The Skyscraper

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Preface

This is a latex version of subtle or important materials I encountered while studying in Peking university. I started this project in the fall of my third undergraduate year, noticing that I have a poor memory and consistently forget what I have already learned thus struggle to check details. So it came to me that I can compile all the proofs of theorems I cannot recall that is hard and subtle yet appearing over and over again. But finally it turns out I want to make it as comprehensive as possible. That's it.

Notice: This is hardly a *readable* book, I use it as a dictionary. It only contains materials that I'm interested in and many proofs are still missing. And maybe I will or maybe I won't complete them.

I constantly adds stuff in this note, and I regularly put them online. You can find the newest version at `https://phacademic.com/files/my-notes.pdf`.

It should be made clear that I took proofs from many different places, so it should not be considered anything in this book originated from me. Until I get a full extensive reference of this note, I have few rights to the texts.

It's true that there is already a great online book StackProject that covers considerably many of the Algebraic Geometry part of this note, but it's TOO long, I haven't finish reading it but I reordered the materials that I learned and keep track of it in my own way. I write it much shorter and omitted easy proofs.

Sincere thanks to Yi Tian for answering all the questions when I was learning Algebraic geometry and p-adic geometry. Without him, These notes can hardly be in shape.

I truly hope these notes can contribute to my study and help anyone who read it, but it comes with no warranty, please use at your own risk.

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Chapter I

Algebra

I.1 Set Theory

1 Cardinal & Ordinal

Def. (1.1.1). A **cardinal number** is an equivalence class, where equivalence and ordering is given by injectives and surjectives. it is used to describe the 'size' of a set.

An **ordinal** is an equivalence class of isomorphic well-ordered transitive (i.e. every element is a subset of itself) sets. Notice that two ordinal can have the same cardinality.

The least ordinal having cardinality α is called the **initial ordinal** of α. The axiom of choice together with(1.1.4) asserts that every cardinal has an initial ordinal.

The first infinite cardinal number or the first initial ordinal is denoted by ω or $\aleph_0$.

Remark (1.1.2). Equivalently we can define cardinal number as an ordinal that is an initial ordinal of some α. Anyway, cardinal number is fewer than ordinal numbers.

Prop. (1.1.3) (Bernstein's Theorem). If there is an injective from A to B and an from B to A, then there is a bijection from A to B. Thus the ordering of the cardinal is well-defined.

Lemma (1.1.4). The ordering of ordinal is by inclusion. The ordering of ordinals is a total ordering and is a well-ordering. Every element of an ordinal is an ordinal, and if an ordinal $\beta \subset \alpha$, then $\beta \in \alpha$. Cf.[Set Theory Jech P108].

Def. (1.1.5). The **cofinality** of an ordinal α is the smallest ordinal δ that is the order type of a cofinal subset of α.

The **cofinality** of or a poset (i.e partially ordered set) α is the is the smallest cardinality δ of a cofinal subset of α.

Cardinal Arithmetics

Def. (1.1.6). The **sum**, **multiplication** and **exponentiation** of two ordinal is the cardinality of the set $A\coprod B$, $A\times B$ or A^B respectively. Note that this is may be smaller than the ordinal sum of the corresponding initial ordinal, because operations of initial ordinals may not be initial, the ultimate reason is that the cardinal case, we can rearrange the order to get a smaller ordinal.

Prop. (1.1.7). $\kappa\times\kappa=\kappa$ for an infinite cardinal. Should use the axiom of choice.

Ordinal Arithmetic

Prop. (1.1.8) (Transfinite Induction/Recursion). If a property defined for a set of ordinals satisfies:

1. $P(0) = 1$.
2. $P(\alpha + 1) = 1$ if $P(\alpha) = 1$.
3. $P(\lambda) = 1$ if $P(\beta) = 1$ for all $\beta < \lambda$.

then P is true for all ordinals.

Transfinite recursion:

Def. (1.1.9). We use infinite recursion to define **addition** of ordinals as

- $\beta + 0 = \beta$
- $\beta + (\alpha + 1) = (\beta + \alpha) + 1$, where $\alpha + 1$ is the successor of α.
- $\beta + \alpha = \sup\{\beta + \gamma | \gamma < \alpha\}$ for a limit ordinal α.

The **multiplication** and **exponentiation** are defined similarly.

Prop. (1.1.10). The addition and multiplication of ordinals are of the order type of $\alpha \coprod \beta$ in adjunction order and $\alpha \times \beta$ in lexicographical order respectively, Cf.[Set Theory Jech P120,122]

Prop. (1.1.11) (Cantor Normal Form). Cf.[Jech Set Theory].

2 Axiomatic Set Theory

I.2 Linear Algebra

References are [Linear Algebra Hoffman] and [线性代数 谢启鸿].

<u>**Basics**</u>

Prop. (2.0.1). All basis of a linear k-vector space has the same cardinality.

Prop. (2.0.2) (Canonical way of Writing a Basis). After so many years, I still find it confusing to write a basis and observing change of basis, so I will write it here:

A vector should always be written vertically, and so a basis should be $\overrightarrow{e} = (e_1, \ldots, e_n)$(horizontal), and a vector with basis $\overrightarrow{a}$(vertical) is in fact $\overrightarrow{e}\overrightarrow{a}$.

A change of basis should be written $\overrightarrow{e'} = ea$, with $a \in GL_n$, and then if an operator has matrix A w.r.t. the basis $\overrightarrow{e}$, it then map in the basis $\overrightarrow{e'}$ $v = \overrightarrow{e'}x = \overrightarrow{e}ax \mapsto \overrightarrow{e}Aax = \overrightarrow{e'}a^{-1}Aax$, so it has matrix $a^{-1}Aa$ w.r.t the basis $\overrightarrow{e'}$.

1 Rank

Prop. (2.1.1). The row rank of a matrix A is the same as the column rank.

Proof: Let A have n rows, the column rank equals $\dim \operatorname{Im} f$, and the row rank is $n - \dim \operatorname{Ker} f$, so by the rank-nullity theorem $\dim \operatorname{Im} f + \dim \operatorname{Ker} f = n$, which is because exact sequence of vector spaces split, the conclusion follows. □

Prop. (2.1.2) (Sylvester's Inequality). For U a $m \times n$ matrix and V a $n \times k$ matrix,

$$\operatorname{Rank}(UV) \geq \operatorname{Rank}(U) + \operatorname{Rank}(V) - n$$

Proof: This comes from $\dim \operatorname{Ker} fg \leq \dim \operatorname{Ker} f + \dim \operatorname{Ker} g$, which is because $\operatorname{Ker} fg = g^{-1}(\operatorname{Ker} f)$. □

2 Dual space

Prop. (2.2.1). For a linear map between two spaces of the matrix form A, the adjoint map between there dual spaces is of the matrix form the transpose of A.

3 Similarity(Linear map)

Prop. (2.3.1). If a linear map has matrix form T in a basis (X_i) and there is another basis (Y_i) that $(Y_i) = (X_i)P$, then it has matrix form PTP^{-1} in the basis (Y_i). IN particular, if T can be diagonalized, with eigenvectors (X_i), then $T = (X_i)D(X_i)^{-1}$.

Prop. (2.3.2). A matrix that $J^2 + 1 = 0$ is similar to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^n$. (Use cyclic decomposition).

Prop. (2.3.3) (Jordan Form). For a matrix over a algebraically closed field, it is similar to a matrix of blocks $\lambda_i I + N, Nx_i = x_i + 1$.

For a real matrix, it is similar to a matrix of blocks of the above form together with $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ on the diagonal and $I_{2\times 2}$ on the upper side.

4 Conguence(Bilinear Form)

Prop. (2.4.1). A symmetric matrix A is orthogonally diagonalizable. Similarly, a skew-symmetric matrix is orthogonally diagonalizable and an (skew)hermitian matrix is unitarily diagonalizable.

Proof: For any real matrix A and any vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle.$$

Now assume that A is symmetric, and $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of A corresponding to distinct eigenvalues λ and μ. Then

$$\lambda\langle \mathbf{x}, \mathbf{y}\rangle = \langle \lambda\mathbf{x}, \mathbf{y}\rangle = \langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x}, \mu\mathbf{y}\rangle = \mu\langle \mathbf{x}, \mathbf{y}\rangle.$$

Therefore, $(\lambda-\mu)\langle \mathbf{x}, \mathbf{y}\rangle = 0$. Since $\lambda - \mu \neq 0$, then $\langle \mathbf{x}, \mathbf{y}\rangle = 0$, i.e., $\mathbf{x} \perp \mathbf{y}$.

Now find an orthonormal basis for each eigenspace; since the eigenspaces are mutually orthogonal, these vectors together give an orthonormal subset of $\mathbb{R}^n$. □

Prop. (2.4.2) (Normal operator). More generally, a normal operator over $\mathbb{C}$ is unitary diagonalizable using resolution of identity(3.8.3) because the spectrum are discrete thus the point projection is orthogonal.

Prop. (2.4.3) (Symplectic Form). Over $\mathbb{R}$, a skew-symmetric matrix are orthogonally congruent to $\mathrm{diag}\left\{\begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}\right\}_i$.

Proof: Choose a α, β and choose their orthogonal complement. □

Cor. (2.4.4). For a matrix that $J^2 + 1 = 0$, by (2.3.2),there is a unique inner product s.t. J is orthogonal and then it is orthogonally congruent to $\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\rangle_n$. (Use cyclic decomposition).

so this J is equivalent to a complex structure, homeomorphic to $O(n)/U(\frac{n}{2})$.

Prop. (2.4.5). Given a bilinear form on a field, the relation of orthogonality is symmetric iff it is symmetric or alternating, i.e. $B(x,x) = 0$.

Proof: Let $w = B(x,z)y - B(x,y)z$, then $B(x,w) = 0$, hence we have $B(w,x) = 0$, that is

$$B(x,z)B(y,x) - B(x,y)B(z,x) = 0.$$

Let $z = x$, then $B(x,x)[B(x,y) - B(y,x)] = 0$.

If some $B(u,v) \neq B(v,u)$ and $B(w,w) \neq 0$, then $B(u,u) = B(v,v) = 0, B(w,v) = B(v,w), B(w,u) = B(u,w)$, Let $x = u$ or v se get $B(w,v) = B(v,w) = 0 = B(w,u) = B(u,w)$. Now $B(u, w+v) \neq B(w+v, u)$, hence $B(w+v, w+v) = 0 = B(w,w)$, contradiction. □

Inner Space

Prop. (2.4.6). For an inner metric on a metric space, it will induce an inner metric on the dual space, that is, asserting the dual basis of an orthonormal basis to be orthonormal. On an arbitrary basis, the matrix on the dual basis is written as A^{-1}. because we can write $A = P^tP$, and the dual basis transformation is like $(P^t)^{-1}$, so the metric matrix is A^{-1}.

Prop. (2.4.7).

5 Determinant

Prop. (2.5.1) (Sylvester's determinant identity). If A and B are matrices of sizes $m\times n$ and $n\times m$, then
$$\det(I_m+AB)=\det(I_n+BA)$$

Proof:
$$\begin{bmatrix}1 & \mathrm{A}\\ \mathrm{B} & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ \mathrm{B} & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1-\mathrm{BA}\end{bmatrix}\begin{bmatrix}1 & \mathrm{A}\\ 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}1 & \mathrm{A}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1-\mathrm{AB} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \mathrm{B} & 1\end{bmatrix}$$
□

Prop. (2.5.2). The determinant of a symplectic matrix $\in Sp(n)$ has determinant 1.

Proof: A symplectic matrix preserves the symplectic structure thus the symplectic form ω, hence ω^n which is $n!$ times the volume form. □

Prop. (2.5.3). $GL_n(\mathbb{C})$ can be embedded into $GL_{2n}(\mathbb{R})$, with determinant $|\det|^2$. And in this way, $U(n)$ is mapped into $O(2n)$. Also, $O(n)$ embeds into $U(n)$ diagonally.

Proof:
$$X+iY\mapsto\begin{bmatrix}X & Y\\ -Y & X\end{bmatrix}\sim\begin{bmatrix}X & Y\\ iX-Y & X+iY\end{bmatrix}\sim\begin{bmatrix}X-iY & Y\\ 0 & X+iY\end{bmatrix}$$
□

Prop. (2.5.4). There is a polynomial Pf s.t. $\det M=\mathrm{Pf}(M)^2$ for a skew-symmetric matrix. This is because a skew symmetric is equal to $A^t\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}^k A$ for A an orthogonal matrix (2.4.1), so it has determinant $(\det A)^2$ and A and depends polynomially on the entries of M.

Cor. (2.5.5).
$$\mathrm{Pf}(A^tMA)=\det A\cdot\mathrm{Pf}(M).$$
Because we only need to consider the sign and it is determined by letting $A=\mathrm{id}$.

6 Minimal and Characteristic Poynomial

Prop. (2.6.1). The linear functor $X\to AX-XC$ is an isomorphism iff the minimal polynomial of A and C has not common factor.

Proof: Notice if $AX=XC$, then we have $P(A)X=XP(C)$ for every polynomial P, in, particular for the minimal polynomials of A and C, thus $P(C)$ is non-invertible and A,C has a characteristic value in common. Conversely, if they have a characteristic value, then we upper trangularize A to see clearly that there is a X that $AX=XC$ (X has only the first row). □

7 Spectral Theory

See also 8

Prop. (2.7.1). a family of commuting diagonalizable operator can be simultaneously diagonalized.

Proof: □

Prop. (2.7.2). in an algebraically closed field, diagonalizable $\iff$ normal. And the eigenvectors are orthogonal to each other.

Proof: □

8 Decompositions

Prop. (2.8.1) (Polar Decomposition). $GL_n(\mathbb{R})$ can be decomposed as $P \cdot O(n)$, where P is a positive symmetric matrix and $O(n)$ the orthogonal matrix. a positive symmetric matrix can be diagonalized, so $GL_n(\mathbb{R})$ have $O(n)$ as deformation kernel.

Similarly, $Sp(2n)$ can be decomposed as $P \cdot U(n)$, because $O(2n) \cap Sp(2n) = U(n)$. And it has $U(n)$ as deformation kernel.

Prop. (2.8.2) (Bruhat Decomposition).

$$GL_n[K] = BWB$$

where W is permutation matrix, B is upper triangular, and the decomposition is a disjoint union。

Proof: Cf.[群与表示 王立中] □

Prop. (2.8.3) (Iwasawa Decomposition).

