Elements of Mathematics

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Preface

This is a latex version of subtle or important materials I encountered while studying in Peking university. I started this project in the fall of my third undergraduate year, noticing that I have a poor memory and consistently forget what I have already learned thus struggle to check details. So it came to me that I can compile all the proofs of theorems I cannot recall that is hard and subtle yet appearing over and over again. But finally it turns out I want to make it as comprehensive as possible. That's it.

Notice: This is hardly a *readable* book, I use it as a reference. It only contains materials that I'm interested in and many proofs are still missing. And maybe I will or maybe I won't complete them.

It should be made clear that I took proofs from many different places, so it should not be considered anything in this book originated from me. Until I get a full extensive reference of this note, I have few rights to the texts.

I truly hope this note can contribute to my study and help anyone who read it.

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Chapter I

Algebra

I.1 Set Theory

1 Cardinal & Ordinal

Def. (1.1.1). A *cardinal number* is an equivalence class, where equivalence and ordering is given by injectives and surjectives. it is used to describe the 'size' of a set. It can only be compared, not operated.

An *ordinal* is an equivalence class of isomorphic well-ordered transitive (i.e. every element is a subset of itself) sets. Notice that two ordinal can have the same cardinality. The ordering of ordinal is by inclusion. Ordinal numbers can apply arithmetics.

The least ordinal having cardinality α is called the initial ordinal of α. The axiom of choice asserts that every cardinal has a initial ordinal.

Prop. (1.1.2) (Bernstein's Theorem). If there is an injective from A to B and an from B to A, then there is a bijection from A to B. Thus the ordering of the cardinal is well-defined.

Lemma (1.1.3). The ordering of ordinals is an total ordering. Every element of an ordinal is an ordinal, and if an ordinal $\beta \subset \alpha$, then $\beta \in \alpha$. Cf.[Set Theory Jech P108].

Def. (1.1.4). The *cofinality* of an ordinal α is the smallest ordinal δ that is the order type of a cofinal subset of α.

The*cofinality* of or a post (i.e partially ordered set) α is the is the smallest cardinality δ of a cofinal subset of α.

Ordinal Arithmetic

Cantor Normal Form

Prop. (1.1.5) (Transfinite Induction/Recurtion).

2 Filters & Ultrafilters

3 Axiomatic Set Theory

I.2 Linear Algebra

1 Similarity

Prop. (2.1.1). A matrix that $J^2+1=0$ is similar to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^n$. (Use cyclic decomposition).

Prop. (2.1.2) (Jordan Form). For a matrix over a algebraically closed field, it is similar to a matrix of blocks $\lambda_i I + N, N x_i = x_i + 1$. A

For a real matrix, it is similar to a matrix of blocks of the above form together with $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ on the diagonal and $I_{2\times 2}$ on the upper side.

2 Conguence

Prop. (2.2.1). A symmetric matrix A is orthogonally diagonalizable. Similarly, a skew-symmetric matrix is orthogonally diagonalizable and an (skew)hermitian matrix is unitarily diagonalizable.

Proof: For any real matrix A and any vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle.$$

Now assume that A is symmetric, and $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of A corresponding to distinct eigenvalues λ and μ. Then

$$\lambda\langle \mathbf{x}, \mathbf{y}\rangle = \langle \lambda\mathbf{x}, \mathbf{y}\rangle = \langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x}, \mu\mathbf{y}\rangle = \mu\langle \mathbf{x}, \mathbf{y}\rangle.$$

Therefore, $(\lambda-\mu)\langle \mathbf{x}, \mathbf{y}\rangle = 0$. Since $\lambda - \mu \neq 0$, then $\langle \mathbf{x}, \mathbf{y}\rangle = 0$, i.e., $\mathbf{x} \perp \mathbf{y}$.

Now find an orthonormal basis for each eigenspace; since the eigenspaces are mutually orthogonal, these vectors together give an orthonormal subset of $\mathbb{R}^n$. □

Prop. (2.2.2) (Normal operator). More generally, a normal operator over $\mathbb{C}$ is unitary diagonalizable using resolution of identity(3.7.3) because the spectrum are discrete thus the point projection is orthogonal.

Prop. (2.2.3). Over $\mathbb{R}$, a skew-symmetric matrix are orthogonally congruent to $\mathrm{diag}\Big\{\begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}\Big\}_i$.

Proof: Choose a α, β and choose their orthogonal complement. □

Cor. (2.2.4). For a matrix that $J^2+1=0$, by (2.1.1),there is a unique inner product s.t. J is orthogonal and then it is orthogonally congruent to $\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rangle_n$. (Use cyclic decomposition).

so this J is equivalent to a complex structure, homeomorphic to $O(n)/U(\frac{n}{2})$.

3 Determinant

Prop. (2.3.1) (Sylvester's determinant identity). If A and B are matrices of sizes $m\times n$ and $n \times m$, then

$$\det(I_m + AB) = \det(I_n + BA)$$

Proof:

$$\begin{bmatrix} 1 & \mathrm{A} \\ \mathrm{B} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathrm{B} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-\mathrm{BA} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{A} \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \mathrm{A} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\mathrm{AB} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathrm{B} & 1 \end{bmatrix}$$

□

Prop. (2.3.2). The determinant of a symplectic matrix $\in Sp(n)$ has determinant 1.

Proof: A symplectic matrix preserves the symplectic structure thus the symplectic form ω, hence ω^n which is $n!$ times the volume form. □

Prop. (2.3.3). $GL_n(\mathbb{C})$ can be embedded into $GL_{2n}(\mathbb{R})$, with determinant $|\det|^2$. And in this way, $U(n)$ is mapped into $O(2n)$. Also, $O(n)$ embeds into $U(n)$ diagonally.

Proof:

$$X+iY \mapsto \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \sim \begin{bmatrix} X & Y \\ iX-Y & X+iY \end{bmatrix} \sim \begin{bmatrix} X-iY & Y \\ 0 & X+iY \end{bmatrix}$$

□

Prop. (2.3.4). There is a polynomial Pf s.t. $\det M = \mathrm{Pf}(M)^2$ for a skew-symmetric matrix. This is because a skew symmetric is equal to $A^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^k A$ for A an orthogonal matrix (2.2.1), so it has determinant $(\det A)^2$ and A and depends polynomially on the entries of M.

Cor. (2.3.5).

$$\mathrm{Pf}(A^t M A) = \det A \cdot \mathrm{Pf}(M).$$

Because we only need to consider the sign and it is determined by letting $A = \mathrm{id}$.

4 Spectral Theory

See also 7

Prop. (2.4.1). a family of commuting diagonalizable operator can be simultaneously diagonalized.

Proof: □

Prop. (2.4.2). in an algebraically closed field, diagonalizable $\iff$ normal. And the eigenvectors are orthogonal to each other.

Proof: □

5 Decompositions

Prop. (2.5.1) (Polar Decomposition). $GL_n(\mathbb{R})$ can be decomposed as $P \cdot O(n)$, where P is a positive symmetric matrix and $O(n)$ the orthogonal matrix. a positive symmetric matrix can be diagonalized, so $GL_n(\mathbb{R})$ have $O(n)$ as deformation kernel.

Similarly, $Sp(2n)$ can be decomposed as $P \cdot U(n)$, because $O(2n) \cap Sp(2n) = U(n)$. And it has $U(n)$ as deformation kernel.

Prop. (2.5.2) (Bruhat Decomposition).

$$GL_n[K] = BWB$$

其中 W 为置换矩阵，B 为上三角矩阵，且分解是不交并。

Proof: Cf.[群与表示 王立中] □

Prop. (2.5.3) (Iwasawa Decomposition).

I.3 Abstract Algebra

1 Field Theory

Prop. (3.1.1).

$$G(\mathbb{Q}[\mu_n]/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*.$$

Proof: We choose a prime p prime to n and show that μ_n^p is conjugate to μ_n.

Let $X^n - 1 = f(X)h(X)$ with f(X) minimal polynomial of μ_n.If $f(\mu_n^p) \neq 0$, then $h(\mu_n^p) = 0$,thus $h(X^p) = f(X)g(X)$.So module p, $X^n - 1$ has a multi root, which is wrong. □

Prop. (3.1.2).

$$\mathrm{Gal}(F_{p^n}/F_p) = \mathbb{Z}/n\mathbb{Z}.$$

Proof: Generated by Frobenius. □

Prop. (3.1.3). A separable extension or an extension having finitely many middle fields has a primitive element.

Proof: □

2 Transcendental extension

Prop. (3.2.1). Let K be an extension of a field k, a **transcendental base** is an algebraically independent set that any element is algebraic over it. Then the number of elements in any algebraically independent set $\leq$ the number of elements in any transcendental base. In particular, given any algebraically independent set $S \subset T$ a set over which K is algebraic, S can be extended to a transcendental base.

Proof: Let $X = \{x_1, \ldots, x_m\}$ transcendental base of minimal number, $S = \{w_1, \ldots, w_n\}$ an algebraically independent set. If $n > m$, we proceed by changing one element a time using induction and prove that K is algebraic over $\{w_1, \ldots, w_r, x_{r+1}, \ldots, x_m\}$, contradiction.

Because w_{r+1} is algebraic over $\{w_1, \ldots, w_r, x_{r+1}, \ldots, x_n\}$, we have a minimal polynomial

$$f = \sum g_j(w_{r+1}, w_1, \ldots, w_r, x_{r+2}, \ldots, x_m) x_{r+1}^j$$

s.t. $f(w_{r+1}, w_1, \ldots, x_m) = 0$ (after possibly renumbering x_i, this x must exists because S is itself algebraically independent). So x_r is algebraic over $\{w_1, \ldots, w_{r+1}, x_{r+2}, \ldots, x_m\}$, hence K is independent over it, too. □

3 Galois Theory

Prop. (3.3.1) (Primitive element). a finite extension E/k is primitive iff there are only finitely middle fields. And if E/k is separable, this is satisfied.

Proof: If k is finite, this is simple. Assume k infinite, for any two elements α, β, consider $k(\alpha + c_i\beta)$, if there is only finitely many middle fields, there exists two that is equal, so $k(\alpha, \beta) = k(\gamma)$. Proceeding inductively, E is primitive.

Conversely, if $k(\alpha) = E$, every middle field corresponds to a divisor of the irreducible polynomial of α. This map is injective, because for any g_F, degree of α over F is the same over the degree over the coefficient field of g_F, so it must be equal to F.

If E/k is separable, Let

$$P(X) = \prod_{i \neq j} (\sigma_i\alpha + X\sigma_i\beta - \sigma_j\alpha - X\sigma_j\beta)$$

for different embedding σ_i, σ_j of $E(\alpha, \beta)$ into k^{al}. Then it is not identically zero, thus there exists c that $\sigma_i(\alpha + c\beta)$ is all distinct, thus generate $K(\alpha, \beta)$. □

Prop. (3.3.2) ((Artin) Galois Main Theorem). Let G be a finite group of automorphisms of K. Then K/K^G is Galois of Galois group G.

Proof: For every element x, set $\{\sigma_1 x, \ldots, \sigma_r x\}$ be distinct conjugates, then $f(X) = \prod_i^r = (X - \sigma_i x)$ shows that K is separable and normal over K^G. And primitive element theorem shows that $[K : K^G] \leq |G|$, so it must equals G. □

Prop. (3.3.3) (Infinite Galois Theorem). The middle fields correspond to the closed subgp of $G(L/K)$.

Proof: The main proof is that $G(L/L^H) = H$ for a closed subgp of $G(L,K)$. If σ fixes L^H but is not in H, because for every finite field M, H corresponds to $M/(M \cap L^H)$, so $\sigma G(L/M) \cap H \neq \varnothing$. So σ is in the closure of H thus in H. □

Prop. (3.3.4) (Normal Basis Theorem). for a finite Galois extension, normal basis exists.

Proof: Finite case:

The Galois group is cyclic, and the linear independent of characters shows that the minimal polynomial of σ is n-dimensional thus equals $X^n - 1$. regard L as a $K[X]$ module thus by (5.2.5) is a direct sum of modules of the form $K[X]/(f(x))$, $f(x)|X^n - 1$ and the minimal polynomial for the action of X is $X^n - 1$. So it is isomorphic to $K(X)/(X^n - 1)$.

Infinite Case:

Let

$$f([X_\sigma]) = \det(t_{\sigma_i,\sigma_j}), \qquad t_{\sigma,\tau} = X_{\sigma^{-1}\tau}$$

□

We see $f \neq 0$ by substituting 1 for X_{id} and 0 otherwise. So it won't vanish for all x if we substitute $X_\sigma = \sigma(x)$ because $[\sigma(x)]$ is pairwisely different. Thus there exists w s.t.

$$\det(\sigma^{-1}\tau(w)) \neq 0.$$

Now if

$$\sum a_\tau \tau(w) = 0, \qquad a_\sigma \in K,$$

act by σ for all σ, we get $[\sigma^{-1}\tau(w)][a_\sigma] = 0$, thus $[a_\sigma] = 0$.

Prop. (3.3.5) (Kummer Theory). There exists an inclusion preserving isomorphism between the lattice of Kummer extensions L of K and the lattice of subgroups of L containing K^n:

$$L \mapsto \Delta = (L^\times)^n \cap K^\times, \qquad \Delta \mapsto K(\sqrt[n]{\Delta}).$$

And $\Delta/(K^\times)^n$ is isomorphic to $\chi(G_{L|K})$.

Proof: Cf.[Neukirch CFT P116]. □

I.4 Representation Theory

1 General Representation

Prop. (4.1.1) (Schur's lemma). If π is an countable dimensional $\mathbb{C}$-representation, then $\mathrm{End}(V) \cong \mathbb{C}$.

Proof: Notice we only have to find an eigenvalue of ϕ, but otherwise $\{(\phi-a)^{-1}$ is uncountable and linearly independent over $\mathbb{C}$, so $\dim(\mathbb{C}(\phi))$ is uncountable, contradiction. □

2 Locally Compact Groups

Prop. (4.2.1) (Brauer-Nesbitt). For a finite group G, if two finite dimensional semisimple representations over a field has the same char poly for every element g of G, then they are isomorphic.

Proof: Just use the irreducible representations are orthogonal and that they have the same and for char p, we can use divide by p and the char poly becomes p-th power and we can do this forever, contradiction. □

Compact Groups

Prop. (4.2.2) (Peter Weyl). For a compact group G, $\{\phi_{ij}(g); \phi(g) = (\phi_{ij}(g)), \phi$ an irreducible character$\}$ is a basis for the Hilbert space $L_2(G)$. Cf.[连续群 Pontryagin 第五章 § 33].

3 Locally Profinite Groups

Def. (4.3.1). A locally profinite group is a topological group that every open neighbourhood of id contains a compact open subgroup of G.

Def. (4.3.2). A representation of a topological gp on a discrete vector space is called **smooth** iff $G \times V \to V$ is continuous and **admissible** iff V^K is finite dimensional for every compact open subset K of G.

Prop. (4.3.3). A smooth irreducible representation is admissible. In fact, this is true for general connected reductive group.

Proof:

□

I.5 Commutative Algebra

1 Adjointness & Exactness

Prop. (5.1.1) (Induced&Coinduced). Given a ring homomorphism $S \to$R.

- $f^*M = N_R$, the restriction.
- $f_!M = R \otimes_S M$ is the induced module, it is left adjoint to restriction.
- $f_*M = \mathrm{Hom}_S(R, M)$ is the coinduced module, it is right adjoint to restriction.

Prop. (5.1.2). S^{-1} is an exact functor from $R-\mathrm{mod}$ to $R-\mathrm{mod}$.

Proof: Suppose that x/s maps to zero in $S^{-1}N$ for some $x/s \in S^{-1}M$. Then by definition there is a $t \in S$ such that $v(xt) = v(x)t = 0$ in M, which means $xt \in \mathrm{Ker(v)}$. By the exactness of $L \to M \to N$, we have $xt = u(y)$ for some y in L. Then x/s is the image of y/st. This proves the exactness. □

Cor. (5.1.3).

$$(R/I)_{\bar{P}} \cong R_P/IR_P$$

in particular,

$$k(R/P) \cong R_P/PR_P$$

Prop. (5.1.4). Taking direct limits commutes with tensor product. (element chasing).

2 Projective & Injective

Prop. (5.2.1). A module over a ring is projective iff it is a direct summand of a free module.

Prop. (5.2.2) (Baer's Criterion). A right $R-$module I is injective iff for every right ideal J of R, every map $J \to I$ can be extended to a map $R \to I$. (Direct from (1.5.6)).

Cor. (5.2.3). A module over a PID is injective iff it is divisible.

Prop. (5.2.4). The category of R-mod has enough injectives by(1.5.2), and it has enough projectives trivially.

Prop. (5.2.5) (Classification of Modules over PID).

1) PID is UFD thus Noetherian.
2) Submodule of a free module over a PID is free.
3) Finitely generated torsion-free module over a PID is free.
4) Finitely generated module over a PID has a primary decomposition $M = \bigoplus_i R/(q_i)$,where (q_i) is primary ideals.

So pojective $\iff$ free $\iff$ torsion-free.

Proof: Cf.[Lang P45] □

Prop. (5.2.6). Any projective module of finite type over $K[X_1, \ldots, X_k]$ is free. (Highly nontrivial).

3 Flatness

Prop. (5.3.1) (Gororov-Lazard). Any flat $A-$module is isomorphic to an direct limit of free modules of finite type.

Prop. (5.3.2). A R-module M is flat iff $\mathrm{Tor}_1(M,N)=0$ for all N, iff $\mathrm{Tor}_i(M,N)=0$ for all $i>0$, because we have: if

$$0\to M_1\to M_2\to M_3\to 0$$

M_2,M_3 flat, then M_1 is flat (Use 9 entry sequence). So $\mathrm{Tor}_{n+1}(M_3,N)=\mathrm{Tor}_n(M_1,N)=0$.

Thus we have the class of flat modules is adapted $-\otimes N$ for all N.

4 Integral Extension

Prop. (5.4.1). Let A a subring of $B, A\to B$ integral. Then:

1.If A is local and p is the maximal ideal of A, then the prime ideals of B lying over p is precisely the maximal ideal of B.

2.There is no inclusion relation between the prime ideals of B lying over a fixed prime ideal of A.

3.The Spec map is surjective.

4.The going-up holds.

Proof: 1:Since for two ring one integral over another, one is a field iff the other is a field.
2:Maximal ideal cannot contain each other.
3:Since $A_p\neq 0$, $B_p\neq 0$, so it has a maximal ideal.
4:From 3. □

Prop. (5.4.2). Let A a subring of $B, A\to B$ integral noetherian. Then:

1.$\dim(A)=\dim(B)$

2.$\mathrm{ht(P)}=\mathrm{ht(P\cap A)}$

3.if going up holds, then $\mathrm{ht(J)}=\mathrm{ht(J\cap A)}$ for any ideal J.

Proof: 1.By the preceding lemma, there is no inclusion relation between prime over a fixed prim, so $\dim(B)\leq\dim(A)$. On the other hands, going-up holds, so $\dim(B)\geq\dim(A)$.
2.follows from (5.5.1)(1) since $\mathrm{ht(P/(P\cap A)B)}=0$ by the preceding lemma.
3.by 2 and surjectiveness of Spec for integral extension. □

5 Dimension

Prop. (5.5.1). Let $A\to B$ noetherian, let $p=P\cap A$,then:

1.$\mathrm{ht(P)}\leq\mathrm{ht(p)}+\mathrm{ht(P/pB)}$,in other words $\dim(B_P)\leq\dim(A_p)+\dim(B_P\otimes k(p))$. Where $k(p)=A_p/pA_p$ and $B\otimes k(p)=B_p/pB_p$.

2.equality holds if going-down holds. For example, if it is flat.

3.if Spec map is surjective and going-down holds, then we have i)$\dim(B)\geq\dim(A)$, and ii)$\mathrm{ht(I)}=\mathrm{ht(IB)}$ for ideal I of A.

4.if going-up holds, then $\dim B\geq\dim(A)$. e.g. B integral over A Cf. (5.4.2)

Proof: Cf.[Commutative Algebra Matsumura (13.B)] □

6 Primary Ring

Def. (5.6.1). A primary ring is a unital ring with only one maximal ideal. Notice that this implys the ring is local. e.g. R/p where p is a primary ideal associated with a maximal ideal m, is primary.

Prop. (5.6.2). An artinian ring is a direct sum of noetherian primary rings and the decomposition is unique.

Proof:
Take a primary decomposition to notice that 0 is a product of maximal ideals, (because of artinian). Then take

$$R_i = \prod_{j\neq i} \mathfrak{m}^{e_i}$$

then:

$$R \cong \bigoplus R/R_i, \qquad R_i \cong R/\mathfrak{m}^{e_i}$$

Notice R_i and $\mathfrak{m}^{e_i}$ coprime and nonintersecting, so take every decomposition of $x = x_i + y_i$ and prove $x = \sum x_i$. The map $R \to R : x \to R/R_i$ has kernel $\sum_{j\neq i} R_i \cong \mathfrak{m}^{e_i}$ by induction. Uniqueness:

Lemma (5.6.3). In a primary ring, there is no nontrivial idempotent element. Because e and $1-e$ will all belong to the same maximal ideal m.

the decomposition gives a way to decompose 1 to sum of idempotent elements and is determines by it. $1 = \sum e_i = \sum f_i$, so $e_j = \sum e_j f_i$. But e_i cannot decompose, so $e_j = e_j f_{i(j)}$, $\exists\ i(j)$. the following is easy to show these two decomposition is the same. □

7 Power Series

Prop. (5.7.1). Let k be a field, then the power series $k[[X_1, \ldots, X_n]]$ is a UFD.

Proof: Cf.[Algebra Lang P209]. □

8 Jacobson Ring

Def. (5.8.1). A commutative ring is called Jacobson if every prime ideal is intersection of maximal ideals. In particular, the Jacobson radical equals the nilradical.

Prop. (5.8.2). The Jacobson radical of R is:

$$J = \{r \in R | 1 + rs \text{ is a unit } \forall \text{s} \in \text{R}\}$$

Prop. (5.8.3). One way is trivial and for the other if r is not in a maximal ideal M, then $(r) + M = (1)$, so contradiction.

Prop. (5.8.4) (Generalized Nullstellensatz). If R is Jacobson and S is a finitely generated R-algebra, then S is Jacobson and the maximal ideal of R intersect with R a maximal ideal and the quotient ring extension is finite, (hence algebraic).

Proof: Cf.[Commutative Algebra Eisenbud P132] □

I.6 Homological Algebra

1 Fundamentals

Functoriality of Resolutions

Prop. (6.1.1) (Horseshoe Lemma). For a exact sequence $0 \to X_1 \to X \to X_2 \to 0$ and a injective resolution of X_1 and X_2, there is a injective resolution of X commuting with them. (Choose them one-by-one, in fact, $I_n = I_n^1 \oplus I_n^2$. Snake lemma told us that the cokernel is an exact sequence, use that to define the next one.

2 Simplicial Method

Prop. (6.2.1) (Eilenberg-Zilber). The three kinds of geometrization of a bisimplicial set is the same: geometrization the diagonal simplicial set, the twice geometrization of left(resp. right) simplicial set.

3 Derived Category

Def. (6.3.1). A class of morphisms S in a category is called localizing if:
- S is closed under composition and has identity.
- for every $s \in S$ and f, there is a $t \in S$ and g, s.t. $f \circ t = s \circ g$(resp. $t \circ f = g \circ s$).
- the existence of a $s \in S$ s.t. $sf = sg$ is equivalent to the existence of $t \in S$ s.t. $ft = gt$.

This will generate a roof-dominating equivalence and make sure it is an equivalence relation.

Prop. (6.3.2). The isomorphisms in $D^*(A)$ is of the form $t \circ s^{-1}$. (look at the homology map they induced).

Prop. (6.3.3). If $\mathcal{B}$ is a full subcategory that $S \cap \mathcal{B}$ is a localizing category of $\mathcal{B}$ and any $s \in S$ can be 'denominated' in one given side (any one is OK) into $\mathcal{B}$, then $\mathcal{B}[S \cap \mathcal{B}^{-1}]$ is a full subcategory of $\mathcal{C}[S^{-1}]$. The proof is easy, use left roof or right roof.

Remark (6.3.4). Remember the translation operator $K[n]$ makes the complex lower n dimensions.

Prop. (6.3.5) (Distinguished Triangle). For any morphism $K^\bullet \to L^\bullet$, there exists a exact sequence of Complexes

$$0 \to K^\bullet \to \mathrm{Cyl}(f) \to C(f) \to 0$$

commuting with (in $K(\mathcal{A})$)

$$0 \to K^\bullet \to \quad L^\bullet \quad \to C(f) \to K^\bullet[1].$$

And $K^\bullet \to L^\bullet \to C(f) \to K^\bullet[1]$ is called a distinguished triangle. And exact triple of complexes in $\mathrm{Kom}(\mathcal{A})$ is quasi-isomorphic to an distinguished triangle.

A distinguished triangle will induce a long exact sequence, for this, just need to verify that the δ-homomorphism coincide with the morphism that $C(f) \to K^\bullet[1]$ induces.

Notice all this can imitate the similar parallel construction in the topology category.

Proof: Cf.[Gelfand P157] □

Def. (6.3.6). A **triangulated category** is an additive category with a T: additive automorphism and an isomorphism class of distinguished triangles satisfying the following axiams:

TR1) $X \xrightarrow{id} X \to 0 \to X[1]$ is distinguished. Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle.

TR2) A triangle is distinguished iff the helix it generate is distinguished.

TR3) Any two consecutive morphisms of two distinguished class can be extended to a morphism of distinguished class.

TR4) Any diagram of the type "upper cap" can be completed to a octahedron diagram.

Prop. (6.3.7). For a distinguished triangle and an object, two long exact sequence exists. In particular, composition of consecutive maps in a distinguished triangle is 0.

Thus the extension of TR3 of two isomorphisms is an isomorphism by 5-lemma. And so the completion in TR2 is unique.

Prop. (6.3.8). For Abelian category A, the categories $K^*(A)$ is triangulated. This is hard to verify, but it solves every problem. Cf[Gelfand P246].

Prop. (6.3.9). K is a triangulated category and a localizing class S compatible with T, i.e. $s \in S \iff T(s) \in S$ and the extension in $TR3$ of f, g in S is in S. Then the localizing category $K[S^{-1}]$ is triangulated.

Cor. (6.3.10). $D(\mathcal{A})$ is a triangulated category. And for a distinguished triangle, the long exact sequence exists. The distinguished triangle is just the obvious one.

Def. (6.3.11). The **derived category** of an Abelian category $D(\mathcal{A})$ represents the universal property that any functor to a category $\mathcal{A} \to \mathcal{C}$ s.t. quasi-isomorphisms is mapped to isomorphisms uniquely factors through $D(\mathcal{A})$.

