (II.8.5.20). There are canonical bijections

$$\{\text{Closed Points of } X_E\} \cong \{\varphi^{d\mathbb{Z}}\text{orbits of } Y_E^0\} \cong ((B \otimes_{E_0} E)^{\varphi^n = \pi} - \{0\})/E^* \cong (G_{LT}(\mathcal{O}_{C^\flat}) - \{0\}/E^*$$

by(II.8.5.9)(II.8.5.19),(II.8.5.12).

Vector Bundles and Base Change

Prop. (II.8.5.21) (Vector Bundles on the Cover). Let $\pi : X_E \to X$ be the covering map, for any vector bundle $\mathcal{E}$ on X_E, $\pi_*(\mathcal{E})$ is a vector bundle on X, and this induces an isomorphism

$$\{X_E\text{-Bundles}\} \cong \{X - \text{Bundles with an } E\text{-action}\}.$$

Now define $\deg(\mathcal{E}) = \deg(\pi_*\mathcal{E})$, and $\text{slope}(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\text{rank}(\mathcal{E})} = \frac{1}{n}\text{slope}(\pi_*\mathcal{E})$.

Then $\mathcal{E}$ is semistable of slope λ iff $\pi_*\mathcal{E}$ is semistable of slope λ/n.

Proof: One direction is clear, for the other, if $\mathcal{F} = \pi_* \mathcal{E}$ is not semistable, choose its HN-filtration, then $\lambda_1 > \lambda/n$. Now the action of E on $\mathcal{F}$ preserves the HN-filtration, thus $\mathcal{F}_1$ is an E-vector bundle, thus by the correspondence above, $\mathcal{F}_1 = \pi_* \mathcal{E}'$ for some subbundle $\mathcal{E}' \subset \mathcal{E}$, and clearly this contradicts the semistablity of $\mathcal{E}$. □

Cor. (II.8.5.22). For any integral number d, n with $n > 0$, there exists a semistable vector bundle on X with rank n and degree d.

Proof: Let E be an extension of $\mathbb{Q}_p$ of degree n, then $\pi_*(\mathcal{O}_{X_E}(d))$ is semistable of rank n and degree d, by(II.8.5.1)(III.4.7.7), because $\mathcal{O}_{X_E}(d)$ is a line bundle(II.8.5.8) so clearly semistable, and it is of degree d because ?. □

Isocrystals and Classification of Semistable Vector Bundle over X

Remark (II.8.5.23). Recall the Dieudonné-Manin Classification(II.10.4.7)(II.10.4.10): Any isocrystal over k is a finite sum of modules pure of slopes λ_i. And if k is alg.closed, then any isocrystal over k has a unique decomposition as sums of E_{λ_i}.

Prop. (II.8.5.24). Let $k = \overline{\mathbb{F}}_p \in C^\flat$, then there is an inclusion $W(k) \to A_{\mathrm{inf}}$, which extends to a map $K \to B$. Now given an isocrystal V over k, denote $\mathcal{E}_V$ the coherent sheaf on X defined by the graded module $\oplus_{n \geq 0} \mathrm{Hom}_K(V, B)^{\varphi = p^n}$. In other words, on an affine open subscheme $U = D(t)$, $\mathcal{E}_V(U) = \{\varphi - \text{equivariant } K\text{-linear maps } V \to B[\frac{1}{t}]\}$.

And when $V = E_{m/n}$ is the simple isocrystal, then $\mathcal{E}_V$ is denoted by $\mathcal{O}(\frac{m}{n})$.

Prop. (II.8.5.25). In fact we have $\mathcal{O}(\frac{m}{n})(U) \cong (B[t^{-1}])^{\varphi^n = p^m} = (\rho_* \mathcal{O}(m))(U)$, where $\rho : X_E \to X$, and E is an unramified extension of $\mathbb{Q}_p$.

Prop. (II.8.5.26) (Classification of Semistable Vector Bundles over X). For every vector bundle on X, the HN-filtration splits non-canonically, and the construction $V \to \mathcal{E}_V$ induces an equivalence of categories between

$$\{\text{Isoclinic Isocrystals of slope } \mu\}^{op} \to \{\text{Semistable vector bundles on } X \text{ of slope } \mu\}$$

Proof: Cf.[FF Curve Johannes].? □

Cor. (II.8.5.27). Any two semistable vector bundles of slope λ over X is isomorphic, and a semistable vector bundle of slope 0 is trivial.

Prop. (II.8.5.28). If $\mathcal{E}, \mathcal{E}'$ be semistable vector bundles on X of slopes μ, μ', then $\mathcal{E} \otimes \mathcal{E}'$ is semistable of slope $\mu + \mu'$.

Proof: We can assume $\mathcal{E} = \rho_* \mathcal{O}_{X_E}(d)$ for an unramified extension $E/\mathbb{Q}_p$ by(II.8.5.27), and then $\mathcal{E} \otimes \mathcal{E}' = \rho_*(\mathcal{O}_X(d) \otimes \rho^* \mathcal{E}')$. Since ρ_*, ρ^* preserves semistability(by(II.8.5.21) and?). So it suffices to prove $\mathcal{O}(d) \otimes -$ preserves semistablity, but this is clear, as $\mathcal{O}(d)$ shifts degree. □

Diamonds Definitions

Def. (II.8.5.29) (Diamond). Let Perf denote the site of perfectoid spaces of characteristic p equipped with the pro-étale topology. A **diamond** X is a sheaf (of sets) on Perf of the form $X = Hom_{Perf}(, Z)/R$, where $Z \in Perf$ and $R \in Z \times Z$ is a reasonable representable equivalence relation.

Prop. (II.8.5.30) (Scholze). Let $\underline{R} = (R, R^+)$ be a Huber pair, then

$$\mathrm{Spd}(\underline{R}) = \mathcal{Z} \mapsto \{\text{untilts of}\mathrm{Spa}(\underline{R}) \text{ over } Z\}$$

is a diamond.

And this construction can be glued to give diamond $X^\diamond$ of any adic space X, which is a sheaf.

Def. (II.8.5.31) (Adic Fargues-Fontaine Curve). Let $\mathcal{Y}$ be the adic space $\mathrm{Spa}(A_{\mathrm{inf}})$ removing the vanishing locus of p and $[t]$, then by what we proved, the Frobenius act totally discontinuous on $\mathcal{Y}$, thus the quotient $\mathcal{X}^{FF}$ is an adic space, the FF-curve.

Prop. (II.8.5.32). There is an isomorphism of diamonds:

$$\mathcal{Y}^\diamond \cong \mathrm{Spd}(C^\flat) \times \mathrm{Spd}(\mathbb{Q}_p), \quad \mathcal{X}^{FF,\diamond} \cong \mathrm{Spd}(C^\flat)/\varphi^{\mathbb{Z}} \times \mathrm{Spd}(\mathbb{Q}_p)$$

More generally, over any Huber pair $\underline{R}$, there is a relative FF-curve which is defined by

$$\mathrm{Spd}(\underline{R}) \times \mathrm{Spd}(\mathbb{Q}_p)/\varphi^{\mathbb{Z}}$$

Proof: $C^\flat$ is a perfectoid of char$p$, so for a perfect Huber pair, $\underline{S}$ point of $C^\flat$ is just a morphism $u : (C^\flat, \mathcal{O}_{C^\flat} \to (S, S^+)$. And a $\mathbb{Q}_p$ point is just a char0 untilts $\underline{T}$ of $\underline{S}$.

So for each pairs (T, u), we need to find a morphism $A_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[t]} \to T$. For this, consider

$$A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat}) \to W(S^+) \cong W(T^\flat) \xrightarrow{\theta_T} T$$

This is a bijection, as we proved in the beginning(II.8.1.2). □

Prop. (II.8.5.33). There is a morphism of ringed spaces $\mathcal{X}^{FF} \to X^{FF}$ that regard $\mathcal{X}^{FF}$ as the rigid analytification of $\mathcal{X}$, so they have the same category of vector bundles and cohomology, prove by Kedlaya-Liu.

6 Applications

Prop. (II.8.6.1). The FF curve X is geometrically simply connected, i.e. the projection defines an isomorphism of étale groups $\pi_1(X) \to \pi_1(\mathrm{Spec}\,\mathbb{Q}_p) = Gal(Q_p)$.

Equivalently, the pullback defines an equivalence of étale sites.

Proof: Let $\widetilde{X} \to X$ be an finite étale morphism, we want to prove that $\widetilde{X} = X \otimes_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}(E)$ for some étale $\mathbb{Q}_p$-algebra. Let $\mathcal{A} = \rho_* \mathcal{O}_{\widetilde{X}}$, and $E = H^0(X, \mathcal{A}) = H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Now it suffices to show $\mathcal{A} = E \otimes_{\mathbb{Q}_p} \mathcal{O}_X$, which shows $\widetilde{X} = X \otimes_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}(E)$, and forces E be an étale $\mathbb{Q}_p$-algebra by fpqc descent?. Equivalently, $\mathcal{A}$ is trivial, and this is equivalent to $\mathcal{A}$ being semistable of slope 0 by(II.8.5.27).

Because ρ is finite étale, the trace pairing $\mathcal{A} \times \mathcal{A} \to \mathcal{A} \xrightarrow{tr} \mathcal{O}_X$ is non-degenerate(check on stalks), which induces an isomorphism $\mathcal{A} \cong \mathcal{A}^\vee$, so $\deg(\mathcal{A}) = 0$, and if $\mathcal{A}$ is not semistable, let $\mathcal{A}'$ be the first term of the HN-filtration of $\mathcal{A}$, then it is of slope $\lambda > 0$, So $\mathcal{A}' \otimes \mathcal{A}'$ is of slope 2λ by(II.8.5.28), so the composite $\mathcal{A}' \otimes \mathcal{A}' \hookrightarrow \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ must by 0(II.8.3.8), which is impossible, because if U is an affine open that $\mathcal{A}$ has a section, then this says $U \otimes_X \widetilde{X}$ has a section s that $s^2 = 0$. But $U \otimes_X \widetilde{X}$ is reduced(check on stalks). □

Cor. (II.8.6.2).

- The projection map induces equivalence of categories between Finite Abelian groups with $Gal(Q_p)$-action and étale Local system on X.
- If M is a finite Abelian group with a $Gal(Q_p)$-action, then
$$H^*(Gal(\mathbb{Q}_p), M) \to H^*_{ét}(X, u^*M)$$
is an isomorphism for $* = 0, 1$.

Proof: 1 is trivial, and 2 Cf. [Lurie P102]. □

7 Weakly Admissible⇒Admissible

Def. (II.8.7.1) (Notations). Let K be a finite extension of $\mathbb{Q}_p$, and $K_0 = W(k)[\frac{1}{p}]$ be the maximal unramified subextension in K, let $C = \widehat{\overline{K}}$ and $F = C^\flat$. Denote by $\infty \in X$ the closed point determined by C, which is just the vanishing locus of the Galois stable line $\mathbb{Q}_p t$, where $t = \log([\varepsilon])$ and $\varepsilon = (1, \xi_p, \xi_{p^2}, \ldots) \in C^\flat$(II.8.2.32).

Notice G_K acts on $\mathbb{Q}_p \log([\varepsilon])$ by the cyclotomic character χ_{cycl}. Recall
$$B^+_{dR} = \widehat{\mathcal{O}}_X(\infty), \quad B^+_{crys}, \quad B_{crys} = B^+_{crys}[t^{-1}], \quad B_e = H^0(X - \{\infty\}, \mathcal{O}_X) = (B[t^{-1}])^{\varphi=\mathrm{id}}$$

Def. (II.8.7.2) (Equivariant Action of G_K on Vector Bundles). Recall an equivariant action of G_K on a bundle $\mathcal{E}$ on X is a data of isomorphisms $\sigma^*(\mathcal{E}) \cong E$ that $c_{\sigma\tau} = c_\tau \circ \tau^*(c_\sigma)$. Notice any equivariant action of G_K on $\mathcal{E}$ induces a semilinear G_K action on $\mathcal{E}^\wedge_\infty = \mathcal{E} \otimes_{\mathcal{O}_X} B^+_{dR}$, and here we require this action is continuous. The category of equivariant G_K-bundles are denoted by $Bun^{G_K}_X$.

Lemma (II.8.7.3). Let V be a finite $\mathbb{Q}_p$-vector space with an action of G_K, then the action is continuous iff the induced action on $V \otimes_{\mathbb{Q}_p} B^+_{dR}$ is continuous.

Proof: This is because the action of G on B^+_{dR} is continuous, and V has the induced topology in $V \otimes_{\mathbb{Q}_p} B^+_{dR}$. □

Cor. (II.8.7.4). By the slope 0 case of the classification of vector bundles on X(II.8.5.26) and the above lemma(II.8.7.3), we see that the functor:
$$\mathrm{Rep}_{\mathbb{Q}_p} G_K \to \mathrm{Bun}^{G_K}_X : V \mapsto V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$$
is fully faithful with essential image the category $Bun^{G_K,sst,0}_X$ of all G_K vector bundles on X that the underlying bundle is semistable of slope 0, i.e. trivial(II.8.5.27).

Prop. (II.8.7.5).

- $K = B^{G_K}$
- $K_0 = B^{G_K}_{crys}$, and the canonical morphism $K \otimes_{K_0} B_{crys} \to B_{dR}$ is injective.
- $\mathbb{Q}_p = B^{G_K}_e$.

Proof: 1: Use the filtration $t^{-n}B^+_{dR}$ on B_{dR}, then the gradeds are just $C_p(n)$, the p-adic closure twisted by χ_{cycl}-action. Then by(II.10.2.10) $H^0(G_K, \mathbb{C}_p(n)) = K$ iff $n = 0$ because χ^n_{cycl} has finite order iff $n = 0$. Hence induction this implies $K = B^{G_K}$.

2: The injectivity of $K \otimes_{K_0} B_{crys} \to B_{dR}$ Cf.[Laurent Fargues and Jean-Marc Fontaine Prop10.2.8].

3: From 2 and notice $B_e = B^{\varphi=\mathrm{id}}_{crys}$. □

Def. (II.8.7.6) (Admissible Representations). Let K be an extension of $\mathbb{Q}_p$, $K_0 = \mathrm{Frac}(W(k))$ be its maximal unramified extension. Let B be a $\mathbb{Q}_p$-algebra that G_K acts continuously that $F = B^{G_K}$ is a field. Given a finite G_K-module V, consider $D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_K}$, where G_K acts diagonally, then V is called B**-admissible** iff $\dim_F D_B(V) = \dim_{\mathbb{Q}_p} V$, equivalently,

$$D_B(V) \otimes_F B \to V \otimes_{\mathbb{Q}_p} B$$

is an isomorphism.

In particular, if $B = B_{dR}$ or B_{crys}, then the representation is called deRham or Crystalline.

Prop. (II.8.7.7) (Crystalline Representation). The functor

$$\mathcal{D} : \mathrm{Rep}_{B_e} G_K \to \varphi - Mod_{K_0} : W \mapsto (W \otimes_{B_e} B_{crys})^{G_K}$$

are left adjoint to the functor

$$\mathcal{V} : \varphi - Mod_{K_0} \to \mathrm{Rep}_{B_e} G_K : (D, \varphi_D) \mapsto (D \otimes_{K_0} B_{crys})^{\varphi_D \otimes \varphi = \mathrm{id}}$$

Moreover, $\mathcal{V}$ is fully faithful, $\mathrm{id} \cong \mathcal{D} \circ \mathcal{V}$, $\mathcal{V} \circ \mathcal{D} \hookrightarrow \mathrm{id}$, and $M \in \mathrm{Rep}_{B_e} G_K$ is in the image of $\mathcal{V}$ iff $\mathcal{V}(\mathcal{D}(M)) \cong M$.

Proof: Cf.[Laurent Fargues and Jean-Marc Fontaine Prop10.2.12]. □

Cor. (II.8.7.8). In particular, a B_e-representation is crystalline iff it is in the image of $\mathcal{V}$. Now we define a Vector bundle $\mathcal{E}$ on X to be crystalline iff the $H^0(X - \{\infty\}, \mathcal{E})$ is crystalline.

Def. (II.8.7.9) (Filtered φ-Modules). A **filtered φ-module** (D, φ_D, Fil) over K is a φ_D-module $(D, \varphi_D) \in \varphi - Mod_{K_0}$ together with a finite filtration Fil on $D_K = D \otimes_{K_0} K$. The category of filtered φ-modules over K is denoted by $\varphi - FilMod_{K/K_0}$.

Prop. (II.8.7.10) (HN-Formalism for Filtered φ-Modules). The category $\varphi - FilMod_{K/K_0}$ is an exact category where exact sequences are those induces exact sequences on gradeds, and the generic fiber functor is $\varphi - FilMod_{K/K_0} \to Vect_{K_0}, (D, \varphi_D, Fil) \mapsto D$.

The rank is defined as usual and $\deg((D, \varphi_D, Fil)) = \deg(D_K, Fil) - \deg(D, \varphi_D)$(II.8.3.3)(II.8.3.4). This is a HN-filtration.

Proof: The proof is clear, the same as that of(II.8.3.3). □

Def. (II.8.7.11) (Weakly Admissible φ-Modules). The cateogory $\varphi - FilMod^{wa}_{K/K_0}$ of **weakly admissible φ-modules** is those that are semistable of slope 0 w.r.t the HH-formalism, it is an Abelian category by(II.8.3.9).

Def. (II.8.7.12) (Rep$_{B_e}G_K$). Denote by $\mathrm{Rep}_{B_e} G_K$ the category of finite locally free B_e-modules M with a semilinear G_K-action that there exists a G_K-invariant B_{dR}^+-lattice $\Gamma \in M \otimes_{B_e} B_{dR}$ that G_K acts continuously.

Lemma (II.8.7.13). Let W be a f.d. K-vector space, then the map:

$$\{\text{Filtrations on } W\} \to \{G_K\text{-stable } B_{dR}^+\text{-lattice in } W \otimes_K B_{dR}\} : Fil \mapsto Fil^0(W \otimes_K B_{dR})$$

is bijective and the inverse is given by $\Gamma \mapsto \{(t^n\Gamma)^{G_K} \subset (B_{dR} \otimes_{B_{dR}^+} \Gamma)^{G_K} = W\}_{n \in \mathbb{Z}}$.

Proof: Cf.[Laurent Fargues and Jean-Marc Fontaine Prop10.4.3].**?** □

Prop. (II.8.7.14). There is a pullback diagram of categories:

$$\begin{tikzcd} \varphi - FilMod_{K/K_0} \arrow[r] \arrow[d, "\mathcal{E}(-)"] & \varphi - Mod_{K_0} \arrow[d, "\mathcal{V}"] \\ Bun_X^{G_K} \arrow[r] & \mathrm{Rep}_{B_e} G_K \end{tikzcd}$$

Where $\mathcal{E}(-)$ maps a φ-filtered module (D, φ_D, Fil) to the bundle that is the bundle $\widetilde{(D, \varphi_D)}$ modified so that the fiber at ∞ is $Fil^0(D_K \otimes_K B_{dR})$.

Proof: 1: By lemma(II.8.7.13), $\varphi - FilMod_{K/K_0}$ is equivalent to a φ-module V with a G_K-stable B_{dR}^+-lattice in $(V \otimes_{K_0} K) \otimes_K B_{dR}^+ = V \otimes_{K_0} B_{dR}^+$.

2: By(II.8.7.7), φ-Mod is a full subcategory of $\mathrm{Rep}_{B_e} G_K$, where the G_K-stable B_{dR}^+-lattice is choose to be $V \otimes_{K_0} B_{dR}^+$.

3: Clearly there is a functor

$$Bun_X^{G_K} \to \mathrm{Rep}_{B_e} G_K : \mathcal{E} \to H^0(X - \{\infty\}, \mathcal{E}).$$

, and(III.3.2.8) says in this case $Bun_X^{G_K}$ is equivalent to a B_e-module with continuous G_K-actions and and a B_{dR}^+-module with continuous G_K-actions that they corresponds as a B_{dR}-module with continuous G_K-actions.

4: The compatibility in 3 just says that the B_{dR}^+-lattice choosen in the definition of(II.8.7.12) just comes from that of 2, so this diagram is clearly a pullback. □

Lemma (II.8.7.15). Let $FilV \in VectFil_K$ and $W = Fil(V \otimes_K B_{dR})$, if $V \otimes B_{dR}^+ = \{e_1, \ldots, e_n\}$ and $Fil^0(V \otimes_K B_{dR}) = \{t^{-a_1}e_1, \ldots, t^{-a_n}e_n\}$, then the Hodge polygon of $FilV$ has slopes $(a_1, \ldots, a_n)$.

Proof: Use(II.8.7.13), notice $(t^a B_{dR}^+)^{G_K} = 0$ for $a > 0$, as in the proof of(II.8.7.5). □

Lemma (II.8.7.16). The functor

$$\mathcal{E}(-) : \varphi - FilMod_{K/K_0} \to Bun_X^{G_K}$$

defined in(II.8.7.14) preserves degree and HN-filtration, where the HN-filtration on the RHS is induces by the HN-filtration on Bun_X by canonicity.

Proof: $\deg(\mathcal{E}((D, \varphi_D, Fil)) = \deg(\mathcal{E}(D, \varphi_D)) - \dim_K[D \otimes_{K_0} B_{dR}^+ : Fil^0(D_K \otimes_K B_{dR})]$
$= \deg(D_K, Fil) - \deg(D, \varphi_D)$.

Now the degree correspond, for the invariance of HN-filtration, it suffices to prove the subobjects are in bijection: Given a subobjects of $\mathcal{E}(V)$, we want to show it is a $\mathcal{E}(V')$, but this is because on the affine open $\mathrm{Spec}(B_e)$, by(II.8.7.7) any subbundle is also crysatalline, i.e. comes from $\varphi - Mod_{K_0}$.
□

Prop. (II.8.7.17) (Weakly Admissible implies Admissible). The category of crystalline Galois representations of G_K is equivalent to the category $\varphi - FilMod_{K/K_0}^{wa}$ of weakly admissible filtered φ-modules for K.

Proof: By definition of weakly admissible and(II.8.7.16), there is a pullback diagram

$$\begin{tikzcd}
\varphi - FilMod^{wa}_{K/K_0} \arrow[r] \arrow[d] & \varphi - FilMod_{K/K_0} \arrow[d, "\mathcal{E}(-)"] \\
\mathrm{Rep}_{\mathbb{Q}_p} G_K \cong Bun_X^{G_K,sst,0} \arrow[r] & Bun_X^{G_K}
\end{tikzcd}$$

Adjunction with(II.8.7.14), we get another pullback diagram

$$\begin{tikzcd}
\varphi - FilMod^{wa}_{K/K_0} \arrow[r] \arrow[d] & \varphi - FilMod_{K/K_0} \arrow[r] \arrow[d, "\mathcal{E}(-)"] & \varphi - Mod_{K_0} \arrow[d, "\mathcal{V}"] \\
\mathrm{Rep}_{\mathbb{Q}_p} G_K \cong Bun_X^{G_K,sst,0} \arrow[r] & Bun_X^{G_K} \arrow[r] & \mathrm{Rep}_{B_e} G_K
\end{tikzcd}$$

But this pullback is just the category of crystalline representations: by(II.8.7.4), for $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$, the condition $\mathcal{V}(\mathcal{D})(M) \cong M$ in(II.8.7.7) is just saying that V is in the image of $\mathcal{V}$ iff

$$(V \otimes_{\mathbb{Q}_p} B_{crys})^{\varphi=\mathrm{id}} \otimes_{B_e} B_{crys} \cong V \otimes_{\mathbb{Q}_p} B_{crys}$$

which is equivalent to V being crystalline. $\square$

II.9 Quadratic Forms over Fields

Basic references are [Quadratic Forms over Fields Y.T.Lam], [Quadratic Forms Clark] and [Algebraic and Geometric Theory of Quadratic Forms].

All fields K in this section has char$\neq 2$.

1 Quadratic Forms

This subsection should be regarded as a continuation of7. In fact, must materials in this subsection are trivial facts.

Def. (II.9.1.1). Given a field K of char$\neq 2$, a **quadratic form** over K is a bilinear form on K^n for some K. It is represented by a symmetric matrix.

The reason that char$K \neq 2$ is because only in this case, a quadratic form q is equivalent to a symmetric bilinear form B, and I will use this equivalence freely.

The determinant det is a function from the set of quadratic forms to $K^*/(K^*)^2$ that is invariant under congruence.

Def. (II.9.1.2). A field is called **quadratically closed** iff $K^2 = K$, or equivalently K has no quadratic extensions.

Def. (II.9.1.3). The category of **quadratic spaces** is a category with objects as finite dimensional spaces with a quadratic form, and its morphisms are isometric embeddings.

Def. (II.9.1.4). A quadratic form is called **universal** if it represents every element of K^*.

<u>**Non-Degeneracy**</u>

Def. (II.9.1.5) (Non-degeneracy). A quadratic space is called **non-degenerate** if $v \mapsto B(v,\cdot)$ is an isomorphism from V to V^*. Notice if $\dim V = \infty$, this cannot happen, because $\dim V^* > \dim V$(I.2.3.2). And in case $\dim V < \infty$, $\dim V = \dim V^*$, so it suffices to show $v \mapsto B(v,\cdot)$ is injective, i.e. if $v \neq 0$, then there is a w that $B(v,w) \neq 0$.

Prop. (II.9.1.6) (Radical Splitting). The **radical** of a quadratic space is defined to be $\mathrm{rad}(V) = V^\perp$. Then for any quadratic form V, there is an orthogonal decomposition $V = \mathrm{rad}(Y) \oplus W$, where W is a non-degenerate form.

Proof: In fact, by the definition, any complement space of $\mathrm{rad}(V)$ in V can be chosen as the orthogonal complement W. □

Prop. (II.9.1.7). If W is a non-degenerate sub-quadratic space of V, then $W \oplus W^\perp = V$.

Proof: Since W is non-degenerate, $W \cap W^\perp = 0$. and for any $v \in V$, $B(v,\cdot) \in W^*$, so by degeneracy, there is a $w \in W$ that $B(v,\cdot) = B(w,\cdot)$, then $z = v - w \in W^\perp$ and $v = w + z$. □

Prop. (II.9.1.8) (Perp and Non-Degeneracy). If V is a non-degenerate quadratic space, then for any non-degenerate subspace W, $\dim W + \dim W^\perp = \dim V$, and $(W^\perp)^\perp = W$.

Proof: The first is immediate from the fact $\dim \mathrm{Ker} + \dim \mathrm{Coker} = \dim V$. The second is by dimensional reason. □

Cor. (II.9.1.9). A subspace W of a non-degenerate quadratic space V is a non-degenerate quadratic space iff $W \cap W^\perp = 0$.

Inner Space

Def. (II.9.1.10). An **inner space** is a f.d. quadratic space that $B(v,v) > 0$ for $v \neq 0$. It is necessarily non-degenerate.

Prop. (II.9.1.11). An inner metric on a vector space will induce an inner metric on the dual space, that is, asserting the dual basis of an orthonormal basis to be orthonormal. On an arbitrary basis, the matrix on the dual basis is written as A^{-1}. because we can write $A = P^t P$, and the dual basis transformation is like $(P^t)^{-1}$, so the metric matrix is A^{-1}.

Diagonalizability

Prop. (II.9.1.12) (Quadratic Form Representable). Any quadratic forms over K of char$\neq 2$ is diagonalizable, and if $\alpha \in K^*$ is represented by K, then it is diagonalizable to a matrix with first entry α.

Proof: Use(I.2.7.4), since in this case, a quadratic form is equivalent to a symmetric form. And if $\alpha = B(v,v)$, then we can choose v in the first place in the proof of(I.2.7.4). □

Cor. (II.9.1.13). Over a quadratically closed field K of char$\neq 2$, any non-degenerate quadratic form is congruent to $x_1^2 + \ldots + x_n^2$.

Proof: Because in this case, we can make $\sum a_i x_i^2$ into $\sum(\sqrt{a_i}x_i)^2$. □

Def. (II.9.1.14). We will use the notation $\langle \alpha_1, \ldots, \alpha_n \rangle$ for the diagonal quadratic form $\sum \alpha_i x_i^2$.

Isotropic and Hyperbolic Spaces

Def. (II.9.1.15) (Isotropic). Given a non-degenerate quadratic space V, a vector v is called **isotropic** if $B(v,v) = 0$. V is itself called **isotropic** if it is non-degenerate and there exists an isotropic vector.

Def. (II.9.1.16) (Hyperbolic). The **hyperbolic plane** $\mathbb{H}$ is the 2-dimensional space with quadratic form $H(x,y) = xy$, which is congruent to $\frac{1}{2}(x^2 - y^2)$.

A quadratic space is called **hyperbolic** if it is isomorphic to a direct sum of hyperbolic planes.

Lemma (II.9.1.17). If V is a non-degenerate isotropic space, then there is an isometric imbedding of the hyperbolic plane into V.

Proof: There is a $u \in V$ that $B(u,w) = 0$. By non-degeneracy, there is a w that $B(u,w) \neq 0$. We may assume $B(u,w) = 1$. Now I claim there is an α that $q(\alpha u + v) = 0$: in fact, $q(\alpha u + v) = 2\alpha B(u,w) + q(w)$. Now $v = \alpha u + w$, then $q(u) = q(v) = 0$, and $B(u,v) = 1$, so it is isomorphic to $\mathbb{H}$. □

Prop. (II.9.1.18). If V is a non-degenerate quadratic space, and $U \subset V$ is an isotropic space with basis $u_1, \ldots, u_m$, then there is another isotropic space V with basis $v_1, \ldots, v_m$ that $B(u_i, v_j) = \delta_{ij}$.

Proof: Use induction on m. The $m = 1$ case is lemma(II.9.1.17) above. If this is true for $n < m$, let $W = \{u_2, \ldots, u_m\}$, then if $W^\perp \subset \{u_1\}^\perp$, then $u_1 \in W$ by(II.9.1.8), contradiction, so there is a $v \in W^\perp$ that $B(u_1, v) \neq 0$, so by the same proof as(II.9.1.17), there is a $\alpha u_1 + v$ that is isotropic, and a $\mathbb{H} \subset W^\perp$, so by(II.9.1.8), $W \subset \mathbb{H}^\perp$, so by induction, we can find in $\mathbb{H}^\perp$ elements $v_2, \ldots, v_m$ that satisfies the requirement. □

Cor. (II.9.1.19). $\langle a, -a\rangle \cong \mathbb{H}$, because it is isotropic, and it has dimension 2.

Cor. (II.9.1.20) (Isotropic Form is Universal). A non-degenerate isotropic space is universal, because hyperbolic plane does.

Cor. (II.9.1.21). A maximal totally isotropic space in a non-degenerate quadratic space V has dimension at most $\frac{1}{2}\dim V$, and equality holds if V is hyperbolic.

Prop. (II.9.1.22) (First Representation Theorem). If q is a non-degenerate quadratic form, then q represents $\alpha \in K^*$ iff $q \oplus \langle -\alpha\rangle$ is isotropic.

Proof: If q represent α, then by(II.9.1.12) shows that q is equivalent to $\langle \alpha, \alpha_1, \ldots, \alpha_n\rangle$, so $q\oplus\langle -\alpha\rangle$ contains a $\langle \alpha, -\alpha\rangle$ which is isomorphic to $\mathbb{H}$ by(II.9.1.17).

Conversely, if $q \oplus \langle -\alpha\rangle$ is isotropic, then there is a $-\alpha x_0^2 + \sum \alpha_i x_i^2 = 0$. If $x_0 \neq 0$, then q represent α, and if $x_0 = 0$, then q is isotropic, thus represent any element(II.9.1.20). □

Cor. (II.9.1.23). The following are equivalent:

- Any n-quadratic form over K is universal.
- Any $(n+1)$-quadratic form over K is isotropic.

Prop. (II.9.1.24) (Isotropy Criterion). For two non-degenerate forms f, g over K, $h = \langle f, -g\rangle$ is isotropic iff there is an $\alpha \in K^*$ that is represented by both f and g.

Proof: Easy, notice to use isotropic form is universal(II.9.1.20). □

2 Witt Theory

Prop. (II.9.2.1) (Witt Cancellation Theorem). If U_1, U_2, V_1, V_2 are quadratic spaces and $V_1 \cong V_2$, $V_1 \oplus U_1 \cong V_2 \cong U_2$, then $U_1 \cong U_2$.

Proof: We may identify $V_1 = V_2 = V$, and $W = U_1 \oplus V = U_2 \oplus V$.

First if V is totally isotropic and U_1 is non-degenerate, then there is a matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ that

$$M^t \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix} M = \begin{bmatrix} 0 & 0 \\ 0 & B_1 \end{bmatrix}$$

So $B_1 = D^t B_2 D$. As B_1 is non-singular, so is D, thus $U_1 \cong U_2$.

Now if V is isotropic but U_1, U_2 are not non-degenerate, then we may assume in their diagonalization, U_1 has less 0s, it has r 0s, then we can extract from both U_i a zero part, thus reducing to the above case.

Now if $\dim V = 1$, $V = \langle a\rangle$, if $a = 0$, then we are done by the above argument, and if $a = 0$, then find $q(x) = a$, then by(II.9.2.13), we can find a $\tau \in O(W)$ that $\tau(V_1) = V_2$, so now U_1, U_2 as the orthogonal complement of V_1, V_2, they are isometric under the map τ.

So now in general, we can cancel V out by moving its diagonal part once a time. □

Cor. (II.9.2.2). If X is a quadratic space and V_1, V_2 are non-degenerate subspaces of X, then any isometry $V_1 \cong V_2$ extends to an isometry of X.

Proof: $V_i \oplus V_i^\perp = X$ by(II.9.1.7). □

Cor. (II.9.2.3). If X is a non-degenerate quadratic space and W is any subspace, then any isometric imbedding $W \to X$ extends to an isomorphic of X.

Proof: [Quadratic Form Clark P16]. □

Cor. (II.9.2.4). If V is a non-degenerate quadratic space, then the group of isometries of X acts transitively on the set of all totally isotropic subspaces of a fixed dimension d.

Prop. (II.9.2.5) (Witt's Decomposition Theorem). For any quadratic space V, there is an orthogonal decomposition

$$V \cong \mathrm{rad}(V) \oplus \bigoplus_1^k \mathbb{H} \oplus V'$$

where V' is not isotropic. Moreover the number $k = I(V)$ which is called the **Witt index** of V and the isometry class of $V' = w(V)$ which is called the **non-isotropic kernel** is independent of the decomposition.

Proof: The existence of the decomposition follows from(II.9.1.6) and an easy induction using(II.9.1.17). The uniqueness is an easy corollary of(II.9.1.17) and Witt's cancellation theorem. □

Cor. (II.9.2.6). The Witt index equals the maximal dimension of a maximal totally isotropic subspace of W, by(II.9.1.18).

Remark (II.9.2.7). This is a good reason that we will only consider non-degenerate quadratic forms from now on.

Cor. (II.9.2.8) (Sylvester's Law of Nullity). Let $q_{r,s} = [r]\langle 1\rangle \oplus [s]\langle 1\rangle$, then any non-degenerate quadratic form q over $\mathbb{R}$ is congruent to exactly one of $q_{r,s}$, and $r - s$ is called the **signature** of q.

Def. (II.9.2.9). Two quadratic forms $q_1 = \langle a_1, \ldots, a_n\rangle$ and $q_2 = \langle b_1, \ldots, b_n\rangle$ are called **simply equivalent** iff there are two indices that $\langle a_i, a_j\rangle \cong \langle b_i, b_j\rangle$. Two quadratic forms are called **chain equivalent** iff there is a chain of simply equivalence between them.

Prop. (II.9.2.10) (Witt's Chain equivalence Theorem). Two diagonal quadratic forms over K are equivalent iff they are chain equivalent.

Proof: Chain equivalent is clearly equivalent, Conversely, by Witt's decomposition theorem, it is easy to reduce to the non-degenerate case.

Now if $q = \langle \alpha_1, \ldots, \alpha_n\rangle \cong q' = \langle \beta_1, \ldots, \beta_n\rangle$, any form $q = \langle \gamma_1, \ldots, \gamma_n\rangle$ that is chain equivalent to q is equivalent to q', so β_1 is represented by it, choose a form that there is a minimal l that β_1 is represented by $\langle \gamma_1, \ldots, \gamma_l\rangle$, we prove that $l = 1$:

if the minimal l is not 1, then $d = \gamma_1 a_1^2 + \gamma_2 a_2^2 \neq 0$(otherwise l can be smaller), so $\langle \gamma_1, \gamma_2\rangle \cong \langle d, \gamma_1\gamma_2 d\rangle$ by(II.9.1.12) and invariance of det. so $q \cong \langle d, \gamma_3, \ldots, \gamma_n, d\gamma_1\gamma_2\rangle$(notice permutation is chain equivalence), and this is smaller, contradiction.

Now $l = 1$, so we may assume $\alpha_1 = \beta_1$, and then Witt's cancellation(II.9.2.1) shows that $\langle \alpha_2, \ldots, \alpha_n\rangle \cong \langle \beta_2, \ldots, \beta_n\rangle$, so we win by induction. □

Orthogonal Group

Prop. (II.9.2.11). The **orthogonal group** of a quadratic form q is the set of matrixes M that $g(Mx) = q(x)$. And it is clear $\det M = \pm$, so we can also define $O^+(V)$ and $O^-(V)$.

Def. (II.9.2.12). A **hyperplane reflection** for a non-isotropic vector v is defines by $x \mapsto x - \frac{2B(x,v)}{q(v)}v$, it is an element in $O(V)$.

Prop. (II.9.2.13). If x, y are two non-isotropic vectors that $q(x) = q(y)$, then there is a $\tau \in O(V)$ that $\tau(x) = y$.

Proof: First notice $q(x+y) + q(x-y) = 2q(x) + 2q(y) = 4q(x) \neq 0$, so one of $x+y, x-y$ is non-isotropic. And it can be easily calculated that $\tau_{x-y}(x) = y$ or $-\tau_{x+y}(x) = y$. □

Prop. (II.9.2.14) (Cartan-Dieudonné). Let V be a non-degenerate quadratic form of dimension n, then every element of the orthogonal group $O(V)$ can be represented as a product of n reflections.

Proof: Cf.[Quadratic Forms Clark P22]. □

3 Witt Ring

Def. (II.9.3.1) (Witt Ring). The **Witt ring** $W(K)$ of K is a commutative ring whose elements are equivalent classes of non-isotropic quadratic forms over K, and the addition is defined by $[q_1] + [q_2] = w[q_1 \oplus q_2]$ and multiplication is defined by $[q_1] \otimes [q_2] = w[q_1 \otimes q_2]$.

Def. (II.9.3.2) (Grothendieck-Witt Ring). There is another ring, the **Grothendieck-Witt ring** $\hat{W}(K)$ which is defined as the ring generated by the semiring of all non-degenerate quadratic forms over K.

Prop. (II.9.3.3). The subgroup $[\mathbb{H}]$ generated by the hyperbolic plane is an ideal of $\hat{W}(K)$. And $\hat{W}(K)/([H]) \cong W(K)$.

Proof: $[\mathbb{H}]$ is an ideal because $[H] \cdot [\langle a_1, \ldots, a_n\rangle] \cong w(\oplus\langle a_i, -a_i\rangle)$ which is 0, by(II.9.1.19). The last proposition is easy. □

4 over Local and Global Fields

II.10 p-adic Hodge Theory

1 Witt Theory (Local Fields Serre)

Complete discrete Valuation Ring

Structure of complete discrete valuation ring will be studied in this subsubsection.

Prop. (II.10.1.1) (0-Type). Let A be a local ring with maximal ideal $\mathfrak{m}$. Now if $A/\mathfrak{m}$ is field of characteristic 0, then A contains a field mapped isomorphically onto k.

Proof: $\mathbb{Z} \to A \to k$ is injective, so $\mathbb{Z}$ is units in A, thus $\mathbb{Q} \in A$, hence A contains a field. Now we show using Zorn's lemma that the maximal field S of A is mapped onto k.

First k is algebraic over $\overline{S}$, this is because if there is a transcendental element $\overline{a}$, then the inverse image a is transcendental over S and $S[a] \cap \mathfrak{a}_1 = 0$, so $S(a) \subset A$. Now for any a that is algebraic over $\overline{S}$, the minipoly has no multiple roots, so it has a lifting by Hensel's lemma, so we are done. This is Cohen's lemma in [Matsumura P206]?. □

Def. (II.10.1.2) (Strict p-Ring). A **p-ring** A is a ring which is complete Hausdorff in the topology defined by the decreasing chain of ideals $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots$ such that $\mathfrak{a}_m\mathfrak{a}_n \subset \mathfrak{a}_{m+n}$ that $k = A/\mathfrak{a}_1$ is a perfect ring of characteristic p.

It is called a **strict p-ring** if moreover $\mathfrak{a}_n = p^n A$ and p is not a zero-divisor of A.

Prop. (II.10.1.3) (Teichmüller Lift). For a p-ring, there exists a unique system of representatives $k \to A$ that $f(\lambda^p) = f(\lambda)^p$, called the **Teichmüller lift**.

For this representative, it is also multiplicative, and if A has char p, then it is also additive. And an element is in the image of f iff it is a p^n-th power for any n.

Proof: For any $\lambda \in k$, the $\lambda^{p^{-n}}$ is unique in k, and if we consider U_n the set of all x^{p^n} where x is a lift of $\lambda^{p^{-n}}$, then U_n is a descending set. Moreover, the diameter converges to 0, because $a \equiv b \mod \mathfrak{a}_1$ implies $a^{p^n} \equiv b^{p^n} \mod \mathfrak{a}_{n+1}$ as $p \in \mathfrak{a}_1$. So it converses to a unique point $f(\lambda)$ in A. And we see that any other f' maps λ to a p^n-th root hence in U_n for any n, hence it map be equal to $f(\lambda)$. The rest is easy. □

Cor. (II.10.1.4) (Equal Characteristic case). If A is a complete discrete valuation ring with residue field k. If k and A have the same characteristic and k is perfect, then $A \cong k[[T]]$.

Cor. (II.10.1.5). When A is a strict p-ring, elements of A can be written uniquely as $\sum f(\alpha_n)p^n$.

Def. (II.10.1.6) ($(0,p)$-type case). When A is a complete DVR with residue field k and quotient field K. If $\mathrm{char}K = 0$ and $\mathrm{char}k = p$, then p goes to 0 in k, so $e = v(p) \geq 1$, called the **absolute ramification index** of A. It is called **absolutely unramified** iff $e = 1$.

Remark (II.10.1.7) (Canonical Strict p-Ring). The canonical strict p-ring is the ring $\hat{S} = \hat{\mathbb{Z}}[X_\alpha^{p^{-n}}]$. Its residue ring is $\mathbb{F}_p[X_\alpha^{p^{-n}}]$ which is perfect. X_i are all Teichmüller lifts, as they has all p^n roots.

Now we consider the $* = + - \times$ in $\hat{S}$. then there are elements $Q_i^* \in \mathbb{F}_p[X_\alpha^{p^{-n}}, Y_\alpha^{p^{-n}}]$ that $x * y = \sum f(Q_i^*)p^i$ where f is the Teichmüller lift.

Prop. (II.10.1.8) (Universal Law of p-Rings). For any p-ring A with residue ring k, the calculation in A is dominated by Q_i^* defined in(II.10.1.7), i.e.

$$\left(\sum f(\alpha_i)p^i\right) * \left(\sum f(\beta_i)p^i\right) = \sum f(\gamma_i)p^i$$

where $\gamma_i = Q_i^*(\alpha_0, \alpha_1, \ldots, \beta_0, \beta_1, \ldots)$.

Proof: There is a map θ from $\hat{S} = \hat{\mathbb{Z}}[X_i^{p^{-n}}, Y_i^{p^{-n}}]$ to A induced by $f(\alpha_i), f(\beta_i)$ as they all has p^{-n}-th roots. Then notice θ induce a $\overline{\theta}$ on residue ring and these two θ commutes with Teichmüller lift, as seen by the definition of the latter. Then the theorem follows immediately. □

Cor. (II.10.1.9). For two p-ring A, A' that A is strict, then any map φ of their residue ring induces a unique ring homomorphism $A \to A'$. In particular, two strict p-ring with the same residue ring is canonically isomorphic.

Proof: We have already seen that ring homomorphism commutes with Teichmüller lift. Now we define

$$g(a) = \sum g(f(\alpha_i))p^i = \sum f(\varphi(\alpha_i))p^i$$

and this is the unique choice. It is a ring homomorphism by universal law of(II.10.1.8). □

Prop. (II.10.1.10) (Witt Vectors of Perfect Rings). For any perfect ring k of char p, there exists uniquely a strict p-ring $W(k)$ that has residue ring k, called the **ring of Witt vectors** with coefficients in k. W is a faithful functor from perfect rings to strict p-rings by(II.10.1.9).

Proof: For a canonical ring $\mathbb{F}_p[X_\alpha^{p^{-n}}]$, $\hat{\mathbb{Z}}[X_\alpha^{p^{-n}}]$ is a strict p-ring. Now arbitrary perfect p-ring is a quotient of $\mathbb{F}_p[X_\alpha^{p^{-n}}]$, so we can construct its strict p-ring $W(k)$ as the quotient of $\hat{\mathbb{Z}}[X_\alpha^{p^{-n}}]$. Uniqueness is by(II.10.1.9).

Notice it is nothing mysterious, it is just the set of all formal sum $\sum f(x_i)p^i$ under the operation defined in(II.10.1.7). See also(II.10.1.13). □

Witt Vectors

A ring homomorphism φ lifting the Frobenius, i.e. $\varphi(x) = x^p + p\delta(x)$. It generate a $\delta-$**ring structure**.

δ-rings form a category and the right adjoint to the forgetful functor is $W(A) = \mathrm{Hom}(\Delta, A)$. Where Δ is the free ring $\mathbb{Z}[e, \delta, \delta^2, \ldots]$. The ring structure of $W(A)$ is that: as φ is a ring homomorphism, there are sum and product formulae for $\delta^n(x+y)$ and $\delta^n(xy)$ in forms of $\delta^n(x)$ and $\delta^n(y)$. So that is the ring structure of $W(A)$.

There is another description of Δ:

Prop. (II.10.1.11) (Ghost Component). Let θ_i be polynomials in δ with integer coefficients that

$$\varphi^n = \theta_0^{p^n} + p\theta_1^{p^{n-1}} + \cdots + p^n\theta_n = W_n(\theta_0, \ldots, \theta_n)$$

In fact $\mathbb{Z}[\theta_0, \theta_i, \ldots, \theta_n] = \mathbb{Z}[e, \delta, \delta^2, \ldots, \delta^n]$.

Proof: Use equation $\varphi \circ \varphi^n = \varphi^n \circ \varphi$ and module $p^n\mathbb{Z}[\theta_0, \theta_1, \ldots, \theta_n]$. □

So there is a map $Z[\varphi] \to \Delta$ inducing an morphism of rings: $W(A) \to \prod_{\mathbb{Z}} A$ that maps

$$(f(\delta^n)) \mapsto (f(\varphi^n))$$

Where the right hand side is the usual addition and multiplication, the left side is the usual coordinate of Witt vector, and $f(\theta_n)$ is called **ghost component**.

This is embedding if A is p-torsion free, and isomorphism iff $\frac{1}{p} \in A$, because θ_n can be presented by φ^n.

Lemma (II.10.1.12) (Formula for p-Rings). For $* = +$ or $\times$, there are integral polynomials $S_*(X_i, Y_i)$ that

$$\left(\sum f(\alpha_i)p^i\right) * \left(\sum f(\beta_i)p^i\right) = \sum f(\gamma_i)p^i$$

where $\gamma_i = Q_i^*(\alpha_0, \alpha_1, \ldots, \beta_0, \beta_1, \ldots)$. And for $+$, when reduced to $\mathbb{F}_p$, Q_i^+ are polynomials in $X_i^{p^{-n}}, Y_i^{p^{-n}}$ for $i \leq n$ and homogenous of degree 1. And

$$Q_i^+ = (X_n + Y_n) + (X_{n-1}^{p^{-1}} + Y_{n-1}^{p^{-1}})R_{n,n-1} + \ldots + (X_0^{p^{-n}} + Y_0^{p^{-n}})R_{n,0}.$$

Proof: We solve S_n by induction. Notice for any lift $\hat{S}_i$ of S_i,

$$f(S_i) \equiv \hat{S}_i(X^{1/p^{n-i}}, Y^{1/p^{n-i}})^{p^{n-i}} \mod p^{n-i+1}$$

so we mod p^{n+1} to solve S_n:

$$S_n \equiv 1/p^n\Big(X_0 + Y_0 + \ldots + p^n X_n + p^n Y_n - \hat{S}_0(X^{1/p^n}, Y^{1/p^n})^{p^n} - \ldots - p^{n-1}\hat{S}_{n-1}(X^{1/p}, Y^{1/p})^p\Big)$$

The rest follows by induciton. ☐

Prop. (II.10.1.13). Notice in Serre book, he presented the Witt vectors in $(f(\theta_n))$ coordinates. In this coordinate, if k is a perfect ring and we let

$$T(\{a_i\}) = \sum f(a_i)^{p^{-i}} p^i,$$

then T is a ring isomorphism from $W(k)$ to the strict p-ring with residue ring k.

Proof: We need to prove this is a ring homomorphism. That on $W(A)$ is to make φ a ring homomorphism, and that on the right is usual. It suffice to prove for the canonical strict p-ring, as seen by the universal law(II.10.1.8).

For this, we let $(\sum X_i^{p^{-i}} p^i) * (\sum Y_i^{p^{-i}} p^i) = \sum f(\psi_i(X_i, Y_i))^{p^{-i}} p^i$, and $W_n(a_i) * W_n(b_i) = W_n(\varphi_i)$, where $\psi_i \in \mathbb{F}_p[X_i, Y_i]$ and $\varphi_i \in \mathbb{Z}[X_i, Y_i]$, they both exist, the latter because of(II.10.1.11).

Then we mod p^{n+1}, and let $X_i = X_i^{p^n}$, $Y_i = Y_i^{p^n}$, so

$$W_n(\varphi_i) = W_n(X_i) * W_n(Y_i) \equiv \sum_{i \leq n} f(\psi_i(X_i^{p^n}, Y_i^{p^n}))^{p^{-i}} p^i \equiv W_n(\psi_i) \mod p^{n+1}$$

Now induction, $\varphi_i \equiv \psi_i \mod p$, then $p^n\varphi_n \equiv p^n\psi_n \mod p^{n+1}$ so this is true for n, too. ☐

Cor. (II.10.1.14). For example, $W(\mathbb{F}_p^n)$ is the unramified extension of $\mathbb{Z}_p$ of degree n. And $W(\overline{F})$ is the completion of the maximal unramified extension of $W(F)$.

Def. (II.10.1.15) (Witt Vectors over Valued Rings). If a perfect ring R itself has a complete valuation v, then we can endow $W(R)$ with a finer topology: we let $w_k(x) = \inf_{i \leq k} v(x_i)$, where $x = \sum p^i f(x_i)$. Now $w_k(x + y) \geq \inf(w_k(x), w_k(y))$ by(II.10.1.12). The **weak topology** of $W(R)$ is defined by the semi-valuations w_k.

Prop. (II.10.1.16). If $a, b \in \mathcal{O}_R) + p^{n+1}W(R)$, then

$$p^n v(a_n - b_n) \geq w_n(a-b) \geq \inf_{k \leq n} p^{-k} v(a_{n-k} - b_{n-k}).$$

So we see that a sequence is Cauchy in $W(R)$ if each coordinate is Cauchy in R, so $W(R)$ is complete in the weak topology.

Proof: Firstly the last proposition follows from the first because we can always multiply by a $f(\alpha)$ to make the first n coordinate in $\mathcal{O}_R$.

The first is nearly an immediate consequence of(II.10.1.12). □

Prop. (II.10.1.17). $\mathcal{O}_{\mathcal{E}} = W(K^{\frac{1}{p^\infty}})$ is a complete ring with maximal ideal $p\mathcal{O}_{\mathcal{E}}$. And $\mathcal{O}_{\mathcal{E}}[\frac{1}{p}] = \mathcal{E}$ is complete ring of character p. And the same construction of $\overline{K^{\frac{1}{p^\infty}}}$ yields the completion of maximal unramified extension of $\mathcal{O}_{\mathcal{E}}$, and the Galois group is the same as G_K.

2 Galois Representations

Lemma (II.10.2.1). If $P(X) \in \overline{F}[X]$ is a monic polynomial of degree n, all of its roots satisfied $val_p(\alpha) \geq c$ for some constant c. We let $q = p^k$ if $n = p^k d, d \neq 1$ or $n = p^{k+1}$.

Then the derivative $P^{(q)}(X)$ has a root β with $val_p(\beta) \geq c$ or in case $n = p^{k+1}$, $val_p(\beta) \geq c - \frac{1}{p^k(p-1)}$.

Proof: Let $P = X^n + a_{n-1}X^{n-1} + \ldots + a_0$, then $val_p(a_i) \geq (n-i)c$. And

$$1/q! \cdot P^{(q)}(X) = \sum_{i=0}^{n-1} C_{n-i}^q a_{n-i} X^{-i-q}.$$

So at lest one root β satisfies

$$val_p(\beta) \geq \frac{1}{n-q}((n-q)c - val_p(C_n^q)) = c - \frac{1}{p^k(p-1)}.$$

□

Lemma (II.10.2.2). If F is a complete valued field and $\alpha \in \overline{F}$, let $\Delta_K(\alpha) = \inf_{g \in G_K} val_p(g(\alpha) - \alpha)$, then there exists $\delta \in K$ that $val_p(\alpha - \delta) \geq \Delta_K(\alpha) - p/(p-1)^2$.

Proof: We strengthen the assertion and use induction on $n = [F(\alpha) : F]$ to prove that there is a δ that $val_p(\alpha - \delta) \geq \Delta_K(\alpha) - \sum_{k=2}^m \frac{1}{p^k(p-1)}$, where p^{m+1} is the largest power of p that $\leq n$.

$n = 1$ is sure, let the minipoly be $P(X)$. By lemma(II.10.2.1), there is a root β of $P^{(q)}$ that $val_p(\beta - \alpha) \geq \Delta_K(\alpha)$ or minus a factor. Then for any σ, $val_p(\sigma(\beta) - \beta) \geq val_p(\sigma(\alpha) - \alpha)$ or minus a factor. Then $\Delta(\beta) \geq \Delta(\alpha)$ or minus a factor. Now $[F(\beta) : F] < n$, so we can use induction hypothesis to get the result. □

Prop. (II.10.2.3) (Ax-Sen-Tate). If F is a complete p-adic field and if $K \in F^{alg}$, then $\widehat{F^{alg}}^{G_K} = \widehat{K}$. Thus $\hat{L}^{G_{L/K}} = \hat{F}$ for any alg.ext L/K.

Proof: Any $\alpha \in \hat{\overline{F}}$ can be written as $\sum \alpha_n$ with $\alpha_n \in \overline{F}$. Then $\Delta_K(\alpha_n) \to \infty$, and α_n can be approximated by $\delta_n \in K$ by lamma(II.10.2.2), thus $\alpha \in \widetilde{K}$. □

Cohomology of G_K action on $\mathbb{C}_p$

K is assumed to be a p-adic number field.

Lemma (II.10.2.4). Giving an $\sigma \in G(K/\mathbb{Q}_p)$, if $x, y \in \mathfrak{m}_{\mathbb{C}_p}$ that $x \equiv y \mod \pi_K^n$, then $[\pi_K]^\sigma(x) \equiv [\pi_K]^\sigma(y) \mod \pi_K^{n+1}$, where f^σ is given by action of σ on the coefficients.

Proof: This is because the coefficients of $[\pi_K]^\sigma$ are divisible by π_K except for degree q, where it is $x^q - y^q = (x-y)(x^{q-1} + x^{q-2}y + \ldots + y^{q-1})$ which is divisible by π_K^{n+1} because the residue field of K is of order q. □

Prop. (II.10.2.5). If we let the action of $\sigma \in G(K/\mathbb{Q}_p)$ on the residue field giving by $\overline{\sigma} : k_K \to \overline{\mathbb{F}}_p : x \mapsto x^{q_\sigma}$, where $q_\sigma = p^{n_\sigma}$ is a p-power, given an element $\eta = (\eta_0, \eta_1, \ldots) \in TG$, we have $\eta^{q_\sigma} \equiv [\pi_K]^\sigma(\eta_{n+1}^{q_\sigma}) \mod \pi_K$, hence the above lemma(II.10.2.4) shows that $[\pi_K^n]^\sigma \eta_n^{q_\sigma} \equiv [\pi_K^{n+1}]^\sigma(\eta_{n+1}^{q_\sigma}) \mod \pi_K^{n+1}$, so $[\pi_K^n]^\sigma(\eta_n^{q_\sigma})$ is a Cauchy sequence, converging to an element μ_σ(don't care about η).

If $g \in G_K$, then $g(\eta_n) = [\chi_K(g)](\eta_n)$, hence take q_σ-th power, $g(\eta_n^\sigma) \equiv [\chi_K(g)]^\sigma(\eta_n^{q_\sigma}) \mod \pi_K$, then

$$[\chi_K(g)]^\sigma[\pi_K^n]^\sigma(\eta_n^{q_\sigma}) \equiv [\pi_K^n]^\sigma g(\eta_n^{q_\sigma}) = g([\pi_K^n]^\sigma \eta_n^{q_\sigma}) \mod \pi_K.$$

hence by limiting, $g(\mu_\sigma) = [\chi_K(g)]^\sigma(\mu_\sigma)$.

Lemma (II.10.2.6).

$$v_p(\mu_\sigma) = \begin{cases} \frac{q_\sigma}{e_K(q-1)} + \frac{1}{e_K} & n(\sigma) \neq 0 \\ \frac{1}{e_K(q-1)} + v_p(\sigma(\pi_K) - \pi_K) & n(\sigma) = 0 \end{cases}$$

Proof: By(III.10.3.19), we know the Newton polygon of $[\pi_K^n]^\sigma$. When $n(\sigma) \neq 0$, $v(\eta_1^{q_\sigma}) = \frac{q_\sigma}{e_K(q-1)} > \frac{1}{e_K(q-1)}$, so the valuation of $[\pi_K]^\sigma(\eta_1^{q_\sigma})$ equals the valuation of its degree 1 term, which is $v(\pi_K \eta_1^{q_\sigma}) = \frac{q_\sigma}{e_K(q-1)} + \frac{1}{e_K}$. Now we have by(II.10.2.5), we have $[\pi_K]^\sigma \eta^{q_\sigma} \equiv [\pi_K^2]^\sigma(\eta_2^{q_\sigma}) \mod \pi_K^2$, and $\frac{q_\sigma}{e_K(q-1)} + \frac{1}{e_K} < 2/e_K$, so valuation already stable at degree 1, and $v(\mu_\sigma) = v([\pi_K]^\sigma(\eta_1^{q_\sigma}))$.

If $q_\sigma = 1$, it's more delicate, because degree 1 and degree q term has the same minimal valuation, so they may jump to higher valuations. Notice $[\pi_K^n](\eta_n) = 0$, so $[\pi_K^n]^\sigma(\eta_n) = ([\pi_K^n]^\sigma - [\pi_K^n])(\eta_n)$. And we have by(II.2.1.17), for $x \in \mathcal{O}_K$, $v(\sigma(x) - x) \geq v(x) + v(\frac{\sigma(\pi_K)}{\pi_K} - 1) + \delta_{v(x),0} v(\pi_K)$, with equality when $v_p(x) = q/e_K$. So by the Newton polygon, the minimum valuation of the coefficient of $[\pi_K^n]^\sigma - [\pi_K^n]$ appear at degree p^{n-1} and possibly p^n. The valuation of η_n is too small($\frac{1}{e_K p^{n-1}(p-1)}$) that we don't need to consider other degrees but can assure that degree p^{n-1} is of minimum valuation, which is $v(\eta_n^{p^{n-1}}) + v(\sigma(\pi_L) - \pi_L) = \frac{1}{e_K(q-1)} + v_p(\sigma(\pi_K) - \pi_K)$. □

Prop. (II.10.2.7). For any $\sigma \in G(K/\mathbb{Q}_p) \backslash \{\mathrm{id}\}$, there is an element $\alpha_\sigma \in \mathbb{C}_p^*$ that $\sigma \circ \chi_K(g) = g(\alpha_\sigma)/\alpha_\sigma$ for all $g \in G_K$, where χ_K is the Lubin-Tate character.

Proof: We let $\alpha_\sigma = \log_{\mathcal{F}_\pi}^\sigma(\mu_\sigma)$, by(II.10.2.6), $1/e_K < \mu_\sigma < \infty$, so by the Newton polygon analysis of $\log_{F_\pi}$ (III.10.3.20), α_σ has the same valuation of μ_σ, in particular, $\alpha_\sigma \neq 0$. Then

$$g(\alpha_\sigma) = \log_{\mathcal{F}_\pi}^\sigma(g(\mu_\sigma)) = (\log_{\mathcal{F}} \circ [\chi_K(g)])^\sigma(\mu_\sigma) = (\chi_K(g) \cdot \log_{\mathcal{F}_\pi})^\sigma(\mu_\sigma) = \sigma(\chi_K(g)) \cdot \alpha_\sigma.$$

□

Cor. (II.10.2.8). $\log_p(\sigma(\chi_K(g))) = g(\log(\alpha_\sigma)) - \log_p(\alpha_\sigma)$.

Def. (II.10.2.9). Let $\psi : G_K \to \Gamma_K \to \mathbb{Z}_p^*$ be a character factoring through Γ_K. Then we can form arepresentation $\mathbb{C}_p(\psi)$ of G_K on $\mathbb{C}_p$ that $\rho(\sigma)(x) = \psi(\sigma)\sigma(x)$. This is an action because G_K acts trivial on $\mathbb{Z}_p^*$.

If $\psi^k = \mathrm{id}$ for some k, then it is trivial on Γ_K^k. Γ_K is an open subgroup of $\mathbb{Z}_p$, so Γ_K^n is of finite index in Γ_K by(II.2.2.5), hence also does its inverse image in G_K. So ψ comes from a finite extension L/K.

Prop. (II.10.2.10). $H^0(G_K, \mathbb{C}_p(\psi)) = K$ if ψ is of finite order, and vanish if ψ is of infinite order.

Proof: Finite case: ψ factor through some G_L, so ψ corresponds to a continuous cocycle w.r.t the discrete topology of $\mathbb{C}_p$. So by(II.3.4.2) there is a $a \in \mathbb{C}_p^*$ that $\psi(\sigma) = \sigma(a)/a$, so $\mathbb{C}_p(\psi) \cong \mathbb{C}_p : x \mapsto ax$. And the result follows from Ax-Sen-Tate, as $K = \hat{K}$.

Infinite case: $H^0(G_K, \mathbb{C}_p(\psi)) \subset H^0(H_K, \mathbb{C}_p(\psi)) = \hat{K}_\infty(\psi)$ by Ax-Sen-Tate and the fact ψ is trivial on H_K. Then for the normalized trace R_n, which commutes with G_K, $g(R_n(x)) = \psi^{-1}(g)R_n(x)$. But $G(K_n/K)$ is finite, so $R_n(x) = \psi^{-N}(g)R_n(x)$ for any g. So $R_n(x) = 0$, otherwise ψ is of finite order. Now $R_n(x) \to x$, so $x = 0$. □

Prop. (II.10.2.11). Now we compute $H^1(G_K, \mathbb{C}_p(\psi))$. There is a inf-res exact sequence

$$0 \to H^1(\Gamma_K, \hat{K}_\infty(\psi)) \to H^1(G_K, \mathbb{C}_p(\psi)) \to H^1(H_K, \mathbb{C}_p(\psi))$$

Then $H^1(H_K, \mathbb{C}_p(\psi)) = 0$. The first two vanish iff ψ is of infinite order, and is a K-vector space of dimension 1 if ψ is of finite order.

Proof: For the first assertion, ψ is trivial on H_K, so $\mathbb{C}_p(\psi) \cong \mathbb{C}_p$ as H_K-representation, so it suffice to show for $\psi = \mathrm{id}$. Let f be a cocycle, as H_K is compact, $f(H_K) \in p^{-k}\mathcal{O}_{\mathbb{C}_p}$ for some integer k. So the lemma below(II.10.2.12) shows that we can move f cohomologouly to higher valuation, i.e. $f(g) = \sum x_i - g(\sum x_i)$, so f is a coboundary.

For the second assertion, we assume $\Gamma_K \neq \mathbb{Z}_p^*$, for this case, see remark(II.10.2.13) below.

let γ be a topological generator of $\Gamma_K = 1 + p^k\mathbb{Z}_2^*$, $k \geq 0$, because $\mathbb{Z}_p^*$ are all topological cyclic groups except for $\mathbb{Z}_2^* \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}_2$, and γ_n be a topological generator of Γ_{F_n} which is also a power of γ. By(II.3.4.12) we know $H^1(\Gamma_K, \hat{K}_\infty(\psi)) = \hat{K}_\infty(\psi)/1 - \gamma$.

For n large, we have a decomposition $\hat{K}_\infty(\psi) = K_n(\psi) \oplus X_n(\psi)$ by(II.2.2.24), and $1 - \gamma_n$ is invertible on $X_n(\psi)$. Now $1 - \gamma_n = (1-\gamma)(1+\gamma+\ldots+\gamma^{k-1})$, so $1-\gamma$ is also invertible in $X_n(\psi)$. And on $K_n(\psi)$, if ψ is of infinite order, then $1-\gamma$ is injective, otherwise $x = \psi(\gamma)^N\gamma^N(x) = \psi(\gamma)^N x$. So it is also surjective because it is a K-linear mapping of K_n. So $\hat{K}_\infty(\psi)/1-\gamma = 0$. If ψ is of finite order then $K_n(\psi) \cong K_n$ as Γ_K-module when n is large enough that γ factors through Γ_{K_n}, by(II.3.4.1). So $K_n/1-\gamma = K_n/\operatorname{Ker}(\mathrm{tr}_{K_n/K}) = K$. □

Lemma (II.10.2.12). If $f : H_K \to p^n\mathcal{O}_{\mathbb{C}_p}$ is a continuous cocycle, then there exists a $x \in p^{n-1}\mathcal{O}_{\mathbb{C}_p}$ that the cohomologous cocycle $g \mapsto f(g) - (x - g(x))$ has values in $p^{n+1}\mathcal{O}_{\mathbb{C}_p}$.

Proof: $p^{n+2}\mathcal{O}_{\mathbb{C}_p}$ is open in $p^n\mathcal{O}_{\mathbb{C}_p}$, so there is a finite extension L/K that $f(H_L) \in p^{n+2}\mathcal{O}_{\mathbb{C}_p}$. By(II.2.2.19), there is a z that $\mathrm{tr}_{L_\infty/K_\infty}(z) = p$, so there is a $y \in p^{-1}\mathcal{O}_{L_\infty}$ that $\mathrm{tr}_{L_\infty/K_\infty}(y) = 1$.

Now for a set of representatives Q of H_K/H_L, denote $x_Q = \sum_{h\in Q} h(y)f(h)$, then for $g \in H_K$, $g(Q)$ is also a set of representative, and $g(x_Q) = \sum_{h\in Q} gh(y)gf(h) = \sum_{h\in Q} gh(y)(f(gh) - f(g)) = x_{g(Q)} - f(g)$, as $\mathrm{tr}(y) = 1$. So $f(g) - (x_Q - g(x_Q)) = x_{g(Q)} - x_Q$. The RHS is in $p^{n+1}\mathcal{O}_{\mathbb{C}_p}$, because: if we let $gh_i = h_{g(i)}a_i$, where $a_i \in H_L$, then $x_{g(Q)} - x_Q = \sum h_{g(i)}(y)f(h_{g(i)}a_i) - \sum h_{g(i)}(y)f(h_{g(i)}) = \sum h_{g(i)}(y)h_{g(i)}(f(a_i))$, which is in p^{n+1} because $h_{g(i)}(y) \in p^{-1}\mathcal{O}_{\mathbb{C}_p}$ and $f(a_i) \in p^{n+2}\mathcal{O}_{\mathbb{C}_p}$ by the choice of L. □

Remark (II.10.2.13). In case $\Gamma_K = \mathbb{Z}_2^*$,

$$0 \to H^1(\{\pm 1\}, K(\psi)) \to H^1(\mathbb{Z}_2^*, \hat{K}_\infty(\psi)) \to H^1(1 + 2\mathbb{Z}_2^*, \mathbb{C}_p(\psi))$$

$H^1(\{\pm 1\}, K(\psi)) = 0$ whether $\psi(-1) = 1$ or -1. And by the same proof as above, possibly replace X_n with X_{n+1}, to remedy the singularity of $p = 2$, $H^1(1 + 2\mathbb{Z}_2^*, \mathbb{C}_p(\psi)) = K$, with generator $[g \mapsto \frac{\chi(g)-1}{\gamma-1}(a)]$ for some a. This cocycle extends to a cocycle of $\mathbb{Z}_2^*$, so the map is surjective.

Prop. (II.10.2.14). The 1-dimensional K-vector space $H^1(G_K, \mathbb{C}_p)$ is generated by the cocycle $[g \mapsto \log_p \chi(g)]$.

Proof: By the proof of(II.10.2.11), we know that $H^1(\Gamma_K, K_n) \xrightarrow{f} H^1(G_K, \mathbb{C}_p)$ is an isomorphism. for $\alpha \in K$, if $\chi(g) = \gamma^k$, then $f(\alpha)(g) = (1 + \gamma + \ldots + \gamma^{k-1})(\alpha) = k\alpha = \alpha \cdot \log_p(\chi(g))/\log_p(\gamma)$. So by continuity, f is a multiple of $[g \mapsto \log_p(\chi(g))]$. □

Lemma (II.10.2.15). And $f \in \mathrm{Hom}(I_K^{ab}, \mathbb{Q}_p)$ is of the form $f(g) = \mathrm{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g))$ for some $\beta_f \in K$.

Proof: By(II.4.1.28), χ_K is a canonical isomorphism $I_K^{ab} \cong \mathcal{O}_K^*$. Any $f \in \mathrm{Hom}(\mathcal{O}_K^*, \mathbb{Q}_p)$ is of the form $f(y) = \mathrm{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p(y))$ for some $\beta_f \in K$, because: by(II.2.2.7), when n is large, $\log_p$ is a bijection between U_K^n and $\pi_K^n \mathcal{O}_K$.

$\pi_K^n \mathcal{O}_K \to \mathbb{Q}_p$ can be extended to a map $K \to \mathbb{Q}_p$ as $\mathbb{Q}_p$ is divisible. Now trace is a invertible bilinear form on K, so the assertion is true on U_K^n for some n, and because U_K^n is of finite index in $\mathcal{O}_K^*$ and $\mathbb{Q}_p$ is of char 0, this is true for all $\mathcal{O}_K^*$. □

Prop. (II.10.2.16). The map $H^1(G_K, \mathbb{Q}_p) \to H^1(G_K, \mathbb{C}_p)$ is given as follows: as $f \in H^1(G_K, \mathbb{Q}_p)$ must factor through G_K^{ab}, if the restriction of f to I_K^{ab} corresponds to β_f, then f maps to $\beta_f[g \mapsto \log_p \chi(g)]$.

Proof: $f(g) = \mathrm{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g))$ on I_K, but this map extends to map on G_K. So $f(g) = \mathrm{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g)) + c(g)$ for a unramified map c on G_K.

Now by(II.3.4.11), $H^1(G, \hat{\mathbb{Q}}_p^{ur}/\mathbb{Q}_p)$ vanish because $H^1(G, \overline{\mathbb{F}}_p)$ vanish(II.3.4.1), so there is a $z \in \hat{\mathbb{Q}}_p^{ur}$ that $c(g) = g(z) - z$. And

$$\mathrm{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g)) = \sum_\sigma \sigma(\beta_f \log_p \chi_K(g)) = \beta_f \,\mathrm{tr}_{K/\mathbb{Q}_p}(\log_p \chi_K(g)) + \sum_\sigma (\sigma(\beta_f) - \beta_f)\sigma(\log_p \chi_K(g)).$$

Notice(II.10.2.7) gives a β_σ that $\sigma(\log_p \chi_K(g)) = g(\beta_\sigma) - \beta_\sigma$, and $\mathrm{tr}_{K/\mathbb{Q}_p}(\log_p \chi_K(g)) = \log_p \chi(g)$ because $(N_{K/\mathbb{Q}_p}\chi_K(g))^{-1} = (\chi(g))^{-1}$, as they both correspond via local CFT to the element in G_K^{ab} which acts by g on L_π and id on K^{ur}. Thus the result. □

Cor. (II.10.2.17). If $\eta : G_K \to \mathbb{Z}_p^*$ is a character and there is $y \in \mathbb{C}_p^*$ that $\eta(g) = g(y)/y$, then there exists a finite Abelian extension L of K that $\eta|_{G_L}$ is unramified, i.e. η is **potentially unramified**.

Proof: Apply $\log_p$, then the image of $f = \log_p \eta$ in $H^1(G_K, \mathbb{C}_p)$ is trivial, so the above proposition shows $\beta_f = 0$, so $\log_p \eta$ is trivial on I_K, so I_K is mapped by η into the μ_p, so $\eta(I_K^{ab} \cap (G_K^{ab})^{p-1}) = 1$. $G_K^{ab} \cong \hat{\mathbb{Z}} \times \mathcal{O}_K^*$, so $(G^{ab})^{p-1}$ is open hence of finite index in G_K^{ab}, so correspond to a finite Abelian extension L. □

Prop. (II.10.2.18). If $G_K \to GL_d(\mathbb{Q}_p)$ is such $\rho(g) = g(M)M^{-1}$ for $M \in GL_d(\mathbb{C}_p)$, then ρ is potentially unramified.

Proof: Cf.[Sen Continuous Cohomology and p-adic Galois representations]. □

3 (φ, Γ)-modules

Basic References are [Fontaine90: Représentations p-adiques des corps locaux],[Fontaine94a: Le corps des périodes p-adiques] and [Fonatine94b: Repésentations p-adiques semi-stables] but I cannot understand French so [Foundations of Theory of (φ, Γ)-modules over the Robba Ring] is used and I'm basically following [Berger Galois representations and (φ, Γ)-modules].

Def. (II.10.3.1) (φ-module). Let M be a A-module and $\sigma : A \to A$ is a ring map. Then an additive map $\varphi : M \to M$ is called **σ-semi-linear** iff $\varphi(am) = \sigma(a)\varphi(m)$ for $a \in A$. A **φ-module** over (A, σ) is just a M with a σ-semi-linear φ.

Giving a A-module M and a $\varphi : M \to M$, there is a map $\Phi : A_\sigma \otimes_A M = M_\sigma \to M : \lambda \otimes m \to \lambda\varphi(m)$, which is a A-module map iff φ is σ-semi-linear.

Prop. (II.10.3.2). We define a ring $A_\sigma[\varphi]$ as the free group $A[X]$ modulo the relation $Xa = \sigma(a)X$ and ring relations in A, then it is a ring. Then a φ-module over (A, σ) is equivalent to a left $A_\sigma[\varphi]$-module.

Cor. (II.10.3.3). Thus we know that the category of φ-modules is a Grothendieck Abelian category $\Phi\mathcal{M}$, and moreover, the kernel as $A_\sigma[\varphi]$-module is the same as the kernel as a A-module.

Def. (II.10.3.4). If A is Noetherian, then a φ-module M is called **étale** iff it is f.g and the corresponding $\Phi : M_\sigma \to M$ in(II.10.3.1) is a bijection. The subcategory of étale φ-modules is denoted by $\Phi\mathcal{M}^{ét}$.

In case when σ is a bijection, Φ is a bijection iff φ is a bijection.

Proof: Note that in this case $M_\sigma \to M$ is a bijection by $\lambda \otimes m \to \sigma^{-1}(\lambda)m$, so the rest is easy. $\square$

Prop. (II.10.3.5). If A is Noetherian and A_σ is flat, then $\Phi\mathcal{M}^{ét}$ is Abelian category.

Proof: 0 is the zero object, the canonical sum&product are clearly étale. And we need to check the kernel and cokernel are etale. But we have an exact sequence $0 \to \mathrm{Ker} \to M \to N \to \mathrm{Coker} \to 0$ so we tensor with A_σ to get a morphism of sequences that $M_\sigma \to M, N_\sigma \to N$ are both bijective, so by 5-lemma, it is bijection at kernel and cokernel, so they are étale. $\square$

Def. (II.10.3.6). If there is a map $\alpha : (A_1, \sigma_1) \to (A_2, \sigma_2)$ that commutes with σs, then we have a **pullback** from $\Phi\mathcal{M}_1$ to $\Phi\mathcal{M}_2$: $\alpha^*(M) = A_2 \otimes_{A_1} M$, and φ is given by $\varphi(a \otimes m) = \sigma_2(a) \otimes \varphi(m)$.

Def. (II.10.3.7) ((φ, Γ)-module). If A has a action of a group Γ that commutes with σ, then a **(φ, Γ)-module** is a φ-module with a semi-linear action of Γ that commutes with φ.

If A, Γ has Hausdorff and complete topologies and the action is continuous, and A is Hausdorff and A_σ flat, then a (φ, Γ)-module M is called **étale** iff it is a étale φ-module and it has a topology that the action of Γ is continuous on M.

(φ, Γ)-modules forms an Abelian category with tensor products, Cf.[Fon90 3.3.2].

4 Isocrystals

Def. (II.10.4.1). We consider a perfect field k and $K = W(k)[1/p]$, K is equipped with the natural σ lifting the Frobenius. We consider φ-modules for $\sigma = \sigma^a$, where $a \in \mathbb{Z}\backslash\{0\}$. We don't care about this a much.

Then a $K - \varphi$-module D is called **effective** if there is a (complete)$W(k)$-lattice M of D that $\varphi(M) \subset M$. In this case, let a_n be the maximum integer that $\varphi^n(M) \subset p^{a_n}M$, then we have $a_{m+n} \geq a_m + a_n$, thus by(IX.2.1.1), we have a_n/n converges to $\sup a_n/n = \lambda$. λ doesn't depend on M because of the cofinality of lattices.

Def. (II.10.4.2).

Lemma (II.10.4.3). Let M be a lattice of D that $\varphi^{h+1}(M) \subset p^{-1}M$, where h is the dimension of D, then D is effective.

Proof: Let $M_j = M + \varphi(M) + \ldots + \varphi^j(M)$, then $M_j/M \subset p^{-1}M/M$, which is a k-vector space of dimension h, then $M_j = M_{j+1}$ for some j, hence M_j is stable under φ. □

Prop. (II.10.4.4). $\lambda \geq 0$ iff D is effective. And $\lambda = s/r$, where $1 \leq r \leq h$.

Proof: If D is effective, then $a_n \geq 0$, conversely, if $a_n \geq 1$, then $M' = M + \varphi(M) + \ldots + \varphi^{n-1}(M)$ is stable under φ, so D is effective.

For the second assertion, we first notice, if $\lambda > 0$, then φ is nilpotent on M/pM, which is a k-vector space of dimension h, then $\varphi^h = 0$ on M/pM, so $\lambda \geq 1/h$.

Now we find s, r that $|r\lambda - s| \leq 1/(h+1)$, and $\widetilde{\varphi} = p^{-s}\varphi^r$ has $|\widetilde{\lambda}| \leq 1/(h+1)$, so(II.10.4.3) shows that $\widetilde{\varphi}$ is effective, hence $\widetilde{\varphi} \geq 0$, and by what we have proved, $\widetilde{\varphi} = 0$, hence it is $\lambda = s/r$. □

Lemma (II.10.4.5). For a φ-stable $W(k)$-lattice of D, one has $M = M_0 \oplus M_{>0}$, where φ is bijection on M_0 and topologically nilpotent on $M_{>0}$.

Proof: We consider M/p^nM, then by(I.3.4.3) under slight modification, we have a decomposition, thus it has a decomposition. And the decompositions for different n are compatible, so it gives a decomposition of M it self. □

Def. (II.10.4.6). A φ-module is called **pure of slope** $\lambda = s/r \in \mathbb{Q}$ if D admits a lattice M on which $p^{-s}\varphi^r$ is a bijection. This is independent of M because λ is independent of M.

Prop. (II.10.4.7) (Dieudonné-Manin). If M is a φ-module, then M is a finite sum of modules pure of slopes λ_i.

Proof: We use the $\widetilde{\varphi}$ as in(II.10.4.4), we see that M has a decomposition $M_0 \oplus M_{>0}$ by(II.10.4.5), and $M_0 \neq 0$ by definition. Then we use induction to get the result. □

Prop. (II.10.4.8). If k is a separably closed field and V is a φ-module with $a \geq 1$, then V has a basis of elements fixed by φ, and $1 - \varphi$ is a surjection.

If A is a ring with $A/pA = k$ a separably close field and V is a φ-module with $a \geq 1$, then V has a basis of elements fixed by φ, and $1 - \varphi$ is a surjection.

Proof: We choose a $e_0 \in V$, and set $e_i = \varphi^i(e_0)$, and suppose $e_d = a_0e_0 + \ldots + a_{d-1}e_{d-1}$, then if we consider the equation $\varphi(b_0e_0 + \ldots + b_{d-1}e_{d-1}) = b_0e_0 + \ldots + b_{d-1}e_{d-1}$, then we need to assure b_{d-1} is a zero of

$$x = a_0^{q^{d-1}}x^{q^d} + a_1^{q^{d-2}}x^{q^{d-1}} + \ldots + a_{d-1}x^q$$

which is separable, so it has a non-zero solution in k, so φ has a fixed point v. By induction, we have $V/k \cdot v$ admits a basis fixed by φ. We know that $1 - \varphi : k \cdot v \to k \cdot v : x \mapsto (x - x^q)$ is surjective, so we can adjust the coefficient of v to get a basis of V fixed by φ. And meanwhile we proved $1 - \varphi$ is surjective.

The second assertion follows from successive approximation, as $x^p - x - a$ always has a root in k. □

Def. (II.10.4.9). When k is pure and separably closed(i.e. k is alg.closed), for $\lambda = s/r$, we define a φ-module over $K = W(k)[1/p]$ $E_\lambda = \oplus_{i+0}^{r-1} K e_i$ that $\varphi(e_i) = e_{i+1}$, and $\varphi(e_{r+1}) = p^s e_0$. In this case, E_λ is irreducible.

Proof: If D is a $W(k)$-lattice stable under φ, then we may assume it is pure of slope d/h by(II.10.4.7), and then we find an element $y = \sum y_i e_i$ fixed by $p^{-d}\varphi^h$, then $p^{sh}\varphi^{rh}(y_i) = p^{rd} y_i$, which by valuation is only possible when $sh = rd$, so $h \geq r$, so D generate E_λ. □

Prop. (II.10.4.10) (Dieudonné-Manin). If k is alg.closed, then any φ-module over K has a unique decomposition as sums of E_{λ_i}.

Proof: By(II.10.4.7) we assume D is pure, then by(II.10.4.8) we find a basis y_i that $\varphi^r(y) = p^s y$, then there is a map $E_\lambda \to D$. Since E_λ is irreducible, this is injective, and we consider all y_i until $E_\lambda^m \to V$ is surjective, then it is an isomorphism(this is like the case of simple modules). □

5 l-adic representations

Prop. (II.10.5.1). Every continuous representation of G_K on a $\mathbb{Q}_l$ vector space (Continuous group morphism to $GL_n(\mathbb{Q}_l)$ has a $\mathbb{Z}_l$ lattice stable under the action.

So the functor $\rho \to \rho \otimes \mathbb{Q}_l$ from $\mathrm{Rep}_{\mathbb{Z}_l}(G_K)$ to the Tannakian natural category $\mathrm{Rep}_{\mathbb{Q}_l}(G_K)$ is essentially surjective.

Proof: Notice that the stablizer of the standard lattice is $GL_n(\mathbb{Z}_l)$ which is open and so the inverse image has a finite coset. And the image of the wild ramification group is finite because it is in $GL_n(\mathbb{F}_l)$. □

Prop. (II.10.5.2) (Grothendieck Monodromy theorem). For a local field K, the étale representation and the Tate module are all potentially semisimple. i.e. semisimple for a finite extension.

II.11 Automorphic Forms(Bump)

Main References are [Automorphic Forms and Representations, Bump] and [A Course given by Liang Xiao].

1 Setups

Def. (II.11.1.1) (General Notations). $\mathcal{H}$ is the Poincaré's upper plane with measure $y^{-2}dxdy$(II.11.1.2), $G = GL(2,\mathbb{R})^+, G_1 = SL(2,\mathbb{R}), K = SO(2,\mathbb{R})$. Then G acts on $\mathcal{H}$ through linear fractional transformation(IV.7.1.5) and fixes the measure. We will denote a, b, c, d the linear functionals on $M(2,\mathbb{R})$ that for $\gamma \in M(2,\mathbb{R})$, $\gamma = \begin{bmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{bmatrix}$.

We will consider right regular action ρ of G on $C^\infty(G)$, and also the left regular action λ. We will write dX for $X \in \mathfrak{g}$ as the representation of Lie algebra of G via ρ, then it commutes with λ. So it induces a map of $U(\mathfrak{g})$ to the ring of left G-invariant differential operators on G(IV.7.6.1).

Also we will use the Lie algebra notations(I.10.2.11):

$$H = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad R = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \quad L = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \quad W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = iH.$$

and the Casimir element in $\mathcal{Z} = Z(U(\mathfrak{g}))$:

$$\Delta = -\frac{1}{4}(H^2 + 2RL + 2LR).$$

Prop. (II.11.1.2) (Iwasawa-Decomposition). Every element of G has a unique representation of the form(I.2.10.3):

$$g = \begin{bmatrix} u & \\ & u \end{bmatrix} \begin{bmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{bmatrix} k_\theta$$

where $k_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. So by(V.6.1.20) and(V.6.1.12) the Haar measure is given by $dg = \frac{du}{u}\frac{dxdy}{y^2}d\theta$.

Prop. (II.11.1.3). In the coordinate(II.11.1.2), we have the following equation:

$$dR = e^{2i\theta}(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}), \quad dL = e^{-2i\theta}(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}), \quad dH = -i\frac{\partial}{\partial\theta}$$

So in particular

$$\Delta = d\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y\frac{\partial^2}{\partial x\partial\theta}.$$

Proof: Cf.[Bump, P155].? □

Def. (II.11.1.4) (Γ Setting). Let Γ be a discrete subgroup of G that the volume of $\Gamma\backslash\mathcal{H}$ is finite, we may also assume that $-1 \in \Gamma \subset SL(2,\mathbb{R})$.

Let χ be a character of Γ, ω be a character of the center $Z(\mathbb{R}) \subset G$(the scalar matrices). Assume that $\omega(-1) = \chi(-1)$.

Def. (II.11.1.5) (Cusps). A **cusp** of Γ is a point in $\mathbb{R} \cup \{\infty\}$ whose stablizer in Γ contains a non-trivial unipotent element. The number of orbits of cusps of Γ under action of Γ is finite.

Proof: The number is finite because**?** □

Def. (II.11.1.6) (Form Spaces). Let $C^\infty(\Gamma\backslash G, \chi, \omega)$ be the space of smooth functions $F : G \to \mathbb{C}$ that

$$F(\gamma g) = \chi(\gamma)F(g), \quad \gamma \in \Gamma, g \in G,$$

$$F(zg) = \omega(z)F(g), \quad z \in Z(\mathbb{R}), g \in G.$$

Let the subspace $C_c^\infty(\Gamma\backslash G, \chi, \omega)$ be those that are compactly supported modulo $Z(\mathbb{R})$.

Def. (II.11.1.7) (Automorphic Forms). Let the space of **automorphic forms** $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ be the subspace of $C^\infty(\Gamma\backslash G, \chi, \omega)$ consisting of K-finite and $\mathcal{Z}$-finite and satisfies the **condition of moderate growth**:

$$|F(g)| < C||g||^N$$

for some $C, N > 0$, where the **height** $||g||$ =length of $(g, \det g^{-1}) \in M_2(\mathbb{R}) \oplus \mathbb{R}$.

Def. (II.11.1.8) (Cuspidal Forms). If ∞ is a cusp of Γ, then Γ contains some $\tau_r = \begin{bmatrix} 1 & r \\ & 1 \end{bmatrix}$, so a $F \in \mathcal{A}(\Gamma\backslash G, \chi, \omega)$ is called **cuspidal** at ∞ iff $\chi(\tau_r) \neq 1$ or

$$\int_0^r F(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g)dx = 0.$$

Notice this is independent of r chosen.

More generally, if a is a cusp of Γ, then choose $\xi \in SL(2, \mathbb{R})$ that $\xi(\infty) = a$, then for $F \in \mathcal{A}(\Gamma\backslash G, \chi, \omega)$, $F'(g) = F(\xi g) \in \mathcal{A}(\Gamma'\backslash G, \chi', \omega)$, where $\Gamma' = \xi^{-1}\Gamma\xi, \chi'(\gamma') = \chi(\xi\gamma'\xi^{-1})$(Because left and right action commutes). Then F is called **cuspidal** at a iff F' is cuspidal at ∞.

The subspace of cuspidal forms is denoted by $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega) \subset \mathcal{A}(\Gamma\backslash G, \chi, \omega)$.

Def. (II.11.1.9) (Behavior at Cusps). If ∞ is a cusp of Γ, then Γ contains some $\tau_r = \begin{bmatrix} 1 & r \\ & 1 \end{bmatrix}$, then a function $f(x + iy)$ is called

- **of moderate growth** at ∞ iff $|f(x + iy)|$ is bounded by a polynomial function of y.
- **decay rapidly** at ∞ iff $|f(x + iy)| \leq y^{-N}$ for some $N > 0$.
- **cuspidal** at ∞ iff either $\chi(\tau_r) \neq 1$ or $\int_0^r f(z + u)du = 0$.

If f is holomorphic on $\mathcal{H}$ then we have a Fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z} = \sum_{n=-\infty}^{\infty} a_n q^n = T(q).$$

Then f is called **meromorphic/holomorphic/vanishes** at the cusp ∞ iff $T(q)$ does at 0.

The general cusp case is reduced to the ∞ case the same way as in(II.11.1.8). Notice this is independent of possible r chosen.

Def. (II.11.1.10) (Fundamental Domain). A **fundamental domain** for Γ is defined to be

Technicalities

Prop. (II.11.1.11) (Gelfand). Let $G = GL(n,\mathbb{R})^+, K = SO(n)$, then $C_c^\infty(K\backslash G/K)$ is commutative.

Proof: Consider the map $\varphi \mapsto \widehat{\varphi} : \widehat{\varphi}(g) = \varphi(g^t)$, then it is an anti-involution of $C_c^\infty(K\backslash G/K)$:

$$(\widehat{\varphi_1 * \varphi_2})(g) = \int_G \varphi_1(g^t h)\varphi_2(h^{-1})dh = \int_G \widehat{\varphi_2}(h^{-t})\widehat{\varphi_1}(h^t g)dh = \int_G \widehat{\varphi_2}(h)\widehat{\varphi_1}(h^{-1}g)dh = (\widehat{\varphi_2} * \widehat{\varphi_1})(g)$$

But we find $\widehat{\varphi} = \varphi$, because we can use(I.2.10.1), $\varphi(g) = \varphi(d) = \widehat{\varphi}(d) = \widehat{\varphi}(g)$. □

Prop. (II.11.1.12). Let $G = GL(2,\mathbb{R})^+, K = SO(2)$, let σ be a character of K, then $C_c^\infty(K\backslash G/K,\sigma)$ is commutative.

Proof: The proof is the same as that of(II.11.1.11), but modified as

$$\widehat{\varphi}(g) = \varphi(\begin{bmatrix} -1 & \\ & 1 \end{bmatrix} g^t \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}).$$

□

Prop. (II.11.1.13) (Harish-Chandra Theorem(Dimension 2 Case)). Let $f \in C^\infty(G)$ be both K-finite and $\mathcal{Z}$-finite, then f is analytic, $U(\mathfrak{g})f$ is an admissible $(\mathfrak{g}, K)$-module, and there exists $\alpha \in C_c^\infty(G)$ that

$$\alpha(kgk^{-1}) = \alpha(g), \quad f * \alpha = f.$$

Moreover if $|f(g)| < C||g||^N$ where $||g||$ is induced from the Euclidean norm of $\mathbb{R}^4$, then all $U(\mathfrak{g})f$ satisfies similar equalities with the same N.

Proof: Because W lies in the Lie algebra of K and $W = iH$, the hypothesis implies Rf is f.d., where $R = \mathbb{C}[\Delta, H]$. Let V be the smallest closed G-invariant subspace of $C^\infty(G)$ containing f and let $V_0 = U(\mathfrak{g})f$.

Notice there is a continuous projection of V onto $V(n)$:

$$E_n\varphi = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\varphi(gk_\theta)d\theta,$$

because f is K-finite, there is an N s.t. $f = \sum_{n=-N}^N E_n f$. Then notice any Df is also K-finite because D is combination of entries in R, L, H. It suffices to show now $E_nDf \in V_0$ for any n: E_nDf can be extracted using polynomial in H, so it is clearly in V_0.

Next we show $V_0 \cap V(n)$ is of f.d.: Let $f_1, \ldots, f_k$ be a basis of Rf, because each f_k is K-finite. so if we use the decomposition of $U(\mathfrak{sl}_2(\mathbb{C}))$**??**, only the R, L shifting of the E_nf will be considered, so each $V_0 \cap V(n)$ is of f.d. and

$$V_0 = \oplus(V_0 \cap V(n)).$$

Now we show f is analytic: because f is $\mathcal{Z}$-finite, there is an equation $P(\Delta)f = 0$ where P is a monic function, and because Δ commutes with E_k and f is finite, it suffices to consider $g(\Delta)E_kf = 0$. Now the $P(\Delta)E_kf = P(\Delta_k)E_kf$, and $P(\Delta_k)$ is an elliptic operator, so E_kf is analytic by(V.8.8.5).

Now we show V is the closure of V_0:

So actually $V(n) \subset V_0$, because $E_nV_0 = V_0 \cap V(n) \subset V(n)$ is dense, and $V_0 \cap V(n)$ is of f.d., so $V(n) \subset V_0$(V.4.1.8) and $V_0 = \oplus V(n)$ is an admissible $(\mathfrak{g}, K)$-module.

Let J be the convolution algebra(because G is unimodular) of α that $\alpha(kgk^{-1}) = \alpha(g)$, then it can be checked that convolution $- * \alpha$ commutes with action of K, so $f * J$ is K-finite so in a f.d. space.

Now we can approximate f by $f * J$: choose a Dirac sequence $\{\alpha_n\} \in C_c^\infty(G)$, we may replace $\{\alpha_n\}$ by the function $\beta_n(g) = \int_K \alpha_n(k^{-1}gk)dk$ to obtain a Dirac sequence in J. Then $f * \alpha_n \to f$ uniformly on compact sets. But $f * J$ is f.d., so there are some $f * \alpha = f$. □

2 Maass Forms

Def. (II.11.2.1) (Right Weight Action). There are right actions of $GL(2,\mathbb{R})^+$ on $C^\infty(\mathcal{H})$ defined to by

$$(f|_k g)(z) = (\frac{c\overline{z}+d}{|cz+d|})^k f(\frac{az+b}{cz+d})$$

Proof: It is an action because**?** □

Def. (II.11.2.2) (Holomorphic Right Weight Action). Besides the wight weight action(II.11.2.1), there is another family of actions:

$$f[\gamma]_k(z) = \deg(\gamma)^{k/2}(cz+d)^{-k} f(\frac{az+d}{cz+d})$$

Def. (II.11.2.3) (Form Spaces). if Γ is a discrete subgroup of $\mathcal{H}$, let $C^\infty(\Gamma\backslash\mathcal{H},\chi,k)$ be the space of smooth functions on $\mathcal{H}$ that

$$f|_k\gamma = \chi(\gamma)f, \quad \gamma \in \Gamma$$

Def. (II.11.2.4) (Maass Operator). A **Maass differential operators** on $C^\infty(\mathcal{H})$ is defined to be

$$R_k = (z-\overline{z})\frac{\partial}{\partial z} + \frac{k}{2}, \quad L_k = -(z-\overline{z})\frac{\partial}{\partial \overline{z}} - \frac{k}{2}$$

and

$$\Delta_k = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + iky\frac{\partial}{\partial x} = -L_{k+2}R_k - \frac{k}{2}(1+\frac{k}{2}) = -R_{k-2}L_k + \frac{k}{2}(1-\frac{k}{2})$$

And Δ_k is a symmetric operator on $L^2(\mathcal{H})$ with domain $C_c^\infty(\mathcal{H})$.

Proof: For the formula:

For the symmetry: composed with the measure, the order2 part is just the ordinary Laplacian, and the order 1 part becomes $iy^{-1}\frac{\partial}{\partial x}$, then notice

$$\int iy^{-1}(\frac{\partial f}{\partial x}\overline{g} + f\frac{\partial \overline{g}}{\partial x})dxdy = i\int d(y^{-1}f\overline{g}dy) = 0$$

□

Prop. (II.11.2.5) (Maass Operator and Weight Action). For $f \in C^\infty(\mathcal{H}), g \in G$,

$$(R_k f)|_{k+2}g = R_k(f|_k g), \quad (L_k f)|_{k-2}g = L_k(f|_k g), \quad (\Delta_k f)|_k g = \Delta_k(f|_k g)$$

Proof: Cf.[Bump, P130]. □

Cor. (II.11.2.6). The operator R_k, L_k, Δ_k maps functions between $C^\infty(\Gamma\backslash\mathcal{H},\chi,k)$ and arises and decreases weights respectively.

Def. (II.11.2.7) (Maass Forms). A **Maass Form** of weight k is an element in $C^\infty(\Gamma\backslash\mathcal{H},\chi,k)$(II.11.2.3) that is an eigenform for Δ_k (of eigenvalue λ) and is of moderate growth at cusps of Γ(II.11.1.9).

3 Irreducible Unitary Representations of $GL(2,\mathbb{R})^+$

$(\mathfrak{g},K)$-Modules of $GL(2,\mathbb{R})$

Prop. (II.11.3.1) (Lie Theory). Let V be an irreducible admissible $(\mathfrak{g},K)$-module for $GL(2,\mathbb{R})^+$, then

- V^k is the space of all vectors $x\in V$ that $Hx=kx$.
- If $x\in V^k$, then $Rx\in V^{k+2}, Lx\in V^{k-2}$.
- If $0\neq x\in V^k$, then $\mathbb{C}x=V^k, \mathbb{C}R^n x=V^{k+2n}, \mathbb{C}L^n V^{k-2n}$ and
$$V=\mathbb{C}x\oplus\bigoplus_{n>0}\mathbb{C}R^n x\oplus\bigoplus_{n>0}\mathbb{C}L^n x.$$
- Suppose $\Delta=\lambda$ on V, then if $x\in V^k$, then
$$LRx=(-\lambda-\frac{k}{2}(1+\frac{k}{2}))x,\quad RLx=(-\lambda+\frac{k}{2}(1-\frac{k}{2}))x.$$

Proof: 1: Let $W=iH$. If $x\in V^k$, then
$$Wx=\frac{d}{dt}\pi(e^{tW})x=\frac{d}{dt}\pi(k_t)x=\frac{d}{dt}e^{ikt}x=ikx$$
thus $Hx=kx$. And we know V decomposes as direct sums of representations of K(V.6.4.5), thus the result follows.

2: clear from(I.10.2.11).

3: Because the RHS is a $\mathfrak{g}$-submodule, by representation of $\mathfrak{sl}_z(\mathbb{C})$(I.10.8.2). And it is also a K-subrepresentation, by item1.

4: By(I.10.8.2). □

Cor. (II.11.3.2) (Non-Discrete Case). For any $\lambda,\mu\in\mathbb{C}$ that $\lambda\neq\frac{k}{2}(1+\frac{k}{2})$ for k even/odd, there exists at most one isomorphism class of irreducible admissible even/odd $(\mathfrak{g},K)$-module V on which Δ, I acts by λ,μ respectively.

Proof: This follows from the classification of representation of $\mathfrak{sl}_2(\mathbb{C})$(I.10.8.2). Notice the action of K is controlled by(II.11.3.1) item1. □

Cor. (II.11.3.3) (Discrete Case). Let $k\geq 1$ be an integer and $\lambda=\frac{k}{2}(1+\frac{k}{2})$. Let V be an irreducible admissible $(\mathfrak{g},K)$-module with parity equals k, Let Σ be the K-types of V, then Σ is one of the following sets:

- $\Sigma^+(k)=\{l\in\mathbb{Z}|l\equiv k \mod 2, l\geq k\}$
- $\Sigma^-(k)=\{l\in\mathbb{Z}|l\equiv k \mod 2, l\leq -k\}$
- $\Sigma^0(k)=\{l\in\mathbb{Z}|l\equiv k \mod 2, -k<l<k\}$

And there are at most one isomorphism class with each Σ.

Proof: This follows from the classification of representation of $\mathfrak{sl}_2(\mathbb{C})$(I.10.8.2). Notice the action of K is controlled by(II.11.3.1) item1. □

Prop. (II.11.3.4) (Existence of $(\mathfrak{g},K)$-Modules). There is a family of Hilbert spaces $H(s_1,s_2,\varepsilon)$ with $s=\frac{1}{2}(s_1-s_2+1), \lambda=s(1-s), \mu=(s_1+s_2)$, that the $(\mathfrak{g},K)$-module corresponding to $H(s_1,s_2,\varepsilon)$ falls into each classes in(II.11.3.2) and(II.11.3.3).

Proof: Cf.[Bump P214]?. □

Prop. (II.11.3.5) (List of Irreducible Admissible $(\mathfrak{g}, K)$-Modules for $GL(2,\mathbb{R})^+$). Every irreducible admissible $(\mathfrak{g}, K)$-module may be realized as the space of K-finite vectors in an admissible representation of G on a Hilbert space. Let $\lambda, \mu \in \mathbb{C}$, and $\varepsilon = 0, 1$.

- If λ is not of the form $\frac{k}{2}(1-\frac{k}{2})$, where $k \equiv \varepsilon \mod 2$, then there exists a unique irreducible admissible $(\mathfrak{g}, K)$-module of parity ε on which Δ, I acts by scalars λ, μ, denoted by $P_\mu(\lambda, \varepsilon)$. These are called the **principal series**.
- If $\lambda = \frac{k}{2}(1-\frac{k}{2})$ for some $1 \leq k \equiv \varepsilon \mod 2$, then there exists three(two for $k = 1$) irreducible admissible $(\mathfrak{g}, K)$-module of parity ε on which Δ, I acts by scalars λ, μ. There K-types are $\Sigma^\pm$, Σ^0 respectively. The K-types $\Sigma^\pm$ are denoted by $D^\pm_\mu(k)$. If $k > 1$, $D^\pm_\mu(k)$ are called **discrete series** and for $k = 1$ they are called **limit of discrete series**.

Proof: This is a consequence of(II.11.3.2)(II.11.3.3) and(II.11.3.4). □

Prop. (II.11.3.6) (List of Irreducible Admissible $(\mathfrak{g}, K)$-Modules for $GL(2,\mathbb{R})$). Every irreducible admissible $(\mathfrak{g}, K)$-module may be realized as the space of K-finite vectors in an admissible representation of G on a Hilbert space. Let $\lambda, \mu \in \mathbb{C}$, and $\varepsilon = 0, 1$.

- The f.d. representations are obtained by tensoring the symmetric powers of the standard representation of G with the 1-dimensional representation of the form $\chi \circ \det$.
- If χ_1, χ_2 are characters of $\mathbb{R}^*$ that $\chi_1\chi_2^{-1}$ is not of the form $y \mapsto \mathrm{sgn}(y)^\varepsilon |y|^{k-1}$, where $k \equiv \varepsilon \mod 2$, then there is a irreducible $(\mathfrak{g}, O(2))$-module $\pi(\chi_1, \chi_2)$.
- If μ is a real number and $k \geq 1$ is an integer, then there are representations $D_\mu(k)$, called **discrete series** if $k \geq 2$ and **limits of discrete series** if $k = 1$.

Proof: Cf.[Bump P219]?. □

Unicity of $(\mathfrak{g}, K)$-Modules

Lemma (II.11.3.7) (Finite Dimensional Case). The only irreducible f.d. unitary representations of the group $GL(n,\mathbb{R})^+$ are the 1-dimensional characters $g \mapsto \det(g)^r$ where r is purely imaginary.

Proof: Such a representation defines a continuous map of $GL(n,\mathbb{R})^+$ into the compact unitary group $U(n)$. Now it induces a Lie algebra map $\mathfrak{sl}_n(\mathbb{R}) \to \mathfrak{u}_m$. This map must be trivial because otherwise this is an embedding because $\mathfrak{sl}_n(\mathbb{R})$ is simple. But this is impossible because the adjoint action of $\mathfrak{sl}_n(\mathbb{R})$ has real eigenvalues but the adjoint action of $\mathfrak{u}_m$ are all purely imaginary?. So the action is trivial on $SL(n,\mathbb{R})^+$, so induces an irreducible representation of $\det(g)$, which is clearly 1-dimensional. □

Lemma (II.11.3.8). Because for a unitary representation $\mathcal{H}$ of G, for $X \in \mathfrak{g}$, we have

$$(Xu, v) = -(u, Xv),$$

so $(Xv, w) = -(v, \overline{X}w)$ when complexified. So

$$(Rv, w) = -(v, Lw)$$

for any $v, w \in \mathcal{H}$, by(II.11.1.1).

Lemma (II.11.3.9) (Principal Series). For the principal series $P_\mu(\lambda, \varepsilon)$ of $GL(2,\mathbb{R})^+$, there exists an irreducible unitary representation in this class if μ is purely imaginary and $\lambda \geq 1/4$ real.

Proof: Cf.[Bump P225].? □

Lemma (II.11.3.10) (Possibilities of Unitary Representations). Let $\mathcal{H}$ be a unitary representation of $GL(2,\mathbb{R})^+$. Assume Δ, I acts by scalarsλ, μ respectively, then

- μ is purely imaginary and λ is real.
- If the $(\mathfrak{g}, K)$-module type of $\mathcal{H}$ is a principal series $P_\mu(\lambda, \varepsilon)$, then $\lambda > 0$, and if $\varepsilon = 1$, $\lambda > 1/4$.

Proof: 1: This follows from(II.11.3.8), as action of I is skew-symmetric and action of Δ is symmetric.

2: By(II.11.3.2), $V^k \neq 0$ for $k \equiv \varepsilon \mod 2$, let $f_k \in V^k$. Because $-4\Delta - H^2 + 2H = 4RL$(II.11.1.1), take $k = \varepsilon$, then

$$(-4\lambda - \varepsilon^2 + 2\varepsilon) f_\varepsilon = 4RLf_\varepsilon.$$

But by(II.11.3.8),

$$(4RLf_\varepsilon, f_\varepsilon) = -4(Lf_\varepsilon, Lf_\varepsilon) < 0$$

thus $4\lambda > 2\varepsilon - \varepsilon^2$. □

Cor. (II.11.3.11) (Reduction of μ). The infinitesimal equivalence class of representations $P_\mu(\lambda, \varepsilon)$ or $D_\mu^\pm(k)$ contains an irreducible unitary representation iff μ is purely imaginary and the corresponding class $P(\lambda, \varepsilon)$ or $D^\pm(k)$ is true.

Proof: μ must be purely imaginary by the proposition. And we may tensoring a unitary representation by a $\deg(g)^r$, it is also unitary iff r is purely imaginary, and this increases the value of action of I by $2r$. □

Lemma (II.11.3.12) (Complementary Series). For μ purely imaginary and $0 < \lambda < 1/4$ and $\varepsilon = 0$, there exists an irreducible unitary representation in this class of the $(\mathfrak{g}, K)$-module $P_\mu(\lambda, 0)$.

Proof: Cf.[Bump, P232]. □

Lemma (II.11.3.13) (Discrete Series). if $k > 1$, then there exists a unitary representation in the infinitesimal equivalence class $D^\pm(k)$, more precisely,

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Lemma (II.11.3.14) (Limits of Discrete Series). There exists a unitary representation in the infinitesimal equivalence class $D^\pm(1)$.

Proof: Cf.[Bump P238]. □

Prop. (II.11.3.15) (List of Irreducible Unitary Representations of $GL(2,\mathbb{R})^+$).

- The 1-dimensional representation $g \mapsto |\deg(g)|^\mu$, where μ is purely imaginary.
- The unitary principal series $P_\mu(\lambda, \varepsilon)$, where μ is purely imaginary, $\varepsilon = 0, 1$ and $\lambda \geq 1/4$.
- The complementary series representations $P_\mu(\lambda, 0)$ where μ is purely imaginary and $0 < \lambda < 1/4$.

- The holomorphic discrete series ($k \geq 2$) and limits of discrete series ($k = 1$) $D_\mu^\pm(k)$, where μ is purely imaginary.

Notice each of these infinitesimal equivalence classes of irreducible representations has a unique representative that is a unitary representation by(I.4.6.17).

Proof: By(I.4.6.7) and(I.4.6.17), the conclusion follows from the classification of $(\mathfrak{g}, K)$-modules(II.11.3.5) and determining which infinitesimal class has a unitary representative, which follows from(II.11.3.7)(II.11.3.10), (II.11.3.9)(II.11.3.11), (II.11.3.12)(II.11.3.13), (II.11.3.14). □

Whittaker Models

Def. (II.11.3.16) (Whittaker Function Space). Let ψ be an additive character on $\mathbb{R}$, denote W the space of smooth functions on $GL(2,\mathbb{R})^+$ that satisfies

$$W(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g) = \psi(x) W(g).$$

A function $f \in W$ is called:

- of **moderate growth** iff in coordinate(II.11.1.2), f is bounded by a polynomial in y as $y \to \infty$.
- **rapidly decreasing** iff in the same coordinate $y^N f \to 0$ as $y \to \infty$ for every $N > 0$.

Lemma (II.11.3.17). Let $\mu, \lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$. Let $W(\lambda, \mu, k)$ be the space of functions $f \in W$(II.11.3.16) on G s.t. $\Delta f = \lambda f$, $If = \mu f$, $f \in (C^\infty(G))^k$ and f is of moderate growth, then $W(\lambda, \mu, k)$ is 1-dimensional, and functions in this space are actually rapidly decreasing and analytic.

Moreover, the operators R, L map $W(\lambda, \mu, k)$ into $W(\lambda, \mu, k+2)$, $W(\lambda, \mu, k-2)$ respectively.

Proof: For $f \in W(\lambda, \mu, k)$, in coordinate(II.11.1.2), we have

$$f(g) = u^\mu \psi(x) e^{ik\theta} w(y), \quad w(y) = f(\begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix})$$

Thus it suffices to study the behavior of $w(y)$. By the expression of Δ(II.11.1.3), if $\psi(x) = e^{iax}$, then

$$w'' + (-a^2 + \frac{k}{2y} + \frac{\lambda}{y^2}) w = 0$$

And this is the Whittaker's equation, and the only moderate growth function is rapidly decreasing and analytic**?**.

For the action of R, L, they preserve W because they are right actions. And Rf, Lf has the same eigenvalue of Δ, I because Δ, I are in the center of $U(\mathfrak{g})$. They shift the weight by(II.11.4.7) and(II.11.2.5). It's left to check the moderate growth condition: it suffices to check $\frac{d}{dy} f$ decays as $y \to \infty$. And this is by direct calculation in [A course of Modern Analysis, Whittaker/Watson(1927)]**?**. □

Prop. (II.11.3.18) (Whittaker Models). Let (π, V) be an irreducible admissible $(\mathfrak{g}, K)$-module for $G = GL(2,\mathbb{R})^+$ or $GL(2,\mathbb{R})$, then there exists at most one space $W(\pi, \psi) \in C^\infty(G)$ consisting of K-finite functions $f \in W$ that is of moderate growth, and is invariant under the action of $U(\mathfrak{g})$ and K, that is infinitesimal equivalent to (π, V).

Functions in $W(\pi, \psi)$ are actually rapidly decreasing and analytic. The space $W(\pi, \psi)$ is called the **Whittaker model** of π, if it exists.

Proof: By(I.4.6.16), Δ, I acts by scalars λ, μ on V. By(II.11.3.1) or(II.11.3.6) if $V^k \neq 0$, then $\dim V^k = 1$. If $V^k \neq 0$, then the image of V^k under the isomorphism with $W(\pi, \psi)$ is in the space $W(\lambda, \mu, k)$. Thus $W(\pi, \psi)$ is the direct sum of the $W(\lambda, \mu, k)$ for all k that $V^k \neq 0$, so uniquely determined by(II.11.3.17). And the rapid decreasing and analytic properties are also consequences of(II.11.3.17). □

Remark (II.11.3.19). This result is true if $\mathbb{R}$ is replaced by $\mathbb{C}$, Cf.[Automorphic Forms on $GL(2)$, Jaccquet/langlands (1970) Thm6.3. P232].

Prop. (II.11.3.20) (Shalika's Local Multiplicity One Theorem). Let $F = \mathbb{R}$ or $\mathbb{C}$. Let ψ be a character of F. We define a character ψ_N on the space of upper triangular unipotent matrices in $GL(n, F)$ by

$$\varphi_N(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1}).$$

Given a unitary irreducible representation V of $GL(n, F)$, a **Whittaker functional** on V^∞ is a continuous linear functional λ on V^∞ that $\lambda(\pi(u)x) = \psi_N(u)\lambda(x)$ for all $u \in N(F), x \in V^\infty$.

Then the dimension of the space of Whittaker functionals on V^∞ is at most 1-dimensional.

Proof:

□

4 $\Gamma\backslash\mathcal{H}$ Compact Case

Def. (II.11.4.1). In this subsection we assume $\Gamma\backslash G_1$ is compact, or equivalently $\Gamma\backslash\mathcal{H}$ is compact, because K is compact. This condition makes every smooth function square integrable.

Def. (II.11.4.2). Denote $C^\infty(\Gamma\backslash G, \chi, 1)$ by $C^\infty(\Gamma\backslash G, \chi)$, and $L^2(\Gamma\backslash G, \chi)$ the space of square integrable functions on $\Gamma\backslash G_1$ with the quotient Haar measure**??**

Prop. (II.11.4.3). The space $C_c^\infty(\Gamma\backslash G, \chi)$ is dense in $L^2(\Gamma\backslash G, \chi)$.

Proof: Cf.[Bump, P140]. □

Cor. (II.11.4.4). The right regular action of G extends to a unitary representation of G on $L^2(\Gamma\backslash G, \chi)$.

Proof: We must verify continuity, and this is clear using the proposition and notice $\Gamma\backslash G_1$ is compact. □

Prop. (II.11.4.5). Notice that if $f, g \in C^\infty(\Gamma\backslash\mathcal{H}, \chi, k)$, then $f\overline{g}$ is invariant under action of Γ, so we may define the Hilbert space $L^2(\Gamma\backslash\mathcal{H}, \chi, k)$ with the norm(II.11.1.1):

$$(f, g) = \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g}(z)\frac{dxdy}{y^2}.$$

Prop. (II.11.4.6). For $f \in C^\infty(\Gamma\backslash\mathcal{H}, \chi, k), g \in C^\infty(\Gamma\backslash\mathcal{H}, \chi, k+2)$,

$$(R_k f, g) = (f, -L_{k+2} g).$$

In particular, $\Delta_k = -L_{k+2}R_k - \frac{k}{2}(1 + \frac{k}{2})$ is symmetric on $L^2(\Gamma\backslash\mathcal{H}, \chi, k)$(unbounded and defined only on $C^\infty(\Gamma\backslash\mathcal{H}, \chi, k)$ now).

Proof: Cf.[Bump P135].**?** □

Prop. (II.11.4.7) (Two Form Spaces Equal). Let $L^2(\Gamma\backslash G,\chi,k)$ be the subspace of $L^2(\Gamma\backslash G,\chi)$ consisting of functions F that $\rho(k_\theta)F=e^{ik\theta}F$. Then there is an isomorphism of Hilbert spaces

$$\sigma_k : L^2(\Gamma\backslash\mathcal{H},\chi,k)\cong L^2(\Gamma\backslash G,\chi,k) : (\sigma_k f)(g)=(f|_k g)(i).$$

And we have(II.11.2.4):

$$\sigma_{k+2}R_k=dR\sigma_k,\quad \sigma_{k-2}L_k=dL\sigma_k,\quad \sigma_k\Delta_k=\Delta\sigma_k.$$

Proof: Check the left action of γ and $Z^+(\mathbb{R})$, and it can be verified that the inverse of σ_k is given by

$$f(z)=F(\begin{bmatrix} y & x\\ & 1\end{bmatrix})$$

More precisely, if coordinates(II.11.1.2),

$$F(u,x,y,\theta)=f(x+iy)e^{ik\theta},\quad f(z)=F(0,x,y,0).$$

then check the Γ-invariance(II.11.2.3) and(II.11.1.6). Finally check that σ_k preserves inner product, which is by(II.11.4.5) and(II.11.1.2).

The equations between R_k,L_k and R,L is easily verified. □

Def. (II.11.4.8) (Holomorphic Modular Forms as Maass Forms). Let $\Gamma\backslash\mathcal{H}$ be compact, then if f is a modular form of weight $k>0$, then $y^{k/2}f(z)$ is a Maass form of weight k with eigenvalue $\frac{k}{2}(1-\frac{k}{2})$ of Δ_k and annihilated by L_k.

Conversely, if a Maass form of weight k can be annihilated by L_k, then it comes from a modular form like above.

And similar things happen for R_k.

Proof: Direct calculation shows $L_k(y^{k/2}f(z))=2iy^{(k+2)/2}\frac{\partial}{\partial\bar z}f(z)$, so the theorem follows. □

The Spectral Problems

Prop. (II.11.4.9). Consider the right regular action on $L^2(\Gamma\backslash G,\chi)$(II.11.4.4), let $\varphi\in C_c^\infty(G)$, then φ can act on $L^2(\Gamma\backslash G,\chi)$ by(V.4.3.24), and:

- $\rho(\varphi)$ is an integral operator, in particular compact. And $\mathrm{Im}(\rho(\varphi))\subset C^\infty(\Gamma\backslash G,\chi)$.
- If $\varphi(g^{-1})=\overline{\varphi(g)}$, then $\rho(\varphi)$ is self-adjoint.
- If $\varphi(k_\theta g)=e^{-tk\theta}\varphi(g)$, then $\mathrm{Im}(\rho(\varphi))\subset C^\infty(\Gamma\backslash G,\chi,k)$.

Proof: 1:

$$(\rho(\varphi)f)(g)=\int_G f(h)\varphi(g^{-1}h)dh=\int_{\Gamma\backslash G_1}\int_{Z^+}\sum_{\gamma\in\Gamma}f(\gamma h)\varphi(g^{-1}\gamma hu)\frac{du}{u}dh=\int_{\Gamma\backslash G_1}f(h)K(g,h)dh$$

where

$$K(g,h)=\int_{Z^+}\sum_{\gamma\in\Gamma}\chi(\gamma)\varphi(g^{-1}\gamma hu)\frac{du}{u}.$$

Because φ is compactly supported, this is a smooth function in g and h, in particular square integrable on the fundamental domain (compact). And $\rho(\varphi)(f)(g)$ is smooth in g because $f\in L_1(\Gamma\backslash G_1)$ as $\Gamma\backslash G_1$ is compact, and $K(g,h)$ is smooth in g.

$2,3$ is easy from 1. □

Lemma (II.11.4.10). If $\mathcal{H}$ is a unitary representation of G on a Hilbert space, and let $f \neq 0 \in \mathcal{H}$, then for any $\varepsilon > 0$, there is a $\varphi \in C_c^\infty(G)$ s.t. $\pi(\varphi)$ is self-adjoint and $|\varphi(\rho)f - f| < \varepsilon$.

Moreover, if $f \in \mathcal{H}^k$ which is the decomposition part for $K = SO(2)$, we can assume $\varphi(k_\theta g) = \varphi(g k_\theta) = e^{-ik\theta}\varphi(g)$. In particular if V^k is f.d., we find a φ that $\pi(\varphi)f = f$, by(II.11.4.9).

Proof: Cf[Bump, P171]. □

Lemma (II.11.4.11). Let H be a nonzero closed G-subrepresentation of $L^2(\Gamma\backslash G, \chi)$, then H decomposes as $\oplus_k H^k$ w.r.t. action of $SO(2)$. And if $H^k \neq 0$, then Δ has a nonzero eigenvector in $H^k \cap C^\infty(\Gamma\backslash G, \chi)$.

Proof: The decomposition is clear from(V.6.4.4). It's left to show Δ has an eigenvalue in $H_k \cap C^\infty(\Gamma\backslash G, \chi)$. By lemma(II.11.4.10) above, for $f_0 \in H^k$, there is a $\varphi \in C_c^\infty(G)$ s.t. $\rho(\varphi)f_0 \neq 0$, and $\varphi(k_\theta g) = e^{-ik\theta}\varphi(g)$. So(II.11.4.9) shows $\rho(\varphi)$ maps H into $H_k \cap C^\infty(\Gamma\backslash G, \chi)$ and induces a compact self-adjoint operator on H_k. So we can choose a f.d. eigenspace of it. Notice Δ commutes with the action $\rho(\varphi)$, so Δ fixes this eigenspace, thus it has an eigenvalue on $H_k \cap C^\infty(\Gamma\backslash G, \chi)$. □

Prop. (II.11.4.12) ($L^2(\Gamma\backslash G, \chi)$ Totally Decomposible). The space $L^2(\Gamma\backslash G, \chi)$ decomposes into a Hilbert space direct sum of subspaces that are invariant and irreducible under the right regular action ρ.

Proof: Let Σ be the set of irreducible invariant subspaces of $L^2(\Gamma\backslash G, \chi)$ that is mutually orthogonal. then choose by Zorn's lemma a maximal one, and we prove the orthogonal complement$\mathcal{H} = 0$ otherwise we construct an irreducible subspace of $\mathcal{H}$.

Let $f \neq 0 \in H$, choose by(II.11.4.10) a $\varphi \in C_c^\infty(G)$ that $\rho(\varphi)$ is compact self-adjoint and $\rho(\varphi)f \neq 0$. So $\rho(\varphi)$ has a non-zero eigenvalue and the eigenspace L is of f.d..

Let L_0 be a minimal nonzero subspace of L that is an intersection of L with a nonzero closed invariant subspace of $\mathcal{H}$, and let V be the intersection of all closed invariant subspaces W of $\mathcal{H}$ that $L_0 = L \cap W$. We show V is irreducible, if not, then $V = V_1 \cap V_2$, and if $0 \neq f_0 \in L_0$, then $f_0 = f_1 + f_2$ and both f_1, f_2 are eigenfunctions of $\rho(\varphi)$ of eigenvalue λ. Now if $f_1 \neq 0$, then by minimality, $V_1 \cap L = L_0$. □

Prop. (II.11.4.13). Let σ be the character on K that $\sigma(k_\theta) = e^{-ik\theta}$, $C_c^\infty(K\backslash G/K, \sigma)$ is commutative by(II.11.1.12), let ξ be a character of it. Let $H(\xi)$ be the subspace of $f \in L^2(\Gamma\backslash G, \chi, k)$ that $\pi(\varphi)f = \xi(\varphi)f$ for $\varphi \in C_c^\infty(K\backslash G/K, \sigma)$.

Then $H(\xi)$ are of f.d. and different $H(\xi), H(\eta)$ are orthogonal that $\oplus_\xi H(\xi) = L^2(\Gamma\backslash G, \chi, k)$.

Proof: Suppose $0 \in f \in H(\xi)$, then by(II.11.4.10), we can find a $\varphi \in C_c^\infty(K\backslash G/K, \sigma)$ s.t. $\rho(\varphi)f \neq 0$. And by hypothesis $\rho(\varphi)f = \xi(\varphi)f$, thus $\xi(\varphi) \neq 0$, and f is an eigenvalue of $\rho(\varphi)$ which is compact and self-adjoint, so the $\xi(\varphi)$-eigenspace of $\rho(\varphi)$ is f.d. and $H(\varphi)$ is contained in this space, thus f.d.

To show the orthogonality, choose $\varphi \in C_c^\infty(K\backslash G/K, \sigma)$ that $\xi(\varphi) \neq \eta(\varphi)$. Considering $\varphi = \varphi_1 + i\varphi_2$, where $\rho(\varphi_1), \rho(\varphi_2)$ are both self-adjoint, then we mays assume φ is self-adjoint. Then $H(\xi), H(\eta)$ are contained in different eigenspaces of $\rho(\varphi)$, so they are orthogonal.

Finally for the direct sum, it suffices to show that if f is orthogonal to all $H(\xi)$, then $f = 0$. Given f, let $\varphi_0 \in C_c^\infty(K\backslash G/K, \sigma)$ be chosen that $\rho(\varphi_0)f$ is near f that $\rho(\varphi_0)f, f$ are not orthogonal(II.11.4.10).

Consider the eigenspace decomposition of $\rho(\varphi_0)$ on $L^2(\Gamma\backslash G, \chi, k)$, then $f = f_0 + f_1 + f_2 + \ldots$, then $\rho(\varphi_0)f = \lambda_1 f + \lambda_2 f + \ldots$. Because f is not orthogonal to $\rho(\varphi_0)f$, thus f_i is not orthogonal to

f for some $i \geq 1$. Let V be the λ_i-eigenvalue of $\rho(\varphi_0)$, then V is f.d. and invariant under $\rho(\varphi)$ for all $\varphi \in C_c^\infty(K\backslash G/K, \sigma)$ because $C_c^\infty(K\backslash G/K, \sigma)$ is commutative(II.11.1.12). So V is a direct sum of elements of the spaces $H(\xi)$, so V is orthogonal to f, contradiction. □

Cor. (II.11.4.14). The space $L^2(\Gamma\backslash\mathcal{H}, \chi, k)$ decomposes into a Hilbert space direct sum of eigenspaces for Δ_k.

Proof: By(II.11.4.7), it suffices to prove for $L^2(\Gamma\backslash G, \chi, k)$ and Δ. Because Δ are in the center of $U(\mathfrak{g})$, $H(\xi)$ are all Δ-invariant. So we finish by the proposition, as each $H(\xi)$ is f.d. so Δ is a self-adjoint operator on $H(\xi)$, because $C_c^\infty(\Gamma\backslash G, \chi)$ is dense(II.11.4.3). So it is a direct sum of Δ-eigenspaces. □

Prop. (II.11.4.15).

- The eigenvalues λ_i of Δ_k on $L^2(\Gamma\backslash\mathcal{H}, \chi, k)$ tends to ∞, and satisfies $\sum \lambda_i^2 < \infty$.
- The laplacian Δ_k has an extension to a self-adjoint operator on the Hilbert space $L^2(\Gamma\backslash G, \chi, k)$.

Proof: Cf.[Bump P185]. □

Prop. (II.11.4.16) (Main Theorem). Let $\chi(-1) = (-1)^\varepsilon$, $\varepsilon = 0, 1$. Now(II.11.4.12) that the representation $\mathcal{H} = L^2(\Gamma\backslash G, \chi)$ decomposes into Hilbert space direct sums of irreducible representations, and Δ acts as real scalars on each irreducible subspace, and μ acts by0. So comparing the classification of representations of G(II.11.3.15), we can list what's the representations appearing in it by looking at eigenvalue λ of Δ:

- There is only one f.d. irreducible unitary subrepresentation of G occurring in $\mathcal{H}$, the trivial representation.
- If $\lambda \neq \frac{k}{2}(1 - \frac{k}{2})$, where $\equiv \varepsilon \mod 2$, then this subrepresentation is isomorphic to $P(\lambda, \varepsilon)$. And let $k \equiv \varepsilon \mod 2$ be any integer, then the multiplicity of $P(\lambda, \varepsilon)$ is equal to the multiplicity of the eigenvalue λ of Δ_k on $L^2(\Gamma\backslash\mathcal{H}, \chi, k)$ because $L^2(\Gamma\backslash\mathcal{H}, \chi, k) \cong L^2(\Gamma\backslash G, \chi, k)$ by(II.11.4.7).
- If $\lambda = \frac{k}{2}(1 - \frac{k}{2})$, where $\equiv \varepsilon \mod 2$, then this subrepresentation is isomorphic to either $D^+(k)$ or $D^-(k)$, and $D^\pm(k)$ have the same multiplicity in $\mathcal{H}$, equal to the dimension of the modular forms of weight k $\dim(M_k(\Gamma, \chi))$(II.12.1.8).

Proof: Only the relation with modular forms need proving. By(II.11.3.3), the multiplicity of $D^+(k)$ equals the dimension of the $\frac{k}{2}(1 - \frac{k}{2})$-eigenspace of Δ_k on $L^2(\Gamma\mathcal{H}, \chi, k)$, and any f in this eigenspace is annihilated by L_k by(II.11.2.4)(II.11.2.6). But(II.11.4.8) shows the dimension of space of functions annihilated by L_k equals the dimension of modular forms of weight k. Notice now complex conjugation interchanges $L^2(\Gamma\backslash\mathcal{H}, \chi, k)$ and $L^2(\Gamma\backslash\mathcal{H}, \chi, -k)$ and $\overline{\Delta_k} = \Delta_{-k}$(II.11.2.4), so the multiplicity of $D^+(k)$ and $D^-(k)$ equal. □

II.12 Modular Forms

Main References are [A First Course in Modular Forms Diamond]. This section is a continuation of the discussion of Automorphic Forms inII.11 and use same notations.

1 Modular Curves

Def. (II.12.1.1). Let $\Gamma(N)$ be the inverse image of $1 \in GL(2, \mathbb{Z}/n)$ in the mapping $GL(2, \mathbb{Z}) \to GL(2, \mathbb{Z}/n)$, then a subgroup Γ of $GL(2, \mathbb{Z})$ is called a **congruent subgroup** iff it contains $\Gamma(N)$ for some N.

Lemma (II.12.1.2). The action of $\Gamma(1)$ on $\mathcal{H}$ is properly discontinuous(IV.1.11.11).

Proof: Cf.[Bump P18]. □

Def. (II.12.1.3) (Fundamental Domain). A **fundamental domain** for a subgroup $\Gamma \in SL(2, \mathbb{R})$ acting discontinuously on $\mathcal{H}$ is an open subset $F \in \mathcal{H}$ that:

- $\cup_{\gamma\in\Gamma}\gamma F = \mathcal{H}$.
- if $z_1, z_2 \in F$ and $\gamma(z_1) = z_2$ for some γ, then $\gamma = \pm 1$ and $z_1 = z_2$.

Prop. (II.12.1.4). The subset $F = \{z \in \mathcal{H} || \operatorname{Re}(z)| < 1/2, |z| > 1\}$ is a fundamental domain for $\Gamma(1)$.

Proof: Cf.[Bump, P19]. □

Def. (II.12.1.5) (Elliptic Points).

Def. (II.12.1.6) (Regular Cusps).

Prop. (II.12.1.7) (Riemann Surface for Γ). The quotient space $\Gamma\backslash\mathcal{H}$ can be compactified to a Riemann surface $X(\Gamma)$ by adjoining f.m. points.

Proof: Cf.[Bump P23]. □

Def. (II.12.1.8) (Holomorphic Modular Forms). Let $k > 0$ and $\chi(-1) = (-1)^k$, then the space of **holomorphic modular forms** $M_k(\Gamma, \chi)$ is the space of all holomorphic functions $\mathcal{H} \to \mathbb{C}$ that satisfies

- $f(\gamma z) = \chi(\gamma)(cz + d)^k f(z)$.
- f is holomorphic at the cusps(II.11.1.9).

Moreover we denote by $S_k(\Gamma, \chi)$ the set of **holomorphic cusp forms** consisting of all $f \in M_k(\Gamma, \chi)$ that vanishes at the cusps.

Let $M(\Gamma) = \oplus_{k\geq 0} M_k(\Gamma)$ as the graded algebra of Modular forms for Γ.

Prop. (II.12.1.9). A holomorphic Modular form is just a holomorphic form on $X(\Gamma)$, and its dimension can be calculated.?

Eisenstein Series for $SL(2,\mathbb{Z})$

Def. (II.12.1.10). In this subsubsection, denote $q = e^{2\pi i z}$.

Prop. (II.12.1.11) (Eisenstein Series for $SL(2,\mathbb{Z})$). Let $k > 2$ be an even integer, define the **Eisenstein series of weight k** to be

$$G_k(z) = \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0\}} \frac{1}{(cz+d)^k}, z \in \mathcal{H}$$

Then $G_k \in M_k(\Gamma(1))$, and

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

And denote $E_k(z) = G_k(z)/2\zeta(k)$ the **normalized Eisenstein series**.

Proof: Cf.[Diamond P5]. □

Prop. (II.12.1.12) (Discriminant Function). We have(II.12.1.11):

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad E_6(z) = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n) q^n$$

so we can define the **discriminant function**

$$\Delta(z) = \frac{1}{1728}(E_4^3 - E_6^2) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \ldots$$

Prop. (II.12.1.13). The discriminant function has no zero on $\mathcal{H}$.

Proof: Cf.[Diamond Chap1.4].? □

Cor. (II.12.1.14) (Modular Function). We can define the **modular function** as

$$j : \mathcal{H} \to \mathbb{C}, j(z) = \frac{(E_4)^3}{\Delta(z)}$$

And it subjects onto $\mathbb{C}$.

Proof: Cf.[Diamond Ex1.1.9].? □

Prop. (II.12.1.15).

$$\Delta(z) = \eta(z)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

Proof: ? □

2 Dimension Formulae

Def. (II.12.2.1) (Notations). In this subsubsection, Γ is a congruence subgroup of $SL_2(\mathbb{Z})$, g is the genus of $X(\Gamma)$, ε_2 the number of elliptic points with period 2, ε_3 the number of elliptic points with period 3, and ε_∞ the number of cusps. ε_∞^{reg} the number of regular cusps and ε_∞^{irr} the number of irregular cusps, then

Prop. (II.12.2.2) (Dimension Formulae for k Even). If $k \geq 0$ is even, then

$$\dim(M_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \lfloor\frac{k}{4}\rfloor\varepsilon_2 + \lfloor\frac{k}{3}\rfloor\varepsilon_3 + \frac{k}{2}\varepsilon_\infty & k \geq 2 \\ 1 & k = 1 \end{cases}$$

and

$$\dim(S_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \lfloor\frac{k}{4}\rfloor\varepsilon_2 + \lfloor\frac{k}{3}\rfloor\varepsilon_3 + (\frac{k}{2}-1)\varepsilon_\infty & k \geq 4 \\ g & k = 2 \\ 0 & k = 0 \end{cases}$$

Proof: Cf.[Diamond P87]. □

Cor. (II.12.2.3) (Modular Forms for $SL(2,\mathbb{Z})$).

$$M(\Gamma(1)) = \mathbb{C}[E_4, E_6], \quad S(\Gamma(1)) = \Delta \cdot M(\Gamma(1)).$$

$$\dim(S_k(\Gamma(1))) = \begin{cases} \lfloor\frac{k}{12}\rfloor - 1 & k \equiv 2 \mod 12 \\ \lfloor\frac{k}{12}\rfloor & \text{otherwise} \end{cases}.$$

$$M_k(\Gamma(1)) = S_k(\Gamma(1)) \oplus \mathbb{C}E_k.$$

Proof: Cf.[Diamond P88]. □

Prop. (II.12.2.4) (Dimension Formulae for k Odd). For k odd, if $-I \in \Gamma$, then $M_k(\Gamma) = S_k(\Gamma) = 0$. And if $-I \notin \Gamma$, then

$$\dim(M_k(\Gamma)) = (k-1)(g-1) + \lfloor\frac{k}{3}\rfloor\varepsilon_3 + \frac{k}{2}\varepsilon_\infty^{reg} + \frac{k-1}{2}\varepsilon_\infty^{irr} \quad \text{if } k \geq 3$$

$$\dim(S_k(\Gamma)) = (k-1)(g-1) + \lfloor\frac{k}{3}\rfloor\varepsilon_3 + (\frac{k}{2}-1)\varepsilon_\infty^{reg} + \frac{k-1}{2}\varepsilon_\infty^{irr} \quad \text{if } k \geq 3$$

And for $k = 1$,

$$\dim(M_1(\Gamma)) \begin{cases} = \varepsilon_\infty^{reg}/2 & \varepsilon_\infty^{reg} > 2g-2 \\ \geq \varepsilon_\infty^{reg}/2 & \varepsilon_\infty^{reg} \leq 2g-2 \end{cases}, \quad \dim(S_1(\Gamma)) = \dim(M_1(\Gamma)) - \varepsilon_\infty^{reg}/2.$$

Proof: Cf.[Diamond P91]. □

Dimension Formulae for $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$

Prop. (II.12.2.5) (Elliptic Points).

Prop. (II.12.2.6) (Dimension Formulae). Lists of Dimension Formulae:

Γ	d	ε_2	ε_3	ε_∞
$\Gamma_0(N), N > 2$	$\frac{2d_N}{N\varphi(N)}$			
$\Gamma_1(2)(\Gamma_0(2))$	3	1	0	2
$\Gamma_1(3)$	4	0	1	2
$\Gamma_1(4)$	6	0	0	3

Proof: Cf.[Diamond P107]. □

3 Eisenstein Series

Def. (II.12.3.1). For any Dirichlet character

4 Hecke Algebra

L-functions and Continuation

Def. (II.12.4.1) (L-functions). Each $f \in M_k(\Gamma_1(N))$ has an associated L-functions: if $f = \sum_{n=0}^\infty a_n q^n$, define

$$L(s, f) = \sum_{n=1}^\infty a_n n^{-s}.$$

5 Twisting

6 Modular Curves as Algebraic Curves

7 Eichler-Shimura Relation

8 Fermat's Last Theorem

Thm. (II.12.8.1) (Fermat's Last Theorem). If $l \geq 5$, then there are no integral solution to the equation $a^l + b^l = c^l$.

Proof:

Step 1 : Frey's Curve: $y^2 = x(x-a^l)(x+b^l)$. It defines an elliptic curve, has semistable reduction with discriminant $\Delta = 16(abc)^l$.

The Galois representation: $\rho_{E,l} : G(\mathbb{Q}) \to Aut(E[l]) \cong GL_2(\mathbb{F}_l)$ is unramified.

Step 2: Taniyama-Shimura-Weil Conjecture: E is associated with a cuspidal modular eigenform $f_E(*q) = \sum a_n q^n, a_n \in \mathbb{Q}$ of level $\Gamma_0(N)$ and weight 2.

This is defined by $a_1 = 1$, and $a_p = p + 1 - |E(\mathbb{F}_p)|$ for p not divisible by N.

By EIchler-Shimura, weight 2 cusp eigenform over $\mathbb{Q}$ can always associated to a Galois representation $\rho_f : G(\mathbb{Q}) \to GL_2(\mathbb{Q}_l)$, Weil did the converse.

Step 3**:** Serre+Ribet's Level-Lowering: there should be a cuspidal modular eigenform f' of level $\Gamma_0(2)$ and weight 2. s.t. $a_n(f') \equiv a_n(f) \mod l, \forall (n,N)=1$.

Step 4**:** But there is no cuspidal form of level $\Gamma_0(2)$ and weight 2, because there $X_0(2)=\mathbb{P}^1$, and $H^0(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1})=0$. □

Prop. (II.12.8.2) (Modular Lifting Theorem). The main tool.

Prop. (II.12.8.3) (Weil's Theorem). modular deformation ring T is isomorphic to the Galois deformation ring $R_{\overline{\rho}}$.

II.13 Automorphic Representations(Bump)

Def. (II.13.0.1) (Notations). We will use the notation that F is a Global field, A is the Adele group of F and A^* is the Idele group of F. A_f is the ring of finite Adeles, $F_\infty = \prod_{v\in S_\infty} F_v$, $A = F_\infty A_f$.

1 Tate's thesis (LLC for GL_1)

Main references are [Fourier Analysis in Number Fields and Hecke's Zeta Functions Tate], [Tate's Thesis Poonen] and [Fourier Analysis on Number Fields].

Topology And Measure of Local Field

Remark (II.13.1.1) (Notations). The basis setting is that K is a local field, and we renormalize the valuation as follows:

$$|k| = \begin{cases} |k| & K=\mathbb{R} \\ |k|^2 & K=\mathbb{C} \\ \frac{1}{(N\mathfrak{p})^{v(k)}} & K \text{ non-Archimedean} \end{cases}$$

where $N\mathfrak{p}$ is the number of elements of the residue field of K.

$\mathcal{O}$ is the ring of integers of K.

δ is the differential of K when K is a number field, or $\delta = \{x|\operatorname{tr}(\operatorname{res}(x\mathcal{O}))=0\}^{-1}$ when K is a function field.

Prop. (II.13.1.2) (Canonical Self Duality of Local Fields). If X is a non-trivial character on the additive group K^+, then for any $\eta\in K^+$, $\xi\mapsto X(\eta\xi)$ is also a character, and

$$F:\eta\mapsto(\xi\to X(\eta\xi))$$

is an isomorphism of topological groups of K^+ and $\widehat{K^+}$. In fact, this character X does exist, by the lemma(II.13.1.3) below.

Proof: First this is clearly a homomorphism of groups, and it is injective, because if $X(\eta\xi)=1$ for all ξ, then $\eta k^+\neq k^+$(because X is nontrivial), so $\eta=0$.

Now the image of F is dense, because if $X(\eta\xi)=1$ for all η, then $\xi=0$, so $\overline{\operatorname{Im}(F)}^\perp=1$. Now $(H^\perp)^\perp=H$ for H closed(V.6.3.21) and use Pontryagin duality(V.6.3.18), so $\operatorname{Im} f$ is dense in $\hat{G}$.

Now F is open and continuous, because: for any $B\in G$ compact, there is a nbhd V of 0 that $|X(V)-1|<\varepsilon$, so there is a nbhd V' that $V'B\subset V$, so if $\eta\in V$, $|X(\eta B)-1|<\varepsilon$, so F is continuous($\hat{K^+}$ has the compact-open topology. And if we choose ξ_0 that $X(\xi_0)\neq 1$, then choose $B=\overline{B(0,\frac{|\varepsilon_0}{\varepsilon})}$ compact, if $|X(\eta B)-1|<|X(\xi_0)-1|$, then $\xi_0\notin\eta B$, which means that $|\eta|<\varepsilon$. This means $F(B(0,\varepsilon))$ contains $V(B,|X(\xi_0)-1|$, so F is open.

So the image of F is a locally compact subgroup of $\hat{G}$, so by(V.6.1.6) it is closed, hence equals G as it is dense, so F is surjective, and is an isomorphism. □

Lemma (II.13.1.3) (Canonical Character of Local Fields). Consider the base field k of K(topologically), which is $\mathbb{R},\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ by Ostrowski(V.3.2.19). Now let

$$\lambda(x) = \begin{cases} -x \mod 1 & k=\mathbb{R} \\ \text{a rational number } \lambda(x) \text{ that } \lambda(x)-x\in\mathbb{Z}_p \text{ in } \mathbb{Q}/\mathbb{Z} & k=\mathbb{Q}_p \\ a_{-1}/p=\operatorname{res}(x)/p(\text{not a right definition}\,?) & k=\mathbb{F}_p((t)) \end{cases}$$

Then λ is a continuous additive function on k. Now let $\Lambda(\xi) = \lambda(\mathrm{tr}_{K/k}(\xi))$, and $X(\xi) = e^{2\pi i\Lambda(\xi)}$. Notice that this is just a rigorous definition of the character $e^{2\pi i\,\mathrm{tr}_{K/k}(\xi)}$.

Cor. (II.13.1.4). $F\eta = e^{2\pi i\Lambda(\eta\xi)}$ is trivial on $\mathcal{O}_K$ is equivalent to $\eta \in \delta^{-1}$, the different of K/k. In other words, adopting the isomorphism of(II.13.1.2), $\mathcal{O}^\perp = \delta, \delta^\perp = \mathcal{O}$.

Proof: Because $\Lambda(\eta\mathcal{O}) = 0$ iff $\mathrm{tr}_{K/k}(\eta\mathcal{O}) \subset \mathcal{O}_k$, which is equivalent to $\eta \in \delta^{-1}$. □

Prop. (II.13.1.5). For a Haar measure μ on K^+, $d\mu(\alpha\xi) = |\alpha| d\mu(\xi)$.

Proof: If $K = \mathbb{R}, \mathbb{C}$, this is routine calculation. If K is non-Archimedean, then by the translation invariance of μ, $\mu(\alpha\mathcal{O}) = \frac{\mu(\mathcal{O})}{N(\alpha)} = |\alpha|\mu(\mathcal{O})$. □

Def. (II.13.1.6) (Self-Adjoint Measure on K^+). Now by Fourier inversion(V.6.3.14), a Haar measure $d\mu$ on K^+ corresponds to a Haar measure on $\widehat{K^+}$, but now by the isomorphism $F: K^+ \to \widehat{K^+}$, $d\alpha$ corresponds to a measure $\widetilde{d\alpha}$ on G. Now anyway, there is a unique $d\mu$ that $d\mu = \widetilde{d\alpha}$, and this is called the **self-adjoint measure** on K^+.

In other words, by(V.6.3.14), this is equivalent $f(\eta) = \int_K \widehat{f}(F(\eta))(\xi, F(\eta))d\mu(\eta)$.

Prop. (II.13.1.7). We can calculate the measure on K^+ as follows:

$$d\mu = \begin{cases} dm & K = \mathbb{R} \\ 2dm & K = \mathbb{C} \\ \text{the measure that } N(\mathcal{O}) = \frac{1}{N(\delta)^{1/2}} & \text{others} \end{cases}$$

Proof: We only calculate for the p-adic fields**?**.

Let $f(\eta) = I_{\mathcal{O}}(\xi)$, then

$$\widehat{f}(F(\eta)) = \int_{\mathcal{O}} e^{-2\pi i\Lambda(\xi\eta)} d\mu(\xi) = \begin{cases} \mu(\mathcal{O}) & \eta \in \delta^{-1} \\ 0 & \text{otherwise} \end{cases} = \mu(\mathcal{O}) I_{\delta^{-1}}(\eta)$$

By(II.13.1.4) and(V.6.4.1). So

$$I_{\mathcal{O}}(\xi) = \int_G \widehat{f}(F(\eta))(\xi, F(\eta))d\mu(\eta) = \int_{\delta^{-1}} \mu(\mathcal{O}) e^{2\pi i\Lambda(\eta\xi)} d\mu(\eta) = \mu(\mathcal{O})\mu(\delta^{-1}) I_{\mathcal{O}}(\eta).$$

So $\mu(\mathcal{O})\mu(\delta^{-1}) = N(\delta)\mu(\mathcal{O})^2 = 1$, which shows the desired result. □

Def. (II.13.1.8). The multiplicative group K^* is also locally compact group. For a quasi-character c of K^*, it is called **unramified** iff $c(\alpha) = 1$ whenever $|\alpha| = 1$.

An unramified quasi-character on K^* is all of the form $|\cdot|^s$ for $s \in \mathbb{C}$.

Proof: An unramified quasi-character is equivalent to a continuous group homomorphism from $\mathrm{val}(K^*) \to \mathbb{Z}$. But $\mathrm{val}(K^*)$ must be isomorphic to $\mathbb{Z}$ or $\mathbb{R}$, so the assertion follows from(V.6.3.2). □

Cor. (II.13.1.9) (Quasi-Character of K^*). Let U be the elements of K^* of norm 1, then there is a continuous morphism from K^* to U: $\widetilde{\alpha} = \alpha|\alpha|$ when α is non-Archimedean or $\widetilde{\alpha} = \alpha/|\alpha|$ if K is Archimedean. So any quasi-character c is of the form $c(\alpha) = c(\widetilde{\alpha})$ times an unramified quasi-character, which is of the form $|\cdot|^s$, where $\mathrm{Re}(s)$ is called the **exponent** of c. Now U is a compact group, so continuous quasi-characters $\widetilde{c}$ on it must be a character.

Now for a character $\widetilde{c}$ of U, if K is non-Archimedean, then by continuity, there is a minimum v that $\widetilde{c}(1 + \mathfrak{p}^v) = 1$, and $\mathfrak{p}^v$ is called the **conductor** of $\widetilde{c}$.

Def. (II.13.1.10). Two quasi-character c_1, c_2 is called **equivalent**, denoted by $c_1 \sim c_2$, if $c_1 = c_2|\cdot|^s$ for some s.

It is clear that if K is Archimedean, then every equivalent class is parametrized by $\mathbb{C}$ and if K is non-Archimedean it is parametrized by $\mathbb{C}/\frac{2\pi i}{\log(N\mathfrak{p})}$.

Def. (II.13.1.11) (Conjugate Quasi-Character). For a quasi-character, we can define the **conjugate** of it as $\hat{c}(\alpha) = |\alpha|c(\alpha)^{-1}$.

Notice if $\sigma(c) \in (0,1)$, then $\sigma(\hat{c}) \in (0,1)$, too.

Def. (II.13.1.12) (Haar Measure on K^*). Notice that if $g(\alpha) \in C_c(K^*)$, then $\frac{g(\alpha)}{|\alpha|} \in C_c(K^+ - 0)$, so if we define $\Phi(g) = \int_{k^+-0} g(\xi)|\xi|^{-1}d\xi$, then

$$\Phi(ag) = \int_{k^+-0} g(a\xi)|a\xi|^{-1}|a|d\xi = \int_{k^+-0} g(a\xi)|a\xi|^{-1}da\xi = \Phi(g).$$

By(II.13.1.5). So By Riesz representation, there is a Haar measure that $\int_{k^*} g(\alpha)d_1\alpha = \int_{k^+-0} g(\eta)|\eta|^{-1}d\xi$, for any $g \in C_c(k^*)$.

But when K is non-Archimedean, renormalize $d\alpha = \frac{N\mathfrak{p}}{N\mathfrak{p}-1}d_1\alpha$, to make $|U| = (N\delta)^{-1/2}$. This is because

$$\int_{\mathcal{O}} d\xi = \sum_{k=0}^{\infty}\int_{\pi^k U} d\xi = (1 + \frac{1}{N\mathfrak{p}} + \frac{1}{N\mathfrak{p}^2} + \ldots)\int_U d\xi = \frac{1}{1-\frac{1}{N\mathfrak{p}}}\int_U d\xi$$

so

$$\int_U d_1\alpha = \int_U |\xi|^{-1}d\xi = \frac{N\mathfrak{p}-1}{N\mathfrak{p}}\int_{\mathcal{O}} d\xi = \frac{N\mathfrak{p}-1}{N\mathfrak{p}}(N\mathfrak{p})^{-1/2}(II.13.1.7)$$

Local ζ-function & Functional Equations

Remark (II.13.1.13) (Good Functions). In this section, we work over the set I of **good functions** that satisfy the following properties:

- $f, \hat{f} \in L^1(K^+)$.
- $f(\alpha)|\alpha|^\sigma, \hat{f}(\alpha)|\alpha|^\sigma \in L^1(k^*)$ for $\sigma > 0$.