Positivity

Prop. (2.8.4) (Farkas' Lemma). For a matrix A, and a vector b, exactly one of the following equation has a solution:

$$\begin{cases} AX = b, X \geq 0 \\ Y^t A \leq 0, Y^t b > 0 \end{cases}$$

Proof: First notice if both have a solution, then $0 \geq Y^t AX > 0$, contradiction. The rest follows form the Hahn-Banach separation theorem. □

Cor. (2.8.5) (Gordan's Theorem). exactly one of the following has a solution:

$$\begin{cases} AX > 0 \\ Y^t A = 0, Y \geq 0, Y \neq 0 \end{cases}$$

Proof: If both have a solution, then $0 = Y^t AX > 0$, contradiction. If the first has no solution, then $A'x = e, z \geq 0$, where $A' = [A, -A, -I]$ has no solution, by Farkas' lemma, there is a solution of $Y^t A' \leq 0$ and $Y^t b = 0$. Which shows that $Y^t A = 0$ and $Y \neq 0$. □

Cor. (2.8.6). For any subspace in $\mathbb{R}^m$, either it has an intersection with the open first quadrant, or its orthogonal complement has an intersection with the closed first quadrant minus 0. (Regard it has the image of a AX).

9 Miscellaneous

I.3 Abstract Algebra

References are [Algebra Lang] and [Finite Groups Issac].

1 Group Theory

Prop. (3.1.1) (Fundamental Isomorphism). For a normal subgroup U of G, $G/HU \cong (G/U)/(H/H\cap U)$.

Free Groups

Prop. (3.1.2) (Nielsen-Schreier). A subgroup of a free group is a free group. Moreover, a subgroup of index m in a free group on n generators is a free group on $1+m(n-1)$ generators.

Proof: A free group is the fundamental group of a wedge sum of circles, and a cover of it is a connected 1-graph. Now the graph has a maximal tree and module the tree gets us a wedge sum of circle. The second statement follows by two ways of counting Euler number χ. □

Sylow Theory

Prop. (3.1.3) (Sylow Theorem). For a finite group of order $|G| = p^k m$.

- There is a Sylow p-group.
- all Sylow p-groups are conjugate.
- the number of Sylow p-groups n_p satisfies: $n_p|m, n_p \equiv 1 \mod p$.

Proof: Consider the action of G on the family of sets of G of order p^k, then we have: the orbits of order prime to p correspond to Sylow groups of G via: $P \mapsto$ left cosets of P. For the third assertion, notice □

Cor. (3.1.4). If $p||G|$, then G has an element of order p.

Split Extension

Prop. (3.1.5). If there is an exact sequence $0 \to Z \to G \to C \to 0$ where $Z \subset C(G)$ and C is cyclic, then G is Abelian. (This is because we can choose an inverse image of a generator of C).

Prop. (3.1.6) (Schur-Zassenhaus). An exact sequence of finite groups $0 \to A \to E \to G \to 0$ must split when $|A|$ and $|G|$ are relatively prime.

Subnormality

Prop. (3.1.7). If a finite group $|G| = \prod p_i$, where p_i are different primes that $\prod p_i$ and $\prod(p_i - 1)$ are coprime, then G is cyclic.

Proof: We prove all the Sylow groups are normal. Choose the maximal Sylow group A_n, then it is normal by Sylow theorem, and other Sylow groups act by conjugation is trivial, hence A_n is in the center. Now consider the quotient, by induction it is cyclic, hence this is a central extension of a cyclic group, hence G is Abelian(3.1.5), so cyclic. □

Commutators

Def. (3.1.8) (Notation).

- $[a, b] = a^{-1}b^{-1}ab$.
- $x^y = y^{-1}xy$.

Prop. (3.1.9). Commutator relation.

Def. (3.1.10). A **metabelian** group is a group G that G' is Abelian.

Prop. (3.1.11). If $G = AB$ where A, B are Abelian, then $[G, G] = [A, B]$ and G is metabelian.

Proof: The first one is easy to verify, the second because if we let $b^{a_1} = a_2b_2$, $a^{b_1} = b_3a_3$, then

$$[a, b]^{a_1b_1} = [a, b^{a_1}]^{b_1} = [a, b_2]^{b_1} = [a^{b_1}, b_2] = [a_3, b_2]$$

and similarly, $[a, b]^{b_1a_1} = [a_3, b_2]$, so we have $[a, b]$ commutes with $[b_1^{-1}, a_1^{-1}]$, which shows $[A, B]$ is Abelian. □

Prop. (3.1.12). If G is a metabelian finite group, then the transfer of $Ver : G \to G'$ is trivial map.

Transfer

Permutation Groups

2 Polynomials

Prop. (3.2.1) (Descartes's Rule of Sign). Let $p(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ be a real polynomial with nonzero a_i, where $A_0 < B_1 < \cdots < b_n$, then the number of positive roots of $p(x)$ is the number of changing signs of $\{a_n\}$ minus $2k$.

Proof: Lemma: when $a_0a_n > 0$, the number of positive roots are even and when $a_0a_n < 0$, it is odd. This is seen by consider $p(0)$ and $p(\infty)$.

Then we consider the derivative p' and use induction. Denote the number of changing sign by $v(p)$ and the number of positive roots by $z(p)$, then if $z_0a_1 > 0$, then $v(p) = v(p')$ and $z(p) \equiv z(p') \mod 2$. Then we have $z(p) \equiv v(p) \mod 2$ and middle value theorem shows that $z(p') \le z(p) - 1$, hence by induction and parity argument, we have $v(p) \ge z(p)$.

If $a_0a_1 < 0$, then the same method shows that $v(p) = v(p') + 1 \ge z(p') + 1 \ge z(p')$ and the have the same parity by the lemma. □

Prop. (3.2.2) (Lagrange Interpolation).

Prop. (3.2.3) (Newton Polynomial).

Prop. (3.2.4) (Eisenstein Polynomial).

Resultant

Def. (3.2.5). Over a commutative ring R, the **resultant** $res(A, B)$ of two polynomials A, B of degree d, e respectively is the determinant of the map $W_e \times W_d \to W_{d+e}$ that $(X, Y) \mapsto AX + BY$, where W_t is the free module of polynomials of degree $< t$.

Prop. (3.2.6). The resultant can be seen as the determinant of the matrix with values the coefficient of A or B in different places, multiplying X^*s with different degree and add to the last row, we can get $A \cdot X^*$s and $B \cdot X^*$s, so: $res(A, B) = AC + BD$ for some C, D.

Now if $R \subset S$ and A, B has common roots in S, then $\mathrm{res}(A, B) = 0$.

Cor. (3.2.7). res is stable under Euclidean division, so it can be seen as a suitable division remainder of the two polynomial.

Prop. (3.2.8). When $R \subset L$ a field and A, B decompose into linear factors in L, let t_i be roots of A and u_j be roots of B, then

$$\mathrm{res}(A, B) = v_0^d w_0^e \prod_{i=1}^{d} \prod_{j=0}^{e} (t_i - u_j)$$

Proof: See the resultant as polynomials of the roots of A and B, then we proved that if they has the same root, then $res = 0$, so it is divisible by $(t_i - u_j)$ for all i, j. Then notice the RHS is homogenous of degree d in u_j and homogenous of degree e in t_i, so does res. So they are equal. □

3 Ring Theory

Def. (3.3.1). A **division ring** is a commutative ring that any nonzero element is invertible.

Bezout Domain

Def. (3.3.2). A **Bezout domain** is an integral domain that any sum of two principal domains is also principal.

Prop. (3.3.3). The localization of a Bezout domain is Bezout. A local ring is Bezout iff it is a valuation ring(by(6.1.9)).

Prop. (3.3.4). Any finite submodule of a free module over a Bezout ring is finite free, Cf.[StackProject 0ASU].

Prop. (3.3.5). Finite locally free R-module is free.

Prop. (3.3.6). torsion free R-module is flat.

Prop. (3.3.7). finitely presented projective R-module is free.

Principal Ideal Domain

Prop. (3.3.8). Any submodule of a free module over a PID is free. Thus a projective module over a PID is free.

Proof: Choose a well ordering on the basis of F, let F_i is the submodule generated by $e_j, j \leq i$. Then $\pi_i(P \cap F_i) \subset R$ is a $a_i R$, thus choose u_i that $p_i(u_i) = a_i$, we have a_i constitute a basis by transfinite induction. □

Prop. (3.3.9) (Classification of Modules over PID).

1) PID is UFD thus Noetherian.
2) Submodule of a free module over a PID is free.
3) Finitely generated torsion-free module over a PID is free.
4) Finitely generated module over a PID has a primary decomposition $M = \bigoplus_i R/(q_i)$,where (q_i) is primary ideals.

So pojective $\iff$ free $\iff$ torsion-free(when f.g.).

Proof: Cf.[Lang P45] □

UFD

Prop. (3.3.10). A Noetherian domain is UFD iff all minimal primes are principal.

Prop. (3.3.11) (Gauss Lemma).

4 Module Theory

Prop. (3.4.1). For an endomorphism T of a R module M, if we denote p the minimal integer that $R(T^p) = R(T^{p+1})$ and q the minimal integer that $N(T^q) = N(T^{q+1})$. Then the morphisms are stable afterward. Then if there is a m, n that $R(T^m) \oplus N(T^n) = X$ for a R-module endomorphism $T \in \mathrm{End}(M)$, then $p, q < \infty$ and they are equal. Moreover, if we know $p, q < \infty$, then we have $R(T^p) \oplus N(T^q) = M$.

Proof: We notice that

$$T^i : N(T^{i+j})/N(T^i) \to R(T^i) \cap N(T^j), \quad T^i : M/(R(T^j) + N(T^i)) \to R(T^i)/R(T^{i+j})$$

are isomorphisms. Thus $R(T^m) \oplus N(T^n) = X$ shows $q \leq m$ and $p \leq n$, thus we have $R(T^p) \oplus N(T^q) = M$, which implies $p \geq q$ and $q \geq p$. thus the result. The rest also follows easily from these isomorphisms. □

Prop. (3.4.2). For a f.g. module over a Noetherian ring, if an endomorphism is surjective, then it is injective.

Proof: The kernel $\mathrm{Ker}(\varphi^i)$ stablize, thus there is a $\mathrm{Ker}(\varphi^i) = \mathrm{Ker}(\varphi^{2i}) \to \mathrm{Ker}(\varphi^i)$ that is also surjective, and it is also zero, thus φ is injective. □

Prop. (3.4.3) (Induced & Coinduced). Given a ring homomorphism $S \to$R.

- $f^*M = M_S$, the restriction.
- $f_!M = M \otimes_S R$ is the induced module, it is left adjoint to restriction.
- $f_*M = \mathrm{Hom}_S(R, M)$ is the coinduced module, it is right adjoint to restriction. (It is a R-mod by $s(f)(t) = f(ts)$.)

Prop. (3.4.4) (Nakayama).

Proof: □

Cor. (3.4.5). If a finite R-module M satisfies $M \otimes_R k(p) = 0$, then there is a $f \notin p$ that $M_f = 0$.

Proof: Because we know that $M_p = 0$, and the support of M is closed(finiteness used). □

Prop. (3.4.6) (Jordan-Horder).

5 Field Theory

Field Extensions

Def. (3.5.1). A family of extensions are called **distinguished** iff it is closed under base change and $k \subset F \subset E$ is separable iff $k \subset F$ and $F \subset E$ are both separable.

Prop. (3.5.2). The family of finite extensions form a distinguished class.
The family of algebraic extensions form a distinguished class.
The family of f.g. extensions form a distinguished class.

Proof: Finite case is trivial. For the alg. extensions, for $k \subset F \subset E$, for any $\alpha \in E$, α satisfies an polynomial function with f.m coefficients in F, the coefficients form a subfield F_0 of F which is finite over k, so $k \subset F_0 \subset F_0(\alpha)$ is a finite tower, so it is finite, hence algebraic. The base change is easy to check.

For f.g. extensions, it suffice to check composition: □

Prop. (3.5.3). For an alg.extension $k \subset E$, any injective field map $E \to E$ over k is an automorphism. (This is because it induce a permutation of any α with its conjugates in E, so it is surjective).

Prop. (3.5.4). For any field K. There exists uniquely an alg.closed field K/k under isomorphism over k.

Proof: Cf.[Algebra Lang P231]. □

Separable Normal & Galois Extensions

Def. (3.5.5). An extension K/k is called **normal extension** iff it satisfied the following equivalent conditions:

- Any embedding of K into k^{alg} induce an automorphism on K.
- K is the splitting field of a family of polynomials in $k[X]$.
- Every irreducible polynomial in $k[X]$ that has a root in K splits in K.

Proof: Cf.[Algebra Lang P237]. □

Prop. (3.5.6). Normal extension are stable under base change and forms a lattice, this is immediate from the first definition of(3.5.5).

Def. (3.5.7). If we define the **separable degree** $[E : k]_s$ of an extension E/k as the number of embedding into a fixed alg.closure, then it commutes with composition and when E/k is finite, $[E : k]_s \le [E : k]$.

Def. (3.5.8). A finite extension is called **separable** iff $[E:k]_s = [E:k]$, an algebraic number α over k is called **separable** iff $k(\alpha)/k$ is separable. A polynomial $f \in k[X]$ is called **separable** iff it has no multiple roots in k^{alg}.

Def. (3.5.9). An extension E/k is called **separable** iff it satisfies the following equivalent conditions:

- every f.g. subfield is separable over k,(this is compatible because subfield of a finite separable extension is separable, by the compatibility of separable degree).
- Every element of E is separable.
- It is generated by a family of separable elements.

Proof: If E/k is separable and $k \subset k(\alpha) \subset E$, then by(3.5.8), $k(\alpha)$ is separable. And if it is generated by a family of separable elements $\{\alpha_i\}$, then any f.g. subfield can are f.g. by elements $\{\alpha_i\}$. Now it is a tower of separable extensions, hence separable by the compatibility of separable degree. □

Prop. (3.5.10). Separable extensions form a distinguished class.

Proof: Cf.[Algebra Lang P241]. □

Prop. (3.5.11) (Primitive element Theorem). A finite extension E/k is primitive iff there are only f.m. middle fields. And if E/k is separable, this is satisfied.

Proof: If k is finite, this is simple. Assume k infinite, for any two elements α, β, consider $k(\alpha + c_i\beta)$, if there is only finitely many middle fields, there exists two that is equal, so $k(\alpha, \beta) = k(\gamma)$. Proceeding inductively, E is primitive.

Conversely, if $k(\alpha) = E$, every middle field corresponds to a divisor of the irreducible polynomial of α. This map is injective, because for any g_F, degree of α over F is the same over the degree over the coefficient field of g_F, so it must be equal to F.