It can be defined as the localization of quasi-isomorphisms, but the class of quasi-isomorphisms is not localizing. But one can prove the quasi-isomorphisms in $K(\mathcal{A})$ is localizing and the localization of quasi-isomorphisms in $K(\mathcal{A})$ is equivalent to $D(\mathcal{A})$. Cf.[Gelfand P159]

Notice that equivalent roofs induce the same map on homology, so the cohomology functor can be regarded defined on $D(\mathcal{A})$.

$$\mathcal{A} \to K(\mathcal{A}) \to K(\mathcal{A})[S^{-1}] = D(\mathcal{A}) \xrightarrow{H^*} \mathcal{A}.$$

Prop. (6.3.12).

4 Acyclic Elements and Derived Functors

Prop. (6.4.1). For a class of objects $\mathcal{R}$ in $\mathcal{A}$ stable under finite direct sum and are adapted to a left exact functor F, i.e. $Kom^+(\mathcal{R})$ is F-acyclic and every object in $\mathcal{A}$ is a subobject of an object from $\mathcal{R}$. Just need to verify the condition of(6.3.3). Similarly for the opposite category.

And in this case $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ is equivalent to $D^+(\mathcal{A})$.

Proof: The hard part is to prove every complex in $K^+(\mathcal{A})$ is quasi-isomorphic to a complex in $K^+(\mathcal{R})$, for this, use direct construciton. Cf.[Gelfand P187]. □

Remark (6.4.2). The adapted class is just the objects that are F-acyclic. (First Prove they must they must contain injectives as a sufficiently large class)Cf.[Gelfand P195].

Lemma (6.4.3). If s is a quasi-isomorphism between an object from $K^+(\mathcal{I})$ to an object from $K^+(\mathcal{A})$, then there exists a reverse map to compose to id_I. Cf.[Gelfand P180]

Cor. (6.4.4). $K^+(\mathcal{I})$ is a saturated subcategory of $D+(\mathcal{A})$. And if $\mathcal{A}$ has enough injectives, this is an equivalence of category. (We only need to verify that the localization of $K^+(\mathcal{I})$ is itself, using the last proposition). In particular, this applies to Grothendieck categories. Cf.[Gelfand P179].

Cor. (6.4.5). If $\mathcal{A}$ contains sufficiently many injectives, then injective objects are adapted to any left exact functor F. (Because acyclic injective complexes is homotopic to id by the lemma).

Prop. (6.4.6). In an Abelian category, the direct summand of a projective object is projective. (The summand has definition in an Abelian category).

Def. (6.4.7) (Derived functor). If a left exact functor between Abelian categories has an adapted class, then by preceding proposition, $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ is equivalent to $D^+(\mathcal{A})$, then we can use to define the **derived functor** $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ satisfying the following universal property:

RF is exact, i.e.respect the distinguished structure, and there is a natural isomorphism

$$\varepsilon_F : Q_{\mathcal{B}}\ K^+F \to RF\ Q_{\mathcal{A}}.$$

Moreover, any other exact $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and a similar transformation must factor through ε_F uniquely. Thus this RF is unique up to natural isomorphism. It is just F^+ on $K^+(\mathcal{R})$.

Notice there is a more general derived functor that use inductive limits in $\hat{\mathcal{A}}$ that it maps $D^*(\mathcal{A})$to $\mathrm{Ind}(D^*(\mathcal{B}))$, and if it has image in the subcategory of representable objects, then it coincide with RF. Similarly for right exact functor F. (This is easy to check)Cf.[Gelfand P198].

Prop. (6.4.8). $\mathcal{A},\mathcal{B},\mathcal{C}$ be Abelian categories and $F : \mathcal{A} \to \mathcal{B}, G : \mathcal{B} \to \mathcal{C}$ are left exact functors. If $F(R_{\mathcal{A}}) \subset R_{\mathcal{B}}$, where $R_{\mathcal{A}}$ is the adapted class of F, then $R(G \circ F)$ and $RG \circ RF$ is natural isomorphic. (The definition of RF is just F on $K^+(R_{\mathcal{A}})$).

Ext & Tor

Prop. (6.4.9). $\mathcal{A}$ is categorically equivalent to the subcategory of $D(\mathcal{A})$ that has only H^0 nonzero. If we define $\mathrm{Ext}^i_{\mathcal{A}}(X,Y)$ as $\mathrm{Hom}_{D(\mathcal{A})}(X[0],Y[i])$, then it is equivalent to the i-term extension of Y by X, and it is an abelian group. We have a

$$\mathrm{Ext}^i_{\mathcal{A}}(X,Y) \times \mathrm{Ext}^i_{\mathcal{A}}(Y,Z) \to \mathrm{Ext}^{i+j}_{\mathcal{A}}(X,Z)$$

by composition or equivalently the conjunction of extensions.

Proof: Cf.[Gelfand P167] □

Prop. (6.4.10). In an Abelian category, the extension $\mathrm{Ext}^1(N,M)$ is equivalent with the extensions with Baer sum.

Proof: We choose a projective resolution $0\to K\to P\to N\to 0$, so $\mathrm{Hom}(K,M)\to\mathrm{Ext}^1(N,M)$ is surjective, so choose a lifting and the pushout $0\to M\to L\to N\to 0$ with be the corresponding extension, Now the Baer sum is easy to verify. ☐

Prop. (6.4.11). Using(6.4.3), we can show when $Y^\bullet\in\mathrm{Ob}\,\mathrm{Kom}^+(\mathcal{I})$ or $X^\bullet\in\mathrm{Ob}\,\mathrm{Kom}^-(\mathcal{P})$,

$$\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet,Y^\bullet)\to\mathrm{Hom}_{D(\mathcal{A})}(X^\bullet,Y^\bullet)$$

is an isomorphism. Then we get that the definition of $\mathrm{Ext}^n(X,Y)$ as $\mathrm{Hom}_{D(\mathcal{A}}(X[0],Y[-n])$ is equivalent to the usual definition. And it also corresponds to the derived functor of $\mathrm{Hom}(X,-)$.

5 Homological Dimension

Prop. (6.5.1). If $\mathcal{A}$ has enough projectives, then the projective dimension of an object X is the length of projective resolutions. (Use resolution and long sequence).

Prop. (6.5.2) (Hilbert Theorem). For an Abelian category $\mathcal{A}$, the category $\mathcal{A}[T]$ is an Abelian category. If $\mathcal{A}$ has enough projectives and have infinite direct sum, then $\mathrm{dhp}_{\mathcal{A}[T]}(X,t)\le\mathrm{dhp}_{\mathcal{A}}(X)+1$ and equality with $t=0$.

Cor. (6.5.3). The Categories *Ab* and $K[X]$-mod have homological dimension 1. $K[X_i,\ldots,X_k]$ has homological dimension k.

6 Spectral Sequence

Def. (6.6.1). A **Spectral Sequence** is a three-dimensional arrange of entries $E_r^{p,q}$ that:

1. $d_r: E_r^{p,q}\to E_r^{p+r,q-r+1}$ that $d_rd_r=0$.
2. $H^{p,q}(E_r^{p,q})\cong E_{r+1}^{p,q}$. And $E_r^{p,q}$ has a direct limit $E_\infty^{p,q}$.
3. There is a complex E^n and a decreasing regular filtration F^pE^n on each E^n and $E_\infty^{p,q}\cong F^pE^{p+q}/F^{p+1}E^{p+q}$.

Prop. (6.6.2) (Spectral Sequence of a Filtered Complex). For a complex $K^\bullet$ and a filtration F^pK^n on K^n, we have a spectral sequence that $E^n=H^n(K^\bullet)$ and $F^pE^n=H^n(F^pK^\bullet)$. If the filtration is finite and regular for all n, we have a convergence to E^n. Cf[Gelfand P203].

There are two examples, the stupid filtration and the canonical filtration, the canonical filtration is natural and factors through $D(\mathcal{A})$.

Prop. (6.6.3) (Spectral Sequence of a Double Complex). A double complex has two natural filtration of the total complex, they defines two spectral sequence, one has $E_{2,x}^{p,q}=H_x^p(H_y^{\bullet,q}(L^{\bullet,\bullet}))$ and the other has $E_{2,y}^{p,q}=H_y^q(H_x^{p,\bullet}(L^{\bullet\bullet}))$. Cf.[Gelfand P209].

If both the filtration is finite and regular, in particular if E is in the first quadrant, so they both converges to $H^n(E)$, this will generate important consequences.

Cor. (6.6.4). If a double complex in the first quadrant has its all column acyclic (3rd-quadrant pointing), then the total complex is acyclic. Thus a morphism of double complex inducing quasi-isomorphism on each column induces a quasi-isomorphism on the total complex.

If a double complex has $H_p(C_{*,q}) = 0, \forall p > 0, q$, then

$$H_n(\mathrm{Tot}C_{*,*}) = H_n(\mathrm{Coker}(C_{1,*} \to C_{0,*})$$

Cor. (6.6.5) (Five lemma).

Cor. (6.6.6) (Snake lemma).

Prop. (6.6.7) (Cartan-Eilenberg Resolution). If $\mathcal{I}_{\mathcal{B}}$ is sufficiently large, for any K in $K^+(\mathcal{B})$ there is a Cartan-Eilenberg resolution, that is, It induces simultaneous injective resolutions of K^n, Z^n, B^n and H^n. Moreover, the resolution for $B^i \to Z^i \to H^i$ and $Z^i \to K^i \to B^{i+1}$ splits.

This is achieved by the functoriality of resolutions, it is natural and induces a functor from $K^+(\mathcal{B})$ to $K^{++}(\mathcal{I}_{\mathcal{B}})$. Cf.1, [Gelfand P210].

For a CE resolution, the spectral sequence can be applied, one side gets us: $K \to \mathrm{Tot}(L)$ is a quasi-isomorphism, so $RG(K) = G(\mathrm{Tot}L)$ in $D(C)$

Cor. (6.6.8) (Grothendieck Spectral Sequence). $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Abelian categories and $F : \mathcal{A} \to \mathcal{B}, G : \mathcal{B} \to \mathcal{C}$ are left exact functors. If $R_{\mathcal{A}} = \mathcal{I}_{\mathcal{A}}$, $R_{\mathcal{B}} = \mathcal{I}_{\mathcal{B}}$, and $F(I_{\mathcal{A}}) \subset I_{\mathcal{B}}$, then there is a spectral sequence with $E_2^{p,q} = R^pG(R^qF(X))$ (to lower right) that converges to $E^n = R^n(G \circ F)(X)$. And this spectral sequence is functorial in X.

In particular, this applies to F is a right adjoint and its left adjoint is exact.

Proof: Let $K = F(I_X) = RF(X)$, and choose the CE resolution of K, because the resolutions for $B^i \to Z^i \to H^i$ and $Z^i \to K^i \to B^{i+1}$ split and G is additive, we have

$$H_x^{p,\bullet}(G(L^{\bullet,\bullet})) = G(H_x^{p,\bullet}(L^{\bullet,\bullet})) = RG(H^p(K))$$

So

$$E_{2,y}^{p,q} = H_y^q(H_x^{p,\bullet}(L^{\bullet\bullet})) = R^pG(H^q(K)) = R^pG(R^qF(X))$$

and

$$E^\bullet = RG(\mathrm{Tot}(L)) = G(\mathrm{Tot}(L)) = RG(K) = RG \circ RF(X).$$

□

Cor. (6.6.9). The low degree parts read:

$$0 \to R^1G(F(A)) \to R^1(G \circ F)(A) \to G(R^1F(A)) \to R^2(G(F(A)) \to R^2(G \circ F)(A).$$

(Check definition). More generally, if $R^pG(R^qF(A)) = 0, 0 < q < n$, then

$$R^mG(F(A)) \cong R^m(G \circ F)(A) \quad m < n$$

And

$$0 \to R^nG(F(A)) \to R^n(G \circ F)(A) \to G(R^nF(A)) \to R^{n+1}(G(F(A)) \to R^{n+1}(G \circ F)(A).$$

7 Group Cohomology and Lie Algebra Cohomology

Prop. (6.7.1). The group cohomology $H^n(G,A)$ is the derived functor of the left exact functor $H^0(G,A) = A^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},A)$, so $H^n(G,A) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},A)$.

A^H is left exact from G-mod to Ab and also left exact from G-mod to G/H-mod because it is adjoint to the inclusion functor: $\mathrm{Hom}_G(X, A^H) = \mathrm{Hom}_{G/H}(X, A^H)$. And it preserves injectives because inclusion is exact**??**.

Prop. (6.7.2) (Serre-Hochschild Spectral Sequence). By Grothendieck Spectral sequence, the relation $A^G = (A^H)^{G/H}$ gives us a spectral sequence E that

$$E_2^{p,q} = H^p(G/H, H^q(H,A)) \Longrightarrow E^n = H^n(G,A).$$

The lower parts give us:

$$0 \to H^1(G/N, A^N) \xrightarrow{inf} H^1(G,A) \xrightarrow{res} H^1(N,A)^{G/N} \xrightarrow{transfer} H^2(G/N, A^N) \xrightarrow{inf} H^2(G,A).$$

I.7 Lie Algebra

Note:[Lie Algebras of Finite and Affine Type Carter] is far more better than [Hymphreys].

1 Main Theorems

Prop. (7.1.1) (Engel). If all elements of L are ad-nilpotent, then L is nilpotent.

Proof: only need to show that If a subalgebra of $GL(n)$ consists of nilpotent elements, then there is a common 0-eigenvector. Use Induction, choose a maximal subalgebra of L, then it must be of codimension 1,$L = K + Fz$. Thus the 0-eigenvector for K is a nonzero subspace, and a 0-eigenvector for z will suffice. □

Prop. (7.1.2) (Lie's theorem). Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a solvable lie algebra. Then there exists a vector $v \in V$ which is a common eigen vector for all $X \in \mathfrak{g}$.

Proof: Idea is to prove by induction on dimension of $\mathfrak{g}$.

Produce a codimension 1 ideal $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g}$ be generated (as a vector space) by $\mathfrak{h}$ and Y.Being a subalgebra of solvable algebra $\mathfrak{g}$, $\mathfrak{h}$ is itself a solvable lie algebra. Apply induction step on $\mathfrak{h}$ and choose $v \in V$ such that v is an eigenvector for all $X \in \mathfrak{h}$.

The idea is to consider set W all common eigenvectors of elements of $\mathfrak{h}$ and produce an eigenvector of Y from this W. Let

$$W = \{v \in V | X(v) = \lambda(X)v \ \forall \ X \in \mathfrak{h} \text{ for a fixed } \lambda(X) \in \mathbb{C}\}.$$

Suppose W is an invariant subspace of Y, we then have restriction map $Y : W \to W$. As we are in complex vector space (algebraically closed) there exists an eigenvector for Y in W say w_0. Thus, w_0 is common eigenvector for all elements of $\mathfrak{g}$.

It remains to show that W is an invariant subspace of Y i.e., $Y(w) \in W$ for all $w \in W$ i.e., given $X \in \mathfrak{h}$, we need to have $X(Y(w)) = \lambda(X)Y(w)$.

Let $w \in W$, we have

$$\begin{aligned} X(Y(w)) &= Y(X(w)) + [X, Y](w) \\ &= Y(\lambda(X)w) + \lambda\left([X, Y]\right) w \\ &= \lambda(X)Y(w) + \lambda\left([X, Y]\right) w \end{aligned}$$

This is almost the same as what we want but with an extra term $\lambda\left([X, Y]\right) w$. Suppose we prove $\lambda\left([X, Y]\right) = 0$ for all $X \in \mathfrak{h}$ then we are done.

Then considers subspace U spanned by elements $\{w, Y(w), Y^2(w), \cdots\}$ and then says that U is invariant subspace of each element of $\mathfrak{h}$ and (assuming n is the smallest integer such that $Y^{n+1}w$ is in the subsapce generated by $\{w, Y(w), \cdots, Y^n(w)\}$) representation of an element Z of $\mathfrak{h}$ with the basis $\{w, Y(w), \cdots, Y^n(w)\}$ is an upper triangular matrix with $\lambda(Z)$ in the diagonal. So, $tr(Z) = n\lambda(Z)$.

So, $tr([X, Y]) = n\lambda([X, Y])$. As $[X, Y] = XY - YX$, we have $\mathrm{tr}([X, Y]) = \mathrm{tr}(XY) - \mathrm{tr}(YX) = 0$. Thus, $\lambda([X, Y]) = 0$ and we are done. □

Def. (7.1.3). A lie algebra is called semisimple if the maximal solvable ideal (Rad L)= 0.

Prop. (7.1.4) (Weyl). Representation of a semisimple lie algebra is completely reducible.

Proof: Cf.[Humphreys P28]. □

Prop. (7.1.5) (Cartan's Criteria for Solvability). If $\mathfrak{g}$ is a Lie algebra $\subset \mathfrak{gl}_n$, then

$$\mathfrak{g} \text{ is solvable } \iff Tr(xy) = 0,\ \forall x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}].$$

Note that a Lie algebra is solvable if the adjoint representation is solvable because the kernel is abelian. Cf.[Humphreys P20]

Prop. (7.1.6) (Cartan Criteria for Semisimplicity). A lie algebra is semisimple $\iff$ the Killing form is non-degenerate. Cf.[Humphreys P22].

Proof: Just show that the kernel of the Killing form is a solvable ideal and that $\mathrm{ad}x \cdot \mathrm{ad}y$ is nilpotent for x in an abelian ideal. □

Prop. (7.1.7). If L is semisimple, then every derivative of L is inner.

Proof: Cf.[Humphreys P23]. □

Prop. (7.1.8) (Abstract Jordan Decomposition). Let L be a semisimple lie algebra and $\phi : L \to GL(v)$ be a representation. If $x = s + n$ is the abstract Jordan decomposition of x, then $\phi(x) = \phi(s) + \phi(n)$ is the usual Jordan decomposition of $\phi(x)$.

Proof: In fact, we only need to prove that if L is a semisimple algebra $\subset \mathfrak{gl}(V)$, then L contains the semisimple and nilpotent element of all its element. Because the image of L is semisimple and the usual Jordan decomposition must be its abstract decomposition. The last assertion is due to the fact that if z is semisimple(nilpotent), then $\mathrm{ad}_{\mathfrak{gl}_n} z$ is semisimple(nilpotent), thus so do $\mathrm{ad}_L z$.

Cf.[Humphreys P27] for the following proof. □

Prop. (7.1.9) (Baker-Campbell-Hausdorff cor).

$$\exp(X)\exp(Y) = \exp(X+Y+1/2[X,Y]+1/12[X,[X,Y]]-1/12[Y,[Y,X]]+\text{higher order terms})$$

Cf.[Hall Lie algebras GTM222 P76].

2 Reductive Lie Algebra

Prop. (7.2.1). A lie algebra is called reductive if $\mathrm{Rad}(L) = Z(L)$.

1. If L is reductive, then L is completely reducible ad L-module.
2. $L = [LL] \oplus Z(L)$.
3. If $L \subset GL(V)$ acting irreducibly on V, then L is reductive with $\dim \mathrm{Rad}(L) \leq 1$. In particular, If $L \in SL(V)$ and $\mathrm{char}F \neq 0$, it must be semisimple. This can be used to prove that all classical algebras are semisimple. And the diagonal matrix will be toral and finding a set of simple roots will suffice to prove that every calssical lie algebra is simple.
4. If L is a completely reducible ad L-module, then L is reductive.
5. If L is reductive, then all finite dimensional representations of L in which $Z(L)$ is represented by semisimple endomorphism are completely reducible.

6. If $[LL]$ is semisimple, then L is reductive.

Proof: (1): Because $L/Z(L)$ is a semisimple lie algebra and $Z(L)$ is mapped to the kernel.

(2): Let $L = M \oplus Z(L)$ as a ad-L module, then $[LL] \subset [MM] \subset M$, but $[LL]$ maps onto $L/Z(L)$ because a semisimple is a sum of simple algebra. So $[LL] = M$.

(3): Cf.[Humphreys P102].

(4): In this way L decompose into $Z(L)$ and simple algebras, so it is reductive.

(5): First simultaneously diagonalize $Z(L)$, then the subspace corresponding to different characters are stable under L. Then decompose w.r.t. $[LL]$ with get the result. (6): Note that the element in $\mathrm{Rad}(L)$ will all be central. $\square$

Prop. (7.2.2). Let L be a simple lie algebra, then any two symmetric associative bilinear forms on L is proportional. Because any of this form corresponds to a L-morphism from L to L^*. In particular, when $L \subset \mathfrak{gl}_n$, the usual trace is proportional to the Killing form.

3 Real Lie Algebra

Def. (7.3.1). A **compact real form** is a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ s.t. $\mathfrak{g}$ is the complexification of $\mathfrak{l}$ and $\mathfrak{l}$ is the lie algebra of a compact simply-connected Lie group.

Prop. (7.3.2). A real Lie algebra is compact iff there exists a inner product s.t.

$$([X, Y], Z) + (X, [Y, Z]) = 0,$$

iff the Killing form is negative definite.

Proof: One direction is easy, just use the average method to find a G-invariant inner product and then take derivative. For the other direction, the identity shows that a complement of an ideal is an ideal so $\mathfrak{g}$ is decomposed into simple lie groups and reduce to the case that $\mathfrak{g}$ is simple. The ideal is to show that $\mathfrak{g} \cong \mathrm{ad}(\mathfrak{g})$ is the whole outer derivative group $\partial(\mathfrak{g})$(the following lemma). So $\mathfrak{g}$ equals to the identity component of $\mathrm{Aut}(\mathfrak{g})$ which is a closed subgroup thus closed but it is also a subgroup of the compact group $O(\mathfrak{g})$ thus it is compact. $\square$

Lemma (7.3.3). If a real semisimple Lie algebra X has a invariant inner product, then every outer derivative is inner.(In fact, this is true by Cartan Criterion for semisimplicity (7.1.7).

Proof: since $\mathrm{ad}(X)$ is skew-symmetric, it's diagonalizable and its eigenvalue is pure imaginary, so the Killing form of X is negative definite. Now choose the complement $\mathfrak{a}$ of $\mathrm{ad}(X)$ in $\partial(X)$, then $\mathfrak{a} \cap X = 0$. Thus for $D \in \mathfrak{a}$, $\mathrm{ad}(D(g)) = [D, \mathrm{ad}(g)] = 0$ for all g in X, so $D = 0$, thus $\mathrm{ad}(X) = \partial(X)$. $\square$

Prop. (7.3.4). -

1. The complexification of the Lie algebra of a connected compact Lie group is reductive.
2. A complex Lie algebra is semisimple iff it is isomorphic to the complexification of the Lie algebra of a simply-connected compact Lie group. i.e. every complex semisimple Lie algebra has a compact real form.

Proof: 1: Because a connected compact Lie group is completely reducible so the does the Lie algebra and so does the complexification. So it is reductive by (7.2.1)4.

2: Cf.[Varadarajan Lie Groups Lie algebras and Their Representations]. The ideal is to find a real form whose corresponding simply-connected group is compact. $\square$

Prop. (7.3.5). If $\mathfrak{g}$ is the Lie algebra of a matrix Lie group G, then:

1. every Cartan subalgebra comes from a maximal commutative subalgebra of a compact real form and any two Cartan subalgebras are conjugate under the Ad-action of G.
2. any two compact real form is conjugate under the Ad-action of G.
3. any two maximal commutative subalgebra of a compact real form is conjugate under the Ad-action of the corresponding compact compact subgroup.

Prop. (7.3.6). A real Lie algebra is semisimple iff its complexification is semisimple. Cf.[Varadarajan].

Cor. (7.3.7). The real Lie algebra of a compact simply-connected group is semisimple.

Note: For the classification of real semisimple Lie algebras, Cf.[李群讲义项武义 §6]

Prop. (7.3.8). If a complex representation of a Lie group admits an invariant bilinear form, then it is non-degenerate and unique. In fact, this is equivalent to a G-map from V to V^*. Thus there is unique invariant inner product in a compact real form by the preceding proposition.

4 Universal Enveloping Algebra

Prop. (7.4.1) (Chevalley). The center of the universal enveloping algebra is isomorphic to the polynomial ring over $\mathbb{C}$ of l elements, where L is a semisimple lie algebra of rank l. In particular, The center for $\mathfrak{sl}_2$ is the algebra generated by the Casimir element $1/2h^2+ef+fe$.

Proof: Because there is a commutative diagram of isomorphisms of algebras:

$$\begin{array}{ccc} S(L)^G & \xrightarrow{\alpha} & P(L)^G \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\phi} \\ S(H)^W & \xrightarrow{\beta} & P(H)^W \end{array}$$

Where P is the polynomial ring$\cong S(L^*)$, the horizontal is Killing isomorphisms and vertical is the restriction maps. Cf.[Carter Theorem 13.32].

The twisted Harish-Chandra map gives an isomorphism of algebras $Z(L)\to S(H)^W$ (It just maps $z\in Z(L)$ to its pure H part and transform every indeterminants h_i to h_i-1). e.g. $z=h^2+2h+1+4fe\in Z(\mathfrak{sl}_2)$ is maped to h^2 in $S(H)$. And $P(H)^W$ is isomorphic to a polynomial ring in l generators over $\mathbb{C}$. $\square$

I.8 Quantum Groups

1 Clifford Algebra

Prop. (8.1.1). Let $Cl_{r,s}$ denote the real Clifford algebra of signature $r-s$, then

$$Cl_{1,0}\cong\mathbb{C},\quad Cl_{0,1}\cong\mathbb{R}\oplus\mathbb{R},\quad Cl_{2,0}\cong\mathbb{H}\subset M(2,\mathbb{C}),\quad Cl_{0,2}\cong R(2)=M(2,\mathbb{R}),$$

And we have

$$Cl_{n+2,0}\cong Cl_{0,n}\otimes Cl_{2,0},\quad Cl_{0,n+2}\cong Cl_{n,0}\otimes Cl_{0,2}.$$

by the mapping $e_i\to e_i\otimes e_1'e_2'$, $e_{n+j}\to 1\otimes e_j'$.

So we have

$$Cl_{n+8,0}\cong Cl_n\otimes\mathbb{R}(16),\quad \mathbb{C}l_{n+2,0}=Cl_{n+2,0}\otimes\mathbb{C}=\mathbb{C}l_{n,0}\otimes\mathbb{C}(2).$$

because $\mathbb{H}\otimes\mathbb{C}=\mathbb{C}(2)$, and

$$\begin{bmatrix} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\\ Cl_{n,0} & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{H}\oplus H & \mathbb{H}(2) & \mathbb{C}(4) & \mathbb{R}(8) & \mathbb{R}(8)\oplus\mathbb{R}(8)\\ \mathbb{C}l_{n,0} & \mathbb{C} & \mathbb{C}\oplus\mathbb{C} & & & & & & \end{bmatrix}$$

The Clifford algebra is a $\mathbb{Z}_2$-graded algebra, $Cl=Cl^0\otimes Cl^1$ and $Cl_{n-1}\cong Cl_n^0$ by the mapping $e_i\to e_i\otimes e_{n+1}$.

Def. (8.1.2). denote $\mathrm{Pin}(n)$ as the group in Cl_n generated by v_i of norm 1. Because $v_i\cdot v_i=-1$, it is a group. And denote $\mathrm{Spin}(n)$ as the subgroup of $\mathrm{Pin}(n)$ generated by even number of v_is.