Clearly I is stable under Fourier transform.

Def. (II.13.1.14) (Zeta Function). The zeta function for f and a quasi-character c of index> 0 is defined to be

$$\zeta(f,c) = \int_{k^*} f(\alpha)c(\alpha)d\alpha$$

Lemma (II.13.1.15). Now as in(II.13.1.10), for an equivalence class of quasi-characters, on the part that has exponent $\sigma > 0$, the function $\zeta(f,c)$ for a fixed f is a holomorphic function.

Proof: First fixed c with exponent $\sigma > 0$, $\zeta(f,c)$ is of the form $\int_{k^*} f(\alpha)\widetilde{c}(\alpha)|\alpha|^s d\alpha$. So to prove that it is holomorphic, use(V.2.3.7) it suffices to show that for any γ near c,

$$\int_\gamma\int_{k^*} f(\alpha)\widetilde{c}(\alpha)|\alpha|^s d\alpha ds = 0.$$

But the integrand converges uniformly because $\sigma > 0$ and the hypothesis of f, so interchanging the integration, it is 0. □

Lemma (II.13.1.16). For any $f, g \in I$,

$$\frac{\zeta(f,c)}{\zeta(\widehat{f},\widehat{c})} = \frac{\zeta(g,c)}{\zeta(\widehat{g},\widehat{c})}.$$

where $\hat{c}$ is the conjugate of c(II.13.1.11).

Proof:

$$\zeta(f,c)\zeta(\widehat{g},\overline{c}) = \int_{k^*} f(\alpha)c(\alpha)\int_{k^*}\widehat{g}(\beta)c^{-1}(\beta)|\beta|d\beta = \int\int f(\alpha)\widehat{g}(\beta)c(\alpha\beta^{-1})|\beta|d\alpha d\beta$$

by Fubini.

$$= \int\int f(\alpha)\widehat{g}(\alpha\beta)c(\beta^{-1})|\alpha\beta|d\alpha d\beta = \int(\int f(\alpha)\widehat{g}(\alpha\beta)|\alpha|d\alpha)|\beta|c(\beta^{-1})d\beta$$

And notice

$$\int_{k^*} f(\alpha)\widehat{g}(\alpha\beta)|\alpha|d\alpha = C\cdot\int_{k^+-0} f(\xi)\widehat{g}(\xi\beta)d\xi = C\cdot\int_{k^+-0}\int f(\xi)\widehat{g}(\xi\beta)d\xi = C\cdot\int\int f(\xi)g(\eta)e^{-2\pi i\Lambda(\xi\beta\eta)}d\eta d\xi$$

which is clearly symmetric in f and g. So the conclusion follows. □

Prop. (II.13.1.17) (Functional Equation in the Local Case). $\zeta(f,c)$ can be extended to a meromorphic function on the whole equivalence class of c, and there is a function $\rho(c)$, meromorphic on the equivalent class of c that $\zeta(f,c) = \rho(c)\zeta(\hat{f},\hat{c})$ for any $f \in I$.

Proof: The strategy is that we only need to calculate explicitly for a good function f and c that $0 < \sigma(c) < 1$ that $\rho(c) = \zeta(f,c)/\zeta(\hat{f},\hat{c})$ extends to a meromorphic function on all equivalent class of c, then the lemma(II.13.1.16) shows $\rho(c)$ is independent of f, so $\zeta(f,c) = \rho(c)\zeta(\hat{f},\hat{c})$ holds for any $f \in I$.

For the calculation of ρ, Cf.[Tate Thesis P316]**?**. □

Prop. (II.13.1.18).

- $\rho(\widehat{c}) = c(-1)/\rho(c)$
- $\rho(\overline{c}) = c(-1)\overline{\rho(c)}$

Proof: 1:

$$\zeta(f,c) = \rho(c)\zeta(\widehat{f},\widehat{c}) = \rho(c)\rho(\widehat{c})\zeta(\widehat{\widehat{f}},c) = \rho(c)\rho(\widehat{c})c(-1)\zeta(f,c)$$

2:

$$\zeta(f,c) = \rho(c)\zeta(\widehat{f},\widehat{c}), \quad \overline{\zeta(f,c)} = \zeta(\overline{f},\overline{c}) = \rho(\overline{c})\zeta(\widehat{\overline{f}},\widehat{\overline{c}})$$

And

$$\widehat{\overline{f}}(\xi) = \int\overline{f}(\eta)e^{-2\pi i\Lambda(\xi\eta)}d\eta = \overline{\int f(\eta)e^{2\pi i\Lambda(\xi\eta)}d\eta} = \overline{\widehat{f}}(-\xi)$$

so

$$\rho(\overline{c})\zeta(\widehat{\overline{f}},\widehat{\overline{c}}) = \rho(\overline{c})c(-1)\zeta(\overline{\widehat{f}},\overline{\widehat{c}}) = \rho(\overline{c})c(-1)\overline{\zeta(\widehat{f},\widehat{c})}$$

Thus $\rho(\overline{c}) = c(-1)\overline{\rho(c)}$. □

Cor. (II.13.1.19). Because when $\sigma(c) = \frac{1}{2}$, $\widehat{c} = \overline{c}$(II.13.1.11), we have $|\rho(c)| = 1$ in this case.

Topology and Measure of Adele and Idele

Def. (II.13.1.20) (Hecke Character). A **Hecke character** is a continuous character of A^*/F^*.

Def. (II.13.1.21).

Riemann-Roch

Analytic Continuation and Functional Equation of ζ-Functions

Hecke L-Functions

2 Representations of $GL(n)$

Prop. (II.13.2.1). $GL(n, A)$ is unimordular.

Proof: Cf.[Bump Ex3.3.3]. □

Prop. (II.13.2.2). The quotient space $GL(n, F)Z(A)\backslash GL(n, A)$ has finite measure.

Proof: Cf.[Bump P295]. □

Tensor Product Theorem

Prop. (II.13.2.3) (Tensor Product Theorem). Let (V, π) be an irreducible admissible representation of $GL(n, A)$, then there exists for each Archimedean place v of F an irreducible admissible $(\mathfrak{g}_\infty, K_v)$-module (π_v, V_v), and for each non-Archimedean place v an irreducible admissible representation (π_v, V_v) of $GL(n, F_v)$ that for a.e. v, V_v contains a non-zero K_v0fixed vector ξ_v^0, and π is the restricted tensor product of the representations π_v.

Proof: Cf.[Bump Chap3.4]. □

3 Local Langlands Correspondence

The basic object of LLC are the Weil group and its representations.
A representation ρ of W_K is called F**-semisimple** iff $\rho(\mathrm{Frob})$ is diagonalizable.

Thm. (II.13.3.1) (LLC for GL_n). The set of
irreducible smooth, admissible representations of $GL_n(K)$
corresponds to
n-dimensional F-semisimple Weil-Deligne representations of W_K.

Cor. (II.13.3.2) (LLC for GL_1).
Local class field theory told us that W_K^{ab} is isometric to K^*, And notice by Schur's lemma, any smooth representation of K^* is 1-dimensional and factors through some U_k.

And a Weil-Deligne representation is now a continuous $W_K^{ab} \to C^*$. but it must factor through some U_K, so these two are equivalent.

most l-adic representation of G_K comes from étale cohomology.

LLC for $GL_2(\mathbb{C})$

4 Global Langlands Correspondence

Chapter III

Algebraic Geometry

III.1 Sites

References are [StackProject].

1 Sites

Def. (III.1.1.1). A **site** is given by a category $\mathcal{C}$ and a set $Cov(\mathcal{C})$ of families of morphisms with fixed target, called the **coverings** of $\mathcal{C}$ that:

- An isomorphism is a covering.
- Coverings of covering is a covering.
- Base change of a covering is a covering.

Sometimes A site is called called a topology, the difference is that the morphism of site is reverse of a morphism of topology.

Def. (III.1.1.2). A **morphism of topologies** $\mathcal{D} \to \mathcal{C}$ is a morphism that preserves covering and base change by covering morphisms. A **morphism of sites** $\mathcal{C} \to \mathcal{D}$ is a morphism u of topologies $\mathcal{D} \to \mathcal{C}$ that u_s(III.1.2.6) is exact.

This exact condition is easy to be satisfied, by(III.1.2.9).

Def. (III.1.1.3). A G**-topological space** is a set X with a family of subsets of X that they form a Grothendieck topology w.r.t inclusions and that covering are all set-theoretic coverings(but not necessarily conversely). These subsets are called **admissible opens** of X and covers are called **admissible covers**. (In another words, a G-topological space is a "topological space without unions").

Def. (III.1.1.4) (Completeness). The completeness of a G-topological space X:

- G0: $\varnothing$ and X are admissible open.
- G1: Let $\{U_i \to U\}$ be an admissible cover, then a subset $V \subset U$ is admissible if $V \cap U_i$ are all admissible.
- G2: Let $\{U_i \to U\}$ be a cover of admissible opens for U admissible, then the cover is admissible if it has an admissible cover as a refinement.

Lemma (III.1.1.5) (Admissible is Local). If G_2 is satisfied for a G-topological space X, then for an admissible covering $\{X_i \to X\}$ and another covering $\{U_i \to X\}$ between admissible opens, it is admissible iff $U_i \cap X_j$ is an admissible covering for X_j for each j. (By composition, $\{U_i \cap X_i \to X\}$ is admissible, and it refines $\{U_i \to X\}$).

Prop. (III.1.1.6) (Glue of Complete G-topological spaces). For sets $\cup X_i = X$, if there are Grothendieck category $\mathcal{I}_i$ on X_i making X_i into a G-topological space, and they all satisfies the completeness conditions G_0, G_1, G_2 of(III.1.1.4). Assume that $X_i \cap X_j$ is $\mathcal{I}_i$-open in X_i and $\mathcal{I}_i, \mathcal{I}_j$ restrict to the same topology on $X_i \cap X_j$, then there is a unique Grothendieck category $\mathcal{I}$ on X making X a G-topological space that:

- X_i is $\mathcal{I}$-open and $\mathcal{I}$ restricts to $\mathcal{I}_i$ on X_i,
- $\mathcal{I}$ satisfies the completeness conditions G_0, G_1, G_2.
- X_i is a $\mathcal{I}$-covering of X.

Proof: By(III.1.1.4) and(III.1.1.5), the uniqueness is straightforward, for the existence,

- check Grothendieck first: Composition, base change.
- check condition 1: by hypothesis, and(III.1.1.4) applied to $X_i \cap X_j \to X_i$(this is admissible because id_{X_i} refined it).
- check condition 2: G_0 obvious, G_1 by if $V \cap U_i \cap X_i$ admissible, then $V \cap X_i$ admissible by admissibility of $U_i \to U$, then V is admissible, G_2: obvious
- check condition 3: because $X_i \cap X_j \to X_i$ is admissible.

□

Def. (III.1.1.7). A G-topological space is called **connected** iff there isn't two nonempty admissible open subset X_1, X_2 that $X_1 \cap X_2 = \varnothing$ and $\{X_1, X_2 \to X\}$ is an admissible cover.

Def. (III.1.1.8). An object U in a site is called **quasi-compact** if for each covering of U, f.m. of them still forms a covering of U. The topology T is called **Noetherian** if each object of T is quasi-compact.

Given a site T, we can define a new site T^f whose coverings are coverings of T that are finite. Then this is truly a site and it is Noetherian.

Topoi

Def. (III.1.1.9). A **topos** is the category of sheaves over a site $\mathcal{C}$. For sites $\mathcal{C}, \mathcal{D}$, a morphism of topoi consists of two natural adjoint morphism $f_* : Sch(\mathcal{C}) \to Sch(\mathcal{D})$ and $f^{-1} : Sch(\mathcal{D}) \to Sch(\mathcal{C})$ that f_* is right adjoint to f^{-1} and f^{-1} is exact.

2 Sheaves on Sites

Def. (III.1.2.1) (Effective Epimorphisms). An epimorphism $\{U_i \to V\}$ in a category is called a **family of effective epimorphisms** if

$$\mathrm{Hom}(V, Z) \to \prod \mathrm{Hom}(U_i, Z) \rightrightarrows \prod \mathrm{Hom}(U_i \times_V U_j, Z)$$

is exact for each Z. Similarly for a **family of universal effective epimorphisms**.

Prop. (III.1.2.2). The set of all families of universal effective epimorphisms in a category forms a Grothendieck topology, called the **canonical topology**. It is the finest topology that all representable presheaves are sheaves.

Topologies that are coarser than the canonical topology are called **subcanonial topology**.

Proof: We only need to verify that family of universal effective epimorphisms is closed under composition. For this, first prove epimorphism, then use epimorphism to prove effectiveness. Universal follows routinely. Cf.[Tamme]. □

Prop. (III.1.2.3). For a subcanonical topology on C, its restriction on a localizing category C/S is subcanonical.

Proof: The only nontrivial part is that the glued morphism is a morphism over S. For this, consider its composition that maps to S, then the uniqueness of the exact sequence(III.1.2.1) will show that it is truly a S-morphism. □

Prop. (III.1.2.4) (Sheafification). The operator F^+ is the presheaf that

$$F^+(U)=\varinjlim \mathrm{Ker}(\prod_i F(U_i)\rightrightarrows\prod_{i,j}F(U_i\times_U U_j))=\check{H}^0(U,F)$$

It is a separated presheaf, i.e. $0\to F(U)\to\prod_i F(U_i)$ and when F is separated, $F\to F^+$ is injective and F^+ is a sheaf. (The problem of separated is that the cover may not be identical in $U_i\times_U U_j$ but only on a cover of it).

The sheafification F^{++} is exact and it is left adjoint to the forgetful functor.

So the forgetful functor is left exact and it preserves injectives. Thus the sheaf cokernel is the shifification of the presheaf kernel, the sheaf kernel is the presheaf kernel.

Proof: The separatedness is simple. For sheaf condition, an element of $F^+(U_i)$ is represented by a covering $\{V_{ij}\to U_i\}$, and there restriction to $U_i\times_U U_j$ coincide by separatedness hence the covering $\{V_{ij}\to U\}$ is an element of $F^+(U)$.

Sh is left exact because $-^+$ is left exact from PAb to PAb by(III.5.2.2) checked on every element U. It is right exact trivially, hence it is exact. □

transfer of sheaves under morphisms

Def. (III.1.2.5) (Pullback & Pushforward of Presheaves). Given a morphism of topologies $f:T\to T'$, which should be regarded as an inverse map. There are maps

$$f^pF'(U)=F'(f(U)):\mathcal{P}'\to\mathcal{P},\quad f_p(F)(U')=\varinjlim_{U_i|U'\to f(U_i)}F(U_i):\mathcal{P}\to\mathcal{P}'$$

Then f_p is left adjoint to f^p, and f^p is exact, checked easily.

Def. (III.1.2.6) (Pullback & Pushforward of Sheaves). Given a morphism of topologies $f:T\to T'$(which should be regarded as an inverse map), there are maps

$$f^s=f^p\circ\iota:\mathcal{S}'\to\mathcal{S},\quad f_s=\sharp\circ f_p\circ\iota:\mathcal{S}\to\mathcal{S}'.$$

f_s is left adjoint to f^s, by adjointness of f_p,f^p and $\sharp,\iota$.

This is dual to the case of usual topology space.

Prop. (III.1.2.7). If a presheaf F on T is represented by $Z\in T$, then f_pF is represented by $f(Z)\in T'$.

Proof: Cf.[Tamme P44]. Use the adjointness of f_p, f^p(III.1.2.5), then

$$\mathrm{Hom}(f_p h_Z, F) = \mathrm{Hom}(h_Z, f^p F) = f^p(F)(Z) = F(f(Z))$$

thus we are done by Yoneda lemma. □

Prop. (III.1.2.8). $(f_p)^\sharp(G) \cong (f_s G)^\sharp$ for any presheaf G.

Proof: It is easy to prove $\mathrm{Hom}((f_p)^\sharp(G), F) = \mathrm{Hom}((f_s G)^\sharp, F)$ for any sheaf F by adjointness. □

Prop. (III.1.2.9) (When is f_s Exact). If $f : T \to T'$ is a morphism of topologies that has final objects and finite fiber products and f respects final objects and finite fiber produces, then $f_s : \mathcal{S} \to S'$ is exact.

Proof: It suffices to show the left exactness of f_p. By definition, $f_p(U') = \varinjlim_{\mathcal{I}_{U'}^{op}} F_{U'}$, where $\mathcal{I}_{U'}$ is the category of all (U, φ) that $U' \to F(U)$ and $F_{U'}$ is the covariant functor $(U, \varphi) \to F(U)$. Now T has fiber products and products, then $\mathcal{I}_{U'}^{op}$ is seen to be cofiltered, so this limit process is exact form $\mathrm{Hom}(\mathcal{I}_{U'}^{op}, \mathcal{A}b)$ to $\mathcal{A}b$ and $F \to F_{U'}$ is clearly exact. □

Prop. (III.1.2.10). For a topology T and an object Z of T, there is a category T/Z as objects over T, and $i : T/Z \to T$ is continuous. Then i^s is exact.

Proof: $R^q i^s(F) = (i^p(\mathcal{H}^q(F)))^\sharp$(III.5.2.15), and $(\mathcal{H}^q(F))^\sharp = 0$(III.5.2.11), so it suffices to show i^p and $\sharp$ commutes. But i^s and $^+$ commutes obviously. □

Prop. (III.1.2.11) (Sheaf Condition is Local). To check sheaf condition for presheaf w.r.t. a topology, it suffice to show that for any covering, there is a refinement covering of it that sheaf condition hold, because by the definition of sheafification functor, $F^+ = F$, so F is a sheaf.

Cor. (III.1.2.12). For two topology on a same category that $\mathcal{I}'$ is finer than $\mathcal{I}$, then any $\mathcal{I}'$-sheaf is a $\mathcal{I}$-sheaf. And if any covering in $\mathcal{I}'$ can be refined by a covering in $\mathcal{I}$, then $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories. In particular, if T is Noetherian, $\mathcal{S}(T)$ and $\mathcal{S}(T^f)$(III.1.1.8) are equivalent.

Prop. (III.1.2.13) (Comparison lemma). Let T' be a fully subcategory of T, $i : T' \to T$ is a morphism of topologies, if each object U of T' and a covering $\{U_i \to U\}$ in T has a refinement $\{U'_j \to U\}$ in T', and each object U of T has a covering $\{U_i \to U\}$ with objects $U_i \in T'$, then i^s, i_s forms an equivalence between sheaves on T and sheaves on T'. $i_s G$ is called the **extension of sheaf**.

Proof: $i^s i_s G(U) = (i_p G)^\sharp(U)$ for $U \in T'$, we show $G(U) \cong i_p(G)(U) \cong (i_p G)^\sharp(U)$. For the first, by definition of i_p, and use the fact (U, id_U) is the initial object. For the second, notice the colimit definition of $^+$, then $(i_p G)^+(U) \cong i_p G(U) = G(U)$.

Now i^s is exact, because $^+$ commutes with i^p for presheaves on T, and $R^q i^s(F) = (i^p \mathcal{H}^q(F))^\sharp$ and $(\mathcal{H}(F))^\sharp = 0$(III.5.2.11).

To prove $i_s i^s F \cong F$, notice $i_s i^s F \cong (i_p i^s F)^\sharp$, and there are commutative diagram $i_p i^s F(U) \to (i_p i^s F)^\sharp(U) \to F(U)$. If we let $U \in T'$, then the first part of the proof applies and shows $i_p i^s \mathcal{F} \cong (i_p i^s F)^\sharp(U)$, and $F(U) = i^s F(U) \cong i_p i^s F(U)$, so $i_s i^s F(U) \cong F(U)$.

Now for any $U \in T$, we can choose a covering $\{U_i \to U\}$ with U_i in T', and then choose a covering $\{U_{ij}^k \to U_i \times_U U_j\}$, then there is a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & i_s i^s F(U) & \longrightarrow & \prod_i i_s i^s F(U_i) & \longrightarrow & \prod_{ijk} i_s i^s F(U_{ij}^k) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(U) & \longrightarrow & \prod_i F(U_i) & \longrightarrow & \prod_{ijk} F(U_{ij}^k) \end{array}$$

The last two vertical map are isomorphisms, so we can use five lemma. □

Cor. (III.1.2.14) (Extending Sheaf). Let T' be a fully subcategory of T, $i : T' \to T$ is a morphism of topologies, and each object U of T' and a covering $\{U_i \to U\}$ in T has a refinement $\{U'_j \to U\}$ in T'. Then $G \cong i^s i_s G$ for any sheaf G on T' and $i^s : \mathcal{S} \to \mathcal{S}'$ is exact. (This is implicit in the proof of the last proposition).

In particular, this applies to the case $T' = T/Z \to T$, the localization category, in with case $i^s F(Z') = F(Z')$ is called the **restriction sheaf**.

3 Modules on Ringed Sites

Basic References are [StackProject Chap 18]

4 Sheaves on Topological Spaces

Prop. (III.1.4.1) (Stalks). Taking stalks is a left adjoint to the skyscraper sheaf from $\mathcal{A}b$ to Ab thus preserves cokernel, moreover it is exact.

Epimorphism and monomorphism can be checked on stalks, so also can be checked on affine opens. Cf.[Hartshorne P63].

Prop. (III.1.4.2). If a sheaf has only one non-vanishing stalk, then it is a skyscraper stalk. (Because the restriction to that point for every open set is an isomorphism).

Def. (III.1.4.3) (Grothendieck's Six Operators).
- the **pushforward** $f_p F$, $f_p F(U) = F(f^{-1}(U))$ sends presheaf to presheaf.
- the **direct image** $f_*\mathcal{F}$, $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ sends sheaf to sheaf.
- the **inverse image** $f^{-1}\mathcal{F}$, $f^{-1}\mathcal{F}(U) = \mathcal{F}(f(U))^\sharp$ sends sheaf to sheaf.
- the **direct image with compact support**:(III.5.5.15).
- the **exceptional inverse image**(special case) $i^!$ for an inclusion of closed subset is $i^!(F)(U) = V \cap Y) = \{s \in \Gamma(V, X) | \operatorname{Supp}(s) \in Y\}$.

Proof: Check that $f_!$ is a sheaf: it is separated clearly, it suffices to show that for a covering $\cup U_i = W$ and $\xi_i \in F(f^{-1}(U_i))$, the section $\xi \in F(f^{-1}(W))$ they generated by sheaf property of F is in $f_! F(W)$. For a compact subset K, there is a finite cover $\cup_i U_i$ of it, thus $K - \cup_{i \neq j} U_j$ is compact in U_j, thus its inverse image is compact in $\operatorname{Supp}(\xi)$. there are f.m. U_j, thus the inverse image of K is compact in $\operatorname{Supp}(\xi)$. □

Prop. (III.1.4.4).
- f^{-1} is left adjoint to f_* by(III.2.1.5) because Ab are just $\mathbb{Z}$-modules. And f^{-1} is exact(**??**.

- $f_!$ is left exact when X, Y are locally compact. And $j_!$ is left adjoint to the functor f^{-1} for an inclusion of open subset $j : U \subset X$.
- $i^!$ is right adjoint to f_* for an inclusion of closed subset $i : Y \to X$, in particular f_* is exact when f is a closed inclusion.

Proof: 2: (III.5.5.15). □

5 Sites over Schemes

Prop. (III.1.5.1). Fiber products exist in the category of schemes.

Proof: Cf.[Hartshorne P88]. You should use glueing(III.1.6.10). □

Zariski Topology

Def. (III.1.5.2). The **Zariski topology** has the the covering of a scheme T as classes of open immersions $\{T_i \to T\}$ that their images cover T.

The **big Zariski site** Sch_{Zar}/S has the objects as all schemes over S.

The **small Zariski site** S_{Zar} has the objects as all open subschemes over S.

The **restricted Zariski site** S_{Zarfp} has the objects as all schemes that are qcqs open subschemes of S.

The **big affine Zariski site** Aff_{Zar}/S has the objects as all schemes affine over S.

These are all topologies because open immersions satisfies base change trick(III.3.4.41).

In particular when the cover has only one element and is affine, the descent datum is equivalent to compatible isomorphisms

$$\varphi_{13} : N \otimes_A B \otimes_A B \xrightarrow{\varphi_{12}} B \otimes_A M \otimes_B \xrightarrow{\varphi_{23}} B \otimes_A B \otimes_A M.$$

Prop. (III.1.5.3). A sheaf w.r.t the small Zariski topology is equivalent to a sheaf on S, trivially, so the sheaf cohomology on Aff_{Zar}/S is equivalent to usual sheaf cohomology on S.

Prop. (III.1.5.4). If X is qs, then $\widetilde{X_{Zar}} \to \widetilde{X_{Zarfp}}$ is an equivalence by i_s and i^s, the same proof as(III.1.5.9).

Étale Topology

Def. (III.1.5.5). The **étale topology** has the covering of a scheme T as classes of étale morphisms that their images cover T.

The **big étale site** $Sch_{étale}/S$ has the objects as all schemes over S.

The **small étale site** $S_{étale}$ has the objects as all schemes that are étale over S.

The **restricted étale site** $S_{étfp}$ has the objects as all schemes that are étale and qcqs over S.

The **big affine étale site** $Aff_{étale}/S$ has the objects as all schemes affine over S.

These are truly topologies because étale is stable under base change and composition.

Prop. (III.1.5.6). Zariski covering is étale, because open immersions are étale.

Prop. (III.1.5.7). An étale covering of a qc scheme can be refined a finite affine étale covering, this is because étale map are open(III.4.4.3). Thus so does all above coverings.

Prop. (III.1.5.8). The restricted étale site of a qc scheme X is Noetherian, because étale map is open, and any object in $X_{\acute{e}tfp}$ is qc.

Prop. (III.1.5.9). If X is qs, then $\widetilde{X_{\acute{e}t}} \to \widetilde{X_{\acute{e}tfp}}$ is an equivalence by i_s and i^s.

Proof: Want to use(III.1.2.13), one condition is satisfied by(III.1.5.7), so it suffice to check any étale scheme X'/X has a covering of qcqs étale schemes over X. For any point $p \in X'$, there is an affine nbhd U' that maps to an affine nbhd U' of X and the ring map is f.p., so $U' \to U$ is étale and f.p, and $U \to X$ is open immersion and qs, it is qc because X is qs and U is qc(III.3.4.27). □

Prop. (III.1.5.10) (Cohomology Big and Small Sites). The inclusion of small sites to the big sites has no infection on the sheaf cohomology, by(III.5.2.19). This is applicable to all topologies τ considered here.

Prop. (III.1.5.11) (Topological Invariance of Étale Site). If $S' \to S$ is universally homeomorphism, then $f^*, f_* : S'_{\acute{e}t} \to S_{\acute{e}t}$ induces an equivalence of categories. Especially for the case of reduced structure.

Proof: Cf.[Étale Cohomology Conrad P18]. □

Smooth Topology

This topology will be shown to be identical to the Étale topology, so it is not so important.

Syntomic Topology

Def. (III.1.5.12). The **syntomic topology** has the covering of a scheme T as classes of syntomic morphisms that their images cover T.

fppf Topology

Def. (III.1.5.13). The **fppf topology** has the covering of a scheme T as classes of flat locally of finite presentation morphisms that their images cover T. (f.f.+locally of f.p.).

The **big Zariski site** Sch_{fppf}/S has the objects as all schemes over S.

The **big affine Zariski site** Aff_{fppf}/S has the objects as all schemes affine over S.

They are all topologies because flatness and finite presentation satisfies base change trick by(III.4.1.2) and(III.4.7.3).

Prop. (III.1.5.14). A syntomic covering is fppf by definition(I.7.3.1).

Prop. (III.1.5.15). A fppf covering of an affine scheme can be refined a finite affine fppf covering, because fppf map are open(III.4.1.7).

fpqc Topology

Def. (III.1.5.16). The **fpqc topology** has the covering of a scheme T as classes of flat morphisms s.t. their images cover T and for any affine open $U \subset T$, the restriction on T can be refined by a finite affine cover of open affine subschemes of the covering(f.f.+qc). It is a topology by(III.4.1.2) and(III.3.4.26).

When the covering consists of affine schemes, it is called **standard fpqc covering**.

Prop. (III.1.5.17). Fppf coverings are fpqc.

Proof: Use(III.4.1.7), we see that fppf covering consists of open morphisms, thus it is qc because affine scheme is quasi-compact. □

Prop. (III.1.5.18). A covering consisting of flat morphisms refined by a fpqc covering is a fpqc covering.

Hence being fpqc is local on the target, because a Zariski cover is a fpqc covering.

If U is a covering consisting of flat morphisms that there is a fpqc covering V that $U \times V \to V$ is a fpqc covering, then U is fpqc, because $U \times V$ does and it refines U.

Remark (III.1.5.19). Defining fpqc sites has inescapable set-theoretic difficulties, thus we don't consider fpqc sites and fpqc cohomologies. Cf.[StackProject 0BBK].

Prop. (III.1.5.20) (Checking Sheaf Condition). A presheaf is a sheaf w.r.t the fpqc topology iff it is a sheaf w.r.t the Zariski topology and satisfies sheaf property w.r.t the single covering $V \to U$ f.f. between affine schemes.

Proof: For any covering $\{X_i \to X\}$, choose an affine open cover U_i of X, then by definition, the pullback cover on U_i can use refined by a finite affine cover $U_{ik} \to U_i$, so The composition covering of $\{U_{ij} \to U_i\}$ and $\{U_i \to X\}$ refines $\{X_i \to X\}$. And sheaf condition for $\{U_{ij} \to U_i\}$ is the same as sheaf condition for $\{\coprod U_{ij} \to U_i\}$. Thus the result follows from (III.1.2.11). □

Prop. (III.1.5.21) (fpqc is Universally Effective). The coverings in X_{fpqc} are families of universal effective epimorphisms, in the category of X-schemes.

Proof: By(III.1.5.20), it suffices to show that any representable presheaf is a sheaf w.r.t Zariski topology and f.f. affine morphisms. The Zariski case follows from(III.1.6.10), for the second, $\operatorname{Spec} B \to \operatorname{Spec} Z$, for any scheme X, the morphism corresponds to $0 \to \operatorname{Hom}(R, A) \to \operatorname{Hom}(R, B) \to Hom(R, B \otimes_A B)$, but this follows immediately from(III.1.6.3), with $M = A$. □

Cor. (III.1.5.22). For $f : Y \to X$ a morphism of schemes, if $Z \in X_\tau$ for the above topologies τ, then $f^*(\operatorname{Hom}_X(-, Z)) \cong \operatorname{Hom}(-, Z \otimes_X Y)$, in other words, the inverse sheaf of a representable sheaf is representable.

Proof: By definition, $f^*(\operatorname{Hom}_X(-, Z))$ is the sheaf associated to the presheaf $f_p(\operatorname{Hom}_X(-, Z))$, which by(III.1.2.7) is just the presheaf represented by $Z \otimes_X Y$, but by the proposition, it is already a sheaf. □

Prop. (III.1.5.23) (Sheaf of $\mathcal{O}_X$-modules). Let M be a Qco sheaf of $\mathcal{O}_X$-modules, then the functor $X' \to \Gamma(X', M \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})$ is an Abelian sheaf on X_{fpqc}, denoted by $\widetilde{M}$. To verify this, use(III.1.5.20), and it suffice to check sheaf condition for f.f. morphisms, and the result follows from fpqc lemma(III.1.6.3).

6 Descent

Basic References are [StackProject Chap34, 10.158].

General Principal

Prop. (III.1.6.1). A property of schemes is called **local** in a topology if for any covering $\{U_i \to S\}$, S has P iff U_i has P. A property of morphisms is called **local** in a topology if for any covering $\{U_i \to S\}$, $X \to S$ has P iff $X \times_S U_i \to U_i$ has P.

For these, we only need to show that it is local in the Zariski topology and check for a standard topology(i.e. affine covering of affine scheme). This is because any covering of an affine scheme can be refined by an affine covering(III.1.5.16) and(III.1.5.17).

Def. (III.1.6.2). A **descent datum** for qco sheaves for a covering in a site is just a family of local Qco sheaves that satisfies the cocycle condition on their intersections.

fpqc Descent

Prop. (III.1.6.3) (fpqc-Poincare Lemma). If a ring map $A \to B$, either has a section $B \to A$, or it is faithfully flat, then the Amitsur complex $s(M)$ for the canonical descent datum (with augmentation):

$$0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M \to \dots$$

with Čech-like maps, is exact.

Proof: Nullhomotopy can be easily constructed, Cf.[Sheaf Cohomology notes P23], the f.f. case can be reduced to the first case by tensoring B to consider$B \to B \otimes_A B$, because it has a section. □

Lemma (III.1.6.4) (Affine fpqc Descent). When $A \to B$ is f.f., there is an equivalence of categories:

$$\{M \in A - mod\} \leftrightarrow \{(N, \varphi) \text{ descent datum}\}$$

giving by $M \to B \otimes_A M$ with the canonical descent datum. and $M \to B \otimes_A H^0(s(N))$.

Proof: Cf.[StackProject 023N]. □

Remark (III.1.6.5). In fact, a descent datum is always effective iff $A \to B$ is universally injective. Cf.[StackProject]. And f.f. extension is u.i.(I.7.1.19).

Prop. (III.1.6.6) (fpqc Descent). If S is a scheme and $\{U_i \to S\}$ is a fpqc covering, then the category of descent datum for the covering is equivalent to the category of qco sheaves on S.

Proof: Faithfulness: if a, b are two morphism of Qco sheaves over S, then for $s \in S$, $s = \varphi_i(u)$ for some $\varphi_i : U_i \to S$, and φ_i is flat, so the stalk map is f.f. by(I.7.1.14), so $a = b$ on every stalk of S.

Fully faithfulness: if we have a morphism $\{\varphi_i\}$ of descent datums of qco sheaves, then for any affine subset V of S, we get a morphism of the pullback descent datum. Then by(III.1.6.4) above, we get a morphism on V. These morphism are compatible on their intersection by the faithfulness just proved, so they gives a morphism on S.

Essentially surjectivity: for a descent datum, pull it back to any affine subscheme V_i of S, then there is a qco sheaf on V_i by (III.1.6.4) above, and there is a canonical isomorphism of their restriction on the intersection, by fully faithfulness just proved, so it gives a qco sheaf on S by Zariski descent(III.1.6.10) and the fact qco is local by(III.2.3.3). □

Cor. (III.1.6.7). For any Qco sheaf $\mathcal{F}$ on S, the functor $(Sch/S)^{op} \to Ab : T \to \Gamma(T, f^*\mathcal{F})$ is a sheaf in the fpqc topology, hence also in the fppf, étale Zariski topology.

Prop. (III.1.6.8). For any Qco sheaf $\mathcal{F}$ on a separated scheme X. If T is a Grothendieck topology on Sch/S containing the Zariski topology and every cover is refined by a fpqc cover by a finite collection of affine schemes, then $H^p(T, X, \mathcal{F}) = H^p(X, \mathcal{F})$. Same as the proof of(III.5.4.2), with the Zariski-Poincare lemma replaced by the fpqc-Poincare lemma.

Prop. (III.1.6.9) (fpqc Descent for morphisms). For a faithfully flat morphism f that is qc, the following property holds for a morphism iff it holds for its base change along f.

1. isomorphism/monomorphism.
2. (quasi-)separated.
3. quai-compact.
4. (locally)of f.t.
5. (locally)of f.p.
6. proper
7. (quasi-)affine.
8. (quasi-)finite.
9. flat.
10. smooth, unramified, étale.
11. (closed/open)immersion.

Proof: Cf.[EGAIV-2, Proposition 2.7.1]. and [StackProject 34.20]. □

Galois Descent

Galois descent is a special case of fpqc descent.

Zariski Descent

Lemma (III.1.6.10) (Zariski Descent). Any descent datum for the Zariski topology is effective. In fact, Qco is no needed. In the same way, we can glue schemes and also morphisms with a fixed target (compatible with the glueing).

Proof: For every open set $V \subset X$, we define the group of sections $\mathcal{F}(V)$ to be a set consisting of all tuples $(s_i)_{i\in I}$ required to obey the compatibility condition:

$$\phi_{ij}(s_i|_{V\cap U_i\cap U_j}) = s_j|_{V\cap U_i\cap U_j} \qquad (*)$$

for all $i, j \in I$. The group addition on $\mathcal{F}(V)$ is the obvious one.

The $\mathcal{F}$ that I defined is guaranteed to be a sheaf, but we also need to satisfy ourselves that the restriction $\mathcal{F}|_{U_k}$ really is isomorphic to the $\mathcal{F}_k$ that we started with, for each $k \in I$. It is here that the cocycle condition is required.

It is easy to write down what the isomorphism $\psi : \mathcal{F}_k \overset{\cong}{\to} \mathcal{F}|_{U_k}$ ought to be. Given an open $V \subset U_k$ and given a section $s \in \mathcal{F}_k$, we would like to define its image under ψ to be

$$\psi(s) = (\phi_{ki}(s|_{V\cap U_i}))_{i\in I}$$

However, we need to be sure that the tuple $(\phi_{ki}(s|_{V\cap U_i}))_{i\in I}$ represents a well-defined element of $\mathcal{F}(V)$. In particular, we must verify that $(\phi_{ki}(s|_{V\cap U_i}))_{i\in I}$ obeys the condition $(*)$, which states that

$$\phi_{ij} \circ \phi_{ki}(s|_{V\cap U_i\cap U_j}) = \phi_{kj}(s|_{V\cap U_i\cap U_j})$$

for any $i, j \in I$. This is true by virtue of the cocycle condition.

This map is obviously injection and it is surjection by virtue of $(*)$. □

7 Stack

III.2 Schemes

1 Ringed Spaces & $\mathcal{O}_X$-Modules

Def. (III.2.1.1). A **ringed space** X is a topological space together with a sheaf of rings $\mathcal{O}_X$. There morphisms are topological maps and a reverse ring map. A $\mathcal{O}_X$**-module** is an Abelian sheaf with a ring module structure compatible with restriction maps. A ringed space is called **local ringed space** iff kts stalks are all local rings.

Prop. (III.2.1.2). Glueing sheaves is available for ringed spaces, similar to(III.1.6.10).

Transfer of Modules

Def. (III.2.1.3).

- the direct image modules: $f_*\mathcal{F}$, $f_*\mathcal{F}(U)=\mathcal{F}(f^{-1}(U))$ sends $\mathcal{O}_X$-module to $\mathcal{O}_Y$-module.
- the pullback of modules:$f^*(\mathcal{F})=f^{-1}\mathcal{F}\otimes_{f^{-1}\mathcal{O}_Y}\mathcal{O}_X$.
- The **tensor** of two modules is the sheaf associated to the tensor of two presheaves.
- for a closed immersion $Y\to X$, there is $i^!:Qco(X)\to Qco(Y)$ that is right adjoint to i_*: $i^!\mathcal{G}=i^*((\mathcal{H}_Z(\mathcal{G}))')$, where $\mathcal{H}_Z(\mathcal{G})$ is the sheaf of sections annihilated by $\mathcal{I}$ and $\mathcal{F}'$ is the maximal Qco sheaf of $\mathcal{F}$.
- For f proper between locally Noetherian scheme, there is a inverse sheaf $f^!\mathcal{G}=\mathcal{H}om_Y(f_*\mathcal{O}_X,\mathcal{G})$, which maps $Qco(Y)$ to $Qco(X)$ by(III.2.3.6) and(III.5.4.11). When f is affine, in particular when it is finite, then $f^!$ is right adjoint to f_* on Qco(III.5.4.12).

Prop. (III.2.1.4). Tensoring is strongly left adjoint to $\mathcal{H}om(\mathcal{F},-)$:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}\otimes\mathcal{F},\mathcal{G})\cong\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{H},\mathcal{G})).$$

(Recall the definition of tensor sheaf).

Prop. (III.2.1.5). f^* is left adjoint to f_* by(I.3.4.5): $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F})\cong\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G},f_*\mathcal{F})$. In fact

$$f_*(\mathcal{H}om(f^*\mathcal{G},\mathcal{F}))=\mathcal{H}om(\mathcal{G},f_*\mathcal{G}).$$

Cor. (III.2.1.6). The f^* may not be exact. f^{-1} is exact, but we tensored with $\mathcal{O}_X$, it is exact when f is flat.

Prop. (III.2.1.7). Tensor commutes with pullbacks, in particular with taking stalks. So tensoring with a locally free sheaf is exact.

Proof: We have

$$\mathrm{Hom}(f^*\mathcal{F}\otimes f^*\mathcal{G},\mathcal{H})=\mathrm{Hom}(\mathcal{F},f_*\,\mathcal{H}om(f^*\mathcal{G},\mathcal{H}))=\mathrm{Hom}(\mathcal{F},\mathcal{H}om(\mathcal{G},f_*\mathcal{H}))=\mathrm{Hom}(f^*(\mathcal{F}\otimes\mathcal{G}),\mathcal{H}).$$

□

Prop. (III.2.1.8). On a ringed space X, for a qc open subset U, $(\oplus\mathcal{F}_i)(U)=\oplus\mathcal{F}_i(U)$. This uses the compactness of U.

Prop. (III.2.1.9). For a closed immersion f, f_* on $\mathcal{O}_X$-mod is fully faithful, with image those killed by $\mathcal{I}$, where $\mathcal{I}$ is the structural kernel, Cf.[StackProject 08KS].

Modules of Finite Type & Finite Presentation

Def. (III.2.1.10). An $\mathcal{O}_X$-module is called of **finite type** iff locally it is a quotient of a finite free sheaf.

Prop. (III.2.1.11). if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}_X$-modules and $\mathcal{F}_1, \mathcal{F}_3$ are of f.t., then $\mathcal{F}_2$ is of finite type.

Proof: Choose a finite generators of $\mathcal{F}_3$ on some nbhd, and shrink it to get inverse image of them in $\mathcal{F}_2$, and choose a smaller nbhd that $\mathcal{F}_1$ is f.g., then on this nbhd, $\mathcal{F}_2$ is f.g. □

Prop. (III.2.1.12). If $\mathcal{G} \to \mathcal{F}$ is surjective on the stalk for a x and $\mathcal{F}$ is of f.t., then it is surjective on a nbhd of x. Thus the support of a f.t. sheaf is closed (look at $0 \to \mathcal{F}$).

Def. (III.2.1.13). A sheaf of module $\mathcal{F}$ is called **of finite presentation** iff locally it is a cokernel of finite free modules. A finitely presented sheaf of modules is Qco, and the pullback of a f.p. sheaf is f.p, by the right adjointness of f^*.

Prop. (III.2.1.14). If $\mathcal{G} \to \mathcal{F}$ is a surjection and $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is of f.t., then the kernel is of finite type. (Use a restriction to smaller nbhd technique, this has the same proof as**??**).

(Quasi-)Coherent Sheaves

Def. (III.2.1.15). A sheaf of module $\mathcal{F}$ is called **quasi-coherent** iff locally it is a cokernel of free modules.

Prop. (III.2.1.16). The pullback of a qco module is a qco, because f^* is right adjoint.

And for a ringed space $(X, \mathcal{O}_X)$ and a $R = \Gamma(X, \mathcal{O}_X)$-module M, we have a coherent sheaf $\mathcal{F}_M$ on X, defined as $\pi^*(M)$, where M is seen as a qco sheaf on (pt, R). It is the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U) \otimes M$.

This construction is a functor from the category of R-module to the category of Qco $\mathcal{O}_X$-modules, and it commutes with colimits because f^* does. And it is left adjoint to Γ by(III.2.1.5):

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{G})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{G})$$

Def. (III.2.1.17) (Coherent Sheaf). On a ringed space X, a **coherent sheaf** is a $\mathcal{O}_X$-module that is of f.t. and on any open set U and for any set of elements of $\Gamma(U, \mathcal{F})$, the kernel of $\oplus \mathcal{O}_U \to \mathcal{F}|_U$ is of f.t.. A coherent sheaf is of finite presentation, by a restriction to nbhd technique, and it is also Qco.

Prop. (III.2.1.18). Any f.t. subsheaf of a coherent sheaf is coherent, by definition. Any kernel of a f.t. sheaf to a coherent sheaf is of f.t.

The category of coherent sheaves is a weak Serre subcategory of $\mathcal{O}_X$-modules. Thus if $\mathcal{O}_X$ is coherent, then a sheaf is coherent iff it is f.p.

Proof: Cf.[StackProject 01BY]. □

Cor. (III.2.1.19). If $\mathcal{G} \to \mathcal{F}$ is injective on some point x, $\mathcal{G}$ is of f.t. and $\mathcal{F}$ is coherent, then it is injective on a nbhd of x.

Proof: By the proposition, the kernel is of f.t., so its support is closed by(III.2.1.12). □

2 Spec and Schemes

Def. (III.2.2.1). The category of schemes is a fully faithful category of the category of ringed spaces that locally isomorphic to Spec A.

Prop. (III.2.2.2). On Spec(A), $\mathcal{O}(D(f)) = A_f$. Cf.[Hartshorne P71]. We can also define it this way and check the sheaf condition.

Cor. (III.2.2.3). For an qcqs scheme X and a Qco module $\mathcal{F}$, $(\Gamma(X, \mathcal{F}))_s \cong \Gamma(X_s, \mathcal{F})$.

Proof: This is the canonical map $f : X \to \operatorname{Spec}\Gamma(X)$ is qcqs, (Notice qc is local on the target). Then $f_*\mathcal{F}$ is Qco on $\operatorname{Spec}\Gamma(X)$ thus the result. □

Prop. (III.2.2.4) (Scheme is Sober). The underlying space of a scheme is sober.

Proof: First prove this for affine scheme, notice that closed irreducible subsets correspond to prime ideal. Then notice the generic point for $Z \cap U$ is the generic point for Z. □

Prop. (III.2.2.5). The closure of a subset T of Spec$(A) = V(\cap p, p \in T)$.

Prop. (III.2.2.6) (Affine Scheme). The Spec operator from $CRing^*$ to Scheme is right adjoint to $X \to \Gamma(X, \mathcal{O}_X)$,

$$\operatorname{Hom}_{Sch}(X, \operatorname{Spec}(A)) \cong \operatorname{Hom}_{Ring}(A, \Gamma(X, \mathcal{O}_X)).$$

Notice the category of schemes is a full subcategory of the category of locally ringed spaces.

Proof: First prove this for $X = \operatorname{Spec}(B)$. Cf.[Hartshorne P73]. Then choose affine cover of X and glue them ($\mathcal{H}om$ is a sheaf). Should notice this is the special case of Global spec with $S = \operatorname{Spec}(\mathbb{Z})$. □

Prop. (III.2.2.7) (Global Spec). There is a S-scheme $f : \mathbf{Spec}_S\,\mathcal{A} \to S$ for every Qco sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ on S that $f^{-1}(U) \cong \operatorname{Spec}\mathcal{A}(U)$. This construction is right adjoint to the direct image map:

$$\operatorname{Hom}_{\mathcal{O}_S-alg}(\mathcal{A}, \pi_*\mathcal{O}_X) \cong \operatorname{Hom}_{Sch/S}(X, \mathbf{Spec}_S\,\mathcal{A}).$$

and defines an equivalence of affine morphisms over S and Qco $\mathcal{O}_S$-algebras. Moreover, this defines an equivalence of the category of $\mathcal{A}$-modules and the category of $\mathcal{O}_{\mathbf{Spec}\,\mathcal{A}}$-modules.

Proof: It suffices to prove for affine opens in S and glue. For this, use the adjointness of $\tilde{}$ and Γ and adjointness for Spec. □

Dimensions

Prop. (III.2.2.8). For any scheme, $\dim\mathcal{O}_x = \operatorname{codim}(\overline{\{x\}}, X)$.

Prop. (III.2.2.9). For an integral scheme of finite type over a field,

$$\dim X = \dim\mathcal{O}_{p,X} = \dim U = tr.\text{deg } K(X)/k$$

for any closed point p and any open subscheme U. (Use closed point are dense(III.3.4.30) and k is universal catenary to prove it is true for some U and all the closed point in it, so other U's because X is irreducible).

Lemma (III.2.2.10). For a Noetherian local ring $(A, \mathfrak{m})$, $\operatorname{Spec} A - \mathfrak{m}$ is affine iff $\dim A \leq 1$.

Proof: if $\dim A = 0$, this is true, if $\dim A = 1$, let $f \in m$ not in any other minimal primes of A, then $\operatorname{Spec} A - m = \operatorname{Spec} A_f$.

Conversely, Cf.[StackProject 0BCR]. □

Prop. (III.2.2.11). Let X be a locally Noetherian scheme, if $U \subset X$ is an open subscheme that $U \to X$ is affine, then every irreducible complements of $X - U$ has codimension≤ 1. And if U is dense, then equality must hold.

Proof: Cf.[StackProject 0BCU]. □

Associated Points

Basic References are [StackProject Chap30].

Def. (III.2.2.12). For a scheme X and a Qco sheaf $\mathcal{F}$ on X, a point is called **associated to** $\mathcal{F}$ iff $\mathfrak{m}_x$ is associated to $\mathcal{F}_x$, which is equivalent to $\mathfrak{m}_x$ are all zero-divisors in M by(I.5.3.9). When $\mathcal{F} = \mathcal{O}_X$, x is called an **associated point of** X.

Prop. (III.2.2.13). If X is locally Noetherian, then an associated prime is equivalent to it is an associated prime of $\Gamma(X, \mathcal{O}_X)$ of $\Gamma(U, \mathcal{F})$ for a nbhd U of x.

Proof: Cf.[StackProject 02OK]. □

Prop. (III.2.2.14). Same results of associated points are parallel to the discussion of associated primes:

- relations of $Ass(\mathcal{F})$ w.r.t exact sequences(I.5.3.6).
- $Ass(\mathcal{F}) \subset \operatorname{Supp}(\mathcal{F})$(I.5.3.7).
- When X is locally Noetherian and $\mathcal{F}$ is coherent, for a quasi-compact open set U of X, the number of associated points in U is finite(I.5.3.7).
- When X is locally Noetherian, $\mathcal{F} = 0$ iff $Ass(\mathcal{F})$ is empty(I.5.3.7).
- When X is locally Noetherian, If $Ass(\mathcal{F}) \subset$ an open subset U, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is injective(I.5.3.12).
- If X is locally Noetherian, then the minimal elements(under specialization) of $\operatorname{Supp}(\mathcal{F})$ are associated points of $\mathcal{F}$. in particular, any generic point of an irreducible component of X is an associated points of X.
- If X is locally Noetherian, then if a map $\varphi : \mathcal{F} \to \mathcal{G}$ that is injective at all the stalks of $Ass(\mathcal{F})$, then φ is injective.

3 (Quasi-)Coherent Sheaves on Schemes

Lemma (III.2.3.1). On an affine scheme $\operatorname{Spec} A$, there is a sheaf $\widetilde{M}$, that is M_f on $\operatorname{Spec} A_f$. To check it is a sheaf, we only need to check to affine coverings, and this is by(III.1.6.3).

Prop. (III.2.3.2) (Quasi-Coherent Sheaves). For any A-module M, there is a sheaf of modules $\mathcal{F}_M$ on $\operatorname{Spec} A$ by(III.2.1.16). This is left adjoint to Γ and defines a functor from the category of A-modules to the category of $\mathcal{O}_{\operatorname{Spec} A}$-modules.

And in fact, this is an equivalence to the category of quasi-coherent sheaves over Spec A because Qco is locally like $\widetilde{M_i}$, by the fact that localization is exact, and(III.1.6.4) shows that locally of the form $\widetilde{M_i}$ must be globally of the form $\widetilde{M}$ and $\Gamma(X, \widetilde{M}) = M$.

This is also an equivalence between f.g. A-modules and coherent sheaves over Spec A, by fpqc descent(III.1.6.4).

Def. (III.2.3.3) (Coherent Sheaves). When X is a locally Noetherian scheme, coherence is equivalent to M_is are f.g. A_i-module. When talking about coherent sheaves over schemes, I tacitly assume the scheme is locally Noetherian.

(Quasi-)coherent is an affine local by(III.2.3.2) and(I.5.1.28).

Prop. (III.2.3.4).

- $(Q)co(X)$ forms a weak Serre subcategory of Mod-$\mathcal{O}_X$.
- Tensor product of two (Q)co sheaf is (Q)co, and locally free if they are locally free (because $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ as tensor product commutes with pullbacks)
- pullback of (qco)coherent sheaves are (qco)coherent. (Local on the affine opens, check $f^*(\widetilde{M}) \cong \widetilde{M \otimes_A B}$. Note in the coherent case, both scheme should be locally Noetherian.

Proof: For the weak Serre subcategory, Coherent case is by(III.2.1.18). For Qco, just need to verify the kernel, cokernel and extension, by the fact that localization is exact. For the extension of Qco, use(III.5.4.3) that the global section is exact, so there is a morphism of exact sequences $\widetilde{\Gamma(X, \mathcal{F})} \to \mathcal{F}$, and five lemma gives the result. □

Prop. (III.2.3.5). If f is qcqs, then the pushforward of of a Qco sheaf is Qco. (Used in(III.5.3.7)).

Proof: The question is local so we let Y be affine, and then X is qcqs, so we cover it with affine opens U_i and their intersections are U_{ijk}. Then we see by sheaf property

$$0 \to f_*\mathcal{F} \to \oplus_i f_*(\mathcal{F}|_{U_i}) \to \oplus_{ijk} f_*(\mathcal{F}|_{U_{ijk}})$$

The last two are Qco because two are maps between affine schemes, so the first is Qco. □

Prop. (III.2.3.6). f_* for f proper maps coherent sheaf to a coherent sheaf. (directly from(III.5.4.27)).

Prop. (III.2.3.7) (Extensions of Coherent Sheaves). On a locally Noetherian scheme, any Qco sheaf is sum of coherent sheaves, so any coherent sheaf on an open subset can be extended to a global coherent sheaf.

Proof: First prove for affine opens, this is true, then we extend by Zorn lemma. The last is because for any section $s \in \Gamma(U)$, we can extend it to a global section of the pushforward sheaf. □

Prop. (III.2.3.8) (Deligne). On a Noetherian scheme X, let $\mathcal{F}$ be a Qco sheaf, $\mathcal{G}$ be a coherent sheaf and $\mathcal{I}$ be a Qco sheaf of ideals correspongding to Z, $U = X - Z$, then we have

$$\varinjlim \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular,

$$\varinjlim \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \cong \Gamma(U, \mathcal{F}).$$

Proof: Cf.[StackProject 01YB]. □

Prop. (III.2.3.9) (Kleinmann). If X is a Noetherian integral separated locally factorial scheme, then every coherent sheaf on X is a quotient of a finite locally free sheaf.

Proof: Cf.[Hartshorne P238]. □

Prop. (III.2.3.10) (Support of Modules). The support of a Qco sheaf of f.t over a scheme is closed(because the support of a single section is closed in every affine open), e.g. coherent sheaf.

This have many consequences applied to kernel and cokernel, for example, a coherent sheaf is locally free iff all its stalk is free (choose a presentation and see kernel and cokernel).

For a flat morphism f, $\mathrm{Supp}(f^*(\mathcal{F})) = f^{-1}(\mathrm{Supp}\,\mathcal{F})$, by(I.5.1.25).

Proof: because for a set of generators x_i of M, $Ann(\mathcal{F}) = \cup Ann(x_i)$, and $Ann(x_i)$ is closed. □

Cor. (III.2.3.11) (Semicontinuity). For a Qco sheaf $\mathcal{F}$ of f.t., $\varphi(y) = \dim_{k(y)(\mathcal{F}\otimes k(y))}$ is an upper semicontinuous function on the scheme.

Proof: By Nakayama, $\varphi(y)$ is equal to the minimal number of generators of the $\mathcal{O}_y$-module $\mathcal{F}_y$. But these generators extends to a nbhd of y, so $\varphi \leq n$ on this nbhd. □

Prop. (III.2.3.12). For X a scheme and any $\mathcal{O}_X$-module $\mathcal{F}$, there is a Qco submodule of $\mathcal{F}$ maximal among all Qco submodules of $\mathcal{F}$. This is because the colimit of Qco sheaves are Qco.

Prop. (III.2.3.13). A f.t. Qco sheaf on a scheme has a minimal closed scheme structure on its support, it is generated locally by the Qco ideal $\mathrm{Ann}_A(M)$(III.3.4.42). And there is a f.t. Qco sheaf $\mathcal{G}$ that $i_*(\mathcal{G}) = \mathcal{F}$.

Proof: Cf.[StackProject 01QY]. □

Devissage of Coherent Sheaves

Lemma (III.2.3.14). Let $\mathcal{F}$ be a coherent sheaf on a Noetherian scheme X, let I be a sheaf of ideals that correspond to Z, then $\mathrm{Supp}(\mathcal{F}) \subset Z$ iff $\mathcal{I}^n\mathcal{F} = 0$ for some n. (This follows easily from Noetherian and(I.5.3.5)).

Lemma (III.2.3.15). If we have a coherent sheaf $\mathcal{F}$ on a Noetherian scheme X, that $\mathrm{Supp}(\mathcal{F}) = Z_1 \cup Z_2$, then we have an exact sequence of coherent sheaves $0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$ that $\mathrm{Supp}(\mathcal{G}_i) \subset Z_i$.

Proof: Let I be the reduced ideal sheaf of Z_1, we use the exact sequence $0 \to \mathcal{I}^n\mathcal{F} \to \mathcal{F} \to \mathrm{Coker} \to 0$, by(III.2.3.14), we can choose n that $\mathrm{Supp}(\mathcal{I}^n\mathcal{F}) \subset Z_2$, thus the result. □

Prop. (III.2.3.16). Let $\mathcal{F}$ be a coherent sheaf on a Noetherian scheme X, then there is a filtration of coherent sheaves that the quotients are pushforward of ideal sheaves on integral subschemes of X. This is analogous to the filtration in the module case.

Proof: We consider the set of these counterexamples and their Supp, then use Noetherian induction, the minimal one if not irreducible, then from(III.2.3.15) we find a filtration for it. Then let the ideal of sheaf be $\mathcal{I}$, then $\mathcal{I}^n\mathcal{F} = 0$, then we should use [StackProject 01YE] to finish to proof. Cf.[StackProject 01YF]. □

Prop. (III.2.3.17). Let P be a property of coherent sheaves on X Noetherian that

- for an exact sequence of sheaves:$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$, if $\mathcal{F}_i$ has P, then $\mathcal{F}$ has P.
- If $\mathcal{F}^{\oplus r}$ has P, then $\mathcal{F}$ has P.
- For every integral closed subscheme Z of X with generic point ξ, there is a coherent sheaf $\mathcal{G}$ that
  1. $\operatorname{Supp}\mathcal{G} \subset Z$.
  2. $\mathcal{G}_\xi$ is annihilated by m_ξ.
  3. For every sheaf of ideal $\mathcal{I}$ on X that $\mathcal{I}_\xi = \mathcal{O}_{X,\xi}$, there is a sheaf $\mathcal{G}' \subset \mathcal{I}\mathcal{G}$ that $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and has P.

Then we have P holds for every coherent sheaf on X.

Proof: Use Noetherian induction, the minimal counterexample should have Supp irreducible by(III.2.3.15) and then we use[StackProject 01YL]. Note this has nothing to do with reducedness. □

4 Projective Space

Def. (III.2.4.1) (Projective Scheme). For a graded ring S, we have a scheme $\mathrm{Proj}(S)$ that consists of homogenous primes of S minus S_+ and the affine cover is $D(f) = \{p|f \not\subseteq p\}$, and $\mathcal{O}(D(f)) = \operatorname{Spec} S_{(f)}$, where $S_{(T)}$ is the degree zero part of $T^{-1}S$. It has $\mathcal{O}_p = S_{(p)}$.

Proof: Define the sheaf using stalks, then we only have to check that $\operatorname{Spec} S_{(f)} \cong$ homogenous $p \in S_f$ by natural intersection of ideals φ. and $S_{(p)} \cong (S_{(f)})_{\varphi(p)}$ for $p \in D(f)$.

We check that for $S_{(f)} \subset S_f$, $p \to p \cap S_{(f)}$ and $p' \to pS$ is natural and inverse to each other. $S_{(f)} \to S_{(p)}$ maps $\varphi(p)$ to invertible, and any $x/a \in S_{(p)}$ can be written as $\frac{xa^{\deg f-1}/f^{\deg a}}{a^{\deg f}/f^{\deg a}}$. □

Prop. (III.2.4.2).

$$\mathrm{Proj}^n_{\mathbb{Z}} \times \operatorname{Spec} A = \mathrm{Proj}^n_A.$$

(Choose the canonical affine open sets to see).

Prop. (III.2.4.3). For two graded ring with the same $S_0 = A$, $\mathrm{Proj}(S \times_A T) \cong X \times_A Y$, where $(S \times_A T)_n = S_n \times_A T_n$ (natural morphism from left to right).

Prop. (III.2.4.4). For a graded S-module, there is a Qco-sheaf $\widetilde{M}$ on $\mathrm{Proj}S$, that $\widetilde{M}_p = M_{(p)}$ and $\widetilde{M}|_{D^+(f)} \cong \widetilde{M_{(f)}}$. the construction is as in(III.2.4.1).

Def. (III.2.4.5) (Relative Proj). The **relative Proj** $\mathcal{S}$ over locally Noetherian Y of a Qco graded $\mathcal{O}_Y$-algebra $\mathcal{S}$ f.g. over $\mathcal{S}_0$ by coherent $\mathcal{S}_1$ is the glueing of locally $\mathrm{Proj}\, S$. $\mathrm{Proj}\,\mathcal{S} \to Y$ is locally projective thus proper. It is equipped with invertible sheaf $\mathcal{O}(1)$ by glueing.

Prop. (III.2.4.6) (Closed Subscheme of Projective Scheme). The closed scheme of $X = \mathbb{P}^n_A$ corresponds to the saturated homogenous ideal $\mathcal{I}_Y$, (i.e. for any s, if there is an n that for any $i, x_i^n s \in \mathcal{I}_Y$, then $s \in I_Y$).

So projective scheme over $\operatorname{Spec} S_0$ corresponds to $\mathrm{Proj}\, S$, where S are f.g. over S_0 by S_1 saturated in the sense above.

Proof: A closed immersion is proper, thus the kernel $\mathcal{I}_Y$ of the structural map is a Qco(III.2.3.4), so it must be an ideal on every affine open, because Qco is affine local. Then we should use(III.2.5.5), $\Gamma_*(\mathcal{I}_Y)$ will suffice. Cf.[Hartshorne Ex2.5.10]. □

Prop. (III.2.4.7). The global section of a projective space $\operatorname{Proj} S \to \operatorname{Spec} S_0$ is just S_0, this is by(III.2.5.5).

Prop. (III.2.4.8). A quasi-projective scheme X over a field k of dimension r can be covered by $r+1$ open affine subsets. This is because there are r hyperplane that intersect X non-empty. This can happen by choosing a hyperplane non-intersecting the generic point of X, otherwise we choose many hyperplane, then their intersection is empty.

Serre Twisting

Def. (III.2.4.9). Define $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) = \mathbb{Z}[\widetilde{X_0, \ldots, X_n}](1)$, this is an invertible sheaf. The invertible **Serre twisting sheaf** $\mathcal{O}(1)$ on $\mathbb{P}^r_Y$ is the pullback of that of $\mathbb{P}^n_{\mathbb{Z}}$ and an invertible **Serre twisting sheaf** of the relative $X = \operatorname{Proj} \mathcal{S}$ over Y is locally the pullback of that of $\mathbb{P}^r_Y$. Giving a Serre twisting sheaf of X over Y, the **Serre twisting sheaf** of $\mathcal{F}$ over X is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Prop. (III.2.4.10). For X projective over $\operatorname{Spec}(A)$, (i.e. $X = \operatorname{Proj}(S)$(III.2.4.6)), $\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ and many other properties involving the Serre twisting, all this boil down to the fact that $(M \otimes_S N)_{(f)} = M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ for $f \in S_1$.

and by virtue of(III.2.4.4), when $X = \operatorname{Proj}(S)$ projective, we have:

- $\widetilde{M}(n) \cong \widetilde{M(n)}$.
- For a graded ring map $S \to T$, we have the corresponding Proj map $f : U \to T$ that $f^*(\widetilde{M}) \cong (\widetilde{M \otimes_S T})|_U$ and $f_*(\widetilde{N}|_U) \cong \widetilde{N_S}$. That's to say, $f^*(\widetilde{M}(n)) = f^*(\widetilde{M})(n)$ and $f_*(\widetilde{M}(n)) = f_*(\widetilde{M})(n)$.

Cor. (III.2.4.11). $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$ for any scheme X projective over Y.

Prop. (III.2.4.12) (Twisting of Proj). With notation as in (III.2.4.5), Let $S' = S * \mathcal{L} : S'_d = S_d \otimes \mathcal{L}^d$, then $\varphi : \operatorname{Proj} S' \to \operatorname{Proj} S$ is an isomorphism that induces

$$\mathcal{O}'(1) \cong \varphi^* \mathcal{O}(1) \otimes \pi'^* \mathcal{L}.$$

Prop. (III.2.4.13). If Y is Noetherian and admits an ample invertible sheaf, then by definition, we have $S_1 \otimes \mathcal{L}^n$ is base point free for some n, thus we have a morphism $\operatorname{Proj} S * \mathcal{L}^n \to \mathbb{P}^N_Y$, so $P = \operatorname{Proj} S$ is H-quasi-projective with $\mathcal{O}(1) = \mathcal{O}_P(1) \otimes \pi^* \mathcal{L}^n$.

Locally Free sheaves

Prop. (III.2.4.14). Pullback and pushforward of locally free sheaves are locally free.

Prop. (III.2.4.15). For a finite locally free sheaf $\mathcal{E}$ on X,

- $\mathcal{E}^{\vee\vee} \cong \mathcal{E}$.
- $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$.
- $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes \mathcal{H} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$ if $\mathcal{F}$ or $\mathcal{H}$ is finite locally free.

- $\mathcal{H}om(\mathcal{E}\otimes\mathcal{F},\mathcal{G})\cong\mathcal{H}om(\mathcal{F},\mathcal{E}^\vee\otimes\mathcal{G})$, by the first and(III.2.1.4).

Proof: We define the map, and verify locally. □

Prop. (III.2.4.16) (Wedge Product). For a exact sequence of locally free sheaves: $0\to F'\to F\to F''\to 0$,

$$\wedge\mathcal{F}'\otimes\wedge\mathcal{F}''\cong\wedge\mathcal{F}.$$

Let $\mathcal{F}$ be a locally free sheaf of rank n, then there is a perfect pairing $\wedge^r\mathcal{F}\otimes\wedge^{n-r}\mathcal{F}\to\wedge\mathcal{F}$ which is a perfect pairing.

Prop. (III.2.4.17). For a exact sequence of locally free sheaves:$0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{L}\to 0$, where L is a line bundle, there is an exact sequence

$$0\to\wedge^r(\mathcal{F}')\to\wedge^r(\mathcal{F})\to\wedge^{r-1}(\mathcal{F}')\otimes\mathcal{L}\to 0$$

This is a special case of [Hartshorne Ex2.5.16c].

Proof: □

Prop. (III.2.4.18). The map $(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))_x\to\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x)$ is an isomorphism in the case when $\mathcal{F}$ is a locally free or when it is of finite presentation, by(I.6.7.6).

Def. (III.2.4.19). A locally free module on schemes can induce a **symmetric vector bundle** $S(\mathcal{E})$, and the section sheaf recovers $E^\vee$. This defines a reverse equivalence of locally free sheaves and vector bundles on X.

When $\mathcal{E}$ is Qco, we can define the **associated projective space bundle** $\mathbb{P}(\mathcal{E})$ as $\mathrm{Proj}\,S(\mathcal{E})$. It is equipped with a Serre twisting sheaf $\mathcal{O}(1)$, which is the glue of locally the Serre sheaf in projective space. There is a surjective morphism $\pi^*(\mathcal{E})\to\mathcal{O}(1)$ (local check).

Prop. (III.2.4.20). Let $g:Y\to X$ by a scheme over X, a morphism $Y\to\mathbb{P}(\mathcal{E})$ over X is equivalent to an invertible sheaf $\mathcal{L}$ on Y and a surjective map $g^*\mathcal{E}\to\mathcal{L}$.

In particular, giving a morphism $X\to\mathbb{P}^n_A$ is equivalent to a base point free invertible sheaf with n generators on X.

Proof: If there is a morphism, it will pullback $\pi^*(\mathcal{E})\to\mathcal{O}(1)$ into $g^*\mathcal{E}\to\mathcal{L}$. For the coverse, construct locally and glue, we have the natural morphisms $A[x_1/x_i,\ldots,x_n/x_i]\to\mathcal{O}_{X_{s_i}}:x_j/x_i\to s_j/s_i$ in a homogenous sense. It is natural hence glue together. For the module, maps $x_i\to s_i$. □

Cor. (III.2.4.21). All automorphisms of $\mathbb{P}^n_k$ is linear.

Proof: The Picard group of $\mathbb{P}^n_k$ is $\mathbb{Z}$ and is generated by $\mathcal{O}(1)$(III.6.1.17), so the automorphism will map $\mathcal{O}(1)$ to $\mathcal{O}(\pm1)$ and $\mathcal{O}(-1)$ has no global section(III.2.5.4). And the globals section is n-dimensional and determines the morphism by the prop. □

Vector Bundles

Def. (III.2.4.22). A **vector bundle** is a locally free sheaf. A vector bundle $\mathcal{E}$ is called a subbundle of another vector bundle $\mathcal{F}$ iff $\mathcal{F}/\mathcal{E}$ is locally free.

5 Invertible Sheaves

Def. (III.2.5.1). An invertible sheaf on a ringed space is a sheaf that $\mathcal{L}\otimes -$ is an equivalence of categories. A locally free sheaf of rank 1 is inveritble and when X is local ringed space, the converse is also true.

Proof: Cf.[StackProject 0B8M]. □

Prop. (III.2.5.2). For any ringed space X, the $\mathrm{Pic}(X)\cong H^1(Z,\mathcal{O}_X^*)$. This is by choosing a locally trivial opens of X and the first Čech cohomology equals sheaf cohomology(III.5.2.11).

Prop. (III.2.5.3). Giving a morphism $X\to\mathbb{P}^n_A$ is essentially equivalent to a base point free invertible sheaf with n generators on X. This follows from(III.2.4.20).

Prop. (III.2.5.4) (Global Section). Let $\mathcal{L}$ be an invertible sheaf over qcqs scheme X, for a Qco module $\mathcal{F}$ let the **global section functor** $\Gamma_*(\mathcal{F})=\oplus\Gamma(X,\mathcal{F}\otimes\mathcal{L}^n)$, then

$$\Gamma_*(\mathcal{F})_{(f)}\cong\mathcal{F}(X_f).$$

where $s\in\Gamma(X,\mathcal{L})$. In particular that if there is a section f of $\mathcal{F}$ on X_s, then for some n, $f\otimes s^n$ is a global section of $\mathcal{F}\otimes\mathcal{L}^n$.

Proof: This is nearly the same as the proof that $(\mathrm{Spec}\,A)_f=\mathrm{Spec}\,A_f$, Cf.[StackProject 01PW]. □

Cor. (III.2.5.5). when $X=\mathrm{Proj}\,S$ projective over $\mathrm{Spec}\,S_0$ and $\mathcal{F}$ Qco, $\widetilde{\Gamma_*(\mathcal{F})}\cong\mathcal{F}$, where $\Gamma_*(\mathcal{F})=\oplus_{n\in\mathbb{Z}}\Gamma(X,\mathcal{F}(n))$, which is a graded S-module. In particular, Γ_* for projective space $\mathbb{P}^n_A$ equals $A[x_1,\ldots,x_n]$.

Ample Invertible Sheaves

Def. (III.2.5.6). On a quasi-compact scheme X, an invertible sheaf $\mathcal{L}$ is called **ample** iff there is a n and sections $s_i\in\Gamma(X,\mathcal{L}^n)$ that X_{s_i} is an affine cover of X.

For a qc morphism $f:X\to Y$, an invertible sheaf on X is called ***f*-ample** iff it is ample restricted to every open subscheme $f^{-1}(V)$, where V are affine open in Y.

On a locally Noetherian scheme X, an invertible sheaf $\mathcal{L}$ is called ***H*-ample** iff for any coherent sheaf $\mathcal{F}$ on X, $\mathcal{F}\otimes\mathcal{L}^n$ is generated by global sections for large n.

Prop. (III.2.5.7). An invertible sheaf $\mathcal{L}$ is (f-)ample iff $\mathcal{L}^m$ is (f-)ample.

Prop. (III.2.5.8). When X is Noetherian, H-ample $\iff$ ample.

Proof: Cf.[StackProject 01Q3], the left to right: For any point, choose a open affine U that $\mathcal{L}$ is free, then the sheaf of ideal for $X-U$ is coherent because X is Noetherian so $I_Y\otimes\mathcal{L}^n$ is generated by global sections thus some s that $p\in\mathrm{Supp}(s)$. So as U is affine, $X_s\subset U$ is affine. Then use finiteness argument. □

Prop. (III.2.5.9). When There is a f-ample sheaf for $f:X\to Y$ qc, then f is separated.

Proof: Being separated is local on the target, so we assume Y is affine, then this follows from [StackProject 01PY]. □

Lemma (III.2.5.10). For an invertible sheaf $\mathcal{L}$ on a qc scheme X, if for each Qco sheaf of ideals $\mathcal{I}\in\mathcal{O}_X$, there is a n that $H^1(X,\mathcal{I}\otimes\mathcal{L}^n)=0$, then $\mathcal{L}$ is ample.

Proof: For any closed pt P, choose an open affine nbhd U that $\mathcal{L}$ is trivial, let $Y=X-U$, by the exact sequence $0\to\mathcal{I}_{Y\cup\{p\}}\to\mathcal{I}_Y\to k(P)\to 0$, we have

$$0\to\mathcal{I}_{Y\cup\{p\}}\otimes\mathcal{L}^n\to\mathcal{I}_Y\mathcal{L}^n\to k(P)\otimes\mathcal{L}^n\to 0.$$

Thus by assumption we have a surjective map $\Gamma(X,\mathcal{I}_Y\otimes\mathcal{L}^n)\to\Gamma(X,k(P)\otimes\mathcal{L}^n)$. Now $k(P)\otimes\mathcal{L}^n$ is A/m_P, so we let $s\in\Gamma(X,\mathcal{I}_Y\otimes\mathcal{L}^n)$ maps to a section in $\Gamma(X,k(P)\otimes\mathcal{L}^n)$ that restricts to $1\in A/m_P$, then $P\in\operatorname{Supp}s\subset U$ is affine. So we find an affine X_s for every closed pt of X, these will cover X. □

Prop. (III.2.5.11) (Serre's Cohomological Criterion of Ample). If X is proper over a Noetherian affine scheme, $\mathcal{L}$ is an invertible sheaf, then the following is equivalent.

- $\mathcal{L}$ is ample
- For each coherent sheaf $\mathcal{F}$, $H^i(X,\mathcal{F}\otimes\mathcal{L}^n)=0$ for n large enough.
- For each Qco sheaf of ideals $\mathcal{I}\in\mathcal{O}_X$, there is a n that $H^1(X,\mathcal{I}\otimes\mathcal{L}^n)=0$

(Notice in this case H-ample $\iff$ ample).

Proof: $1\to 2$: Because $\mathcal{L}^m$ is H-very ample for some m, thus X is projective, then we use Serre theorem(III.5.4.29).

$3\to 1$: (III.2.5.10). □

Prop. (III.2.5.12). $f:X\to Y$, let $\mathcal{L}$ be f-ample on X and $\mathcal{M}$ ample on Y, then $\mathcal{L}\otimes f^*\mathcal{M}^n$ is ample for n large.

Proof: Cf.[StackProject 0892]. □

Cor. (III.2.5.13). If $f:X\to Y$ is quasi-affine, then the pullback of an ample invertible sheaf is ample. This is because quasi-affine $\iff$ $\mathcal{O}_X$ is f-ample.

Prop. (III.2.5.14) (Pullback of Ampleness). If $f:Y\to X$ is finite and surjective morphism between schemes proper over a Noetherian affine scheme, then for an invertible sheaf $\mathcal{L}$ on X, $\mathcal{L}$ is ample iff $f^*\mathcal{L}$ is ample.

Proof: One direction follows from(III.2.5.13), For the other we use Serre criterion(III.2.5.11) and devissage(III.2.3.17). We only verify 3: By(III.3.4.35), there exists such coherent sheaf $f_*\mathcal{F}$ for any integral subscheme, and for a any Qco sheaf of ideals $\mathcal{I}$, $\mathcal{I}f_*\mathcal{F}=f_*(f^{-1}\mathcal{I}\mathcal{F})$ because f is affine, thus

$$H^p(X,\mathcal{I}f_*\mathcal{F})=H^p(X,f_*(f^{-1}\mathcal{I}\mathcal{F}\otimes\mathcal{L}^n))=H^p(Y,f^{-1}\mathcal{I}\mathcal{F}\otimes\mathcal{L}^n)$$

by projection formula, and f is affine. This vanish for n large. □

Prop. (III.2.5.15). If $i:Z\to X$ is a closed immersion that induce homeomorphism on topology between Noetherian schemes, then $\mathcal{L}$ is ample iff $i^*\mathcal{L}$ is ample.

In particular, this applies to $X_{red}\to X$.

Proof: Cf.[StackProject 09MS]. □

Prop. (III.2.5.16). Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let n_0 be an integer. If $H^p(X,\mathcal{L}^{-n})=0$ for $n\geq n_0$ and $p>0$, then X is affine.

Proof: Cf.[StackProject 0EBD]. □

Very Ample Invertible Sheaves

Def. (III.2.5.17). A **very ample** invertible sheaf on X/Y quasi-projective over Y is the pullback along some immersion of $\mathcal{O}(1)$ of $\mathrm{Proj}(\mathcal{E})$ for some Qco module $\mathcal{E}$ over Y, Cf.(III.2.4.9). It is called H**-very ample** iff $\mathcal{E}$ is trivial. Notice when X is proper, this immersion must be closed by(III.3.5.3).

When S is affine and X/S is of f.t., then very ample is equivalent to H-very ample.

Proof: Cf.[StackProject 02NP]. □

Prop. (III.2.5.18) (Ample and H-Very Ample). Let X/S be locally of f.t., then for any ample invertible sheaf $\mathcal{L}$ over X, every $\mathcal{L}^m$ for m large is H-very ample.

Proof: As in the proof of(III.2.5.8), we see that there are f.m affine opens X_{s_i} that cover X refining a inverse image of affine cover of S, we can make them the same degree then by f.t., there are f.m generators $\{c_{ij}\}$(III.2.5.4). So consider the projective space $A[x_i, c_{ij}]$, X is closed immersed into an open subscheme of P^N_S. Cf.[StackProject 01VS]. □

Prop. (III.2.5.19). If X/S is qc, then f-very ample implies f-ample.

Proof: Cf.[StackProject 01VN]. □

Prop. (III.2.5.20) (Serre). When $f : X \to S$ is of f.t. and S is affine, $\mathcal{L}$ is $(H\text{-})$ample $\iff$ $\mathcal{L}$ is f-relative ample $\iff$ $\mathcal{L}^n$ is $(H\text{-})$very ample for some(all large)n. (All these follow from propositions above).

Prop. (III.2.5.21). A proper scheme that has a (H)-very ample invertible sheaf is projective, because the image of a proper scheme is proper.

Prop. (III.2.5.22). When X is Noetherian and has an H-ample invertible sheaf, any coherent sheaf is a quotient of a a finite direct sum of $\mathcal{O}(-n)$.

Proof: This is because X is qc and $\mathcal{F}(n)$ is globally generated for some n. So for any pt p we find f.m. section that generate the stalk, then by coherence, there is a nbhd that generate the stalk, and the compactness shows that there is f.m that generate the stalk, thus $\mathcal{O}_X^N \to \mathcal{F}(n)$ surjective, then we tensor it with $\mathcal{O}_X(-n)$. □

6 Sheaf of Differentials

Def. (III.2.6.1). The diagonal map $\Delta : X \to X \times_Y X$ is an immersion hence an isomorphism onto the image. So we use the locally sheaf of ideal $\mathcal{I}$ corresponding to $\Delta(X)$ to get the **sheaf of differentials** $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ on X. It is a $\mathcal{O}_X$-module on X.

It is a Qco sheaf because pullback of Qco is Qco, and when $X \to Y$ is locally of f.t. and Y is locally Noetherian, X and $X \otimes_Y X$ is also locally Noetherian thus $\Omega_{X/Y}$ is coherent.

By(VI.2.1.2)(VI.2.1.4) $\Omega_{X/Y}$ can also be constructed by locally $\widetilde{\Omega_{B/A}}$ and glue because it is functorial. And we see form this that it is compatible with base change of schemes. From this we see the stalk of $\Omega_{X/Y}$ at p is $\Omega_{X_p/Y_{f(p)}}$.

Prop. (III.2.6.2) (Jacobi-Zariski Sequence). Let $f : X \to Y$ and $g : Y \to Z$, then there is an exact sequence of sheaves on X:

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Immediate from(VI.2.1.6).

Prop. (III.2.6.3). Let $f : Z \to X$ be closed immersion and $g : X \to Y$, then there is an exact sequence of sheaves on Z:

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

Immediate from(VI.2.1.5).

Prop. (III.2.6.4). The stalk of the differential sheaf $\Omega_{X/k}$ at a rational point x of a scheme over a field k is just the Zariski cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$(III.2.8.5).

Proof: Using the Jacobi exact sequence(VI.2.1.6) on an affine nbhd $\operatorname{Spec} A$ of x for A and $\mathfrak{m}_x$. Then we verified that there is a right inverse of $A/\mathfrak{m}^2 \to k(x) = x$, then it follows that $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong \Omega_{A/k} \otimes_A k(x) = \Omega_{A_{\mathfrak{m}_x}/k}$ which is the stalk of $\Omega_{X/k}$ by(III.2.6.1). □

Prop. (III.2.6.5) (Regular and Differential). For an irreducible algebraic separated scheme X over a perfect field k, $\Omega_{X/k}$ is a locally free sheaf of rank $n = \dim X$ iff X is a regular. Plus the condition of separatedness, X will be a regular variety.

By the same method, we can show that an integral scheme of f.t. over k perfect has an open dense subset U that is regular.

Proof: It suffice to consider closed point by**??**, the perfect,irreducible and f.t. conditions are here to use(VI.2.1.11), and a coherent sheaf is locally free iff its stalks are free(III.2.3.10).

For the second assertion, we consider the stalk of $\Omega_{X/k}$ at the generic point, it is $\Omega_{K/k}$, which is free by(VI.2.1.10). So by(III.2.3.10) again there is an open dense nbhd of the generic point that Ω is free hence all the points in it are regular. □

Prop. (III.2.6.6). If $X = \mathbb{P}^n_A$ over $Y = \operatorname{Spec} A$, then there is an exact sequence

$$0 \to \Omega_{X/Y} \to (\mathcal{O}_X(-1))^{n+1} \to \mathcal{O}_X \to 0.$$

This is because locally the kernel is generated by $(e_j - (x_j/x_i)e_i)/e_i = d((x_j/x_i))$.

When A is a field k, this sequence is locally free by(III.7.1.17), so taking dual we get:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to \mathcal{T}_X \to 0.$$

Taking highest exterior power we get $\omega_X \cong \mathcal{O}_X(-n-1)$.

7 Limits of Schemes

Prop. (III.2.7.1) (Fiber Products). Fiber products exist in the category of schemes, also arbitrary products exist, so arbitrary limits exist in the category of schemes(I.8.1.26).

Proof: For affines schemes, this is just the the tensor product, by(III.2.2.6). For general scheme, choose an affine cover S_i of the base scheme and choose affine covers of the inverse image of S_i, then the universal property of fibered product can be used to glue to construct a scheme(III.1.6.10), and it is easily verified to be the fibered product of this diagram. □

Cor. (III.2.7.2). The equalizer of two morphisms from X to Y exists, it is a locally closed subscheme of X, and it is a closed subscheme of X if Y is separated.

Proof: because it is the base change of $\Delta : Y \to Y \times Y$(I.8.1.26), then use(III.3.4.47). □

Def. (III.2.7.3) (Formal Completion). For a locally Noetherian scheme and a Qco sheaf of ideal I on it corresponding to a closed scheme Y, there is a **Formal completion of X along I** defined the ringed space with the glue of locally the functorial completion of A along I on the topological space Y,(I.5.5.12)[Hartshorne P194]. In fact, any coherent sheaf on X can be completed along Y.

A Locally ringed space $\widetilde{X}$ is called Locally Noetherian formal scheme if it is locally a formal complete of some X along I. A sheaf of $O_{\widetilde{X}}$-modules is called coherent iff it is locally the completion of a sheaf of coherent module.

8 Others

Def. (III.2.8.1) (Frobenius). For a scheme over $\mathbb{F}_p$, the **absolute Frobenius** is the unique morphism of schemes $\mathrm{Frob}_{p,X} : X \to X$ that is $x \to x^p$ on the sections. The Frobenius is functional in X.

For a scheme X over a scheme S over $\mathbb{F}_p$, consider the base change $X^{(p,S)} \to X$ of $\mathrm{Frob}_{p,S}$ by $X \to S$, the functionality of Frobenius and the universal property of base change gives us a morphism $F_{X/S} : X \to X^{(p,S)}$, which is called the **relative Frobenius of X over S**.

For a scheme X over $\mathbb{F}_q$, $q = p^n$, then **geometric Frobenius** is defined to be $\pi_X = \mathrm{Frob}_X^n$, as a morphism of schemes over $\mathbb{F}_q$. More generally, if X is a scheme over a scheme S over $\mathbb{F}_q$, and there is a scheme X_0 over $\mathbb{F}_q$ that $X_0 \otimes_{\mathbb{F}_q} S = X$, then the **geometric Frobenius** is defined to be the extension of scalers of the geometric Frobenius π_{X_0} to S, i.e, the map $\pi_X : X \to X$ induced by $X \to X_0 \xrightarrow{\pi_{X_0}} X_0$.

Remark (III.2.8.2). If $S = \operatorname{Spec} R$ and $X = \operatorname{Spec} R[X_1, \ldots, X_n]/I$, then $X^{(p)} = \operatorname{Spec} R[X_1, \ldots, X_n]/I^{(p)}$, and $F_{X/S}$ is given by $r \to r, X_i \to X_i^p$.

Cor. (III.2.8.3). Frobenius is the relative Frobenius over $\mathbb{F}_p$. Relative Frobenius is universal homeomorphism, because Frobenius is universal homeomorphism(I.5.2.11) and use definition, in particular, it is integral.

The relative Frobenius has nice functoriality properties. It is functorial for schemes over S, and for $X \to S$ and $T \to S$, $\mathrm{Frob}_{X_T/T}$ is the base change of $\mathrm{Frob}_{X/S}$ by $(X_T)^{(p,T)} \to X^{(p,S)}$. The composition $\mathrm{Frob}_{X/S}^n$ has two definitions, they are equal, Cf.[StackProject 0CCG].

Prop. (III.2.8.4). If X is a scheme over a field k of char p, then $X^{(p)}$ is reduced iff X is geometrically reduced. This follows from(III.3.3.2).

Geometry of schemes

Prop. (III.2.8.5) (Points of Schemes). For schemes T, X, a **T-point for X** is defined to be a $T \to X$. If $T = \operatorname{Spec} A$, it is also called an A-point.

For $A = K$, this correspond to points of X with $k(x) \subset K$, for $A = k[\varepsilon]/\varepsilon^2$, this correspond to a rational point x and an element in the dual of the $k(x)$-space $\mathfrak{m}_x/\mathfrak{m}_x^2$, i.e. the **Zariski tangent space**. (notice the local map).

III.3 Properties of Schemes(Hartshorne)

Basic References are [Algebraic Geometry Hartshorne] and [Hartshorne Solution 田翊].

1 Basic Scheme Properties

Affine Local Properties of Schemes

Lemma (III.3.1.1) (Nike's Trick). In a scheme X and $x \subset \operatorname{Spec} A \cap \operatorname{Spec} B$, x has an open nbhd in $\operatorname{Spec} A \cap \operatorname{Spec} B$ that are distinguished in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$.

Proof: Choose a nbhd of x that is distinguished in $\operatorname{Spec} A$, then because distinguished of distinguished is distinguished, we may assume $i : \operatorname{Spec} A \subset \operatorname{Spec} B$. Now let $f \in B$ be an element that $D(f) \subset \operatorname{Spec} A$, then I claim $D(i^*(f)) = D(f)$, this will finish the proof, but this is equivalent to $i^{-1}(\operatorname{Spec} B_f) = \operatorname{Spec} A_{i^*(f)}$, which is true for ideal-theoretical reason. □

Prop. (III.3.1.2) (Affine Communication Theorem). A property P of affine open subsets is called **affine local** if: $\operatorname{Spec}(A)$ has $P \Rightarrow$ all $\operatorname{Spec}(A_f)$ has P, and any cover of $\operatorname{Spec}(A_{f_i})$ has $P \Rightarrow \operatorname{Spec}(A)$ has P. Notice a stalk-wise property is obviously affine-local.

Now if we call X has $\widetilde{P}$ if $X = \bigcup_i \operatorname{Spec} A_i$ that A_i has P. Then the following is equivalent:

- all open affine subscheme of X has P.
- all open subscheme of X has $\widetilde{P}$.
- X has a cover of open subschemes that has $\widetilde{P}$.
- X has $\widetilde{P}$.

Proof: $1 \to 2 \to 3 \to 4$ is obvious, only need to prove $4 \to 1$: if $X = \cup \operatorname{Spec} A$, for any open affine subscheme of X, by(III.3.1.1), it can be covered by distinguished opens that are also distinguished in some $\operatorname{Spec} A_i$, so by hypothesis it has P. □

Remark (III.3.1.3). When proving locality of morphism properties using affine communication theorem, one usually resort to1.

Cor. (III.3.1.4) (List of affine local properties). (not complete)

- (Locally)Noetherian.
- (Geometrically-)Reducedness. (Stalk-wise)

Proof:

- Cf.[Hartshorne P83].
- Reducedness: if there is an affine cover that is reduced, then the stalks will be like R_P is reduced if R is reduces. And if the stalks are all reduced, then a nilpotent element will be 0 in every local set, thus 0 because $\mathcal{O}$ is a sheaf. The geometrically regular is equivalent to stalk regular by(III.3.3.5), so it is also stalkewise.

□

Irreducible

Def. (III.3.1.5). A scheme is called **irreducible** iff its underlying topological space is irreducible.

Prop. (III.3.1.6) (Nearly Affine Local). For a scheme, the following are equivalent:

1. It is irreducible.
2. There is an affine cover U_i of X that U_i are all irreducible and $U_i \cap U_j \neq \varnothing$.
3. Every affine open subset of X is irreducible.

Proof: A scheme is sober(III.2.2.4), if X is irreducible, then X has a unique generic pt η that $\overline{\{\eta\}} = X$, then $2, 3$ all holds. If 2 holds, then for a decomposition $X = Z_1 \cup Z_2$, any U_i belongs to Z_1 or Z_2, so it is easy to see $Z_1 = X$ or $Z_2 = X$. If 3 holds, then choose an affine cover U_i of X, then $U_i \cap U_j \neq 0$, otherwise $U_i \coprod U_j$ is affine and not irreducible, contradiction, so 2 holds. □

Cor. (III.3.1.7). The fiber product of irreducible schemes is irreducible, because .

Reduced

Def. (III.3.1.8). A scheme is called **reduced** if $\mathcal{O}_X(U)$ is reduced for every open set U. Reduced is a stalk-wise property(III.3.1.4), and it suffices to check reducedness at closed pts, because the localization of a reduced ring is reduced.

Prop. (III.3.1.9) (Reduction). There is a $X_{red} \to X$ associated tot every scheme, it is $\mathbf{Spec}(\mathcal{O}_X/\mathcal{N})$ where $\mathcal{N}$ is the sheaf of nilpotent elements. This construction is right adjoint to the forgetful functor by the adjoint property of **Spec**(III.2.2.7). $X_{red} \to X$ is an closed immersion.

It's useful to change to X_{red} when the proposition only involve topology because X_{red} has the same topology as X. A map can induce a map on their reduced structure.

Prop. (III.3.1.10) (Induced Reduced Scheme Structure). There is an **induced reduced scheme structure** on a closed subset Y of a scheme X, it is the **Spec** of the $\mathcal{O}_X$-algebra of $\mathcal{O}_X(U)/\cap p_i, (i \in Y)$. It has the universal property that any map morphism from a reduced scheme X to Y factors through the closed subscheme of the closure of its image. (By virtue of reducedness)..

Integral

Def. (III.3.1.11). A scheme X is called **integral** if $\mathcal{O}_X(U)$ are all integral. This is equivalent to reduced and irreducible. So a scheme is integral iff there is an integral open affine cover that are pairwise-intersect(III.3.1.5).

Proof: If X is irreducible and reduced, then so does any affine subscheme $\mathrm{Spec}\, R$, so R is integral as (0) is the generic prime. Conversely, if X is reduced, then any affine subscheme $\mathrm{Spec}\, R$ is integral so reduced, and is irreducible by the presence of prime (0). □

Cor. (III.3.1.12). The projective space over an integral scheme is integral. (Check the affine covers are dense). The projective space $P^n_{\mathbb{Z}}$ is integral.

Noetherian

Def. (III.3.1.13). A scheme is called **locally Noetherian** if it can be covered by open affine schemes of noetherian rings. It is called **Noetherian** if moreover it is quasi-compact.

(Locally)Noetherian is affine local(III.3.1.4).

Prop. (III.3.1.14). Any locally closed subscheme of a (locally)Noetherian scheme is (locally)Noetherian.

Proof: Cf.[StackProject 02IK]. □

Prop. (III.3.1.15). For a closed subscheme, the collection of its irreducible components is locally finite in X, because a Noetherian space has f.m. irreducible components.

Prop. (III.3.1.16). Let k'/k be a f.g. field extension, then a scheme X over k is locally Noetherian iff $X_{k'}$ is locally Noetherian.

Proof: Locally Noetherian is affine local, so the problem is totally ring-theoretic. If $X_{k'}$ is Noetherian, then so does X by ff descent(I.7.2.1). If X is Noetherian, then so does $X_{k'}$ by(I.5.1.12). □

Jacobson

Def. (III.3.1.17). An scheme is called **Jacobson** iff its underlying topological space is Jacobson(IV.1.13.16). In particular, an affine scheme $\operatorname{Spec} R$ is Jacobson iff R is Jacobson(I.5.7.4).

So by(IV.1.13.17), being Jacobson is a local property.

Cohen-Macaulay

Def. (III.3.1.18). A scheme is called C.M. iff all its stalks is C.M. local.

2 Normal & Regular

Def. (III.3.2.1). A scheme is called **normal** if all its stalk is normal domain, so all its affine sections are normal ring. It is called **regular** iff all its stalk is regular local ring, i.e. all affine opens are regular rings. Regular only have to be checked at close pt by(I.6.5.9).

Prop. (III.3.2.2). Regular scheme is C.M, by(I.6.5.13).

Prop. (III.3.2.3). For an integral scheme X, there is a $X_{nom} \to X$ which is $\mathbf{Spec}(\mathcal{O}_{X,nom})$, any dominant morphism f from a normal integral scheme to X will factor through X_{nom}. (Use the adjointness for **Spec** and notice f maps generic to generic.

Prop. (III.3.2.4). For a curve, normal is equivalent to regular. This is because for a Noetherian local domain of dim 1, principal $\iff$ normal $\iff$ regular $\iff$ DVR.

Cor. (III.3.2.5). A Noetherian Normal scheme is regular in codimension 1.

Prop. (III.3.2.6). A Noetherian connected regular scheme is irreducible, since it has f.m. closed components and they cannot intersect, because at the intersection pt, an affine nbhd has multiple minimal primes, thus the local ring also has multiple stalk, thus not integral, not regular.

Dedekind Scheme

Def. (III.3.2.7) (Dedekind Scheme). A **Dedekind scheme** is an integral Noetherian normal scheme of dimension 1.

Prop. (III.3.2.8). Let X be a Dedekind scheme and $x \in X$ is a closed pt, let $\widehat{X} = \mathrm{Spec}(\widehat{\mathcal{O}}_{X,x}) \to X$ be the completion of X at x, then there is a pullback of categories:

$$\begin{array}{ccc} Bun_X & \longrightarrow & Bun_{X-\{x\}} \\ \downarrow & & \downarrow \\ Bun_{\widehat{X}} & \longrightarrow & Bun_{\widehat{X}-\{x\}} \end{array}$$

Proof: We may study locally near x, then we can assume that X is affine. Now shrink X even more, we can assume that x is defined by a single $f \in A$(localized at the maximal ideal defined by x), then we finish by(I.5.5.23). □

3 Geometrical properties

Def. (III.3.3.1).

- A scheme X is called **geometrically integral/reduced/separated/irreducible**... over a field k iff for any field extension k'/k, $X_{k'}$ is integral/reduced/separated/....
- A locally Noetherian scheme is called **geometrically regular** iff for any f.g. field extension K/k, X_K is regular. It is stalkwise by(III.3.3.5).

Prop. (III.3.3.2) (Geometrically Reduced). For a scheme X over a field k, the following are equivalent:

1. X is geometrically reduced.
2. For every reduced k-scheme Y, the product $X \otimes_k Y$ is reduced.
3. All stalks are geometrically reduced ring.
4. X is reduced and for every maximal point η of X, the residue field $k(\eta)$ is separable over k.
5. $X_{k^{per}}$ is reduced.
6. X_K is reduced for every finite purely inseparable field extension K/k.
7. $X_{k^{1/p}}$ is reduced.

Proof: As reduced is local, these all follows from(I.6.6.2). □

Prop. (III.3.3.3) (Geometrically Irreducible). For a scheme X over a field k, the following are equivalent:

- For every irreducible k-scheme Y, the product $X \otimes_k Y$ is irreducible.
- X is geometrically irreducible.
- $X_{k^{sep}}$ is irreducible.
- X is irreducible and if η is the generic pt of X, then k is separably closed in $k(\eta)$.
- X_K is irreducible for any finite separable extension K/k.

Proof: Cf.[Gortz P136]. □

Cor. (III.3.3.4) (geometrically Integral). For a scheme X over a field k, the following are equivalent:

- For every integral k-scheme Y, the product $X \otimes_k Y$ is irreducible.
- X is geometrically integral.
- $X_{\overline{k}}$ is integral.
- X is integral and if η is the generic pt of X, then k is alg.closed in $k(\eta)$ and $k(\eta)/k$ is separable.
- X_K is irreducible for any finite extension K/k.

Proof: Cf.[Gortz P136]. □

Prop. (III.3.3.5). Let X be a locally Noetherian scheme over a field k, then X is geometrically regular iff the local ring $\mathcal{O}_{X,x}$ is geometrically regular over k. And it suffice to check for finite purely inseparable field extensions k'/k.

Proof: For a finite purely inseparable field extension, $\mathcal{O}_{X,x} \otimes_k k'$ is also a local ring because their spectra are the same(I.5.2.11), so $\mathcal{O}_{X,x}$ is geometrically regular by(I.6.6.4). Conversely, if $\mathcal{O}_{X,x}$ is regular, then for any field extension k'/k, stalks of $X_{k'}$ are localization of stalks of $\mathcal{O}_{X,x} \otimes_k k'$, so it is regular by(I.6.5.9). □

Prop. (III.3.3.6) (Partially Stable Under Base Change). If k'/k is a f.g. field extension, then $X_{k'}$ is geometrically regular over k' iff X_k is geometrically regular over k.

Proof: One direction is trivial, for the other, Cf.[StackProject 038W]. □

4 Basic Morphism Properties

Base Change Trick

Prop. (III.3.4.1). If a property P of morphisms satisfy:

- Closed immersion has P.
- Stable under base change and composition.

Then

- it is stable under product.
- $g \circ f : X \to Y \to Z$ has $P + g$ separated $\Rightarrow f$ has P.
- it is stable under f_{red}. (Notice $X_{red} \to X$ is closed immersion).

Proof: For the product, we may assume one of them is identity and use composition, but then the product is just base change, so it has P.

For the second, factorize $f : X \to Z \times_Z Y \to Y$, the first is base change of $Y \to Y \times_Z Y$(I.8.1.29), so it has P because g is separable, and the second map is a base change of $X \to Z$, so it has P, so f has P.

$X_{red} \to X \to Y = X_{red} \to Y_{red} \to Y$ has P because $X_{red} \to X$ is closed immersion, and $Y_{red} \to Y$ is separable because closed immersion is separable(checked directly), so by what has been proved, $X_{red} \to Y_{red}$ has P. □

Prop. (III.3.4.2). Lists of properties satisfying the base change trick(not complete):

1. Universal closed morphism.
2. Affine morphism.
3. Quasi-affine morphism.
4. Closed immersions.
5. Quasi-compact morphism.
6. (Quasi-)Separatedness.
7. Proper.
8. (Locally) of Finite Presentation.
9. Unramified.

Proof:

1. Trivial.
2. Trivial.
3.
4. For closed immersion, check locally, for open immersion, notice that $U \times_W V \to X \times_S Y$ is open immersion.
5. Trivial.
6. Closed immersion is separated is checked directly. Composition: For $X \to Y \to Z$, $X \to X \times_Y X \to X \times_Z X$, the second one is closed immersion(or quasi-compact) by(III.3.4.48), so this follows from that of closed immersion and qc. Base change: The diagonal commutes with base change(I.8.1.27), so this follows from that of closed immersion and qc.
7. Because universally closed and separatedness both do(III.3.4.2).
8. By(I.6.7.8).
9.

□

Local Properties of Morphisms

Our fundamental tool is (III.3.1.2).

Prop. (III.3.4.3) (List of properties affine local on the target). (All the property besides the H-projectiveness is local on the target).

1. All properties defined by a ring map property local on the target(I.5.1.28) .
2. Isomorphism, injective, surjective, open, closed.
3. Quasi-compactness.
4. (Open/Closed)immersions.
5. (Quasi-)Separateness.
6. Finite morphism.
7. Integral morphism.

Proof:

1. Trivial.
2. Only isomorphism need proving, Cf.[Hartshorne Ex 2.2.17].
3. Because affine open is compact and $(\operatorname{Spec} A)_f$ is also compact.
4. Because open and closed are local on the target and check closedness on stalks.
5. Use criterion(III.3.4.51).
6. Cf.[StackProject 02JL].
7. Cf.[StackProject 02JK].

□

Prop. (III.3.4.4) (List of properties affine local on the source). (not complete)

1. All properties defined by a ring map property local on the source(I.5.1.28) .
2. Openness.
3. (Locally)Finite presentation.

Proof:

1. Trivial.
2. Trivial.
3. By(I.6.7.8).

□

Valuation Criteria

Lemma (III.3.4.5) (Valuation Criteria Lemma). If X is a scheme and $x \to y$ is a specialization of pts, then for any field extension $K/k(x)$, there is a valuation ring A and a morphisms $\operatorname{Spec} A \to X$ that maps the generic pt η to x and the unique closed pt to y, and $k(\eta)/k(x) \cong K/k(x)$.

Proof: There is a morphism $\mathcal{O}_{X,y} \to k(x) \to K$, so there is a valuation ring A in K with field of fraction K that dominate $\mathcal{O}_{X,y}$, by(V.3.2.1). Then this what we desire. □

Prop. (III.3.4.6) (Valuation Criteria). The valuation criterion for $\operatorname{Spec}(k) \to \operatorname{Spec}(R)$ where R is a valuation ring: Given a morphism $X \to S$,

1. If it is qc, then it is universally closed iff there is at least one lifting.
2. it is separated iff it is quasi-separated and there is at most one lifting.
3. it is proper iff it is finite type, quasi-separated and lifting exists uniquely.

Proof:

1. Firstly, in this case, by(III.3.4.11), it suffices to prove that: any base change of f satisfies going-up iff it has has least one lifting. For this, Cf.[StackProject 01KE].
2. If it is separated, then if there are two lifting, then consider their equalizer, it is a closed subscheme of $\operatorname{Spec} A$ by(III.2.7.2), and it contains the generic pt, so it equals X, as desired. Conversely, if there are at most two lifting, then we want to prove the diagonal is closed. But by (III.3.4.47) and(III.3.4.44) and the valuation criterion for u.c., it suffices to prove the existence of a lifting for the diagonal(III.3.4.6). But in fact, a valuation digram for the

diagonal correspond to two lifting of a valuation criterion for $X \to S$, so they are the same, i.e. $\operatorname{Spec} A \to X \times_S X$ can lifts along the diagonal.

3. follows from the above two.

□

Injective Morphism

Prop. (III.3.4.7). For a morphism of schemes, the following are equivalent:

- It is universally injective.
- It is injective and the residue field extension are all purely inseparable.
- The diagonal map is surjective.
- For any field K, $\mathrm{Hom}(\operatorname{Spec} K, X) \to \mathrm{Hom}(\operatorname{Spec} K, S)$ is injective.

Proof: Cf.[StackProject 01S4]. □

Closed Map

Prop. (III.3.4.8). Let $A \to B$ noetherian. Then going-up holds $\iff$ Spec map is closed.

Proof: going-up is equivalent to $f^*(V(q)) = V(f^*(q))$, $\forall q$ prime. Use primary decomposition of $\sqrt{I}$, $V(I) = \bigcup V(q_i)$. □

Prop. (III.3.4.9) (Universal Closed). Universal closedness is local on the basis and satisfies the base change trick(III.3.4.2).

Prop. (III.3.4.10). If g is surjective, then $f \circ g$ is universally closed iff f is universally closed (because surjective is S.u.B).

Prop. (III.3.4.11) (Closed Map and Specialization). The image of a quasi-compact morphism is closed iff it is stable under specialization. And it is a closed map iff specialization lifts along f.

Proof: For the first, the question is local, so reduce to Y affine, and then X is qc$= \cup U_i$, then we can replace X by an affine $\coprod U_i$, then reduce to the affine case(I.5.2.8).

For the second, for any closed subset of X with its induced reduced structure, the restriction to it is still qc and specialization lifts, so we prove the image is closed. Now the image is stable under specialization, so the result follows from the first assertion. □

Affine Map

Lemma (III.3.4.12). Isomorphism is local on the target(III.3.4.3)

Prop. (III.3.4.13). X is affine if there is a finite set of elements $f_i \in \Gamma(X, \mathcal{O}_X)$ that generate the unit ideal and X_{f_i} is affine.

Proof: First prove that $X_{f_i} \cap X_{f_j} = X_{f_i f_j}$ is affine because affine intersect X_{f_i} is affine. Second, prove $\Gamma(X_f, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)_f$, finally glue them to get a map $X \to \operatorname{Spec}(A)$ and use(III.3.4.12). X is affine scheme if $X \to \operatorname{Spec}(\Gamma(X))$ is affine. □

Cor. (III.3.4.14). Affineness is affine local on the target, and it satisfies the base change trick(III.3.4.2).

Prop. (III.3.4.15) (Serre Criterion of Affiness). For a qc scheme $(X, \mathcal{O}_X)$, it is isomorphic to an affine scheme as a ringed space $\iff$ X is $(Co)h$-acyclic $\iff$ $H^1(X, \mathcal{I}) = 0$ for every Qco sheaf of ideals $\mathcal{I}$.

Proof: The case of affine scheme is proven by(III.5.4.1) and(III.5.4.2). The converse: For every point p, choose an open affine nbhd U, let $Y = X - U$, by the exact sequence

$$0 \to \mathcal{I}_{Y\cup\{p\}} \to \mathcal{I}_Y \to k(P) \to 0,$$

we have a surjective map $\Gamma(X, \mathcal{I}_Y) \to \Gamma(X, k(P))$ thus there is a $f \in A = \Gamma(X, \mathcal{O}_X)$ that $P \in X_f \subset U$ is affine. So using(III.3.4.13), we only have to show that for f.m f_i, they generate $\Gamma(X, \mathcal{O}_X)$. This is by considering the kernel F of $\mathcal{O}_X^r \to \mathcal{O}_X : (a_1, \ldots, a_r) \to \sum f_i a_i$, and there is a filtration on F, the quotient of which are all coherent sheaves because kernel and cokernel are Qco, and there by induction and hypothesis, $H^1(X, F) = 0$, thus the result. □

Cor. (III.3.4.16). If X is qcqs, then if $H^1(X, \mathcal{I}) = 0$ for every Qco sheaf of ideals $\mathcal{I}$ of f.t., then X is an affine scheme. (Because by(III.5.2.18), it we can use colimit to show that $H^1(X, \mathcal{I}) = 0$ for Qco sheaf of ideals).

Cor. (III.3.4.17). For a Noetherian scheme X, X is affine iff X_{red} is affine.

Proof: The canonical exact sequence(III.5.3.2) reads: $0 \to \mathcal{N}\mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$, so iff X_{red} is affine, then we have $H^i(\mathcal{F}) \cong H^i(\mathcal{N}\mathcal{F})$, and notice $\mathcal{N}^k = 0$ for some k. □

Cor. (III.3.4.18). For a Noetherian reduced scheme X, X is affine iff each irreducible component is affine. (The same as the above, notice that $\prod p_i = 0$, for the minimal primes of A). (The reducedness can be dropped by the last proposition).

Lemma (III.3.4.19). If a morphism $X \to Y$ is a homeomorphism onto a closed subset of Y, then f is affine.

Proof: Cf.[StackProject 04DE]. □

Quasi-affine

Def. (III.3.4.20). A scheme is called **quasi-affine** iff it is quasi-compact and isomorphic to an open subscheme of an affine scheme. A morphism is called **quasi-affine** iff the inverse of any affine scheme is quasi-affine.

Prop. (III.3.4.21). Quasi-affine morphism is separated and qc by(III.3.4.51).

Prop. (III.3.4.22). Quasi-affine is local on the target and satisfies the base change trick. Cf.[StackProject 01SN].

Prop. (III.3.4.23). A scheme is quasi-affine iff $\mathcal{O}_X$ is ample. Cf.[StackProject 01QE].

Cor. (III.3.4.24). A morphism f is quasi-affine iff $\mathcal{O}_X$ is f-ample.

Dominant

Prop. (III.3.4.25). A quasi-compact morphism of schemes:$X \to S$ is dominant if every generic point of irreducible components of S is in the image of f.(Use quasi-compactness to reduce to the affien case). In particular, if X, S is affine, dominant is equivalent to image of f contains minimal primes and equivalent to the kernel is in the nilradical. (Because the closure of image $= V(\text{Ker})$).

Quasi-Compact

Def. (III.3.4.26). A morphism $f : X \to S$ is called quasi-compact if the inverse image of affine open is quasi-compact.

Quasi-compactness is local on the target and satisfies the base change trick(III.3.4.2).

Prop. (III.3.4.27). Let $f : X \to Y$, $g : Y \to Z$. If $g \circ f$ is quasi-compact and g is qs, then f is qc.

Proof: Factor it through $X \to X \times_Z Y \to Y$. The second map is a base change of $X \to Z$ hence qc, the first map is a section of $X \times_Z Y \to X$, which is a base change of $Y \to Z$, hence qs, so by(III.3.4.49), the first map is also qc. □

Prop. (III.3.4.28). For a field extension k'/k, a scheme X over k is qc iff $X_{k'}$ is qc.

Proof: One direction is trivial, the other is by fpqc descent(III.1.6.9). □

Finite Type

Def. (III.3.4.29). A morphism $f : X \to S$ is called of **locally finite type** if for there exists an affine open cover $\{\mathrm{Spec}(B_i)\}$ of S that $f^{-1}(U_i)$ has an affine open cover of spec of finite generated B_i-algebras. It is called **finite type** if moreover it is quasi-compact.

A scheme over a field k is called **(locally)algebraic** iff it is (locally) of finite type over $\mathrm{Spec}\, k$.

(Locally)Finite type is affine local on the target and on the source, and satisfies the base change trick(III.3.4.2).

Prop. (III.3.4.30) (Locally Algebraic Scheme is Jacobson). For a scheme locally of finite type over a field k, the set of closed points X_0 is dense in every closed subset of X, Because it is a Jacobson space by(IV.1.13.17) and(I.5.7.9).

Moreover, the residue field at a closed stalks is finite over k by(I.5.7.9).

Prop. (III.3.4.31) (Chevalley). A qc morphism locally of f.p. maps locally constructible subset to locally constructible subset.

Proof: We prove $f(E) \cap U_i$ is constructible for every U_i affine open in X. The inverse image of U_i is qc, hence a locally constructible set is constructible by(IV.1.13.9). So we reduce to the affine case(I.5.2.1). □

Cor. (III.3.4.32). As in the proposition, if the image is dense in Y, then it contains an open dense subscheme of Y.

Proof: Cf.[GAGA Serre P8] and [StackProject 005K]. □

Finite & Integral Map

Def. (III.3.4.33). A morphism $f : X \to S$ is called **finite** if it is affine and the inverse image of an affine cover is finite module.

Finiteness is affine local on the target and satisfies the base change trick(III.3.4.2).

A morphism $f : X \to S$ is called **quasi-finite** if it is of finite-type and the inverse of a point is a discrete hence finite set.

A morphism $f : X \to S$ is called **integral** if it is affine and the inverse image an affine cover is integral ring extension.

Integral is affine local on the target and satisfies the base change trick(III.3.4.2).

Prop. (III.3.4.34). A locally f.t. integral morphism is finite, trivially.

Lemma (III.3.4.35). For $f : Y \to X$ finite surjective and X locally Noetherian, for every integral subscheme Z of X with generic point ξ, there is a coherent sheaf $\mathcal{F}$ on Y that the support of $f_*\mathcal{F}$ is Z and $(f_*Z)_\xi$ is annihilated by $\mathfrak{m}_\xi$.

Proof: W consider an inverse image of $\xi = \xi'$, and let $Z' = \overline{\{\xi'\}}$ with the induced reduced structure, then let $\mathcal{F} = i_*\mathcal{O}_{Z'}$ on Y, $\mathcal{F}$ is coherent, then we need to show that $(f_*\mathcal{F})_\xi$ is annihilated by m_ξ. This is because it factors through Z.**?** Cf[StackProject 01YO]. $\square$

Prop. (III.3.4.36) (Chevalley). If $f : Y \to X$ is integral surjective, Y is affine, then X is affine.

Proof: Cf.[StackProject 05YU]. $\square$

Lemma (III.3.4.37). If $f : Y \to X$ is finite surjective, Y is affine, then X is affine.

Prop. (III.3.4.38) (Integral and Affine u.c.). Integral map is equivalent to u.c. and affine.

Proof: Integral is stable under base change. If it is integral, then it is closed by(I.5.4.6). Conversely, Cf.[StackProject 01WM]. $\square$

Immersions

Def. (III.3.4.39). An **immersion** is a closed immersion followed by an open immersion. A open immersion followed by a closed immersion can be written as a closed immersion followed by an open immersion, but not reversely. The reverse happens if the immersion is quasi-compact or the source is reduced (use the reduced induced structure) Cf.[StackProject 01QV].

Prop. (III.3.4.40). Open and closed immersions are affine local on the target(III.3.4.3).

Prop. (III.3.4.41). Closed immersion satisfies the base change trick(III.3.4.2). Open immersion are stable under base change and composition.

Prop. (III.3.4.42). The closed subscheme of a scheme corresponds to Qco $\mathcal{O}_X$-ideals. Hence the closed subscheme of Spec A corresponds to the quotients A/I.

Proof: The closed immersion is qcqs, so it maps $\mathcal{O}_Y$ to $i_*(\mathcal{O}_Y)$ Qco(III.2.3.5), thus the kernel of $\mathcal{O}_X \to i_*(\mathcal{O}_Y)$ is Qco. Conversely, for a Qco sheaf of ideals, $Y = \mathbf{Spec}_X(\mathcal{O}_X/I)$ for the Qco $\mathcal{O}_X$-algebra $\mathcal{O}_X/I$. $\square$

Def. (III.3.4.43) (Scheme Theoretic Image). For a morphism $f : X \to Y$, there is a closed scheme called **scheme-theoretic image** that is the smallest subscheme of Y that f factors through Z. This is by considering the kernel of the structural map, and the kernel has a maximal Qco sheaf of ideal $\mathcal{I}$(III.2.3.12).

For an immersion of schemes, the scheme-theoretic image of the immersion is called the **scheme-theoretic closure**.

Prop. (III.3.4.44) (Immersion to be Closed). An immersion is a closed immersion iff the image is closed.

Proof: Cf.[StackProject 01IQ]. $\square$

Universal Homeomorphism

Prop. (III.3.4.45). A morphism is a universally homeomorphism iff it is integral, surjective and universally injective.

Proof: A universally homeomorphism is affine by(III.3.4.19). It is clearly u.c, so it is integral by(III.3.4.38). Conversely, it is integral hence u.c, and universally bijective, so it is universally homeomorphism. □

Cor. (III.3.4.46). The reduction $X_{red} \to X$ is a universal homeomorphism, as closed immersion is u.c..

Separatedness

Def. (III.3.4.47). A map $f : X \to Y$ is called **separated** if the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. It is called **quasi-separated** if the diagonal is quasi-compact.

In fact Δ is always an immersion because maps between affine scheme is separated so $\Delta(X)$ is closed in $\cup U_{ij} \otimes_{V_i} U_{ij}$ where U, V are affine open, hence it suffice to check the image is closed.

Prop. (III.3.4.48). By(I.8.1.29), for $X \to S$ and $Y \to S$, the map $X = X \times_Y Y \to X \times_S Y$ is an immersion. It is closed immersion if $Y \to S$ is separated, and it is qc if $Y \to S$ is quasi-separated.

Cor. (III.3.4.49). If $s : S \to X$ is a section of $f : X \to S$, the above proposition applies to this case, because $S = S \times_X X \to S \times_S X = X$.

Prop. (III.3.4.50). (Quasi-)Separateness is local on the target because closed immersion and quasi-compact is local on the target(III.3.4.3).

(Quasi-)Separatedness satisfies base change trick by(III.3.4.2).

Prop. (III.3.4.51). A morphism is quasi-separated iff for any two affine open that mapped to an affine open, their intersection is quasi-compact. This is because quasi-compact is local on the target.

A morphism is separated iff for any two affine open that mapped to an affine open, their intersection is affine and $\mathcal{O}(U) \otimes_{\mathcal{O}(W)} \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is surjective. This is because closed immersion is local on the target.

Cor. (III.3.4.52). A locally Noetherian scheme is quasi-separated.

Cor. (III.3.4.53). If $g \circ f$ is (quasi-)separated, then so is f.

Cor. (III.3.4.54). If X is (quasi-)separated, then $X \to Y$ is (quasi-)separated.

Prop. (III.3.4.55). monomorphism is separated because the diagonal map is isomorphism(I.8.1.30), so immersions are separated as they are monomorphisms in the category of schemes (because of surjectiveness on the stalk).

Prop. (III.3.4.56). Affine morphism is separated (Check closed immersion directly).

5 Proper & Projective

Prop. (III.3.5.1). A morphism that is separated, finite-type and universally closed is called **proper**.

proper is local on the target, because all these three properties do.

Prop. (III.3.5.2). The class of proper morphisms satisfies the base change trick(III.3.4.1), by valuation criterion(III.3.4.6) and fibered products tricks.

Proof: Closed immersion is proper because it is f.t. and is affine so separated(III.3.4.47), and it is universally closed because immersions are stable under base change(III.3.4.41). □

Prop. (III.3.5.3) (Image of Proper Map). If $X \to Y$ is morphism between separated schemes f.t over S, then if X is proper, then the image is closed (base change trick) and is proper in its scheme-theoretic structure(III.3.4.10). Notice proper is qc and use(III.3.4.43).

Cor. (III.3.5.4) (Connected Proper to Affine Constant). A morphism from a connected proper scheme to an Noetherian affine scheme $\operatorname{Spec} A$ is constant.

Proof: Because the image is proper and use(III.5.4.28), so the global section A of its image is a finite module over $\operatorname{Spec} k$ thus Artinian so has finitely many point(I.5.1.18). So it is discrete. But X is connected, thus it is constant. □

Prop. (III.3.5.5). Finite morphism is proper, by(III.3.4.38)(III.3.4.56).

Projective Morphism

Def. (III.3.5.6). A **projective** morphism $X \to Y$ is a closed immersion $X \to \operatorname{Proj}(\mathcal{E})$ for some Qco f.t. module $\mathcal{E}$. A ***H*-projective** $\to Y$ is a closed immersion $X \to \mathbb{P}^n_Y$. A H-quasi-projective morphism is a H-projective morphism composed with an open immersion. Some proposition about projective is written before the language of Hartshorne so I may not have changed them to the more general projective notion yet.

Prop. (III.3.5.7). H-(Quasi-)Projectiveness satisfies the base change trick(III.3.4.1). (because Segre embedding is closed). Disjoint union of f.m. projective morphisms is projective (embed into the Segre embedding).

Cor. (III.3.5.8). Projective morphism is locally projective and locally projective is proper[StackProject 01WC], because closed immersion is proper and $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is u.c. by valuation criterion. Cf.[Hartshorne P103]. And a quasi-projective morphism is of f.t. and separated(III.3.4.55).

Prop. (III.3.5.9). Projective scheme over $\operatorname{Spec} A$ is of the form $\operatorname{Proj} S$ where $S_0 = A$ and S is f.g over S_0 by S_1(III.2.4.6).

Prop. (III.3.5.10) (Chow's Lemma). Let $X \to S$ be separated of f.t over a Noetherian S, then there is a birational, proper, surjective $X' \to X$ that X' is quasi-projective.

X is proper iff X' can be projective. And if X is integral(irreducible,reduced), X' can be chosen to be so.

Proof: Basic idea: reduce the the irreducible case, and use f.t. to generate a local quasi-projectives, then the closure of the image of $U \to X \times_S P_1 \times_S \ldots \times_S P_n$ will suffice. □

III.4 More Properties of Schemes

1 Flatness

Def. (III.4.1.1). For a morphism $f : X \to Y$ of ringed spaces, a $\mathcal{O}_X$-module $\mathcal{F}$ is called **flat** over Y iff its stalk is flat as a $\mathcal{O}_{Y,f(x)}$-module. f is called **flat** iff $\mathcal{O}_X$ is flat, it is called **faithfully flat** iff moreover it is surjective.

For a Qco sheaf $\mathcal{F}$, this is equivalent to $\Gamma(U, \mathcal{F})$ is flat over A for every U that mapped to $\operatorname{Spec} A \subset Y$ by(I.7.1.22).

Prop. (III.4.1.2). Flatness is local on the target, it is stable under base change, composition. A coherent $\mathcal{O}_X$ module is flat over X iff it is locally free.(I.7.1.7)(I.7.1.18).

Prop. (III.4.1.3). For a morphism $f : X \to S$ locally of f.p., and a Qco sheaf on X that is locally of finite presentation, the set of points that $\mathcal{F}$ is flat over S is open.

Proof: Cf.[StackProject 0399]. □

Prop. (III.4.1.4). For a flat morphism of ringed space, f^* is exact, because it is f^{-1} followed by tensoring with $\mathcal{O}_X$, check on stalks.

Prop. (III.4.1.5). A finite morphism $f : X \to S$ with S locally Noetherian is flat iff $f_*(\mathcal{O}_X)$ is locally free, Cf.[StackProject 02KB].

Prop. (III.4.1.6). Generalization lifts along a f.f. morphism.

Proof: We can find an affine nbhd, then choose a nbhd of the inverse image, then a generalization in an affine open is a true generalization, so it reduce to the affine case. The rest follows from going-down(I.7.1.23). □

Prop. (III.4.1.7) (Flatness and Openness). A flat morphism locally of f.p. is (universally)open, hence it is qc.

And a qc f.f. morphism of schemes is submersive.

Proof: We need only consider they are both affine. Then the assertion follows from(I.7.1.24).

For the second, by(III.4.1.6), a subset whose inverse image is closed is stable under specialization(surjectiveness used), then the complement is closed by(III.3.4.11) □

Prop. (III.4.1.8) (Flat Points are Open). For a morphism $f : X \to Y$ of f.t. of Noetherian schemes, the set of points of X that f is flat is open in X.

Proof: Cf.[EGA3,11.1.1]. □

Prop. (III.4.1.9) (Flat Family and Hilbert Polynomial). For $\underline{X/T \text{ projective}}$, where T is an integral Noetherian scheme and $X \subset \mathbb{P}^n_T$. Then for each point T, X_t is a closed subscheme of $\mathbb{P}^n_{k(t)}$, so we can consider its Hilbert Polynomial P_t. Then X/T is flat iff P_t is independent of T.

Proof: $P_t(m) = \dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t}(m))$ for m large by(III.5.4.18). And we may let $X = \mathbb{P}^n_T$ and prove for any coherent sheaf $\mathcal{F}$. Moreover, we may let T be a affine local Noetherian, because flatness is local and we only need to compare Hilbert polynomial with the generic point. Now we prove a stronger assertion: The following are equivalent:

- $\mathcal{F}$ is flat over T.

- $H^0(X,\mathcal{F}(m))$ is a free A-module of finite rank, for m large.
- The Hilbert polynomial P_t of $\mathcal{F}_t$ on $X_t=\mathbb{P}^n_{k(t)}$ is independent of t.

$1\to 2$: Use the canonical cover and Čech cohomology, then we notice when m is large, $H^0(X,\mathcal{F}(m))$ is a kernel of the Čech resolution, so it is flat. And it is also finite by(III.5.4.28). Then it is free because it is flat by(I.7.1.7).

$2\to 1$: Let $M=\oplus_{m\geq m_0}H^0(X,\mathcal{F}(m))$, then $\widetilde{M}=\mathcal{F}$(III.2.5.5), notice that the truncation doesn't affect.

$2\to 3$: It suffice to prove that for any $t\in T$, when m is large,

$$H^0(X_t,\mathcal{F}_t(m))\cong H^0(X,\mathcal{F}(m))\otimes_A k(t).$$

For this, we may use(III.5.4.31) to pass to the localization and assume t is the closed pt of T. Then $A\to k(t)$ is surjective and we may let $A^q\to A\to k\to 0$, then by(III.5.4.30), we have $H^0(X_t,\mathcal{F}_t(m))$ is the cokernel of $H^0(X,\mathcal{F}(m))^q\to H^0(X,\mathcal{F}(m))$, but this cokernel is $H^0(X,\mathcal{F}(m))\otimes_k$ because tensoring is right-adjoint, so we are done.

$3\to 2$: We have the rank of $H^0(X,\mathcal{F}(m))$ at the generic and closed point of T are the same(still use $H^0(X_t,\mathcal{F}_t(m))\cong H^0(X,\mathcal{F}(m))\otimes_A k(t)$.) Now(I.7.11.1) gives $H^0(X,\mathcal{F}(m))$ is free. It is f.g. automatically. □

Cor. (III.4.1.10). For a flat morphism to a connected scheme T, the dimension, degree, and arithmetic genus of the fibers are independent of t.

Proof: By(III.5.4.20)and (III.7.2.5). □

Def. (III.4.1.11). For a surjective map of varieties $f:X\to T$ over an alg.closed field k, its fiberes over closed points with induced reduced structure $X_{(t)}$ is called a **algebraic family of varieties parametrized by** T if

1. $f^{-1}(t)$ is irreducible of dimension $\dim X-\dim T$ for every closed point t.
2. If ζ is the generic point of $f^{-1}(t)$, then $F^\sharp\mathfrak{m}_t$ generates the maximal ideal $\mathfrak{m}_\zeta\subset\mathcal{O}_{\zeta,X}$.

Prop. (III.4.1.12). if $X_{(t)}$ is an algebraic family of normal varieties over an alg.closed field k parametrized by a nonsingular curve T, then it is a flat family of schemes.

Proof: By(III.7.3.16), $X\to T$ is flat. So what we need to do is to prove X_t is reduced so $X_t=X_{(t)}$. Let $A=\mathcal{O}_{x,X}$, let u_t be a uniformizer of $\mathcal{O}_{t,T}$, then A/tA is the local ring of x on X_t. By hypothesis X_t is irreducible so tA has a unique minimal prime p in A, and t generate the maximal ideal of A_p by hypothesis. The local ring of $X_{(t)}$ is A/p, so A/p is normal by hypothesis. Then the result follows from(I.6.5.7). □

Cor. (III.4.1.13) (Igusa). Let $X_{(t)}$ be an algebraic family of normal varieties in $\mathbb{P}^n_k$ for k alg.closed parametrized a variety T, then the Hilbert polynomials of $X_{(t)}$ are independent of t.

Proof: ? Why is X/T projective? Cf.[Hartshorne P265]. □

Relative Dimension

Def. (III.4.1.14). A morphism of schemes which is locally of f.t. is called of **relative dimension** n iff all fibers X_s are equidimensional of dimension n.

Prop. (III.4.1.15). If $f : X \to Y$ is a morphism of schemes locally of f.t., then $\dim_x(X_s) = \dim \mathcal{O}_{X_s,x} + tr.deg_{k(s)}k(x)$.

Proof: Cf.[StackProject 02FX]. □

Prop. (III.4.1.16). If $f : X \to Y, g : Y \to S$ are locally of f.t., then $\dim_x(X_s) \leq \dim_x(X_y) + \dim_y(X_s)$. Moreover, equality holds if $\mathcal{O}_{X_s,x}/\mathcal{O}_{Y_s,y}$ is flat.

Proof: Cf.[StackProject 02JS]. □

Cor. (III.4.1.17). If $f : X \to Y, g : Y \to Z$ are of relative dimension m and n, and f is flat, then $g \circ f$ is of relative dimension $m + n$.

Prop. (III.4.1.18). For a morphism $X \to S$ locally of f.t, and its base change $X' \to S'$, then $\dim_x(X_s) = \dim_{x'}(X'_{s'})$.

Proof: Cf.[StackProject 02FY]. □

Cor. (III.4.1.19) (Relative Dimension and Base Change). The base change of a morphism locally of f.t. of relative dimension n is still of relative dimension n.

Prop. (III.4.1.20). For a morphism $f : X \to Y$ between locally Noetherian schemes which is flat and locally of f.t and of relative dimension n, then if $y = f(x)$, we have $\dim_x(X_y) = \dim_x(X) - \dim_y(Y)$.

Proof: Shrinking the nbhd, we may assume $\dim_x(X) = \dim X$ and $\dim_y(Y) = \dim Y$ and X, Y are affine. Now f is locally of f.p. and flat, so it is open(III.4.1.7). So we may assume f is surjective. Then $\dim \mathcal{O}_{X,a} = \dim \mathcal{O}_{Y,b} + \dim \mathcal{O}_{X_b,a} = \mathcal{O}_{Y,b} + n$ by(I.5.6.7), then taking supremum(I.5.6.2), the result follows. □

Cor. (III.4.1.21). For a morphism of schemes that is flat and of f.t., if Y is irreducible, then X is equidimensional of dimension $\dim Y + n$ iff X_y is equidimensional of dimension n for every $y \in Y$. This follows immediately from the proposition and(III.3.4.30).

Proof: The proof highly relies on(III.2.2.9).

$1 \to 2$: For $Z \subset X_y$ an irreducible component, choose a closed pt x of Z not contained in any other irreducible component, then

$$\dim_x Z = \dim_x X - \dim_y Y = \dim X - \dim \overline{\{x\}} - \dim Y + \dim \overline{\{y\}}.$$

The two closures are of the same dimension because by(I.5.7.9), their quotient field extension is finite.

$2 \to 1$: Now for an irreducible component of X, choose a closed pt x of Z not contained in any other irreducible component, then the result is immediate. □

2 Smoothness

Def. (III.4.2.1). A morphism $f : X \to Y$ of schemes is called **smooth** if there is an open affine cover $\{U_i\}$ of S and an open affine cover V_{ij} of $f^{-1}(\{U_i\})$ that the ring map is smooth. A **standard smooth morphism** is the Spec map of a standard smooth ring map.

Smoothness is local on the source and target(I.7.4.15). Smoothness is stable under base change and composition(I.7.4.15).

Lemma (III.4.2.2). For a smooth morphism $X \to S$, the morphism of differential $\Omega_{X/S}$ is locally free and $\dim_x \Omega_{X/S} = \dim_x(X_{f(x)})$(local dimension(IV.1.13.20)).

Proof: We can assume that $X \to S$ is standard smooth, so by the proof in(I.7.4.14), $\Omega_{X/S}$ is free of dimension $n-c$, and also standard smooth is relative global complete intersection(I.7.4.12), so $U_{f(x)}$ is equidimensional of dimension $n-c$, thus the result. □

Prop. (III.4.2.3) (Fiberwise and Stalkwise). For a morphism $X \to S$ locally of f.p., the following are equivalent:

- It is smooth at a point $x \in X$ over $s \in S$.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $X_{f(x)}/k(x)$ is smooth at x, by(I.7.4.20). Moreover, using**??**, we even only have to check for the geometric fibers.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $\Omega_{X/S,x}$ can be generated by $\dim_x(X_{f(x)})$ elements, by(III.4.2.2) and(I.7.4.25).
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} k(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} k(x)$ can be generated by $\dim_x(X_{f(x)})$ elements, by Nakayama, because $\Omega_{X/S,x}$ is of f.p. by(VI.2.1.7).

Prop. (III.4.2.4). Open immersion is smooth. Smooth morphism is syntomic hence flat. Smooth morphism is locally of f.p. Hence smooth morphism is universally open(III.4.1.7).

Smooth morphism is locally standard smooth(I.7.4.14).

Prop. (III.4.2.5) (Smooth Morphism is Open). If $X \to Y$ is smooth and a morphism $Y \to S$, then there is an exact sequence of sheaves(III.2.6.2)(I.7.4.7):

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

Prop. (III.4.2.6). If $Z \to X \to S$, Z/S is smooth and $Z \to X$ is an immersion, then there is an exact sequence of sheaves(III.2.6.3)(I.7.4.8):

$$0 \to \Omega_{C/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

Prop. (III.4.2.7). If $X \to Y \to S$, and $X \to Y$ is surjective, flat and locally of f.p., $X \to S$ is smooth, then $Y \to S$ is smooth.

Proof: Cf.[StackProject 05B5]. □

Prop. (III.4.2.8). By(III.4.2.3)(III.4.2.2), A morphism is smooth of relative dimension n is equivalent to fppf+fibers equidimensional of dimension n and $\Omega_{X/S}$ is locally free of dimension n.

Smooth over Fields

Prop. (III.4.2.9). Let X be a scheme algebraic over a field k. If $\Omega_{X/k}$ is locally free, and k is of char 0 or k is perfect and X is reduced, then X is smooth over k.

Proof: Cf.[StackProject 04QN], [StackProject 04QP]. □

Prop. (III.4.2.10) (Smooth over Field and Geo.Regular). For a scheme algebraic over a field k, X is geometrically regular iff it is smooth over k.

Proof: The question is local around x, so may assume X is affine. Then it is smooth at x, then all its base change is smooth at x(III.4.2.1), and the stalk is regular by(I.7.4.25), so it is geometrically regular at x.

Conversely, if X is geometrically regular, then for any point $x \in X$, $k(x)$ is f.g. over k, so there is a finite purely inseparable extension k'/k that the compositum $k'k(x)$ is separable over k'. Then by(I.5.2.11), $\mathrm{Spec}\, A \otimes_k k'$ is homeomorphic to $\mathrm{Spec}\, A$, so there is a unique prime p' of $X_{k'}$ over X, and its residue field is $k'k(x)$. So by(I.7.4.26), as $k'k(x)/k'$ is separable, $X_{k'}$ is smooth over k' at p'. And f.f. descent for smoothness(I.7.2.1) says X is also smooth over k at p. □

Cor. (III.4.2.11) (Hartshorne Definition). By(III.4.2.3) and the above proposition, a morphism between schemes algebraic over a field k is smooth of relative dimension n iff f is flat and every fiber of f is geometrically regular of dimension n.

Prop. (III.4.2.12) (Variety Smooth Locus Open Dense). Let X be a locally algebraic scheme over a field k that is geometrically reduced, then it contains an open dense subset that is smooth over k.

Proof: The problem is local, so we may assume X is affine, consider its irreducible components, all their intersections can be removed, because they are nowhere dense, so we may assume X is irreducible. So X is integral, let η be the generic pt, then $k(\eta)$ is geometrically reduced(III.3.3.2), so $k(\eta)/k$ is separable, by(I.7.9.3). Then choose an affine subscheme $\mathrm{Spec}\, A \subset X$, then A is smooth at (0) over k, by(I.7.4.26), then by definition, it is smooth on some dense open subscheme of X. □

3 Unramified

More advanced materials to learn at [StackProject Chap40].

Def. (III.4.3.1). A morphism is called **(G-)unramified** iff there is an open affine cover U_i and an open affine cover of $f^{-1}(U_i)$ that the induced ring map is (G-)unramified. Equivalently, $\Omega_{X/S} = 0$ and it is locally of f.t.(f.p.).

(G-)unramifiedness is local on the source and target(III.3.4.3)(III.3.4.4). (G-)unramifiedness is stable under base change and composition(I.7.5.4). Moreover, Unramifiedness satisfied the base change trick.

Prop. (III.4.3.2). An unramified map is locally quasi-finite.

Proof: Cf.[StackProject 02V5]. □

Prop. (III.4.3.3) (Fiberwise). A morphism is (G-)unramified iff it is locally of f.t.(f.p.) and all the fibers X_s are disjoint unions of spectra of finite separable extensions of $k(p)$.

Proof: By(I.7.5.7), Notice $pS_q = qS_q$ is equivalent to every q is minimal in X_p, which is equivalent to X_p is discrete. □

Cor. (III.4.3.4) (Unramified over Fields). A scheme over a field k is unramified iff it is a disjoint union of spectra of finite separable extensions of k, because locally of f.p. is trivially satisfied.

Prop. (III.4.3.5). A morphism $X \to S$ is (G-)unramified iff it is of f.t.(f.p.) and the diagonal is an open immersion.

Proof: If it is unramified, then it is an open immersion by(I.7.5.10). Conversely, $\Omega_{X/S}$ is just the conormal sheaf of the diagonal map, so it is zero. □

Cor. (III.4.3.6). Let X, Y be schemes over S, if f, g are two maps from X to Y, then if Y/S is unramified and f, g are equal on a pt x of X(both on image and residue field), then there is a nbhd of x that f, g are equal.

Proof: This follows as $\Delta_{Y/S}$ is open immersion, so the set that f, g are equal is open in X. □

Prop. (III.4.3.7) (Stalkwise and Fiberwise). For a morphism locally of f.t.(f.p.), the following are equivalent:

- It is (G-)unramified at a point x,
- The fiber $X_{f(x)}$ is smooth over $k(f(x))$ at x.
- $\Omega_{X_{f(x)},x} = 0$.
- $\Omega_{X_s/s,x} \otimes \Omega_{\mathcal{O}_{X_s,x}} k(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} k(x) = 0$.(I.7.5.6).
- $\mathfrak{m}_x \mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)/k(s)$ separable.(I.7.5.7).

Prop. (III.4.3.8). If $X \to Y \to S$, X/S is unramified, then X/Y is unramified. And if X/S is G-unramified and Y/S is of f.t., then X/Y is G-unramified.(By(III.4.7.5)and(III.2.6.2)).

Cor. (III.4.3.9) (Unramified Points Base Change). If f is of f.t.(f.p.), then the set of points that f is unramified is stable under base change by the above proposition.

Noetherian Case

4 Étale

More advanced materials to learn at [StackProject Chap40].

Def. (III.4.4.1). A morphism $f : X \to Y$ of schemes is called **étale** if there is an open affine cover $\{U_i\}$ of S and an open affine cover V_{ij} of $f^{-1}(\{U_i\})$ that the ring map is étale. A **standard étale morphism** is the Spec map of a standard étale ring map.

étale is local on the source and target(I.7.6.5). Étale is stable under base change and composition(I.7.6.5).

Prop. (III.4.4.2). Étale at a point x is equivalent to smooth and unramified at a x(I.7.6.4).

étale at a point x is equivalent to flat and G-unramified at that point, by(I.7.6.11). So Étale over field is equivalent to G-unramified, because over a field it is obviously flat.

Étale at a point x is equivalent to locally standard étale at that point(I.7.6.17).

A morphism is étale iff it is smooth of dimension 0, by definition(III.4.2.8).

Étale is equivalent to flat, locally of f.p. and formally unramified, by(I.7.6.11).

Cor. (III.4.4.3). Étale map is smooth, hence syntomic, flat.

Étale map is universally open because it is flat and lcoally of f.p.(III.4.1.7).

Prop. (III.4.4.4). If X, Y are étale over S, then any map $X \to Y$ is étale.(I.7.6.13).

Prop. (III.4.4.5) (Fiberwise). A morphism of schemes is étale iff it is flat, locally of f.p., and every fiber X_s is a disjoint union of spectra of finite separable field extensions of $k(s)$.

Proof: Follows from(III.4.4.2)(III.4.3.3) and(I.7.6.10). □

Cor. (III.4.4.6). A scheme is étale over a field k iff it is a disjoint union of spectra of finite separable field extensions.

Prop. (III.4.4.7). If $X \to Y$ is smooth at x, then there exist a nbhd of x that it factors through $U \xrightarrow{\pi} \mathbb{A}^d_V \to V$, where π is étale.

Proof: Any standard smooth morphism can be factorized as an étale map over a polynomial algebra, as easily seen. □

Prop. (III.4.4.8) (Stalkwise and Fiberwise). For a morphism locally of f.p., the following are equivalent:

- It is étale at a point x.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ if flat and $X_{f(x)}/k(x)$ is smooth at x.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ if flat and $X_{f(x)}/k(x)$ is unramified at x.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $\Omega_{X_{f(x)},x} = 0$.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $\Omega_{X_s/s,x} \otimes \Omega_{\mathcal{O}_{X_s,x}} k(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} k(x) = 0$.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $\mathfrak{m}_x\mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)/k(s)$ separable.

By (III.4.3.7) and(III.4.2.3).

Prop. (III.4.4.9). If $X \to Y \to S$, and $X \to Y$ is surjective, flat and locally of f.p., $X \to S$ is étale, then $Y \to S$ is étale.

Proof: Cf.[StackProject 05B5]. □

Def. (III.4.4.10) (Étale Neighborhood). For a point $s : \operatorname{Spec} k \to X$, an étale nbhd of s in X is defined to be an étale map $U \to X$ that s factors through U.

Prop. (III.4.4.11). For a morphism $f : Y \to X$ of schemes étale over field k, then f is surjective iff $Y(k_s) \to X(k_s)$ is surjective.

Proof: If $Y \to X$ is surjective, then**?** □

<u>**Noetherian Case**</u>

5 Zariski's Main Theorem

References are [StackProject Chap36.38].

Prop. (III.4.5.1) (Zariski's Main Theorem). For a morphism $X \to S$ that is quasi-finite and separated, if S is qcqs, Then there is a factorization $X \to T \to S$ that $X \to T$ is a qc open immersion and $T \to S$ is finite.

Proof: Cf.[StackProject 05K0]. □

Prop. (III.4.5.2) (Chevalley). Finite $\iff$ quasi-finite+proper.**?**

Proof: The fiber of $f : X \to S$ is $\operatorname{Spec}(k(y) \otimes_A B)$, which is Artinian (I.5.1.18), so it has finitely many primes. Finite morphism is proper because it is integral(III.3.4.38).

For the converse, one should use Zariski's Main Theorem. □

6 Complete Intersection

Should be refreshed with intrinsic definition of locally complete intersection, Cf.[StackProject].

Def. (III.4.6.1) (Locally Complete Intersection). A closed subscheme Y of a nonsingular variety X over a field k is called **locally complete intersection** iff Y is locally generated by $r = \mathrm{codim}(Y, X)$ elements. By(I.6.4.12) Y is C.M.. In particular, by(III.7.1.17), a regular variety is always a locally complete intersection.

Def. (III.4.6.2). A variety Y of codimension r in $\mathbb{P}^n_k$ is a **strict complete intersection** iff $\mathcal{I}_Y$ can be generated by r elements. It is called a **set-theoretic complete intersection** iff it can be written as an intersection of r hypersurfaces.

Prop. (III.4.6.3). A local complete intersection has its ideal sheaf $\mathcal{I}$, then $\mathcal{I}/\mathcal{I}^2$ locally free by(I.6.4.11).

Prop. (III.4.6.4). If Y is a complete intersection in $\mathbb{P}^n_k$ of hypersurfaces of degree $d_1, \ldots, d_r$, then $\omega_Y = \mathcal{O}_Y(\sum d_i - n - 1)$.

Proof: Use the exact sequence $0 \to \mathcal{O}_Z(n-d) \to \mathcal{O}_Z \to i_*\mathcal{O}_Y \to 0$ and(III.7.1.18). □

Prop. (III.4.6.5). For a complete intersection of dimension q, $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$. And the natural map $\Gamma(P, \mathcal{O}_P(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is a surjection for every n. In particular, Y is connected, and the arithmetic genus $p_a(Y) = \dim H^q(Y, \mathcal{O}_Y)$.

Proof: We use induction, the case $Y = P$ follows from(III.5.4.14), let $Y = Z \cap H$, where H has degree d, then

$$0 \to \mathcal{O}_Z(n-d) \to \mathcal{O}_Z \to i_*\mathcal{O}_Y \to 0$$

thus use long exact sequence. The rest is easy. □

Cor. (III.4.6.6). If Y is a nonsingular hypersurface of degree d in $\mathbb{P}^n$, then $p_g(Y) = C^n_{d-1}$. If Y is a non-singular curve which is an intersection of two non-singular hypersurface of degree d, e in $\mathbb{P}^3_k$, then $p_g(Y) = \frac{1}{2}de(d+e-4)+1$.

Proof: Use the long exact sequence to reduce to $\mathbb{P}^n_k$. Cf.[Hartshorne Ex2.8.4]. □

7 More Properties of Schemes

Universal Catenary Ring

Def. (III.4.7.1). A scheme S is called **universally catenary** iff S is locally Noetherian and every scheme locally of f.t. over S is catenary.

Universally catenary is a local property, this follows from(IV.1.13.25).

Prop. (III.4.7.2). A locally Noetherian scheme is universally catenary iff all its stalks are universally catenary. Cf.[StackProject 02JA].

Morphism of Finite Presentation

Def. (III.4.7.3). A morphism between schemes $f : Y \to X$ is called **of locally finite presentation** iff for any point $x \in X$, there is an open affine mapped into an open affine that the ring map is of finite presentation. It is called **of finite presentation** iff moreover it is qcqs.

locally finite presentation is local on the source and target and it is stable under composition and base change but it doesn't satisfies the base change trick by(III.3.4.4)(III.3.4.3) and(III.3.4.2)

Prop. (III.4.7.4). When the target is locally Noetherian, (locally)finite type and (locally)finite presentation is equivalent.

Prop. (III.4.7.5). For $f : X \to Y$ over S, if X/S is locally of f.p. and Y/S is locally of f.t., then f is locally of f.p.. If moreover X is of f.t. and Y is qs, then f is of f.t..

Proof: The first follows from(I.6.7.9), the second needs to check qcqs. Qc follows from(III.3.4.27). □

Finite Locally Free Morphism

Def. (III.4.7.6) (Finite Locally Free). A morphism $f : X \to Y$ is called **finite locally free of rank** d iff it is affine, and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module of rank d.

Finite locally freeness is stable under composition and base change, and it is local on the target.

Cor. (III.4.7.7). If f is finite locally free of rank n, then for any locally free sheaf E of rank k on X, f_*E is locally free of rank nk.

Prop. (III.4.7.8). f is finite locally free iff it is finite flat and of f.p.. In particular, when Y is locally Noetherian, this is equivalent to f is finite flat.

Proof: Both notions are local on the target, so we reduce to the ring case, which is by(I.6.1.4). □

III.5 Cohomology

basic References are [StackProject], [Hartshorne Algebraic Geomeetry] and [Sheaf Cohomology Anonymous] and

1 Acyclic Sheaves

Def. (III.5.1.1). An Abelian sheaf on a site is called **flask** if it satisfies the following equivalent conditions:

- It is acyclic for the forgetful functor ι,
- It is acyclic for any $\check{H}^0(\{U_i \to U\}, -)$
- It is acyclic for all $\Gamma(U, -)$.

Also the class of flask sheaves are adapted to ι.

An Abelian sheaf on a site is called **flasque** iff it is acyclic for all $\mathrm{Mor}(S, -)$ for any S a sheaf of sets, which is obviously flask.

It is called **flabby** iff for any $U \to X$, $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective;

Proof: $1 \iff 3$ is by(III.5.2.9), $3 \to 2$ use Čech to sheaf1(III.5.2.10).

$2 \to 1$: suffices to check(I.9.3.3) for ι, should use ι takes injective to injective, $\check{H}^0(\{U_i \to U\}, -)$ commutes with finite sum and the fact that $\check{H}^1 = H^1$ and long exact sequence. □

Prop. (III.5.1.2). Flabby sheaf is flask. By the way, injective sheaves in the $\mathcal{O}_X$-module category are flabby by(I.8.2.27).

Proof: Just need to verify(I.9.3.3). Injectives are flabby, so it is sufficiently large.

For an exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of sheaves, if $\mathcal{F}$ is flabby, then H is just the presheaf cokernel. (It reduces to $\check{H}^1(\{U_i \to U\}, F) = 0$, and this is done by Zorn's lemma). Thus if $\mathcal{F}$ is flabby, $\mathcal{G}$ is flabby iff $\mathcal{H}$ is flabby(by five lemma). □

Prop. (III.5.1.3). For a morphism os topologies $f : T \to T'$, if F' is a flask sheaf on T', then $f^s F'$ is also flask.

Proof: Notice $H^q(\{U_i \to U\}, f^s F) = H^q(\{f(U_i) \to f(U)\}, F)$. □

Prop. (III.5.1.4). Filtered colimits of flabby sheaves is flabby. (This is because filtered colimits is exact).