If E/k is separable, Let

$$P(X) = \prod_{i \neq j} (\sigma_i\alpha + X\sigma_i\beta - \sigma_j\alpha - X\sigma_j\beta)$$

for different embedding σ_i, σ_j of $E(\alpha, \beta)$ into k^{alg}. Then it is not identically zero, thus there exists c that $\sigma_i(\alpha + c\beta)$ is all distinct, thus generate $K(\alpha, \beta)$. □

Prop. (3.5.12). Automorphisms of a field L are linearly independent over L.

Inseparable Extensions

Prop. (3.5.13). Any irreducible polynomial of fields of characteristic 0 is separable and if char$= p$, then all roots have the same multiplicity and thus $[k(\alpha):k] = p^n[k(\alpha):k]$ for some n.

Proof: All roots have the same multiplicity because there are Galois actions. If the multiplicity is not 1, the derivative f' is zero, otherwise f is not irreducible. Then $f(X) = g(X^p)$. We can choose $f(X) = h(X^{p^n})$ with h separable, then $[k(\alpha):k(\alpha^{p^n})] = p^n$, thus the result. □

Def. (3.5.14). The **inseparable degree** $[E:k]_i$ is defined as the quotient $[E:k]/[E:k]_s$. An algebraic element α is called **purely inseparable** over k iff there is a n that $\alpha^{p^n} \in k$.

Def. (3.5.15). An extension is called **purely inseparable** if it satisfies the following equivalent conditions:

- $[E:k]_s = 1$.
- Every element α of E is purely inseparable over k.
- For every $\alpha \in E$, the irreducible equation of α over k is of type $X^{p^n} - a$.
- It is generated by a family of purely inseparable elements.

Proof: Cf.[ALgebra Lang P249]. □

Prop. (3.5.16). Purely Inseparably extensions form a distinguished class.

Proof: Cf.[Algebra Lang P250]. □

Prop. (3.5.17). If E/k is algebraic and E_0 be the maximal separable extension contained in E, then E/E_0 is purely inseparable. And if E/k is normal, then E_0/k is normal, too.

Proof:
By the proof of(3.5.13), any α has a p^n that α^{p^n} that α^{p^n} is separable, hence it is purely inseparable over E_0 by(3.5.15). E_0/k is normal because any σ maps E to itself, and E_0 to $\sigma(E_0) \in E$ separable, hence $\sigma(E_0) \subset E_0$. □

Def. (3.5.18). A field is called **perfect** iff there are no purely inseparable extensions of it. From the third definition of(3.5.15), this is equivalent to $x \to x^p$ is an automorphism of K, where p is the characteristic of K.

Prop. (3.5.19). For any field k of char p, there is a unique purely inseparable field extension k^{perf}/k that k^{perf} is perfect, called the **perfect closure** of k. It is generated by adding all the p^n-th roots to k.

6 Transcendental extension

Prop. (3.6.1). Let K be an extension of a field k, a **transcendental base** is an algebraically independent set that any element is algebraic over it. Then the number of elements in any algebraically independent set $\leq$ the number of elements in any transcendental base. In particular, given any algebraically independent set $S \subset T$ a set over which K is algebraic, S can be extended to a transcendental base.

Proof: Let $X = \{x_1, \ldots, x_m\}$ transcendental base of minimal number, $S = \{w_1, \ldots, w_n\}$ an algebraically independent set. If $n > m$, we proceed by changing one element a time using induction and prove that K is algebraic over $\{w_1, \ldots, w_r, x_{r+1}, \ldots, x_m\}$, contradiction.

Because w_{r+1} is algebraic over $\{w_1, \ldots, w_r, x_{r+1}, \ldots, x_n\}$, we have a minimal polynomial

$$f = \sum g_j(w_{r+1}, w_1, \ldots, w_r, x_{r+2}, \ldots, x_m) x_{r+1}^j$$

s.t. $f(w_{r+1}, w_1, \ldots, x_m) = 0$ (after possibly renumbering x_i, this x must exists because S is itself algebraically independent). So x_r is algebraic over $\{w_1, \ldots, w_{r+1}, x_{r+2}, \ldots, x_m\}$, hence K is independent over it, too. □

7 Galois Theory

Prop. (3.7.1) (Artin). If G is a monoid and K is a field, any distinct characters of G in K are linearly independent over K.

Proof: Consider the minimal length of linear combination that is 0, then we substitute a suitable z in it, then we can can cancel a character, contradicting the minimality. □

Cor. (3.7.2). If α_i are different elements in K and there are element a_i that $\sum a_i\alpha_i^v = 0$ for every $v \geq 0$, then $a_i = 0$ for all n. (Seen as characters from $\mathbb{Z}_{\geq 0} \to K$).

Prop. (3.7.3) (Artin Algebraic Independence). Let K be an infinite field and σ_i be distinct elements of a finite group of automorphisms of K, then σ_i are alg.indepent over K.

Proof: Cf.[Algebra Lang P311]. □

Prop. (3.7.4) ((Artin)Galois Main Theorem). Let G be a finite group of automorphisms of K. Then K/K^G is Galois of Galois group G.

Proof: For every element x, set $\{\sigma_1 x, \ldots, \sigma_r x\}$ be distinct conjugates, then $f(X) = \prod_i^r = (X - \sigma_i x)$ shows that K is separable and normal over K^G. And primitive element theorem shows that $[K : K^G] \leq |G|$, so it must equals G. □

Prop. (3.7.5) (Infinite Galois Theorem). The middle fields correspond to the closed subgp of $G(L/K)$.

Proof: The highlight is that $G(L/L^H) = H$ for a closed subgp H of $G(L/K)$. If σ fixes L^H but is not in H, because for every finite field M, $H \cdot G(L/M)$ corresponds to $M/(M \cap L^H)$, so $\sigma G(L/M) \cap H \neq \varnothing$. So σ is in the closure of H thus in H. □

Prop. (3.7.6) (Normal Basis Theorem). For a finite Galois extension, normal basis exists.

Proof: Finite case: The Galois group is cyclic, and the linear independent of characters shows that the minimal polynomial of σ is n-dimensional thus equals $X^n - 1$. Regard L as a $K[X]$ module thus by (3.3.9) is a direct sum of modules of the form $K[X]/(f(x))$, $f(x)|X^n - 1$ and the minimal polynomial for the action of X is $X^n - 1$. So it must be isomorphic to $K(X)/(X^n - 1)$.

Infinite Case: Let
$$f([X_\sigma]) = \det(t_{\sigma_i,\sigma_j}), \qquad t_{\sigma,\tau} = X_{\sigma^{-1}\tau}$$
We see $f \neq 0$ by substituting 1 for X_{id} and 0 otherwise. So it won't vanish for all x if we substitute $X_\sigma = \sigma(x)$ because $[\sigma(x)]$ is pairwisely different. Thus there exists w s.t.
$$\det(\sigma^{-1}\tau(w)) \neq 0.$$
Now if
$$\sum a_\tau \tau(w) = 0, \qquad a_\sigma \in K,$$
act by σ for all σ, we get $[\sigma^{-1}\tau(w)][a_\sigma] = 0$, thus $[a_\sigma] = 0$. □

Prop. (3.7.7) (Kummer Theory). Let K be a field containing the n-th roots of unity, a **Kummer extension** L/K of order n is one that the Galois group is Abelian and of exponent n. There exists an inclusion preserving isomorphism between the lattice of Kummer extensions L of K and the lattice of subgroups of L containing K^n:
$$L \mapsto \Delta = (L^\times)^n \cap K^\times, \qquad \Delta \mapsto K(\sqrt[n]{\Delta}).$$
And $\Delta/(K^\times)^n$ is isomorphic to $\mathrm{Hom}_{cont}(G_{L|K}, \mathbb{Q}/\mathbb{Z})$.

Proof: Notice the composite of two Kummer extension is an extension, so we consider the maximal Kummer extension L, then $K^* \subset (L^*)^n$, because otherwise we can add a $\sqrt[n]{a}$, this is another Kummer extension.

We use the exact sequence $1 \to \mu_n \to L^* \xrightarrow{n} (L^*)^n \to 0$, then the profinite cohomology exact sequence says

$$1 \to K^* \to (L^*)^n \cap K^* \xrightarrow{\delta} H^1_{cts}(G, \mu_n) \to H^1_{cts}(G, L^*) = 1$$

And G acts trivially on $\mu_n \subset K^*$, then $H^1_{cts}(G, \mu_n) = \mathrm{Hom}_{cts}(G_{L|K}, \mathbb{Q}/\mathbb{Z})$. δ maps $a \mapsto \chi_a(\sigma) = \sigma(\sqrt[n]{a})/\sqrt[n]{a} \in \mu_n$.

Thus if we let L be the maximal Kummer extension, then by Galois theory, Kummer extensions of K corresponds to closed subgroups of G, subgroups of $\mathrm{Hom}_{cts}(G, \mathbb{Q}/\mathbb{Z})$ correspond to subgroups of $K^*/(K^*)^n$. These two correspond by the intersection of the kernel of all them, because they correspond for finite subgroup and open subgroup. And closed subgroups are intersection of open subgroups, and any open subgroup containing it must contain an open subgroup of chosen form, by compactness. Thus they correspond. □

Prop. (3.7.8). $\mathrm{Gal}(F_{p^n}/F_p) = \mathbb{Z}/n\mathbb{Z}$. and is generated by the Frobenius.

Brauer Groups

Prop. (3.7.9). The **Brauer group** $\mathrm{Br}(K)$ is defined as the profinite cohomology $H^2(G(K_s/K), K_s^*)$. For a Galois extension L/K, $\mathrm{Br}(L/K)$ is defined as $H^2(G(L/K), L^*)$. Then by(3.4.2) we have

$$\mathrm{colim}\, \mathrm{Br}(L/K) = \mathrm{Br}(K).$$

Cf.[Neukirch Cohomology of Number Fields Chap6.3].

Prop. (3.7.10). The **Brauer group** $\mathrm{Br}(K)$ is defined as the profinite cohomology $H^2(G(K_s/K), K_s^*)$. For a Galois extension L/K, $\mathrm{Br}(L/K)$ is defined as $H^2(G(L/K), L^*)$. Then by(3.4.2) we have

$$\mathrm{colim}\, \mathrm{Br}(L/K) = \mathrm{Br}(K).$$

And by Hochschild-Serre spectral sequence and by Hilbert's multiplicative theorem90:$H^1(H, K_s^*) = 0$, we have the low term:

$$0 \to \mathrm{Br}(L/K) \xrightarrow{inf} \mathrm{Br}(K) \xrightarrow{res} \mathrm{Br}(L)^G \to H^3(G(L/K), L^*) \to H^3(K, K_s^*).$$

So $\mathrm{Br}(L/K)$ is the kernel of $\mathrm{Br}(K) \to \mathrm{Br}(L)$.

Cor. (3.7.11). $\mathrm{Br}(\mathbb{F}_q)$=0 for finite fields, because the finite Galois extension are cyclic and unramified.

In fact, the Brauer group has semisimple algebraic interpretations. Cf.[Milne].

8 Invariant Theory

Prop. (3.8.1). Any symmetric polynomial is a polynomial of the fundamental symmetric polynomials.

Proof: Set the first coordinate to 0, then the rest is a polynomial of the fundamental symmetric polynomials by induction, then we have $f = a + x_1 b$, thus $x_1 \dots x_n | x_1 b$, thus use induction. □

Prop. (3.8.2). Any polynomial on the entries of matrixes $M_n(k)$ that is invariant under conjugation is generated by coefficients of $\det(\lambda I + X)$ and can also be generated by $\mathrm{tr}(X^k)$.

Proof: We notice that the matrixes having disjoint eigenvalues is dense in $M_n(k)$, thus the restriction of the polynomial on these matrixes is a symmetric polynomial(3.8.1) thus identical to a polynomial described above. Hence they are equal. □

Prop. (3.8.3). For any polynomial on the entries of matrixes $M_n(k)$ that $f(BA) = f(A)$ for $B \in O(n)$, there is a polynomial F that $f(A) = F(A^*A)$. Cf.[Heat Equation and the Index Theorem Atiyah P323].

Prop. (3.8.4) (Weyl). Any linear map f from $(\mathbb{R}^m)^{\otimes n}$ to R that is $O(m)$-equivariant is a linear combinations of maps of the form:

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \langle v_{i_1}, v_{i_2}\rangle\langle v_{i_3}, v_{i_4}\rangle \cdots \langle v_{n-1}, v_n\rangle.$$

Where $i_1, \ldots, i_n$ is a permutation of $1, 2, \cdots, n$ when n is even and when n is odd, f must be 0.

Proof: Cf,[Heat Equation and the Index Theorem]. □

I.4 Representation Theory

1 Semisimple Algebras

Basic references are [StackProject Chap11] and [Algebra Lang Chap17].

Def. (4.1.1). A R-module E is called **simple** iff it has no submodules other than 0 and E.

Prop. (4.1.2) (Shur's lemma). For a simple module E, $\mathrm{End}_R(E)$ is a division ring, this is because the kernel and image are all 0 or E.

Def. (4.1.3). A module E is called **semisimple** iff it satisfies the following equivalent conditions:

- It is a sum of simple modules.
- It is a direct sum of simple modules.
- Any submodule F of E has a complement in E.

Proof: Cf.[Algebra Lang P645]. □

Cor. (4.1.4). Any submodule and quotient module of a semisimple module is semisimple. This is because any submodule is a direct sum of simple modules contained in it, and it has a complement.

Def. (4.1.5). A ring is called **semisimple** iff it is a semisimple module over itself, it is called **simple** iff it is simple module over itself.

2 Linear Representation of Finite Groups

Basic references are [Serre Linear Representations of Finite Groups].

Results for Arbitrary Groups

Prop. (4.2.1). Any f.g.(i.e. f.g. over $F[G]$) representation of a group has an irreducible quotient.

Proof: Use Zorn's lemma for the set of proper G-subspaces of U, the combine of a chain of proper G-space is proper, because it is f.g., so it has a maximal proper G-space, so the quotient is irreducible. □

Prop. (4.2.2) (Schur's lemma). If π is a countable dimensional irreducible $\mathbb{C}$-representation, then $\mathrm{End}(V)\cong\mathbb{C}$.

Proof: Notice we only have to find an eigenvalue of ϕ, but otherwise $\{(\varphi-a)^{-1}\}$ is uncountable and linearly independent over $\mathbb{C}$, so $\dim(\mathbb{C}(\varphi))$ is uncountable, contradiction. □

Prop. (4.2.3). By(3.4.3), the induced and coinduced representation is that of $\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}-$ and $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G],-)$. If $[G:H]$ is finite, then induced is the same as coinduced.

Proof: Choose a left coset representation of H, then check $x\otimes a\to f: hx^{-1}\mapsto ha$ is an isomorphism Cf.[Weibel P172]. □

Finite Case

Prop. (4.2.4). If G is a finite p-group and A is a nonzero p-torsion G-module, then $A^G \neq 0$.