So the conjugation action $-Ad=-v\cdot v=$reflection w.r.t v maps $\mathrm{Pin}(n)$ to $O(n)$ and $\mathrm{SPin}(n)$ to $SO(n)$.

Prop. (8.1.3). The kernel of this mapping is $\{\pm\}$ when n is even. And this is a double covering of $So(n)$ and $O(n)$ and it is nontrivial because $\{\pm\}$ is connected by $(\cos te_1+\sin te_2)(\cos te_1-\sin te_2)$.

Proof: Let $\alpha=e_i\beta+\gamma$, then $\beta,\gamma\in Cl^0$ and so $\alpha=ce_1\dots e_n+d$, and c can happen only when n is odd. □

So for n even, there are two representations of $\mathrm{Spin}(n)\subset Cl(n)^0$ and for n odd only one, called the Spin representations.

Chapter II

Number Theory

II.1 Algebraic Number Theory

1 Basics

Prop. (1.1.1). The ring of integers in the cyclotomic field is generated by the roots of identity.

Proof: First consider the case n a prime power. Because $d(1,\zeta,\cdots,\zeta^{d-1}) = \pm l^s$, $l^s\mathcal{O} \subset \mathbb{Z}[\zeta] \subset \mathcal{O}$. Because p totally splits, $\mathcal{O} = \mathbb{Z}[\zeta] + \pi\mathcal{O}$, thus $\mathcal{O} = \mathbb{Z}[\zeta] + \pi^t\mathcal{O}$. Choose $t = s\phi(n)$ yields $\mathbb{Z}[\zeta] = \mathcal{O}$.

Then for different p, the fields are disjoint and the discriminant are pairwisely coprime, thus by (2.11) in Neukirch, the products of the integral basis form an integral basis. □

Prop. (1.1.2) (Krasner's Lemma). In a separable extension, if $|\beta - \alpha| < |\alpha_i - \alpha|$ for any conjugate α_i of α, then $K(\alpha) \subset K(\beta)$.

Prop. (1.1.3) (Ostrowski). Any non-trivial value on $\mathbb{Q}$ is equivalent to v_p or $|\ |$. Thus any complete Archimedean field is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

2 Ramification Theory

Prop. (1.2.1). If a prime $\mathfrak{p}$ splits completely in two separable extension LM of $\mathbb{K}$, then it also splits completely in the composite LM.

Proof: We use the language of valuation. The extension of a valuation v of K corresponds to the set of equivalent classes of algebra map from L to $\overline{K_v}$ module conjugacy over K_v. So We only need to show that two different maps of LM are not conjugate over K_v. But the restrict of them to L or M is different, thus non-conjugate over K_v. □

Cor. (1.2.2). A prime splits completely in a separable extension L if it splits completely in the Galois closure N of L.

Proof: This is because the Galois closure is the composite of the conjugates of L.

But it also can be proven directly : Set $H = \mathrm{Gal}(N/L)$,$\mathcal{P}$ a prime of N over $\mathfrak{p}$, then

$$H\backslash G/G_{\mathcal{P}} \longrightarrow \{\text{Primes of } L \text{ over } \mathfrak{p}\}, \qquad H\sigma G_{\mathcal{P}} \mapsto \sigma\mathcal{P} \cap L$$

is a bijection. So it splits completely in $L \iff G_{\mathcal{P}}$ is trivial $\iff$ it splits completely in N by counting numbers. □

Prop. (1.2.3). A prime p splits in $\mathbb{Z}[\xi_n]$ iff $p \equiv 1 (\mathrm{mod}\ n)$.

Proof: First, if it splits, then $f = 1$, Because the ring of integers is $\mathbb{Z}[\xi_n]$, so $X^n - 1$ splits in $\mathbb{F}_p$(1.1.1), thus $p \equiv 1 (\mathrm{mod}\ n)$. And if $p \equiv 1 (\mathrm{mod}\ n)$, it is unramified and $X^n - 1$ splits in $\mathbb{F}_p$, so $f = 1$. □

Prop. (1.2.4). The profinite group $\mathbb{Q}_p^{\mathrm{tame}}$ is $\widehat{\mathbb{Z}} \ltimes \Delta_p$. Which is the profinite group generated by the relationship $\sigma\tau\sigma^{-1} = \tau^p$, where σ is a lift of Frobenius. Which means that it is the limit of finite quotients of the group $\langle \sigma\tau\sigma^{-1} = \tau^p \rangle$.

Proof: Cf.[Local Fields Clark]. □

Prop. (1.2.5) (Hasse-Arf Theorem). For a complete discrete valuation field K and an abelian extension L of K, the jump in the upper numbering of higher ramification group G^v must happen at integers.

Proof: Cf.[Local Fields, Serre] □

3 Completion

Prop. (1.3.1). Any infinite separable algebraic extension of a complete field is not complete.

Proof: We use Krasner's lemma. By Ostrowski theorem, we can assume it is non-Achimedian. Choose an infinite linearly independent basis of decreasing value rapidly enough, then we can see the field generated by the limit contains all the partial sums, contradiction. □

II.2 Class Field Theory

II.3 Langlands Program

1 Local Langlands Correspondence

The basic object of LLC are the Weil group and its representations.
A representation ρ of W_K is called F**-semisimple** iff $\rho(\text{Frob})$ is diagonalizable.

Prop. (3.1.1). A

Thm. (3.1.2) (LLC for $GL_n(\mathbb{C})$). The set of
irreducible smooth, admissible representations of $GL_n(K)$
corresponds to
n-dimensional F-semisimple Weil-Deligne representations of W_K.

Cor. (3.1.3) (LLC for $GL_1(\mathbb{C})$).

Local class field theory told us that W_K^{ab} is isometric to K^*, And notice by Schur's lemma, any smooth representation of K^* is 1-dimensional and factors through some U_k.

And a Weil-Deligne representation is now a continuous $W_K^{ab} \to C^*$. but it must factor through some U_K, so these two are equivalent.

most l-adic representation of G_K comes from étale cohomology.

LLC for $GL_2(\mathbb{C})$

II.4 Witt Theory (Local Fields Serre)

1 Witt Vectors

A ring φ lifting the Frobenius, i.e. $\varphi(x) = x^p + p\delta(x)$. It generate a $\delta-$ring structure.

This is a category and The right adjoint to the forgetful functor is $W(A) = \mathrm{Hom}[\Delta, A]$. Where Δ is the free ring $\mathbb{Z}[e, \delta, \delta^2, \ldots]$. and it add and product in the way of Leibniz rule. There is another description of Δ:

Prop. (4.1.1). Let θ_i be polynomials in δ with integer coefficients that

$$\varphi^n = \theta_0^{p^n} + p\theta_1^{p^{n-1}} + \cdots + p^n\theta_n$$

In fact

$$\mathbb{Z}[\theta_0, \theta_i, \ldots, \theta_n] = \mathbb{Z}[e, \delta, \delta^2, \ldots, \delta^n]$$

Proof: Use equation $\varphi \circ \varphi^n = \varphi^n \circ \varphi$ and module $p^n\mathbb{Z}[\theta_0, \theta_1, \ldots, \theta_n]$. □

So there is a map

$$Z[\varphi] \to \Delta$$

inducing an morphism of rings:

$$W(A) \to \prod_{\mathbb{Z}} A$$

that maps

$$(f(\delta^n)) \mapsto (f(\varphi^n))$$

Where the right hand side is the normal addition and multiplication, the left side is the usual coordinate of Witt vector, and $f(\theta_n)$ is called ghost component.

Prop. (4.1.2). Notice in Serre book, he presented the Witt vectors in $(f(\theta_n))$ coordinates. In this coordinate, if k is a perfect ring and we let

$$T(A) = \sum a_i^{p^{-i}} p^i,$$

then T is an ring homomorphism from $W(k)$ to the strict p-ring with residue ring k.

Cor. (4.1.3). For example, $W(\mathbb{F}_p^n)$ is the unramified extension of $\mathbb{Z}_p$ of degree n. And $W(\overline{F})$ is the completion of the maximal unramified extension of $W(F)$.

This is embedding if A is p-torsion free, and isomorphism iff $\frac{1}{p} \in A$.

Prop. (4.1.4). $\mathcal{O}_{\mathcal{E}} = W(K^{\frac{1}{p^\infty}})$ is a complete ring with maximal ideal $p\mathcal{O}_{\mathcal{E}}$. And $\mathcal{O}_{\mathcal{E}}[\frac{1}{p}] = \mathcal{E}$ is complete ring of character p. And the same construction of $\overline{K^{\frac{1}{p^\infty}}}$ yields the complete of maximal unramified extension of $\mathcal{O}_{\mathcal{E}}$, and the Galois group is the same as G_K.

II.5 Abelian Variety(Mumford)

An Abelian variety A is a smooth projective variety with a group structure.

Prop. (5.0.1). For a field K of characteristic p, then $A(K^{\text{sep}}$ is an Abelian group and its l^n torsion is isomorphic to $(\mathbb{Z}/l^n\mathbb{Z})^{2g}$ and its p^n torsion is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^r$.

Prop. (5.0.2). There is an isomorphism

$$H_t^m(\Lambda_{K^{\text{sep}}}, \mathbb{Q}_l) \cong \bigwedge_{\mathbb{Q}_l}^m (V_l(A))^*.$$

Cf.[Grothendieck Monodromy theorem].

II.6 Étale Cohomology

II.7 p-adic Hodge Theory

1 l-adic representations

Prop. (7.1.1). Every continuous representation of G_K on a $\mathbb{Q}_l$ vector space (Continuous group morphism to $GL_n(\mathbb{Q}_l)$) has a $\mathbb{Z}_l$ lattice stable under the action. (notice that the stablizer of the standard lattice is $GL_n(\mathbb{Z}_l)$ which is open and so the inverse image has a finite coset. And the image of the wild ramification group is finite because it is in $GL_n(\mathbb{F}_l)$.

So the functor $\rho \to \rho \otimes \mathbb{Q}_l$ from $\mathrm{Rep}_{\mathbb{Z}_l}(G_K)$ to the Tannakian natural category $\mathrm{Rep}_{\mathbb{Q}_l}(G_K)$ is essentially surjective.

Prop. (7.1.2) (Grothendieck Monodromy theorem). For a local field K, the étale representation and the Tate module are all potentially semisimple. i.e. semisimple for a finite extension.

Chapter III

Geometry

III.1 Topology

1 Connected Component

Prop. (1.1.1). Let X be a topological space, $x \in X$, C is a connected component of x, i.e. a maximal connected subset containing x. Define A to be the intersection of all the open-and-closed sets that contain x (also called the pseudo-component sometimes). Then $A = C$, if X is normal.

Proof: Assume A splits into two components B, D. Since A is closed, B and D are both closed, because X is normal there are disjoint open neighborhoods U and V around B and D, respectively. The open sets U and V cover the intersection of all clopen neighborhoods of A, so cause X is compact, there must exist a finite number of clopen sets around A, say $A_1, \ldots, A_n$ such that $U \cup V$ covers $K = \bigcap_1^n A_i$.

Note that K is clopen. We can assume that $x \in U$. It is not difficult to see that $K \cap U$ is clopen and does not contain all of A, contradicting the definition of A. $\square$

Prop. (1.1.2). A noetherian topological space has only finitely many connected components.

Proof: Let $\mathcal{C}$ be the family of closed subset that has infinitely many component, then there is a minimal element, but it is not connected, one of the component has infinitely many component and be smaller. $\square$

2 Covering Map

Prop. (1.2.1). if X and Y are Hausdorff spaces, $f : X \to Y$ is a local homeomorphism, X is compact, and Y is connected, then f a covering map.

Proof: First, f is surjective (using the connectedness), and that for each $y \in Y$, $f^{-1}(y)$ is finite. Because X is compact, there exists a finite open cover of X by $\{U_i\}$ such that $f(U_i)$ is open and $f|_{U_i} : U_i \to f(U_i)$ is a homeomorphism. For $y \in Y$, let $\{x_1, \ldots, x_n\} = f^{-1}(y)$ (the x_i all being different points). Choose pairwise disjoint neighborhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$, respectively (using the Hausdorff property).

By shrinking the U_i further, we may assume that each one is mapped homeomorphically onto some neighborhood V_i of y.

Now let $C = X \setminus (U_1 \cup \cdots \cup U_n)$ and set

$$V = (V_1 \cap \cdots \cap V_n) \setminus f(C)$$

V should be an evenly covered nbhd of y. □

Prop. (1.2.2). If $\pi : \tilde{B} \to B$ is a local onto homeomorphism with the property of lifting arcs. Let $\tilde{B}$ be arcwise connected and B simply connected, then π is a homomorphism.

Proof: only need to prove injective. If p_1 and p_2 map to the same point, then they can be connected, and the image is a loop thus contractable, contradiction. □

Cor. (1.2.3). If $\tilde{B}$ is locally arcwise connected and B is locally simply connected, then π is a covering map.(choose the connected component)

Prop. (1.2.4). a simply connected manifold is orientable. (Use the orientable double cover).

3 Paracompactness

Prop. (1.3.1). If X is regular, then TFAE:

1. Each open cover of X has an open locally finite refinement.
2. Each open cover of X has a locally finite refinement.
3. Each open cover of X has a closed locally finite refinement.
4. Each open cover of X is even. i.e. for any cover, there is an open nbhd V of diagonal of $X \times X$ such that $\forall x$, $V[x] = \{y|(x,y) \in V\}$ refines the cover.
5. Each open cover of X has an open σ-discrete refinement.
6. Each open cover of X has an open σ-locally finite refinement.

If this is satisfied, then X is called **paracompact**.

Proof: $6 \to 2$:Just minus every open set the part of open sets that appeared in families that ordered before it. $2 + 4 \to 1$:Use the lemma below, we can transform the cover $\mathcal{A}$ into $V[\mathcal{A}] \cap U_A$ which is an open locally finite cover
Cf.[General Topology Kelley] □

Lemma (1.3.2). If X satiesfies 4, let U be a nbhd of diagonal of $X \times X$, then their exists a symmetric nbhd of diagonal s.t. $V \circ V \subset U$, where $U \circ V = \{(x,z)|(x,y) \in U, (y,z) \in V, \exists y\}$.

Proof: $\forall x$ in X, there is a nbhd s.t. $W[x] \times W[x] \subset U$, this is an open cover, so there is a nbhd R of diagonal s.t. $R[x]$ refines it. Hense $R[x] \times R[x] \subset U$. Let $V = R \cap R^{-1}$, $V \circ V$ is the union of sets $V[x] \times V[x]$, so $V \circ V \subset U$. □

Lemma (1.3.3). In the preceding proposition, if X satisfies 4, Let $\mathcal{A}$ be a locally finite(resp. discrete i.e. intersect only one) family of subsets of X, then use the last lemma, there is a nbhd V of diagonal of $X \times X$ such that $V[A] = \{y|(x,y) \in V, \exists x \in A\}$ is locally finite(resp. discrete).

Proof: Choose for every pt a nbhd satisfy the property, then it is an open cover. Choose a diagonal nbhd U for the property 4, then choose coordinate symmetric nbhd V of diagonal s.t. $V \circ V \subset U$. If $V[x]$ intersect $V[A]$, then $V \circ V[x]$ intersect A. Done. □

Prop. (1.3.4). A regular paracompact space is normal.

Proof: The family consisting of two closed is locally discrete, by preceding lemma,there exists a V s.t.$V[A]$,$V[B]$ open and non-intersecting. □

Prop. (1.3.5). For a connected, Hausdorff, locally euclidian space, paracompact, second countable and a compact exhaustion is equivalent.

Proof: Cf.[Paracompactness and second countable]. □

Prop. (1.3.6). A metric space is paracompact.

Prop. (1.3.7). A compact Hausdorff space is paracompact.

Prop. (1.3.8) (Partition of unity). In a paracompact space, given any open cover, there exists a partition of unity $\{\rho_i\}$ that ρ_i has compact support and $\mathrm{supp}\rho_i \subset U_i$.

4 Normal (T4)

Prop. (1.4.1) (Urysohn lemma). Let X be normal, A and B two closed subset of X, then there exists a continuous map from X to $[0, 1]$ that maps A to 0 and B to 1.

Proof: Use the countability of rational numbers to construct a family of U_q s.t.

$$p < q \Rightarrow \bar{U}_p \subset U_q$$

Then choose $f(x) = \inf\{p \in \mathbb{Q} | x \in U_p\}$, then this f meets the requirement. □

Prop. (1.4.2) (Tietze extension). If X is normal and Y is a closed subspace, then any continuous function f on Y can be extended to a continuous function on X.

5 Compact-Open Topology

Prop. (1.5.1). The **compact-open topology** on X^Y is the topology generated by subbasis of $(K, U) =$ all f that maps K to U, for K compact and U open. When Y is compact and X a metric space, this coincides with the uniform topology.

Prop. (1.5.2).

- $X^Y \times Y \to X$ is continuous if Y is locally compact.
- $\mathrm{Map}(Y \times X, Z) \cong \mathrm{Map}(Z, X^Y)$.

6 Baire Space

Prop. (1.6.1) (Baire Category Theorem). Every complete metric space & locally compact Hausdorff space is a Baire space, i.e. not countable union of subsets whose closure have no interior point.

Proof: Choose consecutively open subsets that doesn't intersect $\overline{E_n}$ to find a limit point. □

III.2 Algebraic Topology

1 Homology and Cohomology

Prop. (2.1.1). The fundamental group of a topological group is abelian.

Proof: This is because π_1 preserves products, so takes group objects to group objects. And the group objects in the category of groups is the abelian groups (1.7.2) □

Prop. (2.1.2) (de Rham). The de Rham cohomological group $H^*_{dR}(X)$ is isomorphic to the singular cohomological group $H^*(X,\mathbb{R})$.

Proof: First, $H^*_{dR}(X)\cong H^*(X,R)$ for the constant sheaf cohomology by(3.1.3), and prove □

Prop. (2.1.3). For two homotopic map between two topological space (Fine enough), they induce the same map on singular (co)homology and de Rham cohomology.

Proof: For singular homology, the combinatorial 'pillariazation' can be constructed that $f-g=k^{n-1}\circ d+d\circ k^n$. And for de Rham cohomology, then a similar k^n can be constructed. Cf.[Gelfand Homological Algebra P52]. □

Prop. (2.1.4). The cellular (co)homology coincides with the singular (co)homology for CW-complex.

Prop. (2.1.5) (Morse Inequality). for any field F,

$$\sum^k(-1)^i\dim H_i(X,F)\leq\sum^k(-1)^ic_i,$$

where c_i is the number of I-dimensional cells. (Use the dimension counting of the long exact sequence).

Prop. (2.1.6) (Poincare Duality).

Cor. (2.1.7).

$$H^*(\mathbb{RP}^n,\mathbb{Z}_2)=\mathbb{Z}_2[X]/X^n,\quad H^*(\mathbb{CP}^n,\mathbb{Z})=\mathbb{Z}[X]/X^n$$

Proof: Use induction and Poincare duality to find that $\alpha*\alpha^{n-1}=\alpha^n$. □

Prop. (2.1.8) (Alexander Duality).

Prop. (2.1.9) (Thom isomorphism). Cf.[姜伯驹同调论].

Prop. (2.1.10) (Gysin Sequence). Cf.[姜伯驹同调论].

Prop. (2.1.11) (Lefschetz Fixed Point Theorem).

Cohomology of Fiber Bundles

Prop. (2.1.12) (Leray-Hirsch). For a fiber bundle and a ring R s.t. $H^n(F,R)$ is f.g free for all n, and there exists classes c_j that constitute a basis for each fiber F, then

$$H^*(B,R)\otimes H^*(F,R)\to H^*(E,R)$$

is an isomorphism of $H^*(B,R)$-modules.

Cor. (2.1.13).
- $H^*(U(n);\mathbb{Z})=\Lambda_{\mathbb{Z}}[x_1,x_3,\ldots,x_{2n-1}]$.
- $H^*(SU(n);\mathbb{Z})=\Lambda_{\mathbb{Z}}[x_3,\ldots,x_{2n-1}]$.
- $H^*(Sp(n);\mathbb{Z})=\Lambda_{\mathbb{Z}}[x_3,x_7,\ldots,x_{4n-1}]$.

Prop. (2.1.14). $H^*(G_n(\mathbb{K}^\infty);\mathbb{Z})$ where $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$ is generated by the symmetric polynomials, where for $\mathbb{R}$ the coefficient is $\mathbb{Z}_2$.

Proof: Use the flag variety and first calculate for ∞. Then use Poincare duality to show it is onto the symmetric polynomials. Cf.[Hatcher P435]. □

Cup Product and Cohomology Operators

Prop. (2.1.15). The cup product will restrict to a relative version:

$$H^*(X,A)\times H^*(X,B)\to H^*(X,A\cup B),$$

This implies that if X is a union of n contractible open set, then the cup product of n-elements vanish. In particular, the cup product in a suspension vanishes.

Prop. (2.1.16) (Steenrod Powers). The total Steenrod squares Sq is map from $H^n(X,\mathbb{Z}_2)\to H^{n+*}(X,\mathbb{Z}_2)$ that:
- it is natural and stable under suspension.
- it is additive.
- $\mathrm{Sq}(\alpha\cup\beta)=\mathrm{Sq}(\alpha)\cup\mathrm{Sq}(\beta)$.
- $Sq^i(\alpha)=\alpha^2$ if $i=|\alpha|$, and 0 if $i>|\alpha|$.

The total Steenrod Powers P is a similar map from $H^n(X,\mathbb{Z}_p)\to H^{n+*}(X,\mathbb{Z}_p)$ that $P^i(\alpha)=\alpha^p$ if $2i=|\alpha|$ and 0 if $2i>\alpha$.

The algebra of powers is generated respectively by elements Sq^{2^k}, and for p it is generated by β and the elements P^{p^k}. (Because of Adem relations) Cf.[Hatcher P497].

2 Fundamental Groups

Prop. (2.2.1) (Van Kampen).

3 CW Complex

Prop. (2.3.1). If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, thus (X, A) has the **homotopy extension property** because we can perform infinite induction on dimension.

Prop. (2.3.2). The loop space ΩX for X a CW complex has CW complex type. In particular, if it has noly finitely many cells for a given dimension, then so does ΩX. Milnor proved this.

Prop. (2.3.3). The homotopy group defines a long exact sequence for triples (X, A, B), in particular for $B = \mathrm{pt}$.

Prop. (2.3.4) (Compression Theorem). If (X, A) is a CW pair that (Y, B) be a pair that $\pi_n(Y, B, y_0)$=0, for any n, then every map (X, A) to $(Y < B)$ is homotopic rel$A$ to a map $X \to B$. (Use extension property to extend by dimension). This shows that the homotopy doesn't depend on higher dimensions, (but might on lower one).

Cor. (2.3.5) (Whitehead Combinatorial Homotopy I). If M and K is dominated by CW complexes, then any map $M \to K$ inducing homotopy group isomorphisms is an homotopic equivalence. If the map is inclusion, then it is a deformation retract. In particular, if M is manifold, then it is dominated by its tubular nbhd, so this theorem is applied.

Proof: For inclusion, use compression, and in general use mapping cylinder and cellular approximation. □

Cor. (2.3.6). If $\pi_n(X) = 0$ for all n and a CW complex X, then X is contractible.

Prop. (2.3.7) (Cellular Approximation Theorem). Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. This makes calculation of homotopy easy. (It suffice to show a map cannot be surjective on a higher dim cell, Cf.[Hatcher P349].

Moreover, Any map of pairs of CW complexes can be deformed to a cellular map. (first deform the small complex, then deform the big by dimension.

Cor. (2.3.8). If a CW complex has only cells of dim> n, then it's homotopy group vanishes for $i < n$. In particular, $\pi_n(S^k) = 0$ for $n < k$.

Prop. (2.3.9) (CW Approximations). There exists a CW approximation for any pair (X, A), that is, induce isomorphism on X and X_0 thus on relative homotopy group. Cf.[Hatcher P353].

If A is CW, then there is a n-connected CW models (Z, A) to (X, A), i.e. $\pi_{\leq n}(Z, A) = 0$ and $Z \to X$ induce isomorphism on $\pi_{>n}$ and injection for π_n. And this approximation is unique up to homotopy equivalence rel$A$, (use relative mapping cylinder and use compression). They act like injective resolutions.

Use Long exact sequence compression and mapping cylinder, we can prove the approximations preserve (co)homology and mapping classes.

Cor. (2.3.10). For any n-connected CW pair (X, A), there exist a homotopic (Z, A) that $Z \backslash A$ has only cells of dimension n.

Cor. (2.3.11) (Whitehead theorem). A f between two simply connected CW complexes that induce isomorphism on homology groups is a homotopy equivalence. (using mapping cylinder, we can assume it's an inclusion, and $\pi_1(Y, X) = 0$, so the theorem shows that $\pi_n(Y, X) = 0$, and use Whitehead).

Prop. (2.3.12). A closed manifold or the interior of a manifold with boundary has a homotopy type of a CW complex of finite type.

Remark (2.3.13). The use of mapping cylinder and relative mapping cylinder is important.

4 Homotopy

Prop. (2.4.1). The universal cover have the same homotopy group $\pi_{>1}$, by lifting property.

Prop. (2.4.2) (Excision Theorem). If A, B are CW-complexes, then if $(A, A\cap B)$ are m-connected and $(B, A\cap B)$ are n-connected, then $\pi_i(A, A\cap B) \to \pi_(A\cup B, A)$ is isomorphism for $i < m+n$, and surjective for $i = m+n$. Cf.[Hatcher P360].

Moreover, if (X, A) is r-connected and A is s-connected, then $\pi_i(X, A) \to \pi_i(X/A)$ is isomorphism for $i \leq r+s$ and surjection for $i = r+s+1$.

Cor. (2.4.3) (Freudenthal Theorem). For $i \leq 2n-2$, $\pi_i(S^n) \cong \pi_{i+1}(S^{n+1}) = \mathbb{Z}$. (Can also be derived considering antipodal point point of S^n by (6.2.9)) and surjective for $i = 2n-1$. In general, $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n+1$.

Proof: Use the suspension, for $n = 1$, we can use Hopf bundle. □

Prop. (2.4.4) (Generalized Hurewicz theorem). If (X, A) is a $(n-1)$-connected pair of spaces, $n \geq 2$, then the Hurewicz map induces isomorphism $\pi_n(X, A)/(\pi_1(A) - id) \to H_k(X, A)$, and $H_k(X, A) = 0, k < n$. And for on π_{n+1}, the Hurwicz map is surjective for $n > 1$.Cf.[Hatcher P390Ex23] for surjectiveness.

Prop. (2.4.5). For a fiber bundle $S \to M \to N$, there is a long exact sequence of homotopy groups:

$$\cdots \to \pi_i(N) \to \pi_{i-1}(S) \to \pi_{i-1}(M) \to \pi_{i-1}(N) \to \cdots$$

Because it has lifting property.