Filtered colimits of injective sheaves over a Noetherian topological space is injective. (Use Baer criterion, then notice every sub-object of $\mathbb{Z}_U$ is finitely generated because it has only f.m. connected component(IV.1.13.2) so it maps to some F_α.

Cor. (III.5.1.5). For an injective Abelian presheaf F on T, $F(U)$ is injective Abelian group for every U, this is because the morphism $i : \mathrm{pt} \to T : \mathrm{pt} \mapsto U$ is exact($i_p A(V) = \oplus_{\mathrm{Hom}(V,U)} A$), hence i^p preserves injectives.

Prop. (III.5.1.6). Let I be an injective module over a Noetherian ring A, then the sheaf $\widetilde{I}$ on $\mathrm{Spec}\, A$ is flabby.

Proof: We have for a Qco module over $\mathrm{Spec}\, A$, $\Gamma(U, \widetilde{M}) \cong \varinjlim \mathrm{Hom}(I^n, M)$(III.2.3.8), so if we have two open set $X - V(a)$ and $X - V(b)$, and a, b radical, then the restriction map is induce by the inclusion $b \subset a$, and it is surjective because I is injective and filtered colimits is exact. □

Lemma (III.5.1.7). A constant sheaf on an irreducible topological space is flabby, thus flask.

Prop. (III.5.1.8). If $\mathcal{I}$ is an injective $\mathcal{O}_X$-mod, then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-mod for U open, this is because $-|_U$ is right adjoint to the exact $j_!$.

Prop. (III.5.1.9). If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then for a coherent locally free sheaf $\mathcal{L}$, $\mathcal{L}\otimes\mathcal{I}$ is also injective, because tensoring with $\mathcal{L}$ is adjoint to tensoring with $\mathcal{L}^\vee$(III.2.4.15), which is exact.

Def. (III.5.1.10). A sheaf of modules $\mathcal{F}$ is called **flat** iff $\mathcal{F}\otimes -$ is an exact functor. This is equivalent to the stalks are all exact, because tensor commutes with stalks and use skyscraper sheaf.

Locally free sheaves are flat. There are enough flat sheaves because $j_!\mathcal{O}_U$ is flat and any sheaf of module is a quotient of sums of these.

2 Cohomology on Site

More Materials to add from [StackProject Chap21].

Čech Cohomology

Def. (III.5.2.1). Let X be a site, we have a canonical complex of presheaves $K(U)_\bullet$ w.r.t. an open covering U that is

$$\cdots \to \bigoplus Z_{U_{i_0 i_1 i_2}} \to \bigoplus Z_{U_{i_0 i_1}} \to Z_{U_{i_0}} \to 0.$$

And for any presheaf of $\mathcal{O}_X$-module $\mathcal{F}$, $\mathrm{Hom}_{\mathcal{O}_X}(K(U)_\bullet,\mathcal{F})$ gives out the Čech complex of $\mathcal{F}$. Hence we have: an injective sheaf is Čech acyclic.

This complex has homotopy 0 unless $i=0$. This is because we have a homotopy: choose a fixed i_0, for a $s\in\Gamma(X,U_{i_1\ldots i_n})$, we map it to $(hs)_{ii_1\ldots i_n}=\delta_{i,i_0}s$.

Prop. (III.5.2.2) (Čech-Cohomology). For any U and a cover in a site, the corresponding Čech cohomology is a derived functor on the category of presheaves on a site.

If we take colimit for coverings, $F\to \check{H}^0(U,F)$ is a left exact functor from presheaves to sets, the derived functors are just the limits $\check{H}^q(U,F)$.

Proof: It suffice to prove the Čech cohomology is universal, for this, we only need to prove the sheave defined in(III.5.2.1) is exact then cech cohomology group vanish for $\mathcal{F}$ injective. We check on every V, then the complex can be classified by its image in $\mathrm{Hom}(V,U)$, after that, if we denote $S(\varphi)=\oplus\,\mathrm{Hom}(V,U_i)$, then the complex is of the form

$$\bigoplus_{\varphi\in\mathrm{Hom}(V,U)}(\ldots\to\oplus_{S(\varphi)\times S(\varphi)}\mathbb{Z}\to\oplus_{S(\varphi)}\mathbb{Z})$$

which is easily to seen to be nullhomotopic.

To check the refinement colimit is exact, we sow that the refinement is independent of the refinement map chosen, in this way, this is obviously a filtered colimit which is exact. And this is the content of(III.5.2.3). □

Lemma (III.5.2.3). The refinement morphism of Čech cohomologies of two coverings doesn't depends on the refinement map chosen.

Proof: For two refinement map, there is a commutative diagram

$$\begin{tikzcd} \prod F(U_i) \arrow[r, "d^0"] \arrow[d, "f-g"] & \prod F(U_i \times_U U_j) \arrow[dl, "\Delta^1"] \\ \prod F(U'_j) \end{tikzcd}$$

so it induce the same map on the kernel. □

Prop. (III.5.2.4) (Non-Abelian Čech). For a exact sequence of sheaves of groups $1 \to A \to B \to C \to 1$, where A is in the center of B, then there is a exact sequence:

$$1 \to H^0(U, A) \to H^0(U, B) \to H^0(U, C) \to H^1(U, A) \to H^1(U, B) \to H^1(U, C) \to H^2(U, A)$$

which is by direct calculation, the last one is the Čech composed with the injection to sheaf cohomology(III.5.2.10). Use the same method.

Comparison Theorem for Čech Cohomology

Prop. (III.5.2.5). If two coverings are refinements of each other, then their Čech cohomology is isomorphic.

Proof: Because the refinement morphism doesn't depends on the refinement map(III.5.2.3). □

Prop. (III.5.2.6) (Comparison Theorem for Čech Acyclicity). If there are two coverings $\mathfrak{U}, \mathfrak{V}$ and a presheaf $\mathcal{F}$, then we can construct a double Čech complex with the (p, q)-term being $\mathcal{F}(U_{i_1,\ldots i_p} \cap V_{j_1,\ldots j_q})$. Then the vertical and horizontal arrays calculate the Čech cohomology $\prod_j H^*(\mathfrak{U}|_{V_{j_1\ldots,j_q}}, \mathcal{F})$, $\prod_i H^*(\mathfrak{V}|_{U_{i_1\ldots,i_q}}, \mathcal{F})$ respectively.

So by Spectral sequence(I.9.5.7), if both higher Čech cohomology group $H^k(\mathfrak{U}|_{V_{j_1\ldots,j_q}}, \mathcal{F})$, $H^k(\mathfrak{V}|_{U_{i_1\ldots,i_q}}, \mathcal{F})$ vanish, i.e., they are both $\mathcal{F}$-acyclic, then $H^*(\mathfrak{U}, \mathcal{F}) \cong H^*(\mathfrak{V}, \mathcal{F})$.

Cor. (III.5.2.7). If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, and $\mathfrak{V}|_{U_{i_1,\ldots,i_p}}$ are all $\mathcal{F}$ acyclic, then $H^*(\mathfrak{U}, \mathcal{F}) \cong H^*(\mathfrak{V}, \mathcal{F})$.

Proof: It suffices to prove $\mathfrak{U}|_{V_{i_1,\ldots,i_q}}$ is $\mathcal{F}$-acyclic. But $\mathfrak{U}|_{V_{i_1,\ldots,i_q}}$ and $\mathrm{id}_{V_{i_1,\ldots,i_q}}$ are refinements of each other, so(III.5.2.5) settles the proof. □

Cor. (III.5.2.8). If $\mathfrak{V}|_{U_{i_1,\ldots,i_p}}$ is $\mathcal{F}$-acyclic, then the covering $H^*(\mathfrak{U} \times \mathfrak{V}, \mathcal{F}) = H^*(\{U_i \cap V_j\}, \mathcal{F}) \cong H^*(\mathfrak{U}, \mathcal{F})$.

Proof: Because $\mathfrak{V}|_{U_{i_1,\ldots,i_p}}$ and $\mathfrak{U} \times \mathfrak{V}|_{U_{i_1,\ldots,i_p}}$ are refinement of each other, so $\mathfrak{U} \times \mathfrak{V}|_{U_{i_1,\ldots,i_p}}$ are $\mathcal{F}$-acyclic by(III.5.2.5), and $\mathfrak{U} \times \mathfrak{V}$ refines $\mathfrak{U}$, so(III.5.2.7) can be applied. □

Derived Cohomology

Prop. (III.5.2.9) (Sheaf-Cohomology-Presheaf). The forgetful functor is right adjoint to the exact shifi functor, the Grothendieck spectral sequence applies to the exact functor $\Gamma(U, -)$ from $\mathcal{P}$ to *Ab* shows its right derived functor is

$$\mathcal{H}^p(F) = R^p\iota(F) : U \to H^p(U, F).$$

Prop. (III.5.2.10) (Čech to Sheaf). For any Abelian sheaf $\mathcal{F}$ on a site, the Grothendieck spectral sequence applied to $\Gamma(U,-)=H^0(\{U_i\to U\},-)\circ\iota=\check{H}^0(U,-)\circ\iota$ gives us:

$$H^p(\{U_i\to U\},\mathcal{H}^q(F))\Rightarrow H^{p+q}(U,F).$$

$$\check{H}^p(U,\mathcal{H}^q(F))\Rightarrow H^{p+q}(U,F).$$

Cor. (III.5.2.11). The Grothendieck spectral sequence applied to forgetful functor and exact $\sharp$ functor shows that $\mathcal{H}^p(F)^{++}=\mathcal{H}^p(F)^\sharp=0$ for $p>0$, so

$$\mathcal{H}^p(F)^+(U)=\check{H}^0(U,\mathcal{H}^p(F))=0\quad p>0.$$

because $\mathcal{H}^p(F)^+$ is separated, See(III.1.2.4).

Thus the low degree of Čech to sheaf says:

$$0\to\underline{\check{H}^1(U,F)\to H^1(U,F)}\to 0\to\check{H}^2(U,F)\to H^2(U,F).$$

Cor. (III.5.2.12). If we have $H^q(U_{i_0i_1\dots i_r},\mathcal{F})=0,q>0$, then $H^p(\{U_i\to U\},\mathcal{F})=H^p(U,\mathcal{F})$. (because $H^p(\{U_i\to U\},\mathcal{H}^q(F))$ vanish for $q>0$).

Cor. (III.5.2.13) (G-torsors). If G is a Abelian sheaf on a topology, then $H^1(U,G)$ classifies G-torsor sheaves on U.

Prop. (III.5.2.14) (Čech Acyclic Čech Comparison). Notice the proof extends to any case if there is a family of open subsets $\mathfrak{G}$ that is:

- closed under intersection.
- any admissible covering has a refinement covering that consists of objects in $\mathfrak{G}$.
- higher Čech cohomology $H^q(U,\mathcal{F})$ vanish, for any $U\in\mathfrak{G}$.

Then $H^p(X,\mathcal{F})=\check{H}^p(X,\mathcal{F})=H^p(\{U_i\to X\},F)$ for $U_i\in\mathfrak{G}$.

Proof: By(III.5.2.12), we only have to show that $H^p(U,F)=0$ for $U\in\mathfrak{G}$. Use induction on p, use Čech to sheaf2: $\check{H}^p(U,\mathcal{H}^q(F))\Rightarrow H^{p+q}(U,F)$. The case $p\neq 0$ is by the fact affine covering is cofinal, and condition3, and induction hypothesis. For $p=0$, use (III.5.2.11). □

Prop. (III.5.2.15) (Higher Direct Image). For $f:T'\to T$ a morphism of topologies and $\mathcal{F}$ a sheaf on T, the trivial spectral sequence for $(\sharp\circ f^p)\circ\iota$(because $\sharp,f^p$ are exact) shows that $R^pf^s\mathcal{F}=(f^p\mathcal{H}^p(\mathcal{F}))^\sharp$. So flask sheaf thus flabby sheaf is acyclic for f^s.

Prop. (III.5.2.16) (Leray Spectral Sequence). For $T''\xrightarrow{g}T\xrightarrow{f}T'$ of topologies, for a sheaf $\mathcal{F}'$ on T', there is a spectral sequence

$$E_2^{p,q}=R^pg^s(R^qf^s(F'))\Rightarrow R^{p+q}(fg)^s(\mathcal{F}'),$$

by Grothendieck spectral sequence applied to $g^sf^s=(fg)^s$. (Use(III.5.1.3) and Grothendieck spectral sequence).

Cor. (III.5.2.17) (Leray Spectral Sequence). Let $f:T\to T'$ be a morphism of topologies and $U\in T$, then for a sheaf F', there is a spectral sequence

$$E_2^{pq}=H^p(U,R^qf^s(\mathcal{F}'))\Rightarrow H^{p+q}(f(U),\mathcal{F}').$$

Letting g be the morphism from pt to T that maps pt to U.

In particular, there is a boundary map $H^p(U,f^s\mathcal{F}')\to H^p(f(U),\mathcal{F}')$.

Prop. (III.5.2.18) (Filtered Colimits). $H^n(U,-)$ commutes with filtered colimits if T is Noetherian topology.

Proof: For $n=0$ the limit presheaf is already a sheaf, because for any finite cover, the Čech complex of the limit sheaf is the limit of Čech sheaves, and direct limit is exact.

And the limit sheaf of flask sheaves are flask, because flask need only be checked for finite covers at this case(because T and T^f have equivalent category of sheaves(III.1.1.8) and definition of flask(III.5.1.1)). Then the limit of exact Čech complexes is exact. So we can use the limit of the flask sheaf resolutions to calculate cohomology, thus the result. □

Comparison Lemma

Prop. (III.5.2.19) (Change to Subtopologies). Let T' be a fully subcategory of T, $i: T' \to T$ is a morphism of topologies, and each object U of T' and a covering $\{U_i \to U\}$ in T has a refinement $\{U'_j \to U\}$ in T'. Then

$$H^p(T'; U, i^s F) \cong H^p(T; U, F), \quad H^p(T'; U, F') \cong H^p(T; U, i_s F')$$

Proof: Use(III.1.2.14), for the first one, use Leray spectral sequence, and the fact i^s is exact. The second follows from the fist because $F' \cong i^s i_s F'$. □

3 Cohomology on Ringed Spaces

There are three basic objects, the derived functor for f_* as an Abelian sheaf, f_* as a $\mathcal{O}_X$-module, $\Gamma(U,-)$ as an Abelian sheaf. Notice that an Abelian group is just a $\mathbb{Z}$-module.

Prop. (III.5.3.1) (Grothendieck). The sheaf cohomology of a sheaf over a Noetherian topological space of dimension n vanish for $k > n$.

Proof: Use(III.5.3.2) and(III.5.4.23) and long exact sequence, we can reduce to the case of X irreducible. Then we induct on dimension. Notice first any sheaf is a filtered colimits of sheaf generated by f.m sections, thus we can use(III.5.2.18) to reduce to f.m sections case. And notice $\mathcal{F}_{\alpha'} \to \mathcal{F}_\alpha \to \mathcal{G}$, then G Is generated by at most $|\alpha| - |\alpha'|$ elements, so reduce to the one section case.

Now it is a quotient sheaf of $\mathbb{Z}$, look at the kernel R. If the kernel is $d\mathbb{Z}$ at the generic pt, then $R|_V \cong \mathbb{Z}$ on some nbhd, and $R|_V/\mathbb{Z}$ supports on a lower dimension set, then we only need to consider the pushout of constant sheaf $\mathbb{Z}_U$.

Now there is an exact sequence $0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0$(III.5.3.2), $\mathbb{Z}$ is flabby(III.5.1.7) so flask, and the conclusion follows by induction. □

Prop. (III.5.3.2) (Canonical Exact Sequence). We have a canonical exact sequences of sheaves of modules:

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0$$

$$0 \to i_* i^!_Y \mathcal{F} \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to 0$$

(check on stalks), which is important to use reduction to calculate sheaf cohomology. The latter induces long exact sequences:

$$0 \to H^0_Y(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}|_U) \to H^1_Y(X, \mathcal{F}) \to \cdots$$

Prop. (III.5.3.3). For $f : X \to Y$, if $\mathcal{I}$ is an injective module on X, then $\check{H}^p(\{U_i \to U\}, f_*\mathcal{I}) = 0$ for every open cover for an open subset U(III.5.1.1). This is because Čech cohomology is a derived functor. (Notice $f_*\mathcal{I}$ may not be injective when f is not flat).

Prop. (III.5.3.4). $H^i(X, -)$ commutes with direct limits if X is a qcqs ringed space.

Proof: Cf.[StackProject 01FF]. □

Cor. (III.5.3.5) (Mayer-Vietoris). For $X = U \cup V$, there is a long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \otimes H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \cdots$$

derived from the Čech to sheaf1 because it has only two column, just wrap out the definition.

Prop. (III.5.3.6). For a subscheme of $\mathbb{P}^2_k$ defined by a d-dimensional homogenous polynomial f that $f(1, 0, 0) \neq 0$, then using the two open affines $\{x_1 \neq 0\}$ and $\{x_2 \neq 0\}$, we see that $\dim H^0(X, \mathcal{O}_X) = 1$, $\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$.

Proof: We need to see that $\sum x_0^{a_i} x_1^p / x_2^q \equiv \sum x_0^{b_j} x_2^s / x_1^t \mod f$, where $a_i, b_j < d$. Just look at the degree of x_0. For H^1, notice $\{x_0^{a+b}/x_1^a x_2^b\}$ where $a + b < d$ forms a basis of H^1. □

Higher Direct Image

Prop. (III.5.3.7). For f a morphisms of ringed spaces, $R^p f_* \mathcal{F} = (f_p \mathcal{H}^p(\mathcal{F}))^\sharp$, by(III.5.2.15).

So flask sheaf thus flabby sheaf is acyclic for f_*. When $\mathcal{F}$ has $\mathcal{O}_X$ structure, injective $\mathcal{O}_X$-modules are flabby(III.5.1.2) thus acyclic as Abelian sheaf, so the higher direct image is the same in the category of $\mathcal{O}_X$-module as in the category SAb.

Cor. (III.5.3.8) (Leray Spectral Sequence). The Leray spectral sequence for f_* is applicable in this case, because it is computable in the category of Abelian sheaves, don't stuck in f^* and impose flatness condition.

Prop. (III.5.3.9) (Projection Formula). Let $f : X \to Y$, and $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module, then we have

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong R^i f_*(\mathcal{F}) \otimes \mathcal{E}.$$

Proof: It suffice to prove for $i = 0$, because then we know that $f^*\mathcal{E}$ and $\mathcal{E}$ are locally free thus flat and preserves injectives(III.5.1.9) and then use Grothendieck spectral sequence.

For $i = 0$, there is a map from the right to the left, and there stalk are both $(f_*(\mathcal{F})_x)^{\operatorname{rank} \mathcal{E}}$, so they are equal. □

Base Change

4 Cohomology on Schemes

Lemma (III.5.4.1) (Zariski-Poincare). A Qco sheaf on an affine scheme X is Čech-acyclic.

Proof: Because the principal affine covers are cofinal in the ordering of covers, we only need to consider principal affine covers. Let $R \to A = \prod R_{f_i}$, then it is f.f., so we can use(III.1.6.3), just notice the higher term is $\prod_{i_0, \ldots, i_n} R_{f_{i_0} \cdots f_{i_n}}$. □

Prop. (III.5.4.2) (Separated Čech Derived Coh Equal). For a Qco sheaf $\mathcal{F}$ on a separated scheme, we have $H^p(X,\mathcal{F}) = \check{H}^p(X,\mathcal{F}) = H^p(\{U_i \to X\}, F)$. for U_i any open affine cover.

Proof: Use(III.5.2.14), the family of affine open subsets of X satisfies the requirement because X is separated and(III.5.4.1), thus the result. □

Cor. (III.5.4.3) (Affine Qco Cohomology Vanish). For a Qco sheaf on an affine scheme, $H^i(X,\mathcal{F})$ vanish for $i > 0$. For a Qco sheaf on a qcqs scheme X, the sheaf cohomology vanish for n large enough. (Use check to sheaf2).

Prop. (III.5.4.4) (Compatibility of Qco and $\mathcal{O}_X$-mod). We have in the category of Qco sheaves, injective objects are flabby sheaves, thus nearly calculating all derived functors are legitimate in the category of Qco sheaves(III.5.1.1).

Proof: We use the Deligne formula(III.2.3.8) and the definition of injective, just need to consider the sheaf of ideals of the corresponding induced reduced structure. □

Prop. (III.5.4.5) (Filtered Colimits). If X is qcqs, then sheaf cohomology on X commutes with filtered colimits. (Follows from(III.1.5.4) the same way(III.8.1.7) follows from(III.1.5.9)).

Prop. (III.5.4.6). In the category of $Qco(X)$, we have two Ext, for $\mathrm{Hom}(\mathcal{F},-)$ and $\mathcal{H}om(\mathcal{F},-)$.

We have
$$\mathcal{E}xt^i_X(\mathcal{F},\mathcal{G})|_U = \mathcal{E}xt^i_U(\mathcal{F}|_U,\mathcal{G}|_U),$$
because both gives a universal delta functor for $\mathcal{G}$. In particular, we have $\mathcal{H}om(\mathcal{F},-)$ is exact for $\mathcal{F}$ locally free.

Prop. (III.5.4.7). Ext and $\mathcal{E}xt$ are universal δ functors in $\mathcal{G}$ and a δ functors in $\mathcal{F}$ using injective resolution of $\mathcal{G}$. (Notice injective are acyclic for $\mathcal{E}xt$ because $\mathcal{I}|_U$ is also injective).

Cor. (III.5.4.8). When X is locally Noetherian and $\mathcal{F}$ is coherent, we have
$$\mathcal{E}xt^i(\mathcal{F},\mathcal{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_x}(\mathcal{F}_x,\mathcal{G}_x).$$

Proof: Check local on an affine open, Use a finite locally free resolution and(III.2.4.18), notice the stalk function is exact. □

Cor. (III.5.4.9). If X is locally Noetherian, suppose that every coherent sheaf is a quotient of a locally free sheaf, we can define the **homological dimension** $hd(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ as the minimal length of locally free resolution of $\mathcal{F}$. Then $hd(\mathcal{F}) \le n \iff \mathcal{E}xt^i(\mathcal{F},\mathcal{G}) = 0$ for every $\mathcal{G}$ and every $i > n$. And $hd(\mathcal{F}) = \sup pd_{\mathcal{O}_{X,x}}\mathcal{F}_x$. This follows easily from(III.5.4.8).

Prop. (III.5.4.10). When $\mathcal{L}$ is a locally free sheaf, we have:
$$\mathcal{E}xt^i(\mathcal{F}\otimes\mathcal{L},\mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F},\mathcal{L}^\vee\otimes\mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F},\mathcal{G})\otimes\mathcal{L}^\vee$$
because there are maps between them(III.2.4.15), and $\mathcal{E}xt$ is local, so check locally. In particular,
$$\mathrm{Ext}^i(\mathcal{F}\otimes\mathcal{L},\mathcal{G}) = \mathrm{Ext}^i(\mathcal{F},\mathcal{L}^\vee\otimes\mathcal{G}).$$

Prop. (III.5.4.11). On a Noetherian affine scheme, if M is f.g., then

$$\mathcal{E}xt^i(\widetilde{M},\widetilde{N})\cong \mathcal{E}xt^i(M,N).$$

So on a locally Noetherian scheme, when $\mathcal{F}$ is coherent and $\mathcal{G}$ Qco, $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})$ is Qco and if moreover $\mathcal{G}$ is coherent, then $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})$ is coherent (because this is true for Ext by free resolution).

Proof: Show that they are both universal effaceable. □

Prop. (III.5.4.12). For f proper between locally Noetherian scheme, there is a inverse sheaf $f^!\mathcal{G}=\mathcal{H}om_Y(f_*\mathcal{O}_X,\mathcal{G})$, which maps $Qco(Y)$ to $Qco(X)$ by(III.2.3.6) and(III.5.4.11). When f is affine, in particular when it is finite, then for $\mathcal{F}$ coherent and $\mathcal{G}$ Qco,

$$f_*\,\mathcal{H}om(\mathcal{F},f^!\mathcal{G})\cong \mathcal{H}om_Y(f_*\mathcal{F},\mathcal{G})$$

and when X,Y is separated and X has the resolution property and f is flat, then

$$\mathrm{Ext}^i(\mathcal{F},f^!\mathcal{G})\cong \mathrm{Ext}^i_Y(f_*\mathcal{F},\mathcal{G})$$

is also an isomorphism.

Proof: The first on e is just local check, for the second one, just use Grothendieck spectral sequence and the fact $f_*\mathcal{O}_X$ is locally free thus $f^!$ is exact. □

Prop. (III.5.4.13) (Kunneth Formula). If X,Y are qcqs over a field k and $\mathcal{F},\mathcal{G}$ be Qco $\mathcal{O}_X,\mathcal{O}_Y$-modules, then there is a canonical isomorphism:

$$H^n(X\times_{\mathrm{Spec}\,k}Y,pr_1^*\mathcal{F}\otimes_{\mathcal{O}_{X\otimes_{\mathrm{Spec}\,k}Y}}pr_2^*\mathcal{G})\cong \oplus_{p+q=n}H^p(X,\mathcal{F})\otimes_k H^q(Y,\mathcal{G}).$$

Proof: Cf.[StackProject 0BEF]. □

Cohomology of Proper & Projective Spaces

Prop. (III.5.4.14). Let $X=\mathbb{P}^r_A$ we have:

- $H^i(X,\mathcal{O}_X(n))=0$ for $0<i<r$ and all n.
- There is a perfect pairing

$$H^0(X,\mathcal{O}_X(n))\times H^r(X,\mathcal{O}_X(-n-r-1))\to H^r(X,\mathcal{O}_X(-r-1))\cong A.$$

- Of course for $i>r$, the cohomology vanish because X is separated. And when $n>0$, $H^r(X,\mathcal{O}_X(n-r-1))=0$.

Proof: X is separated, we use Cech cohomology, the second one is easy, $(x_0x_1\ldots x_r)^{-1}$ forms a basis of H^r.

For the first one, induction on r,Cf.[Hartshorne P225]. □

Prop. (III.5.4.15). Let $X=\mathbb{P}^n_k$ and $0\le p,q\le n$, then $H^q(X,\Omega^p_X)=0$ for $p\ne q$ and when $p=q$, $H^q(X,\Omega^p_X)=k$.

Proof: By(III.2.4.17) and(III.2.6.6), there is an exact sequence $0\to\Omega^q\to\wedge^q\mathcal{O}(-1)\to\Omega^{q-1}\to 0$, and the middle has vanishing q-th cohomology by(III.5.4.14), thus we can induct and (III.5.4.14) gives the result. □

Def. (III.5.4.16) (Euler Characteristic). Let X be proper over a field k and $\mathcal{F}$ be coherent, then the **Euler characteristic** of $\mathcal{F}$ is defined to be:

$$\chi(\mathcal{F}) = \sum(-1)^i \dim_k H^i(X, \mathcal{F}).$$

It is definable by(III.5.4.2), and It is clearly an additive functor on $Coh(X)$.

Prop. (III.5.4.17). For a proper scheme X over a field k and $\mathcal{L}_i$ be invertible sheaves on X. Then for any coherent sheaf $\mathcal{F}$ on X,

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r})$$

is a polynomial in $(n_1, \ldots, n_r)$ of total degree at most $\dim \operatorname{Supp} \mathcal{F}$.

Proof: Cf.[StackProject 0BEM]. □

Cor. (III.5.4.18) (Hilbert Polynomial). For a projective scheme over a field k and a coherent sheaf $\mathcal{F}$, there is a polynomial **Hilbert polynomial** P that $\chi(\mathcal{F}(n)) = P(n)$, and $\deg P \leq \dim \operatorname{Supp}(\mathcal{F})$.

This Hilbert polynomial is compatible with the definition in(III.7.2.5), because by(III.5.4.29), the higher cohomology group vanishes for n large, so $\chi(\mathcal{F}(n)) = \Gamma(\mathcal{F}(n)) = \Gamma_*(\mathcal{F})_n$.

Prop. (III.5.4.19). Let $f : Y \to X$ be morphism between schemes proper over field k and $\mathcal{F}$ a coherent sheaf, then we have

$$\chi(Y, \mathcal{F}) = \sum(-1)^i \chi(X, R^i f_* \mathcal{F}).$$

This comes from the Leray spectral sequence.

Def. (III.5.4.20). Thr **arithmetic genus** of a proper scheme of dimension r over a field is defined as $p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1) = (-1)^r(P_X(0) - 1)$. In particular, when X is a curve over a field k, then $p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$(III.7.1.12).

Prop. (III.5.4.21) (Asymptotic Riemann-Roch). If X is a proper scheme over a field k of dimension d and $\mathcal{L}$ is an ample invertible sheaf, then $\dim \Gamma(X, \mathcal{L}^n) \sim cn^d$, Cf.[StackProject 0BJ8].

Prop. (III.5.4.22). Let X be H-projective over a Noetherian affine scheme and $\mathcal{F}$, $\mathcal{G}$ be coherent, then for n large,

$$\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \cong \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n)).$$

Proof: This true for $i = 0$, so let $i > 0$. When $\mathcal{F} = mclO_X$, this is true by(III.5.4.29), and hence true for $\mathcal{F}$ locally free(III.5.4.10), and for $\mathcal{F}$ general, choose a locally free surjective $\mathcal{E} \to \mathcal{F}$ with kernel $\mathcal{G}$, then for n large, there is an exact sequence

$$\operatorname{Hom}(\mathcal{E}, \mathcal{G}(n)) \to \operatorname{Hom}(\mathcal{R}, \mathcal{R}(n)) \to \operatorname{Ext}^1(\mathcal{F}, \mathcal{G}(n))$$

and $\operatorname{Ext}^i(\mathcal{R}, \mathcal{G}(n)) \cong \operatorname{Ext}^{i+1}(\mathcal{F}, \mathcal{G}(n))$. And similarly for $\mathcal{E}xt$. When proving $i = 1$, we need to use(III.5.4.30) to choose n even larger to get the corresponding global section exact sequence. □

Higher Direct Image

Cor. (III.5.4.23) (Sheaf Cohomology Commutes with Affine Map). For f affine and $\mathcal{F}$ Qco, we have $H^n(Y,\mathcal{F}) = H^n(X, i_*\mathcal{F})$ (because $R^i f_*\mathcal{F}(U) = 0$(by Serre criterion(III.3.4.15) and(III.5.3.7)) and then use(III.5.2.16)).

Prop. (III.5.4.24) (Higher Direct Image is Qco and Local). If f is qcqs then $R^n f_*$ maps Qco to Qco, $\underline{R^p f_*\mathcal{F}(U) = H^p(\widetilde{f^{-1}(U)},\mathcal{F})}$.

Proof: check local on affine open of Y, both side are δ-functors from $Qco(X)$ to Mod_Y, injectives in $Qco(X)$ are flabby, thus both are effaceable. We only need to show $f_*\mathcal{F} = \widetilde{\Gamma(X,\mathcal{F})}$, and this is(III.2.3.5). Cf.[Hartshorne P251]. □

Prop. (III.5.4.25). For a qcqs morphism $f : X \to S$, if S is qc, there is a N that for every base change f' of f, we have $R^n f'_*\mathcal{F} = 0$ for every $\mathcal{F}$ Qco and $n \geq N$.

Proof: We check local on affine open and use(III.5.4.24), choose an finite affine cover of X, their intersection are all f.m. affine opens. Then local on a base change, the number of affine opens are the same. So when n is large enough, using Cech to Sheaf2, we have the cohomology vanish.(This uses the fact that the intersection of affine opens are separated and(III.5.4.2). □

Cor. (III.5.4.26). For a qcqs scheme X, the cohomology vanish for n large. And when X is separated and can be covered by r affine opens, then N can chosen to be r.

Prop. (III.5.4.27) (Proper Pushout of Coherent). If $f : X \to Y$ is proper and Y locally Noetherian, then $R^n f_*$ maps coherent to coherent.

Proof: Cf.[StackProject 02O5]. □

Cor. (III.5.4.28). If X is proper over a Noetherian affine scheme, its global section is a f.g. A-module.

Prop. (III.5.4.29) (Serre). If $X \to Y$ is a projective morphism of Noetherian schemes and $\mathcal{F}$ be a coherent sheaf on X, then we have $R^i f_*(\mathcal{F}(n)) = 0$ for $i > 0$ and n large enough.

For this it suffice to prove the local case: If X is projective scheme over a Noetherian affine scheme, $H^i(X,\mathcal{F}(n)) = 0$.

Proof: Because $i_*\mathcal{F}$ is coherent on $\mathbb{P}^r_A$, we reduce to the case $X = \mathbb{P}^r_A$. The conclusion is true for $\mathcal{O}_X(n)$ by(III.5.4.14), and for general $\mathcal{F}$, we use descending induction on i, choose a $\oplus\mathcal{O}_X(n_i) \to \mathcal{F} \to 0$ with kernel $\mathcal{R}$(III.2.5.22), then

$$H^i(X, \oplus\mathcal{O}_X(n_i + n)) \to H^i(X,\mathcal{F}(n)) \to H^{i+1}(X,\mathcal{R}(n)),$$

and the left term vanish for n large(III.5.4.14) thus the result. □

Cor. (III.5.4.30). For any finite exact sequence of coherent sheaves on a H-projective scheme over a Noetherian affine scheme, if tensoring it with $\mathcal{O}(n)$ for large n, the resulting global section is exact.

Kunneth Formula

Base Change

Prop. (III.5.4.31) (Flat Base Change). For a Cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

if g is flat and f is qcqs, then for every Qco sheaf $\mathcal{F}$ on X with base change $\mathcal{F}'$, there is an canonical isomorphism

$$g^*R^if_*\mathcal{F} \cong R^if'_*\mathcal{F}'$$

when S, S' is affine, this reads:

$$H^i(X, \mathcal{F}) \otimes_A B \cong H^i(X', \mathcal{F}').$$

Proof: First we show the existence canonically such a morphism. Choose an injective resolution $\mathcal{I}$ of $\mathcal{F}$ and an injective resolution J of $(g')^*\mathcal{F}$, then we $g_*\mathcal{J}$ is also injective because (g') is exact for flat morphism. Now the canonical map $\mathcal{F} \to (g')_*(g')^*\mathcal{F}$ gives rise to a morphism of complexes from $\mathcal{I} \to (g')_*\mathcal{J}$, thus giving a morphism of complexes $f_*\mathcal{I} \to f_*g'_*\mathcal{J} = g_*f'_*\mathcal{J}$, which gives by adjointness a morphism $g^*f_*\mathcal{I} \to f'_*\mathcal{J}$.

By(III.5.4.24), we only need to check the results affine opens, so let S, S' be affine open. If X is separated, then the cohomology equals Čech cohomology, and the Čech cohomology of $\mathcal{F}'$ is just the cohomology of the Čech complex tensored with B, so it commutes with taking cohomology because B is A-flat.

Now if X is only qs, then we choose an open affine cover(finite) $\{U_i\}$, then all the intersections of these U_i are separated. Now we use Čech-to-sheaf2 spectral sequence(III.5.2.10), then by what we proved for separated case, there is an isomorphism of spectral sequences E_2, so their limit are the same. □

Cor. (III.5.4.32). Let $X \to Y$ be qcqs and Y affine, then for any $y \in Y$, let X_y be the fiber, then $H^i(X_y, \mathcal{F}_y) \cong H^i(X, \mathcal{F} \otimes k(y))$.

Proof: The only problem is to reduce to the case that $Y' = \{y\}$ with the induced reduced structure, because then $\operatorname{Spec} k(y) \to Y$ is flat. All we care is the fiber over y, and $X' = X \times_Y Y'$ is a closed scheme of X, $\mathcal{F}$ pullbacks to $\mathcal{F} \otimes_A A/p_y$, so $H^i(X', \mathcal{F} \otimes_A A/p_y \otimes k(y)) \cong H^i(X, \mathcal{F} \otimes k(y))$.
□

Semicontinuity Theorem

Prop. (III.5.4.33). If X is projective over a Noetherian affine scheme $\operatorname{Spec} A$, and $\mathcal{F}$ is a coherent sheaf on X which is flat over Y, then if we define $T^i(M) = H^i(X, \mathcal{F} \otimes_A M)$ as a functor from A-modules to A-modules, then they form a δ-functor as $\mathcal{F}$ is flat.

And there is a complex $L^\bullet$ of f.g. free A modules bounded above that $T^i(M) \cong h^i(L^\bullet \otimes_A M)$.

Proof: Firstly, the Ćech complex satisfies the requirement, but it is not finite free. But, its cohomology equals $H^i(\mathcal{F})$(III.5.4.2), so(I.8.6.1) can be used. □

Prop. (III.5.4.34). T^i is left exact iff $\mathrm{Coker}\, d^{i-1}$ is a projective A-module, iff it is representable by a finite A-module.

Proof: Denote $W^i = \mathrm{Coker}\, d^{i-1}$, then $\mathrm{Coker}\, d^{i-1} \otimes_A M = W^i \otimes M$, because tensoring is right exact. Thus $T^i(M) = \mathrm{Ker}(W^i \otimes M \to L^{i+1} \otimes M)$. Then for $M' \subset M$, there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T^i(M) \arrow[r] \arrow[d, "\alpha"] & W^i \otimes M' \arrow[r] \arrow[d, "\beta"] & L^{i+1} \otimes M' \arrow[d, "\gamma"] \\ 0 \arrow[r] & T^i(M) \arrow[r] & W^i \otimes M \arrow[r] & L^{i+1} \otimes M \end{tikzcd}$$

γ is injective, so using spectral sequence, its clear α is injective iff β is injective, i.e. W^i is flat, which is equivalent to finite projective(I.6.1.4).

To prove T^i is representable, let $Q = \mathrm{Coker}(L^{i+1,*} \to W^{i,*})$, then Q is finite because W^i is finite(I.6.1.11), and $0 \to \mathrm{Hom}(Q, M) \to \mathrm{Hom}(W^{i,*}, M) \to \mathrm{Hom}(L^{i+1,*}, M)$, but by(I.6.1.12), the last two are just $W^i \otimes M$ and $L^{i+1} \otimes M$, $\mathrm{Hom}(Q, M) = T^i(M)$ by what has already be proved. □

Prop. (III.5.4.35). T^i is right exact iff the morphism $T^i(A) \otimes_A M \to T^i(M)$ are all isomorphism.

Proof: Because T^i and $\otimes$ commutes with direct limit, it suffices to prove for M finite. In this case, choose a finite presentation $A^r \to A^s \to M \to 0$, then there is a diagram

$$\begin{tikzcd} T^i(A) \otimes A^r \arrow[r] \arrow[d] & T^i(A) \otimes A^s \arrow[r] \arrow[d] & T^i(A) \otimes M \arrow[r] \arrow[d] & 0 \\ T^i(A^r) \arrow[r] & T^i(A^s) \arrow[r] & T^i(M) & \end{tikzcd}$$

The first two vertical arrows are isomorphisms, so if T^i is right exact, so does the third vertical arrow. Conversely, if $T^i(A) \otimes_A M \to T^i(M)$ are all isomorphisms, then by a similar diagram, there $T^i(M) \to T^i(M')$ are surjective for $M \to M'$ surjective. □

Cor. (III.5.4.36). T^i is exact iff it is right exact and $T^i(A) = H^i(\mathcal{F})$ is a finite projective A-module.

Proof: Because in this case $T^i(M) \cong T_i(A) \otimes_A M$, so it is exact iff $T^i(A)$ is flat, and it is also finite, so it is equivalent to projective(I.6.1.4). □

Def. (III.5.4.37). For a prime p of A, we define $T^i_p(N) = H^i(L^\bullet_p \otimes N)$, then T^i is (left/right)exact iff T^i_p are all (left/right)exact(exact sequence is stalkwise(I.5.1.27).

Prop. (III.5.4.38). If T^i is (left/right)exact at a point y, then the same is true on a nbhd of y.

Proof: From(III.5.4.34), $(\mathrm{Coker}\, d^{i-1})_y$ is a finite projective A_p module, so it is free. Now $\mathrm{Coker}\, d^{i-1}$ is a coherent sheaf, so it is free at a nbhd of y, so the same is true on a nbhd of y. Now right exactness of T^i is equivalent to left exactness of T^{i+1}, and exact is left exact+right exact, so we are done. □

Prop. (III.5.4.39) (Semicontinuity of Cohomology). Let $X \to Y$ be a projective morphism of locally Noetherian schemes and $\mathcal{F}$ is a coherent sheaf on X, flat over Y, then for each i, $h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$ is an upper semicontinuous function on Y.

Proof: The question is local on Y, so we may assume Y is affine Noetherian. By(III.5.4.31), $H^i(y,\mathcal{F}) = \dim_{k(y)} T^i(k(y))$. And as in the proof of(III.5.4.34), $T^i(M) = \mathrm{Ker}(W^i \otimes M \to L^{i+1} \otimes M)$, and $W^i \to L^{i+1} \to W^{i+1} \to 0$ is exact, so $0 \to T^i(k(y)) \to W^i \otimes k(y) \to L^{i+1} \otimes k(y) \to W^{i+1} \otimes k(y) \to 0$, and counting dimension, $h^i(y,\mathcal{F}) = \dim W^i \otimes k(y) + \dim W^{i+1} \otimes k(y) - \dim L^{i+1} \otimes k(y)$. Notice the last term is constant as L^{i+1} is free A-module and the first two terms are upper semi-continuous by(III.2.3.11), thus $h^i(y,\mathcal{F})$ is upper-semicontinuous. □

Cor. (III.5.4.40) (Grauert). If Y is integral and $h^i(y,\mathcal{F})$ is constant on Y, then $R^i f_*(\mathcal{F})$ is locally free on Y and $R^i f_*(\mathcal{F}) \otimes k(y) \cong H^i(X_y, \mathcal{F}_y)$.

Proof: Following the above proof, we get $\dim W^i \otimes k(y)$ and $\dim W^{i+1} \otimes k(y)$ are all constant. This implies that W^i and W^{i+1} are all locally free, so T^i and T^{i+1} are both left exact, so T^i is exact. So $T^i(A)$ is finite projective by(III.5.4.36). So $R^i f_X(\mathcal{F})$, as equal to $widetildeT^i(A)$, is locally free. The last assertion follows from(III.5.4.32) and(III.5.4.35). □

Prop. (III.5.4.41). To check T^i is right exact, it suffice to check $T^i(A) \otimes k(y) \to T^i(k(y))$ is surjective.

Proof: Cf.[Hartshorne P289]. □

Theorem of Formal Functions

Basic References are [StackProject 29.20].

Prop. (III.5.4.42).

5 Topological Sheaves

Acyclic sheaves

Def. (III.5.5.1). An Abelian sheaf on a paracompact Hausdorff topological space X is called
soft iff is and $\forall$ closed $V, \mathcal{F}(X) \to \mathcal{F}(V)$ is surjective. A flabby sheaf is soft.
fine iff the sheaf of rings $\mathrm{Hom}(\mathcal{F},\mathcal{F})$ is soft.
Fine and soft are local properties (Use Zorn's lemma to construct one-by-one).

Prop. (III.5.5.2). For a sheaf of *unital rings* over a paracompact Hausdorff space X, the following are equivalent,

1. it is a soft sheaf.
2. for any disjoint closed sets V, W, there is a section of X that is 0 on V, and 1 on W.
3. it possesses a partition of unity.
4. it is a fine sheaf.

Note any soft sheaf possesses a partition of unity.

Proof: $1 \iff 2$ is easy and $1 \to 3$ is the to choose a closed locally finite subcover and use Zorn's lemma to construct one-by-one. For $3 \to 1$, notice a closed section can extend to a slightly larger nbhd.

Because for a sheaf of rings $\mathcal{F}$, a partition of unity is equivalent to a partition of unity $\mathrm{Hom}(\mathcal{F},\mathcal{F})$, so 34 are equivalent because 13 are equivalent. □

Cor. (III.5.5.3).

- Note that a fine sheaf possesses a decomposition of section because the previous proposition applies to $\mathrm{Hom}(\mathcal{F},\mathcal{F})$, and a partition of unity in $\mathrm{Hom}(\mathcal{F},\mathcal{F})$ yields a decomposition of section in $\mathcal{F}$. Thus a fine sheaf is soft. (extend to a small nbhd and use partition of unity).
- The sheaf of modules over a soft sheaf of rings is soft, by partition of unity.
- The continuous function sheaf on a paracompact Hausdorff space or the smooth function sheaf on a smooth manifold is fine, thus any smooth module is fine (Use bump function).

Prop. (III.5.5.4). Soft sheaf, e.g. fine sheaf is adapted to $\Gamma(X,-)$. (Similar as in(III.5.1.2), notice flabby is soft and the others are the same as before).

Prop. (III.5.5.5). Let X be a locally compact space of finite compact dimension, when S is a soft sheaf, and one of S and $\mathcal{F}$ is flat, then $S\otimes_k\mathcal{F}$ is soft. Cf.[Cohomology of Sheaves Iversen P319].

Prop. (III.5.5.6). Over a locally compact space of finite dimension, any flat sheaf $\mathcal{F}$ on X has a resolution of soft flat sheaves, Cf.[Gelfand P232].

Comparison Theorems

Lemma (III.5.5.7) (Poincare Lemma). For a smooth manifold X of dimension n, there is an exact sequence

$$0\to\mathbb{R}_X\xrightarrow{d}\Omega^0\xrightarrow{d}\Omega^1\xrightarrow{d}\ldots\xrightarrow{d}\Omega^n\to 0$$

Lemma (III.5.5.8) ($\overline{\partial}$-Poincare Lemma). If X is a complex manifold of dimension n, there are exact sequences:

$$0\to\Omega^p_{hol}\xrightarrow{\overline{\partial}}\Omega^{p,0}\xrightarrow{\overline{\partial}}\ldots\xrightarrow{\overline{\partial}}\Omega^{p,n-p}\to 0$$

Proof: Cf.[Sheaf Cohomology P21]. □

Cor. (III.5.5.9). If X is a complex manifold of dimension n, there are exact sequences:

$$0\to\underline{\mathbb{C}}\xrightarrow{d}\Omega^0_{hol}\xrightarrow{d}\Omega^1_{hol}\xrightarrow{d}\ldots\xrightarrow{d}\Omega^n_{hol}\to 0$$

Proof: This follows from the Poincare lemma(III.5.5.7) and ∂-Poincare lemma(III.5.5.8), by applying the Spectral sequence(I mean in the category of sheaves). □

Cor. (III.5.5.10) (Holomorphic Cohomology). For a complex manifold X,

$$H^p(X,\underline{\mathbb{C}})=H^p(X,\Omega^\bullet_{hol}).$$

Prop. (III.5.5.11) (De Rham). For a smooth manifold and an Abelian group G,

$$H^*_{dR}(X,G)\cong H^*(X,\underline{G})$$

Where the right is constant sheaf cohomology. (Use the fact that smooth sheaf is fine so adapted to sheaf cohomology(III.5.5.4), and Poincare lemma(III.5.5.7)).

Prop. (III.5.5.12) (Singular). For a locally contractible topological space,

$$H^p_{sing}(X,G)\cong H^p(X,\underline{G}).$$

Proof: Shifification of the singular cochain complex is a flabby presheaf resolution of $\underline{G}$ because it is locally contractible, check on stalks. Then we only have to prove $C^\bullet(X) \to (C/V)^\bullet(X)$ is quasi-isomorphism, where V is the presheaf of locally vanishing cochain. It suffice to prove $V^\bullet(X)$ is exact.

For any i-cocycle φ, for any $i-1$-complex σ, use barycentric subdivision, we can construct a c_σ whose boundary is σ and other simplexes on which ϕ vanishes, so we have the coboundary of $\eta : \sigma \to \varphi(c_\sigma)$ is φ. □

Prop. (III.5.5.13) (Dolbeault). For a complex bundle on a complex manifold,

$$H^{p,q}_{\overline{\partial}}(X, \mathcal{E}) \cong H^q(M, \Omega^p_{hol} \otimes_{\mathcal{O}_X} \mathcal{E}),$$

where the left is Dolbeault cohomology and the right is sheaf cohomology. (Use the fact that smooth sheaf is fine so adapted to sheaf cohomology(III.5.5.4), and $\overline{\partial}$-Poincare lemma(III.5.5.8)).

Moreover, there is a spectral sequence of

$$E_1^{p,q} = H^{p,q}_{\overline{\partial}}(X) \Rightarrow E^n = H^n_{dR}(X, \mathbb{R}) \times_{\mathbb{R}} \mathbb{C}.$$

Prop. (III.5.5.14) (Cartan). The class of Coh-Acyclic subsets of an analytic space is exactly the Stein manifold.

Cohomology with Proper Support

References are [Cohomology of Sheaves Iversen].

Prop. (III.5.5.15). For a morphism of locally compact spaces, we can define a **direct image of proper support**:

$$f_!(\mathcal{F})(U) = \{s \in \Gamma(f^{-1}(U), \mathcal{F}) | \operatorname{Supp}(s) \to U \text{ proper}\}$$

This is a subsheaf of $f_*\mathcal{F}$ and it is left exact. we denote $\Gamma_c(X, \mathcal{F})$ as the group $f_!(\mathcal{F})$ where $f : X \to \text{pt}$. And the stalk $f_!(\mathcal{F})_y = \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$ Cf.[Gelfand P224 P225].

Prop. (III.5.5.16). Soft sheaf is adapted to $f_!$ when X, Y are locally compact. Cf.[Gelfand P226]. So we can use soft resolution to define $R^i f_!$, in particular, when $Y = \text{pt}$, we denote it by $H^i_c(X, \mathcal{F})$. Using(III.5.5.15), we get the stalk of $R^i f_!(\mathcal{F})$ at y is just $H^i_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$.

Def. (III.5.5.17). The **compact dimension** of a locally compact topological space is the smallest n that $H^i_c(X, \mathcal{F}) = 0$ for $i > n$. It is also the maximal length of minimal soft resolution.

$\dim_c \mathbb{R}^n = n$, and when Y is an open or closed subset of X, $\dim_c Y \le \dim_c X$. $\dim_c$ is local in the sense if every point has a nbhd of dimension$\le n$, then $\dim_c X \le n$. Cf.[Iversen].

Locally Profinite Space Case

Prop. (III.5.5.18) (Base Change). If $\pi : X \to Y$ is a map between two locally profinite spaces with locally constant sheaves, then $\pi_!$ is exact, and for a pullback diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\tau'"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Z \arrow[r, "\tau"] & Y \end{tikzcd}$$

we have $\tau^{-1}(\pi_! \mathcal{F}) = \pi'_!(\tau')^{-1}\mathcal{F}$.

Cor. (III.5.5.19). If X is locally profinite space, then for any open subset $U \subset X$, $Z = X - U$, there is an exact sequence:

$$0 \to \Gamma_c(U, \mathcal{F}) \to \Gamma_c(X, \mathcal{F}) \to \Gamma_c(Z, \mathcal{F}) \to 0$$

Proof: This the the exact functor $\pi_!$ for $\pi : X \to \mathrm{pt}$ applied to the canonical exact sequence(III.5.3.2). □

III.6 Topics in Schemes

1 Divisors

Weil Divisors

We consider divisors on a locally Noetherian integral scheme. Cartier divisor and Picard Group are more general.

Def. (III.6.1.1). A **prime Weil divisor** is a single closed integral subscheme of codimension 1. A **Weil divisor** is on a locally Noetherian integral scheme is a formal combination of prime Weil divisors that is locally finite. The collection of Weil Divisors are denoted by $\mathrm{Div}(X)$.

Prop. (III.6.1.2) (Principal Weil Divisor). For a rational function $f \in \mathcal{K}$ on a locally Noetherian integral scheme X, for and prime divisor Z with generic pt η, we can define $\mathrm{ord}_Z(f) = \mathrm{ord}_{\mathcal{O}_{X,\eta}}(f)$(I.5.1.17). It is multiplicative, and the closed integral subschemes Z that $\mathrm{ord}_Z(f) \neq 0$ is locally finite.

So, we can define the **principle Weil divisor** $\mathrm{div}(f) = \sum_Z \mathrm{ord}_Z(f)[Z]$.

Proof: There is an open subset U that $f \in \Gamma(U, \mathcal{O}_X^*)$, so all Z are irreducible components of $X - U$, which is locally finite because X is locally Noetherian and(III.3.1.15). □

Def. (III.6.1.3). The **Weil divisor class group** $\mathrm{Cl}(X)$ of a locally Noetherian integral scheme is defined to be $\mathrm{Div}(X)$ modulo principal Weil divisors.

Prop. (III.6.1.4). If X is a Noetherian integral separated scheme regular in codimension 1, then so does $X \times \mathrm{Spec}\,\mathbb{Z}[T]$ and $X \times \mathbb{P}^n_{\mathbb{Z}}$(local check), and $\mathrm{Cl}(X \times \mathrm{Spec}\,\mathbb{Z}[T]) = \mathrm{Cl}(X)$ and $\mathrm{Cl}(X) \times \mathbb{P}^n_{\mathbb{Z}} = \mathbb{Z} \times \mathrm{Cl}(X)$. Cf.[Hartshorne P134].

Prop. (III.6.1.5). For A a Noetherian domain, it is a UFD iff $X = \mathrm{Spec}(A)$ is normal and $\mathrm{Cl}(X) = 0$.

Proof: We only have to show minimal primes of A is principal iff minimal primes of A is a principal divisor. This is done by (I.6.5.6) and (I.3.3.11). □

Cor. (III.6.1.6). The divisors on $\mathbb{P}^n_k$ is is locally defined by a function, this is because the affine opens are UFD.

Prop. (III.6.1.7) (Weil Group of $\mathbb{P}^n_k$). A hypersurface of degree d in P^n_k is equivalent to dH, where H is the surface $x_0 = 0$. This is because irreducible hypersurface of P^n_k correspond to a homogeneous prime ideal of height 1 which is principal. So $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$. Cf.[Hartshorne P132].

Cartier Divisors

Def. (III.6.1.8). A **Cartier divisor** on a scheme is an element in $\Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$. An **effective Cartier divisor** is a Cartier divisor that is locally defined as $\{(U_i, f_i)\}$ where $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$, it is equivalent to a closed subscheme locally defined by a single element.

Notice by definition, $\mathcal{K}$ is the localization w.r.t. non-zero-divisors, and f_i is invertible in $\mathcal{K}^*$ so f_i must be non-zero-divisors.

The **Cartier group** CaCl is the quotient of $\Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$.

Prop. (III.6.1.9) (Weil-Cartier). For an integral normal locally Noetherian scheme, the (effective)Cartier divisor is the same thing as (effective)Weil divisor.(Immediate from(III.6.1.16) and(III.6.1.13)).

This in particular applies to non-singular varieties, by(I.6.5.12).

Cor. (III.6.1.10). If X is Noetherian and the diagonal map is affine, for a dense affine open U, if all the stalk of $X-U$ are UFD, then U is the complement of an effective Cartier divisor.

Proof: The irreducible complements of $X-U$ is finite and has codimension 1 by(III.2.2.11) because $U\to X$ is affine, and it is an effective Cartier divisor by(III.6.1.9)., so their sum will suffice. □

Picard Group

Def. (III.6.1.11). For any ringed space, the **Picard group** is the group of isomorphism classes of invertible sheaves on X, under the tensor operation.

The Picard group is seen via Cech cohomology isomorphic to $H^1(X,\mathcal{O}_X^*)$ by(III.5.2.11).

Def. (III.6.1.12). For a Cartier divisor on a scheme X, we can define $\mathcal{L}(D)$ the **sheaf associated to** D as the sub $\mathcal{O}_X$-module of $\mathcal{K}$ locally generated by (f_i^{-1}), where $D=(f_i)$ locally.

Prop. (III.6.1.13) (Cartier-Pic). For an integral scheme X, the homomorphism $\mathrm{CaCl}(X)\to\mathrm{Pic}(X):D\to\mathcal{L}(D)$ is an isomorphism. (It is alway injective, as it is in fact the δ-functor of the exact sequence of sheaves $0\to\mathcal{O}_X^*\to\mathcal{K}^*\to\mathcal{K}^*/\mathcal{O}_X^*\to 0$.)

Proof: It suffices to show any invertible sheaf can embed into the constant sheaf, tensor with K and restrict to the stalk of the generic point, i.e. there is a compatible choice of homomorphisms into $K(X)$. □

Cor. (III.6.1.14). For an integral separated Noetherian scheme X that is locally factorial, by(III.6.1.9), a Weil divisor is equivalent to an effective line bundle, so giving an integral closed subscheme E of X, $\mathcal{L}(E)$ can be defined, and there is an exact sequence of sheaves on X:

$$0\to\mathcal{L}(-E)\to\mathcal{O}_X\to i_*\mathcal{O}_E\to 0$$

Def. (III.6.1.15) (Weil divisor of an invertible module). For X a locally Noetherian integral scheme and $\mathcal{L}$ an invertible module, if $s\in\Gamma(X,\mathcal{K}_X(L))$ is a meromorphic section of $\mathcal{L}$, for any prime Weil divisor Z with generic pt η, define the $\mathrm{ord}_Z(s)=\mathrm{ord}_{\mathcal{O}_{X,\eta}}(s/s_\eta)$, for any s_η a generator of $\mathcal{L}_\eta$ over $\mathcal{O}_{X,\eta}$. This is independent of s_η chosen.

The prime Weil divisors that $\mathrm{ord}_Z(s)\neq 0$ is locally finite, the same as in(III.6.1.2). And any two different sections s_i defines Weil divisors up to a difference of $\mathrm{div}(f)$(III.6.1.2). So we can define the **Weil divisor class associated to** $\mathcal{L}$ as $\sum\mathrm{ord}_Z(s)[Z]$ for any meromorphic section s of $\mathcal{L}$.

It is easy to verify that this induces a homomorphism from $\mathrm{Pic}(X)$ to $\mathrm{Cl}(X)$.

Prop. (III.6.1.16) (Cl-Pic). For a normal integral Noetherian scheme, the above map $\mathrm{Pic}(X)\to\mathrm{Cl}(X)$(III.6.1.15) is an injection. It is an isomorphism iff all local rings of X are UFD.

Proof: If it is not injective, then some meromorphic section on $\mathcal{L}$ has no associated Weil divisors, then it suffices to show $\mathcal{L}$ is trivialized by s. Consider on an affine subscheme $\mathrm{Spec}\,A$, then $\mathrm{ord}_{A_\mathfrak{p}}(s)=0$ for each minimal prime $\mathfrak{p}$ of A, but $A_\mathfrak{p}$ is DVR by(I.6.5.11) and Serre Criterion(I.6.5.19), so $s\in A_\mathfrak{p}^*$ for each minimal prime $\mathfrak{p}$, so $s\in A^*$ by(I.6.5.6). This will show s trivialize $\mathcal{L}$.

For the surjectiveness, Cf.[StackProject 0BE9]. □

Remark (III.6.1.17). Take $\mathbb{P}_k^n$ for example, the hyperplane $x_0 = 0$ defines a Cartier divisor (x_0/x_i) on U_i, thus it define the subsheaf of $\mathcal{K}^*$ generated by (x_i/x_0) on U_i, thus it is isomorphic to the Serre sheaf $\mathcal{O}(1)$ by multiplication by x_0. The Picard group of $\mathbb{P}_k^n$ are generated by $\mathcal{O}(1)$(III.6.1.7).

Prop. (III.6.1.18). For an integral normal projective scheme of dimension≥ 2 over an alg.closed field, then the support of an effective ample divisor is connected.

Proof: We may assume the divisor is very ample, denote $\mathcal{O}_X(1) = \mathcal{L}(D)$, let Y_q be the closed subscheme corresponding to the divisor qD, then there is an exact sequence

$$0 \to \mathcal{O}_X(-q) \to \mathcal{O}_X \to i_*\mathcal{O}_{Y_q} \to 0$$

(III.6.1.14), so for q large,(III.6.4.7) shows that $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_{Y_q})$ is surjective. But the former is k by(III.7.1.12) and the second contains k, thus the latter is also k, thus it is connected. □

2 Blowing Up

Prop. (III.6.2.1). On a locally Noetherian scheme, the **blowing up** $\widetilde{X}_I$ along a closed scheme (Corresponding to a coherent sheaf) is defined as $\mathrm{Proj}(\oplus I^d)$. It has the universal property that any morphism $Z \to X$ that pulls back Y to an effective Cartier divisor uniquely factors through $\widetilde{X}_I$.

Proof: Notice first an effective devisor is equivalent to an invertible sheaf of ideal. And any morphism $Z \to X$ pulls back I to the image of $f^{-1}I \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to f^{-1}I \cdot \mathcal{O}_Z$. This is just $\mathcal{O}(1)$ on $\widetilde{X}_I$ so invertible.

For the construction, the local uniqueness will implie the existence. Notice locally $\widetilde{X}_I$ is projective over X. Now because the $Z \to X$ pulls back I to an invertible sheaf and it is generated by $f^{-1}(a_i)$, we use**??** to get another $Z \to \mathrm{Proj}_X^n$ and it factors through the closed subscheme $\widetilde{X}_I$. If there is another morphism g, then $f^{-1}I \cdot \mathcal{O}_Z = g^{-1}(\pi^{-1}I \cdot \mathcal{O}_{\widetilde{X}_I}) \cdot \mathcal{O}_Z = g^{-1}(\mathcal{O}_{\widetilde{X}_I}) \cdot \mathcal{O}_Z$ surjective, and a surjective morphism between two invertible sheave is an isomorphism, and they are both ideal sheaves, thus is the same, so this morphism is unique as it is determined by the map on $\mathcal{O}_X$**??**. □

Cor. (III.6.2.2). If the sheaf of ideal is itself invertible, then the blowing up is an isomorphism by the universal property. In particular, on the open set $U = X - Y$, $I_U \cong \mathcal{O}_U$, so $\pi^{-1}(U) \cong U$.

Cor. (III.6.2.3). $\pi : \widetilde{X}_I \to X$ is birational, proper thus surjective. If X is a (complete)variety, then so does $\widetilde{X}_I$.

Prop. (III.6.2.4) (Strict Transformation). Same notation as before, for any locally Noetherian scheme $Z \to X$, we have the pullback sheaf $J = i^{-1}(I) \cdot \mathcal{O}_Z$ on Z, so $\widetilde{Z}_J \to X$ factors through $\widetilde{X}_I$. This a pullback diagram. (Recall the definition of fiber product, we only need to check for Z,X affine and glue. For this, check $B \otimes_A (\oplus I^d) \to \oplus (IB)^d$ defines the fiber map).

Prop. (III.6.2.5). If X is H-(quasi-)projective, then so does $\widetilde{X}_I$ and π is H-projective(III.2.4.13). And any birational projective morphism from another variety Z to X comes from a blowing-up.

Proof: Cf.[Hartshorne P166]. □

Blowing up along a regular variety

Prop. (III.6.2.6). If X is a regular variety over k and Y is a regular closed subvariety defined by $\mathcal{I}$, then the blowing up along $\mathcal{I}$ is also regular, and the inverse image Y' of Y is locally principal in it. In fact, $Y' \to Y$ is isomorphic to $\mathbb{P}(\mathcal{I}/\mathcal{I}^2)$, the projective space associated to the locally free bundle $\mathcal{I}/\mathcal{I}^2$ on Y, and the normal sheaf $\mathcal{N}_{Y'/X'} \cong \mathcal{O}_{\mathbb{P}(\mathcal{I}/\mathcal{I}^2)}(-1)$.

Proof: (Imagine the blowing up of $\mathbb{A}^2$ along $\{0\}$). $X' \cong \operatorname{Proj} \oplus \mathcal{I}^d$ and $Y' \cong \operatorname{Proj} \oplus \mathcal{I}^d/\mathcal{I}^{d+1}$. Then since Y is regular, (I.6.4.12) tells us $\mathcal{I}$ is locally generated by a regular sequence and(I.6.4.11) tells $Y' = \mathbb{P}(\mathcal{I}/\mathcal{I}^2)$. Y' is regular by(I.6.5.10), and then(I.6.5.14) shows that X' is regular also. For the normal sheaf, the defining sheaf $\mathcal{I}' = \mathcal{O}_{X'}$ and then $\mathcal{I}'/\mathcal{I}'^2 = \mathcal{O}_Y(1)$, thus $\mathcal{N}_{Y'/X'} \cong \mathcal{O}_{\mathbb{P}(\mathcal{I}/\mathcal{I}^2)}(-1)$. $\square$

Prop. (III.6.2.7). In a blowing up along a regular variety of codimension $r \geq 2$, There is an isomorphism $\operatorname{Pic} X' \cong \operatorname{Pic} X \oplus \mathbb{Z}$ induced by the Weil divisor exact sequence of $Y' \subset X'$. This is because $r \geq 2$ and (III.6.2.2).

We also have $\omega_{X'} = f^*\omega_X \otimes \mathcal{L}((r-1)Y')$ because $\mathcal{L}(Y') = \mathcal{O}(-1)$ and $\omega_Y \cong \omega_X \otimes \mathcal{L}(D) \otimes \mathcal{O}_Y$ by(III.7.1.18), so it suffice to prove $\omega_{Y'} \cong f^*\omega_X \otimes \mathcal{O}_{Y'}(-r)$. For this, notice for a closed pt of Y, the fiber is a $\mathbb{P}^{r-1}$ because $\mathcal{I}/\mathcal{I}^2$ is locally free of rank r by(III.7.1.17) and the functoriality of $\mathcal{O}(1)$.

3 Derived Category of Schemes

Def. (III.6.3.1). For a ringed space X,

We denote $K(\mathcal{O}_X)$ the complexes of $\mathcal{O}_X$ modules modulo quasi-isomorphisms and $D(\mathcal{O}_X)$ the derived category of Mod-$\mathcal{O}_X$.

We denote $K(Qco(X))$ the complexes of Qco $\mathcal{O}_X$-modules modulo quasi-isomorphisms and $D(Qco(X))$ the derived category of Qco $\mathcal{O}_X$-modules.

We denote $K(Coh(X))$ the complexes of coherent $\mathcal{O}_X$-modules modulo quasi-isomorphisms and $D(Coh(X))$ the derived category of coherent $\mathcal{O}_X$-modules.

Perfect Complex and Pseudo-Coherent Module

Def. (III.6.3.2). A complex of $\mathcal{O}_X$-modules is called **strictly perfect** if it is finite and every term is a direct summand of a finite free sheaf.

Prop. (III.6.3.3). Every mapping from a strictly perfect complex to an acyclic complex has a cover of open sets that on each open set the map is nullhomotopic.

Proof: This is true for a single direct summand of a finite free sheaf, and we can use induction to prove, Cf.[StackProject 08C7]. $\square$

Cor. (III.6.3.4). The strictly perfect complex is fake "K-projective" object in $K(\mathcal{O}_X)$. Note it is note technically K-projective, but it has all the properties of K-projective when proven, noticing the fact it is irrelevant when taken shifification.

Def. (III.6.3.5). We say an object $K^\bullet$ in $K(\mathcal{O}_X)$ **perfect** if there is a an open cover that on each open set there is a quasi-iso $K_i^\bullet \to K^\bullet|_{U_i}$ with $K_i^\bullet$ strictly perfect.

This is equivalent to $K^\bullet$ is locally represented by perfect objects in $D(\mathcal{O}_X)$ by the fact that perfect object is fake K-projective.

Prop. (III.6.3.6). When X is local ringed space, perfectness is equivalent to the fact that it is locally a finite free $\mathcal{O}_{U_i}$-module. This is because direct summand of a finite free module is free, Cf.[StackProject 0BCI].

Resolution Property

Def. (III.6.3.7). A scheme X is said to have **resolution property** iff every Qco $\mathcal{O}_X$-module of f.t. is a quotient of a locally free sheaf.

Prop. (III.6.3.8). If X is Noetherian scheme and has an ample invertible sheaf, then X has the resolution property(III.2.5.22). In fact, every coherent sheaf is a quotient of a finite direct sum of $\mathcal{O}(-n)$.

Prop. (III.6.3.9). If X is qc regular scheme with an affine diagonal, then X has the resolution property, Cf.[StackProject 0F8A]. Conversely, if X is qcqs with the resolution property, then X has affine diagonal. Cf.[StackProject 0F8C].

Prop. (III.6.3.10) (Kleiman). If X is a qc irreducible and locally factorial scheme with affine diagonal map, then X has the resolution property.

Proof: By(III.6.1.10), we have an basis of the form X_s for $s \in \Gamma(X, \mathcal{L})$ for various invertible sheaves, then for any coherent sheaf, it is generated by f.m. sections in $\Gamma(U_i, \mathcal{F})$ and $U_i = X_s$ for $s \in \Gamma(X, \mathcal{L})$, and for each of them, we can use(III.2.5.4), we can extend these to global sections on $\Gamma(\mathcal{F} \otimes \mathcal{L}_i^{n_i})$ for n_i large. Then tensoring $\mathcal{L}_i^{-n}$, we find a $\oplus L_i^{-n_i} \to \mathcal{F}$ surjective. □

Prop. (III.6.3.11). When X has the resolution property, $\mathcal{E}xt^\bullet(-, \mathcal{G})$ is an universal δ-functor for every Qco $\mathcal{G}$, this is because locally free sheaf is adapted to $\mathcal{E}xt^\bullet(-, \mathcal{G})$ by(III.5.4.7), so we can calculate $\mathcal{E}xt(\mathcal{F}, \mathcal{G})$ using a finite locally free resolution of $\mathcal{F}$.

4 Duality for Schemes

References are [Hartshorne Residues and Duality]. The following materials are at a low level, should be refreshed with [StackProject Chap46].

Serre Duality Theorem

Def. (III.6.4.1). Let X be a proper scheme of dimension n over a field k, then a **dualizing sheaf** for X is a coherent sheaf ω_X together with a trace map $H^n(X, \omega_X) \to k$ that for every coherent sheaf $\mathcal{F}$,

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \to k$$

is a perfect pairing. In other words, ω_X represents the functor $\mathcal{F} \to (H^n(X, \mathcal{F}))^\vee$.

Prop. (III.6.4.2). If X is proper over a field k, then there exists uniquely a dualizing sheaf.

Proof: □

Lemma (III.6.4.3). For $X = \mathbb{P}^n_k$, the canonical sheaf ω_X is the dualizing sheaf. Moreover, there is a perfect pairing

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \to k$$

Proof: In this case, $\omega_X = \mathcal{O}_X(-n-1)$. For $i=0$, when $\mathcal{F} = \mathcal{O}_X(n)$, then this follows from (III.5.4.14)and(III.5.4.10). And X has the resolution property(III.6.3.8), so we can write $\mathcal{F}$ as a quotient of two finite direct sum of $\mathcal{O}(-n)$. then the long exact sequence gives us the result as H^{n+1} vanish.

For $i>0$, both side are universal δ-functors, so we show they are both coeffaceable. write $\mathcal{F}$ as a quotient of two finite direct sum of $\mathcal{O}(-n)$ for n large, then $\mathrm{Ext}^i(\mathcal{O}(-n),\omega) = H^i(X,\omega(n)) = 0$ for $i>0$. And $H^{n-i}(X,\mathcal{O}_X(-n)) = 0$ by(III.5.4.14). □

Cor. (III.6.4.4). If X is a closed subscheme of $\mathbb{P}^n_k$ of codimension r, then X has a dualizing sheaf $\omega^0_X = \mathcal{E}xt^r_P(i_*\mathcal{O}_X,\omega_P)$.

Proof: It suffices to prove that $\mathrm{Hom}_X(\mathcal{F},\omega_X) \cong \mathrm{Ext}^r_P(i_*\mathcal{F},\omega_P)$, then the above proposition will give the desired result together with the fact pushforward commutes with sheaf cohomology.

For this, we choose an injective resolution $\mathcal{I}^\bullet$ of ω_X and let $\mathcal{J}^\bullet = \mathcal{H}om_P(\mathcal{O}_X,\mathcal{I}^\bullet)$. Then $\mathcal{J}^\bullet$ are injective $\mathcal{O}_X$-modules because $\mathrm{Hom}_X(\mathcal{F},\mathcal{H}om_P(\mathcal{O}_X,\mathcal{I}^\bullet)) = \mathrm{Hom}_P(\mathcal{F},\mathcal{I}^\bullet)$. And by the lemma(III.6.4.5) below, $\mathcal{J}^\bullet$ is exact up to $r = \operatorname{codim} X$, so it splits and $\omega_X = \operatorname{Coker}\mathcal{J}^n$ hence $\mathrm{Hom}(\mathcal{F},\omega_X) = \mathrm{Ext}^r_P(\mathcal{F},\omega_P)$. □

Lemma (III.6.4.5). Let X be a closed subscheme of $\mathbb{P}^n_k$ of codimension r, then $\mathcal{E}xt^i(\mathcal{O}_X,\omega_{\mathbb{P}^n}) = 0$ for $i<r$.

Proof: Since $\mathcal{F}^i = \mathcal{E}xt^i(\mathcal{O}_X,\omega_P) = 0$ is a coherent sheaf, it suffice to show that $\Gamma(P,\mathcal{F}^i(q)) = 0$ for q large enough. But this equals $\mathrm{Ext}^i_P(\mathcal{O}_X,\omega_P(q))$, which is the dual of $H^{n-i}(P,\mathcal{O}_X(-q)) = H^{n-i}(X,\mathcal{O}_X(-q))$ which vanish by Grothendieck vanishing theorem. □

Prop. (III.6.4.6). Let X be projective of dimension n over a field k and ω^0_X be the dualizing sheaf, then for $\mathcal{F}$ coherent, there is a natural map

$$\mathrm{Ext}^i(\mathcal{F},\omega^0_X) \to (H^{n-i}(X,\mathcal{F}))^\vee$$

And the following are equivalent:

- For any $\mathcal{F}$ locally free on X, $H^i(X,\mathcal{F}(-q)) = 0$ for $i<n$ and q large.
- $H^i(X,\mathcal{O}_X(-q)) = 0$ for $i<n$ and q large.
- This is an isomorphism of δ-functors.
- X is C.M. and equidimensional.

Proof: Notice the left side is an universal δ-functor in $\mathcal{F}$ by(III.6.3.11), so the map exist, and

$2\to 3$: This implies that the right is also universal by(III.6.3.8).

$3\to 1$: For $\mathcal{F}$ locally free, by(III.5.4.10),

$$H^i(X,\mathcal{F}(-q)) = (\mathrm{Ext}^{n-i}(\mathcal{F}(-q),\omega_X))^\vee = (H^{n-i}(X,\mathcal{F}^\vee\otimes\omega_X(q)))^\vee$$

which is 0 for q large.

$4\to 1$: Embed X in $P = \mathbb{P}^N_k$, for $\mathcal{F}$ locally free, since X is catenary, equidimensional is equivalent to $\dim\mathcal{F}_x = n$ for all closed pt x, and C.M. says depth $\mathcal{F}_x = n$. Thus by(I.6.5.16), $pd_{\mathcal{O}_{P,x}}\mathcal{F}_x = N-n$. Thus $\mathcal{E}xt^k_P(\mathcal{F},-)$ vanish for $k>N-n$ checked on stalks.

Now $H^i(X,\mathcal{F}(-q))$ is dual to $\mathrm{Ext}^{N-i}_P(\mathcal{F},\omega_P(q))$ by the proof of(III.6.4.4), which is isomorphic to $\Gamma(P,\mathcal{E}xt^{N-i}_P(\mathcal{F},\omega_P(q)))$ for q large by(III.5.4.22), so it vanish when $i<n$ by what we proved.

$1 \to 4$: The same as the proof of $4 \to 1$, then for $i < n$,

$$\Gamma(P, \mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P(q))) = 0 = \Gamma(P, \mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P)(q))$$

for q large, so $\mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P) = 0$ as it is coherent. Then the stalk is $\mathrm{Ext}_{\mathcal{O}_{P,x}}^{N-i}(\mathcal{O}_{X,x}, \mathcal{O}_{P,x})$, so $pd_{\mathcal{O}_{P,x}}\mathcal{F}_x \leq N - n$ by(I.6.5.17), so depth $\mathcal{O}_{X,x} \geq n$, we must have equality, thus X is C.M. and equidimensional, as it suffice to check at closed pts. □

Cor. (III.6.4.7) (Enriques-Severi-Zariski). Let X be a normal projective scheme that every irreducible component has dimension≥ 2, then for any locally free sheaf $\mathcal{F}$ on X, $H^1(X, \mathcal{F}(-q)) = 0$ for q large.

Proof: Just notice that $\dim \mathcal{F}_x \geq 2$, and Serre criterion shows depth $\mathcal{F}_x \geq 2$, the rest is the same as $4 \to 1$ in the proof of(III.6.4.6). □

Prop. (III.6.4.8) (Dualizing Sheaf on Regular Variety). When X is a closed subscheme of $P = \mathbb{P}_k^n$ which is a local complete intersection of dimension r, then $\omega_X^0 = \omega_P \otimes \wedge(\mathcal{I}/\mathcal{I}^2)^{-1}$, which is an invertible sheaf on X. Notice $\mathcal{I}/\mathcal{I}^2$ is locally free by(III.4.6.3).

In particular, when X is regular projective over a field k, then ω_X is just the canonical sheaf(III.7.1.18)**??**

Proof: Cf.[Harthorne P245]. The basic idea is to use the free resolution of Koszul complex for the stalk of $\mathcal{O}_X$ to calculate $\omega_X = \mathcal{E}xt^r(\mathcal{O}_X, \omega_P)$. It depends on the regular sequence, and the transition of $(\mathcal{I}/\mathcal{I}^2)^{-1}$ neutralize this. □

Cor. (III.6.4.9) (Serre Duality). If X is a regular projective variety, then for any locally free sheaf $\mathcal{F}$, by(III.3.2.2)(III.6.4.6)(III.6.4.8)(III.5.4.10), there is an isomorphism:

$$H^i(X, \mathcal{F}) \cong (H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X))^\vee.$$

Cor. (III.6.4.10). For a projective regular variety over a field k, $H^n(X, \omega_X) = k$, by(III.7.1.12).

Cor. (III.6.4.11). Let X be a regular projective variety of dimension n over an alg.closed field k, $\Omega = \Omega_{X/k}$ is locally free by(III.2.6.5), thus by(III.2.4.16), $\Omega^{n-p} \cong (\Omega^p)^\vee \otimes \omega_X$. So by(III.6.4.9):

$$H^q(X, \Omega^p) \cong (H^{n-q}(X, \Omega^{n-p}))^\vee.$$

Topological Sheaves

Prop. (III.6.4.12) (Global Verdier Duality). If $f : X \to Y$ is a map between locally compact space with finite dimension, then there exists a functor $f^! : D^+(SAb_Y) \to D^+(SAb_X)$ that

$$R\,\mathrm{Hom}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong R\,\mathrm{Hom}(\mathcal{F}, f^!\mathcal{G}^\bullet).$$

In particular, $f^!$ is right adjoint to $Rf_!$. Cf.[Gelfand P228].

There is also a local form of Verdier duality, which implies the global version by taking global section, Cf.[Cohomology of Sheaves Iversen P330].

Prop. (III.6.4.13). When $X \to Y$ is an inclusion of open subset, $f_!$ is just $j_!$ defined in(III.1.4.3) and $f^!$ is the restriction. When it is an inclusion of closed subset of locally compact spaces, it is the direct image f_* and $f^!$ is the $j^!$ previously defined in(III.1.4.3). They are not barely defined on $D^+(SAb)$ but on SAb.

Prop. (III.6.4.14). We consider the case where $f : X \to \mathrm{pt}$, and let $G = \mathbb{Z}$, denote $f^!(\mathbb{Z})$ by $\mathcal{D}_X^\bullet$, called the **dualizing complex**, then there is a duality:

$$R\,\mathrm{Hom}(R\Gamma_c(X, \mathcal{F}^\bullet), \mathbb{Z}) \cong R\,\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{D}_X^\bullet).$$

for $\mathcal{F}^\bullet \in D^+(SAb_X)$.

Prop. (III.6.4.15). When X is a n dimensional topological manifold with boundary, then $\mathcal{D}_X^\bullet = \omega_X[n]$, where the sheaf ω_X is defined by

$$\Gamma(U, \omega_X) = \mathrm{Hom}_{Ab}(H_c^n(U, \mathbb{Z}), \mathbb{Z}).$$

Cf.[Gelfand P234]. If we replace $\mathbb{Z}$ by a field k, then w_X is the sheaf of k-orientations of $\mathrm{int}(X)$, thus the constant sheaf when X is oriented or char $k = 2$?.

In particular, place k in dimension i then we get an isomorphism

$$\mathrm{Hom}_k(H_c^i(X, \mathcal{F}), k) = \mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X)$$

(because k is a field thus injective). Gelfand even gives an interpretation of this pair in [Gelfand P236].

And if $\mathcal{F} = \omega_X$ and X oriented or char$k = 2$, we have $\mathrm{Ext}^{n-i}(k_X, k_X[n]) = H^{n-i}(X, k_X)$ using the adjointness of constant sheaf, so we get the Poincare duality:

$$H_c^i(X, k_X)^\vee \cong H^{n-i}(X, k_X).$$

Prop. (III.6.4.16). Compact cohomology commute with colimits, Cf.[Cohomology of Sheaves Iversen P173].

5 Deformation Theory

Basic references are [StackProject Chap36].

Def. (III.6.5.1) (Thickening). We call X' a **thickening** of a X iff X is a closed subscheme of X' that their underlying topological space are the same. Morphisms of thickenings are defined routinely.

A thickening is said to **have order** n iff the ideal sheaf $\mathcal{I}$ satisfies $\mathcal{I}^{n+1} = 0$.

Base change and composition of a (order n)thickening is also a (order n)thickening, because closed immersion and surjective do.

Def. (III.6.5.2). Let X be a scheme algebraic over a field k and $\mathcal{F}$ is a coherent sheaf on X, then a **infinitesimal extension** of X by the sheaf $\mathcal{F}$ is a scheme X' over k that has a sheaf of ideals $\mathcal{I}$ that $\mathcal{I}^2 = 0$ and $(X', \mathcal{O}_{X'}/\mathcal{I}) \cong (X, \mathcal{O}_X)$, and moreover, $\mathcal{I}$ with the $\mathcal{O}_X$-structure is isomorphic to $\mathcal{F}$.

There is a trivial extension, that is $(X', \mathcal{O}_{X'}) \cong (X, \mathcal{O}_X \oplus \mathcal{F})$, where the multiplication is $(a, f)(a', f') = (aa', af' + a'f)$.

Def. (III.6.5.3) (Deformation). Let X be a scheme algebraic over a field k, an **infinitesimal deformation** of X is a scheme X' flat over $D = k[t]/(t^2)$ that $X' \otimes_D k = X$. A infinitesimal deformation is a first order thickening, by(III.6.5.1).

If Y is a closed subscheme of X, then we define the **infinitesimal deformation of Y in X** to be a closed subscheme $Y' \subset X \otimes_k D$ which is flat over D and $Y' \otimes_D k = Y$.

A scheme algebraic over a field k is called **rigid** if it has no infinitesimal deformations.

Prop. (III.6.5.4) (Affine Case). Any thickening of an affine scheme is affine. (Immediate from(III.3.4.36)).

Prop. (III.6.5.5). Let X be a nonsingular variety over an alg.closed field k, infinitesimal deformation of X is the same as an infinitesimal extension of X by the sheaf $\mathcal{O}_X$. Thus we get the set of infinitesimal deformations of X is parametrized by $H^1(X, \mathcal{T}_X)$, by(III.6.5.7)below.

Proof: For an infinitesimal deformation, tensoring $\mathcal{O}_{X'}$ with the exact sequence $0 \to k \xrightarrow{t} D \to k \to 0$, we get(by flatness)

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_{X'} \to \mathcal{O}_X \to 0,$$

, and conversely, an extension is locally free(because it is f.g. so flat over D is equivalent to free). □

Prop. (III.6.5.6). If X is an affine regular scheme algebraic over an alg.closed field k, then any extension by coherent sheave is trivial.

Proof: For any infinitesimal extension, the morphism $X \to X'$ is a closed immersion and surjection, so X' is also affine by(III.6.5.4), $= \operatorname{Spec} A'$. Now the rest follows from(VI.2.5.7). □

Cor. (III.6.5.7) (Infinitesimal Extension and Cohomology). Let X be a nonsingular variety over an alg.closed field k, then the set of infinitesimal extensions by a coherent sheaf $\mathcal{F}$ is parametrized by $H^1(X, \mathcal{F} \otimes \mathcal{T}_X)$.

If Y is a closed subscheme of X, then the set of infinitesimal deformation of Y in X is parametrized by $H^0(Y, \mathcal{N}_{Y/X})$.

Proof: By the proposition, we know that an infinitesimal extension s locally isomorphic to $(U, \mathcal{O}_X(U) \otimes \mathcal{F}(U))$, by a section $\mathcal{F}(U) \to \mathcal{O}_{X'}(U)$.

But there is a twist, because there can because different sections. But the different sections different at a $\mathrm{Hom}_{\mathcal{O}_X(U)}(\Omega_{\mathcal{O}_X(U)/k}, \mathcal{F}(U)) = (\mathcal{T} \otimes \mathcal{F})(U)$. These forms a Čech cocycle for $\mathcal{F} \otimes \mathcal{T}_X$, and the converse is also true. Finally, use the fact that X is separated so Čech and sheaf cohomology coincide.

For the subscheme, □

Formal Smoothness

6 Intersection Theory

[Hartshorne Ex2.6.2] might be useful.

Setup

Def. (III.6.6.1). The setup is a universally catenary locally Noetherian scheme S endowed with a dimension function δ.

Def. (III.6.6.2). For X/S locally of f.t., the function

$$\delta(x) = \delta(f(x)) + tr.deg_{k(f(x))} k(x)$$

is a dimension function on X. Cf.[StackProject 02JW].

For a closed subscheme Z of X, let $\dim_\delta(Z) = \sup dim_\delta(\eta)$ where η are generic pts of irreducible components of Z.

Def. (III.6.6.3). A **cycle** on a scheme X locally of f.t. over S is a formal sum of integral closed subschemes of X that is locally finite.

Prop. (III.6.6.4) (Cycle associated to a Closed Subscheme). For a closed subscheme Z of a scheme X locally of f.t. over S, if $\dim_\delta(Z) = k$ and $\eta \in Z$ has dimension k, then η is a generic pt of an irreducible component Z' of Z, and $m_{Z,Z'} = \mathrm{length}_{\mathcal{O}_{X,\eta}} \mathcal{O}_{Z,\eta}$ is finite.

So we may define the k**-cycle associated to** Z as: $[Z]_k = \sum_{Z' \subset Z} m_{Z,Z'}[Z']$.

Proof: $m_{Z,Z'}$ is finite because $\mathrm{length}_{\mathcal{O}_{Z,\eta}} \mathcal{O}_{Z,\eta}$ is finite because it is Noetherian and have 0 dimension(I.5.1.18), so also $\mathrm{length}_{\mathcal{O}_{X,\eta}} \mathcal{O}_{Z,\eta}$ is finite. The sum is locally finite by(III.3.1.15). □

Pushforward and Pullback

Prop. (III.6.6.5). For $f : X \to Y$ between schemes integral and locally of f.t. over S, if $\dim_\delta X = \dim_\delta Y$, then either $f(X)$ not dominant or the function field extension is finite. (Because the generic stalk has tr.deg 0). The extension degree d is called the **degree** of f.

Rational Equivalence

Chern Classs

Prop. (III.6.6.6) (Grothendieck-Riemann-Roch).

Proper Intersection

Chow Ring

Prop. (III.6.6.7) (Bezout). The Chow ring of $\mathbb{P}^n_k$ is isomorphic to $\mathbb{Z}[x]/(x^{n+1})$. The degree of an irreducible closed variety corresponds to the coefficient of it.

III.7 Varieties

Basic references are [StackProject] and [Hartshorne].

1 Varieties

Classical variety

Prop. (III.7.1.1). the underlying space of a scheme is sober, Cf.(III.2.2.4).

Prop. (III.7.1.2). For k alg.closed, the soberization functor t induce a fully faithful functor from $\mathrm{Var}(k) \to \mathrm{Sch}(k)$ that maps to quasi-projective integral schemes over k. It maps projective varieties to projective integral schemes. And this functor preserves fiber products **?**.

Proof: We assign the irreducible closed subsets space $t(X)$ and show that this embeds X in $t(X)$, and for an affine variety $(V, \mathcal{O}_V)$, the regular function sheaf is isomorphic to the pullback sheaf on $t(V) = \mathrm{Spec}(A)$.

By definition $t(X)$ is quasi-projective, and for a closed subscheme of $\mathbb{P}^n_k$, the closed pt of any closed subscheme are dense so $t(V)$ is homeomorphic to X. And because they are both reduced, they are isomorphic. So it is essentially surjective.

It is fully faithful because the closed point are equivalent to $k(x) = k$ and is dense in a f.t scheme over k so it maps closed pt to closed pt. □

Prop. (III.7.1.3). The soberization of a classical variety X is regular at a closed point iff the local defining functions has rank $n - \dim X$.

Proof: Consider the space of closed point of X, they correspond to classical points because k is alg.closed. Let $a_p = (x_1 - a_1, \ldots, x_n - a_n)$ and b be the locally defining ideal. Then the differential defines an isomorphism of vector space $a_p/a_p^2 \cong k^n$, and the local ring at p is $m/m^2 \cong (a_p/b)/(a_p/b)^2 \cong a_p/(b + a_p^2)$. The rank of the defining functions is $b + a_p^2/a_p^2$. Counting dimension gives us the result. (Use (III.2.2.9) also). □

Abstract Variety

Def. (III.7.1.4) (Abstract Variety). An **abstract variety** is a geo.integral separated scheme algebraic over a field k. An **prevariety** is an integral separated scheme algebraic over a field k.

A classical variety is an abstract variety because quasi-projective is f.t. and separated(III.3.5.8). It is called **complete** if it is also proper(i.e. universally closed).

A curve over k is an abstract variety of dimension 1. It is called **non-singular** iff all the local rings are regular local.

Cor. (III.7.1.5). An abstract variety is birational to an integral H-quasi-projective scheme. A complete variety is birational to an integral projective scheme by Chow's lemma(III.3.5.10)(III.3.5.3).

Prop. (III.7.1.6). By valuation criterion, for a complete variety, every valuation of the function fields of K/k dominate a unique point of X. So the points of X correspond to valuations of K/k (valuation ring is the maximal local ring).

Prop. (III.7.1.7) (Extension of Morphism). Let X, Y be schemes over S, X is Noetherian and Y is proper. If there is a morphism from an open subset U of X to Y, and there is a point x in the closure of U with the stalk being a valuation ring, then the morphism can be extended to an open set containing x.

Proof: Cf.[StackProject 0BX7]. □

Prop. (III.7.1.8) (Nagata's Theorem). Any abstract variety can be embedded as an open subset of a complete variety.

Prop. (III.7.1.9) (Product of Varieties). The product of two varieties over k is also a variety.

Proof: It is geometrically integral by(III.3.3.4), it is separated because separatedness is stable under composition and base change(III.3.4.2). So does properness. □

Prop. (III.7.1.10). The following categories are equivalent.

- The category of varieties (curves) over k with dominant rational morphisms.
- The dual category of f.g. field extensions over k (of trans.dimension 1).

Proof: Cf.[StackProject 0BXN]. □

Prop. (III.7.1.11). To verify two morphisms f, g between two varieties X and Y are equivalent, it suffices to prove that they are equivalent on the k-point. Because the equalizer is a closed subscheme of X(III.2.7.2), and it contains all closed pts of X, so it must by X, as X is reduced. Thus checking identity of two morphisms between varieties is enough to check on the closed pts $X(\overline{k}) \to Y(\overline{k})$.

Prop. (III.7.1.12) (Global Section). If X is geometrically reduced, connected and proper over a field k, then $\Gamma(X, \mathcal{O}_X) = k$. In particular, this is true for a complete variety over a field k.

Proof: Cf.[StackProject 0BUG]. □

Prop. (III.7.1.13) (Check Varieties on Geometric Points). A nice property of varieties is that identity of two morphisms of products of varieties can be checked at the closed pts $X(\overline{k})$, by(III.7.1.11) and(III.7.1.9).

Also surjective and injective of Qco sheaves need only be checked at closed pts by(III.2.3.10)(III.3.4.30).

Linear System

Prop. (III.7.1.14). A **complete linear system** on a prevariety is the set of effective divisors linearly equivalent to D_0.

When X is a variety, the equivalent divisors correspond to projective space of $\Gamma(X, \mathcal{L}(D_0))$,

Proof: Any divisor equivalent to D_0 defines a global section on $\mathcal{L}(D_0)$. And $\Gamma(X, \mathcal{O}_X^*) = k^*$ by(III.2.4.7). □

Canonical Sheaves

Def. (III.7.1.15) (Canonical Sheaves). For a geometrically regular(smooth) variety X over a field k and Y a local complete intersection of codimension r, by(III.4.2.2) and(III.4.2.10) and(III.4.6.3), $\Omega_{X/k}$ and $\Omega_{Y/k}$ are locally free, and $\mathcal{I}/\mathcal{I}^2$ is locally free, so we can define:

- The **canonical sheaf** $\omega_X = \wedge^n \Omega_{X/k}$ on X.
- The **tangent sheaf** $\mathcal{T}_X = (\Omega_{X/k})^{-1}$ on X.
- The **conormal sheaf** $\mathcal{I}/\mathcal{I}^2$ on Y.
- The **normal sheaf** $\mathcal{N}_{Y/X} = (\mathcal{I}/\mathcal{I}^2)^{-1}$ on Y.

Prop. (III.7.1.16) (Kodaira-Spencer map). There is another characterization of tangent vector fields. (Note: this should be a special case of Prop8.5.9 in [FGA]).

Let X be a variety over k and $S = k[\varepsilon]$ the dual numbers. Then $H^0(X, \mathcal{T}_X) \cong \mathrm{Aut}^{(1)}(X_S/S)$, where $\mathrm{Aut}^{(1)}(X_S/S)$ means that the automorphisms of X_S over S that is identity on X(inclusion to X_S induced by $\mathrm{Spec}\, k \subset \mathrm{Spec}\, S$).

Proof: First the case $X = \mathrm{Spec}\, A$ is affine, then because $H^0(X, \mathcal{T}_X) = \mathrm{Hom}_k(\Omega_{A/k}, A) = \mathrm{Der}(A, A)$, so this is equivalent to $\mathrm{Der}(A, A) \cong$ automorphisms of $A[\varepsilon]$ that is identity under pass to quotients to A. For this, a $d \in \mathrm{Der}(A, A)$ is mapped to $a + b\varepsilon \mapsto a + b\varepsilon + d(a)\varepsilon$. This is checked to be a ring morphism, and any desired morphism are like these.

The above construction is natural and functorial in A, so it glue together to give the global case.
□

Prop. (III.7.1.17) (Geo.Regular and Conormal Sheaf). **?** Let X be a regular variety over an alg.closed field k, then an irreducible closed subscheme Y of X is regular iff $\Omega_{Y/k}$ is locally free and(III.2.6.3) is exact on the left.

In this case, $\mathcal{I}$ is locally generated by r elements and $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf of rank r on Y by(III.4.6.3).

Proof: Cf.[Hartshorne P178]. Should has something to do with(III.4.2.2),(III.4.2.10) and(III.2.6.5). □

Prop. (III.7.1.18) (Adjunction Formulas). For a nonsingular variety X over an alg.closed field k and Y a nonsingular subvariety of codimension r, from(III.7.1.17), $0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$.

Taking the highest exterior power(III.2.4.16), we get:

$$\omega_Y = \omega_X \otimes \wedge^r \mathcal{N}_{Y/X} = \omega_X \otimes (\wedge \mathcal{I}/\mathcal{I}^2)^{-1}$$

In particular, if $r = 1$ then Y is a divisor D in X, the canonical sheaf

$$\omega_Y \cong \omega_X \otimes \mathcal{L}(D) \otimes \mathcal{O}_Y, \quad \omega_{\mathbb{P}^n_k/k} = \mathcal{O}(-n-1)(III.2.6.6).$$

because $\mathcal{I}_Y \cong \mathcal{L}(-Y)$ in this case so $\mathcal{I}_Y/\mathcal{I}_Y^2 = L(-Y) \otimes \mathcal{O}_Y$.

Taking dual, we get:

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X \otimes \mathcal{O}_Y \to \mathcal{N}_{Y/X} \to 0$$

Prop. (III.7.1.19) (Geometric Genus). For a regular proper variety over a field k, the **geometric genus** p_g is defined as the rank of the global section of the invertible canonical sheaf $\omega_X = \wedge^n \Omega_{X/k}$. It is a birational invariance. With the same methods, we can prove the rank of global sections of any other functorially defined bundles of Ω_X is birational invariance, e.g. Hodge numbers.

Proof: For any rational map $U \to Y$, there is a subset $V \in U$ and a local isomorphism V and $f(V)$, that will define an isomorphism of global sections. Because a nonzero section of an invertible sheaf cannot vanish on a dense open set $f(V)$, the morphism of global sections is injective into $\Gamma(U, \mathcal{O}_U)$. Now we find a U that $\operatorname{codim}(X - U) > 1$, then we can use(I.6.5.6) to get $\Gamma(U) = \Gamma(X)$, then $p_g(X) \geq p_g(X')$, and the converse is also true. For this, we use valuation criterion of properness, then for any codimension 1 point, the stalk is a DVR, thus we find a $\operatorname{Spec} \mathcal{O}_p \to X'$, this extends to a nbhd of p because X' is of f.t.. □

Cor. (III.7.1.20). By the exact sequence(III.2.6.6) for $\mathbb{P}^n_k$ and(III.2.4.15), we have $\omega_{\mathbb{P}^n_k} \cong \mathcal{O}(-n-1)$, so it has no global section, $p_g(\mathbb{P}^n_k) = 0$. Hence every rational variety over a field k, i.e. one that is birational to $\mathbb{P}^n_k$, has geometric genus 0.

Complete Variaties

Lemma (III.7.1.21) (Rigidity Lemma). Let X, Y be varieties over a field K. If X is complete and $f : X \times Y \to Z$ that is constant with value $z \in Z(k)$ on some $y \in Y(k)$, then f factors through the projection $pr_Y : X \times Y \to Y$.

Proof: Notice that checking equality of morphisms can be checked on geometric points(III.7.1.13), so we assume $k = \overline{k}$. Now for any $x \in X(k)$, let $g(y) = f(x_0, y)$, then we want to show $f = g \circ pr_Y$. (?I think this step is not necessary).

Let U be affine open in Z, then because X is universally closed, pr_Y is closed, so $V = pr_Y(f^{-1}(Z - U))$ is closed in Y. But if $P \notin V$, then $f(X \times P) \subset V$, but X is complete and connected, so $f(X \times P)$ is constant(III.3.5.4). Now $f = g \circ pr_Y$ on an non-empty subset of $X \times Y$, which is irreducible(III.7.1.9), so this is true on all of $X \times Y$. □

Prop. (III.7.1.22). Let X, Y be varieties that X is complete. If L, M are two line bundles on $X \times Y$ that $L|_y = M|_y$ for all closed points $y \in Y$, then there exists a line bundle N on Y that $L \cong M \otimes pr_Y^*(N)$.

Proof: For any y closed in Y, $L_y \otimes M_y^{-1}$ is trivial on X_y, thus $H^0(X_y, L_y \otimes M_y^{-1}) = k(y)$ by(III.7.1.12). Thus $pr_{Y,*}(L \otimes M^{-1})$ is locally free of rank 1, hence a line bundle.

Now $p^* p_*(L \otimes M^{-1}) \cong L \otimes M^{-1}$, because it is isomorphism on the fibers, so by Nakayama, it is surjective as sheaves, thus isomorphism by comparing rank. □

Cor. (III.7.1.23). If in addition to the above proposition, $L_x \cong M_x$ for some $x \in X$, then $L \cong M$.

Prop. (III.7.1.24). If X is complete variety over a field k and Y is a k-scheme, and L is a line bundle over $X \times Y$, then there is a closed subscheme $Y_0 \hookrightarrow Y$ that is maximal subscheme of Y that $L|_{Y_0}$ is trivial, i.e. $L|_{X \times Y_0}$ is a line bundle $pr_{Y_0}^*(N)$ for some line bundle N over Y_0.

Proof: Cf.[Abelian Variety, van Der Geer, 6.4]. □

Prop. (III.7.1.25) (Theorem of the Cube). If X, Y are complete varieties over a field k, and Z is a connected, locally Noetherian k-scheme, if x, y, z are points of X, Y, Z respectively, and L is a line bundle on $X \times Y \times Z$ that is trivial on $x \times Y \times Z, X \times y \times Z, X \times Y \times z$, then L is trivial.

Proof: A field extension is faithfully flat, thus a line bundle is trivial iff its base change of fields is trivial**?**. Thus we can assume that x, y, z are both rational points.

Let Z_0 be the maximal closed subscheme of Z that L_z is trivial on Z_0. We show that Z_0 is open, thus it is all of Z: If $\zeta \in Z'$, let $I \subset \mathcal{O}_{Z,\zeta}$ be the ideal defining Z', we show $I = (0)$. If not, then because $\cap \mathfrak{m}^n = 0$ by Krull's theorem(locally Noetherian used), there is an n that $I \subset \mathfrak{m}^n, I \nsubseteq \mathfrak{m}^{n+1}$. Now let $a_1 = (I, \mathfrak{m}^{n+1})$, and $\mathfrak{m}^{n+1} \subset \mathfrak{a}_2 \subset a_1$ that $\dim_{k(\zeta)}(a_1/a_2) = 1$, and let $Z_i \subset \operatorname{Spec} \mathcal{O}_{Z,\xi}$ be the closed subscheme defined by a_i. Let L_i be the restriction of L on $X \times Y \times Z_2$. If we show that L_2 is trivial, then Z_2 is contained in Z_0, which is contradiction because $I \nsubseteq a_2$.

For this, notice that L_1 is trivial, and to show that L_2 is trivial, it suffices to lift a non-vanishing global section s of L_1 to L_2, because Z_1, Z_2 has the same underlying set.

For this, notice that the obstruction of the lifting is a $\xi \in H^1(X \times Y, \mathcal{O}_{X\times Y})$**?**. But now the conditions show that ξ is zero under the pullback along $x \times Y \hookrightarrow X \times Y$ and $X \times y \hookrightarrow X \times Y$. So by Kunneth formula(III.5.4.13) and(III.7.1.12), ξ vanishes. □

2 Projective Variety

Prop. (III.7.2.1) (Bertini). Any regular projective variety over k alg.closed with f.m singular pt has a hyperplane that intersect it with a regular variety. These hyperplanes form an open dense subset of the complete linear system $|H|$ of $\mathbb{P}^n_k$, Cf.[Hartshorne P179].

Cor. (III.7.2.2). When $\dim X \geq 2$, this is even a regular variety by(III.6.1.18)and(III.3.2.6).

Prop. (III.7.2.3) (Affine Dimension Theorem). For two affine variety Y, Z of dimension r, s in $\mathbb{A}^n_k$ over fields, there intersection has every component dim $\geq r + s - n$.

Proof: The theorem follows from Krull's theorem(I.5.6.9) when $Y = H$, and for the general case, notice $Y \cap Z \cong (Y \times Z) \cap \Delta$ in $\mathbb{A}^n \times \mathbb{A}^n$. □

Cor. (III.7.2.4) (Projective Dimension Theorem). For two projective variety Y, Z of codimension r, s in $\mathbb{P}^n_k$ over fields, there intersection has every component of codimension$\leq r + s$.

Proof: First prove this for $Y = H$, then we can either induct or directly from the theorem above. For this, we just use Krull's theorem(I.5.6.9). □

Degree of Projective Varieties

Basic References are [Hartshorne I.7].

Def. (III.7.2.5) (Hilbert Polynomial). For a scheme projective over a field k of dimension r, we define the **Hilbert polynomial** P_Y as the Hilbert polynomial of its homogenous coordinate ring $\Gamma_*(Y)$. It has dimension r by(I.5.5.8).

The **degree** of Y is defined as the $r!$ times the leading coefficients of P_Y.

Prop. (III.7.2.6).

- The degree is a positive integer.

- If $Y = Y_1 \cup Y_2$ and $\dim Y_1 \cap Y_2 < r$, then $\deg Y = \deg Y_1 + \deg Y_2$.
- If H is a hypersurface whose ideal is generated by a homogeneous polynomial of degree d, then $\deg H = d$.

Proof: Cf.[Hartshorne P52]. □

Prop. (III.7.2.7). For a variety of degree k and a general linear space, the intersection has k points.

3 Curves

Basic references are [Hartshorne Chap4] and [StackProject Algebraic Curves].
Notice a criterion for a theorem to be put into this subsection is that it has condition $\dim X \leq 1$.

Def. (III.7.3.1). A **(pre)curve** is a (pre)variety over a field k of dimension 1.

Prop. (III.7.3.2). A Noetherian separated scheme of dimension 1 has an ample invertible sheaf.

Proof: First reduce to the case when X is reduced. This is because(III.7.3.25) shows this invertible sheaf is a pullback of a sheaf of X and(III.2.5.15) shows this sheaf is ample.
Second we reduce to the case X is integral. Cf.[StackProject 09NX]. □

Cor. (III.7.3.3) (Complete Curve Projective). A proper scheme of dimension 1 over a field k is H-projective, by(III.7.3.2) and(III.2.5.18).

Prop. (III.7.3.4). A separated scheme of f.t. of dimension≤ 1 over a field k is a H-projective scheme $\overline{X}$ called the **completion** of X minus f.m. closed pts. And when X is reduced, the stalk are discrete valuation rings at these closed pts. Cf.[StackProject 0BXV,0BXW].

Cor. (III.7.3.5). A morphism of varieties $X \to Y$ with X a curve and Y proper over a field k factors through the completion $\overline{X}$ of X by(III.7.1.7).

Prop. (III.7.3.6) (Affine or Projective). A precurve over a field k is either affine(not proper) or H-projective(proper).

Proof: Cf.[StackProject 0A27]. □

Cor. (III.7.3.7). Let X be a separated scheme algebraic over a field k. If $\dim X \leq 1$ and no irreducible component of X is proper of dimension 1, then X is affine.

Proof: Let X_i be f.m. irreducible components of X, then they are precurves in the induced reduced structure, so they are affine by(III.7.3.6). Now $\coprod X_i \to X$ is a finite surjective morphism, so X is affine by(III.3.4.36). □

Cor. (III.7.3.8). Two birationally equivalent complete curve are isomorphic. Thus if a curve is birationally equivalent to another complete curve, then it is an open immersion, by(III.7.3.4).

Prop. (III.7.3.9) (Non-constant Morphism Finite). Let $f : X \to Y$ be a morphism of schemes over a field k that Y is separated and X is proper of dimension ≤ 1. If the image of every irreducible component of X is not a pt, then f is finite.

Proof: Cf.[StackProject 0CCL]. □

Lemma (III.7.3.10). For an Noetherian integral scheme of dimension 1, there is an isomorphism $\mathcal{K}/\mathcal{O}_X \to \sum_p i_*(\mathcal{K}/\mathcal{O}_p)$.

Proof: Check on stalks, this is because closed subsets are finite. □

Nonsingular Curves

Lemma (III.7.3.11) (Extension of Morphism). Rational map from a non-singular curve to a proper variety can be extended to a morphism. This is a consequence of(III.7.1.7).

Prop. (III.7.3.12) (Category of Non-singular Complete Curves). The category of non-singular complete curves over a field k with non-constant morphisms is the opposite category of f.g. field extenision of k of trans.deg 1.

Proof: First a non-constant morphism maps the generic pt to the generic pt, thus inducing a map of function fields, and a map of there function fields induce a birational map by(III.7.1.10), and this extends to a morphism by(III.7.3.11).

It's left to show that any these fields is a function fields, for this, Cf.[StackProject 0BY1]. □

Cor. (III.7.3.13). It follows from this that two birational equivalent non-singular proper curve over a field is isomorphic.

Cor. (III.7.3.14) (Non-singular Projective Model). Comparing this and(III.7.1.10), we see that every curve over k correspond to a unique non-singular proper curve over k with the same function field, which is called the **non-singular projective model**.

Prop. (III.7.3.15) (Flatness and Associated Points). $f : X \to Y$ with Y integral and regular of dimension 1. Then f is flat iff every associated prime of X is mapped to the generic point of Y.

In particular when X is reduced, this is equivalent to every irreducible component of X dominants Y, by(I.5.3.14).

Proof: If x is mapped to a closed pt of Y, then $\mathcal{O}_{y,Y}$ is a DVR, let t be a uniformizer, then t is not a zero-divisor, and $f^\sharp(t) \in \mathfrak{m}_x$ is also not a zero-divisor. So x is not an associated point.

Conversely, to show f is flat, if y is the generic point, then $\mathcal{O}_{y,Y}$ is a field, so it is flat. When y is a closed pt, $\mathcal{O}_{y,Y}$ is a DVR, so by(I.3.3.6), we need to show that it is torsion free. If it is not, then $f^\sharp(t)$ must be a zero-divisor for a uniformizer t of $\mathcal{O}_{y,Y}$. But then it is contained in some associated prime p of $\mathcal{O}_{x,X}$(I.5.3.8). Now p is mapped to y, which is a contradiction. □

Cor. (III.7.3.16). If $f : X \to Y$ is a dominant morphism from a variety to a nonsingular curve over k, then f is flat.

Cor. (III.7.3.17). Let Y be integral and regular of dimension 1 and P a closed pt. X is a closed subscheme in $\mathbb{P}^n_{Y-P}$ that is flat over $Y - P$, then there is a unique closed subscheme $\overline{X}$ closed in $\mathbb{P}^n_Y$ that is flat over Y and restrict to X on $\mathbb{P}^n_{Y-P}$.

Proof: Choose the scheme-theoretic closure of X in $\mathbb{P}^n_Y$. Cf[Hartshorne P258]. □

Cor. (III.7.3.18). Combining this with(III.7.3.9), we say that a morphism between two non-singular curves are finite flat.

Prop. (III.7.3.19). A projective non-singular curve of degree d in $\mathbb{P}^n_k$, where $d \leq n$ not contained in any $\mathbb{P}^{n-1}_k$ is isomorphic to the n-tuple embedding, and $d = n$.

This has easy generalization to surfaces and higher dimensions.

Proof: (III.6.1.17)shows $\mathcal{O}_X(1) \cong \mathcal{O}(d)$ over $\mathbb{P}^1_k$, and the restriction of global sections is injective. So the global section is an isomorphism, and it defines the embedding up to a linear automorphism. □

Cor. (III.7.3.20) (Genera Equal). For a complete curve over a field k,

$$p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X)$$

by Serre duality(III.7.1.19) and(III.5.4.20).

So whenever talking about a complete curve, I will just call it genus without discriminant.

Prop. (III.7.3.21). Any complete regular curve is rational iff it is has genus 0.

Proof: Cf.[Hartshorne P297]. □

Divisors on Curves

Def. (III.7.3.22). If X is a locally Noetherian integral scheme of dimension 1, then the Weil divisors of X are just locally finite formal sums of closed pts of X.

If X is Noetherian integral algebraic over a field k of dimension 1, then the sum is in fact finite, we we can define the **degree** of a divisor $D = \sum n_P P$ as $\deg(D) = \sum n_P[k(P); k]$.

The **canonical divisor** K of X is the Weil divisor associated to the canonical sheaf $\mathcal{K}_X$, up to equivalence.

If X is a regular, then $\deg(D) = \deg(\mathcal{L}(D))$, by(III.7.3.31).

Prop. (III.7.3.23). For a finite morphism f between two non-singular curves over alg.closed field, e.g. dominant morphism between complete non-singular curves, $\deg f^*D = \deg f \cdot \deg D$. This is because f is finite locally free(III.7.3.18), thus this follows from [StackProject 02RH].

Prop. (III.7.3.24). An element $\notin k$ in the function fields of a projective non-singular curve over an alg.closed k defines a inclusion $k(f) \subset K(X)$ thus a morphism from X to P^1_k(III.7.1.10), and $(f) = \varphi^*(\{0\} - \{\infty\})$.

Prop. (III.7.3.25). If $Z \to X$ is a closed immersion and $\dim X \leq 1$, then Pic $X \to$ Pic Z is a surjection.

Proof: Use the exact sequence $0 \to (1 + \mathcal{I})\mathcal{O}_X^* \to \mathcal{O}_X^* \to i_*\mathcal{O}_Z^* \to 0$, $\dim X \leq 1$ and the Grothendieck vanishing theorem gives the desired result, also notice i is affine. □

Prop. (III.7.3.26). For a 1-dimensional integral scheme proper over k and a function $f \in K(X)^*$,

$$\sum_{x \text{ closed}} [k(x) : k]\mathrm{ord}_{\mathcal{O}_x}(f) = 0.$$

Cf.[StackProject 02RU].

Riemann-Roch

Def. (III.7.3.27) (Degree). The **degree** of a locally free sheaf $\mathcal{E}$ of rank n on a proper scheme X of dimension ≤ 1 over a field k is defined to be $\deg(\mathcal{E}) = \chi(X, \mathcal{E}) - n\chi(X, \mathcal{O}_X)$, where χ is the Euler characteristic(III.5.4.16).

If X is integral(hence complete precurve), then this definition can extends to any coherent sheaves $\mathcal{F}$, if we define $\mathrm{rank}(\mathcal{F}) = \dim_{k(\eta)} \mathcal{F}_\eta$.

Prop. (III.7.3.28). The degree function is stable under base change of fields, additive, and stable under birational equivalence of proper scheme X of dimension ≤ 1 over a field k.

Proof: The base change follows from flat base change(III.5.4.31), the additivity follows from that of rank and Euler characteristic(III.5.4.16).

For the birational equivalence, Cf.[StackProject 0AYU]. $\square$

Prop. (III.7.3.29) (Riemann-Roch). Let D be a Weil divisor on a complete regular precurve X of genus g, if $l(D) = H^0(X, \mathcal{L}(D))$, $l(D)$ is finite by(III.5.4.27).

$$l(D) - l(\omega_X - D) = \deg D + 1 - g.$$

Proof: As X is H-projective by(III.7.3.6), so using Serre duality(III.6.4.9), it suffices to show $\chi(\mathcal{L}(D)) = \deg D + 1 - g$.

if $D = 0$, then this follows by(III.7.1.12). Any divisor is a sum of closed pts of X, so we can use induction. By(III.6.1.14), if P is a closed pt and D is a Weil divisor, then

$$0 \to \mathcal{L}(D) \to (D + P) \to k(P) \to 0$$

so by additivity, $\chi(\mathcal{L}(D + P)) = \chi(\mathcal{L}(D)) + [k(P) : k]$, so the induction is finished. $\square$

Cor. (III.7.3.30). If D is an effective Cartier divisor on a proper scheme of dimension≤ 1 on a field k, then for any locally free sheaf $\mathcal{E}$ of rank n, $\deg(\mathcal{E}(D)) = n \deg(D) + \deg(\mathcal{E})$.

Proof: Cf.[StackProject 0AYY]. $\square$

Cor. (III.7.3.31). $\deg(D) = \deg(\mathcal{L}(D))$, in fact, this is just equivalent to Riemann-Roch.

Cor. (III.7.3.32). It is clear that if $l(D) > 0$, then $\deg D > 0$. So for any D with $\deg X > 0$, for n large, $l(nD) = n \deg D - g + 1$.

Cor. (III.7.3.33). $\deg K = 2g - 2$.

Proof: Apply Riemann-Roch with $D = K$, then $l(K) = g$ and $l(0) = 1$, so $g - 1 = degK + 1 - g$, thus the result. $\square$

Prop. (III.7.3.34) (Ampleness and Degree). For an invertible $\mathcal{O}_X$-module $\mathcal{L}$ over a precurve proper over a field k, $\mathcal{L}$ is ample iff $\deg(\mathcal{L}) > 0$.

Proof: Cf.[StackProject 0B5X]. $\square$

Cor. (III.7.3.35). Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module over a proper scheme of dimension≤ 1 over k, let C_i be the irreducible components of X of dimension 1, then $\mathcal{L}$ is a ample iff $\deg(\mathcal{L}|_{C_i}) > 0$ for all i.

Proof: Consider $(\coprod_i C_i) \coprod (\coprod_j x_j)$ where x_i are closed pts of X, then by(III.2.5.14), $\mathcal{L}$ is ample iff the pullback is ample. And use the last proposition and the fact on a pt it is obviously ample. $\square$

Prop. (III.7.3.36). a non-singular curve in $\mathbb{P}^2_k$ where char $k \neq 0$ is projectively isomorphic to $xy - z^2$ if it has a rational point. (Use Riemann-Roch to show that $O(p)$ has a nontrivial section which gives a isomorphism to P^1). And in fact the assertion can be checked directly.

Residues

Prop. (III.7.3.37). Let X be a complete regular curve over an alg.closed field k, K be the function field, then for any closed pt P, there is a unique k-linear map $res_P : \Omega_{K/k} \to k$ with the following properties:

- $\mathrm{res}_P(\tau) = 0$ for $\tau \in \Omega_P$, where Ω_P is the stalk of the canonical sheaf at P.
- $\mathrm{res}_P(f^n df) = 0$ for $f \in K$ and $n \neq -1$.
- $\mathrm{res}_P(f^{-1}df) = v_P(f)$, where v_P is the valuation associated to P.

Prop. (III.7.3.38) (Residue Theorem). For every $\tau \in \Omega_{K/k}$, we have $\sum \mathrm{res}_P\, \tau = 0$.

Picard Scheme of Curves

Basic References are [StackProject Chap43].

4 Surfaces

Prop. (III.7.4.1). Any birational transformation of non-singular surfaces will be factorized into f.m blowing-ups and blowing-downs of points.

Resolution of Surfaces

Cf.[StackProject Chap51].

5 Others

Prop. (III.7.5.1). Variety is triangulable.

Proof: Cf.[Hironaka Triangulation of Algebraic Sets]. □

III.8 Étale Cohomology

Basic references are [Étale Cohomology Fulei], [StackProject] and [Etale Cohomology Tamme].

1 Basics(Tamme Level Stuff)

Prop. (III.8.1.1) (Zariski-Étale Comparison). Considering the inclusion $\varepsilon : X_{Zar} \to X_{ét}$ of topologies, for any Abelian sheaf F on $X_{ét}$, there is a Leray spectral sequence(III.5.2.16)

$$E_2^{pq} = H^p_{Zar}(X, R^q\varepsilon^s(F)) \Rightarrow H^{p+q}_{ét}(X, F).$$

Def. (III.8.1.2) (Pushforward & Pullback). Denote $\widetilde{X_{ét}}$ as the category of sheaves on $X_{ét}$. For a morphism of schemes $X \to Y$, there is a morphism of topologies $f_{ét} : Y_{ét} \to X_{ét}$, and we define

$$f_* = (f_{ét})^s : \widetilde{X_{ét}} \to \widetilde{Y_{ét}}, \quad f^* = (f_{ét})_s : \widetilde{Y_{ét}} \to \widetilde{X_{ét}}$$

f^* is called the inverse image, it is exact because $f_{ét}$ preserves fiber product and final object(III.1.2.9). So it is a morphisms of sites $X_{ét} \to Y_{ét}$. $f^*G(X')$ equals the colimit over all Y' that $X' \to Y' \times_Y X$, equivalently, all $X' \to Y'$ over Y, by definition(III.1.2.6).

Cor. (III.8.1.3). For a $f : X' \to X$ étale, f^* induce a morphism of topoi, that $F/X'(Z') = f^*F(Z') = F(Z')$, and $H^q(X_{ét}; Z', F) \cong H^q(X'_{ét}; Z', F/X')$, by(III.5.2.19) and(III.1.2.14).

Prop. (III.8.1.4). If Z is étale over X, then the canonical morphism

$$f^* \mathrm{Hom}_X(-, Z) \to \mathrm{Hom}_Y(-, Z \times_X Y)$$

is an isomorphism.

Proof: By definition, $f^* \mathrm{Hom}_X(-, Z)$ is the sheaf associated to the presheaf $f_p \mathrm{Hom}_X(-, Z)$(III.1.2.6), which is identical to the presheaf $\mathrm{Hom}_Y(-, Z)$ on $Y_{ét}$, but it is already a sheaf(III.1.5.21). □

Prop. (III.8.1.5) (Leray Spectral Sequence). For any Abelian sheaf on $X_{ét}$ and any étale scheme Y'/Y, there is a Leray spectral sequence(III.5.2.17):

$$E_2^p = H^p(Y', R^q f_*(F)) \Rightarrow H^{p+q}(Y' \times_Y X, F)$$

Prop. (III.8.1.6) (Leray Spectral Sequence). If $f : X \to Y, Y \to Z$ is a morphism of schemes, then for any sheaf on $X_{ét}$, there is a Leray spectral sequence(III.5.2.16)

$$E_2^{pq} = R^p g_*(R^q f_*(F)) \Rightarrow R^{p+q}(gf)_*(F)$$

Prop. (III.8.1.7) (Commutes with Colimits). If X is qcqs, then by(III.1.5.8)(III.1.5.9) and(III.5.2.18), $H^q_{ét}(X, -)$ commutes with filtered colimits.

Field Case

Prop. (III.8.1.8) (Etále Site on Fields). The functor $f: X' \to X'(k_s)$ is an equivalence of topologies from the small étale site $(\mathrm{Spec}(k))_{ét}$ to the canonical topology T_G on the category of G-sets, where $G = G(k_s/k)$.

In particular, any Abelian sheaf on $(\mathrm{Spec}(k))_{ét}$ is representable by**?**.

Proof: First f maps a family of morphisms of schemes to a covering iff this family is a covering itself. This is because both are defined by set-theoretical surjectivity, and this is by(III.4.4.11).

Next we need to show this is an equivalence of categories. f has a left adjoint g because $X' \to \mathrm{Hom}_G(U, X'(k_s))$ is representable for any G-set U, because any G-set is equivalent to disjoint sums of G/H, and both category has arbitrary sums, so it suffice to prove for G/H, but this is represented by $\mathrm{Spec}\, k'$, where k' is the fixed field of H.

To prove $fg \cong \mathrm{id}$ and $gf \cong id$, they commutes with direct sums, so the first one is true because $G/H \to fg(G/H)) = \mathrm{Spec}(k_s)(k)$ is an isomorphism, and the second follows from(III.4.4.6) as all étale schemes over field k is a disjoint union of spectra of finite separable field extensions of k. □

Cor. (III.8.1.9). By(II.3.3.1), $F \to \varinjlim_{k\subset k'\subset k_s} F(\mathrm{Spec}\, k')$ is an equivalence between the category of Abelian sheaves on $(\mathrm{Spec}\, k)_{ét}$ to the category of continuous G-modules. So

$$H^q_{ét}(\mathrm{Spec}\, k, F) \cong H^q(G, \varinjlim_{k\subset k'\subset k_s} F(\mathrm{Spec}\, k_s)).$$

And if k is separably closed, then $(\mathrm{Spec}\, k)_{ét}$ is equivalent to $\mathcal{A}b$, and $H^p(\mathrm{Spec}(k), \mathcal{F}) = 0$ for $p > 0$.

Stalks

Def. (III.8.1.10) (Stalk). By(III.8.1.8), for any scheme X and a geometric point P, the section functor $F \to F(P)$ is an equivalence of categories from $(\mathrm{Spec}\, k)_{ét}$ to $\mathrm{A}b$, thus for any $F \in X_{ét}$, we can define the **stalk** $F_P = u^* F(P)$.

Prop. (III.8.1.11). For any geometric point P of X,

- the stalk map is exact and commutes with colimits.
- For any morphism $u: P' \to P$ of geometric points over X, $\mathcal{F}_P \cong F_{P'}$.
- If $X \to Y$ is a morphism, then $(f^*F)_P \cong F_P$.

Proof: 1: taking stalk is a composition of f^* and taking section over P(which is an equivlence), so it is exact and commutes with colimits(III.8.1.2).

2, 3: Trivial. □

Prop. (III.8.1.12) (Stalk is Defined Naturally). By the definition of f^*(III.1.2.6), if X' be an étale nbhd of P in X, i.e. $P \to X' \to X$, then

$$(f_{ét})_p(F(P)) = \varinjlim_{X'} F(X')$$

and $F_P = f^* F(P)$, thus there is a natural map $\varinjlim_{X'} F(X') \to F_P$.

Then we have:

$$\varinjlim_{X'} G(X') \to (G^\sharp)_P$$

for any presheaf G on $X_{ét}$.

Proof: Firstly $(f^\cdot)^\sharp(G) \cong (f^*G)^\sharp$ by(III.1.2.8). Then it suffices to prove that $G(P) \to G^\sharp(P)$ is an isomorphism for any presheaf G on $P_{\acute{e}t}$. But this is because $P_{\acute{e}t}$ is just $\mathcal{A}b$(III.8.1.8), and $P \xrightarrow{\mathrm{id}} P$ is cofinal in the category of coverings of P. $\square$

Cor. (III.8.1.13). For a morphism of schemes $X \to Y$ and P is a geometric point of Y, then

$$R^p f_*(F)_P \cong \varinjlim_{P \in Y'} H^p(X \times_Y Y', F).$$

Cor. (III.8.1.14). For $X = \operatorname{Spec} k$, the equivalence(III.8.1.8) of $X_{\acute{e}t}$ with continuous G-modules are just induced by taking the stalk at $\operatorname{Spec} k_s$.

Prop. (III.8.1.15) (Exactness and Stalks). The exactness, injectivity and surjectivity of maps of sheaves $F' \to F$ on $X_{\acute{e}t}$ can be checked on stalks.

Proof: It suffices to prove the isomorphism case, because taking stalks are exact(III.8.1.11) and other maps can be characterized by isomorphisms.

Monomorphism: suppose not, if $s \in F'(X')$ is mapped to 0, by taking base change, we can assume $X' = X$, and then $0 = v(s)_{\overline{x}} = v_{\overline{x}}(s_{\overline{x}})$, thus $s_{\overline{x}} = 0$ by assumption. Now by(III.8.1.12), for any x there is an étale nbhd of x that s vanishes on it. So we find an étale covering of X that s vanishes, thus $s = 0$ because F' is a sheaf.

Epimorphism: Similarly, for any $v \in F(X')$, we can pass to the base change and assume $X' = X$, then find for each x a nbhd that comes from some $v(s_x)$, and they glue together to be a global section of $F'(X)$. $\square$

Prop. (III.8.1.16) (Finite Morphism is Exact). For a finite morphism f, f_* are exact on étale topos.

Proof: Check on stalks, $\square$

Artin-Schreier Theory and Kummer Theory

Prop. (III.8.1.17) (Étale Sheaves of $\mathcal{O}_X$-Modules). Recall by(III.1.5.23) if M is a Qco $\mathcal{O}_X$-sheaf, then $\widetilde{M}$ is a fpqc sheaf on X, in particular an étale sheaf on X. Now the edge map of the Zariski-étale comparison for $\widetilde{M}$ is an isomorphism:

$$H^p_{Zar}(X, M) \cong H^p_{\acute{e}t}(X, \widetilde{M})$$

In particular, $H^p_{\acute{e}t}(X, (\mathbb{G}_a)_X) \cong H^p(X, \mathcal{O}_X)$, and the etale cohomology for Qco sheaves vanishes on affine schemes.

Proof: We show that $R^p\varepsilon^s(\widetilde{M}) = 0$ for $p > 0$, Cf.[Tamme P109]. Not hard. $\square$

Prop. (III.8.1.18) (Artin-Schreier Sequence). Let X be a scheme that has char p, let $F : (G_a)_X \to (G_a)_X$ be the Frobenius map, and let $P = \mathrm{id} - F$, then there is an **Artin-Schreier exact sequence**

$$0 \to (\mathbb{Z}/p\mathbb{Z})_X \to (\mathbb{G}_a)_X \xrightarrow{P} (\mathbb{G}_a)_X \to 0$$

Proof: If $s \in \mathcal{O}_{X'}$ is in the kernel, then $s = s^p$, so it is locally constant and comes from the map $\mathbb{Z}/p\mathbb{Z} \to \mathcal{O}_{X'}$. Conversely, for any $s \in \mathcal{O}_{X'}$, it suffices to find an *tale* cover that s is a p-th power in $\mathcal{O}_{X'_i}$. For this, it suffices to notice that for any p-ring A, $A[t]/(t^p - t - s)$ is free of rank p and étale over A. $\square$

Cor. (III.8.1.19). If X has char p, then by the long exact sequence and(III.8.1.17), there is an exact sequence

$$0 \to H^0(X, \mathcal{O}_X)/P(H^0(X, \mathcal{O}_X)) \to H^1(X, (\mathbb{Z}/p\mathbb{Z})) \to H^1(X, \mathcal{O}_X)^F \to 0$$

where the last one is the fixed elements.

Cor. (III.8.1.20). If $X = \operatorname{Spec} A$ and $pA = 0$, then $H^q(X, (\mathbb{Z}/p\mathbb{Z})_X) = A/P(A)$ for $p = 0$ and vanish for $p > 0$.

Cor. (III.8.1.21). If k is separably closed field of char$p$ and X is a reduced proper k-scheme, then $H^1(X, (\mathbb{Z}/p\mathbb{Z})_X) = (H^1(X, \mathcal{O}_X))^F$.

Prop. (III.8.1.22) (Hilbert's Theorem 90). $H^1(X, (\mathbb{G}_m)_X) \cong \operatorname{Pic}(X)$. Equivalently, $H^1(X, \mathcal{O}_X^*) \to H^1_{\acute{e}t}(X, (\mathbb{G}_m)_X)$ is an isomorphism.

Proof: Using the lower five term of the Leray spectral sequence(III.8.1.1), it will suffice to prove that $R^1\varepsilon^s(\mathbb{G}_m)_X = 0$. For this, Cf.[Tamme P107]. □

Prop. (III.8.1.23) (Kummer Sequence). If n is invertible on X, then there is an exact sequence

$$0 \to (\mu_n)_X \to (\mathbb{G}_m)_X \xrightarrow{n} (\mathbb{G}_m)_X \to 0$$

Proof: The proof is similar to that of Artin-Schreier sequence(III.8.1.18), noticing that $A \to A[t]/(t^n - s)$ is an étale map. □

Cor. (III.8.1.24). If n is invertible on X,then there is an exact sequence

$$0 \to H^0(X, \mathcal{O}_X^*)/n \to H^1(X, (\mu_n)_X) \to {}_n\operatorname{Pic}(X) \to 0$$

Cor. (III.8.1.25). If $X = \operatorname{Spec} A$ and n is invertible in A, then $H^1(X, (\mu_n)_X) \cong A^*/(A^*)^n$.?

Strict Henselization

Def. (III.8.1.26).

Torsion Sheaves

Def. (III.8.1.27) (Torsion Sheaf). An Abelian sheaf $\mathcal{F}$ on a topology is called a **torsion sheaf** iff it is associated to a presheaf of torsion Abelian groups. Equivalently, the canonical morphism $\lim_n {}_n\mathcal{F} \to F$ is an isomorphism.

Proof: It $\mathcal{F} = P^\sharp$, then consider $0 \to {}_nP \to P \xrightarrow{n} P \to 0$. Because $\sharp$ is exact, ${}_n\mathcal{F} = ({}_nP)^\sharp$. Because $\sharp$ commutes with inductive limits and $P = \lim_n {}_nP$, $\mathcal{F} = \lim_n {}_n\mathcal{F}$.

Conversely, if $\mathcal{F} = \lim_n {}_n\mathcal{F}$, then $\mathcal{F} = \lim_n ({}_nP)^\sharp = (\lim_n {}_nP)^\sharp$ which is presheaf of torsion Abelian groups. □

Remark (III.8.1.28). For a torsion sheaf, $\mathcal{F}(U)$ need not be torsion Abelian, but this is the case if U is quasi-compact, Cf.[Tamme P146].

Prop. (III.8.1.29) (Being Torsion is Local). An Abelian sheaf F on $X_{\acute{e}t}$ is a torsion sheaf iff all stalks F_x are torsion groups.

Proof: Use the definition of torsion sheaf $\mathcal{F} = \lim_n {}_n\mathcal{F}$ and the fact isomorphisms are checked on stalks(III.8.1.15) and stalk maps are exact(III.8.1.11). □

Prop. (III.8.1.30).
- If $X \to Y$ is a morphism of schemes and F is a torsion sheaf on Y, then f^*F is torsion sheaf on X.
- If $X \to Y$ is a qcqs morphism of schemes and F is a torsion sheaf on X, then $R^q f_* F$ are torsion sheaves on Y.
- In particular, if X is qcqs and F is a torsion sheaf on X, then $H^q_{ét}(X, F)$ are torsion for all q.

Proof: 1: This follows immediately from(III.8.1.29) and(III.8.1.11).

2: For any $y \in Y$, $(R^q f_* F)_{\overline{y}} \cong H^q(\overline{X}, \overline{F})$, where $\overline{X} = X \otimes_Y \overline{Y}$ and $\overline{Y}$ is the strict localization of Y in $\overline{y}$ by**?**. Now $\overline{F}$ is torsion sheaf by item1, and $\overline{X} \to \overline{Y}$ is also qcqs with $\overline{Y}$ being affine, so $H^q(\overline{X}, \overline{F})$ is torsion by item3, so $R^q f_* F$ is torsion by(III.8.1.29).

3: By(III.8.1.7), in this case, $H^p_{ét}(X, -)$ commutes with filtered colimits, so we can replace F by ${}_nF$. Then multiplying by n is zero on F, so also it is zero on $H^p_{ét}(X, F)$, so $H^p_{ét}(X, F)$ is torsion. □

Prop. (III.8.1.31). If X is Noetherian scheme and x is a point of X, let $i : \mathrm{Spec}(k(x)) \to X$ be the structure map, then
- for any Abelian sheaf F on $\mathrm{Spec}(k(x))_{ét}$, the sheaves $R^p i_* F$ are torsion sheaf for $p > 0$.
- $H^p_{ét}(X, i_* F)$ are torsion for all $p > 0$.

Proof: 1: Cf.[Tamme P148]. Uses strict Henselization.

2: Consider the Leray spectral sequence $H^p_{ét}(X, R^q i_* F) \Longrightarrow H^{p+q}_{ét}(\mathrm{Spec}(k(x)), F)$, the left term vanishes for $p \geq 0, q > 0$ by item1 and(III.8.1.30), and the right hand side vanish for $p + q > 0$ by(III.8.1.30), then it can be checked that $H^p_{ét}(X, i_* F)$ are torsion for $p > 0$. □

Prop. (III.8.1.32). For a closed subscheme $i : Y \subset X$, $R^p i^!$ preserves torsion sheaves.

Proof: Cf.[Tamme P148]. □

Prop. (III.8.1.33). For a regular Noetherian scheme X, $H^q_{ét}(X, (\mathbb{G}_m)_X)$ are torsion for $q \geq 2$.

Proof: Cf.[Tamme P149]. □

Prop. (III.8.1.34). For a torsion sheaf F, define $F(l) = \varinjlim_n {}_{l^n}F$, so it is a l-torsion sheaf, and in fact

$$\oplus F(l) = F$$

this is because this is true at the stalks, because stalks is exact and commutes with colimits(III.8.1.11).

So if X is qcqs, then $H^p(X, F) \cong H^p(X, F(l))$, which is the primary decomposition of $H^p(X, F)$.

Def. (III.8.1.35) (Cohomological Dimension). If X is qcqs, then we define the **cohomological l-dimension** of X as the smallest number $cd_l(X) = n$ that $H^p(X, F)(l) = 0$ for all $p > n$ and F torsion sheaf on X, and define the **cohomological dimension** of X as the smallest number $cd(X) = n$ that $H^p(X, F) = 0$ for all $p > n$ and F torsion sheaf on X. Equivalently, $cd(X) = \sup_l \{cd_l(X)\}$.

Prop. (III.8.1.36). If X is an algebraic scheme over a field k of char$p$, then

$$cd_l(X) \le \begin{cases} 2\dim X + cd_l(k) & l \neq p \\ \dim X + 1 & l = p \end{cases}.$$

Proof: □

Cor. (III.8.1.37). If k is separably closed, then $cd(X) \le 2\dim X$.

Proof: □

Prop. (III.8.1.38) (Artin Vanishing theorem). If X is an affine algebraic scheme over a separably closed field k, then $cd(X) \le \dim X$.

Proof: Cf[Milne Étale Cohomology P153]. □

<u>**Locally Constructible Sheaves**</u>

Prop. (III.8.1.39). $\mu_{n,X}$ is étale over X iff n is prime to the characteristic of all local residue fields of X. (Only unramifiedness is concerned, and it is fiberwise(I.7.5.6). And we can compute the Kahler differential of $k[T]/(T^n-1)$ vanish iff $n \neq 0$ in k.

In this case, μ_n is locally isomorphic to $(\mathbb{Z}/n\mathbb{Z})_X$, because for any affine open $U = \operatorname{Spec} A$, $U' = \operatorname{Spec} A[t]/(t^n-1) \to U$ is étale and surjective(I.7.1.13) and U' has all n-th roots of unity, so $(\mu_n)_{\operatorname{Spec} U'} \cong (\mathbb{Z}/n\mathbb{Z})_{\operatorname{Spec} U'}$.

Prop. (III.8.1.40). If G is a commutative, finite and étale group scheme on X, the sheaf G_X represented by G is locally finite on $X_{ét}$.

Conversely, any locally constant sheaf on $X_{ét}$ is represented by a unique commutative étale group scheme over X, and it is finite if F has finite stalks.

Proof: Cf.[Tamme P152]. □

Def. (III.8.1.41). An Abelian sheaf is called **finite** iff all its stalks are finite.

Def. (III.8.1.42) (Constructible Sheaf). An Abelian sheaf F on $X_{ét}$ is called **constructible** if each affine open subset U has a decomposition into f.m. constructible reduced subschemes U_i of U that F/U_i are locally constant and finite(i.e. stalks are finite) for each i.

Prop. (III.8.1.43) (Properties of Constructible Sheaves).

- If F is an Abelian sheaf on $X_{ét}$ that X has a finite decomposition into constructible reduced subschemes X_i that F/X_i are locally constant, then F is constructible. The converse is also true if X is qcqs.
- Constructible is a local property.
- Constructibility is stable under pullback, pushout and finite direct limits.
- Constructibility is stable $j_!$ for a qc étale map.
- Subsheaves of a constructible sheaf is constructible.

Proof: Cf.[Tamme P155]. [Conrad L3 P2], [Étale Cohomology and Weil Conjecture P42]**?** □

Prop. (III.8.1.44) (lcc Sheaves and Finite Étale Schemes). The functor $X \mapsto \mathrm{Hom}_S(-, X)$ defines an equivalence of categories between the category of finite étale S-schemes to the category of locally constant finite sheaves.

Proof: The Yoneda functor is fully faithful, thus we need to show the essentially surjectivity. Notice first $\mathrm{Hom}_S(-, X)$ is locally constant finite: we can restrict to an open subset of S that the fiber are of fixed order n, and $X \to X \times_S X$ is étale and a closed immersion, thus $X \times_S X = X \coprod Y$, and Y is finite étale over X through π_1. Now by induction on the order of the fiber, $Y = X \otimes \Sigma'$ locally. So $X = S \times \Sigma$ locally, which means X represents the constant sheaf $\underline{\Sigma}$ locally.

To show that every locally constant finite étale sheaf is represented by a finite étale scheme, Cf.[Conrad, P19]. □

Prop. (III.8.1.45). If G is a commutative étale group scheme over X, then the sheaf G_X represented by G is constructible iff G is f.p. over X.

Proof: □

Prop. (III.8.1.46) (Locally Constancy and Stalks). If $\mathcal{F}$ is a finite sheaf over a Noetherian scheme S, then $\mathcal{F}$ is locally constant iff all the specialization maps for geometric points $\mathcal{F}_{\overline{s}} \to \mathcal{F}_\eta$ are bijective.

Proof: If $\mathcal{F}$ is locally constant, because the conclusion is local, we may assume $\mathcal{F}$ is constant, then $\mathcal{F}_{\overline{s}} \to \mathcal{F}_\eta$ are all identities.

Conversely, for any geometric point s, $\Sigma = \mathcal{F}_s$ is finite by definition, thus there is an étale nbhd U of s that the map $\underline{\Sigma} \to \mathcal{F}$ induces an isomorphism on s-stalks, so this is an isomorphism for any geometric point linked to s by specialization, in particular the generic point of the irreducible component containing s and all the points in this irreducible component, thus $\mathcal{F}$ is constant on an open nbhd of s(because X is Noetherian thus has f.m. irreducible components), so $\mathcal{F}$ is locally constant because X is Noetherian. □

Prop. (III.8.1.47) (Constructible Sheaves are Noetherian). The constructible sheaves of Abelian groups are exactly the Noetherian objects in the category of torsion sheaves.

2 Étale Fundamental Group

Basic references are [StackProject Chap53] and [Fulei Chap3].

Def. (III.8.2.1). A **geometric point** is a map $\overline{x} : \mathrm{Spec}(k_s) \to X$ where k_s is separably closed.

Lemma (III.8.2.2) (Rigidity Lemma). If $f, g : S' \to S''$ are two S-morphisms where S'' is a separated étale S-scheme and $(S', \overline{s}')$ is a pointed scheme that $f(\overline{s}') = g(\overline{s}')$, then $f = g$.

Proof: The diagonal $S'' \to S'' \otimes_S S''$ is a closed immersion and also étale hence open(III.4.4.3), so the diagonal is an clopen subset. And now $f \times g : S' \to S'' \otimes_S S''$ intersects the diagonal, and S' is connected, so f, g are identical on the diagonal. □

Def. (III.8.2.3) (Galois Cover). If $(S, \overline{s})$ is a pointed connected scheme, $S' \to S$ is a finite étale cover of degree n, then there are at most n point over $\overline{s}$, so by(III.8.2.2), $|\mathrm{Aut}(S'/S)| \leq n$. If the equality holds, then we call S'/S a **Galois cover** and define $Gal(S'/S) = \mathrm{Aut}(S'/S)^{op}$.

Prop. (III.8.2.4). If $S' \to S$ is a connected finite étale cover, then there is a finite étale cover $S'' \to S'$ that $S'' \to S$ is Galois.

Proof: Cf.[SGA1, Exp.V, §2 – §4]. □

Def. (III.8.2.5) (Étale Fundamental Group). For any two finite Galois étale cover $S'/S, S''/S$, if there is a S-morphism $S'' \to S'$, then it induces a morphism of Galois groups because the Galois group of S' acts transitively on the fiber over a closed point. And it is surjective by the same reason for S''.

Then we define the **étale cohomology group**

$$\pi_1(S,x) = \varprojlim_{(S',\overline{x}')} Gal(S'/S)$$

Prop. (III.8.2.6) (Fundamental Group and Covers). For X connected smooth scheme and $\overline{x} \to X$ a geometric point, there is a profinite group $\pi_1(X,\overline{x})$ that there is a correspondence:

$$\{\text{finite étale covers } Y \to X\} \leftrightarrow \{\text{Finite sets with a continuous action of } \pi_1(X,x)\}$$

Proof:

□

Prop. (III.8.2.7). Let (S,s) be a connected scheme, then the functor $S' \mapsto S'_{\overline{s}}$ induces an equivalence of categories between the finite étale covers $S' \to S$ with the category of finite discrete $\pi_1(X,x)$-sets.

Proof: We may assume S' is connected, then use(III.8.2.4) to find a Galois cover $S'' \to S'$ that S''/S is Galois, then clearly there is a bijection

$$Gal(S''/S)/Gal(S''/S') \cong S'(s')$$

And any transitive discrete $\pi_1(X,x)$-sets arise this way.

To prove the essentially surjectivity and fully faithfulness, ? □

Cor. (III.8.2.8). The étale fundamental group is independent of the base point $\overline{s}$ chosen.

Proof: This is because for two profinite groups, if the categories of their finite sets are equivalent, then they are isomorphic?. □

Cor. (III.8.2.9) (Locally Constant Sheaves and Fundamental Group). By(III.8.1.43), if X is a connected scheme and $\overline{x}$ be a geometric point of X, then there is an equivalence of categories between finite locally constant Abelian sheaves on X and finite $\pi_1(X,\overline{x})$-modules.

Prop. (III.8.2.10). For k alg.closed, $\pi_1(\mathbb{P}^1_k) = 0$.

Proof: □

Prop. (III.8.2.11) (Arithmetic Geometric Exact Sequence). If X_0 is a variety over $\mathbb{F}_q$, then there is an exact sequence

$$1 \to \pi_1(X,\overline{x}) \to \pi_1(X_0,\overline{x}) \to G(\overline{k}/k) \to 1.$$

3 Curve Case

Prop. (III.8.3.1). If X is separated f.t. scheme of dimension≤ 1 over a separably closed field k, and $\mathcal{F}$ is a torsion sheaf on $X_{ét}$, then $H^i_{ét}(X,\mathcal{F})=0$ for $i>2$, and if $\mathcal{F}$ is constructible, then $H^i_{ét}(X,\mathcal{F})$ are finite.

Moreover if X is affine and $\mathcal{F}$ is locally killed by n not divisible by char$k$ or X is proper and sections of $\mathcal{F}$ are locally p-torsions with $p=$char$k>0$, then $H^2_{ét}(X,\mathcal{F})=0$.

Proof: Cf.[Conrad L4 P4] and [Tamme]. □

Lemma (III.8.3.2). If X is a connected smooth projective curve over an alg.closed field k of char$p>0$ and n is not divisible by p, M is a finite Abelian group, then

$$H^1_{ét}(X,\mu_n)\cong \mathrm{Pic}(X)[n],\quad H^2_{ét}(X,\mu_n)=\mathbb{Z}/n\mathbb{Z},\quad H^0(X,\underline{M})=M,\quad H^{\geq 2}_{ét}(X,\mu_n)=0;$$
$$H^1_{ét}(X,\mathbb{Z}/p\mathbb{Z})\text{ is finite},\quad H^2_{ét}(X,\mathbb{Z}/p\mathbb{Z})=0,\quad H^{\geq 2}_{ét}(X,\mathbb{Z}/p\mathbb{Z})=0.$$

Proof: Cf.[Conrad L4 P6] □

<u>Duality</u>

<u>l-adic Cohomology</u>

4 Base Changes

Def. (III.8.4.1) (Base Change Morphism). By Leray spectral sequence(III.5.2.16), there are edge morphisms $R^pg_*(f_*F)\to R^p(gf)_*(F)$ and $R^p(gf)_*(F)\to g_*(R^pf_*(F))$. So if there is a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y \end{array},$$

there is a morphism $F\to v'_*v'^*F$, and

$$R^pf_*F\to R^pf_*(v'_*v'^*F)\to R^p(fv')_*(v'^*F)=R^p(vf')_*(v^*F)\to v_*(R^pf_*(v'^*F)).$$

Hence by adjointness a morphism

$$v^*(R^pf_*(F))\to R^pf'_*(v'^*F))$$

called the **base change morphism**.

Prop. (III.8.4.2) (Proper Base Change). If there is a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

that f is proper, then for any **torsion Abelian sheaf** $\mathcal{F}$, the base change map

$$g^*R^qf_*\mathcal{F}\to R^qf'_*(g'^*\mathcal{F})$$

is an isomorphism.

Proof: Cf.[Conrad L6]. □

5 Cohomology with Compact Support

Cf.[Weil1, P18].

Lemma (III.8.5.1). Extension by 0 commutes with pullback Cf.[KF Lemma4.9].

Def. (III.8.5.2). For a scheme X any étale sheaf $\mathcal{F}$ on X, and any morphism $f : X \to Y$, if it can be extended to a proper morphism $f^c : X^c \to Y$ where $j : X \to X^c$ is an open dense subscheme, then we define the **higher direct image with compact support** as

$$Rf_! = Rf^c_* j_! : D^+(X, tor) \to D^+(Y, tor).$$

And this is the case of $Y = \operatorname{Spec} k$ and we define $H^i_c(X, \mathcal{F}) = H^i(\operatorname{Spec} k, Rf_!\mathcal{F})$.

Prop. (III.8.5.3). Any separated morphism of f.t. $X \to Y$ that Y is qcqs has higher direct image with compact support. And it is independent of the compactification chosen.

Thus $f_!$ can be defined even in case Y is not qcqs, because we can define it on each affine opens U_i of S and glue them together by uniqueness.

Proof: The compactification exists because of Nagata compactification. For the uniqueness, notice for any two compactification, we can find a common compactification that dominates them both?, so using lemma(III.8.5.4), we easily show they are isomorphic. □

Lemma (III.8.5.4) ($i_!$ and Higher Pushforward). If there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"] & \overline{X} \arrow[d, "\overline{f}"] \\ Y \arrow[r, hook, "j"] & \overline{Y} \end{tikzcd}$$

where i, j are open immersion and $f, \overline{f}$ are proper, then there is a natural transformation $j_! f_* \to \overline{f}_* i_!$ that induces a natural transformation

$$j_! Rf_* \to R\overline{f}_* i_!,$$

which is an isomorphism iff $\overline{f}$ is proper.

Proof: If this is a Cartesian diagram, then the natural transformation is give by

$$j_! f_* \to \overline{f}_* \overline{f}^* j_! f_* \cong \overline{f}_* i_! f^* f_* \to \overline{f}_* i_!.$$

(the second isomorphism is by(III.8.5.1)). The rest is by proper base change Cf.[KF P88].

The general case is also easily reduced to the Cartesian case.? □

Prop. (III.8.5.5) (Proper Map Induces Map on Proper Pushforward). If $g : Y \to X$ is a proper morphism between schemes separated of f.t. over a Noetherian scheme S, then for any étale Abelian sheaf $\mathcal{F}$ on X, there is a canonical mop

$$g : R(f_1)_!(\mathcal{F}) \to R(f_2)_!$$

Proof: Choose a compactification $X_2 \xrightarrow{j} \overline{X}_2$, then choose a compactification $X_1 \xrightarrow{i} \overline{X}_1$ of $i \circ g$, now

Cf.https://math.stackexchange.com/questions/3120833/proper-morphism-induces-a-map-between-compact-support-etale-cohomology-groups ?. □

Prop. (III.8.5.6) (Properties of Compact Pushforward).