Proof: We may consider A generated by a single element. Because A is p-torsion, $|A| = p^n$ for some n. Now consider the orbit, then if the orbit is not a single element, then its order is divisible by p, so $|A^G|$ is divisible by p. But 0 is fixed, so $A^G \neq 0$. □

Prop. (4.2.5) (Maschke's Theorem). If F is a field of char p and G is a finite group of order prime to p, then for any representation U of $F[G]$ and a submodule V, there exists a complement of V in U.

Proof: Choose an arbitrary projection π of U to V, and let $\rho(v) = 1/|G| \sum g^{-1}\pi(g(v))$, then it can be checked ρ commutes with G-actions, thus its kernel is also a G-modules, and it is identity on V, so $U = V \oplus k \operatorname{Ker} \rho$. □

3 Locally Compact Groups

Prop. (4.3.1) (Brauer-Nesbitt). For a finite group G, if two finite dimensional semisimple representations over a field has the same char poly for every element g of G, then they are isomorphic.

Proof: Just use the irreducible representations are orthogonal and that they have the same and for char p, we can use divide by p and the char poly becomes p-th power and we can do this forever, contradiction. □

Compact Groups

Prop. (4.3.2).

Prop. (4.3.3) (Peter Weyl). For a compact group G, $\{\phi_{ij}(g); \phi(g) = (\phi_{ij}(g)), \phi$ an irreducible character$\}$ is a basis for the Hilbert space $L_2(G)$. Cf.[连续群 Pontryagin 第五章 § 33].

4 Locally Profinite Groups

Def. (4.4.1). A smooth representation (π, V) of a locally profinite group G is called **semisimple** iff it satisfies the following equivalent conditions:

- It is the sum of its irreducible G-subspaces.
- It is a direct sum of irreducible G-representations.
- Any G-subspace of V has a complement in V.

Proof: □

Prop. (4.4.2). If G is profinite, then Any smooth representation is semisimple, because for any element v, G_v is compact open thus of finite index in G, then the orbit of v is finite and some open normal subgroup H of G_v fixes all the orbits of v, so it is a finite representation of the finite group G/H, hence by Mackey's theorem(4.2.5), it is a finite sum of irreducible representations.

So if G is locally profinite, then any G-representation is K-semisimple for compact open subgroup K of G.

Def. (4.4.3). A representation is called **admissible** iff for any compact open subgroup K of G, V^K is a finite G-module.

Prop. (4.4.4). A smooth irreducible representation is admissible. In fact, this is true for general connected reductive group.

Proof:

□

I.5 Commutative Algebra(Matsumura)

Also referenced [Weibel Homologocal Algebra Ch4].

1 Atiyah Level Stuff

Localization

Prop. (5.1.1) (Localization is exact). S^{-1} is an exact functor from $R-\text{mod}$ to $R-\text{mod}$. Because it is a filtered colimit,(7.1.32).

Cor. (5.1.2). $(R/I)_{\bar{P}} \cong R_P/IR_P$, in particular, $k(R/P) \cong R_P/PR_P$.

Prop. (5.1.3). Let k be a field, then the power series $k[[X_1,\ldots,X_n]]$ is a UFD.

Proof: Cf.[Algebra Lang P209]. □

Prop. (5.1.4) (Prime Avoidance). If finitely many primes cover an ideal, then one of them cover it.

Proof: Assume otherwise, use induction. For two primes, use $x+y$, for r primes, choose $x \notin p_i, i<r$, then $x \in p_r$, and choose $y \in JI_1\ldots I_{r-1}$ and $y \notin y_r$, then $x+y$ suffice. □

Def. (5.1.5). A map between two local rings are called **local ring map** iff it maps non-invertible elements to non-invertible elements, equivalently, $f^{-1}(\mathfrak{m}_S)=\mathfrak{m}_R$.

Noetherian

Prop. (5.1.6). Subring, quotient ring, finitely generated module, localization and power series are Noetherian, hence graded algebra of a A by an ideal I is Noetherian.

Proof: Only need to prove $A[X]$ and $A[[X]]$. For an ascending chain of ideal I_j of $A[X]$, we consider the coefficients ideal $I_{i,j}$ of X^i of I_j, then there are only f.m. different $I_{i,j}$s, so we have I_j stabilize as well.

Similarly for $A[[X]]$, we prove any ideal I is f.g. Consider the lowest terms coefficient ideal at degree i, then it is ascending and stabilize, then a set of generators as a whole generate I.

□

Prop. (5.1.7). When A is Noetherian and is quipped with I-adic topology, then I is f.g., and there is surjective ring map $A[[X]] \to A^*$ the completion, mapping to the generators of I, hence the completion is Noetherian. (It is surjective can be seen by the Cauchy sequence construction of completion).

Prop. (5.1.8). If R is Noetherian and M is a f.g. R-module, then there is a filtration $\{M_i\}$ of M that the quotients are all isomorphic to $R_{\mathfrak{p}_i}$ where $\mathfrak{p}_i$ are primes.

Proof: M is generated by x_i, so $(x_1) \cong R/I_i$, and so we modulo x_i, then the result follows by induction. So we may assume $M=R/I$. We use Noetherian condition to choose a maximal element J that is a counterexample, then J is not a prime, so there are $a,b \notin J$ that $ab \in J$. Then we have a filtration $0 \subset aR/(J\cap aR) \subset R/J$. Notice $R/(J+bR) \to aR/(J\cap aR) \to 0$, and the second quotient is $R/(J+aR)$, so they all can be factorized. □

Artinian Ring

Prop. (5.1.9). A ring A is called **Artinian** iff the length of A as a A-module is finite. This is equivalent to it is Noetherian of dimension 0.

So a f.d algebra over a field is Artinian. Artinian ring has finitely many primes.

Proof: Cf.[StackProject 00KH]. □

Prop. (5.1.10). An Artinian ring is a direct sum of Noetherian primary rings and the decomposition is unique.

Proof: Take a primary decomposition to notice that 0 is a product of maximal ideals, (because of artinian). Take $R_i = \prod_{j\neq i} \mathfrak{m}^{e_i}$ then:

$$R \cong \bigoplus R/R_i, \qquad R_i \cong R/\mathfrak{m}^{e_i}$$

Notice R_i and $\mathfrak{m}^{e_i}$ coprime and nonintersecting, so take every decomposition of $x = x_i + y_i$ and prove $x = \sum x_i$. The map $R \to R : x \to R/R_i$ has kernel $\sum_{j\neq i} R_i \cong \mathfrak{m}^{e_i}$ by induction.

Uniqueness: Use(5.6.15). The decomposition gives a way to decompose 1 to sum of idempotent elements and is determines by it. $1 = \sum e_i = \sum f_i$, so $e_j = \sum e_j f_i$. But e_i cannot decompose, so $e_j = e_j f_{i(j)}$, $\exists\ i(j)$. the following is easy to show these two decomposition is the same. □

Tensor Product, Limits and Colimits

Prop. (5.1.11). If X is f.g module over a Noetherian ring, then $\mathrm{Hom}(X, -)$ commutes with direct sums and $X \otimes -$ commutes with direct products.

Proof: Write X as a cokernel of free modules. □

Prop. (5.1.12). For a ring map $R \to S$, let q be a prime in S, $p = q \cap R$, then $(M \otimes_R S)_q = M_p \otimes_{R_p} S_q$ for any R-module M.

Proof: $(M\otimes_R)_q = M \otimes_R S_q = M \otimes_R R_p \otimes_{R_p} S_q = M_p \otimes_{R_p} S_q$. □

Prop. (5.1.13). The tensor product of two integral domain over alg.closed field is also an integral domain, Cf.[StackProject 05P3].

Local Properties

Def. (5.1.14). A property P of rings or modules over a ring is called **local property** iff X has P iff X_{f_i} all has P for a covering $(f_1, \ldots, f_n) = 1$.

A property of morphisms of rings is called **local on the target** iff $R \to S$ has P iff $R_{f_i} \to S_{f_i}$ has P for a covering $(f_1, \ldots, f_n) = 1$ in R.

Prop. (5.1.15) (Stalkwise Properties). For a ring R,

1. Trivial is stalkwise for modules over R, it is even stalkwise. Hence so does injectivity and surjectivity because localization is exact.
2. Flatness for modules over R.
3. Flatness for rings over R.
4. Formal unramifiedness for rings over R, both on the target and source.

Proof:

1. It suffice to prove an element is trivial on every localization then it is 0. For this, consider the annihilator $Ann(x)$, it is not contained in any maximal ideal so it contains 1.
2. We use the definition(6.2.1). Notice $(IM)_\mathfrak{p} = I_\mathfrak{p} M_\mathfrak{q}$ and every ideal of $R_\mathfrak{p}$ is of the form $I_\mathfrak{p}$. Then use the fact injective is stalkwise(5.1.15).
3. We use the definition(6.2.1). Notice $(I \otimes_R S)_\mathfrak{q} = I_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ for all primes q of S and $p = q \cap R$. And every ideal of $R_\mathfrak{p}$ is of the form $I_\mathfrak{p}$. Then use the fact injective is stalkwise(5.1.15)
4. Because formally unramified is equivalent to $\Omega_{R/S} = 0$(6.6.1), so we get the result by functorial properties of $\Omega_{S/R}$(1.2.3) and triviality is stalkwise(5.1.15).

□

Prop. (5.1.16) (Local Properties). For a fixed ring R,

1. Finite is a local property for modules over R.
2. F.p. is a local property for modules over R.
3. Noetherian is a local property for rings.
4. F.g. is a local property for rings over R, both on the source and target.
5. F.p. is a local property for rings over R, both on the source and target.
6. Smoothness for rings over R, both on the target and source.

Proof: Cf.[StackProject 00EO].

1.
2.
3.
4. If S_{g_i} is finite type over R, choose $\sum h_i g_i = 1$. Let S_{g_i} be generated by $y_{ij}/g_i^{n_{ij}}$, then we claim S equals the subalgebra generated by generated by g_i, h_i, y_{ij}. For this, notice g_i generate the unit ideal in S', and $S'_{g_i} \to S_{g_i}$ is surjective by definition, so $S' \to S$ is surjective because surjection is stalkwise(5.1.15). It is local on the target because it is local on source and stable under composition and $R \to R_f$ is f.g..
5. By f.t. is local property(5.1.16), we know S is f.g. over R. The rest Cf.[StackProject 00EP]. It is local on the target by what we have already proved.
6. (6.5.13).

□

2 Projective

References are [Projective Modules].

Prop. (5.2.1). Localization and tensor product preserves projective because they are left adjoints. And when tensoring f.f. map, then the converse is also true(6.3.1).

Prop. (5.2.2). A module over a ring is projective iff it is a direct summand of a free module, in particular, it is flat. Moreover, there is a free module Q that $P \oplus Q = F$ free.

Proof: For the second assertion, we can choose an arbitrary Q that $P\oplus Q$ free, and see $\bigoplus_{\mathbb{N}}(P\oplus Q)$ is free. □

Prop. (5.2.3) (Projective over Local Ring). A projective module over a local ring or a PID is free.

Proof: Local ring case: We only prove for P finite, the infinite case is proved in [Kaplansky. Projective modules]. Choose minimal number of generator, then $R^m = P\oplus N$, pass to the quotient field, we have $k^m = P/mP \oplus N/mN$. P/mP has rink m by Nakayama, thus $N/mN = 0$, thus $N = 0$ by Nakayama.

PID case: directly from(3.3.8). □

Prop. (5.2.4) (Finite Projective,Locally Free,Flat,F.P.). Let M be a R-module, the following are equivalent:

1. M is finite projective.
2. M is f.p. and flat.
3. M is f,p. and all its localizations at (maximal)primes are free.
4. M is finite locally free.
5. M is finite and locally free.
6. M is finite and all its localizations at primes are free and the function $p \to \dim_{k(p)} M\otimes_R k(p)$ is a locally constant function on $\operatorname{Spec} R$.

Proof: $1\to 2$: $M\otimes K = R^m$ for some K and m, so K is finite and $M = R^m/K$ is f.p. And M is flat because it is a summand of R^n(6.2.3).

$2\to 4$: For any prime p, choose a basis for the $k(p)$-vector space $M\otimes k(p)$, then by Nakayama, their inverse image generate M_g for some $g\notin p$(3.4.5), and the kernel K of this generation is finite because M_g is f.p. And $K\otimes k(p) = 0$ by the flatness of M_g. Then by Nakayama again there is a $g'\notin p$ that $M_{gg'} = 0$(3.4.5).

$4\to 3$: Because f.p. is local(5.1.16).

$3\to 2$: Because flatness is trivial.

$4\to 5$: Because finite is local(5.1.16).

$5\to 4, 4\to 6$: Trivial.

$6\to 4$: Cf.[StackProject 00NX]

$2+3+4+5+6\to 1$: Cf.[StackProject 00NX].

Consider the stalk, it is all free by(5.2.1) and(5.2.3), thus by(2.1.29), it is locally free. □

Cor. (5.2.5) (Partially Stalkwise). If P is fo f.p., then finite projectiveness is a stalkwise property for P.

Cor. (5.2.6) (Projective and Flat). A finite module over a Noetherian ring is projective iff it is flat.

Cor. (5.2.7). If M is finite projective, then the canonical map $\operatorname{Hom}(M,N)\otimes L \to \operatorname{Hom}(M, N\otimes L)$ is an isomorphism.

Proof: By proposition above M is f.p. and finite locally free, so by(5.12.6) and tensor commutes with localization, we can check locally, where M is finite free so the isomorphism is obvious. □

Duality of Projective Modules

Prop. (5.2.8) (Basis Criterion of Projectiveness). An A-module P is projective iff there are elements x_i in P and f_i in P^* that for any x, $f_i(x) = 0$ a.e. i, and $\sum f_i(x)x_i = x$. Moreover, P is finite projective iff there are f.m. of them.

Proof: If P is projective, as a summand of a free module, then we can choose the coordinates of the inclusion map as f_i, and choose the image of the quotient map of the coordinate as x_i. The converse is verbatim. □

Cor. (5.2.9). If P is projective, then $P \to P^{**}$ is injective, and if P is finite projective, then it is an isomorphism.

Proof: If $f(x) = 0$ for all $f \in P^*$, then the proposition says $x = 0$. And if P is finite projective, it can be seen x_i, f_i forms a "basis" of P^*(finiteness used), so f_i generate P^*, and similarly x_i generate P^*, so $P \to P^{**}$ is surjective. □

Cor. (5.2.10). If P is projective over R, then $P^* \neq 0$.