Prop. (2.4.6). $\pi_{i+1}(M) \cong \pi_i(\Omega(M))$, where Ω is the loop space. More generally,

$$\langle \Sigma X, K\rangle = \langle X, \Omega K\rangle.$$

Prop. (2.4.7). If K_n is an Ω-spectrum, then the functors $X \mapsto h^n(X) = \langle X, K_n\rangle$ define a reduces cohomology theory on the category of basepointed CW complexes, i.e. it satisfies the long exact sequence for $A \to X \to X/A$ and wedge axiom. Cf.[Hatcher P397].

Proof: Use(2.4.6) and there is a Cofibration sequence:

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \dots$$

□

Prop. (2.4.8). Every map can be decomposed as a homotopy equivalence followed by a fibration, by the construction of homotopy fibers. Cf.[Hatcher P407].

Prop. (2.4.9). The homotopic direct limit of a family of homotopy equivalence is a homotopy equivalence. Cf.[Morse Theory Milnor].

Prop. (2.4.10). for $i \le 2m$, $\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i-1}U(m)$, and

$$\pi_{i-1}U(m) \cong \pi_{i-1}U(m+1) \cong \cdots .$$

and for $j \ne 1$, $\pi_j U(m) \cong \pi_j SU(m)$.

Similarly, $\pi_i \Omega_1(2m) \cong \pi_{i+1}O(2m)$ for $i \le n-4$. (6.2.10),Cf[Morse Theory Milnor Prop23.4].

Cor. (2.4.11) (Bott Periodicity theorem for Unitary Groups). The stable homotopy group $\pi_i U$ has period 2. $\pi_{2k+1}U \cong 0$ and $\pi_{2k}U \cong \mathbb{Z}$.

Proof: Use the last proposition and long exact sequence to show that for $1 \le i \le 2m$,

$$\pi_{i-1}U = \pi_{i-1}U(m) \cong \pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i+1}SU(2m) \cong \pi_{i+1}U.$$

Notice that $U(m) \to U(2m)/U(m) \to G_m(\mathbb{C}^{2m})$ □

Prop. (2.4.12) (Bott Periodicity for O). For the infinite dimensional orthogonal space O, $\Omega_8(16r) \cong O(r), \Omega_4(8r) \cong Sp(2r)$. So $\Omega_8 \cong O$ and $\Omega_4 O \cong Sp$. Thus by (2.4.6),

$$\pi_i(O) = \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z} \ldots, \quad \pi_i(Sp) = 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z} \ldots$$

respectively. (Use (6.2.11)) Cf.[Morse Theory Prop24.7].

Obstruction Theory

Prop. (2.4.13) (Towers). There are Whitehead Towers and Postnikov Towers for a CW complex X.

$$\cdots \to Z_2 \to Z_1 \to Z_0 \to X \to \cdots \to X_2 \to X_1 \to X_0$$

Z_n annihilate $\pi_{\le n}(X)$, X_n remains only $\pi_{\le n}(X)$. The towers can be chosen to be fibrations, with fibers $K(\pi_n X, n)$ by(2.4.8).

Prop. (2.4.14) (Obstructions). If a connected abelian CW complex $X(\pi_1(X)$ abelian and action on higher homotopy trivial) and (W, A) satisfies $H^{n+1}(W, A; \pi_n X) =$ for all n, then $A \to X$ can extend to a map $M \to X$.

Proof: Cf.[Hatcher P417]. □

Cor. (2.4.15). A map between Abelian CW complexes that induce isomorphisms on homology is a homotopy equivalence.

Proof: Notice that $\pi_1(X)$ acts trivially on $\pi_(Y, X)$ and use Hurewicz. □

Classifying spaces

Def. (2.4.16). The **classifying space** for a topological group is a space BG with a weakly contractable universal cover EG that EG is a G-fiber bundle on BG. $[X, BG] \cong G$-bundles on X. And BG is Abelian if G is Abelian.

For an discrete Abelian group A, we denote $K(A, 0) = A$, $K(A, n+1) = B(K(A, n))$. Then $H^n(X, A) = [X, K(A, n)]$. Notice that $\pi_{n+1}(BG) = \pi_n(G)$. It is called **Eilenberg-Maclane spaces**, i.e. it has only a nontrivial homotopy group $\pi_n(K(A, n)) = A$. The compression theorem shows $K(G, n)$ is unique up to homotopy.

EM complexes can also be built by a set of $n, n+1$ cells and use cells of dim$\ge n+2$ to kill other homotopy.

Prop. (2.4.17) (Examples).

- $K(\mathbb{Z},1)=S^1=U(1)$, $K(\mathbb{Z},2)=\mathbb{CP}^\infty$, Because $S^\infty\to\mathbb{CP}^\infty$ is a contractable covering and use the fiber sequence.
- $B(\mathbb{Z}/2)=\mathbb{RP}^\infty$, and $B(\mathbb{Z}/n\mathbb{Z})=S^\infty/(\mathbb{Z}/n)$.
- $BSU(2)=\mathbb{HP}^\infty$.
- $B(\mathbb{Z}^{2g})=$ torus of genus g.
- $BO(n),BU(n),BSp(n)$ are respectively the Grassmannian of n-planes in the infinite dimensional real, complex and quaternion vector spaces, because we have
$$O(n)\to V_n(\mathbb{R}^\infty)\to G_n(\mathbb{R}^\infty).$$
and similarly for $\mathbb{C}$ and $\mathbb{H}$, and $V_n(\mathbb{R}^\infty)$ is contractible by linear homotopy and Schmidt orthogonalization.
- there are fiber bundles
$$S^0\to BSO(n)\to BO(n)$$
and similarly others, because they both have the flag variety $V_n(R^\infty)$ as EG.

III.3 Differential Manifold

1 Simplifications

Prop. (3.1.1). For every vector field $X$and every point $X(p) \neq 0$, there exists a coordinate nbhd $(x_1, \ldots, x_{n-1}, t)$ such that $X = \frac{\partial}{\partial t}$.

2 Differential Forms

Lemma (3.2.1).

$$[X, Y] = \frac{\partial}{\partial t}(d(\phi_{-t})Y)|_{t=0}$$

Proof: For any function f,set $g(t, q) = \frac{f(\phi_t(q)) - f(q)}{t}$,$g(0, q) = Xf(q)$. Then g is differentiable (because $g(t, q) = \int_0^1 Xf(\phi_{ts}(p))ds$, and:

$$\begin{aligned}
\lim_{t\to 0} d(\phi_{-t})Yf(p) &= \lim \frac{Yf(p) - Y(f\phi_{-t})(\phi_t(p))}{t} \\
&= \lim \frac{Yf(p) - Yf(\phi_t p) - Y(tg(-t, \phi_t(p))}{t} \\
&= ((XY - YX)f)(p) \\
&= [X, Y]f(p)
\end{aligned}$$

□

Prop. (3.2.2) (Lie formula).

$$L_X(g(Y, Z)) = L_X(g)(Y, Z) + g(L_X Y, Z) + g(Y, L_X Z).$$

Prop. (3.2.3) (Derivative formula).

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

.

Prop. (3.2.4) (Cartan's magic formula).

$$L_X\omega = \iota_X(d\omega) + d(\iota_X\omega)$$

$$\iota([X, Y]) = [L_X, \iota_Y]$$

Proof: Notice that four of them are derivatives (check because $\iota_X(w \wedge v) = \iota_X w \wedge v + (-1)^{|w|} w \wedge \iota_X v)$. So by induction, we only has to verify them on dimension 0 and 1. □

Prop. (3.2.5) (Stoke's theorem).

$$\oint_\Omega d\omega = \oint_{\partial\Omega} i^*\omega.$$

In a 3-dimensional Riemanian manifold, If we set:

$$df = \omega^1_{\mathrm{grad}_f}, \quad d\omega^1_A = \omega^2_{\mathrm{curl}_A}, \quad d\omega^2_A = (\nabla A)\omega^3,$$

Then:

$$f(y) - f(x) = \int_l \mathrm{grad} f \cdot dl.$$

$$\int_l A \cdot dl = \oint_S \mathrm{curl} A \cdot dn.$$

$$\oint_U \nabla \cdot F dV = \oint_{\partial U} F \cdot ndS.$$

Prop. (3.2.6). Lie bracket commutes with derivative.$[df(X), df(Y)] = df([X,Y])$. (Use $XY - YX$ to see).

Prop. (3.2.7) (Frobenius Theorem). If X is an involutive distribution on a manifold M, then there is a unique maximal integration manifold passing through it. Where a distribution is involutive if it is closed under Lie bracket.

Proof: The key to the proof is to prove that involutive is equivalent to integrable, i.e. flat locally as $\{\frac{\partial}{\partial x_i}\}$ for some local coordinate. Cf.[李群讲义 项武义 P226] □

Cor. (3.2.8). X, Y in a Lie algebra commute iff their corresponding vector fields commute.

Def. (3.2.9) (Hodge Star Operator). given a n-form ω, the Hodge star operator $*-$ is an operator from $\bigwedge^k V \to \bigwedge^{n-k} V$ such that:

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \omega.$$

3 Transversality

Prop. (3.3.1) (Parametric Transversality Theorem). Suppose N and M are smooth manifolds, $X \subset M$ is an embedded submanifold, and F_s is a smooth family of maps from N to M. If the map $F : N \times S \to M$ is transverse to X, then for almost every s, the map $Fs : N \to M$ is transverse to X. Cf.[Smooth Manifold Lee T6.35].

Proof: □

Prop. (3.3.2) (Transversality Homotopy Theorem). Suppose N and M are smooth manifolds and $X \subset M$ is an embedded submanifold. Every smooth map $f : N \to M$ is homotopic to a smooth map $g : N \to M$ that is transverse to X. Cf.[Smooth Manifold Lee T6.36].

Proof: embed M into a R^k and take a tubular neighbourhood, then we can construct a $N \times S^k$ transversal to M. □

Cor. (3.3.3). For a vector bundle over a compact manifold, there exists a global section transversal to the zero section, in particular, if $\dim E > M$, then it has no zero.

Proof: choose a finite trivializing cover that there closure is compact and choose a compact subcover, find finitely many sections to assure $C^N \times X \to E$ is transversal, and use parametric trnasversality theorem to prove there is a section that is transversal. □

Cor. (3.3.4). There is a vector field on compact manifold of only isolated zeros. And a vector bundle over a k dimensional curve splits to components of dimension no bigger than k. Determined by its Chern class.

4 Flow

Prop. (3.4.1) (Isotopy Extension Theorem). Let M be a manifold and A be a compact subset. Then an isotopy $F: A\times I\to M$ can be extended to an diffeotopy of M.

Proof: Consider $F(A\times I)\subset M\times I$ is a compact set, and $TM\times I\to M\times I$ is a vector bundle. The time lines generate a section $F(A\times I)\to TM\times I$, so (7.1.2) guarantees an extension $M\times I\to TM\times I$, and because manifolds are locally compact, this section can be chosen to be compactly supported, then the flow it generates is a diffeotopy. $\square$

III.4 Differential Geometry

1 Different Coordinates

Prop. (4.1.1). In a polar coordinate,

$$g_{11} = 1, g_{12} = 0, g_{22} = |\frac{\partial f}{\partial \theta}|^2, \quad K = -\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}}$$

And $\sqrt{g_{22}} \sim \rho$. (Use the formula relating Jacobi Field with curvature)

2 Moving Frame Method

Prop. (4.2.1) (Theorema Egregium).

;'l

III.5 Riemann Geometry

1 Fundamentals(Do Carmo)

- Christoffel symbol: $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$.

$$\Gamma_{ij}^m = 1/2 \sum_k \{g_{jk,i} + g_{ki,j} - g_{ij,k}\} g^{km}.$$

- the geodesic equation:

$$\frac{Dt}{dt}(\frac{d\gamma}{dt}) = \ddot{x}_k + \sum_{i,j} \Gamma_{ij}^k \dot{x}_i \dot{x}_j = 0 \quad \forall k$$

- Geodesic is a solution only depends on the metric, so a local isometry preserves geodesics.
- Geodesic flow: the flow on TM whose trajectories are $t \mapsto (\gamma(t), \gamma^{'}(t))$, where γ is a geodesic on M.
- **(The smoothness of geodesics)** for every point p, there exists a nbhd V and a C^∞ mapping
$$\gamma : (-\delta, \delta) \times V \times B(0, \epsilon) \to M,$$
s.t. $\gamma(t, q, v)$ is the geodesic passing through p with velocity v.
- $$\frac{D}{\partial u}\frac{\partial s}{\partial v} = \frac{D}{\partial v}\frac{\partial s}{\partial u}.$$
- **(Totally normal nbhd)** For any point p, there exists a nbhd W and a number $\delta > 0$ s.t. for every $q \in W$, $\exp_q$ is a diffeomorphism on $B_\delta(0)$ and $\exp_q(B_\delta(0)) \supset W$. Thus, fine cover exists in every smooth manifold.
- (Gauss Lemma) In a normal nbhd, the vectors orthogonal to geodesics is mapped under $(d\exp_p)_v$ to vectors orthogonal to geodesics.
- a locally minimizing piecewise differentiable curve is a geodesic.(Choose normal nbhd and use polar coordinate).
- Killing field is which generates an infinitesimal isometry. X is killing $\iff L_X(g) = 0 \iff \langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ for all Y, Z (Killing equation).
- A Killing field is a Jacobi field along geodesics. (by Calculation).
- The singularities of a Killing field is a submanifold and will generate a vector field along a geodesic sphere of the orthogonal component.
- (**Geodesic Frame**) In a neighborhood of every point p, there exists n vector fields, orthonormal at each point, and $\nabla_{E_i} E_j(p) = 0$.(Choose normal nbhd and parallel a orthonormal basis to every point.
- gradient: $\langle \mathrm{grad} f(p), X\rangle = X(f)(p)$.
- divergence: $\mathrm{div} X(p)$ =trace of the linear map $Y(p) \to \nabla_Y X(p) = \sum_i \langle \nabla_{E_i} X, E_i\rangle$. It measures the variation of the volume and it depends only on the point.

- Hessian: $\mathrm{Hess} f$ is a self-adjoint operator that $(\mathrm{Hess} f)Y = \nabla_Y \mathrm{grad} f$ as well as a symmetric form $(\mathrm{Hess} f)(X,Y) = \langle (\mathrm{Hess} f)X, Y\rangle$.
- Laplace: $\Delta f = \mathrm{div}\ \mathrm{grad} f = \mathrm{trace}\ \mathrm{Hess} f = \sum_i E_i(E_i(f))$.
- in a geodesic frame,
$$\mathrm{grad} f(p) = \sum_{i=1}^{n} (E_i(f))E_i$$
$$\mathrm{div} X(p) = \sum_{i=1}^{n} E_i(f_i)(p), \mathrm{where} X = \sum_i f_i E_i.$$
$$\Delta f = \sum_i E_i(E_i(f))(p).$$
- $\Delta(f\cdot g) = f\Delta g + g\Delta f + 2\langle \mathrm{grad} f, \mathrm{grad} g\rangle$.
- $d(i(X)m) = (\mathrm{div} X)m$. where m is the volume form.
- (Hopf theorem) If f is a differentiable function on a compact orientable manifold with $\Delta f \geq 0$, then f is constant.
- curvature tensor $R(X,Y) : Z \mapsto \nabla_y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$.
- The curvature tensor is determined by its sectional curvature, thus if M is isotropic at a point p (The sectional curvature depends only on the point), then $R(X,Y,W,Z) = K_0(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)$
- **Ricci curvature** $\mathrm{Ric}_p(x) = \frac{1}{n-1}\sum\langle R(x,z_i)x, z_i\rangle$, for x a unit vector, where z_i is an orthonormal basis orthogonal to x. $\mathrm{Ric}(x) = \mathrm{Ric}(x,x)$, where $\mathrm{Ric}(x,y)$ is the symmetric form of $\frac{1}{n}$ of trace of the map $z \to R(x,z)y$.
- **scalar curvature** $K(p) = 1/n \sum \mathrm{Ric}_p(z_i)$, where z_i is an orthonormal basis.
-
$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})V. \quad \text{(obvious because } \frac{\partial}{\partial_s}, \frac{\partial}{\partial_t} \text{ commutes)}$$
- sectional curvature $K(X,Y) = \langle R(X,Y)X, Y\rangle$.
- curvature tensor only depends on the point and
$$R(X,Y,Z,W) = R(Z,W,X,Y), \quad R(X,Y,Z,W) = R(X,Y,W,Z).$$
- Levi-Civita connection: symmetric and compatible with the Riemann metric.
$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \qquad \nabla_X Y - \nabla_Y X = [X,Y].$$
It satisfies:
$$\langle Z, \nabla_Y X\rangle = 1/2\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [X,Y],Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle\}.$$
- Covariant differential $\nabla R(Y_i, Z) = Z(R(Y_i)) - \sum_i R(\nabla_Z Y_i, Y_j)$.
- (Bianchi Identity)$\sum_{(X,Y,Z)} R(X,Y)Z = 0$.
- (Second Bianchi Identity)$\sum_{(Z,W,T)} \nabla R(X,Y,Z,W,T) = 0$.

- (Schur's Theorem) Let M be a manifold of dimension $n \geq 3$, suppose M is isotropic , then M has constant curvature. (Use the second Bianchi Identity and geodesic frame).
- Jacobi field equation along a geodesic γ: $D^2 J(t) + R(\dot{\gamma(t)}, J(t))\dot{\gamma}(t) = 0$. It is defined by its initial condition $J(0)$ and $J^{'}(0)$. It can be used to detect the sectional curvature、 the critical point of $\exp_p$ and calculate variation of energy.
- The Jacobi field along a point with initial velocity 0 all has the form $J(t) = (d\exp_p)_{t\dot{\gamma}(0)}(tJ^{'}(0))$. Corollary: the Jacobi transport from p to q is an isomorphism iff p and q is not conjugate.
- for general Jacobi field,
$$\langle J(t), \dot{\gamma}(t)\rangle = \langle J^{'}(0), \dot{\gamma}(0)\rangle t + \langle J(0), \dot{\gamma}(0)\rangle.$$
- If J is a Jacobi field $J(t) = (d\exp_p)_{tv}(tw), |v| = |w| = 1$, then
$$|J(t)| = t - \frac{1}{6}K_p(v, w)t^3 + o(t^3).$$
- $B(X, Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$. It is bilinear and symmetric.
- $H_\eta(x, y) = \langle B(x, y), \eta\rangle$. Thus $B(x, y) = \sum H_i(x, y)E_i$ for an orthonormal frame E_i in $\mathfrak{X}(U)^{\perp}$.
- $S_\eta(x) = -(\overline{\nabla}_x \eta)^T$. It satisfies:$\langle S_\eta(x), y\rangle = H_\eta(x, y) = \langle B(x, y), \eta\rangle$. It is self-adjoint. When codimension 1, it is the derivative of the Gauss mapping.
- (**Gauss Formula**): let x, y be orthonormal tangent vector. Then:
$$K(x, y) - \overline{K}(x, y) = \langle B(x, x), B(y, y)\rangle - |B(x, y)|^2.$$
- An immersion is called **geodesic** at p if the second fundamental form S_η is zero for all η, (which means $\nabla_X Y$ has no normal component). It is called **minimal** if the trace of S_η is zero.
- An immersion is called umbilic if there exists a normal unit field η s.t. $\langle B(X, Y), \eta\rangle(p) = \lambda(p)\langle X, Y\rangle$.
- If the ambient space has constant sectional curvature and the immersed manifold is totally umbilic, then λ is constant.
- mean curvature tensor of immersion $f = 1/n \sum_i(\mathrm{tr}\ S_i)E_i = 1/n\ \mathrm{tr}\ B$. It is zero if f is minimal.
- normal connection $\nabla^{\perp}_X \eta = (\overline{\nabla}_X \eta)^N = \overline{\nabla}_X \eta + S_\eta(X)$.
- (Gauss equation)
$$\langle \overline{R}(X, Y)Z, T\rangle = \langle R(X, Y)Z, T\rangle - \langle B(Y, T), B(X, Z)\rangle + \langle B(X, T), B(Y, Z)\rangle.$$
- (Ricci equation)
$$\langle \overline{R}(X, Y)\eta, \zeta\rangle - \langle R^{\perp}(X, Y)\eta, \zeta\rangle = \langle [S_\eta, S_\zeta]X, Y\rangle.$$
- (Codazzo equation)
$$\langle \overline{R}(X, Y)Z, \eta\rangle = (\overline{\nabla}_Y B)(X, Z, \eta) - (\overline{\nabla}_X B)(Y, Z, \eta). \qquad \text{(Lie bracket)}$$

Prop. (5.1.1) (Hopf-Rinow theorem). The following definitions is equivalent to completeness.

1. $\exp_p$ is defined for all of $T_p(M)$.
2. The closed and bounded sets of M are compact.
3. M is complete as a metric space.
4. M is $\sigma-$compact and if $q_n \notin K_n$, $d(p, q_n) \to \infty$.
5. The length of any divergent (compact escaping) curve is unbounded.

and if M is complete, then for any q, there exists a minimizing geodesic. And any compact submanifold of a complete manifold is complete.

Prop. (5.1.2) (Hadamard theorem). M a complete simply connected Riemann manifold of sectional curvature ≤ 0, then $\exp_p : T_pM \to M$ is an isomorphism of M to $\mathbb{R}^n$.(negative sectional curvature to show exp is a local isomorphism, complete to show it is a covering map)

- For any two manifold of the same constant curvature and any two orthogonal basis, there is a local isometry (It is locally isotropic).
- Any complete manifold with a sectional curvature is like $\tilde{M}/\Gamma$, where $\tilde{M}$ is $\mathbf{H}^n, \mathbf{R}^n$ or $\mathbf{S}^n$.
- Every compact orientable surface of genus $p > 1$ can be provided with a metric of constant negative curvature.

Prop. (5.1.3) (Liouville Theorem). Any conformal mapping for an open subset of $\mathbb{R}^n, n > 2$ is restriction of a composition of isometry, dilations and/or inversions, at most once.

- Energy
$$E(s) = \int_0^a |\frac{\partial f}{\partial t}(s,t)|^2 dt.$$
- A minimizing geodesic must minimize energy.
- (**First Variation of Energy**)
$$1/2E^{'}(0) = -\int_0^a \langle V(t), D\dot{c}(t)\rangle dt + \langle V(a), \dot{c}(a)\rangle - \langle V(0), \dot{c}(0)\rangle.$$
A piecewise differentiable curve is a geodesic iff every proper variation has first derivative 0.
- (**Second Variation of Energy**) If γ is a geodesic,
$$1/2E^{''}(0) = \int_0^a \{\langle DV(t), DV(t)\rangle - \langle R(\dot{\gamma}, V)\dot{\gamma}, V\rangle\} dt + \langle D_s V(a), \dot{\gamma}(a)\rangle - \langle D_s V(0), \dot{\gamma}(0)\rangle.$$
- a variation is equivalent to a vector field along the curve, and a variation that $f_s(t)$ are all piecewise geodesics corresponds to a piecewise Jacobi field(Choose a normal partition).

Prop. (5.1.4) (Bonnet-Mayer). M a complete manifold of Ricci curvature $\mathrm{Ric}_p(v) \geq \frac{1}{r^2}$, Then M is compact and have diameter $\leq \pi r$.

Cor. (5.1.5). M is a complete manifold of Ricci curvature $\geq \delta > 0$, then the universal cover is compact thus $\pi_z(M)$ is finite.

Prop. (5.1.6) (Synge). f is an isometry of a compact oriented manifold M^n of positive sectional curvature, f alter orientation by $(-1)^n$, then f has a fixed pt.

Cor. (5.1.7). M a compact manifold of positive sectional curvature, then

1. If M is orientable and n is even, then M is simply connected. So If M is compact and even dimension, then $\pi(M) = 1$ or $\mathbb{Z}_2$.
2. If n is odd, then M is orientable.

(Use the universal cover and covering transformation.)

Prop. (5.1.8) (Index Lemma). Among the piecewise differentiable vector fields along a geodesic without conjugate point or without focal point, with initial value 0 and fixed end value, the Jacobi field attain minimum of the index form:

$$I_a(V,V) = \int_0^a \{\langle DV(t), DV(t)\rangle - \langle R(\dot{\gamma}, V)\dot{\gamma}, V\rangle\}dt.$$

Cor. (5.1.9). $I_l(J,J) = \langle J, J'\rangle(l)$ for a Jacobi field.

Prop. (5.1.10). a focal point is a critical value of $exp^{\perp}$. For an embedded manifold, the focal point equals $x + 1/t\eta$, where η is a vertical vector and t is a principal value of $S_e ta$.

Prop. (5.1.11) (Rauch Comparison theorem). Let M and $\tilde{M}$ be manifolds, dim$\tilde{M} \geq$ dim M. If J and $\tilde{J}$ be two Jacobi fields along geodesics γ and $\tilde{\gamma}$ that $|J'(0)| = |\tilde{J}'(0)|$. If $\tilde{\gamma}$ has no conjugate point or focal point free and $\tilde{K}(\tilde{x}, \dot{\tilde{\gamma}}(t)) \geq K(x, \dot{\gamma})$ for any vector $x, \tilde{x}$, then $|\tilde{J}| \leq |J|$.

Cor. (5.1.12). If the sectional curvature of M satisfies: $0 < L \leq K \leq H$, then the distance between any two conjugate points satisfies: $\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}$.

Cor. (5.1.13) (Bishop theorem). Let M be complete manifold such that $\mathrm{Ric}_M \geq H$, let $\tilde{M}(H)$ be a complete simply connected manifold of constant sectional curvature H, then $\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(B_r(\tilde{p}))$.

Prop. (5.1.14). If two manifold M and M' satisfy $K \leq K'$, then in a normal nbhd of a point p in M and a nbhd of p' that exp is nonsingular, the transformation of a curve c shortens length.

Note that this is not Toponogov thoerem, because if you try to map from a large curvature manifold to a small curvature, then you cannot guarantee that the mapped curve is the shortest.

Cor. (5.1.15). In a complete simply connected manifold of non-positive curvature,

$$A^2 + B^2 - 2AB\cos\gamma \leq C^2$$

thus $\alpha + \beta + \gamma \leq \pi$.

Prop. (5.1.16) (Moore theorem). Let $\overline{M}$ be a complete simply connected manifold of sectinoal curvature $\overline{K} \leq -b \leq 0$, M a compact manifold of sectional curvature satisfying $K - \overline{K} \leq b$. If dim$\overline{M} <$ dim$M$, M cannot be immersed into $\overline{M}$.(use Hadamard theorem to choose the furthest geodesic and calculate the second variation of energy and use Gauss formula).

Cor. (5.1.17). Let $\overline{M}$ be a complete simply connected manifold of sectinoal curvature $\overline{K} \leq 0$, M a compact manifold of sectional curvature satisfying $K \leq \overline{K}$. If dim$\overline{M}$ < dim$M$, M cannot immerse into $\overline{M}$.

Remark (5.1.18). There exist complete surfaces with $K \leq 0$ in $\mathbb{R}^3$, but the hyperbolic surface cannot be immersed into $\mathbb{R}^3$(Hilbert Theorem).

Prop. (5.1.19) (Morse Index theorem). The index of the the index form $I_a(V, W)$ on the space of vector fields 0 at the endpoints, equal to the number of points conjugate to $\gamma(0)$ in $[0, a)$.