- (Base Change) If there is a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow{f'} & & \downarrow{f} \\ S' & \xrightarrow{g} & S \end{array}$$

  that f is separated of f.t., then there is a natural isomorphism

$$g^* R f_! \cong R(f'_!) h^*$$

- (Composition) For two separated morphisms of f.t, $R(f_1 f_2)_! = R f_{1,!} R f_{2,!}$, which induces Leray spectral sequence.
- (Excision) Let $f : X \to S$ be a separated morphism of f.t, and $\mathcal{F}^\bullet \in D^+(X, tor)$. Let $Z \subset X$ be a closed subscheme and $U = X - Z$, then there is a long exact sequence

$$\cdots \to R^p (f_U)_! (\mathcal{F}^\bullet|_U) \to R^p f_! \mathcal{F}^\bullet \to R^p (f_Z)_! (\mathcal{F}^\bullet|_Z) \to R^{p+1} (f_U)_! (\mathcal{F}^\bullet|_U) \to \cdots$$

Proof: Cf.[Conrad L10 P3].

1: Choose a compactification of f, then it suffices to show $R f^c_*$ and $j_!$ both commutes with base change, which is by proper base change(III.8.4.2) and(III.8.5.1).

2: Two compactification can be splinted, and use(III.8.5.4).

3: Use the long exact sequence applied to the exact sequence(checked on stalks(III.8.1.15))

$$0 \to j_! j^* F \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

□

Prop. (III.8.5.7) (Proper Pushforward to Direct Image). There is a natural map from $R^i f_! \to R^i f_*$, which is induced by

$$R^i f_! = R^i f^c_* j_! \to R^i f^c_* j_* \to R^i f_*$$

where the second one is edge map of Leray spectral sequence.

In particular, there is a map $H^i_{c,\acute{e}t}(X, \mathcal{F}) \to H^i_{\acute{e}t}(X, \mathcal{F})$.

Prop. (III.8.5.8) (Vanishing Result). If $f : X \to S$ is separated of f.t. and let $d = \sup_{s \in S} \dim X_s$, then if $\mathcal{F} \in D(X, tor)$ satisfies ${}^p H^p(\mathcal{F}) = 0$ for $p \geq r$, then

$$R^p f_! \mathcal{F} = 0, \quad p \geq r + 2d.$$

Proof: Cf.[Conrad L10 P4]. □

Prop. (III.8.5.9) (Projection Formula). If $X \to S$ is a quasi-projective morphism, $\mathcal{F} \in D^-(S)$ and $\mathcal{G} \in D(X)$, then we have a natural isomorphism

$$\mathcal{F} \otimes^L R f_! \mathcal{G} \cong R f_! (f^* \mathcal{F} \otimes^L \mathcal{G})$$

Proof: We may pass to the compactification, as $j_!$ commutes with f^*?. □

Finiteness Theorems

Prop. (III.8.5.10). If $f : X \to S$ is a separated morphism of f.t., S is Noetherian, and $\mathcal{F}$ is a constructible Abelian sheaf on X whose torsion is order is invertible in S, then $R^p f_! F$ are all constructible on Y.

Proof: Cf.[Conrad L10 P5]. □

Cor. (III.8.5.11). If X is a proper scheme over a separably closed field k and F is a constructible Abelian sheaf on $X_{ét}$, then $H^q(X, F)$ are finite for all $p \geq 0$.

Lemma (III.8.5.12). If $X \to S$ is smooth and proper, and F is a locally constant finite Abelian sheaf with torsion order invertible on S, suppose S is Noetherian, then all specialization maps for $R^p f_* \mathcal{F}$ are isomorphisms.

Proof: Cf.[Conrad L10 P5]. □

Prop. (III.8.5.13). If $X \to S$ is smooth and proper, and F is a locally constant finite Abelian sheaf(III.8.1.41) with torsion order invertible on S, then $R^p f_* \mathcal{F}$ are locally constant finite sheaves for any $p \geq 0$.

Proof: By(III.8.1.44), we may assume $\mathcal{F} = \underline{X'}$ for some finite étale scheme $X' \to X$. By Noetherian descent**?** together with proper base change, we may reduce to the case S is Noetherian. Thus by(III.8.5.10), $R^p f_* \mathcal{F} = R^p f_! \mathcal{F}$ are constructible, and(III.8.5.12) shows that the stalk maps are isomorphisms. So(III.8.1.46) shows that $R^p f_* \mathcal{F}$ are locally constant finite. □

Comparison Theorems

6 Poincare Duality

Prop. (III.8.6.1) (Poincare Duality). If $\mathcal{F}$ is lisse, X is smooth separated of dimension d, then we have a perfect pairing

$$H^n_{ét}(X_{\overline{k}}, \mathcal{F}) \times H^{2d-n}_{ét}(X_{\overline{k}}, \mathcal{F}^\vee(d)) \to H^{2d}_{ét}(X, \overline{\mathbb{Q}}_l(d)) \xrightarrow{tr_X} \overline{\mathbb{Q}}_l$$

which is Galois Invariant.

Proof: Cf.[Conrad L12-13].**?** Use Duality, Cf.[Weil 2Bhatt P5]. □

Cor. (III.8.6.2) (Weak Lefschetz). Is a consequence of poincare duality and localization sequence.

Proof: Cf.[Weil 1 Proof P19]. □

III.9 Weil 2 Proof

Basic References are [Conrad Seminar note in Princeton], [Seminar on Gross-Zagier over Function Fields, Lei Fu], [Seminar notes on Weil 2 Bhatt], [Weil conjectures Perverse Sheaves and l-adic Fourier Transform Kiehl/Weissauer].

1 l-adic Étale Cohomology

Def. (III.9.1.1) (Notations). p is a prime, $q = p^r$ is a p-power, $k = \mathbb{F}_q$ is a finite field, X_0 is a separated algebraic scheme over k, $X = X_0 \otimes_k \overline{k}$ is its base change.

Fix a mixed characteristic complete DVR $(\Lambda, \mathfrak{m})$ with residue field finite of char$=l \neq p$, and K its quotient field. Let A be a Noetherian ring that is I-adically complete Hausdorff, let $A_n = A/I^n$.

Artin-Rees Formalism

Def. (III.9.1.2). The **pre Artin-Rees category** of A-modules has objects $M^\bullet = (M_n) \in \mathbb{Z}$ which are projective systems of A-modules with $M_n = 0$ for $n << 0$, and the morphisms in this category are the elements of the set

$$\mathrm{Hom}_{A-R}(M^\bullet, N^\bullet) = \lim \mathrm{Hom}(M^\bullet[d], N^\bullet)$$

An object $M^\bullet$ in the Artin-Rees category is called a **null system** if for some ≥ 0 the map $M_{n+} \to M_n$ vanishes for all n.

Prop. (III.9.1.3). The pre Artin-Rees category is an Abelian category, and the null systems form a Weak Serre subcategory. Then we define the **Arin-Rees category** as the quotient category.

Proof: □

Prop. (III.9.1.4). If the kernel and cokernel of two systems are all null systems, then they induce isomorphism on inverse limit.

Proof: Cf.[Conrad L15, P5]. □

Def. (III.9.1.5). An object $M^\bullet$ in the A-R category is called **Artin-Rees I-adic** if it is represented by a system M_n that $M_n = 0$ for $n < 0$ and M_n is finite over A_n, $M_{n+1} \otimes_{A_{n+1}} A_n \to M_n$ is an isomorphism for $n \geq 0$.

Prop. (III.9.1.6). The full subcategory of Artin-Rees I-adic modules is an Abelian category, and it is equivalent to the category of finite A-modules by the stalk functor.

l-adic Sheaves

Def. (III.9.1.7). The **Artin-Rees category of Λ-sheaves** on X_0, **strict $\mathfrak{m}$-adic sheaves** are defined as before. It is called **constructible $\mathfrak{m}$-adic sheaf** iff it is isomorphic to a system that $\mathcal{F}_n$ are all constructible.

It is called **lisse $\mathfrak{m}$-adic** if it is isomorphic to a strict $\mathfrak{m}$-adic system that $\mathcal{F}_n$ are all locally constant finite Λ_n-modules.

Prop. (III.9.1.8). Constructibility of Artin-Rees Λ-Sheaves are étale local, and stratification local.

Proof: Cf.[Conrad L15, P10].? □

Prop. (III.9.1.9) (Constructible and Lisse). Let $\mathcal{F}$ be a constructible $\mathfrak{m}$-adic sheaf, then there is a stratification of X that $\mathcal{F}$ is locally constant finite on each stratum.

Proof: By the stalk criterion of locally constant finite(III.8.1.46), a constructible extension of locally constant finite sheaves is also locally constant finite. So by the exact sequence $1 \to l^{n-1}\mathcal{F}_n \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to 1$, iff we show there is a stratification that all $l^{n-1}\mathcal{F}_n$ are locally constant finite, then by induction all $\mathcal{F}_n$ are locally constant finite. But then $l^{n-1}\mathcal{F}_n$ is a descending chain of quotients of $\mathcal{F}_1$, thus the kernel is ascending thus stablizes because $\mathcal{F}_1$ is constructible(III.8.1.47), so there are only f.m. such $l^n\mathcal{F}_n$, so there is a common stratification. □

Cor. (III.9.1.10). Exactness of complexes of constructible sheaves can be checked at stalks.

Proof: Cf.[Conrad L16 P3]. □

Prop. (III.9.1.11). Constructible $\mathfrak{m}$-adic sheaves are Noetherian: Ascending chain of subsheaves stablizes.

Proof: Cf.[Conrad L16 P3]. □

Prop. (III.9.1.12) (Direct Pushforward of $\mathfrak{m}$-adic Sheaves). For a constructible $\mathfrak{m}$-adic sheaf $\mathcal{F}$ and a compatifiable morphism $X_0 \to S_0$, we can define R^if_* and $R^if_!$ termwisely, and we have $R^if_*\mathcal{F}$ is a constructible sheaf, hence also does $R^if_!$.

Proof: Cf.[Weil Conjecture and Étale sheaves, P128]. □

$\overline{\mathbb{Q}}_l$-Sheaves

Def. (III.9.1.13). For a finite extension $E/\mathbb{Q}_l$, the category of **E-sheaves** are the category of constructible $\mathcal{O}_E$-sheaves with the homomorphism given by

$$\mathrm{Hom}_E(\mathcal{F},\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_E}(\mathcal{F},\mathcal{G}) \otimes_{\mathcal{O}_E} E.$$

We may write $\mathcal{F} \otimes E$ for this object, but the tensor is fake.

The category of $\overline{\mathbb{Q}}_l$-sheaves are the direct limit of categories of E-sheaves for $E/\mathbb{Q}_l$ finite.

Def. (III.9.1.14) (Tate Twist Sheaf). The **Tate weist sheaf** $\mathbb{Q}_l(1)$ is defined to be the lisse $\overline{\mathbb{Q}}_l$ sheaf of the limit system of locally constant finite sheaves μ_n of rank 1. It is invertible, thus we denote its dual by $\mathbb{Q}_l(-1)$. For any $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{F}$, denote $\mathcal{F}(1)$ to be the sheaf $\mathcal{F} \otimes \overline{\mathbb{Q}}_l(1)$.

Prop. (III.9.1.15) (Lisse $\overline{\mathbb{Q}}_l$-Sheaves and $\pi_1(X_0,\overline{x})$). Assume X_0 is connected, then for a geometric point $\overline{x}$ of X_0, the functor $\mathcal{F} \to \mathcal{F}_{\overline{x}}$ induces an equivalence between the category of lisse $\overline{\mathbb{Q}}_l$-sheaves to the category of continuous f.d. representations of $\pi_1(X_0,\overline{x})$ over $\overline{\mathbb{Q}}_l$.

Proof: By(I.4.7.3) any representation of $\pi_1(X_0,\overline{x})$ is in fact an $\mathcal{O}_E$-action for some $E/\mathbb{Q}_l$ finite. So by the equivalence(III.8.2.6)(III.8.1.44) and taking limit using(III.9.1.6), we get the result about $\mathcal{O}_E$-sheaves and representations of $\pi_1(X_0,\overline{x})$ over $\mathcal{O}_E$. □

Cor. (III.9.1.16). We can call a lisse $\overline{\mathbb{Q}}_l$-sheaf **irreducible/semisimple** if its corresponding representation is. It is called **geometrically irreducible/semisimple** if $\mathcal{F} = (\mathcal{F}_0)_{\overline{k}}$ is irreducible, or equivalently, its corresponding representation is irreducible as a $\pi_1(X,\overline{x})$-representation.

2 Frobenius Morphisms

Def. (III.9.2.1) (Frobenius). Let $q = p^r$, for a $k = \mathbb{F}_q$ scheme X_0 with base change X,

- The **absolute Frobenius** for X_0 or X is the automorphism $\varphi_{r,X} = \varphi^r : X \to X$ that is q-th power on $\mathcal{O}_X$.
- $F_X = \mathrm{id}_{X_0} \times_k \varphi_{\overline{k}/k}^{-1}$ is called the **geometric Frobenius**.
- $Fr_X = \varphi_{X_0} \times_k \mathrm{id}_{\overline{k}} : X \to X$, which is $\overline{k}$-linear.
- Let U be a X_0-scheme, then the **relative Frobenius** $F_{U/X_0} : U \to \varphi_{X_0}^{-1}(U)$ is defined by the universal property of the base change of U by F_X.

Prop. (III.9.2.2). $Fr_X = \varphi_{r,X} \circ F_X : X \to X$.

Proof: Easy. □

Prop. (III.9.2.3). F_{U/X_0} is a universal homeomorphism. In particular, if $U \to X_0$ is étale, then it is an isomorphism.

Proof: Because $U \to X, X \times_{\varphi_X, X} U \to X$ are both étale, F_U/X_0 is étale. And from the fact both both φ_{X_0} and φ_{U_0} are universally bijective, we see F_{U_0/X_0} is universally bijective. So it must be an isomorphism?. □

Cor. (III.9.2.4) (Frobenius action on Sheaves). For any étale sheaf $\mathcal{F}$ on X, we have an isomorphism $\mathcal{F} \cong (Fr_X)_*\mathcal{F}$ which is the inverse of the isomorphism

$$(Fr_X)_*(\mathcal{F})(U) = \mathcal{F}(Fr_X^{-1}(U)) \xrightarrow{F_{U/X}^*} \mathcal{F}(U),$$

and its adjoint $Fr_X^*\mathcal{F} \to \mathcal{F}$ is denoted by $\mathrm{Frob}_{\mathcal{F}}$.

Then $\mathrm{Frob}_{\mathcal{F}}$ commutes with tensor product and it is an isomorphism.

Proof: The adjoints are isomorphism because $(Fr_X)_*, (Fr_X)^*$ induces equivalence of categories of étale site(III.1.5.11). □

Remark (III.9.2.5). Notice this reverse in the definition of $\mathrm{Frob}_{\mathcal{F}}$.

Prop. (III.9.2.6) (Compatibility of $\mathrm{Frob}_{\mathcal{F}}$ with Higher Direct Image). If $X \to S$ is a separated morphism between $\mathbb{F}_p$-schemes of f.t., so we have a Cartesian diagram about Fr_X and Fr_S. Then the composition

$$Fr_S^* R^i f_* \mathcal{F} \to R^i f_* Fr_X^* \mathcal{F} \xrightarrow{R^i f_*(\mathrm{Frob}_{\mathcal{F}})} R^i f_* \mathcal{F}$$

is just $\mathrm{Frob}_{R^i f_* \mathcal{F}}$.

Proof: Cf.[Conrad L18 P4].? □

Cor. (III.9.2.7) (Compatibility of $\mathrm{Frob}_{\mathcal{F}}$ with Proper Pushforward). If $X \to S$ is a separated morphism of f.t. between k-schemes and $\mathcal{F}$ is a torsion Abelian sheaf on $X_{ét}$, then the morphism

$$Fr_S^* R^i f_! \mathcal{F} \to R^i f_! Fr_X^* \mathcal{F} \xrightarrow{R^i f_!(\mathrm{Frob}_{\mathcal{F}})} R^i f_! \mathcal{F}$$

is just $\mathrm{Frob}_{R^i f_! \mathcal{F}}$.

Proof: Choose a compactification, the $j_!$ doesn't matter, so we finish by(III.9.2.6). □

Prop. (III.9.2.8) (Frobenius Action on Compact Cohomology). As Fr_X is proper because it is finite, by(III.8.5.5) it induces a map $H^i_c(X,\mathcal{F}) \to H^i_c(X, Fr^*_X\mathcal{F})$, which by composing with $\mathrm{Frob}_{\mathcal{F}}$, gives us an endomorphism $Fr^*_{\mathcal{F}} : H^i_c(X,\mathcal{F}) \to H^i_c(X,\mathcal{F})$, which is called the Frobenius action on $H^i_c(X,\mathcal{F})$.

Similarly, there are isomorphisms $F^*_X\pi^*\mathcal{G}_0 \cong (\pi\circ F_X)^*\mathcal{G}_0 = \pi^*\mathcal{G}_0$, $Fro_{\mathcal{G}} : \varphi^*_{r,X}\mathcal{G} \cong \mathcal{G}$, so we can define the action of F_X or $\varphi_{r,X}$ on $H^i_c(X,\mathcal{G})$:

Then the action of $\varphi_{r,X}$ on $H^i_c(X,\mathcal{F})$ is in fact identity because it induces an isomorphsim on étale site(III.1.5.11). In particular, (III.9.2.2) show that F^*_X agrees with the Frobenius action for $H^i_c(X,\mathcal{F})$, so we can calculate with either one of them, and denoted by F^*_X.

3 Weil Sheaf

Fundamental Groups

Def. (III.9.3.1) (Weil Group). For a

Weil Sheaves

Def. (III.9.3.2) (Weil Sheaf). By Galois descent, the pullback induces an equivalence of categories between the category of constructible $\overline{\mathbb{Q}}_l$-sheaves on X_0 to the category of constructible $\overline{\mathbb{Q}}_l$-sheaves on X with an specified $G(X/X_0) = G(\overline{k}/k) \cong \widehat{\mathbb{Z}}$-actions. In practice, sometimes it is hard to verify the action of $\mathbb{Z}\in\widehat{\mathbb{Z}}$ is continuous, which leads to the following definition:

A **Weil sheaf** $\mathcal{G}_0$ on an algebraic scheme X_0 over k consists of a constructible $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{G}$ on X and an isomorphism $F_{\mathcal{G}_0} : F^*_X\mathcal{G}\to\mathcal{G}$. A **lisse Weil sheaf** is a Weil sheaf that G is lisse.

For any constructible $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{F}_0$ on X_0, the canonical $F^*_X\pi^*\mathcal{G}_0 \cong (\pi\circ F_X)^*\mathcal{G}_0 = \pi^*\mathcal{G}_0$ makes $\mathcal{F}$ into a Weil sheaf.

Prop. (III.9.3.3) (Weil Sheaf and Representation). When X_0 is geometrically connected, the functor $\mathcal{G}_0 \mapsto (\mathcal{G}_0)_{\overline{x}}$ defines an equivalent between the category of Weil sheaves on X_0 and the category of continuous $\overline{\mathbb{Q}}_l$-representations of $W(X_0,\overline{x})$. And the correspondence defined in(III.9.1.15) is a subcorrespondence of this.

Thus the notion of **geometric irreducible/semisimple** is definable for Weil sheaves.

Proof: Because by the correspondence(III.9.1.15), $\mathcal{G}$ corresponds to a representation of $\pi_1(X,\overline{x})$, and $\pi_1(X,\overline{x})$ acts trivially on the Galois cover X/X_0. Now a representation of $W(X_0,\overline{x})$ is equivalent to an automorphism $\rho(\sigma)$(where $\sigma\in W(X_0,\overline{x})$ satisfies $\deg(\sigma)$ corresponds to the geometric Frobenius) that $\rho(\sigma)\rho(\pi_1(X,\overline{x}))\rho(\sigma^{-1}) = \rho(\sigma\pi_1(X,\overline{x})\sigma^{-1})$, which is equivalent to an isomorphism $F^*_X\mathcal{G}\to\mathcal{G}$. □

Prop. (III.9.3.4). The constructions like $R^if_*, R^if_!, f^*$ is functorial thus is definable in the category of Weil sheaves by(III.8.1.43). And the specified isomorphism $F_{\mathcal{G}_0} : F^*_X\mathcal{G}\to\mathcal{G}$ gives us an action F^*_X of F_X on $H^i_c(X,\mathcal{G})$, just like in(III.9.2.8).

Similarly, there is an action of F_X on $\mathcal{G}_{\overline{x}}$ for each $x\in|X_0|$.

Prop. (III.9.3.5) (Weil Sheaf and Eigenvalues). If X_0 is geometrically connected, a lisse Weil sheaf $\mathcal{G}_0$ on X_0 is an ordinary $\overline{\mathbb{Q}}_l$-sheaf iff some deg1 element σ in $W(X,\overline{x})$ acts on $\mathcal{G}_{0\overline{x}}$ with eigenvalues which are l-adic units.

Proof: This is purely a Galois representation problem, concerning whether the representation of $W(X_0, \overline{x})$ can be extended to a representation of $\pi_1(X_0, \overline{x})$, and it is a continuity problem.

Firstly the representation of $\pi_1(X, \overline{x})$ stabilizes a lattice O_E^n for some $E/\mathbb{Q}_l$ finite, and then extends E to contain coefficients of $\rho(\sigma)$ and even its rational form. Then notice $\pi_1(X_0, \overline{x})$ is the profinite completion of $W(X_0, \overline{x})$, thus it suffices to see if the image $\rho(W(X_0, \overline{x}))$ is compact, and this is equivalent to eigenvalues of $\rho(\sigma)$ are units. $\square$

Prop. (III.9.3.6) (Determinential Criterion). If X_0 is normal and geometrically connected, then a irreducible lisse Weil sheaf on X_0 is an actual $\overline{\mathbb{Q}}_l$-sheaf iff its determinant bundle is.

Proof: Use geometric monodromy group. Cf.[Conrad L19 P7].

First assume that $\mathcal{G}_0$ is geometrically irreducible, then(III.9.5.4) shows that there is a nonzero power $\sigma^m = gz$ where $g \in G_{geo}(\overline{\mathbb{Q}}_l)$ and $z \in Z(G(\overline{\mathbb{Q}}_l))$. Now G_{geo} is a semisimple algebraic group(III.9.5.4), so the determinential character maps $G_{geo}(G(\overline{\mathbb{Q}}_l))$ to a finite group, because connected semisimple algebraic group has no nontrivial character as $[G, G] = G$**?**. So the determinant of g is an l-unit, and $\det(\sigma^m) = \det(z)$ is a unit. But z is a scalar by Schur's lemma, thus z is an l-adic unit. Now it suffices to show the eigenvalue of g are all l-units.

Now consider $\rho(\pi_1(X, \overline{x}))$ is a compact group in $\mathrm{End}(V)$, thus it generates a finite $\mathcal{O}_E$-submodule A , which is full-rank lattice in $\mathrm{End}(V)$ by Jacobson density theorem**?** and the fact ρ is absolutely irreducible. g normalized A, because σ and z both normalizes $\rho(\pi_1(X, \overline{x}))$, so the eigenvalue of the conjugate action of g are all l-units, but its eigenvalue are of the form $\lambda_i\lambda_j^{-1}$ where λ_k are eigenvalues of g, so this together with the fact $\det(g)$ is l-units shows that all λ_i are l-units.

For the general case, Cf.[Conrad L19 P7].**?** $\square$

Cor. (III.9.3.7) (Filtration of Weil Sheaf). If X_0 is normal and geometrically connected, then for any lisse Weil sheaf $\mathcal{G}_0$, there is some $b \in \overline{\mathbb{Q}}_l^*$ and a lisse Weil sheaf $\mathcal{F}_0$ that $\mathcal{G}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_b$, where $\mathcal{L}_b$ is the Weil sheaf corresponding to the character $W(X_0, \overline{x}) \to \overline{\mathbb{Q}}_l^* : x \mapsto b^{\deg(x)}$, which is a pull back from $\operatorname{Spec} \mathbb{F}_q$.

More generally, for any lisse Weil sheaf, there is a filtration that each quotient is of the form $\mathcal{F}_0^{(i)} \otimes \mathcal{L}_{b_i}$ for some $b_i \in \overline{\mathbb{Q}}_l^*$ and $\mathcal{F}_0^{(i)}$ lisse $\overline{\mathbb{Q}}_l$-sheaves.

Proof: Just choose $b = \chi_{\det}(\sigma)^{1/n}$, where $\deg(\sigma) = 1$, then

$$\wedge(\mathcal{G}_0 \otimes \mathcal{L}_{b^{-1}}) \cong \wedge(\mathcal{G}_0) \otimes \mathcal{L}_{\chi_{\det}(\sigma)}^{-1}$$

which has unit eigenvalues thus is a lisse $\overline{\mathbb{Q}}_l$-sheaf. $\square$

Grothendieck-Lefschetz Trace Formula

Def. (III.9.3.8) (L-Function). Given a constructible K-sheaf $\mathcal{F}$ on $X_{ét}$, its L-function is defined to be

$$L(X, \mathcal{F}, t) = \prod_{x \in |X|} \det(1 - F_x t^{d_x} | \mathcal{F}_x)^{-1} \in 1 + t\Lambda[[t]].$$

Prop. (III.9.3.9) (Grothendieck-Lefschetz Trace Formula). For a separated morphism of f.t. k-schemes $X \to S$, if $\mathcal{F}$ is any constructible K-sheaf $\mathcal{F}$ on $X_{ét}$, then we have

$$L(X, \mathcal{F}, t) = \prod_{n=0}^{2\dim X} L(S, R^n f_! \mathcal{F}, t)^{(-1)^n}.$$

In particular, for $S = \operatorname{Spec} k$, we have

$$L(X_0, \mathcal{F}, t) = \prod_{n=0}^{2\dim X_0} \det(1 - F_X^* t | H^n_{c,\acute{e}t}(X, \mathcal{F}))^{(-1)^{n+1}}$$

Notice by(III.8.5.8), the higher proper pushforward just vanish.

Proof: First take an open subscheme $U \to S$ and $Z = S - U$, consider $f_U : X_U \to U, f_Z : X_Z \to Z$, then we can use the excision long exact sequence for compact pushforward(III.8.5.6), we can use Noetherian induction to reduce to the case that X, S are both separated.

Now we reduce to the absolute case $S = \operatorname{Spec} k$: If the absolute case is true, then it suffices to prove that

$$\prod_{s\geq 0} \det(1 - F_X^* t | H^s_{c,\acute{e}t}(X_{\overline{k}}, \mathcal{F}))^{(-1)^s} = \prod_{n,m\geq 0} \det(1 - F_X^* t | H^m_{c,\acute{e}t}(S_{\overline{k}}, R^n f_! \mathcal{F}))^{(-1)^{m+n}}$$

And for this, use Leray spectral sequence, which is Frobenius equivariant by(III.9.2.7) and a determinential Euler characteristic of spectral sequences, Cf.[Conrad L18, P9].

The $S = \operatorname{Spec} k$ case is done in(III.9.3.13). □

Cor. (III.9.3.10). The L-function is a rational.

Remark (III.9.3.11) (Name of the Trace Formula). Using the formula

$$\det(1 - Ft|V)^{-1} = \exp(\sum_{i\geq 1} tr(F^i)\frac{t^i}{i})$$

for each endomorphism $F \in \operatorname{End}(V)$, we can unwinding the equation that it is equivalent to

$$\chi(F_X^* | H^*_{c,\acute{e}t}(X_{\overline{k}}, \mathcal{F})) = \sum_{x\in X(k)} \operatorname{tr}(F_x | \mathcal{F}_x),$$

for any $k = \mathbb{F}_{q^n}$, where $\chi(F_X^* | H^*_{c,\acute{e}t}(X_{\overline{k}}, \mathcal{F})) = \sum_{n\geq 0} (-1)^n \operatorname{tr}(F_X^* | H^i_{c,\acute{e}t}(X_{\overline{k}}, \mathcal{F}))$.

Proof: Cf.[Conrad L18 P8]. □

Lemma (III.9.3.12) (Weil Trace Formula). If C is a smooth projective curve over $k = \overline{\mathbb{F}_q}$ and $\psi : C \to C$ is an endomorphism, then

$$\Delta \cdot \Gamma_\psi = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\psi^* | H^i(C, \overline{\mathbb{Q}}_l)).$$

Prop. (III.9.3.13) (General Trace Formula for Frobenius). Let X_0 be a variety over $k = \mathbb{F}_q$ and $K_0 \in D_{perf}(X_0)$, then

$$\sum_{x\in X(k)} \operatorname{tr}(F_x | K_x) = \sum_i (-1)^i \operatorname{tr}(F_X^*, H^i_c(X, K)).$$

Prop. (III.9.3.14) (Grothendieck-Lefschetz Trace Formula for Weil Sheaves). For any Weil sheaf $\mathcal{F}_0$ over X_0, define

$$L(X_0,\mathcal{F}_0,t)=\prod_{x\in|X_0|}\det(1-t^{d_x}F_x|\mathcal{F}_{\overline{x}})^{-1}$$

Then we have

$$L(X_0,\mathcal{F}_0,t)=\prod_{i=0}^{2\dim X_0}\det(1-tF_X^*|H_c^i(X,\mathcal{G}))^{(-1)^{i+1}}$$

Proof: Use the filtration in(III.9.3.7), notice that the trace is additive for a filtration, so we can reduce to the case $\mathcal{G}_0=\mathcal{F}_0\otimes\mathcal{L}_b$ and $\mathcal{G}_{\overline{x}}=\mathcal{F}_{\overline{x}}\otimes\mathcal{L}_{b,\overline{x}}$, then the Euler factor is

$$\det(1-t^{d_x}F_x|\mathcal{F}_{\overline{x}})$$

and the cohomology factor is

$$\det(1-t\mathcal{F}_X^*|H_c^i(X,\mathcal{F}\otimes\mathcal{L}_b))=\det(1-tbF_X^*|H_c^i(X,\mathcal{F}))$$

where the projection formula(III.8.5.9) is used, noticing the $\mathcal{L}_b$ is pulled back from $\operatorname{Spec}\mathbb{F}_q$. □

4 Weights and Purity

Determinantial Weights

Prop. (III.9.4.1) (Structure of Weil Group of Curve). If X_0 is a geometrically connected smooth curve over $\mathbb{F}_q$, then the image of $\pi_1(X,\overline{x})$ in $W(X_0,\overline{x})^{ab}$ is a product of a finite group and a pro-p group.

Proof: Let K be the function field of X_0, $\overline{X_0}$ be the regular completion of X_0, with $S_0=\overline{X_0}-X_0$, then we have an isomorphism $\pi_1(\overline{X_0},\overline{x})\cong G_K$ Cf.[Étale Cohomology Lei Fu P136]?. So we can use global class field theory:

$$\begin{tikzcd}
 & \pi_1(\overline{X},\overline{x})^{ab} \arrow[r] \arrow[d] \arrow[dr] & \pi_1(\overline{X_0},\overline{x})^{ab} \arrow[r] & G_k\cong\widehat{\mathbb{Z}} \arrow[r] & 0 \\
0 \arrow[r] & I_K \arrow[r] \arrow[d, "\cong"] & W(\overline{X_0},\overline{x})^{ab}\cong W(K,k) \arrow[r] \arrow[u] \arrow[d, "\cong"] & W(k)\cong\mathbb{Z} \arrow[r] \arrow[u] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & K^*\backslash(A_K^*)^1/\prod_v\mathcal{O}_v^* \arrow[r] & K^*\backslash A_K^*/\prod_v\mathcal{O}_v^* \arrow[r] & q^{\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

So the image of $\pi_1(\overline{X},\overline{x})$ factors through $\pi_1(\overline{X},\overline{x})\to\pi_1(\overline{X},\overline{x})^{ab}\to K^*\backslash A_K^*/\prod_v\mathcal{O}_v^*$ which is the class number of K, is finite.

In this diagram, $W(X_0,\overline{x})$ corresponds to $K^*\backslash A_K^*/\prod_{v\notin S_0}\mathcal{O}_v^*$, so

$$0\to\operatorname{Ker}(W(X_0,\overline{x})\to W(\overline{X_0},\overline{x}))\to\operatorname{Im}(\pi_1(X_0,\overline{x}))\to\operatorname{Im}(\pi_1(\overline{X_0},\overline{x}))\to 0$$

But the kernel is a quotient of $\prod_{v\in S_0}\mathcal{O}_v^*$, which is a pro$-p$ group times a finite group, so finally $\operatorname{Im}(\pi_1(X_0,\overline{x}))$ is a product of a pro-p-group times a finite group. □

Lemma (III.9.4.2) (Curve Rank 1 case). If X_0 is a geometrically connected smooth curve over $\mathbb{F}_q$ and $\chi : W(X_0, \overline{x}) \to \overline{\mathbb{Q}}_l^*$ be a continuous character, then there exists a $c \in \overline{\mathbb{Q}}_l^*$ that χ is a product of a character of finite order and the character $\sigma \mapsto c^{\deg(\sigma)}$.

In particular, the Weil sheaf corresponding to χ is punctually ι-pure of weight $2\log_q |\iota(c)|$.

Proof: By(I.4.7.3), the image of χ is in $\mathcal{O}_E^*$ for some $E/\mathbb{Q}_l$ finite, so use(III.9.4.1), it has an open subgroup which is pro-p and pro-l so trivial, thus $\pi_1(X_0, \overline{x})$ is mapped to a finite group.

In particular there is an n that $\chi^n = \mathrm{id}$ on $\pi_1(X_0, \overline{x})$, so there is some b that $\chi^n = b^{\deg(\sigma)}$, hence if c is an n-th roots of b and we let $\chi' = \chi/c^{\deg(\sigma)}$, then $\chi'^n = 1$. □

Cor. (III.9.4.3) (Rank 1 Lisse Sheaf is Pure). If X_0 is a geometrically connected smooth curve, then any lisse Weil sheaf of rank 1 is pure.

Def. (III.9.4.4) (Determinential Weight). Let $\mathcal{F}_0$ be a lisse Weil sheaf on a geometrically connected smooth scheme X_0, and $0 = \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_0$ be a filtration of lisse sheaves that the quotients are irreducible, we define the **determinential ι-weights** of $\mathcal{F}_0$ to be that of the ι-weights of the top wedge products of the successive quotients divided by their ranks, which exists by(III.9.4.3).

Notice that the determinential ι-weights are unchanged when $\mathcal{F}_0$ is replaced by its semisimplification $\mathcal{F}_0^s = \oplus_{i\geq 0}(\mathcal{F}_i/\mathcal{F}_{i-1})$.

Purity

Def. (III.9.4.5) (Purity). For an embedding $\overline{\mathbb{Q}}_l \to \mathbb{C}$, a constructible sheaf $\mathcal{F}$ on a k-scheme X is called **ι-pure of weight** w if for any closed point $x \in X$, the $\overline{\mathbb{Q}}_l$-eigenvalues of F_x on $\mathcal{F}_x$ satisfies $|\iota(\alpha_i)| = (q^{d_x})^{w/2}$. It is called **pure of weight** w iff for any closed point $x \in X$, the $\overline{\mathbb{Q}}_l$-eigenvalues are q^{d_x}-Weil numbers of weight w, i.e. ι-pure for any embedding $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$.

It is said to be **(ι-)mixed** with weights $w_1, \ldots, w_n$ if it has a successive quotients of constructible $\overline{\mathbb{Q}}_l$-sheaves that are pure of weight w_i respectively.

Prop. (III.9.4.6). $\mathbb{Q}_l(1)$ is pure of weight -2, thus $\mathbb{Q}_l(r)$ is pure of weight $-2r$. This is because the geometric Frobenius F_x acts by $1/q^{d_x}$-th power, which is additively multiplying by $(q^{d_x})^{-2/2}$.

Prop. (III.9.4.7) (Permanence Properties).

- $f_0 : X_0 \to Y_0$ is a morphism, and $\mathcal{G}_0$ is a Weil sheaf on Y_0, then if $\mathcal{G}_0$ is ι-pure, then $f_0^*\mathcal{G}_0$ is also ι-pure, and the converse is also true if f is surjective.
- If $f_0 : X_0 \to Y_0$ is finite, and $\mathcal{G}_0$ is a Weil sheaf on X_0, then

Proof: 1 is because the stalk corresponds.

2: This is because the stalks can be calculated, by**?**. □

Semicontinuity of Weights

Def. (III.9.4.8) (Purity). A Weil sheaf $\mathcal{G}$ on X_0 is called **pure of weight** w if for any closed point $x \in X$, the $\overline{\mathbb{Q}}_l$-eigenvalues of $F_{\mathcal{G}}$ on the stalks $\mathcal{F}_x$ are all q^{d_x}-Weil numbers of weight w.

Def. (III.9.4.9) (Maximal Weight). For a general Weil sheaf $\mathcal{G}_0$ on X_0, we can also define the **maximal ι-weight** of $\mathcal{G}_0$ as

$$w(\mathcal{G}_0) = \sup_{x\in|X_0|} \sup_{\alpha_i} 2\log_{N(x)}(|\iota(\alpha_i)|).$$

Lemma (III.9.4.10). $|X_0(k_n)| = O(q^{n\dim X})$

Proof: We can pass to the reduced structure of X_0, then we can use excision to pass to the integral case. Then choose an open affine dense subset U_0 of X_0, then by Noetherian normalization, it factors through a finite map $f : U_0 \to \mathbb{A}_{k_n}^{\dim X_0}$, so

$$|U(k_n)| \leq (\deg f)q^{n\dim X_0}$$

Then we can use induction on dimension, because $\dim(X_0 - U_0) < \dim X_0$. □

Lemma (III.9.4.11). Let $\mathcal{G}_0$ be a Weil sheaf on X_0 and β be a real number that $\beta \geq w(\mathcal{G}_0)$, then the L-function

$$\iota(L(X_0, \mathcal{G}_0, t)) = \prod_{x\in|X_0|} \iota(\det(1 - t^{d_x}F_x, \mathcal{G}_{0,\overline{x}})^{-1})$$

converges for $|t| < q^{-\beta/2-\dim X_0}$ and has no zero or pole there.

Proof: We can show that it has no zero or pole using the fact that the logarithmic derivative has no poles(when it is convergent). We suppress the isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ and calculate:

$$\frac{d\log}{dt}L(X_0, \mathcal{G}_0, t) = \sum_{x\in|X_0|}\sum_{n\geq 1} d_x(\mathrm{tr}(F_x^n))t^{d_xn-1}(III.9.3.11) = \sum_{n\geq 1}\sum_{x\in|X_0|, d_x|n} d_x(\mathrm{tr}(F_x^{n/d_x}))t^{n-1}$$

Notice by assumption on β, $|\,\mathrm{tr}(F_x^{n/d_x})| \leq rq^{n\beta/2}$, where $r = \max_{x\in|X_0|}\dim_{\overline{\mathbb{Q}}_l}\mathcal{G}_{0,x}$ is finite because it has a stratification by(III.9.1.9), so

$$|\frac{d\log}{dt}L(X_0, \mathcal{G}_0, t)| \leq \sum_{n\geq 1}\sum_{x\in|X_0|, d_x|n} d_xrq^{n\beta/2}t^{n-1} = \sum_{n\geq 1}|X_0(k_n)|rq^{n\beta/2}t^{n-1}$$

converges for $|t| < q^{-\beta/2-\dim X_0}$ by(III.9.4.10). □

Lemma (III.9.4.12) (Semicontinuity of Weights for Curves). If X_0 is a smooth geometrically irreducible curve over k and $U_0 \xrightarrow{j_0} X_0$ be a nonempty open with $S_0 = X_0 - U_0$. Let $\mathcal{G}_0$ be a Weil sheaf on X_0 s.t. the restriction $j_0^*\mathcal{G}_0$ is lisse and $H_S^0(X, \mathcal{G}) = 0$, then $w(j_0^*(\mathcal{G}_0)) \leq \beta$ implies $w(\mathcal{G}_0) \leq \beta$.

Proof: For any point x, consider an affine open subset of X_0, then reduce to the affine case, and because $H_S^0(X, \mathcal{G}) = 0$ and the excision sequence(III.8.5.6), we have $\mathcal{G} \hookrightarrow j_*j^*\mathcal{G}$, so the weights of $\mathcal{G}_0$ are no more than that of $j_*j^*\mathcal{G}$, and replacing $\mathcal{G}_0$ with $j_{0*}j_0^*\mathcal{G}_0$, we can assume $\mathcal{G}_0 = j_{0*}j_0^*\mathcal{G}_0$. Then

$$H_c^0(X, \mathcal{G}) = H_c^0(X, j_*j^*\mathcal{G}) = H_c^0(U, j^*\mathcal{G}) = 0$$

by Poincare duality and the fact j_* is exact because it is finite.

Now by Grothendieck-Lefschetz trace formula,

$$L(X_0, \mathcal{G}_0, t) = L(U_0, j_[^*(\mathcal{G}_0), t) \cdot \prod_{s\in|S_0|}\det(1 - t^{d_s}F_s, \mathcal{G}_{\overline{s}})^{-1} = \frac{\det(1 - F_Xt|H_c^1(X, \mathcal{G}))}{\det(1 - F_Xt|H_c^2(X, \mathcal{G}))}$$

Denote $\mathcal{F}_0 = j_0^*\mathcal{G}_0$, then

$$H_c^2(X, \mathcal{G}) = H_c^2(U, \mathcal{F}) = (\mathcal{F}_{\overline{x}})_{\pi_1(U,\overline{x})}(-1)$$

So the weights of eigenvalues of F_X on $H_c^2(X,\mathcal{G}) \leq$ weights of $\mathcal{F}+2$, hence the L-function converges for $|t| < q^{-\beta/2-1}$. Now the LHS has $L(U_0, j_0^*(\mathcal{G}_0), t)$ converges for $|t| < q^{-\beta/2-1}$ because $w(\mathcal{F}_0) \leq \beta$, and so for the points in S_0, we also have $\det(1 - t^{d_s}F_s, \mathcal{G}_{\overline{s}})$ has no zero there, which means they have weights$\leq \beta + 1$. Now consider replacing $\mathcal{G}_0$ with $\mathcal{G}_0^{\otimes k}$ and let $k \to \infty$, then their weights$\leq \beta$. □

Prop. (III.9.4.13) (Semicontinuity of Weights). Let X_0 be normal geometrically $\mathcal{G}_0$ be a lisse sheaf on X_0 and $j_0 : U_0 \to X_0$ be an open dense subscheme, then

- $w(\mathcal{G}_0) = w(j_0^*\mathcal{G}_0)$.
- If $j_0^*(\mathcal{G}_0)$ is ι-pure of weights β, then $\mathcal{G}_0$ is also ι-pure of weights β.
- Let X_0 be irreducible and normal, and $\mathcal{G}_0$ is irreducible, then if $j_0^*\mathcal{G}_0$ is ι-mixed, then $\mathcal{G}_0$ is ι-pure.

Proof: 1: The weights is local so we may assume X_0 is irreducible, and then for any closed point x, we can connect it with U_0 with a curve(choose an affine open and use Noetherian Normalization to choose an irreducible component of an arbitrary curve in $\mathbb{A}^n$). Notice $H_S^0(X,\mathcal{G}) = 0$ because it is lisse thus $H^0(X,\mathcal{G})$ is determined by stalk thus $H^0(X,\mathcal{G}) \to H^0(U,\mathcal{G}|_U)$ is injective. So we finish by the curve case(III.9.4.12).

2: Apply item1 to $\mathcal{G}_0$ and $\mathcal{G}_0^\vee$.

3: It is ι-mixed so it has ι-pure Weil sheaf constituents. Now by(III.9.1.9) we can find an open dense U_0 that restriction to U_0 has constituents ι-pure lisse sheaves. But it is also irreducible because $\pi_1(U_0,\overline{a}) \to \pi_1(X_0,\overline{a})$ is surjective?, so it is ι-pure and item2 shows $\mathcal{G}_0$ is ι-pure. □

L^2-Norms and Maximal Weights

Def. (III.9.4.14). As in(III.9.9.1), for any Weil sheaf $\mathcal{G}_0$, we have a a function

$$f^{K_0} : X_0(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_l : x \mapsto \sum_i (-1)^i \operatorname{tr}(F_x^{n/d_x}|(\mathcal{H}^i(K_0))_{\overline{x}}),$$

Fix an arbitrary isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$, we can consider the usual L^2-norm for functions on $X_0(k_n)$, denoted by $(f,g)_n$.

Def. (III.9.4.15). Notice the equation form(III.9.4.11) can be rewritten as

$$\frac{d\log}{dt}L(X_0,\mathcal{G}_0,t) = \sum_{n\geq 1}\sum_{x\in|X_0|,d_x|n} d_x(\operatorname{tr}(F_x^{n/d_x}))t^{n-1} = \sum_n (f^{\mathcal{G}_0},1)_n t^{n-1}.$$

Now we define another closed related function

$$\varphi^{\mathcal{G}_0}(t) = \sum_n ||f^{\mathcal{G}_0}||_n^2 t^{n-1},$$

which works better with Fourier transform we are about to define later.

Lemma (III.9.4.16). There is a constant C that $||f^{\mathcal{G}_0}(x)||^2 \leq Cq^{n(w(\mathcal{G}_0)+\dim X_0)}$, so $\varphi^{\mathcal{G}_0}(t)$ converges for $|t| \leq q^{-w(\mathcal{G}_0)-\dim X_0}$.

Proof: The proof is similar to that of(III.9.4.11) thus omitted. □

Def. (III.9.4.17) (Norm of a Weil Sheaf). Define the Norm of a Weil sheaf as

$$||\mathcal{G}_0|| = \sup\{\rho| \limsup_n \frac{||f^{\mathcal{G}_0}||_n^2}{q^{n(\rho+\dim X_0)}} > 0\}$$

Then $q^{-||\mathcal{G}_0||-\dim X_0}$ is just the radius of convergence of the function $\varphi^{\mathcal{G}_0}(t)$, and $||\mathcal{G}_0|| \le w(\mathcal{G}_0)$ by lemma(III.9.4.16) above.

Prop. (III.9.4.18) (Radius of Convergence). Let $\mathcal{G}_0$ be a ι-mixed sheaf on an algebraic scheme X_0 of dimension 1, then If X_0 is a smooth curve, and $H_c^0(X,\mathcal{G}) = 0$, then $||\mathcal{G}_0|| = w(\mathcal{G}_0) = \beta$.

Proof: It suffices to show $w(\mathcal{G}_0) \le ||\mathcal{G}_0||$. First notice we can assume X_0 is reduced because the nilpotents corredsponds to zero Frobenius eigenvalues, and also it is connected, because the function $f^{\mathcal{G}_0}$ is additive in X. Now we study by cases:

1: If $\mathcal{G}_0$ is a lisse ι-pure sheaf on a smooth affine curve X_0, we may assume $\mathcal{G}_0 \neq 0$, then $\mathcal{G}_0 \otimes \overline{\mathcal{G}_0}(III.9.6.2)$ is ι-real of weight 2β and $(f^{\mathcal{G}_0\otimes\overline{\mathcal{G}_0}}, 1)_n = ||f^{\mathcal{G}_0}||_n^2$, so $\varphi^{\mathcal{G}_0}(t)$ is just the logarithmic derivative of the L-function $L(X_0, \mathcal{G}_0 \otimes \overline{\mathcal{G}_0}, t)$, thus(III.9.4.11) shows its convergence radius$\ge q^{-\beta-1}$. And notice the H_c^0 terms vanish so the poles can only appear as the zeros of H_c^2 term, so(III.9.6.3) shows the poles of the $L(X_0, \mathcal{G}_0 \otimes \overline{\mathcal{G}_0}, t)$ has weight $2\beta + 2$, thus the poles can only appear on $|t| = q^{-\beta-1}$.

Now consider each local Euler factor $\det(1 - F_x t^{d_x}|(\mathcal{G}_0 \otimes \overline{\mathcal{G}_0})_x)^{-1}$ has non-negative coefficients, they have poles because $\mathcal{G}_0 \neq 0$, and their poles have weight β because of purity, thus their product also has (real)poles, by previous argument, the pole has weight $\beta+1$, thus it has convergence radius at most $q^{-\beta-1}$, so we are done.

2: If $\mathcal{G}_0$ is a ι-mixed, consider its semisimplification $\mathcal{G}_0^{ss} = \mathcal{F}_0 \oplus \mathcal{H}_0$, where $\mathcal{F}_0$ is ι-pure of weight $w(\mathcal{G}_0)$, and $w(\mathcal{H}_0) \le w(\mathcal{F}_0)$.

Then $f^{\mathcal{G}_0} = f^{\mathcal{F}_0} + f^{\mathcal{H}_0}$, and

$$\varphi^{\mathcal{G}_0}(t) = \varphi^{\mathcal{F}_0}(t) + \sum_{n\ge 1} 2\,\mathrm{Re}(f^{\mathcal{F}_0}, f^{\mathcal{H}_0})_n t^{n-1} + \varphi^{\mathcal{H}_0}(t)$$

then by item1 $\varphi^{\mathcal{F}_0}(t)$ has convergence radius $q^{-w(\mathcal{G}_0)-1}$, and by(III.9.4.16) $\varphi^{\mathcal{H}_0}(t)$ has radius at least $q^{-w(\mathcal{H}_0)-1} > q^{-w(\mathcal{G}_0)-1}$, and by Cauchy inequality the middle term satisfies

$$|2\,\mathrm{Re}(f^{\mathcal{F}_0}, f^{\mathcal{H}_0})_n| \le 2||f^{\mathcal{F}_0}||_n ||f^{\mathcal{H}_0}||_n \le Cq^{n(w(\mathcal{F}_0)+w(\mathcal{H}_0)/2+1)}$$

So the middle term has convergence radius$>$ $q^{-w(\mathcal{G}_0)-1}$, so their sum has convergence radius $q^{-w(\mathcal{G}_0)-1}$. □

5 Geometric Monodromy

Def. (III.9.5.1) (Notations). Let X_0 be a geometrically connected normal scheme over $k = \mathbb{F}_q$ in this subsection.

Def. (III.9.5.2) (Geometric Monodromy Group). Let $\mathcal{G}_0$ be a Weil-sheaf associated to a representation $(V, \rho) = GL(\mathcal{G}_{\overline{x}})$ of $W(X_0, \overline{x})$, the **geometric monodromy group** G_{geo} associated to $\mathcal{G}_0$ is the Zariski Closure of $\rho(\pi_1(X, \overline{x})) \subset GL(V)$.

Every element in $\rho(W(X_0, \overline{x}))$ normalizes G_{geo} by continuity, so choosing an arbitrary generator $\sigma \in W(\overline{k}/k)$, we have an action of $W(\overline{k}/k)$ on G_{geo}. Define $G = W(\overline{k}/k) \ltimes G_{geo}$ the **arithmetic monodromy group** of $\mathcal{G}_0$.

Lemma (III.9.5.3). If G_{geo} is connected, then there is a positive integer N that the semidirect sequence

$$1 \to G_{geo} \to \deg^{-1}(N\mathbb{Z}) \xrightarrow{\deg} N\mathbb{Z} \to 1$$

is direct, i.e. $\deg^{-1}(N\mathbb{Z}) \cong G_{geo} \times \mathbb{Z}$.

Proof: Choose a $\deg(g) = 1$. The representation G_{geo} splits as a characters of $Z(G)$, and then some g^n stablizes these characters, hence stablizes $Z(G)$, which then it descends to an action on G_{adj}, whose automorphism is the automorphism of the Dynkin diagram**?**, so finite, so some g^m fixes G_{adj} after changing a semidirect product, thus induces a map $\mathrm{Hom}(G_{adj}, Z(G))$, but G_{adj} is semisimple(III.10.2.8), so the connected component is mapped to 1 in $Z(G)$(III.10.2.10), so there are only f.m. such homomorphism, showing g^k is 1, so the product is exact for $N = k$. □

Prop. (III.9.5.4) (Geometric Monodromy Group is Semisimple). Let $\mathcal{G}_0$ be a geometrically semisimple lisse Weil sheaf(III.9.3.3), then

- G_{geo} and G^0_{geo} are semisimple algebraic group.
- Let $Z = Z(G(\overline{\mathbb{Q}}_l))$, then the map $\psi : Z \to W(\overline{k}/k)$ has finite kernel and cokernel. In particular, Z contains an element of finite degree, and it is surjective after a finite base change of fields.

And notice in fact if $\mathcal{G}_0$ is semisimple, then it is automatically geometrically semisimple by(I.4.2.4).

Proof: 1: L G_{geo} is semisimple iff G^0_{geo} is semisimple. Pass to a finite étale covering, we may assume $G_{geo} = G^0_{geo}$. Let $R(G^0_{geo})$ be the radical and $R_u(G^0_{geo})$ be the unipotent radical, then R is normal in G^0 and G^0 is normal in G, so by(I.4.2.4) $V = GL(\mathcal{G}_{\overline{x}})$ is irreducible $R(G^0_{geo})$ representation, but it is solvable, so V is a direct sum of 1-dimensional representations, and $R_u(G^0_{geo})$ is trivial, in particular G^0_{geo} is reductive. So it is semisimple if the maximal Abelian quotient G^{ab}_{geo} is finite(III.10.2.10).

let T_1 be the maximal central torus of G^0_{geo}, then lemma(III.9.5.3) shows after a finite base change of fields, we may assume $G = G_{geo} \times \mathbb{Z}$, consider the composite $W(X_0, \overline{x}) \to G_{geo} \times \mathbb{Z} \to G_{geo} \to G^{ab}_{geo}$, then $\pi_1(X, \overline{x})$ is Zariski dense in G^{ab}_{geo}, and (III.9.4.3) shows clearly G^{ab}_{geo} has no maximal torus thus finite.

2: $\mathrm{Ker}\,\psi \subset Z(G_{geo}(\overline{\mathbb{Q}}_l))$ is finite since G_{geo} is semisimple. To find an element in $Z(G)$ of positive degree, we may use the same method as before to find an element ζ that commutes with G^0_{geo}, and pass to a power, we may assume it acts trivially on G_{geo}/G^0_{geo}.

For any $g \in G_{geo}$, consider $vp_g(n) = g\zeta^n g^{-1}\zeta^{-n} \in G^0_{geo}$, so $\varphi_g(m+n) = \varphi_g(n)\zeta^n\varphi_g(m)\zeta^{-n} = \varphi_g(n)\varphi_g(m)$, thus it is a homomorphism, and if $g' \in G^0_{geo}$, then

$$\varphi_g = \varphi_{g(g^{-1}g'g)} = \varphi_{g'g} = g'\varphi_g(g')^{-1}$$

so φ_g has image in $Z(G^0_{geo})$, which is finite, so $\varphi_g(n) = 1$ for some n, then ζ^n commutes with G_{geo} so $\zeta^n \in Z(G)$. □

Cor. (III.9.5.5) (Weights and Center Element Actions). Let $\mathcal{G}_0$ be a semisimple lisse Weil sheaf on X_0, if $z \in Z(G(\overline{\mathbb{Q}}_l))$ satisfies $\deg(z) = n \neq 0$, which exists by(III.9.5.4), then if z acts on V with eigenvalues α_i, then $\frac{2}{n}\log_q(|\iota(\alpha_i)|)$ is just the determinential ι-weights of $\mathcal{G}_0$.

Proof: z is in the center, thus by Shur's lemma, it acts on each irreducible part of $\mathcal{G}_0$ by a constant. Thus the determinential weights are clear, by definition. □

Cor. (III.9.5.6) (Properties of Determinential Weights). Let X_0 be a smooth and geometrically connected curve, $\mathcal{F}_0, \mathcal{G}_0$ be lisse Weil sheaves on X_0, then

- If α_i are the determinential ι-weights of $\mathcal{F}_0$ and β_j be that of $\mathcal{G}_0$, then $\alpha_i + \beta_j$ are those of $\mathcal{F}_0 \otimes \mathcal{G}_0$ with multiplicity.
- For $\gamma \in \mathbb{R}$, let $r(\gamma)$ be the sum of ranks of all irreducible constituents of $\mathcal{F}_0$ which have determinential weight γ w.r.t ι, then the determinential weights of $\wedge^r \mathcal{F}_0$ are the numbers $\sum_\gamma n(\gamma)\gamma$ with $\sum n(\gamma) = r$ and $0 \leq n(\gamma) \leq r(\gamma)$, $n(\gamma) \in \mathbb{Z}$ with multiplicity.

Proof: Firstly notice the determinential weight is unchanged when we change $\mathcal{F}_0, \mathcal{G}_0$ to their semisimplification $\mathcal{F}_0^{ss}, \mathcal{G}_0^{ss}$(III.9.4.4). And notice $(\mathcal{F}_0 \otimes \mathcal{G}_0)^{ss} = ((\mathcal{F}_0)^{ss} \otimes (\mathcal{G}_0)^{ss})^{ss}$, thus the determinential weights of $\mathcal{F}_0 \otimes \mathcal{G}_0$ are also unchanged. Similarly for the wedge product.

2: We may assume $\mathcal{F}_0, \mathcal{G}_0$ are irreducible, and let $\mathcal{H}_0 = (\mathcal{F}_0 \otimes \mathcal{G}_0)^{ss}$, $G_{geo}^{\oplus}, G_{geo}^{ss}$ be the geometric monodromy group of $\pi_1(X, \overline{x})$ in $GL(\mathcal{F}_{\overline{x}} \oplus \mathcal{G}_{\overline{x}})$ and $GL(\mathcal{H}_{\overline{x}})$ correspondingly, then $G_{geo}^{\oplus} \to G_{geo}^{ss}$ is surjective because they are both the geometric monodromy group of $\mathcal{H}_{\overline{x}}$. So also $G^{\oplus} \to G^{ss}$ is surjective. So if g be an element in the center of $G^{\oplus}$ that has nonzero degree, then it maps to the center of G^{ss} of nonzero degree. And the action of g on each factor $\mathcal{F}_x, \mathcal{G}_x$ is a constant, so action on H_x is also a constant, so we are done.

3: Easy from2. □

6 Real Sheaves

Def. (III.9.6.1) (ι-Real Sheaf). Let $\mathcal{F}_0$ be a Weil sheaf on X_0, then $\mathcal{F}_0$ is called **ι-real** if for any $x \in |X_0|$, the characteristic polynomial $\iota(\det(1 - F_x t, \mathcal{F}_{\overline{x}})$ of F_x real coefficients.

Prop. (III.9.6.2). Any ι-pure Weil sheaf of weight w is a direct sum of a ι-real ι-pure Weil sheaf. In fact, $\mathcal{F}_0 \oplus \mathcal{F}_0^\vee(-w) = \mathcal{F}_0 \oplus \overline{\mathcal{F}_0}$ is ι-real.

Lemma (III.9.6.3) (Eigenvalue of Cohomology and Stalk in Curve case). Let X_0 be a smooth geometrically connected curve over $\mathbb{F}_q$, $\mathcal{F}_0$ is a lisse Weil sheaf on X_0, then the eigenvalues of F_X on $H^0(X, \mathcal{F})$ or $H_c^2(X, \mathcal{F})$ is related to the determinential weights of $\mathcal{F}_0$ and the eigenvalue of F_x on $\mathcal{F}_{\overline{x}}$.

Proof: Let $V = \mathcal{F}_{\overline{x}}$, then

$$H^0(X, \mathcal{F}) = V^{\pi_1(X,\overline{x})}, \quad H_c^2(X, \mathcal{F}) = V_{\pi_1(X,\overline{x})}(-1).$$

Then the base change sheaf of the sheaf $V^{\pi_1(X,\overline{x})}$ or $V_{\pi_1(X,\overline{x})}(-1)$ on $\operatorname{Spec} k$ is the maximal subsheaf/quotient lisse sheaf of $\mathcal{F}_0$ that is constant on X. Then it has determinential weights just the action of F_X on the stalk by(III.9.5.5), which are also determinential weights of $\mathcal{F}_0$ by(III.9.5.6). □

Lemma (III.9.6.4) (Rankin-Selberg Method). Let X_0 be a smooth geometrically connected curve over $\mathbb{F}_q$, $\mathcal{F}_0$ be a lisse Weil sheaf on X_0, and w be the largest determinential weight of $\mathcal{F}_0$, then for any $x \in |X_0|$, $w_{N(x)}(\alpha) \leq w$.

Proof: By the arbitraryness of x, we can replace X_0 by an affine open nbhd of x. Then $H_c^0(X, \mathcal{G}) = 0$ by Artin vanishing(III.8.1.38). By Grothendieck trace formula,

$$\prod_{x \in |X_0|} \iota \det(1 - t^{d_x} F_x | \otimes^{2k} \mathcal{F}_{\overline{x}})^{-1} = \frac{\iota \det(1 - tF_X^* | H_c^1(X, \otimes^{2k} \mathcal{F}))}{\iota \det(1 - tF_X^* | H_c^2(X, \otimes^{2k} \mathcal{F}))}$$

Now the weight of root t_0 of $\det(1-tF_X^*|H_c^2(X,\otimes^{2k}\mathcal{F}))$ has weight$\leq$ determinential weight of $\mathcal{G}_0^{\otimes 2k}+2(III.9.6.3)\leq 2kw+1$(III.9.5.6), so $|t_0|\geq q^{-k\beta-1}$.

Now by the formula(III.9.3.11) and noticing $\mathrm{tr}(F_x^n,\otimes^{2k}\mathcal{F}_{\overline{x}})=(\mathrm{tr}(F_x,\mathcal{F}_{\overline{x}}))^{2k}$, so $(1-t^{d_x}F_x|\otimes^{2k}\mathcal{F}_{\overline{x}})^{-1}$ has non-negative coefficients, which means their convergence radius are no less than $q^{-k\beta-1}$, equivalently, $(1-t^{d_x}F_x|\otimes^{2k}\mathcal{F}_{\overline{x}})$ has no zeros with eigenvalue$<q^{-k\beta-1}$.

So for any eigenvalue α of F_x acting on $\mathcal{F}_{\overline{x}}$, $|\iota(\alpha^{-2k/d_x})|\leq q^{-k\beta-1}$, or equivalently,

$$|\iota(\alpha)|^2\leq N(x)^{\beta+1/k}.$$

Now let $k\to\infty$, we are done. □

Lemma (III.9.6.5) (Real Sheaf Mixed Curve case). Let X_0 be smooth geometrically irreducible curve over k and $\mathcal{G}_0$ be an ι-lisse Weil sheaf on X_0, then all irreducible constituents of $\mathcal{G}_0$ is ι-pure, and their ι-weights coincides with their determinential weights.

Proof: For $\beta\in\mathbb{R}$, let $\mathcal{F}_0(\beta)$ be the sum of constituents of $\mathcal{F}_0$ of determinential weight β, and let $n(\beta)=\mathrm{rank}(\mathcal{F}_0(\beta))$, then we need to show that $w_{N(x)}(\alpha_i(\beta))=\beta$ for any eigenvalue of $\mathcal{F}_x$ on $\mathcal{F}(\beta)_{\overline{x}}$.

By definition of determinential weight, for each γ, we have $\sum w_{N(x)}(\alpha_j(\gamma))=n(\gamma)\gamma$. Now let $N=\sum_{\gamma>\beta}n(\gamma)$, then any determinential weight of $\wedge^{N+1}\mathcal{F}_0$ has weight$\leq\beta+\sum_{\gamma>\beta}n(\gamma)\gamma$: This is clear by(III.9.5.6) as the determinential weights of $\wedge^{N+1}\mathcal{F}_0$ is of the form $\sum_\gamma a(\gamma)\gamma$ that $0\leq a(\gamma)\leq\gamma$ and $\sum_\gamma a(\gamma)=N+1$.

But now $\alpha_i(\beta)\prod_{\gamma>\beta}\prod_{i=1}^{n(\gamma)}\alpha_j(\gamma)$ is an eigenvalue of $(\wedge^{N+1}\mathcal{F}_0)_{\overline{x}}$, but by lemma(III.9.6.4), $w_{N(x)}(\alpha_i(t))\leq t$. Thus we must have equality $w_{N(x)}(\alpha_i(\beta))=\beta$. □

Prop. (III.9.6.6) (Real Sheaf is Mixed). Let X_0 be an algebraic scheme over $\mathbb{F}_q$, then

- Any ι-real Weil sheaf on X_0 is ι-mixed.
- If X_0 is irreducible and normal, any irreducible constituent of a lisse of an ι-real sheaf is ι-pure.

Proof: Cf.[Bhatt P28], [KW, P36].

We have the following devissages:

- Choose an open subset $j_0:U_0\hookrightarrow X_0, S_0=X_0-U_0$ and consider the fundamental excision sequence(III.8.5.5), we can reduce to an open affine subscheme $U_0\subset X_0$.
- We may base change to a finite field extension.?
- So we may reduce to the case X_0 is smooth, irreducible affine, and $\mathcal{G}_0$ is lisse, with all the irreducible constituents geometrically irreducible(by base change, because they are geometrically semisimple(III.9.5.4)). And we may assume $\dim X_0>1$ because the curve case is proven.
- Change k to the alg.closure of k in the function field of X_0, we can assume X_0 is geometrically irreducible by(III.3.3.3).

Embed X_0 in some projective space $\mathbb{P}_0^N$, then by a suitable Bertini theorem, the linear subspaces of codimension $\dim X-1$ that intersects X with a non-empty smooth irreducible curve C_L is dense in the Grassmannian. Now the closed points in any C_L is a pure-point for the any irreducible component $\mathcal{F}_0$ of $\mathcal{G}_0$ of the same weights. Now let L vary, then there is a dense subset of a finite extension of X_0 that $\mathcal{F}_0$ is pure. So we are done. □

7 Deligne's Purity Theorem

Prop. (III.9.7.1) (Deligne's Purity Theorem). If $f : X_0 \to Y_0$ is a separated morphism of algebraic scheme over $\mathbb{F}_q$, and $\mathcal{F}$ is a constructible $\overline{\mathbb{Q}}_l$-sheaf on X that is ι-mixed weights$\leq n$, then for any integer $i \geq 0$, the sheaf $R^i f_! \mathcal{F}$ is also ι-mixed of weights$\leq n+i$. Moreover, each ι-weight of of $R^i f_! \mathcal{F}$ is equivalent modulo $\mathbb{Z}$ to an ι-weight of $\mathcal{F}$.

Proof: This follows from(III.9.7.6). □

Cor. (III.9.7.2). If X is a smooth separated algebraic k-scheme, $\mathcal{F}$ is mixed of weight $\geq n$, then $H^i_{ét}(X, \mathcal{F})$ is mixed of weights $\geq n+i$.

Proof: Use Poincare duality(III.8.6.1), we know $H^{2d-n}_{c,ét}(X_{\overline{k}}, \mathcal{F}^\vee(d))$ is the Galois dual representation of $H^n_{ét}(X_{\overline{k}}, \mathcal{F})$, and $\mathcal{F}^\vee(d)$ is still a lisse sheaf pure of weight $-w-2d$, thus Deligne's purity theorem(III.9.7.1) shows that $H^{2d-n}_{c,ét}(X_{\overline{k}}, \mathcal{F}^\vee(d))$ has weight $\leq (-w-2d)+(2d-n) = -w-n$, thus we are done. □

Cor. (III.9.7.3) (Weil's Conjecture). Let X be a smooth separated algebraic k-scheme, and $\mathcal{F}$ is a lisse $\overline{\mathbb{Q}}_l$-sheaf which is pure of weight w, then the image of $H^n_{c,ét}(X_{\overline{k}}, \mathcal{F})$ in $H^n_{ét}(X, \mathcal{F})$ is pure of weight $w+n$.

Proof: The morphism $H^n_{c,ét}(X_{\overline{k}}, \mathcal{F}) \to H^n_{ét}(X, \mathcal{F})$ defined in(III.8.5.7) is compatible with Frobenius, so from(III.9.7.2) we know the image has weights$\geq w+n$, so combined with Deligne's purity theorem(III.9.7.1), we know it is pure of weight $w+n$. □

Cor. (III.9.7.4). If $f_0 : X_0 \to Y_0$ is a smooth proper map of schemes of f.t. and $\mathcal{F}_0$ is ι-pure of weight β, then $R^i f_{0,*} \mathcal{F}_0$ ι-pure of weight $\beta + i$.

Proof: Use proper base change of(III.8.5.6) to reduce to the case of(III.9.7.3). Notice in the proper case, $Rf_* = Rf_!$. □

Cor. (III.9.7.5) (Riemann Hypothesis). If X is smooth proper k-scheme, then $H^n_{ét}(X, \overline{\mathbb{Q}}_l)$ is pure of weight n.

Reduction to Curve case

Prop. (III.9.7.6). Deligne's purity theorem can be reduced to case that X_0 is a smooth geometrically connected affine curve$\subset \mathbb{A}^1_{\mathbb{F}_q}$ and $\mathcal{F}_0$ a lisse $\overline{\mathbb{Q}}_l$-sheaf.

Proof: We have the following Devissages for Deligne's theorem:

- It is trivial in case f_0 is quasi-finite. This is because of(III.8.5.8), as the fiber has dimension 0.
- We can replace X_0 by an affine open $U_0 \subset X_0$ by Noetherian induction and excision sequence(III.8.5.6), which commutes with Frobenius action.
- If the conclusion is true for g_0, h_0, then it is true for $f_0 = g_0 \circ h_0$, this follows from the Leray spectral sequence(III.8.5.6), which is Frobenius equivariant by(III.9.2.7).
- We can replace Y_0 with an affine open $U_0 \subset Y_0$: If the image f_0 is not dense, then trivial, if it is dense, then choose any affine open U_0, then it suffices to prove for $f_0 : f_0^{-1}(U_0) \to Y_0$ by item2, then then by item3 it suffice to prove for $f_0 : f_0^{-1}(U_0) \to U_0$, because $U_0 \hookrightarrow Y_0$ is quasi-finite and use item1.

Now we claim we can reduce to the case of $f_0 : X_0 \to Y_0$ surjective affine smooth with the fibers being geometrically irreducible curves: By devissage2 and 4, we may assume X_0, Y_0 is affine, thus f_0 is affine. Take a generic point η of Y_0, then $(X_0)_\eta \to \operatorname{Spec} k(\eta)$ is affine hence by Noetherian normalization(I.5.6.8) there is a finite map $X_\eta \to \mathbb{A}^n_{k(\eta)}$, and this spread out to a finite morphism $f_0^{-1}(U_0) \to U_0$ for some affine open $U_0 \subset Y_0$ because f_0 is of f.t.. Then by Devissage1 and 3 we are reduced to the case $A^1_{Y_0} \to Y_0$. Now by(III.9.1.9), there is an affine open $U_0 \subset A^1_{Y_0}$ that $\mathcal{F}_0|_{U_0}$ is lisse, so by Devissage2 we may change X_0 to U_0.

That is we reduced to the case that $\mathcal{F}_0$ is lisse and X_0 is open in $A^1_{Y_0}$ so f_0 is smooth affine, in particular open(III.4.2.5), so we can replace Y_0 by $f(X_0)$ and assume f_0 is surjective. Then the fiber are all geometrically irreducible curves.

Then the assertion about weights are clear from proper base change(III.8.5.6) and the curve case.

For the ι-mixedness, we may use(III.9.6.2) and(III.9.6.6) to reduce to showing that $R^i f_!$ maps ι-real sheaf to ι-real sheaf.

for a geometric point $\overline{x} \to x \to X_0$, let $C \to C_0$ be the fiber, which is affine irreducible, so $H^0_c(X, \mathcal{F}) = 0$ by Poincare duality(III.8.6.1) and Artin vanishing theorem(III.8.1.38), so

$$\iota L(C_0, \mathcal{G}_0, t) = \frac{\iota \det(1 - tF_X^* | H^1_c(C, \mathcal{G}|_C))}{\iota \det(1 - tF_X^* | H^2_c(C, \mathcal{G}|_C))}$$

by Grothendieck-Lefschetz formula(III.9.3.14). Now we can use Poincare duality and the definition that $\mathcal{G}_0$ is pure of weight β, we know $H^2_c(C, \mathcal{G}|_C)$ is pure of weight $\beta+2$, by(III.9.6.3). And $H^2_c(C, \mathcal{G}|_C)$ has weights smaller than $\beta + 1$ by the curve case, so the two polynomial is coprime, and both has constant coefficient 1, which shows they are both real. And then by proper base change(III.8.5.6), this just says $R^i f_{0!}\mathcal{G}_0$ is ι-real. □

Third Reduction

Prop. (III.9.7.7). The final proof of Weil conjecture by proving(III.9.7.6).

Proof: We have the following devissages:

- We only need to check for $H^1_c(X, \mathcal{F})$, because H^0_c vanish by Poincare duality(III.8.6.1) and Artin vanishing theorem(III.8.1.38) and , $H^2_c(X, \mathcal{F})$ is dealed with in(III.9.6.3).
- We are free to pass to finite base change.
- We may assume $\mathcal{F}_0$ is geometrically irreducible: By(III.9.5.4), all the irreducible constituents of $\mathcal{F}_0$ are geometrically semisimple, so pass to a finite base change, we may assume that its irreducible filtration is just the geometric irreducible filtration, then because $H^0_c = 0$, H^1_c is left exact.
- We can assume that $\mathcal{F}_0$ can be extended to a lisse sheaf on ∞. This is because we can choose a closed point and move it to ∞ by using Möbius transform, after a finite base change.
- We can assume $\mathcal{F}_0$ is not geometrically constant: if $\mathcal{F}_0 \cong \underline{\overline{\mathbb{Q}}_l}$, then let $i : U_0 \to P^1_{\mathbb{F}_q}$ and $Z_0 = P^1_{\mathbb{F}_q} - U_0$, then there is a short exact sequence

  $$0 \to j_{0!}(\underline{\overline{\mathbb{Q}}_l}) \to j_*(\underline{\overline{\mathbb{Q}}_l}) \to Q \to 0$$

  where Q is supported at S, so its higher compact cohomology vanish, and weights of $H^0(\mathbb{Q}) = \prod_{s \in S} (j_{0*}(\mathcal{F}_0))_{\overline{s}}$ is no more than the maximal weight of $\underline{\overline{\mathbb{Q}}_l}$ on X_0, which is 0, by semicontinuity of weights for curves(III.9.4.12). And $j_*(\overline{\mathbb{Q}}_l)$ is also geometrically constant, thus its cohomology is $\mathrm{Pic}(\mathbb{P}^1)[n] = 0$ by(III.8.3.2), so $H^1(\overline{P^1}, j_{0!}(X_0))$ has weights zero.

The actual proof will use the following lemma(III.9.7.8). After that, notice by(III.9.9.8)

$$(T_\psi(G_0))|_{\{0\}} = R\Gamma_c(\mathbb{A}^1, \mathcal{G})[1] = R\Gamma_c(U, \mathcal{F})[1] = H^1_c(U, \mathcal{F})$$

Then to understand the Frobenius eigenvalues of $H^1_c(U, \mathcal{F})$, it suffices to understand the weights of $T_\psi(\mathcal{G}_0)$, i.e.

$$w(T_\psi(\mathcal{G}_0)) \le w + 1$$

Then we use(III.9.4.18), notice the condition is satisfied by lemma(III.9.7.8), so $w(T_\psi(\mathcal{G}_0)) = ||T_\psi(\mathcal{G}_0)||$, and also $w(G_0) = ||G_0||$ for the same reason as $H^0_c(\mathbb{A}^1, \mathcal{G}) = H^0_c(U, \mathcal{F}) = 0$ by Poincare duality. Now(III.9.9.12) gives the result.

□

Lemma (III.9.7.8) (Key Assertions of Weil Proof). If $\mathcal{G}_0 = j_{0!}(\mathcal{F}_0)$ where $j_0 : U_0 \hookrightarrow A^1_{\mathbb{F}_q}$, $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_l$ is a fixed non-trivial additive character, then

- $T_\psi(\mathcal{G}_0)$ is a sheaf placed at degree 0.
- $H^0_c(\mathbb{A}^1, T_\psi(\mathcal{G}_0)) = 0$.
- $T_\psi(\mathcal{G}_0)$ is ι-mixed.

Proof: 1: By(III.9.9.8), we need to show $H^i(\mathbb{A}^1, \mathcal{G}_0 \otimes \mathcal{L}(\psi_a)) = 0$ for $i \ne 1$, and this is equivalent to

$$H^i(\mathbb{A}^1, j_!\mathcal{F} \otimes \mathcal{L}(\psi_a)) = H^i(U, F \otimes \mathcal{L}(\psi_a)) = 0.$$

Notice by vanishing resultproper-pushforward-to-direct-image-sheanomqsk, only need to show $i = 0, i = 2$, $i = 0$ case is done by Poincare duality(III.8.6.1) and Artin vanishing(III.8.1.38) because it is smooth and $\mathcal{F}$ is lisse.

$H^2_c(\mathcal{G}_0 \otimes \mathcal{L}(\psi_a)) = V_{\rho\otimes\chi_a}|_{\pi_1(U,\overline{x})}(-1)$ by(III.9.6.3), and $\rho \otimes \chi_a$ irreducible as ρ does, so if $V_{\rho\otimes\chi_a}|_{\pi_1(U,\overline{x})} \ne 0$, then $\rho \otimes \psi_a$ is trivial representation. Then $\mathcal{G} \cong \mathcal{L}_{\psi_{-a}}$ on $\mathbb{P}^1_{\overline{k}}$ as an étale sheaf on $\mathbb{A}^1 \cup \{\infty\} = \mathbb{P}^1$ by our reduction, so we have the character ψ_{-a} factors through $\pi_1(\mathbb{P}^1, \overline{x})$, i.e.

$$\begin{tikzcd} \pi_1(\mathbb{A}^1, \overline{x}) \arrow[r] \arrow[d] & \pi_1(\mathbb{P}^1, \overline{x}) = 0 \arrow[d] & \\ \pi_1(\mathbb{A}^1_0, \overline{x}) \arrow[r] & \pi_1(\mathbb{P}^1_0, \overline{x}) \arrow[r, "\psi_{-a}"] & \overline{\mathbb{Q}}_l \end{tikzcd}.$$

But this is in contradiction with the fact the Artin-Schreier cover is geometrically irreducible**?**.

2: Denote $T_\psi(\mathcal{G}_0) = \mathcal{K}_0$, then by(III.9.9.7) and Fourier inversion(III.9.9.10):

$$H^0_c(\mathbb{A}^1, \mathcal{K}) = \mathcal{H}^{-1}((T_{\psi^{-1}}(\mathcal{K}_0))_0) = \mathcal{H}^{-1}(T_{\psi^{-1}} \circ T_\psi(j_{0!}(\mathcal{F}_0))_0) = \mathcal{H}^{-1}(j_{0!}(\mathcal{F}_0)(-1))_0 = 0$$

because $\mathcal{F}_0$ is placed at degree 0.

3: To show ι-mixed, the only thing we can do it show it is embedded in a ι-real sheaf: Consider the ι-real sheaf

$$\mathcal{H}_0 = \pi^{2*}(j_{0!}\mathcal{F}_0) \otimes m^*(\mathcal{L}(\psi)) \oplus \pi^{2*}(j_{0!}\mathcal{F}_0^\vee) \otimes m^*(\mathcal{L}(\psi^{-1}))(-w)$$

Then

$$(R^i\pi_!^1(\mathcal{H}_0))_{\overline{x}} = H^i(\{\overline{x}\}\times\mathbb{A}^1,\mathcal{H}_0) = H^i(j_{0!}\mathcal{F}_0\otimes\mathcal{L}(\psi_x))\oplus H^i(j_{0!}\mathcal{F}_0^\vee\otimes\mathcal{L}(\psi_{x^{-1}}^{-1}))(-w)$$

which we proved to vanish for $i\neq 1$. So using Poincare duality on $\{\overline{x}\}\times\mathbb{A}^1$,

$$\det(1-tF_x^{d_x}|(R^1\pi_!^1(\mathcal{H}_0)_{\overline{x}}) = \det(1-tF_X|H_c^1(\{\overline{x}\}\times\mathbb{A}^1,H_0|_{\{\overline{x}\}\times\mathbb{A}^1}) = \prod_{y\in\mathbb{F}_{q^n}}\det(1-tF_y^{d_y}|(\mathcal{H}_0)_{\overline{(x,y)}})^{-1}$$

which is real, so by(III.9.6.5), the direct summand $T_\psi(\mathcal{G}_0)$ is ι-mixed. $\square$

Remark (III.9.7.9). If we use the machinery of perverse sheaf and show that Fourier transform commutes preserves perversity, then item1, 2 will be a direct consequence, Cf.[Bhatt notes, P39]. In fact, this is just the bigger picture, given in [Weil conjectures Perverse Sheaves and l-adic Fourier Transform Kiehl/Weissauer].

8 Semisimplicity and Hard Lefschetz

Prop. (III.9.8.1) (Semisimplicity Theorem). If X_0 is smooth and $\mathcal{F}_0$ is a lisse and ι-pure $\overline{\mathbb{Q}}_l$-sheaf, then $\mathcal{F}_0$ is semisimple, thus geometrically semisimple by(III.9.5.4) .

Proof: Let $\mathcal{F}'$ be the sum of irreducible lisse subsheaves of $\mathcal{F}$, then it is the largest semisimple subsheaf of $\mathcal{F}$. It is stable under $G(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, thus can be descended to a lisse subsheaf $\mathcal{F}_0'$ of $\mathcal{F}_0$, and let $\mathcal{F}''=\mathcal{F}_0/\mathcal{F}_0'$, we want to show the exact sequence

$$0\to\mathcal{F}_0'\to\mathcal{F}_0\to\mathcal{F}_0''\to 0$$

splits geometrically. Notice this exact sequence defines an element in $\mathrm{Ext}^1_X(\mathcal{F}'',\mathcal{F}') = H^1(X,(\mathcal{F}'')^\vee\otimes\mathcal{F}')$?. $\mathcal{F}_0$ is pure, hence so does $(\mathcal{F}_0'')^\vee\otimes\mathcal{F}_0'$, thus $H^1(X,(\mathcal{F}'')^\vee\otimes\mathcal{F}')$ is ι-mixed of weights≥ 1. But the exact sequence is compatible with Frobenius action, thus it defines a Frobenius fixed element, which then must vanish. $\square$

Cor. (III.9.8.2). If $f:X\to Y$ is proper between smooth sheaves, then the sheaves $R^if_*\underline{\mathbb{Q}_l}$ are semisimple.

Prop. (III.9.8.3) (Hard-Lefschetz). Cf.[Bhatt P42].

9 Fourier Transformation

Sheaf to Functions Correspondence

Def. (III.9.9.1) (Sheaf to Functions Correspondence). For a complex $K_0\in D^b_{cons}(X_0,\overline{\mathbb{Q}}_l)$, we can associate a function

$$f^{K_0}:X_0(\mathbb{F}_{q^n})\to\overline{\mathbb{Q}}_l: x\mapsto\sum_i(-1)^i\,\mathrm{tr}(F_x^{n/d_x}|(\mathcal{H}^i(K_0))_{\overline{x}})$$

Prop. (III.9.9.2). We can use Grothendieck formula for a constructible sheaf(III.9.3.9) to relate the function f^{K_0} to the compact cohomologies of $\mathcal{H}^iK_0$, and we can translate many know theorems:

- $f^{f^*K_0}=f^{K_0}\circ f$.

- $$f^{K_0}\cdot f^{T_0}=f^{K_0\otimes^L T_0}?$$
- (Base Change)(III.8.5.6) asserts that given a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

  then it says in case Y' is a closed point of Y,

$$f^{Rf_!K_0}(y)=\sum_{x\in X_y(\mathbb{F}_{q^n})}f^{K_0}(x)$$

  where $y\in Y(\mathbb{F}_{q^n})$, and more generally

$$\sum_{x'\in X'_{y'}}f^{K_0}(g'(x'))=\sum_{x\in X_{g(y')}}f^{K_0}(x)$$

- The projection formula(III.8.5.9) turns out to say something trivial:

$$\sum_{x\in X_y}(f^{K_0}(f(x))\cdot f^{T_0}(x))=f^{K_0}(y)\cdot(\sum_{x\in X_y}f^{T_0}(x))$$

Artin-Schreier Sheaf

Def. (III.9.9.3) (Notations). For an arbitrary character $\psi:\mathbb{F}_q\to\overline{\mathbb{Q}}_l^*$, it can be extended to $\mathbb{F}_{q^n}$ by

$$\mathbb{F}_{q^n}\xrightarrow{\mathrm{tr}}F_q\xrightarrow{\psi}\overline{\mathbb{Q}}_l^*$$

Prop. (III.9.9.4) (Artin-Schreier Sheaf). Let $y^{q^n}-y-x\in\mathbb{A}_0^2$ be the finite Galois cover of $\mathbb{A}_0^1$ via x coordinates, with the Galois group isomorphic to $\mathbb{F}_q$ with $1\mapsto(x\mapsto x+1)$. Then we get a surjection $\pi_1(\mathbb{A}_0^1,\overline{x})\to\mathbb{F}_q$, when composed with ψ, we get a rank1 étale sheaf $\mathcal{L}_0(\psi)$ called the **Artin-Schreier sheaf** on $\mathbb{A}_0^1$.

Prop. (III.9.9.5). $f^{\mathcal{L}_0(\psi)}(x)=\psi(-x)$.

Proof: If $k(x)=\mathbb{F}_{q^n}$, then consider the arithmetic Frobenius $\sigma:(x,y)\mapsto(x^{q^n},y^{q^n})$, then if $y^{q^n}-y=x$, then we have

$$y^q=y+x,\quad y^{q^2}=y^q+x^q=y+x^q+x,\ldots,\quad y^{q^n}=y+x+x^q+\ldots+x^{q^{n-1}}=y+\mathrm{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x)$$

So in the correspondence(III.9.9.4), we know $F_{\overline{x}}$ acts on $\mathcal{L}_\psi$ by multiplication by $\psi(\mathrm{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x))=\psi(x)$, so the geometric Frobenius acts by $\psi(-x)$. □

Def. (III.9.9.6) (Deligne-Fourier Transform). Consider the multiplication map $\mathbb{A}_0\times\mathbb{A}_0'\to\mathbb{A}_0$, let the sheaf $\mathcal{L}(\psi)$ be placed at A_0, and $K_0\in D_c^b(\mathbb{A}_0',\overline{\mathbb{Q}}_l)$ be placed at $\mathbb{A}_0'$, then define the **Deligne-Fourier transform**

$$T_\psi:D_c^b(\mathbb{A}_0',\overline{\mathbb{Q}}_l)\to D_c^b(\mathbb{A}_0,\overline{\mathbb{Q}}_l):K_0\mapsto R\pi_!^1(\pi^{2*}K_0\otimes^L m^*\mathcal{L}_0(\psi))[1]$$

Lemma (III.9.9.7). We have $f^{T_\psi K_0}(x) = -\sum_{y\in\mathbb{F}_{q^n}} f^{K_0}(y)\psi(-xy)$ for any $x \in F_{q^n}$.

Proof: Use(III.9.9.2), we have

$$\begin{aligned} f^{T_\psi K_0}(x) &= \sum_{y\in\mathbb{F}_{q^n}} f^{(\pi^{2*}K_0 \otimes^L m^*\mathcal{L}_0(\psi))[1]}((x,y)) \\ &= -\sum_{y\in\mathbb{F}_{q^n}} f^{\pi^{2*}K_0}((x,y)) \cdot f^{m^*\mathcal{L}_0(\psi)}((x,y)) \\ &= -\sum_{y\in\mathbb{F}_{q^n}} f^{K_0}(y)\psi(-xy). \end{aligned}$$

□

Prop. (III.9.9.8). Let a be a geometric point of $\mathbb{A}^1_0$, then

$$(T_\psi(K_0))_a = R\Gamma_c(K \otimes^L \mathcal{L}(\psi_a))[1]$$

where $\psi_a : \mathbb{F}_{q^n} \to \overline{\mathbb{Q}}_l^*$ maps $x \mapsto \psi(ax)$. In particular, $\mathcal{H}^i((T_\psi(K_0))_0) = H^i_c(\mathbb{A}^1, K)$, so we placed the complex into a family of deformations.

Proof: By base change(III.8.5.6),

$$(T_\psi(K_0))_a = R\Gamma_c((\pi^{2*}K_0 \otimes^L m^*\mathcal{L}_0(\psi)|_{\{a\}\times\mathbb{A}^1})[1] = R\Gamma_c(K \otimes^L \mathcal{L}(\psi_a))[1].$$

□

Lemma (III.9.9.9). If $\delta_0 = i_{0*}\overline{\mathbb{Q}}_l$ be the skyscraper sheaf, where $i_0 : \{0\} \hookrightarrow \mathbb{A}^1$, then

$$T_\psi(\overline{\mathbb{Q}}_l[1]) = \delta_0(-1).$$

Proof: For the Artin-Schreier cover $P : x \mapsto x^q - x$, we have

$$P_*\overline{\mathbb{Q}}_l \cong \oplus_{x\in\mathbb{F}_q}\mathcal{L}(\psi_x)\textbf{?}$$

and P is finite thus proper and P_* is exact(III.8.1.16), so using the Leray spectral sequence(III.8.5.6), we can calculate

$$H^1_c(\mathbb{A}^1, \mathcal{L}(\psi_x)) = 0 = H^1_c(\mathbb{A}^1, \overline{\mathbb{Q}}_l), \quad H^2_c(\mathbb{A}^1, \mathcal{L}(\psi_x)) = 0(III.8.5.8), \quad H^2_c(\mathbb{A}^1, \mathcal{L}(\psi_x)) = \delta_0(x)\overline{\mathbb{Q}}_l(-1)(III.9.6.3)$$

So

$$(R\pi^1_!(m^*\mathcal{L}_0(\psi)[1])[1])_x = R\Gamma_c(\mathcal{L}(\psi_x))[2] = \delta_0(-1).$$

□

Prop. (III.9.9.10) (Fourier Inversion). $T_{\psi^{-1}}T_\psi K_0 = K_0(-1)$.

Proof: Consider

$$\begin{tikzcd} \mathbb{A}^1_0 \times \mathbb{A}^1_0 \times \mathbb{A}^1_0 \arrow[r, "\pi^{23}"] \arrow[d, "\pi^{12}"] & \mathbb{A}^1_0 \times \mathbb{A}^1_0 \arrow[r, "\pi^2"] \arrow[d, "\pi^1"] & \mathbb{A}^1_0 \\ \mathbb{A}^1_0 \times \mathbb{A}^1_0 \arrow[r, "\pi^2"] \arrow[d, "\pi^1"] & \mathbb{A}^1_0 & \\ \mathbb{A}^1_0 & & \end{tikzcd}$$

And we will use the following Cartesian diagrams:

$$\begin{tikzcd} \mathbb{A}_0^3 \arrow[r, "\alpha:(x,y,z)\mapsto(y,z-x)"] \arrow[d, "\pi^{13}"] & \mathbb{A}_0^2 \arrow[d, "\pi^2"] \\ A_0^2 \arrow[r, "\beta:(x,z)\mapsto z-x"] & A_0^1 \end{tikzcd}, \qquad \begin{tikzcd} A_0^1 \arrow[r] \arrow[d, "\Delta"] & * \arrow[d, "i_0"] \\ A_0^2 \arrow[r, "\beta"] & A_0^1 \end{tikzcd}$$

Then

$$T_{\psi^{-1}}T_\psi K_0 \qquad\qquad (T_{\psi^{-1}}T_\psi f)(x)$$

$$= R\pi_!^1(\pi^{2*}R\pi_!^1(\pi^{2*}K_0\otimes m^*\mathcal{L}_0(\psi))\otimes m^*\mathcal{L}_0(\psi^{-1}))[2] \qquad = \sum_y(\sum_z f(z)\psi(-yz))\psi(xy)$$

By base change$(III.8.5.6)$:

$$= R\pi_!^1(R\pi_!^{12}\pi^{23*}(\pi^{2*}K_0\otimes m^*\mathcal{L}_0(\psi))\otimes m^*\mathcal{L}_0(\psi^{-1}))[2]$$

By projection formula$(III.8.5.6)$:

$$= R\pi_!^1R\pi_!^{12}(\pi^{23*}(\pi^{2*}K_0\otimes m^*\mathcal{L}_0(\psi)\otimes \pi^{12*}m^*\mathcal{L}_0(\psi^{-1})))[2] \qquad = \sum_y\sum_z f(z)\psi(-yz)\psi(xy)$$

Combine the character:

$$= R\pi_!^1R\pi_!^{12}(\pi^{23*}\pi^{2*}K_0\otimes \alpha^*m^*\mathcal{L}_0(\psi))[2] \qquad = \sum_y\sum_z f(z)\psi(-y(z-x))$$

Change order of summation:

$$= R\pi_!^1R\pi_!^{13}(\pi^{13*}\pi^{2*}K_0\otimes \alpha^*m^*\mathcal{L}_0(\psi))[2] \qquad = \sum_z\sum_y f(z)\psi(-y(z-x))$$

By projection formula:

$$= R\pi_!^1(\pi^{2*}K_0\otimes R\pi_1^{13}\alpha^*m^*\mathcal{L}_0(\psi))[2] \qquad = \sum_z f(z)\sum_y \psi(-y(z-x))$$

By base change:

$$= R\pi_!^1(\pi^{2*}K_0\otimes \beta^*R\pi_!^2(m^*\mathcal{L}_0(\psi))[2] = R\pi_!^1(\pi^{2*}K_0\otimes\beta^*T_\psi\overline{\mathbb{Q}}_l[-1])[2]$$

By$(III.9.9.9)$:

$$= R\pi_!^1(\pi^{2*}K_0\otimes\beta^*\delta_0[-2])[2] = R\pi_!^1(\pi^{2*}K_0\otimes\beta^*\delta_0(-1)) \qquad = \sum_z f(z)q^n\delta_0(z-x)$$

Use base change and noticing$i_0$is finite thus proper and exact:

$$= R\pi_!^1(\pi^{2*}K_0\otimes R\Delta_!\overline{\mathbb{Q}}_l(-1)) \qquad = \sum_z\sum_{x=z} q^n$$

By projection formula:

$$= R\pi_!^1R\Delta_!(\Delta^*\pi^{2*}K_0\otimes\overline{\mathbb{Q}_l})(-1) \qquad = q^n\sum_{\{z|z=x\}} f(z)$$

$$= K_0(-1) \qquad = q^nf(x)$$

□

Prop. (III.9.9.11) (Plancherel Formula).

$$||f^{T_\psi(K_0)}||_n = q^{n/2}||f^{K_0}||_n.$$

Proof: By definition and using(III.9.9.7),

$$\begin{aligned} ||f^{T_\psi(K_0)}||_n^2 &= \sum_{x \in \mathbb{F}_{q^n}} f^{T_\psi(K_0)}(x)\overline{f^{T_\psi(K_0)}(x)} \\ &= \sum_{x,y,z} f^{K_0}(y)\overline{f^{K_0}(z)}\psi(-xy)\psi(xz) \\ &= q^n \sum_{z=y} f^{K_0}(y)\overline{f^{K_0}(z)} \\ &= q^n(f,f)_n \end{aligned}$$

□

Cor. (III.9.9.12). Notice by the definition of norm of a Weil sheaf $\mathcal{G}_0$, we have

$$||T_\psi(K_0)|| \leq ||K_0|| + 1$$

III.10 Group Schemes

1 Group Schemes

Basic References are [StackProject Chap38].

Def. (III.10.1.1). A **group scheme** is a group object in the category of schemes(I.8.1.35). that satisfy the supposed identities.

A **open/closed subgroup scheme** of a group scheme G/S is an open/closed subscheme of G/S that the restriction of multiplication $m : G \times G \to G$ on $H \times H$ factors through H.

We call a group scheme **smooth/flat/separated/**... iff G/S is **smooth/flat/separated/**....

We have the left(right)translation for an elements in $G(R)$, equivalently, a natural transformation on G, and base change $(G \otimes_R R')(T'_{R'}) = G(T'_R)$

Remark (III.10.1.2) (Yoneda Interpretation). We do not need to verify all the relations, whenever we have a natural group structure on all the set $\mathrm{Hom}(T, G)$, we immediately recover the map $m : G \times G \to G$ as $pr_1 pr_2$ in $G(G \times G)$, $inv : G \to G$ as id^{-1} in $G(G).u : S \to G$ as 1 in $G(S)$, by Yoneda lemma.

Lemma (III.10.1.3). A bialgebra over a field k is direct limit of bialgebras of f.t. over k.

Prop. (III.10.1.4). Affine group schemes over a field is reduced. And it is smooth over k. Cf.[Jacob Stix P5].

Prop. (III.10.1.5). Common group schemes include

- $\mathbb{G}_a = \mathbb{Z}[T]$
- $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$
- $\mu_n = \mathbb{Z}[T]/(T^n - 1)$
- $\mathbb{GL}_n = \mathbb{Z}[T_{ij}][1/\det]$
- $D(g) = \operatorname{Spec} Z[g]$ for a commutative group g.
- The **constant group scheme** $g_{\mathbb{Z}} = \coprod_{\sigma \in g} \operatorname{Spec} \mathbb{Z}$, where m maps the component of (g, g') to the component of gg'. It represents the functor : $T \to$ the group of locally constant functions $T \to G$.

Def. (III.10.1.6) (Character Group Scheme). A **character** of a group scheme G is a homomorphism of group sheaves of Sch/S from G to $\mathbb{G}_m$, it is equivalent to a non-vanishing section χ of G that $m^*\chi = pr_1\chi \cdot pr_2\chi$ multiplication as sections. This is a subgroup of $\mathbb{G}_m(G)$.

A **character group scheme** of G is one that represent the functor $T \to \mathrm{Hom}_{Gr/T}(G_T, \mathbb{G}_{m,T})$. This will induce a compatible pairing $G(T) \times G'(T) \to \mathbb{G}_m(T)$, which gives a map $\mathbb{G}_{mS} \to G \times G'$.

Prop. (III.10.1.7) ($g_{\mathbb{Z}}$ and $D(g)$). $\Gamma(g_S, \mathcal{O}_{X_S}) =$ group homomorphism from g to $\Gamma(S, \mathcal{O}_S)$. So we see that a character group scheme(III.10.1.6) of g_S is equivalent to a group homomorphism $g \to \mathbb{G}_m(S)$, equivalent to $D(g)(S)$. So $D(g)_S$ is the character group scheme of g_S.

Conversely, g_S is also the character group of $D(g)_S$, because the the composition gives a pairing

$$D(X)(T) \times X(T) \to \mathbb{G}_{mT}$$

This gives an isomorphism from g_S to the character of $D(g)_S$, Cf.[Tate Finite Flat Group Scheme].

Prop. (III.10.1.8) (Semi-Direct Product). If X, Y are group schemes over S, and there is a natural map of group schemes $X \to \mathrm{Aut}(Y)$, then we can define their semi-direct product $Y \rtimes X$, because of(III.10.1.2).

Prop. (III.10.1.9). For a group scheme G/S, there is a Cartesian diagram:

$$\begin{array}{ccc} G & \xrightarrow{\Delta_{X/S}} & G \times_S G \\ \downarrow & & \downarrow{\scriptstyle (g,g')\mapsto m(i(g),g')} \\ S & \xrightarrow{\quad e \quad} & G \end{array}$$

This can be seen by a testing scheme T.

Cor. (III.10.1.10). G/S is (quasi-)separated iff e is qc(closed immersion).

Prop. (III.10.1.11). If G/S is a flat group scheme, T/S is a scheme and $\psi : T \to G$ is a morphism. Then $T \times_S G \to T \times_S G : (t, g) \mapsto m(\psi(t), g)$ is flat. In particular, m is flat.

Proof: Notice $T \times_S G \to T \times_S G : (t, g) \to (t, m(\psi(t), g))$ is an isomorphism, and the desired morphism is this composed with the projection, which is base change of $T \to S$, so it is flat. □

(Algebraic) Group Schemes over Fields

Prop. (III.10.1.12). Any group scheme over a field of char 0 is reduced.

Proof: Cf.[StackProject 047O]. □

Prop. (III.10.1.13) (Cartier's Theorem). A locally algebraic group scheme over a field of char 0 is smooth.

Proof: Cf.[StackProject 047N]. □

Prop. (III.10.1.14). For a locally algebraic group scheme over a perfect field, if it is reduced, then it is smooth.

Proof: Cf.[StackProject 047P]. □

Prop. (III.10.1.15). An algebraic group scheme over a field k is quasi-projective.

Proof: Cf.[StackProject 0BF7]. □

Prop. (III.10.1.16). For a locally algebraic group scheme G over a field k, its center is a closed subgp scheme of G.

Proof: Cf.[StackProject 0BF8]. □

2 Linear Algebraic Group

Def. (III.10.2.1). Let K be a field of char0, then a **linear algebraic group** over K is a closed subgroup scheme of $GL_n(K)$ for some n.

Notice a linear group scheme is automatically smooth, by Cartier theorem(III.10.1.13).

Def. (III.10.2.2) (Unipotent Linear Group Scheme). A **unipotent linear algebraic group** is a closed subgroup of the upper triangular subgroup of some $U_n \subset GL_n(K)$ for some n.

For a linear algebraic group G, it has a maximal connected smooth normal unipotent subgroup $R_u(G)$, which is called the **unipotent radical** of G.

Proof: □

Prop. (III.10.2.3). The unipotent radical of G commutes with finite base change of fields.

Proof: □

Def. (III.10.2.4) (Reductive Algebraic Group). A linear algebraic group over K where $\mathrm{char}K = 0$ is called **reductive** if $\mathrm{Rep}(G)$ is semisimple.

This condition is equivalent to $R_u(G_{\overline{K}}) = \{1\}$.

Proof: □

Prop. (III.10.2.5). A connected smooth commutative linear algebraic group G is of the form $U \times T$ where U is unipotent and T is a torus.

Proof: □

Def. (III.10.2.6) (Radical). The **radical** $R(G)$ of a linear algebraic group G is the maximal smooth connected solvable normal subgroup of G. G is called **semisimple** iff $R(G_{\overline{K}}) = \{1\}$. Semisimple linear algebraic group is reductive, because U_n is solvable.

Prop. (III.10.2.7). The radical commutes with finite base change of fields.

Prop. (III.10.2.8). If G is reductive, then $[G, G]$ is semisimple. And $G/Z(G)$ is semisimple.

Prop. (III.10.2.9) (Maximal Central Torus). If G is a connected, reductive linear algebraic group over a perfect field K, then $R(G) = Z(G)^0_{red}$, which is the maximal central torus by(III.10.2.5). In particular, G is semisimple if and only if $Z(G)$ is finite.

Proof: □

Prop. (III.10.2.10) (Maximal Quotient). If G is reductive, then $G/[G, G]$ is finite iff $Z(G)$ is finite iff G is semisimple.

Proof: □

3 Formal Groups

Basic References are [Cartier Theory of Commutative Formal Groups Zink]

Formal Power Series

Prop. (III.10.3.1) (Automorphisms). If $\alpha \in R^*$ and F_i are power series that the degree 1 term of (F_i) is invertible, then there are unique power series G_i that $G \circ F = \mathrm{id}$ and $F \circ G = \mathrm{id}$.

Proof: Use induction to find G that $F \circ G = id$. Then the degree 1 terms of G is also invertible, thus there are $G \circ H = \mathrm{id}$, now $F = H$ and the proof is finished. □

Def. (III.10.3.2) (Formal Logarithm). The **formal exponential** and **formal logarithm** is defined to be elements in $\mathbb{Q}[[x]]$:

$$\exp(x) = \sum_{n\geq 0}\frac{x^n}{n!},\quad \log(x) = \sum_{n>0}\frac{(-1)^{n+1}}{n}x^n.$$

They satisfies $\exp\circ\log = \log\circ\exp = \mathrm{id}$.

Proof: It suffices to prove $\exp\circ\log = \mathrm{id}$, because log must has an inverse in $\mathbb{Q}[[x]]$ by(III.10.3.1). Then? □

Prop. (III.10.3.3). In $\mathbb{Q}[[x]]$,

$$\exp(x) = \prod_{d>0}(\frac{1}{1-x^d})^{\frac{\mu(d)}{d}}.$$

Proof: Taking log, we prove its convergence and equality at once:

$$\sum_{d>0}\log((\frac{1}{1-x^d})^{\frac{\mu(d)}{d}}) = \sum\frac{\mu(d)}{d}\log(\frac{1}{1-x^d}) = \sum\frac{\mu(d)}{d}\sum_{d'>0}\frac{x^{dd'}}{d'} = \sum_{n>0}\frac{x^n}{n}\sum_{n|d}\mu(d) = x.$$

□

Formal Group Law

Def. (III.10.3.4). A **formal group law** of dimension n over a commutative ring R is a set of n power series $G = (G_1,\ldots,G_n)$ in $K[[X_1,\ldots,X_n,Y_1,\ldots,Y_n]]$ that

$$G(X,0) = G(0,X) = X,\quad G(G(X,Y),Z) = G(X,G(Y,Z)).$$

Note this immediately induce an inverse $\mathrm{inv}(X)$ that $G(X,\mathrm{inv}(X)) = G(\mathrm{inv}(X),X) = 0$. This can be constructed noticing $G(X,Y) = X+Y+o(X,Y)$.

A morphism of formal groups is a vector of power series $\varphi(X)$ that $\varphi(G(X,Y)) = H(\varphi(X),\varphi(Y))$.

A **formal R-module** is a formal group G over R together with a ring homomorphism $R\to\mathrm{End}_R(G)$ that $[a](X) = aX+\ldots$.

Prop. (III.10.3.5). $\mathbb{G}_a$ is the one-dimensional formal group with $\mathbb{G}_a(X,Y) = X+Y$, $\mathbb{G}_m$ is the one-dimensional formal group with $\mathbb{G}_m(X,Y) = X+Y+XY$. Over a $\mathbb{Q}$-algebra K, there is an isomorphism between $\mathbb{G}_a$ and $\mathbb{G}_m$ giving by $X\to\exp(X)-1$.

Def. (III.10.3.6). A continuous K-linear mapping $D: K[[X]]\to K[[X]]$ is called a **differential operator** of degree N iff

$$L_D: K[[X,Z]]\to K[[X]]: \sum p_\alpha(X)Z^\alpha\to\sum p_\alpha(X)D(X^\alpha)$$

vanish on J^{N+1}, where $J = (X_i - Z_i)$.

It can be shown D is of degree N if $fD - Df$ is of degree $N-1$ for all f, Cf.[Cartier Theory of Commutative Formal Groups Zink P20].

Prop. (III.10.3.7). There is a representation $G(X+Y) = \sum D_\alpha g(X)Y^\alpha$, and every D_α is a differential operator of degree $|\alpha|$. And D_α forms a basis for the differential operators.

Prop. (III.10.3.8) ($\mathbb{Q}$-Theorem). Any commutative connected formal group over Λ a $\mathbb{Q}$-algebra is a direct sum of $\hat{\mathbb{G}}_a$, Cf.[Cartier Theory of Commutative Formal Groups Zink P19].

1-dimensional Formal Groups

Def. (III.10.3.9). For a 1-dimensional formal group $\mathcal{F}$ over R, the **invariant differential** is a differential form $\omega = P(T)dT \in R[[T]]dT$ that $\omega \circ F(T,S) = \omega$. It is called **normalized** if $P(0) = 1$.

There exists uniquely an invariant differential, it is given by $F_X(0,T)^{-1}dT$.

Proof: We need to check $F_X(0, F(T,S))^{-1} F_X(T,S) = F_X(0,T)^{-1}$, and this is just $F(U, F(T,S)) = F(F(U,T), S)$ differentiated at U and let $U = 0$.

Conversely, if ω is an invariant differential, then $P(F(T,S))F_X(T,S) = P(T)$, let $T = 0$, then $P(S) = P(0)F_X(0,S)^{-1}$. □

Prop. (III.10.3.10). For a morphism $f : \mathcal{F} \to \mathcal{G}$ of 1-dimensional formal groups over R, $\omega_{\mathcal{G}} \circ f = f'(0)\omega_{\mathcal{F}}$.

Proof: We only need to show that $\omega_{\mathcal{G}} \circ f$ is an invariant differential for $\mathcal{F}$ and then compare their constant coefficients. For this, notice

$$\omega_{\mathcal{G}} \circ f(F(T,S)) = \omega_{\mathcal{G}}(G(f(T), f(S))) = \omega_{\mathcal{G}}(f(T)) = \omega_{\mathcal{G}} \circ f(T).$$

□

Def. (III.10.3.11). When R has characteristic 0, the **formal logarithm** $\log_{\mathcal{F}}$ for a 1-dimensional formal group is the integration of invariant differential $\int_0^T \omega_{\mathcal{F}} = T + c_1/2T^2 + \cdots$.

Then the **formal power exponential** is the the unique power series $\exp_{\mathcal{F}}$ that is the inverse of $\log_{\mathcal{F}}$. It exists uniquely by(III.10.3.1).

Prop. (III.10.3.12). For R char$=0$ and an 1 dimensional formal group $\mathcal{F}$ over R, $\log_{\mathcal{F}} : \mathcal{F} \to \mathbb{G}_a$ is an isomorphism of formal groups over $R \otimes_{\mathbb{Z}} \mathbb{Q}$.

And if $\mathcal{F}$ is a formal R-module, then it is an isomorphism of R-modules, because from(III.10.3.10) that $\omega_{\mathcal{F}} \circ [a] = a\omega_{\mathcal{F}}$, thus $\log_{\mathcal{F}} \circ [a] = a \cdot \log_{\mathcal{F}}$.

Proof: From $\omega_{\mathcal{F}}(F(T,S)) = \omega_{\mathcal{F}}(T)$, we get that $\log_{\mathcal{F}}(F(T,S)) = \log_{\mathcal{F}}(S) + \log_{\mathcal{F}}(T)$. So it is a homomorphism. Now the inverse $\exp_{\mathcal{F}}$ is already given, so it is an isomorphism. □

Cor. (III.10.3.13). A 1-dimensional formal group over a ring R that has no torsion nilpotents is commutative.

Proof: We only prove for R torsion free. $F(T,S) = \exp_{\mathcal{F}}(\log_{\mathcal{F}}(T) + \log_{\mathcal{F}}(S))$. □

Lubin-Tate Formal Group

Def. (III.10.3.14). For a p-adic number field K with a uniformizer π_K with residue field $\mathbb{F}_q$, a **Lubin-Tate power series** for π_K is a $\varphi(X) \in \mathcal{O}_K[[X]]$ that $\varphi(X) \equiv \pi_K X \mod X^2$ and $\varphi(X) \equiv X^q \mod \pi_K$.

A **Lubin-Tate module** G over $\mathcal{O}_K$ is a formal $\mathcal{O}_K$-module that $[\pi_K](X)$ is a Lubin-Tate power series.

Prop. (III.10.3.15). Given a p-adic number field K with residue field $\mathbb{F}_q$, we consider the set ξ_π of all Lubin-Tate power series for π.

If $f, g \in \xi_\pi$ and $L(X) = \sum a_i X_i$ be a linear form, then there exists a unique power series $F(X)$ that $F(X) \equiv L(X)$ mod degree 2 and $f(F(X)) = F(g(X_1), \ldots, g(X_n))$.

Proof: Choose F consecutively, if $F_{r+1} = F_r + \Delta_r$, then must

$$\Delta \equiv \frac{f(F_r(X)) - F_r(g(X))}{\pi^{r+1} - \pi} \text{ mod degree } (r+2).$$

This has coefficient in $\mathcal{O}$ because $f \equiv g \equiv Z^q \bmod \pi$. □

Cor. (III.10.3.16). If we let $f = g, L = X + Y$ to get F_f and $f, g, L = aX$ to get $a_{f,g}$, then

- $F_f(X,Y) = F_f(Y,X)$.
- $F_f(F_f(X,Y),Z) = F_f(X, F_f(Y,Z))$.
- $a_{f,g}(F_g(X,Y)) = F_f(a_{f,g}(X), a_{f,g}(Y))$.
- $a_f b_f(Z) = (ab)_f(Z)$.
- $(a+b)_f(Z) = F_f(a_f(Z), b_f(Z))$.
- $\pi_f(Z) = f(Z)$.

all follow from the unicity of the last proposition.

Cor. (III.10.3.17) (Existence of Lubin-Tate Module). We get a commutative formal $\mathcal{O}$-module F_f for every f. And this group can act on $\mathfrak{p}_L$ for an alg.ext L/K. The set of zeros $\Lambda_{f,n}$ of f^n in L, as the elements annihilated by π^n, is a submodule of $\mathfrak{p}_L^{(f)}$.

And $u_{g,f}$ for any unit $u \in \mathcal{O}$ defines an isomorphism between F_f and F_g, thus this formal group only depends on π, called F_π. Hence $L_{f,n} = K(\Lambda_{f,n})$ only depends on π, with Galois group $G_{\pi,n}$.

Prop. (III.10.3.18) (Different Uniformizers). Now consider different π, it is proven that F_π and $F_{\pi'}$ are isomorphic, but the coefficient in $\mathcal{O}_{\hat{T}}$ where T is the maximal unramified extension.

Thus $L_{\pi,n}$ and $L_{\pi',n}$ may not be isomorphic, but $T \cdot L_{\pi,n} = T \cdot L_{\pi',n}$ since $\hat{T} \cdot L_{\pi,n} = \hat{T} \cdot L_{\pi',n}$ and both of them is the algebraic closure of K in it.

Proof: Cf.[Neukirch CFT P105]. □

Lemma (III.10.3.19). The Newton polygon of $[\pi_K^n]/\pi_K^n$ has vertices

$$(1,0), (q, -1/e_K), (q^2, -2/e_K), \ldots.$$

Proof: Notice $[\pi_K^n]$ has no infinite edge of negative slope because all its coefficient are in $\mathcal{O}_K$. Now look at its roots, it has a root 0, and $q-1$ roots of valuation $v_p(\pi_K)/(q-1)$, $q(q-1)$ roots of valuation $v_p(\pi_K)/q(q-1)$, and so on. So by factor out these roots, $[\pi_K^n]/\pi_K^n$ is left with a power series whose Newton polygon is a single line, which shows the desired result. □

Prop. (III.10.3.20). The formal logarithm of the Lubin-Tate formal group F_π satisfies:

$$\log_{\mathcal{F}_\pi}(T) = \varinjlim [\pi_{\mathcal{F}}^n]/\pi_{\mathcal{F}}^n.$$

Proof: By(III.10.3.12) we have

$$\log_{\mathcal{F}}(T) = \log_{\mathcal{F}}([\pi_{\mathcal{F}}^n])/\pi_{\mathcal{F}}^n = ([\pi_K^n] + a_2/2[\pi_K^n]^2 + \ldots)/\pi_K^n$$

and for any degree n, the coefficient of $[\pi_K^{2n}]/\pi_K^{2n}$ is bounded below by a $c(n)$, so $[\pi_K^{2n}]/\pi_K^n$ converges to 0, thus the result. □

Cor. (III.10.3.21). The Newton polygon of $\log_{\mathcal{F}}(T)$ has vertices $(1,0), (q, -1/e_K), (q^2, -2/e_K), \ldots$.

The discussion is continued at1.

4 Cartier Theory

Basic References are [Cartier Theory of Commutative Formal Groups Zink].

5 Quotients of Group Schemes

Def. (III.10.5.1) (Categorical Quotient). Let C be a category with finite products, G be a group object in C. A **left action** ρ of G on an element X is a morphism $G\times X\to X$ that induces for each T a group structure $G(T)\times X(T)\to X(T)$.

Given a left action $G\times X\to X$, a morphism $X\to Y$ is called the **categorical quotient** of X iff Y is the coequalizer of $G\times X\underset{pr_2}{\xrightarrow{\rho}} X$. It is called the **universal categorical quotient** of X iff its base change over S is the categorical quotient for each element $S\in C$, in the category $C_{/S}$.

Prop. (III.10.5.2). If G is proper and flat group scheme of f.t over a locally Noetherian basis S, and $G\times X\to X$ is a strictly free action of G on a quasi-projective scheme of

6 Finite Flat Group Schemes

Def. (III.10.6.1) (Cartier Duality). There is a Cartier duality on the category of finite flat affine commutative group schemes over $\operatorname{Spec} R$. This is because a finite flat module is locally free(III.4.1.5), thus $A^{\vee\vee}=A$ for a R-algebra A.

Prop. (III.10.6.2). When $G=\operatorname{Spec} A$ over R, $A^\vee$ represent the character group scheme of G. Cf.[Jakob P10].

Prop. (III.10.6.3). If G is a finite flat commutative group scheme over R of constant order n, then multiplication by n kills the group. Cf.[Jakob P12].

7 p-divisible Groups

Def. (III.10.7.1). Let Λ be a local complete Noetherian ring and A_Λ^f be the category of finite length Artinian Λ-algebra,

Then a **Λ-formal functor** is a functor $A_\Lambda^f\to\mathcal{S}ets$.

The **formal completion** of a functor $A_\Lambda\to\mathcal{S}ets$ is its restriction on A_Λ^f. We denote the formal completion of $\operatorname{Spec} A$ by $\mathrm{Spf}A$.

Then a **Λ-formal scheme** is a filtered colimits of functors $\varinjlim \mathrm{Spf}A_i$, or equivalently a profinite Λ-algebra $A=\varprojlim A_i$ with profintie topology.

A **Λ-formal group** is a Λ-formal scheme with values in groups.

A **formal Lie group** over Λ is a connected formally smooth Λ-formal group. It is necessarily isomorphic to $\mathcal{G}=\mathrm{Spf}\Lambda[[X_1,\ldots,X_n]]$ where $n=\dim\mathcal{G}$.

A **p-divisible formal Lie group** is a commutative formal Lie group $\mathcal{G}=\mathrm{Spf}\Lambda[[X_1,\ldots,X_n]]$ that multiplication by $p:[p]^*$ is a finite flat morphism on $\Lambda[[X_1,\ldots,X_n]]$.

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III.11 Abelian Variety(van Der Geer)

Basic References are [StackProject], [Abelian Variety Mumford], [Abelian Varieties notes Conrad], [Abelian Varieties Milne] and [Abelian Variety van der Geer].

1 Basics

Def. (III.11.1.1). A **group variety** over a field k is a k-variety that is also a group scheme, with the identity $e \in X(k)$(Notice a group can not be empty).

Def. (III.11.1.2) (Abelian Variety). An **Abelian variety** A over a field k is a group variety over k that is a complete variety over k(i.e. proper and geometrically integral).

For each field extension K/k, A_K is an Abelian variety over K, because proper and geometrically integral is sable under base change.

Prop. (III.11.1.3) (Group Variety Smooth). A group variety is smooth over k. In particular, it is also geo.regular.

Proof: As the smooth locus of a variety is open dense(III.4.2.12), and it has a smooth closed pts by(III.3.4.30). The closed pts are transitive by translation, so all points of X is smooth. □

Prop. (III.11.1.4) (Tangent Bundle Trivial). For a group variety over a field k, $T_{X,e}$ is the tangent space at T, then there is a natural isomorphism $\Omega_{X/k} \cong T^*_{X,e}\otimes\mathcal{O}_X$. Also true for $\mathcal{T}_X$(because $\Omega_{X/k}$ is locally free as X is smooth(III.11.1.3)(III.7.1.15)).

Proof: Use a dual number characterization of tangent spaces and tangent vector fields(III.2.8.5)(III.7.1.16), then notice a tangent vector $\tau \in T_{X,e}$ is a $S = k[\varepsilon]$-point of X, then right translation gives a translation $X_S \to X_S$ that is invariant on X, which gives a tangent vector on X.

So there is a map $T_{X,e} \otimes \mathcal{O}_X \to \Omega_{X/k}$. To check isomorphism, both are locally free of the same rank, so it suffices to show it is surjective. But on closed pts, pass to Nakayama, this is clearly true, so it is surjective by(III.7.1.13). □

Cor. (III.11.1.5). If X is an Abelian variety, any global tangent vector on a group variety is left invariant.

Proof: Because $\Gamma(X, \mathcal{O}_X) = k$(III.7.1.12), so $\Gamma(X, \mathcal{O}_X \otimes T_{X,e}) = T_{X,e}$ are all generated by left invariant vectors(left and right translation commutes). □

Cor. (III.11.1.6). Any morphism from a $\mathbb{P}^1$ to a group variety is constant.

Proof: Cf.[Abelian Varieties Geer P8]. □

Prop. (III.11.1.7). Abelian variety is projective, by(III.10.1.15) and(III.3.5.3).

Prop. (III.11.1.8). The analytification of a complex Abelian variety is a complex tori with a Riemann form. And the reverse is also true.

Proof: The analytification is compact because an Abelian variety is proper(IV.8.6.2). It is a smooth manifold by(IV.8.6.2)(III.11.1.3). It is connected by(IV.8.6.2) and the fact it is projective(III.11.1.7). It is then a compact complex Lie group, which is then a complex tori by(IV.7.4.1). Then notice it is projective, so by(IV.9.8.10) it has a Riemann bilinear form. □

Prop. (III.11.1.9) (Rigidity Theorem). Let $f : X \to Y$ be a morphism of Abelian varieties, then it is a group homomorphism followed by a translation $t_{f(e_X)}$.

Proof: Set $y = i_Y(f(e_X))$ and consider $t_y \circ f$, then $h(e_x) = e_Y$, and consider the morphism:

$$g : X \times X \xrightarrow{(h \circ m_X)\times(i_Y \circ m_Y \circ(h\times h))} Y \times Y \xrightarrow{m_Y} Y,$$

then $g(e_X, X) = g(X, e_X) = e_Y$, so the rigidity lemma(III.7.1.21) shows g is constant with value e_Y. Thus $h \circ m_X = m_Y \circ (h \times h)$, thus a group homomorphism. □

Cor. (III.11.1.10) (Abelian Variety is Commutative).

- Let X be a variety over k, then there is at most one one structure of Abelian variety on X.
- The group law of an Abelian variety is commutative, justifying the name.

Remark (III.11.1.11). The completeness of X is essential for the proof. In fact, there are many non-commutative algebraic groups, like GL_n.

From now, use the additive notation for Abelian varieties.

Proof: 1: If there are two structure $(m, i), (n, j)$, then consider $X \times X \to X \to X : (x, y) \mapsto m(x, y)(n(x, y)^{-1})$, then it is constant on $e_X \times X$ and $X \times e_X$, thus it is constant, so $m = n$. And $i = j$ is also clear by the associativity.

2: The inverse i is a group homomorphism by(III.11.1.9), thus it is Commutative. □

Prop. (III.11.1.12). If X is an Abelian variety over a field k, then a rational map from another smooth k-variety V extends to a morphism $V \to X$.

Proof: Cf.[Van de Geer P14]. □

2 Line Bundles

Prop. (III.11.2.1). As Abelian varieties are regular, $CL(X) \cong Pic(X)$, by(III.6.1.9).

Prop. (III.11.2.2) (Theorem of the Cube). If X is an Abelian variety, and L is a line bundle over X, then

$$\Theta(L) = p_{123}^*(L) \otimes p_{12}^*(L^{-1}) \otimes p_{13}^*(L^{-1}) \otimes p_{23}^*(L^{-1}) \otimes p_1^*(L) \otimes p_2^*(L) \otimes p_3^*(L) \otimes$$

is trivial.

Proof: This is trivial on $0 \times X \times X, X \times 0 \times X, X \times X \times 0$, so it is trivial, by(III.7.1.25). □

Cor. (III.11.2.3). There is a form of morphisms from a scheme to X, just by considering $(f, g, h) : Y \times Y \times Y \to X \times X \times X$.

Cor. (III.11.2.4) (Theorem of the Square). Let X be an Abelian variety that L is a line bundle, then for any $x, y \in X(k)$,

$$t_{x+y}^* L \otimes L \cong t_x^* L \otimes t_y^* L.$$

Proof: Apply the theorem of the cube(III.11.2.3) for $f : \mathrm{id}_X$ and g, h the function with constant value x, y. □

Cor. (III.11.2.5). For a line bundle L on an Abelian variety X, then the map $\varphi_L : X \to \mathrm{Pic}(X) : x \mapsto [t_x^* L \otimes L^{-1}]$ is a homomorphism.

Cor. (III.11.2.6). For any line bundle L, $[n]^* L \cong L^{n(n+1)/2} \otimes (-1)^* L^{n(n-1)/2}$.

Proof: Use(III.11.2.3) in case $f = [n], g = [1], h = [-1]$, then we have:

$$n^* L^2 \otimes (n+1)^* L^{-1} \otimes (n-1)^* L^{-1} \cong (L \otimes (-1)^* L)^{-1}.$$

So we can use induction. □

p-divisible Groups

Prop. (III.11.2.7). For a field K of characteristic p, then $A(K^{sep})$ is an Abelian group and its l^n torsion is isomorphic to $(\mathbb{Z}/l^n\mathbb{Z})^{2g}$ and its p^n torsion is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^r$.

Prop. (III.11.2.8). There is an isomorphism

$$H_t^m(\Lambda_{K^{\mathrm{sep}}}, \mathbb{Q}_l) \cong \bigwedge_{\mathbb{Q}_l}^m (V_l(A))^*.$$

Cf.[Grothendieck Monodromy theorem].

Isogenies

Def. (III.11.2.9) (Isogenies). A homomorphism $f : X \to Y$ between Abelian varieties is called an isogeny iff it satisfies the following equivalent conditions:

- f is surjective and $\dim X = \dim Y$.
- $\mathrm{Ker}\, f$ is a finite group scheme and $\dim X = \dim Y$.
- f is finite flat and surjective.

If f is an isogeny, then we define $\deg f = [K(X) : K(Y)]$.

Proof: Cf.[van der Geer P72]. □

Cor. (III.11.2.10). Isogenies are stable under composition, and degree is multiplicative.

Def. (III.11.2.11) (Separable Isogenies). An isogeny $f : X \to Y$ is called **separable isogeny** iff it satisfies the following conditions:

- $K(X)/K(Y)$ is separable.
- f is étale.
- $\mathrm{Ker}\, f$ is an étale group scheme.

Proof: Cf.[van der Geer P73]. □

Def. (III.11.2.12).

3 Algebraically Closed Field Case

All Abelian variety X in this subsection is over an alg.closed field k.

Prop. (III.11.3.1). There is a closed pt 0 in X that corresponds to 0 in the group $X(k)$, if we denote Ω_0 the cotangent space at 0, it is the stalk of the differential $\Omega_{X/k}$ at 0(III.2.6.4).

4 Mordell-Weil Theorem

Lemma (III.11.4.1) (Weak Mordell-Weil Theorem). Let K be a number field and X be an Abelian variety over K, then $X(K)/nX(K)$ is finite for any $n \geq 1$.

Lemma (III.11.4.2). There is a symmetric bilinear scalar product $X(K) \times X(K) \to R$ that $\langle x, x\rangle \geq 0$, and $\{x | \langle x, x\rangle < C\}$ is finite for all $C > 0$.

Prop. (III.11.4.3) (Mordell-Weil Theorem). The group $X(K)$ of rational points of an Abelian variety X is f.g.

Proof: Let $n > 1$, choose a generator x_i for $X(K)/nX(K)$. Now □

5 Elliptic Curves

Basic References are [Arithmetic on Elliptic Curves Silverman] and [Advanced Topics in the Arithmetic of Elliptic Curves Silverman].

Materials that need to be added in the Algebraic Geometry Part

Prop. (III.11.5.1). Prop4.2, 4.3 in [Silverman1] needs clarification.

Def. (III.11.5.2) (Elliptic Curves). An **elliptic curve** is a complete regular curve of genus 1, together with a rational pt.

Prop. (III.11.5.3). If X is an Abelian variety of dimension 1, then X is an elliptic curve. The converse is also true, by(III.11.5.4).

Proof: By(III.11.1.4)(III.11.1.3), the tangent space $T_{X/k}$ is trivial of rank 1, thus it is a curve of genus 1 by(III.7.3.20). □

Prop. (III.11.5.4) (Explicit Embedding of Elliptic Curves). If E is an elliptic curve, consider a rational point $P \in E(k)$. Now Riemann-Roch tells use $l(nP) = \deg(nP) = n$ for $n \geq 1$. Now $L(P) = k$, and choose a basis $1, x$ for $L(2P)$, and extend it to a basis $1, xy$ of $L(3P)$. Since $L(6P) = 6$, there is a linear relation between the seven elements $1, x, x^2, xy, y^2, x^3$. And y^2, x^3 must occur by observing the pole order at P. Thus by rescaling, we can write the relation as

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

So x, y defines a rational map of E to $\mathbb{P}^2 : a \mapsto (1, a(x), a(y))$. This map extends to an embedding of E into $\mathbb{P}^2$, by(III.7.3.11).

To define an Abelian structure on E, first notice that $E(k) \to \mathrm{Cl}^0(E)$ is an isomorphism. Using Riemann-Roch, it is injective because $l(Q) = 1$ for $Q \in E(k)$, and for any divisor A of degree 0, $L(A + P) > 0$, so there exists an effective divisor that is equivalent to $A + P$, but this must be a rational point $Q \in E(k)$. Thus we can endow E with a group structure inherited from $\mathrm{Cl}^0(E)$. This makes E an Abelian variety, by(III.11.5.5).

Prop. (III.11.5.5). The group actions defined in(III.11.5.4) are all morphisms.

Proof: Cf.[Silverman1 Chap3.2]. □

III.12 FGA Explained

References are [FGA Explained] and [StackProject].

1 Fibered Category

2-Categories

Def. (III.12.1.1).

Stacks

Prop. (III.12.1.2). If $\mathcal{F},\mathcal{G}$ are prestacks over a topological space X, if there is a morphism $\eta:\mathcal{F}\to\mathcal{G}$ that satisfies:

- $\mathcal{F}$ is a stack and $\mathcal{G}$ is separated.
- η is isomorphism on stalks.
- $\eta(U):\mathcal{F}(U)\to G(U)$ is fully faithful.

Then η is an equivalence of prestacks. In particular, $\mathcal{G}$ is also a stack.

Proof: Let $\mathcal{H}$ be the stackification of G, then $\mathcal{F}\to\mathcal{G}\to\mathcal{H}$ is a morphism of stacks that is isomorphism on the stalk, so it is an isomorphism. But G is separated, so for any open U, $\mathcal{F}(U)\to\mathcal{G}(U)$, $\mathcal{G}(U)\to\mathcal{H}(U)$ is fully faithful, and their composition is an equivalence, thus both of them are equivalences. □

Chapter IV

Geometry

IV.1 Topology

Basic references are [Topology Munkres].

1 Basics

Def. (IV.1.1.1). A function from X to $\mathbb{R}\cup\{-\infty,\infty\}$ is called **upper semicontinuous** iff $f^{-1}(<a)$ are all open. It is called **lower semicontinuous** iff $f^{-1}(>a)$ are all open.

Def. (IV.1.1.2). A topological space is called **separable** if it has a countable dense subset.

Filter Langrange

Def. (IV.1.1.3) (Convergence and Filter). For a filter $\mathcal{F}$ on a topological space, $\mathcal{F}$ **converges** to a point y iff any open set containing y is in $\mathcal{F}$.

If X is a set and Y is a topological space and $X \to Y$ is a function, then $y \in Y$ is a $\mathcal{F}$**-limit of** $\mathcal{F}$ if $f_*\mathcal{F}$ converges to Y.

Prop. (IV.1.1.4) (Ultrafilter Convergence Theorem). Let X, Y be a topological space, then:

1. Y is compact iff any ultrafilter on Y has a limit point.
2. Y is Hausdorff iff any ultrafilter on Y has at most one limit point.
3. a function $f : X \to Y$ is continuous iff for any filter on X converging to x, the filter $f_*(\mathcal{F})$ converges to y.

Proof:

1. If Y is compact but every point is not a limit point, then for any x, there is an open set U_x that $U_x \notin \mathcal{F}$, but then f.m. of them covers Y, which is in $\mathcal{F}$, so one of them must be in $\mathcal{F}$ by(I.1.6.6), contradiction.
   Conversely, if $\cup_I U_i = X$ but no finite union of them cover X, then $X - U_i$ satisfies the finite intersection property, so there is an ultrafilter containing all $X - U_i$ by(I.1.6.3) and(I.1.6.4). Then clearly any point x is not a limit point of $\mathcal{F}$.
2. If Y is Hausdorff and x, y are both limit point of a filter $\mathcal{F}$, then there are two non-intersecting nbhd of them in $\mathcal{F}$, so its intersection $\varnothing \in \mathcal{F}$, contradiction.

Conversely, if x, y are two point that their nbhds both intersect, then their nbhds together satisfies the finite intersection property, so there is an ultrafilter containing all of them, by(I.1.6.3) and(I.1.6.4), thus converging to both x and y.

3. This is an easy consequence considering the filter of all the nbhd containing x.

□

Connected Component

Prop. (IV.1.1.5) (Clopen Subsets and Connected Components). Let X be a topological space, $x \in X$, C is a connected component of x, i.e. a maximal connected subset containing x. Define A to be the intersection of all the open-and-closed sets that contain x (also called the pseudo-component sometimes). Then $A = C$, if X is normal.

Proof: Assume A splits into two components B, D. Since A is closed, B and D are both closed, because X is normal there are disjoint open neighborhoods U and V around B and D, respectively. The open sets U and V cover the intersection of all clopen neighborhoods of A, so cause X is compact, there must exist a finite number of clopen sets around A, say $A_1, \ldots, A_n$ such that $U \cup V$ covers $K = \bigcap_1^n A_i$.

Note that K is clopen. We can assume that $x \in U$. It is not difficult to see that $K \cap U$ is clopen and does not contain all of A, contradicting the definition of A. □

Cor. (IV.1.1.6). For a compact Hausdorff topological space X and a point $x \in X$, the connected component of X containing x is the intersection of all compact open neighborhoods of x, because X is normal(IV.1.6.1).

Def. (IV.1.1.7). A space is called **totally disconnected** iff any connected subset of X contains only one point.

Prop. (IV.1.1.8). A subspace of a totally disconnected space is totally disconnected, because totally disconnected is equivalent to the only connected subsets are pt sets.

2 Compactness

Def. (IV.1.2.1) (Compact Space). A topological space is called **compact** or **quasi-compact** iff any open covering of it has a finite sub-covering. A subspace of a topological space is called **precompact** if its closure is compact.

Prop. (IV.1.2.2) (Alexander Subbase Theorem). A topological space is compact iff the closed subsets has the finite intersection property(I.1.6.2). In fact, it suffices to show that the family of complements of a subbasis of open sets has the finite intersection property.

Proof: Cf.[StackProject 08ZP]. □

Prop. (IV.1.2.3) (Tychonoff). An arbitrary direct product of compact topological groups is compact.

Proof: We prove the finite intersection property. If A is a family of subsets that any finite intersection of closure of them is nonempty, then consider a maximal family $\mathfrak{D}$ of subsets containing A that any finite intersection of closures of them is nonempty, it exists by Zorn's lemma. Consider the projection of $\mathfrak{D}$ onto a coordinate, then by Hypothesis, it has an intersection x_α. Now we want to show $x = (x_\alpha)$ belongs to each $D \in \mathfrak{D}$.

If U_β is any subbasis element containing x, then U_β intersect each $\mathfrak{D}$ because $x_\beta \in \pi_\beta(D)$, so it is in $\mathfrak{D}$, by maximality of $\mathfrak{D}$. So the finite intersections are also in $\mathfrak{D}$, so all local basis of x are in $\mathfrak{D}$. This means that local basis intersect each element of $\mathfrak{D}$, that is, all closure of elements in $\mathfrak{D}$ contains x. □

Def. (IV.1.2.4) (Sequentially Compact). A subset A in a space X is called **sequentially compact** iff any sequence of points in A has a convergent subsequence in X. It is called **self sequentially compact** if it is sequentially compact in itself.

Prop. (IV.1.2.5). $f : X \to Y$, X is compact and Y is Hausdorff, then for a descending chain Y_i of closed subsets of X,

$$f(\bigcap_n Y_n) = \bigcap_n f(Y_n).$$

Proof: The left side is compact, so if $x \notin f(\bigcap_n Y_n)$, there is a closed subsets $x \in T$ that $T \cap f(\bigcap_n Y_n)$=0, so $f^{-1}(T)\bigcap_n Y_n = 0$, so $f^{-1}(T) \cap Y_n = 0$ for some n, hence $x \notin f(Y_n)$. □

Prop. (IV.1.2.6). A locally compact second countable space X is σ-compact.

Proof: For every point, there is an open nbhd that the closure is compact. Then we find a basis B_i in this nbhd, then its closure are also compact. Then we have countable compact subsets that cover X. □

Prop. (IV.1.2.7) (Fixed Point Theorem). If X is a compact metric space M, T is a continuous map $X \to X$ that $d(x, y) < d(Tx, Ty)$, then T has a unique fixed point in X.

Proof: The uniqueness is obvious, for the existence, first notice T is obviously continuous, so consider $d(x, Tx)$, this is a continuous function on M, so it contains a minimum value, if it not 0, then $d(Tx, T^2x) < d(x, Tx)$, which is a contradiction. □

Def. (IV.1.2.8). A map is called **proper** if the inverse image of compact subsets are compact.

Prop. (IV.1.2.9). If X is compact and Y is Hausdorff, then a continuous map $f : X \to Y$ is proper.

Cor. (IV.1.2.10). A continuous bijective map from a compact space to a Hausdorff space is a homeomorphism.

Stone-Čech Compactification

Def. (IV.1.2.11) (Stone-Čech Compactification). The **Stone-Čech Compactification** β is defined to be a functor from the category of sets to the category of compact Hausdorff space that is left adjoint to the forgetful functor.

The construction of $\beta(X)$ is as follows: βX =the set of all ultrafilters on X, and the topology is generate by $U_A = \{\mathcal{F}|A \in \mathcal{F}\}$ as a basis of clopen subsets. For a map $f : X \to Y$, the map $\beta X \to \beta Y$ is given by f_*.

Proof: First βX is a compact Hausdorff space: it is compact because if there are sets A_i that any ultrafilter contains at least one of them, then f.m. of them must cover X, otherwise $X - A_i$ satisfies the finite intersection property thus is contained in some ultrafilter, by(I.1.6.3) and(I.1.6.4), contradiction, Then by(I.1.6.6) shows that any ultrafilter contains one of them. It is Hausdorff because for any two different ultrafilter, there must be an A that $A \in \mathcal{F}_1$ and $X - A \in \mathcal{F}_2$. f_* is continuous because $f^{-1}(U_A) = U_{f^{-1}(A)}$.

Now for any map $f : X \to Y$ where Y is a topological space, map an ultrafilter $\mathcal{F}$ to the unique limit point of $f_*\mathcal{F}$ in Y(existence and uniqueness by(IV.1.1.4)). This map is continuous from βX to Y because for any open set $V \subset Y$, $U_{f^{-1}(V)}$ is mapped into V. And for any $\beta X \to Y$ continuous, consider $X \to Y$ which maps x to the image of the principle ultrafilter $\mathcal{F}_x$ in Y.

This two map are mutually converses to each other, first for a $f : X \to Y$, $X \to \beta X \to Y$ is f itself, because the pushout of the principle ultrafilter clearly converges to $f(x)$. And for a $\beta X \to Y$, if $\mathcal{F}$ doesn't map to $\lim f_*\mathcal{F}$ but mapped to some t, then by definition, there is a nbhd U of t that $f^{-1}(U) \notin \mathcal{F}$, but by continuity, there is a $\mathcal{F} \in U_B$ mapped into U. But then $B \in f^{-1}(U)$, otherwise if $x \in B - f^{-1}(U)$, then $\mathcal{F}_x$ is mapped to U, contradiction, so $f^{-1}(U)$ containing B is also in $\mathcal{F}$, contradiction. □

Lemma (IV.1.2.12). In fact, the spaces in the image of the Stone-Čech compactification are all profinite spaces.

Proof: As shown before, for any two different ultrafilter, there must be an A that $A \in \mathcal{F}_1$ and $X - A \in \mathcal{F}_2$, U_A is open and closed. □

Prop. (IV.1.2.13) (Stone Representation Theorem). The Stone-Čech compactification β gives an equivalent of categories from the category of Boolean algebras to the category of profinite spaces.

Proof: βB is a profinite space by lemma(IV.1.2.12), and B can be recovered from βB as the Boolean algebra of all clopen subsets of βB, because βB is compact. This is a inverse isomorphism because**?**. □

Compact-Open Topology

Prop. (IV.1.2.14). The **compact-open topology** on X^Y is the topology generated by subbasis of $(K, U) =$ all f that maps K to U, for K compact and U open. When Y is compact and X a metric space, this coincides with the uniform topology.

Prop. (IV.1.2.15).
- $X^Y \times Y \to X$ is continuous if Y is locally compact.
- $\mathrm{Map}(Y \times X, Z) \cong \mathrm{Map}(Z, X^Y)$.

Profinite Space(Stone space)

Def. (IV.1.2.16) (Profinite Space). A space is called **profinite** if it is a cofiltered limit of discrete topological spaces.

A profinite space is the same thing as a totally disconnected, compact Hausdorff topological space. Thus a closed subspace of a profinite space is profinite.

Proof: The profinite spaces are clearly totally disconnected, compact Hausdorff(by Tychonoff).

Conversely, if it is totally disconnected and compact Hausdorff, let $\mathcal{I}$ be the set of clopen decompositions $X = \coprod_I U_i$ of X, then for each $I \subset \mathcal{I}$, there is a map $X \to I$, and there is a partial order on the decompositions of X. We show that the map $X \to \lim_{I \subset \mathcal{I}} I$ is a homeomorphism. It is injective by(IV.1.1.7)(IV.1.1.5)(IV.1.6.1). It is surjective by compactness of X, and it is clearly open, thus homeomorphism by(IV.1.2.10). □

Cor. (IV.1.2.17). A cofiltered limit of profinite spaces is profinite.

Prop. (IV.1.2.18). Any open covering of a profinite space has a clopen disjoint subcover.

Cor. (IV.1.2.19). By(IV.1.2.16), we may assume that $X = \lim_{i \in I} X_i$, where X_i is finite. Let f_i be the projection, as the limit is filtered, a fundamental family of nbhds of a point (x_i) $f_i^(x_i)$, Then for each covering, we may assume it is finite $X = \cup_{i \in I} f_i^{-1}(x_i)$, choose a $j > i$ for each i, as I is cofiltered, then $X = \coprod_{x \in X_j} f_j^{-1}(x)$ satisfies the desired property.

Locally Profinite Space

Def. (IV.1.2.20) (Locally Profinite Space). A space is called **locally profinite** iff it is a totally disconnected, local compact Hausdorff topological space.

Prop. (IV.1.2.21). A locally closed subsets of a locally profinite space is locally profinite. And compact subsets are profinite.

Proof: Closed subsets are clearly locally profinite, for the open subsets, it is also locally compact. □

Cor. (IV.1.2.22). Any open covering of a compact subsets of a locally profinite space has an clopen disjoint subcover, by(IV.1.2.18).

3 Real Numbers

Cf[Set Theory Jech Chap10] or [Topology Munkres].

Arithmetic of Real numbers

Prop. (IV.1.3.1). As in(I.1.3.13), the set of real numbers is defined as an ordered set that is the completion of the ordered set of rational numbers $\mathbb{Q}$. And it can be endowed with a field structure, making it an ordered field.

Proof: Cf.[Set Theory Jech P175]. □

Topology

Prop. (IV.1.3.2). $\mathbb{R}^n$ satisfies the Heine-Borel property.

Borel Set

Def. (IV.1.3.3). Let U be a ultrafilter on a set I and $\{a_I\}$ be a bounded sequence of real numbers. Then a number a is called the U**-limit** of $\{a_I\}$ is for every $\varepsilon > 0, \{i \in I||a_i - a| < \varepsilon\} \in U$.

There is at most one limit, because $\{i \in I||a_i - a| < \varepsilon\}, \{i \in I||a_i - b| < \varepsilon\}$ will be disjoint hence cannot both be in U.

Prop. (IV.1.3.4) (Generalized Limit). Let U be an ultrafilter on $\mathbb{N}$, then for any bounded sequence of real numbers $\{a_n\}$, $\lim_U a_n$ exists. i.e. There is a functional from l^∞ to $\mathbb{R}$.

And if $\{a_n\}$ has a limit pt a in the usual sense, then $\lim_U a_n = a$ for any non-principal ultrafilter U, because any $\{i \in I||a_i - a| < \varepsilon\}$ is cofinite hence in U(I.1.6.8).

Proof: Let $A_x = \{n|a_n < x\}$. Then A_x is monotone, then we can choose $c = \sup\{x|A_x \notin U\}$(I.1.3.11). And it is easily verified that $c = \lim_U a_n$. □

Cor. (IV.1.3.5) (Density Measure). There exists a measure m on $\mathbb{N}$ that $m(A) = d(A)$ for each set $A \subset \mathbb{N}$ that has a density $d(A)$.

Proof: Let U be a non-principal ultrafilter on $\mathbb{N}$(I.1.6.8), let $m(A) = \lim_U \frac{A(n)}{n}$. It is clearly additive and monotone. And it equals the density by(IV.1.3.4) □

4 Covering Space

Prop. (IV.1.4.1). For a connected and locally connected space, it has a universal cover, and the fundamental group acts on it continuously and properly. (Define the universal cover as the homotopy equivalence class of lines starting from a base point).

Prop. (IV.1.4.2). if X and Y are Hausdorff spaces, $f : X \to Y$ is a local homeomorphism, X is compact, and Y is connected, then f a covering map.

Proof: First, f is surjective (using the connectedness), and that for each $y \in Y$, $f^{-1}(y)$ is finite. Because X is compact, there exists a finite open cover of X by $\{U_i\}$ such that $f(U_i)$ is open and $f|_{U_i} : U_i \to f(U_i)$ is a homeomorphism. For $y \in Y$, let $\{x_1, \ldots, x_n\} = f^{-1}(y)$ (the x_i all being different points). Choose pairwise disjoint neighborhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$, respectively (using the Hausdorff property).

By shrinking the U_i further, we may assume that each one is mapped homeomorphically onto some neighborhood V_i of y.

Now let $C = X \setminus (U_1 \cup \cdots \cup U_n)$ and set

$$V = (V_1 \cap \cdots \cap V_n) \setminus f(C)$$

V should be an evenly covered nbhd of y. □

Prop. (IV.1.4.3). If $\pi : \tilde{B} \to B$ is a local onto homeomorphism with the property of lifting arcs. Let $\tilde{B}$ be arcwise connected and B simply connected, then π is a homomorphism.

Proof: only need to prove injective. If p_1 and p_2 map to the same point, then they can be connected, and the image is a loop thus contractable, contradiction. □

Cor. (IV.1.4.4). If $\tilde{B}$ is locally arcwise connected and B is locally simply connected, then π is a covering map.

Proof: Choose the connected components of a simply connected nbhd of a point p and use(IV.1.4.3). □

Prop. (IV.1.4.5). a simply connected manifold is orientable. (Use the orientable double cover).

5 Paracompactness

Prop. (IV.1.5.1). If X is regular, then TFAE:

1. Each open cover of X has an open locally finite refinement.
2. Each open cover of X has a locally finite refinement.
3. Each open cover of X has a closed locally finite refinement.
4. Each open cover of X is even. i.e. for any cover, there is an open nbhd V of diagonal of $X \times X$ such that $\forall x$, $V[x] = \{y|(x, y) \in V\}$ refines the cover.
5. Each open cover of X has an open σ-discrete refinement.
6. Each open cover of X has an open σ-locally finite refinement.

If this is satisfied, then X is called **paracompact**.

Proof: $6 \to 2$:Just minus every open set the part of open sets that appeared in families that ordered before it. $2 + 4 \to 1$:Use the lemma below, we can transform the cover $\mathcal{A}$ into $V[\mathcal{A}] \cap U_A$ which is an open locally finite cover
Cf.[General Topology Kelley] □

Lemma (IV.1.5.2). If X satiesfies 4, let U be a nbhd of diagonal of $X \times X$, then their exists a symmetric nbhd of diagonal s.t. $V \circ V \subset U$, where $U \circ V = \{(x, z)|(x, y) \in U, (y, z) \in V, \exists y\}$.

Proof: $\forall x$ in X, there is a nbhd s.t. $W[x] \times W[x] \subset U$, this is an open cover, so there is a nbhd R of diagonal s.t. $R[x]$ refines it. Hense $R[x] \times R[x] \subset U$. Let $V = R \cap R^{-1}$, $V \circ V$ is the union of sets $V[x] \times V[x]$, so $V \circ V \subset U$. □

Lemma (IV.1.5.3). In the preceding proposition, if X satisfies 4, Let $\mathcal{A}$ be a locally finite(resp. discrete i.e. intersect only one) family of subsets of X, then use the last lemma, there is a nbhd V of diagonal of $X \times X$ such that $V[A] = \{y|(x, y) \in V, \exists x \in A\}$ is locally finite(resp. discrete).

Proof: Choose for every pt a nbhd satisfy the property, then it is an open cover. Choose a diagonal nbhd U for the property 4, then choose coordinate symmetric nbhd V of diagonal s.t. $V \circ V \subset U$. If $V[x]$ intersect $V[A]$, then $V \circ V[x]$ intersect A. Done. □

Prop. (IV.1.5.4). A regular paracompact space is normal.

Proof: The family consisting of two closed is locally discrete, by preceding lemma,there exists a V s.t.$V[A]$,$V[B]$ open and non-intersecting. □

Prop. (IV.1.5.5). For a connected Hausdorff locally euclidian space, the condition of paracompact, second countable and a compact exhaustion is equivalent.

Proof: Cf.[Paracompactness and second countable]. □

Prop. (IV.1.5.6). A metric space is paracompact.

Prop. (IV.1.5.7). A compact Hausdorff space is paracompact.

Prop. (IV.1.5.8) (Partition of unity). In a paracompact space, given any open cover, there exists a partition of unity $\{\rho_i\}$ that ρ_i has compact support and supp$\rho_i \subset U_i$.

6 Separation Axioms

Hausdorff

Hausdorffization

Cf.[the Hausdorff Quotient].

Regular

Completely Regular

Normal (T4)

Prop. (IV.1.6.1). A compact Hausdorff space is normal.

Prop. (IV.1.6.2) (Urysohn lemma). Let X be normal, A and B two closed subset of X, then there exists a continuous map from X to $[0,1]$ that maps A to 0 and B to 1.

Proof: Use the countability of rational numbers to construct a family of U_q s.t.

$$p < q \Rightarrow \bar{U}_p \subset U_q$$

Then choose $f(x) = \inf\{p \in \mathbb{Q} | x \in U_p\}$, then this f meets the requirement. □

Cor. (IV.1.6.3) (Tietze extension). If X is normal and Y is a closed subspace, then any continuous function f on Y can be extended to a continuous function on X.