Cor. (5.2.11). In the meanwhile of the proof, we already get: if P is finite projective, then P^* is finite projective, by(5.2.8).

Cor. (5.2.12). If P is finite projective, the the map $P \otimes M \to \mathrm{Hom}(P^*, M)$ is an isomorphism.

Proof: In(5.2.7), let $N = R$ and let $M = P^*$, then use the fact $P \cong P^{**}$. □

Prop. (5.2.13). Any finite projective module over $K[X_1, \ldots, X_k]$ is free. (Highly nontrivial).

Proof: □

Prop. (5.2.14). $\prod^{\mathbb{N}} \mathbb{Z}$ is not free thus not projective over $\mathbb{Z}$(5.2.3). And

$$\mathrm{Hom}(\prod^{\mathbb{N}} \mathbb{Z}, \mathbb{Z}) = \bigoplus^{\mathbb{N}} \mathbb{Z}.$$

Proof: Cf.https://wildtopology.wordpress.com/2014/07/02/the-baer-specker-group/. □

3 Injective

Prop. (5.3.1) (Baer's Criterion). A right $R-$module I is injective iff for every right ideal J of R, every map $J \to I$ can be extended to a map $R \to I$. (Directly from (7.1.28)).

Cor. (5.3.2). A module over a PID is injective iff it is divisible.

Cor. (5.3.3). A is injective iff $\mathrm{Ext}^1(R/I, A) = 0$ for every ideal I of R.

Prop. (5.3.4). The category of R-mod has enough injectives by(7.1.32), and it has enough projectives trivially.

Prop. (5.3.5). If I is an injective A-module, then for any ideal α of A, $\Gamma_\alpha(I) = \{m | \alpha^n m = 0\}$ for some n is injective.

Proof: Use Baer criterion, for any ideal b of A, it is f.g. so there is a n that $\phi(\alpha^n b) = 0$, and Artin-Rees tells us that $\phi(\alpha^N \cap b) = 0$ for some N. So we have an extension of ϕ over $b/b \cap \alpha^N$ to $A/\alpha^N \to I$, and this obviously factor through $\Gamma_\alpha(I)$, so it is done. □

Prop. (5.3.6). For an injective module A-module I, $I \to I_f$ is surjective.

Proof: we have the sheaf of modules $\widetilde{I}$ is flabby(4.1.6), thus the map to the stalk is surjective. □

Pontryagin Duality

Basic references are [Weibel Homological Algebra].

Def. (5.3.7). The **Pontryagin dual** $M^\vee$ of a left R-module M is the right R-module $\mathrm{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$, where $(fr)(b) = f(rb)$.

It is easily verified that if $A \neq 0$, then $A^\vee \neq 0$, and $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, thus the Pontryagin dual is faithfully exact.

Prop. (5.3.8). *M is flat R-module iff $M^\vee$ is an injective right R-module (Because $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is exact).*

4 Homological Dimension

Def. (5.4.1). For a R-mod A, the **projective dimension** $\mathrm{pd}(A)$ is the minimal length of a projective resolution of A. The **injective dimension** $\mathrm{id}(A)$ is the minimal length of a injective resolution of A. The **flat dimension** $\mathrm{fd}(A)$ is the minimal length of a flat resolution of A.

Prop. (5.4.2). *If R is Noetherian, then $fd(A) = pd(A)$ for every f.g. module A.*

Proof: Use(5.4.3), we see that if we choose a syzygy and look at the n-th term, then it is f.p and flat, so we have it is projective by(6.2.9). □

Lemma (5.4.3) (pd). *If $\mathrm{Ext}^{d+1}(A, B) = 0$ for every B, then for every resolution*

$$0 \to M \to P_{d-1} \to \ldots, P_1 \to P_0 \to A \to 0$$

where P_k is projective, then M is projective. Hence we have $pd(A) \leq d$. (Use dimension shifting, the following two are the same).

Lemma (5.4.4) (id). *If $\mathrm{Ext}^{d+1}(A, B) = 0$ for every A, then for every resolution*

$$0 \to B \to P_0 \to \ldots, P_{n-1} \to M \to 0$$

where P_k is injectives, then M is injective. Hence we have $id(B) \leq d$

Lemma (5.4.5) (fd). *If $\mathrm{Tor}_{d+1}(A, B) = 0$ for every B, then for every resolution*

$$0 \to M \to F_{d-1} \to \ldots, F_1 \to F_0 \to A \to 0$$

where F_k is flat, then M is flat. Hence we have $fd(A) \leq d$

Prop. (5.4.6) (Global Dimension Theorem). *The following are the same for any ring R and called the **left global dimension** of R:*

1. $\sup\{id(B)\}$
2. $\sup\{pd(A)\}$
3. $\sup\{pd(R/I)\}$
4. $\sup\{d : \mathrm{Ext}^d_R(A, B) \neq 0$ for some module A,B.

Proof: This follows from(5.4.3),(5.4.4) and (5.3.3). □

Prop. (5.4.7). A $\mathbb{Z}$ has global dimension 1 because injective is equivalent to divisible, and this shows that a quotient of an injective is injective.

Prop. (5.4.8) (Tor Dimension Theorem). The following are the same for any ring R and called the **Tor dimension** of R:

1. $\sup\{fd(A)\}$ for A a left module.
2. $\sup\{fd(B)\}$ for B a right module.
3. $\sup\{pd(R/I)\}$ for I a left ideal.
4. $\sup\{pd(R/J)\}$ for J a right ideal.
5. $\sup\{d : \mathrm{Tor}_d^R(A, B) \neq 0$ for some module A,B.

Proof: This follows from(5.4.5) applied to R and R^{op} and also(6.2.1). □

Prop. (5.4.9) (Change of Rings). Let $S \to R$ be a ring map, let A be a R-mod, then we have $pd_S(A) \leq pd_R(A) + pd_S(R)$.

Proof: Use the Cartan-Eilenberg resolution and the total complex has length $pd_R(A) + pd_S(R)$. □

5 Spectrum

Prop. (5.5.1). If I is an ideal of R that $I = I^2$, and I is f.g., then $V(I)$ is open and closed in $\mathrm{Spec}\, R$, and $V(I) = R_e$ for some idempotent e.

Proof: By Nakayama, there is a a $f = 1 + i$ that $fI = 0$. So $i + i^2 = 0$ and $f^2 = f$. $V(I) = D(f)$(easily checked). Consider the map $R \to R_f$, it is surjective because f is idempotent, and it has kernel I(easily checked). □

Going-up and down

Prop. (5.5.2). Integral injective ring extension satisfies going-up(5.7.1). Flat ring map satisfies going-down(6.2.22).

Prop. (5.5.3). Going-up and Going-down are stable under composition.

Prop. (5.5.4). If the image of the Spec map of a ring map is closed under specialization, then this image is closed.

Proof: Cf.[StackProject 00HY]. □

Prop. (5.5.5). Going-up is equivalent to Spec closed by(5.5.4) because we can restrict to a closed subset. If Spec map is open, then going-down holds.

Prop. (5.5.6). For $R \subset S$, all the minimal primes of R are in the image of the Spec map of a minimal prime of S.

Proof: Localize w.r.t. to the minimal prime $\mathfrak{p}$, then it is a local ring with only one prime. And $S_\mathfrak{p}$ is nonzero because localization is exact, so it has a maximal ideal $\mathfrak{q}$. Now we choose a minimal prime of S contained in $\mathfrak{q}$, then it is also mapped to $\mathfrak{p}$. □

6 Support and Associated Primes

Def. (5.6.1). The **support** $\text{Supp}(M)$ of a module M is the set of all p that $M_p \neq 0$. When M is f.g., $\text{Supp}(M) = V(\text{Ann}(M))$.

The **associated primes** $\text{Ass}(M)$ of a A-module M is the set of $p = \text{Ann}(m)$. I is called **unmixed** if primes in $\text{Ass}(A/I)$ don't contain each other, and of the same height.

Prop. (5.6.2). The support of a nonzero module is not empty.

Prop. (5.6.3). If $0 \to N \to M \to Q \to 0$, then we have $\text{Supp}(M) = \text{Supp}(N) \cup \text{Supp}(Q)$, this is because localization is exact.

Prop. (5.6.4). For E, F f.g over a ring A, $\text{Supp}(E \otimes F) = \text{Supp}(E) \cap \text{Supp}(F)$, this is because on a local ring A_p, $E \neq 0, F \neq 0 \to E \times F \neq 0$, which can be seen by passing to the residue field and use Nakayama.

Prop. (5.6.5). Let A be Noetherian and I be an ideal, then $I^n M = 0$ for some n iff $\text{Supp}(M) \subset V(I)$.

Proof: If $I^n M = 0$, then if $I \nsubseteq P$, then $M_P = 0$. Conversely, we have a filtration of M, and by(5.6.3) we have all the P_i include I, so I^n annihilate M. □

Prop. (5.6.6) (Associated Primes and Exact Sequence). Note that $P \in \text{Ass}(M)$ iff M contains a submodule isomorphic to A/P. So for an exact sequence $0 \to M_1 \to M \to M_2$, $\text{Ass}(M) \subset \text{Ass}(M_1) \cup \text{Ass}(M_2)$ and $\text{Ass}(M_1) \subset \text{Ass}(M)$. Hence for a f.g module over a Noetherian ring, $\text{Ass}(M)$ is finite by(5.1.8).

Prop. (5.6.7) (Associated Primes and Support). $\text{Ass} M \subset \text{Supp}\, M$, and when R is Noetherian, their minimal elements are the same.

In particular, $Ass(M)$ is not empty by(5.6.2), and $\text{Ass}(A/I)$ contains all the minimal primes over I.

Proof: If $\mathfrak{p} = Ann(m)$, then m is nonzero in $M_{\mathfrak{p}}$, so $M_{\mathfrak{p}}$ is nonzero, i.e. $\mathfrak{p} \in \text{Supp}(M)$.

For the second assertion, we first prove for M finite, and then write any module as sum of finite submodules, and use the fact Supp and ass are all unions of those of the submodules. Cf.[StackProject 02CE]. □

Prop. (5.6.8) (Associated Primes and Zero-divisors). When R is Noetherian and M a R-module, the union of the associated primes of M is the set of zero-divisors in M.

Proof: Elements in associated points are zero-divisors obviously, and conversely, if $xm = 0$, then $x \in Ann(m)$ and $Ann(m)$ has an associated point $\mathfrak{q}$ by(5.6.7). Now x must be in $\mathfrak{q}$ and $\mathfrak{q}$ is also an associated point of M by(5.6.7). □

Cor. (5.6.9). Use the prime avoidance(5.1.4), we can prove if R is Noetherian and M is a finite R-module, then $I \subset \mathfrak{p}$ for some $\mathfrak{p} \in Ass(M)$ iff I consists of zero-divisors.

Prop. (5.6.10) (Associated Primes and Maps). For a ring map $\varphi : R \to S$ and a S-module M, then $\text{Spec}(\varphi)(Ass_S(M)) \subset Ass_R(M)$, and equal if S is Noetherian.

Proof: We prove it is equal. If $\mathfrak{p} = Ann_R(m)$, then we let $I = Ann_S(m)$, then $R/\mathfrak{p} \subset S/I \subset M$, so by(5.5.6), there is a minimal prime of S over I that are mapped to $\mathfrak{p}$, now this prime is in $Ass(S/I)$ by(5.6.7) and also in $Ass_S(M)$ by(5.6.6). □

Prop. (5.6.11). If R is Noetherian and M is a R-module, then $M \to \prod_{\mathfrak{p}\subset Ass(M)} M_{\mathfrak{p}}$ is injective.

Proof: Notice $(x) \subset M$, if (x) is nonzero, then there is an associated prime $\mathfrak{p}$ of (x)(5.6.7), then it is an associated prime of M, and then $(x)_{\mathfrak{p}} \subset M_{\mathfrak{p}}$ is not zero, contradiction. □

Prop. (5.6.12). An associated point that is not minimal among them is called a **embedded point**. An embedded point correspond to a nilpotent element, because $px = 0$ is contained in every minimal element but p is not, so x is contained in every minimal prime ideal.

Cor. (5.6.13) (Reduced Ring No Embedded Primes). A reduced ring has no embedded primes, because it has no nilpotent elements. Hence all its associated primes are just the minimal primes.

Primary Decomposition

Def. (5.6.14). For R Noetherian, a R-module M is called **coprimary** iff it has only one associated primes. A submodule N of M is called p**-primary** iff $Ass(M/N) = \{p\}$. A ring is called p**-primary** iff (0) is p-primary.

Notice coprimary is equivalent to the following: if $a \in A$ is a zero divisor for M, then for each $x \in M$, there is a n that $a^n x = 0$, i.e. **locally nilpotent**.

Proof: If M is p-primary, if $x \in M$ is nonzero, then $Ass(Rx) = \{p\}$, so p is the unique minimal element of $\mathrm{Supp}(Rx) = V(Ann(x))$ by(5.6.7). So p is the radical of $Ann(x)$, i.e. $a^n x = 0$ for some n(6.9.1).

Conversely, we know the ideal p of locally nilpotent elements equals the union of the associated primes(5.6.8), so if $q \in AssM = Ann(x)$, then by definition, $p \subset q$. So $p = q$, and thus $AssM = \{p\}$. □

Lemma (5.6.15). A primary ring has no nontrivial idempotent element, because e and $1-e$ will all belong to the same minimal ideal p.

Lemma (5.6.16). The intersection of p-primary submodules are p-primary. (Because there is a injection $M/Q_1 \cap Q_2 \to M/Q_1 \oplus M/Q_2$).

Lemma (5.6.17) (Associated Prime and Primary Decomposition). If $N = \cap Q_i$ is an irredundant primary decomposition and if Q_i belongs to p_i, then we have $Ass(M/N) = \{p_1, \ldots, p_r\}$.

Proof: There is a injection $M/N \to M/Q_1 \oplus \ldots M/Q_r$ which shows $Ass(M/N) \subset \{p_1, \ldots, p_r\}$. And for the inverse, notice $Q_2 \cap \ldots \cap Q_r/N$ is a submodule of M/Q_1, which shows $Ass(Q_2 \cap \ldots \cap Q_r/N) = \{p_1\}$ by(5.6.7). □

Prop. (5.6.18). If N is a p-primary submodule of a R-module M, and p' is a prime ideal, then

- $N_{p'} = M_{p'}$ if $p \not\subseteq p'$.
- $N = M \cap N_{p'}$ if $p \subset p'$.

Proof: Cf.[Matsumura P55]. □

Cor. (5.6.19). For a irredundant primary decomposition $N = \cap Q_i$, if Q_1 corresponds to p_1 and p_1 is minimal in $Ass(M/N)$, then $Q_1 = M \cap N_{p_1}$. In particular, the minimal prime part of a irredundant primary decomposition is uniquely determined.