Cor. (5.1.20). If γ is minimizing, γ has no conjugate points on $(0, a)$, γ has a conjugate point, it is not minimizing.

Prop. (5.1.21) (Cartan). in any nontrivial homotopy class in a compact manifold , there exists a closed geodesic.

Prop. (5.1.22) (Morse). If M is complete with non-negative sectional curvature, then $\pi_1(M)$ have no finite non-trivial cyclic group and $\pi_k(M) = 0$.

Proof: because universal cover of M is contractible, so the higher homotopy group vanish and $H^k(M) = H^k(\pi_1(M))$,**?** so if a subgroup is finite cyclic, its homology is periodic, contradiction. □

Prop. (5.1.23) (Preissman). For a compact manifold with $K < 0$, any nontrivial abelian subgroup of π_1 is infinite cyclic.

Prop. (5.1.24). If M is compact and $K < 0$, $\pi_1(M)$ is not abelian.

Assuming M complete,

- The cut point of p along γ is the maximum $\gamma(t)$ s.t. $d(p, \gamma(t)) = t$. It is either the first conjugate point of p or the intersection of two minimizing geodesics.
- Conversely, if a point is a conjugate point of p or is intersection of two geodesics of equal length, then there is a cut point before it. So, if intersection of two minimizing geodesics happens, it must happen before the occurrence of conjugate point.
- thus the cut point relation is reflexive, and if $q \in M \setminus C_m(p)$, then there exists a unique minimizing geodesic joining p and q.
- $M \setminus C_m(p)$ is homeomorphic to an open ball through exp.
- the distance of p to the cut locus is continuous, thus $C_m(p)$ is closed.
- If M is complete and there is a p which has a cut point for every geodesic, then M is compact.
- for q the closest of $C_m(p)$ to p, either there exists a minimizing geodesic and q is conjugate to p or there is to minimizing geodesic connecting at q.

Prop. (5.1.25). The index of a geodesic will decrease when transferred to a manifold of smaller sectional curvature K.

Lemma (5.1.26) (Klingenberg). (P236)Let M be a complete manifold of sectional curvature $K \geq K_0$, let γ_0, γ_1 be two homotopic geodesics from p to q, then there exists a middle curve γ_s s.t.

$$l(\gamma_0) + l(\gamma_1) \geq \frac{2\pi}{\sqrt{K_0}}.$$

Prop. (5.1.27) (Klingenberg). Let M be a simply connected compact manifold of dimension $n \geq 3$ such that $\frac{1}{4} < K \leq 1$, then $i(M)$(The infimum of distance to the cut locus)$\geq \pi$.

Cor. (5.1.28). If M is a compact orientable manifold of even dimension satisfying $0 < K \leq 1$, then $i(M) \geq \pi$.

Prop. (5.1.29) (1/4-pinch Sphere Theorem). Let M be a compact simply connected manifold satisfying $0 < 1/4K_{\max} < K \leq K_{\max}$, then M is homeomorphic to a sphere.

(Use Klingenberg Theorem, this is a special case of diameter geodesic sphere theorem). Cf.(5.2.9).

It can be shown that in this case, this sphere is even diffeomorphic to S^n using Ricci flow.

Remark (5.1.30). $0 < 1/4K_{\max} < K$ cannot be changed to $\geq$. In fact, the Funibi-Study metric on CP^n has sectional curvature $1 \geq K \geq 4$. Cf. **??**

Prop. (5.1.31). In a complete manifold, if there is a sequence of points $\{p_i\}$ converging to a point p, choose for each point a minimal geodesic, then a subsequence of them will converge to a minimal geodesic to p.

Proof: The convergence is by smoothness and of exp and Hadamard. The minimality is by comparing distance. □

2 Comparison Theorems

Hess$\rho(X,Y)$ where ρ is the distance to a fixed point, is important.

Prop. (5.2.1). Hess$\rho(X,Y)$ is positive definite on the tangent space of the geodesic sphere within the injective radius, and its principal value is $|\frac{J'}{J}|$ for a Jacobi field in that direction. And it is zero on the normal direction.

So there would be a Riccati comparison theorem on the eigenvalue of $\Pi_2 : \lambda' \leq -K - \lambda^2$,Hess$(\rho)$ is bounded.

Proof: Notice that

$$\text{Hess}\rho(X,Y) = (\nabla_X \text{grad}\rho, Y) = XY\rho - (\nabla_X Y)\rho$$

so if choose a normal geodesic γ of initial vector X, then

$$\text{Hess}\rho(X,X) = X\langle\dot{\gamma}, d\rho\rangle - (\nabla_X\dot{\gamma})\rho = X\langle\dot{\gamma}, d\rho\rangle = \langle\dot{\gamma}, d\langle\dot{\gamma}, d\rho\rangle\rangle = E''(0)$$

$$= I_q(X,X) = ((\nabla_{\dot{\gamma}}X)(q), X(q)) = \frac{\langle J', J\rangle}{|J|^2}$$

□

Prop. (5.2.2) (Toponogov). Let$M$ be a complete manifold with $K \geq H$.

If a hinge satisfies γ_1 is minimal and $\gamma_2 \geq \frac{\pi}{\sqrt{H}}$ if $H > 0.$, then on M^H the same hinge has smaller distance of endpoints than this hinge

Proof: Cf.[Cheeger Comparison Theorems in Riemannian Geometry P42]. And there is another triangle version: For a minimal geodesic triangle, the comparison triangle has smaller angles. NOTE this theorem cannot be derived from Rauch Comparison Theorem. □

Critical Point for Distance Function

Prop. (5.2.3). The critical point for distance function on a complete manifold is that for every direction v, there is a minimal geodesic γ s.t. $\langle \gamma'(l), v\rangle \leq \frac{\pi}{2}$.

The set of regular point is open and there exists a smooth gradient like vecter field (i.e. acute angle with every minimal geodesic) on this open subset .

Prop. (5.2.4) (Berger's Lemma). A maximal point for the distance function is a critical point.

Proof: If not, choose a convergent point v of the minimal geodesics with endpoint in a curve of that direction, then exp near v will generate a Jacobi field with endpoint Jacobi is the sam of that direction. So the distance will increase by $\cos\theta$ along that direction, contradiction. □

Prop. (5.2.5) (Soul Lemma). Let M is a Riemannian manifold and A is a closed submanifold. If $dist(A, -)$ has no critical point on $D(A, R)\backslash A$, then $B(A, R)$ is diffeomorphic to the normal bundle of $A \to M$.

Proof: A has a normal exp radius ϵ, and we can vary the gradient-like vector field to be identical to the normal vector near A, and use Morse lemma (the flow) to get a diffeomorphism. □

Cor. (5.2.6) (Disk Theorem). If A is a point then M is diffeomorphic to a disk.

Lemma (5.2.7) (Generalized Schoenflies Theorem). Easy to do, just use the fact that exp is continuous to find a boundary sphere depending continuously on the direction (both p and q).

Prop. (5.2.8) (Sphere Theorem). If M is a closed manifold and has a distance function with only one critical point (the furthest one), then M is homeomorphic to a twisted ball.

Proof: There exists a ϵ and r that $B(q, \epsilon)$ and $B(p, r)$ covering M, (Use the convergent point argument). Then use the generalized Schoenflies theorem. □

Prop. (5.2.9) (Diameter Sphere Theorem). If a closed manifold M satisfies sec$M \geq K > 0$, and diam$(M) > \frac{\pi}{2\sqrt{K}}$, then M is homeomorphic to S^n.

Proof: First, if there are two maximal distance point, then use Toponogov to show contradiction. Second, at other points x,

$$\angle pxq > \frac{\pi}{2}$$

(Regular domain) because of Toponogov and The formula

$$\cos\tilde{\alpha} = \frac{\cos l - \cos l_1 \cos l_2}{\sin l_1 \sin l_2}.$$

So the geodesic direction $\overrightarrow{xq}$ will serve as a geodesic-like vector field (might need paracompactness). □

Prop. (5.2.10) (Critical Principle). In a complete manifold M of sectional curvature $> K$, if q is a critical point of p, then for any point x with $d(p,x) > d(p,q)$ and any minimal geodesic from p to x, the $\angle xpq$ is smaller than the $\cosh_K^{-1}(\frac{d(p,x)}{d(p,q)})$.

Proof: Use Toponogov for the hinge xpq. Then notice that there is a different minimal geodesic from $p \to q$ that makes the $\angle pqx < \pi/2$ by the definition of critical point, thus there is another Toponogov inequality, this two inequality contradicts. □

Cor. (5.2.11). For a complete open manifold whose K are lower bounded, then it is homeomorphic to the interior of a manifold with boundary. (Use Soul lemma, otherwise there will be a sequence of critical point whose angles are big).

Prop. (5.2.12) (Soul Theorem). If M is an open manifold with non-negative sectional curvature, then there is a totally geodesic submanifold S that M is diffeomorphic to the normal bundle over S.

Proof: Use the ray construction to get a totally convex compact subset, hence it is a manifold or with boundary , if it has boundary, then find to set of maximal distance to the distance to boundary, the distance to the boundary is a convex function, so it is a smaller totally geodesic manifold. So a S without boundary must exist and this constitutes a stratification, all the level set is strongly convex. Thus all point outside S is not critical, hence the soul lemma applies. Cf.[GeJian Comparison theorems in Riemannian Geometry Lecture7]. □

Prop. (5.2.13) (Soul Conjecture). For an open(non-compact) manifold M with a point p s.t. sectional curvature at p are all positive, then M is diffeomorphic to R^n.

III.6 Differential Topology

1 Differential Topology

Prop. (6.1.1) (transversality).

Prop. (6.1.2) (Sard Theorem). The set of critical values is of measure zero in the image manifold.

Prop. (6.1.3) (Hopf Index theorem). In a compact manifold, any vector field V with isolated zeros has sum of its index equal to $\chi(M)$. Where the index of a singularity is the mapping degree of V on a surrounding sphere.

2 Morse Theory(Milnor)

Prop. (6.2.1) (Morse Lemma). In a non-degenerate critical point of f, there is a coordinate that

$$f = f(p) + x_1^2 + \cdots + x_{n-\lambda}^2 - y_1^2 - \cdots - y_\lambda^2.$$

Proof: Just extract the first order part out and reform the bilinear form one-by-one. Cf.[Milnor Morse Theory lemma 2.2]. □

Prop. (6.2.2). If f is a smooth function that $f^{-1}([a,b])$ is compact and have no critical points, then M^a is a deformation retracts of M^b using $\mathrm{grad} f/|\mathrm{grad} f|^2$.

Prop. (6.2.3) (Morse Main Lemma). If f is a smooth function with p a non-degenerate critical point and λ downward pointing direction. If for some $f^{-1}([c-\epsilon, c+\epsilon])$ is compact, then $M^{c+\epsilon}$ is homotopic to $M^{c-\epsilon}$ gluing a λ dimensional cell.

Proof: Cf.[Milnor Prop3.2]. □

Prop. (6.2.4). For an embedded manifold and almost all point p, the distance to p is a morse function. (Use Sard theorem and degenerate $\iff$ p is a focal point.

Cor. (6.2.5). smooth manifold has CW type; on a compact manifold any vector field with discrete singular points has its index sum equal to $\chi(M)$ (Hopf-Rinow), and there exists one.

Prop. (6.2.6). for $\Omega(p,q)^c$ the path space of energy $< c$, the piecewise geodesic path space B (piece fixed), the energy function is smooth and B^a is compact and is the deformation contraction of $\mathrm{int}\Omega^a$ for $a < c$. E has the same critical point and same index and nullity on B and Ω^c. (Just geodesicrize any path in Ω.

So for two point not conjugate in B^a, Ω^a has a finite CW complex type and a λ-dimensional cell for every geodesic of index λ in B^a.

Prop. (6.2.7) (Morse Main Theorem). If p and q are not conjugate along any geodesic, then $\Omega(p,q)$ has a countable CW complex type and has a λ-cell for every geodesic of index λ.

If M has nonnegative Ricci curvature, then M has only finite cell for every dimension.

Proof: Cf.[Milnor Morse Theory Prop17.3]. □

Cor. (6.2.8). The path space homotopy type only depend on the homotopy type of M (use the two homotopy to id to get a composition of homotopy of the two path space), so one can get the information of path space of M by looking at the homotopy type of M.

Prop. (6.2.9) (Minimal Geodesics). If p, q in a complete manifold M has distance $\sqrt{d}$ and the minimal geodesics form a topological manifold, and if all non-minimal geodesic has index $\geq \lambda$, then for $0 \leq i < \lambda$, $\pi_i(\Omega, \Omega^d) = 0$.

Lemma (6.2.10). In $SU(2m)$, the minimal geodesic from I to $-I$ is homeomorphic to Grassmannian $G_m(\mathbb{C}^{2m})$ and and non-minimal geodesic has index $\geq 2m+2$.

Similarly, The space of minimal geodesic from I to $-I$ in $O(2m)$ is homeomorphic to the space of complex structures in $\mathbb{R}^{2m}$, and any non-minimal geodesic has index $\geq 2m-2$.

Proof: Cf.[Milnor Morse Theory Lemma23.1 Lemma24.4]. □

Lemma (6.2.11). Ω_{k+1} is homotopic to the space of minimal geodesics in Ω_k from J to $-J$. (The same way, calculate the index of geodesics from J to $-J$ and use (6.2.9)). Cf.[Milnor Morse Theory Prop24.5] for definition of Ω_{k+1}.

3 Affine Connection and Chern-Weil Theory

Prop. (6.3.1) (Transformation map). In two coordinates $\overline{e} = ea$ for $a : U \to GL(r, \mathbb{R})$, $d_A = d + \omega, d + \overline{\omega}$ respectively, $\Omega = d\omega + \omega \wedge \omega$. Then:

$$\overline{\omega} = a^{-1}\omega a + a^{-1}da, \quad \overline{\Omega} = a^{-1}\Omega a$$

Prop. (6.3.2) (Bianchi's Identity). A affine connection on E looks like $d_A = d+\omega$, where $\omega \in \Omega^1(\operatorname{End} E)$. And $F_A = d_A \circ d_A \in \Omega^2(\operatorname{End}(E))$ satisfies

$$d_A F_A = dF_A + [\omega, F_A] = 0.$$

Prop. (6.3.3). The curvature of a (affine) connection d_A is $d_A \circ d_A \in \Omega^2(\operatorname{End}(E))$, and it is called flat if $d_A \circ d_A = 0$. For a flat connection (i.e with trivial curvature tensor), there is a bundle isomorphism (Gauge transform) that transforms d_A into natural d.

Proof: Because $d_{gA}(s) = g d_A(g^{-1}(s))$,

$$d_{gA} = d - dg \cdot g^{-1} + g \cdot \omega \cdot g^{-1}.$$

Solve this PDE direvtly. (Cf.[Topics in Geometry Xie Yi week3]).

□

Cor. (6.3.4). For a flat connection, the parallel transportation only depends on the homotopy type of the loop, thus gives an action of $\pi(X)$ on $SO(T_p(X))$ (or $SU(T_p(X))$. (because it is locally constant). And in this way, connections module gauge equivalence (preserving matric) equals representation of $\pi(X)$ module conjugations. The reverse map is giving by principal bundle.

Prop. (6.3.5). The connection action $d_A = d + \omega$ on a vector bundle E induces connection on relevant bundles. the action on dual bundle is by

$$d_A(s^*) = ds^* + \omega^t(s^*) = ds^* + s^* \circ \omega.$$

And the connection on $\operatorname{End} E$ by

$$d_A(\alpha) = d\alpha + [\omega, \alpha]$$

And they act on $\Omega^*(E)$ by Leibniz rule thus the formula looks the same. (Note that the convention is the matrix and composition act their way, and assume ω are always at left, so for example, $[\alpha, \alpha] = 2\alpha \wedge \alpha$).

Prop. (6.3.6) (Chern-Weil). For any connection on E, the map from invariant polynomial ring to $H^*(X) : P \mapsto [P(\Omega)]$ is a ring homomorphism independent on the connection.

The invariant polynomial ring is generated by coefficients f_k of the $\det(1+t\Omega)$ polynomial and also generated by the $\operatorname{tr}(\Omega^k)$ polynomials. Cf.[Loring Tu Appendix].

For a complex line bundle of degree r over a complex manifold,

$$\det(1 - \frac{1}{2\pi i} F_A) = 1 + c_1 + \ldots + c_n$$

gives out the Chern class, because it satisfies the axioms of Chern class (7.2.1).

For a real line bundle of degree r,

$$\det(1 - \frac{1}{2\pi i} F_A) = 1 + p_1 + \ldots + p_{\lfloor \frac{r}{2} \rfloor}$$

gives out the Pontrjagin class, where $p_k \in H^{4k}(X)$. (Notice the Ω can be chosen to be skew-symmetric thus for odd k, so the classes $\operatorname{tr}(\Omega^k) \in H^{2k}(X)$ vanish).

For an oriented real bundle of degree $2r$, the ω and thus Ω can be chosen to be skew-symmetric and the transformation matrix in $SO(2r)$, then

$$\operatorname{Pf}(\frac{1}{2\pi}\Omega) \in H^{2r}(X)$$

is well-defined and closed and gives the Euler class $e(E)$ (recall $e(E)^2 = p_r(E)$). (Use $\operatorname{Pf}^2 = \det$ to get that $[\frac{\partial \operatorname{Pf}}{\partial \Omega_{ij}}]^t$ commutes with Ω, then calculate $d\operatorname{Pf}(\Omega) = 0$).

There are relations between c_i and $\operatorname{tr}(F_A^k)$, they can be derived by considering diagonal elements.

Cor. (6.3.7).

$$c_1(E) = c_1(\wedge^{\dim E} E).$$

Direct from the formula.

Cor. (6.3.8) (Whitney Product Formula).

$$c(E \oplus F) = c(E)c(F), \quad p(E \oplus F) = p(E)p(F)$$

Directly form the product connection on $E \oplus F$.

Prop. (6.3.9) (Chern Character). The Chern character

$$ch(E) = [\mathrm{tr}\exp(\frac{i}{2\pi}F_A)]$$

satisfies $ch(E \oplus F) = ch(E) + ch(F)$ and $ch(E \otimes F) = ch(E)ch(F)$. So it defines a ring homomorphism from $K(X)$ to $H^*(X)$.

Prop. (6.3.10) (Chern-Gauss-Bonnet). For a $2n$-dimensional orientable manifold M,

$$\int_M e(TM) = \chi(M).$$

Prop. (6.3.11) (Hirzebruch Signature Formula). On a $4n$-dimensional orientable manifold M, the Poincare duality defines a bilinear pairing $H^{2n}(M) \times H^{2n}(M) \to \mathbb{R}$, its signature $\sigma(M)$ is given by:

$$\sigma(M) = \int_M L_n(p_1, \ldots, p_n).$$

Where L_n is the degree n part of the Taylor expansion of $\prod_{i=1}^r \dfrac{\sqrt{p_i}}{\tanh p_i}$.

Cor. (6.3.12). For a 4-dimensional M which is a boundary of a manifold, its signature is 0.

Proof: By Stokes theorem, if M is a boundary of a manifold, then all its Pontryagin numbers, i.e. $\int_M \prod p_i^{n_i}, \sum n_i = n$, vanish. □

Prop. (6.3.13) (Riemann-Roch). for a n-dimensional complex line bundle E over a Riemann Surface M, let

$$\chi(M, E) = \sum_{q=0}^n (-1)^q \dim H^q(M, E), \quad \deg L = \int_M c_1(E).$$

then

$$\chi(M, L) = \deg L - g + 1.$$

Prop. (6.3.14) (Hirzebruch-Riemann-Roch). For a n-dimensional complex line bundle E over a complex manifold M,

$$\chi(M, E) = \int_M [\mathrm{ch}(E)\mathrm{td}(TM)]_n.$$

Where $\chi(M, E)$ is defined as in (6.3.13), ch is the Chern character and $\mathrm{td}(TM)$ is the Todd polynomial, i.e. Taylor expansion of $\prod_{i=1}^r \dfrac{t_i}{1 - e^{-t_i}}$ applied to $c_i(TM)$.

Prop. (6.3.15). For a vector bundle and a flat connection d_A on a manifold, i.e. $d_A^2 = 0$, we have a deRham like cohomology, and there is a sheaf of flat sections.

$$H^*(X, A) = H^*(X, E).$$

4 Young-Mills Euqation & Gromov-Witten Equation

Def. (6.4.1). The Young-Mills functional on connections on a compact oriented space:

$$YM(A)^2 = ||F_A||^2 = -\int_X \mathrm{tr}(F_A \wedge F_A)$$

it is a critical point when $d_A \star F_A = 0$ and $d_A F_A = 0$.

Prop. (6.4.2) (2-dim Case). $\star F \in \Omega^0(su(E))$ is parallel thus it its characteristic spaces is orthogonal and a stable under parallel transport. So an irreducible YM $SU(n)$-connection must by flat, thus correspond to irreducible $SU(n)$ representation of $\pi_1(X)$.

Prop. (6.4.3) (4-dim Case). $** = (-1)^{2*2} = \mathrm{id}$ on $\Omega^2(E)$ so $\Omega^2(E) = \Omega^+ \oplus \Omega^-$. We have

$$||F_A^+||^2 + ||F_A^-||^2 \geq ||F - A^-||^2 - ||F - A^+||^2 = \int_X \mathrm{tr}(F_A \wedge F_A) = 8\pi^2 c_2(E)$$

Cf.[谢毅 Lecture5]. So it attains minimum at the connection that $*F_A = -F_A$ and $d_A F_A = 0$.(Anti-self-dual equation).

Prop. (6.4.4) (Anti-Self-Dual Connection on Complex Line Bundle). For a $U(1)$-bundle, $d_A F_A = dF_A$, so F_A is harmonic, thus $c_1(L) = [\frac{-1}{2\pi i} F_A] \in H^2(X, \mathbb{Z}) \cap \mathcal{H}^2_-(X, \mathbb{R})$, In fact, this is equivalent to the existence of a anti-self-dual connection on this bundle.

If this is the case, then we have the ASD-connections module Gauge equivalence is isomorphic to $H^1(X, \mathbb{R}/H^1(X, \mathbb{Z}) = T^{b_1(X)}$.

Proof: Because a gauge is just a $X \to S^1$, and its connected component thus equals $[X, S^1] = H^1(X, \mathbb{Z})$ (MacLane space), and its identity is just the map that is homotopic to id. and $d(gA) = dA - g^{-1}dg = dA - idu$, for $g = exp(iu)$, so $\Omega^1/\mathcal{G} = H^1(X, \mathbb{R}/H^!(X, \mathbb{Z}) = T^{b_1(X)}$. □

5 Atiyah-Singer Theory

III.7 Vector Bundle & K-Theory

1 Fundamentals

Prop. (7.1.1). A vector bundle can have its transform map $\in O(n)$ (or $U(n)$) by constructing a riemannian metric on it. And for every local trivialization, we choose the metric on it compatible with the given metric, thus the transform map is $\in O(n)$ (or $U(n)$).

Prop. (7.1.2) (Tietze extension general). For a Hausdorff paracompact(hence normal) space X and a paracompact subspace Y, every section on Y can be extended to a section on X. (For every point of Y, find a local trivialization and an even smaller open set. Use Tietze extension to extend localy to this nbhd, then use partition of unity to unify all).

Prop. (7.1.3). For a continuous family of maps from a paracompact Hausdorff space Y to a Hausdorff paracompact space X, then the pullback bundle is isomorphic.

Proof: Consider the space $Y \times I$ and the pullback bundle E, then for every t_0, consider a new bundle $\mathrm{Hom}(E, \pi_1^* E_{t_0})$, then Y has a section id, this section by the last proposition can be extended, so it spans the vector space for nearby t (because of paracompactness), thus is an isomorphism because it is a locally invertible vector bundle homomorphism. □

2 Chern Class

Prop. (7.2.1). Axioms for Chern classs for complex bundles:

- $c(E) = 1 + c_1(E) + \ldots + c_n(E)$, $n = \deg(E)$.
- $f^*(c(E)) = c(f^*(E))$.
- $c(E \oplus F) = c(E)c(F)$.
- On the tautological bundle over $\mathbb{CP}^k$, $c(\eta) = 1 + c_1(\eta)$ and $\int_{\mathbb{CP}^k} c_1(\eta) = -1$. There is a Affine connection definition of Chern class.

Proof: □

Prop. (7.2.2). For a complex line bundle, the first Chern class characterize them. We have by the Affine definition that $c_1(E) = c_1(\wedge^{\dim E} E)$, so $c_1(E) = 0 \iff \wedge^{\dim E} E$ is trivial $\iff E$ is orientable.

3 Thom isomorphism

Prop. (7.3.1) (Thom Class). For a vector bundle, we can compactify its bundles to get a (D^n, S^n)-bundle, if there is a Thom class that induce a generator $H^n(D^n, S^n)$ on every fiber. Then the relative Leray-Hirsch will give that c induces an isomorphism $H^i(B, R) \to H^{i+n}(E, E', R)$. For $\mathbb{Z}_2$ coefficient there exists a Thom class, and for orientable bundle there exists a $\mathbb{Z}$-Thom class. Notice that fiber bundle over a simply connected base is orientable.

Prop. (7.3.2). Similarly, for a orientable fiber bundle $S^{n-1} \to E \to B$, make it a $D^n \to E' \to B$ bundle, then E' is homotopy equivalent to B so there is a Gysin sequence

$$\to H^{i-n}(B) \xrightarrow{*e} H^i(B) \to H^i(E) \to H^{i-n+1}(B) \to$$

Where the Euler class e is chosen to commute with the Thom isomorphism.

4 Principal Bundles & Spin Geometry

Prop. (7.4.1) (Homogenous Space). If G is a Lie group and H is a closed subgroup, then the quotient $H\backslash G$ can be given a structure of a G-homogenous space and $G \to H\backslash G$ is a principal H-bundle.

Prop. (7.4.2). The projection $S^{2n+1} \to \mathbb{C}P^n$ is a principal S^1-bundle.

Prop. (7.4.3) (Spin Structure Obstruction). For a oriented real bundle, its transformation map can be chosen to be in $SO(n)$, and constitute a Cech Cohomology $H^1(X, SO(n))$, and by exact sequence of

$$0 \to \pm 1 \to \mathrm{Spin}(n) \to SO(n),$$

this can be lifted to a $H^1(X, \mathrm{Spin}(n))$ iff its image w in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ is 0. and then its inverse image will be parametrized by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

We have $w = w_2$, the Whitney class.

Proof: First prove that if $E \oplus R^n$ is spin, then E is spin, and then pull $H^2(X, \mathbb{Z}/2\mathbb{Z})$ into $H^2(\mathrm{sk}_2(X), \mathbb{Z}/2\mathbb{Z})$, this in a injection, and the homology is natural, so we only have to prove this for $\mathrm{sk}_2(X)$. But E on $\mathrm{sk}_2(X)$ can decompose into a E' of dimension on more than 2, and for this, we see E is Spin iff it is the square of another bundle, so w and w_2 are the same. □

Prop. (7.4.4). For a Spin bundle E, the Spin-principal bundle with the Spin representation will generate a bundle S called the **Spinor bundle**. And the Ad action of $\mathrm{Spin}(n)$ on $Cl_{n,0}$ will generate a bundle $Cl(E)$. The $\mathrm{Spin}(n)$ actions are compatible, so this bundle can act on the spinor bundle.