Proof: □

7 Metric Space

Complete Metric Space

Def. (IV.1.7.1). A set E in a metric space is called **totally bounded** iff for every $\varepsilon > 0$, there exists a finite set F that $E \subset B(F, \varepsilon)$. This definition is compatible with that in the case of a topological vector space when it is metrizable.

Prop. (IV.1.7.2). The closure of a totally bounded set in a metric space is totally bounded.

Proof: For each $\varepsilon > 0$, choose a finite set F that $E \subset B(F, \varepsilon/2)$, then $\overline{E} \subset B(F, \varepsilon)$. □

Prop. (IV.1.7.3). A totally bounded metric space X is separable.

Proof: $\cup N_n$ is dense and countable in X, where N_n is a finite $1/n$-net of X. □

Prop. (IV.1.7.4) (Hausdorff). Let X be a metric space, then:

1. A sequentially compact subset M is totally bounded and the converse is true if X is complete.
2. A subset M is compact iff it is self-sequentially compact iff it is closed and sequentially compact.
3. M is precompact iff it is sequentially compact.

Proof:

1. If M is not totally bounded, then for some $\varepsilon > 0$, we can choose consecutively a sequence of points x_i that $d(x_i, x_j) \geq \varepsilon$, this cannot has a convergent subsequence in X.
   Conversely, if M is totally bounded, choose a $1/k$-net for each k, then for any sequence in M, there is a y_i that some infinite subsequence $\{x_n^{(1)}\} \subset B(y_1, 1)$, and consecutively find infinite subsequences $\{x_n^{(m)}\} \subset B(y_k, 1/k)$, then finally choose the diagonal, then it is a Cauchy sequence.

2. If it is compact, given a sequence, if no point is a convergent point, then each point has a nbhd that contains at most one point of the sequence. Then by compactness, there are at most f.m. points, contradiction. A compact set must has the convergent point in itself because it is closed as M is Hausdorff.
   Conversely, if it is self-sequentially compact, then it is totally bounded by 1. so if M is not compact, then for each n it has $1/n$-net N_n, then there is at least one x_n that $B(x_n, 1/n)$ cannot by covered by f.m. of the covering, The sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that is convergent to x. But $x \in M$ is in some open cover, so $B(x_{n_k}, 1/n_k)$ is contained in some open cover, contradiction.
   That closed and sequentially compact is equivalent to self sequentially compact is obvious.

3. If it is precompact, then it is sequentially compact by 2, conversely, if x_i is a sequence in $\overline{M}$, then choose $|y_n - x_n| \leq 1/n$, so some sequence y_{n_k} is convergent to $y_0 \in \overline{M}$, so x_{n_k} also converges to y_0. So $\overline{M}$ is self-sequentially compact, so it is compact by 2.

$\square$

Cor. (IV.1.7.5) (Arzela-Ascoli). For M compact Hausdorff, $F \subset C(M)$ is a sequentially compact(precompact, by(IV.1.7.4)) subset iff it is uniformly bounded and equicontinuous.

Proof: As $C(M)$ is complete metric space, sequentially compact is equivalent to totally bounded. If it is totally bounded, then it is clearly uniformly bounded, and for every $\varepsilon > 0$, find a $\varepsilon/3$-net for F, which means f.m. functions in F that any other function is $\varepsilon/3$-close to one of them. So they are equicontinuous.

Conversely, if it is uniformly bounded and equicontinuous, for every $\varepsilon > 0$, find a finite covering of M that for any two points x, y in one cover of them, $|f(x) - f(y)| < \varepsilon/3$ for all $f \in F$. Then choose for each covering a point x_i, consider $f : F \to \mathbb{C}^n : \varphi \mapsto (\varphi(x_1), \ldots, \varphi(x_n))$, then the image is bounded, hence precompact by(IV.1.3.2), so it is totally bounded by(IV.1.7.4). So we can choose a $\varepsilon/3$-net φ_k for x_i simultaneously, and it is by $\varepsilon/3$ argument that these φ_k is a ε-net for F. $\square$

Prop. (IV.1.7.6) (Fixed point theorem). If X is a complete metric space and $f : X \to X$ satisfies $d(f(x), f(y)) \leq \lambda d(x, y)$ for some $0 \leq \lambda < 1$, then f has a unique fixed point in X. If X is moreover compact, then and f that $d(f(x), f(y)) < d(x, y)$ will have a unique fixed point.

Proof: $x + f(x) + f^2(x) + \cdots$ is the fixed point. And uniqueness is easy. For compact case, notice the image $\operatorname{Im} f^n$ is a descending chain, it must stable to some T. If $x, y \in Y$ attains the diameter of Y, and let $x = f(X), y = f(Y)$, where $X, Y \in T$, then $d(x, y) < d(X, Y) \leq d(x, y)$, contradiction. $\square$

Prop. (IV.1.7.7) (Dilation Closed). If X, Y are metric spaces that X is complete metric space, then if $f : X \to Y$ is continuous function that is a dilation, i.e. $d(f(x_1), f(x_2)) \geq d(x_1, x_2)$, then $f(X)$ is closed.

So a continuous dilation map on a complete metric space is a closed map.

Proof: If $y \in \overline{f(X)}$, then because Y is metric, there are x_n that $y = \lim f(x_n)$. Thus $\{f(x_n)\}$ is Cauchy in Y, and $\{x_n\}$ is Cauchy too. So there is a $x = \lim x_n$, and clearly $f(x) = y$. □

Compact Metric Space

Lemma (IV.1.7.8) (Lebesgue Number Lemma). For any open covering U_i of a compact metric space X, there exists a $\delta > 0$ that any subset X of diameter smaller than δ is contained in some U_i.

Proof: If X is in the covering U_i, then there is nothing to prove, otherwise, it suffices to assume the covering is a finite covering, let $C_i = X - U_i$, and let $f(x) = \frac{1}{n}\sum d(x, C_i)$. Notice $f(x) > 0$, because $x \notin C_i$ for some C_i. Now it is also continuous, so it has a minimal value$> \delta > 0$.

Now if B has diameter smaller than δ, then if $x_0 \in B$, $\delta < f(x_0) < d(x, C_{i_0})$, where $d(x, C_{i_0})$ is the maximal among $d(x, C_k)$, and $B \in B(x, \delta)$, thus $B \in U_{i_0}$. □

Prop. (IV.1.7.9) (Uniform Continuity Theorem). If $f : X \to Y$ is a continuous map between two metric spaces that X is compact, then f is uniformly continuous.

Proof: Take an open covering of Y with balls $B(y_i, \varepsilon/2)$ of diameter $\varepsilon/2$, and consider their inverse image, then choose the lebesgue number δ for this covering(IV.1.7.8), we see that for any $d(x, y) < \delta$, $d(f(x), f(y)) < \varepsilon$. □

Prop. (IV.1.7.10). If f is an isometry of a compact metric space X, then it is a bijection thus an homeomorphism.

Proof: It is clearly injective. If it is not surjective, then choose a $x \notin \mathrm{Im}(f)$, because $\mathrm{Im}(f)$ is compact hence closed in X, $d(x, \mathrm{Im}(f)) = \varepsilon > 0$. Now consider the minimal N that we can cover X with open subsets of diameter smaller than ε, this N exists because X is compact. Now if U_i covers X, then the one that contains x cannot intersect with $\mathrm{Im}(f)$. But $f^{-1}(U_i)$ is an open cover of X with smaller numbers of open subsets, contradiction. □

8 Baire Space

Def. (IV.1.8.1). A subset of a topological space X is called of **first category** if it is contained in some countable union of closed subsets of X having no interior point. It is called of **second category** if it is not of first category.

A **Baire space** is a topological space that any nonempty open subsets of X is of second category.

Prop. (IV.1.8.2) (Baire Category Theorem). Every complete metric space & locally compact Hausdorff space is a Baire space.

Proof: Choose consecutively (precompact)open subsets that doesn't intersect $\overline{E_n}$ to find a limit point. □

9 Uniform Space

10 Manifold

Def. (IV.1.10.1). A **manifold** of dimension n is a Hausdorff topological space that is locally subsets of R^n and it is second countable. By(IV.1.5.5), the last condition is equivalent to say it is paracompact.

11 Topological Groups

Def. (IV.1.11.1). A **topological group** is a group object in the category of groups(I.8.1.35).

Prop. (IV.1.11.2) (Separating Axioms). For a topological group G, the following are equivalent:

- e is a closed pt.
- G is T_1.
- G is Hausdorff(T_2).
- G is regular.
- G is completely regular

Proof: □

Prop. (IV.1.11.3). Hausdorff topological group is completely regular.

Proof: Use a sequence of neighbourhood of identity to construct a uniform metric on G. Then set $\phi(x) = \min\{1, 2\sigma(a,x)\}$. Cf.[Abstract Harmonic Analysis Ross §8.4] □

Prop. (IV.1.11.4). For a compact subset K and a nbhd U of e in a topological group, there exists a nbhd V of e that $xVx^{-1} \subset U$ for any $x \in K$.

Proof: For any x, there exists a nbhd W_x of x and a nbhd V_x of e that $txt^{-1} \in U$ for any $t \in W_x$ and $y \in V_x$. Let f.m. W_{x_i} cover K, then $V = \cap V_{x_i}$ satisfies the condition. □

Prop. (IV.1.11.5). A compact open nbhd of e in a Hausdorff topological group contains an open subgroup of G.

Proof: Cf.[Etale Cohomology Fulei P147] □

Totally Disconnectedness

Prop. (IV.1.11.6). A compact topological group is totally disconnected iff the intersection of all compact open nbhds of e equals $\{e\}$.

Proof: If it is totally disconnected, then $\{1\}$ is closed, so G is Hausdorff(IV.1.11.2), so by(IV.1.1.6), the assertion is true. Conversely, if the intersection of all compact open nbhds of e equals $\{e\}$, then $\{1\}$ is closed because G is a group. □

Prop. (IV.1.11.7). A precompact nbhd of a e in a totally disconnected topological group contains a compact open subgroup.

Proof: Cf.[Etale Cohomology Fulei P147]. □

Locally Profinite Groups

Def. (IV.1.11.8). A **locally profinite group** is a topological group that is Hausdorff, locally compact and totally disconnected. A profinite group is locally profinite, and any compact open subgroup of a locally profinite group is profinite.

A p-adic number field F is a locally profinite group, so does $GL_n(F)$.

Cor. (IV.1.11.9). A closed subgroup of a locally profinite group is locally profinite, and a quotient group is locally profinite.

Proof: The proof is very similar to that of(II.3.2.4), as the result of(IV.1.1.6) remains true, because any connected nbhd of e is contained in any compact open subgroup. □

Group Action

Def. (IV.1.11.10). An **action** of a topological group G on a topological group H is defined to be a continuous map $\rho : G \times H \to H$ that $(g_1 g_2)h = g_1(g_2 h)$.

Def. (IV.1.11.11) (Proper Discontinuous Group Action). A group action is called **proper discontinuous** iff any elements $x, y \in H$ there are nbhds U_x, U_y that $\{g \in G | g(U_x) \cap U_y \neq \varnothing\}$ is finite.

Prop. (IV.1.11.12). If G acts proper discontinuously on a topological space H, then for any compact subsets $K_1, K_2 \in H$, $\{g \in G | K_2 \cap g(K_1) \neq \varnothing\}$ is finite.

Proof: Notice for any two points we can find nbhds that f.m. g intersects these two nbhds, so we can use the compactness to find f.m. pair of nbhds to cover K_2, and then use these nbhds to cover K_1 and finite the proof. □

12 Hausdorff Geometry

Def. (IV.1.12.1). The **Hausdorff distance** for two subset $Y_1, Y_2 \in X$ is the

$$d_X^H(Y_1, Y_2) = \inf\{\varepsilon | Y_2 \subset B(Y_1, \varepsilon), Y_1 \subset B(Y_2, \varepsilon)\}.$$

The **Gromov-Hausdorff metric** for two metric space is

$$d^{GH}(X_1, X_2) = \inf\{d_Z^H(i_1(X_1), i_2(X_2))\}$$

where i_1, i_2 are isometry of X_1, X_2 into a metric space Z.

This metric makes the set of all compact metric space into a complete Hausdorff space $\mathcal{MET}$.

Def. (IV.1.12.2). A map from X to Y is called a ε **approximation** iff $B(f(X), \varepsilon) = Y$ and $|d(x, y) - d(f(x), f(y))| \leq \varepsilon$.

We have: if there is a ε approximation, then $d^{GH}(X, Y) \leq 3\varepsilon$, and if $d^{GH}(X, Y) \leq \varepsilon$, there is a 3ε approximation.

Prop. (IV.1.12.3). The set of isometries of

13 Spaces from Algebraic Geometry

Noetherian Space

Prop. (IV.1.13.1). A Noetherian space is quasi-compact and all subsets of it in the induced topology is Noetherian hence quasi-compact.

Proof: Let $T \subset X$, for a chain of closed subsets $Z_i \cap T$ of T, $Z_1, Z_1 \cap Z_2, \ldots$ stabilize in X, hence the chain stabilize in T. □

Prop. (IV.1.13.2). A Noetherian space has only f.m. irreducible component, hence it has only f.m. connected components.

Proof: Let $\mathcal{C}$ be the family of closed subset that has infinitely many component, then there is a minimal object, but it is not irreducible, one of the component has infinitely many components and be smaller. □

Specialization & Generalization

Def. (IV.1.13.3). A map f of spaces is said to satisfy the **going-up property** iff specialization lifts along f. It is said to satisfy the **going-down property** iff generalization lifts along f.

Prop. (IV.1.13.4). A closed map satisfies going-up.

Proof: If $y \to y'$, $f(x) = y$, consider $f(\overline{\{x\}})$, it is closed and contains y, so it contains y', thus the result. □

Constructible Set

Def. (IV.1.13.5) (Retrocompact Subset). A subset of X is called **retrocompact** if the inclusion map is quasi-compact, i.e. the inverse image of any quasi-compact open set is quasi-compact open.

Def. (IV.1.13.6). A subset of X is called **constructible** if it is a finite union of sets of the form $U \cap V^c$ where U, V are open and retrocompact in X. In the case when X is Noetherian, by(IV.1.13.1), all subsets are retrocompact hence constructible sets are just union of locally closed subsets of X.

A set of X is called **locally constructible** if locally it is constructible.

Prop. (IV.1.13.7). Constructible subsets of X forms a Boolean algebra.

Proof: Cf.[StackProject 005H]. □

Prop. (IV.1.13.8) (Constructible and Subsets).
- If U is open in X, then for any E constructible in X, $E \cap U$ is constructible in U.
- If U is retrocompact open and E is constructible in U, then E is constructible in X.

Proof: Easy. □

Prop. (IV.1.13.9). A locally constructible set is constructible on every quasi-compact subset.

Prop. (IV.1.13.10). Any constructible subsets of X is retrocompact.

Proof: It suffices to prove $U_i \cap V_i^c \cap W$ is quasi-compact for W quasi-compact, but this is because it is a closed subspace of the quasi-compact subspace $U_i \cap W$. □

Cor. (IV.1.13.11). An open subset of X is constructible iff it is retrocompact, a closed subset of X is constructible iff its complement is retrocompact.((IV.1.13.7) used).

Def. (IV.1.13.12) (Constructible topology). The **constructible topology** X_{cons} on a quasi-compact space X is generated by the U, U^c, where U is a quasi-compact open.

Notice that the space is quasi-compact, so the constructible topology is the coarsest topology that every constructible open are both open and closed.

Prop. (IV.1.13.13). Let X be quasi-compact and quasi-separated, then any constructible subset of X is quasi-compact. In particular, if Y is closed in X, then Y is constructible iff it is quasi-compact.

Proof: For $Y = \cup_{i=1}^{n}(U_i - V_i)$, with U_i, V_j quasi-compact open in X, then $U_i - V_i$ is closed in U_i thus quasi-compact, and then Y is quasi-compact. □

Irreducible

Def. (IV.1.13.14). A space is called **irreducible** iff there are no two nonempty nonintersecting open subsets. Thus an open subset of an irreducible set is dense and irreducible.

Prop. (IV.1.13.15). If Y is irreducible in X, then $\overline{Y}$ is also irreducible.

Proof: Any two nonempty open sets of $\overline{Y}$ must intersect Y thus must intersect. □

Jacobson Space

Def. (IV.1.13.16). Let X be a space and X_0 the set of closed pts of X, then X is called **Jacobson** iff $\overline{Z \cap X_0} = Z$ for every closed subset Z of X. This is equivalent to every non-empty locally closed subset of X contains a closed pt.

Thus there is a correspondence between closed subsets of X_0 and closed subsets of X, so they have the same Krull dimension.

Prop. (IV.1.13.17). Being Jacobson is local. And for an open covering U_i of X, $X_0 = \cup U_{i,0}$.

Proof: Firstly, if $X = \cup U_i$ where U_i are Jacobson, $X_0 \cap U_i = U_{i,0}$. One direction is trivial, for the other, let x be closed in U_i, then consider $\{x\} \cap U_j$. If $x \notin U_j$, this is empty, if $x \in U_j$, consider $T = \{x\} \cup (U_j - U_i \cap U_j)$, then T is closed in U_j, so by hypothesis, closed pts of U_j are dense in T, so x must be closed in U_j, so x is closed in X. Now clearly X is Jacobson.

Conversely, if X is Jacobson, for a closed subset Z of U_i, $X_0 \cap \overline{Z}$ is dense in $\overline{Z}$, so $X_0 \cap Z$ is dense in Z, then clearly U_i is Jacobson. □

Cor. (IV.1.13.18). If X is Jacobson, then any locally constructible sets of X is Jacobson. And its closed pts are closed in X.

Proof: By the proposition, we only have to prove for constructible sets. For $T = \cup T_i$ where T_i is locally closed, then a locally closed set in T has a non-empty intersection $T \cap T_i$ which is also locally closed for some i.

Hence is has a closed pt in X hence in T, so T is Jacobson. The second assertion is implicit in the proof. □

Prop. (IV.1.13.19). If X is Jacobson, then an open set U of X is compact iff $U \cap X_0$ is compact, hence an open set U is retrocompact iff $U \cap X_0$ is retrocompact.

Hence the constructible sets of X correspond to the constructible sets of X_0.

And Irreducible closed subsets correspond to irreducible subsets of X_0.

Krull Dimension

Def. (IV.1.13.20). The **Krull dimension** of a topological space is the length of the longest chain of closed irreducible subsets.

The **local dimension** $\dim_x(X) = \min\{\dim U | x \in U \subset X \text{ open in X}\}$.

Prop. (IV.1.13.21). If $Y \subset X$, then $\dim Y \leq \dim X$, because the closure of any chain of Y is a chain of X by(IV.1.13.15).

For an open covering of X, $\dim X = \sup \dim U_i$, because for any chain of closed irreducible subsets, if U_i contains the minimal one, then $\dim U_i =$ length of this chain.

Prop. (IV.1.13.22). $\dim X = \sup \dim_x(X)$.

Proof: The right is smaller than the left, and for any chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of irreducible closed subset of X, if I choose a point $x \in Z_0$, then $\dim_x(X) \geq n$. □

Prop. (IV.1.13.23). In case $X = \operatorname{Spec} A$ for a Noetherian ring A, $\dim X = \sup \dim A_p$, because A is of finite**?**

Catenary space

Def. (IV.1.13.24). A space X is called **catenary** iff for any inclusion of irreducible closed subsets of X, their codimension is finite and every maximal chain of irreducible closed subsets has the same dimension. This is equivalent to $\operatorname{codim}(X, Y) + \operatorname{codim}(Y, Z) = \operatorname{codim}(X, Z)$.

Prop. (IV.1.13.25). Catenary is a local property. Cf.[StackProject 02I2].

Sober Space

Def. (IV.1.13.26). A space X is called **sober** if every irreducible closed subset has a unique generic point.

Prop. (IV.1.13.27). A sober space is T^1. Conversely, a finite T_0 space is sober.

Proof: The first assertion is because if $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$, then $\overline{\{x\}} = \overline{\{y\}}$, and this irreducible closed subset has two generic point, contradiction.

If the space is finite, then for a closed irreducible subset $T = \{x_1, \ldots, x_n\}$, $T = \cup \overline{\{x_i\}}$, as it is irreducible, $T = \overline{\{x_i\}}$ for some x_i, and i is unique as it is T^1, so X is sober. □

Prop. (IV.1.13.28) (Soberization). There is a left adjoint t to the forgetful functor from the Sober spaces. $t(X)$ consists of irreducible closed subsets of X, and use $t(Y)$ for Y closed as closed subsets. for a map $f : X \to Z$ to a sober space Z, the extension maps the generic point of an irreducible Y to the generic point of the closure of $f(Y)$.

Def. (IV.1.13.29) (Zariski Space). A Noetherian Sober space is called a **Zariski space**.

Dimension Function

The dimension function is usually considered when the space is sober.

Def. (IV.1.13.30). On a topological space, we consider the specialization relation, a **dimension function** δ on X is one that if y is a specialization of x, then $\delta(y) < \delta(x)$, and if it is a direct specialization, then $\delta(y) = \delta(x) - 1$.

14 Spectral Spaces

References are [StackProject 5.23] and [Adic Spaces].

Def. (IV.1.14.1) (Spectral Space). A space is called **spectral** iff it is quasi-compact, quasi-separated, sober and the quasi-compact opens form a basis for the topology.

A space is called **locally spectral** iff it has an open covering by spectral spaces.

A map $f : X \to Y$ between two locally spectral spaces is called **spectral** if for any open spectral spaces $U \subset f^{-1}(V)$, $f : U \to V$ is quasi-compact.

The Constructible Topology

Prop. (IV.1.14.2). If X is a finite T_0 space, then it is spectral and every subset of X is constructible.

Proof: Cf.[Adic Space Morel, P26]. □

Prop. (IV.1.14.3). If X is a spectral space, then the constructible topology(IV.1.13.12) is Hausdorff, totally disconnected and quasi-compact.

Proof: The space is sober hence T_0(IV.1.13.27), and then the constructible topology is Hausdorff and totally disconnected.

To show quasi-compactness, it suffices to show that the family $\mathcal{C}$ of quasi-compact open and complement quasi-compact open subsets has the finite intersection property(IV.1.2.2). Notice elements in $\mathcal{C}$ are all quasi-compact. Now if there is a family that has the finite intersection property by has intersection 0, by Zorn's lemma, there is a maximal one of them, $\mathcal{B}$. Now let Z be the intersection of all the closed subsets in $\mathcal{B}$, then it is non-empty as X is quasi-compact. And we claim Z is irreducible: otherwise $Z = Z_1 \cup Z_2$, thus there are quasi-compact open sets U_1, U_2 that $U_1 \cap Z_1 \neq \varnothing, U_1 \cap Z_2 = \varnothing, U_2 \cap Z_2 \neq \varnothing, U_2 \cap Z_1 = \varnothing$. then let $B_i = X - U_i$, then B_1, B_2 cannot by added to $\mathcal{B}$ by maximality, so there is a finite intersection T_1, T_2 that $B_i \cap T_i = 0$. But then $Z \cap T_1 \cap T_2 = 0$, but Z is an intersection of closed subsets, thus some finite intersection of closed subsets in $\mathcal{B}$ will $\cap T_1 \cap T_2 = 0$, contradiction.

So now Z is irreducible, but then for every element $B \in \mathcal{B}$, $Z \cap B$ contains the generic point of Z, thus the intersection of B is not empty, contradiction. □

Cor. (IV.1.14.4). Let X be a spectral space, then

- The constructible topology is finer than the original topology.
- A subset X is constructible iff it is clopen in the constructible topology of X.
- If U is open in X, then the constructible topology induces the constructible topology on U.

Proof: Cf.[Adic Space, Morel P28]. □

Prop. (IV.1.14.5). If $E \subset X$ is closed in the constructible topology, then it is a spectral space with the induced topology.

Proof: Cf.[StackProject 0902]. □

Prop. (IV.1.14.6). For a set E closed in the constructible topology in a spectral space,

- every point of $\overline{E}$ is a specialization of elements in E. Thus if E is stable under specialization, then it is closed.

- If E is also open in the constructible topology and stable under generalization, then it is open.

Proof: Cf.[StackProject 0903]. □

Prop. (IV.1.14.7). For a map between spectral spaces $f : X \to Y$, the following are equivalent:
- f is spectral.
- f is quasi-compact.
- $f : X_{cons} \to Y_{cons}$ is continuous.

And if this is true, then $f : X_{cons} \to Y_{cons}$ is proper.

Proof: $1 \to 2 \to 3$ is trivial, an open subset of X is quasi-compact iff it is clopen in the constructible topology(IV.1.14.4), so $3 \to 2$. For $2 \to 1$, notice that if $U \subset f^{-1}(V)$ are open spectrals, and $W \subset U$ is quasi-compact open, then $f^{-1}W \cap U$ is quasi-compact open, because X is quasi-separated.

Finally, f is proper because X_{cons}, Y_{cons} is compact Hausdorff(IV.1.14.3), then use(IV.1.2.9). □

Criterion of Spectralness

Lemma (IV.1.14.8). Let X be a quasi-compact T_0 space that is has a subbasis consisting of quasi-compact open subsets that is stable under finite intersections. Let X' be the topology generated by the quasi-compact open subsets and their complements, then the following are equivalent:
- X is spectral.
- X' is compact Hausdorff, and its topology has a basis consisting of open and closed subsets.
- X' is quasi-compact.

Proof: Cf.[Adic Space Morel P30]. □

Prop. (IV.1.14.9) (Hochster's Criterion of Spectrality). Let $X' = (X_0, \mathcal{T}')$ be a quasi-compact topological space, and let $\mathcal{U}$ be the family of clopen subsets of $\mathcal{T}'$, let $\mathcal{T}$ be the topology generated by $\mathcal{U}$, let $X = (X_0, \mathcal{T})$.

Then if X is T_0, then it is spectral, and every element of $\mathcal{U}$ is quasi-compact open in X, and $X' = X_{cons}$.

Proof: Cf.[Adic Space Morel, P29]. □

Prop. (IV.1.14.10) (Spectral and Inverse Limit). A space is spectral iff it is an direct limit of finite sober(T^1(IV.1.13.27)) spaces.

Proof: Cf.[StackProject 09XX]. □

Prop. (IV.1.14.11). A spectral space is exactly the underlying space of spectrum of a ring.

Proof: Cf.[M. Hochster. Prime ideal structure in commutative rings, Thm6]. □

Cor. (IV.1.14.12). Every quasi-compact irreducible scheme is homeomorphic to an affine scheme.

IV.2 Riemannian Geometry

Basic references are [Riemannian Geometry Do Carmo], [Geometric Analysis Jost] and [Differentia Geometry Loring Tu].

1 $\mathbb{R}^3$-Geometry

Different Coordinates

Prop. (IV.2.1.1). In a polar coordinate,

$$g_{11}=1, g_{12}=0, g_{22}=|\frac{\partial f}{\partial \theta}|^2, \quad K=-\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}}$$

And $\sqrt{g_{22}}\sim\rho$. (Use the formula relating Jacobi Field with curvature)

Moving Frame Method

Prop. (IV.2.1.2) (Theorema Egregium).

$$R_{1212}=K(g_{11}g_{22}-g_{12}^2)$$

Which is a special case of the definition of curvature.

Prop. (IV.2.1.3) (Gauss-Bonnet). Let M be a compact oriented 2-dimensional Riemannian manifold, then

$$\chi(M)=\frac{1}{2\pi}\int_M K\ \mathrm{Vol}.$$

Proof: Should be an direct corollary of(IV.3.7.6). □

Topology and Geometry

Prop. (IV.2.1.4). Every compact orientable surface of genus $p>1$ can be provided with a metric of constant negative curvature.

Remark (IV.2.1.5) (Hilbert Theorem). There exist complete surfaces with $K\leq 0$ in $\mathbb{R}^3$, but the hyperbolic surface cannot be immersed into $\mathbb{R}^3$.

2 Basics

Prop. (IV.2.2.1). If the metric tensor on the tangent space is g in a coordinate, then it is g^{-1} in the cotangent space. (Follows from**??**).

3 Connections

Def. (IV.2.3.1) (Affine Connection). An **affine connection** on a vector bundle E is a map $D:\Gamma(E)\to\Gamma(E)\otimes\Gamma(T^*M)$ that satisfies differential-like properties, it can be written as $D=d+\omega$, with $\omega\in\Omega^1(\mathrm{End}(E))$.

Prop. (IV.2.3.2) (Transformation Law). In two coordinates $\overline{e} = ea$ for $a : U \to GL(r, \mathbb{R})$, $d_A = d + \omega, d + \overline{\omega}$, then $\overline{\omega} = a^{-1}\omega a + a^{-1}da$.

Moreover, giving any locally compatible $d + \omega$,$\omega \in \Omega^1(\mathfrak{g})$ in the sense above, then for any G-associated bundle E, where G has lie algebra $\mathfrak{g}$, there is a connection that locally looks like $d + \omega$, (where $\mathfrak{g}$ embeds into $\mathfrak{gl}(E)$).

Cor. (IV.2.3.3) (Local Nature of Connection). From the description of connection given above, it's easy to say if the is a local connection that satisfies these transformation laws, then it generate a global connection. So by partition of unity(IV.1.5.8), connection exists in any vector bundle over a manifold.

Cor. (IV.2.3.4) (Simplification). $d_{gA}(s) = gd_A(g^{-1}(s))$, So for any connection d_A and any point x_0, there is a gauge transformation that makes $d_A = d$ at x_0.

Proof: Just need to have $s(x_0) = \mathrm{id}$, $ds(x_0) = -A(x_0)$. this is possible because $A \in \Omega^1(\mathrm{Ad}E)$ which is the fiber of the frame bundle, use exp. □

Prop. (IV.2.3.5) (Induced connections). The connection action $d_A = d+\omega$ on a vector bundle E induces connection on many relevant bundles. the action on dual bundle is by

$$d_A(s^*) = ds^* - \omega^t(s^*) = ds^* - s^* \circ \omega.$$

And the connection on $\mathrm{End}\, E$ by

$$d_A(\alpha) = d\alpha + [\omega, \alpha] = [\nabla, \alpha]$$

And they act on $\Omega^*(E)$ by Leibniz rule thus the formula looks the same. (Note that the convention is section write on the left of the differential forms, so for example, $[\omega, \omega] = 2\omega \wedge \omega$).

Proof: Cf.[Jost P110]. □

Cor. (IV.2.3.6). For a line bundle L, for a connection on it with curvature Ω, the induced on the dual line bundle L^* has connection $-\Omega$. (because $\Omega = d\omega$ and $\omega' = -\omega$).

Prop. (IV.2.3.7) (Second Bianchi's Identity). A affine connection on E looks locally like $d_A = d + \omega$, where $\omega \in \Omega^1(\mathrm{End}\, E)$. And $F_A = d_A \circ d_A \in \Omega^2(\mathrm{End}(E))$ satisfies

$$d_A F_A = dF_A + [\omega, F_A] = 0.$$

Proof: Notice $dF_A = dd\omega + d(\omega \wedge \omega) = d\omega \wedge \omega - \omega \wedge d\omega$, and $\omega(d\omega + \omega \wedge \omega) - (d\omega + \omega \wedge \omega)\omega = \omega \wedge d\omega - d\omega \wedge \omega$. □

Def. (IV.2.3.8) (Christoffel Symbol). The **Christoffel symbol** of a connection is defined by the equations: $\nabla_{X_i} X_j = \sum_k \Gamma^k_{ij} X_k$.

The **geodesic equations** is $\dfrac{D}{dt}(\dfrac{d\gamma}{dt}) = \ddot{x}_k + \sum_{i,j} \Gamma^k_{ij} \dot{x}_i \dot{x}_j = 0 \quad \forall k.$

Def. (IV.2.3.9). The **torsion tensor** of a connection ∇ on TM is defined as $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. The connection is called **torsion-free** if $T = 0$. This is equivalent to $\Gamma^k_{i,j} = \Gamma^k_{j,i}$.

A connection is called **metric** if it preserves metric. i.e. $\nabla g = 0$.

Proof: T is a tensor because it is skew-symmetric, and

$$T(fX,Y) = f\nabla_X Y - f\nabla_Y X - df(Y)X - (f[X,Y] - df(Y)X) = fT(X,Y),$$

where(IV.3.2.5) is used. □

Prop. (IV.2.3.10). If ∇ is torsion-free connection on TM, then its induced connection on T^*M satisfies

$$(d\alpha)(v_1,\ldots,v_k) = \sum(-1)^i (D_{v_i}\alpha)(v_1,\ldots,\hat{v}_i,\ldots,v_k).$$

Proof: □

Def. (IV.2.3.11) (Curvature Tensor). The **curvature** of a (affine) connection d_A is $F_A = d_A \circ d_A \in \Omega^2(\mathrm{End}(E))$. The curvature tensor it induced is

$$F_A(Z)(X,Y) = R(X,Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]}Z.$$

In particular, the curvature depends only on the point, and locally $F_A = d\omega + \omega\wedge\omega$

In two coordinates $\overline{e} = ea$ for $a : U \to GL(r,\mathbb{R})$,$\overline{F_A} = a^{-1}F_A a$.

The connection is called **flat** if $F_A = 0$.

Proof: To verify the equation, check first the left side is pointwise, and the third component of the right side assures it is pointwise, too, thus we can check for a local coordinate vector field($[X_i, X_j] = 0$), then because $\nabla s = \sum_i \nabla_i s dx_i$,

$$\nabla^2 s = \nabla(\sum_i \nabla_i s dx_i) = \sum_{ij}\nabla_j\nabla_i s dx_j dx_i = \sum_{i<j}(\nabla_i\nabla_j - \nabla_j\nabla_i)s dx_i \wedge dx_j$$

□

Prop. (IV.2.3.12) (Flat coordinate). A connection on TM assumes near every point a flat coordinate, i.e. $\nabla(\partial/\partial x^i) = 0$, iff it is flat and torsion-free.

Proof: One side is easy because its Christoffels vanish. On the other side, use integrability theorems (V.8.6.2). Cf.[Jost P115]. □

Prop. (IV.2.3.13).

$$\Delta\langle\varphi,\varphi\rangle = 2(\langle D^*D\varphi,\varphi\rangle - \langle D\varphi, D\varphi\rangle).$$

Proof: Cf.[Jost P118]. □

Prop. (IV.2.3.14). For a flat connection, there is a bundle isomorphism (Gauge transform) that transforms d_A into natural d.

Proof: Because $d_{gA}(s) = g d_A(g^{-1}(s))$,$d_{gA} = d - dg\cdot g^{-1} + g\cdot\omega\cdot g^{-1}$. Solve this PDE direvtly. (Cf.[Topics in Geometry Xie Yi week3]). □

Cor. (IV.2.3.15). For a flat connection, by(IV.2.3.14), the parallel transportation only depends on the homotopy type of the loop, thus gives an action of $\pi(X)$ on $SO(T_p(X))$ (or $SU(T_p(X))$. (because it is locally constant).

In this way, connections module gauge equivalence (preserving matric) equals representation of $\pi(X)$ module conjugations. The reverse map is giving by principal bundle.

Proof: □

Levi-Civita Connection

Def. (IV.2.3.16) (Levi-Civita Connection). The Levi-Civita connection is the unique connection on M that is metric and torsion-free:

$$X\langle Y,Z\rangle=\langle\nabla_XY,Z\rangle+\langle Y,\nabla_XZ\rangle,\qquad \nabla_XY-\nabla_YX=[X,Y].$$

It satisfies:

$$\langle Z,\nabla_YX\rangle=1/2\{X\langle Y,Z\rangle+Y\langle Z,X\rangle-Z\langle X,Y\rangle-\langle[X,Z],Y\rangle-\langle[Y,Z],X\rangle-\langle[X,Y],Z\rangle\}.$$

Then

$$\Gamma_{ij}^m=1/2\sum_k\{g_{jk,i}+g_{ki,j}-g_{ij,k}\}g^{km}$$

Thus geodesic is a solution that only depends on the metric(IV.2.3.8), so a local isometry preserves geodesics.

Prop. (IV.2.3.17). Now the Lie derivative has the form:

$$L_X(S)(Y_1,\ldots,Y_p)=\nabla_X(S)(Y_1,\ldots,Y_p)+\sum_{i=1}^pS(Y_1,\ldots,Y_{i-1},\nabla_{Y_i}X,\ldots,Y_p).$$

The exterior derivative d and its adjoint d^* has the form:

$$d\omega(Y_i)=\sum(-1)^p\nabla_{Y_i}\omega(\check{Y_p}),\quad d^*\omega(Y_i)=-\sum\nabla_{e_j}\omega(e_j,Y_i)$$

where e_i is an orthonormal basis. Cf.[Jost P140].

Prop. (IV.2.3.18) (Covariant Differential Symmetry). For a parametrized surface: $s:(u,v)\to M$,

$$\frac{D}{\partial u}\frac{\partial s}{\partial v}=\frac{D}{\partial v}\frac{\partial s}{\partial u}.$$

Proof:

$$\frac{D}{\partial u}\frac{\partial s}{\partial v}=\frac{D}{\partial u}(\sum\frac{\partial s_i}{\partial v}X_i)=\frac{\partial^2s_i}{\partial u\partial v}+\sum\frac{\partial s_i}{\partial v}(\sum\frac{\partial s_j}{\partial u}\nabla_jX_i)=\frac{\partial^2s_i}{\partial u\partial v}+\sum_{ij}\frac{\partial s_i}{\partial v}\frac{\partial s_j}{\partial u}\nabla_jX_i$$

But now the Levi-Civita connection is symmetric, thus $\nabla_jX_i=\nabla_iX_j$, showing the symmetry in u and v. □

Lemma (IV.2.3.19) (Gauss). Let $p\in M$ and $v\in T_pM$ s.t. $\exp_p v$ is defined, $w\in T_pM$, then

$$\langle(d\exp_p)_v(v),(d\exp_p)_v(w)\rangle=\langle v,w\rangle.$$

Proof: Cf.[Do Carmo P69]. □

Prop. (IV.2.3.20) (Geodesic Locally Minimizing). In a normal nbhd of p, the geodesic starting at p is the minimal line.

Proof: And curve $c(t)$ can be written as $\exp_p(r(t)v(t)) = f(r(t), t)$, where $f(s,t) = \exp_p(sv(t))$, so by Gauss lemma, $\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle = 0$. Now $dc/dt = \partial f/\partial r r'(t) + \partial f/\partial t$, so

$$|dc/dt|^2 = |r'(t)|^2 + |\partial f/\partial t|^2 \geq |r'(t)|^2.$$

Integrate this will give us the desired result. □

Prop. (IV.2.3.21) (Totally normal nbhd). For any point p, there exists a nbhd W and a number $\delta > 0$ s.t. for every $q \in W$, $\exp_q$ is a diffeomorphism on $B_\delta(0)$ and $\exp_q(B_\delta(0)) \supset W$. Thus, fine cover exists in every smooth manifold, because Riemannian metric exists on these manifolds.

Proof: Cf.[Do Carmo P72]. □

- (**Geodesic Frame**) In a neighborhood of every point p, there exists n vector fields, orthonormal at each point, and $\nabla_{E_i} E_j(p) = 0$.(Choose normal nbhd and parallel a orthonormal basis to every point. (WARNING: this is not a flat coordinate, it only helps when dealing with point-wise properties).

Def. (IV.2.3.22) (Killing Fields). A **Killing field** is a vector field which generates an infinitesimal isometry. X is killing $\iff L_X(g) = 0 \iff \langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0$ for all Y, Z, which is called the **Killing equation**.

Proof: Use Lie formula,

$$L_X(g)(Y,Z) = X\langle Y, Z \rangle - \langle [X,Y], Z \rangle - \langle Y, [X,Z] \rangle$$

. and Levi-Civita connection is torsion-free. □

Prop. (IV.2.3.23). Let M be a compact Riemannian manifold of even dimension with positive sectional curvatures, then every Killing field on M has a singularity.

Proof: Cf.[Do Carmo P104]. □

Def. (IV.2.3.24) (Geometric Differential Notions).

- The **gradient** is defined to be $\langle \mathrm{grad} f(p), X \rangle = X(f)(p)$.
- The **divergence** is defined to be $\mathrm{div} X(p) =$trace of the linear map $Y(p) \to \nabla_Y X(p) = \sum_i \langle \nabla_{E_i} X, E_i \rangle$. It measures the variation of the volume and it depends only on the point.
- The **Hessian** is defined to be $\mathrm{Hess} f$ is a self-adjoint operator that $(\mathrm{Hess} f)Y = \nabla_Y \mathrm{grad} f$ as well as a symmetric form $(\mathrm{Hess} f)(X,Y) = \langle (\mathrm{Hess} f)X, Y \rangle$.
- The **Laplacian** is defined to be $\Delta f = \mathrm{div}\ \mathrm{grad} f = \mathrm{trace}\ \mathrm{Hess} f$.

Prop. (IV.2.3.25). In a geodesic frame,

$$\mathrm{grad} f(p) = \sum_{i=1}^n (E_i(f))E_i$$

$$\mathrm{div} X(p) = \sum_{i=1}^n E_i(f_i)(p), \mathrm{where} X = \sum_i f_i E_i.$$

$$\Delta f = \sum_i E_i(E_i(f))(p).$$

Cor. (IV.2.3.26).
$$\Delta(f\cdot g)=f\Delta g+g\Delta f+2\langle \mathrm{grad} f,\mathrm{grad} g\rangle,$$
because these only depends on the point.

Prop. (IV.2.3.27). $d(i(X)m)=(\mathrm{div}X)m$. where m is the volume form.

Proof: Choose a geodesic frame E_i, θ_i is a dual form of E_i, let $X=\sum f_iE_i$, then $\iota(X)m=\sum_i(-1)^{i+1}f_i\theta_i$, so
$$d(\iota(X)m)=\sum(-1)^{i+1}df_i\wedge\theta_i+\sum(-1)^{i+1}f_i\wedge d\theta=(\sum E_i(f_i))m+\sum(-1)^{i+1}f_i\wedge d\theta.$$
Notice that $d\theta_i=0$, because $d\theta_k(E_i,E_j)=E_i\theta_k(E_j)-E_j\theta_k(E_i)-\theta_k([E_i,E_j])=0$(IV.3.2.8), as it is a geodesic frame. And $\sum E_i(f_i)=\mathrm{div}(X)$(IV.2.3.25). □

Prop. (IV.2.3.28) (Hopf theorem). If f is a differentiable function on a compact orientable manifold with $\Delta f\geq 0$, then f is constant.

Proof: Let $\mathrm{grad}(f)=X$, then
$$\int_M\Delta f dm=\int_M\mathrm{div}(X)dm=\int_M d(\iota(X)dm)=0.$$
So $\Delta f=0$. Now
$$0=\int_M\Delta(f^2/2)dm=\int_M f\Delta f dm+\int_M|\mathrm{grad}(f)|^2dm$$
by(IV.2.3.26), thus $\mathrm{grad}(f)=0$, so f is constant. □

Def. (IV.2.3.29) (Riemannian Curvatures).
- The **sectional curvature** $K(X,Y)=\frac{\langle R(X,Y)X,Y\rangle}{|X\wedge Y|^2}$.
- The **Ricci curvature** $\mathrm{Ric}(x)=\mathrm{Ric}(x,x)$, where $\mathrm{Ric}(x,y)$ is the symmetric form of $\frac{1}{n}$ of trace of the map $z\to R(x,z)y$.
  Thus $\mathrm{Ric}_p(x)=\frac{1}{n-1}\sum\langle R(x,z_i)x,z_i\rangle$, for x a unit vector, where z_i is an orthonormal basis orthogonal to x.
- The **scalar curvature** $K(p)=1/n\sum\mathrm{Ric}_p(z_i)$, where z_i is an orthonormal basis.

The curvatures only depends on the point(IV.2.3.11).

Lemma (IV.2.3.30).
$$\frac{D}{\partial t}\frac{D}{\partial s}V-\frac{D}{\partial s}\frac{D}{\partial t}V=R(\frac{\partial f}{\partial s},\frac{\partial f}{\partial t})V.\quad\text{(obvious because }\frac{\partial}{\partial_s},\frac{\partial}{\partial_t}\text{ commutes)}$$

Proof:

□

Prop. (IV.2.3.31) (Sectional Curvature Define Curvature). The curvature tensor is determined by its sectional curvature.

In particular, if M is isotropic at a point p (The sectional curvature depends only on the point), then
$$R(X,Y,W,Z)=K_0(\langle X,W\rangle\langle Y,Z\rangle-\langle X,Z\rangle\langle Y,W\rangle).$$

Proof: Cf.[Do Carmo P95], should use the cyclicity of the first three terms. □

Prop. (IV.2.3.32) (Bianchi Identities). Recall the covariant differential $\nabla R(Y_i, Z) = Z(R(Y_i)) - \sum_i R(\nabla_Z Y_i, Y_j)$(IV.2.3.5).

- (Bianchi Identity)$\sum_{(X,Y,Z)} R(X,Y)Z = 0$.
- (Second Bianchi Identity)$\sum_{(Z,W,T)} \nabla R(X,Y,Z,W,T) = 0$.

Proof: 1: Cf.[Do Carmo P91], should reduce to Jacobi identity.
2: □

Prop. (IV.2.3.33) (Schur's Theorem). Let M be a manifold of dimension $n \geq 3$, suppose the sectional curvature only depends on p, then M has constant curvature.

Proof: Use the second Bianchi Identity and geodesic frame and(IV.2.3.31).Cf.[Do Carmo P106]. □

Def. (IV.2.3.34) (Eisenstein Curvature). A manifold M is called a **Eisenstein manifold** iff its Ricci curvature $\lambda(p)$ only depends on the point. Then

- If M is connected and Eisenstein of dimension≥ 3, then it has constant Ricci curvatures everywhere every direction.
- If M is connected and Eisenstein of dimension3, then it has constant sectional curvatures.

Proof: 1: Cf.[Do Carmo P108].
2: Now it has constant Ricci curvature, then

$$R_{1212} + R_{1313} = \lambda = R_{1212} + R_{2323} = R_{1313} + R_{2323}.$$

So we can solve these curvatures out. □

Prop. (IV.2.3.35) (Riemannian Curvature Identities).

-
- $R(X,Y,Z,W) = R(Z,W,X,Y), \quad R(X,Y,Z,W) = R(X,Y,W,Z)$.

Proof: Cf.[DO Carmo P91]. □

- $B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$. It is bilinear and symmetric.
- $H_\eta(x,y) = \langle B(x,y), \eta\rangle$. Thus $B(x,y) = \sum H_i(x,y)E_i$ for an orthonormal frame E_i in $\mathfrak{X}(U)^\perp$.
- $S_\eta(x) = -(\overline{\nabla}_x \eta)^T$. It satisfies:$\langle S_\eta(x), y\rangle = H_\eta(x,y) = \langle B(x,y), \eta\rangle$. It is self-adjoint. When codimension 1, it is the derivative of the Gauss mapping.
- (**Gauss Formula**): let x, y be orthonormal tangent vector. Then:

$$K(x,y) - \overline{K}(x,y) = \langle B(x,x), B(y,y)\rangle - |B(x,y)|^2.$$

- An immersion is called **geodesic** at p if the second fundamental form S_η is zero for all η, (which means $\nabla_X Y$ has no normal component). It is called **minimal** if the trace of S_η is zero.
- An immersion is called umbilic if there exists a normal unit field η s.t. $\langle B(X,Y), \eta\rangle(p) = \lambda(p)\langle X, Y\rangle$.

- If the ambient space has constant sectional curvature and the immersed manifold is totally umbilic, then λ is constant.
- mean curvature tensor of immersion $f = 1/n \sum_i (\mathrm{tr}\ S_i)E_i = 1/n\ \mathrm{tr}\ B$. It is zero if f is minimal.
- normal connection $\nabla^{\perp}_X \eta = (\overline{\nabla}_X \eta)^N = \overline{\nabla}_X \eta + S_\eta(X)$.
- (Gauss equation)

$$\langle \overline{R}(X,Y)Z,T\rangle = \langle R(X,Y)Z,T\rangle - \langle B(Y,T),B(X,Z)\rangle + \langle B(X,T),B(Y,Z)\rangle.$$

- (Ricci equation)

$$\langle \overline{R}(X,Y)\eta,\zeta\rangle - \langle R^{\perp}(X,Y)\eta,\zeta\rangle = \langle [S_\eta,S_\zeta]X,Y\rangle.$$

- (Codazzo equation)

$$\langle \overline{R}(X,Y)Z,\eta\rangle = (\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta). \qquad \text{(Lie bracket)}$$

Parallel Transportation

Def. (IV.2.3.36) (Parallel Transportation).

Def. (IV.2.3.37) (Holonomy Group). The **holonomy group** $Hol_x(g)$ of a Riemannian manifold M w.r.t to the Levi-Civita connection is defined to be the subgroup of $O(T_x(M))$ induced by the parallel transportation along a loop. If M is connected, For different points, holonomy groups are conjugate, so holonomy group is defined up to conjugation.

Prop. (IV.2.3.38) (Trivial Holonomy Group). If M is a Riemannian manifold and the holonomy group is trivial, then for any $X,Y,Z \in X(M)$, $R(X,Y)Z = 0$.

Proof: Cf.[Do Carmo P105]. □

Prop. (IV.2.3.39) (Berger). in fact, the groups that can be realized as a holonomy group of a simply connected complete Riemannian manifold can be classified.

Proof: Cf.[Complex geometry Daniel P214]. □

Def. (IV.2.3.40). The **Geodesic flow** for a connection on TM is the flow on TM whose trajectories are $t \mapsto (\gamma(t),\gamma'(t))$, where γ is a geodesic on M.

Prop. (IV.2.3.41) (The smoothness of geodesics). For every point p, there exists a nbhd V and a C^∞ mapping

$$\gamma : (-\delta,\delta)\times V \times B(0,\epsilon) \to M,$$

s.t. $\gamma(t,q,v)$ is the geodesic passing through p with velocity v.

Complete manifold

Prop. (IV.2.3.42) (Hopf-Rinow theorem). The following is equivalent definition of **completeness**.

1. $\exp_p$ is defined for all of $T_p(M)$.
2. The closed and bounded sets of M are compact.
3. M is complete as a metric space.

4. M is $\sigma-$compact and if $q_n \notin K_n$, $d(p, q_n) \to \infty$.

5. The length of any divergent (compact escaping) curve is unbounded.

and if M is complete, then for any q, there exists a minimizing geodesic. In particular, any compact submanifold of a complete manifold is complete.

Proof: Cf.[Do Carmo P147]. □

- For any two manifold of the same constant curvature and any two orthogonal basis, there is a local isometry (It is locally isotropic).
- Any complete manifold with a sectional curvature is like $\tilde{M}/\Gamma$, where $\tilde{M}$ is $\mathbf{H}^n, \mathbf{R}^n$ or $\mathbf{S}^n$.

Prop. (IV.2.3.43) (Cartan). in any nontrivial homotopy class in a compact manifold , there exists a closed geodesic.

4 Jacobi Field and Comparison Theorems

Def. (IV.2.4.1) (Jacobi Field). The **Jacobi field equation** along a normalized geodesic γ is defined to be

$$D^2 J(t) + R(\dot{\gamma(t)}, J(t))\dot{\gamma}(t) = 0.$$

It is defined by its initial condition $J(0)$ and $J^{'}(0)$. It can be used to detect the sectional curvature, the critical point of $\exp_p$ and calculate variation of energy.

Prop. (IV.2.4.2) (Constant Curvature Case). On a manifold with constant curvature K, the Jacobi field equation for a vector field J normal to γ is equivalent to

$$D^2 J(t) + KJ(t) = 0.$$

Proof: Use(IV.2.3.31), we have

$$\langle R(\gamma', J)\gamma', T\rangle = K\{\langle\gamma', \gamma'\rangle\langle J, T\rangle - \langle\gamma', T\rangle\langle J, \gamma'\rangle\} = K\langle J, T\rangle$$

So $R(\gamma', J)\gamma' = KJ$. □

Prop. (IV.2.4.3). The Jacobi field along a point with initial velocity 0 all has the form

$$J(t) = (d\exp_p)_{t\dot{\gamma}(0)}(tJ^{'}(0)).$$

Proof: Cf.[Do Carmo P113]. Should use uniqueness theorem of ODE. □

Cor. (IV.2.4.4) (Conjugate Points). If two points p, q are connected by a geodesic γ, and $q = \exp_p(v_0)$, then p, q are called **conjugate** along γ, if there is a non-trivial Jacobi field on γ that $J(p) = J(q) = 0$.

Then q is conjugate to p iff v_0 is the critical point of $\exp_p$, and the multiplicity of conjugacy is equal to the kernel of $(\exp_p)_{v_0}$.

Prop. (IV.2.4.5). For a Jacobi field J along γ, $\langle J(t), \gamma'(t)\rangle$ is linear in t.

Proof: Take second derivatives. □

- If J is a Jacobi field $J(t) = (d\exp_p)_{tv}(tw)$, $|v| = |w| = 1$, then

$$|J(t)| = t - \frac{1}{6}K_p(v, w)t^3 + o(t^3).$$

Prop. (IV.2.4.6). There are no conjugate points on a Riemannian manifold of non-positive curvature.

Proof: Cf.[Do Carmo P119]. □

Prop. (IV.2.4.7) (Killing Field is everywhere Jacobi). A Killing field is a Jacobi field along geodesics.

And if $X(p) = 0$, then X is tangent to the geodesic sphere near p, because X preserves length.

Proof: □

Energy Analysis

Def. (IV.2.4.8) (Energy). The **energy** of a geodesic γ is defined to be

$$E(s) = \int_0^a |\frac{\partial f}{\partial t}(s,t)|^2 dt.$$

Prop. (IV.2.4.9). A minimizing geodesic must minimize energy.

- (**First Variation of Energy**)

$$1/2E^{'}(0) = -\int_0^a \langle V(t), D\dot{c}(t)\rangle dt + \langle V(a), \dot{c}(a)\rangle - \langle V(0), \dot{c}(0)\rangle.$$

  A piecewise differentiable curve is a geodesic iff every proper variation has first derivative 0.

- (**Second Variation of Energy**) If γ is a geodesic,

$$1/2E^{''}(0) = \int_0^a \{\langle DV(t), DV(t)\rangle - \langle R(\dot{\gamma}, V)\dot{\gamma}, V\rangle\}dt + \langle D_s V(a), \dot{\gamma}(a)\rangle - \langle D_s V(0), \dot{\gamma}(0)\rangle.$$

- a variation is equivalent to a vector field along the curve, and a variation that $f_s(t)$ are all piecewise geodesics corresponds to a piecewise Jacobi field(Choose a normal partition).

Prop. (IV.2.4.10) (Rauch Comparison theorem). Let M and $\tilde{M}$ be manifolds, dim$\tilde{M} \geq$ dim M. If J and $\tilde{J}$ be two normal Jacobi fields along geodesics γ and $\tilde{\gamma}$ that $|J(0)| = |J'(0)| = 0$ and $|J^{'}(0)| = |\tilde{J}^{'}(0)|$. If $\tilde{\gamma}$ has no conjugate point or focal point free and $\tilde{K}(\tilde{x}, \dot{\tilde{\gamma}}(t)) \geq K(x, \dot{\gamma})$ for any vector $x, \tilde{x}$, then $|\tilde{J}| \leq |J|$.

Cor. (IV.2.4.11) (Injectivity Radius Estimate). If the sectional curvature of M satisfies: $0 < L \leq K \leq H$, then the distance between any two conjugate points satisfies: $\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}$.

Prop. (IV.2.4.12). If two manifold M and $M^{'}$ satisfy $K \leq K^{'}$, then in a normal nbhd of a point p in M and a nbhd of $p^{'}$ that exp is nonsingular, the transformation of a curve c shortens length.

Note that this is not Toponogov thoerem, because if you try to map from a large curvature manifold to a small curvature, then you cannot guarantee that the mapped curve is the shortest.

Cor. (IV.2.4.13). In a complete simply connected manifold of non-positive curvature,

$$A^2 + B^2 - 2AB\cos\gamma \leq C^2$$

thus $\alpha + \beta + \gamma \leq \pi$.

Prop. (IV.2.4.14) (Moore theorem). Let $\overline{M}$ be a complete simply connected manifold of sectinoal curvature $\overline{K} \leq -b \leq 0$, M a compact manifold of sectional curvature satisfying $K - \overline{K} \leq b$. If $\dim\overline{M} < \dim M$, M cannot be immersed into $\overline{M}$.(use Hadamard theorem to choose the furthest geodesic and calculate the second variation of energy and use Gauss formula).

Cor. (IV.2.4.15). Let $\overline{M}$ be a complete simply connected manifold of sectinoal curvature $\overline{K} \leq 0$, M a compact manifold of sectional curvature satisfying $K \leq \overline{K}$. If $\dim\overline{M} < \dim M$, M cannot immerse into $\overline{M}$.

Lemma (IV.2.4.16) (Klingenberg Lemma). Let M be a complete manifold of sectional curvature $K \geq K_0$, let γ_0, γ_1 be two homotopic geodesics from p to q, then there exists a middle curve γ_s s.t.

$$l(\gamma_0) + l(\gamma_s) \geq \frac{2\pi}{\sqrt{K_0}}.$$

Proof: Assume $l(\gamma_0) < \frac{2\pi}{\sqrt{K_0}}$, otherwise we are done. Then by Rauch comparison(IV.2.4.10), the $\exp_p : T_pM \to M$ has no critical point in the open ball B of radius $\pi/\sqrt{K_0}$. Now we want to lift γ_s to T_pM. It is clear that we cannot lift γ_1, because otherwise it is not a curve. Hence for every $\varepsilon > 0$, there is a curve $\alpha_{t(\varepsilon)}$ that can be lifted and has a point with distance smaller than ε to the boundary ∂B, otherwise the s that can be lifted will be open and closed in $[0, 1]$, thus containing 1.

So now if we choose a sequence of lifts curves γ_s converging to the boundary, then s has a convergent point, then we have $l(\gamma_0) + l(\gamma_{t_0}) \geq \frac{\pi}{\sqrt{K_0}}$. □

Prop. (IV.2.4.17) (Klingenberg). Let M be a simply connected compact manifold of dimension $n \geq 3$ such that $\frac{1}{4} < K \leq 1$, then $i(M)$(The infimum of distance to the cut locus)$\geq \pi$.

Cor. (IV.2.4.18). If M is a compact orientable manifold of even dimension satisfying $0 < K \leq 1$, then $i(M) \geq \pi$.

Prop. (IV.2.4.19) (1/4-pinch Sphere Theorem). Let M be a compact simply connected manifold satisfying $0 < 1/4K_{\max} < K \leq K_{\max}$, then M is homeomorphic to a sphere.

(Use Klingenberg Theorem, this is a special case of diameter geodesic sphere theorem). Cf.(IV.2.4.29).

It can be shown that in this case, this sphere is even diffeomorphic to S^n using Ricci flow.

Remark (IV.2.4.20). $0 < 1/4K_{\max} < K$ cannot be changed to $\geq$. In fact, the Funibi-Study metric on CP^n has sectional curvature $1 \geq K \geq 4$. Cf. **??**

Hess$\rho(X, Y)$ where ρ is the distance to a fixed point, is important.

Prop. (IV.2.4.21). Hess$\rho(X, Y)$ is positive definite on the tangent space of the geodesic sphere within the injective radius, and its principal value is $|\frac{J'}{J}|$ for a Jacobi field in that direction. And it is zero on the normal direction.

So there would be a Riccati comparison theorem on the eigenvalue of $\Pi_2 : \lambda' \leq -K - \lambda^2$,Hess$(\rho)$ is bounded.

Proof: Notice that

$$\mathrm{Hess}\rho(X, Y) = (\nabla_X \mathrm{grad}\rho, Y) = XY\rho - (\nabla_X Y)\rho$$

so if choose a normal geodesic γ of initial vector X, then

$$\begin{aligned}\mathrm{Hess}\rho(X,X) &= X\langle\dot\gamma, d\rho\rangle - (\nabla_X\dot\gamma)\rho = X\langle\dot\gamma, d\rho\rangle = \langle\dot\gamma, d\langle\dot\gamma, d\rho\rangle\rangle = E''(0)\\ &= I_q(X,X) = ((\nabla_{\dot\gamma}X)(q), X(q)) = \frac{\langle J', J\rangle}{|J|^2}\end{aligned}$$

□

Prop. (IV.2.4.22) (Toponogov). Let$M$ be a complete manifold with $K\geq H$.

If a hinge satisfies γ_1 is minimal and $\gamma_2\geq\frac{\pi}{\sqrt{H}}$ if $H>0.$, then on M^H the same hinge has smaller distance of endpoints than this hinge

Proof: Cf.[Cheeger Comparison Theorems in Riemannian Geometry P42]. And there is another triangle version: For a minimal geodesic triangle, the comparison triangle has smaller angles. NOTE this theorem cannot be derived from Rauch Comparison Theorem. □

__Critical Point for Distance Function__

Prop. (IV.2.4.23). The critical point for distance function on a complete manifold is that for every direction v, there is a minimal geodesic γ s.t. $\langle\gamma'(l), v\rangle\leq\frac{\pi}{2}$.

The set of regular point is open and there exists a smooth gradient like vector field (i.e. acute angle with every minimal geodesic) on this open subset .

Prop. (IV.2.4.24) (Berger's Lemma). A maximal point for the distance function is a critical point.

Proof: If not, choose a convergent point v of the minimal geodesics with endpoint in a curve of that direction, then exp near v will generate a Jacobi field with endpoint Jacobi is the sam of that direction. So the distance will increase by $\cos\theta$ along that direction, contradiction. □

Prop. (IV.2.4.25) (Soul Lemma). Let M is a Riemannian manifold and A is a closed submanifold. If $dist(A,-)$ has no critical point on $D(A,R)\backslash A$, then $B(A,R)$ is diffeomorphic to the normal bundle of $A\to M$.

Proof: A has a normal exp radius ϵ, and we can vary the gradient-like vector field to be identical to the normal vector near A, and use Morse lemma (the flow) to get a diffeomorphism. □

Cor. (IV.2.4.26) (Disk Theorem). If A is a point then M is diffeomorphic to a disk.

Lemma (IV.2.4.27) (Generalized Schoenflies Theorem). Easy to do, just use the fact that exp is continuous to find a boundary sphere depending continuously on the direction (both p and q).

Prop. (IV.2.4.28) (Sphere Theorem). If M is a closed manifold and has a distance function with only one critical point (the furthest one), then M is homeomorphic to a twisted ball.

Proof: There exists a ϵ and r that $B(q,\epsilon)$ and $B(p,r)$ covering M, (Use the convergent point argument). Then use the generalized Schoenflies theorem. □

Prop. (IV.2.4.29) (Diameter Sphere Theorem). If a closed manifold M satisfies sec $M\geq K>0$, and $\mathrm{diam}(M)>\frac{\pi}{2\sqrt{K}}$, then M is homeomorphic to S^n.

Proof: First, if there are two maximal distance point, then use Toponogov to show contradiction. Second, at other points x,

$$\angle pxq > \frac{\pi}{2}$$

(Regular domain) because of Toponogov and The formula

$$\cos\tilde{\alpha} = \frac{\cos l - \cos l_1 \cos l_2}{\sin l_1 \sin l_2}.$$

So the geodesic direction $\overrightarrow{xq}$ will serve as a geodesic-like vector field (might need paracompactness). □

Prop. (IV.2.4.30) (Critical Principle). In a complete manifold M of sectional curvature $> K$, if q is a critical point of p, then for any point x with $d(p,x) > d(p,q)$ and any minimal geodesic from p to x, the $\angle xpq$ is smaller than the $\cosh_K^{-1}(\frac{d(p,x)}{d(p,q)})$.

Proof: Use Toponogov for the hinge xpq. Then notice that there is a different minimal geodesic from $p \to q$ that makes the $\angle pqx < \pi/2$ by the definition of critical point, thus there is another Toponogov inequality, this two inequality contradicts. □

Cor. (IV.2.4.31). For a complete open manifold whose K are lower bounded, then it is homeomorphic to the interior of a manifold with boundary. (Use Soul lemma, otherwise there will be a sequence of critical point whose angles are big).

Prop. (IV.2.4.32). ray construction and Line construction**?**

Prop. (IV.2.4.33) (Soul Theorem). If M is an open manifold with $K \geq 0$, then there is a totally geodesic submanifold S that M is diffeomorphic to the normal bundle over S.

Proof: Use the ray construction to get a totally convex compact subset, hence it is a manifold or with boundary , if it has boundary, then find to set of maximal distance to the distance to boundary, the distance to the boundary is a convex function, so it is a smaller totally geodesic manifold. So a S without boundary must exist and this constitutes a stratification, all the level set is strongly convex. Thus all point outside S is not critical, hence the soul lemma applies. Cf.[GeJian Comparison theorems in Riemannian Geometry Lecture7]. □

Prop. (IV.2.4.34) (Perelman). There is a distance non-increasing contraction unto the soul, and it must be just the projection along the normal bundle. Moreover, for any geodesic on the soul and a parallel vector field in the normal bundle along it, it spans a flat surface (by Rauch comparison).

Cor. (IV.2.4.35) (Soul Conjecture). For an open(non-compact) complete manifold M with $K \geq 0$, if it has a point p s.t. sectional curvature at p are all positive, then M is diffeomorphic to R^n. (It's enough to show that its soul is a point, otherwise for any point, it must has a direction that is flat, $K = 0$).

5 Curvature Inequalities and Topology

Sectional Curvature

Prop. (IV.2.5.1) (Hadamard theorem). M a complete simply connected Riemann manifold of sectional curvature ≤ 0, then $\exp_p : T_pM \to M$ is an isomorphism of M to $\mathbb{R}^n$.(negative sectional curvature to show exp is a local isomorphism, complete to show it is a covering map)

Prop. (IV.2.5.2) (Liouville Theorem). Any conformal mapping for an open subset of $\mathbb{R}^n, n > 2$ is restriction of a composition of isometry, dilations and/or inversions, at most once.

Prop. (IV.2.5.3) (Positive Curved, Closed Geodesic not Minimal). If M is an even dimensional orientable Riemannian manifold with positive sectional curvature, let $\sigma : [0,1] \to M$ be a closed geodesic curve, then there exists an $\varepsilon > 0$ that parametrized closed curves $F; [0,1] \times (-\varepsilon, \varepsilon) \to M$ near σ with lengths less than that of σ.

Proof: Cf.[Solution to Yau Test Geometry Individual2013 Prob5]. □

Prop. (IV.2.5.4) (Synge). f is an isometry of a compact oriented manifold M^n of positive sectional curvature, f alter orientation by $(-1)^n$, then f has a fixed pt.

Proof: Cf.[Do Carmo P203]. □

Cor. (IV.2.5.5). M a compact manifold of positive sectional curvature, then

1. If M is orientable and n is even, then M is simply connected. So If M is compact and even dimension, then $\pi(M) = 1$ or $\mathbb{Z}_2$.
2. If n is odd, then M is orientable.

(Use the universal cover and covering transformation.)

Conjecture (IV.2.5.6) (Hopf Conjecture). If M is a compact Riemannian manifold of even dimension that $K > 0$, then it has positive Euler characteristic.

Morse Index

Prop. (IV.2.5.7) (Index Lemma). Among the piecewise differentiable vector fields along a geodesic without conjugate point or without focal point, with initial value 0 and fixed end value, the Jacobi field attain minimum of the index form:

$$I_a(V,V) = \int_0^a \{\langle DV(t), DV(t)\rangle - \langle R(\dot{\gamma}, V)\dot{\gamma}, V\rangle\}dt.$$

Cor. (IV.2.5.8). $I_l(J,J) = \langle J, J'\rangle(l)$ for a Jacobi field.

Prop. (IV.2.5.9). a focal point is a critical value of $exp^{\perp}$. For an embedded manifold, the focal point equals $x + 1/t\eta$, where η is a vertical vector and t is a principal value of $S_e ta$.

Prop. (IV.2.5.10) (Morse Index theorem). The index of the the index form $I_a(V,W)$ on the space of vector fields 0 at the endpoints, equal to the number of points conjugate to $\gamma(0)$ in $[0,a)$.

Cor. (IV.2.5.11). If γ is minimizing, γ has no conjugate points on $(0,a)$, γ has a conjugate point, it is not minimizing.

Prop. (IV.2.5.12) (Morse). If M is complete with non-negative sectional curvature, then $\pi_1(M)$ have no finite non-trivial cyclic group and $\pi_k(M) = 0$.

Proof: because universal cover of M is contractible, so the higher homotopy group vanish and $H^k(M) = H^k(\pi_1(M))$, so if a subgroup is finite cyclic, its homology is periodic, contradiction. □

Prop. (IV.2.5.13) (Preissman). For a compact manifold with $K < 0$, any nontrivial abelian subgroup of π_1 is infinite cyclic.

Prop. (IV.2.5.14). If M is compact and $K<0$, $\pi_1(M)$ is not abelian.

Assuming M complete,

- The cut point of p along γ is the maximum $\gamma(t)$ s.t. $d(p,\gamma(t))=t$. It is either the first conjugate point of p or the intersection of two minimizing geodesics.
- Conversely, if a point is a conjugate point of p or is intersection of two geodesics of equal length, then there is a cut point before it. So, if intersection of two minimizing geodesics happens, it must happen before the occurrence of conjugate point.
- thus the cut point relation is reflexive, and if $q\in M\setminus C_m(p)$, then there exists a unique minimizing geodesic joining p and q.
- $M\setminus C_m(p)$ is homeomorphic to an open ball through exp.
- the distance of p to the cut locus is continuous, thus $C_m(p)$ is closed.
- If M is complete and there is a p which has a cut point for every geodesic, then M is compact.
- for q the closest of $C_m(p)$ to p, either there exists a minimizing geodesic and q is conjugate to p or there is to minimizing geodesic connecting at q.

Prop. (IV.2.5.15). The index of a geodesic will decrease when transferred to a manifold of smaller sectional curvature K.

Prop. (IV.2.5.16). In a complete manifold, if there is a sequence of points $\{p_i\}$ converging to a point p, choose for each point a minimal geodesic, then a subsequence of them will converge to a minimal geodesic to p.

Proof: The convergence is by smoothness and of exp and Hadamard. The minimality is by comparing distance. □

Ricci Curvature

Prop. (IV.2.5.17) (Ricci Comparison). Volume comparison, Laplacian Comparison, Mean Curvature comparison. Cf.[葛健 Week13].

Prop. (IV.2.5.18) (Bishop-Gromov). Let M be an open manifold with $\mathrm{Ric}\geq H$, let $\tilde{M}(H)$ be a complete simply connected manifold of constant sectional curvature H, then

$$\mathrm{Vol}(B_r(x))\leq \mathrm{Vol}(B_r(\tilde{p})),\quad \frac{\mathrm{Vol}(B_R(x))}{\mathrm{Vol}(B_r(x))}\leq\frac{\mathrm{Vol}(B_R(\tilde{p}))}{\mathrm{Vol}(B_r(\tilde{p}))}.$$

Cf.[葛健 Week13].

Prop. (IV.2.5.19) (Bonnet-Myer). M a complete manifold of Ricci curvature $\mathrm{Ric}_p(v)\geq\frac{1}{r^2}$, Then M is compact and have diameter $\leq\pi r$.

And if the identity is achieved, $M\cong\mathbb{S}^n$.

Proof: Use Laplacian comparison $\Delta r\leq(n-1)\cot r$. Cf.[葛健 week13]. □

Cor. (IV.2.5.20) (Positive Ricci Finite Fundamental Groups). M is a complete manifold of Ricci curvature $\geq\delta>0$, then the universal cover is compact thus $\pi_1(M)$ is finite. This can be seen as an obstruction for a compact manifold to have positive Ricci curvature.

Cor. (IV.2.5.21) (Calabi-Yau). For an open manifold with non-negative Ricci curvature, for any point, $\mathrm{Vol}(B(p,r)) \geq c_p r$.

Prop. (IV.2.5.22) (Milnor). Let M be an open manifold of non-negative Ricci curvature of dimension n, then any f.g. subgroup of $\pi_1(M)$ has polynomial growth $\leq n$. Milnor conjectured that $\pi_1(M)$ in fact is f.g..

Prop. (IV.2.5.23) (First Betti Number Theorem). There is a number $f(n,\lambda,D)$, $f(n,0,D) = n$, $f(n,\lambda,D) = 0$ for $\lambda > 0$ that for a manifold of diameter $\leq D$ and Ricci curvature $\geq \lambda$, $b_1(M) \leq f(n,\lambda,D)$.

Cor. (IV.2.5.24) (Splitting Theorem). The universal cover of a compact RIemannian manifold with non-negative Ricci curvature splits isometrically as a product $\widetilde{M} = N \times \mathbb{R}^k$ where N is a compact manifold manifold.

Scalar Curvature

IV.3 Geometric Analysis

Basic references are [Smooth Manifold Lee], [Differential Topology Pollack](Good) and [Geometric Analysis Jost].

All manifolds in this section is tacitly assumed to be smooth.

1 Smooth Manifolds

Prop. (IV.3.1.1) (Global Rank Theorem). Let $F : M \to N$ be a map of manifolds of constant tank, then:

- if it is an injection, then it is a submersion.
- if it is an surjection, then it is a submersion.
- if it is a bijection, then it is an diffeomorphism.

Proof: Cf.[Lee Smooth Manifold P83]. □

Prop. (IV.3.1.2) (Local Embedding Theorem). If $F; M \to N$ is a smooth morphism of manifolds, then it is a smooth immersion iff it is locally a smooth embedding on the target.

Proof: Cf.[Lee Smooth Manifolds P87]. □

Submanifolds

Def. (IV.3.1.3) (Submanifolds). For a manifold M, an **immersed submanifold** $S \subset M$ is an immersion of topological spaces that is a smooth morphism of manifolds. An **embedded submanifold** $S \in M$ is an S in the induced topology that is smooth embedding of manifold.

Remark (IV.3.1.4). Examples of immersed submanifolds that is not an embedded submanifolds are the 8-figure and the dense curve in a torus. However, an immerse submanifold is locally embedded on the source. This follows immediately from(IV.3.1.2).

Prop. (IV.3.1.5). If M is a compact manifold, then any injective immersion $f : M \hookrightarrow M$ is an embedding of submanifolds.

Proof: The topology of S is equivalent to the induced topology by(IV.1.2.9). □

Prop. (IV.3.1.6). If S is an immersed submanifold in M, if $N \to M$ is a smooth morphism of manifolds that has image in S, if $N \to S$ is continuous, then $N \to S$ is smooth.

Proof: Cf.[Lee Smooth Manifold P112]. □

Cor. (IV.3.1.7) (Restricting Codomain of Smooth Morphism). If S is an embedded submanifold in M, if if $N \to M$ is a smooth that has image in S, then $N \to S$ is smooth.

Proof: Because in this, S has the induced topology, so it is easily seen that $N \to S$ is continuous. □

Sard's Theorem

Lemma (IV.3.1.8) (Invariance of Measure Zero Sets). If $S \in \mathbb{R}^n$ has measure zero, then for any smooth map $g : \mathbb{R}^n \to \mathbb{R}^m$, $g(S)$ has measure zero.

Thus the notion of measure zero is definable for arbitrary smooth manifolds.

Lemma (IV.3.1.9). Cf.[Pollack Appendix A].

Def. (IV.3.1.10). For a map of schemes $f : X \to Y$, a point $y \in Y$ is called **critical** iff df_x is not surjective, for some $x \in f^{-1}(y)$, otherwise it is called a **regular value**.

Prop. (IV.3.1.11) (Regular Value Theorem). If y is a regular value for a map $f : X \to Y$, then $f^{-1}(Y)$ has a natural submanifold structure.

Prop. (IV.3.1.12) (Stack of Records Theorem). If y is regular value of a map $f : X \to Y$, where X is compact and $\dim X = \dim Y$, then f is a covering map locally on $f^{-1}(U)$ for some nbhd U of y.

Proof: □

Prop. (IV.3.1.13) (Sard Theorem). For a map $X \to Y$ of smooth manifolds, the set of critical values is of measure zero Y.

Proof: Cf.[Pollack Appendix A]. □

Prop. (IV.3.1.14) (Whitney Embedding Theorem). Any k-dimensional manifold M can be embedded into $\mathbb{R}^{2k+1}$.

Proof: Cf.[Pollack P51]. □

Cor. (IV.3.1.15) (Whitney Immersion Theorem). Any smooth manifold M of dimension k can be immersed into $\mathbb{R}^{2k}$.

Proof: □

Tangent and Cotangent Bundle

Def. (IV.3.1.16). The tangent bundle is defined to be

Simplifications

Prop. (IV.3.1.17). For every vector field X and every point $X(p) \neq 0$, there exists a coordinate nbhd $(x_1, \ldots, x_{n-1}, t)$ such that $X = \frac{\partial}{\partial t}$.

1-dimensional Smooth Manifold with Boundaries

Prop. (IV.3.1.18). Any smooth manifold of dimension 1 with boundary is isomorphic to $[0, 1]$ or S^1.

Proof: Cf.[Pollack Appendix]. □

Cor. (IV.3.1.19). The boundary of any smooth manifold of dimension 1 consists of points of even number.

2 Differential Forms

Prop. (IV.3.2.1) (Frobenius Theorem). If X is an involutive distribution on a manifold M, then there is a unique maximal integration manifold passing through it. Where a distribution is involutive if it is closed under Lie bracket.

Proof: The key to the proof is to prove that involutive is equivalent to integrable, i.e. flat locally as $\{\frac{\partial}{\partial x_i}\}$ for some local coordinate. Cf.[李群讲义 项武义 P226] □

Cor. (IV.3.2.2). X, Y in a Lie algebra commute iff their corresponding vector fields commute.

Interior and Exterior Derivatives

Lie Derivatives

Def. (IV.3.2.3). The **Lie bracket** of two vector fields X, Y is defined to be $[X,Y](f) = (XY - YX)f$, then if $X = \sum a_i \partial/\partial x_i$, $Y = \sum b_i \partial/\partial x_i$, then $[X,Y] = \sum (X(b_i) - Y(a_i))\partial/\partial x_i$.

Lemma (IV.3.2.4). $[X,Y] = \frac{\partial}{\partial t}(d(\phi_{-t})Y)|_{t=0}$.

Proof: For any function f,set $g(t,q) = \frac{f(\phi_t(q))-f(q)}{t}$,$g(0,q) = Xf(q)$. Then g is differentiable (because $g(t,q) = \int_0^1 Xf(\phi_{ts}(p))ds$, and:

$$\begin{aligned}
\mathrm{lim}_{t\to 0}\, d(\phi_{-t})Yf(p) &= \lim \frac{Yf(p) - Y(f\phi_{-t})(\phi_t(p))}{t} \\
&= \lim \frac{Yf(p) - Yf(\phi_t p) - Y(tg(-t,\phi_t(p))}{t} \\
&= ((XY - YX)f)(p) \\
&= [X,Y]f(p)
\end{aligned}$$

□

Prop. (IV.3.2.5). $[fu, v] = f[u,v] - df(u)v$.

Proof: Direct from the definition(IV.3.2.3). □

Prop. (IV.3.2.6). Lie bracket commutes with pushforward: $[df(X), df(Y)] = df([X,Y])$.

Proof: (Use $XY - YX$ to see).? □

Prop. (IV.3.2.7) (Lie Derivative). We can define the **Lie derivative** of any differential forms, to make sure that formally satisfies the Leibniz rule. For example:

$$L_X(g(Y,Z)) = L_X(g)(Y,Z) + g(L_X Y, Z) + g(Y, L_X Z).$$

Prop. (IV.3.2.8) (Derivative formula).

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

Proof: □

Prop. (IV.3.2.9) (Cartan's magic formula).

$$L_X\omega = \iota_X(d\omega) + d(\iota_X\omega)$$

$$\iota([X,Y]) = [L_X, \iota_Y]$$

Proof: Notice that four of them are derivatives (check because $\iota_X(w\wedge v) = \iota_X w\wedge v + (-1)^{|w|}w\wedge \iota_X v$). So by induction, we only has to verify them on dimension 0 and 1. □

Prop. (IV.3.2.10) (Stoke's theorem).

$$\oint_\Omega d\omega = \oint_{\partial\Omega} i^*\omega.$$

In a 3-dimensional Riemanian manifold, If we set:

$$df = \omega^1_{\mathrm{grad}f},\quad d\omega^1_A = \omega^2_{\mathrm{curl}A},\quad d\omega^2_A = (\nabla A)\omega^3,$$

Then:

$$f(y) - f(x) = \int_l \mathrm{grad}f\cdot dl.$$

$$\int_l A\cdot dl = \oint_S \mathrm{curl}A\cdot dn.$$

$$\oint_U \nabla\cdot F dV = \oint_{\partial U} F\cdot ndS.$$

Proof: □

Hodge Star

Def. (IV.3.2.11) (Hodge Star Operator). given a volume-form ω on a vector space, the Hodge star operator $*-$ is an operator from $\bigwedge^k V \to \bigwedge^{n-k} V$ such that:

$$\alpha\wedge(*\beta) = \langle\alpha,\beta\rangle\omega.$$

On a closed oriented Riemannian manifold, given a volume form ω, the star operator satisfies:

$$(\alpha,\beta) = \int_M \langle\alpha,\beta\rangle\omega = \int_M \alpha\wedge *\beta.$$

And $** = (-1)^{p(n-p)}$ on $\Omega^p M$.

Def. (IV.3.2.12). For a operator d on $\Omega^* M$, we define the adjoint $d^* = (-1)^{n(p+1)+1} * d*$ on Ω^p, which satisfies the adjoint property by calculation:

$$(d^*\alpha,\beta) = (\alpha, d\beta).$$

The laplacian $\Delta = d^*d + dd^*$. It can be verified that Δ commutes with $*$ and d.

3 Differential Topology

Transversality

Def. (IV.3.3.1) (Transversality).

Prop. (IV.3.3.2) (Transversal Stable under Pertabations). The property of transversal for a map $f : X \to Y$ for a compact manifold X to a fixed submanifold Z of Y is stable under smooth deformation.

Proof: We can assume the submanifold is defined by a slice, so the transversality is in fact equivalent to locally submersion in the vertical direction. Thus it is clearly stable under deformation. □

Prop. (IV.3.3.3). If a smooth map $f : X \to Y$ is transversal to a submanifold $Z \subset Y$ of codimension r, then the preimage $f^{-1}(Z)$ is a submanifold of X of codimension r.

Proof: Cf.[Pollack P28]. □

Cor. (IV.3.3.4). If two submanifolds are transversal at every point is again a submanifold, and the codimension is the sum of them.

Prop. (IV.3.3.5) (Parametric Transversality Theorem). Suppose N and M are smooth manifolds, $X \subset M$ is an embedded submanifold, and F_s is a smooth family of maps from N to M. If the map $F : N \times S \to M$ is transverse to X, then for almost every s, the map $F_s : N \to M$ is transverse to X.

Proof: Cf.[Smooth Manifold Lee T6.35]. □

Prop. (IV.3.3.6) (Transversality Homotopy Theorem). Suppose N and M are smooth manifolds and $X \subset M$ is an embedded submanifold. Every smooth map $f : N \to M$ is homotopic to a smooth map $g : N \to M$ that is transverse to X.

Proof: Embed M into an R^k and take a tubular neighbourhood, then we can construct a $N \times D^k$ transversal to M. Cf.[Smooth Manifold Lee T6.36]. □

Prop. (IV.3.3.7) (Transversality Extension Theorem). Let X is a manifold with boundary and $C \subset X$ is a closed subscheme, Z is a closed submanifold of Y. If $f : X \to Y$ is a smooth map that is transversal to Z on C and transversal to Z on $C \cap \partial X$, then there is a map $g : X \to Y$ that is homotopic to f, and $g = f$ on a nbhd of C.

Proof: Cf.[Pollack P72]. □

Intersection Numbers Modulo 2

Prop. (IV.3.3.8) (Intersection Number Modulo 2). Let X be a compact manifold, and Z is an closed submanifold of Y, where $\dim X + \dim Z = \dim Y$, then for any smooth map $f : X \to Y$ transversal to Z, define $I_2(f, Z)$ as the number of points of $f^{-1}(Z)$ modulo 2.

Prop. (IV.3.3.9) (Boundary Theorem). If X is the boundary of a smooth manifold W, Z is a closed subscheme of Y that $\dim X + \dim Z = \dim Y$. If $g : X \to Y$ is a map of smooth manifolds that can be extended to $W \to Y$, then $I_2(g, Z) = 0$.

Proof: Use extension theorem(IV.3.3.7), (IV.3.3.3) and(IV.3.1.19). □

Cor. (IV.3.3.10). Let X be a compact manifold, and Z is an closed submanifold of Y, where $\dim X + \dim Z = \dim Y$, then for any smooth maps $f, g : X \to Y$ transversal to Z. If f is homotopic to g, then $I_2(f, Z) = I_2(g, Z)$.

Proof: Immediate from boundary theorem(IV.3.3.9). □

Prop. (IV.3.3.11) (Mod 2 Degree of Maps). If X, Y are manifolds of the same dimension and X is compact, then $I_2(f, \{y\})$ is the same for each $y \in Y$, called the **mod 2 degree of** f. This number is 0 for the boundary of a map, by(IV.3.3.9).

Proof: Cf.[Pollack P80]. □

Orientable Intersection Numbers

Prop. (IV.3.3.12) (Preimage Orientation). Let X, Y is orientable and Z is an orientable closed subscheme in Y. If $f : X \to Y$ is transversal to Z, then the orientation of Z, Y, Z defines canonically an orientation on $f^{-1}(Z)$, called the **preimage orientation** of $f^{-1}(Z)$.

Def. (IV.3.3.13) (Intersection Number). If X is an orientable smooth manifold, Z is an orientable closed subscheme of an orientable manifold Y that $\dim X + \dim Z = \dim Y$. If $g : X \to Y$ is a map of smooth manifolds that is transversal to Z, then we defined the $I(g, Z)$ to be the sum of the orientations of $f^{-1}(Z)$.

Lemma (IV.3.3.14) (Boundary Theorem). If X is the boundary of an orientable compact smooth manifold W, Z is an orientable closed subscheme of an orientable manifold Y that $\dim X + \dim Z = \dim Y$. If $g : X \to Y$ is a map of smooth manifolds that is transversal to Z and can be extended to $W \to Y$, then $I(g, Z) = 0$.

Proof: The same as the proof of(IV.3.3.9). □

Prop. (IV.3.3.15). Homotopic transversal maps always have the same intersection number w.r.t Z.

Prop. (IV.3.3.16) (Degree of Maps). If X, Y are orientable manifolds of the same dimension and X is compact, then $I_2(f, \{y\})$ is the same for each $y \in Y$, called the **degree of** f. This number is 0 for a boundary map, by(IV.3.3.14).

Proof: The same as that of(IV.3.3.11). □

Cor. (IV.3.3.17). The only finite group G that can act freely on S^{2n} is $\mathbb{Z}/2\mathbb{Z}$ or 1.

Proof: Consider the degree map, then it is a homomorphism from G to $\mathbb{Z}$, thus the image is just ± 1. Now it is by Lefschetz fixed point theorem that $\deg(g) = -1$ for $g \neq 1$, thus it is injective to $\pm$. □

Prop. (IV.3.3.18) (General Intersection Number). The intersection number can be generalized to the case that $g : Z \to Y$ is an arbitrary map of the complementary dimension, and we can define $I(f, g)$. Then:

- f, g are transversal iff $f \times g$ are transversal to Δ_Y.

- $I(f, g) = (-1)^{\dim Z}(f \times g, \Delta_Y)$.

Proof: This a simple local tangent vector calculation. □

Cor. (IV.3.3.19). If $f' \sim f, g' \sim g$, then $I(f, g) = I(f', g')$ if they are definable. This is because $f \times g \sim f' \times g'$.

Prop. (IV.3.3.20). $I(f, g) = (-1)^{\dim X \cdot \dim Z} I(g, f)$. This is obvious from the definition.

Cor. (IV.3.3.21). This shows that the intersection number of a odd-dimensional orientable submanifold of an orientable submanifold with itself is 0. If this fails, then the ambient space is not orientable, for example the Möbius band with the central circle.

Prop. (IV.3.3.22). The Euler character of an orientable compact manifold Y equals the intersection of the diagonals $I(\Delta, \Delta)$.

Proof: For this, we use the Poincare-Hopf theorem(IV.3.3.24). It is clear that on a triangulation, we can place a source on the center of each face/edge/..., thus producing a smooth vector fields, thus it is clear the sum of their indices equals both the combinatorial Euler character and the defined character. □

Cor. (IV.3.3.23). The Euler character of an odd dimensional compact manifold Y is 0.

Prop. (IV.3.3.24) (Poincare-Hopf Index theorem). In a compact manifold M, any vector field V with isolated zeros has sum of its index equal to $\chi(M)$. Where the index of a singularity is the mapping degree of V on a surrounding sphere.

Proof: Should use Euler character defined in(IV.3.3.22), Cf.[Pollack]. □

4 Flow

Prop. (IV.3.4.1) (Isotopy Extension Theorem). Let M be a manifold and A be a compact subset. Then an isotopy $F : A \times I \to M$ can be extended to an diffeotopy of M.

Proof: Consider $F(A \times I) \subset M \times I$ is a compact set, and $TM \times I \to M \times I$ is a vector bundle. The time lines generate a section $F(A \times I) \to TM \times I$, so (IV.6.1.2) guarantees an extension $M \times I \to TM \times I$, and because manifolds are locally compact, this section can be chosen to be compactly supported, then the flow it generates is a diffeotopy. □

5 Spin Structure

Prop. (IV.3.5.1) (Spin Structure Obstruction). For a oriented real bundle, its transformation map can be chosen to be in $SO(n)$, and constitute a Cech Cohomology $H^1(X, SO(n))$, and by exact sequence of

$$0 \to \pm 1 \to \mathrm{Spin}(n) \to SO(n),$$

this can be lifted to a $H^1(X, \mathrm{Spin}(n))$ iff its image w in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ is 0. and then its inverse image will be parametrized by $H^1(X, \mathbb{Z}/2\mathbb{Z})$ (By the non-commutative spectral sequence of Čech).

We have $w = w_2$, the Whitney class, (Just need to reduce to sk_2X and in this case, check they both equivalent to the bundle can be lifted). Cf.[XieYi 几何学专题]. Or we can use the Postnikov system of $BO(n)$(IV.4.6.2).

Proof: First prove that if $E \oplus R^n$ is spin, then E is spin, and then pull $H^2(X, \mathbb{Z}/2\mathbb{Z})$ into $H^2(\mathrm{sk}_2(X), \mathbb{Z}/2\mathbb{Z})$, this in a injection, and the homology is natural, so we only have to prove this for $\mathrm{sk}_2(X)$. But E on $\mathrm{sk}_2(X)$ can decompose into a E' of dimension on more than 2, and for this, we see E is Spin iff it is the square of another bundle, so w and w_2 are the same. □

Prop. (IV.3.5.2). For a Spin bundle E, the Spin-principal bundle with the Spinor representation**??** will generate a bundle S called the **Spinor bundle**. And the Ad action of $\mathrm{Spin}(n)$ on $Cl_{n,0}$ will generate a **Clifford bundle** $Cl(E)$. The $\mathrm{Spin}(n)$ actions are compatible, so the Clifford bundle can act on the spinor bundle. bundle. The act of the chirality operator on the Spinor bundle will generate two half spinor bundles $S^{\pm}$. Then TM will maps $S^{\pm} \to S^{\mp}$ for n even,(because of anti-commutative with Γ).

Prop. (IV.3.5.3) (Spin^c-structure). The group Spin^c is the covering space of $SO(n) \times S^1$ (n>2) that corresponds to the group of elements mod 0 mod 2 in $\mathbb{Z}_2 \times \mathbb{Z}$, i.e. $\mathrm{Spin}(n) \times S^1/\{\pm 1\}$.

For example, $\mathrm{Spin}^c(4) = \{(A_1, A_2) \in U(2) \times U(2) | \det A_1 = \det A_2\}$, and $\mathrm{Spin}^C(3) = U(2)$.

Then a $SO(n)$ bundle can be lift to be a Spin^c-bundle if the line bundle determined by S^1 is determine the same w_2 as it, i.e. $w_2 = c_1(L)$ mod 2, This is equivalent to w_2 is in the image of $H^2(X, \mathbb{Z})$, and this is equivalent to the Bockstein image of it is zero.

Use a variant of Wu's formula: $w_2(TM)[\alpha] = \alpha \cdot \alpha (\mathrm{mod}\ 2)$ for M orientable of dimension 4, we have any orientable manifold of dimension 4 has a Spin^c-structure. Cf.[XieYi 几何学专题 Homework3].

There is a connection on the Clifford bundle and on the Spinor bundle induced by the Levi-Civita connection of M(IV.2.3.2). This is compatible with the Clifford action. and it is also metric because the connection 1-form is in $\mathfrak{so}(n)$ because the action of $SO(n)$ preserves metric.

6 Young-Mills Euqation & Seiberg-Witten Equation

Def. (IV.3.6.1) (Yong-Mills). The Young-Mills functional on connections A on a bundle E on a compact oriented space:

$$YM(A)^2 = ||F_A||^2 = -\int_X \mathrm{tr}(F_A \wedge *F_A)$$

it is a critical point when $d_A \star F_A = 0$ and $d_A F_A = 0$.

Prop. (IV.3.6.2) (2-dim Case). $\star F \in \Omega^0(su(E))$ is parallel thus its characteristic spaces is orthogonal and a stable under parallel transport. So an irreducible YM $SU(2)$-connection must by flat, thus correspond to irreducible $SU(2)$ representation of $\pi_1(X)$.

Prop. (IV.3.6.3) (4-dim Case). $** = (-1)^{2*2} = \mathrm{id}$ on $\Omega^2(E)$ on E a $SU(n)$-bundle, so $\Omega^2(E) = \Omega^+ \oplus \Omega^-$. We have

$$||F_A^+||^2 + ||F_A^-||^2 \geq ||F_A^-||^2 - ||F_A^+||^2 = \int_X \mathrm{tr}(F_A \wedge F_A) = 8\pi^2 c_2(E)$$

Cf.[谢毅 Lecture5]. So it attains minimum at the connection that $*F_A = \pm F_A$ and $d_A F_A = 0$.((Anti)self-dual((anti)instanton)) depending on the sign of $c_2(E)$.

Prop. (IV.3.6.4) (Anti-Instanton Connection on Complex Line Bundle). For a $U(1)$-bundle, $d_A F_A = dF_A$, so F_A is harmonic, thus $c_1(L) = [\frac{-1}{2\pi i}F_A] \in H^2(X,\mathbb{Z}) \cap \mathcal{H}^2_-(X,\mathbb{R})$, In fact, this is equivalent to the existence of a anti-self-dual connection on this bundle.

If this is the case, then we have the ASD-connections module Gauge equivalence is isomorphic to $H^1(X,\mathbb{R})/H^1(X,\mathbb{Z}) = T^{b_1(X)}$.

Proof: Because a gauge is just a $X \to S^1$, and its connected component thus equals $[X, S^1] = H^1(X,\mathbb{Z})$ (MacLane space), and its identity is just the map that is homotopic to id. and $d(gA) = dA - g^{-1}dg = dA - idu$, for $g = exp(iu)$, so $\Omega^1/\mathcal{G} = H^1(X, \mathbb{R}/H^!(X,\mathbb{Z}) = T^{b_1(X)}$. □

Lemma (IV.3.6.5) (Weizenbock Formula). On a Riemannian manifold M, the Laplace operator has the form:

$$\Delta = -\nabla^2_{e_i e_i} - \xi^i \wedge \iota(e_j) R(e_i, e_j)$$

where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$.

$$\int |\mathcal{D}_A \varphi|^2 = \int |\nabla_A \varphi|^2 + \frac{1}{4} R|\varphi|^2 + \frac{1}{2}\langle F_A^+ \varphi, \varphi\rangle.$$

If M is a spin manifold, then the Dirac operator D satisfies:

$$D^2 = -\nabla^2_{e_i e_i} + \frac{1}{4}R$$

where R is the scalar curvature form on M. If M is a $Spin^c$ manifold with a $Spin^c$ connection ∇_A, then the Dirac operator satisfies

$$D_A^2 = -\nabla^2_{A, e_i e_i} + \frac{1}{4}R + \frac{1}{2}F_A$$

Cf.[Geometric Analysis Jost P143,153].

Prop. (IV.3.6.6) (Seiberg-Witten). The Seiberg-Witten equation functional for a unitary connection A on the determinant bundle of a Spin^c structure of M and a section of $\mathcal{S}^+$ is:

$$\begin{aligned} SW(\varphi, A) &= \int \Big(|\nabla_A \varphi|^2 + |F_A^+|^2 + \frac{R}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^2\Big) Vol. \\ &= \int \Big(|\mathcal{D}_A \varphi|^2 + |F_A^+ - \frac{1}{4}\langle e_j e_k \varphi, \varphi\rangle e^j \wedge e^k|^2\Big) Vol \end{aligned}$$

So the Seiberg-Witten equation is the lowest topological possible value of the Seiberg-Witten functional. It writes:

$$\mathcal{D}_A \varphi = 0, \quad F_A^+ = \frac{1}{4}\langle e_j e_k \varphi, \varphi\rangle e^j \wedge e^k.$$

Cf.[Jost Chapter 7].

Cor. (IV.3.6.7). If a compact oriented Spin^c manifold M has nonnegative scalar curvature, then the only possible solution is $\varphi = F_A^+ = 0$. (See from the equivalence of forms of Seiberg-Witten functional.)

7 Chern-Weil Theory

Prop. (IV.3.7.1) (Chern-Weil). An **Invariant polynomial** of a the entries of $M_n(k)$ is one that is invariant under the conjugation action(I.3.2.17).

For any connection on E, the **Chern-Weil** map CW from invariant polynomial ring to $H^*(X)$: $P \mapsto [P(\Omega)]$ is a ring homomorphism independent on the connection A.

There are relations between c_i and $\mathrm{tr}(\Omega^k)$, they can be derived formally by considering diagonal elements.

Proof: To prove $P(\Omega)$ is closed, notice by(I.3.2.17), it suffice to show $\mathrm{tr}(\Omega^k)$ is closed. By(IV.2.3.7), $d\,\mathrm{tr}(\Omega^k) = \mathrm{tr}(\omega \wedge \Omega^k - \Omega^k \wedge \omega) = 0$, which is zero because Ω is of even dimension.

For the independence of connections, use(IV.4.1.16). For two connection ∇_i, $\nabla = t\nabla_0 + (1-t)\nabla_1$(you can smooth it) is a connection on the vector bundle π^*E on $M \times I$, and the section 0 and 1 induces the connection ∇_0 and ∇_1. Thus s_0^* and s_1^* are the same map, thus $CW_M(p) = s_i^* CW_{M\times I}(p)$ are all the same map. □

Cor. (IV.3.7.2). For a complex line bundle of degree r over a complex manifold,

$$\det(1 - \frac{1}{2\pi i}F_A) = 1 + c_1 + \ldots + c_r$$

gives out the **Chern class**, because it satisfies the axioms of Chern class (IV.6.4.1). In other words, $c_k = \mathrm{tr}((-\frac{1}{2\pi i}F_A)^k)$.

For a real line bundle of degree r,

$$\det(1 - \frac{1}{2\pi i}F_A) = 1 + p_1 + \ldots + p_{\lfloor \frac{r}{2} \rfloor}$$

gives out the **Pontryagin class**, where $p_k \in H^{4k}(X)$. (Notice the ω thus Ω can be chosen to be skew-symmetric, thus for odd k the classes $\mathrm{tr}(\Omega^k) \in H^{2k}(X)$ vanish).

For an oriented real bundle of degree $2r$, the ω and thus Ω can be chosen to be skew-symmetric and the transformation matrix in $SO(2r)$, then

$$\mathrm{Pf}(\frac{1}{2\pi}\Omega) \in H^{2r}(X)$$

is well-defined and closed and gives the **Euler class** $e(E)$ (recall $e(E)^2 = p_r(E)$). (Use $\mathrm{Pf}^2 = \det$ to get that $[\dfrac{\partial \mathrm{Pf}}{\partial \Omega_{ij}}]^t$ commutes with Ω, then calculate $d\mathrm{Pf}(\Omega) = 0$).

Proof: In fact, the construction is natural w.r.t the connection because connection can be pulled back and summed. Then the only task is the normality, which is direct calculation on $\mathbb{CP}^1$. □

Cor. (IV.3.7.3).

$$c_1(E) = c_1(\wedge^{\dim E} E).$$

Direct from the formula.

Cor. (IV.3.7.4) (Whitney Product Formula).

$$c(E \oplus F) = c(E)c(F), \quad p(E \oplus F) = p(E)p(F)$$

Directly form the product connection on $E \oplus F$.

Prop. (IV.3.7.5) (Chern Character). The Chern character

$$ch(E) = [\mathrm{tr}\exp(\frac{i}{2\pi}F_A)]$$

satisfies $ch(E \oplus F) = ch(E) + ch(F)$ and $ch(E \otimes F) = ch(E)ch(F)$ by simple calculation. So it defines a ring homomorphism from $K(X)$ to $H^*(X)$.

Prop. (IV.3.7.6) (Chern-Gauss-Bonnet). For a $2n$-dimensional orientable manifold M,

$$\int_M e(TM) = \chi(M).$$

Prop. (IV.3.7.7). For a vector bundle and a flat connection d_A on a manifold, i.e. $d_A^2 = 0$, we have a deRham like cohomology, and there is a sheaf of flat sections.

$$H^*(X, A) = H^*(X, E).$$

8 Index Theorems(Atiyah-Singer)

References are [Heat equation and the Index Theorem Atiyah] and [Index Theorem].

Prop. (IV.3.8.1) (Gilkey). For a natural transformation ω from the functor $p : M \to$ the Riemannian structure on M to the functor $q : M \to k$-forms on M, if it is homogenous of weight 0 w.r.t to metric g(i.e. $\omega(\lambda^2 g) = \omega(g)$) and in local coordinates it has the coefficients of $\omega(g)$ generated by g_{ij} and $\det g^{-1}$ and their derivatives , then is is a polynomial of Pontryagin classes of the given dimension. (not only up to homology).

Proof: Cf.[Heat equation and the Index Theorem Atiyah P284]. □

Prop. (IV.3.8.2) (Gilkey Generalized). For a natural transformation ω from the functor $p : M \to$ Riemannian structures on M with a Hermitian bundle E with a Hermitian connection and the functor $q : M \to k$-forms on M, if it is homogenous of weight $(0, 0)$ w.r.t to metric g, h and the Hermitian structure(i.e. $\omega(\lambda^2 g, \mu^2 \xi) = \omega(g, \xi)$) and in local coordinates it has the coefficients of $\omega(g, \xi)$ generated by $g_{ij}, h_{ij}, \det h^{-1}, \det g^{-1}$ and Γ_k^{ij}(the connection form) and their derivatives, then is is a polynomial of Pontryagin classes and Chern classes of E of the given dimension. (not only up to homology).

Proof: Cf.[Heat equation and the Index Theorem Atiyah P290]. □

Cor. (IV.3.8.3). For a natural transformation ω from the functor $p : M \to$ Hermitian bundle E on M with a Hermitian connection and the functor $q : M \to k$-forms on M, if it is homogenous of weight 0 w.r.t to metric h and the Hermitian structure(i.e. $\omega(\mu^2 \xi) = \omega(\xi)$) and in local coordinates it has the form $\omega(g, \xi)$ generated by $h_{ij}, \det h^{-1}$ and Γ_k^{ij}(the connection form) and their derivatives, then is is a polynomial of Chern classes of E of the given dimension. Because when composed with the forgetful functor, it gives a transformation as above. And it is obviously independent of g.

Prop. (IV.3.8.4) (Hodge). For any differential operator A from a vector bundle E to a vector bundle F, we form two operators AA^* and A^*A, then they are both self adjoint elliptic operators, let these corresponding eigenspace be $\Gamma_\lambda(E)$ and $\Gamma_\lambda(F)$, then A and A^* define an isomorphism between $\Gamma_\lambda(E)$ and $\Gamma_\lambda(F)$.

Proof: □

Prop. (IV.3.8.5) (Hirzebruch Signature Formula). On a $4n$-dimensional orientable manifold M, the Poincare duality defines a bilinear pairing $H^{2n}(M)\times H^{2n}(M)\to\mathbb{R}$, its signature $\sigma(M)$ is given by:

$$\sigma(M)=\int_M L_n(p_1,\ldots,p_n).$$

Where L_n is the degree n part of the Taylor expansion of $\prod_{i=1}^r\dfrac{\sqrt{x_i}}{\tanh\sqrt{x_i}}$ in terms of the symmetric polynomial.

Proof: We consider the operator $\tau:\alpha\mapsto i^{l+p(p-1)}*\alpha$, $\tau^2=1$, thus Γ^* is decomposed into two eigenspaces of τ. We define the **signature operator** A as the restriction of $\Delta=d-\tau d\tau$ to Γ_+. Δ anti commutes with τ thus maps Ω_+ to Ω_-, then we have $\operatorname{Ker}A=\operatorname{Ker}\Delta\cap\Omega_+$, which is the positive harmonic forms H_+. So

$$\mathrm{Ind}A=\dim H_+-\dim H_-.$$

And we notice the positive and negative harmonic forms neutralize each other unless on the $2n$-forms, so only need to consider them. In fact, if we consider $4n+2$ manifolds, then τ is pure imaginary and the conjugation neutralize even the $2n+1$ forms, so there are no signature.

Now the inner product $\alpha\to\int\alpha\wedge*\alpha$ is positive definite for a real form α, so this index of A is just the signature of the intersection form defined by cup product. □

Cor. (IV.3.8.6). For a $4n$-dimensional M which is a boundary of a manifold, its signature is 0.

Proof: By Stokes theorem, if M is a boundary of a manifold, then all its Pontryagin numbers, i.e. $\int_M\prod p_i^{n_i},\sum n_i=n$, vanish. □

Prop. (IV.3.8.7) (Generalized Hirzebruch Signature Formula). Let M be a $2l$ dimensional smooth manifold and E be a Hermitian bundle over M, then The index of the generalized signature operator is giving by

$$\mathrm{Ind}A_\eta=2^l\cdot ch(E)L(p_1,\ldots,p_l).$$

where $L(M)(p_i)=\prod\dfrac{x_i/2}{\tanh x_i/2}$.

Prop. (IV.3.8.8) (Hirzebruch-Riemann-Roch). For a n-dimensional complex line bundle L over a compact Kähler manifold M,

$$\chi(M,L)=\int_M[\mathrm{ch}(E)\mathrm{td}(T^{1,0}M)]_n.$$

Where $\chi(M,L)=\sum_{q=0}^n(-1)^q\dim H^q(M,E)$, ch is the Chern character(IV.3.7.5) and $\mathrm{td}(T^{1,0}M)$ is the Todd polynomial, i.e. Taylor expansion of $\prod_{i=1}^r\dfrac{t_i}{1-e^{-t_i}}$ in terms of the symmetric polynomial, applied to $c_i(T^{1,0}M)$.

Cor. (IV.3.8.9) (Riemann-Roch). For a n-dimensional complex vector bundle E over a Riemann Surface M, let $\deg E=\int_M c_1(E)$, then

$$\chi(M,E)=H^0(M,E)-\dim H^1(M,E)=\deg E+\mathrm{rk}(E)(1-g).$$

Cf.[Index Theorem P115].

Hodge Theory

Prop. (IV.3.8.10) (Hodge). By(V.8.8.12), if we investigate the Laplace operator Δ_d on a compact orientable Riemannian manifold, we get that

$$\Omega^i = \mathcal{H}^i \oplus \operatorname{Im}\Delta_d = \mathcal{H}^i \oplus \operatorname{Im} d \oplus \operatorname{Im} d^*.$$

Thus H^i can be uniquely represented by elements of $\mathcal{H}^i$.

Proof: It suffice to prove Δ_d is self-adjoint elliptic.

$\operatorname{Im}\Delta_d \subset \operatorname{Im} d \oplus \operatorname{Im} d^*$, and the result follows if we show $\mathcal{H}^i, \operatorname{Im} d, \operatorname{Im} d^*$ are orthogonal. In fact, let ω be harmonic, then $(\omega, d^*\xi) = (d\omega, \xi) = 0$, $(\omega, d\eta) = (d^*\omega, \eta) = 0$, $(d\eta, d^*\xi) = (dd\eta, \xi) = 0$. □

Cor. (IV.3.8.11) (Poincare Duality for deRham Cohomology). If M is a n-dimensional oriented Riemannian manifold, then

$$H^p_{dR}(M) \cong H^{n-p}_{dR}(M)$$

Induced by $*$, because $** = \pm 1$ and $*$ commutes with Δ_d(IV.3.2.12), so it induce an isomorphism $\mathcal{H}^p \cong \mathcal{H}^{n-p}$.

Moreover, $*$ in fact induces a perfect pairing:

$$H^k_{dR}(M) \times H^{n-k}(M) \to \mathbb{R}$$

induced by the map

$$* : \mathcal{H}^k(M) \times \mathcal{H}^{n-k}(M) \to \mathbb{R} : (\alpha, \beta) \mapsto \int_M \alpha \wedge *\beta$$

As $\int_M \alpha \wedge *\alpha = ||\alpha||^2 \neq 0$.

Prop. (IV.3.8.12). On a compact complex manifold, the formal adjoint of $\overline{\partial}$ is $*\partial*$. (By direct calculation). Also $d^* = (-1)^{n(p+1)+1} * d* = - * d*$.

Prop. (IV.3.8.13) (Hodge). Given a compact Hermitian complex manifold (X, J, g) and a holomorphic line bundle E over it, there is a Hermitian metric on $A^{p,q}E$, and an operator $\overline{\partial}$ on it. Then $\overline{\partial}$ has a formal adjoint $\overline{\partial}^*$, and $\Delta_{\overline{\partial}_E}$ can be defined. Let $\mathcal{H}^{p,q}_E$ be the kernel of $\Delta_{\overline{\partial}}$ on $A^{p,q}E$, called the E**-valued** (p,q)**-forms**, then there is a orthonormal decomposition

$$A^{p,q}E = \mathcal{H}^{p,q}_E \oplus \operatorname{Im}\Delta_{\overline{\partial}_E} = \mathcal{H}^{p,q}_E \oplus \operatorname{Im}\overline{\partial}_E \oplus \operatorname{Im}\overline{\partial}^*_E$$

And thus $\mathcal{H}^{p,q}(X, E) \cong H^{p,q}_{\overline{\partial}}(X, E)$.

Proof: It suffice to prove $\Delta_{\overline{\partial}_E}$ is self-adjoint elliptic. The rest is verbatim as the proof of(IV.3.8.10). □

Cor. (IV.3.8.14) (Hodge). In case $E = \mathcal{O}_X$, $\mathcal{H}^{p,q}(X) \cong H^{p,q}_{\overline{\partial}}(X)$.

Cor. (IV.3.8.15) (Kodaira-Serre Duality). For a Hermitian line bundle over a compact Hermitian complex manifold X, from Hodge theorem and(IV.8.5.2), we get

$$H^p(X, \Omega^q(E)) \cong H^{n-p}(X, \Omega^{n-q}(E^*))$$

induced by $\overline{*}_E$ and $\overline{*}_{E^*}$.

9 Knots and Links

Prop. (IV.3.9.1) (Linking Number). For two knots A, B in $\mathbb{R}^n$, we can choose a $D \cong D^2$ with boundary A, then define their linking number as the intersection number of D with B.

This can be extended to higher dimensions.

IV.4 Algebraic Topology

1 Homology and Cohomology

Def. (IV.4.1.1). The **singular cohomology** of a topological space with coefficients R is the cohomology groups of the Moore complex of $R[\mathrm{Sing}X]$(VI.1.1.4).

Prop. (IV.4.1.2) (Homotopy Axiom for Singular Cohomology). For two homotopic map between two topological space, they induce the same map on singular (co)homology.

Proof: For singular homology, the combinatorial 'pillariazation' can be constructed that $f - g = k^{n-1} \circ d + d \circ k^n$. □

Prop. (IV.4.1.3). The cellular (co)homology coincides with the singular (co)homology for CW-complex.

Prop. (IV.4.1.4) (Morse Inequality). for any field F,

$$\sum^k (-1)^i \dim H_i(X, F) \leq \sum^k (-1)^i c_i,$$

where c_i is the number of I-dimensional cells. (Use the dimension counting of the long exact sequence).

Prop. (IV.4.1.5) (Universal Coefficient Theorem). See (I.8.3.12).

Cor. (IV.4.1.6). A map between topological spaces that induce isomorphism on arbitrary homology group induce isomorphisms on cohomology groups.

Prop. (IV.4.1.7) (Poincare Duality). For X a closed manifold, if X is oriented or $\mathrm{char} k = 2$, then there is an isomorphism

$$H_i(X, k) \cong H^{n-i}(X, k)$$

which follows immediately from(III.6.4.15) and(III.5.5.12). (Should also attain the compact cohomology case if know the relation of compact sheaf cohomology better).

Cor. (IV.4.1.8). If X is a compact manifold of odd dimension, then $\chi(X) = 0$, using mod 2 Euler characteristic.

Cor. (IV.4.1.9).

$$H^*(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2[X]/X^n, \quad H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[X]/X^n$$

Proof: Use induction and Poincare duality to find that $\alpha * \alpha^{n-1} = \alpha^n \neq 0$. □

Prop. (IV.4.1.10) (Brouwer Fixed Point Theorem). If f is a continuous map from D^n to D^n, then f has a fixed pt in D_n.

Proof: If f does not has a fixed pt, consider the intersection of the ray $f(x)x$ with S^{n-1}, then it depends continuously on x, and this defines a function from D^n to S_n that is identity on S^n, but then this will induce a map $H^*(S^{n-1}) \to H^*(D^n) \to H^(S^{n-1})$ which is impossible. □

Prop. (IV.4.1.11) (Alexander Duality).

Prop. (IV.4.1.12) (Thom isomorphism). Cf.[姜伯驹同调论].

Prop. (IV.4.1.13) (Gysin Sequence). Cf.[姜伯驹同调论].

Prop. (IV.4.1.14) (Lefschetz Fixed Point Theorem).

deRham Cohomology

Prop. (IV.4.1.15) (De Rham). For a smooth manifold and an Abelian group G,

$$H^*_{dR}(X,G) \cong H^*(X,G)$$

Where the right is constant sheaf cohomology.(III.5.5.11).

Prop. (IV.4.1.16) (Homotopy Axiom for deRham Cohomology). For two homotopic map between two smooth manifold, they induce the same map on deRham Cohomology.

Proof: We only have to prove the case of $M \times \mathbb{R} \to M$, where any constant section map induces an isomorphism $H^*_{dR}(M \times I) \cong H^*_{dR}(M)$. Because any homotopy is a morphism $M \times I \to N$ where f and g are the sections 0 and 1.

For the zero section, we define $K : a + bdt \mapsto \int_0^t b$. This is the desired homotopy, Cf.[Differential Forms in Algebraic Topology Bott Tu]. □

Cohomology of Fiber Bundles

Def. (IV.4.1.17). A **Serre fibration** is the right lifting class of $D^n \to D^n \times I$ for every n. This is equivalent to: for any homotopy of ∂D^n and a image D^n, there is a homotopy of D^n.

Prop. (IV.4.1.18) (Leray-Hirsch). For a fiber bundle $F \to E \to B$ and a ring R s.t. $H^n(F,R)$ is f.g free for all n, and there exist classes c_j of $H^*(E)$ that constitute a basis for each fiber F, then

$$H^*(B,R) \otimes H^*(F,R) \to H^*(E,R)$$

is an isomorphism of $H^*(B,R)$-modules.

Cor. (IV.4.1.19).
- $H^*(U(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_1, x_3, \ldots, x_{2n-1}]$.
- $H^*(SU(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3, \ldots, x_{2n-1}]$.
- $H^*(Sp(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3, x_7, \ldots, x_{4n-1}]$.

Prop. (IV.4.1.20). $H^*(G_n(\mathbb{K}^\infty);\mathbb{Z})$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is generated by the symmetric polynomials, where for $\mathbb{R}$ the coefficient is $\mathbb{Z}_2$.

Proof: Use the flag variety and first calculate for ∞. Then use Poincare duality to show it is mapped onto the symmetric polynomials. Cf.[Hatcher P435]. □

Prop. (IV.4.1.21) (Leray-Serre). For a Serre fibration, especially fiber bundle, $F \to E \to B$, that B is simply connected, then there is a spectral sequence

$$E_2^{pq} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E) \quad E_2^{pq} = H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)$$

Cor. (IV.4.1.22) (Wang Sequence). When $B = S^n$, there is a long exact sequence:

$$\cdots \to H_q(F) \to H_q(E) \to H_{q-n}(F) \to H_{q-1}(F) \to H_{q-1}(E) \to \cdots$$

Cor. (IV.4.1.23) (Gysin Sequence). When $F = S^n$, there is a long exact sequence:

$$\cdots \to H_{p-n}(B) \to H_p(E) \to H_p(B) \to H_{p-n-1}(B) \to H_{p-1}(E) \to \cdots$$

Cup Product and Cohomology Operators

Prop. (IV.4.1.24). The cup product will restrict to a relative version:

$$H^*(X,A)\times H^*(X,B)\to H^*(X,A\cup B),$$

This implies that if X is a union of n contractible open set, then the cup product of n-elements vanish. In particular, the cup product in a suspension vanishes.

Prop. (IV.4.1.25) (Steenrod Powers). The total Steenrod squares Sq is a map from $H^n(X,\mathbb{Z}_2)\to H^{n+*}(X,\mathbb{Z}_2)$ that:

- it is natural and stable under suspension.
- it is additive.
- $\mathrm{Sq}(\alpha\cup\beta)=\mathrm{Sq}(\alpha)\cup\mathrm{Sq}(\beta)$.
- $Sq^i(\alpha)=\alpha^2$ if $i=|\alpha|$, and 0 if $i>|\alpha|$.

The total Steenrod Powers P is a similar map from $H^n(X,\mathbb{Z}_p)\to H^{n+*}(X,\mathbb{Z}_p)$ that $P^i(\alpha)=\alpha^p$ if $2i=|\alpha|$ and 0 if $2i>\alpha$.

The algebra of powers is generated respectively by elements Sq^{2^k}, and for p it is generated by β and the elements P^{p^k}. (Because of Adem relations) Cf.[Hatcher P497].

2 Fundamental Groups

Prop. (IV.4.2.1). The fundamental group of a topological group is abelian.

Proof: This is because π_1 preserves products, so takes group objects to group objects. And the group objects in the category of groups is the abelian groups (I.8.1.37) □

Prop. (IV.4.2.2) (Van Kampen). If X is a union of path-connected subsets A_α all containing x_0 that $A_\alpha\cap A_\beta$ and $A_\alpha\cap A_\beta\cap A_\gamma$ are all path-connected, then $*\pi_1(A_\alpha)/\sim$ where $\sim$ is generated by $i_*(\pi_1(A_\alpha\cap A_\beta))\in\pi_1(A_\alpha)\sim i_*(\pi_1(A_\alpha\cap A_\beta))\in A_\beta$ for every α,β, Cf.[Hatcher P52].

3 Applications

Prop. (IV.4.3.1). For a compact connected manifold M with boundary, there doesn't exist a retraction of M onto ∂M.

Proof: We may assume ∂M is connected, otherwise clearly there is no retraction. Let M be of dimension n, it suffices to show that $H_{n-1}(\partial M,\mathbb{Z}_2)\to H_{n-1}(M,\mathbb{Z}_2)$ is 0. So it suffices to show that $H_n(M,\partial M,\mathbb{Z}_2)\to H_{n-1}(\partial M,\mathbb{Z}_2)$ is surjective. For this, use Lefschetz Duality, both these two homology group is isomorphic to $\mathbb{Z}_2$. But $H_n(M,\mathbb{Z}_2)=0$, thus it is an isomorphism. □

4 CW Complex

Prop. (IV.4.4.1). If (X,A) is a CW pair, then $X\times\{0\}\cup A\times I$ is a deformation retract of $X\times I$, thus (X,A) has the **homotopy extension property** because we can perform infinite induction on dimension.

Prop. (IV.4.4.2). The loop space ΩX for X a CW complex has CW complex type. In particular, if it has noly finitely many cells for a given dimension, then so does ΩX. Milnor proved this.

Prop. (IV.4.4.3). The homotopy group defines a long exact sequence for triples (X, A, B), in particular for $B = \mathrm{pt}$.

Prop. (IV.4.4.4) (Compression Theorem). If (X, A) is a CW pair that (Y, B) be a pair that $\pi_n(Y, B, y_0)$=0, for any n, then every map (X, A) to $(Y < B)$ is homotopic rel$A$ to a map $X \to B$. (Use extension property to extend one dimension a time). This shows that the homotopy doesn't depend on higher dimensions, (but might on lower one).

Cor. (IV.4.4.5) (Whitehead Combinatorial Homotopy I). If M and K is dominated by CW complexes, then any weak homotopy equivalence is an homotopy equivalence. If the map is an inclusion, then it is a deformation retract. In particular, if M is manifold, then it is dominated by its tubular nbhd, so this theorem is applied.

Proof: For inclusion, use compression, and in general use mapping cylinder and cellular approximation. □

Cor. (IV.4.4.6). If $\pi_n(X) = 0$ for all n and a CW complex X, then X is contractible.

Def. (IV.4.4.7). A morphism is called a **weak homotopy equivalence** iff it induces isomorphism on homotopy groups on every dimension.

Prop. (IV.4.4.8). A weak homotopy equivalence induce isomorphism on all homology and cohomology. And also $[K, A] \cong [K, B]$ and $\langle K, A\rangle = \langle K, B\rangle$ for every finite CW complex K.

Proof: Pass to the mapping cylinder, the homotopy case follows easily from the compression lemma(IV.4.4.4), and the cohomology follows from universal coefficient theorem(IV.4.1.5).

We may use (reduced) mapping cylinder to assume $A \to B$ is an injection, then compression shows surjectivity, and the relative case for homotopy also show injectivity. □

Prop. (IV.4.4.9) (Cellular Approximation Theorem). Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. This makes calculation of homotopy easy. (It suffice to show a map cannot be surjective on a higher dim cell, Cf.[Hatcher P349].

Moreover, Any map of pairs of CW complexes can be deformed to a cellular map. (first deform the small complex, then deform the big by dimension.

Cor. (IV.4.4.10). The cellular approximation makes the computation of homotopy theoretically easier, but the difficulty comes from the complexity of the homotopy group of the sphere. If a CW complex has only cells of dim> n, then it's homotopy group vanishes for $i < n$. In particular, $\pi_n(S^k) = 0$ for $n < k$.

Prop. (IV.4.4.11) (CW Approximations). If A is CW, then there is a n-connected CW models (Z, A) to (X, A), i.e. $\pi_{\leq n}(Z, A) = 0$ and $Z \to X$ induce isomorphism on $\pi_{>n}$ and injection for π_n, moreover it can be constructed from A by attaching cells of dimension greater than n. Cf.[Hatcher P353].

Thus there exists a CW approximation for any space A, thus there exists a CW approximation for any pair (X, X_0), (first approximate X_0 and use the mapping cylinder to get a embedding) that is, induce isomorphism on $\pi_n X$ and $\pi_n X_0$ and on relative homotopy group.

Use long exact sequence, compression and mapping cylinder, we can prove the approximations preserve (co)homology and mapping classes.

And this approximation is unique up to homotopy equivalence rel$A$, (use relative mapping cylinder and use compression). They act like injective resolutions. Cf.[Hatcher P55].

Cor. (IV.4.4.12). For any n-connected CW pair (X,A), there exist a homotopic $(Z,A)\cong(X,A)$ rel A that $Z\backslash A$ has only cells of dimension greater than n.

Proof: Choose the n-connected approximation as above. The map induce and isomorphism on $\pi_{>n}$ by definition and on $\pi_{<n}$ because $\pi_i(A)\to\pi_i(Z)$ and $\pi_i(A)\to\pi_i(X)$ are isomorphisms. And on π_n, it is injective by definition and surjective because $\pi_n(A)\to\pi_n(Z)\to\pi_n(X)$ is isomorphism. Hence we know that the collapsed mapping cylinder at is weak-homotopy equivalent to Z, thus it deforms into Z by(IV.4.4.5), thus $Z\to X$ rel A by(IV.4.5.1). □

Cor. (IV.4.4.13) (Whitehead theorem). A f between two simply connected CW complexes that induce isomorphism on homology groups is a homotopy equivalence. (using mapping cylinder, we can assume it's an inclusion, and $\pi_1(Y,X)=0$, so the theorem shows that $\pi_n(Y,X)=0$, and use Whitehead(IV.4.4.5)).

Prop. (IV.4.4.14). A closed manifold or the interior of a manifold with boundary has a homotopy type of a CW complex of finite type.

Remark (IV.4.4.15). The use of mapping cylinder and relative mapping cylinder is important.

5 Homotopy

Prop. (IV.4.5.1). A map $X\to Y$ is a homotopy equivalence iff the mapping cylinder deformation retracts onto X.

Prop. (IV.4.5.2). The universal cover have the same homotopy group $\pi_{>1}$, by lifting property.

Prop. (IV.4.5.3) (Excision Theorem). If A,B are CW-complexes, then if $(A,A\cap B)$ are m-connected and $(B,A\cap B)$ are n-connected, then $\pi_i(A,A\cap B)\to\pi_i(A\cup B,A)$ is isomorphism for $i<m+n$, and surjective for $i=m+n$. Cf.[Hatcher P360].

Moreover, if (X,A) is r-connected and A is s-connected, then $\pi_i(X,A)\to\pi_i(X/A)$ is isomorphism for $i\leq r+s$ and surjection for $i=r+s+1$.

Cor. (IV.4.5.4). For $n>1$, $\pi_n(\bigvee_\alpha S^\alpha)$ is free Abelian with $\pi_n(S^n)$ as generators. This is because $(\prod_\alpha S^n,\bigvee_\alpha S^n)$ is $(2n-1)$-connected thus use excision, because $\pi_n\prod_\alpha S^n$ is easy to calculate.

Cor. (IV.4.5.5) (Freudenthal Theorem). For $i<2n-1$, $\pi_i(S^n)\cong\pi_{i+1}(S^{n+1})$ and . (Can also be derived considering antipodal point point of S^n by (IV.4.7.11)) and surjective for $i=2n-1$. In general, this holds when X is $(n-1)$-connected. Thus we have $\pi_n(S^n)=\mathbb{Z}$.

Proof: Use the suspension, $\pi_i(X)\cong\pi_{i+1}(C_+X,X)\to\pi_{i+1}(SX,C_-X)\cong\pi_{i+1}(SX)$.

for $n=1$ for the homotopy of sphere, we can use Hopf bundle. □

Prop. (IV.4.5.6) (Generalized Hurewicz theorem). If (X,A) is a $(n-1)$-connected pair of spaces, $n\geq 2$, then the Hurewicz map induces isomorphism

$$\pi_n(X,A)/(\pi_1(A)\text{action})\cong H_k(X,A),$$

and $H_k(X,A)=0,k<n$. And on π_{n+1}, the Hurwicz map is surjective for $n>1$.Cf.[Hatcher P390Ex23] for surjectiveness.

Prop. (IV.4.5.7) (Fiber Bundle). For a fiber bundle $S \to M \to N$, there is a long exact sequence of homotopy groups:

$$\cdots \to \pi_i(N) \to \pi_{i-1}(S) \to \pi_{i-1}(M) \to \pi_{i-1}(N) \to \cdots$$

Because it has lifting property.

Prop. (IV.4.5.8). $\pi_{i+1}(M) \cong \pi_i(\Omega(M))$, where Ω is the loop space. More generally,

$$\langle \Sigma X, K\rangle = \langle X, \Omega K\rangle.$$

Prop. (IV.4.5.9). The homotopic direct limit of a family of homotopy equivalence is a homotopy equivalence. Cf.[Morse Theory Milnor].

Prop. (IV.4.5.10). for $i \leq 2m$, $\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i-1}U(m)$, and

$$\pi_{i-1}U(m) \cong \pi_{i-1}U(m+1) \cong \cdots .$$

and for $j \neq 1$, $\pi_j U(m) \cong \pi_j SU(m)$.

Similarly, $\pi_i \Omega_1(2m) \cong \pi_{i+1}O(2m)$ for $i \leq n-4$. (IV.4.7.12),Cf[Morse Theory Milnor Prop23.4].

Cor. (IV.4.5.11) (Bott Periodicity theorem for Unitary Groups). The stable homotopy group $\pi_i U$ has period 2. $\pi_{2k+1}U \cong 0$ and $\pi_{2k}U \cong \mathbb{Z}$.

Proof: Use the last proposition and long exact sequence to show that for $1 \leq i \leq 2m$,

$$\pi_{i-1}U = \pi_{i-1}U(m) \cong \pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i+1}SU(2m) \cong \pi_{i+1}U.$$

Notice that $U(m) \to U(2m)/U(m) \to G_m(\mathbb{C}^{2m})$ □

Prop. (IV.4.5.12) (Bott Periodicity for O). For the infinite dimensional orthogonal space O, $\Omega_8(16r) \cong O(r), \Omega_4(8r) \cong Sp(2r)$. So $\Omega_8 \cong O$ and $\Omega_4 O \cong Sp$. Thus by (IV.4.5.8),

$$\pi_i(O) = \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}\ldots, \quad \pi_i(Sp) = 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}\ldots$$

respectively. (Use (IV.4.7.13)) Cf.[Morse Theory Prop24.7].

Prop. (IV.4.5.13). Homotopy Fibers.

6 Obstruction Theory & General Cohomology Theory

Towers

Prop. (IV.4.6.1) (Towers). There are Whitehead Towers and Postnikov Towers for a CW complex X.

$$\cdots \to Z_2 \to Z_1 \to Z_0 \to X \to \cdots \to X_2 \to X_1 \to X_0$$

Z_n annihilate $\pi_{\leq n}(X)$, X_n remains only $\pi_{\leq n}(X)$. The towers can be chosen to be fibrations, with fibers $K(\pi_n X, n)$ by(VI.1.3.16).

Prop. (IV.4.6.2). There is a Postnikov towers of :

$$B\mathrm{String}(n) \to B\mathrm{Spin}(n) \to BSO(n) \to BO(n)$$

with corresponding obstructions $w_1(X), w_2(X)$ and $p_1(X)/2$.

Prop. (IV.4.6.3) (Obstructions). If a connected abelian CW complex X ($\pi_1(X)$ abelian and action on higher homotopy trivial) and (W, A) satisfies $H^{n+1}(W, A; \pi_n X) =$ for all n, then $A \to X$ can extend to a map $M \to X$.

Proof: Cf.[Hatcher P417]. □

Cor. (IV.4.6.4). A map between Abelian CW complexes that induce isomorphisms on homology is a homotopy equivalence.

Proof: Notice that $\pi_1(X)$ acts trivially on $\pi_1(Y, X)$ and use Hurewicz. □

Eilenberg-Maclane Space

Prop. (IV.4.6.5) (Generalized Cohomology). If K_n is an Ω-spectrum, i.e. $K_n \cong \Omega K_{n+1}$ weak equivalence, then the functors $X \mapsto h^n(X) = \langle X, K_n \rangle$ define a reduces cohomology theory on the category of basepointed CW complexes, i.e. it satisfies the long exact sequence for $A \to X \to X/A$ and wedge axiom. Cf.[Hatcher P397].

Proof: Use(IV.4.5.8) and there is a Cofibration sequence:

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \dots$$

□

Def. (IV.4.6.6). For a discrete Abelian group G, an **Eilenberg-Maclane spaces** $K(G, n)$ is a space having only one nontrivial homotopy group $\pi_n(K(G, n)) = G$.

It can be constructed by $K(A, 0) = A$, $K(A, n+1) = B(K(A, n))$(IV.6.3.4). Note $K(G, 1)$ is constructed the same as by(VI.1.2.31).

Alternatively, it can also be constructed by first use(IV.4.5.4) and then use higher cells to kill higher homotopies.

Prop. (IV.4.6.7). The homotopy typw of a CW complex $K(G, n)$ is unique, thus $\Omega(K(G, n)) \cong K(G, n-1)$ hence $H^n(X, A) \cong [X, K(A, n)]$(IV.4.6.5) and this isomorphism is generated by a distinguished class of $H^n(K(G, n), G)$.

Proof: Cf.[Hatcher P366]. □

Prop. (IV.4.6.8). $K(\mathbb{Z}, 1) = S^1 = U(1)$, $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$, Because $S^\infty \to \mathbb{CP}^\infty$ is a contractable covering.

7 Morse Theory & Floer Homology

Morse Theory(Milnor)

Def. (IV.4.7.1) (Non-Degenerate Critical Point). For a smooth map $f : X \to \mathbb{R}$, a critical point is called a **non-degenerate critical point** iff the Hessian matrix is non-singular at x.

The notion of non-degenerate critical is independent of the coordinate chosen.

Proof: Cf.[Pollack P42]. □

Prop. (IV.4.7.2) (Non-Degenerate Critical General). Non-degenerate critical points are the general situation in the following sense: For a manifold $M \subset \mathbb{R}^n$, for any smooth function f on M, consider the functions $f_a = f + \sum a_i x_i$, then for almost all (a_i), all critical points of f_a is non-degenerate.

Proof: Cf.[Pollack P43] □

Prop. (IV.4.7.3) (Morse Lemma). In a non-degenerate critical point of f, there is a coordinate that
$$f = f(p) + x_1^2 + \cdots + x_{n-\lambda}^2 - y_1^2 - \cdots - y_\lambda^2.$$

Proof: Just extract the first order part out and reform the bilinear form one-by-one. Cf.[Milnor Morse Theory lemma 2.2]. □

Prop. (IV.4.7.4). If f is a smooth function that $f^{-1}([a,b])$ is compact and have no critical points, then M^a is a deformation retracts of M^b using $\mathrm{grad}f/|\mathrm{grad}f|^2$.

Prop. (IV.4.7.5) (Morse Main Lemma). If f is a smooth function with p a non-degenerate critical point and λ downward pointing direction. If for some $f^{-1}([c-\epsilon, c+\epsilon])$ is compact, then $M^{c+\epsilon}$ is homotopic to $M^{c-\epsilon}$ gluing a λ dimensional cell.

Proof: Cf.[Milnor Prop3.2]. □

Prop. (IV.4.7.6). For an embedded manifold and almost all point p, the distance to p is a morse function. (Use Sard theorem and degenerate $\iff$ p is a focal point.

Cor. (IV.4.7.7). smooth manifold has CW type; on a compact manifold any vector field with discrete singular points has its index sum equal to $\chi(M)$ (Hopf-Rinow), and there exists one.

Prop. (IV.4.7.8). for $\Omega(p,q)^c$ the path space of energy $< c$, the piecewise geodesic path space B (piece fixed), the energy function is smooth and B^a is compact and is the deformation contraction of $\mathrm{int}\Omega^a$ for $a < c$. E has the same critical point and same index and nullity on B and Ω^c. (Just geodesicrize any path in Ω.

So for two point not conjugate in B^a, Ω^a has a finite CW complex type and a λ-dimensional cell for every geodesic of index λ in B^a.

Prop. (IV.4.7.9) (Morse Main Theorem). If p and q are not conjugate along any geodesic, then $\Omega(p,q)$ has a countable CW complex type and has a λ-cell for every geodesic of index λ.

If M has nonnegative Ricci curvature, then M has only finite cell for every dimension.

Proof: Cf.[Milnor Morse Theory Prop17.3]. □

Cor. (IV.4.7.10). The path space homotopy type only depend on the homotopy type of M (use the two homotopy to id to get a composition of homotopy of the two path space), so one can get the information of path space of M by looking at the homotopy type of M.

Prop. (IV.4.7.11) (Minimal Geodesics). If p,q in a complete manifold M has distance $\sqrt{d}$ and the minimal geodesics form a topological manifold, and if all non-minimal geodesic has index $\geq \lambda$, then for $0 \leq i < \lambda$, $\pi_i(\Omega, \Omega^d) = 0$.

Lemma (IV.4.7.12). In $SU(2m)$, the minimal geodesic from I to $-I$ is homeomorphic to Grassmannian $G_m(\mathbb{C}^{2m})$ and and non-minimal geodesic has index $\geq 2m+2$.

Similarly, The space of minimal geodesic from I to $-I$ in $O(2m)$ is homeomorphic to the space of complex structures in $\mathbb{R}^{2m}$, and any non-minimal geodesic has index $\geq 2m-2$.

Proof: Cf.[Milnor Morse Theory Lemma23.1 Lemma24.4]. □

Lemma (IV.4.7.13). Ω_{k+1} is homotopic to the space of minimal geodesics in Ω_k from J to $-J$. (The same way, calculate the index of geodesics from J to $-J$ and use (IV.4.7.11)). Cf.[Milnor Morse Theory Prop24.5] for definition of Ω_{k+1}.

IV.5 Differential Forms in Algebraic Geometry(Bott-Tu)

This section is dedicated to the analysis of algebraic geometry, using the tool of differential forms.

Basic references are [Differential Forms in Algebraic Geometry Bott-Tu].

1 Basics

Prop. (IV.5.1.1) (Cohomological Generator of Sphere). Let $v : x \mapsto x/|x|$ be the outward-pointing vector field on $\mathbb{R}^n - \{0\}$, then the differential form

$$\iota(v)(dm) = \frac{1}{|x|}\sum x_i(-1)^{i-1}dx_1 \wedge dx_2 \wedge \ldots \widehat{dx_i} \wedge \ldots dx_n$$

restricts to a differential form on S^{n-1} that is a generator of $H^{n-1}(S^{n-1})$.

Proof: First we calculate $d(\iota(v)(dm)) = \mathrm{div}(v)dm = \frac{n-1}{|r|}dm$(IV.2.3.27). Consider using Stoke's formula:

$$\int_{\partial B_1} \iota(v)(dm) - \int_{\partial B_\varepsilon} \iota(v)(dm) = \int_{B(0,1)-B(0,\varepsilon)} \frac{n-1}{|r|}dm = 0$$

Leting $\varepsilon \to 0$, $\int_{\partial B_\varepsilon} \iota(v)(dm)$ converges to 0 as $|\iota(v)(dm)|$ is bounded and $V(B_\varepsilon)$ converges to 0. And using the polar coordinate $dm = r^{n-1}drd\omega$, the right hand side is just

$$\int_{S^{n-1}}\int_0^1 (n-1)r^{n-2}drd\omega = V(S^{n-1}).$$

□

Prop. (IV.5.1.2) (Degree Formula). If $f : X \to Y$ is an arbitrary map of two compact oriented manifolds of dimension k, then for any k-form ω on Y,

$$\int_X f^*\omega = \deg(f)\int_Y \omega.$$

Proof: deg is defined in(IV.3.3.16). Cf.[Pollack P188]**?**. □

Prop. (IV.5.1.3) (Hopf Invariant). Let $n > 1$, given a map $S^{2n-1} \to S^n$, let α be a generator of $H^n)S^n$, then $f^*\alpha = d\omega$ on S^{2n-1} for some ω. Define the **Hopf invariant** of f to be $H(f) = \int_{S^{2n-1}} \omega \wedge d\omega$, then:

- The definition of Hopf invariant is independent of ω chosen.
- For odd n, the Hopf invariant is 0.
- Homotopic maps f, g have the same Hopf invariant.

Proof: 1: If $d\omega = d\omega'$, then

$$\int_{S^{2n-1}} \omega' \wedge d\omega' - \int_{S^{2n-1}} \omega \wedge d\omega = \int_{S^{2n-1}} (\omega' - \omega) \wedge d\omega = \pm\int_{S^{2n-1}} d((\omega - \omega') \wedge \omega) = 0.$$

2: If n is odd, then ω is of even dimensional, thus $\omega \wedge d\omega = \frac{1}{2}d(\omega \wedge \omega)$, so $H(f) = 0$ by Stokes.

3: If $F : S^{2n-1} \times I \to S^n$ is a homotopy of f, g, then $F^*\alpha = d\omega$ for some ω on $S^{2n-1} \times I$. Thus consider

$$H(f) - H(g) = \int_{S^{2n-1}} \omega_1 \wedge d\omega_1 - \int_{S^{2n-1}} \omega_0 \wedge d\omega_0 = \int_{\partial(S^{2n-1}\times I)} \omega \wedge d\omega = \int_{S^{2n-1}\times I} d\omega \wedge d\omega,$$

But $d\omega \wedge d\omega = F^*(\alpha \wedge \alpha)$, and $\alpha \wedge \alpha = 0$. □

IV.6 Vector Bundle & K-Theory

1 Fundamentals

Prop. (IV.6.1.1). A vector bundle can have its transform map $\in O(n)$ (or $U(n)$) by constructing a riemannian metric on it. And for every local trivialization, we choose the metric on it compatible with the given metric, thus the transform map is $\in O(n)$ (or $U(n)$).

Prop. (IV.6.1.2) (Tietze extension general). For a Hausdorff paracompact(hence normal) space X and a paracompact subspace Y, every section on Y can be extended to a section on X. (For every point of Y, find a local trivialization and an even smaller open set. Use Tietze extension to extend locally to this nbhd, then use partition of unity to unify all).

Prop. (IV.6.1.3) (Homotopy Invariance of Vector Bundles). For a continuous family of maps from a paracompact Hausdorff space Y to a Hausdorff paracompact space X, then the pullback bundle is isomorphic.

Proof: Consider the space $Y \times I$ and the pullback bundle E, then for every t_0, consider a new bundle $\mathrm{Hom}(E, \pi_1^* E_{t_0})$, then Y has a section id, this section by the last proposition can be extended, so it spans the vector space for nearby t (because of paracompactness), thus is an isomorphism because it is a locally invertible vector bundle homomorphism. □

Prop. (IV.6.1.4) (Splitting Principle). For a vector bundle $E \to X$, there is a space $Y \to X$ that p^* is injective on $H^*(-, \mathbb{Z})$ and p^*E splits as a sum of line bundles. This proposition is useful when proving theorems about characteristic classes.

Proof: It suffice to find a Y that p^*E has a subbundle, then choose its orthogonal part, and use induction. For this, choose $Y = P(E)$, then Y has a tautological bundle, which is a subbundle of p^*E, and Y is fibered over X with fiber $\mathbb{P}^n$, and we want to use Leray-Hirsch, so check the fact $H^*(\mathbb{P}^n)$ is free and generated by the first Chern class, by(IV.6.4.1) and(III.5.4.14). And Chern class is functorial, so the powers of Chern class of f^*E will generate the cohomology ring of any stalks. □

Prop. (IV.6.1.5) (Global Sections). For a vector bundle over a compact manifold, there exists a global section transversal to the zero section, in particular, if $\dim E > M$, then it has no zero.

Proof: choose a finite trivializing cover that there closure is compact and choose a compact subcover, find finitely many sections to assure $C^N \times X \to E$ is transversal, and use parametric transversality theorem(IV.3.3.5) to prove there is a section that is transversal. □

Cor. (IV.6.1.6) (Vector Field with Isolated Zeros). There is a vector field on compact manifold of only isolated zeros. And a vector bundle over a k dimensional curve splits to components of dimension no bigger than k. Determined by its Chern class.

2 Thom isomorphism

Prop. (IV.6.2.1) (Thom Class). For a vector bundle E over base B, we can compactify its bundles to get a (D^n, S^n)-bundle, if there is a Thom class that induce a generator $H^n(D^n, S^n)$ on every fiber. Then the relative Leray-Hirsch will give that c induces an isomorphism $H^i(B, R) \to H^{i+n}(E, E', R)$. For $\mathbb{Z}_2$ coefficient there exists a Thom class, and for orientable bundle there exists a $\mathbb{Z}$-Thom class. Notice that fiber bundle over a simply connected base is orientable.

Prop. (IV.6.2.2). Similarly, for a orientable fiber bundle $S^{n-1} \to E \to B$, make it a $D^n \to E' \to B$ bundle, then E' is homotopy equivalent to B so there is a Gysin sequence

$$\to H^{i-n}(B) \xrightarrow{\smile e} H^i(B) \to H^i(E) \to H^{i-n+1}(B) \to$$

Where the Euler class e is chosen to commute with the Thom isomorphism.

3 Principal Bundles

Basic reference is [Principal Bundles and Classifying Space].

Def. (IV.6.3.1). A **principal bundle** or G-bundle is a bundle P with G-fibers that the transition function is right G-map, i.e. left multiplication by some $g_{\alpha\beta}$. a associated bundle of a representation $G \to \mathrm{End}(V)$ is the total space of $P \times V$ module the equivalence $[gg_0, v] = [g, g_0 v]$. The corresponding transition function is just the left action by $g_{\alpha\beta}$.

Prop. (IV.6.3.2) (Homogenous Space). If G is a Lie group and H is a closed subgroup, then the quotient $H\backslash G$ can be given a structure of a G-homogenous space and $G \to H\backslash G$ is a principal H-bundle.

Proof: □

Prop. (IV.6.3.3). The projection $S^{2n+1} \to \mathbb{C}P^n$ is a principal S^1-bundle.

Classifying Space

Def. (IV.6.3.4). The **classifying space** for a topological group G is a CW complex BG with a weakly contractable universal cover EG that EG is a G-fiber bundle on BG.

$\pi_{n+1}(BG) = \pi_n(G)$ by(IV.4.5.7).

Prop. (IV.6.3.5). $[X, BG] \cong G$-bundles on X. And BG is Abelian if G is Abelian. Thus the classifying space BG is unique up to homotopy equivalence because they all represent the functor from the CW homotopy category to the set of G-bundles on it.

Proof: Cf.[Principal Bundles and Classifying Space P13]. □

Cor. (IV.6.3.6). If $H \to G$ is a homomorphism of topological groups, then any H-principal bundle can be made into a G-bundle by right tensor G. Thus there is a map $BH \to BG$ by Yoneda lemma. In other words, there is a **classifying functor** θ from the category of topological space to the category of homotopy class of CW complexes.

Prop. (IV.6.3.7) (Examples).

- $B(\mathbb{Z}/2) = \mathbb{RP}^\infty$, and $B(\mathbb{Z}/n\mathbb{Z}) = S^\infty/(\mathbb{Z}/n)$.
- $BSU(2) = \mathbb{HP}^\infty$.
- $B(\mathbb{Z}^{2g})$ = torus of genus g because torus has the upper half plane as universal cover, this can be seen observing only has to satisfy the sum of inner angle is π.
- $BO(n), BU(n), BSp(n)$ are respectively the Grassmannian of n-planes in the infinite dimensional real, complex and quaternion vector spaces, because we have

$$O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty).$$

  and similarly for $\mathbb{C}$ and $\mathbb{H}$, and $V_n(\mathbb{R}^\infty)$ is contractible by linear homotopy and Schmidt orthogonalization. In particular, $B(\mathbb{Z}/2) = \mathbb{RP}^\infty$ and $BS^1 = \mathbb{CP}^\infty$.

Def. (IV.6.3.8). A subgroup of a topological group G is called **admissible** if $G \to G/H$ is a H-bundle.

Prop. (IV.6.3.9). If H is an admissible subgroup of G, then there is a homotopy fiber sequence $G/H \to BH \to BG$.

Proof: Cf.[Principal Bundles and Classifying Space P22]. □

Cor. (IV.6.3.10). There are homotopy equivalences $\Omega BK \cong K$ and $B\Omega K \cong K$.

Prop. (IV.6.3.11). If H is an admissible normal subgroup of G, then there is a homotopy fiber sequence $BH \to BG \to B(G/H)$.

Cor. (IV.6.3.12).
- there are fiber bundles $S^0 \to BSO(n) \to BO(n)$ and similarly for $\mathbb{C}$ and $\mathbb{H}$.
- there are fiber bundles $S^n \to BO(n) \to BO(n+1)$.
- there are fiber bundles $U(n)/T^n \to (\mathbb{CP}^\infty)^n \to BU(n)$, and where $U(n)/T^n$ is the variety of complete flags in $\mathbb{C}^n$.
- for a discrete group $H \subset G$, $BH \to BG$ is a covering map.
- there are fiber bundles $BSO(n) \to BO(n) \to \mathbb{RP}^\infty$ and similarly for $\mathbb{C}$ and $\mathbb{H}$.
- there are fiber bundles $\mathbb{RP}^\infty \to BSpin(n) \to BSO(n)$.

Prop. (IV.6.3.13). $H_*(BG,\mathbb{Z}) \cong H_*(G,\mathbb{Z})$ and $H^*(BG,\mathbb{Z}) \cong H^*(G,\mathbb{Z})$.

Proof: Because EG is weakly contractible, $S_*(EG)$ is a free $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$ and $S_*(EG)_G$ is identified with $S_*(BG)$. The rest is easy. □

4 Characteristic Classes

References are [Cohomology of Classifying Space Toda] and [Characteristic Classes Milnor].

Def. (IV.6.4.1). Axioms for **Chern classes** for complex bundles:
- $c(E) = 1 + c_1(E) + \ldots + c_n(E) \in H^*(X,\mathbb{Z})$, $n = \deg(E)$.
- $f^*(c(E)) = c(f^*(E))$.
- $c(E \oplus F) = c(E)c(F)$.
- On the tautological bundle over $\mathbb{CP}^1$, $c(\eta) = 1 + c_1(\eta)$ and $\int_{\mathbb{CP}^k} c_1(\eta) = -1$. There is an affine connection definition of Chern class.

Prop. (IV.6.4.2). There exists uniquely a natural transformation $c : Vect_{\mathbb{C}}(X) \to H^*(X,\mathbb{Z})$ satisfying these axioms. (For this, it suffice to calculate the cohomology ring of $BGL_n(\mathbb{C})$, Cf.[Cohomology of Classifying Space Toda].

Prop. (IV.6.4.3) (First Chern Class Map). A complex line bundle can be seen as an element of $H^1(X,\underline{\mathbb{C}}^*)$, by(III.5.2.11), by the exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathbb{C} \xrightarrow{exp(1\pi i-)} \mathbb{C}^* \to 0$$

($\mathbb{C}$ is sheaf of smooth functions from X to $\mathbb{C}$)which gives a map $H^1(X,\mathbb{C}^*) \to H^2(X,\mathbb{Z})$, called the **first Chern class map**. It is called so because it gives the first Chern class of this complex line bundle. It is also an isomorphism because $\mathbb{C}$ is fine sheaf so acyclic.