Proof: By the above proposition, there are elements u_i of Q_i, i$\neq 1$ that are mapped to units in M_{p_1}, so we have $Q \cdot u_2 u_3 \ldots u_r$ is mapped onto $(Q_1)_{p_1}$. Then $Q_1 = M \cap (Q_1)_{p_1} = M \cap N_{p_1}$. □

Prop. (5.6.20). If R is Noetherian and M is a R-module, there are p-primary submodules $Q(p)$ for each $p \in AssM$ that $(0) = \bigcap_{p \in AssM} Q(p)$.

Proof: For a $p \in AssM$, we seek $Q(p)$ to be the maximal submodule N that $p \notin AssN$. This has a maximal ideal because of Zorn and the fact $Ass(\cup N_\lambda) = \cup Ass(N_\lambda)$. Then We have $Ass(M/Q(p)) = \{p\}$, otherwise there is another p', then there is a $Q'/Q(p) \cong A/p'$. Now Q' is bigger than $Q(p)$. Finally, $(0) = \bigcap_{p \in AssM} Q(p)$ because it has no associated primes. □

Cor. (5.6.21). If M is f.g., then any submodule has a primary decomposition. (Notice M has only f.m. associated primes).

Def. (5.6.22). For a prime ideal in a Noetherian ring, The n**-th symbolic power** $p^{(n)}$ of p is defined to be the p-primary component of p^n, who has only one minimal prime(hence one associated prime). The symbolic power is giving by $p^n A_p \cap A$ by(5.6.19).

7 Integral Extension

Prop. (5.7.1). Let A a subring of $B, A \to B$ integral. Then:

1. If A is local and p is the maximal ideal of A, then the prime ideals of B lying over p is precisely the maximal ideal of B.
2. There is no inclusion relation between the prime ideals of B lying over a fixed prime ideal of A.
3. The Spec map is surjective.
4. The going-up holds.

Proof:

1. Since for two ring one integral over another, one is a field iff the other is a field.
2. Localize at the prime p, then we see that xaximal ideal of B_p cannot contain each other.
3. For any prime p of A, since $A_p \neq 0$, $B_p \neq 0$, so it has a maximal ideal.
4. Localize and use 3.

□

Prop. (5.7.2). Let A a subring of $B, A \to B$ integral noetherian. Then:

1. $\dim(A) = \dim(B)$
2. $\mathrm{ht(P)} = \mathrm{ht(P \cap A)}$
3. If going up holds, then $\mathrm{ht(J)} = \mathrm{ht(J \cap A)}$ for any ideal J.

Proof: 1:By the preceding lemma, there is no inclusion relation between prime over a fixed prime, so $\dim(B) \leq \dim(A)$. On the other hands, going-up holds, so $\dim(B) \geq \dim(A)$.

2:Follows from (5.9.5)(1) since $\mathrm{ht(P/(P \cap A)B)} = 0$ by the preceding lemma.

3:by 2 and surjectiveness of Spec for integral extension. □

8 Graded Ring & Completion

Cf.[Matsumura Ch11].

Def. (5.8.1) (Hilbert-Serre). Let A be an Artinian ring and $B = A[X_i]$. For a f.g. graded B-module $\oplus M_n$, we have $l(M_n)$ is a polynomial of n for n big, called the **Hilbert Polynomial**. Its degree is $\operatorname{Supp} M$.

Proof: The case when A is a field follows from(4.4.17). Cf.[Hartshorne P51]. □

Prop. (5.8.2) (Hilbert Polynomial and Dimension). For a Noetherian local ring A, the Hilbert polynomial of a f.g. module M w.r.t $\mathfrak{m}$ has degree $\dim M$. And $\dim M$ is the smallest integer r s.t. there exists $x_1, \ldots, x_r$ that $l(M/x_1M + \ldots, x_rM) < \infty$. Cf.[Mat P76].

<u>**Completion**</u>

Prop. (5.8.3) (Artin-Rees). For A Noetherian and I an ideal, let $N \subset M$ be <u>finite</u> A-module, then

$$I^n M \cap N = I^{n-r}(I^r M \cap N)$$

hence the I-adic topology on M induce the I-adic topology on N.

Cor. (5.8.4) (Intersection Theorem). Notation as above, let $N = \cap^\infty I^n M$, then $IN = N$. So if $I \subset \operatorname{rad}(A)$, Nakayama tells us $N = 0$. This can be used to use induction to prove some theorem.

Cor. (5.8.5) (Krull). For A Noetherian, if $I \subset \operatorname{rad}(A)$ or A is a domain, then $\cap^\infty I^n = 0$.

Prop. (5.8.6). Let the topology on a A-module be defined by countable filtration of submodules, then iff M is complete, then M/N is complete in the quotient topology.

Proof: Write $x_{i+1} - x_i = y_i + z_i$ with $y_n \in M_n$ and $z_n \in N$, then the image of the limit of $\sum y_i$ is the limit of $\overline{x_i}$. □

Prop. (5.8.7). For a local ring map of two power series map, it is an isomorphism iff its Jacobian is invertible.

Def. (5.8.8). The **completion** of a topological A-module is a functor $\varphi : M \to M'$ that are left adjoint to the forgetful functor from the category of complete Hausdorff A-modules. It is defined as composition of the Hausdorffization functor followed by $\lim M/M_n$ with the topology like that of profinite groups. The completion is right exact. For left exactness, otice the limit process is exact, so only the Hasudorffization can go astray.

Prop. (5.8.9). The completion of a submodule $N \subset M$ is the closure of $\varphi(N)$ (By direct construction). The completion of M/N is M^*/N^* because it is right exact.

Cor. (5.8.10). If N is open in M then $M/N \cong M^*/N^*$ because M/N is discrete hence complete Hausdorff.

Prop. (5.8.11). When N is finite, $0 \to N^* \to M^* \to (M/N)^* \to 0$ is exact, because the Hasfdorffization of N embeds in that of M by Artin-Rees.

Prop. (5.8.12). When A is Noetherian and M is finite A-module, then the natural map $M \otimes_A A^* \to M^*$ is an isomorphism (use M is finite presentation and tensor & completion is right exact), and five lemma.

Cor. (5.8.13). When A is Noetherian, A^*/A is flat (because flatness is check for finite module), and when A is complete Hausdorff, any finite module M is complete Hausdorff and hence any its submodule is complete thus closed in it. Hence the the completion of a submodule $N \subset M$ is $\varphi(N)A^*$ in $M^* = MA^*$. In fact this implies complete Hausdorff adic-ring is Zariski.

Prop. (5.8.14). A Noetherian I-adic ring is called **Zariski ring** if it satisfies the following equivalent conditions:

- Every finite module is Hausdorff in the I-adic topology.
- Every submodule in a finite module is closed in the I-adic topology.
- Every ideal is closed.
- $I \subset \mathrm{rad} A$.
- A^*/A is f.f.

Hence every complete Hausdorff ring is Zariski.

Proof: $1 \to 2$:apply it to the submodule M/N.

$3 \to 4$:If $I \not\subseteq m$, then $I^n + m = A$, thus $\overline{M} = A$, contradiction.

$4 \to 1$: by intersection theorem(5.8.4).

$4 \to 5$: for any maximal ideal m, $I \subset m$ so it is open, thus $A^*/mA^* = A/m \neq 0$ by(5.8.10) thus f.f. by(6.2.11).

$5 \to 1$:by(6.2.12), for any m maximal, there is a maximal ideal m' lying over m, so $IA^* \subset m^*$ by(5.8.13), thus $I \subset m$, hence $I \subset \mathrm{rad} A$. □

Cor. (5.8.15). In a Zariski ring A, maximal ideals are open, thus $A/m \cong A^*/mA^*$ by(5.8.10), thus $\mathrm{Spec}\, A^* \to \mathrm{Spec}\, A$ is bijection on closed pt.

Prop. (5.8.16) (Cohen Structure Theorem). If A is a complete local ring containing a field k that the residue field is separably generated over k, then there is a field K containing k that is a Cohen ring, i.e. complete local ring with a prime number as a uniformizer, that has the same residue field as A.

9 Dimension

Def. (5.9.1). For a A-module M, $\dim(M)$ is defined as $\dim(A/\mathrm{Ann}(M))$.

The **height** of an ideal I in A is defined as the height of the minimal prime ideal over I.

Prop. (5.9.2). For a Noetherian ring A, $\dim A = \sup \dim A_p$, because X is Noetherian hence has f.m. minimal primes.

Def. (5.9.3). A ring is called **universally catenary** if all its f.g. algebra is catenary, i.e. the dimension behave well.

Dedekind domain, e.g. field is universally catenary, so f.g. domain over fields is catenary.

Prop. (5.9.4). If A is a Noetherian local ring with maximal ideal m, then $\dim A \leq \dim_k m/m^2$. Cf.[Matsumura P78].

Prop. (5.9.5) (Dimension Extension Formula). Let $A \to B$ Noetherian, let $p = P \cap A$,then:

- ht(P) $\leq$ ht(p) + ht(P/pB),in other words $\dim(B_P) \leq \dim(A_p) + \dim(B_P \otimes k(p))$. Where $k(p) = A_p/pA_p$ and $B \otimes k(p) = B_p/pB_p$.

- equality holds if going-down holds. For example, if it is flat.
- if Spec map is surjective and going-down holds, then we have i)$\dim B \geq \dim A$, and ii)ht(I) = ht(IB) for ideal I of A.
- if going-up holds, then $\dim B \geq \dim A$. e.g. B integral over A Cf. (5.7.2)

Proof: Cf.[Commutative Algebra Matsumura (13.B)]. □

Lemma (5.9.6). For a Noetherian ring A, $\dim A[X] = \dim A + 1$.

Proof: Cf.[Matsumura P83]. □

Prop. (5.9.7). If $R \to S$ is a map of Noetherian rings, if the going down property holds, then $\dim S_q = \dim R_p + \dim S_q/pS_q$ for a prime q of S over p.

Proof: Cf.[StackProject 00ON]. □

Prop. (5.9.8) (Noetherian Normalization Theorem). If A is a f.g. algebra over a field. then there are r alg. independent elements y_i that A is integral over $k[y_i]$. Hence $\dim A =$ tr.deg A because integral extension has the same dimension(5.7.2).

Proof: Cf.[Commutative Algebra Matsumura P91]. □

Cor. (5.9.9) (Krull's Height Theorem). In a Noetherian domain, the height of an ideal generated by n elements is at most n.

Proof: Cf.[StackProject 0BBZ]. □

10 Depth & Cohen-Macaulay Ring

Prop. (5.10.1) (Rees). For a f.g. module M and $IM \neq M$,

$$\mathrm{depth}_I(M) = \min\{i | \mathrm{Ext}^i_A(A/I, M) \neq 0\} = \min\{i | \mathrm{Ext}^i_A(N, M) \neq 0\}$$

where $\mathrm{depth}_I(M)$ is the length of the maximal M-regular sequence in I, N is a finite A-module with $\mathrm{Supp}(N) \subset V(I)$.

Proof: If No elements of I are M-regular, then $i \subset \cup \mathrm{Ass}(M)$ thus in one of them, so $\mathrm{Hom}_{A_p}(k, M_p) \neq 0$, and we have $N_p/PN_p = N \otimes_A k_p$ nonzero by Nakayama, thus $\mathrm{Hom}_k(N \otimes_A k_p, k_p) \neq 0$, thus $\mathrm{Hom}_{A_p}(N_p, M_p) = (\mathrm{Hom}_A(N, M))_p \neq 0$, so $\mathrm{Ext}^0_A(N, M) \neq 0$. Other dimensions follows by induction, consider the cokernel of $M \xrightarrow{a_1} M$.

Conversely, use induction, then we have an injection $\mathrm{Ext}^i_A(N, M) \xrightarrow{a_1} \mathrm{Ext}^i_A(N, M)$ for $i < n$. And the condition shows that $I \subset \sqrt{\mathrm{Ann}(M)}$, so $a_1^r N = 0$, thus the result. □

Cor. (5.10.2). Two maximal regular sequence in a f.g. module have the same length.

Cor. (5.10.3). For a module M over a Noetherian ring A, we know $\Gamma_I(M) = \{m | I^n m = 0 \text{ for some } n\}$, and H^n_I is its right derived functor, then we have $\mathrm{depth}_I(M) \geq n \iff H^i_I(M) = 0$ for $i < n$. (Because derived functor commutes with colimits, consider $N = A/I^k$).

Lemma (5.10.4) (Ischebeck). For a Noetherian local ring A, if M, N are finite modules, then we have $\mathrm{Ext}^i_A(N, M) = 0$ for $i < \mathrm{depth}(M) - \dim N$. Cf.[Matsumura P104].

Prop. (5.10.5). Let A be a local ring and M is finite A-module, then $\operatorname{depth}(M) \le \dim A/P \le \dim M$ for every $P \in \operatorname{Ass}(M)$. (Because $\operatorname{Hom}(A/P, M) \neq 0$.)

Prop. (5.10.6) (Auslander-Buchsbaum Formula). For a local ring R, if M is a finitely generated R-mod, if $pd(M) < \infty$, then we have $\operatorname{depth}(R) = \operatorname{depth}(M) + \operatorname{pd}(M)$. Cf.[Weibel P109].

Prop. (5.10.7). For a A-module M, if $x_1, \ldots, x_n$ is an M-regular sequence in A, then the Koszul complex has higher homology vanish and $H_0 = M/\sum x_i M$.

Proof: Cf.[Hartshorne P135]. □

Cohen-Macaulay

Def. (5.10.8). For A Noetherian local, a f.g. A-module M is called **Cohen-Macaulay** if $\operatorname{depth}(M) = \dim M$. In view of(5.10.5), this is equivalence to $\operatorname{depth}(M) = \dim A/P$ for all $P \in \operatorname{Ass}(M)$.

A localization of a C.M local ring is C.M, so we call a ring **C.M.** if all its localization at primes are C.M.

Prop. (5.10.9). A ring R is called **Gorenstein** iff $id_R R < \infty$. A Gorenstein local ring is C.M. In this case, $\operatorname{depth}(R) = id_R R = \dim R$, and $\operatorname{Ext}^q_R(R/m, R) \neq 0 \iff q = \dim R$. Cf.[Weibel P107].

Prop. (5.10.10). A ring is C.M. iff for all ideals, the associated primes of A/I all have the same height as I, i.e. unmixed.

Prop. (5.10.11). If a local ring is C.M. and $I = (x_1, \ldots, x_r)$ is a regular sequence, then there is an isomorphism $(A/I)[t_1, \ldots, t_r] \to \operatorname{gr}_t A = \oplus I^n/I^{n+1}$. In particular, I/I^2 is a free A/I module.