Prop. (7.4.5) (Spin^C-structure). The group Spin^C is the covering space of $SO(n) \times S^1$ (n>2) that corresponds to the group of elements mod 0 mod 2 in $\mathbb{Z}_2 \times \mathbb{Z}$, i.e. $\mathrm{Spin}(n) \times S^1/\{\pm 1\}$.

Then a $SO(n)$ bundle can be lift to be a Spin^C-bundle if the line bundle determined by S^1 is determine the same w_2 as it, i.e. $w_2 = c_1(L)$ mod 2, this can be achieved by choosing the det bundle.

III.8 Symplectic Geometry

1 Basics

Cf.[Methods in Classical Mechanics Arnold Chapter8],[辛几何讲义范辉军].

Prop. (8.1.1). A hamiltonian phase flow preserves the symplectic form. $g^{t*}\omega = \omega$.

Proof: by Cartan's magic formula,

$$\frac{d}{dt}(g^t)^*\omega = L_X\omega = \iota_X(d\omega) + d(\iota_X\omega) = d(\iota_X\omega)$$

because ω is closed. And by definition, $d(\iota_X\omega)(\eta) = \omega(J\mathrm{d}H, \eta) = \langle dH, \eta\rangle$, so $d(\iota_X\omega) = dH$, Thus the theorem. $\square$

For the following Cf.[辛几何讲义范辉军 lecture3].

Prop. (8.1.2) (Moser's Stability). If ω_t is a smooth family of cohomologous forms on a closed manifold M, then there exists an isotopy Ψ_t s.t.

$$\Psi_t^*(\omega_t) = \omega_0.$$

Prop. (8.1.3) (Relative Moser Stability). If M is a closed manifold and S is a compact submanifold, then if two closed 2-form equals on S, then there is an open neighborhood N_0, N_1 of S and a diffeomorphism $\Psi : N_0 \to N_1$ that

$$\Psi|S = \mathrm{id},\ \ \Psi^*\omega_1 = \omega_0.$$

Cor. (8.1.4) (Darboux's Theorem). Every symplectic form ω on M is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.

Proof: Choose $S = \mathrm{pt}$ and uses relative Moser stability. $\square$

III.9 Lie Groups & Symmetric spaces

1 Main Theorems

Prop. (9.1.1). For a Lie group G, for any lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there exists uniquely a connected Lie subgroup H s.t. $\mathfrak{h}$ is the lie algebra of H.

Proof: By (3.2.7), there is a maximal connected manifold H corresponding to $\mathfrak{h}$, we only need to show that it is a group. But the left invariance of $\mathfrak{h}$ shows that $HH \subset H$ because H is maximal. □

Cor. (9.1.2). If G_1 is a simply connected Lie group and G_2 is a connected Lie group, then any Lie algebra homomorphism $\tilde{h} : \mathfrak{g}_1 \to \mathfrak{g}_2$ can be lifted to a Lie group homomorphism.

Proof: use the image of $\tilde{h}$: $\Gamma(\tilde{h}) \subset \mathfrak{g}_1 \times \mathfrak{g}_2$, the prop shows that there is a Lie group in $G_1 \times G_2$ for $\Gamma(\tilde{h})$. It is isomorphic to G_1 because the Lie algebra is the same and both are connected, thus a covering map and G_1 is simply connected. □

Prop. (9.1.3) (Closed Subgroup Theorem). If H is a closed subgroup of a Lie group G, then there exists uniquely a differential structure s.t. H is a Lie subgroup of G. Cf[Helgason Symmetric Spaces].

Prop. (9.1.4) (Ado). Any finite dimensional Lie algebra can be embedded in some $\mathfrak{gl}(n, \mathbb{C})$.

Cor. (9.1.5). From the preceding propositions, it follows that the category of finite dimensional Lie algebras is equivalent to the category of simply connected Lie groups.

2 Generals

Prop. (9.2.1). A connected Lie group is second countable.

Proof: This follows from the fact that a Lie group is a manifold hence locally compact and it is a union of their products. □

Prop. (9.2.2). A continuous homomorphism between Lie groups is smooth.

Proof: use exp coordinates. □

Prop. (9.2.3). Any connected Lie group has a compact subgroup as deformation contraction.

Prop. (9.2.4).

$$SU(2) = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}, \alpha, \beta \in \mathbb{C}$$

is isomorphic to the group of unit quaternions and diffeomorphic to S^3.

3 Classical Groups

Fundamental Groups

Prop. (9.3.1).

- $SU(n-1) \to SU(n) \to S^{2n-1}$ gives us $SU(n)$ is simply connected.

$$\pi_1(Sp(2n)) = \pi_1(U(n)) = \mathbb{Z}$$

  and the determinant induces an isomorphism onto $\pi_1(S^1)$. In fact, this is used to define the Maslov index.

- $SO(n-1) \to SO(n) \to S^{n-1}$ gives us $\pi_1(SO(n)) \cong \pi_1(SO(3))$. And $SU(2)$ as the unit sphere in $\mathbb{H}$ maps to $SO(3)$ via the conjugation:$\mathrm{Ad}(z): w \mapsto zw\bar{z}$ has kernel ± 1, so $SO(3)$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$.

Generals

Prop. (9.3.2). As in (9.3.1) $SU(2)$ is a universal covering of $SO(3)$ and so does Spin(3)(8.1.3), so $SU(2) \cong \mathrm{Spin}(3)$.

Exponential Map

Prop. (9.3.3). The exponential map for $GL_n(\mathbb{C})$ and $U(n)$ is surjective and the image of the exponential map for $GL_n(\mathbb{R})$ is $GL_n(\mathbb{R})^2$.

Proof: Use Jordan Decomposition (Real). For complex case, it is unitary diagonalizable. □

4 Analysis

Lemma (9.4.1). Bi-invariant metric exists in a compact manifold.

Proof: Because the Haar measure on a compact metric is bi-invariant. Choose a Riemann metric and set

$$\langle V, W\rangle = \int_{G\times G} \langle L_{\sigma *}R_{\tau *}(V), L_{\sigma *}R_{\tau *}(W)\rangle d\mu(\sigma)d\mu(\tau).$$

Note that L_* and R_* commute. □

Prop. (9.4.2). If G is a Lie group with a bi-invariant metric, then

$$2\nabla_X Y = [X, Y], \quad \langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle,$$

$$\nabla_X Y = 1/2[X, Y], \quad R(X, Y)Z = 1/4[[X, Y], Z], \quad K(\sigma) = 1/4\Big|[X, Y]\Big|^2.$$

So its curvature is non-negative, and all 1-parameter subgroups are geodesic.

Prop. (9.4.3). A bi-invariant Lie group with $\mathfrak{g}$ having trivial center is compact and $\pi_1(G)$ finite.

Proof: From Myer Theorem because the Ricci curvature has a positive lower bound. □

Proof: Cf.[Morse Theory Milnor Prop20.5]. □

Prop. (9.4.4) (Structure theorem for bi-invariant Lie group). A simply connective Lie group with a bi-invariant metric is equal to $G^{'} \times R^k$, $G^{'}$ compact.

Proof: Because the orthogonal complement of the center of $\mathfrak{g}$ is a Lie algebra, G is like $G^{'} \times R^k$, and a simply connected abelian Lie group is R^k **?**. □

5 Symmetric space

Prop. (9.5.1). A **symmetric space** is that for every point p, there is a isometry reversing the geodesics passing p. A manifold is called **locally symmetric** if $\nabla R = 0$. Locally symmetric is equivalent to the fact that every local reversing map is an isometry. A symmetric space is complete because two folding is an extension of geodesic.

Prop. (9.5.2). A Lie group with a bi-invariant metric is a symmetric space.

Prop. (9.5.3). The conjugate points in a symmetric space is easy to calculate, they are $\exp(\frac{\pi k}{\sqrt{e_i}}V)$, counting multiplicity, where e_i is the eigenvalue of the sself-adjoint operator $K_V(W) = R(V, W)V$ at p.

III.10 Hyperbolic Geometry

Prop. (10.0.1). 双曲圆盘的保距同构都是由 Mobius 变换给出的。因为任何三点为半径做圆就可以确定出每一个点。Cf.[双曲几何 刘毅]

III.11 Complex Geometry

Prop. (11.0.1). the Funibi-Study metric on CP^n has sectional curvature $1 \geq K \geq 4$. Cf.[Do Carmo P188].

Chapter IV

Analysis

IV.1 Real Analysis

1 Basics

Prop. (1.1.1). A function f is real analytic on an open set iff there is a extension to a complex analytic function to an open set. And this is equivalent to: For every compact subset, there is a constant C that for every positive integer k, $|\frac{d^k f}{dx^k}(x)| \leq C^{k+1}k!$.

Proof: Use Lagrange residue(中值定理) to show that it will converge to f. □

Prop. (1.1.2) (convergences). There are three different kinds of convergences.

Prop. (1.1.3) (Bounded Convergence Theorem).

Prop. (1.1.4) (Riesz Representation Theorem). on $C_c(X)$ for a LCH space X,

If I is a positive linear functional, there is a unique regular (both inner and outer) Radon measure μ on X such that $I(f) = \int f d\mu$. Moreover,

$$\mu(U) = \sup\{I(f) : f < U\} \text{ for } U \text{ open},$$

$$\mu(K) = \inf\{I(f) : f > \chi(K)\} \text{ for } K \text{ compact}.$$

If I is a continuous linear functional, there is a unique regular countably additive complex Borel measure μ on X that $I(f) = \int f d\mu$.

So if X is compact, $M(X)$ the space of Borel measures on X is the dual spcae of $C(X)$.

Proof: Cf.[Real Analysis Folland]. □

Prop. (1.1.5). For a pair of Hilbert basis $\{e_i\}$ of $L^2(M)$ and $\{f_j\}$ of $L^2(N)$, $\{e_i \otimes f_j\}$ gives a basis for $L^2(M \times N)$. (Use Fubini).

2 Approximations

Prop. (1.2.1). The polynomial functions are dense in $C[-1, 1]$.

Proof: We only have to prove that $|x|$ can be approximated, because then all piecewise linear function can. For this, Taylor expand $\sqrt{1 + (x^2 - 1)}$. (or we can use Stone-Weierstrass). □

Prop. (1.2.2) (Stone-Weierstrass Approximation). If a unital C^*-algebra of continuous functions on a compact Hausdorff space separates points, then it is dense in $C(X)$.

Proof: This is a consequence of Bishop theorem(3.4.14) □

Prop. (1.2.3). for $1 \leq p < +\infty$,continuous functions are dense in $L^p(\mathbb{T})$, but not for $p = \infty$.

Proof: 用有限阶梯函数逼近，再用内闭逼近，再用 Tietze 扩张。 □

Prop. (1.2.4) (Approximate Identity). A family of $L^\infty(\mathbb{T})$ functions $\{\Phi_N\}$ are called an approximate identity if:

1. $\int_0^1 \Phi_N(x)dx = 1$.
2. $\sup\int_0^1 |\Phi_N(x)|dx < \infty$.
3. For any $\delta > 0$, $\int_{|x|>\delta} |\Phi_N(x)|dx \to 0$ as $N \to +\infty$.

For any approximate identity, if $f \in C(\mathbb{T})$ or $L^p(\mathbb{T})$ for $1 \leq p < +\infty$, then $\Phi_N * f \to f$.

Proof: Use uniform continuity and also use continuous approximation (1.2.3). □

Cor. (1.2.5). for $1 \leq p < +\infty$,trigonometric polynomials are dense in $L^p(\mathbb{T})$ and $C(\mathbb{T})$, but not for $p = \infty$. So $e^{2\pi inx}$ forms an orthogonal basis in $L^2(\mathbb{T})$.

Thus, the Parseval's identity holds.

Proof: Just use the fact that fejer kernels are an approximate identity. □

3 Convolution

Prop. (1.3.1). Convolution with a smooth function makes the function smooth, in particular,$\frac{\partial}{\partial x}(f * g) = \frac{\partial f}{\partial x} * g$.

Prop. (1.3.2) (Young's Inequality). $||f * g||_r \leq ||f||_p||g||_q$ for all $1 \leq r, p, q \leq \infty$ and

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

In particular, $||K * f||_p \leq ||K||_1||f||_p$.

Proof: BY Riesz representation, it suffices to show that: for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$,

$$\int\int f(x)g(y-x)h(y) \leq ||f||_p||g||_q||h||_r.$$

write the LHS as

$$\int\int (f^p(x)g(y-x)^q)^{1-\frac{1}{r}}(f^p(x)h^r(y))^{1-\frac{1}{q}}(g^q(y-x)h^r(y))^{1-\frac{1}{p}}$$

and use Holder inequality. □

4 Measures

Prop. (1.4.1) (Radon-Nikodym). If two σ-finite measures v, μ on a measurable space satisfies v is absolutely continuous w.r.t μ, then there is a μ-integrable function f such that

$$dv = f d\mu.$$

Cor. (1.4.2). Special case of the Freudenthal spectral theorem **??**.

IV.2 Complex Analysis

1 Topology

Prop. (2.1.1). A first differentiable conformal map in $\mathbb{C}$ is holomorphic or anti-holomorphic. Cf.[Ahlfors P74]. In higher dimension, conformal is equivalent to $\langle df_p(v_1), df_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle$.

Prop. (2.1.2). The roots of a polynomial depends continuously on the coefficients. (Use Rouch Principle).

2 Theorems

Prop. (2.2.1) (Uniformization Theorem). Any connected Riemann Surface is the quotient by a discrete subgroup of $\mathbb{C}, \mathbb{H}$ or $\mathbb{P}^1$.

Proof: $\square$

Prop. (2.2.2) (Runge's Theorem). Let K be a compact subset of $\overline{\mathbb{C}}$ and let f be a function which is holomorphic on an open set containing K. If A is a set containing at least one complex number from every bounded connected component of $\overline{\mathbb{C}}\backslash K$, then there exists a sequence of rational functions which converges uniformly to f on K and all the poles of the functions are in A.

Proof: $\square$

Prop. (2.2.3) (Mergelyan's theorem). If K is compact in $\mathbb{C}$ and f is a continuous function on K that is holomorphic in $\mathrm{int}(K)$, then f can be uniformly approximated by polynomials.

Prop. (2.2.4) (Montel's Theorem). Sets of holomorphic functions bounded in the topology of $H(\Omega)$, inter convex uniform convergence, is sequentially compact.

Proof: $\square$

IV.3 Functional Analysis

1 Various Spaces and Duality

For a bounded connected open set Ω,

- The space $C^k(\Omega)$ is the Banach space of C^k functions u on $\overline{\Omega}$ with the norm
$$||u|| = \max_{|\alpha|\le k} \max_{x\in\overline{\Omega}} |\partial^\alpha u(x)|.$$
It is complete because the limits of u^α is compatible.
- **Sobolev Space** $H^{m,p}(\Omega)$ is the completion of a subspace $C^k(\Omega)$ with the norm
$$||u||_{m,p} = (\sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha u(x)|^p \,\mathrm{dx}).$$
And we denote $H^{m,p}(\Omega)$ by $H^m(\Omega)$.
- $C_0^k(\Omega)$ is the subspace of $C^k(\Omega)$ that have support in Ω. Its completion $H_0^m(\Omega)$ is a closed subspace of $H^m(\Omega)$.
- $C(\Omega)$ in the topology of compact convergence is a Fréchet space. It is not locally convex.
- $H(\Omega)$ the space of holomorphic functions in Ω is a closed subspace of $C(\Omega)$ thus is a Fréchet space. Montel's theorem says exactly that $H(\Omega)$ is Heine-Borel.
- $C^\infty(\Omega)$ in the topology defined by seminorms $p_N(f) = \max\{|D^\alpha f(x)| : x \in K_N, |\alpha| \le N\}$, is a metrizable Fréchet space thus locally convex and it has the Heine-Borel property by Arzela-Ascoli.
- $D(K)$ is the closed subspace of functions on Ω with support in K, thus a Fréchet space.

Prop. (3.1.1) (Dual Spaces).
- For a σ-finite measure μ on a measurable space X, for $1 \le p < \infty$
$$L^p(X,\Omega,\mu)^* = L^q(X,\Omega,\mu).$$
- $C[0,1]^* = BV[0,1]$ and $C[X]^* = M(X)$, the space of complex measure on compact X with the norm of total variance, by Riesz representation theorem(1.1.4).

2 Topological Vector Space

Def. (3.2.1). there are different topology in the space of operators on a Hilbert space.
Norm operator topology: $||H_i - H|| \to 0$.
Strong operator topology: $\forall u, ||(H_i - H)u|| \to 0$.
Weak operator topology: $\forall u, v, \langle H_i(u), v\rangle \to \langle H(u), v\rangle$

Def. (3.2.2). A space is called a F**-space** if its toopology is induced by a complete invariant metric.
A locally convex F-space is called a **Fréchet space**.

Def. (3.2.3). A **sublinear functional** is a function p that $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for $\lambda > 0$.

A **seminorm** is a non-negative function p that $p(x+y) \leq p(x)+p(y)$ and $p(\alpha x) = |\alpha| p(x)$ for all complex α.

Prop. (3.2.4) (Minkowski Functional). The set of seminorms/sublinear functionals correspond 1-to-1 with convex balanced absorbing open sets containing 0 through Minkowski functional. and it is uniformly continuous iff 0 is an interior point.

Prop. (3.2.5). A separating family of seminorms is equivalent to a convex balanced local basis at 0. And it generate a metric making the space a Fréchet space. Cf.[Rudin P27].

Prop. (3.2.6) (Hausdorff). In a complete space, a subset M is sequentially compact iff it is totally bounded. (Use the diagonal method).

In a metric space, a subset M is sequentially compact iff its closure is compact. Hence in Fréchet space, a closed subset is compact iff it is totally bounded.

Cor. (3.2.7) (Arzela-Ascoli). $F \subset C(M)$ M compact is a sequentially compact subset iff it is uniformly bounded and equicontinuous.

Prop. (3.2.8). In a locally bounded space, if E is totally bounded, then $\mathrm{co}(E)$ is totally bounded. Thus in a Fréchet space, the closed convex closure of a compact set is compact.

Prop. (3.2.9). A topological vector space is locally compact iff it is f.d., and there is only one topology on a finite dimensional space and it is complete. Cf.[Rudin P17].

Cor. (3.2.10). A f.d subspace in a F-space is complete thus closed.

Prop. (3.2.11) (Schauder Fixed Point Theorem). If C is a closed convex subset in a normed space and $T : C \to C$ has sequentially complete image, e.g. C is compact, then T has a fixed point.

Proof: Use a $1/n$-net and construct a contraction to their convex hull. Then use Brouwer fixed point theorem to find $Tx_n = x_n$, and choose a convergent point x to show $Tx = x$. □

Reflexive and Separable (张恭庆)

Prop. (3.2.12) (Banach). For a norms space X, if X^* is separable, then X is separable.

Proof: Choose a countable dense set in X^*, then there projection to the unit sphere $\{g_n\}$ is dense, and choose for each of them a x_n that $||x_n|| = 1$ and $g_n(x_n) > \frac{1}{2}$. Use Hahn to show span of $\{x_n\}$ is dense in X. □

Prop. (3.2.13) (Pettis). Closed subspace of a reflexive normed space is reflexive.

Prop. (3.2.14). If a subspace of normed space X is separable or reflexive, then its unit sphere in X^* is weak*-sequentially compact.

Closed Range Theorem

Prop. (3.2.15). Let T be continuous mapping between Banach spaces X and Y, then $T(X) = Y \iff ||T^*y^*|| \geq \delta||y^*|| \iff T^*$ is one-to-one and $R(T^*)$ is closed(By Banach theorem).

$R(T)$ is closed in Y iff $R(T^*)$ is closed in X^*.

Proof: Cf.[Rudin P100]. □

3 Completeness

Prop. (3.3.1) (Banach-Steinhaus). Γ is a collection of continuous linear mapping between two TVS, then if the set B of x that $\Gamma(x)$ is bounded is a second category set in X, then $B = X$ and Γ is equicontinuous (thus maps bounded sets to bounded sets).

Similarly, if for a compact convex set K in X, if a set Γ of continuous linear mapping is bounded for every $x \in K$, then Γ is equicontinuous on K.

Proof: For an open set of 0, choose a balanced nbhd U s.t. $\overline{U}+\overline{U} < W$, set $E = \cap_{\Lambda\in\Gamma}\lambda^{-1}(\overline{U})$, then $B \subset \bigcup_{i=1}^{\infty} nE$, so by Baire, E has a interior point thus has a nbhd V s.t. $\Gamma(V) \subset \overline{U}+\overline{U} \subset W$. □

Cor. (3.3.2) (Uniform Boundedness Theorem). If a set Γ of continuous linear mapping from a F-space X to Y satisfies $\Gamma(x)$ is bounded for every x, then Γ is equicontinuous.

Prop. (3.3.3) (Open Mapping theorem). If a continuous linear mapping T from a F-space X to Y is surjective and $R(T)$ is of second category, then it is a surjective open mapping and Y is a F-space.

Proof: We can show that $T(B(0, \frac{1}{2^n})$ are all of second category, so $\overline{T(B(0, \frac{1}{2^n}))}$ is open, thus for a $y \in T(B(0, 1)$ we can consecutively choose $x_n \in B(0, \frac{1}{2^n})$ s.t. $y - \sum_{i\leq n} T(x_i) \in \overline{T(B(0, \frac{1}{2^n}))}$. So by completeness of X, $y \in T(B(0.1)$, thus it is open. □

Cor. (3.3.4) (Banach). If a continuous T between F-spaces is a bijection, then it has a continuous inverse.

Cor. (3.3.5). If a F-space is complete w.r.t two different topologies and one is stronger than the other, then they are equivalent.

Cor. (3.3.6) (Closed Graph Theorem). If T is a closed linear mapping between two F-spaces, i.e. the graph of it is closed, then it is continuous. (Because the metric induced by the graph is stronger than the original one). This is very useful when proving some map is continuous.

Prop. (3.3.7). Any symmetric operator on a Hilbert space is continuous. (Because $x_n \to 0$ implies $Tx_n \to 0$ weakly, so we can use closed graph theorem.

Cor. (3.3.8). If the image of a continuous linear mapping T between F-spaces has finite codimensional image, then the image is closed. (Construct $\mathbb{C}^n \oplus X/N(T) \to Y$, by Banach it is a homeomorphism.

4 Convexity

Hahn-Banach

Prop. (3.4.1) (Real Hahn). For a sublinear functional p on a real linear space X and a subspace X_0, if a functional f satisfies $f(x) < p(x)$ on X_0, then it can be extended to a functional on X with the same condition. (Use Zorn's lemma)

Prop. (3.4.2) (Complex Hahn). For a seminorm p (i.e. it can attain 0) on a complex linear space X and a subspace X_0, if a functional f satisfies $f(x) < p(x)$ on X_0, then it can be extended to a functional on X with the same condition.

Proof: Let $g(x) = \operatorname{Re} f(x)$ and extend it and set $f(x) = g(x) - ig(ix)$. □

Prop. (3.4.3) (Hahn). In a normed space X, a bounded linear functional on a subspace X_0 can extend to a bounded functional on X with the same norm.

Cor. (3.4.4). For every $x \neq 0$, there is a continuous functional f of norm 1 that $f(x) = ||x||$. So continuous functionals can separate points. Thus the inclusion from X to X^{**} is an isometry into and the conjugation $T*$ form $L(X, Y)$ to $L(Y^*, X^*)$ is an isometry into a closed subspace.

Prop. (3.4.5) (Geometric Hahn).

- If E_1 and E_2 are two convex set that $E_1 \cap E_2 = \emptyset$ and E_1 is open, then there is a continuous linear functional that separate them, i.e $f(E_1) < f(E_2)$. (use the continuous sublinear functional for $E_1 - E_2$).
- In a locally compact TVS, if E_1 is convex compact and E_2 is convex closed, then there is a functional that separate them. Thus for a set E and a point x, $x \in \overline{\operatorname{span} E} \iff$ for all f that $f(E) = 0$, $f(x) = 0$.

Cor. (3.4.6). If a sequence $\{x_n\}$ weak converge to x in a normed space, then convex combination of $\{x_n\}$ strongly converge to x, i.e. $x \in \overline{\operatorname{co}}(\{x_n\})$. The weak closure of a convex set in a locally convex space equals the original closure.

Prop. (3.4.7). For a bounded operator T,

$$\overline{R(T)} = N(T^*)^\perp, \text{ Thus } \overline{R(T^*)} = N(T)^\perp$$

Prop. (3.4.8). For a finite dimensional space in a Banach space, the projection exists. (construct the dual functional for a basis and extends it to a functional using Hahn.

Prop. (3.4.9) (Banach-Alaoglu theorem). For a nbhd V of 0, the set

$$K = \{f |\ |fx| \leq 1,\ \forall x \in V\}$$

is weak*-compact.

Proof: The point is that the weak*-topology coincides with the pointwise convergence topology. And that topology is embedded in a compact space(Tychonoff) and K is a closed subspace of that space. □

Prop. (3.4.10). In a locally convex space, bounded $\iff$ weakly bounded. Cf.[Rudin Prop3.18].

Prop. (3.4.11). For a commuting family of continuous affine maps from K to K where K is a compact convex set in a TVS, then there is a fixed point.

Proof: Consider the semigroup generated by these maps together with their average, they have a common image, and for this image, consider $p = \frac{1}{n}(I + T + t^2 + \ldots + T^{n-1})(x)$, then $p - Tp = \frac{1}{n}(x - T^n x) \in \frac{1}{n}(K - K)$, thus $p = Tp$ for all T. $\square$

Cor. (3.4.12) (Invariant Hahn). For a commuting family Γ of operators on a normed space and Y a invariant space, then for any Γ-invariant continuous functional y^* has a Γ-invariant Hahn extension. (Consider the action of T^* on all the Hahn extension of f, use Alaoglu).

Krein-Milman theorem

Prop. (3.4.13). For a compact convex set in a TVS that is weak-Hausdorff, then $K = \overline{\mathrm{co}}(\mathrm{Extreme}(K))$.

If K is a compact set in a locally convex space, then $K \subset \overline{\mathrm{co}}(K) = \overline{\mathrm{co}}(E(K))$.

Proof: Show that every extreme set contains a extreme point, and use the geometric Hahn, because the extreme value point for any functional on K is a extreme set. $\square$

Prop. (3.4.14). If K is a compact set in a locally convex space X and if $\overline{\mathrm{co}}(K)$ is also compact, e.g in a Fréchet space, then every extreme point of $\overline{\mathrm{co}}(K)$ lies in K.

Cor. (3.4.15). There is a Bishop theorem that derive Stone-Weierstrass theorem, proved using Krein-Milman.