Proof: Only have to prove they are equal in $H^2(X,\mathbb{C})$. We choose a totally convex covering U_i of X by(IV.2.3.21), then it is a fine cover, so by(III.5.2.12) the Čech cohomology and sheaf cohomology equal.

Use the Chern-Weil map definition of the Chern class, a connection on a line bundle satisfies $\nabla e_\alpha = \omega_\alpha e_\alpha$, and if $e_\beta = e_\alpha g_{\alpha\beta}$, then $\omega_\beta = g_{\alpha\beta}^{-1}\omega_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} = \omega_\alpha + d(\log g_{\alpha\beta})$. So $\Omega_\alpha = d\omega_\alpha$ locally, and the first Chern class is giving by Ω_α in $H^2(X,\mathbb{C})$.

Then we need to understand the deRham isomorphism. For the exact sequence $0\to\underline{\mathbb{C}}\to\mathcal{A}^0\to\mathcal{A}^1\to\ldots$, it has a splitting: $0\to\underline{\mathbb{C}}\to\mathcal{K}^1\to 0$ and $0\to\mathcal{K}^1\to 0\to\mathcal{A}^1\to\mathcal{K}^2\to 0$, this gives

$$0\to H^1(X,\mathcal{K}^1)\xrightarrow{\delta} H^2(X,\underline{\mathbb{C}})\to 0,\quad \mathcal{A}^1(X)\to\mathcal{K}^2(X)\xrightarrow{\delta} H^1(X,\mathcal{K}^1)\to 0.$$

because $\mathcal{A}^k$ are fine sheaves. The composite of them is just the de Rham isomorphism(Here we are identifying $H^2(X,\underline{\mathbb{C}})$ to $H^2(X,\mathbb{C})$ by(III.5.5.12)). Tracking the lifting, we notice Ω is mapped to the cocycle $\{\log g_{\alpha\beta} + \log g_{\beta\gamma} - \log g_{\alpha-\beta}\}$, which is exactly the image of the first Chern class map. □

Cor. (IV.6.4.4). Complex line bundles are characterized by the first Chern class up to smooth isomorphism, because $H^1(X,\mathbb{C}^*)\to H^2(X,\mathbb{Z})$ is an isomorphism.

Def. (IV.6.4.5). Axioms for **Stiefel-Whitney classes** for real bundles:

- $w(E) = 1 + w_1(E) + \ldots + w_n(E) \in H^*(X,\mathbb{Z}/2\mathbb{Z})$, $n = \deg(E)$.
- $f^*(w(E)) = w(f^*(E))$.
- $w(E\oplus F) = w(E)w(F)$.
- On the tautological bundle over $\mathbb{RP}^1$, $w(\eta) = 1 + w_1(\eta)$ and $\int_{\mathbb{CP}^k} c_1(\eta) = -1$.

Def. (IV.6.4.6). The Pontryagin class is defined as $p_k(E) = (-1)^k c_k(E_{\mathbb{C}}) \in H^{4k}(X,\mathbb{Z})$.

Def. (IV.6.4.7). Axioms for **Euler classes** for orientable real bundles:

- if E has non-vanishing section, then $e(E) = 0$.
- $f^*(w(E)) = w(f^*(E))$.
- $w(E\oplus F) = w(E)w(F)$.
- for the opposite orientation $\overline{E}$, $e(\overline{E}) = -e(E)$.

Definition via Classifying Space

Prop. (IV.6.4.8).

$$H^*(BO(n),\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1,w_2,\ldots,w_n].$$

$$H^*(BSO(n),\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_2,\ldots,w_n].$$

Cf.[Cohomology of Classifying Space Toda P82].

Prop. (IV.6.4.9).

$$H^*(BU(n),R) = R[c_1,c_2,\ldots,c_n].$$

$$H^*(BSU(n),R) = R[c_2,\ldots,c_n].$$

Cf.[Cohomology of Classifying Space Toda P81].

Prop. (IV.6.4.10). For R of characteristic $\neq 2$,

$$H^*(BSO(2n+1), R) = R[p_1, p_2, \ldots, p_n].$$

$$H^*(BSO(2n), R) = R[p_1, p_2, \ldots, p_n, e], e^2 = p_n.$$

Cf.[Cohomology of Classifying Space Toda P81].

Prop. (IV.6.4.11). There are maps $t : SO(n) \to U(n)$, and it will induce a map of classifying spaces, and induce

$$p_k = (-1)^k Bt^*(c_{2k}).$$

There are maps $O(n) \xrightarrow{i} U(n) \xrightarrow{j} SO(2n)$, and it will induce a map of classifying spaces, and induce

$$Bi^*(c_k) = w_k^2, \quad Bj^*(w_{2k}) = c_k.$$

There are maps $k : U(n) \to SO(m)$ $m = 2n$ or $2n+1$, then for a field R of characteristic $\neq 2$,

$$Bk^*(p_k) = \sum_{i+j=k} (-1)^i c_i c_j, \quad Bk^*(e) = c_n.$$

Cf.[Cohomology of Classifying Space Toda P81].

Applications

Prop. (IV.6.4.12). Note that $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty = B(K(\mathbb{Z}, 1)) = BS^1$(IV.6.3.7) thus it is also the classifying space of $U(1)$, thus we have $H^2(X, \mathbb{Z}) \cong [X, K(\mathbb{Z}, 2)] \cong$complex line bundles on X. Similarly, we have $H^1(X, \mathbb{Z}/2\mathbb{Z}) \cong$ real line bundles on X.

IV.7 Lie Groups & Symmetric spaces

Basic references are [Lie Groups Beyond an Introduction Knapp], [Smooth Manifold Lee].

1 Basics

Prop. (IV.7.1.1). A connected Lie group is second countable.

Proof: This follows from the fact that a Lie group is a manifold hence locally compact and it is a union of their products. □

Prop. (IV.7.1.2). A continuous homomorphism between Lie groups is smooth.

Proof: use exp coordinates. □

Prop. (IV.7.1.3). Any connected Lie group has a compact subgroup as deformation contraction.

Prop. (IV.7.1.4).

$$SU(2) = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}, \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1$$

is isomorphic to the group of unit quaternions and diffeomorphic to S^3.

Actions of Lie Groups

Def. (IV.7.1.5) (Linear transformation). The group $GL(2,\mathbb{C})$ acts on $\mathbb{C}$ by $\gamma(z) = \frac{az+b}{cz+d}$ where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. it can be checked that this is a continuous group action.

Also in the same way the group $GL(2,\mathbb{R})$ or $SL(2,\mathbb{R})$ acts on the upper plane, by(V.2.2.5). This action is transitive and the stabilizer of i is $SO(2,\mathbb{R})$, thus we have $\mathcal{H} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$.

Proof: □

2 Lie Theory

Def. (IV.7.2.1). A **Lie group** is a topological group endowed with a smooth manifold structure that the multiplication and inversion are smooth. Notice it suffices to prove that $(g,h) \to gh^{-1}$ is smooth. The left and right translation are all smooth morphisms hence diffeomorphisms.

A homomorphism of Lie groups is a smooth morphism that is also a group homomorphism. By translation invariance, a group homomorphism always has constant ranks, so a homomorphism of Lie groups that is a bijection is an isomorphism by global rank theorem(IV.3.1.1).

Prop. (IV.7.2.2) (Universal Covering Lie Group). If G is a connected Lie group, then its universal covering is also a Lie group, and the covering map is a Lie group homomorphism. Moreover, the group structure is unique.

Proof: Cf.[Lee Smooth Manifold P154]. □

Prop. (IV.7.2.3) (Lie Subgroup). A **Lie subgroup** is a closed submanifold that is also a subgroup of G. It is automatically a Lie group in the induced structures.

Proof: We need to show the $(g,h)\to gh^{-1}$ is smooth. They are smooth from $H\times H\to G$, and**?**
□

Prop. (IV.7.2.4) (Analytic Structure). Any smooth Lie group structure has a unique real analytic structure that is compatible with the smooth structure. So it is not important to distinguish between a smooth Lie group and an analytic Lie group, and call it **analytic group** if it is a connected Lie group.

Proof: □

Prop. (IV.7.2.5) (Lie Algebra of a Lie Group). The tangent space of a Lie group at e is a Lie algebra with the Lie bracket.

Proof: □

Cor. (IV.7.2.6). A homomorphism of Lie groups induces a morphism of their Lie algebras via the tangent space.

Proof: □

Prop. (IV.7.2.7) (Baker-Campbell-Hausdorff cor).

$$\exp(X)\exp(Y)=\exp(X+Y+1/2[X,Y]+1/12[X,[X,Y]]-1/12[Y,[Y,X]]+\text{higher order terms})$$

Proof: Cf.[Hall Lie algebras GTM222 P76]. □

Lemma (IV.7.2.8). Let H be a Lie subgroup of G and $g\notin H$, then there is a smooth function Φ on G that $\Phi(xh)=\Phi(x)$ for any $x\in G,h\in H$, and $\Phi(H)=0$, yet $\Phi(g)\neq 0$.

Proof: Because H is closed, there is a nbhd U of g disjoint from H and a function φ supported in U. Then $\Phi(x)=\int_H\varphi(xh)dh$ satisfies the desired condition. □

Lemma (IV.7.2.9) (Lie Subgroup Subalgebra). If H is a Lie subgroup of G connected, then for $X\in\mathfrak{g}$, the following are equivalent:

- X is in the tangent space of H.
- $\exp(tX)\in H$ for any $t\in\mathbb{R}$.

Proof: The second implies the first obviously, for the converse, if $\exp(tX)\notin H$, consider the function Φ defined in(IV.7.2.8), then consider $\varphi(t)=\Phi(\exp(tX))$, so $\varphi(0)=0$ but $\varphi(t)\neq 0$. However, we can prove $\frac{\partial\varphi}{\partial t}=0$, because Φ is constant on left H-cosets, which is a contradiction. □

Lemma (IV.7.2.10) (Adjoint Representation). For a Lie group G, there is an action Ad of G on $\mathfrak{g}$, which is induced by $Ad(g)X=\frac{\partial}{\partial t}g\exp(tX)g^{-1}$. Then The Lie algebra map of Ad is just ad.

Proof:

$$d(Ad)(X)Y=\frac{\partial}{\partial s}\frac{\partial}{\partial t}\exp(sX)\exp(tY)\exp(-sX)=[X,Y]$$

by(IV.7.2.7). □

Prop. (IV.7.2.11). If H is a Lie subgroup of G connected, then the Lie algebra $\mathfrak{h}$ of H is a Lie subalgebra of $\mathfrak{g}$.

Proof: It suffices to show that if $X, Y \in \mathfrak{h}$, then $[X, Y] \in \mathfrak{h}$. Notice that for $h \in H$, $Ad(h)Y \in \mathfrak{h}$, so take a derivative, we get $[X, Y] \in \mathfrak{h}$ by(IV.7.2.10). □

Prop. (IV.7.2.12) (Lie Subalgebra Subgroup). For a Lie group G, for any lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there exists uniquely a connected Lie subgroup H s.t. $\mathfrak{h}$ is the lie algebra of H.

Proof: By (IV.3.2.1), there is a maximal connected manifold H corresponding to $\mathfrak{h}$, we only need to show that it is a group. But the left invariance of $\mathfrak{h}$ shows that $HH \subset H$ because H is maximal. □

Cor. (IV.7.2.13). If G_1 is a simply connected Lie group and G_2 is a connected Lie group, then any Lie algebra homomorphism $\tilde{h} : \mathfrak{g}_1 \to \mathfrak{g}_2$ can be lifted to a Lie group homomorphism.

Proof: use the image of $\tilde{h}$: $\Gamma(\tilde{h}) \subset \mathfrak{g}_1 \times \mathfrak{g}_2$, the prop shows that there is a Lie group in $G_1 \times G_2$ for $\Gamma(\tilde{h})$. It is isomorphic to G_1 because the Lie algebra is the same and both are connected, thus a covering map and G_1 is simply connected. □

Prop. (IV.7.2.14) (Closed Subgroup Theorem). If H is a closed subgroup of a Lie group G, then there exists uniquely a differential structure s.t. H is a Lie subgroup of G. Cf.[Helgason Symmetric Spaces].

Prop. (IV.7.2.15) (Ado). Any finite dimensional Lie algebra can be embedded in some $\mathfrak{gl}(n, \mathbb{C})$.

Proof: □

Cor. (IV.7.2.16). From the preceding propositions, it follows that the category of finite dimensional Lie algebras is equivalent to the category of simply connected Lie groups.

3 Classical Groups

For more classical groups, Cf.[Classical Groups Baker].

Fundamental Groups

Prop. (IV.7.3.1).

- $SU(n-1) \to SU(n) \to S^{2n-1}$ gives us $SU(n)$ is simply connected.
$$\pi_1(Sp(2n)) = \pi_1(U(n)) = \mathbb{Z}$$
and the determinant induces an isomorphism onto $\pi_1(S^1)$. In fact, this is used to define the Maslov index.
- $SO(n-1) \to SO(n) \to S^{n-1}$ gives us $\pi_1(SO(n)) \cong \pi_1(SO(3))$. And $SU(2)$ as the unit sphere in $\mathbb{H}$ maps to $SO(3)$ via the conjugation:$\mathrm{Ad}(z) : w \mapsto zw\bar{z}$ has kernel ± 1, so $SO(3)$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$.

Generals

Prop. (IV.7.3.2). As in (IV.7.3.1) $SU(2)$ is a universal covering of $SO(3)$ and so does Spin(3)**??**, so $SU(2) \cong \mathrm{Spin}(3)$.

Prop. (IV.7.3.3). The symplectic group $Sp(2n, \mathbb{R}) = Sp(2n, \mathbb{C}) \cap U(n, n, \mathbb{C}) = Sp(n, \mathbb{H})$. And

$$Sp(2n) \cap O(2n) = Sp(2n) \cap GL(n, \mathbb{C}) = O(2n) \cap GL(n, \mathbb{C}) = U(n) = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}, \quad X + iY \in U(n).$$

Exponential Map

Prop. (IV.7.3.4). The exponential map for $GL_n(\mathbb{C})$ and $U(n)$ is surjective and the image of the exponential map for $GL_n(\mathbb{R})$ is $GL_n(\mathbb{R})^2$.

Proof: Use Jordan Decomposition. For complex case, notice the logarithm of $(cI+N), c\neq 0$ is definable for N nilpotent.

For the real case, its elementary form is $x-a$ or $(x-a)^2+b^2$. For the quadratic part, we choose a generator $v_1=1, v_2=-\frac{x-a}{b}$, then $xv_1=av_1-bv_2, xv_2=bv_1+av_2$. So in this basis A it is of the form $\begin{bmatrix} a & b\\ -b & a\end{bmatrix}$. Now let $a=r\cos\theta, b=r\sin\theta$, then $A=\exp(\theta\ln r\cdot\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix})$. □

4 Compact Lie Groups

Prop. (IV.7.4.1). Any compact connected complex Lie group is Abelian. And it is a complex tori.

Proof:

□

Prop. (IV.7.4.2) (Compact Lie Group and Representations). A compact group G is a Lie group iff it has a faithful f.d. representation.

Proof: If it has a faithful f.d. representation, then by(V.6.4.3) we can assume $G\subset U(n)$ compact hence closed, thus a Lie subgroup by(IV.7.2.14).

Conversely, if G is a Lie group, then we can choose a small nbhd of e that contains no subgroup of G(choose $\exp(\frac{1}{2}V)$ where exp is an diffeomorphism on V). Consider kernel K_π for irreducible representations π of G, then $\cap_\pi K_\pi=\varnothing$ by Gelfand-Raikov(V.6.2.16), in particular $\cap_\pi(K_\pi-U)=\varnothing$. But $G-U$ is compact, hence there are f.m. π_i that $\cap_i K_{\pi_i}\in U$, but by definition of U, $\cap_i K_{\pi_i}=0$, which gives a f.d. faithful representation of G, by(V.6.4.5). □

5 Real Reductive Groups

Remark (IV.7.5.1). Recall the definition of a reductive group(III.10.2.4). Let $G_\mathbb{R}$ be a reductive group.

Compact Real Algebraic Groups

Def. (IV.7.5.2) (Compact Real Algebraic Groups). A real algebraic group G over $\mathbb{R}$ is called **compact** iff $G(\mathbb{R})$ is a compact Lie group.

Def. (IV.7.5.3) (Relevant Group). A compact real algebraic group is called **relevant** iff the map $\pi_0(G(\mathbb{R}))\to\pi_0(G)$ is surjective, where the RHS is the algebro-geometric group of connected components.

Lemma (IV.7.5.4). Let Z be an affine variety over $\mathbb{R}$, let X be a subset of $Z(\mathbb{R})$, and let I_X be the ideal of regular functions on Z that vanishes at X, then $X'=V(I_X)$ satisfies $X\subset X'(\mathbb{R})$. Also by construction, X' is relevant and X intersects real points of every connected components of X'.

Now if Z is acted on by a compact Lie group K and X is a single K-orbit, then $X\cong X'(\mathbb{R})$.

Proof: Cf.[Gaitsgory P17]. □

Prop. (IV.7.5.5). The functor $G \mapsto G(\mathbb{R})$ is an equivalence of categories from the category of relevant real compact groups to the category of compact Lie groups.

Proof: For the fully faithfulness: given a map $\varphi : G_1(\mathbb{R}) \to G_2(\mathbb{R})$, we need to show it comes from a unique algebraic group homomorphism. Let $K \subset G_1(\mathbb{R}) \times G_2(\mathbb{R})$ be the graph of φ, then let Γ be the subgroup of $G_1 \times G_2$ corresponding to K in(IV.7.5.4), then it suffices to prove that the map $\Gamma \to G_1$ is an isomorphism. It is an isomorphism after passing to real points, so isomorphism on the level of Lie algebras. And then it is an isomorphism, because both groups are relevant?. □

Cor. (IV.7.5.6). The proof actually works only if G_1 is relevant compact real group. So if we choose $G_2 = GL(n, \mathbb{C})_R$, then by adjointness there is a bijection

$$\mathrm{Hom}_{AlgGrp/\mathbb{C}}(G_{\mathbb{C}}, GL(n, \mathbb{C})) \cong \mathrm{Hom}_{LieGrp}(G(\mathbb{R}), GL_n(\mathbb{C})).$$

That is, their complex representations correspond.

Complex Reductive Algebraic Groups

Prop. (IV.7.5.7). If G is a real reductive group, then its complexification $G_{\mathbb{C}}$ is complex reductive.

Proof: It suffices to show that $Rep(G(\mathbb{C}))$ is semisimple. For this, notice $Rep(G)$ is semisimple by definition, so it suffices to show for any representation V of $G_{\mathbb{C}}$, if W is a G-invariant subspace, then W is also W is $G_{\mathbb{C}}$-invariant. But the invariance condition is a vanishing of some matrix coefficients, they vanish on G so also vanish on $G_{\mathbb{C}}$. □

Def. (IV.7.5.8) (Real Form). A **real form** on a complex reductive Algebraic group is an anti-linear group isomorphism $\sigma : G \to G$ that $\sigma^2 = 1$. It is called **compact** iff G^σ is compact real, and it is called **relevant** iff G^σ is relevant compact(IV.7.5.3).

Prop. (IV.7.5.9) (Polar Decomposition). If G is a complex algebraic group and $K \in G(\mathbb{C})$ is a compact Lie subgroup. Assume that

- $\mathfrak{g} \cong \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$.
- K intersects non-trivially every connected components of $G(\mathbb{C})$.

Then the group G contains a unique real structure σ that $K = G(\mathbb{C})^\sigma$. And if $\mathfrak{p} \subset \mathfrak{g}$ be the subspace $\{\xi \in \mathfrak{g} | \sigma(\xi) = -\xi\}$, then the map

$$k \times \mathfrak{p} \to G(\mathbb{C}) : (k, p) \mapsto k \cdot \exp(p)$$

is a diffeomorphism.

Proof: Cf.[Gaitsgory P18]. □

Cor. (IV.7.5.10). If we denote $P = \exp(\mathfrak{p})$, then $P \subset \widetilde{P} = \{g \in G(\mathbb{C}) | \sigma(g) = g^{-1}\}$, and there is a diffeomorphism:

$$\coprod_{k \in K, k^2 = 1} \{k\} \times P \cong \widetilde{P}.$$

Cor. (IV.7.5.11). In the situation of(IV.7.5.9), G is reductive, by(IV.7.5.7).

Cor. (IV.7.5.12). $K \to G(\mathbb{C})$ is a homotopy equivalence.

Maximal Compact Subgroup

Cor. (IV.7.5.13). For any compact subgroup $K' \subset G(\mathbb{R})$, there exists an element $g \in P^\tau$ s.t. $Ad_g(K') \in K^\tau$.

Proof: Cf.[Gaitsgory P25]. $\square$

6 Representations of Lie Groups

Prop. (IV.7.6.1) (Regular Representations). For a connected Lie group G, consider its left and right regular action λ, ρ on $C^\infty(G)$(V.6.1.1). We will write dX for $X \in \mathfrak{g}$ as the representation of Lie algebra of G via ρ, then it commutes with λ. So it induces a map of $U(\mathfrak{g})$ to the ring of left G-invariant differential operators on G.(I.10.7.1).

Prop. (IV.7.6.2) (Center element Bi-invariant). If G is a connected Lie group with Lie algebra $\mathfrak{g}$ and $D \in Z(U(\mathfrak{g}))$, then the differential operator D defined in(IV.7.6.1) is invariant under both left and right regular representations of G.

Proof: The left invariance is general from(IV.7.6.1), for the right invariance, Because G is connected, it suffices to prove invariance for a nbhd of identity of G, thus suffices to prove

$$\rho(g_t)D = D\rho(g_t), \quad g_t = \exp(tX).$$

For this, let $\varphi(g,t) = (\rho(g_t)D - D\rho(g_t))(g)$ and take derivative w.r.t t, then it reads:

$$\partial/\partial t\varphi(g,t) = (DdX\rho(g_t)f - dX\rho(g_t)Df)(g) = dX\varphi(g,t)$$

because dX commutes with D. And also $\varphi(g,0) = 0$, so by lemma**??**, $\varphi(g,t) = 0$ for any t, g. $\square$

Lemma (IV.7.6.3). ?? If G is a connected Lie group with Lie algebra $\mathfrak{g}$, and $\varphi \in C^\infty(G \times \mathbb{R})$ satisfies

$$\frac{\partial}{\partial t}\varphi(g,t) = dX\varphi(g,t)$$

for some $X \in \mathfrak{g}$ and $\varphi(g,0) = 0$, then $\varphi = 0$.

Proof: Cf.[Bump AutoForm P151]. $\square$

Cor. (IV.7.6.4). If $G = GL(2,\mathbb{R})^+$, then $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$, and the Casimir element(I.10.7.11) $\Delta = -1/2(1/2h^2 + ef + fe)$ corresponds to a bi-invariant differential operator on $C^\infty(G)$, and it is called the **Laplace-Beltrami operator**.

Differential Vectors

Def. (IV.7.6.5) (Smooth Vectors). Let V be a continuous representation of G on a locally convex TVS. we define the space V^∞ of **smooth vectors** of V as $V^\infty = \cap_n V^n$ where

$$V^n = \{v \in V^{n-1}, T_\xi(v) \in V^{n-1}, \forall \xi \in \mathfrak{g}\}.$$

And V^∞ is given the inverse limit topology.

Prop. (IV.7.6.6). There is a canonically defined continuous map

$$C^n(G)^* \times V^{m+n} \to V^m : \mu \to T_\mu$$

which is uniquely defined by the requirement that for any $v \in V^{m+n}, u^* \in (V^m)^*$,

$$(u^*, T_\mu(v)) = (\mu, (u^*, F^v)).$$

Cor. (IV.7.6.7) (Action of Distribution on Smooth Vectors). There is a continuous action of $Distr_c(G)$ on V^∞: $\mu \mapsto T_\mu$ compatible with the convolution structure on $Distr_c(G)$.

Proof: □

Lemma (IV.7.6.8). If $f \in C_c^k(G)$, then $f\mu_{Haar} \in (C(G))^*$. Then for any $v \in V$, $T_{f\mu_{Haar}}(v) \in V^k$(V.4.3.24).

Proof: It can be verified that

$$X(T_{f\mu_{Haar}}(v)) = T_{X(f)\mu_{Haar}}(v).$$

for any $X \in \mathfrak{g}$. □

Prop. (IV.7.6.9). If $f \in C_c^\infty(G)$, then for any $v \in V$, the vector $T_{f\mu_{Haar}}(v) \in V^\infty$.

Cor. (IV.7.6.10) (Smooth Vectors Dense). V^∞ is dense in V.

Proof: Choose a Dirac sequence $\{f_n\}$, then $T_{f_n\mu_{Haar}}v \in V^\infty$ converges to v(V.4.3.24). □

Cor. (IV.7.6.11). If V is a f.d. vector space, then $V^\infty = V$. Equivalently, the image of

$$MC_V : \mathrm{End}(V) \to C(G)$$

lies in $C^\infty(G)$(V.6.4.8).

7 Analysis

Bi-invariant Metric

Lemma (IV.7.7.1). Bi-invariant metric exists in a compact manifold.

Proof: Because the Haar measure on a compact metric is bi-invariant. Choose a Riemann metric and set

$$\langle V, W\rangle = \int_{G\times G} \langle L_{\sigma*}R_{\tau*}(V), L_{\sigma*}R_{\tau*}(W)\rangle d\mu(\sigma)d\mu(\tau).$$

Note that L_* and R_* commute. □

Prop. (IV.7.7.2). A Lie group G possesses a bi-invariant metric iff the metric at the origin e satisfies

$$\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$$

Proof: □

Cor. (IV.7.7.3). If G is a Lie group with a bi-invariant metric, then

$$\nabla_X Y = 1/2[X,Y], \quad R(X,Y)Z = 1/4[[X,Y],Z], \quad K(\sigma) = 1/4\Big|[X,Y]\Big|^2.$$

So it has non-positive sectional curvature, and its curvature is non-negative, and <u>all 1-parameter subgroups are geodesic.</u>

Proof: □

Cor. (IV.7.7.4). A bi-invariant Lie group with $\mathfrak{g}$ having trivial center is compact and $\pi_1(G)$ finite.

Proof: The Ricci curvature has a positive lower bound, otherwise for some $X, [X,Y]=0$ for all Y, thus X is in the center. Hence we use Myer theorem(IV.2.5.20). □

Prop. (IV.7.7.5) (Structure theorem for bi-invariant Lie group). A simply connective Lie group with a bi-invariant metric is equal to $G' \times R^k$, G' compact.

Proof: Because the orthogonal complement of the center of $\mathfrak{g}$ is a Lie algebra, G is like $G' \times R^k$, and a simply connected abelian Lie group is R^k **?**. □

8 Symmetric space

Prop. (IV.7.8.1). A **symmetric space** is that for every point p, there is a isometry reversing the geodesics passing p. A manifold is called **locally symmetric** if $\nabla R = 0$. Locally symmetric is equivalent to the fact that every local reversing map is an isometry. A symmetric space is complete because two folding is an extension of geodesic.

Prop. (IV.7.8.2). A Lie group with a bi-invariant metric is a symmetric space.

Proof: □

Prop. (IV.7.8.3). The conjugate points in a symmetric space is easy to calculate, they are $\exp(\frac{\pi k}{\sqrt{e_i}}V)$, counting multiplicity, where e_i is the eigenvalue of the sself-adjoint operator $K_V(W) = R(V,W)V$ at p.

IV.8 Complex Geometry

Basic References are [Voisin], [Principle of Algebraic Geometry Griffith/Harris], more advanced stuffs to add from [Kähler Geometry] and [Complex Geometry Daniel]. Even more advanced is [Demailly Complex Analytic and Differential Geometry].

1 Complex Manifold

Def. (IV.8.1.1). A **complex manifold** is an even dimensional manifold that the transformation matrix is holomorphic.

Prop. (IV.8.1.2) (Adjunction Formula). The normal sheaf of a submanifold $Y \subset X$ is defined the same as the case of nonsingular varieties(III.7.1.18), then the same is true of the adjunction formula:

$$\mathcal{K}_Y \cong \mathcal{K}_X \otimes \det \mathcal{N}_{Y/X}$$

In case Y is of codimension 1, $\mathcal{N}_{Y/X} \cong \mathcal{L}(Y)|_Y = \mathcal{O}_Y(Y)$.

Prop. (IV.8.1.3) (Remmert's Theorem). A non-compact manifold admits a proper holomorphic embedding into $\mathbb{C}^N$ for some N iff it is a Stein manifold.

Prop. (IV.8.1.4) (Siegel). Let X be a compact complex manifold of dimension n, then the field $K(X)$ of meromorphic functions on X has transcendence degree$\leq n$ over $\mathbb{C}$. And in case $tr.d.K(X) = \dim X$, it is a f.g. field extension of $\mathbb{C}$. Then we define the **algebraic dimension** of a compact connected complex manifold X to be $a(X) = \operatorname{tr}.d.K(X)$.

Proof: It suffices to show that given any meromorphic functions $f_1, \ldots, f_{n+1}$, there is an algebraic relation between them.

Now for each x, there is a nbhd U_x that any f_i writes as the quotient of two holomorphic functions $\frac{g_{i,x}}{h_{i,x}}$. And assume $W_x \subset V_x \subset \overline{V_x} \subset U_x$ are the metric balls $B(x, \frac{1}{2}) \subset B(x, 1)$. As X is compact, there are N x_k that $X = \cup W_{x_k}$.

As on the intersections, $\frac{g_{i,k}}{h_{i,k}} = \frac{g_{i,l}}{h_{i,l}}$, any we can assume they are all prime, so $\frac{g_{i,k}}{g_{i,l}} = \varphi_{i,kl}$ is a unit. Let $\varphi_{kl} = \prod_i \varphi_{i,kl}$, as X is compact, let $C = \max_{k,l} \varphi_{kl} \geq 1$.

For any homogenous polynomial $F \in \mathbb{C}[X_1, \ldots, X_{n+1}]$ of deg m, let $G_k = F(\frac{g_{1,k}}{h_{1,k}}, \ldots, \frac{g_{n,k}}{h_{n,k}})(\prod_i h_{ik})^m$. Then G_k are holomorphic and $G_k = \varphi_{lk}^m G_l$ on the intersection. Now I claim for any $M > 0$, there is a F that G_k vanishes vanishes up to at least order M at x_k.

For this, just consider the dimension of all homogenous polynomials of degree m is C_{m+n+1}^m, and the number of desired equations of elements needs to be vanish is $N \cdot C_{m'-1+n}^{m'-1}$, so this always can be achieved when m is sufficiently large.

By Schwartz lemma(V.2.8.4), $|G_k(x)| \leq (\frac{1}{2})^{m'} C'$, where $C' = \max\{|G_k(x)|| k = 1, \ldots, n, x \in \overline{V_k}\}$.

If $C' = |G_k(x)|$, and $x \in W_l$, then $C' = |G_l(x)||\varphi_{lk}^m(x)| \leq \frac{C'}{2^{m'}} \cdot C^m$. If for some m, m', $C^m < 2^{m'}$, then this shows $C' = 0$ which will finish the proof.

Look back at the condition of m, m', $C_{m+n+1}^m > N \cdot C_{m'-1+n}^{m'-1}$ can be achieved together with $m < \lambda m'$ for any λ, because the left hand is degree $n+1$ in m and the right hand is degree n in m'. $\square$

Almost Complex Structure

Def. (IV.8.1.5). For M a real orientable manifold of dimension $2n$, an **almost complex structure** is a bundle map $J : TM \to TM$ satisfying $J^2 = -1$.

A complex manifold has an almost complex structure, just define $J(\partial/\partial x_i) = \partial/\partial y_i$ and $J(\partial/\partial y_i) = -\partial/\partial x_i$.

Def. (IV.8.1.6). J will define a bundle map on $T^*M \to T^*M$, and it has two eigenvalues $\pm i$, denoted by $T^{*1.0}M$ and $T^{*0,1}M$. The **formal differential forms** $\wedge^k T^*M \cong \sum \wedge^{p,k-p} T^*M$. ∂ is defined to be $\pi_{p+1,q} \circ d$ on $\wedge^{p,q}T^*M$, and $\overline{\partial}$ is defined to be $\pi_{p,q+1} \circ d$.

Def. (IV.8.1.7) (Integrability). An almost complex structure is called **integrable** iff it satisfies the following equivalent conditions:

- $d\alpha = \partial\alpha + \overline{\partial}\alpha$.
- $d\alpha = \partial\alpha + \overline{\partial}\alpha$ is true for $\alpha \in \mathcal{A}^{1,0}(X)$
- $[T^{0,1}X, T^{0,1}X] \subset T^{0,1}X$.
- $\overline{\partial}^2 f = 0$ for functions f.

Proof: 1 $\iff$ 3 is because by(IV.3.2.8), if $u, v \in T^{0,1}X$,

$$d\alpha(u,v) = u(\alpha(v)) - v(\alpha(u)) - \alpha([u,v]) = -\alpha([u,v]).$$

3 $\iff$ 4 is because by(IV.3.2.8), if $\alpha = \overline{\partial}f$ and $u, v \in T^{0,1}X$, then

$$\overline{\partial}^2 f(u,v) = u(\overline{\partial}f(v)) - v(\overline{\partial}f(u)) - \overline{\partial}f([u,v]) = u(df(v)) - v(df(u)) - \overline{\partial}f([u,v]) = \partial f([u,v])$$

□

Prop. (IV.8.1.8) (Nirenberg-Newlander). Given an almost complex manifold (M, J), it is integrable iff it comes from a complex structure.

Proof: Cf.[Foundation of Differential Geometry Kobayashi Chap9.2]. □

Blowing-up

Blowing-up serves as a way to magnify local properties to global ones.

Remark (IV.8.1.9). Cf.[Complex Geometry P98] for blowing up along an arbitrary subvariety.

Def. (IV.8.1.10) (Blowing-up along Point). For a nbhd U of 0 in $\mathbb{C}^n$, we can define the **blowing-up** $\pi : \widetilde{U} \to U$: $\widetilde{U}$ is the subset of $U \times \mathbb{CP}^n$ consisting of $(z, [l])$ that $z \subset [l]$. Then $\pi^{-1}(U - \{0\}) \cong U - \{0\}$ holomorphically.

For a complex manifold M and a point x, then choose a local coordinate centered at x, then we can form the blowing-up, because it is holomorphism away from x, so it can glue with the rest of M and form a knew manifold $\widetilde{M}$, called the **blowing-up** of M along x.

Notice this is independent of the coordinate chosen, because if $f(U)$ is a new coordinate of U, then $\pi'^{,-1} f \pi : \widetilde{U} - \{x\} \to \widetilde{U}' - \{x\}$ is a holomorphism, and it can be extended to $\widetilde{U} \to \widetilde{U}'$ by setting $f(x, [l]) = (x, [(\frac{\partial f_i}{\partial z_j})(0) l])$.

Prop. (IV.8.1.11) (Exceptional Divisor). Let E be $\pi^{-1}(x)$ for a blowing-up, called the **exceptional divisor**. Often the line bundle $\mathcal{O}_{\widetilde{X}}(E)$ associated with it is called denoted by E.

There are canonical coordinates near E: let $\widetilde{U}_i$ be $\widetilde{U}-\{(l_i=0)\}$, then endow $\widetilde{U}_i$ with the coordinate $z(i)=(l_j/l_i,\ldots,z_i,\ldots,l_n/l_i)$, it is holomorphic to $\mathbb{C}^n$. π in this coordinate is written as $(z(1),\ldots,z(n))\mapsto(z(i)z(1),\ldots,z(i),\ldots,z(n)z(i))$.

The transition function can be written, it is

$$\varphi_j\circ\varphi_i^{-1}((z(i)_1,\ldots,z(i)_n))=(\frac{z(i)_1}{z(i)_j},\ldots,\frac{1}{z(i)_j},\ldots,z(i)_iz(i)_j,\ldots,\frac{z(i)_n}{z(i)_j}).$$

Notice it is somewhat tricky because it has two different coordinates.

The defining function of E in this coordinate is $(z(i))=(z_i)$. So the line bundle $\mathcal{O}_{\widetilde{X}}(E)$ has transition function $g_{ij}=z(i)/z(j)$, and it can be thought of as the line bundle that has line $[l]$ at the point $(z,[l])\in\widetilde{U}$. So it is kindof tautological, in fact its restriction on $E\cong\mathbb{CP}^{n-1}$ is just the tautological line bundle.

Prop. (IV.8.1.12). The canonical line bundle $\mathcal{K}_{\widetilde{X}}=\pi^*\mathcal{K}_X+(n-1)E$, where n is the dimension of X.

Proof: Away from E, the π is a holomorphism, so It suffices to compare the two transition function of the two canonical maps near E using the coordinates in(IV.8.1.11), with the local section given by $dz_1\wedge\ldots\wedge dz_n$ and $dz(i)_1\wedge dz(i)_n$ respectively. On $\widetilde{U}_i$, locally $dz_1\wedge\ldots\wedge dz_n$ is pulled by π^* to the trivial bundle on U', and by calculation, $dz(j)_1\wedge dz(j)_n=z(i)_j^{n-1}dz(i)_1\wedge dz(i)_n$, so $\mathcal{K}_{\widetilde{X}}-(n-2)E$ has a global section $z(i)_i^{n-1}dz(i)_1\wedge dz(i)_n$, so it is also trivial on $\widetilde{U}$, so $\mathcal{K}_{\widetilde{X}}=\pi^*\mathcal{K}_X+(n-1)E$ is true. $\square$

2 Deformation of Complex Structures

Cf.[Kähler Geometry] and [Complex Geometry Chap6], should be completed as soon as possible.

Calabi-Yau Manifolds

3 Coherent Sheaves and Analytic Spaces

Cf.[Demailly] and [GAGA Serre].

Analytic Subvarieties

Def. (IV.8.3.1) (Analytic subvariety). An **analytic subvariety** is a closed subset of a complex manifold that is locally defined by f.m. holomorphic functions. The **regular points** of an analytic subvatiety locally defined by k functions is the points that $\mathrm{rank}(\frac{\partial(f_1,\ldots,f_k)}{\partial(z_1,\ldots,z_n)})=k$.

Prop. (IV.8.3.2) (Proper Mapping Theorem). If U,M are complex manifolds and $M\subset U$ is an analytic subvariety, then if $f:U\to N$ is a holomorphic mapping whose restriction on M is proper, then $f(M)$ is an analytic subvariety of N.

Proof: Cf.[Griffith/harris P395]. $\square$

Def. (IV.8.3.3). An **analytic space of** $\mathbb{C}^n$ is an analytic subvariety of $\mathbb{C}^n$. On an analytic space, there is a sheaf of holomorphic functions $\mathcal{H}_U$. So we can define **holomorphic map** φ as continuous functions that maps holomorphic germs to holomorphic germs, which is equivalent to the coordinates of φ are all holomorphic.

Def. (IV.8.3.4). An **analytic space** is a Hausdorff space X with a structure sheaf $\mathcal{H}_X$ that is locally isomorphic to an analytic set. Morphisms are continuous maps that are locally holomorphic. Sub-analytic spaces are defined as usual.

An **analytic module** is just a module over the sheaf $\mathcal{H}_X$. For a sub-analytic space Y, we have a sheaf of ideals $\mathcal{A}_Y$ which is the sheaf of germs vanishing at Y, and $\mathcal{H}_X/\mathcal{A}_Y$ is a sheaf of X that is zero outside Y, and we identify it with $\mathcal{H}_Y$.

The products of analytic spaces can be defined, and it has the product topology, unlike the case of schemes.

Prop. (IV.8.3.5). The structure sheaf of an analytic space is coherent, and the sheaf of ideals of a sub analytic space is coherent.

Proof: First prove for X is an open subset of $\mathbb{C}^n$, Cf.[GAGA Serre P4]. And by definition $\mathcal{A}_X$ is a $\mathcal{O}_X$-module of f.t., and it is also coherent?, so $\mathcal{H}_X$ is coherent. $\mathcal{A}_Y$ is coherent because it is a kernel of $\mathcal{H}_X \to \mathcal{H}_Y$. □

4 Positive Current

5 Hermitian Vector Bundles

Def. (IV.8.5.1). A **holomorphic vector bundle** is a vector bundle on a complex manifold that the transition function is holomorphic. A **Hermitian vector bundle** is a holomorphic vector bundle endowed with a Hermitian metric. Any holomorphic vector bundle has a Hermitian structure, by partition of unity method.

Prop. (IV.8.5.2) (Hodge Star for Hermitian bundles). If E is a Hermitian vector bundle over a compact complex manifold of complex dimension n, we define a conjugate-linear operator $\overline{*} : A^{p,q}(X) \to A^{n-p,n-q}(X) : \eta \mapsto *\overline{\eta}$, and a conjugate-linear functor $\tau E \to E^*$ induced by the Hermitian metric on E.

Then we can define $\overline{*}_E : A^{p,q}(E) \to A^{n-p,n-q}E : \eta \otimes s \mapsto \overline{*}(\eta) \otimes \tau(s)$. It can be checked that

$$(\alpha, \beta) * 1 = \alpha \wedge *_E \beta,$$

$$\overline{\partial}^*_E = -\overline{*}_{E^*}\overline{\partial}_{E^*}\overline{*}_E, \quad \overline{*}_E \Delta_{\overline{\partial}_E} = \Delta_{\overline{\partial}_{E^*}}\overline{*}_{E^*}, \quad \overline{*}_{E^*}\overline{*}_E = (-1)^{p+1} \text{ on } \Omega^{p,q}(E).$$

Hermitian Metric

Def. (IV.8.5.3). The complexified tangent bundle $T_{\mathbb{C}}M$ is defined as $TM \otimes_{\mathbb{R}} \mathbb{C}$, the **holomorphic tangent bundle** $T^{1,0}M$ and anti-holomorphic bundle $T^{0,1}M$ are defined to be the vectors generated resp. by $\partial/\partial z_i$ and $\partial/\partial \overline{z_i}$. The **holomorphic cotangent bundle** and anti-holomorphic cotangent bundle is defined to be the covectors generated by dz_i and $d\overline{z}_i$.

Def. (IV.8.5.4). A metric on TM is called **Hermitian** iff it is J-invariant, that is $g(Ju, Jv) = g(u, v)$. If g is Hermitian, then it can be checked that $g(T^{0,1}, T^{0,1}) = 0 = g(T^{1,0}, T^{1,0})$. A metric extend by linearity to a bilinear form on $T_{\mathbb{C}}M$. And if we define $(Z,W) = g(Z,\overline{W})$, then it is a Hermitian metric on $T^{1,0}$(non-degenerate because g is, and g trivial on $T^{0,1}$ and $T^{1,0}$). Conversely, the same construction shows a Hermitian metric on $T^{1,0}$ is equivalent to a Hermitian metric on TM, that's what the name means.

Def. (IV.8.5.5). Given a Hermitian metric g on TM, define the **Kahler form** ω_g as $\omega_g(u,v) = g(Ju,v)$. Then it is a real 2-form on M.

Notice $g(u,v) = \omega_g(u,Jv)$, so g can be constructed by ω_g, iff ω_g is positive(IV.9.6.1).

Picard Group

Def. (IV.8.5.6). The group of isomorphisms of holomorphic line bundles on a complex manifold X is denoted by $Pic_{\mathbb{C}}(X)$.

Prop. (IV.8.5.7) (Picard Group). For a connected space X, there is an exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{f \to e^{1\pi i f}} \mathcal{O}_X^* \to 0$, and it induces a map $Pic_{\mathbb{C}}(X) = H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$, which is a just the first Chern class(same proof as in(IV.6.4.3)).

WARNING: in this case it is not necessarily isomorphism, not as in the case of topological line bundles.

in particular, The image of the first Chern class is trivial in $H^2(X,O_X)$.

Def. (IV.8.5.8). The dual of the universal line bundle on $\mathbb{CP}^n$ is called the **hyperplane line bundle**, denoted by H or $\mathcal{O}(1)$.

Prop. (IV.8.5.9). $Pic_{\mathbb{C}}(\mathbb{CP}^n) \cong \mathbb{Z}$, with $\mathcal{O}(1)$ as a generator.

Proof: As $\mathbb{CP}^n$ is Kähler, use(IV.9.5.2), then $H^{0,k}(X,\mathbb{C}) \cong H^k(X,\mathcal{O}_X) = H^k(X,\mathcal{K}_X \otimes \mathcal{O}(2)) = 0$ for $k \geq 1$ by Kodaira vanishing(IV.9.7.3), and then $NS(X) = H^{1,1}(X) = H^2(X,\mathbb{Z}) = Z$ by Lefschetz $(1,1)$-form theorem(IV.9.5.3). It remains to prove $c_1(\mathcal{O}(1))$ is the generator, for this, Cf.[Demailly P280]. $\square$

Prop. (IV.8.5.10). Let S_d be the set of homogenous polynomials of degree d, then

$$H^0(\mathbb{CP}^n,\mathcal{O}(d)) = \begin{cases} S_d & d \geq 0 \\ 0 & d < 0 \end{cases}$$

Proof: This is because it is sections that satisfy $f_\alpha([z]) = (\frac{z_\beta}{z_\alpha})^k f_\beta([z])$, which says f_α glue together to give a holomorphic function homogenous of degree k on $\mathbb{C}^n - \{0\}$, which extends to a function on $\mathbb{C}^n$ by(V.2.8.3), then it is easy to see it is a homogenous polynomial using the power series expansion. $\square$

Def. (IV.8.5.11) (Neron-Severi Group). For a compact complex manifold, the **Neron-Severi group** $NS(X)$ is the image of $Pic_{\mathbb{C}}(X) \to H^2(X,\mathbb{R})$. $rank_{\mathbb{R}}(NS(X))$ is called the **Picard number** of X.

There is a good description of $NS(X)$ in case X is Kahler, See Lefschetz theorem(IV.9.5.3).

Chern Connection

Prop. (IV.8.5.12) (Chern Connection). Given a Hermitian holomorphic bundle $E \to M$ on a complex manifold, there is a unique **Chern connection** ∇ on E, that ∇ is holomorphic(i.e. the connection matrix is holomorphic w.r.t a homomorphic frame), and it is compatible with the Hermitian metric.

Proof: Write out the requirement: if $H = h_{ij}$ is the matrix of the Hermitian metric, so H is Hermitian, and we need $dh_{ij} = (\nabla e_i, e_j) + (e_i, \nabla e_j) = \sum_k \omega_{ik} h_{kj} + \sum \overline{\omega_{jk}} h_{ik}.\omega$ is holomorphic, so must

$$\partial H = \theta H, \quad \overline{\partial} H = H \overline{\theta}^t.$$

But $H^t = \overline{H}$ so these two equations are equivalent and $\theta = \partial H H^{-1}$. □

Cor. (IV.8.5.13). The curvature of the Chern connection is $\Omega = \overline{\partial}(\partial(h)h^{-1})$. In particular, it is a skew-symmetric matrix of $(1,1)$-forms. If it is of dimension 1, then $\Omega = \overline{\partial}\partial \log h$.

Proof: Ω is locally $d\omega + \omega \wedge \omega$, so if we choose a unitary basis, then ω is skew-symmetric by definition and $\omega \wedge \omega$ is also skew-symmetric, so Ω is skew-symmetric. The calculation is direct calculation. □

Prop. (IV.8.5.14). The transformation matrix of a complex manifold is holomorphic, so it is possible to define globally $\overline{\partial}$ operator. And locally on a nbhd, ∂ is defined as $d - \overline{\partial}$.

Prop. (IV.8.5.15) (Normal Coordinate). For a Hermitian vector bundle E over a complex manifold X, given any coordinate frame (z_j), there exists a holomorphic frame (e_λ) that

$$\langle e_\lambda, e_\mu \rangle = \delta_{\lambda,\mu} - \sum c_{jk\lambda\mu} z_j \overline{z}_k + O(|z|^3)$$

where $c_{ij\lambda\mu}$ is the coefficient of the Chern connection Ω. Such a coordinate is called the **normal coordinate frame** of E at x.

Proof: Cf.[Demailly P270]. □

Def. (IV.8.5.16) (Dolbeault Cohomology). The **Dolbeault cohomology group** $H^{p,q}_{\overline{\partial}}(X, \mathcal{E})$ of a holomorphic vector bundle $\mathcal{E}$ over a complex manifold X is defined to be the q-th cohomology group of the complex

$$0 \to \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \Omega^{p,n-p} \to 0$$

and $H^{p,q}_{\overline{\partial}}$ is defined to be $H^{p,q}_{\overline{\partial}}(X, \mathbb{C}_X)$. By(III.5.5.13), $H^{p,q}_{\overline{\partial}}(X, \mathcal{E}) \cong H^q(M, \Omega^p_{hol} \otimes_{\mathcal{O}_X} \mathcal{E})$.

6 GAGA

Basic Reference are [GAGA Serre].

Analytification of Algebraic Varieties and Sheaves

Prop. (IV.8.6.1) (Analytification). For any variety over $\mathbb{C}$, any open affine subset is isomorphic to an analytic space of $\mathbb{C}^n$, hence can be given an analytic structure $X^{an} \to X$, called the **analytification of** X. This is because algebraic isomorphisms are analytic isomorphism. Moreover, for any morphism of algebraic variety, the scheme is

X^{an} is Hausdorff because analytification preserves products and morphisms, and separability of X shows that $\Delta(X)$ is closed in $X \times X$, hence it is also closed in the analytification.

Notice X^{an} and X have in fact the same underlying sets.

Prop. (IV.8.6.2) (Transfer of Properties).

- X^{an} is locally compact and σ-compact.
- X is smooth over $\mathbb{C}$ iff ^{an} is a complex manifold.
- A morphism $f : X \to Y$ is proper iff $X^{an} \to Y^{an}$ is proper. In particular, X is complete(proper) iff X^{an} is compact.
- If X is projective and connected, then X^{an} is connected iff X is connected.

Proof: The first is because X is qc hence covered by f.m. affine subsets hence second-countable and use(IV.1.2.6). X^{an}/X is flat because completion of Noetherian rings are flat(I.5.5.17).

For proper, Cf.[GAGA Serre P8].

If S is smooth, then by Jacobian criterion(I.7.4.13), the Jacobian of local defining function is not zero every where, so its analytification is clearly smooth.

The last assertion in fact follows from(IV.8.6.15), as $H^0(X, \mathcal{O}_X) = H^(X^{an}, \mathcal{O}_X)$. □

Remark (IV.8.6.3). There is in fact a more general analytification for any scheme locally of finite type over $\mathbb{C}$. That is, we define it as the right adjoint to the forgetful functor from analytic spaces to local ringed spaces. Where an analytic space is a local ringed space that locally has immersions into $\mathbb{C}^n$. Should consult [Grothendieck EGA1-7].

Proof: Notice the schemes that have an analytification is stable under open subscheme, closed subscheme and products, and we can make a glue a large space from open subschemes by the unicity. So we only need to consider $\operatorname{Spec}\mathbb{C}[T]$, whose analytification is $\mathbb{C}$. □

Prop. (IV.8.6.4). There is a natural map from $\mathcal{O}_x$ to $\mathcal{H}_x$ that maps $\mathfrak{m}_x$ to $\mathfrak{m}_x\mathcal{H}_x$, thus inducing a map $\hat{\theta} : \hat{\mathcal{O}}_x \to \hat{\mathcal{H}}_x$. This is an isomorphism. In particular, $\theta : \Omega_x \to \mathcal{H}_x$ is injective.

Moreover, if Y is a locally closed subscheme of X, then the local ideal of functions vanishing at Y maps to $\mathcal{A}_x(Y)$, and $\mathcal{A}_x(Y)$ is generated by $\theta(\mathcal{I}_x(Y))$. Moreover, $\mathcal{H}_{x,Y} = \mathcal{H}_x/\mathcal{I}_x(Y)$.

Proof: [GAGA Serre P6]. □

Cor. (IV.8.6.5). The inclusion $\mathcal{O}_x \subset \mathcal{H}_x$ is flat ring extension, by(I.7.1.16) and the fact $\hat{A}/A$ is flat. And $\dim \mathcal{O}_x = \dim \mathcal{H}_x$ because $\dim A = \dim \hat{A}$(I.5.6.10).

Cor. (IV.8.6.6). Given an open and dense subscheme U of an algebraic variety X over $\mathbb{C}$, U^{an} is dense in X^{an}.

Proof: Consider the complement Y, if U^{an} is not dense in X^{an}, then there exists a x that $\mathcal{A}_x(Y) = 0$, so by(IV.8.6.4), $\mathcal{I}_x(Y) = 0$, so Y is not dense near x, contradiciton. □

Cor. (IV.8.6.7). For a morphism f of algebraic varieties over $\mathbb{C}$, $\overline{f(X)^{an}} = \overline{f(X)}^{an}$.

Proof: By Chevalley theorem(III.3.4.32), there is a open dense subscheme U of $\overline{f(X)}$ that is contained in $f(X)$, then(IV.8.6.6) shows U^{an} is dense in $\overline{f(X)}^{an}$, so $\overline{f(X)}^{an} \subset \overline{f(X)^{an}}$. The converse is obvious. □

Def. (IV.8.6.8) (Analytification of Sheaves). Denote for a sheaf $\mathcal{F}$ over X F' the inverse image sheaf over X^{an} pulled back along $X^{an} \to X$. Define the **analytification of** $\mathcal{F}$ $\mathcal{F}^{an}$ as the sheaf $\mathcal{F}' \otimes_{\mathcal{O}'_X} \mathcal{H}_X$.

Prop. (IV.8.6.9). $\mathcal{F} \to \mathcal{F}^{an}$ is exact from the category of sheaves on X to the category of analytic sheaves on X^{an}, $\mathcal{F}' \to \mathcal{F}^{an}$ is injective, and it maps coherent sheaves to coherent analytic sheaves.

Proof: The first two follows from the fact that $\mathcal{H}_X$ is flat over $X^{an} \to X$(IV.8.6.5). For the last assertion, notice if $\mathcal{O}_X^p \to \mathcal{O}_X^q \to \mathcal{F} \to 0$, then $\mathcal{H}_X^p \to \mathcal{H}_X^q \to \mathcal{F}^{an} \to 0$, so it is coherent because $\mathcal{H}_X$ is coherent(IV.8.3.5)(III.2.1.18). □

Prop. (IV.8.6.10). Let $i : Y \to X$ be a closed subscheme, then for a coherent sheaf $\mathcal{F}$ on Y, $(i^{an})_*\mathcal{F}^{an} \cong (i_*\mathcal{F})^{an}$.

Proof: These two sheaves are both 0 outside Y^{an}, consider a point of Y, their stalks are respectively $\mathcal{F}_x \otimes_{\mathcal{O}_{x,X}} \mathcal{H}_{x,X}$ and $\mathcal{F}_x \otimes_{\mathcal{O}_{x,Y}} \mathcal{H}_{x,Y}$. By(IV.8.6.4) we notice

$$\mathcal{H}_{x,Y} = \mathcal{H}_{x,X}/\mathcal{I}_x(Y)\mathcal{H}_{x,X} = \mathcal{H}_{x,X} \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{x,Y}.$$

So this two are equal by associativity of tensor product. □

Prop. (IV.8.6.11). By Leray Spectral sequence(III.5.3.8), for an analytic sheaf $\mathcal{G}$, there is a boundary map $H^k(X,\mathcal{G})H^k(X^{an},\mathcal{G})$. So for a sheaf $\mathcal{F}$ on X, there is a map

$$\varepsilon : H^k(X,\mathcal{F}) \to H^k(X, an_*\mathcal{F}^{an}) \to H^k(X^{an},\mathcal{F}^{an})$$

Equivalence between Algebraic Variety and Analytic Spaces

Remark (IV.8.6.12) (GAGA Principle). Any global analytic object on a projective variety over $\mathbb{C}$ is algebraic.

Prop. (IV.8.6.13). Let X, Y be algebraic varieties over $\mathbb{C}$ and $f : X \to Y$ is morphism that is bijective, if f^{an} is an analytic isomorphism, then f is an isomorphism.

Proof: Cf.[GAGA Serre P9]. □

Cor. (IV.8.6.14). Let X, Y be algebraic varieties over $\mathbb{C}$, iff $f : X^{an} \to Y^{an}$ is holomorphic map and the image of f in $X^{an} \times Y^{an} = (X \times Y)^{an}$ comes from an algebraic subscheme, then f comes from an algebraic morphism. (Because $X^{an} \to \Gamma(X)$ is an analytic isomorphism).

Prop. (IV.8.6.15) (GAGA). Let X be a projective scheme in $\mathbb{P}^n_{\mathbb{C}}$, then: $\mathcal{F} \to \mathcal{F}^{an}$ defines an equivalence of categories between the coherent sheaves on X and coherent analytic sheaves on X^{an} that preserves cohomology groups.

Proof: Cf.[GAGA Serre P13]. □

Remark (IV.8.6.16). This theorem can be generalized to the case that X is proper over $\mathbb{C}$, Cf.[SGA1,Chap12].

Cor. (IV.8.6.17) (Chow). Any analytic subvariety of $\mathbb{CP}^n$ is projective algebraic.

Proof: Cf.[GAGA Serre P20]. □

Prop. (IV.8.6.18).

- Any meromorphic function on an algebraic variety $V \subset \mathbb{CP}^n$ is rational.
- Any meromorphic differential form on a smooth variety is algebraic.

- Any holomorphic map between smooth varieties can be given by rational maps.
- Any holomorphic vector bundle on a smooth variety is algebraic, i.e. transition function can be made rational.

Cf.[Griffith/Harris P168,170].

Cor. (IV.8.6.19). If the analytification of a variety X is a compact complex manifold, i.e. X is smooth(IV.8.6.2), then $K(X) = K(X^{an})$, as they are both morphism to $\mathbb{P}^1$.

Applications

Cf.[GAGA Serre].

Prop. (IV.8.6.20) (Generalized Riemann Existence Theorem). Let X be a normal scheme of finite type over $\mathbb{C}$. Given any finite morphism of analytic spaces(i.e. proper and has finite fibers) form a normal complex analytic space $f : X' \to X^{an}$, then there is a unique normal scheme X' and a finite morphism $g : X' \to X$ that $g^{an} = f$.

Proof: Cf.[SGA1, Chap12]. □

Cor. (IV.8.6.21) (Algebraic Fundamental Group).

7 Moishezon Manifolds

Def. (IV.8.7.1) (Moishezon Manifold). A compact complex manifold is called a **Moishezon manifold** iff $tr.d.K(X) = \dim X$, by(IV.8.1.4) this is the highest degree it can have. When X is an analytification of an algebraic variety X^{an}, $K(X^{an}) = K(X)$ by(IV.8.6.19), so Moishezon is a necessary condition for a compact complex manifold to be algebraic.

Prop. (IV.8.7.2) (Artin). The category os smooth proper algebraic spaces over $\mathbb{C}$ is equivalent to the category of Moishezon manifolds.

Proof: □

Prop. (IV.8.7.3) (Moishezon). Every Moishezon manifold that is Kähler is projective algebraic.

Proof: Cf.[Moishezon On n-dimensional compact varieties with n algebraically independent meromorphic functions]. □

Cor. (IV.8.7.4). Use all these above and Kodaira Embedding(IV.9.8.6), we have the following applications of compact complex manifdols:

- Hodge iff projective algebraic.
- Projective algebraic is abstract algebraic, abstract algebraic is Moishezon.
- Hodge manifold is Kähler.
- Kähler+Moishezon is projective algebraic.

Algebraic Compact Complex Manifold

Prop. (IV.8.7.5). If X_t is an algebraic family of nonsingular projective varieties over $\mathbb{C}$ parametrized by a variety T, then the functions $h^i(X_t, \mathcal{O}_y)$ are constant for all i.

IV.9 Kähler Geometry

Basic References are [Voisin], [Principle of Algebraic Geometry Griffith/Harris], more advanced stuffs to add from [Kähler Geometry] and [Complex Geometry Daniel]. Even more advanced is [Demailly Complex Analytic and Differential Geometry].

1 Kähler Metric

Def. (IV.9.1.1). The metric g is called **Kähler** iff ω_g is closed. In which case, it is called the **Kähler class** of g in $H^2_{dR}(M)$. A complex manifold with a Kähler metric is called a **Kähler manifold**.

If $g_{ij} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j})$, then $\omega_g = \sum_{ij} g_{ij} dz_i \wedge d\overline{z}_j$. Then the condition of ω_g being closed can in fact be written in derivatives of g.

Prop. (IV.9.1.2). If g is Hermitian, then ω_g is real, non-degenerate and $\frac{1}{n!}\omega^n$ is a volume form on M. In particular, if ω is Kähler, then it is a symplectic form.

Proof: If $g = \sum \varphi_i \otimes \overline{\varphi}_i$, then $\omega = i \sum \varphi_i \wedge \overline{\varphi}_i$, so it is clear that $\overline{\omega} = \omega$. ω is non-degenerate as g is. The last assertion follows from(I.2.7.6). □

Cor. (IV.9.1.3). If M is a compact Kähler manifold, then its even dimensional cohomology group doesn't vanish**??**.

Remark (IV.9.1.4). Notice there are notions like almost Hermitian and almost Kähler, similar to the definition of Hermitian and Kähler, but they are just defined using an almost complex structure on M. And a almost Kähler structure is Kähler iff $\nabla J = 0$, Cf.[Foundation of Differential Geometry Kobayashi].

Remark (IV.9.1.5) (Examples of Kähler Manifolds).

- If $M = \mathbb{R}^{2n}$, $g = \sum dx_i \wedge dx_i + \sum dy_i \wedge dy_i$, then $\omega_g = \frac{i}{2} \sum dz_i \wedge d\overline{z}_i$ is Kähler.
- The metric $\omega_g = \sum dz_i \wedge dz_i$ on a complex tori $\mathbb{C}^n/\Lambda$ is Kähler.
- Any compact Riemann surface is Kähler, because $d\omega$ is a 3-form so vanish.
- if $M = B(0,1) \in \mathbb{C}^n$ and $\omega_g = i\partial\overline{\partial} \log \frac{1}{1-|z|^2}$, then it is Kähler.
- The product metric on the product space $M \times N$ of two Kähler manifold is Kähler.
- A submanifold of a Kähler manifold is Kähler, as the Kähler form is the pullback of the Kähler form of the large manifold.

Prop. (IV.9.1.6) (Fubini-Study Metric). The **Fubini-Study metric** form on $\mathbb{CP}^n$ is defined locally to be $i\partial\overline{\partial}|s|^2$, for any local lifting of the projection $\mathbb{C}^n - \{0\} \to \mathbb{CP}^n$. This doesn't depend on the lifting, as $\partial\overline{\partial}(\log f + \log \overline{f}) = 0$, so they glue together to be a global form on $\mathbb{CP}^n$. It can be checked, ω is translation invariant and on the coordinate $(1, w_1, \ldots, w_n) \to (w_1, \ldots, w_n)$, $\omega|_{(0,\ldots,0)} = \sum dw_i \wedge dw_i$, so it is positive definite.

Cor. (IV.9.1.7). Any projective manifold is Kähler.

Prop. (IV.9.1.8). the Fubini-Study metric on $\mathbb{CP}^n$ has sectional curvature $1 \leq K \leq 4$.

Proof: Cf.[Do Carmo P188]. □

Prop. (IV.9.1.9) (Kähler Normal Coordinate). For a Hermitian metric g on M, g is Kähler iff for any point p of M, there is a holomorphic coordinate centered at p, $\omega_g = \sum g_{ij} dz_i \wedge d\bar{z}_j$ satisfying $g_{ij}(p) = 0$ and $dg_{ij}(p) = 0$. This coordinate is called **Kähler normal coordinate**. (Notice this is different from Darboux theorem, because this coordinate should be holomorphic).

Proof: Cf.[Complex Geometry P210]. □

2 Geometry of Kähler Manifolds

Prop. (IV.9.2.1). Let (M, J, g) be a Kähler manifold, then the complexification of the Levi-Civita connection of g restricts to the Chern connection on $T^{1,0}M$.

Proof: Cf.[Complex Geometry note 石亚龙 48] and [Complex geometry Daniel Chap4.A]. □

Prop. (IV.9.2.2). For a Kähler manifold, $\nabla J = 0$.

Proof: The problem depends only on first derivative, so choosing a Kähler normal nbhd(IV.9.1.9), we may choose J to be constant, so obviously $\nabla J(p) = 0$, P is arbitrary, so $\nabla J = 0$. □

Cor. (IV.9.2.3). $\nabla(JX) = J\nabla X$, so $R(X, Y)JZ = JR(X, Y)Z$, thus

$$\langle R(JX, JY)Z, W\rangle = \langle R(Z, W)JX, JY\rangle = \langle R(Z, W)X, Y\rangle = \langle R(X, Y)Z, W\rangle,$$

so $R(JX, JY)Z = R(X, Y)Z$.

Prop. (IV.9.2.4). The curvature tensor of the complexified Levi-Civita connection on a Kähler manifold can be calculated in terms of $\partial_i, \overline{\partial}_j$, Cf.[Complex Geometry note 石亚龙 50].

3 Kähler Identities

Let X be a compact complex Kähler manifold.

Def. (IV.9.3.1). Introduce some operators:

- $d^c = i(\overline{\partial} - \partial)$, then $dd^c = 2i\partial\overline{\partial}$.
- The **Lefschetz operator** $L(\eta) = \omega \wedge \eta$. Λ is defined as the formal adjoint of L as $A^{p,q}$ is an inner space. In fact, $\Lambda = \pm * L *$.
- $h = (k - n)$ on $\mathcal{A}^k(X)$.

Prop. (IV.9.3.2). $[L, \Lambda] = p + q - n$ on (p, q)-forms.

Proof: The problem doesn't depends on the derivatives, so using the Kähler normal coordinate(IV.9.1.9), it suffice to prove for $\mathbb{C}^n$, for this, Cf.[Griffith/Harris P120] or [Complex Geometry P34]. □

Prop. (IV.9.3.3) (Kähler Identities).

$$[\Lambda, \partial] = i\overline{\partial}^*, \quad [\Lambda, \overline{\partial}] = -i\partial^*.$$

Proof: The second one follows from the first because ω is a real form. For the first, notice only first derivation are involved, so by using the Kähler normal coordinate, it suffice to prove for $\mathbb{C}^n$, and this is by [Complex Geometry 石亚龙 P61]. □

Cor. (IV.9.3.4).

$$[\Lambda, d^c] = d^*, \quad [\Lambda, d] = -d^{c*}.$$

Prop. (IV.9.3.5). Δ_d commutes with both L and Λ.

Proof: L commutes with d because ω is closed, so taking adjoints, Λ commutes with d^*. Now by Kähler identities,

$$\Lambda\Delta_d = \Lambda(dd^* + d^*d) = -d^{c*}d^* + dd^*\Lambda - dd^{*c} + d^*d\Lambda = \Delta_d\Lambda.$$

So taking adjoints, Δ_d also commutes with L. □

Prop. (IV.9.3.6). In the Kähler case, $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$.

Proof:

$$\Delta_d = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) = \Delta_\partial + \Delta_{\overline{\partial}} + (\partial\overline{\partial}^* + \overline{\partial}^*\partial) + (\overline{\partial}\partial^* + \partial^*\overline{\partial})$$

So it suffice to prove $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$(so $\overline{\partial}\partial^* + \partial^*\overline{\partial} = 0$ by conjugation), and $\Delta_\partial = \Delta_{\overline{\partial}}$. For the first, use Kähler identities, then

$$i(\partial\overline{\partial}^* + \overline{\partial}^*\partial) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = 0$$

For the second, using Kähler identities,

$$i\Delta_{\overline{\partial}} = \overline{\partial}[\Lambda, \partial] + [\Lambda, \partial]\overline{\partial} = \overline{\partial}\Lambda\partial + \partial\overline{\partial} - \Lambda\overline{\partial}\partial - \partial\Lambda\overline{\partial}$$

and the same is miraculous true for Δ_∂, so the result is true. □

4 Hodge Theory

Prop. (IV.9.4.1) (Hodge Decomposition of compact Kähler Manifold). For a compact Kähler manifold X,

$$H^r_{dR}(X, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}_{\overline{\partial}}(X) \cong \bigoplus_{p+q=r} H^q(X, \Omega^p)$$

and $\overline{H^{p,q}_{\overline{\partial}}(X)} \cong H^{q,p}_{\overline{\partial}}(X)$. Moreover, this decomposition doesn't depends on the Kähler metric.

Proof: (IV.9.3.6) shows that Δ_d maps $A^{p,q}$ to $A^{p,q}$, so $\mathcal{H}^{p+q}_d \cap A^{p,q} = H^{p,q}_{\overline{\partial}}(X)$. The last assertion is seen using the Δ_d definition.

If chosen two different Kähler metric g, g', there $\mathcal{H}^{p,q}(X, g) \cong H^{p,q}(X) \cong H^{p,q}(X, g')$. If α, α' be g, g' $\overline{\partial}$-harmonic respectively, so by definition $\alpha - \alpha' = \overline{\partial}\gamma$ for some γ, and they are both d-harmonic, so $d\overline{\partial}\gamma = 0$, and $\overline{\partial}\gamma$ is g-orthogonal to $\mathcal{H}^k(X, g)$ by Hodge decomposition for $\overline{\partial}$ with metric g, so by Hodge theorem for d with metric g, $\partial\gamma$ is d-exact, so $[\alpha] = [\alpha']$. □

Cor. (IV.9.4.2). Betti number $b_r = \sum_{p+q=r} h^{p,q}$, $h^{p,q} = h^{q,p}$. In particular, b_{2k+1} is always even.

Cor. (IV.9.4.3) (Holomorphic Form on Kahler Manifold is Closed). $\mathcal{H}^{p,0}_{\overline{\partial}}(X) = H^0(X, \Omega^p)$. Now a $(p, 0)$-form is automatically $\overline{\partial}^*$-closed, so it is $\overline{\partial}$-harmonic iff it is holomorphic. So we conclude any holomorphic p-form on a Kähler manifold is d-closed, even d-harmonic.

Lemma (IV.9.4.4) ($\partial\overline{\partial}$-lemma). A closed differential form η on a compact Kähler manifold M is d-exact iff it is ∂-exact iff it is $\overline{\partial}$-exact iff it is $\partial\overline{\partial}$-exact.

Proof: Now $\Delta_d, \Delta_{\overline{\partial}}, \Delta_{\partial}$ are all the same, By Hodge theorem, it suffice to prove, if a form is orthogonal to $\mathcal{H}^{p,q}(X)$, then it is $\partial\overline{\partial}$-exact(this implies other exactness).

Noe η is d-closed hence ∂ and $\overline{\partial}$-closed, then $\eta = \partial\gamma$ for some γ, and then $\gamma = \overline{\partial}\beta + \overline{\partial}^*\beta' + \beta''$ for β'' harmonic. So $\eta = \partial\overline{\partial}\beta + \partial\overline{\partial}^*\beta'$, and then $\overline{\partial}\eta = \overline{\partial}\overline{\partial}^*\partial\beta = 0$, but then inner product with $\partial\beta$ shows $\overline{\partial}^*\partial\beta = 0$, so $\eta = \partial\overline{\partial}\beta$. □

Cor. (IV.9.4.5) (Kodaira-Serre Duality). By(IV.3.8.15), For a Hermitian line bundle over a compact Hermitian complex manifold X, from Hodge theorem and(IV.8.5.2), we get

$$H^p(X, \Omega^q(E)) \cong H^{n-p}(X, \Omega^{n-q}(E^*))$$

induced by $\overline{*}_E$ and $\overline{*}_{E^*}$. Moreover, there is a perfect pairing

$$H^p(X, \Omega^q(E)) \times H^{n-p}(X, \Omega^{n-q}(E^*)) \to \mathbb{C}$$

induced by

$$\mathcal{H}^{p,q}(X, E) \times \mathcal{H}^{n-p,n-q}(X, E^*) \to \mathbb{C} : (\alpha, \beta) \mapsto \int_X \alpha \wedge \overline{*}_E \beta$$

In fact, $\int_X \alpha \wedge \overline{*}_E \alpha = ||\alpha||^2 \neq 0$.

Prop. (IV.9.4.6). Holomorphic 1-forms on a compact complex surface is closed.?

Prop. (IV.9.4.7) (Hard Lefschetz Theorem). For a compact Kähler manifold M, the map

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism,(notice it is defined because L commutes with d).

Define the **primitive cohomology class** $P^{n-k}(M) = \mathrm{Ker}\, L^{k+1}$ on H^{n-k}, then

$$H^m(M) = \oplus_k L^k P^{m-2k}(M).$$

Proof: Cf.[Griffith/Harris P122], using representation theory of $\mathfrak{sl}_2$. □

Prop. (IV.9.4.8) (Hodge-Riemann Bilinear Relation). Let (X, ω) be a Kähler manifold, if $\alpha \neq 0 \in H^{p,q}(X)$ is a primitive cohomology class, then

$$i^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} \int_X \alpha \wedge \overline{\alpha} \wedge \omega^{n-p-q} > 0$$

Proof: Cf.[Griffith/Harris] or [Complex Geometry Daniel P138]. □

Cor. (IV.9.4.9). For a compact Kähler manifold of complex dimension $2m$,

$$\mathrm{sgn}(X) = \sum_{p,q=0}^{m} (-1)^p h^{p,q}(m)$$

Proof: Cf.[Complex Geometry Daniel P140]. □

Prop. (IV.9.4.10) (Hirzebruch-Riemann-Roch). By(IV.3.8.8), for a n-dimensional complex line bundle L over a compact Kähler manifold M,

$$\chi(M,L)=\int_M[\mathrm{ch}(E)\mathrm{td}(T^{1,0}M)]_n.$$

Where $\chi(M,L)=\sum_{q=0}^n(-1)^q\dim H^q(M,E)$, ch is the Chern character(IV.3.7.5) and $\mathrm{td}(T^{1,0}M)$ is the Todd polynomial, i.e. Taylor expansion of $\prod_{i=1}^r\dfrac{t_i}{1-e^{-t_i}}$ in terms of the symmetric polynomial, applied to $c_i(T^{1,0}M)$.

Cor. (IV.9.4.11) (Riemann-Roch). By(IV.3.8.9), for a complex vector bundle E over a Riemann Surface M, let $\deg E=\int_M c_1(E)$, then

$$\chi(M,E)=H^0(M,E)-\dim H^1(M,E)=\deg E+\mathrm{rk}(E)(1-g).$$

Cor. (IV.9.4.12). For other examples of corollaries of Hirzebruch-Riemann-Roch theorem, Cf.[Complex Geometry P232].

Formality of Complex Kähler Geometry

Cf.[Complex Geometry Daniel Chap3.A].

5 Jacobian and Abanese

Prop. (IV.9.5.1) (Complex Tori). The **complex tori** is defined to be $X=\mathbb{V}/\Gamma$, where $V\cong\mathbb{C}^n$ and Γ is a complete lattice. It is a Kähler manifold. And $H^2(X,\mathbb{R})\cong V^*\wedge V^*$.

Proof: For the last assertion, use the fact that the cotangent bundle is trivial(III.11.1.4). □

Lemma (IV.9.5.2). if X is compact Kahler, then the natural map $H^k(X,\mathbb{C})\to H^k(X,\mathcal{O}_X)$ is just the projection onto the $(0,k)$-part. In particular, the image is in $H^{0,k}(X)$.

Proof: By Hodge decomposition, the definition of Dolbeault cohomology and the commutative diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] & \mathcal{A}^0(X) \arrow[r, "d"] \arrow[d, "="] & \mathcal{A}^1(X) \arrow[r, "d"] \arrow[d, "\pi_{0,1}"] & \mathcal{A}^2(X)\dots \arrow[d, "\pi_{0,2}"] \\
\mathcal{O}_X \arrow[r] & \mathcal{A}^0(X) \arrow[r, "\bar\partial"] & \mathcal{A}^1(X) \arrow[r, "\bar\partial"] & \mathcal{A}^2(X)\dots
\end{tikzcd}$$

□

Prop. (IV.9.5.3) (Lefschetz theorem on $(1,1)$-forms). By(IV.8.5.7), we know that the image of $Pic_{\mathbb{C}}(X)\to H^2(X,\mathbb{Z})$ is trivial in $H^2(X,\mathcal{O}_X)$. If X is compact Kähler, there is Hodge decomposition(IV.9.4.1) $H^2(X,\mathcal{O}_X)=H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X)$.

So if we define $H^{1,1}(X,\mathbb{Z})=H^2(X,\mathbb{Z})\cap H^{1,1}(X)$, then the image of $Pic_{\mathbb{C}}(X)$ is contained in $H^{1,1}(X,\mathbb{Z})$ by(IV.9.5.2), and it is also surjective, this is to say, $NS(X)=H^{1,1}(X)$

Proof: Because by the long exact sequence of(IV.8.5.7) and(IV.9.5.2) again, an $\alpha\in H^2(X,\mathbb{C})$ is in $H^{1,1}(X,\mathbb{Z})$ iff α is in the image of $Pic_{\mathbb{C}}(X)\to H^{1,1}(X,\mathbb{Z})$. □

Cor. (IV.9.5.4). The image of $H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X)$ is a lattice. In particular, it is isomorphic to $\mathbb{Z}^{b_1(X)}$.

Proof: $H^1(X,\mathbb{Z})$ is a lattice in $H^1(X,\mathbb{R})$, and $H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O}_X) = H^{0,1}(X,\mathbb{C})$ is an isomorphism, because $H^{0,1}(X,\mathbb{C})$ are conjugate to $H^{1,0}(X,\mathbb{C})$ and $H^1(X,\mathbb{R})$ is real. □

Def. (IV.9.5.5) (Jacobian). The **Jacobian** $\mathrm{Jac}(X)$ of a compact Kähler manifold X is defined to be $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$, so it is a complex torus of dimension $b_1(X)$ by(IV.9.5.4), it is also the kernel of the first Chern class map by the long exact seqence(IV.8.5.7), i.e.

$$0 \to \mathrm{Jac}(X) \to Pic(X) \xrightarrow{c} NS(X) \to 0$$

Def. (IV.9.5.6) (Albanese). The **Albanese** $\mathrm{Alb}(X)$ of a compact Kähler manifold X is defined to be the complex torus $H^0(X,\Omega^1_X)^*/H_1(X,\mathbb{Z})$, where

$$H^1(X,\mathbb{Z}) \to H^0(X,\Omega_X)^* : [\gamma] \mapsto \Big(u \mapsto \int_\gamma u\Big)$$

(Notice this is well-defined because by(IV.9.4.3) any $u \in H^0(X,\Omega^1_X)$ is closed).

Fix a base point x_0 of X, the **Albanese map** $\mathrm{Alb} : X \to \mathrm{Alb}(X)$ is defined to be

$$x \mapsto \Big(u \mapsto \int_{x_0}^x u\Big)$$

It is holomorphic and functorial in (X,x_0). It is just the so called **Abel-Jacobi map** in case when X is a Riemann surface.

6 Positivity

Def. (IV.9.6.1) (Positive Line Bundle). A 2-form ω on a Hermitian complex manifold M is called **positive** iff $\omega(u,Ju) \geq 0$ for $u \neq 0 \in TM$, which is equivalent to $-i\omega(v,\overline{v}) > 0$ for all $v \in T^{1,0}X$.

A holomorphic vector bundle is called **(Griffith-)positive** iff there exists a Hermitian metric on it that the curvature form Ω for the Chern connection(IV.8.5.12) satisfies $h(\Omega(s),s)(v,\overline{v}) > 0$ for all $s \in E$ and $v \in T^{1,0}X$.

The pullback of a positive line bundle along an immersion is positive.

Prop. (IV.9.6.2) (Positivity on Kähler Manifolds). On a compact Kähler manifold, being positive is a topological property for line bundles. It is equivalent to the first Chern class of L can be represented by a positive form in $H^2_{dR}(M)$.

Proof: $c_1(L) = [\frac{i}{2\pi}\Omega]$, so one direction is trivial, and if $c_1(L) = [\frac{i}{2\pi}\theta]$, choose an arbitrary Hermitian metric h on L, then by $\partial\overline{\partial}$-lemma(IV.9.4.4), $\theta = \Omega + \overline{\partial}\partial\rho$ for some smooth function ρ. Then $e^\rho h$ has $\Omega = \theta$ by formula(IV.8.5.13). □

Cor. (IV.9.6.3). On a compact Kähler manifold, if L is positive, then for any other Hermitian line bundle L', $kL + L'$ is positive.

Prop. (IV.9.6.4). The hyperplane line bundle $\mathcal{O}(1)$(IV.8.5.8) is positive.

Proof: The hyperplane line bundle is dual to the tautological line bundle. The metric on the tautological line bundle is given by locally $g_i = \frac{1}{|z_i|^2}\sum|z_i|^2$. It is compatible with the transition map, and then by(IV.8.5.13), the Chern curvature is

$$\overline{\partial}\partial(\frac{1}{|z_i|^2}\sum|z_i|^2) = \overline{\partial}\partial(\sum|z_i|^2).$$

So by(IV.2.3.6)the curvature of the hyperplane line bundle times i is just the Fubini-Study metric form(IV.9.1.6), so it is positive. □

Prop. (IV.9.6.5). For $\widetilde{X}\to X$ the blowing-up of X at a point x, If L is a positive line bundle on X, then for any integer n, there exists a $k>0$ that $\pi^*L^k - nE$ is a positive line bundle on $\widetilde{X}$, where E is the exceptional divisor.

Proof: Involves explicit metric calculation, Cf.[Kodaira Embedding Theorem P11] and [Complex Geometry P249].. □

7 Kodaira Vanishing Theorem

Prop. (IV.9.7.1) (Nakano Identities). For a holomorphic vector bundle over a compact Kähler manifold (M,ω) with Hermitian metric h, introduce operators L and Λ as before. If we denote the $(1,0)$ and $(0,1)$-part of the Chern connection on E by D' and $D''=\overline{\partial}$, then

$$[\Lambda,\overline{\partial}] = -iD'^*,\quad [\Lambda, D'] = i\overline{\partial}^*$$

Proof: The question is local, choose normal coordinate frame at x(IV.8.5.15), then by the formula of Chern connection(IV.8.5.13), $\nabla_E = d+A$, $A(x)=0$, and $\nabla_{E^*} = d+B$, $B(x)=0$. so

$$[\Lambda,\overline{\partial}_E] + iD'^* = [\Lambda,\partial] + i\partial^* + [\Lambda, A^{0,1}] + iB^{0,1}$$

where the usual Kähler identities(IV.9.3.3) are used. Then it is zero when evaluated at x, Cf.[Demailly Complex Analytic and Differential Geometry P329]. □

Cor. (IV.9.7.2) (Bochner-Kodaira-Nakano Identity).

$$\Delta_{\overline{\partial},E} - \Delta_{D',E} = i[\Omega,\Lambda]$$

Proof:

$$-i\Delta_{D',E} = D'[\Lambda,\overline{\partial}] + [\Lambda,\overline{\partial}]D' = D'\Lambda\overline{\partial} - D'\overline{\partial}\Lambda + \Lambda\overline{\partial}D' - \overline{\partial}\Lambda D'$$

and similar calculation for $i\Delta_{\overline{\partial},E}$, so

$$i\Delta_{\overline{\partial},E} - i\Delta_{D',E} = \Lambda(\overline{\partial}D' + D'\overline{\partial}) - (\overline{\partial}D' + D'\overline{\partial})\Lambda = -[\Omega,\Lambda].$$

□

Prop. (IV.9.7.3) (Kodaira-Akizuki-Nakano Vanishing Theorem). If L is a positive line bundle on a compact Kähler manifold M, then

$$H^p(M,\Omega^q(L)) = 0$$

for $p+q>n$. In particular, $H^q(M,\mathcal{K}_X\otimes L) = 0$ for $q>0$.

Proof: By Hodge theorem(IV.3.8.13), it suffice to prove there are no harmonic (p,q)-forms$\in \mathcal{H}^{p,q}(X,L)$ on L.

As $i\Omega = \omega$ is positive, we may endow M with the metric ω, then by(IV.9.7.2) and(IV.9.3.2), $\Delta_{\overline{\partial}} - \Delta_{D'} = [L,\Lambda] = p+q-n$ on $A^{p,q}$.

So if $s \in \mathcal{H}^{p,q}(X,L)$, then $(\Delta_{\overline{\partial}}s - \Delta_{D'}s, s) = (p+q-n)||s||^2 \geq 0$, but $(\Delta_{\overline{\partial}}s - \Delta_{D'}s, s) = -(\Delta_{D'}s,s) = -||D's||^2 - ||D'^*s||^2 \leq 0$, so $s=0$. □

Cor. (IV.9.7.4) (Serre's Theorem). Let L be a positive line bundle on a compact complex Kähler manifold X, then for any holomorphic vector bundle E, for m large, $H^q(X, L^m \otimes E) = 0$.

Proof: Same notation as in the proof of(IV.9.7.3), choose Hermitian structure on E and L and their Chern connections by ∇_E, ∇_L, the corresponding Chern connection on $E \otimes L^m$ is denoted by ∇, and make sure $\frac{i}{2\pi}F_{\nabla_L}$ is the Kähler form ω, then for any harmonic form $\alpha \in \mathcal{H}^{p,q}(X, E\otimes L^m)$, by(IV.9.7.2), $\frac{i}{2\pi}([\Lambda, F_\nabla](\alpha),\alpha) \geq 0$, but $\frac{i}{2\pi}F_\nabla = \frac{i}{2\pi}F_{\nabla_E} + m\omega$, so

$$0 \leq \frac{i}{2\pi}([\Lambda, F_{\nabla_E}](\alpha),\alpha) + m(n-p-q)||\alpha||^2$$

Notice $|([\Lambda, F_{\nabla_E}](\alpha),\alpha)|$ has a bound by Schwartz inequality, then if $p+q>n$ and m sufficiently large, α must by 0. In this case $\mathcal{H}^{p,q}(X, E\otimes L^m) = 0$, but $\mathcal{H}^{0,q}(X, \mathcal{K}_X \otimes E \otimes L^m) \subset \mathcal{H}^{n,q}(X, E\otimes L^m)$, so it is 0. Now we've proved $H^q(X, \mathcal{K}_X \otimes E \otimes L^m) = 0$ for any E if m is large. But E is arbitrary, so the conclusion is true. □

Cor. (IV.9.7.5) (Grothendieck's Lemma). Every holomorphic line bundle E over $\mathbb{CP}^1$ is uniquely isomorphic to a finite direct sum of $\mathcal{O}(a_i)$.

Proof: If E has rank 1, this is the content of(IV.8.5.9), so use induction on rank of E. Choose a maximal a that $\mathrm{Hom}(\mathcal{O}(a), E) = H^0(CP^1, E(-a)) \neq 0$. This a exists because Serre's Theorem(IV.9.7.4) shows that $H^1(\mathbb{CP}^1, E(-a)) = 0$ for a sufficiently small, and Riemann-Roch(IV.3.8.9) shows that $\chi(\mathbb{CP}^1, E(-a)) = \deg E + \mathrm{rk}(E)(1-a)$ is positive for a sufficiently small, so $H^0(CP^1, E(-a)) \neq 0$. Conversely, if a is sufficiently large, then $H^0(\mathbb{CP}^1, E(-a)) \cong H^1(\mathbb{CP}^1, E^*(a-2)) = 0$(Notice $\mathcal{K}_{\mathbb{CP}^n} = \mathcal{O}(-n-1)$).

So now there is an exact sequence of sheaves

$$0 \to \mathcal{O}(a) \xrightarrow{s} E \to E_1 \to 0$$

I claim E_1 is also a vector bundle, because s never vanishes, otherwise if it vanish at some x, then we can divide by a linear factor $s_x \in H^0(\mathbb{CP}^1, \mathcal{O}(1))$ to get a map $\mathcal{O}(a+1) \to E$, contradicting the maximality. So by induction $E_1 = \oplus\mathcal{O}(a_i)$, then I claim $a_i \leq a$, because otherwise $H^0(\mathbb{CP}^1, E_1(-a-1)) \neq 0$, and by the exact sequence $0 \to \mathcal{O}(-1) \to E(-a-1) \to E_1(-a-1) \to 0$, $H^0(\mathbb{CP}^1, E(-a-1)) \neq 0$, contradiction.

Then we want to show the above sequence splits, this is equivalent to

$$0 \to E_1^*(a) \to E^*(a) \to \mathcal{O} \to 0$$

splits, and his follows from the fact $H^1(\mathbb{CP}^1, E_1^*(a)) = H^1(\mathbb{CP}^1, \oplus\mathcal{O}(a-a_i)) = 0$, by Serre duality. So there is a section lifting $\mathcal{O} \to E^*(a)$, which splits the sequence. □

Prop. (IV.9.7.6) (Weak Lefschetz Theorem). Let X be a compact Kähler manifold and Y be a submanifold that the line bundle $\mathcal{L}(Y)$ is positive, then the canonical restriction map $H^k(X,\mathbb{C}) \to H^k(Y,\mathbb{C})$ is isomorphism for $k \leq n-2$ and injective for $k = n-1$.

Proof: In fact, using Hodge decomposition, it suffices to prove on the level of $H^q(X,\Omega_X^p)$. Tensoring the exact sequence
$$0\to\mathcal{O}_X(-Y)\to\mathcal{O}_X\to\rangle_* i_Y^* O_X\to 0$$
with Ω_X^p and taking the cohomology. By Serre duality and Kodaira vanishing(IV.9.7.3), the map $H^q(X,\Omega_X^p)\to H^q(X,\Omega^p i_* i_Y^* O_X)$ is isomorphism for $p+q<n-1$ and injection for $p+q=n-1$.

Next consider the exact sequence $0\to TY\to TX\to\mathcal{N}_{Y/X}\to 0$. By(III.2.4.17) there is an exact sequence
$$0\to\wedge^p TY\to\wedge^p TX|_Y\to\wedge^{q-1}TY\mathcal{N}_{Y/X}\to 0$$
Taking dual and applying adjunction formula(IV.8.1.2), it becomes:
$$0\to\Omega_Y^{q-1}\otimes\mathcal{O}(-N)\to\Omega_X^q|Y\to\Omega_Y^q\to 0$$
Taking cohomology and use Serre duality and Kodaira vanishing as before, the result follows, and the composition is also true. $\square$

Remark (IV.9.7.7). There is a topological proof of weak Lefschetz theorem in [Bott On a theorem of Lefschetz].

8 Kodaira Embedding Theorem

Prop. (IV.9.8.1) (Kodaira map). For a holomorphic line bundle L on a compact complex manifold M, if $s_0,\ldots,s_n$ be a basis of $H^0(X,L)$, we try to define a map from M to $\mathbb{CP}^n: x\to[s_0(x),\ldots,s_n(x)]$. This is independent of the change of coordinates because $g_{\alpha\beta}$ is invertible, and it is definable iff L is basepoint-free. This map is holomorphic where it is definable.

Def. (IV.9.8.2). For a holomorphic vector bundle L on a compact complex manifold X, L is called

- **semi-ample** iff for m large, L^m is basepoint-free.
- **very ample** iff L is basepoint-free and the Kodaira map $\iota_L: X\to\mathbb{CP}^N$ is a holomorphic embedding.
- **ample** iff for m large, L^m is very ample.

Lemma (IV.9.8.3) (Cohomological Method for Very Ampleness). For the above Kodaira map to be a holomorphic embedding, it suffice to show that the map is definable, injective and surjective on cotangent space. For these, it is equivalent to $H^0(X,L)\to L_x$ surjective, $H^0(X,L)\to L_x\oplus L_y$ surjective, and $L\otimes\mathcal{I}_x\to L_x\otimes T^{1,0*}(X)_x$ surjective. And they are true if
$$H^1(X,L\otimes\mathcal{I}_x)=0,\quad H^1(X,L\otimes\mathcal{I}_{x,y})=0,\quad H^1(X,L\otimes\mathcal{I}_x^2)=0.$$
respectively.

Proof: Basepoint-free at x is easily seen to be equivalent to $H^0(X,L)\to L_x$ surjective. And there is an exact sequence of sheaves:
$$0\to L\otimes\mathcal{I}_x\to L\to L_x\to 0$$
where L_x means the skyscraper sheaf. So $H^1(X,L\otimes\mathcal{I}_x^2)=0$ induces the result.

Injective is easily seen to be equivalent to $H^0(X,L)\to L_x\oplus L_y$ surjective. And there is an exact sequence of sheaves:

$$0\to L\otimes\mathcal{I}_{x,y}\to L\to L_x\oplus L_y\to 0$$

where $\mathcal{I}_{x,y}$ is the sheave of functions vanishing at x and y, and $L_x\oplus L_y$ means the skyscraper sheaf. So $H^1(X,L\otimes\mathcal{I}_{x,y})=0$ induces the result.

For the surjection on cotangent spaces, given any point x, choose a basis $s_1,\ldots,s_n$ of sections in $H^0(X,L)$ vanishing at x, and by basepoint-free, there is a s_0 not vanishing at x, then on a coordinate, the Kodaira map is given by $x\to(s_1/s_0,\ldots,s_n/s_0)$, then it need to be checked $d_x(s_i/s_0)=d_x(x_i)/s_0$ span $T^{1,0*}(X)_x$. But there are exact sequences of sheaves:

$$0\to L\otimes\mathcal{I}_x^2\to L\otimes\mathcal{I}_x\xrightarrow{d_x}L_x\otimes T_x^{1,0*}\to 0$$

where d_x is given by $d_x(s\otimes f)=s(x)\otimes d_x(f)$(by the universal property of skyscraper sheaf, it suffice to give a map $(L\otimes\mathcal{I}_x\to L_x\otimes T_x^{1,0*})$, notice this is independent of the coordinate because $d_x(s_\alpha)=d_x(g_{\alpha\beta}s_\beta)=g_{\alpha\beta}d_x(s_\beta)$, as s_α vanishes at x, so this is truly a sheaf map, and its kernel is $L\otimes\mathcal{I}_x^2$. So $H^1(X,L\otimes\mathcal{I}_x^2)=0$ induces the result. □

Prop. (IV.9.8.4). A holomorphic line bundle L on a compact Kähler manifold is ample iff it is positive.

Proof: If L is ample, then L^m is the pullback of the hyperplane bundle by the Kodaira map. The hyperplane line bundle is positive by(IV.9.6.4), so L^m is positive with the induced metric, so L is also positive given the m-th roots of the induced metric (notice the metric of line bundle is just locally a number compatible with transition map).

Conversely, using(IV.9.8.3), we want to find a L^k that $H^1(X,L^k\otimes\mathcal{I}_x)=0,H^1(X,L^k\otimes\mathcal{I}_{x,y})=0,H^1(X,L^k\otimes\mathcal{I}_x^2)=0$. First notice it suffice to prove for single points when k is sufficiently large, because the holomorphic embedding is an open property and X is compact so a sufficiently large k will suffice.

Consider the blowing-up $\widetilde{X}$ at a point x, there is a commutative diagram

$$\begin{tikzcd}
H^0(X,L^k) \arrow[r] \arrow[d, "\pi^*"] & L_x^k \arrow[d, "\cong"] \\
H^0(\widetilde{X},\pi^*L^k\otimes L_x^k) \arrow[r] & H^0(E,\mathcal{O}_E)\otimes L_x^k
\end{tikzcd}$$

The right vertical map is isomorphism as $E\cong\mathbb{CP}^n$, so $H^0(E,\mathcal{O}_E)=\mathbb{C}$. The left exact sequence is also isomorphism: it is injective because π is surjective, and it is surjective because: if $\dim X=1$, then $\pi=\mathrm{id}$ so trivially true, and if $\dim X\geq 2$, then because $\pi:\widetilde{X}-E\cong X-\{x\}$, any holomorphic function on $\widetilde{X}$ induces a holomorphic function on $X-\{x\}$ and by Hartog's theorem(V.2.8.3), it comes from a holomorphic function on X.

Now the second horizontal line is part of the cohomology exact sequence of(III.6.1.14)

$$0\to\pi^*L^k\otimes\mathcal{O}_{\widetilde{X}}(-E)\to\pi^*L^k\to\pi^*L^k|_E\to 0$$

So it is reduced to prove $H^1(\widetilde{X},\pi^*L^k\otimes\mathcal{O}(-E))=0$, but by(IV.8.1.12), $\pi^*L^k-E=\pi^*L^k-E+\mathcal{K}_{\widetilde{X}}-\pi^*\mathcal{K}_X-(n-1)E=\mathcal{K}_{\widetilde{X}}+(\pi^*L^k-E)+\pi^*(L^k-\mathcal{K}_X)$, and by(IV.9.6.5)(IV.9.6.3) the last two are positive when k is large, so the conclusion follows from Kodaira vanishing(IV.9.7.3).

The proof of $H^1(X, L^k \otimes \mathcal{I}_{x,y}) = 0$ is verbatim, just use blowing-up at two different points.

To prove $H^1(X, L^k \otimes \mathcal{I}_x^2) = 0$, consider the blowing-up $\widetilde{X}$ at a point x, notice there is a commutative diagram

$$\begin{array}{ccc} H^0(X, L^k \otimes \mathcal{I}_x) & \xrightarrow{d_x} & L_x^k \otimes T^{1,0*}X_x \\ \big\downarrow{\scriptstyle \pi^*} & & \big\downarrow{\scriptstyle \cong} \\ H^0(\widetilde{X}, \pi^* L^k - E) & \longrightarrow & L_x^k \otimes H^0(E, -E) \end{array}$$

In fact this comes from the two commuting exact sequences twisted with $\pi^* L^k$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\mathcal{I}_x^2 & \longrightarrow & \pi^*\mathcal{I}_x & \xrightarrow{d_x} & \pi^* T^{1,0*}X_x & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(-2E) & \longrightarrow & \mathcal{O}(-E) & \longrightarrow & \mathcal{O}_E(-E) & \longrightarrow & 0 \end{array}$$

The second line is(III.6.1.14) and the fact a section vanishing at x lifts to a section vanishing at E thus equivalent to a section in the twisted sheaf $- \otimes \mathcal{O}(-E)$. These two exact sequences commutes because

Back to the commutative diagram, the above argument also shows that the first vertical map is isomorphism. To show the second vertical map is isomorphism, notice by(IV.8.1.11) $\mathcal{O}(-E)$ is just the hyperplane line bundle on E, so $H^0(E, -E) \cong T^{1,0*}X_x$, we need to know the vertical map is the natural map $V^* \to H^0(\mathbb{P}(V), \mathcal{O}(1))$. This in fact need some careful calculation using coordinates in(IV.8.1.11).?

Now the map d_x is surjective iff the second horizontal map is surjective, with is part of the cohomology exact sequence of

$$0 \to \pi^* L^k \otimes \mathcal{O}_{\widetilde{X}}(-2E) \to \pi^* L^k \mathcal{O}_{\widetilde{X}}(-E) \to \pi^* L^k|_E \to 0$$

So it is reduced to prove $H^1(\widetilde{X}, \pi^* L^k \otimes \mathcal{O}(-wE)) = 0$, which is by Kodaira vanishing theorem the same reason as before. □

Cor. (IV.9.8.5) (Kodaira Embedding Theorem). If a compact complex manifold M has a positive line bundle, then it is projective.

Cor. (IV.9.8.6) (Hodge Manifold). A compact Kähler manifold X is a projective submanifold iff it has a closed positive $(1,1)$-form ω whose cohomology class $[\omega]$ is rational/integral(i.e. in $H^2(X, \mathbb{Q})$). In fact, a Kähler manifold with a Hodge metric is called a **Hodge manifold**. So Hodge manifolds are just those Kähler manifolds that are projective.

Proof: if ω is rational, then a multiple of it is integral, then there is a L that $c_1(L) = k[\omega]$ by Lefschetz theorem on $(1,1)$-forms(IV.9.5.3), so L is positive by(IV.9.6.2), so X is projective. Conversely, the Chern class of the pullback of the hyperplane line bundle is positive rational(IV.9.6.4)(IV.9.5.3). □

Cor. (IV.9.8.7). if $\widetilde{X}$ is the blowing-up of a Kähler manifold X at a point x, then if X is projective, then $\widetilde{X}$ is also projective, because by(IV.9.6.5) $\pi^* L^k - E$ is positive for k large.

Cor. (IV.9.8.8). For a finite unbranched cover of compact Kähler manifolds $\widetilde{X} \to X$, $\widetilde{X}$ is projective iff X is projective.

Proof: A positive rational closed $(1,1)$-form on X pull backs to a positive rational closed $(1,1)$-form on $\widetilde{X}$, and it can even be pulled forward: $\omega' = \sum_{y\in\pi^{-1}(x)}(\pi^{-1})^*\omega(y)$, then it is also positive closed. It is rational because $\int_X \omega'\wedge\eta = \frac{1}{d}\int_{X'}\omega\wedge\pi^*\eta$, where $\widetilde{X}\to X$ is branched of degree d. □

Cor. (IV.9.8.9). If X is projective, then the map $Div(X)\to Pic(X): D\to\mathcal{L}(D)$ is surjective.

Proof: In fact, it suffice to show any line bundle E has a meromorphic section s, thus $L = \mathcal{L}(div(s))$. But X has a positive line bundle, so L^k+E and L^k are very ample thus clearly effective, with sections s_1 and s_2, so s_1/s_2 is a section of E. □

Cor. (IV.9.8.10) (Riemann Bilinear Form). For a complex variety V/Λ, it is projective iff there is a **Riemann form** on V, that is, an alternating bilinear form $\omega: V\times V\to\mathbb{R}$ that:

- $\omega(iu,iv)=\omega(u,v)$.
- $\omega(v,iv)>0$ for $v\neq 0$.
- $\omega(u,v)\in\mathbb{Z}$ for $u,v\in\Gamma$.

Proof: Use(IV.9.5.1). The conditions are just equivalent to ω is an integral positive Kähler form. □

Def. (IV.9.8.11). For a Kähler manifold X, the **Kähler cone** K_X is defined to be the set of closed real positive $(1,1)$-forms. Then K_X is an open convex cone in $H^{1,1}(X)\cap H^2(X,\mathbb{R})$. Then(IV.9.8.6) says X is projective iff $\mathcal{K}_X\cap H^2(X,\mathbb{Z})\neq 0$.

9 Fujiki manifolds

10 Riemann Surfaces

Basic references are [黎曼曲面 伍鸿熙].

Prop. (IV.9.10.1). A compact Riemann surface is Kähler, so by Hodge decomposition(IV.9.4.1),

$$H^1(M,\mathbb{C})\cong H^0(M,\Omega^1)\oplus\overline{H^0(M,\Omega^1)},$$

so the number of holomorphic 1-forms on M is equal to $b_1(M)/2=g$, the topological genus.

Prop. (IV.9.10.2) (Riemann Existence Theorem). Any compact Riemann Surface is a Hodge manifold, thus projective algebraic, by(IV.9.8.6) and Chow's lemma(IV.8.6.17).

Proof: Because $H^{1,1}(X)=H^2(X,\mathbb{Z})$, so it clearly contains integral classes. And it is positive because there is a basis generated by any Hermitian metric on X. So the theorem follows from(IV.9.8.6). □

Cor. (IV.9.10.3). In fact the same argument shows that any Kähler manifold with $H^{0,2}(X)=0$ is projective.

Prop. (IV.9.10.4) (Riemann Hurewitz). If $f: M\to N$ is a non-constant holomorphic map between two compact Riemann surfaces, then

$$2g_M-2=2\deg f(2g_N-2)+\sum_{p\in M}(v_p-1).$$

Proof: Cf.[黎曼曲面导引梅加强 P106]. □

Prop. (IV.9.10.5) (Weil Theorem). f, g are meromorphic functions on a compact Riemannian manifold that $(f), (g)$ has no common points, then

$$\prod f(p)^{v_p(g)} = \prod g(p)^{v_p(f)}.$$

Proof: Cf.[黎曼曲面导引梅加强 P143]. □

Chapter V

Analysis

V.1 Real Analysis

Basic references are [Folland Real Analysis], [Set Theory Jech].

1 Measures

Def. (V.1.1.1) (Measure).

Def. (V.1.1.2). A **Borel measure** is a measure defined on the σ-algebra generated by open sets.

A Borel measure μ is called **inner regular** iff $\mu(E) = \inf\{\mu(K)|K \subset E \text{ compact}\}$ for every Borel set E. It is called **outer regular** iff $\mu(E) = \sup\{\mu(U)|E \subset U \text{ open}\}$.

A **Radon measure** is a Borel measure that is finite on compact set, outer regular on Borel sets, inner regular on open sets.

Prop. (V.1.1.3) (Radon-Nikodym). If two σ-finite measures v, μ on a measurable space satisfies v is absolutely continuous w.r.t μ, then there is a μ-integrable function f such that

$$dv = f d\mu.$$

Cor. (V.1.1.4). Special case of the Freudenthal spectral theorem (V.5.4.18).

Prop. (V.1.1.5) (Riesz Representation Theorem). on $C_c(X)$ for a LCH space X,

- If I is a positive linear functional, there is a unique regular (both inner and outer) Radon measure μ on X such that $I(f) = \int f d\mu$. Moreover,

$$\mu(U) = \sup\{I(f) : f < U\} \text{ for } U \text{ open},$$

$$\mu(K) = \inf\{I(f) : f > \chi(K)\} \text{ for } K \text{ compact}.$$

- If I is a continuous linear functional, there is a unique regular countably additive complex Borel measure μ on X that $I(f) = \int f d\mu$.

In particular if X is compact, $M(X)$ the space of Borel measures on X is the dual spcae of $C(X)$.

Proof: Cf.[Real Analysis Folland P212]. □

Measurable Functions

Prop. (V.1.1.6) (convergences). There are three different kinds of convergences:

- **almost everywhere convergence** iff $f_n(x) \to f(x)$ a.e.
- **almost uniform convergence** iff for any $\delta > 0$, there is a measurable subset E_δ that f_n convergent to f uniformly on $E - E_\delta$.
- **convergence in measure** iff $\lim_{k\to\infty} m(\{x \in E \| f_n(x) - f(x)| > \varepsilon\}) = 0$.

Prop. (V.1.1.7) (Relations between Convergences).

- (Egoroff) If $m(E) < \infty$ and f_k converges to f a.e. then f_k converges to f almost uniformly.
- If f_k converges to f almost uniformly, then f_k converges to f in measure.
- (Riesz) If f_k converges to f in measure, then there is a subsequence f_{n_k} that converges to f a.e..

Proof: 1: Cf.[实变函数周明强 P113].
2: Trivial.
3: Cf.[实变函数周明强 P118]. □

2 Differentiation

Lemma (V.1.2.1) (Vitali Covering Theorem). Let $\mathcal{C}$ be a collection of balls in $\mathbb{R}^n$, and let $U = \cup_{B\in\mathcal{C}} B$. Then if $c > m(U)$, then there exists disjoint $B_1, \ldots, B_k \in \mathcal{C}$ that $\sum_{i=1}^k m(B_k) > 3^{-n}c$.

Lemma (V.1.2.2). If $f \in L^1_{loc}$ and $A_r f(x) = \frac{1}{m(B(r,x))}\int_{B(r,x)} f(y)dy$, then $A_r f$ is continuous in both r and x.

Proof: Cf.[Folland P96]. □

Prop. (V.1.2.3). If $f \in L^1_{loc}$, then $\lim_{r\to 0} A_r f(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$.

Proof: Cf.[Folland P97]. □

3 Integrations

Def. (V.1.3.1). A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is called **locally integrable** if $\int_K |f(x)|dx < \infty$ for every bounded measurable set K of $\mathbb{R}$. The set of locally integrable function is denoted by $L^1_{loc}(\mathbb{R}^n)$.

Prop. (V.1.3.2). A function f is real analytic on an open set of $\mathbb{R}$ iff there is a extension to a complex analytic function to an open set of $\mathbb{C}$. And this is equivalent to: For every compact subset, there is a constant C that for every positive integer k, $|\frac{d^k f}{dx^k}(x)| \le C^{k+1}k!$.

Proof: Use Lagrange residue(中值定理) to show that it will converge to f. □

Prop. (V.1.3.3) (Monotone-convergence-theorem).

Prop. (V.1.3.4) (Dominant Convergence Theorem).

Prop. (V.1.3.5). The set E of nowhere differentiable functions are of second category in $C[0, 1]$, and its complement set is of first category.

Proof: let A_n be the sets of functions f that there exists a s that for any $|h| \leq 1/n$, $|\frac{f(s+h)-f(s)}{h}| \leq n$. It is easy to see that $C[0,1] - E \subset \cup_n A_n$, so it suffices to show each A_n is of first category.

Firstly A_n is closed, because if $s \notin A_n$, then for any s, there is a $|h_s| \leq 1/n$ that $|f(s+h_s)-f(s)| > n|h_s|$. So by continuity, there is a $\varepsilon_s > 0$ and some nbhd J_s of s that $|f(\sigma - h_s) - f(\sigma)| > n|h_s| + 2\varepsilon_s$ for all $\sigma \in J_s$. Then there are f.m. J_{s_i} that covers $[0,1]$, so let $\varepsilon = \min\{\varepsilon_i\}$, then if $||g - f|| < \varepsilon$, then $g \notin A_n$.

And A_n has no interior point, because for any $f \in A_n$, f can be approximated by a polynomial g, by Stone-Weierstrass theorem(V.1.5.1), and by Mean-value theorem, there is a M that $|g(s+h) - g(s)| \leq M|h|$ for all s and $|h| < 1/n$. So if p is a pairwise-linear function that $||p||$ is small and the slopes of p are bigger than $M + n$, then $g + p$ is near f but $g + p \notin A_n$.

Finally, E is of second category by Baire theorem(IV.1.8.2). □

Prop. (V.1.3.6). For a pair of Hilbert basis $\{e_i\}$ of $L^2(M)$ and $\{f_j\}$ of $L^2(N)$, $\{e_i \otimes f_j\}$ gives a basis for $L^2(M \times N)$. (Use Fubini).

Proof: Cf.[泛函分析讲义张恭庆 P91]. □

Prop. (V.1.3.7) (Fubini-Tonelli). For two σ-finite measure space, if $f \in L^+(X \times Y)$, then $f_x \in L^+(Y)$ and $f^y \in L^+(X)$, and $\int_{X\times Y} f dxdy = \int_Y \int_X f dxdy = \int_X \int_Y f dydx$.

If $f \in L^1(X \times Y)$, then $f_x \in L^1(Y)$ and $f^y \in L^1(X)$, a.e. and the product formula is definable and holds.

Proof: Cf. [Folland P67]. □

4 L^p-space

Lemma (V.1.4.1) (Hoder's Lemma). if $\sum x_i = 1, x_i \geq 0$, then for any $a_i \geq 0$

$$\prod a_i^{x_i} \leq \sum a_i x_i.$$

Proof: □

Prop. (V.1.4.2) (Holder's Inequality). Let (S, Ω, μ) be a measure space, and $1 \leq p, q \leq \infty$ satisfies $1/p + 1/q = 1$, then

$$||fg||_1 \leq ||f||_p ||g||_q$$

More generally, if $\sum_{i=1}^n 1/p_i = 1$, then

$$|| \prod f_i ||_1 \leq \prod ||f_i||_{p_i}$$

Proof: The both sides are homogenous for f_i, so we may assume $||f_i||_{p^i} = 1$, then use Hoder's Lemma(V.1.4.1) for $x_i = q/p_i$. □

Prop. (V.1.4.3). For a σ-finite measure μ on a measurable space X, for $1 \leq p < \infty$

$$L^p(X, \Omega, \mu)^* = L^q(X, \Omega, \mu).$$

Proof: Firstly, Holder inequality(V.1.4.2) shows that a $g \in L^q(X, \Omega, \mu)$ is truly defines a functional by $f \mapsto \int fg d\mu$. Conversely, if given a functional F, define a measure $v(E) = F(\chi_E)$ for all measurable set $E \in \Omega$. It is countably additive: first it is finitely additive, and if E_n is a descending sequence of measurable sets that $\cap E_n = \varnothing$, then

$$v(E_n) \leq ||F||||\chi_{E_n}||_{L^p} = ||F||\mu(E_n)^{\frac{1}{p}} \to 0.$$

(where we used the fact $p < \infty$). And it is clearly absolutely continuous w.r.t. μ.

So by Radon-Nikodym(V.1.1.3), there is a measurable function g that $v(E) = \int_E g d\mu$. Now for all simple function f, $F(f) = \int f(x)g(x)$. Next we want to prove $||g||_{L^q} \leq ||F||$, because in this way, because any measurable function f can be approximated by simple functions f_i in L^p norm(V.1.5.5), so

$$|\int (f(x) - f_i(x))g(x)d\mu| \leq ||f - f_i||_p ||g||_q \leq ||f - f_i||_p ||g||_q$$

So $F(f) = \lim F(f_i) = \lim \int f_i g d\mu = \int f g d\mu$.

To prove this, if $1 < p$, then let $E_t = \{x||g(x)| \leq t\}$, and $f = \chi_{E_t}|g|^{q-2}g$, then

$$\int_{E_t} |g|^q d\mu = \int f g d\mu = F(f) \leq ||F||||f||_{L^p} = ||F||(\int_{E_t} |g|^q d\mu)^{\frac{1}{p}}$$

which is equivalent to $||g\chi E_t||_{L^q} \leq ||F||$. Let $t \to \infty$, then the monotone convergence theorem(V.1.3.3) gives us the result.

If $p = 1$, then $q = \infty$. For any $\varepsilon > 0$, let $A = \{x||g(x)| > ||F|| + \varepsilon\}$, $E_t = \{x||g(x)| \leq t\}$, and let $f = \chi_{E_t \cap A}\,\mathrm{sign}(g)$, then $||f||_{L^1} = \mu(E_t \cap A)$, and

$$\mu(E_t \cap A)(||F|| + \varepsilon) \leq \int_{A \cap E_t} |g| d\mu = \int f g d\mu \leq ||F||\mu(E_t \cap A)$$

If $\mu(A) \neq 0$, then let $t \to \infty$, this is a contradiction. So $||g||_\infty \leq ||F||$. □

5 Approximations

Prop. (V.1.5.1) (Stone-Weierstrass Approximation). If a unital C^*-algebra A of continuous functions on a compact Hausdorff space separates points, then it is dense in $C(X)$.

Proof: This is a consequence of Bishop theorem(V.4.3.18) because in this case the real functions in A separate points, so all A-antisymmetric sets consists of one point. □

Cor. (V.1.5.2). The polynomial functions are dense in $C[-1, 1]$.

Prop. (V.1.5.3) (Simple Function Approximation).

- If $f(x)$ is a non-negative measurable function on E, then there is an ascending sequence of simple functions $\varphi_n(x)$ that converges to f point-wise.
- If $f(x)$ is a measurable function on E, then there is a sequence of simple functions φ_n that $|\varphi_k(x)| \leq |f(x)|$, and converges to f pointwise.
- If $f(x)$ is bounded, then the convergence can be chosen to be uniform.

Proof: Cf.[实变函数周明强 P110]. □

L^p-Approximation

Prop. (V.1.5.4). for $1 \leq p < +\infty$, $C(X)$ are dense in $L^p(X)$ for a Radon measure, but not for $p = \infty$.

Proof: Use finite stair approximation and then inner regular approximation and then Tietz extension. □

Prop. (V.1.5.5). Any function in L^p can be approximated by simple functions in L^p norm.

Prop. (V.1.5.6) (Lusin). If f is almost everywhere finite on E, then for any $\delta > 0$, there is a closed subset $F \subset E$ that f is continuous function on F.

Proof: First if f is a simple function $f = \sum_{i=1}^n c_i \chi_{E_i}$, then for each E_i, choose a closed subset $F_i \subset E_i$ that $m(E_i - F_i) < \frac{\delta}{n}$, and then $\cup F_i$ satisfies the required condition.

Now if f is arbitrary, let $g(x) = \frac{f(x)}{1+|f(x)|}$ to make it bounded, then by(V.1.5.3), there is a sequence of simple functions φ_k converging to f, and for each k, we choose a closed subset F_k that $m(E - F_k) < \frac{\delta}{2^k}$, so if we let $F = \cap F_k$, then φ_k are all continuous on F, so by the uniform convergence, f is also continuous on F. □

Cor. (V.1.5.7). If f is measurable function on E that is a.e. finite, then for any δ, there is a continuous function g that $m(\{x \in E | f(x) \neq g(x)\}) < \delta$. And if E is bounded, g can be chosen to be compactly supported.

Proof: Now that there is a closed subset F that $m(E - F) < \delta$ and f is continuous on F, we can use Tietze extension(IV.1.6.3), there is a function g that equals f on F.

If $E \subset B(0, R)$, the we can choose a bump function to multiply with g. □

Prop. (V.1.5.8) (Approximate Identity). A family of $L^\infty(\mathbb{T})$ functions $\{\Phi_N\}$ are called an approximate identity if:

1. $\int_0^1 \Phi_N(x)dx = 1$.
2. $\sup \int_0^1 |\Phi_N(x)|dx < \infty$.
3. For any $\delta > 0$, $\int_{|x|>\delta} |\Phi_N(x)|dx \to 0$ as $N \to +\infty$.

For any approximate identity, if $f \in C(\mathbb{T})$ or $L^p(\mathbb{T})$ for $1 \leq p < +\infty$, then $\Phi_N * f \to f$.

Proof: Use uniform continuity and also use continuous approximation (V.1.5.4). □

Cor. (V.1.5.9). for $1 \leq p < +\infty$,trigonometric polynomials are dense in $L^p(\mathbb{T})$ and $C(\mathbb{T})$, but not for $p = \infty$. So $e^{2\pi i n x}$ forms an orthogonal basis in $L^2(\mathbb{T})$.

Thus, the Parseval's identity holds.

Proof: Just use the fact that Fejer kernels are an approximate identity. □

Prop. (V.1.5.10). For a integrable function u that has compact support, $u_\delta = j_\delta * u$ is a smooth function of compact support that $||u_\delta - u||_{C^k} \to 0$ when $u \in C^k$. Where j_δ is the scaling of a smooth function of compact support. So Smooth function of compact support are dense in C_0^k.

Prop. (V.1.5.11). $D(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$.

Proof: Use the fact that C_0 are dense in L^p by(V.1.5.10). And $f_\delta \to f$ in L^p norm for $f \in C_0$. So we can use the three-part argument applied to $D_\alpha u$ to get $D_\alpha(u_\delta) \to D_\alpha u$ in L^p norm for $|\alpha| \leq m$. Thus the result. □

6 Convolution

Prop. (V.1.6.1). Convolution with a smooth function makes the function smooth, in particular, $\frac{\partial}{\partial x}(f * g) = \frac{\partial f}{\partial x} * g$.

Prop. (V.1.6.2) (Young's Inequality). $||f * g||_r \leq ||f||_p ||g||_q$ for all $1 \leq r, p, q \leq \infty$ and

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

In particular, $||K * f||_p \leq ||K||_1 ||f||_p$.

Proof: By Riesz representation(V.1.1.5), it suffices to show that: for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$,

$$\int\int f(x)g(y-x)h(y) \leq ||f||_p ||g||_q ||h||_r.$$

write the LHS as

$$\int\int (f^p(x)g(y-x)^q)^{1-\frac{1}{r}} (f^p(x)h^r(y))^{1-\frac{1}{q}} (g^q(y-x)h^r(y))^{1-\frac{1}{p}}$$

and use Holder inequality for three functions(V.1.4.2). □

7 Special Functions

Gamma Function

Bessel Function

V.2 Complex Analysis

Basic References are [Complex Analysis Ahlfors], [复变函数简明教程 谭小江 武胜健], [Complex Analysis Stein] and [复分析导引 李忠].

1 Topology

Def. (V.2.1.1). A **region** is defined to be a nonempty connected open set of $\mathbb{C}$.

Prop. (V.2.1.2). The roots of a polynomial depends continuously on the coefficients. (Use Rouch Principle).

2 Basics

Def. (V.2.2.1). We introduce the following notation:

$$\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}),\quad \frac{\partial}{\partial \overline{z}}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}),\quad dz=dx+idy\quad d\overline{z}=dx-idy.$$

Then dz is dual to $\frac{\partial}{\partial z}$ and $d\overline{z}$ is dual to $\frac{\partial}{\partial \overline{z}}$. And for any function f,

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial \overline{z}}d\overline{z}.$$

Def. (V.2.2.2) (Cross Ratio). For any three pts $z_1,z_2,z_3\in\overline{\mathbb{C}}$, there is a unique linear transformation that maps them to $1,0\infty$. In fact, the linear transformation is just $Sz=\frac{z-z_3}{z-z_4}/\frac{z_2-z_3}{z_2-z_4}$.

Then for any for point z_1,z_2,z_3,z_4, the **cross ratio** $(z_1,z_2,z_{3,4})$ is the image of z_1 under the linear transformation that carries z_2,z_3,z_4 to $1,0,\infty$.

Prop. (V.2.2.3). The cross ratio is invariant under linear transformation, and it is real iff z_1,z_2,z_3,z_4 are colinear or cocycle.

Proof: The first is because there is only one linear transformation that maps z_2,z_3,z_4 to $1,0,\infty$.

For the second, notice by(V.2.2.2), $\arg(z_1,z_2,z_{3,4})=\arg\frac{z_1-z_3}{z_1-z_4}-\arg\frac{z_2-z_3}{z_2-z_4}$, and this is real iff $\angle z_4z_2z_3=\angle z_4z_1z_3$ or $\pi-\angle z_4z_1z_3$, which is equivalent to cocycle. For other degenerate cases, we need some other argument. □

Cor. (V.2.2.4). A linear transformation maps colinear/cocycle points to colinear/cocycle points.

Lemma (V.2.2.5). If $a,b,c,d\in\mathbb{R}$, then

$$\mathrm{Im}(\frac{az+b}{cz+d})=\frac{ad-bc}{|cz+d|^2}\,\mathrm{Im}(z).$$

Analytic Function

Def. (V.2.2.6) (Analytic Function). A function on $\mathbb{C}$ are called **analytic** or holomorphic if $\frac{\partial f}{\partial \overline{z}}=0$. Equivalently, $\frac{\partial f}{\partial x}=-i\frac{\partial f}{\partial y}$, i.e. f has the same derivative vertically and horizontally, hence in every direction.

Prop. (V.2.2.7) (Regularity Problems). For the equation $(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})f=0$ or $(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})u=0$, the function is elliptic, so by(V.8.8.4), the solution of the equation is automatically smooth, so no smoothness condition need to be added.

Lemma (V.2.2.8). An analytic function in a region Ω whose derivative vanishes must be a constant function.

Proof: □

Prop. (V.2.2.9) (Uniqueness). If the zeros of a holomorphic function f has a convergent point in the domain of definition, then $f=0$.

Proof: □

Conformal Mapping

Prop. (V.2.2.10). A 1st-differentiable conformal map in $\mathbb{C}$ is holomorphic or anti-holomorphic. In higher dimension, conformal is equivalent to $\langle df_p(v_1), df_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle$.

Proof: Cf.[Ahlfors P74]. □

3 Complex Integration

Lemma (V.2.3.1) (Technical Lemma). If f is analytic on a rectangle R minus f.m. points ζ_i and if $\lim_{z\to\zeta_i}(z-\zeta_i)f(z)=0$, then $\int_{\partial R} f(z)dz=0$.

Proof: Cf.[Ahlfors P109-112] □

Prop. (V.2.3.2) (Cauchy Theorem). if Ω is a simply connected region in $\mathbb{C}$, f is holomorphic in Ω minus f.m. points ζ_i, $\lim_{z\to\zeta_i}(z-\zeta_i)f(z)=0$, and continuous on $\overline{\Omega}$, then $\int_{\partial\gamma} f(z)dz=0$ for any curve γ in Ω.

Proof: Fix a $z_0\in\Omega$, then for any $z\in\Omega$, choose a horizontal and vertical path γ from z_0 to z, then let $F(z)=\int_\gamma f(z)dz$, then the above lemme(V.2.3.1) shows that $F(z)$ is independent of the path chosen, and it is clear that $F(z)$ has derivatives in both direction so holomorphic by definition(V.2.2.6). So clearly $\int_{\partial\Omega} f(z)dz=0$, by an uniformly continuity argument. □

Prop. (V.2.3.3) (Index of a Point w.r.t a Curve). If γ is a piecewise C^1 curve that doesn't pass a point a, then $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is an integer $n(\gamma,a)$, called the **index of a w.r.t γ**.

And this index function is constant on each connected component of $\mathbb{C}-\gamma$, and 0 on the unbounded component.

Proof: Cf.[Ahlfors P115]. □

Cor. (V.2.3.4) (Cauchy Integral Formula). if Ω is a simply connected region in $\mathbb{C}$, f is holomorphic in Ω and continuous on $\overline{\Omega}$, then for $a\notin\gamma$,

$$n(\gamma,a)f(a)=\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}dw.$$

Proof: Consider the function $F(z)=\frac{f(z)-f(a)}{z-a}$, then it is analytic for $z\neq a$, and at a it satisfies the condition of Cauchy theorem(V.2.3.2), so $\int_\gamma F(z)dz=0$ which is $\int_\gamma \frac{f(z)dz}{z-a}=f(a)\int\frac{dz}{z-a}$, and use(V.2.3.3). □

Prop. (V.2.3.5) (Generating Analytic Functions). If $\varphi(\zeta)$ is continuous on an arc γ, then the function

$$F_n(\zeta) = \int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta - z)^n}$$

is analytic in each of the regions defined by γ, and its derivative satisfies $F_n'(z) = nF_{n+1}(z)$.

Proof: Cf.[Ahlfors P121]. □

Cor. (V.2.3.6) (Higher Derivations). As any analytic function f can be written as $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}dw$. by Cauchy integral theorem(V.2.3.4), its derivatives are all analytic, and satisfeis:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}$$

Cor. (V.2.3.7) (Morera's Theorem). If f is continuous on a region Γ and if $\int_\gamma f dz = 0$ for any closed curve γ in Ω, then $f(z)$ is analytic in Ω.

Proof: There is an analytic function F that $F' = f$, by the same method of the proof of(V.2.3.2), so f is analytic, by(V.2.3.6). □

Cor. (V.2.3.8) (Cauchy Estimate). If f is holomorphic on a disk $B(a,r)$, and $|f| \leq M$ on the boundary, then $|f^{(n)}(a)| \leq Mn!r^{-n}$.

Cor. (V.2.3.9) (Liouville). Any bounded holomorphic function on $\mathbb{C}$ is constant.

Proof: if $|f(z)| \leq M$, then the Cauchy estimate shows that $|f'(a)| \leq Mr^{-1}$, letting r tends to ∞, then $f'(a) = 0$ for all a, thus f is constant. □

Cor. (V.2.3.10) (Mean Value Property). If f is holomorphic on the unit disk D, then $|f(0)| \leq \int_D |f(z)|dxdy$.

Proof: $|f(0)| \leq \frac{1}{2\pi}\int f(re^{i\theta})d\theta$, so if multiplied by rdr and integrate, then

$$|f(0)| \leq \int\int f(re^{i\theta})rdrd\theta = \int\int f(z)dxdy.$$

□

Def. (V.2.3.11). The space $\mathcal{H}^2(D)$ is the space of all holomorphic functions on the unit disk that is also L^2. This is a Hilbert space in the L^2 norm, and $\varphi_n(z) = \sqrt{\frac{n}{\pi}}z^{n-1}$ is a basis. So using(V.5.2.5), it is clear that the reproducing kernel of $\mathcal{H}^2(D)$ is $K(z,w) = \frac{1}{\pi(1-z\overline{w})^2}$.

Proof:

□

Local Properties of Analytic Functions

Prop. (V.2.3.12). An analytic function is an open map from $\mathbb{C}$ to $\mathbb{C}$.

The Residue

Prop. (V.2.3.13) (Rouche's Theorem).

4 Theorems

Prop. (V.2.4.1) (Uniformization Theorem). Any connected Riemann Surface is the quotient by a discrete subgroup of $\mathbb{C}, \mathcal{H}$ or $\mathbb{P}^1$.

Proof: □

Prop. (V.2.4.2) (Runge's Theorem). Let K be a compact subset of $\overline{\mathbb{C}}$ and let f be a function which is holomorphic on an open set containing K. If A is a set containing at least one complex number from every bounded connected component of $\overline{\mathbb{C}} \backslash K$, then there exists a sequence of rational functions which converges uniformly to f on K and all the poles of the functions are in A.

Proof: □

Prop. (V.2.4.3) (Mergelyan's theorem). If K is compact in $\mathbb{C}$ and f is a continuous function on K that is holomorphic in $\mathrm{int}(K)$, then f can be uniformly approximated by polynomials.

Prop. (V.2.4.4) (Weierstrass Theorem). For a ascending sequence of regions $\Omega_1 \subset \Omega_2 \subset \ldots$, $\cup_n \Omega_n = \Omega$, and f_n is analytic on Ω_n, and $f_n(z)$ converges to a function $f(z)$ in the compact-open topology, then $f(z)$ is also analytic, and moreover, $f'(z)$ converges to $f'(z)$ in the compact-open topology.

Proof: The analyticity follows from Morera's theorem(V.2.3.7) as the integration on a closed curve commutes with uniform convergence, the same argument applied to the limit of equations

$$f_n'(z) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{f_n(\zeta) d\zeta}{(\zeta - z)^2}$$

shows that the derivative also converges, and uniformly on $\overline{B(0,\rho)}$ for $\rho < r$. □

Cor. (V.2.4.5) (Hurwitz). Cf.[Ahlfors P178].

Prop. (V.2.4.6) (Montel's Theorem). Sets of holomorphic functions bounded in the topology of $H(\Omega)$, inter convex uniform convergence, is sequentially compact.

Proof: □

5 Series and Product Developments

Series

Prop. (V.2.5.1) (Abel). Any power series $a_0 + a_1 z + \ldots + a_n z^n + \ldots$ has a **circle of convergence** R that:

- The series converges absolutely for every $|z| < R$, if $\rho < R$, then the convergence is uniform for $|z| \leq \rho$.
- If $|z| > R$, the terms are unbounded, and the series diverges.

Moreover, in $|z| < R$, the sum of this series is an analytic function.

Proof: Cf.[Complex Analysis Ahlfors P38]. □

Prop. (V.2.5.2) (Hadamard's Formula). In the last proposition(V.2.5.1), $1/R = \limsup \sqrt[n]{a_n}$

Proof: Cf.[Complex Analysis Ahlfors P38]. □

Prop. (V.2.5.3). Any holomorphic function f defined on the punctured disk $0 < |z| < 1$ is of the form

$$f(z) = \sum a_n z^n$$

where $\limsup_{n>0} |a_n|^{1/n} \leq 1$, and $\lim_{n\to\infty} |a_{-n}|^{1/n} = 0$.

Partial Fractions and Factorizations

Entire Functions

Riemann Zeta Function

6 Harmonic Function

Def. (V.2.6.1) (Harmonic Function). A real-valued function on a region Ω is called **harmonic** iff it is C^1 and has second order derivatives and $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ holds, as said before in(V.2.2.7), this is a elliptic function, so u is automatically smooth.

Prop. (V.2.6.2). Analytic-function-norm-is-analyis-harmno if $f(z)$ is an analytic function and on a nbhd Ω $f(z) \neq 0$, then $\log |f(z)|$ is a harmonic function on Ω.

Proof:

□

Prop. (V.2.6.3) (Poisson Formula). For u harmonic for $|z| < \rho$ and continuous for $|z| \leq \rho$,

$$u(z) = \frac{1}{2\pi} \int \frac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} u(\rho e^{i\theta}) d\theta = \mathrm{Re}\Big[\frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{\zeta - z}{\zeta + z} u(\zeta) \frac{d\zeta}{\zeta}\Big].$$

for $|z| < \rho$.

In particular, the bracketed part is a analytic function for $|z| < R$, so for any analytic function on $|z| \leq \rho$ by(V.2.3.5), for $|z| < R$:

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{\zeta - z}{\zeta + z} \mathrm{Re}(f(z)) \frac{d\zeta}{\zeta} + iC.$$

Proof: Cf.[Ahlfors P168]. □

Prop. (V.2.6.4) (Schwarz's Theorem). Cf.[Ahlfors P169].

Def. (V.2.6.5) (Mean-Value Property). A real valued function u on a region Ω is said to have the **mean-value property** iff

$$u(z_0) = \frac{1}{2\pi} \int_0^{1\pi} u(z_0 + re^{i\theta}) d\theta$$

whenever $B(z_0, r) \subset \Omega$.

Lemma (V.2.6.6) (Harmonic Mean Value). If u is a Harmonic function between two concentric circles, then the arithmetic mean of it over circles $|z| = r$ is a linear function of $\log r$:

$$\frac{2\pi}{\int_{|z|=r}} ud\theta = \alpha \log r + \beta.$$

In particular, if u is harmonic in the disk, then by continuity, $\alpha = 0$, and the mean value is a constant.

Proof: Cf.[Ahlfors P165]. □

Prop. (V.2.6.7) (Harmonicity and Mean-Value Property). A harmonic function satisfies the mean-value property, and conversely, and continuous function satisfying the mean-value property is harmonic.

Proof: Harmonic function satisfies the mean-value property by(V.2.6.6). Conversely, for any z_0, by Schwarz's theorem(V.2.6.4), there is a harmonic function $v(z)$ that is harmonic in $B(z_0, \rho)$ and equals $u(z)$ on $\partial B(z_0, \rho)$. Now the maximal and minimal principles apply to $u - v$, thus $u = v$ is harmonic. □

Cor. (V.2.6.8) (Maximum Principle). If u is a harmonic function, then it attains neither maximum nor minimum at its region of definition.

Prop. (V.2.6.9) (Hadamard's Three Circle Theorem). Let $f(z)$ be analytic in the annulus $r_1 < |z| < r_2$, and continuous on the boundary, if $M(r)$ denotes the maximum of $|f(z)|$ for $|z| = r$, then:

$$M(r) \leq M(r_1)^\alpha M(r_2)^{1-\alpha}$$

where $\alpha = \log(r_2/r) : \log(r_2/r_1)$.

Proof: Apply the maximum principle(V.2.6.8) for

$$g(z) = \log|f(z)| - \log(M(r_1))(\log(r_2/|z|) : \log(r_2/r_1)) - \log(M(r_2))(1 - \log(r_2/|z|) : \log(r_2/r_1)),$$

it is harmonic by**??**, then $g(z) \leq 0$ on $|z| = r_1$ and $|z| = r_2$, so $g(z) \leq 0$ on all the annulus. □

Cor. (V.2.6.10). Let $f(z)$ be analytic in the annulus $r_1 < |z| < r_2$, then the function

$$s \mapsto \max_{z=e^s} |f(z)|$$

is convex on the interval $[\log(a), \log(b)]$.

Prop. (V.2.6.11) (Reflection Principle).

Prop. (V.2.6.12) (Harnack's Inequality). For a positive harmonic function u on $B(0, \rho)$,

$$\frac{\rho - |z|}{\rho + |z|} u(0) \leq u(z) \leq \frac{\rho + |z|}{\rho - |z|} u(0).$$

Proof: By Poisson formula,

$$u(z) = \frac{1}{2\pi}\int \frac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} u(\rho e^{i\theta}) d\theta$$

for $|z| < \rho$, so the conclusion follows from the obvious inequality

$$\frac{\rho - |z|}{\rho + |z|} \leq \frac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} \leq \frac{\rho + |z|}{\rho - |z|}$$

and Mean-value property(V.2.6.7). □

Cor. (V.2.6.13). If E is a compact subset of a region Ω, there is a constant M, depending on E, Ω that for any positive harmonic function $u(z)$ on Ω, $u(z_2) \leq Mu(z_1)$ for any $z_1, z_2 \in E$.

Proof: This is an easy consequence of Harnack inequality and the compactness of E. □

Cor. (V.2.6.14) (Harnack's Principle). Consider a sequence of functions $u_n(z)$, each harmonic in a region Ω_n, and there is a region Ω that every point has a nbhd that is contained in all but f.m. Ω_n, and in this nbhd $u_n(z) \leq u_{n+1}(z)$ for n large, then either $u_n(z)$ tends to $+\infty$ in the compact open topology, or they tends to a harmonic function in compact open topology.

Proof: The uniform continuity follows easily from Harnack's inequality, and for the harmonicity of the limit function $u(z)$ is a consequence of the Poinsson formula. □

Dirichlet Problem

7 Elliptic Function

8 Multi-Variable case

Basics

Should cover the part from [Complex Analytic and Differential Geometry Demailly], [Principle of Algebraic Geometry Griffith/Harris] and [Complex Geometry Daniel].

Def. (V.2.8.1). A function is called **holomorphic** in several variables iff it is holomorphic for each indeterminate.

Def. (V.2.8.2). The **polydisc** $B_\varepsilon(a)$ in $\mathbb{C}^n$ is defined to be the set $\{z||z_i - a_i| < \varepsilon_i\}$.

Prop. (V.2.8.3) (Hartog's Extension Theorem). If K is a compact subset in an open domain Ω of $\mathbb{C}^n(n \geq 2)$ and $\Omega - K$ is connected, then any holomorphic function on $\Omega - K$ extends to a holomorphic function on Ω.

Proof: □

Prop. (V.2.8.4). Let $\varepsilon = (\delta, \ldots, \delta)$ and f be a holomorphic function on the polydisc $\overline{B_\varepsilon(0)}$. Then if f vanishes of order k at the origin and $|f(z)| \leq C$, then

$$f(z) \leq C(\frac{|z|}{\delta})^k$$

for all $z \in \overline{B_\varepsilon(0)}$.

Proof: Fix $z \in \overline{B_\varepsilon(0)} \neq= 0$, consider the one-variable function $g_z(w) = w^{-k} f(w \cdot \frac{z}{|z|})$, then g_z is holomorphic and $|g_z(w)| \leq \delta^{-k} C$ for $|w| = \delta$. So maximal principle implies that $g_z(w) \leq \delta^{-k} C$ for all $|\omega| \leq \delta$. Hence $|z|^{-k}|f(z)| = |g_z(|z|)| \leq \delta^{-k} C$. □

V.3 General Functional Analysis

Basic references are [Rudin Functional Analysis],[Nonarchimedean Functional Analysis].

This section only contains theorems that are applicable to both Archimedean and non-Archimedean valuations. For theorems specialized to non-Archimedean valuations, SeeII.1, for theorems specialized to Archimedean valuations, SeeV.4. Many propositions in Functional Analysis can be transplanted in the general case, but I haven't finish yet.

The major problem is convex is not definable, so Hahn-Banach fail, causing many to fail.

Topological Rings and Continuous Valuations

1 Valuations

This subsubsection should in fact be moved to the p-adic analysis section.

Def. (V.3.1.1). Let Γ be a totally ordered Abelian group, then a **convex subgroup** of A is a subgroup Δ that if $a < b < c$ and $a, c \in \Delta$, then $b \in \Delta$. Notice this is in fact equivalent to if $0 < c \in \Delta$, then $0 < b < c$ are also in Δ.

Prop. (V.3.1.2) (Height). The set of all convex subgroups of Γ is well-ordered, and its ordinal is called the **height** of Γ.

For a valuation, its **rank** is defined as the height of its value group.

Proof: If Δ_1, Δ_2 don't contains each other, let $a \in \Delta_1 - \Delta_2$ and $b \in \Delta_2 - \Delta_1$, then changing $\pm a, \pm b$, we may assume $0 < a < b$, so $a \in \Delta_2$, contradiction. □

Prop. (V.3.1.3) (Height 1 Case). Let Γ be a totally ordered Abelian group, then the following are equivalent:

1. $\mathrm{ht}(\Gamma) = 1$.
2. for all $a, b \in \Gamma$ that $a > 0$ and $b \geq 0$, there is an integer n that $b \leq na$.
3. there exists an injection from Γ to $\mathbb{R}$.

Proof: $3 \to 1$ is easy.

$1 \to 2$: Consider the convex subgroup generated by a, then it is Δ by height condition, so b must by in it, i.e. $b \leq na$ for some n.

$2 \to 3$: Choose an $a > 0$, let the injection φ given by $\varphi(b) = \sup\{\frac{n}{k} | na \leq kb\}$ for $b > 0$ and extends to negative elements.

It is easily verified that $\varphi(c) + \varphi(b) \leq \varphi(c+b)$, and if $\varphi(c) + \varphi(b) < \varphi(c+b)$, choose a rational approximation of them, and multiply to get integers, then if $k(c+b) \leq \varphi(c+b)ka > \varphi(b)a + \varphi(c)a + a$, then either $kc \geq \varphi(c)ka + a$ or $kb \geq \varphi(b)ka + a$, contradiction.

So this map is truly a morphism of ordered Abelian groups, and it is injective because if $b > 0$, then by 2, there must be an n that $a \leq nb$, so $\varphi(b) \geq 1/n$. □

Microbial Valuations

Lemma (V.3.1.4).

Prop. (V.3.1.5) (Microbial Valuation). For a valuation ring $A \subset K$, a $f \neq 0 \in A$ is called **topologically nilpotent** iff $f^n \to 0$ in the valuation topology of A. The following are equivalent:

- The topology on K coincides with a rank 1 topology.
- There exists a nonzero topologically nilpotent element in K.
- R has a prime ideal of height 1.

If this is the case, then the valuation defined by A is called **mirobial**.

And in this case, for any topological nilpotent element ϖ, $K = R[\varpi^{-1}]$, and $\varpi^r \in R$ for some r, and the topology on R is ϖ^r-adic. And if $\mathfrak{p}$ is a prime ideal of rank 1, then the valuation ring $R_{\mathfrak{p}}$ is of rank 1, and defines the same topology on R.

Proof: $1 \to 2$: if there is a rank 1 valuation $|-|'$ that defines the same topology as R, then any $|x|' < 1$ will be a topological nilpotent element by(V.3.1.6).

$2 \to 3$: if ϖ is a topological nilpotent element, then $\mathfrak{p} = \sqrt{(\varpi)}$ is a prime ideal, and it is minimal, because if there is another $\mathfrak{q} \subsetneq \mathfrak{p}$, then $\varpi \notin \mathfrak{q}$, but $\mathfrak{p} \subset (\varpi^n)$ by induction: because $(\varpi) \nsubseteq \mathfrak{q}$, $\mathfrak{q} \subset (\varpi)$, and if $x \in \mathfrak{q}$, then $x = \varpi^n y$, and $\varpi \notin \mathfrak{q}$, so $y \in \mathfrak{q} \subset (\varpi^n)$, so $x \in (\varpi^{n+1})$. Now $\mathfrak{q} = 0$ because ϖ is topological nilpotent.

$3 \to 1$: It suffices to prove that the valuation defined by $R_{\mathfrak{p}}$ is the same as the topology of R. But this is true in general, just notice that the valuation topology of a nontrivial valuation is also defined by $B(a, \gamma]$.

The final remark is clear as $x\varpi^n \in R \iff |\varpi^n| \leq |x^{-1}|$. □

Lemma (V.3.1.6). Let R be a valuation ring, if $x \in R^*$ is topologically nilpotent, then $|x| < 1$, and the converse is also true if R has rank 1.

Proof: if $|x| \geq 1$, then $x^{\mathbb{N}} \notin B(0,1)$, so it is not topologically nilpotent. And if R has rank 1, $|x| < 1$, then for any $\delta \neq 0$, there is some n that $|\delta^{-1}| < |x^{-n}|$(V.3.1.3), so $|x^m| < |\delta|$ for m large, thus x is topologically nilpotent. □

Prop. (V.3.1.7) (Constructing Microbial Valuations). If A is a valuation ring and $f \in A$ is a non-zero non-unit, then the f-adic Hausdorffization $\overline{A} = A/\cap_n f^n V$ and the completion $\widehat{A}$ are all microbial. And $\overline{A} \to \widehat{A}$ is faithfully flat that preserves pseudo-uniformizers.

Proof: Easy, Cf.[Bhatt Perfectoid Spaces, P63]. □

2 Valuation Ring

Def. (V.3.2.1). In a field K, the **valuation ring** is the maximum elements in the dominating ordering of local rings, where B **dominates** A iff $A \subset B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$.

Prop. (V.3.2.2). A local ring A in a field K is dominated by a valuation ring with fractional field K.

Proof: Note that the dominating relation satisfies the condition of the Zorn's lemma, so it suffices to prove that A is not maximal if its fractional field is not K. Let $t \notin K_0 = \mathrm{frac}A$. If t is transcendental over K_0, then $A[t]$ with the maximal ideal $(\mathfrak{m}, t)$ dominate A. If t is algebraic over K_0, then there is a a that at is integral over A, hence by(I.5.4.5) there is a maximal ideal of $A[at]$ above A, which proves the lemma. □

Prop. (V.3.2.3) (Valuation Ring Criterion). A is a valuation ring with field of fraction K iff for any $x \in K$, x or x^{-1} is in A.

Proof: If A is a valuation ring, then for $x \notin A$, we know that $A[x]$ is a local ring, hence there is no prime over $\mathfrak{m}$ otherwise $A[x]_\mathfrak{p}$ is a bigger local ring, so we see $\mathfrak{m}A[x] = A[x]$, i.e. $1 = \sum t_i x^i$, so x^{-1} is integral over A. Now $A[x^{-1}]$ has a $\mathfrak{m}'$ over $\mathfrak{m}$, so $A = A[x^{-1}]_{\mathfrak{m}'}$, which shows $x^{-1} \in A$.

Conversely, when A is not K, then A is not field by(V.3.2.2), then it has a non-zero maximal ideal, but only one, otherwise we can choose x, y that $x/y, y/x \notin A$. And A is maximal because if there is a $A \subset A'$, and a $x \in A'$, then if $x \notin A$, then $x^{-1} \in A$, hence also in $\mathfrak{m}_A$, so it is in $\mathfrak{m}_{A'}$, but now x^{-1} cannot be in A', contradiction. □

Cor. (V.3.2.4). For $K \subset L$ subfield, if A is a valuation ring of L, then $A \cap K$ is a valuation ring of K. And if L/K is algebraic and A is not a field, then $A \cap K$ is not a field.(This is because the primes of A are all over 0 so cannot contain each other(I.5.4.5) so A is a field).

Cor. (V.3.2.5). The quotient A/p at a prime is a valuation ring, and any localization of valuation ring is a valuation ring, by this criterion.

Prop. (V.3.2.6) (Valuation Ring is Normal). Valuation ring is normal, because for x algebraic over A, either $x \in A$, or x is a combination of x^{-1} thus in A.

Cor. (V.3.2.7) (Integral Closure and Valuation Ring). The integral closure of a subring in a field K is the intersection of valuation rings containing A.

Proof: Valuation ring is integrally closed, so it suffices to prove if x is not algebraic over A, then there is a valuation ring of A not containing x. This is because $x \notin B = A[x^{-1}]$ otherwise x is integral over A. Now x^{-1} is not a unit in B, hence $x \in p \in B$, hence B_p is dominated by some valuation ring V, and $x \notin V$ because $x^{-1} \in \mathfrak{m}_V$. □

Prop. (V.3.2.8) (Valuation Ring and Valuation). A valuation ring A is equivalent to a field K with a surjective valuation map to a totally valued abelian group Γ that $A = v^{-1}(\{x \geq 0\})$.

Proof: These are definitely valuation rings, and if A is a valuation ring by(V.3.2.3), then we set $\Gamma = K^*/A^*$, where A^* is the invertible elements of A and $x \leq y$ iff $y/x \in (A - \{0\})/A^*$. This is totally ordered by(V.3.2.3). □

Cor. (V.3.2.9) (Rank and Dimension). A valuation ring of rank n has Krull dimension n, because clearly the convex subgroups of Γ is in bijection with ideals of A.

Prop. (V.3.2.10) (Bezout Domain and Valuation Ring). A valuation ring is equivalent to a Bezout local domain.

Proof: One way is because the element of minimum valuation generate the ideal. Conversely, for $f, g \in A$, $(f, g) = (h)$, so $f = ah, g = bh$, and $h = cf + dg$, then $ab + cd = 1$, hence a or b is a unit, so $f/g \in A$ or $g/f \in A$. By(V.3.2.3), A is a valuation ring. □

Prop. (V.3.2.11). A valuation ring is Noetherian iff it is discrete valuation iff it is PID.

Proof: Only need to prove Noetherian then $\Gamma = \mathbb{Z}$. we know ideals of Γ of the form $\{x | x \geq \gamma\}$, where $\gamma > 0$ has a maximal element, so there is a minimal element bigger than 0, so $\Gamma \cong \mathbb{Z}$. □

Prop. (V.3.2.12). In a fixed field, any inclusion relation of two valuation ring is given by localization.

Proof: Just localize at the image of the maximal ideal $\mathfrak{m}_B \cap A$, then they are valuation rings(V.3.2.5) that dominate each other, thus they are the same by definition(V.3.2.1). □

Valuations of Rank 1

Def. (V.3.2.13). A valuation is called **non-Archimedean** iff $|x+y| \leq \max\{|x|,|y|\}$. It is called **archimedean** iff it is not non-Archimedean.

Prop. (V.3.2.14). A valuation is non-Archimedean iff $\{|n||n \in \mathbb{N}\}$ is bounded.

Proof: If it is non-archimedean, then clearly by induction and $n = 1+(n-1)$ $|n| \leq 1$. Conversely, if $|n| \leq M$, then consider $|x+y|^n = |(x+y)^n| \leq \sum |C_n^k|x^k y^{n-k}| \leq M \max\{|x|,|y|\}$, so letting n be large, clearly $|x+y| \leq \max\{|x|,|y|\}$. □

Cor. (V.3.2.15). Any valuation on a field of char$\neq 0$ is non-Archimedean.

Prop. (V.3.2.16) (Equivalent Valuations). Two valuation on a field is equivalent iff $|x|_1 < 1 \Rightarrow |x|_2 < 1$ and is equivalent to $|x|_1 = |x|_2^s$ for some $s > 0$.

Proof: if two valuation are equivalent, then $x^n \to 0$ in τ_1 iff $x^n \to 0$ in τ_2, so $|x|_1 < 1 \Rightarrow |x|_2 < 1$.

If $|x|_1 < 1 \Rightarrow |x|_2 < 1$, then let y be an element that $|y|_1 > 1$, then any element $|x| = |y|^\alpha$ for some $\alpha \in \mathbb{R}$. Let $\frac{n_i}{m_i}$ converges to α from above, then $|\frac{x^{n_i}}{y^{m_i}}|_1 < 1$, so $|\frac{x^{n_i}}{y^{m_i}}|_2 < 1$, so $|x|_2 \leq |y|_2^\alpha$. Similarly, $|x|_2 \geq |y|_2^\alpha$, so $|x|_2 = |y|_2^\alpha$. So $|x|_1 = |x|_2^s$ for some $s > 0$.