Prop. (5.10.12). Let A is a Noetherian local ring and M a f.g. module, if a set of elements $(x_1, \ldots, x_r)$ forms a regular sequence for M, then $\dim M/(x_1, \ldots, x_r) = \dim M - r$. The converse is also true when A is C.M. If this is the case, then $A/(x_1, \ldots, x_r)$ is also C.M.

Proof: By(5.8.2), we have $<$, for the converse, $\operatorname{Supp}(M/fM) = \operatorname{Supp}(M) \cap \operatorname{Supp}(A/fA) = \operatorname{Supp}(M) \cap V(f)$, and when f is M-regular, $V(f)$ doesn't contain any $\operatorname{Ass}(M)$ thus no minimal elements of $\operatorname{Supp}(M)$, so $\dim(M/fM) < \dim M$, thus we have $>$.

When A is C.M.: □

11 Normal Ring & Regular Local Ring

Serre Conditions R_k & S_k

Def. (5.11.1). A ring is called R_k iff for all prime p of height $\le k$, A_p is regular.

A ring is called S_k iff $\operatorname{depth}(A_p) \ge \min(k, \operatorname{ht}(p))$ for all prime p.

A module M is called S_k iff $\operatorname{depth}(M_p) \ge \min(k, \dim(\operatorname{Supp}(M_p)))$ for all prime p.

Prop. (5.11.2).

- M is S_1 iff M has no associated embedded primes. Cf.[StackProject 031Q].
- A Noetherian ring is reduced iff it is R_0 and S_1. Cf.[StackProject 031R].
- (Serre Criterion)A Noetherian ring is normal iff it is R_1 and S_2.
- A ring is C.M. iff it is $S_{\mathbb{N}}$.

Cor. (5.11.3). A normal ring is regular in codimension 1.

Normal Ring

Def. (5.11.4). A ring is called **normal** iff all its stalks are integrally closed domain. It is called **completely normal** iff all almost normal elements are in A, i.e. $\{u|\exists a, au^n \in A\ \forall n\} \in A$. For Noetherian ring, these notion are the same.

The **normalization** of an integral domain is the alg.closure of it in its quotient field. It commutes with localization.

Prop. (5.11.5). A normal domain is just an integrally closed domain. A normal ring is a finite direct product of integrally closed domains.

Proof: The localization of an integral domain is integral, and if it is not integral, and x is a root of a monic polynomial, then $x \in \cap A_p$ for any p, so $x \in A$.

For the second assertion: □

Prop. (5.11.6). A is completely normal $\Rightarrow$ $A[X]$ and $A[[X]]$ is completely normal. A is a normal ring then $A[X]$ is a normal ring. Hence it works well for A Noetherian. (Induction on the coefficient, Cf.[Matsumura P116]).

Prop. (5.11.7). Principal ideals in a Noetherian normal domain is unmixed and $A = \cap_{\text{ht } p=1} A_p$. Cf.[Matsumura P124].

Prop. (5.11.8) (Hironaka). Let A be a local Noetherian domain that is a localization of an algebra of f.t. over a field k. Let $t \in A$ that

- tA has only on eminimal associated prime ideal p.
- t generate the maximal ideal of A_p.
- A/p is normal.

Then $p = tA$ and A is normal.

Proof: Cf.[Hartshorne P264]. □

Regular Ring

Def. (5.11.9). A Noetherian local ring is called **regular** iff it $\operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$. This is equivalent to $\operatorname{gr} A \cong k[X_1, \ldots, X_d]$ by(5.8.2).

Localization of a regular local ring at primes are regular local, Cf.[Matsumura P139]. Hence we can call a ring is called **regular** iff all its localization at primes are regular local.

Prop. (5.11.10). If A is regular, then $A[X_1, \ldots, X_n]$ is regular, and $A[[X_1, \ldots, X_n]]$ is regular, Cf.[Matsumura P176].

Prop. (5.11.11). A regular local ring of dim 1 is the same as a principal DVR.

Prop. (5.11.12). A Noetherian local ring of dim 1 is normal iff it is regular. i.e. integrally closed iff principal. Cf.[Matsumura P124].

Prop. (5.11.13) (Auslander-Buchsbaum). A regular local ring is UFD. A priori it is an normal domain.

Proof: Cf.[Matsumura P142],[Weibel P106]. □

Prop. (5.11.14). A regular local ring Gorenstein hence C.M.

Prop. (5.11.15). If a quotient of a Noetherian local ring by a non-zerodivisor is regular, then it is itself regular.

Prop. (5.11.16) (Serre). A Noetherian local ring A is regular iff the global dimension of A is finite. Cf.[Mat P139].

Prop. (5.11.17). For A a regular local ring and M a f.g. A-module,

$$pd(M) + \text{depth } M = \dim A.$$

Cf.[Hartshorne P237].

Cor. (5.11.18). For a f.g. module M over a regular local ring A, $pd(M) \leq n$ iff $\text{Ext}^i(M, A) = 0$ for all $i > n$.

Proof: This is because we can use dimension shifting to show $\text{Ext}^i(M, N) = 0$ for all N f.g., then(5.4.3) says that $pd(M) \leq n$. □

Local Complete Intersection

Basic references are [StackProject 10.133, 23.8].

Def. (5.11.19). A f.g. k-algebra S is called a **complete intersection** if $S = k[X_1, \ldots, X_n]/(f_1, \ldots, f_c)$ with $\dim S = n - c$. It is called a **local complete intersection** if it is locally a complete intersection. Notice by Krull's theorem(5.9.9), this is equivalent to it is equidimensional of dimension $n - c$.

Relative complete intersection definition, Cf.[StackProject 00SP].

To be well-defined, we in fact need the following lemma(5.11.20).

Lemma (5.11.20). For a f.g. k-algebra, (Local)Complete intersection is stable under localization.

Proof: Cf.[StackProject 00SA]. □

Prop. (5.11.21). For a f.g. k-algebra S and a field extension $k \to K$, S is a local complete intersection iff $S \otimes_k K$ is a local complete intersection.

12 Finitely Presented

Finite Presented Module

Def. (5.12.1). A module is called **finitely presented** iff it is like R^m/R^n.

Finite presentation is stable under base change because tensoring is right exact.

Prop. (5.12.2). For a surjective module map $F \to M$, if F is f.g. and M is f.p., then the kernel is f.g.

Proof: use the diagram
$$\begin{tikzcd} & R^m \arrow[r] \arrow[d, "\alpha"] & R^n \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \text{Ker} \arrow[r] & F \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$
. and snake lemma, then image and cokernel of α are all finite, then Ker is finite. □

Prop. (5.12.3). For $0 \to M_1 \to M_2 \to M_3 \to 0$, if M_1, M_3 are f.p., then so does M_2. This is because we can find a compose a diagram of $R^* \to M_*$, and look at the kernel.

Prop. (5.12.4). If $R \to S$ is a f.g. ring map and a S-module M is f.p. over R, then it is f.p. over S.

Proof: Let $S = R[x_1, \ldots, x_n]$, and $M = R[y_1, \ldots, y_m]/(\sum a_{ij}y_j), 1 \leq i \leq t$, then as M is a S-module, we let $x_i y_j = \sum a_{ijk}y_k$, and forms a quotient $S^{mn+t} \to S^m \to N \to 0$, where S^{mn+t} corresponds to the relations $\sum a_{ij}y_j$ and $x_i y_j - \sum a_{ijk}y_k$. Then there is a surjective A-module map $N \to M$, and we check it is injective: if $z = \sum b_j y_j$ are mapped to 0, where $b_j \in S$, then we can transform z into the shape $\sum c_j y_j$, where $c_j \in R$ by relations $x_i y_j - \sum a_{ijk}y_k$. Thus it is zero by definition. □

Prop. (5.12.5). Any module is a direct limit of f.p. modules. This can be seen by considering all f.g. submodules and f.m relations between them.

Prop. (5.12.6) (FP and Localization). For M f.p., $S^{-1}\operatorname{Hom}_R(M, N) = \operatorname{Hom}_{S^{-1}R}\operatorname{Hom}(S^{-1}M, S^{-1}N)$ for any R-module N. (Use the presentation and Hom is left exact).

Finitely Presented Ring Map

Def. (5.12.7). A ring map is called **of finite presentation** iff it is a quotient of a free algebra by a free algebra.

Prop. (5.12.8). Finite presentation is stable under composition(choose a presentation form to see) and base change because tensoring is right exact.

It is local on the source and target by(5.1.16).

Prop. (5.12.9). If $g \circ f : R \to S' \to S$ is of finite presentation and f is of finite type, then g is of finite presentation.

Proof: Let $S' = R[y_1, \ldots, y_a]$ and $S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$, then let $h_i(X) \cong y_i$ in S, then $S = S'[X_1, \ldots, X_n]/(f_1, \ldots, f_m, h_i - y_i)$. □

Prop. (5.12.10). For S f.p. over R, then the kernel of any surjective ring map $R[X_1, \ldots, R_n] \xrightarrow{\alpha} S$ is f.g..

Proof: Let $S = R[Y_1, \ldots, Y_m]/(f_1, \ldots, f_k)$, then if $\alpha(X_i) \cong g_i(Y)$, then $\alpha : R[X_1, \ldots, R_n] \to R[X_1, \ldots, X_m, Y_1, \ldots, Y_m]/(f_1, \ldots, f_k, X_i - g_i)$. And the Y_i are in the image, thus we let Y_i are mapped onto by $h_j(X)$, then $\operatorname{Ker}\alpha = (f_i(h_j(X)), X_i - g_i(X))$. □

Prop. (5.12.11). If S is f.p. over R that S has a presentation $S = R[X_1, \ldots, X_n]/I$ that I/I^2 is free over S, then S has a presentation $R[X_1, \ldots, X_m]/(f_1, \ldots, f_c)$ that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is freely generated by $f_1, \ldots, f_c$.

Proof: Cf.[StackProject 07CF]. □

Prop. (5.12.12) (Chevalley). The Spec map of a f.p. ring map maps constructible sets to constructible sets.

Proof: Cf.[StackProject 00FE]. □

I.6 Commutative Algebra(StackProject)

1 Valuation Ring

Def. (6.1.1). In a field K, the **valuation ring** is the maximum elements in the dominating ordering of local rings, where B **dominates** A iff $A \subset B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$. Cf.[Atiyah].

Prop. (6.1.2). A local ring A in a field K is dominated by a valuation ring with fractional field K.

Proof: Note that the dominating relation satisfies the condition of the Zorn's lemma, so it suffices to prove that A is not maximal if its fractional field is not K. Let $t \notin K_0 = fracA$. If t is transcendental over K_0, then $A[t]$ with the maximal ideal $(\mathfrak{m}, t)$ dominate A. If t is algebraic over K_0, then there is a a that at is integral over A, hence by(5.7.1) there is a maximal ideal of $A[at]$ above A, which proves the lemma. □

Prop. (6.1.3) (Valuation Ring Criterion). A is a valuation ring with field of fraction K iff for any $x \in K$, x or x^{-1} is in A.

Proof: If A is a valuation ring, then for $x \notin A$, we know that $A[x]$ is a local ring, hence there is no prime over $\mathfrak{m}$ otherwise $A[x]_\mathfrak{p}$ is a bigger local ring, so we see $\mathfrak{m}A[x] = A[x]$, i.e. $1 = \sum t_i x^i$, so x^{-1} is integral over A. Now $A[x^{-1}]$ has a $\mathfrak{m}'$ over $\mathfrak{m}$, so $A = A[x^{-1}]_{\mathfrak{m}'}$, which shows $x^{-1} \in A$.

Conversely, when A is not K, then A is not field by(6.1.2), then it has a non-zero maximal ideal, but only one, otherwise we can choose x, y that $x/y, y/x \notin A$. And A is maximal because if there is a $A \subset A'$, and a $x \in A'$, then if $x \notin A$, then $x^{-1} \in A$, hence also in $\mathfrak{m}_A$, so it is in $\mathfrak{m}_{A'}$, but now x^{-1} cannot be in A', contradiction. □

Cor. (6.1.4). For $K \subset L$ subfield, if A is a valuation ring of L, then $A \cap K$ is a valuation ring of K. And if L/K is algebraic and A is not a field, then $A \cap K$ is not a field.(This is because the primes of A are all over 0 so cannot contain each other(5.7.1) so A is a field.

Cor. (6.1.5). The quotient A/p is a valuation ring, and any localization of valuation ring is a valuation ring, by this criterion.

Prop. (6.1.6). Valuation ring is normal, because for x algebraic over A, either $x \in A$, or x is a combination of x^{-1} thus in A.

Cor. (6.1.7) (Integral Closure and Valuation Ring). The integral closure of a subring in a field k is the intersection of valuation rings containing A.

Proof: We need to show that if x is not algebraic over A, then there is a valuation ring of A not containing x. This is because $x \notin B = A[x^{-1}]$ otherwise x is integral over A. Now x^{-1} is not a unit in B, hence $x \in p \in B$, hence B_p is dominated by some valuation ring V, and $x \notin V$ because $x^{-1} \in \mathfrak{m}_V$. □

Prop. (6.1.8) (Valuation Ring and Valuation). A valuation ring A is equivalent to a field K with a surjective valuation map to a totally valued abelian group Γ that $A = v^{-1}(\{x \geq 0\})$.

Proof: These are definitely valuation rings, and if A is a valuation ring by(6.1.3), then we set $\Gamma = K^*/A^*$, where A^* is the invertible elements of A and $x \leq y$ iff $y/x \in (A - \{0\})/A^*$. This is totally ordered by(6.1.3). □

Prop. (6.1.9) (Bezout Domain and Valuation Ring). A valuation ring is equivalent to a Bezout local domain.

Proof: One way is because the element of minimum valuation generate the group. Conversely, for $f, g \in A$, $(f, g) = (h)$, so $f = ah, g = bh$, and $h = cf + dg$, then $ab + cd = 1$, hence a or b is a unit, so $f/g \in A$ or $g/f \in A$. By(6.1.3), A is a valuation ring. □

Prop. (6.1.10). A valuation ring is Noetherian iff it is discrete valuation iff it is PID.

Proof: Only need to prove Noetherian then $\Gamma = \mathbb{Z}$. we know ideals of Γ of the form $\{x | x \geq \gamma\}$, where $\gamma > 0$ has a maximal element, so there is a minimal element bigger than 0, so $\Gamma \cong \mathbb{Z}$. □

2 Flatness

Prop. (6.2.1). Flatness need only be checked for finite modules, and it is equivalent to $\mathrm{Tor}_1(M, A/I) = 0$ for any f.g. ideal I(i.e $I \otimes M \to M$ is injective). This is all because tensor product commutes with colimit.