5 Sobolev Space

6 Banach Algebra

Def. (3.6.1). For a bounded operator from on X complete, then a λ is called a:

- **point spectrum** $\rho(A)$ if $(\lambda I - A)^{-1}$ doesn't exists;
- **continuous spectrum** if $R(\lambda I - A) \neq X$ but $\overline{R(\lambda I - A)} = X$.
- **residue point** if $\overline{R(\lambda I - A)} \neq X$.
- **regular point** if $(\lambda I - A)^{-1}$ exists and is continuous, i.e. $R(\lambda I - A) = X$;

denote $\sigma(A) = \mathbb{C} \backslash \rho(A)$ the spectrum of A.

Prop. (3.6.2). $\mathbb{C} \backslash \sigma(T)$ is an open set and $\lambda \to (\lambda I - T)^{-1}$ is a holomorphic function on $\mathbb{C} \backslash \sigma(T)$.

Thus for every bounded operator T, $\sigma(T)$ is not empty, otherwise this holomorphic function is bounded.

Cor. (3.6.3) (Gelfand-Mazur). If in a Banach space where all the nonzero element is invertible, then it is isomorphic to $\mathbb{C}$.

Prop. (3.6.4). Notice $(I-T)$ is invertible for $||T||<1$ and the derivative can be calculated by definition.

Cor. (3.6.5). The spectrum of an element of a Banach algebra is continuous.

Prop. (3.6.6). In a Banach algebra A, $e-xy$ is invertible iff $e-yx$ is invertible, thus $\sigma(xy)\cup\{0\}=\sigma(yx)\cup\{0\}$.

Proof: Use power expansion to get an expression and prove it is the inverse. □

Prop. (3.6.7) (Gelfand). The spectrum radius

$$r_\sigma(A)=\lim_{n\to\infty}||A^n||^{\frac{1}{n}}=\inf||A^n||^{\frac{1}{n}}.$$

So $\sigma(A)$ is compact.

Proof: Use Hadamard radius formula and for the other side, use the fact that $|f(\frac{A^n}{(r_A+\varepsilon)^{n+1}})|$ is bounded, so by uniform boundedness theorem, $\frac{||A||^n}{(r_A+\varepsilon)^{n+1}}<M$ for all n. And $\lambda\in\sigma(A)$ implies $\lambda\in\sigma(A^n)$ thus the limit is well-defined. □

Cor. (3.6.8). For Banach algebra B and its closed subalgebra A, $\sigma_A(x)$ is obtained from $\sigma_B(x)$ by filling some holes. So when $\sigma_B(x)$ doesn't separate $\overline{\mathbb{C}}$ or $\sigma(A)$ has empty interior, then $\sigma_A(x)=\sigma_B(x)$. Cf.[Rudin P256].

Symbolic Calculus

Prop. (3.6.9). For a Banach algebra A. For a domain Ω in $\mathbb{C}$, define A_Ω as the set of x that $\sigma(x)\in\Omega$, it is an open set by(3.6.5), then:

$$f\mapsto\tilde{f}(x)=\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda e-x)^{-1}d\lambda$$

for any contour Γ that surrounds $\sigma(x)$, is a continuous algebra isomorphism of $H(\Omega)$ into the set of A-valued functions on A_Ω with the compact-open topology.

We have $\widetilde{g\circ f}=\tilde{g}\circ\tilde{f}$.

Proof: The nontrivial part is that this map is multiplicative, but for this we can use Runge's theorem to approximate any function on $\sigma(x)$. □

This theorem makes it possible to implant complex analysis to the study of Banach Algebra.

Cor. (3.6.10). $\exp(x)$ is defined on A and is continuous. If $\sigma(x)$ doesn't separate 0 from ∞, then $\log(x)$ is defined but might not be continuous.

Prop. (3.6.11) (Spectral Mapping Theorem). $\tilde{f}(x)$ is invertible in A iff $f(\lambda)\neq 0$ on $\sigma(x)$. Thus we have $\sigma(\tilde{f}(x))=f(\sigma(x))$.

Prop. (3.6.12). If f doesn't vanish identically on any component of Ω, then $f(\sigma_p(T))=\sigma_p(\tilde{f}(T))$. Cf.[Rudin P266].

Commutative Banach Algebra

Prop. (3.6.13). For A a commutative Banach algebra, the set of maximal ideals has codimension 1 corresponds to kernels of complex homomorphisms to $\mathbb{C}$. (Consider the quotient space and use Gelfand-Mazur). Note that a complex homomorphism is all continuous because $\lambda e - x$ maps to nonzero.

$\lambda \in \sigma(x)$ iff there is a complex homomorphism h s.t. $h(x) = \lambda$. (Because x is invertible iff it is not contained in any proper ideal of A.

Prop. (3.6.14) (Gelfand Transform). The set Δ of maximal ideals of a commutative Banach algebra is a compact Hausdorff space w.r.t to the weak*-topology and the Gelfand transform: $x \mapsto \hat{x}(h) = h(x)$ is a map of A into $C(\Delta)$. And the range of $\hat{x}$ equals $\sigma(x)$, so $||\hat{x}|| = \rho(x) \le ||x||$.(Use Alaoglu).

Prop. (3.6.15). For $A = C(X)$ where X is compact Hausdorff, Δ is homeomorphic to X. (otherwise it has finite $f_i \neq 0$, then $\sum |f_i|^2$ is positive thus invertible but maps to 0). In fact, for a space X, $\Delta(C(X))$ is the stone-$\check{C}$ech compactification of X.

Prop. (3.6.16). For $A = L^\infty(m)$, the spectrum of f is just the essential range of f.

Lemma (3.6.17). If $\hat{A} \subset C(\Delta)$ with a chosen topology that makes it compact, and A separate points, then the topology of it is the same of the weak*-topology. (Compact to Hausdorff).

Prop. (3.6.18). The algebra $L^1(\mathbb{R}^n) + \delta$ with the multiplication by convolution has the spectrum $R^n \cap \{\infty\}$. (Use $L^{p*} = L^q$ and see when will it be homomorphism).

B^*-Algebra and Hilbert space

Prop. (3.6.19). A closed convex subset in a Hilbert space has a unique element that attains the minimum norm.

Proof: Choose a sequence approaching the infimum and use parallelogram identity to show it is a Cauchy sequence. □

Cor. (3.6.20). The orthogonal complement of a closed subspace of a Hilbert space exists. and the projection on to a closed subspace exists. This is a good trait of Hilbert space.

Prop. (3.6.21). Linear functionals on a Hilbert space is all of the form $x \mapsto (x, z)$ (Choose a orthogonal of the kernel).

Prop. (3.6.22). For a Hilbert space, the adjoint operation serves as an involution and makes $B(H)$ into a B^*-algebra, i.e. $||T^*T|| = ||T||^2$. (In fact, $||T|| = ||T^*|| = \sup\{(Tx, y)|||x||, ||y|| \le 1\}$).

Prop. (3.6.23). Any B^*-algebra is isomorphic to a closed subspace of $B(H)$ for some Hilbert space. Cf.[Rudin P338].

Prop. (3.6.24) (Gelfand-Naimark). A commutative B^*-algebra A is a Banach algebra with involution s.t. $||xx^*|| = ||x||^2$. Then the Gelfand transform is an isomorphism from A to $C(\Delta)$ with $||x|| = ||\hat{x}||_\infty$ and $\widehat{x^*} = \overline{\hat{x}}$.

Proof: First use $||xx^*|| = ||x||^2$ to prove that a hermitian element is mapped to real function, and use Stone-Weierstrass to show that the image is dense, then let $y = xx^*$ and $||y^{2^m}|| = ||y||^{2^m}$ to prove $||\hat{x}|| = ||x||$, so its image is closed. □

Now we want to use commutative algebra methods in the non-commutative case, there are two ways.

Prop. (3.6.25). For a commutative set of elements S in A, Γ the set of elements that commute with S, then $B = \Gamma(\Gamma(S))$ is commutative and contains S. And $\sigma_B(x) = \sigma_A(x)$ for $x \in B$.

Proof: Because $S \subset \Gamma(S)$, $\Gamma(\Gamma(S)) \subset \Gamma(S)$, thus $\Gamma(\Gamma(S))$ is commutative. And if $xy = yx$, then $x^{-1}y = yx^{-1}$. □

Cor. (3.6.26). In a Banach algebra, if x, y commutes, then

$$\sigma(x + y) \subset \sigma(x) + \sigma(y), \quad \sigma(xy) \subset \sigma(x)\sigma(y).$$

(because $\sigma(x)$ is just the range of $\hat{x}$ on Δ that x, y generated).

The second method applies to normal elements:

Prop. (3.6.27). In a Banach algebra with an involution, a set S is called **normal** if it is commutative and $S^* = S$. A maximal normal set B is a closed subalgebra and $\sigma_B(x) = \sigma_A(x)$.

Proof: Cf.[Rudin P294]. □

Cor. (3.6.28). In a B^*-algebra A,
- Hermitian elements have real spectra.
- If x is normal, then $\rho(x) = ||x||$.
- If $u, v \geq 0$, then $u + v \geq 0$, i.e. $\sigma(u + v) \subset [0, \infty)$.
- $yy^* \geq 0$. Thus $e + yy^*$ is invertible.

Proof: Cf.[Rudin P295]. □

Prop. (3.6.29). In a Banach algebra with an involution, a **positive functional** is such that $F(xx^*) \geq 0$. It has the following properties.
- $F(x^*) = \overline{F(x)}$ and $|F(xy^*)|^2 \leq F(xx^*)F(yy^*)$. (Use Swartz like trick).
- $|F(x)|^2 \leq F(e)F(xx^*) \leq F(e)^2\rho(xx^*)$, because $e = ee^*$. Thus $|F(x)| \leq F(e)\rho(x)$ for every normal x (By the last prop), so $||F|| = F(e)$ if A is commutative.

Cf.[Rudin P297].

Prop. (3.6.30). If A is a commutative Banach algebra with an involution that $h(x^*) = \overline{h(x)}$, then The map

$$\mu \to F(x) = \int_\Delta \hat{x} d\mu$$

is a one-to-one correspondence between the convex set of μ that $\mu(\Delta) \leq 1$ to the convex set K of positive functionals on A of norm ≤ 1, i.e. $F(e) \leq 1$, so maps the extreme points, i.e. the point mass to extremes points, thus the extreme points of K is exactly Δ. This can be used to prove **Bochner's theorem**.

Proof: Use the last prop to show that there is a functional on $C(\Delta)$ and use Riesz representation. It is positive and by Stone-Weierstrass, it is unique. □

7 Spectral Theory

Resolution of Identity

Def. (3.7.1). A **resolution of identity** on a Hilbert space H for a σ-algebra on a set Ω is a E that:

1. $E() = 0, E(\Omega) = 1$.
2. $E(\omega)$ is self-adjoint projection.
3. $E(\omega' \cap \omega) = E(\omega')E(\omega)$.
4. $E(\omega \cup \omega') = E(\omega) + E(\omega')$ for disjoint ω, ω'.
5. $E_{x,y}(\omega) = (E(\omega)x, y)$ is a complex measure on E.

Thus for any x, $\omega \to E(\omega)x$ is a countably additive H-valued measure.

This will generate an isometric*-isomorphism Ψ of the Banach algebra $L^\infty(E)$ onto a closed normal subalgebra A of $B(H)$. (Define on simple function first).

$$\Psi(f) = \int_\Omega f dE, \quad (\Psi(f)x, y) = \int_\Omega f dE_{x,y}$$

Prop. (3.7.2) (Spectral Decomposition). For any closed B^*-algebra A of $B(H)$, there is a unique resolution E of identity on the Borel subsets of Δ that the inverse of Gelfand transform extends to an isometric *-isomorphism Φ of the algebra $L^\infty(E)$ to a closed subalgebra B containing A. Cf.,[Rudin P322]. In fact, $B = \Gamma(\Gamma(A))$ is normal by Fuglede theorem(3.7.8).

Cor. (3.7.3) (Generalized Symbolic Calculus). If T is a commutative B^*-algebra that x, x^* topologically generate A, then the map

For a normal operator T and the closed normal B^*-algebra it generates, we have $\hat{x}$ maps $\Delta \cong \sigma(x)$ and the inverse of Gelfand transform (by Naimark) gets us a map $\Psi : C(\sigma(x)) \to A$ that $\Psi(z) = x$, $\Psi(\overline{z}) = x^*$. And this can be extended to a resolution of identity on the Borel set of $\sigma(T)$ that maps $L^\infty(m)$ to $B(H)$. $||\Psi(f)|| = ||f||_\infty$.

Cor. (3.7.4). If T is normal, then

1. $||T|| = \sup\{|(Tx, x)| \ ||x|| = 1\}$.
2. T is self-adjoint iff $\sigma(T)$ is real.
3. T is unitary iff $|\sigma(T)| = 1$.

Proof: For 1, use the fact that $||T|| = \rho(T) = ||z_0||$ for some $z_0 \in \rho(T)$, then use Urysohn to show $E(U) \neq 0$ for a open U near x, then there are x_0 that $E(U)x_0 = x_0$ and use $f = z - z_0$ to show that this x_0 get near $||T||$. □

Normal Operator on Hilbert Space

Prop. (3.7.5) (Normal Operators). An operator is normal iff $||Tx|| = ||T^*x||$. So we have $N(T) = N(T^*)$ thus $\sigma_p(T^*) = \overline{\sigma_p(T)}$. And different eigenspaces are orthogonal.

An operator is unitary iff $R(U) = H$ and $||Ux|| = ||x||$ for every x. (Because an operator is defined by its value (Tx, y).

Prop. (3.7.6). For a normal operator T on a Hilbert space, $N(T) = R(T)^\perp$, so T is invertible iff there is a δ that $||Tx|| = ||T^*x|| \geq \delta||x||$.

Prop. (3.7.7) (Polar Decomposition). A positive operator is self-adjoint and has positive spectrum, they have a positive square root. (use the last prop).

So polar decomposition exists in $B(H)$ and normal operator has commuting decomposition. Thus two similar normal operator are unitarily equivalent, (use Fuglede).

Prop. (3.7.8) (Fuglede). If N_1 and N_2 are normal operators and A is a bounded linear operator on a Hilbert space such that $N_1 A = AN_2$, then $N_1^* A = AN_2^*$.

Proof: For any $S \in B(H)$, $\exp(S - S^*)$ is unitary thus $||\exp(S - S^*)|| = 1$, $\exp(N_1)A = A\exp(N_2)$. So we have

$$||\exp(\lambda N_1^*)T\exp(-\lambda N_2^*)|| \leq ||T||$$

because λN_i is normal. Thus by Liouville, compare the coefficients of λ, we get the result. □

Prop. (3.7.9). An operator $A \in B(H)$ has the same spectrum w.r.t all the closed *-algebras of $B(H)$.

Proof: Because AA^* is self-adjoint thus has real thus doesn't separate $\mathbb{C}$ thus it is invertible in any closed *-algebra of $B(H)$. so does $T^{-1} = T^*(TT^*)^{-1}$. □

Prop. (3.7.10). For T normal and E its spectral decomposition, then if $f \in C(\sigma(T))$ and $\omega_0 = f^{-1}(0)$, then $N(f(T)) = R(E(\omega_0))$.

Proof: $\chi_E f = 0$, and let $\omega_n = f^{-1}([1/(n-1), 1/n))$, then $E(\omega_n)x = 0(f(T)x = 0)$, so countable additivity shows that $E(\sigma\backslash\omega_0)x = 0$, so $E(\omega_0)x = x$. □

Cor. (3.7.11).

1. $N(T - \lambda I) = R(\{\lambda\})$.
2. every isolated point of $\sigma(T)$ is point spectra, because this point is open thus is $E(\{x\}) \neq 0$.
3. if $\sigma(T)$ is countable, then every $x \in H$ has a unique orthogonal decomposition $x = \sum E(\lambda_i)x$ and $T(E(\lambda_i)x) = \lambda_i E(\lambda_i)x$.

Prop. (3.7.12). A normal compact operator $T \in B(H)$ is compact iff $\sigma(T)$ has no limit point and $\dim N(T - \lambda I) < \infty$ for $\lambda \neq 0$. In particular, a compact operator is a limit of f.d. operators.

Cor. (3.7.13) (Spectral Theorem). A compact normal operator (in particular a normal operator on a f.d linear space) is unitarily diagonalizable. (Use resolution of identity(3.7.11)).

Cor. (3.7.14) (Hilbert-Schimidt). For a symmetric compact operator A on a Hilbert space H, there is a set of orthonormal basis that A is diagonal on it. And of course, its eigenvalue is real and converges to 0.

Cor. (3.7.15). The Hermite functions $C_n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2}$, as the eigenvector of $\hat{H} = x^2 - \frac{d^2}{dx^2}$, forms a complete basis for $L^2(\mathbb{R})$.

Prop. (3.7.16) (Freudenthal Spectral Theorem).

Normal Compact Operator

Compact Operator & Fredholm Operator(张恭庆)

Def. (3.7.17). An operator between Banach spaces is called **compact** if it maps bounded set to sequentially compact(Closure compact) set.

Prop. (3.7.18). The space of compact operator is a closed subspace of $L(X, Y)$. (Use Hausdorff theorem(3.2.6) to show a limit is totally bounded). Thus the limit of f.d. operators is compact.

If one of A or B is compact and the other is continuous, then AB is compact, because continuous maps bounded to bounded and compact to compact.

Prop. (3.7.19). T is compact $\iff$ T^* is compact.

Proof: We need only to show that $T^*y_n^*$ has a uniformly convergent subsequence on the unit sphere, and we use Arzela-Ascoli because $\overline{T(B(0,1))}$ is compact. For the other half, use reflexive. □

Prop. (3.7.20) (Riesz-Fredholm). For a compact operator A, let $T = I - A$. Then:
1. $R(T)$ is closed. So $R(T) = N(T^*)^\perp$.
2. $\sigma(T) = \sigma(T^*)$.
3. $\text{codim}R(T) = \dim N(T) = \dim N(T^*) < \infty$. Equivalently, $\sigma(A)\backslash\{0\} = \sigma_p(A)\backslash\{0\}$.
4. $\sigma(A)$ has at most one convergent point 0 (it must attain 0 if X is a infinite-dimensional). Hence it at most countable spectrum.

Proof: For 3,4, it suffices to find a convergent series that cannot converge. □

Prop. (3.7.21) (Jordan Decomposition for Compact Operators). For a compact operator A and all the non-zero eigenvalues λ_i, we can find space

$$\bigoplus_{i=1}^{\infty} N((\lambda_i - A)^{p_i})$$

on which A has a Jordan decomposition.

Def. (3.7.22) (Fredholm Operator). A bounded operator between Banach space is called a Fredholm operator if $\dim N(T) < \infty$ and $\text{codim}R(T) < \infty$. It necessarily has closed image. The index is defined as $\text{ind}(T) = \dim N(T) - \text{codim}R(T)$, thus for a compact operator A, $I - A$ has index 0.

Prop. (3.7.23) (Characterization of Fredholm Operator). If T is Fredholm from X to Y iff there exist a bounded S from Y to X that $S_1T = I - A_1, TS_2 = I - A_2$, where A_1, A_2 is compact. S_1 and S_2 can be chosen the same, so S is Fredholm as well.

Cor. (3.7.24). Fredholm operators constitute an open set in $L(X, Y)$, and it is closed under composition. and index is an open map on it. $\text{ind}(T_1T_2) = \text{ind}(T_1) + \text{Ind}(T_2)$ (Direct calculation).

Proof: Use the fact that composition with a compact operator is compact and Notice $I - S$ is invertible for $||S|| < 1$. □

Cor. (3.7.25). If T is Fredholm and A is compact, then $T+A$ is compact, and $\mathrm{ind}(T+A) = \mathrm{ind}(T)$.

So the Fredholm operator is the set of operators 'invertible module compact ones'.

IV.4 Abstract Harmonic Analysis(Folland)

1 Locally Compact Groups

Prop. (4.1.1). Topological group is completely regular.

Proof: Use a sequence of neighbourhood of identity to construct a uniform metric on G. Then set $\phi(x) = min\{1, 2\sigma(a,x)\}$. Cf.[Abstract Harmonic Analysis Ross §8.4] □

Prop. (4.1.2). Locally compact group (Hausdorff) is normal. In particular, Dirac Sequence exists.

Proof: Notice that by choosing a precompact symmetric open neighbourhood U of identity, there exists a σ-compact clopen subgroup H. So H can σ-locally refine every open cover, thus G can, too. So by (1.3.1) G is paracompact. As a topological group, G is regular, thus G is normal by (1.3.4). □

2 Analysis on Locally compact groups

Prop. (4.2.1). The dual group G^* can be regarded as the spectrum of $L^1(G)$:

$$\xi \mapsto \Big(f \mapsto \int \overline{(x,\xi)} f(x)dx\Big),$$

and in this way, the Fourier transform is just the Gelfand transform from $L^1(G)$ to $C(\hat{G})$. Its range is a dense space of $C_0(\hat{G})$.

Prop. (4.2.2). There is another map from $M(\hat{G})$ to bounded continuous functions on G:

$$\mu \mapsto \Big(\phi_\mu : x \mapsto \int (x,\xi)d\mu(\xi)\Big).$$

This is a norm decreasing injection.

Prop. (4.2.3). $\widehat{(f*g)} = \widehat{f}\cdot\widehat{g}$, so if $f,g \in L^2(G)$, $\widehat{(fg)} = \widehat{f}*\widehat{g}$. Cf.[Folland Abstract Harmonic Analysis].

Def. (4.2.4). A function of **positive type** on a closed compact group G is a function $\phi \in L^\infty(G)$ that defines a positive linear functional on the B^*-algebra $L^1(G)$.

We set $P = P(G) =$ the set of continuous functions of positive type on G and $P_0(G) = \{\phi | ||\phi||_\infty \leq 1\}$. By Alaoglu, $P_0(G)$ is a weak*-compact set.

Prop. (4.2.5) (Bochner's Theorem). If $\phi \in P(G)$, there is a unique positive $\mu \in M(\widehat{G})$ s.t. $\phi = \phi_\mu$.

Proof: We have the map defined in(4.2.2) maps into $P_0(G)$ and it is weakly*continuous, so maps the compact convex set of positive measures that $\mu(\hat{G}) \leq 1$ to a compact convex set. And the image contains all the extreme point of P_0, i.e. characters of G and 0. So by Krein-Milman, this map is surjective. Cf. [Folland Abstract Harmonic Analysis Prop4.19]. □

Cor. (4.2.6) (Herglotz). A numerical sequence $\{a_n\}$ is positive iff there is a positive measure $\mu \in M(T)$ s.t. $a_n = \widehat{\mu}(n)$.

Prop. (4.2.7). The set of regular Borel probability measures on a compact X is weak*-compact in $C(X)^*$. (Use Alaoglu).

IV.5 Harmonic Analysis

1 Fourier Analysis on $\mathbb{R}^n$

Prop. (5.1.1). If $f \in L^1(\mathbb{T})$ is absolutely continuous, then $\widehat{(f')}(n) = 2\pi i n \cdot \widehat{f}(n)$.

Prop. (5.1.2). $f \in L^1(\mathbb{T})$ is determined by its Fourier coefficients.

IV.6 Differential Equations

1 ODE-Fundamentals

Prop. (6.1.1).

$$x^{(2)} = f(x)$$

It can be solved.

Proof:

$$x^{'}x^{(2)} = f(x)x^{'}$$

$$\frac{1}{2}(x^{'})^2 = \int^x f(t)dt$$

□

2 ODE-Theorems

Prop. (6.2.1) (Existence and Uniqueness of ODE of Lipschitz Type). If $F(t,x)$ defined on $[-h,.h]\times[\eta-\delta,\eta+\delta]$ is a function that is locally Lipschitz: that is, $\exists\delta, L$, s.t. if $|t|\leq h, |x_i-\eta|\leq\delta$, then

$$|F(t,x_1)-F(t,x_2)|\leq L|x_1-x_2|.$$

Then the initial value problem:

$$(Tx)(t) = \eta + \int_0^t F(\tau, x(\tau))d\tau$$

has a unique solution on the interval $[-h,h]$ if $h<\min\{\delta/M, 1/L\}$, where M is the maximum of F on $[-h,.h]\times[\eta-\delta,\eta+\delta]$. Because T is a contraction.

Prop. (6.2.2) (Existence of ODE of continuous Type (Caratheodory)). If $F(t,x)$ defined on $[-h,.h]\times[\eta-\delta,\eta+\delta]$ is a continuous function, then

$$(Tx)(t) = \eta + \int_0^t F(\tau, x(\tau))d\tau$$

has a unique solution on the interval $[-h,h]$ if $h<\delta/M$, where M is the maximum of F on $[-h,.h]\times[\eta-\delta,\eta+\delta]$. (Use Schauder fixed point theorem and Arzela-Ascali).

Prop. (6.2.3) (Existence Theorem for Complex Differential Equations). Let $f(z,\mathbf{w})$ be a holomorphic vector function in a domain $D\subset\mathbb{C}^{n+1}$, then the initial value problem

$$\mathbf{w}^{'} = f(z,\mathbf{w}),\quad w(z_0) = w_0$$

has exactly on holomorphic solution locally (Thus on a simply connected domain).

Cor. (6.2.4). So a holomorphic high-order ODE for a complex variable can be solves. And luckily it can be solved even $\overline{z}$ appears (just regard it as a constant).Δ

Proof: Cf.[Ordinary Differential Equations, P110]. □

Prop. (6.2.5). For the equation:

$$\frac{d\mathbf{y}}{d\mathbf{x}} = \mathbf{A}\mathbf{y},$$

One solution basis is:

$$\begin{cases} e^{\lambda_1 x}\mathbf{P}_1^{(1)}(x), \cdots, e^{\lambda_1 x}\mathbf{P}_{n_1}^{(1)}(x); \\ \cdots\cdots\cdots \\ e^{\lambda_s x}\mathbf{P}_1^{(d)}(x), \cdots, e^{\lambda_s x}\mathbf{P}_{n_s}^{(1)}(x); \end{cases}$$

Where

$$\mathbf{P}_j^{(i)}(x) = \mathbf{r}_{j0}^{(i)} + \frac{x}{1!}\mathbf{r}_{j1}^{(i)} + \cdots,$$

where $\mathbf{r}_{j0}^{(i)}$ is a basis of solution of $(\mathbf{A} - \lambda_i I)^n \mathbf{x} = 0$, and $\mathbf{r}_{k+1}^{(i)} = (\mathbf{A} - \lambda_i I)\mathbf{r}_k^{(i)}$.

Proof: Cf.[常微分方程丁同仁定理 6.6]. □

Cor. (6.2.6). For the equation:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0,$$

If the characteristic equation has s different roots $\lambda_1, \ldots, \lambda_s$ and corresponding multiplicities $n_1, \ldots, n_s$, then:

$$\begin{cases} e^{\lambda_1 x}, xe^{\lambda_1 x}, \cdots, x^{n_1-1}e^{\lambda_1 x}; \\ \cdots\cdots\cdots \\ e^{\lambda_s x}, xe^{\lambda_s x}, \cdots, x^{n_s-1}e^{\lambda_s x}; \end{cases}$$

is a solution basis.