If $|x|_1 = |x|_2^s$ for some $s > 0$, then these two valuations are clearly equivalent. □

Cor. (V.3.2.17) (Weak Approximation). If $|\cdot|_1, \ldots, |\cdot|_n$ be pairwise inequivalent valuations on K, then for any $a_1, \ldots, a_n \in K$ and $\varepsilon > 0$, there is an $x \in K$ that $|x - a_i|_i < \varepsilon$.

Proof: As $|\cdots|_1, |\cdots|_n$ are inequivalent, there are α, β that $|\alpha|_1 < 1, |\alpha|_n \geq 1$, $|\beta|_n < 1, |\beta|_1 \geq 1$ by(V.3.2.16), so let $y = \beta/\alpha$, then $|y|_1 > 1$, $|y|_n < 1$.

Now we prove by induction that there is an α that $|\alpha|_1 > 1, |\alpha|_i < 1$ for $i = 2, \ldots, n$. the case $n = 2$ is done, for general n, if the z for $n-1$ satisfies $|z|_n \leq 1$, then $z^m y$ will do, for m large. if $|z| > 1$, then the sequence $|t_m|_i = |\frac{z^m}{1+z^m}|_i$ converges to 1 for $i = 1, n$ and converges to 0 for $i = 2, \ldots, n-1$, so $t^m y$ will do, for m large. □

Prop. (V.3.2.18) (Gelfand). Any field with an Archimedean valuation K is a subfield of $\mathbb{C}$.

Proof: We consider its completion. when it contains $\mathbb{C}$, this is a corollary of**??**, otherwise, we consider $K \otimes \mathbb{C}$, then it is a finite dimensional module over K thus also complete. □

Prop. (V.3.2.19) (Ostrowski).
1. Any non-trivial value on $\mathbb{Q}$ is equivalent to v_p or $|\ |$. Thus any complete Archimedean field is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ by(V.3.2.18).
2. Any non-trivial valuation on $\mathbb{F}_q(t)$ is of the form $|\cdot|_p$ or $|\cdot|_\infty$, where p is an irreducible polynomial in $\mathbb{F}_q[t]$.

Proof: 1: if it is non-Archimedean, then $|n| \leq 1$, and it is not trivial, so there is a minimal p that $|p| < 1$. Then p is easily seen to be a prime. Then for any $(a,p) = 1$, $a = dp + r$, so $|r| = 1$, so $|a| = 1$.

And if it is Archimedean, then we prove that in $\mathbb{N}$, $|m| = m^\lambda$ for some λ: Let $F(n) = |n|$ and $f = \log_2 F$, then $f(m+n) \leq \max\{f(m), f(n)\} + 1$, and if $m = \sum_{i=1}^r d_i n^i$, then $f(m) \leq r(1+f(n)) + a_n$, where $a_n = \sup\{f(k)|k < n\}$. And $r \leq \log m/\log n$, so

$$\frac{f(m)}{\log m} \leq \frac{a + f(n)}{\log n} + \frac{b}{\log n}$$

then letting $m \to m^k, k \to \infty$, and then let $n \to n^k, k \to \infty$, we get $\frac{f(m)}{\log m} \le \frac{f(n)}{\log n}$, for any m, n.

2: Any valuation on $\mathbb{F}_q(t)$ is non-Archimedean(V.3.2.15), and $|n| = 1$ if $(n,p) = 1$, because $n^{p-1} = 1$. Similarly, if there is a minimal hence irreducible P that $|P| < 1$, then use induction and $Q = sP + r$ for some s, r of degree$< \deg Q$, so $|Q| = 1$ for all $(Q,P) = 1$. Otherwise, $|P| \le 1$ for all P, then $|t| > 1$, otherwise $|\cdot|$ is trivial, so it is easy by induction that $|F(t)| = |t|^{\deg F}$. □

Lemma (V.3.2.20) (Continuity of Roots). For a separable polynomial f over a valued alg.closed field $\overline{K}$, there is a ε that every polynomial g that are closed enough to f, the roots of g is closed to roots of f respectively.

Proof: This is easy to see by decomposition as each root of f is close to a root of g. f, g have the same degree so the roots correspond to each other. □

Prop. (V.3.2.21) (Fundamental Inequality). if (K,v) is a valued field and L/K be a field extension of degree n, if w_i are the valuations of L above v, then

$$\sum e(w_i/v)(f(w_i/v) \le [L:K].$$

The equality holds when v is discrete and L/K is separable.

Proof: Cf.[Clark note Theorem4]. □

Def. (V.3.2.22). A field K is called **spherically complete** iff each descending chain of balls has a nonempty intersection.

3 Topological Vector Space

Def. (V.3.3.1). A **topological vector space**(TVS) over a complete valued field k is a k-vector space that the addition and scalar multiplication is continuous.

Remark (V.3.3.2). If the field k is not of char 0, then we fix a sequence of elements $\{a_n\}$ that $\lim |a_n| = \infty$. This will be applied for example in the proof of Banach-Steinhaus theorem, but we will just write n instead of a_n.

Prop. (V.3.3.3). For subsets K, C of a TVS X that K is compact and C is closed, there is a nbhd V that $(K+V) \cap (C+V) = \varnothing$.

Proof: For each $x \in K$, there are symmetric nbhd V_x that $(x + V_x + V_x + V_x) \cap C = \varnothing$. Then $(x + V_x + V_x) \cap (C + V_x) = \varnothing$. Because K is compact, there are f.m. x_i that $K \subset \cup(x_i + V_{x_i})$, so let $V = \cap V_{x_i}$, then $(K+V) \cap (C+V) = \varnothing$. □

Cor. (V.3.3.4) (Closed Subbasis). Every nbhd of 0 in a TVS contains a closure of another nbhd of 0. (Apply the above proposition for $K = \{0\}$.

Def. (V.3.3.5). A subset containing 0 is called **balanced** iff $kU = U$ for each $|k| = 1$.

Prop. (V.3.3.6) (Balanced Subbasis). Every nbhd U of 0 in a TVS contains a balanced nbhd of 0. By(V.3.3.4), we can even assume $\overline{V} \subset U$.

Proof: Since scalar multiplication is continuous, there is a $\delta > 0$ and a nbhd V that $\alpha V \subset U$, for each $|\alpha| < \delta$. Then let $W = \cup_{|\alpha|<\delta} \alpha V$. □

Def. (V.3.3.7). A space is called a F**-space** if its toopology is induced by a complete invariant metric. F-space is of second Baire category by(IV.1.8.2)

A locally convex F-space is called a **Fréchet space**.

A TVS is said to satisfy **Heine-Borel** iff every closed and bounded subset of X is compact.

Def. (V.3.3.8). A **seminorm** on a vector space X is a real-valued function p that $p(x+y) \leq p(x)+p(y)$, and $p(\alpha x) = |\alpha|p(x)$ for $\alpha \in k$. It is called a **norm** if moreover $p(x) = 0 \iff x = 0$.

A family of seminorms $\{p_i\}$ on X is called **separating** iff for each x, at least one p_i satisfies $p_i(x) \neq 0$.

Prop. (V.3.3.9). A TVS is metrizable by a translation-invariant metric iff it has a countable basis.

Proof: One direction is trivial, for the other, Cf.[Rudin P18]. □

Prop. (V.3.3.10). If a subspace Y of a TVS X is a F-space, then it is closed in it.

Proof: Choose an invariant metric d compatible with its topology, Let U_n be a nbhd of X that $U_n \cap Y = B(0, 1/n)$, and choose a symmetric nbhd V_n of X that $V_n + V_n \subset U_n$, and $V_{n+1} \subset V_n$.

If $y \subset \overline{Y}$, then for any $y_n \subset Y \cap (y + V_n) = E_n$, then $y_n - y_m \subset U_{\min\{m,n\}} \cap Y = B(0, 1/n)$, so it is a Cauchy sequence in Y, hence all E_n has a unique element y_0 in common. Now if we intersect each V_n by a nbhd W of X, the same argument shows that there is a unique element y_W in $Y \cap (y + W \cap V_n)$, and this must by just y_0, but $y - y_W \subset W$, so we must have $y = y_0 \subset Y$. □

Def. (V.3.3.11). A set E in a TVS is called **totally bounded** if for every nbhd V of 0, there is a finite set F that $E \subset F + V$.

4 Completeness

Prop. (V.3.4.1) (Banach-Steinhaus). Γ is a collection of continuous linear mapping between two TVS, if the set B of x that $\Gamma(x)$ is bounded is a second category set in X, then $B = X$ and Γ is equicontinuous (thus maps bounded sets to bounded sets).

Proof: For an open balanced nbhd W of 0, choose a balanced nbhd U s.t. $\overline{U} + \overline{U} < W$(V.3.3.6), set $E = \cap_{\Lambda \in \Gamma} \Lambda^{-1}(\overline{U})$, then $B \subset \bigcup_{i=1}^{\infty} nE$, so by Baire theorem(IV.1.8.2), E has a interior point thus has a nbhd V s.t. $\Gamma(V) \subset \overline{U} + \overline{U} \subset W$. Thus we are done. □

Cor. (V.3.4.2) (Uniform Boundedness Theorem). If a set Γ of continuous linear mappings from a F-space X to Y satisfies $\Gamma(x)$ is bounded for every x, then Γ is equicontinuous.

Cor. (V.3.4.3). If A_n is a sequence of continuous linear mapping from X to Y, if X is a F-space, then $\lim A_n = A$ iff $||A_n||$ is bounded and $\lim A_n x = Ax$ for x in a dense subset of X.

Proof: One direction is immediate from Banach-Steinhaus, the converse is an easy $\varepsilon/3$-technique. □

Prop. (V.3.4.4) (Open Mapping theorem). If a continuous linear mapping T from a F-space X to Y satisfies $R(T)$ is of second category, then it is a surjective open mapping and Y is a F-space.

In fact, we only need T be defined on a subspace $D(T)$ and it is **closed** in the sense the graph of it is closed.

Proof: $V_n = T(B(0, \frac{r}{2^n}))$ are all of second category, because $\cup_n nV_n = R(T)$, so $\overline{V_n}$ has an interior by definition. Then also it contains a nbhd of V because $\overline{V_{n+1}} + \overline{V_{n+1}} \subset \overline{V_n}$.

Now we show $\overline{V_{n+1}} \subset V_n$, this will show T is open. thus for a $y \in \overline{V_n}$ since $\overline{T(V_{n+1})}$ contains a nbhd of 0, we can consecutively choose $x_i \in B(0, \frac{r}{2^{n+i}})$ s.t. $y - \sum_{i=1}^n T(x_i) \in \overline{T(B(0, \frac{1}{2^{n+i+1}}))}$. So by completeness of X and closedness, $\sum x_i$ converges to some $x \in D(T)$, and $Tx = y \in V_n$.

And an open linear mapping must be surjective. hence $Y \cong X/N(T)$, so Y is also a F-space. □

Cor. (V.3.4.5) (Banach). If a continuous map T between F-spaces is a bijection, then it has a continuous inverse.

Cor. (V.3.4.6). If a F-space is complete w.r.t two different topologies and one is stronger than the other, then they are equivalent.

Cor. (V.3.4.7). For every operator between F-spaces that has closed image, we have $X/N(T) \cong R(T)$.

Cor. (V.3.4.8) (Closed Graph Theorem). If T is a closed linear mapping between two F-spaces, i.e. the graph of it is closed, then it is continuous, because the metric induced by the graph is stronger than the original one, and use Banach(V.3.4.5).

The graph is closed is equivalent to if $x_i \to x$ and $Tx_i \to y$, then $y = Tx$. This is very useful when proving some map is continuous.

Cor. (V.3.4.9). If A, B, C are F-spaces, and $f : A \to B, g : B \to C$, if gf, g is continuous and g is injective, then f is continuous.

Proof: Use closed graph theorem, if $x_i \to x, f(x_i) \to z$, then $gf(x_i) \to g(z)$, so $gf(x) = g(z)$, so $f(x) = z$. □

Cor. (V.3.4.10) (Finite Codimensional Image). If the image of a continuous linear mapping T between F-spaces has finite codimensional image, then the image is closed and complemented.

Proof: It has finite codimension, so we can construct $K^n \oplus X/N(T) \to Y$, by Banach theorem(V.3.4.5) it is a homeomorphism, and the image of $X/N(T)$ corresponds to the image, so the image is closed. □

Prop. (V.3.4.11) (Separate Continuous). If a bilinear map $B : X \times Y \to Z$ where X is a F-space is separately closed, then $B(x_n, y_n)$ converges to $B(x_0, y_0)$.

Proof: Use Banach-Steinhaus to prove $B_{y_n}(x)$ is equicontinuous, then use $B(x_n - x_0, y_n) + B(x_0, y_n - y_0)$. ? □

5 Dual Space

Prop. (V.3.5.1). If X, Y are normed spaces then $L(X, Y)$ is also a normed space with the metric $||\Lambda|| = \sup\{||\Lambda x|||x| \leq 1\}$. And if Y is Banach, then $L(X, Y)$ is also Banach. The proof is easy.

In particular, if $Y = K$, then X^* is a Banach space.

Prop. (V.3.5.2). For a bounded operator T,

$$\overline{R(T)} = N(T^*)^\perp, \text{ Thus } \overline{R(T^*)} = N(T)^\perp$$

In particular, using Hahn Banach, $R(T)$ is dense in Y iff T^* is injective, T is injective iff T^* is weak*-dense in X^*.

Prop. (V.3.5.3). Let $\Lambda_1, \ldots, \Lambda_n, \Lambda$ are linear functionals on a vector space X, let $N = \cap \operatorname{Ker} f_i$, the following are equivalent:

1. $\Lambda = \sum \alpha_i \Lambda_i$.
2. $|\Lambda x| \leq \gamma |\Lambda_i x|$.
3. $\operatorname{Ker} \Lambda \subset N$.

Proof: Only need to show $3 \to 1$: define $\pi : X \to k^n : x \mapsto (\Lambda_1 x, \ldots, \Lambda_n x)$, then by hypothesis $f(\pi_i(x)) = \Lambda(x)$ defined a linear functional on $\pi(X)$. This can be extended to a functional on $k^n : F(u_1, \ldots, u_n) = \sum \alpha_i u_i$, so Λ is a linear combination of Λ_i. □

Weak Convergence

Def. (V.3.5.4) (Operator Topologies). There are three topologies on $L(X)$ for a normed space X:

- norm topology: $||T_i - T|| \to 0$.
- strong topology: $\forall x \in X, ||(T_i - T)x|| \to 0$.
- weak topology: $\forall x \in X, f \in X^*$, $\lim f(T_n x) = f(Tx)$.

Prop. (V.3.5.5) (Weak Convergence and Bounded). In a normed space X, if $x_n \to x$ weakly iff $\{x_n\}$ is bounded and $\lim f(x_n) = f(x)$ for a dense subset $f \in M^* \subset X^*$.

Proof: This follows from(V.3.4.3), as X^* is a Banach space, by(V.3.5.1). □

6 Banach Space

Def. (V.3.6.1). For K complete valued field, a complete normed(valued) K-vector space is called a **Banach space**.

A K-algebra with a complete K-algebra norm is called a **Banach algebra**.

Prop. (V.3.6.2). The dual space of a Banach space is a Banach space. (Immediate from(V.3.5.1)).

Prop. (V.3.6.3). if A is a Banach space as well as a Topological group, then there is a norm on A which induce the same topology and makes A into a Banach algebra.

Proof: embed A into $L(A)$ by left multiplication, which is injective, and $||x|| = ||xe|| = ||M_x e|| \leq ||M_x||||e||$, so its inverse is continuous. Now if we show the image $\tilde{A}$ is closed in $L(A)$, then the open mapping theorem will show that $A \cong \widetilde{A}$, and $\widetilde{A}$ is clearly a Banach algebra.

To show it is closed, if $T = \lim T_i$, notice $T_i(y) = T_i(e)y$, so take a limit, $T(y) = T(e)y = M_{T(e)}y$. □

Cor. (V.3.6.4). Every f.d. Banach algebra is isomorphic to an algebra of matrices. In particular, if $xy = e$, then $yx = e$.

Proof: Embed A into $L(A)$. □

Remark (V.3.6.5) (Inequivalent Banach Norms). There exists two complete norm on a vector space that is inequivalent. For this, just choose a banach space X, and notice if we can choose a discontinuous bijection $X \to X$, then the induced metric is also complete, and it cannot by equivalent by Banach theorem(V.3.4.5). For this, choose a infinite dimensional Banach space over $\mathbb{C}$, and choose a $\mathbb{C}$-basis x_i for it, and choose a sequence x_n and maps x_n to nx_n, the rest are invariant, then this is not continuous.

Hilbert Space

Def. (V.3.6.6). there are different topologies in the space of operators on a Hilbert space $\mathcal{H}$.
Norm operator topology: defined by the norm $||T||$.
Strong operator topology: defined by the separating seminorms $T \mapsto ||Tu||, u \in \mathcal{H}$.
Weak operator topology: defined by the separating seminorms $T \mapsto (Tu, v), u, v \in \mathcal{H}$.

Prop. (V.3.6.7). The strong and weak operator topology coincides on the unitary operators on $\mathcal{H}$. The sets of unitary operators that is continuous in this two topology is denoted by $U(\mathcal{H})$.

Proof: If T_n converges to T in the weak operator topology, then

$$||(T_n - T)u||^2 = ||Tu||^2 + ||T_n u||^2 - 2\,\mathrm{Re}(T_n u, Tu).$$

The right hand side is clearly bounded by the weak seminorms, so the two topologies coincide. □

Prop. (V.3.6.8) (Hilbert Basis). If H is a Hilbert space and $S = \{e_\alpha\}$ is an orthonormal basis in H, then the following are equivalent:

1. For any x, $x = \sum(x, e_\alpha)e_\alpha$,(notice the sum are in fact infinite sum).
2. There is a no nonzero element x that is orthogonal to all e_α.
3. **Parseval equality** holds: $||x||^2 = \sum |(x, e_\alpha)|^2$.

If these are true, then S is called a **Hilbert basis** of H, a Hilbert basis always exists, by Zorn's lemma.

Proof: $1 \to 2$: if $(x, e_\alpha) = 0$ for all e_α, then by 1, $x = \sum(x, e_\alpha)e_\alpha = 0$.
$2 \to 3$: Notice $y = x - \sum(x, e_\alpha)e_\alpha$ is orthogonal to all e_α, and

$$||y|| = ||x||^2 - \sum |(x, e_\alpha)|^2,$$

so Parseval equality holds.
$3 \to 1$: $||x - \sum(x, e_\alpha)e_\alpha|| = 0$. □

Prop. (V.3.6.9). Any symmetric operator on a Hilbert space is continuous.

Proof: Because $x_n \to 0$ implies $Tx_n \to 0$ weakly, so we can use closed graph theorem(V.3.4.8).**?**
□

7 Nuclear Maps and Spaces

V.4 Archimedean Functional Analysis

Reference: [Rudin Functional Analysis]. [Rudin Functional Analysis Chap11,13] needs to be revised.

This section contains functional analysis in characteristic 0. By Ostrowski theorem(V.3.2.19), the base field is jsut R or $\mathbb{C}$.

1 Topological Vector Space

Def. (V.4.1.1). A **sublinear functional** is a function p that $p(x+y) \leq p(x)+p(y)$ and $p(\lambda x) = \lambda p(x)$ for $\lambda > 0$.

A **seminorm** is a non-negative function p that $p(x+y) \leq p(x)+p(y)$ and $p(\alpha x) = |\alpha| p(x)$ for all complex α.

Def. (V.4.1.2). A **absorbing set** is a convex set A that $\cup_{k>0} kA = X$. A convex nbhd of 0 is clearly absorbing.

Def. (V.4.1.3) (Minkowski Functional). For an absorbing set A, the **Minkowski functional** μ_A is defined to be $\mu_A(x) = \inf\{t > 0, x/t \in A\}$. It satisfies:

- $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$.
- $\mu_A(kx) = k\mu_A(x)$ if $k > 0$.
- μ_A is a seminorm if A is balanced.
- If $B = \{x|\mu(x) < 1\}, C = \{x|\mu(x) \leq 1\}$, then $B \subset A \subset C$ and $\mu_A = \mu_B = \mu_C$.

Proof: Cf.[Rudin P27]. □

Cor. (V.4.1.4) (Seminorm and Absorbing set). A seminorm on X is exactly the Minkowski functional of a balanced absorbing set W, but the set may not be unique. and it is uniformly continuous iff 0 is an interior point.

Proof: If p is a seminorm, then $\{x|p(x) < 1\}$ is convex, balanced and absorbing by definition(V.3.3.8). The converse is by(V.4.1.3). The last assertion is easy. □

Prop. (V.4.1.5) (Minkowski Functional and Separating Seminorms). If $\mathfrak{B}$ is a convex local base in a TVS X, then the Minkowski functionals of elements of $\mathfrak{B}$ forms a separating family of seminorms.

Conversely, a separation family P of seminorms on a vector space defines a convex balanced local base for a topology τ that is locally convex. And in this topology, a sequence converges iff $p(x_i - x) \to 0$ for $p \in \mathfrak{P}$, a set is bounded if each p is bounded on it.

Proof: For any $V \in \mathfrak{B}$, $V = \{x \in X|\mu_V(x) < 1\}$, because V is open and convex. (V.4.1.3) shows each μ_V is a seminorm, and it is continuous because it is bounded on V. And they are separating because $\mathfrak{B}$ is a local base.

Defined $V(p,n) = \{x \in X|p(x) < 1/n\}$, and let these be a local subbasis at 0, and make it a topology by translation. This is checked to be a locally convex TVS. For the last assertion, if E is bounded, then $E \subset kV(p,1)$ for k large, so p is bounded on E, and conversely, for each nbhd U, there are p_i and M_i and $\cap V(p_i, M_i) \subset U$, so $E \subset kU$ for n large. □

Prop. (V.4.1.6). If $\mathfrak{P}$ is a family of countable separating family of semi-norms on X, then the topology defined in(V.4.1.5) is in fact metrizable, by a metric $d(x,y) = \sum \frac{1}{2^k}\frac{p_i(x-y)}{1+p_i(x-y)}$.

Finite Dimensional Subspace

Prop. (V.4.1.7) (Finite Dimensional and Locally Compact). There is only one topological vector space structure on a finite dimensional $\mathbb{C}$-vector space and it is complete. A TVS is locally compact iff it is f.d.

Proof: Cf.[Rudin P17].

For the second assertion, if is locally compact, then 0 has a nbhd V that is precompact, so bounded, hence $2^{-n}V$ forms a local basis. the compactness of $\overline{V}$ shows there are f.m. x_i that $\overline{V} \subset \cup(x_i + \frac{1}{2}V)$. Let Y be the subspace spanned by x_i, then it is of f.d, thus closed. Now $V \subset Y + \frac{1}{2}V$, so $\frac{1}{2}V \subset Y + \frac{1}{4}V$, hence continuing this way, $V \subset \cap(Y + 2^{-n}V)$, so $V \subset \overline{Y} = Y$. But then $Y = X$. □

Cor. (V.4.1.8) (Finite Subspace Closed). A f.d subspace in a TVS over $\mathbb{C}$ is closed, because it must be a F-space, hence it is closed by(V.3.3.10).

Prop. (V.4.1.9) (Finite Subspace in Banach Space). For a finite dimensional space V in an Archimedean Banach space, there is a continuous projection onto it. In particular, any finite dimensional space in an Archimedean Banach space is complemented.

Also finite codimensional subspace in any Banach space is complemented by(V.3.4.10).

Proof: Choose a basis e_i for V, consider the dual basis f_i. Because a finite dimensional space only has one topology(V.4.1.7), these f_i are bounded on V. Extend them to bounded functional on X, then consider $p(x) = \sum f_i(x)e_i$, then this is a continuous projection onto V. □

2 Various Spaces and Duality

For a bounded connected open set Ω,

- **Sobolev Space** $W^{m,p}(\Omega)$ is the completion of a subspace of $C^\infty(\Omega)$ with the norm
$$||u||_{m,p} = (\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha u(x)|^p \,\mathrm{dx})^{1/p}.$$
for $m > 0$. And we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. It is also a subspace of $L^p(\Omega)$ that satisfies this, without completion(V.7.3.1).
- $C_0^\infty(\Omega)$ is the subspace of $C^\infty(\Omega)$ that have compact support in Ω. Its completion $W_0^{m,p}(\Omega)$ is a closed subspace of $W^{m,p}(\Omega)$. And we denote $W_0^{m,2}(\Omega)$ by $H_0^m(\Omega)$ and the dual space of $H_0^m(\Omega)$ by $H^{-m}(\Omega)$.
- $C(\Omega)$ in the topology of compact convergence is a Fréchet space. It is not locally convex.
- $H(\Omega)$ the space of holomorphic functions in Ω is a closed subspace of $C(\Omega)$ thus is a Fréchet space. Montel's theorem says exactly that $H(\Omega)$ is Heine-Borel.
- $\mathcal{H}^2(D)$ the space of holomorphic functions on the unit disk D that is also L^2. It has the L^2 norm.
- $C^\infty(\Omega)$ in the topology defined by seminorms $p_N(f) = \max\{|D^\alpha f(x)| : x \in K_N, |\alpha| \leq N\}$, is a Fréchet space thus locally convex and it has the Heine-Borel property by Arzela-Ascoli.
- $D(K)$ is the closed subspace of smooth functions on Ω with support in K, thus a Fréchet space with Heine-Borel property.

- $D(\Omega)$ is the space of smooth functions with support in Ω. It has the topology generated by translation of basis consisting of convex balanced sets W that $W \cap D(K)$ is open for every compact K. This makes $D(\Omega)$ into a locally convex TVS, Cf.[Rudin P152]. It has Heine-Borel property(V.7.1.1).

Dual Spaces

Prop. (V.4.2.1).
- For a σ-finite measure μ on a measurable space X, for $1 \le p < \infty$

$$L^p(X, \Omega, \mu)^* = L^q(X, \Omega, \mu).$$

(V.1.4.3)

- $C[0,1]^* = BV[0,1]$ and $C[X]^* = M(X)$, the space of complex measure on compact X with the norm of total variance, by Riesz representation theorem(V.1.1.5).

3 Convexity

Prop. (V.4.3.1). Every convex nbhd of 0 contains a balanced convex nbhd of 0. By(V.3.3.4), we can even assume $\overline{V} \subset U$.

Proof: If U is convex, choose W as in(V.3.3.6), then $W \subset A = \cap_{|\alpha|=1} \alpha U$ because it is balanced. Then $W \subset A^\circ$ and A° is open and balanced, satisfying the requirement. □

Prop. (V.4.3.2). For a compact convex set K in a TVS X, if a set Γ of continuous linear mapping is bounded for every $x \in K$, then Γ is equicontinuous on K.

Proof: The proof is similar to that of Banach-Steinhaus(V.3.4.1). For K compact convex, the same argument shows that there is a nbhd V that $K \cap (x_0 + V) \subset nE$, fix $p > 1$ that $K \subset x_0 + pV$, then for any $x \in K$, consider $z = (1 - p^{-1})x_0 + p^{-1}x$, then $z \in K$ as K is convex and $z - x_0 = p^{-1}(x - x_0) \in V$, so $z \in nE$, and since $x = pz - (p-1)x_0$, $\Lambda x \in pnW$ for each $\Lambda \in \Gamma$, so Γ is equicontinuous. □

Hahn-Banach

Prop. (V.4.3.3) (Real Hahn). For a sublinear functional p on a real linear space X and a subspace X_0, if a functional f satisfies $f(x) \le p(x)$ on X_0, then it can be extended to a functional Λ on X that $|\Lambda(x)| \le |p(x)|$.

Proof: Use Zorn's lemma, if the maximum extension is not on the whole space but on M, choose $x_1 \in X - M$, we want to define $f(x_1)$. Now let $M_1 = \{x + tx_1 | x \in M\}$. Since for $x, y \in M$, $f(x) + f(y) = f(x+y) \le p(x+y) \le p(x-y) + p(x_1 + y)$, so

$$f(x) - p(x - x_1) \le p(y + x_1) - f(y)$$

for $x, y \in M$. Let the maximum of the left side be α, and define $f(x_1) = \alpha$, then it is clear $f(z) \le p(z)$ still. □

Prop. (V.4.3.4) (Complex Hahn). For a seminorm p (i.e. it can attain 0) on a complex linear space X and a subspace X_0, if a functional f satisfies $|f(x)| \le p(x)$ on X_0, then it can be extended to a functional on X with the same condition.

Proof: Let $g(x) = \operatorname{Re} f(x)$ and extend it by Hahn and set $f(x) = g(x) - ig(ix)$, then f is complex linear, and for any x, for some $|\alpha| = 1$, $|f(x)| = f(\alpha x) = g(\alpha x) \leq p(\alpha x) = p(x)$. □

Cor. (V.4.3.5) (Hahn). In a normed space X, a bounded linear functional on a subspace X_0 can be extended to a bounded functional on X with the same norm.

Cor. (V.4.3.6) (Extending Functional Preserving Norm). If X is a normed space and N is a closed subspace, if x_0 satisfies $d = d(x_0, M) > 0$, then here is a continuous functional f that $f(x) = 0$ and $f(x_0) = d$ and $||f|| = 1$.

Proof: Define $f(m + \alpha x) = |\alpha| d$ on $\operatorname{span}\{M, x\}$, then $f(m + \alpha x) = |\alpha| d = |\alpha| d(x_0, M) \leq |\alpha| ||\frac{x'}{\alpha} + x_0|| = ||x' + \alpha x|| = ||x||$. So $||f|| \leq 1$, so we can use Hahan-Banach to extend it to a functional on X. □

Prop. (V.4.3.7) (Geometric Hahn).

- If E_1 and E_2 are two convex set that $E_1 \cap E_2 = \emptyset$ and E_1 has interior point, then there is a continuous linear functional that separate them, i.e $\operatorname{Re} f(E_1) < \operatorname{Re} f(E_2)$. (The interior point is here to assure f is continuous).
- In a locally convex TVS, if E_1 is convex compact and E_2 is convex closed, then there is a real functional that separate them, i.e. $\operatorname{Re} f(E_1) < \gamma_1 < \gamma_2 < \operatorname{Re} f(E_2)$. Thus for a set E and a point x, $x \in \overline{\operatorname{span} E} \iff$ for all f that $f(E) = 0$, $f(x) = 0$.

Proof: The complex case follows from the real case, so assume it is real. Consider $a_0 \in E_1, b_0 \in E_2$, let $x_0 = a_0 - b_0$ and let $C = E_1 - E_2 + x_0$, then C is a convex nbhd of 0. Let p be the Minkowski functional of C, then p is sublinear by(V.4.1.3) and $p(x_0) \geq 1$. Let $f(tx_0) = t$ on the subspace M generated by x, then it extends to a functional Λ that≤ 1 on C, thus it is bounded by 1 on $C \cap (-C)$, hence continuous. For any $a \in E_1, b \in E_2$, because $\lambda(x_0) = 1$ and $a - b + x_0 \in C$ open, $\Lambda a < \Lambda b$.

For the second, There is a convex nbhd V of 0 that $E_1 + V \cap B = \varphi$, so the above argument applied with $E_1 + V$ and B shows that there is a f that separate them. And $f(E_1 + V)$ is open and $f(E_1)$ is compact, so the conclusion follows. □

Cor. (V.4.3.8) (Banach-Saks). The weak closure of a convex set in a locally convex metric space equals the original closure.

Thus if a sequence $\{x_n\}$ weakly converges to x in a metrizable locally convex space, then a convex combination of $\{x_n\}$ strongly converge to x, i.e. $x \in \overline{\text{co}}(\{x_n\})$, because metric space is first countable.

Proof: A weak closed set is closed, and to show the closure is weakly closed, use(V.4.3.7). □

Prop. (V.4.3.9). If A_i are compact convex sets in a TVS X, then $co(A_1 \cup \ldots \cup A_n)$ is compact.

Proof: Firstly, the image K of $S \times A_1 \times \ldots \times A_n$, $(s_1, \ldots, s_n) \times (a_1, \ldots, a_n) \mapsto \sum s_i a_i$ is closed, where $S = \{0 \leq x_i, \sum x_i = 1\}$. And we show K is just the convex closure: it contains all A_i, and it is convex because each A_i is. □

Prop. (V.4.3.10). In F-space, a closed subset is compact iff it is totally bounded by(IV.1.7.4).

Prop. (V.4.3.11). In a locally convex space, if E is totally bounded, then $\operatorname{co}(E)$ is totally bounded. Thus in a Fréchet space, if K is compact, then $\overline{co}(K)$ is compact.

Proof: For a nbhd U of 0, choose a convex nbhd V that $V+V\subset U$, then $E\subset F+V$ for some finite set F, hence $co(E)\subset co(F)+V$. But $co(F)$ is compact by(V.4.3.9). So $co(F)\subset F_1+V$ for some finite set F_1, then $co(E)\subset F_1+U$.

If K is compact, then it is totally bounded, and then $co(K)$ is totally bounded and $\overline{co(K)}$ is totally bounded by(IV.1.7.2), so it is compact by(V.4.3.10). □

Prop. (V.4.3.12) (Weakly Bounded and Locally Convex). In a locally convex space, bounded $\iff$ weakly bounded.

Proof: One direction is trivial, for the other, suppose E is weakly bounded and U is a closed nbhd of 0. Because X is locally convex, there is a convex, balanced nbhd of 0 that $\overline{V}\subset U$(V.4.3.1). Now $\overline{V}=V^{**}$ the polar(V.4.4.1) by(V.4.3.8).

Now V^* is weak*-compact and $|\Lambda(x)|\leq\gamma(\Lambda)$ for each $\Lambda\in X^*$ for some $\gamma(\Lambda)$ because E is weakly bounded. So we can use(V.4.3.2) to show that $|\Lambda x|\leq\gamma$ for some γ and all $\Lambda\in V^*$. So we have $\gamma^{-1}E\subset\overline{V}\subset U$. This proves that E is bounded. □

Prop. (V.4.3.13) (Markov-Kakutani Fixed Point Theorem). For a commuting family $\mathcal{F}$ of continuous affine maps from K to K where K is a compact convex set in a TVS, then there is a fixed point in K for all maps in $\mathcal{F}$.

Proof: Consider the semigroup $\mathcal{F}^*$ generated by these maps together with their average, it is also commutative because they are all affine. For any $f,g\in\mathcal{F}^*$, $f(K)\cap g(K)\supset f\circ g(K)$, so by finite intersection property, there is a point in $p\in K$ in all $f(K)$.

For this p, consider $p=\frac{1}{n}(I+T+t^2+\ldots+T^{n-1})(x)$, then $p-Tp=\frac{1}{n}(x-T^nx)\in\frac{1}{n}(K-K)$. as K is bounded and n is arbitrary, this means that $p=Tp$ for all T. □

Cor. (V.4.3.14) (Invariant Hahn). For a commuting family Γ of operators on a normed space and Y an invariant space, then for any Γ-invariant continuous functional f on Y, it has a Γ-invariant Hahn extension.

Proof: We may assume $||f||=1$. Let K be all extensions of f that has norm≤ 1. K is obviously convex, and it is weak*-compact by Banach-Alaoglu. The adjoint action of T is checked to be continuous in the weak*-topology, so by(V.4.3.13), there is some $F\in K$ that is invariant under Γ. □

Krein-Milman theorem

Prop. (V.4.3.15) (Krein-Milman Theorem). For a compact convex set in a TVS that is weak-Hausdorff(X^* separate points), then $K=\overline{\mathrm{co}}(\mathrm{Extreme}(K))$.

If K is a compact set in a locally convex space, then $K\subset\overline{\mathrm{co}}(E(K))=\overline{\mathrm{co}}(K)$.

Proof: First show that every compact extreme set S of K contains an extreme point. Notice arbitrary intersection of compact extreme sets of K is compact extreme, because compact is closed, because X is Hausdorff. And for any functional $\Lambda\in X^*$, the maximal value point in K is compact extreme. Now we use Zorn's lemma to find a minimal compact extreme set in S, then it must be a point because X^* separate points.

Now use the weak topology Hahn(V.4.4.2), if $\overline{co}(E(K))\subset K$ is not K, then it is compact, then we can find a functional that separate $\overline{co}(E(K))$ and some point of K. This is a contradiction because the extreme value point for any functional on K is an extreme set.

In the locally convex case, the convexity of K is not needed, and we can show using geometric Hahn(V.4.3.7) instead that, $K\subset\overline{co}(K)$. □

Prop. (V.4.3.16) (Milman's Theorem). If K is a compact set in a locally convex space X and if $\overline{\text{co}}(K)$ is also compact(e.g in a Fréchet space(V.4.3.11)), then every extreme point of $\overline{\text{co}}(K)$ lies in K.

Proof: $\square$

Def. (V.4.3.17). For a compact Hausdorff space S and an algebra in $C(S)$, a subset E is called **A-antisymmetric** iff every $f \in A$ that is real on E is constant on E. There are in fact maximal A-antisymmetric subsets of S.

Prop. (V.4.3.18) (Bishop Theorem). If A is a closed subalgebra of $C(S)$. If $g \in C(S)$ satisfies $g|_E \in A|_E$ for every maximal A-antisymmetric set E, then $g \in A$. This theorem is a generalization of Stone-Weierstrass approximation.

Proof: The annihilator $A^\perp$ of A consists of all regular complex Borel measure μ on S that $\int f d\mu = 0$ for all $f \in A$ by Riesz representation(V.1.1.5). ?Cf.[Rudin P122]. $\square$

Prop. (V.4.3.19) (Schauder Fixed Point Theorem). If C is a closed convex subset in a metrizable TVS and continuous $T : C \to C$ has sequentially compact image (e.g. C is compact and X is locally convex hence X^* separate points), then T has a fixed point.

Proof: As $T(C)$ is sequentially compact, for each n, there is a $1/n$-net $N_n = \{y_i\} \subset T(C)$, let $E_n = \text{span}\{N_n\}$.

Define a map $T(C) \to co(N_n) : I_n(y) = \sum y_i\lambda_i(y)$, where $\lambda_i(y) = \frac{m_i(y)}{\sum m_i(y)}$, and $m_i(y) = 1 - n||y - y_i||$ if $y \in B(y_i, 1/n)$, and 0 otherwise.

Now $||I_n(y) - y|| = ||\sum(y_i - y)\lambda_i(y)|| \le \sum ||y_i - y||\lambda_i(y) \le \frac{1}{n}$ for each $y \in T(C)$. As C is convex, $co(N_n) \subset C$, if we let $T_n = I_n \circ T$, then T_n has a fixed pt x_n in $co(N_n)$ by Brower fixed pt theorem(IV.4.1.10).

As $T(C)$ is sequentially compact and C is closed, there is a subsequence Tx_{n_k} that converges to $x \in C$. And then

$$||x_{n_k} - x|| = ||I_nTx_{n_k} - x|| \le ||I_nTx_{n_k} - Tx_{n_k}|| + ||Tx_{n_k} - x|| < \frac{1}{n} + ||Tx_{n_k} - x||$$

so $x_{n_k} \to x$, and then by continuity, $Tx = x$. $\square$

Vector-valued Integration

Def. (V.4.3.20) (Vector-Valued Integration). Given a measure space (Q, μ) and X is an Archimedean TVS on which X^* separate points. If f is a function from M to X that $\Lambda \circ f$ are integrable w.r.t μ for any $\Lambda \in X^*$. The **integration** $\int_M f d\mu$ of f w.r.t Q is an element y that

$$\Lambda y = \int_Q (\Lambda f) d\mu$$

for any $\Lambda \in X^*$.

Prop. (V.4.3.21). If X is an Archimedean TVS on which X^* separate points, (Q, μ) is a Radon measure on a locally compact Hausdorff space that μ is compactly supported, and f is continuous that $\overline{co}(f(Q))$ is compact(e.g. when X is Fréchet(V.4.3.11)), then the integral $y = \int_Q f d\mu$ exists, and belongs to the closed linear span of the range of H. Moreover if μ is positive and $\mu(Q) = 1$, then $y \in \overline{co}(f(Q))$.

Proof: Cf.[Rudin P78]. □

Cor. (V.4.3.22). If Q is compact Hausdorff, X is Archimedean Banach and $f : Q \to X$ is continuous, then

$$||\int_Q f d\mu|| \leq \int_Q ||f|| d\mu$$

Proof: Let $y = \int_Q f d\mu$. By(V.4.3.6), there is a $||\Lambda|| \leq 1$ that $\Lambda y = ||y||$, so

$$\Lambda y = ||y|| = \int_Q \Lambda f d\mu = |\int_Q \Lambda f d\mu| \leq \int_Q |\Lambda f| d\mu \leq \int_Q ||f|| d\mu$$

□

Prop. (V.4.3.23). If X is an Archimedean TVS on which X^* separate points, Q is a compact subset of X, and $\overline{co}(Q)$ is compact, then $y \in \overline{co}(Q)$ iff there is a regular Borel measure μ on Q that $y = \int_Q x d\mu(x)$.

Proof: Cf.[Rudin P79]. □

Prop. (V.4.3.24) (Continuous Action Extends to Measure). For a fixed map $f : Q \to X$, assume X is a Fréchet space, then the integration functor in(V.4.3.21) induces a continuous map

$$Meas_c(Q) \to X : \mu \mapsto \int_\mu f$$

that maps δ_x to $f(x)$. And this map is a morphism of associative algebras.

Proof: It suffices to verify continuity: for any seminorm ρ, by convexity,

$$\rho(\int_\mu f) \leq (\mu, \rho(f)),$$

thus for $\mu \in U$ satisfying $(\mu, \rho(f)) < 1$, $\rho(\int_\mu f) < 1$. This proves continuity.

For the algebra structure, **?**. □

Prop. (V.4.3.25) (Vector Valued Integration Stronger). If V is a Banach space and μ is a Radon measure on a locally compact Hausdorff space X. If $g \in L^1(\mu)$ and $H : X \to V$ is bounded and continuous, then $\int gH d\mu$ exists and belongs to the closed linear span of the range of H, and

$$||\int gH d\mu|| \leq \sup_{x\in X} ||H(x)|| \int |g(x)|\mu(x)$$

Proof: Clearly $\varphi(gH) \in L^1(\mu)$ for all $\varphi \in V^*$. And μ is Radon, so there is a sequence $\{g_n\} \in C_c(X)$ that converges to g in L^1, so $\int g_n H d\mu$ is integrable by(V.4.3.21), and

$$\int ||g_n(x)H(x) - g_m(x)H(x)dx|| d\mu(x) \leq \int |g_n(x) - g_m(x)| d\mu(x) \to 0$$

thus this is a Cauchy sequence, converging to some y. Now for any $\varphi \in V^*$,

$$\varphi(y) = \lim \varphi(\int g_n H d\mu(x)) = \lim \int \varphi \circ (g_n H) d\mu = \int \varphi \circ (gH) d\mu$$

The last equality uses boundedness again.

Moreover, each $\int g_n H d\mu$ belongs to the closed range of H by(V.4.3.21), hence so does $\int gH d\mu$. And last assertion is also from(V.4.3.21). □

Holomorphic Functions

Def. (V.4.3.26) (Holomorphic Functions). Let Ω be an open set in $\mathbb{C}$, and X be a TVS over $\mathbb{C}$, then A function $f:\Omega\to X$ is called:

- **weakly holomorphic** if Λf is holomorphic for any $\Lambda\in X^*$.
- **strongly holomorphic** if $\lim_{w\to z}\frac{f(w)-f(z)}{w-z}$ exists for every $z\in\Omega$.

A strongly holomorphic function is clearly weakly holomorphic, and the converse is true when X is Fréchet space, by the following proposition(V.4.3.27).

Prop. (V.4.3.27) (Weak and Strong Holomorphic). Let Ω be an open set in $\mathbb{C}$, and X be a Fréchet space over $\mathbb{C}$, then f is strongly continuous, and the Cauchy integral formula(V.2.3.4) holds for f, and f is strongly holomorphic.

Proof: We may assume $0\in\Omega$, then Let $B(0,2r)\subset\Omega$ and Γ the boundary of $B(0,2r)$, since Λf is holomorphic,

$$\frac{(\Lambda f)(z)-(\Lambda f)(0)}{z}=\frac{1}{2\pi i}\int_\Gamma\frac{(\Lambda f)(\zeta)}{(\zeta-z)\zeta}d\zeta\quad 0<|z|<2r.$$

Therefore $\{\frac{f(z)-f(0)}{z}|0<|z|\leq r\}$ is weakly bounded, so it is also bounded by(V.4.3.12), so f is strongly continuous.

The integral exists by(V.4.3.21), so f satisfies Cauchy integral formula because it satisfies this when acting with any functional Λ, and X^* separate points.

For the last assertion, Cf.[Rudin P84]. □

Prop. (V.4.3.28) (Liouville's Theorem). If X is a TVS over $\mathbb{C}$ on which X^* separate points and $f:\mathbb{C}\to X$ is weakly holomorphic and $f(\mathbb{C})$ is weakly bounded, then f is constant.

Proof: Immediate from Liouville's theorem(V.2.3.9). □

4 Duality Theory

Prop. (V.4.4.1) (Banach-Alaoglu). For a nbhd V of 0 in a TVS X, the set

$$K=\{f|\ |fx|\leq 1,\ \forall x\in V\}$$

is weak*-compact in X^*, which is called the **polar** V^* of V.

Proof: Consider the Minkowski function γ of V, then for each $\Lambda\in K$, $|\Lambda x|\leq\gamma(x)$. If we consider the space $P=\prod_{x\in X}[-\gamma(x),\gamma(x)]$, then P is compact by Tychonoff(IV.1.2.3).

The point is that the weak*-topology coincides with the pointwise convergence topology on K, because they have the same generating subbasis. If we show K is a closed subspace of P, this will finish the proof that K is weak*-compact. For this, consider any f_0 in its closure, then for each $x,y\in X$, $\alpha,\beta\in K$, there is a $f\in K$ that is close to f_0 at x,y and $\alpha x+\beta y$. So f_0 is linear. Similarly we can show $|f_0(x)|\leq 1$ for $x\in V$, so $f_0\in K$. □

Prop. (V.4.4.2). If X is a TVS that X^* separate points(e.g. locally convex), then the weak topology X_w is a locally convex space, and $(X_w)^*=X^*$.

Proof: If Λ is a functional that is continuous in X_w-topology, then $|\lambda x|<1$ for some set defined by elements in X^*, so by(V.3.5.3), $\Lambda=\sum\alpha_i\Lambda_i$ which is continuous w.r.t the original topology. □

Prop. (V.4.4.3) (Hahn Weak Topology case). If X is a TVS that X^* separate points, then if A, B are disjoint nonempty, compact convex sets in X, then there is a $\Lambda \in X^*$ that separate A and B, i.e. $\operatorname{Re} f(E_1) < \gamma_1 < \gamma_2 < \operatorname{Re} f(E_2)$.

Proof: Let X_w be X with the weak topology, then the sets A and B are compact in X_w as it's weaker. And they are also closed because X_w is Hausdorff. X_w is convex, so we can use geometric Hahn(V.4.3.7). Now $(X_w)^* = X^*$, so the chosen functional is also continuous in the original topology. □

Prop. (V.4.4.4) (Dual Banach Space). For a normed space X, $x \in X$ can be seen as functional on X^*, of norm exactly $||x||$. And the closed ball B^* of the dual space X^* is weak*-compact.

Proof: The first assertion is because of(V.4.3.6), the last assertion is because of Banach-Alaoglu(V.4.4.1). □

Prop. (V.4.4.5) (Adjoint Norm). For X, Y normed, the adjoint of $T : X \to Y$ satisfies $||T^*|| = ||T||$.

Proof: Use(V.4.4.4), $||T|| = \sup\{|\langle Tx, y^*\rangle| | ||x|| \leq 1, ||y^*|| \leq 1\} = ||T^*||$. □

Prop. (V.4.4.6) (Closed Range Theorem). Let T be continuous mapping between Banach spaces X and Y, let U, V be open balls of X, Y particularly. then the following are equivalent:

1. $||T^*y^*|| \geq \delta||y^*||$ for some δ.
2. $\delta V \subset \overline{T(U)}$.
3. $\delta V \subset T(U)$, i.e. T^{-1} is continuous.
4. $T(X) = Y$.
5. T^* is one-to-one and $R(T^*)$ is closed in X.

Proof: $1 \to 2$: If $||T^*y^*|| \geq \delta||y^*||$, first prove $\delta V \subset \overline{T(U)}$. If $y_0 \notin \overline{T(U)}$, since $\overline{T(U)}$ is convex closed and balanced, geometric Hahn shows that there is a y^* that $|y^*(y)| \leq 1$ for every $y \in T(U)$, and $|y^*(y_0)| > 1$. Then if follows $||T^*y^*|| \leq 1$. So

$$\delta < \delta|y^*(y_0)| \leq \delta||y_0||||y^*|| \leq ||y_0||||T^*y^*|| \leq ||y_0||$$

This shows $\delta V \subset \overline{T(U)}$.

$2 \to 3$: may assume $\delta = 1$. Then $\overline{V} \subset \overline{T(U)}$. Then for every $y \in Y$ and every $\varepsilon > 0$, there is a x that $||x|| \leq ||y||$ and $||y - Tx|| < \varepsilon$. For any $y_1 \in V$, pick $\varepsilon_n > 0$ that $\sum \varepsilon_n < 1 - ||y_1||$, then choose $||x_n|| \leq ||y_n||$ that $||y_n - Tx_n|| < \varepsilon_n$, and let $y_{n+1} = y_n - Tx_n$. Then is verified that $x = \sum x_n \in U$ and Tx.

$3 \to 1$: $||T^*y^*|| = \sup\{|\langle x, T^*y^*\rangle| |x \in U\} \geq \sup\{|\langle y, y^*\rangle| |y \in V\} = \delta||y^*||$.

$3 \iff 4$: By Open mapping theorem.

$4 \to 5$: T^* is injective by(V.3.5.2). By open mapping theorem, T^* is a multiple of a dilation, so $R(T^*)$ is closed by(IV.1.7.7).

$5 \to 4$: $R(T)$ is dense in Y by(V.3.5.2), and it is closed by the proposition(V.4.4.7) below. □

Prop. (V.4.4.7) (Closed Range Theorem). If X, Y are Banach spaces and $T \in L(X, Y)$, the following are equivalent:

1. $R(T)$ is closed in Y.

2. $R(T^*)$ is weak*-closed in X^*.
3. $R(T^*)$ is closed in X.

Proof: $1 \to 2$: As $N(T)^\perp$ is the weak*-closure of $R(T^*)$, it suffices to prove $N(T)^\perp \subset R(T^*)$. As $R(T)$ is complete, the open mapping theorem applies to $X \to R(T)$, showing that each $y \in R(T)$ corresponds to an element $x \in X$ that $Tx = y$ and $||x|| \leq K||y||$.

For $x^* \in N(T)^*$, define a functional Λ on $R(T)$ by $\Lambda Tx = \langle x, x^* \rangle$, this is well-defined, and $|\Lambda y| = \Lambda Tx| \leq K||y||||x^*||$. So it is continuous and by Hahan=Banach some continuous functional $y^* \in Y^*$ extends Λ. Then $\langle Tx, y^* \rangle = \Lambda Tx = \langle x, x^* \rangle$, so $x^* = T^* y^*$ is in the image of T^*, so we are done.

$3 \to 1$: let $Z = \overline{R(T)}$. RT is dense in Z, so(V.3.5.2) shows $T^* : Z^* \to X^*$ is injective. And for each $z^* \in Z*$, there is an extension y^* by Hahn-Banach, and then $\langle x, T^* y^* \rangle = \langle Tx, y^* \rangle = \langle Tx, z^* \rangle = \langle x, T^* z^* \rangle$, so $T^*(Y^*) = T^*(Z^*)$, which is closed by hypothesis.

Now use open mapping theorem for $Z^* \to R(T^*)$, then there is a c that $c||z^*|| \leq ||T^* z^*||$. So $T : X \to Z$ is surjective, by(V.4.4.6). So $R(T) = Z$ is closed. □

Prop. (V.4.4.8). In a normed space, iff $x_n \to x$ weakly, then $\underline{\lim}||x_n|| \geq ||x||$.

Proof: Choose a functional that $||f|| = 1$ and $|f(x)| = 1$ by(V.4.3.6), then use the definition of weak convergence. □

Prop. (V.4.4.9) (Eberlein-Smulian). For a set A in a Banach space X, A is weak*-sequentially compact iff its weak precompact.

Proof: **?**

We prove here that the case that the closed unit ball of a reflexive Banach space is weakly*-self sequentially compact.

To prove this, first we show that a bounded sequence has a subsequence that is weak*-convergent in X. Let $X_0 = \overline{\text{span}\{x_n\}}$, then X_0 is reflexive by(V.4.4.13), and it is separable, so X_0^* is separable by(V.4.4.11). Then the result follows from(V.4.4.14).

Finally, the weak limit x is in the closed unit ball, by(V.4.4.8). □

Reflexive and Separable

Def. (V.4.4.10) (Reflective Banach Space). If X is a Banach space, there is an isometric immersion of X onto a closed subspace of X^{**}(closed because X is complete). X is called **reflexive** iff $X \cong X^{**}$.

Prop. (V.4.4.11) (Separability Banach). For a normed space X, if X^* is separable, then X is separable.

Proof: Choose a countable dense set in X^*, then their projection to the unit sphere S^* $\{g_n\}$ are dense in S^*(easily checked), and choose for each of them a x_n that $||x_n|| = 1$ and $g_n(x_n) > \frac{1}{2}$.

Now I claim x_n are dense in X, i.e. $X_0 = \overline{\text{span}\{x_n\}} = X$. If this is not the case, then there is a $||x|| = 1$ not in X_0, so by(V.4.3.6), there is a f that $f(X_0) = 0$ and $||f|| = 1$ and $f(x) = 1$. Then $||g_n - f|| = \sup_{||x||=1}\{|g_n(x) - f(x)|\} \geq |g_n(x_n) - f(x_n)| = |g_n(x_n)| \geq 1/2$, contradicting the fact g_n is dense in S^*. □

Prop. (V.4.4.12) (Duality Exact). If X is a closed subspace of a normed space Y, and Y/X is the quotient field, then $(Y/X)^*$ is a closed subspace of Y^*, and X^* is the quotient.

Proof: $(Y/X)^* \to Y^*$ is clearly injective, and the X^* are all functionals on Y modulo the functionals that vanish on X. □

Cor. (V.4.4.13) (Pettis). Closed subspace and quotient space of a reflexive normed space is reflexive.

Proof: Use the fact that $0 \to X \to Y \to Z \to 0$ induces an exact sequence $0 \to X^{**} \to Y^{**} \to Z^{**} \to 0$, and there is a map $X \to X^{**}$, so we can use snake lemma(as modules). □

Prop. (V.4.4.14) (Separable Ball Weak*-Sequentially Compact). If a normed space X is separable, then the closed unit ball of X^* is weak*-sequentially compact.

Proof: Let x_n be a countable dense subset of X, then by diagonal method, for each bounded sequence of $f_n \in X^*$, there is a subsequence f_{n_k} that $f_{n_k}(x_m)$ converges for each x_m. Then by(V.3.4.3), f_{n_k} converges to some $f \in X^*$. So the theorem is finished. □

Prop. (V.4.4.15) (Reflexive Ball Weak*-Sequentially Compact). In a reflexive Banach space X, then a set in X is bounded iff it is weak*-sequentially compact.

Proof: If it is reflexive, then the unit ball is weak*-compact by Alaoglu, so it is weak*-sequentially compact by Eberlein(V.4.4.9). Conversely, if it is weak*-sequentially compact, then its closure is weak*-compact, thus bounded. □

Prop. (V.4.4.16). A closed convex set of a reflexive Banach space attains minimal norm.

Proof: By Hahn, a closed convex set is weakly closed. let $d = \inf\{||x||\}$, then if $d \le ||x_n|| < d+q/n$, then $\{x_n\}$ is bounded, so by(V.4.4.15) it is weak-sequencially compact(V.4.4.9), thus some $x_n \to x$ weakly. and use(V.4.4.8), x must attains minimal norm d. □

5 Compact Operator & Fredholm Operator

Def. (V.4.5.1). An operator between Banach spaces is called **compact** if it maps bounded set to sequentially compact(equivalently precompact or totally bounded(IV.1.7.4)) set. It is necessarily continuous because the norm function is continuous thus $||Tx||$ is bounded on the unit ball.

Prop. (V.4.5.2) (Examples of Compact Operators).

- $Lu(x) = \int_X K(x,y)u(y)dy$ for X compact and $K \in C(X \times X)$. This is a compact operator on $C(X)$ by Arzela-Ascali(IV.1.7.5).
- $Lu(x) = \int_\Omega K(x,y)u(y)dy$ for $K(x,y) \in L^2(\Omega \times \Omega)$. This is a compact operator on $L^2(\Omega)$.

Proof: For the second, because $L^2(\Omega)$ is reflexive, we only need to show this is totally continuous(V.4.5.4). For this, if $u_n \to 0$ weakly, then $f(x) = \int K(x,y)u_n(y)dy \to 0$, a.e.. and $||u_n||$ is bounded by(V.3.5.5). Then it is verified that $f(x)$ is bounded. Now we use dominant convergence, $||Au_n||^2_{L^2} = \int |f(x)|^2 dx \to 0$. □

Prop. (V.4.5.3) (Compact Operator).

1. For a continuous operator, it has f.d. image iff it is compact and the image is closed.
2. The space of compact operator is a closed subspace of $L(X,Y)$. Thus the limit of f.d. operators is compact.

3. If one of A or B is compact and the other is continuous, then AB is compact, because continuous maps bounded to bounded and compact to compact.

Proof:

1. A finite dimensional space is closed by(V.4.1.8), and a finite dimensional space is Heine-Borel(IV.1.3.2), so it maps closed ball to precompact set, as it is continuous. Conversely, if it is compact and the image is closed, then it is an open map to its image, by open mapping theorem, and the image is locally compact because T is compact, so it has finite dimension(V.4.1.7).
2. $S+T$ is continuous because sum of precompact set is precompact. To show it is closed subspace, Use totally bounded definition, for T is the closure, let $||S-T||<r$, then if $S(x_i)$ is a r-net for $S(B(0,1))$, then $T(x_i)$ is a $3r$-net for $T(B(0,1))$.
3. Because continuous function preserves both boundedness and (pre)compactness.

□

Prop. (V.4.5.4) (Compact and Totally Convergence). Let $x_n\to x$ weakly, if T is compact, then $Tx_n\to T_n$ strongly. The converse is true when X is reflexive. In particular, this applies to Hilbert space.

Proof: Assume the contrary, if Tx_n doesn't converge to Tx, there is a subsequence x_{n_k} that $||Tx_{n_k}-Tx||\geq\varepsilon_0$. Now $\{x_n\}$ is bounded by(V.3.5.5), so by T compact, there is a subsubsequence $Tx_{n_k}\to z$ strongly. But because $x_{n_i}\to x$ weakly, $Tx_{n_i}\to Tx$ weakly because T is continuous, and thus $z=Tx$.

The converse is by Eberlein(V.4.4.9), because the bounded x_n has a weak convergent subsequence, and it is mapped to convergent sequence by T. □

Prop. (V.4.5.5). T is compact $\iff$ T^* is compact.

Proof: We need only to show that $T^*y_n^*$ has a uniformly convergent subsequence on the unit sphere, but for this it suffice to prove y_n^* is sequentially compact in $C(\overline{T(B(0,1))})$. And we use Arzela-Ascoli because $\overline{T(B(0,1))}$ is compact. For the other half, use the double dual space. □

Lemma (V.4.5.6). If there is a chain of closed subspaces $M_1\subset M_2\subset\ldots$ that $T(M_n)=M_n$ and $(T-\lambda_nI)M_n\subset M_{n-1}$ for some $\lambda_n\in\sigma(A)-B(0,r)$, then T is not compact.

Proof: There are $y_n\in M_n$ that $||y_n||\leq 1$ and $||y_n-x_n||\geq 1/2$ for $x\in M_{n-1}$, so if $m<n$, $||Ty_m-Ty_n||=||\lambda y_n-(Ty_m-(T-\lambda_n)y_n)||\geq\frac{|\lambda_n|}{2}\geq\frac{r}{2}$, so Ty_n has no convergent subsequence. □

Lemma (V.4.5.7). If A is compact and $T=1-A$, then if T is not injective, then it is not surjective. And for any $r>0$, $\sigma_p(A)-B(0,r)$ is a finite set.

Proof: We use(V.4.5.6). If $R(T)=X$, then let $M_n=N(T^n)$, then $M_0\neq 0$ because there is a $Tx_0=0$, and $M_n\subset M_{n+1}$ because there is a $T^nx_{n+1}=x_0$, so $x_{n+1}\in M_{n+1}-M_n$.

If $\sigma_p(A)-B(0,r)$ is infinite, then choose M_n to be generated by n eigenvectors, then it is clear that a chain like above will be found. □

Lemma (V.4.5.8). If A is compact and $T=1-A$, then $R(T)$ is closed.

Proof: it suffices to show $T^{-1}:R(T)\to X/N(T)$ is continuous, if this is not the case, then there is a sequence $||x_n||=1$ but $Tx_n\to 0$. But A is compact, so there is a subsequence that $Ax_{n_k}\to z$, so $x_{n_k}\to z$. So $Tz=0$ so $z=0$, but then $x_{n_k}\to 0$, contradiction. □

Prop. (V.4.5.9) (Riesz-Fredholm). For a compact operator $A \in L(X)$, let $T = I - A$. Then:

1. $0 \in \sigma(A)$ if X is not f.d.
2. T is Fredholm of index 0(V.4.5.16). Equivalently, $\sigma(A)\backslash\{0\} = \sigma_p(A)\backslash\{0\}$(because either T not injective or T is surjective).
3. $\sigma(A)$ has at most one convergent point 0 (it must attain 0 if X is a infinite-dimensional). Hence it has at most countable spectrum.

Proof: 1: If 0 is a regular value, then T is invertible, thus $T^{-1}T = \mathrm{id}$ is compact, thus X has f.d.(V.4.1.7).

2 : Firstly $\dim N(T) < \infty$. This is because $T|_{N(T)} = \mathrm{id}_{N(T)}$, so it is compact iff it is f.d.(V.4.5.3). by [Rudin P108]**?**.

3: By(V.4.5.7). □

Prop. (V.4.5.10) (Lomonosov's Invariant Subspace Theorem). If X is an infinite dimensional complex Banach space, and $T \neq 0$ is a compact operator in $L(X)$, then there is a proper closed subspace M of X that is invariant under S for any S that commutes with T.

In particular, if S commutes with some compact operator T, then S has an invariant closed subspace.

Proof: If Γ is the subspace of all operators that commutes with T, then it is a subalgebra of $L(X)$, and for each $y \in X$, let $\Gamma(y) = \{Sy | S \in \Gamma\}$, then $S(\Gamma(y)) \subset \Gamma(y)$, then so does the closure of $\Gamma(y)$. So if the proposition is false, then $\Gamma(y)$ is dense in X for each y.

Pick x_0 that $Tx_0 \neq 0$, then there is an open ball B of x_0 that $||Tx|| \geq \frac{1}{2}||Tx_0||$ and $||x|| \geq \frac{1}{2}||x_0||$ for $x \in B$. Now our assumption shows that for every $y \neq 0$, there is a nbhd W of it that maps into B by some $S \in \Gamma$(notice Γ is a subspace).

Now $K = \overline{T(B)}$ is compact because T is compact, so there are f.m. open sets W_i whose union cover K, and $S_i(W_i) \in B$, where $S_i \in \Gamma$. Now let $\mu = \max\{||S_i||\}$. Consider $Tx_0 \in K$, so there is a $S_{i_1}Tx_0 \in B$, then $TS_{i_1}Tx_0 \in K$, so there is a $S_{i_2}TS_{i_1}Tx_0 \in B$. Continuing this way, we get

$$\frac{1}{2}||x_0|| \leq ||x_N|| \leq \mu^N ||T^N||||x0||,$$

so by Gelfand theorem(V.5.1.8), $\rho(T) > 0$, so there is a eigenvalue λ of T(by(V.4.5.9)) that $N(T-\lambda I)$ is finite dimensional, so not equal to X, and it is clearly invariant under Γ. □

Prop. (V.4.5.11) (Jordan Decomposition for Compact Operators). For a compact operator A and all the non-zero eigenvalues λ_i, we can find a <u>subspace</u>

$$\bigoplus_{i=1}^{\infty} N((\lambda_i - A)^{p_i}), \quad \lambda_i \neq 0$$

on which A has a Jordan decomposition.

Proof: Let $T = 1 - A$, By**??**, we only have to prove there are some m, n that There is a p that $N(T^p) = N(T^{p+1})$ and a q that $R(T^q) = R(T^{q+1})$, because then we have a decomposition $X = N((T-\lambda I)^p) \oplus R((T-\lambda I)^p)$, and all these $N((T-\lambda I)^p)$ are pairwisely disjoint.

Now $q < \infty$, because if $R(T) \supset R(T^2) \supset \ldots$, because T^k is of the form $1 +$ compact operator, $R(T^k)$ are all closed by(V.4.5.8), so by(V.4.5.6), this is impossible.

For p, use Riesz-Fredholm(V.4.5.9),

$$\dim N(T^q) = \mathrm{codim} R(T^q) = \mathrm{codim} R(T^{q+1}) = \dim N(T^{q+1}) < \infty$$

So $p \leq q < \infty$. □

Prop. (V.4.5.12). If X, Y are Banach spaces and $T, K \in L(X, Y)$, K is compact and $R(A) \subset R(K)$, then A is compact.

Proof: Use(V.3.4.9), then we can lift the function to a map $\widetilde{T} : X \to X/N(K)$, which is also continuous, so $T = \widetilde{K} \to \widetilde{T}$ is also compact. □

Schauder Basis

Def. (V.4.5.13) (Schauder Basis). Let X be a Banach space, a sequence e_n is called a **Schauder basis** iff for any $x \in X$, there is a unique sequence $C_n(x)$ that $x = \lim \sum_{n=1}^N C_n(x)e_n$. Notice in this case X is automatically separable.

Prop. (V.4.5.14). If X has a Schauder basis, then $C_n(x)$ are continuous functional on X.

Proof: Consider the module $||x||_1 = \sup ||S_N x||$, Firstly, it is complete, because $||x|| = \lim ||S_N x|| \leq ||x||_1$, so if there is a Cauchy sequence $\{x_i\}$ in $|\cdot|_1$, then it is a Cauchy sequence in $|\cdot|$, then it converges to some x. Now $C_N(x) = S_N(x_i) - S_{N-1}(x_i)$ are all Cauchy sequence, uniform in N, so they converges to some sequence c_N.

It is left to verify that $s_N = \sum_{i=1}^N c_n e_n$ converges to x, because then it is easy to verify that $\lim ||x_i - x||_1 = 0$. For this, choose N_1 large that $||x_n - x|| \leq \varepsilon$ for $n \geq N_1$, and choose N_2 large that $||S_k(x_n) - s_k|| \leq \varepsilon$ for all k and $n \geq N_2$. Then for $x_{N_1+N_2}$, there is a N_3 that $||S_n x_{N_1+N_2} - x_{N_1+N_2}|| \leq \varepsilon$, so $||x - s_k|| \leq ||x - x_{N_1+N_2}|| + ||S_k x_{N_1+N_2} - x_{N_1+N_2}|| + ||S_k(x_{N_1+N_2}) - s_k|| \leq 3\varepsilon$ for k large.

Now by Banach(V.3.4.6), $||x||_1 \leq M||x||$ for some M, so $|C_n(x)e_n| \leq 2M||x||$ and C_n is continuous. □

Prop. (V.4.5.15). If X has a Schauder basis, then any compact operator is a limit of operators of f.d. range.

Proof: Let $S_N(x) = \sum_{n=1}^N C_n(x)e_n$, it is continuous by(V.4.5.14). And it converges, so $||S_N|| \leq M$, by Banach-Steinhaus(V.3.4.1).

For any compact operator, we want to find f.d. range operator T_i that $T_i \to T$. For this, given any $\varepsilon > 0$, because $\overline{T(B(0,1))}$ is compact, there are operators that is a ε/M^2-net y_i, then choose N large enough that $|S_N y_i - y_i| \leq \varepsilon/M^2$, then for any x, there is a y_i that $|Tx_i - y_i| < \varepsilon/M^2$, so $|S_N T x_i - S_N y_i| < \varepsilon/M$, and then $|S_N T x - T x_i| < \varepsilon$, and notice $S_N T$ has f.d. range. □

Fredholm Operator

Def. (V.4.5.16) (Fredholm Operator). A bounded operator between Banach space is called a **Fredholm operator** if $\dim N(T) < \infty$ and $\mathrm{codim} R(T) < \infty$. It necessarily has closed image by(V.3.4.10), so $R(T) = N(T^*)^\perp$(V.3.5.2).

The **index** of a Fredholm operator is defined as $\mathrm{ind}(T) = \dim N(T) - \mathrm{codim} R(T)$, thus for a compact operator A, $I - A$ has index 0, by(V.4.5.9).

Prop. (V.4.5.17). For a Fredholm operator between Banach space, we have

$$X = N(T) \oplus R(T) \quad Y = Y/R(T) \oplus R(T)$$

and $X/N(T) \cong R(T)$.

Proof: Because $R(T)$ and $N(T)$ are finite/cofinite hence closed and complemented by(V.4.1.9). If $X = N(T) \oplus M_1$ and $Y = R(T) \oplus M_2$, then $M_1 \cong X/N(T)$, $X/N(T) \cong R(T)$ and $M_2 \cong Y/R(T)$ by Banach theorem. □

Prop. (V.4.5.18) (Characterizating Fredholm Operator). T is Fredholm from X to Y iff there exist a bounded S_1, S_2 from Y to X that $S_1 T = I - A_1, TS_2 = I - A_2$, where A_1, A_2 is compact. If this is the case, S_1 and S_2 can be chosen the same as S, then S is called the **regulator** of T, and S is Fredholm as well.

So the Fredholm operator is the set of operators invertible 'modulo compact ones'.

Proof: By(V.4.5.17) $T : X/N(T) \cong R(T)$, and there is a projection of $\pi : Y$ onto $R(T)$. Thus we composed them to get a $S = T^{-1} \circ \pi : Y \to X$. And ST and TS are both 1 minus a projection with f.d. image, hence compact(V.4.5.3).

For the converse, $R(T) \supset R(1 - A_2)$ is of finite codimention because $1 - A_2$ is Fredholm, and $N(T) \subset N(1 - A_1)$ is of finite dimension because $1 - A_1$ is Fredholm. □

Cor. (V.4.5.19). The set of Fredholm operators is closed under composition. Index is a locally constant function on it, and $\mathrm{ind}(T_1 T_2) = \mathrm{ind}(T_1) + \mathrm{Ind}(T_2)$.

Proof: There is a long exact sequence(use(I.8.3.10) in the category of vector spaces)

$$0 \to \operatorname{Ker} T_2 \to \operatorname{Ker} T_1 T_2 \to \operatorname{Ker} T_2 \to \operatorname{Coker} T_2 \to \operatorname{Coker} T_1 T_2 \to \operatorname{Coker} T_1 \to 0.$$

which shows the composition and index is additive.

For the openness and locally constancy, use(V.4.5.18), when adding a small R, $S(T + R) = 1 - A_1 + SR$, and if when $||R|| < ||S||^{-1}$, $E_1 = (I + RS)^{-1}$ is bounded, so $E_1 S(T + R) = I - E_1 A_1$, and similarly does $(T + R)SE_2$, so $T + R$ is Fredholm. And $\operatorname{ind} E_1 + \operatorname{ind} S + \operatorname{ind}(T + R) = \operatorname{ind}(1 - E_1 A_1) = 0$, and $\operatorname{ind} E_1 = 0$ because it is invertible, and $\operatorname{ind} S + \operatorname{ind} T = \operatorname{ind}(1 - A_1) = 0$, so $\operatorname{ind} T = \operatorname{ind}(T + R)$. □

Cor. (V.4.5.20). If T is Fredholm and A is compact, then $T + A$ is Fredholm, and $\operatorname{ind}(T + A) = \operatorname{ind} T$, so ind is in fact defined on the quotient of $L(X, Y)$ by compact operators.

Proof: It is Fredholm by(V.4.5.18), and we notice $S(T + A)$ and ST are both 1 minus compact operators, thus(V.4.5.19) and(V.4.5.9) gives the result. □

Cor. (V.4.5.21). If T is Fredholm, then T^* is Fredholm, and $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$.

Proof: The first follows from(V.4.5.18) and(V.4.5.5). For the second, use the fact $R(T)$ and $N(T)$ are all closed. □

6 Unbounded Operators

V.5 Archimedean Banach Algebra

1 Banach Algebra

Def. (V.5.1.1). For a bounded operator $A \in L(X)$ where X is Banach space, a $\lambda \in \mathbb{C}$ is called a:

- **point spectrum** if $\lambda I - A$ is not injective;
- **continuous spectrum** if it is not a point spectrum and $R(\lambda I - A) \neq X$ but $\overline{R(\lambda I - A)} = X$.
- **residue point** if it is not a point spectrum and $\overline{R(\lambda I - A)} \neq X$.
- **regular point** if $\lambda I - A$ is injective and $R(\lambda I - A) = X$, in which case $(\lambda I - X)^{-1}$ is continuous, by Banach.

denote $\sigma(A) = K -$ regular points of A the **spectrum** of A, and $\rho(A) = \sup\{|\lambda||\lambda \in \sigma A\}$ is called the **spectral radius** of A.

Prop. (V.5.1.2). If A is a Banach algebra and x is invertible in A, and $h \in A$ satisfies $||h|| < \frac{1}{2}||x^{-1}||^{-1}$, then $x + h$ is also invertible and

$$||(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}|| \leq 2||x^{-1}||^3||h||^2$$

Proof: $x+h = x(e+x^{-1}h)$ and $||x^{-1}h|| < \frac{1}{2}$, so $x+h$ is invertible by(V.5.1.14), and $||(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}|| = ||[(e+x^{-1}h)^{-1} - e + x^{-1}h]x^{-1}|| \leq 2||x^{-1}||^3||h||^2$ also by(V.5.1.14). □

Cor. (V.5.1.3). If A is a Banach algebra, then the invertible elements $G(A)$ is an open subset of A, and the mapping $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$ onto $G(A)$.

Prop. (V.5.1.4). For $T \in L(X)$ where X is Banach space, $\mathbb{C}\backslash\sigma(T)$ is an open set and $\lambda \to (\lambda I - T)^{-1}$ is a holomorphic function on $\mathbb{C}\backslash\sigma(T)$.

Thus for every bounded operator T on a Banach space, $\underline{\sigma(T) \text{ is not empty}}$.

Proof: The first assertion is by(V.5.1.3), for the second, let $f(\lambda) = (\lambda e - x)^{-1}$ is defined on $\Omega = \mathbb{C} - \sigma(x)$ and(V.5.1.2) shows

$$||f(\mu) - f(\lambda) + (\mu - \lambda)f^2(\lambda)|| \leq 2||f(\lambda)||^3|\mu - \lambda|^2$$

so $\lim_{\mu\to\lambda} \frac{f(\mu)-f(\lambda)}{\mu-\lambda} = -f^2(\lambda)$, which means that f is strongly holomorphic in Ω.

Now if $|\lambda| > ||x||$, then $|f(\lambda)| = |\lambda^{-1}e + \lambda^{-2}x + \ldots| \leq \frac{1}{|\lambda|-||x||}$, so $\sigma(x)$ cannot be empty by Liouville theorem(V.4.3.28). □

Cor. (V.5.1.5) (Gelfand-Mazur). If in a Banach algebra A over $\mathbb{C}$, all the nonzero element is invertible, then it is isomorphic to $\mathbb{C}$.

Proof: Any nonzero element x has a nonempty spectrum, so there is a $\lambda(x)$ that $x - \lambda(x)e$ is not invertible, so it must be 0. That is, the mapping $\mathbb{C} \to A : \lambda \mapsto \lambda e$ is bijective, so is is isomorphism by Banach. □

Prop. (V.5.1.6). Notice $(I - T)$ is invertible for $||T|| < 1$ and the inverse can be calculated by definition.

In particular, for a Banach algebra A and any $x \in A$, when $\lambda > ||x||$, $e - \lambda^{-1}x$ is invertible, so the spectrum of x is bounded. Now that its complement is open as the inverse image of $G(A)$ by $\lambda \mapsto \lambda e - x$, so $\underline{\text{the spectrum of } x \text{ is compact}}$.

Cor. (V.5.1.7) (Spectrum is Continuous). The spectrum of an element of a Banach algebra is continuous, i.e. if $\sigma(x) \subset \Omega$ for some open subset $\Omega \subset \mathbb{C}$, then there is a $\delta > 0$ that $\sigma(x+y) \subset \Omega$ for $||y|| < \delta$.

Proof: As $||(\lambda e - x)^{-1}||$ is a continuous function of λ in the complement of σ, and since the norm tends to 0 as $\lambda \to \infty$, there is a M that $||(\lambda e - x)^{-1}|| < M$ for all $\lambda \notin \Omega$. Now if $||y|| < 1/M$ and $\lambda \notin \Omega$, then $\lambda e - (x+y) = (\lambda e - x)[e - (\lambda e - x)^{-1} y]$ is invertible. □

Prop. (V.5.1.8) (Gelfand). The spectrum radius $\rho(A) = \lim_{n\to\infty} ||A^n||^{\frac{1}{n}} = \inf ||A^n||^{\frac{1}{n}} \leq ||A||$.

This formula is remarkable, as the LHS depends only on the algebraic structure, and the RHS depends on the metric structure.

Proof: For $r > \rho(x)$
$$x^n = \frac{1}{2\pi i}\int_{\Gamma_r} \lambda^n f(\lambda) d\lambda.$$
Let $M(r) = \max ||f(re^{i\theta})||$, then $||x^n|| \leq r^{n+1} M(r)$, hence $\limsup ||x^n||^{1/n} \leq r$, so $\limsup ||x^n||^{1/n} \leq \rho(x)$.

For the converse, if $\lambda \in \sigma(x)$, then $\lambda^n \in \sigma(x^n)$, because $\lambda^n e - x^n = (\lambda e - x)(\lambda^{n-1} e + \ldots + x^{n-1})$, and this two commutes. So $|\lambda^n| \leq ||x^n||$, so $\rho(x) \leq \inf ||x^n||^{1/n}$. □

Prop. (V.5.1.9). $\sigma(A) = \sigma(A^*)$.

Proof: It suffices to show if T is invertible iff T^* is invertible. If T is invertible, then T^* is invertible with inverse $(T^{-1})^*$. Conversely, if T^* is invertible, then T^{**} is invertible, so, as the restriction of T^{**}, T is injective and image is closed. If the image is not X, then there is a f that vanish on the image, so $T^* f = 0$, but then $f = 0$. □

Prop. (V.5.1.10). In a Banach algebra A, $e - xy$ is invertible iff $e - yx$ is invertible, thus $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.

Proof: Let $z = (e - xy)^{-1}$, then we claim $e + yzx$ is just the inverse of $e - yx$: $(e - yx)(e + yzx) = e - yx + yzx - yxyzx = e$ and $(e + yzx)(e - yx) = e + yzx - yx - yzxyx = e$. □

Lemma (V.5.1.11). If A is a Banach algebra and $x_n \in G(A)$ converges to $x \notin G(A)$, then $||x_n^{-1}|| \to \infty$.

Proof: If $||x_n^{-1}|| < M$, choose n that $||x_n - x|| < 1/M$, then $||e - x_n^{-1} x|| = ||x_n^{-1}(x_n - x)|| < 1$, so $x_n^{-1} x$ is invertible, so x is invertible. □

Prop. (V.5.1.12). For Banach algebra B and its closed subalgebra A, $\sigma_A(x)$ is obtained from $\sigma_B(x)$ by filling some holes. So when $\sigma_B(x)$ doesn't separate $\overline{\mathbb{C}}$ or $\sigma(A)$ has empty interior, then $\sigma_A(x) = \sigma_B(x)$.

Proof: Cf.[Rudin P256]. □

Prop. (V.5.1.13). if A is a Banach algebra over $\mathbb{C}$ that $||x||||y|| \leq M||xy||$ for some fixed M, then A is isomorphic to $\mathbb{C}$.

Proof: If y is a boundary pt of $G(A)$, $y = \lim y_n$, then $||y_n^{-1}|| \to \infty$. But $||y_n||||y_n^{-1}|| \leq M||e||$, so $y_n \to 0$, so $y = 0$.

But any boundary point of $\sigma(x)$ gives a boundary point $\lambda e - x$ of $G(A)$, so $x = \lambda e$, so $A \cong \mathbb{C}$. □

Complex Homomorphism

Prop. (V.5.1.14). Suppose A is a Banach algebra over $\mathbb{C}$, $x \in A$ satisfies $||x|| < 1$, then $||(e-x)^{-1} - e - x|| \leq \frac{||x||^2}{1-||x||}$, and $|\varphi(x)| < 1$ for any complex homomorphism φ on A. In particular, any complex homomorphism is continuous.

Proof: $||(e-x)^{-1} - e - x|| = ||x^2 + x^3 + \ldots|| \leq \sum_{n=2}^{\infty} ||x||^n = \frac{||x||^2}{1-||x||}$.

For the second, notice $e - \lambda^{-1}x$ is invertible for each $|\lambda| \geq 1$, so $1 - \lambda\varphi(x) \neq 0$, so $\varphi(x) \neq \lambda$. □

Prop. (V.5.1.15) (Gleason-Kahane-Zelazko). If φ is a linear functional on a Banach algebra A over $\mathbb{C}$, if $\varphi(e) = 1$ and $\varphi(x) \neq 0$ for every invertible element $x \in A$, then φ is a complex homomorphism.

Proof: Cf.[Rudin P251]. □

Symbolic Calculus

Prop. (V.5.1.16) (Symbolic Calculus). For a Banach algebra A. For a domain Ω in $\mathbb{C}$, define A_Ω as the set of x that $\sigma(x) \in \Omega$, it is an open set by(V.5.1.7), then:

$$f \mapsto \tilde{f}(x) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda e - x)^{-1} d\lambda$$

for any contour Γ that surrounds $\sigma(x)$, is a continuous algebra isomorphism of $H(\Omega)$ into the set of A-valued functions on A_Ω with the compact-open topology.

We have $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$.

Proof: The nontrivial part is that this map is multiplicative, but for this we can use Runge's theorem to approximate any function on $\sigma(x)$. □

This theorem makes it possible to implant complex analysis to the study of Banach Algebra.

Cor. (V.5.1.17). $\exp(x)$ is defined on A and is continuous. If $\sigma(x)$ doesn't separate 0 from ∞, then $\log(x)$ is defined but might not be continuous.

Prop. (V.5.1.18) (Spectral Mapping Theorem). $\tilde{f}(x)$ is invertible in A iff $f(\lambda) \neq 0$ on $\sigma(x)$. Thus we have $\sigma(\tilde{f}(x)) = f(\sigma(x))$.

Prop. (V.5.1.19). If f doesn't vanish identically on any component of Ω, then $f(\sigma_p(T)) = \sigma_p(\tilde{f}(T))$. Cf.[Rudin P266].

Commutative Banach Algebra

Lemma (V.5.1.20). For A a commutative Banach algebra, the set of maximal ideals has codimension 1 corresponds to kernels of complex homomorphisms to $\mathbb{C}$. (Consider the quotient space and use Gelfand-Mazur). Note that a complex homomorphism is all continuous because $\lambda e - x$ maps to nonzero.

$\lambda \in \sigma(x)$ iff there is a complex homomorphism h s.t. $h(x) = \lambda$. (Because x is invertible iff it is not contained in any proper ideal of A.

Proof:

□

Prop. (V.5.1.21) (Gelfand Transform). The **spectrum** Δ_A of a unital commutative Banach algebra A is defined to be the set Δ of maximal ideals of A. It is a locally compact Hausdorff space w.r.t to the weak*-topology and the Gelfand transform: $x \mapsto \hat{x}(h) = h(x)$ is a continuous map of A into $C(\Delta)$. And the range of $\hat{x}$ equals $\sigma(x)$, so $||\hat{x}|| = \rho(x) \leq ||x||$.

Proof: First we prove it is compact Hausdorff: As $\sigma(A) = \{h \in \text{closed ball of } A^* | h(e) = 1, h(xy) = h(x)h(y)\}$ which is a closed subset of the closed ball of A^*, so it is compact Hausdorff. The rest is clear and follows from(V.5.1.20). □

Prop. (V.5.1.22). For $A = C(X)$ where X is compact Hausdorff, Δ is homeomorphic to X. (otherwise it has finite $f_i \neq 0$, then $\sum |f_i|^2$ is positive thus invertible but maps to 0). In fact, for a space X, $\Delta(C(X))$ is the stone-Čech compactification of X.

Prop. (V.5.1.23). For $A = L^\infty(m)$, the spectrum of f is just the essential range of f.

Lemma (V.5.1.24). If $\hat{A} \subset C(\Delta)$ with a chosen topology that makes it compact, and A separate points, then the topology of it is the same of the weak*-topology. (Compact to Hausdorff).

Prop. (V.5.1.25). The algebra $L^1(\mathbb{R}^n) + \delta$ with the multiplication by convolution has the spectrum $R^n \cap \{\infty\}$. (Use $L^{p*} = L^q$ and see when will it be homomorphism).

2 Hilbert spaces

Prop. (V.5.2.1) (Optimal Approximation). A closed convex subset in a Hilbert space has a unique element that attains the minimum norm.

Proof: Assume $0 \notin C$, so let $d = \inf_{z \in C} ||z|| > 0$, then there are x_n that $d \leq ||x_n|| \leq d + 1/n$. It suffices to show that x_n is a Cauchy sequence, because then it has a convergent point in C. Now

$$||x_n - x_m||^2 = 2(||x_n||^2 + ||x_m||^2) - 4||\frac{x_n + x_m}{2}||^2 \leq 2[(d+1/n)^2 + (d+1/m)^2] - 4d^2 \to 0.$$

For the unicity, if $||x_1|| = ||x_2|| = d$, then

$$||x_1 - x_2||^2 = 2(||x_1||^2 + ||x_2||^2) - 4||\frac{x_1 + x_2}{2}||^2 \leq 4d^2 - 4d^2 = 0.$$

□

Cor. (V.5.2.2) (Orthogonal Decomposition). The orthogonal complement of a closed subspace of a Hilbert space exists. and the projection on to a closed subspace exists. This is a good trait of Hilbert space.

Proof: For any element x, let y be the optimal approximation(V.5.2.1) of x, then $z = x - y$ is orthogonal to y. □

Prop. (V.5.2.3) (Riesz). Linear functionals on a Hilbert space over $\mathbb{C}$ are all of the form $x \mapsto (x, z)$ (Choose an orthogonal of the kernel). In other words, Hilbert spaces are reflexive.

Proof: Choose a x_0 orthogonal to $N(f)$ by(V.5.2.2) and $||x_0|| = 1$, then any $x = \alpha x_0 + y$ where $y \in N(f)$. Inner product with x_0, we get $\alpha = (x, x_0)$, so $f(x) = \alpha f(x_0) = (x, \overline{f(x_0)} x_0)$. □

Cor. (V.5.2.4). For a $T \in L(H)$, $||T|| = \sup\{(Tx, y) | ||x|| \leq 1, ||y|| \leq 1\}$.

Proof: Use(V.4.3.6) to find for each x a functional f that $|f(Tx)| = ||Tx||$, then use Riesz theorem. □

Cor. (V.5.2.5) (Reproducing Kernel). For a Hilbert space H, if elements of H are all complex valued functions on a set S, and $J_x : f \mapsto f(x)$ is continuous functional for H, then there is a function $K(x,y)$ on $S \times S$ that $K_y(x) = K(x,y) \in H$, and $f(y) = (f, K_y)_H$, called the **reproducing kernel**.

And if e_α is a basis for H, then $K(x,y) = \sum e_\alpha(x)\overline{e_\alpha}(y)$.

Proof: For any y, there is a $K_y \in H$ that $f(y) = (f, K_y)_H$ by Riesz representation. If we let $K(x,y) = (K_y, K_x) = K_y(x)$, then this is the desired kernel.

If e_α is a basis, then $K_x = (K_x, e_n)e_n = \overline{e_n(x)}e_n$, so by Parseval equality, $K(x,y) = \sum e_\alpha(x)\overline{e_\alpha(y)}$. □

Prop. (V.5.2.6). Let H be a Hilbert basis, then a sequence x_n converges to x iff x_n converges to x weakly and $||x||_n \to ||x||$.

Proof: One direction is trivial, for the other, notice that $||x_n - x||^2 = ||x_n||^2 + ||x||^2 - 2\,\mathrm{Re}(x, x_n)$ which converges to 0. □

Prop. (V.5.2.7) (Lax-Milgram Theorem). If $a(x,y)$ is a sesquilinear form on a Hilbert space H over $\mathbb{C}$ that $|a(x,y)| \leq M||x||||y||$, then there is a unique continuous operator $A \in L(H)$ that $a(x,y) = (x, Ay)$. If moreover $|a(x,x)| \geq \delta||x||^2$, then A is bijective and $||A^{-1}|| \leq \frac{1}{\delta}$.

Proof: For any y, $x \mapsto a(x,y)$ is a continuous functional, so by Riesz theorem(V.5.2.3), there is a z that $a(x,y) = (x, Ay)$.

Now Ay depends linearly on y, and $||Ay|| = \sup |a(x,y)|/||x|| \leq M||y||$.

If $|a(x,x)| \geq \delta||x||^2$, then A is clearly injective, and $R(A)$ is closed, because for any $z = \lim Av_n$, it is easily verified that v_n is a Cauchy sequence. And $R(A)^\perp = 0$, because if $(w, Av) = 0$ for any $v \in H$, then $\delta||w||^2 \leq |a(w,w)| = 0$. A^{-1} exists by Banach theorem, and $\delta||x||^2 \leq |a(x,x)| = (x, Ax)| \leq ||x||||Ax||$, so $\delta||x|| \leq ||Ax||$. □

Cor. (V.5.2.8) (Involution on Hilbert Space). For a Hilbert space over $\mathbb{C}$, for any $T \in L(H)$, there is an operator $T^* \in L(H)$ that $(Tx, y) = (x, T^*y)$, which is called the **formal adjoint** or involution of T. Notice it is defined on H, not on H^*.

Moreover, $||T|| = ||T^*|| = ||T^*T||^{1/2}$.

Proof: Use Lax-Milgram for $a(x,y) = (Tx, y)$. For the last assertion, notice

$$||Tx||^2 = (Tx, Tx) = (T^*Tx, x) \leq ||T^*T||||x||^2,$$

so $||T|| \leq ||T^*T||^{1/2}$. □

Cor. (V.5.2.9) (Variational Inequality). If H is a Hilbert space that $a(x,y)$ is an anti-symmetric bilinear function that $\delta||x||^2 \leq a(x,x) \leq M||x||^2$, then if $u_0 \in H$, and C is a closed convex subset of X, the function

$$f : x \mapsto a(x,x) - \mathrm{Re}(u_0, x)$$

attains minimum at C.

Proof: Similar to the proof of(V.5.2.1). $f(x) \geq \delta||x||^2 - ||u_0||||x||$ is bounded below on C. if x_n is a sequence that converse to the infimum d, then

$$\begin{aligned} a(x_n - x_m, x_n - x_m) &= 2(a(x_n, x_n) + a(x_m, x_m)) - 4a(\frac{x_n + x_m}{2}, \frac{x_n + x_m}{2}) \\ &= 2(f(x_n) + f(x_m)) - 4f(\frac{x_n + x_m}{2}) \leq 2[(d + 1/n)^2 + (d + 1/m)^2] - 4d^2 \to 0. \end{aligned}$$

So x_n is a Cauchy sequence by the condition, and it contains a unique minimum. □

3 B^*-algebra

Def. (V.5.3.1). A B^***-algebra** is a Banach algebra with an involution s.t. $||xx^*|| = ||x||^2$.
Any B^*-algebra is isomorphic to a closed subspace of $B(H)$ for some Hilbert space.

Proof: Cf.[Rudin P338]. □

Prop. (V.5.3.2). For a Hilbert space, the adjoint operation serves as an involution and makes $B(H)$ into a B^*-algebra by(V.5.2.8).

Prop. (V.5.3.3) (Gelfand-Naimark). For a commutative B^*-algebra, the Gelfand transform $x \mapsto \hat{x}$ is an isomorphism from A to $C(\Delta)$ with $||x|| = ||\hat{x}||_\infty$ and $\widehat{x^*} = \overline{\hat{x}}$.

Proof: First use $||xx^*|| = ||x||^2$ to prove that a Hermitian element is mapped to real function, and use Stone-Weierstrass to show that the image is dense, then let $y = xx^*$ and $||y^{2^m}|| = ||y||^{2^m}$ to prove $||\hat{x}|| = ||x||$, so its image is closed. □

Cor. (V.5.3.4). If A is a commutative B^*-algebra that contains an element x s.t polynomials of x, x^* are dense in A, then $\hat{x}$ is an isomorphism from Δ_A to $\sigma(x)$, in particular, the Gelfand transform is an isomorphism from $C(\sigma(x))$ to A.

Proof: Cf.[Rudin P290]. □

Now we want to apply commutative algebra methods in the non-commutative case, there are two ways.

Prop. (V.5.3.5). For a commutative set of elements S in A, Γ the set of elements that commute with S, then $B = \Gamma(\Gamma(S))$ is commutative, closed and contains S. And $\sigma_B(x) = \sigma_A(x)$ for $x \in B$.

Proof: Because $S \subset \Gamma(S)$, $\Gamma(\Gamma(S)) \subset \Gamma(S)$, thus $\Gamma(\Gamma(S))$ is commutative. And if $xy = yx$, then $x^{-1}y = yx^{-1}$, so the inverse, if exists, are in B. □

Cor. (V.5.3.6). In a Banach algebra, if x, y commutes, then

$$\sigma(x + y) \subset \sigma(x) + \sigma(y), \quad \sigma(xy) \subset \sigma(x)\sigma(y).$$

Proof: Because $\sigma(x)$ is just the range of $\hat{x}$ on Δ_A where $A = \Gamma(\Gamma(\{x, y\}))$(V.5.3.5)(V.5.1.21). □

The second method applies to normal elements:

Def. (V.5.3.7) (Normal). In a Banach algebra with an involution, a set S is called **normal** if it is commutative and $S^* = S$. An element x is called:

- **normal** iff x commutes with x^*.

- **unitary** iff $x^* = x^{-1}$.
- **Hermitian** iff $x^* = x$.
- **positive** iff $x = x^*$ and $\sigma(x) \subset [0, \infty)$.

Prop. (V.5.3.8). A maximal normal set B in A is a closed subalgebra and $\sigma_B(x) = \sigma_A(x)$ for $x \in B$.

Proof: Cf.[Rudin P294]. □

Cor. (V.5.3.9) (Normalness and Spectra). In a B^*-algebra A,

- Hermitian elements have real spectra.
- If x is normal, then $\rho(x) = ||x||$.
- If $u, v \geq 0$, then $u + v \geq 0$.
- $yy^* \geq 0$. Thus $e + yy^*$ is invertible.

Proof: Cf.[Rudin P295]. □

Prop. (V.5.3.10). In a Banach algebra with an involution, a **positive functional** is such that $F(xx^*) \geq 0$. It has the following properties.

- $F(x^*) = \overline{F(x)}$ and $|F(xy^*)|^2 \leq F(xx^*)F(yy^*)$. (Use Swartz like trick).
- $|F(x)|^2 \leq F(e)F(xx^*) \leq F(e)^2\rho(xx^*)$, because $e = ee^*$. Thus $|F(x)| \leq F(e)\rho(x)$ for every normal x by(V.5.3.9), so $||F|| = F(e)$ if A is commutative.

Proof: Cf.[Rudin P297]. □

Prop. (V.5.3.11) (Positive Functional and Measure). If A is a commutative Banach algebra with an involution that $h(x^*) = \overline{h(x)}$, then The map

$$\mu \to F(x) = \int_\Delta \hat{x} d\mu$$

is a one-to-one correspondence between the convex set of measures μ that $\mu(\Delta) \leq 1$ to the convex set K of positive functionals on A of norm ≤ 1, i.e. $F(e) \leq 1$, so maps the extreme points, i.e. the point mass to extremes points, thus the extreme points of K is exactly Δ. This can be used to prove Bochner's theorem**??**

Proof: Use the last prop to show that there is a functional on $C(\Delta)$ and use Riesz representation. It is positive and by Stone-Weierstrass, it is unique. □

von Neumann Algebras

Def. (V.5.3.12) (von Neumann Algebra). A **von Neumann Algebra** is a B^*-algebra of operators in $L(\mathcal{H})$ that contains the identity and is closed in the weak operator topology(V.3.5.4).

Prop. (V.5.3.13) (von Neumann Density Theorem). Let A be a non-degenerate $*$-subalgebra of $L(\mathcal{H})$, then A is dense in $(A^c)^c$ in the strong operator topology.

Proof: Cf.[Folland Abstract Harmonic Analysis P30]. □

4 Spectral Theory on Hilbert Spaces

The most useful tool is the general symbolic calculus for normal operators.

Resolution of Identity

Def. (V.5.4.1). A **resolution of identity** on a Hilbert space H for a σ-algebra on a set Ω is a E that:

1. $E(\varnothing) = 0, E(\Omega) = 1$.
2. $E(\omega)$ is self-adjoint projection.
3. $E(\omega' \cap \omega) = E(\omega')E(\omega)$.
4. $E(\omega \cup \omega') = E(\omega) + E(\omega')$ for disjoint ω, ω'.
5. $E_{x,y}(\omega) = (E(\omega)x, y)$ is a complex measure on E.

Thus for any x, $\omega \to E(\omega)x$ is a countably additive H-valued measure.

This will generate an isometric*-isomorphism Ψ of the Banach algebra $L^\infty(E)$ onto a closed normal subalgebra A of $B(H)$. (Define on simple function first).

$$\Psi(f) = \int_\Omega f dE, \quad (\Psi(f)x, y) = \int_\Omega f dE_{x,y}$$

Proof: Cf.[Rudin P319]. □

Prop. (V.5.4.2) (Spectral Decomposition for Normal Algebra). For any closed normal algebra A of $B(H)$, there is a unique resolution E of identity on the Borel subsets of Δ_A that the inverse of Gelfand transform extends to an isometric *-isomorphism Φ of the algebra $L^\infty(E)$ to a closed subalgebra B containing A.

In fact, $B = \Gamma(\Gamma(A))$ is normal by Fuglede theorem(V.5.4.10).

Proof: Cf.[Rudin P322]. □

Cor. (V.5.4.3) (Generalized Symbolic Calculus for Normal Operator). For a normal operator T and the minimal closed commutative B^*-algebra A it generates, then the inverse of Gelfand transform gets us a map $\Psi : C(\sigma(x)) \to A$ that $\Psi(z) = x$, $\Psi(\overline{z}) = x^*$, by(V.5.3.4).

Then the above proposition says there is a resolution of identity on the Borel set of $\sigma(T)$ that Ψ extends to a function that maps $L^\infty(m)$ to $B(H)$ and $||\Psi(f)|| = ||f||_\infty$.

Cor. (V.5.4.4) (Normal and Invariant Subspace). Any closed normal algebra A has many invariant subspaces, just choose a decomposition of Borel sets $\Delta_A = \omega \coprod \omega'$, then $R(E(\omega)) \oplus R(E(\omega')) = H$.

In particular, any normal operator has an invariant subspace.

Normal Operators on Hilbert Space

Lemma (V.5.4.5). For a Hilbert space H and $T \in L(H)$, T is defined by values (Tx, x).

Proof: If $(Tx, x) = 0$, then $(Tx, y) + (Ty, x) = 0$, so $-i(Tx, y) + i(Ty, x) = 0$, solving $(Tx, y) = 0$ for all x, y, so $T = 0$. □

Prop. (V.5.4.6) (Normal Operators).

1. An operator is normal iff $||Tx|| = ||T^*x||$. So $N(T) = N(T^*)$ thus $\sigma_p(T^*) = \overline{\sigma_p(T)}$, and $R(T)$ is dense iff T is injective. And different eigenspaces are orthogonal.
2. An operator is unitary iff $R(U) = H$ and $||Ux|| = ||x||$ for every x. (Because an operator is defined by its value (Tx, y).

Proof: $||Tx||^2 = (T^*Tx, x)$, $||T^*x|| = (TT^*x, x)$, and they are equal iff T, T^* commutes by(V.5.4.5). In particular, for different eigenvectors, $\alpha(x, y) = (Tx, y) = (xT^*y) = (x, \overline{\beta}y) = \beta(x, y)$.

For unitary, one way is obvious, for the other, if $||Ux|| = ||x||$, then $(U^*Ux, x) = (x, x)$, so $U^*U = id$ by(V.5.4.5), and U is a bijection. So it is invertible. □

Cor. (V.5.4.7). For a normal operator T on a Hilbert space T is invertible iff there is a δ that $||Tx|| = ||T^*x|| \geq \delta||x||$.

Proof: T is injective iff $R(T)$ is dense, and if $||Tx|| = ||T^*x|| \geq \delta||x||$, then $R(T)$ is closed by(IV.1.7.7), so it is invertible by Banach theorem. □

Prop. (V.5.4.8). If T is normal, then
1. $||T|| = \sup\{|(Tx, x)|\ ||x|| = 1\}$.
2. T is self-adjoint iff $\sigma(T)$ is real.
3. T is unitary iff $|\sigma(T)| = 1$.

Proof: For 1, $||T|| = \rho(T) = ||z_0||$ for some $z_0 \in \rho(T)$ by Naimark(V.5.3.3), then Urysohn lemma to show $E(U) \neq 0$ for a open U near x(because otherwise there is a continuous function supported in U that are mapped to 0), then there are $||x_0|| = 1$ that $E(U)x_0 = x_0$.

Consider now $f = (z - z_0)i_U(z)$, then $f(T)(x_0) = Tx_0 - \lambda_0 x_0$, so

$$(Tx_0, x_0) - \lambda_0| = |(f(T)x_0, x_0)| \leq ||f(T)|| = ||f|| \leq \varepsilon$$

This shows that $||T|| = \sup\{|(Tx, x)|\ ||x|| = 1\}$.

For $2, 3$, by generalized symbolic calculus(V.5.4.3), $\hat{T} = \lambda$ on σ and $\hat{T^*} = \overline{\lambda}$ on σ, so they are equal iff $\sigma(T)$ is real, and $TT^* = I$ iff $\lambda\overline{\lambda} = 1$ on $\sigma(T)$. □

Prop. (V.5.4.9) (Decomposition of Operators). Every operator $S \in L(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a linear combination of two self-adjoint operator and a linear combination of four unitary operator.

Proof: The first assertion is easy as $S = (S + S^*)/2 + (S - S^*)/2$. Now any self-adjoint operator is a multiple of a self-adjoint operator of norm $||S|| \leq 1$, so $1 - S^2$ is positive, and we have $S = \frac{1}{2}(f_+(S) + f_-(S))$, where $f_\pm(s) = s \pm i\sqrt{1 - s^2}$. □

Prop. (V.5.4.10) (Fuglede). If N_1 and N_2 are normal operators and A is a bounded linear operator on a Hilbert space such that $N_1 A = A N_2$, then $N_1^* A = A N_2^*$.

Proof: For any $S \in B(H)$, $\exp(S - S^*)$ is unitary thus $||\exp(S - S^*)|| = 1$, $\exp(N_1)A = A\exp(N_2)$. Because $\exp(M)T = T\exp(N)$, if we let $U_1 = \exp(M^* - M), U_2 = \exp(N - N^*)$, then

$$||\exp(N_1^*)T\exp(-N_2^*)|| = ||U_1 T U_2|| \leq ||T||$$

because λN_i is normal. Now

$$||\exp(\lambda N_1^*)T\exp(-\lambda N_2^*)|| = ||U_1 T U_2|| \leq ||T||$$

also holds, thus by Liouville, $\exp(\lambda N_1^*)T\exp(-\lambda N_2^*) = T$. Compare the coefficients of λ, we get the result. □

Prop. (V.5.4.11). An operator $T \in B(H)$ has the same spectrum w.r.t all the closed B^*-algebras of $B(H)$ containing it.

Proof: If T is invertible, because TT^* is self-adjoint thus has real spectrum(V.5.4.8) so doesn't separate $\mathbb{C}$ thus it is invertible in any closed B^*-algebra of $B(H)$(V.5.1.12). so does $T^{-1} = T^*(TT^*)^{-1}$. □

Prop. (V.5.4.12). For T normal and E its spectral decomposition, then if $f \in C(\sigma(T))$ and $\omega_0 = f^{-1}(0)$, then $N(f(T)) = R(E(\omega_0))$.

Proof: $\chi_{\omega_0} f = 0$, so $f(T)R(E(\omega_0)) = 0$, and if we let $\omega_n = f^{-1}([1/(n-1), 1/n))$, and let $f_n(\lambda) = 1/f(\lambda)\chi_{\omega_n}$, then $f_n(T)f(T) = E(\omega_n)$, so if $f(T) = 0$, then $E(\omega_n)x = 0$, so countable additivity shows that $E(\sigma\backslash\omega_0)x = 0$, so $E(\omega_0)x = x$. These shows the desired result. □

Cor. (V.5.4.13).
1. $N(T - \lambda I) = R(\{\lambda\})$.
2. every isolated point of $\sigma(T)$ is point spectrum, because this point is open thus is $E(\{x\}) \neq 0$ by Urysohn lemma.
3. if $\sigma(T)$ is countable, then every $x \in H$ has a unique orthogonal decomposition $x = \sum E(\lambda_i)x$ and $T(E(\lambda_i)x) = \lambda_i E(\lambda_i)x$.

Prop. (V.5.4.14) (Normal Compact Operator). A normal operator $T \in B(H)$ is compact iff $\sigma(T)$ has no limit point except possibly 0 and $\dim N(T - \lambda I) < \infty$ for $\lambda \neq 0$.

In particular, a normal compact operator is a limit of f.d. operators

Proof: One direction is general, by(V.4.5.9), for the other, it is a limit of operators of finite dimensional range by general symbolic calculus(V.5.4.3). □

Cor. (V.5.4.15) (Spectral Theorem). A compact normal operator (in particular a normal operator on a f.d linear space) is unitarily diagonalizable.

Proof: it suffices to find a basis of eigenvectors, but this is easy, just by(V.5.4.13). □

Cor. (V.5.4.16) (Hilbert-Schimidt). For a self-adjoint compact operator A on a Hilbert space H, there is a set of orthonormal basis that A is diagonal on it. And of course, its eigenvalues are real and can only converges to 0(V.5.4.8).

Prop. (V.5.4.17). For a normal compact operator $T \in L(H)$, then:
1. T has an eigenvalue $|\lambda|$ that $|\lambda| = ||T||$.
2. $f(T)$ is compact if $f \in C(\sigma(T))$ and $f(0) = 0$.
3. $f(T)$ is not compact if $f \in C(\sigma(T))$ and $f(0) \neq 0$ and $\dim H = \infty$.

Proof: 1: The spectrum of maximal norm is isolated(V.4.5.9) hence a point spectrum by(V.5.4.13). And $|\lambda| = ||T||$ by symbolic calculus(V.5.4.3).

2: Cf.[Rudin P330].

3: The 2 sill show that $f(0)I - f$ is compact, If f is compact, then $f(0)I$ is compact, so $\dim H < \infty$(V.4.5.3). □

Prop. (V.5.4.18) (Freudenthal Spectral Theorem).

Prop. (V.5.4.19) (Positive Equivalent Definition). A $T \in L(H)$ is positive, i.e. $T = T^*$ and $\sigma(T) \subset [0, \infty)$ iff $(Tx, x) \geq 0$.

Proof: If $(Tx, x) \geq 0$, then $(Tx, x) = (x, Tx) = (T^*x, x)$, so $T = T^*$ by(V.5.4.5), so $\sigma(T)$ is real(V.5.4.8), and for $\lambda > 0$,

$$\lambda||x||^2 = (\lambda x, x) \leq ((T + \lambda I)x, x) \leq ||(T + \lambda I)x||||x||,$$

so $T + \lambda I$ is invertible by(V.5.4.7), so $\sigma(T) \subset [0, \infty)$.

Conversely, if T is positive, then it is normal, so $(Tx, x) = \int_{\sigma(T)} \lambda dE_{x,x} \geq 0$. □

Prop. (V.5.4.20) (Polar Decomposition).

1. Every positive operator T has a positive square root, which is invertible if T is.
2. Polar decomposition exists in $B(H)$: Any $T \in L(H)$ invertible has a unique decomposition $T = UP$ where U is unitary and P is positive. And $||Px|| = ||Tx||$ for all x.
3. Any normal operator has commuting decomposition UP, where U, P, T commutes.

Proof: 1: Use general symbolic calculus, then $S = \sqrt{\lambda}(T)$ is the square root of T. If T is invertible, then $S^{-1} = T^{-1}S$.

2: $(T^*Tx, x) = (Tx, Tx) \geq 0$, so T^*T is positive(V.5.4.19), so let $P = \sqrt{T}$, then it is also invertible, and $U = TP^{-1}$ is unitary.

3: Use general symbolic calculus, let $p(\lambda) = |\lambda|$, $u(\lambda) = \lambda/|\lambda|$ if $\lambda \neq 0$, and $u(0) = 0$. Then $T = UP$, and they are commutative. □

Cor. (V.5.4.21) (Similar Normal Operator). Similar normal operators are unitarily equivalent.

Proof: It suffices to show that if $M = TNT^{-1}$, and $T = UP$ is the polar decomposition, then $M = UNU^{-1}$. Fuglede(V.5.4.10) shows $M^*T = TN^*$, so $NP^2 = NT^*T = T^*MT = T^*TN = P^2N$, so N commutes with any functions $f(P)$, in particular P. Hence $M = (UP)N(UP)^{-1} = UNU^{-1}$. □

Prop. (V.5.4.22) (Ergodic Theorem). If $U \in L(H)$ is unitary and $x \in H$, then the average $A_n x = \frac{1}{n}(x + Ux + \ldots + U^{n-1}x)$ converges to some $y \in H$.

Proof: Define $a_n = \frac{1}{n}(1 + \lambda + \ldots + \lambda^{n-1})$ on the unit circle, and $b(1) = \chi_{\{1\}}$, then if $y = b(U)x$, then $||y - A_n x||^2 = \int_{\sigma(U)} |b - a_n|^2 dE_{x,x}(\lambda)$. But this integral converges to 0 by dominated convergence theorem. □

5 Trace Class and Hilbert-Schmidt Operator

Cf.[Psudo-Differential Operator notes]. and [Folland Appendix].

Def. (V.5.5.1) (Trace-Class). Let $\mathcal{H}$ be a Hilbert space, an operator $T \in L(\mathcal{H})$ is called a positive **trace-class** iff T is orthogonalizable with eigenvectors e_α and non-negative λ_α that $\lambda_\alpha < \infty$(notice this implies that the α that $\lambda_\alpha \neq 0$ is countable). More generally, an operator T is called a **trace-class** iff $\sqrt{T^*T}$(V.5.4.20) is a positive trace-class.

Clearly, it is a limit of f.d. range operators, so trace-class is clearly compact.

Prop. (V.5.5.2). if T is a positive trace-class, then for any orthogonal basis x_α of $\mathcal{H}$, $\sum(Tx_\alpha, x_\alpha) = \mathrm{tr}(T)$.

Proof: Clear. □

Prop. (V.5.5.3). if T is a positive trace-class and S is an operator, then if x_α is an orthogonal basis of $\mathcal{H}$, then the sum $\sum(STx_\alpha, x_\alpha)$ is absolutely convergent, and is independent of the basis chosen.

Proof: Let e_α be the basis of eigenvectors of T of eigenvalues λ_α, then

$$(STx_n, x_n) = \sum_j (x_n, e_j)(STe_j, x_n) = \sum_j \lambda_j (x_n, e_j)(Se_j, x_n)$$

And

$$\sum_n \sum_j \lambda_j |(x_n, e_j)(Se_j, x_n)| \leq \sum_j \lambda_j ||e_j|| ||Se_j|| \leq ||S|| \sum \lambda_\alpha < \infty.$$

Moreover:

$$\sum_n (STx_n, x_n) = \sum_n \sum_j ((x_n, e_j)STe_j, x_n) = \sum_n \sum_j (STe_j, (e_j, x_n)x_n) = \sum_j (STe_j, e_j).$$

□

Cor. (V.5.5.4). If T is a trace class, then T is compact, and if x_α is an orthogonal basis of $\mathcal{H}$, then $\sum(Tx_\alpha, x_\alpha)$ absolutely converges, and is independent of the basis chosen.

Proof: Use polar decomposition $T = VP$(V.5.4.20) and the definition(V.5.5.1) and(V.5.5.3). □

Cor. (V.5.5.5) (Singular Trace). If T is a trace class, then we defined the general **trace** of T to be $\sum(Tx_\alpha, x_\alpha)$, for any orthogonal basis x_α of $\mathcal{H}$.

Prop. (V.5.5.6). The set of all trace-classes is a two-sided *-ideal of $L(\mathcal{H})$, and for any $S \in L(\mathcal{H})$, $\mathrm{tr}(ST) = \mathrm{tr}(TS)$.

Proof: Let S, T be trace-classes, $S = V|S|, T = W|T|$, $S + T = X|S + T|$, then $|S + T| = X^*(S + T)$ is compact, so it has an orthogonal eigenbasis e_n by(V.5.4.16), so $\sum(|S + T|x_n, x_n) = \sum(X^*V|S|x_n, x_n) + \sum(X^*W|S|x_n, x_n) < \infty$. So $S + T$ is a trace class.

Now if U is unitary and T is a trace-class, then $(UT)^*UT = T^*T$, so UT is a trace class. and $(TU)^*TU = U^{-1}TU$ has $|UT| = U^{-1}|T|U$, so UT is also a trace class. Moreover, $\mathrm{tr}(TU) = \sum(TUx_\alpha, x_\alpha) = \sum(UTUx_\alpha, Ux_\alpha) = \mathrm{tr}(UT)$.

Now every $S \in L(\mathcal{H})$ is a linear combination of four unitary operator(V.5.4.9), so the proposition is true, and if T is a trace-class, then $T = V|T|$, and $T^* = |T|V^*$ is also a trace-class. □

Def. (V.5.5.7) (Hilbert-Schimidt Operator). An operator T is called **Hilbert-Schimidt** iff T^*T is a trace-class. It follows from(V.5.5.4) that T is Hilbert-Schimidt iff $\sum ||Tx_\alpha||^2 < \infty$ for some/all basis x_α of $\mathcal{H}$.

Every Hilbert-Schimidt operator is compact, because T^*T does, so does $|T| = \sqrt{T^*T}$, so does $ZT = U|T|$.

Prop. (V.5.5.8). If T is Hilbert-Schimidt then so does T^*. If S, T are both Hilbert-Shimidt, then ST is a trace-class.

Proof:

$$\sum ||Tx_\alpha||^2 = \sum_\alpha \sum_\beta |(Tx_\alpha, x_\beta)|^2 = \sum_\alpha \sum_\beta |(T^* x_\beta, x_\alpha)|^2 = \sum ||Tx_\beta||^2$$

For the second assertion, Cf.[Folland Abstract Functional Analysis P278]. □

Prop. (V.5.5.9). The operator $Lu(x) = \int_\Omega K(x,y)u(y)dy$ for $K(x,y) \in L^2(\Omega \times \Omega)$ defined in(V.4.5.2) is a Hilbert-Schimidt operator on $L^2(\Omega)$.

V.6 Abstract Harmonic Analysis(Folland)

basis references are [Folland Abstract Harmonic Analysis].
All representations in this section are assumed to be over $\mathbb{C}$.

1 Locally Compact Groups

Def. (V.6.1.1). On a topological group G, the **left regular action** and **right regular action** are defined as follows: $L_y f(x) = (y^{-1}x), R_y f(x) = f(xy)$.

Prop. (V.6.1.2). If $f \in C_c(G)$, then f is left and right uniformly continuous. Equivalently, $G \to C_c(G) : y \mapsto R_y(f)$ and $y \mapsto L_y(f)$ is continuous group homomorphism from $G \to C_c(G)$.

Proof: Cf.[Folland Abstract Harmonic Analysis P38]. □

Prop. (V.6.1.3). Locally compact Hausdorff group is normal.

Proof: Notice that by choosing a precompact symmetric open neighbourhood U of identity, there exists a σ-compact clopen subgroup H. So H can σ-locally refine every open cover, thus G can, too. So by (IV.1.5.1) G is paracompact. As a topological group, G is regular, thus G is normal by (IV.1.5.4). □

Prop. (V.6.1.4) (Diract sequence). For a locally compact Hausdorff group G , a **Dirac sequence** is a sequence $f_n \in C_c(G)$ that $f_n \to \delta_1$ in the weak topology of $Meas_c(G)$.

Dirac sequence exists.

Proof: □

Prop. (V.6.1.5). Every locally compact group G has a subgroup G_0 that is clopen and σ-compact.

Proof: Let U be a symmetric precompact nbhd of 1 in G, then let $U_n = U^n$, then $\overline{U_n} \subset U_{n+1}$, so let $G_0 = \cup_n U^n = \cup \overline{U_n}$, then it is open because each U_n does, and compact because each $\overline{U_n}$ does. □

Prop. (V.6.1.6). If G is locally compact, and H is subgroup that is locally compact in the induced topology, then H is closed in G.

Proof: Cf.[Folland P110]. □

Integration on Locally Compact Groups

Prop. (V.6.1.7) (Haar Measure). A left(right) **Haar measure** on a topological group G is a non-zero Radon measure(V.1.1.2) μ on G that satisfies $\mu(xE) = \mu(E)(\mu(Ex) = \mu(E))$.

Every Locally compact group G possesses a unique left Haar measure μ.

Proof: Cf.[Folland Abstract Harmonic Analysis P41-44]. □

Def. (V.6.1.8) (Modular Function). For a left Haar measure μ on a locally compact group G, $\mu_x(E) = \mu(Ex)$ is also a left Haar measure, so there is a $\Delta(x)$ that $\mu_x = \Delta(x)\mu$. Then the function Δ is a group homomorphism from G to $\mathbb{R}^+$, which is called the **modular function** of G.

G is called **unimodular** iff $\Delta = 1$, i.e. a left Haar measure is also a right Haar measure. Obviously, a locally compact Abelian group is unimodular.

Prop. (V.6.1.9). Δ is a continuous group homomorphism from G to $\mathbb{R}^+$, and

$$\int R_y f d\mu = \Delta(y^{-1}) \int f d\mu.$$

equivalently, $d\mu(xy_0) = \Delta(y_0)d\mu(x)$.

Proof: Cf.[Folland P51].

For the continuity of Δ, because $y \mapsto R_y(f)$ is continuous for each f, so $y \mapsto \int R_y f d\lambda$ is continuous, as μ is Radon measure, so by the equation just proved, Δ is continuous. □

Prop. (V.6.1.10). For a compact group K of G, Δ is trivial on K. So compact group is unimodular, and if $G/[G,G]$ is compact, then it is also unimodular.

Proof: These all follows from the fact that a compact subgroup of $\mathbb{R}^+$ is $\{1\}$, and $\mathbb{R}$ is Abelian. □

Prop. (V.6.1.11) (Lie Group Case). The Haar measure in the case of Lie groups.

Proof: Cf.[Folland Abstract Analysis P45,52].**?** □

Cor. (V.6.1.12). Any Abelian, reductive or nilpotent Lie group G is unimodular.

Prop. (V.6.1.13) (Involution Measure). If μ is a left Haar measure and ρ is defined by $\rho(E) = \rho(E^{-1})$, then ρ is a right Haar measure, and $d\mu(x^{-1}) = \Delta(x^{-1})d\mu(x)$.

Proof: Notice

$$\int R_y(f)(x)\Delta(x^{-1})d\mu(x) = \Delta(y)\int f(xy)\Delta((xy)^{-1})d\mu(x) = \int f(x)\Delta(x^{-1})d\mu(x),$$

so $\Delta(x^{-1})d\mu(x)$ is a right Haar measure, hence $cd\mu(x^{-1})$ for some c. If $c \neq 1$, we let U be a precompact symmetric nbhd U of 1 that $|\Delta(x^{-1}) - 1| \leq \frac{1}{2}|c-1|$ on U. But then $|c-1|\mu(U) = |\int_U (\Delta(x^{-1}) - 1)d\mu(x)| \leq \frac{1}{2}|c-1|\mu(U)$, contradiction. □

Convolution

Prop. (V.6.1.14). The convolution makes $L^1(G)$ a Banach *-algebra, with involution given by $f^*(x)dx = \overline{f(x^{-1})}d(x^{-1})$, thus $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$(V.6.1.13).

Homogenous Spaces

Def. (V.6.1.15) (Notations). If G is a locally compact group with left Haar measure dx and H is a closed subgroup with left Haar measure $d\xi$, let $q : G \to G/H$ be the quotient map.

Prop. (V.6.1.16). If G is a σ-compact locally compact group and S is a transitive G-space that is locally compact and Hausdorff, then if $s_0 \in S$ and $\mathrm{Stab}(s_0) = H$, then $G/H \cong S$ as G-spaces.

Proof: Cf,[Folland P60]. □

Lemma (V.6.1.17). If $E \subset G/H$ is compact, then there is a compact $K \subset G$ that $q(K) = E$.

Proof: Choose a precompact nbhd V of 1 in G, since q is open, the set $q(xV)$ is an open cover of E, so there are f.m. x_i that $E \subset \cup q(x_i V)$. Then let $K = q^{-1}(E) \cap (\cup x_i \overline{V})$, this will suffice. □

Prop. (V.6.1.18). There is a map $P : C_c(G) \to C_c(G/H) : Pf(xH) = \int_H f(x\xi)d\xi$. It is checked that Pf is continuous and this map is well-defined.

$\mathrm{Supp}(Pf) \subset q(\mathrm{Supp}\, f)$, and if $\varphi \in C_c(G/H)$, then $P((\varphi \circ q)f) = \varphi Pf$.

Prop. (V.6.1.19) (Quotient Measure). If G is a locally compact group and H is a closed subgroup, then there is a G-invariant positive Radon measure μ on G/H iff $\Delta_G|_H = \Delta_H$. And if this is the case, then this measure is unique up to constant, and if suitably chose, satisfies:

$$\int_G f(x)dx = \int_{G/H} Pf d\mu = \int_{G/H}\int_H f(x\xi)d\xi d\mu(xH).$$

Proof: Cf.[Folland Abstract Analysis P62]. □

Cor. (V.6.1.20). If G is a unimodular locally compact group and P, K be closed subgroups s.t. $P \cap K$ is compact and $G = PK$. Let $d_L p, d_R k$ be the left and right Haar measure on P, K respectively, then the Haar measure on G is given by

$$\int_G f(g)dg = \int_P \int_K f(pk)d_L p d_R k.$$

Proof: Consider $H = P \times K$ and $M = P \cap K$ embedded diagonally in H, then there is a homeomorphism $H/M \cong G$ given by $(p, k) \mapsto pk^{-1}$. Then we can verify that this formula does satisfies the formula for the quotient measure in(V.6.1.19), so by uniqueness it is true. □

2 Representations

Def. (V.6.2.1) (Representation of Locally Compact Groups). Usually we consider representations on a Hilbert space. A **unitary representation** of a locally compact group on a Hilbert space $\mathcal{H}$ is defined to be a homomorphism from G to the group $U(\mathcal{H})$ of unitary representations of $\mathcal{H}$ continuous in the strong operator topology(V.3.6.6). Notice by(V.3.6.7), this is equivalent to the unitary and continuous in the weak operator topology.

Def. (V.6.2.2) (Intertwining Operators). if π_1, π_2 are unitary representations of G, then the space $C(\pi_1, \pi_2)$ of **intertwining operators** of π_1, π_2 as:

$$C(\pi_1, \pi_2) = \{T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2} : T\pi_1(x) = \pi_2(x)T, \quad \forall x \in G\}.$$

And denote $C(\pi_1, \pi_1)$ by $C(\pi_1)$.

Lemma (V.6.2.3). If $\mathcal{H}_\pi$ is a representation of G, M is a closed subspace. Let P be the orthogonal projection onto M, then M is invariant under π iff $P \in C(\pi)$.

Proof: If $P \in C(\pi)$ and $v \in M$, then $\pi(x)v = \pi(x)Pv = P\pi(x)v \in M$, so M is π-invariant. Conversely if M is π-invariant, then so does $M^\perp$, so $\pi(x)Pv = \pi(x)v = P\pi(x)v$, and also for $v \in M^\perp$, so $\pi(x)P = P\pi(x)$, for any x. □

Prop. (V.6.2.4) (Schur's Lemma).

- A unitary representation π of G is irreducible iff $C(\pi)$ consists only of scalar multiples of identity.
- If π_1, π_2 are non-equivalent irreducible unitary representations of G, then $C(\pi_1, \pi_2) = 0$.

Proof: 1: if π is reducible, then it contains a non-trivial projection by lemma(V.6.2.3). Conversely, if $T\neq cI\in C(\pi)$, then we consider $A=\frac{1}{2}(T+T^*), B=\frac{1}{2i}(T-T^*)$, then at least one of them are not cI. But they are normal, thus by Symbolic calculus(V.5.4.3) any $\chi_E(A)$ for $E\subset\mathbb{R}$ non-trivial Borel is non-trivial and commutes with π, so $\mathcal{H}_\pi$ is reducible by(V.6.2.3) again.

2: It can be verified that if $T\in C(\pi_1,\pi_2)$, then $T^*\in C(\pi_2,\pi_1)$, thus $TT^*=cI, T^*T=cI$. so $T=0$ or $c^{-1/2}T$ is unitary. And in the latter case, if $T_1,T_2\in C(\pi_1,\pi_2)$, then $T_1^{-1}T_2\in C(\pi_1)=cI$, thus $C(\pi_1,\pi_2)$ is of dimension 1. □

Cor. (V.6.2.5). if G is Abelian, then any irreducible representation of G is 1-dimensional.

Proof: If π is a representation of G, then any $\pi(x)$ commutes with π, thus $\pi(x)=c_xI$ for some c_x, so every subspace of $\mathcal{H}_\pi$ is irreducible, thus $\dim\mathcal{H}=1$. □

Prop. (V.6.2.6) (Unitary Representation and $L^1(G)$-Representation). Any unitary representation of G determines a representation of $L^1(G)$ by

$$f\mapsto\int f(x)\pi(x)dx$$

This is a non-degenerate *-representation of $L^1(G)$.

And Conversely, any non-degenerate *-representation of $L^1(G)$ arises from a unitary representation of G.

Proof: If π is a unitary representation and $f\in L^1$, let $\pi(f)=\int f(x)\pi(x)dx$, where the integral is in the weak sense(V.4.3.25), and it satisfies $||\pi(f)||\leq||f||_1$.

Cf.[Folland P79-81]. □

Functions of Positive Type

Def. (V.6.2.7) (Positive Type Function). A function of **positive type** on a closed compact group G is a function $\varphi\in L^\infty(G)$ that defines a positive linear functional on the B^*-algebra $L^1(G)$. In other word,

$$\int f(x)\overline{f(y)}\varphi(y^{-1}x)dydx\geq 0,\quad\forall f\in L^1(G).$$

We denote by $P(G)$ the set of continuous functions of positive type on G.

Prop. (V.6.2.8). If φ is of positive type, then so does $\overline{\varphi}$. (Easy calculation).

Prop. (V.6.2.9). If π is a unitary representation of G and $u\in\mathcal{H}_\pi$, then $\varphi(x)=(\pi(x)u,u)\in P$.

Proof: φ is continuous by definition, so if $f\in L^1$, then

$$\int\int f(x)\overline{f(y)}\varphi(y^{-1}x)d\mu(x)d\mu(y)=\int\int(f(x)\pi(x)u,f(y)\pi(y)u)dxdy=||\pi(f)u||^2\leq 0$$

□

Prop. (V.6.2.10) (Cyclic Representations and Functions of Positive Type). Any function of positive type arises from a irreducible representation and a cyclic vector ε as in(V.6.2.9)

Proof: Cf.[Folland P83-85]. □

Cor. (V.6.2.11). If φ is a function of positive type, then φ can be chosen to be continuous.

Cor. (V.6.2.12). If $\varphi \in P$, then $||\varphi||_\infty = \varphi(1)$, and $\varphi(x^{-1}) = \overline{\varphi(x)}$.

Proof: $\varphi(x) = (\pi(x)u, u)$ for some representation π and $u \in \mathcal{H}$, so $|\varphi(x)| \leq ||u||^2 = \varphi(1)$ and $\varphi(x^{-1}) = (\pi(x^{-1})u, u) = (u, \pi(x)u) = \overline{\varphi(x)}$. □

Def. (V.6.2.13). We set:

- $P_0(G) = \{\varphi | ||\varphi||_\infty \leq 1\} = \{\varphi(1) = 1\}$.
- $P_1(G) = \{\varphi | ||\varphi||_\infty = 1\} = \{0 \leq \varphi(1) \leq 1\}$.

By Banach-Alaoglu, $P_0(G)$ and $P_1(G)$ are a weak*-compact set.

Prop. (V.6.2.14) (Extreme Points of P_1). A $\varphi \in P_1$ is an extreme point iff the representation it corresponds is irreducible. And $E(P_0) = E(P_1) \cup \{0\}$.

Proof: Cf.[Folland P86]. □

Prop. (V.6.2.15) (Two Topologies Coincide). On P_1, the compact-open topology coincides with that of the weak*-topology.

Proof: Cf.[Folland Abstract Harmonic Analysis P80]. □

Prop. (V.6.2.16) (Gelfand-Raikov). If G is a locally compact group, then the irreducible representations of G separate points of G.

Proof: Cf.[Folland Abstract Analysis P91]. □

3 Locally Compact Abelian Group

Dual Group

Def. (V.6.3.1) (Dual Group). These 1-dimensional unitary representations of G are all 1-dimensional by(V.6.2.5). Then we called the group of all continuous homomorphism of G into S^1 the **dual group** $\widehat{G}$ of G.

An element of $\widehat{G}$ is called a **character** of G, denoted by ξ. And a continuous homomorphism from G to $\mathbb{C}$ is called a **quasi-character**.

The topologies on $\widehat{G}$ that makes it into a LCA group is given in(V.6.3.5).

Remark (V.6.3.2). $\widehat{\mathbb{R}} \cong \mathbb{R}$, and the quasi-characters of $\mathbb{R}$ are all of the form $x \to e^{sx}$ for $s \in \mathbb{C}$.

Proof: If $\varphi \in \widehat{\mathbb{G}}$, then $\varphi(0) = 1$, and there is an $a > 0$ that $\int_0^a \varphi(t)dt \neq 0 = A$. Now $A\varphi(x) = \int_x^{x+a} \varphi(t)dt$, so taking derivative,

$$\varphi'(x) = \frac{\varphi(x+a) - \varphi(x)}{A} = \frac{\varphi(a) - 1}{A}\varphi(x),$$

which shows $\varphi(x) = e^{sx}$ for some $s \in \mathbb{C}$. □

Prop. (V.6.3.3) (Dual Group as Spectrum of $L^1(G)$). The dual group G^* can be regarded as the spectrum of $L^1(G)$, i.e. multiplicative homomorphism of $L^1(G)$:

$$\xi \mapsto \Big(\xi(f) = \int \overline{(x, \xi)} f(x) dx\Big).$$

and in this way, the Fourier transform is just the Gelfand transform from $L^1(G)$ to $C(\widehat{G})$. Its range is a dense space of $C_0(\widehat{G})$.

Proof: First, ξ is multiplicative because

$$\xi(f*g)=\int\int f(y)g(y^{-1}x)(x,\xi)dydx=\int\int f(y)g(x)(xy,\xi)dydx=\xi(f)\xi(g).$$

Conversely, any continuous functional on L^1 is like $\varphi(f)=\int f(x)\varphi(x)dx$ for some $\varphi\in L^\infty$, and it is multiplicative, so

$$\varphi(f)\int\varphi(x)g(x)=\varphi(f)\varphi(g)=\varphi(f*g)=\int\int\varphi(y)f(yx^{-1})g(x)dxdy=\int\varphi(L_x(f))g(x)dx$$

So $\varphi(x)=\frac{\varphi(L_x(f))}{\varphi(f)}$, a.e., for any f. so $\varphi(x)$ can be chosen to be continuous, as $x\to L_x(f)$ is continuous(V.6.1.2). And clearly φ is multiplicative. $\square$

Cor. (V.6.3.4). $\widehat{G}\subset P_1(G)$, because $\int(f^**f)\varphi d\mu=|\Phi(f)|^2\geq 0$.

Cor. (V.6.3.5) (Dual Group as a LCA Group). Now we can give $\widehat{G}$ the compact-open topology, then the group operation is clearly continuous, and the topology coincides with that inherited by the weak*-topology of the L^∞ by(V.6.2.15), so $\widehat{G}\cup\{0\}$ is a compact Hausdorff space because $\widehat{G}\subset P_1(G)$ and it is the subset of L^∞ that $\{h(xy)=h(x)h(y)\}$ which is weak*-closed hence weak*-compact. In particular, $\widehat{G}$ is a locally compact topological group.

Prop. (V.6.3.6) (Duality between Discrete Groups and Compact Groups). if G is discrete, then $\widehat{G}$ is compact, if G is compact, then $\widehat{G}$ is discrete.

Proof: if G is discrete, then there is a unit δ in $L^1(G)$, which is 1 on e and 0 otherwise. So The spectrum of G is compact by(V.5.1.21).

If G is compact, then $1\in L^1$, so $U=\{f\in L^\infty||f|>\frac{1}{2}\}$ is weak*-open, but $U\cap\widehat{G}=\{1\}$ by(V.6.4.1), so $\widehat{G}$ is discrete. $\square$

Fourier Transform

Prop. (V.6.3.7) (Fourier Transform). The **Fourier transform** on G is defined to be the map:

$$\mathcal{F}f(\xi)=\widehat{f}(x)=\int f(x)\overline{(x,\xi)}$$

which is a norm-decreasing *-homomorphism form $L^1(G)$ to $C_0(\widehat{G})$, and its range is a dense subspace of $C_0(\widehat{G})$.

Equivalently, the Fourier transform is just the Gelfand transform of $L^1(G)$(V.5.1.21) composed with an inverse map.

Proof: Cf.[Folland Abstract Harmonic Analysis P102]. $\square$

Prop. (V.6.3.8). $\widehat{(f*g)}=\widehat{f}\cdot\widehat{g}$, so if $f,g\in L^2(G)$, $\widehat{(fg)}=\widehat{f}*\widehat{g}$.

Proof: Cf.[Folland Abstract Harmonic Analysis P102]. $\square$

Prop. (V.6.3.9). There is another map from $M(\widehat{G})$ to bounded continuous functions on G:

$$\mu\mapsto\Big(\varphi_\mu:x\mapsto\int(x,\xi)d\mu(\xi)\Big).$$

This is a norm decreasing injection from $M(\widehat{G})$ to $L^\infty(G)$, and if μ is positive, then φ_μ are all functions of positive type.

Proof: It suffices to prove injectivity, but if $\varphi_\mu = 0$, then $0 = \int\int f(x)(x,\xi)d\mu(\xi)dx = \int \widehat{f}(\xi^{-1})d\mu(\xi)$ for all $f \in L^1(G)$, so but this shows $\mu = 0$ because of(V.6.3.7) and Riesz representation.

For the positive type, notice that

$$\int\int f(x)\overline{f(y)}vp_\mu(y^{-1}x)dxdy = \int\int\int f(x)\overline{f(y)(y,\xi)}(x,\xi)d\mu(\xi)dxdy = \int |\widehat{f}(\xi)|^2 d\mu(\xi) \geq 0$$

□

Prop. (V.6.3.10) (Bochner's Theorem). If $\varphi \in P(G)$, there is a unique positive $\mu \in M(\widehat{G})$ s.t. $\varphi = \varphi_\mu$.

Proof: We have the map defined in(V.6.3.9) injects $M(\widehat{G})$ into $P(G)$(norm-decreasing), so it suffices to prove the existence. For this, we may assume $\varphi \in P_0(G)$. Let M_0 be the set of positive measure $\mu \in M(\widehat{G})$ that $\mu(\widehat{G}) \leq 1$, then M_0 is weak*-compact in $M(\widehat{G})$. Now

$$\int f(x)\varphi_\mu(x)dx = \int\int f(x)d\mu(\xi)dx = \int \widehat{f}(\xi^{-1})\mu(x)$$

so the mapping $\mu \to P_0$ must be continuous w.r.t their weak*-topologies, so the image is a compact convex subset of P_0. But the image contains all characters and 0(by taking the point mess), which are the extreme points of P_0, by(V.6.2.14), so it contains all the P_0, by Krein-Milman(V.4.3.15). □

Cor. (V.6.3.11) (Herglotz). A numerical sequence $\{a_n\}$ is positive iff there is a positive measure $\mu \in M(T)$ s.t. $a_n = \widehat{\mu}(n)$.

Cor. (V.6.3.12). So (V.6.3.10) together with(V.6.3.9) says that the image $B(G)$ of all φ_μ are just the linear span of $P(G)$. And we denote $B^p(G) = B(G) \cap L^p(G)$.

Prop. (V.6.3.13). The set of regular Borel probability measures on a compact X is weak*-compact in $C(X)^*$. (Use Alaoglu).

Prop. (V.6.3.14) (Fourier Inversion Formula). (special case of(V.6.3.19)) If $f \in B^1$, then $\widehat{f} \in L^1(\widehat{G})$, and if the Haar measure $d\xi$ of $\widehat{G}$ is suitably normalized w.r.t. the Haar measure of G, then $d\mu_f(\xi) = \widehat{f}(\xi)d\xi$, i.e. $f(x) = \int(x,\xi)\widehat{f}(\xi)d\xi$. This measure $d\xi$ is called the **dual measure** of dx.

Proof: Cf.[Folland Abstract Harmonic Analysis P105]. □

Cor. (V.6.3.15). If $f \in L^1(G) \cap P$, then $\widehat{f} \geq 0$, as $d\mu_f(\xi) = \widehat{f}(\xi)d\xi$ and μ_f is positive, by Bochner's theorem(V.6.3.10).

Prop. (V.6.3.16). If μ is the counting measure on a discrete group, then its dual measure satisfies $|\widehat{G}| = 1$, and if G is compact and $|G| = 1$, then the dual measure is the counting measure on $\widehat{G}$.

Proof: First(V.6.3.6) should be noticed. If G is compact and $|G| = 1$, then if $g = 1$, then $\widehat{g} = \chi_{\{1\}}$, so $g(x) = \sum(x,\xi)\widehat{g}(\xi)$, which shows the dual measure is counting measure by definition(V.6.3.14).
□

Prop. (V.6.3.17) (Plancherel Theorem). The Fourier transform on $L^1(G) \cap L^2(G)$ extends uniquely to an isomorphism from $L^2(G)$ to $L^2(\widehat{G})$.

Proof: Cf.[Folland P108]. □

Pontryagin Duality

Prop. (V.6.3.18) (Pontryagin Duality). For a locally compact Abelian group G, $G \to \widehat{\widehat{G}}$ is an isomorphism of topological groups.

Proof: Cf.[Folland Abstract Harmonic Analysis P110]. □

Cor. (V.6.3.19) (Fourier Inversion Theorem). If $f \in L^1(G)$ and $\widehat{f} \in L^1(\widehat{G})$, then $f(x) = \widehat{\widehat{f}}(x^{-1})$, i.e. $f(x) = \int (x,\xi)\widehat{f}(\xi)d\xi$ a.e..

Proof: As
$$\widehat{f}(\xi) = \int \overline{(x,\xi)} f(x)dx = \int (x^{-1},\xi)f(x)dx = \int (x,\xi)f(x^{-1})dx,$$
so by definition $\widehat{f} \in B^1(\widehat{G})$, and $d\mu_{\widehat{f}}(x) = f(x^{-1})dx$. Then by(V.6.3.14), $f(x^{-1}) = \widehat{\widehat{f}}(x)$. □

Cor. (V.6.3.20) (Fourier Uniqueness Theorem). If $u, v \in M(G)$ satisfy $\widehat{u} = \widehat{v}$, then $u = v$. In particular, if $f, g \in L^1(G)$ and $\widehat{f} = \widehat{g}$, then $f = g$.

Proof: By(V.6.3.9)(norm decreasing), μ is uniquely determined by $\varphi_\mu(\xi) = \widehat{\mu}(\xi^{-1})$ by Fourier inversion. □

Prop. (V.6.3.21) (Duality of Subgroups). $(H^\perp)^\perp = H$ for closed subgroup H of a locally compact Abelian group G.

Proof: Suffices to prove $(H^\perp)^\perp \subset H$. If $x_0 \notin H$, then Gelfand-Raikov shows that there is a character η on G/H that $\eta(q(x_0)) \neq 1$, so $x_0 \notin (H^\perp)^\perp$. □

Prop. (V.6.3.22). If H is a closed subgroup of G, then there are natural isomorphisms of LCA groups:
$$\Phi : \widehat{(G/H)} \cong H^\perp, \quad \Psi : \widehat{G}/H^\perp \cong \widehat{H}$$

Proof: Φ is clearly algebraic isomorphism. If $|\eta(q(K)) - 1| < \varepsilon$, then $|\eta(K) - 1| < \varepsilon$, so Φ is continuous in the compact-open topology. Similarly, to show Φ is open, it suffices to show a compact subset of G/H has a compact inverse image in G, but this is just(V.6.1.17).

Now for Ψ, notice $\widehat{\widehat{G}/H^\perp} \cong (H^\perp)^\perp \cong H$ by(V.6.3.21), so by Pontryagin duality theorem, $\widehat{G}/H^\perp \cong \widehat{H}$. □

Prop. (V.6.3.23) (Poisson Summation Formula).

4 Compact Group

Cf.[群表示论 notes] and [Folland Abstract Analysis Chap5].

Basics

Prop. (V.6.4.1). Integration of a nontrivial character on a compact group G w.r,t. the Haar measure is 0.

Proof: $\int f(x)d\mu(x) = \int f(yx)d\mu(yx) = f(y)\int f(x)d\mu(x)$. Now choose a y that $f(y) \neq 1$. □

Prop. (V.6.4.2). If G is compact and the Haar measure is normalized that $|G| = 1$, then $\widehat{G}$ is a set of orthonormal basis of $\mathcal{L}^2(G)$.

Proof: It is an easy consequence of(V.6.4.1) that $\widehat{G}$ are mutually orthogonal. And if $f \in L^2$ is orthogonal to each ξ, then $0 = \int f\overline{\xi}dx = \widehat{f}(\xi)$ for all $\xi \in \widehat{G}$, so by Plancherel theorem, $f = 0$. □

Prop. (V.6.4.3) (F.d. Representation is Unitary). If V is a f.d. representation of G compact, then there is an inner product on V that the action of G is unitary.

Proof: Choose an arbitrary inner product $(-,-)_0$ on V, then consider

$$(u, v) = \int_G (\rho(x)u, \rho(x)v)_0 dg$$

which is π-invariant because G is compact hence unimodular. □

Lemma (V.6.4.4). Cf.[Folland P135].

Prop. (V.6.4.5) (Irreducible Representations of Compact Groups). If G is compact group, then very irreducible representation of G is of f.d., and every unitary representation of G is a direct sum of irreducible unitary subrepresentations.

Proof: If π is an irreducible representation, choose T as in lemma(V.6.4.4), then by Schur's lemma, $T = cI$, so $\dim \mathcal{H}_\pi < \infty$, by(V.4.1.7).

For the direct sum, by taking orthogonal complements and Zorn's lemma, it suffices to show any unitary representation π has an irreducible representation. Choose T as in(V.6.4.4), then T is compact nonzero self-adjoint, so by Riesz-Fredholm(V.4.5.9) it has a finite-dimensional eigenspace, which is π-invariant, and it clearly has an irreducible subrepresentation by taking orthogonal complements. □

Def. (V.6.4.6). Denote $\widehat{G}$ the set of unitary representations of G, called the **dual space** of G.

Matrix Coefficients and Peter-Weyl Theorem

Def. (V.6.4.7) (K-Finite Vectors). For an irreducible representation ρ of K, denote $V^\rho = \rho \otimes \mathrm{Hom}_K(\rho, V)$ the ρ-isotypic component in V. And let $V^{K-\text{fin}} = \oplus_\rho V^\rho$ the space of K**-finite vectors** in V(V.6.4.5).

Def. (V.6.4.8) (Matrix Coefficients). Firstly $C(G)$ is a representation of $G \times G$ by $((g_1, g_2)f)(g) = f(g_1^{-1}gg_2)$.

For a f.d. representation V of a topological group G, we can view $\mathrm{End}(V)$ as a representation of $G \times G$ via

$$(g_1, g_2)S = T_{g_1}ST_{g_2^{-1}}$$

There is a **matrix coefficient map**:

$$MC_V : \mathrm{End}(V) \to C(G), \quad MC_V(S)(k) = MC_{S,V}(g) = \mathrm{tr}(ST_{g^{-1}}|V)$$

is a map of $K \times K$-representations.

Prop. (V.6.4.9) (Orthogonality Conditions). Let V_1, V_2 be f.d. continuous irreducible representations of K, then
$$\int_k MC_{S_1,V_1}(k)MC_{S_2,V_2}(k) = 0$$
unless $V_2 \cong V_1^*$. And in this case,
$$\inf_k MC_{S_1,V}(k)MC_{S_2,V^*}(k) = \frac{1}{\dim V}\operatorname{tr}(S_1 \circ S_2^*|V)$$

Proof: Cf.[Gaitsgory P3]. □

Def. (V.6.4.10) (Characters). Let V is a f.d. continuous representation of K, let $\chi_V = MC_V(\mathrm{Id}_V)$, it is called the **character** of V, and if V is irreducible, define $\xi_V = \dim V \chi_V$.

Prop. (V.6.4.11). Let V, W be irreducible, then
$$\int_K \chi_V(k)\chi_W(k^{-1}) = \int_K \chi_V(k)\chi_{W^*}(k) = \int_K \chi_V(k)\overline{\chi_W(k)}$$
so by**??** this equals 1 if $V \cong W$ and 0 otherwise.

Prop. (V.6.4.12). Then for V, W irreducible, $T_{\xi_V} \in \operatorname{End} W$ is Id_W if $W \cong V$ and zero otherwise.

Proof: Cf.[gaitsgory P4]. □

Cor. (V.6.4.13). $\xi_V * \xi_W = 0$ unless $W \cong V$ and $\xi_V * \xi_V = \xi_V$.

Proof: Cf.[gaitsgory P4]. □

Prop. (V.6.4.14). If ρ is an irreducible f.d representation of K and V is a continuous representation of K, then for any $S \in \operatorname{End}(\rho^*)$, the image of
$$MC_{S,\rho}\mu_{Haar} \in meas(K)$$
acting on V(V.4.3.24) belongs to V^ρ.

Proof: Cf.[Gaitsgory P7]. □

Cor. (V.6.4.15). For an irreducible representation ρ of K, the element $\xi_\rho\mu_{Haar} \in Meas(K)$ acts in any continuous representation V as a projection with image equal to V^ρ.

Proof: Directly from the proposition and**??**. □

Prop. (V.6.4.16) (Peter-Weyl).

Prop. (V.6.4.17). For any continuous representation K, the subset V^{K-fin} is dense in V.

Proof: For any $v \in V$, choose a Dirac sequence f_n, then $T_{f_n\mu_{Haar}}v \to v$. Then by Peter-Weyl(V.6.4.11), we can choose K-finite functions g_n that $||g_n - f_n||_{L^2} < \frac{1}{n}$. Then g_n also converges to δ_1 in the weak topology. Thus
$$T_{g_n\mu_{Haar}}v \to v$$
and(V.6.4.9) shows $T_{g_n\mu_{Haar}}v \in V^{K-fin}$. □

Cor. (V.6.4.18). Matrix coefficients of f.d. representations are dense in $C(K)$. (Immediate from the proposition and Peter-Weyl theorem(V.6.4.11).

V.7 Harmonic Analysis

1 Distributions

Def. (V.7.1.1). The space $D(\Omega)$ of **test functions** has the induced topology coincides with that of $D(K)$, and any bounded subsets are in some $D(K)$, thus it is complete and has Heine-Borel because $D(K)$ does.

The space of continuous linear functionals of $D(\Omega)$ is called the space of **distributions** $D'(\Omega)$. It is equivalence to the restriction to every $D(K)$ is continuous, Cf.[Rudin P155]. The **order** of a distribution Λ is the minimal N that $|\Lambda\varphi| \leq C_K||\varphi||_N$ for every $\varphi \in D(K)$, it might be ∞.

Def. (V.7.1.2). The **differentiation** of a distribution Λ is defined as $D^\alpha\Lambda(\varphi) = (-1)^{|\alpha|}\Lambda(D^\alpha\varphi)$. The multiplication by a smooth function f is defined by $f\Lambda(\varphi) = \Lambda(f\varphi)$. Then

$$D^\alpha(f\Lambda) = \sum_{\beta\leq\alpha} C_{\alpha\beta}(D^{\alpha-\beta}f)(D^\beta\Lambda).$$

Support of a Distribution

Def. (V.7.1.3). The **support** $\mathrm{Supp}(\Lambda)$ of a distribution is the complement of the open sets U that $\Lambda(f) = 0$ for any f with support in U.

If $\mathrm{Supp}(\Lambda)$ is compact, then Λ has finite order and $|\Lambda\varphi| \leq C||\varphi||_N$ for some N, and Λ extends uniquely to a continuous linear functional on $C^\infty(\Omega)$.

Proof: This is because its support is compact so we can choose a smooth ψ that $= 1$ on $\mathrm{Supp}\,\varphi$ and has support in $W \subset \Omega$. Then by (V.7.1.1), there is a C that $|\Lambda(\psi\varphi)| < C||\psi\varphi||_N$, and Leibniz rule will give us the result. □

Prop. (V.7.1.4). If the support of a Λ is a pt p (thus has finite order m), then it is a linear combination of $D^\alpha\delta_p$,$|\alpha| \leq m$. (use approximate identity and show the kernel of Λ is contained in the kernel of $D^\alpha\delta_p$.