Proof: Cf.[常微分方程丁同仁 P198]. □

Prop. (6.2.7) (Lyapunov). Consider the Lyapunov stability of an autonomous system of the form:

$$\frac{dx}{dt} = Ax + o(|x|),$$

Then:

1. If A has a eigenvalue whose real part is positive, then the trivial solution is not weak stable.
2. If all eigenvalues of A has negative real part, then the trivial solution is strong stable.

3 PDE

Direct Solution

Prop. (6.3.1) (Characteristic Line). Consider a 1-dimensional parabolic equation:

$$p_t + c(p, x, t)p_x = r(p, x, t)$$

Let $P(t) = p(X(t), t)$, this equation is equivalent to

$$P_t = r(X(t), t, P(t)), \quad X_t = c(X(t), t).$$

an ODE equation.

Chapter V

Algebraic Geometry

V.1 Basic Notions

1 Connectedness

Prop. (1.1.1). Spec(A) is connected $\iff$ $A = A_1 \times A_2$ $\iff$ A has no nontrivial idempotent element.

Proof: $I + J = 1$. Thus, $x + y = 1$ for some $x \in I$, $y \in J$. Now, since $IJ \subset \sqrt{0}$, we have $(xy)^m = 0$ for some m. Then, we have

$$1 = (x+y)^{2m} = \underbrace{x^{2m} + \cdots + \binom{2m}{m+1} x^{m+1} y^{m-1}}_{e_1} + \underbrace{\binom{2m}{m} x^m y^m + \cdots + y^{2m}}_{e_2}$$

and these elements e_1, e_2 satisfy the relations

$$e_1 e_2 = 0, \quad e_1^2 = e_1, \quad e_2^2 = e_2.$$

$\square$

2 Closed Map

Prop. (1.2.1). Let $A \to B$ noetherian. Then going-up holds $\iff$ Spec map is closed.

Proof: going-up is equivalent to

$$f^*(V(q)) = V(f^*(q)) \quad \forall q \quad \text{prime}$$

and using primary decomposition of $\sqrt{I}$, $V(I) = \bigcup V(q_i)$ $\square$

3 Finite Type

Def. (1.3.1). Call a scheme $X \to \mathrm{Spec}(R)$ finite type over R if X is quasi-compact and $\Gamma(U, \mathcal{O}_X)$ is finite generated over R for every open set U.

Prop. (1.3.2). finite type is a affine local property. Cf.[Red Book Mumford P87].

4 Reduced

Def. (1.4.1). Call a scheme is Reduced if $\Gamma(U, \mathcal{O}_X)$ is reduced for every open set U.

Prop. (1.4.2). Reducedness is an affine local and stalkwise property.

Proof: if there is an affine cover that is reduced, then the stalks will be like R_P is reduced if R is reduces. And if the stalks are all reduced, then a nilpotent element will be 0 in every local set, thus 0 because $\mathcal{O}$ is a sheaf. □

V.2 Schemes

1 Sheaves

Prop. (2.1.1) (Glueing Sheaves). We have a space X and a cover for this space $\{U_i\}_{i\in I}$. Further, we are given a family of sheaves indexed by the same set I, $\{\mathcal{F}_i\}_{i\in I}$ on each of the U_i. We are given isomorphisms

$$\phi_{ij}:\mathcal{F}_i|_{U_i\cap U_j}\longrightarrow \mathcal{F}_j|_{U_i\cap U_j}$$

along with the so-called cocycle condition

$$\phi_{ik}=\phi_{jk}\circ\phi_{ij}\quad\text{on}\quad U_i\cap U_j\cap U_k.$$

The task is to construct a sheaf on X compatible with these local sheaves. In the same way, we can glue schemes and also morphisms with a fixed target (compatible with the glueing).

Proof: It's more intuitive to define the glued-up sheaf $\mathcal{F}$ like this:

For every open set $V\subset X$, we define the group of sections $\mathcal{F}(V)$ to be a set consisting of all tuples $(s_i)_{i\in I}$, where each s_i is a section in $\mathcal{F}_i(V\cap U_i)$, and where the s_i's are required to obey the compatibility condition:

$$\phi_{ij}(s_i|_{V\cap U_i\cap U_j})=s_j|_{V\cap U_i\cap U_j}\qquad(*)$$

for all $i,j\in I$. The group addition on $\mathcal{F}(V)$ is the obvious one.

The $\mathcal{F}$ that I defined is guaranteed to be a sheaf, regardless of whether we impose the cocycle condition. It comes with a natural restriction map, making it a presheaf, and it also obeys all the gluing conditions necessary for it to be a sheaf. The cocycle condition is not needed to verify any of these things.

However, we also need to satisfy ourselves that the restriction $\mathcal{F}|_{U_k}$ really is isomorphic to the $\mathcal{F}_k$ that we started with, for each $k\in I$. It is here that the cocycle condition is required.

It is easy to write down what the isomorphism $\psi:\mathcal{F}_k\xrightarrow{\cong}\mathcal{F}|_{U_k}$ ought to be. Given an open $V\subset U_k$ and given a section $s\in\mathcal{F}_k$, we would like to define its image under ψ to be

$$\psi(s)=(\phi_{ki}(s|_{V\cap U_i}))_{i\in I}$$

However, we need to be sure that the tuple $(\phi_{ki}(s|_{V\cap U_i}))_{i\in I}$ represents a well-defined element of $\mathcal{F}(V)$. In particular, we must verify that $(\phi_{ki}(s|_{V\cap U_i}))_{i\in I}$ obeys the condition $(*)$, which states that

$$\phi_{ij}\circ\phi_{ki}(s|_{V\cap U_i\cap U_j})=\phi_{kj}(s|_{V\cap U_i\cap U_j})$$

for any $i,j\in I$. This is true by virtue of the cocycle condition.

This map is obviously injection and it is surjection by virtue of $(*)$. $\square$

Prop. (2.1.2) (Stalks). Taking stalks is right adjoint thus preserves kernel on PAb, and it is exact on Ab. And the epimorphism and monomorphism can be checked on stalks.

Prop. (2.1.3). The category of presheaves and sheaves on a site is a Grothendieck category.

Proof: For the presheaf, the only problem is the existence of generator, for that, just construct a family of presheaves and sum them. Take $Z_U(V) = \oplus_{\mathrm{Mor}(V,U)}\mathbb{Z}$, then $F(U) = \mathrm{Hom}(Z_U, F)$(Similar to Yoneda Lemma). So they are a family of generators.

For the sheaf, the shifification is exact**??**, it follows that coimage is the image. colimits in ShA is the shifification of the colimits in PShA. Sh is exact so filtered limits is exact, and Z_U^{++} is a family of generators, Z_U^{++} represents the functor $\Gamma(U,-)$. □

Prop. (2.1.4). The category of sheaves of sets on the site of the G-sets S_G is equivalent to S_G.

Prop. (2.1.5). Fiber Products exist in the category of schemes.

Proof: Cf.[Hartshorne P88]. You should use (2.1.1). □

Prop. (2.1.6). The category of coherent sheaves over a ringed space is an Abelian category.

transfer of sheaves under morphisms

Def. (2.1.7).

- the **direct image** $f_\bullet\mathcal{F}$, $f_\bullet\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$.
- the inverse image $f^\bullet\mathcal{F}$, $f^\bullet\mathcal{F}(U) = Sh(\mathcal{F}(f(U)))$.
- the pullback sheaf $f^*(\mathcal{F}) = f^\bullet\mathcal{F} \otimes_{f^\bullet\mathcal{O}_Y} \mathcal{O}_X$

$f^\bullet$ or f^* is left adjoint to $f_\bullet$. And $f^\bullet$ is exact (Check on stalks).

Prop. (2.1.8) (Sheafification). The operator F^+ is the sheaf that

$$F^+(U) = \lim_{\rightarrow}\ker(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j))$$

takes presheaf to a separated presheaf, i.e. $0 \to F(U) \to \prod_i F(U_i)$ and a separated presheaf to a sheaf. (The problem of separated is that the cover may not be identical in $U_i \times_U U_j$ but only on a cover of it.

The sheafification F^{++} is exact and it is left adjoint to the forgetful functor, so the forgetful functor is left exact and it preserves injectives, Thus the sheaf cokernel is the shifification of the presheaf kernel, the sheaf kernel is the presheaf kernel.

Proof: Sh is left exact because F^+ is left exact Cf.[Tamme]. □

2 Group Schemes

Prop. (2.2.1) (Cartier Duality).

3 Curves

Prop. (2.3.1). a non-singular curve in $P^2(k)$ where $\mathrm{char} k \neq 0$ is projectively isomorphic to $xy - z^2$ if it has a rational point. (Use Riemann-Roch to show that $O(p)$ has a nontrivial section which gives a isomorphism to P^1). And in fact the assertion can be checked directly.

V.3 Cohomology

1 Sheaf cohomology

Acyclic Sheaves

A sheaf is called
flasque iff for any open $U, \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective;
soft iff X is paracompact Hausdorff and $\forall$ closed $V, \mathcal{F}(X) \to \mathcal{F}(V)$ is surjective.

Prop. (3.1.1). For a sheaf of *rings* over a paracompact Hausdorff space X, the following are equivalent,

1. it is a soft sheaf.
2. for any disjoint closed sets V, W, there is a section of X that is 0 on V, and 1 on W.
3. it possesses a partition of unity.
4. it is a fine sheaf. (See definition below).

Proof: $1 \iff 2$ is easy and $1 \to 3$ is the to choose a closed locally finite subcover and use Zorn's lemma to construct one-by-one. For $3 \to 1$, notice a closed section can extend to a slightly larger nbhd.

Because for a sheaf of rings $\mathcal{F}$, a partition of unity is equivalent to a partition of unity $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$, so 34 are equivalent because 13 are equivalent. □

fine iff it is a sheaf of abelian group on a paracompact Hausdorff space and the sheaf of rings $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$ is soft.

Note that a fine sheaf possesses a decomposition of section because the previous proposition applies to $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$, and a partition of unity in $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$ yields a decomposition of section in $\mathcal{F}$.

Thus a fine sheaf is soft. (extend to a small nbhd and use partition of unity).

A flasque sheaf is soft.

The sheaf of modules over a soft sheaf of rings is soft.

Fine, flasque and soft, are local properties. (Use Zorn's lemma to construct one-by-one).

The continuous function sheaf on a paracompact Hausdorff space or the smooth function sheaf on a smooth manifold is fine, thus any smooth module is fine (Use bump function).

For a exact sequence $0 \to \mathcal{F} \to G \to H \to 0$, if F is flasque, then $0 \to \mathcal{F}(U) \to G(U) \to H(U) \to 0$ is exact for every open U.(It reduces to $\check{H}^1(F) = 0$, and this is done by Zorn's lemma). Thus if $\mathcal{F}$ is flasque, $\mathcal{G}$ is flasque iff $\mathcal{H}$ is flasque (Snake lemma). Similar for soft sheaf (choose a closed locally finite subcover and notice a restriction of a soft sheaf on a closed subset is soft).

flasque sheaf is acyclic; thus soft sheaf is acyclic (Use flasque resolution and induction).

Prop. (3.1.2) (Dolbeault). On a smooth manifold,

$$H^{p,q}(X, \mathcal{E}) \cong H^q(M, \Omega^p \otimes_{\mathcal{O}_X} \mathcal{E}),$$

where the left is Dolbeault cohomology and the right is sheaf cohomology. (Use the fact that smooth sheaf is soft 1 and Poincare-lemma).

Prop. (3.1.3) (De Rham). For a smooth manifold and an Abelian group G,

$$H^*_{dR}(X, G) \cong H^*(X, G)$$

Where the right is constant sheaf cohomology. (Use the fact that smooth sheaf is soft 1 and Poincare-lemma).

Prop. (3.1.4) (Serre). An open subset U of X is Coh-acyclic iff $\mathcal{O}_{X|U}$ is isomorphic to an affine scheme as a ringed space.

Prop. (3.1.5) (Cartan). The class of Coh-Acyclic subsets of an analytic space is exactly the Stein manifold.

V.4 Curves

Chapter VI

Higher Algebra

VI.1 Category

1 Exactness

Prop. (1.1.1). In a category that can apply diagram chasing, taking direct limits is exact. In particular, in an Abelian category.

Prop. (1.1.2). In an Abelian category, the functor $X \mapsto \mathrm{Hom}(X, Y)$ and $X \mapsto \mathrm{Hom}(Y, X)$ is both left exact. Note that left and right is seen on the image.

2 Adjointness

Prop. (1.2.1). A right adjoint functor is left exact and it preserves injectives if its left adjoint is exact.

A left adjoint functor is right exact and it preserves projectives if its right adjoint is exact.

Prop. (1.2.2). Any presheaf on a small category is a colimit of representable sheaves h_X. (Consider all $h_X \to \mathcal{F}$ and take colimit, prove it is isomorphism).

Prop. (1.2.3) (Kan Extension). For a cocomplete category $\mathcal{D}$, there is a natural bijection between left adjoint functor $\widehat{\mathcal{C}} \to \mathcal{D}$ and functors $\mathcal{C} \to \mathcal{D}$ by Yoneda embedding.

Prop. (1.2.4). The sheaf Γ functor is right adjoint to the constant sheaf functor over arbitrary site.

Prop. (1.2.5). The inclusion functor is right adjoint to the shifification functor over arbitrary site.

Prop. (1.2.6). The forgetful functor is right adjoint to the Shifification functor, and shifification is exact, so it preserves injectives.

3 Injective & Projective

4 Abelian Category

Prop. (1.4.1) (Axioms for Abelian Category).

- **A1**:$\mathrm{Hom}(X, Y)$ is an Abelian group.
- **A2**:There exists a zero object.
- **A3**:There exists a canonical sum and product with projections.

(Satisfying this three is called a additive category.)

- **A4**:Coimage equals image.

Prop. (1.4.2). The $\mathrm{Hom}(X, -)$ operator is left exact in Abelian category, because kernel is a kind of limit.

Prop. (1.4.3). Axiom A3 asserts the good existence of product and sum of objects as we wanted, and it can be used to prove that monomorphism and epimorphism are stable under pushout and pullback.

Prop. (1.4.4). equalizer and finite product derives finite limit, thus finite limits and finite colimits exists in Abelian categories.

Prop. (1.4.5) (Mitchell's embedding theorem). If $\mathcal{A}$ is a small category, then there exists a unital ring R, not necessary commutative and a fully faithful and exact functor $\mathcal{A} \to R-$mod.

Prop. (1.4.6) (Examples). If A is an Abelian category, then A^C is an Abelian category.

5 Grothendieck Abelian Category

Prop. (1.5.1) (Axioms for Grothendieck Abelian Category).

- **AB3**: It is an Abelian category and arbitrary direct sums exists. (Thus colimits over small categories exists.)
- **AB5**: Filtered colimits over small categories are exact. This is equivalent to {Any family of subobjects $\{A_i\}$ of A indexed by inclusion can induce a morphism $\sum A_i \to B$ (internal sum)} Cf.[Tamme].
- **GEN**: It has a generator, that is, an object U s.t. for any proper subobject $N \subsetneq M$, there is a map $U \to M$ that doesn't factor through N.

Cor. (1.5.2). The category of R-modules is a Grothendieck Abelian category with generator R.

Prop. (1.5.3). The category of Abelian presheaves on a site is a Grothendieck topology.

Proof: The only problem is the existence of generator, for that, just construct a family of presheaves and sum them. Take $Z_U(V) = \oplus_{\mathrm{Mor}(V,U)}\mathbb{Z}$, then $F(U) = \mathrm{Hom}(Z_U, F)$(Similar to Yoneda Lemma). So they are a family of generators. □

Prop. (1.5.4). The category of Abelian sheaves on a site is a Grothendieck topology.

Prop. (1.5.5). The category of abelian sheaves satisfies AB4 and AB3*, but doesn't satisfy AB4*, i.e. not every limit of epimorphisms is epimorphism.

Proof: Consider the zero extension constant sheaf on $B(0, \frac{1}{n})$. □

Prop. (1.5.6) (Injectives). In a Grothendieck Abelian category with generator U, an object is injective iff it is extendable over subobjects of U. Cf.[Sheaf Cohomology Notes1.2]

Prop. (1.5.7). Grothendieck Abelian category has enough injectives. Cf.[Sheaf Cohomology Notes1.2].

6 Category Equivalence

Prop. (1.6.1). A Functor $\mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it's fully faithful and essentially surjective.Hom

Proof: There exist an object $G(X) \in \mathcal{C}$ and an isomorphism $\xi_X : FG(X) \to X$ for every $X \in \mathcal{D}$. Because F is fully faithful, there exists a unique morphism $G(f) : G(X) \to G(Y)$ such that $F(G(f)) = \xi_Y^{-1} \circ f \circ \xi_X$ for every morphism $f : X \to Y$ in $\mathcal{D}$. Thus we obtain a functor $G : \mathcal{C} \to \mathcal{D}$ as well as a natural isomorphism $\xi : F \circ G \to \mathrm{Id}_{\mathcal{D}}$. Moreover, the isomorphism $\xi_{F(Z)} : FGF(Z) \to F(Z)$ decides an isomorphism $\eta_Z : GF(Z) \to Z$ for every $Z \in \mathcal{C}$.This yields a natural isomorphism $\eta : G \circ F \to \mathrm{Id}_{\mathcal{C}}$. □

Prop. (1.6.2) (Yoneda Lemma). $X \mapsto (Y \mapsto \mathrm{Hom}(Y, X))$ is a fully faithful embedding from $\mathcal{C}$ to $\hat{\mathcal{C}} = \mathrm{Func}(\mathcal{C}^\circ, \mathrm{Set})$.

Prop. (1.6.3). For an Abelian category $\mathcal{A}$ satisfying AB3 (i.e arbitrary sum exists), An object P of $\mathcal{A}$ is called a projective generator if the functor $h' : X \mapsto \mathrm{Hom}_{\mathcal{A}}(P, X)$ is exact and and strict: $h'(X) = 0 \to X = 0$. Then h' determines an equivalence from $\mathcal{A}$ to mod-R, where $R = \mathrm{Hom}_{\mathcal{A}}(P, P)$.

Similarly, if $\mathcal{A}$ is an Abelian Noetherian category and P is a projective generator, then R is Noetherian and $\mathcal{A}$ is equivalent to the category of finitely generated R-categories.

Proof: Essentially surjective: construct using direct limit and cokernel.

Notice that $h'(X) \cong h'(X') \to X \cong X'$ by strictness and A4 axiom. So let $X = \mathrm{Coker}(P^{\oplus I}, P^{\oplus J})$,

$$\begin{aligned}
\mathrm{Hom}(h'(X), h'(Y)) &= \mathrm{Hom}(\mathrm{Coker}(h'(P^{\oplus J}), h'(P^{\oplus I}), h'(Y)) \\
&= \ker(\mathrm{Hom}(h'(P^{\oplus J}), h'(Y)) \to \mathrm{Hom}(h'(P^{\oplus I}), h'(Y))) \\
&= \ker(h'(Y^{\Pi I}) \to h'(Y^{\Pi J})) \\
&= \mathrm{Hom}(X, Y)
\end{aligned}$$

□

Cor. (1.6.4) (Morita Equivalence). The following are equivalent:

1. categories A-mod and B-mod are equivalent.
2. categories mod-A and mod-B are equivalent.
3. There exist a finitely generated projective generator P of mod-A that $B \cong \mathrm{End}_A P$.

7 General Category

Prop. (1.7.1) (Eckmann-Hilton argument). If $\circ$ and $\otimes$ is two unital binary operator that commutes:$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$, then they are equal and in fact commutative and associative. Cf.[Wiki].

Prop. (1.7.2). The group objects in the category of groups is abelian groups.

Proof: By Eckmann-Hilton argument, the category multiplication is the same as the group multiplication, so the unit is obviously the sam unit, thus the inverse. So the commutativity of m with inverse implies that it is abelian. □

Prop. (1.7.3). One should notice that the group object structure in any category (m, id, i, X definition) is equivalent to a group structure on $\mathrm{Hom}(Y, X)$ that are preserve under composition with morphisms.

VI.2 Higher Category

1 Kan Complex

Prop. (2.1.1). The fact that any simplicial set X is a colimit of $\Delta[n]$ (1.2.3) is important in proving properties of constructions of simplicial set.

Prop. (2.1.2). A simplicial group is a Kan complex. In particular, simplicial abelian group and simplicial R-module are Kan complexes.

Proof: Cf.[Simplicial Homology Theory Jardine P12] □

2 Simplicial Set

Prop. (2.2.1). The Nerve functor N is a fully faithful functor from the category of small categories to the category of simplicial sets.

Prop. (2.2.2). A natural transformation will induce homotopic nerve map. thus a pair of adjoint functors will induce a simplicial homotopy between their nerve.

3 ∞-Algebras

Prop. (2.3.1).

- The category of functors from the $E_\infty = Fin_*$ (pointed finite sets) to Cat that

$$X([n]) \xrightarrow{\prod_{i=2}^n X(\rho^i)} \prod_{i=1}^n X([1]) \quad n \geq 0$$

  is equivalent to the category of symmetric unital monoidal categories $(X([1]))$.

- The category of functors from the Δ^{op} to Cat that

$$X([n]) \xrightarrow{\prod_{i=2}^n X(\rho^i)} \prod_{i=1}^n X([1]) \quad n \geq 0$$

  is equivalent to the category of symmetric unital monoidal categories $(X([1]))$. And it is symmetric iff it factors through Cut:$\Delta^{op} \to Fin_*$.

VI.3 Simplicial Homotopy Theory

1 Cyclic Homology Theory(欧阳恩林)

Dold-Kan Correspondence

Prop. (3.1.1). The normalized Moore complex gives an equivalence between the category of simplicial abelian group and the category of chain complex of abelian groups. And $NA_*, A_*, (A/DA_*)$ are all homotopically equivalent, $A_* \cong NA_* \oplus DA_*$. Cf.[Wiki hyperref] https://web.archive.org/web/20160913201635/http://people.fas.harvard.edu/ amathew/doldkan.pdf.

Proof: □

2 Homotopy Algebra

3 Topological Cyclic Homology(Scholze)

VI.4 Derived Algebraic Geometry

Chapter VII

Theoretical Physics

VII.1 Hamiltonian Mechanics

1 TBA

Prop. (1.1.1). Yang-Mills Field.

VII.2 Fluid Dynamics

VII.3 Quantum Mechanics

1 Schrodinger Equation

Prop. (3.1.1) (Axioms). The Schrodinger equation can be derived from the Dirac-von Neumann axioms:

The state of particals is a countable dimensional Hilbert space, and

- The observables of a quantum system are defined to be the (possibly unbounded) self-adjoint operators A on $\mathbb{H}$.
- The state φ of the quantum system is a unit vector of $\mathbb{H}$, up to scalar multiples.
- The expectation value of an observable A for a system in a state φ is given by the inner product $\langle\varphi, A\varphi\rangle$.
- (Unitarity) the time evolution of a quantum state according to the Schrodinger equation is mathematically represented by a unitary operator $U(t)$ (depends only on the state an relative time)(one-parameter subgroup).

Now that $\varphi(t)=\hat{U}(t)\varphi(t_0)$, so $\hat{U}(t)\varphi(t_0)=e^{-i\hat{\mathcal{H}}t}$, $\hat{\mathcal{H}}$ hermitian.

So now take derivative w.r.t t, we get $i\frac{d\varphi}{dt}=\hat{\mathcal{H}}\varphi$. By quantum correspondence principle, it is possible to derive the expression of $\hat{\mathcal{H}}$ by classical methods.

Prop. (3.1.2). The solution of a Schrodinger equation for a non Relativistic particle is assumed to be a Schwartz function (Vanish fast enough at infinity). The coefficients is assumed smooth enough to guarantee at least uniqueness and existence locally.

Prop. (3.1.3). The wave function on the (p,t) coordinates is the Fourier Transform of the wave function on the (x,t) coordinates, because the eigenstate of the p-operator $i\hbar\frac{\partial}{\partial x}$ is e^{ikx}, the coefficients of which is the value (probability) of the wave function of the (p,t) coordinates.

Prop. (3.1.4) (Schrodinger Uncertainty Principle). Set $\sigma_A=\sqrt{\langle\hat{A}^2\rangle-\langle\hat{A}\rangle^2}$, then:

$$\sigma_A^2 2\sigma_B^2\geq\left|\frac{1}{2}\langle\hat{A}\hat{B}+\hat{B}\hat{A}\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle\right|^2+\left|\frac{1}{2i}\langle[\hat{A},]\hat{B}]\rangle\right|^2.$$

(Derived from definition and Schwarz inequality, Cf.[Wiki]).

Cor. (3.1.5) (Heisenberg Uncertainty Principle).

$$\sigma_x\sigma_p\geq\frac{\hbar}{2}$$

Proof:

$$[x,i\hbar\frac{\partial}{\partial x}]=i\hbar$$

. □

Prop. (3.1.6) (Spectral Decomposition). In Quantum physics, one need to use spectral decomposition of the Hamiltonian operator. But at most cases, there are only countably many eigenstate and the eigenvalue has a lower bound and tends to infinity. In this case, $(\hat{H}+A)^{-1}$ is a compact operator thus by spectral theorem(3.7.13) the eigenstate of $\hat{H}$ forms a set of complete basis.

2 Calculations

Prop. (3.2.1) (Virial Theorem). For a system that $V(r) \sim r^n$, the average kinetic energy and the average potential energy has the relation :

$$2\langle T\rangle = n\langle V\rangle.$$

3 Spin

VII.4 Quantum Field Theory

VII.5 General Relativity

1 Basics

Prop. (5.1.1) (Maxwell's Equation). Normal Maxwell's equation reads:

$$\begin{cases} \mathrm{div}E = q & \text{(Coulomb's law)} \\ \mathrm{div}H = 0 & \text{(Gaussian law)} \\ \mathrm{curl}E = -\frac{1}{c}\frac{\partial H}{\partial t} & \text{(Faraday's law)} \\ \mathrm{curl}H = j + \frac{1}{c}\frac{\partial E}{\partial t} & \text{(Ampère-Maxwell law)} \end{cases}$$

where E is the magnetic field, H is the electric field, q the charge density, j the electric current.

In Minkowski space, we define the electromagnetic 2-form

$$F = \frac{1}{2}F_{\alpha\beta}dx^\alpha \wedge dx^\beta,$$

where $F_{i0} = E_i, \quad Fij = H_k$, and electric current J, $J^i = -j^i, J^0 = q$.

Maxwell's equation can be re-written as:

$$d^*F = J \qquad dF = 0.$$

Where $d^* = *d*$.

Proof: The Minkowski space is flat, the equivalence can be seen by direct calculation. □

VII.6 String Theory

Chapter VIII

Unknown

VIII.1 TBA

- regularity theorem for elliptic operator.
- facts about linear algebra.
- a right Kan fibration which is a weak equivalence is a trivial fibration.
- smooth irreducible representations of Weil group is admissible.
- fundamental class relation with Weil group
- conductor of a Weil representation is an integer
- $\mathcal{O}_{\mathfrak{p}} \cong \mathbb{Z}_p^d$.