Proof: Cf.[Rudin P165]. □

Prop. (V.7.1.5). For any distribution Λ, there exist continuous functions g_α in $C^\infty(\Omega)$ that each compact K intersects support of f.m g_α and $\Lambda = \sum D^\alpha g_\alpha$. When Λ has finite order, we can use only f.m g_α.

Proof: use partition of unity. Then for a compact K, find a compact-open W, then find a bump function between $K \subset W$, thus reduce to the case of $D_{\overline{W}}$. For the rest, Cf.[Rudin P169]. □

Convolution on $\mathbb{R}^n$

Denote $D = D(\mathbb{R}^n), D' = D'(\mathbb{R}^n)$.

Def. (V.7.1.6). The **translation** of a distribution u is defined as $(\tau_x u)(\varphi) = u(\tau_{-x}\varphi)$, where $\tau_x\varphi(y) = \varphi(y - x)$.

The **convolution** of a test function with a distribution u is defined as $(u * \varphi)(x) = u(\tau_x\check{\varphi})$, where $\check{\varphi}(y) = \varphi(-y)$.

Prop. (V.7.1.7) (Special Case of (V.7.1.10)). For $u \in D', \varphi \in D, \psi \in D$,

- $\tau_x(u*\varphi)=(\tau_x u)*\varphi=u*(\tau_x\varphi)$.
- $u*\varphi\in C^\infty$ and $D^\alpha(u*\varphi)=(D^\alpha u)*\varphi=u*(D^\alpha\varphi)$.
- $u*(\varphi*\psi)=(u*\varphi)*\psi$.

If u has compact support, then(V.7.1.3) shows that u can extend to C^∞, thus convolution is defined for $\varphi\in C^\infty$ and the first two formulae still hold, and when $\psi\in D$,

$$u*\psi\in D,\quad u*(\varphi*\psi)=(u*\varphi)*\psi=(u*\psi)*\varphi$$

Proof: Cf.[Rudin P171], [Rudin P174]. □

Cor. (V.7.1.8). $L:\varphi\mapsto u*\varphi$ is a continuous linear map into C^∞ that commutes with τ_x. And any these map comes from a $u*$:let $u=(L\check{\varphi})(0)$.

Proof: It is continuous because of of closed graph theorem(V.3.4.8), $\lim(u*\varphi_i)(x)=\lim u(\tau_x\check{\varphi})=u(\tau_x\check{\varphi})$. □

Cor. (V.7.1.9). When $u,v\in D'$ and one of them has compact support, then similar to(V.7.1.8), $L\varphi=u*(v*\varphi)$ is a continuous linear map that commutes with τ_x, so there is a unique **convolution distribution** $u*v$ that $(u*v)*\varphi=u*(v*\varphi)$. This convolution is compatible with the previous one when $v\in D$.

Prop. (V.7.1.10) (Convolution of Distributions). For $u,v,w\in D'$,

- if one of u,v has compact support, then $u*v=v*u$, and $\mathrm{Supp}(u*v)\subset\mathrm{Supp}(u)+\mathrm{Supp}(v)$.
- if two of three of u,v,w has compact support, then $(u*v)*w=u*(v*w)$.
- $D^\alpha u=(D^\alpha\delta)*u$.
- if one of u,v has compact support, then $D^\alpha(u*v)=(D^\alpha u)*v=u*(D^\alpha v)$.

Proof: Cf.[Rudin P177]. □

Def. (V.7.1.11). A **approximate identity** here is a $h\in D$ that $h_k(x)=k^nh(kx)$. Then we will have $\lim\varphi*h_j=\varphi$ for $\varphi\in D$, $\lim u*h_j=u$ in D'.

2 Fourier Analysis on $\mathbb{R}^n$

Def. (V.7.2.1). We denote the normalized notation $\mathbb{R}^n$ as $dm=(2\pi)^{-n/2}dx$ and $D_\alpha=1/i^{|\alpha|}D^\alpha$, this will simplify notations. The **Fourier transform** here of a function $f\in L^1(\mathbb{R}^n)$ is the function $\hat{f}$ that $\hat{f}(t)=\int_{\mathbb{R}^n}fe_{-t}dm_n=(f*e_t)(0)$.

See(V.7.2.13) for general Fourier transform.

Prop. (V.7.2.2). For $f\in L^1(\mathbb{R})$,

$$\widehat{\tau_x f}=e_{-x}\hat{f},\quad \widehat{e_{-x}f}=\tau_x\hat{f},$$

$$\widehat{f*g}=\hat{f}\hat{g},\quad \widehat{f(x/\lambda)}(t)=\lambda^n\hat{f}(\lambda t).$$

(Note $||f*g||_1\le||f||_1||g||_1$).

Def. (V.7.2.3). The class of **Shwarts functions** $\mathcal{S}$ is defined as smooth functions on $\mathbb{R}^n$ that

$$\sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^N|(D_\alpha f)(x)|<\infty.$$

Lemma (V.7.2.4). Let $f = e^{-1/2|x|^2}$, then $f \in \mathcal{S}$, $\hat{f} = f$ and $f(0) = \int \hat{f}$. (reduce to the 1 dimensional case, in which case, $f' + xy = 0$, and $\hat{f}$ also satisfies this).

Lemma (V.7.2.5). For $f, g \in L^1$, Fubini theorem shows $\int \hat{f}g = \int f\hat{g}$.

Prop. (V.7.2.6) (Classical Fourier Transform).

- $\mathcal{S}$ is a Fréchet space in the topology defined by these norms.
- multiplication by $g \in \mathcal{S}$ and derivations are continuous linear map from S to S (direct calculation).
- $\widehat{P(D)f}(t) = P(t)\hat{f}(t)$ and $\widehat{Pf} = P(-D)\hat{f}$.
- The Fourier transform is a continuous linear one-to-one automorphism of S, and $\Psi^2 g = \check{g}$.

Proof: 1:

2:

3: use(V.7.1.10) for the first one, and for the second one, should use definition of derivative and dominated convergence.

4:$\Psi f \in \mathcal{S}$ by 3, and it is continuous by closed graph theorem. By(V.7.2.5) and(V.7.2.2), $\int \hat{f}(t)g(t/\lambda) = \int f(t/\lambda)\hat{g}(y)$. If $\hat{f}, \hat{g} \in L^1$, dominant convergence shows $g(0)\int \hat{f} = f(0)\int \hat{g}$. So we only need one f that $f(0) = \int \hat{f}$, $f = e^{-1/2|x|^2}$ will suffice(V.7.2.4). Hence $g(0) = \int \hat{g}$ for every such g, and the conclusion follows by translation(V.7.2.2), and(V.7.2.8) also follows. □

Cor. (V.7.2.7). If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$, and $||\hat{f}||_\infty \leq ||f||_1$, because $\mathcal{S}$ is dense in $L^1(\mathbb{R}^n)$.

Prop. (V.7.2.8) (Inversion Theorem). If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then $\check{f} = \Psi^2 f$ a.e.

Proof: In(V.7.2.5), let $g \in \mathcal{S}$ and substitute $g = \Psi g$ and use Fubini, we get $\check{f} - \Psi^2 f$ is orthogonal to every $\mathcal{S}$, then every continuous function with compact support by(V.1.5.10). Thus they equal a.e. □

Cor. (V.7.2.9). If $f, g \in \mathcal{S}$, then $\widehat{fg} = \hat{f} * \hat{g}$ (apply Fourier one time and use(V.7.2.2)), and thus $f * g \in \mathcal{S}$.

Prop. (V.7.2.10) (Fourier-Plancherel). If $f, g \in \mathcal{S}$, then

$$\int f\overline{g} = \int \overline{g}(x)\hat{f}(t)e^{ixt} = \int \hat{f}(t)\int \overline{g}(x)e^{ixt} = \int \hat{f}\overline{\hat{g}}$$

by inversion formula. And $\mathcal{S}$ is dense in L^2, thus it extends to a linear isometry of $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. This coincides with the Fourier transform on $L^1 \cap L^2$.

Prop. (V.7.2.11). D injects into $\mathcal{S}$ and is dense.(Notice they both are complete, but the subspace topology are different)(Use scaling, Cf.[Rudin Functional Analysis P189]). So we call a distribution **tempered** iff it comes from a continuous functional of $\mathcal{S}$.

From(V.7.1.3), we know any distribution with compact support is tempered. By Holder, every $f \in L^p(\mathbb{R}^n), p \geq 1$ is tempered distribution, and every polynomial or functions of polynomial growth are tempered distribution.

$$D \subset \mathcal{S} \subset L^2 = (L^2)^\vee \subset \mathcal{S}' \subset D'.$$

$\mathcal{S}, \mathcal{S}'$ is complete(V.3.6.2).

Prop. (V.7.2.12). A $f \in \mathcal{S}'$ iff $f = \sum_{|\alpha|\leq m} D_\alpha(u_\alpha(1+|x|^2)^{m/2})$ for some m, where $u_\alpha \in L^2(\mathbb{R}^n)$.

Proof: In fact,

$$||\varphi||'_m = (\sum_{|\alpha|\leq m} \int (1+|x|^2)^m |D_\alpha \varphi|^2 \,\mathrm{dx})^2$$

is an equivalent set of norms of $\mathcal{S}'$, Cf.[泛函分析张恭庆 P182]. And each of them defines a Hilbert space. So by Riesz we get the result. □

Prop. (V.7.2.13) (Generalized Fourier Transform). For a tempered distribution $u \in \mathcal{S}'$, we define the **Fourier transformation** as the tempered distribution $\hat{u}(\varphi) = u(\hat{\varphi})$. It is easily verified that it is compatible with previously defined Fourier transform when seen as tempered distributions by?? In particular, this is defined for compactly supported distribution, $L^p(\mathbb{R}^n), p \geq 1$ and smooth functions of polynomial growth(V.7.2.11).

Prop. (V.7.2.14). $\widehat{P(D)u} = P\hat{u}$ and $\widehat{Pu} = P(-D)\hat{u}$. And The Fourier transformation is a continuous linear isometry of $\mathcal{S}'$ in the weak* topology.

Cor. (V.7.2.15). $\hat{1} = \delta$, thus $\hat{P} = P(-D)\delta$ and $P(\hat{D})\delta = P$. Now(V.7.1.4) tells us a distribution is the Fourier transform of a polynomial iff it has support in the origin.

Prop. (V.7.2.16) (Convolution of Tempered Distributions). Let $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$, then

- $u * \varphi \in C^\infty$ of polynomial growth and $D^\alpha(u * \varphi) = (D^\alpha u) * \varphi = u * (D^\alpha \varphi)$.
- $u * (\varphi * \psi) = (u * \varphi) * \psi$.
- $\widehat{u * \varphi} = \hat{\varphi}\hat{u}, \hat{u} * \hat{\varphi} = \hat{\varphi u}$.
- If P is a polynomial and $g \in \mathcal{S}$, then $D^\alpha u, Pu$ and gu are all tempered.

Proof: Cf.[Rudin Functional Analysis P195] for the first 3. □

Paley-Wiener Theory

Prop. (V.7.2.17). For $\varphi \in D(\mathbb{R}^n)$ that has support in rB, the You-Know-How defined $\hat{\varphi}(z)$ is an entire function of several variable and satisfies:

$$|\varphi'(z)| \leq \gamma_N(1+|z|)^{-N} e^{r|\operatorname{Im} z|}.$$

For $N \geq 0$. Conversely, any such function correspond to a $\varphi \in D(\mathbb{R}^n)$ that has support in rB.

Proof: Cf.[Rudin P198]. □

Prop. (V.7.2.18) (Fourier-Laplace transformation). For $u \in D'(\mathbb{R}^n)$ that has support in rB, of order N, the $\hat{u}(z) = u(e_{-z})$ is an entire function of several variable and satisfies:

$$|f(z)| \leq \gamma(1+|z|)^N e^{r|\operatorname{Im} z|}.$$

Conversely, any such function correspond to a $u \in D'(\mathbb{R}^n)$ that has support in rB.

Proof: Cf.[Rudin P199]. □

3 Sobolev Space

Def. (V.7.3.1). For $1 \le p < \infty$, the **Sobolev space** $W^{m,p}(\Omega)$ is the space of functions u that $D^\alpha u \in L^p(\Omega)$ for every $|\alpha| \le m$, with the norm $||u|| = \sum_{|\alpha|<m} \int_\Omega |D^\alpha u(x)|^2 dx$. The **Sobolev space** $W_0^{m,p}(\Omega)$ is the completion of the subspace $C_0^\infty(\Omega)$.

Prop. (V.7.3.2) (Meyers-Serrin). The Sobolev space $W^{m,p}(\Omega)$ is the completion of $u \in C^\infty(\Omega)$ that $D^\alpha u \in L^p(\Omega)$ for every $|\alpha| \le m$.

Proof: Choose a countable partition of unity ψ_k, then as in the proof of(V.1.5.11), we can choose δ_k small enough and $||\psi u - (\psi u)_{\delta_k}|| < \varepsilon/2^k$ and $\varphi = \sum(\psi u)_{\delta_k}$ is definable. □

Prop. (V.7.3.3). We denote $H^m(\Omega) = W^{m,2}(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$ and $H^{-m}(\Omega) = (H_0^m(\Omega))^*$ when m is an integer. Notice derivative is not applicable for $H^{-m}(\Omega)$ unless $\Omega = \mathbb{R}^n$.

When $\Omega = \mathbb{R}^n$, $D(\mathbb{R}^n)$ is dense in $W^{m,p}$(V.1.5.11), thus $W_0^{m,p} = W^{m,p}$. Define the **Sobolev space**

$$H^s = \{u|(1+|y|^2)^{s/2}|\hat{u}| \in L^2\}$$

H^s is a Hilbert space and $H^s \subset \mathcal{S}'$ for every s (use Holder to show $\hat{u} \in \mathcal{S}'$). H^m coincides with previously defined H^m when m is a positive integer thus also negative-integer. A linear operator on $H = \cup H^s$ is said to have **order** t if it maps every H^s continuously into H^{s-t}.

Proof: By Plancherel,

$$||\varphi||'_m = (\sum_{|\alpha|\le m} ||D_\alpha u||_2^2)^{1/2} \quad \text{and} \quad (\int (1+|x|^2)^m |\hat{u}|^2)^{1/2}$$

are equivalence norms on H^m. □

Lemma (V.7.3.4) (Poincare Inequality). For Ω bounded, on $C_0^m(\Omega)$ the $W^{m,p}$ norm is controlled by L^p norms of its mth order derivatives .

Proof: We may assume $\Omega \subset \prod_{i=1}^n [0,a]$, then for any $u \in W^{m,p}$, $u(x) = \int_0^{x_1} D^1 u(t, x_2, \ldots, x_n) dt$, so by Holder inequality,

$$|u(x)| \le a^{1/q} (\int_0^a |D^1 u|^p dx_1)^{1/p}.$$

so

$$\int_\Omega |u(x)|^p dx \le a^q \int_\Omega |D^1 u|^p dx_1.$$

Doing the same for all other derivatives, we can see the norm is controlled by the highest(m-th) order norms. □

Prop. (V.7.3.5). When $t < s$, $H^s \subset H^t$. And H^s are isometric to H^t by $\hat{v} = (1+|y|^2)^{t/2}\hat{u}$ and is of order t. D^α is of order $|\alpha|$. If $f \in \mathcal{S}$, then $u \to fu$ is an operator of order 0.

Every distribution of compact support is in some H^s(V.7.1.3), in particular $D(\Omega)$.

Proof: Cf.[Rudin P217]. □

Prop. (V.7.3.6) (Sobolev Embedding Theorem). On a manifold of dimension n which is compact with Lipschitz boundary or complete of positive injective radius and bounded sectional curvature,

- if $k > l$ be integers and
$$\frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{l}{n}$$
then $W^{k,p}(\text{int}(M)) \subset W^{l,q}(M)$ continuously.
- if
$$\frac{1}{p} - \frac{k}{n} = -\frac{r+\alpha}{n}$$
then $W^{k,p}(\text{int}(M)) \subset C^{r,\alpha}(M)$ continuously.

Proof: Cf.[Evans P290]. □

Cor. (V.7.3.7) (Gagliardo–Nirenberg–Sobolev). On a manifold of dimension n which is compact with Lipschitz boundary or a complete of positive injective radius and bounded sectional curvature, if $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ (Sobolev conjugate), then $W^{1,p}(\text{int}(M)) \subset L^{p^*}(M)$ continuously.

Cor. (V.7.3.8). On a manifold of dimension n which is compact with Lipschitz boundary or a complete of positive injective radius and bounded sectional curvature, if $m > n/2$, then $W^{m,2}(\text{int}(M)) \subset C(\overline{\Omega})(M)$ continuously. And the functions in $W^{m,2}_0$ are continuous and vanish at the boundary, by C_0 approximation.

Proof: The $\mathbb{R}^n$ case can be directly proved: because we have the equivalent norm(V.7.3.3), $\hat{u} \in L^2$ thus $u \in L^2$, and
$$\int |\hat{u}| \leq (\int (1+|x|^2)^m |\hat{u}|^2)^{1/2} \cdot \Big(\int 1/(1+|x|^2)^m\Big)^{1/2}.$$
We have $\hat{u} \in L^1$, thus inversion formula applies that u is continuous and $||u||_\infty \leq ||\hat{u}||_1 \leq C||u||_{H^m}$. □

Cor. (V.7.3.9). $\cap_s H^s = C^\infty(M)$.

Prop. (V.7.3.10) (Rellich-Kondrechov). On a compact manifold with C^1 boundary of dimension n, if $k > l$ and
$$\frac{1}{p} - \frac{k}{n} < \frac{1}{q} - \frac{l}{n}$$
then $W^{k,p} \subset W^{l,q}$ completely continuously.

Proof: Cf.[Distributions and Operators P199], [Evans P290]. □

Cor. (V.7.3.11). On a bounded extension domain of $\mathbb{R}^n$, $W^{1,p} \subset L^p$ completely continuously.

Proof: We prove the $p = 2$ case. For a sequence u_m in $W^{1,2}$, we have $||u_m - u_p||_2 = ||U_m - U_p||_2 = ||\hat{U}_m - \hat{U}_p||_2$. By(V.4.4.9), there is a subsequence that $\hat{U}_m$ pointwise converge. Notice they are uniformly bounded, Now apply two region argument, for $|x| < r$, use Lebesgue dominant convergence, and for $|x| > r$, use $\int (1+|x|^2)|\hat{U}_m - \hat{U}_p|^2$ is bounded to conclude $||u_m - u_p||_2 \to 0$. □

Prop. (V.7.3.12). $u \in D'(\Omega)$ is a locally H^s $\iff$ $\psi u \in H^s$ for every $\psi \in D(\Omega)$ $\iff$ $D_\alpha u$ is locally L^2 for every $|\alpha| \leq s$.

Thus every smooth function is locally H^s for every s.

Proof: $1 \to 2$ use partition of unity, $2 \to 1$ easy, and $2, 3$ are all equivalent to $D_\alpha(\psi u) \in L^2$ for every $\psi \in D(\Omega)$. by Leibniz+Plancherel or (V.7.3.5). □

Prop. (V.7.3.13). If $r > p + n/2$, then if a function f on Ω has all the distribution derivative $D^k_i f$ locally $L^2, = g_{is}$, for $0 \leq k \leq r$, then $f \in C^p(\Omega)$ a.e.

Cor. (V.7.3.14). If $u \in D'(\Omega)$ is locally H^s, then $u \in C^{s-n/2}(\Omega)$. Thus $\cap$locally $H^s = C^\infty(\Omega)$.

Holder Space

Def. (V.7.3.15). Holder space $C^{k,\alpha}(\Omega)$ is the subspace of $C^k(\Omega)$ with the norm

$$||f||_{C^{k,\alpha}} = ||f||_{C^k} + \max_{|\beta|=k} \sup_{x\neq y\in\Omega} \frac{|D^\beta f(x) - D^\beta f(y)|}{||x-y||^\alpha}.$$

4 Fourier Analysis on $\mathbb{T}^n$

Prop. (V.7.4.1). If $f \in L^1(\mathbb{T})$ is absolutely continuous, then $\widehat{(f')}(n) = 2\pi i n \cdot \widehat{f}(n)$.

Prop. (V.7.4.2). $f \in L^1(\mathbb{T})$ is determined by its Fourier coefficients.

V.8 Differential Operators

1 ODE-Fundamentals

Prop. (V.8.1.1).

$$x^{(2)} = f(x)$$

It can be solved.

Proof:

$$x^{'}x^{(2)} = f(x)x^{'}$$
$$\frac{1}{2}(x^{'})^2 = \int^x f(t)dt$$

□

Prop. (V.8.1.2) (Wronsky).

2 ODE-Theorems

Prop. (V.8.2.1) (Existence and Uniqueness of ODE of Lipschitz Type). If $F(t,x)$ defined on $[-h,.h]\times[\eta-\delta,\eta+\delta]$ is a function that is locally Lipschitz: that is, $\exists\delta, L$, s.t. if $|t|\leq h, |x_i-\eta|\leq\delta$, then

$$|F(t,x_1)-F(t,x_2)|\leq L|x_1-x_2|.$$

Then the initial value problem:

$$(Tx)(t) = \eta + \int_0^t F(\tau, x(\tau))d\tau$$

has a unique solution on the interval $[-h,h]$ if $h<\min\{\delta/M, 1/L\}$, where M is the maximum of F on $[-h,.h]\times[\eta-\delta,\eta+\delta]$. Because T is a contraction.

Prop. (V.8.2.2) (Existence of ODE of continuous Type (Caratheodory)). If $F(t,x)$ defined on $[-h,.h]\times[\eta-\delta,\eta+\delta]$ is a continuous function, then

$$(Tx)(t) = \eta + \int_0^t F(\tau, x(\tau))d\tau$$

has a unique solution on the interval $[-h,h]$ if $h<\delta/M$, where M is the maximum of F on $[-h,.h]\times[\eta-\delta,\eta+\delta]$. (Use Schauder fixed point theorem and Arzela-Ascali).

Prop. (V.8.2.3) (Existence Theorem for Complex Differential Equations). Let $f(z,\mathbf{w})$ be a holomorphic vector function in a domain $D\subset\mathbb{C}^{n+1}$, then the initial value problem

$$\mathbf{w}' = f(z,\mathbf{w}),\quad w(z_0) = w_0$$

has exactly on holomorphic solution locally (Thus on a simply connected domain).

Cor. (V.8.2.4). So a holomorphic high-order ODE for a complex variable can be solves. And luckily it can be solved even $\overline{z}$ appears (just regard it as a constant).Δ

Proof: Cf.[Ordinary Differential Equations, P110]. □

Prop. (V.8.2.5). For the equation:

$$\frac{d\mathbf{y}}{d\mathbf{x}} = \mathbf{A}\mathbf{y},$$

One solution basis is:

$$\begin{cases} e^{\lambda_1 x}\mathbf{P}_1^{(1)}(x), \cdots, e^{\lambda_1 x}\mathbf{P}_{n_1}^{(1)}(x); \\ \cdots\cdots\cdots \\ e^{\lambda_s x}\mathbf{P}_1^{(d)}(x), \cdots, e^{\lambda_s x}\mathbf{P}_{n_s}^{(1)}(x); \end{cases}$$

Where

$$\mathbf{P}_j^{(i)}(x) = \mathbf{r}_{j0}^{(i)} + \frac{x}{1!}\mathbf{r}_{j1}^{(i)} + \cdots,$$

where $\mathbf{r}_{j0}^{(i)}$ is a basis of solution of $(\mathbf{A} - \lambda_i I)^n \mathbf{x} = 0$, and $\mathbf{r}_{k+1}^{(i)} = (\mathbf{A} - \lambda_i I)\mathbf{r}_k^{(i)}$.

Proof: Cf.[常微分方程丁同仁定理 6.6]. □

Cor. (V.8.2.6). For the equation:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y^{'} + a_n y = 0,$$

If the characteristic equation has s different roots $\lambda_1, \ldots, \lambda_s$ and corresponding multiplicities $n_1, \ldots, n_s$, then:

$$\begin{cases} e^{\lambda_1 x}, xe^{\lambda_1 x}, \cdots, x^{n_1 - 1}e^{\lambda_1 x}; \\ \cdots\cdots\cdots \\ e^{\lambda_s x}, xe^{\lambda_s x}, \cdots, x^{n_s - 1}e^{\lambda_s x}; \end{cases}$$

is a solution basis.

Proof: Cf.[常微分方程丁同仁 P198]. □

Prop. (V.8.2.7) (Lyapunov). Consider the Lyapunov stability of an autonomous system of the form:

$$\frac{dx}{dt} = Ax + o(|x|),$$

Then:

1. If A has a eigenvalue whose real part is positive, then the trivial solution is not weak stable.
2. If all eigenvalues of A has negative real part, then the trivial solution is strong stable.

Stum-Liouville

Prop. (V.8.2.8) (Stum-Liouville). The eigenvalue BVP problem of L-S equation:

$$Lu = (pu')' + qu = \lambda u, \quad a_1 u(a) + a_2 u'(a) = 0, b_1 u(b) + b_2 u'(b) = 0, \sigma(x) > 0.$$

can be solved by the method of Green's function. For the function:

$$G(x,s) = \begin{cases} Cu_1(x)u_2(s), & x < s \\ Cu_2(x)u_1(s), & x > s \end{cases}$$

for some C, where u_1 is a solution of the L-S equation with boundary value at a, and u_2 with boundary value at b that are linear independent (This happens when the homogenous equation has no solution). It satisfies: $LG(x,s) = \delta(x-s)$ and satisfies the boundary conditions.

Because L is self-adjoint, we have:

$$Gf(x) = \int f(s)G(x,s)ds, LG = \mathrm{id}, GL = \mathrm{id}$$

thus the eigenvalues of L is the reciprocal of the eigenvalues of G, and G is a compact self-adjoint operator on $L^2(\sigma, \mathbb{R})$, so by spectral theorem, the eigenvectors are countable and form an orthonormal basis.

And when the homogenous problem do have a solution ϕ, then we have:$Lu = f$ has a solution iff $(f, \phi) = 0$. one way is simple and the other way is because we solve the initial problem of ODE and find that it automatically satisfies the boundary condition. Cf.[Stum Liouville Theory].

Prop. (V.8.2.9). More generally, if there the boundary is mixed of $u(a), U'(a), u(b), u'(b)$, the solution of

$$Lu = (pu')' + qu = 0, B_1(u) = \alpha, B_2(u) = \beta.$$

has a unique solution for any α, β iff the homogenous equation has only non-trivial solution. (Because the solution space is of 2 dimensional.

Prop. (V.8.2.10) (Stum Seperation Theorem).

Prop. (V.8.2.11) (Stum Comparison Theorem). If $y'' + K_i(x)y = 0$ are equations. If $y_i(0) = 0$ and $|y_1'(0)| = |y_2'(0)|$, then if $K_1(x) \geq K_2(x)$, then $y_1(x) \geq y_2(x)$ until $y_2(x)$ is zero. (directly from(IV.2.4.10)).

3 Linear PDE

Def. (V.8.3.1). For a linear PDE with constant coefficients $P(D)u = v$, the **fundamental solution** is a distribution $E \in D'(\mathbb{R}^n)$ that $P(D)E = \delta$. This is important because if v is a distribution with compact support, $P(D)(E * v) = (P(D)E) * v = \delta * v = v$(V.7.1.10), so $u = E * v$ is a distribution solution.

Prop. (V.8.3.2). When $v \in D'(\mathbb{R}^n)$ has compact support, $P(D)u = v$ has a solution u with compact support iff $Pg = \hat{v}$ has a solution g entire. In this case, $g = \hat{u}$ for some distribution u, and u has support in the convex hull of the support of v.

Proof: Use(V.7.2.18), and some bound relation between g and Pg. Cf.[Rudin Functional Analysis P212]. □

Prop. (V.8.3.3). The fundamental solution always exist when for PDE of constant coefficients.

Proof: For a $\varphi \in D(\mathbb{R}^n)$, there is at most one ψ that $\psi = P(D)\varphi$ because $\hat{\psi} = P\hat{\varphi}$ and they are entire function. Thus the task is to verify the functional $u : P(D)\varphi \to \varphi(0)$ is continuous and extend to a distribution $u \in D'(\mathbb{R}^n)$. Cf.[Rudin Functional Analysis P215]. □

4 Differential Operator on Manifolds

Prop. (V.8.4.1) (Index Theorem P109). has a nice definition of symbol of a differential operator on a manifold as a map form $\mathrm{Sym}^m T^*M \otimes \mathbb{C} \to \mathrm{Hom}(E, F)$.

5 Pseudo-Differential Operator

Def. (V.8.5.1). Denote the **Japanese bracket** $[x] = (1 + |x|^2)^{1/2} \sim 1 + |x|$.

Motivated by the formula $(\widehat{P\hat{f}})^\vee = P(D)f$ for $f \in \mathcal{S}$ and polynomial P of ξ with coefficients smooth functions of x**??** we define the **symbol class** $S^{\mu,\beta}$ as the space of smooth functions $a : \mathbb{R}^{2n} \to \mathbb{C}$ that

$$|D_{x,\alpha}D_{\xi,\beta}a(x,\xi)| \leq C_{\alpha,\beta}[x]^\mu[\xi]^{m-|\beta|}$$

and denote $S^m = S^{0,m}$.

We denote the **symbol class** $\mathcal{A}^v$ as the space of smooth functions $a : \mathbb{R}^{2n} \to C$ that $|D_\alpha a| \leq C_\alpha[x+\xi]^v$ for any α. So $S^{\mu,m} \subset \mathcal{A}^{|\mu|+|m|}$

And we define the **pseudo-differential operator of symbol** a:

$$(a(x,D)u)(x) = \int_\xi e^{ix\xi}a(x,\xi)\hat{u}$$

Moreover, we can define the **amplitude function** $p(x,y,\xi)$ and define

$$Pu(x) = \int e^{i(x-y)\xi}p(x,y,\xi)u(y)dy.$$

Def. (V.8.5.2). We define the space S^d of **polyhomogenous symbols of degree** d as the set of all symbols in $S^d_{0,1}$ that there exists a set of p_{d-l} homogenous in ξ of degree $d-l$ that $p = \sum p_{d-l}$ modulo an operator in $S^{-\infty}$. Note that when p_{d-l} is homogenous of degree $d-l$, then it is automatically in $S^{d-l}_{0,1}$.

Def. (V.8.5.3). A ψdo operator a is called **elliptic** if $\sigma(a) \in S^m$ and $\sigma(a) \geq [\xi]^{-m}$ for ξ big enough.

Prop. (V.8.5.4) (Peetre's Inequality). For all $v \in \mathbb{R}$, there is a constant C that

$$[X+Y]^v < C[X]^v[Y]^v.$$

Proof: For $v > 0$, just as normal. For $v < 0$, use $X = (X+Y) + (-Y)$ applied to $-v$. □

Prop. (V.8.5.5). The mapping $a(x,\xi) \times u(x) \mapsto a(x,D)u$ is continuous from $\mathcal{A}^v \times \mathcal{S} \to \mathcal{S}$, thus also continuous from $S^{\mu,m} \times \mathcal{S} \to \mathcal{S}$. Cf.[Pseudo Differential Operator P28].

Lemma (V.8.5.6) (Schur Test). For a function K on $\mathbb{R}^{2n}$ and $u \in L^p(\mathbb{R}^n)$, let $||K||_1 = \sup_x \int |K(x,y)|dy$ and $||K(x,y)||_2 = \sup_y \int |K(x,y)|dx$. Let $Au(x) = \int K(x,y)u(y)dy$, then

$$||Au||_{l^p} \leq ||K||_1^{1-1/p}||K||_2^{1/p}||u||_{L^p}.$$

by Holder.

Prop. (V.8.5.7) (Calderón-Vaillancourt). There is a constant C, N_{CV} that for $u \in \mathcal{A}^0$ and $\varphi \in \mathcal{S}$,

$$||Op(u)\varphi||_{L^2} \leq C \max_{|\alpha|+|\beta| \leq N_{CV}} ||\partial_x^\alpha D_{\beta,\xi}u||_{L^\infty}||\varphi||_{L^2}.$$

This in particular applies to $u \in S^0$.

Proof: Cf.[Calderon-Vaillancourt]. □

Cor. (V.8.5.8). S^m maps H^s to H^{s-m}. Because by symbolic calculus(V.8.5.10), we have

$$Op([\xi]^{s-m})Op(u)Op([\xi]^{-s}) = Op(b) \in S^0,$$

thus $Op(u) = Op([\xi]^{m-s})Op(b)Op([\xi]^s)$ maps H^s into H^{s-m}.

Symbolic Calculus

Def. (V.8.5.9) (Semiclassical Operator). For $a \in S^{\mu,m}$ and $h \in (0,1]$, we denote $a_h(x,\xi) = a(x,h\xi)$, it is also in $S^{\mu,m}$.

Prop. (V.8.5.10) (Composition). If $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$, there is a pseudo-differential operator $(a\#b)(h) \in S^{\mu_1+\mu_2,m_1+m_2}$ for every $h \in (0,1]$ that

$$Op(a_h)Op(b_h) = Op((a\#b)(h)_h)$$

and for all $J > 0$, $(a\#b)(h)$ can be written as

$$a\#b(h) = \sum_{j<J} h^j \Big(\sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b \Big) + h^J r_J^\#(a,b,h)$$

where $r_J^\#(a,b,h) \in S^{\mu_1+\mu_2,m_1+m_2-J}$ and it is bilinear of a,b and equicontinuous independently of h.

Proof: Cf.[Pseudo Differential Operator P36]. □

Prop. (V.8.5.11) (Adjoint). If $a \in S^{\mu,m}$ and $u,v \in \mathcal{S}$, there is a pseudo-differential operator $a*(h)$ for every $h \in (0,1]$ that

$$(u, Op(a_h)v) = (Op(a^*(h)_h)u, v)$$

in the L^2 norm and for all $J > 0$, $a^*(h)$ can be written as

$$a^*(h) = \sum_{j<J} h^j (\sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{a}) + h^J r_J^*(a,h)$$

where $r_J^*(a,h) \in S^{\mu,m-J}$ and it is anti-linear of a and equicontinuous independently of h.

Proof: Cf.[Pseudo Differential Operator P30]. □

Def. (V.8.5.12). For $u \in \mathcal{S}'$, we define the action of $a(x,\xi)$ on u by

$$(Op(a_h)u)(\overline{\varphi}) = u(\overline{Op(a^*(h)_h)\varphi}).$$

This is compatible with the definition on $\mathcal{S}$.

6 General PDE

Direct Solution

Prop. (V.8.6.1) (Characteristic Line). Consider a 1-dimensional parabolic equation:

$$p_t + c(p,x,t)p_x = r(p,x,t)$$

Let $P(t) = p(X(t),t)$, this equation is equivalent to

$$P_t = r(X(t),t,P(t)), \quad X_t = c(X(t),t).$$

an ODE equation.

Prop. (V.8.6.2). A set of equations:

$$\frac{\partial}{\partial x^i}\mu = A_i\mu$$

where μ is a n-vector. It has a solution iff

$$[A_i, A_j] = \frac{\partial}{\partial x^i}A_i - \frac{\partial}{\partial x^i}A_j.$$

Proof: □

Cor. (V.8.6.3). This seems to be able to derive Frobenius integrability theorem, but I cannot figure it out.

7 Analysis on Manifolds

Prop. (V.8.7.1) (Peetre's Theorem). For a linear operator from $C^\infty(M)$ to $C^\infty(M)$ that $Supp(Lu) \subset Supp(u)$ where M is a compact manifold, then on every compact subset of a coordinate chart L looks like a differential operator of finite order.

Proof: The first thing is to prove on a chart Ω, L is continuous on $C_0^\infty(\Omega)$. In fact, it suffice to show it is continuous from $C_0^\infty(\Omega)$ to $C_0^0(\Omega)$ because we can apply to $D_\alpha L$. For this, Cf.[Pseudo Differential Operator P86].

Then we have $|Lu| \leq C\max_{|\alpha\leq m}\sup_K|D_\alpha\varphi|$ for every $\varphi \in C_0(K)$. And the functional $\varphi \to (L\varphi)(x)$ is a distribution supported on x, thus by(V.7.1.4), it is of the form

$$Lu(x) = \sum_{|\alpha|\leq m} a_\alpha(x)D_\alpha\varphi(x).$$

We need to show a_α is smooth, which we choose a bump function χ to show a_0 is smooth and then choose $x_i\chi$ applied to $L\varphi - a_0\varphi$ to show a_i is smooth, etc. □

Prop. (V.8.7.2). The property of ψdo of order d is preserved under diffeomorphism, Cf.[Distributions and Operators P195], giving us the possibility to define ψdo differential operator on manifolds, and the principal symbol variate in this way that it forms a function from the cotangent bundle to the $M_n(\mathbb{C})$. And the Sobolev space is defined by the property that all of its restrictions on a atlas are Sobolev, using the partition of unity.

Prop. (V.8.7.3). All the norms of different are commensurable up to constant factor w.r.t. each other, so it doesn't quite matter with different norms.

Prop. (V.8.7.4). The parametrix exists for an elliptic operator on manifolds. Cf.[Distributions and Operators P207].

8 Elliptic Operator

Prop. (V.8.8.1). Elliptic operator is a Fredholm operator. And the kernel and cokernel are smooth functions, so it is also a Fredholm operator on $C^\infty(\Omega)$.

Proof: It suffice to find a left and right inverse modulo compact operators, and in fact we find it module $S^{-\infty}$. Since $S^{-\infty}$ are all compact operators, i.e. it has a parametrix. Cf.[Distributions and Operators, P184]. □

Prop. (V.8.8.2) (Garding Inequality). For an elliptic operator of order d on $\Gamma(E)$,

$$||f||_{H^s} \leq C(||f||_{H^{s-d}} + ||Pf||_{H^{s-d}})$$

Proof: □

Cor. (V.8.8.3) (Elliptic Regularity Theorem). The inverse image of a smooth function under an elliptic operator is a smooth function, because the intersection of $H^s(E)$ is $C^\infty(E)$.

Cor. (V.8.8.4) (Elliptic Regularity Theorem). For $L = \sum_{|\alpha|\leq N} f_\alpha D_\alpha$, where $f_\alpha \in C^\infty(\Omega)$ and the equation $Lu = v$ for distributions u and $v \in D'(\Omega)$, when v is locally H^s, u is locally H^{s+N}. Thus if $v \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$ by(V.7.3.12)(V.7.3.14).

Proof: We prove the case when L has leading coefficients constant. For every $\varphi \in D(\Omega)$ that is 1 on some open ball B, φu has compact support thus in some H^t< and then we use a sublemma that says if ψ is 1 on the support of φ, then if ψu is in H^t, where $t \leq s + N - 1$, then $\varphi u \in H^{t+1}$. In this way, we can shrink the nbhd to reach H^{s+N}. The proof of the lemma is in [Rudin Functional Analysis P220]. □

Prop. (V.8.8.5) (Analytic Ellipticity theorem). Suppose L is an analytic elliptic differential operator on a domain $M \subset \mathbb{R}^n$, then every solution to $L\varphi = 0$ is analytic.

Proof: □

Prop. (V.8.8.6). The formal adjoint of an elliptic operator is an elliptic operator.

Proof: □

Cor. (V.8.8.7). The index of an elliptic operator, regarded as an operator form $L_s \to L_{s-d}$ doesn't depend on s, because all the kernel of P and P^* are smooth.

Prop. (V.8.8.8). For an elliptic operator, It has a inverse, the Green function which is a compact operator, so it has countable eigenfunctions consisting of smooth functions on L^2 with eigenvalues converging to ∞. Moreover, the eigenvalues satisfy $|\lambda_n| \geq Cn^\delta$ for some δ, C.

Proof: We prove for P self-adjoint. Use(V.8.8.1), $\operatorname{Ker} P$ is all smooth, so there is a map $P(H^{-2d}) \to P(H^{-d})$ which is bijective thus an isomorphism by Banach. So the inverse of this isomorphism composed with the Sobolev embedding $H^{-d} \to L^2$ is a compact operator G. we notice that this map has the same eigenfunctions as P, thus the result from that of compact operators.

For the second assertion, if suffice to prove $\dim N(\lambda) \leq C\lambda^M$. Using Garding inequality and Sobolev embedding, we have for $f \in N(\lambda)$, $||f||_{C^0} \leq C(1+\lambda^l)||f||_{L^2}$ for large l. So if we choose an orthonormal basis f_i, then $|a_i f_i(x)| \leq C(1+\lambda^l)|\sqrt{\sum |a_i|^2}$. Let $a_i = f_i(x)$ and integrate over M, we get the desired result. □

Cor. (V.8.8.9). For a self-adjoint elliptic operator P which is not a constant, $L^2(E)$ has a basis consisting of eigenfunctions of P.

Cor. (V.8.8.10) (Stum-Liouville). This can be used to solve for example eigenvalue problem for Liouville's equation:

$$(pu')' + qu = \lambda\sigma u.$$

where p and σ are positive. Cf.(V.8.2.8).

Cor. (V.8.8.11). The Hermite functions $C_n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2}$, as the eigenvector of $\hat{H}=x^2-\frac{d^2}{dx^2}$, forms a complete basis for $L^2(\mathbb{R})$. Because it is e^{-x^2} times the solution of the operator $(e^{-x^2}F')'-e^{-x^2}F$.

Prop. (V.8.8.12). For a formally self-adjoint elliptic operator P of degree d on E, $\Gamma(E)=\operatorname{Im}P\oplus P(\Gamma(E))$.

Proof: We know that $L^2(E)=P(H^dE)\oplus\operatorname{Ker}P$, and $\operatorname{Ker}P$ are all smooth by(V.8.8.3), so $\Gamma(E)=\operatorname{Ker}P\oplus P(H^dE)\cap\Gamma(E)$. Now use Garding's inequality(V.8.8.2), the intersection is just $P(\Gamma(E))$, thus the result. □

Prop. (V.8.8.13) (Asymptotic Heat Equation). In this case we have the series

$$h_t(A^*A)=\sum_\lambda e^{-\lambda t}\dim\Gamma_\lambda(E)$$

converges and h_t has an asymptotic expansion

$$h_t=\sum_{k\geq -n}t^{k/2m}U_k(A^*A)$$

where $n=\dim M$ and $U_k=\int_M\mu_k$ for a differential form on M. Cf.[Heat Equation and the Index Theorem P297].

By the proposition above, the eigenspaces of eigenvalue non-zero neutralize, so $\operatorname{Ind}A=h_t(A^*A)-h_t(AA^*)$, so

$$\operatorname{Ind}A=U_0(A^*A)-U_0(AA^*)=\int_M\mu_0(A^*A)-\mu_0(AA^*).$$

The proof consists of the following propositions,

Prop. (V.8.8.14). Using the fact that an elliptic operator has a countable basis, for an elliptic operator P, when $t>0$, we let $K(t,x,y)=\sum_n e^{-t\lambda_n}\Phi_n(x)\overline{\Phi}_n(y)$, then

$$e^{-tP}f(x)=\int K(t,x,y)f(y)dy.$$

$K(t,x,y)$ is smooth. and the trace of e^{-tA^*A} is exactly $h_t(A^*A)$ as in the last proposition. And the trace is just $\int_M K(t,x,x)$, as can be easily seen.

Proof: Use Garding inequality and(V.8.8.8), we can show $||K||_{C^k}$ is bounded. □

Chapter VI

Higher Algebra

VI.1 Simplicial Homotopy Theory

References are [Jardine Simplicial Homotopy Theory].

1 Simplicial Category

Simplicial Set

Def. (VI.1.1.1). The category of simplicial objects Δ consists of $[n]$ for each $n \geq 0$ and there maps are order-preserving maps. A **simplicial object** in A is a functor from $\Delta^{op} \to A$. A **cosimplicial object** in A is a functor from $\Delta \to A$. $\Delta[n]$ is the simplicial set $\Delta^n([m]) = \mathrm{Hom}([m],[n])$.

Prop. (VI.1.1.2). The fact that any simplicial set X is a colimit of Δ^n (I.8.1.13) is important in proving properties of constructions of simplicial set.

Def. (VI.1.1.3). The **nerve** of a category C is a simplicial category with $NC_n = \mathrm{Hom}([n], C)$, i.e. composable arrows of morphisms of length n. It is a fully faithful functor from the category of small categories to the category of simplicial sets.

Def. (VI.1.1.4). The **geometrization** of a simplicial object is

$$|X| = \varinjlim_{\Delta[n]\to X} \Delta_n.$$

The **singular functor** maps a topological space X to a simplicial object $\mathrm{Sing}Y_n = \mathrm{Hom}(\Delta_n, Y)$. The geometrization functor is left adjoint to the singular functor (use colimit definition of X). This is just the Kan adjoint in(I.8.1.14).

Moreover, the geometrization as a functor from $\Delta_{Set} \to CGHaus$ preserves finite limits. Cf.[Jardine P9].

The three kinds of geometrization of a bisimplicial set is the same: geometrization the diagonal simplicial set, the twice geometrization of left(resp. right) simplicial set.

Def. (VI.1.1.5). A morphism of simplicial set is called **Kan fibration** iff it has right lifting property w.r.t all $\Lambda^n_k \to \Delta^n$. So a morphism between topological spaces $X \to Y$ is a Serre fibration iff $S(X) \to S(Y)$ is a Kan fibration(VI.1.1.4).

Def. (VI.1.1.6). A **groupoid** is a category that every morphism is invertible. The nerve of a groupoid is a Kan fibration, because we only need to consider dimension< 3.

Prop. (VI.1.1.7). A surjection of simplicial groups is a Kan fibration. In particular, simplicial abelian group and simplicial R-module are Kan complexes.

Proof: Cf.[Simplicial Homology Theory Jardine P12] □

Prop. (VI.1.1.8). The bar resolution BG is a Kan fibration for every group G.

Prop. (VI.1.1.9). A principal G fibration, i.e. $X \to X/G$ where X is a simplicial object of G-sets that G acts freely on X_n, is a Kan fibration.

Def. (VI.1.1.10). A class of monomorphisms in Δ_{Set} is called **saturated** iff it contains all isomorphisms, closed under pushout, retraction, countable composition and arbitrary direct sum.

Def. (VI.1.1.11). The saturated class generated by either of the following three class of monomorphisms is called **anodyne**:

1. $\Lambda^n_k \to \Delta[n]$, $0 \le k \le n$.
2. $(\Delta[1] \times \partial\Delta[n]) \cup (\{e\} \times \Delta[n]) \to \Delta[1] \times \Delta[n]$, $e = 0$ or 1.
3. $(\Delta[1] \times Y) \cup (\{e\} \times X) \to \Delta[1] \times X$, $e = 0$ or 1, for any $Y \subset X$.

Proof: 2 and 3 are equivalence because any inclusion comes from attaching cells(VI.1.1.3). For 1 and 2, Cf.[Jardine P17]. □

Prop. (VI.1.1.12). A natural transformation will induce homotopic nerve map. thus a pair of adjoint functors will induce a simplicial homotopy between their nerve.

Prop. (VI.1.1.13) (Koszul Resolution). The Koszul Complete or the sequence r_i is the tensor complex $K[r;R] = K[r_1,R]\otimes_R \cdots \otimes_R K[r_n;R]$, where $K[x;R] = 0 \to R \xrightarrow{r} R \to 0$.Cf.[Weibel P111].

Prop. (VI.1.1.14) (Chevalley-Eilenberg Resolution).

2 Cyclic Homology Theory(欧阳恩林)

<u>**Combinatorial Category**</u>

Def. (VI.1.2.1). The **Segal category** Fin_* is the category of pointed finite sets. A morphism is called **inert** iff $|f^{-1}(\{i\})| = 1$ for all $i \neq *$. It is called **active** iff $f^{-1}(\{*\}) = \{*\}$.

A morphism can be uniquely factorized as a composition gh, where h is inert and g is active.

Prop. (VI.1.2.2). There is a morphism $\mathrm{Cut} : \Delta^{op} \to \mathrm{Fin}_*$ where we interpret $[n] \in \mathrm{Fin}_*$ as the set of cut in $[n]$, and

$$\mathrm{Cut}(\alpha)(i) = \begin{cases} j & \text{if there are } j \text{ s.t. } \alpha(j-1) < i \le \alpha(j) \\ 0 & \text{otherwise} \end{cases}$$

Prop. (VI.1.2.3). The category of functors from the $E_\infty = \mathrm{Fin}_*$ to $\mathcal{C}at$ that

$$X([n]) \xrightarrow{\prod_{i=2}^n X(\rho^i)} \prod_{i=1}^n X([1]) \quad n \ge 0$$

and $X([0])$ is the final object, is equivalent to the category of symmetric unital monoidal categories with base category $(X([1]))$. (Because the commutativity of morphisms encodes the fact that the tensor action is symmetric).

Similarly, the category of functors from the Δ^{op} to Cat that

$$X([n]) \xrightarrow{\prod_{i=2}^{n} X(\rho^i)} \prod_{i=1}^{n} X([1]) \quad n \geq 0$$

is equivalent to the category of symmetric unital monoidal categories $(X([1]))$. And it is symmetric iff it factors through Cut:$\Delta^{op} \to Fin_*$.

Def. (VI.1.2.4). The **Conne cyclic category** Δ_C is a category containing Δ that $\mathrm{Aut}_{\Delta C}([n])$ is C_{n+1}. And every morphism $[n] \to [m]$ in Δ_C can be uniquely written as the form φg, where $\varphi \in \mathrm{Hom}_{\Delta}([n],[m])$ and $g \in \mathrm{Aut}_{\Delta_C}([n])$.

Δ_C^{op} is isomorphic to Δ_C Cf.[杨恩林循环同调 P31], thus Δ and Δ^{op} are all subcategories of Δ_C.

Def. (VI.1.2.5). The category Δ_S is the category that $\mathrm{Aut}_{\Delta_S}([n]) \cong S^n$ and every morphism $[n] \to [m]$ in Δ_S can be uniquely written as the form φg, where $\varphi \in \mathrm{Hom}_{\Delta}([n],[m])$ and $g \in \mathrm{Aut}_{\Delta_S}([n])$.

Def. (VI.1.2.6). For a category C, a **cyclic object** in C is a functor $\Delta_C^{op} \to C$.

For example, the functor that maps $[n]$ to C_{n+1} and the functor maps to the pull back of the order of the cyclic, is a cyclic object.

Simplicial Homology

Def. (VI.1.2.7) (Moore Complex). Giving a simplicial object in an Abelian category, we can have a **Moore chain complex** with Čech-like differenticals. $\partial_n = \sum_1^n (-1)^i d_i$. And we have $\partial^2 = 0$.

Proof: Should use $d_i d_j = d_{j-1} d_i$ for $i < j$. □

Def. (VI.1.2.8). The **normalization** of a Simplicial Abelian group M is the chain complex

$$NM : \cdots \to NM_n \xrightarrow{(-1)^n d_n} NM_{n-1} \to \cdots$$

where $NM_n = \bigcap_{i=0}^{n-1} \mathrm{Ker}(d_i) \in M_n$. This is a chain complex because $d_{n-1}d_n = d_{n-1}d_{n-1}$ is 0 on NM_n. In fact NM is preserved by all injections.

The **degenerate complex** of a Moore complex DM is the chain complex that $D_n = \sum_{i=0}^{n-1} s_i M_{n-1}$ is a sub chain complex of M by the relation of d_i, s_j.

Prop. (VI.1.2.9). The simplicial homology of the Moore complex of the bar resolution BG of group homology with coeeficient in R is just the group homology $H_n(G,R)$ for the trivial module R. And it has the same homology with the geometrization $|BG|$.

Lemma (VI.1.2.10). $A_* \cong NA_* \oplus DA_*$ as a complex, $NA_*, A_*, (A/DA_*)$ are all homotopically equivalent.

Proof: We define similarly $N_k A_*$ and $D_k A_*$ and induct on k, our conclusion is the case $k = n-1$. When $k = 0$, $\mathrm{Im}\, d_0 \oplus \mathrm{Ker}\, s_0 A_n = A_n$ because $d_0 s_0 = id_{n-1}$ thus $A_{n-1} \xrightarrow{s_0} A_n$ is a split injection.

There are two split exact rows by simplicial relations:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{k-1}A_{n-1} & \xrightarrow{s_k} & N_{k-1}A_n & \xrightarrow{1-s_kd_k} & N_kA_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_{n-1}/D_{k-1}A_{n-1} & \xrightarrow{s_k} & A_n/D_{k-1}A_n & \longrightarrow & A_n/D_kA_n & \longrightarrow & 0 \end{array}$$

The first one split because it has a right section, the second one split because it has a left section. So by induction, $N_kA_n \to A_n/D_kA_n$ is an isomorphism, thus $N_kA_n \oplus D_kA_n = A_n$ because it splits.

For the homotopy equivalence, Cf.[Jardine P150]. □

Prop. (VI.1.2.11) (Dold-Kan Correspondence). The normalized Moore complex NA_* gives an equivalence between

$$\text{simplicial Abelian group} \cong \text{chain complex of Abelian groups.}$$

Proof: We define a functor that maps a chain complex to a simplicial Abelian group as follows: $\sigma(C)_n = \bigoplus_{[n]\to[k]\text{surjects}} C_k$, and a morphism $\sigma_n \to \sigma_m$ for a morphism $[m] \to [n]$ is defined as follows: For $[n] \to [k]$ surjects, write $[m] \to [n] \to [k]$ as $[m] \to [r] \xrightarrow{\psi} [k]$ where $[m] \to [r]$ surjects and $[r] \to [k]$ injects, thus maps $a \in C_k$ in σC_n to $\psi^*(a) \in C_r$ in σC_m, where ψ^* is zero unless $\psi = d^n : \Delta[n-1] \to \Delta[n]$. This is natural and defines a simplicial Abelian group because of the unicity of the the canonical decomposition. There is a natural map from $\sigma(N\mathcal{A})$ to $\mathcal{A}$.

Now the task is to show that $\sigma(NA) \cong A$ and $N(\sigma C) \cong C$. We has $N(\sigma C)_n = C_n$ because d^iC_n is 0 for $i \neq n$ and the other components are all degeneracies thus are not in $N(\sigma C)_n = C_n$ by(VI.1.2.10).

Then we prove $\sigma(NA) \cong A$. It is a surjection by(VI.1.2.10) and induction. For the injectivity, if $(a_\varphi) \neq 0$ is mapped to 0, a_{id_n} is 0 by(VI.1.2.10). And we choose an ordering on the $\varphi : [n] \to [k]$ by dominating, and suppose ψ is a minimal one. Now choose a section ξ of ψ that ξ is the maximal section, thus $\varphi\xi$ cannot by id_k for any other φ. Now by induction we have $a_\psi = 0$, contradiction. □

Prop. (VI.1.2.12). There is a functor form a R-algebra S to a trivial simplicial R-algebra $s(S)$, it is a fully faithful embedding and π_0 is left adjoint to it. (The action of A_n on $s(S)_n$ is $A_n \to A_{n-1} \to \cdots \to A_0 \to S$).

Hochschild Homology

Def. (VI.1.2.13). For a R-algebra A and a (A, A)-bimodule L, there is a simplicial module $C(A, L)$ called the **Hochschild complex** of A with coefficient in M, with $M_n = L \otimes A^n$ that

$$d_i(m, a_1, \ldots, a_n) = \begin{cases} (m_0a_1, a_2, \ldots, a_n) & i = 0 \\ (a_nm_0, a_1, \ldots, a_{n-1}) & i = n \\ (m_0, a_1, \ldots, a_ia_{i+1}, \ldots, a_n) & \text{otherwise} \end{cases}$$

$$s_j(m, a_1, \ldots, a_n) = (m, a_1, \ldots, a_{j-i}, 1, a_{j+1}, \ldots, a_n)$$

When $L = A$, this is even a cyclic module, denoted by $C(A, A)$.

Def. (VI.1.2.14). The homology group of the Moore complex associated to the Hochschild complex is called the **Hochschild homology** $H_n(A, M)$. And we denote the homology of $C(A, A)$ as $HH_*(A)$. $H_n(A, M)$ is a $Z(A)$ module by the action of $Z(A)$ on M and HH_* defines a functor $\mathcal{A}lg_R \to {}_R\mathrm{Mod}$.

Prop. (VI.1.2.15). For a commutative ring R and a symmetric R-bimodule M, there is a spectral sequence

$$E^2_{pq} = \mathrm{Tor}^R_p(H_p(A,A),M) \Rightarrow H_{p+q}(A,M).$$

Cor. (VI.1.2.16). For A commutative and a symmetric (A,A)-module M,$HH_0(A,A) = A^{ab}$ and $HH_1(A,A) \cong \Omega^1_{A/R}$ giving by $a \otimes x \mapsto adx$ by direct calculation. Thus we have $H_1(A,M) = M \otimes_A A^{ab}$ and $H_1(A,M) = M \otimes_A \Omega^1_{A/R}$. And if M is flat, $H_n(A,M) = M \otimes_A H_n(A,A)$.

Prop. (VI.1.2.17) (Hochschild-Kostant-Rosenberg). The isomorphism $\Omega^1_{A/R} \cong HH_1(A)$ extends to a graded ring map

$$\Psi : \Omega^*_{A/k} \to H_*(A,A)$$

. If A/R be smooth algebra and R Noetherian, then Ψ is an isomorphism of graded algebra. Cf.[Weibel P322], [阳恩林循环同调 P133].

Def. (VI.1.2.18) (Tsygan's Double Complex). For a cyclic objec M in an Abelian category, let t_* be the cyclic morphism and $\partial_n = \sum_{i=0}^n (-1)^i d_i$, $\partial'_n = \sum_{i=0}^{n-1}(-1)^i d_i$, $N_n = \sum_{k=0}^n ((-1)^n t_n)^k$, then there is a double complex $CC(M)$:

$$\begin{tikzcd}
{} \arrow[d, "\partial"] & {} \arrow[d, "-\partial'"] & {} \arrow[d, "\partial"] & \\
M_1 \arrow[d, "\partial"] & M_1 \arrow[l, "1-(-1)^1 t"'] \arrow[d, "-\partial'"] & M_1 \arrow[l, "N"'] \arrow[d, "\partial"] & \arrow[l, "1-(-1)^1 t"'] \\
M_0 & M_0 \arrow[l, "1-(-1)^0 t"'] & M_0 \arrow[l, "N"'] & \arrow[l, "1-(-1)^0 t"']
\end{tikzcd}$$

That the column are 2-cyclic. Cf.[Weibel P337]. The first column is called the **Hochschild complex of** M: $C^h(M)$, the second column is called **acyclic complex of** M(VI.1.2.19) $C^a(M)$. And we can even augment a cokernel column on the left, which is the complex of M modulo the cyclic action, called the **Conne complex** $C^\lambda(M)$.

We define the **Cyclic Homotopy Group** $HC_n(M) = H_n(\mathrm{Tot}CC(M))$ and when M is the cyclic module $C(A)$(VI.1.2.13), denote $CC(C(A)) = CC(A)$, $HC_n(A) = HC_n(C(A))$.

Lemma (VI.1.2.19). The second column is exact and $h = t_{n+1}s_n$ is a null-homotopy. Cf.[阳恩林循环同调 P122].

Lemma (VI.1.2.20). Notice the rows are in fact a group homology $\mathrm{Hom}(\mathbb{Z}/(n+1)\mathbb{Z}, M_n)$, thus when $\mathbb{Q} \in R$, we have the rows are acyclic because the group homology is killed by $|G|$**??**, thus $HC_*(M) \cong H^\lambda_*(M)$ are isomorphisms by spectral sequence.

Prop. (VI.1.2.21) (Conne SBI Sequence). For a cyclic module M, there is a long exact sequence

$$\cdots \to HH_n(M) \xrightarrow{I} HC_n(M) \xrightarrow{S} HC_{n-2}(M) \xrightarrow{B} HH_{n-1}(M) \to \cdots$$

Proof: shift the diagram 2 column right, then there is an exact sequence of double complexes and notice the second column is exact(VI.1.2.19), thus we have the kernel is quasi-isomorphic to $C^h(M)$. So the sequence follows. □

Cor. (VI.1.2.22). $HC_0(A) = HH_0(A) = A^{ab}$.

When A is commutative, $HC_1(A) = \mathrm{Coker}(HC_0(A) \xrightarrow{B} HH_1(A)) = \Omega^1_{A/R}/dA$ as a R module, because we can verify that $B(a) = a \otimes 1 - 1 \otimes A$.

Cor. (VI.1.2.23). For a morphism of two cyclic objects, $HH_*(M)\cong HH_*(M')$ iff $HC_n(M)\cong HC_n(M')$. (Use five lemma).

Def. (VI.1.2.24). A **mixed complex** (M,b,B) is a complex with $b:M_n\to M_{n-1}$ and $B:M_n\to M_{n+1}$ that makes M into a double chain complex. And there is a **Conne double complex** associated with this mixed complex. And similarly there is a same SBI sequence associated to the following diagram:

$$\begin{tikzcd}
\downarrow & \downarrow & \downarrow \\
M_2 \arrow[d, "b"] & M_1 \arrow[l, "B"'] \arrow[d, "b"] & M_0 \arrow[l, "B"'] \\
C_1 \arrow[d, "b"] & C_0 \arrow[l, "B"'] & \\
C_0 & &
\end{tikzcd}$$

From a cyclic object M, we notice that the $2k$-th column is acyclic(VI.1.2.19), thus there is a snake-like connection homomorphism B that makes M into a mixed complex BM. Cf.[Weibel P344]. And the Conne double complex will compute the same cyclic homology with previous defined cyclic homology, Cf.[Weible P345].

Notice for this B, B_* on homology is exactly the composition BI.

Prop. (VI.1.2.25). Let R be a unital commutative ring and A is a commutative R-algebra and M is a A-module, then there is a natural morphism

$$M\otimes_A\Omega^n_{A/R}\xrightarrow{\varepsilon_n}H_n(A,M)\xrightarrow{\pi_n}M\otimes_A\Omega^n_{A/R}.$$

such that $\pi_n\circ\varepsilon_n=n!$.

We first define a map $\varepsilon_n:M\otimes\wedge^nA\to H_n(A,M)$ that

$$\varepsilon_n(m,a_1,\dots,a_n)=\sum\operatorname{sgn}(\sigma)(m,a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(n)})$$

then define $\varepsilon_n(m\otimes xda_1\wedge\dots\wedge da_n)=\varepsilon_n(mx,a_1,\dots,a_n)$. And we verify that this map is well-defined and maps into $Z_n(C(A,M))$,Cf.[阳恩林循环同调 P99].

Then we define $\pi_n(m,a_1,\dots,a_n)=m\otimes da_1\wedge\dots\wedge da_n$ and verify easily that this vanish on $B_n(C(A,M))$. And it is easy to verify $\pi_n\circ\varepsilon_n=n!$.

Prop. (VI.1.2.26). When A is a unital R-algebra, there is a commutative diagram

$$\begin{tikzcd}
\Omega^n_{A/R} \arrow[r, shift left, "(n+1)d"] \arrow[r, shift right, "d"'] \arrow[d, shift right, "\pi_n"'] & \Omega^{n+1}_{A/R} \arrow[d, shift right, "\pi_{n+1}"'] \\
HH_n(A) \arrow[u, shift right, "\varepsilon_n"'] \arrow[r, "B_*"] & HH_{n+1}(A) \arrow[u, shift right, "\varepsilon_{n+1}"']
\end{tikzcd}$$

Proof: We notice $B=(1-(-1)^nt)sN$:

$$(m,a_1,\dots,a_n)\mapsto\sum_{i=0}^n(-1)^{in}(1,a_i,\dots,a_n,m,a_1,\dots,a_{i-1})-\sum_{i=0}^n(-1)^{in}(a_i,1,a_{i+1},\dots,a_n,m,a_1,\dots,a_{i-1}).$$

Cf.[阳恩林循环同调 P128]. □

Cor. (VI.1.2.27). For a commutative unital R-algebra A, there is a functorial $\varepsilon_n : \Omega^n_{A/R}/d\Omega^{n-1}_{A/R} \to HC_n(A)$ making the following diagram commutative:

$$\begin{tikzcd}
{} \arrow[r, "0"] & \Omega^{n-1}/d\Omega^{n-2} \arrow[r, "d"] \arrow[d, "\varepsilon_{n-1}"] & \Omega^n \arrow[r] \arrow[d, "\varepsilon_n"] & \Omega^n/d\Omega^{n-1} \arrow[r, "0"] \arrow[d, "\varepsilon_n"] & \Omega^{n-2}/d\Omega^{n-3} \arrow[r] \arrow[d, "\varepsilon_{n-2}"] & \cdots \\
{} \arrow[r] & HC_{n-1} \arrow[r, "B"] & HH_n \arrow[r, "I"] & HC_n \arrow[r, "S"] & HC_{n-2} \arrow[r, "B"] & \cdots
\end{tikzcd}$$

which is induced by the cokernel. Cf.[阳恩林循环同调 P130]. When $\mathbb{Q} \in R$, ε_n is a split injection.

Prop. (VI.1.2.28). When $\mathbb{Q} \in R$, $\frac{1}{n!}\pi_n$ induces a morphism of mixed complexes $(BA, \partial, B) \to (\Omega^*_{A/R}, 0, d)$ by(VI.1.2.25), thus there is a natural map

$$HC_n(A) \to \Omega^n_{A/R}/d\Omega^{n-1}_{A/R} \bigoplus_{i>0} H^{n-2i}_{dR}(A).$$

Prop. (VI.1.2.29) (Morita Invariance). $Tr : HH_*(M_r(A), M_r(M)) \cong HH_*(A, M)$ by the trace and inclusion functors. Cf.[阳恩林循环同调 Morita Invariance]. In particular, there is an isomorphism $HH_*(M_r(A)) \cong HH_*(A)$, thus also $HC_*(M_r(A)) \cong HC_*(A)$ by(VI.1.2.21).

Prop. (VI.1.2.30) (Karoubi). BG is a cyclic group, and then the cyclic homology group $HC_n(G, A) \cong \bigoplus_{k\geq 0} H_{n-2k}(G, A)$. Cf.[Weibel P339].

Simplicial Homotopy

Prop. (VI.1.2.31). For a Kan fibration X, there can be defined a homotopy groups π_n that they agree with $\pi_i(|X|)$ thus also $\pi_i(S|X|)$, Cf.[Weibel P263]. Thus we see that $|BG|$ is truly the Eilenberg-Maclane spaces BG.

3 Model Category

References are [Simplicial Homotopy Theory Jardine] and [Model Category and SImplicial Methods Goerss].

Def. (VI.1.3.1). A **model category** is a category C with three classes of morphisms:fibrations, cofibrations and weak equivalences that satisfy the following axioms.

- M0: C is closed under finite limits and colimits.
- M1: We have a lifting property with a cofibration i and fibration p when either of them is a weak equivalence.
- M2: Any map f can be factored as pi where i is cofibration and p is a fibration and assure any of them be a weak equivalence, i.e. trivial (co)fibration.
- M3: Fibration is stable under composition, base change and isomorphism is a fibration. Dually for cofibrations.
- M4: The base change of a trivial fibration is a weak equivalence. Dually for cofibration.
- M5: If two of f, g, fg is weak equivalence, then so is the third.

The definition is dual, i.e., if A is a model category, then so is A^{op}.

It is called a **closed model category** iff moreover it satiesfies

- M6: (co)fibration, weak equivalence is closed under retract.

It is called **simplicial model category** iff all $\mathrm{Hom}(X,Y)$ are simplicial sets and it satisfies:

- SM7: If $i : U \to V \in Cof$ and $p : X \to Y \in F$, then the induced map
$$\mathrm{Hom}(V,X) \xrightarrow{(i^*,p_*)} \mathrm{Hom}(U,X) \times_{\mathrm{Hom}(U,Y)} \mathrm{Hom}(V,Y)$$
is a fibration, and trivial iff any of i,p is trivial.

Prop. (VI.1.3.2). A model category satisfies M6 iff:
fibration$= r$(trivial cofibrations),
cofibration$= l$(trivial fibrations),
weak equivalence$= uv$, where $v \in l(F)$ and $u \in r(Cof)$.

Proof: If these are satisfied, M6 is easy: a retract of an isomorphism is an isomorphism, so $\gamma(f)$ is an isomorphism and(VI.1.3.3) shows $f \in r(Cof)$ thus a weak-equivalence.

Conversely, notice for a diagram $\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ \downarrow_i & \overset{s}{\nearrow} & \downarrow_p \\ Z & \xrightarrow{u} & Y \end{array}$ induce p as a retraction of u. straightforward for (co)fibrations and for $f = uv$, the same diagram proves u, v are all weak-equivalences. □

Prop. (VI.1.3.3). Let p be a fibration in C_{cf}, then $p \in r(Cof)$ iff $\gamma(p)$ is an isomorphism, Cf.[Quillen 5.2]. So if conditions of(VI.1.3.2) are satisfied (i.e. C is a closed model category), $\gamma(f)$ is an isomorphism iff f is a weak equivalence by the charaterization of weak-equivalence of(VI.1.3.2).

Prop. (VI.1.3.4). A **cylinder object** for an object A is a C with $X \coprod X \xrightarrow{i} C \xrightarrow{j} X$, where $i \in Cof$ and $j \in W$ and ji is the codiagonal map.

Dually, a **path object** for Y is a P with $Y \xrightarrow{q} P \xrightarrow{p} Y \times Y$ where $q \in W$ and $p \in F$ and pq is the diagonal map. They are named because $C = A \times I$ and $P = Y^I$ is the prototype and we will write this way often.

Two morphisms $f, g : X \to Y$ are called **left homotopic** iff there is a cylinder object $X \coprod X \to X \times I$ with $X \times I \to Y$ that induce $(f,g) : X \coprod X \to Y$. Dually for right homotopic.

Lemma (VI.1.3.5). If A is Cof and $A \times I$ is a cylinder object for A, then $\partial_i : A \to A \times I$ are trivial fibrations. (Because it's pushout of Cof and $\sigma \circ \partial_i = \mathrm{id}_A$).

Cor. (VI.1.3.6) (Covering Homotopy Theorem). If A is Cof and, then $\partial_i : A \to A \times I$ has left lifting property w.r.t. all fibrations.

Cor. (VI.1.3.7) (Homotopy Extension Theorem). If B is fibrant, then $\sigma_i : B^I \to B$ has right lifting property w.r.t. all cofibrations.

Prop. (VI.1.3.8). If A is Cof, the left homotopy is an equivalence relation on $\mathrm{Hom}(A,B)$.

Proof: For this, the only problem is transitivity, so we construct a glueing A'' as the pushout of $\partial_1 : A \to A \times I$ and $\partial_0' : A \to A \times I'$. $A'' \to A$ is a weak equivalence by M4,M5 and(VI.1.3.5). $A \coprod A \to A''$ is a Cof because it is composition of two pushouts. So it is a cylinder object. □

Path object and cylinder object exists by M2.

Prop. (VI.1.3.9). If A is Cof and $f, g \in \mathrm{Hom}(A,B)$, then

1. f, g are left homotopic iff they are right homotopic.
2. If f, g are right homotopic, then $s \to B^I$ can be chosen to be trivial Cof.
3. If f, g are right homotopic, then so does $uf \sim ug$ or $fv \sim gv$. Thus if A is Cof, hence there is a map: $\pi^r(A, B) \times \pi^r(B, C) \to \pi^r(A, C)$.
4. For a $X \to Y \in TF$, $\pi^l(A, X) \to \pi^l(A, Y)$ is a bijection.

And dual argument hold for B fibrant.

Proof:

1. Cf.[Quillen Homotopical Algebra 1.8].
2. factorize $B \to B^I$ to $B \to B^{I'} \to B^I$ where $B \to B^{I'} \in TCof$ and $B^{I'} to B^I \in W$, so $B^{I'}$ is also a cylinder object and the homotopy $A \to B^I$ can be lifted to $A \to B^{I'}$.
3. there is a diagram $\begin{array}{ccc} B & \xrightarrow{su} & C^I \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle (d_0,d_1)} \\ B^I & \xrightarrow{(d_0u,d_1u)} & C \times C \end{array}$ which has a lifting φ, then composed with $A \to B^I$ will give the desired homotopy.
4. the map is well-defined, it is surjective because of lifting property, and it is injective because $A \coprod A \to A \times I \in Cof$ so the homotopy can be lifted to X.

$\square$

Def. (VI.1.3.10). Let C_c, C_f, C_{cf} denote the full subcategory of cofibrant, fibrant and cofibrant-fibrant objects. And we define πC_c as the category module right homotopy equivalence between morphisms, dually for πC_f. Notice for C_{cf}, left homotopy is equivalent to right homotopy by(VI.1.3.9), so πC_{cf} is full subcategory for both πC_c and πC_f.

Def. (VI.1.3.11). The **localization** of a category is defined as usual, and the **homotopy category** hC for a model category C is the localization of C w.r.t. to the class of weak equivalences.

Lemma (VI.1.3.12). A functor $C \to B$ that maps weak equivalence to isomorphisms will map all left homotopic or right homotopic morphisms to the same morphism (look at the definition of cylinder object). Thus it induces a functor $\gamma : \pi C_{cf} \to hC$, and similarly $\gamma_f : \pi C_f \to hC_f$ and $\gamma_c : \pi C_c \to hC_c$.

Prop. (VI.1.3.13). $\pi C_{cf} \cong hC \cong hC_c \cong hC_f$. So hC_c injects into πC_c and is right adjoint to γ_c. hC_f injects into πC_f and is left adjoint to γ_c, Cf.[Quillen Homotopical Algebra 1.13].

Examples

Prop. (VI.1.3.14) (Serre-Quillen). The category $\mathcal{T}op$ is a closed model category with Serre fibrations, weak homotopy equivalence and cofibrations defined as the left lifting class of trivial fibrations.

Cor. (VI.1.3.15). $\partial D^n \to D^n$ is an cofibration, hence all inclusion of CW complexes are cofibration. All topologocal space are fibrant.

Proof: Use mapping cylinder, we can regard it as an injection and then use compression lemma.
?

$\square$

Cor. (VI.1.3.16). Every map can be decomposed as a homotopy equivalence followed by a fibration, by the construction of homotopy fibers. Cf.[Hatcher P407].

Prop. (VI.1.3.17) (Derived Category Model). If $\mathcal{A}$ is an Abelian category with enough injectives, then $K^+(\mathcal{A})$ is a closed model category with Fibration= epimorphisms with Ker in $K^+(\mathcal{I})$, cofibration=monomorphisms, weak equivalence=quasi-isomorphisms.

Prop. (VI.1.3.18). The category C of semi-simplicial sets is a closed model category with fibrations=Kan fibrations, cofibration= injective maps, weak equivalence= maps which induce homotopy equivalence on geometrizations.

Prop. (VI.1.3.19) (Kan Model). The category of Simplicial sets $\mathcal{S}et_\Delta$ is a model category with cofibrations=monomorphisms and fibrations=Kan fibrations, weak equivalence= which induce homotopy equivalence of their geometrizations.

Proof: Cf.[Jardine P62]. □

Prop. (VI.1.3.20). The singular functor and the geometrization functor defines an equivalence of categories between $h(\mathcal{S}et_\Delta)$ and $h(\mathcal{T}op)$. Cf.[Jardine P63].

Prop. (VI.1.3.21) (Joyal). The

VI.2 André-Quillen Cohomology

Basic references are [Andre-Quillen Cohomology of Commutative Algebras Iyenger]. See also [Quillen Cohomology of Commutative Rings] and [Quillen On the (Co-)homology of Commutative Rings].

1 Kahler Differentials

Def. (VI.2.1.1) (Kähler Differential). Let $S \to R$ a ring map, $\mathrm{Der}_S(R, M)$ is defined as the set of S-mod maps $R \to M$ that satisfies Leibniz rule and vanish on S. Then the **Kahler Differential** $\Omega_{R/S}$ is defined as a R-module that $\mathrm{Der}_S(R, M) \cong \mathrm{Hom}_S(\Omega_{R/S}, M)$. In particular, $\mathrm{Der}_S(R, R)$ is the R-dual of $\Omega_{R/S}$.

Prop. (VI.2.1.2). One construction is by the free group generated by elements of R module some relations.

It can also be constructed as follows: there are two ring maps λ_i from S to $R \otimes_R S$, and one map ε from $S \otimes_R S$ to S. Let $I = \mathrm{Ker}\,\varepsilon$ as a R module by λ_1, then $I/I^2 \cong \Omega_{S/R}$ by (VI.2.1.6) that

$$0 \to I/I^2 \to \Omega_{S\otimes_R S/S} \otimes_{S\otimes_R S} S \to \Omega_{S/S} \to 0.$$

So $I/I^2 \cong \Omega_{S/R} \otimes_S (S \otimes_R S) \otimes_{S\otimes_R S} S \cong \Omega_{S/R}$. And it can be verified that $a \otimes 1 - 1 \otimes a$ corresponds to da.

Prop. (VI.2.1.3) (Adjointness). Let RngMod be the category of ring morphisms ($A \to B$) and RngMod be the category of modules over rings ($R \to M$), then the Kähler differential is a morphism from RngMor to RngMod, and there is a right adjoint to it: for a module over a ring $B \to M$, let $D_B(M) = B \oplus M$ with the ring structure $(b, m)(b', m') = (bb', bm' + b'm)$, then it is a B-ring, and right adjoint to $\Omega_{-/-}$.

Proof: Cf.[Perfectoid Spaces Masullo P113]. □

Cor. (VI.2.1.4) (Functoriality). From the first construction, we can see directly that for a family of morphisms $R_i \to S_i$,

$$\Omega_{\mathrm{colim}\, S_i/\,\mathrm{colim}\, R_i} = \mathrm{colim}\, \Omega_{S_i/R_i}.$$

In particular, we have:

$$T^{-1}\Omega_{B/A} = \Omega_{T^{-1}B/A}, \quad \Omega_{S^{-1}B/S^{-1}A} = S^{-1}\Omega_{B/A}.$$

Moreover, we have

$$\Omega_{S/R} \otimes_R R' = \Omega_{S\otimes_R R'/R'}, \quad (S \otimes_R \Omega_{T/R}) \oplus (T \otimes_R \Omega_{S/R}) \cong \Omega_{S\otimes_R T/R}$$

by universal property.

Prop. (VI.2.1.5) (Jacobi-Zariski Sequences). For a sequence of commutative rings:$A \to B \to C$, there is an exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules. It has a left inverse and splits iff any derivation B/A to a C-module can functorially be extended to a C/A derivation. This is true when C/B is smooth(I.7.4.7).

Proof: Taking Hom with an arbitrary C-module M, by universal property, we need to check the exactness of $0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$, which is easy. □

Prop. (VI.2.1.6) (Second Exact Sequence). (This is a special case of(VI.2.2.5)). If $S' = S/I$, then there is an exact sequence of R'-modules:

$$I/I^2 \to \Omega_{S/R} \otimes_S S' \to \Omega_{S'/R} \to 0.$$

Where $f \in I$ is mapped to $df \otimes 1$ and it has a left inverse and splits iff $S/I^2 \to S'$ has a right inverse.

And in fact $\Omega_{S/R} \otimes S' \cong \Omega_{(S/I^2)/R} \otimes S'$.

Proof: For a S/I-module M, we check:

$$0 \to \mathrm{Der}_R(S/I, M) \to \mathrm{Der}_R(S, M) \to \mathrm{Hom}_{S/I}(I/I^2, M)$$

To prove $\Omega_{S/R} \otimes S' \cong \Omega_{(S/I^2)/R} \otimes S'$, we apply Hom for a S'-module M.

So to prove the left exactness, we may assume $I^2 = 0$. If we have an inverse $\Omega_{S/R} \otimes_S S' \to I$. then it gives a derivation $D : A \to I$ that is identity on I, so $a - D(a)$ gives a R-ring map $S \to S$ that is trivial on I(because $I^2 = 0$). Hence it gives a $S/I \to S$ that is inverse to the projection.

For the converse, if $d : S/I \to S$ is a right inverse, then $a - d(\overline{a})$ is a derivation $S \to I$, which is identity on I, so it gives a inverse map $\Omega_{S/R} \otimes_S S' \to I$ by universal property. □

Cor. (VI.2.1.7). If $R \to S$ is of f.p., then $\Omega_{S/R}$ is of f.p. over S. If $R \to S$ is of f.t., then $\Omega_{S/R}$ is of f.t. over S. (Follows from the second exact sequence(VI.2.1.6) and(VI.2.1.8)).

Cor. (VI.2.1.8) (Examples).

- $\Omega_{A[X_1,\ldots,X_n]/A} = A[X_1, \ldots, X_n]\{dX_1, \ldots, dX_n\}$ (use the differential operator and universal property).
- If $S = A[X_i]/\{f_j\}$, then $\Omega_{S/A} = S[dX_i]/\{df_j\}$ by exact sequence2.
- $\Omega_{A[X_i]/k} = \Omega_{A/k} \otimes_A A[X_i] \oplus A[X_i]\{dX_1 \ldots, dX_n\}$ because any derivative of A/k can be extended to derivative of B/k by acting on the coefficients.
- (Standard Étale Algebra)For $A = R[x]_g/(f)$, where f' has image invertible in $A, \Omega_{A/R} = 0$.
- The differential for the inclusion $k[y^2, y^3] \to k[y]$ is $k[y]/(2y, 3y^2)\{dy\}$.

Cor. (VI.2.1.9). If S/I is a field k that embeds in S, then $I/I^2 \cong \Omega_{S/k} \otimes_S k$.

Prop. (VI.2.1.10). Let $k \subset K \subset L$ be fields, and L/K f.g., then

$$\dim_L \Omega_{L/k} \geq \dim_K \Omega_{K/k} + tr.\deg(L/K).$$

Equality holds if L/K is separably generated, i.e. separable over a transcendental basis. If $K = k$, then the equality hold iff L/k is separably generated. In particular, when L/k separable field extension, $\Omega_{L/k} = 0$, e.g. when k is perfect.

Proof: Consider extension by one element at a time, Cf.[Matsumura P190]. □

Prop. (VI.2.1.11). Let B be a Noetherian local ring containing its residue field k and k is perfect, then $\Omega_{B/k}$ is a free B-module of rank $\dim B$ iff B is regular. [Hartshorne Ex.2.8.1] has a generalization of this fact.

Proof: One way is by(VI.2.1.9). Conversely, if B is regular, then it is integral(I.6.5.12), so $\Omega_{B/k} \otimes_B K = \Omega_{K/k}$(VI.2.1.4) is of K-dimension $tr.\deg K/k = \dim B$, where K is the quotient field of B, and $\Omega_{B/k} \otimes k \cong m/m^2$ is of k-dimension $\dim B$ once again. These two facts shows that $\Omega_{B/k}$ is free B-module of rank $\dim B$ by(I.7.11.1). □

2 Naive Cotangent Complex

Def. (VI.2.2.1). The **naive cotangent complex** of a ring map $R \to S$ is defined as the complex $NL_{S/R} = (I/I^2 \to \Omega_{R[S]/R} \otimes_{R[S]} S)$ as in(VI.2.1.5), where $I = \mathrm{Ker}(R[S] \to S)$, I/I^2 is in degree -1 and $\Omega_{R[S]/R} \otimes_{R[S]} S$ in degree 0.

So it has homology $H^0 = \Omega_{S/R}$.

The naive cotangent complex is the canonical truncation of cotangent complex at degree 1.

For a ring map $R \to S$, if we choose another presentation $\alpha : P \to S \to 0$ where P is a polynomial algebra over R, then we denote $NL(\alpha) = NL_{P/S}$.

Prop. (VI.2.2.2). For a morphism of ring morphisms $(R \to S) \to (R' \to S')$, if there is a morphism of presentations, then we get by functoriality(VI.2.1.4) a S-module morphism $\Omega_{P/R} \otimes_P S \to \Omega_{P'/R'} \otimes_{P'} S'$ and also a map $I/I^2 \to (I')/(I')^2$ for the kernel, so we get a morphism $NL(\alpha) \to NL(\alpha')$. The morphism constructed is compatible with composition.

In particular, we get a morphism $NL_{S/R} \to NL_{S'/R'}$.

Prop. (VI.2.2.3) (Polynomial Replacement). For a morphism of ring morphisms $(R \to S) \to (R' \to S')$, let α, α' be two presentations, then there exists morphism of presentations, and different morphisms induce homotopic maps $NL_{S/R} \to NL_{S'/R'}$.

Proof: Cf.[StackProject 00S1]. In fact, any surjective formally smooth representation will give the naive cotangent complex, up to quasi-isomorphism**??**. $\square$

Cor. (VI.2.2.4). If $A = R[X_i]$ be a polynomial algebras, then $NL_{A/R}$ is homotopic to $(0 \to \Omega_{B/A})$ because $A \to A$ is a presentation with zero kernel.

If $R \to A$ is surjective with kernel I, then $NL_{A/R}$ is homotopic to $(I/I^2 \to 0)$.

Prop. (VI.2.2.5) (Jacobi-Zariski Sequence). Let $A \to B \to C$ be a ring map. Choose a presentation $\alpha : P \to B$ for B/A with kernel I, a presentation $\beta : Q \to C$ for C/B with kernel J, a presentation $\gamma : R \to C$ for the induced representation C/A with kernel K, then there is an exact sequence of complexes:

$$\begin{tikzcd}
 & I/I^2 \otimes_B C \arrow[r] \arrow[d] & K/K^2 \arrow[r] \arrow[d] & J/J^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{P/A} \otimes_B C \arrow[r] & \Omega_{R/A} \otimes C \arrow[r] & \Omega_{Q/B} \otimes C \arrow[r] & 0
\end{tikzcd}$$

Applying snake lemma, we get

$$H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

Proof: Cf.[StackProject 00S2]. $\square$

Prop. (VI.2.2.6) (Flat Base Change). For a $R \to S$ and a flat R-ring R', for a presentation α of S/R, tensoring R' gives a presentation α' of S'. Then $NL(\alpha) \otimes_R R' = NL(\alpha) \otimes_S S' = NL(\alpha')$. This is because flatness implies kernel commutes with tensoring. In particular, $NL_{S/R} \otimes_R R' \to NL_{S'/R'}$ is a homotopy equivalence.

Prop. (VI.2.2.7) (Colimit). NL commutes with colimit. (Because the kernel commutes with colimits).

Cor. (VI.2.2.8) (Localization). Let $A \to B$ be a ring map, for a multiplicative set S of B, we have $NL_{B/A} \otimes_B S^{-1}B$ is quasi-isomorphic to $NL_{S^{-1}B/A}$.

Proof: Because it commutes with colimit, it suffice to prove for $S = f$, and this is the content of lemma(VI.2.2.9) below. □

Lemma (VI.2.2.9). If $A \to B$ is a ring map and $\alpha : P \to B$ is a presentation of B with kernel I, then $\beta : P[X] \to B_g : X \to 1/g$ is a presentation of B_g with kernel $J = I + (gX - 1)$. Then we have

- $J/J^2 = (I/I^2)_g \oplus B_g(fX - 1)$.
- $\Omega_{P[X]/A} \otimes_{P[X]} B_g = \Omega_{P/A} \otimes_P B_g \oplus B_g dX$.
- $NL(\beta) \cong NL(\alpha) \otimes_B B_g \oplus (B_g \xrightarrow{g} B_g)$.

Hence $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ is a homotopy equivalence.

Proof: Cf.[StackProject 08JZ]. □

3 Cotangent Complex

Def. (VI.2.3.1). The Kähler differential operator $\Omega_{-/A}$ is an left adjoint(VI.2.1.3), the idea of cotangent complex is to construct its right derive, but the category of A-algebras is not Abelian.

Alternatively, consider the category of A-algebras and the category of sets, there is a forgetful functor G, and a free algebra operator $F : S \to A[S]$, and for any such two operator, the unit and counit maps will give us a simplicial A-algebra $P_A(B)$, with $(P_A(B))_n = (FG)^n(B)$, and the boundary and coboundary maps are given by units and counits.

Now we form the Kähler differential operator termwise, as it is functorial, this gives us a $P_A(B)$-module $\Omega^1_{P_A(B)}$, and we can form the simplicial B-module

$$\Omega^1 = \Omega^1_{P_A(B)} \otimes_{P_A(B)} B,$$

where B is regarded as a trivial simplicial ring, and by Dold-Kan correspondence(VI.1.2.11), the normalized Moore complex gives an equivalence between simplicial category of A-modules and the category of A-module complexes, so the normalized Moore complex(nerves) will give us a complex of B-modules $L_{B/A} = N_*(\Omega^1)$, called the **cotangent complex**, it will be regarded as an element of $D^{\leq 0}(B)$.

Prop. (VI.2.3.2). $H^0(L_{B/A}) = \Omega_{B/A}$, thus can be calculated directly from the generator-relation definition of Kähler differential.

Prop. (VI.2.3.3) (Functoriality). The formation of Kähler differential commutes with arbitrary colimit as it is a left adjoint, and the formation of normalized Moore complex commutes is by choosing kernel, so the formation of cotangent complex commutes with filtered colimits, both in A and B. Especially, it commutes with taking stalks, hence the sheaf of cotangent complexes of a map between schemes can be constructed as in the case of Kähler differentials, and it is a Qco sheaf.

Prop. (VI.2.3.4). By the Dold-Kan correspondence, we will say that two simplicial A-modules are quasi-isomorphic iff their normalized nerves are quasi-isomorphic. Then $P_A(B) \to B$ is a quasi-isomorphic resolution of B, where B is the trivial complex.

Proof: There is a homotopy d between id and 0 for $n \leq 0$, where

$$d_n : F(GF)^n G(B) \to F(FG)^n \circ GFG(B)$$

using counit map, and on degree $0, -1$, it is $A[A[B]] \xrightarrow{\partial_1 - \partial_2} A[B] \to B \to 0$, which is clearly 0, so this is a zero map.

Thus $Tot(P_A(B)) \cong B$, and $N_*(A) \cong Tot(A)$ by Dold-Kan correspondence, so we are done. □

Prop. (VI.2.3.5) (The fundamental Distinguished Triangle). If $\mathcal{T}$ is a ringed topos and $A \to B \to C$ are morphisms of sheaves of rings over $\mathcal{T}$, then there is a morphism of distinguished triangle in $D^{\leq 0}(T)$:

$$L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B}$$

where $- \otimes^L_B C$ is the derived change of rings.

Proof: Cf.[Foundations of Perfectoid Gemetry P133]. The basis idea is to choose a resolution $P_A(B)$, then choose a bisimplicial resolution $Q(C)_{*,*}$ over $P_A(B)_*$, so it uses a bisimplicial version of the cotangent complex, so the notation is complicated, but the idea is finally reduced to use the Jacobi-Zariski sequence(VI.2.1.5) of Kähler differentials. □

Prop. (VI.2.3.6). The cotangent complex can be calculated using any polynomial resolution of B.

Proof:

□

Prop. (VI.2.3.7) (Properties of Cotangent Complexes).

- If B is a polynomial A-algebra, then $L_{B/A} \cong \Omega^1[0]$.
- (Kunneth Formula) If B, C are Tor independent over A, then

$$L_{B \otimes C/A} \cong (L_{B/A} \otimes_A C) \oplus (L_{C/A} \otimes_A B).$$

- (Flat Base Change)If B, C are Tor independent over A, $L_{B/A} \otimes_A C \cong L_{B \otimes_A C/C}$.
- If $A \to B$ is étale, then $L_{B/A} \cong 0$. In particular, if $B \to C$ is étale, then $L_{C/A} \cong L_{B/A} \otimes_B C$, by distinguished triangle(VI.2.3.5).
- If $A \to B$ is smooth, then $L_{B/A} \cong \Omega^0_{B/A}$.

Proof:

- Calculate the cotangent complex using $B \to B$(VI.2.3.6).
- The complex $P_{B/A} \otimes P_{C/A}$ is a polynomial resolution of $B \otimes_A C$, by Tor independence. And the formation of nerves are compatible with tensor products. Then we reduced to the corresponding properties of Kähler differentials(VI.2.1.4).
- The same as Kunneth formula. In fact, both of them has more general form using simplicial cotangent complex, Cf.[Foundations of Perfectoid Geometry, P138].
- Cf.[Foundations of Perfectoid Geometry, P141], in fact, étale is equivalent to $L_{B/A} = 0$.
- The cotangent complex is local, so we may assume it is standard smooth, so it factors as $A \to A[X_1, \ldots, X_k] \xrightarrow{g} B$, where g is étale, so using the distinguished triangle and polynomial case, the result follows.

□

4 Deformations

Prop. (VI.2.4.1) (Invariance of Étale Site under Infinitesimal Thickening). Let A be a ring, consider the following category:$\mathcal{C}_A$ of flat A-algebras B that $L_{B/A} = 0$, then if $\widetilde{A} \to A$ is surjective with nilpotent kernel, then the base change defines an isomorphism of categories $\mathcal{C}_{\widetilde{A}} \cong \mathcal{C}_A$.

By(VI.2.3.7), $L_{B/A}$ vanish is equivalent to being étale, thus the properties characterize the invariance of étale site under infinitesimal thickening.

Proof: ? □

Prop. (VI.2.4.2) (Relative Perfect Case). If A is a ring of char$p$ and B is an A-algebra which is relatively perfect, i.e. $B^{(1)} = B \otimes_{A,\mathrm{Frob}} A \to B$ is an isomorphism, then $L_{B/A} = 0$.

Proof: Notice for any A-algebra C, the relative Frobenius induces zero map $L_{C^{(1)}/A} \to L_{C/A}$, because by using the canonical polynomial resolution, $d(x^p) = px^{p-1}xs = 0$. Now the relative Frobenius is an isomorphism $B^{(1)} \to B$, thus induces an isomorphism $L_{B^{(1)}/A} \to L_{B/A}$ by Functoriality, thus $L_{B/A} = 0$. □

Cor. (VI.2.4.3). There is an equivalence of categories of $\mathcal{C}_n =$ flat $\mathbb{Z}/p^n$-algebras that A/p is perfect and $\mathcal{C}_1 =$ perfect rings over $\mathbb{Z}/p$.

moreover, taking limit, this is even equivalent to the category of flat p-adically complete $\mathbb{Z}_p$ algebras that A/p is perfect. Which is just the construction of Witt vectors.

Proof: It suffices to show that $\mathcal{C}_n \subset \mathcal{C}_{\mathbb{Z}/p^n}$: By(VI.2.4.2) and flat base change(VI.2.3.7), $L_{A/(\mathbb{Z}/p^n)} \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p \cong L_{(A/p)/(\mathbb{Z}/p)} = 0$, so $L_{A/(\mathbb{Z}/p^n)} \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p^k \cong 0$ by induction, and so $L_{A/(\mathbb{Z}/p^n)} \cong 0$.

For the last assertion, it is flat because it is torsion-free, which is because if $p(x_n) = 0$, then by $0 \to p^n\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^{n+1} \xrightarrow{p} \mathbb{Z}/p^n \to 0$ and the flatness of A_{n+1}, $x_{n+1} \in p^n A_{n+1}$, thus $x_n = 0$, and $x = 0$. □

Prop. (VI.2.4.4). Using a more careful analysis of Cotangent complex(embedded deformation), we can show that if $A \to B \in \mathcal{C}_A$ and there is a infinitesimal deformation $C \to C'$ of A-algebra, then a map $B \to C'$ can be lifted to an A-algebra map $A \to C$.

In particular, taking inverse image, we get that

$$\mathrm{Hom}_{\mathbb{F}_p}(A, B/p) \cong \mathrm{Hom}_{\mathbb{Z}_p-alg}(W(A), B).$$

which is the usual adjointness of the Witt vector construction.

5 Algebra Extension

Cf.[Perfectoid Geometry Appendix B].

Def. (VI.2.5.1) (Algebra Extensions). Let $A \to B$ be a ring map and M be a B-module, then an **A-algebra extension** of B by M is a short exact sequence of A-modules $0 \to M \to B' \to B \to 0$ that B' is an A-algebra with M being an ideal of it.

The set of such extensions are denoted by $\mathrm{Exal}_A(B, M)$.

Prop. (VI.2.5.2). $\mathrm{Exal}_A(B, M)$ is a group under Baer sum, where the sum of two extension is the extension given by pushout, i.e. $(B_1 \oplus B_2)/\{(m, -m) | m \in M\}$. Moreover, it is a B-module, where the multiplication is the pushout along multiplication of b on M.

Prop. (VI.2.5.3). There is a trivial extension given by $D_B(M) = B \oplus M$(VI.2.1.3), and the automorphism of $D_B(M)$ is isomorphic to $\mathrm{Der}_A(B, M)$ via $d \mapsto \mathrm{id} \oplus d$.

Proof: Cf.[Foundations of Perfectoid Spaces Masullo P118]. □

Prop. (VI.2.5.4). Let $A \to B \to C$ be ring maps, then for any C-module M, there is an exact sequence

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M) \xrightarrow{\partial} \mathrm{Exal}_B(C, M) \to \mathrm{Exal}_A(C, M) \to \mathrm{Exal}_A(B, M)$$

functorial in M. Where ∂ is given by(VI.2.5.3).

Proof: Cf.[Foundations of Perfectoid Spaces Masullo P119]. □

Prop. (VI.2.5.5). Let $A \to B$ be a ring map or a map of sheaves of rings, and let M be a B-module, then there is an isomorphism of B-modules that is natural in M:

$$\mathrm{Exal}_A(B, M) = \mathrm{Ext}^1_B(NL_{B/A}, M).$$

Proof: Cf.[Foundations of Perfectoid Spaces, P127]. □

Infinitesimal Deformation

Def. (VI.2.5.6). An **infinitesimal deformation** of a f.g. k-algebra is defined as a algebra A' flat over $D = k[t]/(t^2)$ that $A' \otimes_D k = A$.

A f.g. k-algebra is called **rigid** if it has no infinitesimal deformations.

Lemma (VI.2.5.7) (Infinitesimal Lifting Property). If A is a f.g. k-algebra that is regular, where k is alg.closed, then for any k-algebra homomorphism $0 \to I \to B' \to B \to 0$ that $I^2 = 0$, we can lift a map $A \to B$ to a map $A \to B'$.

Proof: Cf.[Hartshorne Ex2.8.6]. □

Cor. (VI.2.5.8). Let A be a f.g. k-algebra, write A as a quotient of a polynomial ring over k with kernel J, then there is an exact sequence $J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0$ by(VI.2.1.5), then we apply $\mathrm{Hom}_A(-, A)$ and let $T^1(A) = \mathrm{Coker}(\mathrm{Hom}_A(\Omega_{P/k} \otimes_A A, A) \to \mathrm{Hom}_A(J/J^2, A))$. Then $T^1(A)$ parametrize infinitesimal deformations of A.

VI.3 Higher Topos Theory

1 ∞-Categories

Def. (VI.3.1.1). An **∞-category** is a simplicial set that has lifting property w.r.t all $\Lambda_i^n \to \Delta^n$, where $0 < i < n$.

2 ∞-Algebras

3 Topological Cyclic Homology(Scholze)

Chapter VII

Logics

VII.1 Model Theory

Basic references are [Model Theory Marker] and [Mathematical Logics Hamilton]. The exercises of [Model Theory Marker] is important.

1 Mathematical Logics

Turing Machine and Computability

Prop. (VII.1.1.1) (Turing Machine). For the ring structure of $\mathbb{N}$, there is a $\mathcal{L}$-formula $\varphi(e,x,s)$ that $\mathbb{N}\models T(e,x,s)$ iff the Turing machine coded with e halts on input x within s steps. So the set of **halting computation** is definable by the formula: $\exists s\varphi(e,x,s)$.

Proof: Cf.[Models of Peano Arithmetic Kaye]. □

Def. (VII.1.1.2) (Recursively Enumerable Sets). A set S of the natural numbers is called **recursively enumerable** iff there is an algorithm that the set of input numbers that halts is exactly S. Equivalently, a recursively enumerable set is a set that there is an algorithm that 'enumerates' the members of S.

Prop. (VII.1.1.3) (Hilbert's 10-th Problem). For any recursively enumerable set $A\subset\mathbb{N}^n$, there is a polynomial $P(X_1,\ldots,X_n,Y_1,\ldots,Y_m)$ that

$$A=\{\overline{x}\in\mathbb{N}^n:\mathbb{N}\models\exists y_1\exists y_2\ldots,\exists y_m p(\overline{x},\overline{y})=0\}$$

Proof: Cf.[M. Davis, J. Matijasevi˘c, and J. Robinson, Hilbert's 10th Problem. Diophantine equations: Positive aspects of a negative solution, in Mathematical Developments from Hilbert's Problems, F. Browder, ed., American Mathematical Societ, Providence, RI, 1976.] □

2 Structure and Theories

Basics

Def. (VII.1.2.1) (Boolean Algebra). A **Boolean algebra** is a set B with **Boolean connectives** $\wedge,\urcorner,0,1$, where:

- $\wedge$ is a symmetric function $B\times B\to B$ that $\varphi\wedge\varphi=\varphi$,

- $\urcorner$ is function $B \to B$ that $\urcorner \circ \urcorner = \mathrm{id}$,
- $0, 1$ are nullary operations that $\urcorner 0 = 1$, and $x \wedge 0 = 0$, and $x \wedge \urcorner x = 0$.

Sometimes $\vee$ is also used, which is defined as $\varphi \vee \psi = \urcorner(\urcorner\varphi \wedge \urcorner\psi)$.

Prop. (VII.1.2.2). The power set of any set is a Boolean algebra withe $\wedge = \cap$ and $\urcorner(A) = X - A$.

In fact, every Boolean algebra can be embedded as a subalgebra of a power set algebra of some set by(IV.1.2.13).

Def. (VII.1.2.3) (Languages and Structures). A **language** $\mathcal{L}$ is a set of symbols, where a **symbol** is indeterminants that are labeled **constant symbol** $\mathcal{C}$, **function symbol**(of finite arity) $\mathcal{F}$ or **relation symbol**(of finite arity) $\mathcal{R}$.

For a language $\mathcal{L}$, a **$\mathcal{L}$-structure** on a set M is an assignment for each constant symbol c an element $c^M \in M$, for each function symbol f of arity n a function $f^M : M^n \to M$, and for each relation symbol R of arity m a subset $R^M \subset M^m$. These c^M, f^M, R^M are called **interpretations** of $\mathcal{L}$.

And there is a natural definition of morphisms of $\mathcal{L}$-structures, and an injective morphism of $\mathcal{L}$-structures is called an embedding or a **structure extension**.

Def. (VII.1.2.4) (Formulae). For a language $\mathcal{L}$, the set of **terms** is the smallest set $\mathcal{T}$ that $c \in T$ for each constant symbol c, $x_i \in \mathcal{T}$ for each variable symbol, and if $t_1, \ldots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$, then the symbol $f(t_1, \ldots, t_{n_f}) \in \mathcal{F}$.

An **atomic formula** is a symbol of the form $t_1 = t_2$ or $R(t_1, \ldots, t_n)$.

The set of **$\mathcal{L}$-formulae** is the smallest set $\mathcal{W}$ that any atomic formula is in $\mathcal{W}$, and if $\varphi, \psi \in \mathcal{W}$, then $\urcorner\varphi, \varphi \wedge \psi$ are in $\mathcal{W}$, and adjunction with the **qualifier symbol** $\exists x_i \varphi$ is in $\mathcal{W}$.

A **free variable in a formula** is a x_i that is not qualified with $\forall$ or $\exists$, otherwise it is called **bound**. A **sentence** is a formula without free variables.

A **theory** is a set of sentences in a language $\mathcal{L}$.

Def. (VII.1.2.5) (Simplifications). $\varphi \to \psi$ is the simplification of $\urcorner\varphi \vee \psi$, $\varphi \leftrightarrow \psi$ is the simplification of $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

Even $\vee$ is the simplification of $\urcorner(\urcorner\varphi \wedge \urcorner\psi)$, and $\forall x_i \varphi$ is the simplification of $\urcorner(\exists x_i \urcorner\varphi)$.

Def. (VII.1.2.6) (First Order Logic). A **first order logic** is a $\mathcal{L}$-strcture that only elements in M are quantified with $\forall$ or $\exists$. For example, "Every bounded subsets has a least upper bound" cannot be expressed as a formulae in a first order logic of $\mathbb{R}$.

Def. (VII.1.2.7) (Models). Definition for **truth** in a $\mathcal{L}$-structure M:Cf.[Model Theory Marker P11].

If a sentence φ is true in the $\mathcal{L}$-structure of M, then we say M satisfies φ and writes $M \models \varphi$.

If T is a theory and the $\mathcal{L}$-structure M satisfies all $\varphi \in T$, then M is called a **model** of T, and writes $M \models T$. A theory T is called **satisfiable** iff there is a model $\mathcal{M}$ for T.

A set of $\mathcal{L}$-structures $\mathcal{K}$ is called an **elementary class** iff there is an $\mathcal{L}$-theory T that $\mathcal{K} = \{\mathcal{M} | \mathcal{M} \models T\}$.

Given a $\mathcal{L}$-structure on M, the **theory of** M is the set of all sentences true in M.

Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are called **elementary equivalent**, denoted by $\mathcal{M} \equiv \mathcal{N}$, if for all $\mathcal{L}$-sentences φ, $\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi$.

Prop. (VII.1.2.8). Suppose $\mathcal{M}$ is a substructure of $\mathcal{N}$ and $\overline{a}$ is a tuple in $\mathcal{M}$. If $\varphi(\overline{v})$ is a quantifier-free formula, then $\mathcal{M} \models \varphi(\overline{a})$ iff $\mathcal{N} \models \varphi(\overline{a})$.

Proof: Cf.[Model Theory P11]. □

Prop. (VII.1.2.9). If $\mathcal{L}$-structures $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$.

Proof: This seemingly trivial proposition still needs proof, and the proof uses induction, just as that of(VII.1.2.8). □

Def. (VII.1.2.10) (Logical Consequence). Let T be an $\mathcal{L}$-theory and φ an $\mathcal{L}$-sentence, then φ is called a **logical consequence** of $\mathcal{T}$, writes $T \models \varphi$, if for any $\mathcal{L}$-structure $\mathcal{M}$ that $\mathcal{M} \models T$, $\mathcal{M} \models \varphi$.

Definable Sets and Interpretability

Def. (VII.1.2.11). Let $\mathcal{M}$ be an $\mathcal{L}$-structure, a subset X of $\mathcal{M}^n$ is called **definable** iff there is an $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and a tuple $\bar{b} \in \mathcal{M}^m$ that $X = \{\bar{a} \in \mathcal{M}^n : \models\varphi(\bar{a}, \bar{b})\}$. Moreover if $A \subset M$, X is called A**-definable** iff $y_i \in A$.

Prop. (VII.1.2.12) (Examples of Definable Sets). The definability of some sets are often nontrivial, using many number theories. For example, Cf.[Marker P20].

Prop. (VII.1.2.13). There is an inductive characterization of definable sets, Cf.[Marker P22].

Prop. (VII.1.2.14). If $\mathcal{M}$ is an $\mathcal{L}$-structure, If $X \subset \mathcal{M}^n$ is A-definable, then very $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes A pointwise will fixes X setwise.

Proof: For an automorphism τ of $\mathcal{M}$, $\mathcal{M} \models \varphi(\bar{b}, \bar{a})$ iff $\mathcal{M} \models \varphi(\tau(\bar{b}), \tau(\bar{a})) = \varphi(\tau(\bar{b}), \bar{a})$. □

Cor. (VII.1.2.15). $\mathbb{R}$ is not definable in $\mathbb{C}$.

Proof: If $\mathbb{R}$ is definable, it is definable over a finite set $A \subset \mathbb{C}$, Let r, s be algebraically independent over $\mathbb{A}$ and $r \in \mathbb{R}, s \notin \mathbb{R}$. This can be done, otherwise $\mathbb{C}$ or $\mathbb{R}$ is finite transcendental over $\mathbb{Q}$, then $|\mathbb{C}| = |\mathbb{Q}|$(I.3.6.2), which is impossible by(I.1.3.13). Then there is an automorphism σ of $\mathbb{C}$ fixing A that $\sigma(r) = s$, so $\mathbb{R}$ is not definable by(VII.1.2.14). □

Def. (VII.1.2.16) (Definably Interpretability). An $\mathcal{L}_0$-structure $\mathcal{N}$ is called **definably interpreted** in an $\mathcal{L}$-structure $\mathcal{M}$ if there is a definable set $X \subset \mathcal{M}^n$ that we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions of X and the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{N}$.

The usual example is that the group structure of $GL_2(K)$ is definably interpreted in the ring structure of a field.

Def. (VII.1.2.17) (Interpretability and Quotient Construction). An $\mathcal{L}_0$-structure $\mathcal{N}$ is called **interpretable** in an $\mathcal{L}$-structure $\mathcal{M}$ iff there is a definable set $X \in \mathcal{M}^n$ and a definable equivalence relation E on X, that we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions of X/E and the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{N}$.

The usual example is that the set structure of a projective space is interpretable in the ring structure of a field.

Prop. (VII.1.2.18). Any structure for a countable language can be interpreted in a graph.

Proof:

□

3 Some Theories Often Used

Def. (VII.1.3.1) (Languages).
- $\mathcal{L}_r$ is defined to be the language of rings.
-

Def. (VII.1.3.2) (Theories).
- DAG is defined to be the theory of non-trivial torsion-free divisible Abelian groups, w.r.t. $\mathcal{L}_r$.
- ACF is defined to be the theory of alg.closed fields, w.r.t $\mathcal{L}_r$.
-

4 Basis Techniques

Def. (VII.1.4.1) (Definitions for Theories). A theory T is said to have the **witness property** iff whenever $\varphi(v)$ is an $\mathcal{L}$-formula with one free variable v, there is a constant symbol c that $T \models (\exists v\varphi(v) \to \varphi(c))$.

A theory T is called **inconsistent** if there is a sentence φ that $T \models \varphi \wedge \neg\varphi$, otherwise it is called **consistent**.

A theory T is called **complete** iff for any $\mathcal{L}$-sentence, either $T \models \varphi$ or $T \models \neg\varphi$.

A theory T is called **decidable** iff there is an algorithm that given an $\mathcal{L}$-sentence, it decides whether $T \models \varphi$.

Def. (VII.1.4.2) (Proof System). A **proof** of a $\mathcal{L}$-sentence φ from a theory T, denoted by $T \vdash \varphi$, is a is Cf.[Mathematical Logic Shoenfield].

Def. (VII.1.4.3) (Recursiveness and Decidablility). A language $\mathcal{L}$ is called **recursive** iff there is an algorithm that decides whether a sequence of symbols is an $\mathcal{L}$-formula.

An $\mathcal{L}$-theory T is called **recursive** iff there is an algorithm that decides whether a given $\mathcal{L}$-sentence is in T.

An $\mathcal{L}$-theory T is called **decidable** iff there is an algorithm that decides whether a given φ satisfies $T \models \varphi$.

Prop. (VII.1.4.4). If $\mathcal{L}$ is a recursive language and T is a recursive $\mathcal{L}$-structure, then $\{\varphi | T \vdash \varphi\}$ is recursively enumerable.

Proof: There is a computable listing $\sigma_1, \ldots, \sigma_n \ldots$ of all the finite sequences of $\mathcal{L}$-formulas, because $\mathcal{L}$ is recursive. Then we can check at each stage iff σ_i is a proof of φ. This involves checking if each formula is in T(checkable because T is recursive) or it is a simple consequences of formulae before it, and finally check the last formula is φ. If σ_i is a proof of φ, then halt, otherwise go on to check σ_{i+1}. □

Prop. (VII.1.4.5). The halting computation set is not computable.

Proof: Cf.[Mathematical Logic Shoenfield]. □

Cor. (VII.1.4.6). The full theory $Th(\mathbb{N})$ of ring structures of $\mathbb{N}$ is undecidable.

Proof: If such an algorithm exists, then we can use it to compute whether the sentence

$$\varphi(e, x) = \exists s T(\underbrace{1 + \ldots + 1}_{e-\text{times}}, \underbrace{1 + \ldots + 1}_{x-\text{times}}, s)$$

is computable. Then this will contradicts the fact that halting computation set is computable(VII.1.4.5). □

Prop. (VII.1.4.7) (Gödel's Completeness Theorem). Let T be an $\mathcal{L}$-theory and φ is an $\mathcal{L}$-sequence, then $T \models \varphi$ iff $T \vdash \varphi$.

Proof:

□

Cor. (VII.1.4.8). A theory T is consistent iff it is satisfiable.

Proof: If T is satisfiable, then it is clearly consistent, and if T is not satisfiable, then there are no models for T, so $T \models \varphi \wedge \urcorner \varphi$ by definition, so $T \vdash \varphi \wedge \urcorner \varphi$ by Gödel's completeness theorem. □

Ultraproducts of Theories

Def. (VII.1.4.9) (Ultraproducts of Theories). If $M_i, i \in I$ is a collection of $\mathcal{L}$-structures and $\mathcal{F}$ is an ultrafilter on I, then the **ultraproduct** $\prod_I M_i/\mathcal{F}$ of M_i is a $\mathcal{L}$-structure defined as:

- The underlying set $M = \prod M_i/\sim$, where $(a_i) \sim (b_i)$ iff $\{i \in I | a_i = b_i\} \in \mathcal{F}$, which is an equivalent relation.
- If c is a constant symbol, then $c^M = (c^{M_i})$.
- If f is a function symbol, then $f^M([a_{i1}], \ldots, [a_{in}]) = [(f^{M_i}(a_{i1}, \ldots, a_{in})]$.
- If R is a relation symbol, then $([a_{i1}], \ldots, [a_{in}]) \in R^M$ iff $\{i \in I | (a_{i1}, \ldots, a_{in}) \in R^{M_i}\} \in \mathcal{F}$.
- A theory T is called **satisfiable** iff there is a $\mathcal{L}$-structure M that $M \models T$.

Prop. (VII.1.4.10) (Los Theorem). Let $M_i, i \in I$ be $\mathcal{L}$-structures and $\mathcal{F}$ be an ultrafilter on I. Let $\varphi(\overline{x})$ be a first-order logic formula in the free variables $\overline{x}$, and let $\overline{[(a_i)]}$ be a tuple of elements from $\prod_I M_i/\mathcal{F}$, then $\prod_I M_i/\mathcal{F} \models \varphi(\overline{[(a_i)]})$ iff $\{i \in I | M_i \models \varphi_i(\overline{a_i})\} \in \mathcal{F}$.

Proof:

□

Cor. (VII.1.4.11). An ultraproduct of models for a theory T is also a model for T.

Cor. (VII.1.4.12) (Non-Standard Model for Th($\mathbb{R}$)). Consider $\mathbb{R}$ in the language $\mathcal{L}_r$, where L_r is the language of rings, (i.e., the ring structure), let $\mathcal{F}$ be a non-principle ultrafilter on $\mathbb{N}$(I.1.6.8), and consider the ultraproduct $\mathcal{R} = \prod_{i \in \mathbb{N}} \mathbb{R}/\mathcal{F}$, which is called an **ultrapower** of $\mathbb{R}$.

Notice each factor satisfies $Th(\mathbb{R})$, so $\mathbb{R}$ also satisfies $Th(\mathbb{R})$.

Cor. (VII.1.4.13). Using ultraproduct construction, we can find a field of characteristic 0 that has exactly one algebraic extension in each degree.

Proof: Just use the field model $\mathbb{F}_p$ for all p and construct their ultraproduct w.r.t. a non-principal ultrafilter. □

Prop. (VII.1.4.14) (Compactness Theorem). A theory T is satisfiable iff every finite subset of T is satisfiable.

Proof: One direction is trivial, for the other, if there is a family of structures $\{M_\Delta\}$ indexed by the collection of all finite subsets of T, with $M_\Delta \models \Delta$ for all $\Delta \in I$ where I is the set of all finite subsets of T.

Then we want to find an ultrafilter $\mathcal{F}$ on I that for all $\varphi \in T$, $\{\Delta | M_\Delta \models \varphi\} \in \mathcal{F}$, then we can use Leo's theorem(VII.1.4.10) to show that $\prod_I M_\Delta/\mathcal{F}$ is a model for all $\varphi \in T$. Now in fact we make pick a ultrafilter over the filter generated by all the $A_\varphi = \{\Delta | \varphi \in \Delta\}$, because $M_\Delta \models \Delta$. In fact, this is the case because A_φ has the finite intersection property trivially. □

Prop. (VII.1.4.15) (Keiler-Shelah Theorem). Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent iff there is an index set I and an ultrafilter $\mathcal{F}$ on I that $\prod_I M/\mathcal{F} \cong \prod_I N/\mathcal{F}$.

Proof: Cf.[C. C. Chang and H. J. Keisler, Model Theory 6.1.15]. □

Compactness Theorem and Complete Theories

See [Marker P35] for another proof of the compactness theorem.

Cor. (VII.1.4.16). If $T \models \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subset T$.

Proof: If not, then $\Delta \cup \{\neg\varphi\}$ is satisfiable for all finite $\Delta \subset T$, so $T \cup \{\neg\varphi\}$ is finitely satisfiable, thus satisfiable by compactness theorem, but this cannot by true because $T \models \varphi$. □

Cor. (VII.1.4.17) (Torsion Elements). Let $\mathcal{L}$ be a language containing $\{\cdot, e\}$, the language of groups, and T is a theory extending the theory of groups, let $\varphi(v)$ be an $\mathcal{L}$-formula. If for any n there is a $G_n \models T$ and G_n has an element of finite order greater than n, then there is an $\mathcal{L}$-structure G that $G \models T$ and G has an element of infinite order.

In particular, there is no formula that defines the torsion elements in any models for T.

Proof: Consider a new language $\mathcal{L}^* = \mathcal{L} \cup \{c\}$, and T^* an $\mathcal{L}^*$-theory that

$$T^* = T \cup \{\varphi(c)\} \cup \{\neg(\underbrace{c \cdots c \ldots c}_{n-\text{times}} = e)\},$$

then the theory T^* is finitely satisfiable by hypothesis, so T^* is satisfiable by compactness theorem. □

Cor. (VII.1.4.18) (Larger than All Natural Number). Consider $\mathcal{L}$ the language of ordered rings, let $\mathrm{Th}(\mathbb{N})$ be the theory of $\mathbb{N}$, then there is an $\mathcal{L}$-structure $\mathcal{M}$ that $\mathcal{M} \models Th(\mathbb{N})$ and $\mathcal{M}$ has an element that is larger than every natural number.

Proof: The same proof as that of(VII.1.4.17), but use

$$T^* = Th(\mathbb{N}) \cup \{\underbrace{1 + \ldots + 1}_{n-\text{times}} < c\}$$

□

Def. (VII.1.4.19). Let κ be an infinite cardinal and T is a theory with models of size κ. T is called **κ-categorical** iff any two models of T of cardinality κ are isomorphic.

Prop. (VII.1.4.20). If T is an $\mathcal{L}$-theory with infinite models, then if κ is an infinite cardinal that $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality κ.

Proof: Cf.[Marker P40]. □

Prop. (VII.1.4.21). The theory of torsion-free divisible Abelian groups is κ-categorical for all $\kappa > \aleph_0$.

Proof: Cf.[Marker P41]. □

Prop. (VII.1.4.22) (Vaught's Test). Let T be a satisfiable theory with no finite models, if L is κ-categorical for some $\kappa \geq |\mathcal{L}|$, then T is complete.

Proof: If there is a sentence φ that $T \not\models \varphi$ and $T \not\models \neg\varphi$, so $T_0 = T \cup \{\varphi\}$ and $T_1 = T \cup \{\neg\varphi\}$ is satisfiable. They both have infinite models by hypothesis, so by(VII.1.4.20) there are models of cardinal κ for T_0 and T_1, but they cannot be isomorphic, contradiction. So T is complete. □

Prop. (VII.1.4.23). If T is a recursive complete satisfiable theory in a recursive language $\mathcal{L}$, then T is decidable.

Proof: Because T is satisfiable, The set of all φ that $\mathcal{M} \models \varphi$ and the set of all φ that $\mathcal{M} \models \neg\varphi$ are disjoint, and their sum is the set of all sentences by completeness. By Gödel's completeness theorem, this is equivalent to $\mathcal{M} \vdash \varphi$ or $\mathcal{M} \vdash \neg\varphi$. Then by(VII.1.4.4), they are both enumerable, so it is decidable by definition. □

Up and Down

Def. (VII.1.4.24) (Elementary Embedding). An $\mathcal{L}$-structure embedding $j : \mathcal{M} \to \mathcal{N}$ is called an **elementary embedding**, denoted by $\mathcal{M} \prec \mathcal{N}$, if for any $\mathcal{L}$-formula φ, $\mathcal{M} \models \varphi(a_1, \ldots, a_n) \iff \mathcal{N} \models \varphi(j(a_1), \ldots, j(a_n))$.

Isomorphisms are elementary embeddings by(VII.1.2.9).

Prop. (VII.1.4.25). If $\mathcal{M}_i$ is a chain of $\mathcal{L}$-structures that $\mathcal{M}_i \prec \mathcal{M}_j$ for any $i < j$, then we can define there union $\mathcal{M} = \cup M_i$. Then $\mathcal{M}$ is an elementary extension of each $\mathcal{M}_i$.

Proof: Use induction on formulas to show that

$$\mathcal{M}_i \models \varphi(\overline{a_i}) \iff \mathcal{M} \models \varphi(\overline{a_i}),$$

for all $\mathcal{L}$-formulas φ. This is true for quantifier-free formula by(VII.1.2.8), and if this is true for φ, ψ, then this is true for $\neg\varphi$ and $\varphi \wedge \psi$. For the sentence $\varphi = \exists v \psi(v, \overline{w})$, if $\mathcal{M}_i \models \psi(b, \overline{a})$ for some b, then so does $\mathcal{M}$. Conversely, if $\mathcal{M} \models \psi(b, \overline{a})$ for some b, then $b \in \mathcal{M}_j$ for some j, then $\mathcal{M}_j \models \varphi$, by the condition, $\mathcal{M}_i \models \varphi$ also. □

Prop. (VII.1.4.26) (Löwenheim-Skolem Theorem). Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $X \subset \mathcal{M}$, then there is a elementary submodel $\mathcal{N}$ that $X \subset \mathcal{N}$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$.

Proof: Cf.[Marker P46]. □

Def. (VII.1.4.27) (Universal Sentence). A **universal sentence** is a sentence of the form $\forall v \varphi(v)$, where φ is quantifier-free.

A $\mathcal{L}$-theory T is said to have a **universal aximatization** iff there is a set of universal $\mathcal{L}$-sentences Γ that $\mathcal{M} \models T$ iff $\mathcal{M} \models \Gamma$.

Prop. (VII.1.4.28). An $\mathcal{L}$-theory T has a universal axiomatization iff for any $\mathcal{N} \subset \mathcal{M}$, if $\mathcal{M} \models T$, then $\mathcal{N} \models T$.

Proof: Cf.[Marker P47]. □

Prop. (VII.1.4.29) (Universal Consequences). If T is an $\mathcal{L}$-theory, let $T_\forall$ be all of the universal sentences φ that $T \models \varphi$. Then $\mathcal{A} \models T_\forall$ iff there is an $\mathcal{M} \models T$ that $\mathcal{A} \subset \mathcal{M}$.

Proof: □

Back and Forth

5 Quantifier Elimination

Def. (VII.1.5.1) (Quantifier Elimination). A theory is said to have **quantifier elimination** if for each formula φ, there is a quantifier free ψ that $T \models \varphi \leftrightarrow \psi$.

Prop. (VII.1.5.2). *DLO*, the theory of dense linear orders without endpoints, has quantifier elimination.

Proof: Cf.[Marker P72]. $\square$

Prop. (VII.1.5.3) (Criterion for Quantifier Elimination). Suppose $\mathcal{L}$ contains a constant symbol c, T is an $\mathcal{L}$-theory, and $\varphi(\overline{a})$ is an $\mathcal{L}$-formula, then the following are equivalent:

- There is a quantifier-free $\mathcal{L}$-formula $\varphi(\overline{v})$ that $T \models \forall\overline{v}(\varphi(\overline{v}) \leftrightarrow \psi(\overline{v}))$.
- If $\mathcal{M},\mathcal{N}$ are models of T, and $\mathcal{A}$ is an $\mathcal{L}$-structure that $\mathcal{A} \subset \mathcal{M} \cap \mathcal{N}$, then for all $\overline{a} \in \mathcal{A}$, $\mathcal{N} \models \varphi(\overline{a}) \iff \mathcal{M} \models \varphi(\overline{a})$.

Proof: Cf.[Marker P73]. $\square$

Def. (VII.1.5.4) (Algebraically Closed Models). A theory T is said to have **algebraically prime models** if for any $\mathcal{A} \models T_\forall$, there is a $\mathcal{M} \models T$ and an embedding $i : \mathcal{A} \subset \mathcal{M}$ that for any other $\mathcal{N} \models T$, any embedding $j : \mathcal{A} \to \mathcal{N}$ factors through i.

Def. (VII.1.5.5) (Simply Closed Model). If $\mathcal{M},\mathcal{N}$ are models of T, and $\mathcal{M} \subset \mathcal{N}$, then $\mathcal{M}$ is called **simply closed** in $\mathcal{N}$, denoted by $\mathcal{M} \prec_s \mathcal{N}$, iff for any quantifier-free formula $\varphi(\overline{a}, w)$ and $\overline{a} \in \mathcal{M}^n$, if $\mathcal{N} \models \exists w\varphi(\overline{a}, w)$, then so does $\mathcal{M}$.

Prop. (VII.1.5.6) (Quantifier Elimination Test). If T is an $\mathcal{L}$-theory that has algebraically prime models, and any inclusion of models for T is simply closed, then T has quantifier elimination.

Proof: Cf.[Marker P77]? $\square$

Lemma (VII.1.5.7). *DAG* has algebraically prime models.

Proof: This is just the alg.closure of the quotient field of an integral domain. $\square$

Lemma (VII.1.5.8). Any inclusion of models for *DAG* is simply closed.

Proof: Cf.[Marker P76]. $\square$

Prop. (VII.1.5.9). *DAG* has quantifier elimination.

Proof: This follows immediately from(VII.1.5.7)(VII.1.5.8)(VII.1.5.6). $\square$

Def. (VII.1.5.10) (Strongly Minimal Theory). A theory T is called **strongly minimal** iff for any $\mathcal{M} \models T$, every definable subset of $\mathcal{M}$ is either finite or cofinite.

Prop. (VII.1.5.11). DAG is strongly minimal.

Proof: Cf.[Marker P78]. $\square$

Def. (VII.1.5.12) (Model Complete). An $\mathcal{L}$-theory T is called **model complete** if $\mathcal{M} \prec \mathcal{N}$ whenever $\mathcal{M} \subset \mathcal{N}$.

Prop. (VII.1.5.13). If T has quantifier elimination, then T is model complete.

Proof: If $\mathcal{M} \subset \mathcal{N}$, let $\varphi(\overline{a})$ be an $\mathcal{L}$-formula, and $\overline{a} \in \mathcal{M}$, then there is a quantifier-free formula $\psi(\overline{v})$ that $\mathcal{M} \models \forall \overline{v}(\varphi(\overline{v}) \leftrightarrow \psi(\overline{v}))$. By(VII.1.2.8), the formula $\psi(\overline{a})$ passes between $\mathcal{M}$ and $\mathcal{N}$, so

$$\mathcal{M} \models \varphi(\overline{a}) \iff \mathcal{M} \models \psi(\overline{a}) \iff \mathcal{N} \models \psi(\overline{a}) \iff \mathcal{N} \models \varphi(\overline{a}).$$

□

Prop. (VII.1.5.14) (Model-Complete and Complete). If T is model-complete and there is a $\mathcal{M}_0 \models T$ that $\mathcal{M}_0$ embeds into every model of T, then T is complete.

Proof: Clearly any model is elementary equivalent to $\mathcal{M}_0$, thus clearly T is complete. □

Cor. (VII.1.5.15). the theory DAG of non-trivial torsion-free divisible Abelian groups is complete, as $(\mathbb{Q}, +, 0)$ embeds in every model of DAG.

Algebraically Closed Fields

Prop. (VII.1.5.16). ACF_p the theory of algebraically closed fields of characteristic p is κ-categorical for all uncountable cardinal κ.

Proof: Cf.[Marker P41]. □

Cor. (VII.1.5.17). ACF_p is complete theory, by(VII.1.5.16) and(VII.1.4.22). This also follows from(VII.1.5.23).

Cor. (VII.1.5.18). ACF_p is decidable, in particular, $\mathrm{Th}(\mathbb{C})$, the first-order theory of the fields of complex numbers, is decidable.

Proof: ACF_p is complete by(VII.1.5.16) and(VII.1.4.22), it is clearly recursive, and it is clearly satisfiable, so use(VII.1.4.23). The last assertion is because ACF_p is κ-categorical, by(VII.1.5.16). □

Prop. (VII.1.5.19) (First order Lefschetz Principle). Let φ be a sentence in the language of rings, the following are equivalent:

1. φ is true in $\mathbb{C}$.
2. φ is true in every(some) alg.closed field of char 0.
3. For p large/there exists arbitrary large p, φ is true in any alg.closed field of char p.

Proof: $1, 2$ are equivalent because ACF_p is complete(VII.1.5.17) and use Gödel's completeness theorem. If 2 is true, then $ACT_0 \models \varphi$, so by(VII.1.4.16), then some $\Delta \models \varphi$. So for p sufficiently large, $ACF_p \models \Delta$, so $ACF_p \models \varphi$ for p large.

If $ACF_0 \not\models \varphi$, then $ACF_0 \models \neg\varphi$ by completeness(VII.1.5.17), so by above argument, $ACF_p \models \neg\varphi$ for p large, contradiction. □

Cor. (VII.1.5.20). Any injective polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ is surjective.

Proof: By Lefschetz principle, it suffices to show this for $(\mathbb{F}_p)^{alg}$ for p large. If this is not true, then choose the coefficients of the coordinate map of f, and the element that is not in the image, then the subfield it generated is algebraic over $\mathbb{F}_p$, so it is finite, and clearly it is surjective. □

Prop. (VII.1.5.21). $ACF_\forall$ is the theory of integral domains.

Proof: Clearly a ring is a subring of an alg.closed field iff it is an integral domain, so the result follows from(VII.1.4.29). □

Prop. (VII.1.5.22). ACF has qualifier-elimination.

Proof: Use(VII.1.5.4), clearly it has algebraically prime models(VII.1.5.21)(VII.1.5.4), and we need to check simply closedness.

For this, If $K \subset L$, notice that a quantifier-free formula φ is just a conjunction of some polynomial functions and negation of polynomial functions, with their coefficients in K. If there are some polynomial, then the solution b of φ in L is algebraic over K, thus in K because K is alg.closed. Now if it is just negations of polynomials, then clearly φ is true in K for some $c \in K$, because K is alg.closed thus infinite. □

Cor. (VII.1.5.23). ACF is model-complete and ACF_p is complete for p a prime or 0.

Proof: The first is by(VII.1.5.13), the second follows from the first by(VII.1.5.14). □

Lemma (VII.1.5.24). Let K be a field, then the subsets of K^n defined by atomic formulas are exactly Zariski closed subsets. And a subset of K^n that is quantifier-definable iff it is a Boolean combination of Zariski closed subsets(constructible). (Clear).

Chapter VIII

Theoretical Physics

VIII.1 Quantum Mechanics

Basic References are [Napkin Evan Chen].

1 Basics

Axiom (VIII.1.1.1) (Axioms). The Schrodinger equation can be derived from the Dirac-von Neumann axioms:

The **state of particles** is a countable dimensional Hilbert space, and

- The **observables** of a quantum system are defined to be the (possibly unbounded)Hermitian operators A on $\mathbb{H}$. Then any continuous observable is unitarily diagonalizable, with real eigenvalues by Hilbert-Schimidt(V.5.4.16).
- The **state** φ of the quantum system is a unit vector of $\mathbb{H}$, up to scalar multiples.
- The expectation value of an observable A for a system in a state φ is given by the inner product $(\varphi, A\varphi)$.
- (Unitarity)The time evolution of a quantum state according to the Schrodinger equation is mathematically represented by a unitary operator $U(t)$ (depends only on the state an relative time)(one-parameter subgroup).

Now that $\varphi(t) = \hat{U}(t)\varphi(t_0)$, so $\hat{U}(t)\varphi(t_0) = e^{-i\hat{\mathcal{H}}t}, \hat{\mathcal{H}}$ hermitian.

So now take derivative w.r.t t, we get $i\frac{d\varphi}{dt} = \hat{\mathcal{H}}\varphi$. By **quantum correspondence principle**, it is possible to derive the expression of $\hat{\mathcal{H}}$ by classical methods.

Def. (VIII.1.1.2). A **qubit** is a state that is complex combination of 0 and 1, i.e. $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$. Notice I very dislike the 'bra-ket' notation, I prefer to think of state just as an element in the Hilbert space, and use any notation I like.

Def. (VIII.1.1.3). The observables on a two dimensional Hilbert space, i.e. a qubit state space, are all combinations of **Pauli Observables** or Pauli matrixes plus I:

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Their corresponding eigenvalues are denoted by

$$\uparrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \downarrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \rightarrow = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \leftarrow = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \otimes = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \odot = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Remark (VIII.1.1.4). The solution of a Schrodinger equation for a non Relativistic particle is assumed to be a Schwartz function (Vanish fast enough at infinity). The coefficients is assumed smooth enough to guarantee at least uniqueness and existence locally.

Prop. (VIII.1.1.5). The wave function on the (p, t) coordinates is the Fourier Transform of the wave function on the (x, t) coordinates, because the eigenstate of the p-operator $i\hbar\frac{\partial}{\partial x}$ is e^{ikx}, the coefficients of which is the value (probability) of the wave function of the (p, t) coordinates.

Prop. (VIII.1.1.6) (Schrodinger Uncertainty Principle). Set $\sigma_A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$, then:

$$\sigma_A^2 2\sigma_B^2 \geq \left| \frac{1}{2}\langle \hat{A}\hat{B} + \hat{B}\hat{A} \rangle - \langle \hat{A} \rangle\langle \hat{B} \rangle \right|^2 + \left| \frac{1}{2i}\langle [\hat{A},]\hat{B}] \rangle \right|^2.$$

Proof: Derived from definition and Schwarz inequality, Cf.[Wiki]. □

Cor. (VIII.1.1.7) (Heisenberg Uncertainty Principle). $\sigma_x \sigma_p \geq \frac{\hbar}{2}$.

Proof:

$$[x, i\hbar\frac{\partial}{\partial x}] = i\hbar.$$

□

Prop. (VIII.1.1.8) (Spectral Decomposition). In Quantum physics, one need to use spectral decomposition of the Hamiltonian operator. But at most cases, there are only countably many eigenstate and the eigenvalue has a lower bound and tends to infinity. In this case, $(\hat{H} + A)^{-1}$ is a compact operator thus by spectral theorem(V.5.4.15) the eigenstate of $\hat{H}$ forms a set of complete basis.

Prop. (VIII.1.1.9) (No-Cloning Theorem).

Calculations

Prop. (VIII.1.1.10) (Virial Theorem). For a system that $V(r) \sim r^n$, the average kinetic energy and the average potential energy has the relation :

$$2\langle T \rangle = n\langle V \rangle.$$

Spin

2 Quantum Computations

Classical Logic Gates

Def. (VIII.1.2.1). A **Boolean function** is a function $f : \mathbb{F}_2^n \to \mathbb{F}_2$.

Def. (VIII.1.2.2) (Classical Logic Gates). There are four classical logic gates, if we let $0, 1 \in \mathbb{F}_2$, then:

- **AND gate**: $(a, b) \mapsto ab$.
- **OR gate**: $(a, b) \mapsto ab + a + b$.
- **NOT gate**: $a \mapsto a + 1$.
- **COPY gate**: $a \mapsto (a, a)$.

Def. (VIII.1.2.3) (Reversible Gates). A gate is called **reversible** iff it is a bijection from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$.

Def. (VIII.1.2.4) (Stimulation). A set of gates is said to be able to **stimulate** a boolean function f iff there is a composition of these gates that maps:

$$(x_1, \ldots, x_{m+n}) \to (g_1(x_1, \ldots, x_{m+n}), \ldots, g_k(x_1, \ldots, x_{m+n}))$$

that if we let $x_{i_1} = a_1, x_{i_2} = a_2, \ldots, x_{i_m} = a_m$ be fixed, where $\{0, 1 \ldots, m+n\} - \{i_1, \ldots, i_m\} = \{j_1, \ldots, j_n\}$(in some order), then $g_1(x_1, \ldots, x_{m+n}) = f(x_{j_1}, \ldots, x_{j_n})$.

A set of gates is called **universal** iff they can stimulates all Boolean function $f : \mathbb{F}_2^n \mapsto \mathbb{F}_2$.

Prop. (VIII.1.2.5) (Classical Gates Universal). The four classical gates are universal. In fact, $AND(x, y) = OR(NOT(x), NOT(y))$, so even AND is disposable, and COPY is not used as well.

Proof: Just use OR gate to juxtapose all possible combinations that are mapped to 1. □

Def. (VIII.1.2.6) (Examples of Reversible Gates).

- The **CNOT gate** is defined to be $CNOT : (a, b) \mapsto (a, a + b)$.
- The **Toffoli gate** is defined to be $CCNOT : (a, b, c) \mapsto (a, b, c + ab)$.

Prop. (VIII.1.2.7). CNOT gate cannot stimulate AND. In particular, CNOT is not universal.

Proof: It can be shown that any Boolean function that can be stimulated by CNOT gate is of the form $(x_1, \ldots, x_n) \mapsto \sum a_i x_i + b$. But AND is of the form $(a, b) \mapsto ab$, which is not of the form, so it is not stimulated by CNOT. □

Prop. (VIII.1.2.8). Toffoli gate is universal.

Proof: It suffices to show it can stimulate AND and NOT, then it can stimulate OR because $OR(x, y) = NOT(NOT(x), NOT(y))$.

AND is outputted in the third bit with $c = 0, a = x, b = y$, NOT is outputted in the third bit with $a = 1 = c, b = x$. □

Quantum Logic Gates

Def. (VIII.1.2.9). A **quantum logic gate** is a unitary matrix. So a quantum logic gate is always reversible.

Prop. (VIII.1.2.10) (Examples of Quantum Gates).

- The **Hadamard gate** H is a rotation on one single qubit given by the matrix $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.
- If a classical gate is reversible and its matrix is unitary, then the same matrix will give out a quantum gate with all entries 0 and 1, called the **quantization** of the classical gate.
- The **Fredkin gate** or CSWAP gate is a three-bit gate defined as the quantization of the gate given by:$(a, b, c) \mapsto (a, a(b + c) + b, a(b + c) + c)$.

3 Quantum Algorithms

Deutsch-Jozsa Algorithm

Prop. (VIII.1.3.1) (Deutsch-Jozsa Algorithm). The **Deutsch-Jozsa problem** is that: given a function $f:\mathbb{F}_2^n\to\mathbb{F}_2$, which is either a constant function or a function that takes half value 0 and half value 1, If we have a box that maps:$(x_1,\ldots,x_n,x)\mapsto(x_1,\ldots,x_n,x+f(x_1,\ldots,x_n))$.

Now there is a **Deutsch-Jozsa algorithm** that can determine if f is a constant function just using the box one time.

Proof: Cf.[Napkin P270]. The circuit is

$$\begin{aligned}(0,0,\ldots,0,1)&\mapsto(H(0),H(0),\ldots,H(0),H(1))\mapsto(H(0),H(0),\ldots,H(0),H(1)+f(H(0),H(0),\ldots,H(0)))\\&\mapsto(H(H(0)),H(H(0)),\ldots,H(H(0)),H(1)+f(H(0),H(0),\ldots,H(0))).\end{aligned}$$

Then we measure all the first n bits in the $|0)/|1)$-basis.

Notice if f is constant, then the first n bits must be all $\pm|00\ldots0)$, so the measure is all 0. And if f is not constant, then the first n bits are entangled, equals the image of

$$\frac{1}{\sqrt{2}^n}\sum_{(a_1,\ldots,a_n)\in\mathbb{F}_2^n}(-1)^{f(a_1\ldots a_n)}|a_1\ldots a_n).$$

after the action of $H^{\otimes n}$, then its coefficient of $|0\ldots0)$ is just 0, so the measure cannot be all 0. □

Quantum Fourier Transform

Def. (VIII.1.3.2) (Discrete Fourier Transform). The **inverse Fourier transform** is defined to be:$(x_0,\ldots,x_{n-1})\mapsto(y_0,\ldots,y_{n-1})$, where

$$y_k=\frac{1}{N}\sum_{j=0}^{N-1}\omega_N^{jk}x_k.$$

This is in fact represented by a van der Waerden matrix of ω_N times $\frac{1}{N}$.

Prop. (VIII.1.3.3) (Fast Fourier Transform). There is a **fast Fourier transform algorithm** that can calculate the Fourier transform in $O(N\log N)$ times.

Proof: □

Def. (VIII.1.3.4) (Quantum Fourier Transform). The **quantum Fourier transform** is a gate represented by a matrix U_{QFT} which is the van der Waerden matrix of ω_N times $\frac{1}{\sqrt{N}}$.

Proof: It suffices to prove this matrix is truly unitary. □

Prop. (VIII.1.3.5) (Tensor Representation). The trick of the quantum Fourier transform lies in its connection with the 2-adic decimal representation:

$$\begin{aligned}U_{QFT}(|x_nx_{n-1}\ldots x_1))=&\frac{1}{\sqrt{N}}(|0)+\exp(2\pi i\cdot 0.x_1)|1))\\&\otimes(|0)+\exp(2\pi i\cdot 0.x_2x_1)|1))\\&\otimes\ldots\\&\otimes(|0)+\exp(2\pi i\cdot 0.x_nx_{n-1}\ldots x_1)|1))\end{aligned}$$

Proof: Direct calculation.? □

Prop. (VIII.1.3.6). Now by the above tensor representation, the thing seems to be beautiful, and it seems the Fourier transform for 2^n data can be done in n^2 steps. And this is true, QFT_n is inductively define, Cf.[Napkin P273].

Shor's Algorithm

Def. (VIII.1.3.7). For a number $M = pq$, where p, q are different odd prime numbers. Then an $x \mod M$ is called **good** iff: $(x, M) = 1$, $r = \mathrm{ord}(x)$ is even, and neither of $x^{r/2} \pm 1$ is divisible by M.

Then at least half of $(\mathbb{Z}/M\mathbb{Z})^*$ is good.

Proof: It suffices to consider a fixed order $2a$, and this is additive in $\mathbb{Z}/(2a, p-1)\mathbb{Z} \times \mathbb{Z}/(2a, q-1)\mathbb{Z}$.?. □

Remark (VIII.1.3.8). If we find a good x for M, then $x^{r/2} \pm 1$ contains separately a prime p or q, so we can use Euclidean algorithm to extract a prime of M. This is just the ideal of Shor's Algorithm.

Prop. (VIII.1.3.9) (Shor's Algorithm). For $M = pq$, we can factor p, q out in $O((\log M)^2)$ time.

Proof: Cf.[Napkin P274]. □

Grover's Algorithm

Prop. (VIII.1.3.10).

Prop. (VIII.1.3.11) (Grover's Algorithm). If there are n items labeled $\{0, \dots, n-1\}$, and there is a marked item w. then there is an algorithm that find w in $O(\sqrt{n})$ times.

Proof: Cf.[Quantum Algorithm MIT P33,35]. □

Chapter IX

Others

IX.1 TO DO List

- a right Kan fibration which is a weak equivalence is a trivial fibration.
- smooth irreducible representations of Weil group is admissible.
- fundamental class relation with Weil group.
- conductor of a Weil representation is an integer.
- Refresh the Bernstein p-adic group representation theory by language of abstract harmonic analysis.

IX.2 Elementary Mathematics

1 Algebra

Prop. (IX.2.1.1). If a_n is a series of real numbers that $a_{m+n} \geq a_m + a_n$, then a_m/m converges to $\lambda = \sup a_n/n$.

Proof: Let $\lambda - a_n/n < \varepsilon$, then for any N large, $N = kn + m$ for $m < n$, so $a_N \geq ka_n + a_m$, so $\liminf a_N/N \geq a_n/n$ for N large. Thus the result follows. □

2 Number Theory

Prop. (IX.2.2.1). $v_p(n!) = \frac{n-c(n)}{p-1}$, where $c(n)$ is the sum of the presentation of n in the p-adic base.

Cor. (IX.2.2.2). $v_p(C_{a+b}^b)$ equals the number of carries when adding a and b in base p.

Prop. (IX.2.2.3) (Cauchy-Davenport). If A, B are two nonempty subsets of $\mathbb{F}_p$, then $|A+B| \geq \min\{p, |A| + |B| - 1\}$.

Proof: If $|A| + |B| > p$, then this is trivial, because $A \cap (B - x) \neq \varnothing$ for all x.

Now if $|A| + |B| \leq p$, and if $A + B \subset C$ with $|C| = |A| + |B| - 2$, define $f = \prod_{c\in C}(x + y - c)$, then $f(a, b) = 0$ for all $a \in A, b \in B$, but the coefficient of the highest degree term $x^{|A|-1}y^{|B|-1}$ is $C_{|A|+|B|-2}^{|A|-1} \neq 0$, so this contradicts combinatorial Nullstellensatz(I.3.2.7). □

Prop. (IX.2.2.4) (Power Lifting). If $a \equiv b \mod p$, then $a^{p^n} \equiv b^{p^n} \mod p^{n+1}$ for all $n > 0$.

Proof: Prove by induction, if $a^{p^n} = b^{p^n} + p^{n+1}c$ for some c, then $a^{p^{n+1}} = b^{p^{n+1}} + p^{n+2}d + t^{p(n+1)}c^p$, so we are done. □

Lemma (IX.2.2.5). $\sum_{x=0}^{p-1} x^k \equiv -1 \mod p$ iff $p - 1|k$, and $\equiv 0 \bmod p$ otherwise.

Proof: Choose a a that $a^k - 1$ is not divisible by p, this is doable iff k is not divisible by $p - 1$, then it is clear. □

Prop. (IX.2.2.6) (Quadratic Legendre Symbol Sum). For add prime p,

$$\sum(\frac{x^2+1}{p}) = -1.$$

Proof: The sum is equivalent modulo p to $\sum_{x=0}^{p-1}(x^2 + ax + b)^{\frac{p-1}{2}}$, which by the lemma(IX.2.2.5) above equivalent to -1 modulo p. Now it can not be $p - 1$, because otherwise there is a solution for $p|x^2 + 1$, and then it can be calculated directly. □

Prop. (IX.2.2.7). The multiplicative group of $\mathbb{Z}/2^n$ is $\mathbb{Z}/2 \times Z/2^{n-2}$ for $n \geq 2$.

Proof: it suffices to show $3^{2^{n-3}} \neq \pm 1 \mod 2^n$, and $3^{2^{n-2}} \equiv 1 \mod 2^n$. □

Lemma (IX.2.2.8). If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ satisfies $a_n \geq 1, a_{n-1} \geq 0$, and $|a_i| \leq H$ for $i \leq n - 2$, then for any root α of P, either $\operatorname{Re}\alpha \leq 0$, or $|\alpha| < \frac{1+\sqrt{1+4H}}{2}$.

Prop. (IX.2.2.9) (b-adic decomposition and Irreducibility). If $b > 2$ and p is a prime, consider the b-adic expansion $p = \sum a_n b^n$, then the polynomial

$$\sum a_n X^n$$

is irreducible over $\mathbb{Z}$.

Proof: If $p(x) = h(x)r(x)$, use the lemma(IX.2.2.8) to show that $r(b)$ and $h(b)$ cannot be 1, thus $h(b)r(b) = p$ cannot happen. □

Algebraic Numbers

Prop. (IX.2.2.10) (Lindemann). If α is an algebraic number, then e^{α} would be a transcendental number.

Proof: □

Cor. (IX.2.2.11). π and e are transcendental numbers.

Proof: $e^{2\pi i} = 1$, and $e^1 = e$. □

3 Combinatorics

Prop. (IX.2.3.1) (Spencer's Lemma). If there is a triangulation of a plane polygon P, for arbitrary 3-color numbering$(0, 1, 2)$ of the vertices, if the number of edges on the boundary with color $(0, 1)$ is odd, then there is a triangle with vertices of pairwise different colors.

Proof: In fact the number of those triangles with vertices of pairwise different colors is odd. In fact, the number of $(0, 1)$-edges on a triangle is odd iff its vertices has pairwise different colors. But the sum of numbers of $(0, 1)$-edges on the triangles are odd, by hypothesis, thus the result. □

Prop. (IX.2.3.2) (Monsky Theorem). A square cannot be divided into m triangles of the same area, where m is odd.

Proof: Choose a 3-coloring on $\mathbb{R}^2$: By(II.1.3.3) there can be a non-Archimedean valuation on R that extends the p-adic valuation on $\mathbb{Q}$, choose the extended 2-adic valuation, then color a point (x, y)

- 0 if $|x|_2 < 1, |y|_2 < 1$,
- 1 if $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.
- 2 if $|y|_2 \geq 1$ and $|y|_2 > |x|_2$.

Then there are two things:

- The coloring is invariant under translation by vectors represented by a point of color 0.
- The valuation of the area of a triangle with vertices of different color is bigger than 1, because we can assume one vertex is the origin, and then its area is $\frac{1}{2}|x_1y_2 - x_2y_1|$, which, because the coloring, must have 2-adic valuation bigger than 1.

Now back to the question, let the square be placed as a unit square, then its area is 1, and it has exactly one $(0, 1)$-edge, so by Spencer's lemma(IX.2.3.1), it has a triangle with vertices of pairwise different colors, so its area A has valuation> 1, but the total area is $mA = 1$ that has valuation 1, so $|m| < 1$, which means that m is even. □

Prop. (IX.2.3.3) (Hindman's Theorem). Whenever the set of natural numbers are colored with f.m. colors, one can find an infinite subset $A \subset \mathbb{N}$ and a color c that whenever $F \subset A$ is finite, the color of the sum of numbers in F is colored c.

Proof: Cf.[W. W. Comfort, Ultrafilters: some old and some new results, Bull. Amer. Math. Soc. 83 (1977) 417–455.] □

Ramsey's Theory

Prop. (IX.2.3.4) (Finite Ramsey's Theorem). For any positive natural number k, r, s, there is a n that if we color the r-subsets of a set with cardinality n into s groups, then there is a subset of cardinal k that all its r-subsets are colored the same.

Proof: Cf.[Set Theory P218]. □

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