Cor. (6.2.2). If $0 \to M' \to M \to M'' \to 0$, then M' and M'' flat implies M is flat.

Prop. (6.2.3). If M is flat then $\mathrm{Tor}_i^A(M, N) = 0$ for all $i > 0$, because we have: if $0 \to M_1 \to M_2 \to M_3 \to 0$ M_2, M_3 flat, then M_1 is flat (Use 9 entry sequence and the fact that Tor is symmetric(8.6.5)). So $\mathrm{Tor}_{n+1}(M_3, N) = \mathrm{Tor}_n(M_1, N) = 0$ by induction.

And a direct summand of a flat module is flat. Thus we have the class of flat modules is adapted $- \otimes N$ for all N (because free is flat).

Prop. (6.2.4) (Flatness and Base Change).

- (Faithfully)Flatness is stable under base change.
- If $R \to S$ is f.f., then M is flat iff its base change is flat.
- Flatness is stable under direct limit because direct limit commutes with tensoring and is exact. $S^{-1}A$ are A-flat because localization is exact.
- If $R \to S$, and a S-module is R-flat and S-f.f., then $R \to S$ is flat.

Proof: Use definition and tensor trick. □

Prop. (6.2.5) (Equational Criterion of Flatness). For a R module M, a relation $\sum f_i x_i = 0$ of elements of M are called **trivial** iff $x_i = \sum a_{ij} y_j$ and $0 = \sum f_i a_{ij}$ for any j. Then M is flat iff all relations of elements of M is trivial.

Proof: Cf.[StackProject 00HK]. □

Prop. (6.2.6) (Gororov-Lazard). Any flat $A-$module is isomorphic to a direct limit of free modules of finite type.

Prop. (6.2.7). A finite module M over a local ring A is flat iff it is free. In particular, finite flat modules over a field are all flat.

Proof: Cf.[StackProject 00NZ]. □

Prop. (6.2.8). A module over a PID is flat iff it is torsion free. (Check(6.2.1), all ideal are principal).

Prop. (6.2.9) (Finite Flat and Projective). Fnitely presented flat module is equivalent to finite projective. (Immediate from(5.2.4)).

Prop. (6.2.10). if M is a flat R-module, then $IM \cap JM = (I \cap J)M$ for ideals of A.

Prop. (6.2.11) (Faithfully Flat). The following are equivalent:

- M is f.f.
- M is flat and for any $N \neq 0$, $N \otimes M \neq 0$.
- M is flat and for any (maximal)prime ideal $\mathfrak{m}$ of A, $k_{\mathfrak{m}} \otimes_R M \neq M$. (When $\mathfrak{m}$ is maximal, this says $\mathfrak{m}M \neq M$).

Proof: $3 \to 2$: any nonzero module has a submodule A/I, and thus $(A/I)M = M/IM \neq 0$.

$2 \to 1$:first show S is a complex if $S \otimes M$ is exact, then $H^*(S) \otimes M = H^*(S \otimes M)$ by flatness, thus $H^*(S) = 0$. □

Flat ring extension

Prop. (6.2.12). The following are equivalent:

- $A \to B$ is f.f.
- It is flat and $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.
- It is flat and Spec map contains all the closed pts.

This follows form(6.2.11) as we see that $\mathfrak{p}$ is in the image of Spec map iff $k_{\mathfrak{p}} \otimes_R S \neq S$.

Cor. (6.2.13). Integral flat injective of rings is f.f..

Cor. (6.2.14). Flat local ring map of local rings is f.f..

Cor. (6.2.15). Direct limits of f.f. rings over R is f.f.

Proof: If is flat by(6.2.4), and for a maximal ideal $\mathfrak{m}$ of R, $S_i/\mathfrak{m}S_i$ is non-zero, hence there direct limit is non-zero because 1 is contained. So $\mathfrak{m}$ is in the image, hence it is f.f. by(6.2.12). □

Prop. (6.2.16). If B is flat over A, then

$$\operatorname{Tor}_i^A(M,N) \otimes B = \operatorname{Tor}^B(M_{(B)}, N_{(B)}), \quad \operatorname{Ext}_i^A(M,N) \otimes B = \operatorname{Ext}^B(M_{(B)}, N_{(B)}).$$

Prop. (6.2.17). If $R \to S$ is (faithfully)flat ring map and M is a (faithdully)flat S-module, then M is a (faithfully)flat R-module. In particular, (faithfully) flatness is stable under composition.

Also (faithfully)Flatness is stable under base change(6.2.4) and local on the target and source by(6.2.21).

Prop. (6.2.18) (Faithfully Flat Injective). A f.f. ring map $R \to S$ is universally injective. In particular, tensoring with R/I, we get $R \cap IS = I$ for an ideal I of R.

Proof: Because $R \to S$ is f.f., we only need to show that $N \otimes_R S \to N \otimes_R S \otimes_R S$ is injective for any N, but this is true because it has a left inverse. □

Prop. (6.2.19). A flat ring map maps a non-zero-divisor to a non-zero-divisor, because if we consider the principal ideal generated by it, then(6.2.1) shows the ideal in M is also injective, so it is not a zero-divisor.

Prop. (6.2.20). If A is Noetherian and I is an ideal, the the I-adic completion $\hat{A}/A$ is flat by(5.8.13).

Prop. (6.2.21) (Flatness is Local). Flatness is stalkwise both on the target and source, thus flatness is local both on the target and the source**??**

Cor. (6.2.22) (Going-down). Going-down holds for flat ring map.

Proof: The ring map $R_{\mathfrak{p}'} \to S_{\mathfrak{q}'}$ is flat by(6.2.21), thus it is f.f. by(6.2.14). Then(6.2.12) says $\mathfrak{p} \subset \mathfrak{p}'$ is in the image. □

Prop. (6.2.23). The Spec map of a ring map $R \to S$ of f.p. that satisfies going-down(e.g. flat), is open.

Proof: $S \to S_f$ satisfies going-down and is of f.p, so we see that $R \to S_f$ satisfies going down. It suffice to prove the image of this map is open. By Chevalley, the image is constructible, and it is stable under specialization. So it is closed by(1.13.3). □

Prop. (6.2.24). The Spec map f of a f.f. ring map is submersive.

Proof: For a T that $f^{-1}(T)$ is closed, we see that $f^{-1}(T) \to T$ satisfies going-down because f does, so its complement is closed under specialization, so it is closed by(5.5.4) it is closed. So T is open. □

3 Faithfully Flat Descent

Prop. (6.3.1) (Faithfully Flat Descent). List of properties that descent through faithfully flat morphism.

1. Projectiveness for modules over a ring.
2. Finiteness for modules over a ring.
3. F.p. for modules over a ring.
4. Flatness for modules over a ring.
5. Formal Smoothness for ring maps.
6. Noetherian for rings over a ring.
7. Reducedness for rings over a ring.
8. Normal for rings over a ring.
9. Regular for rings over a ring.

Proof:

1. Cf.[StackProject 05A9].
2. Cf.[StackProject 03C4].
3. Cf.[StackProject 03C4].
4. Cf.[StackProject 03C4].
5. Use criterion(6.5.3), we see by flatness that the sequence $I/I^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$ commutes with flat base change, and when it is f.f., then use(1.2.3) and descent for projectiveness(6.3.1) that $\Omega_{S/R}$ is projective, so it is a split exact sequence.

6. Cf.[StackProject 033E].
7. Trivial as $S \to S'$ is f.f. hence injective(6.2.18).
8. Cf.[StackProject 033G].
9. Cf.[StackProject 07NG].

□

4 Syntomic

Def. (6.4.1). A ring map is called **syntomic** iff it is of f.p., flat and the fibers are all local complete intersection rings.

5 Smooth

Formally Smoothness

Def. (6.5.1). A ring map $R \to S$ is called **formally smooth** if for every R-ring A and an ideal I of A that $I^2 = 0$, a map $S \to A/I$ can extends to a map $S \to A$.

Formal smooth is stable under base change and composition, by universal arguments. A polynomial algebra is formally smooth.

Prop. (6.5.2). Giving a presentation $S = P/J$ where P is formally smooth(e.g. polynomial algebra), S is formally smooth iff there is a map $S \to P/J^2$ that is right converse to the obvious projection.

Proof: One way is from the definition of formally smooth applied to P/J^2 and J. Conversely, for any A and I, we notice the map $P \to S \to A/I$ can be lifted to $P \to A$, and J is mapped to I, so J^2 is mapped to 0, so we have a map $P/J^2 \to A$. Then $S \to P/J^2 \to A$ is the lifting. □

Cor. (6.5.3). If $P \to S$ is a presentation of S/R by polynomial algebra with kernel I, then S/R is formal smooth iff

$$0 \to I/I^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

is split exact as in(1.2.5).(by(1.2.5)).

Cor. (6.5.4) (Equivalence Definition). S/R is formally smooth iff $NL_{S/R}$ is quasi-isomorphic to a projective S-module at degree 0.

Proof: If S/R is formally smooth, then choose a presentation will suffice by(6.5.3). The converse is also true by projectiveness and(6.5.3). □

Cor. (6.5.5). If C/B is formally smooth, then the Jacobi-Zariski sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

as in(1.2.4) is split exact, by(1.2.15). In particular, any derivation of B to a C-module can be extended to a derivation C to a C-module.

Cor. (6.5.6). If $A \to B \to C$ with $A \to C$ formally smooth and $B \to C$ surjective with kernel I, then there is an split sequence

$$0 \to I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

by(1.2.15).

Standard Smooth Algebra

Def. (6.5.7). A **standard smooth algebra** over R is a algebra $S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_c)$, where $c \leq n$ and $J(\frac{f_1,\ldots,f_c}{X_1,\ldots,X_c})$ is invertible in S.

Prop. (6.5.8) (Standard Smooth Localization). If $R \to S$ is standard smooth, then $R \to S_g$ is standard smooth, and $R_f \to S_f$ is standard smooth(because stable under base change(6.5.9)).

Proof: For localization at $g \in S$, let h be an inverse image of g in $R[X_1, \ldots, X_n]$, then $S_g = R[X_1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_c, X_{n+1}h - 1)$, and it is standard smooth. □

Prop. (6.5.9). Standard smoothness is stable under base change and composition.

Proof: For base change, notice the Jacobi matrix is the base change of the Jacobi matrix, so it is also invertible. For composition, write out the presentation, the determinant is the product of the presentation. □

Prop. (6.5.10). A standard smooth algebra is a relative global complete intersection.

Proof: Cf.[StackProject 00T7]. □

Prop. (6.5.11) (Jacobian Criterion). For a f.p. ring $S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_c)$, S/R is standard smooth in a nbhd of q iff the Jacobian matrix has rank c at q, i.e. $J(\frac{f_1,\ldots,f_c}{X_1,\ldots,X_c})$ is not in q for some permutation of $X_1, \ldots, X_n$.

Proof: If $h = J(\frac{f_1,\ldots,f_c}{X_1,\ldots,X_c}) \notin q$, let $S_h = R[X1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_c, X_{n+1}g - 1)$ is a standard smooth algebra. Conversely, $S_h = R[X1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_c, X_{n+1}g - 1)$ is standard smooth for some $g \notin q$, so $J(\frac{f_1,\ldots,f_c}{X_1,\ldots,X_c}) \notin q$. □

Smooth

Def. (6.5.12) (Smooth Ring Map). A ring map $R \to S$ is called **smooth** if it satisfies the following equivalent conditions:

- It is of f.p. and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module placed at degree 0. In other words,
$$0 \to I/I^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$
is exact and $\Omega_{S/R}$ is finite S-projective. By(1.2.13), we only need to prove for a single presentation of S.
- It is locally standard smooth.
- It is formally smooth and of f.p..

Proof: $1 \to 3$: by(6.5.4). $3 \to 1$: By(6.5.4), $\Omega_{S/R}$ is f.p. and projective, so it is finite projective.

At this point we already know that the first definition is stable under base change and composition, because f.p. and formally smoothness both do(6.5.1).

And also the first definition is local on source because f.p. does(5.1.16) and NL commutes with localization(1.2.18) so we can use the local properties of triviality(5.1.15) and finite projectiveness(5.2.4).

Now it is also local on the source because it is stable under base change and composition and $R \to R_{f_i}$ does by locality on the source.

$2 \to 1$:Now the property are all local on source. It suffices to prove a standard smooth map is smooth: its naive cotangent complex is $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \to S[dX_1, \ldots, dX_n]$, and it is a split injection by linear algebra, and $\Omega_{S/R} = S[dX_{c+1}, \ldots, dX_n]$, so it is a smooth ring map.

$1 \to 2$:We need to prove, assuming the first definition, it is locally standard smooth. For this, Cf.[StackProject 00TA]. □

Cor. (6.5.13). Smoothness is stable under composition and base change. Smoothness is local on the source and target(In particular, $R \to R_f$ is smooth). (Already proved in the proof of(6.5.12)).

Cor. (6.5.14). A smooth map is syntomic, hence flat.

Proof: Cf.[StackProject 00TA]. □

Prop. (6.5.15). A smooth ring map $R \to S$ is a base change of smooth ring map over a ring f.g. over $\mathbb{Z}$.

Proof: Use the equivalence definition(6.5.2), we know that there is a map

$$S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_c) \to R[X_1, \ldots, X_n]/(f_1, \ldots, f_c)^2,$$

which if we write $\sigma(X_i) = h_i$, then must satisfy

$$f_i(h_1, \ldots, h_n) = \sum a_{ijk} f_j f_k.$$

Then we consider the subalgebra generated by f_i, h_i, a_{ijk}, then by the same reason, they form a smooth algebra over $\mathbb{Z}$, and its tensor with R gives out S. □

Cor. (6.5.16). The lifting property in the definition of formally smooth is true for any I that is locally nilpotent, when S/R is moreover smooth.

Proof: By the proposition, we can retract to a f.g. algebra S_0 over $\mathbb{Z}$, then the image of S_0 is a f.g. algebra A_0. Then A_0 is Noetherian and so I_0 is nilpotent, then we can use finite induction to find lifting to $S_0 \to A_0$ over $\mathbb{Z}$. □

Prop. (6.5.17) (Stalkwise). If $R \to S$ is f.p., then it is smooth iff it is S_q/R_p is smooth for every (maximal)prime q of S and p under it.

Proof: Because of f.p., we only need to check triviality of $H_1(NL)$ and finite projectivity of $\Omega_{S/R}$(fp used). But both triviality and fintie projectivity is stalkwise(5.1.15). (Notice $R \to R_p$ is smooth). □

Cor. (6.5.18) (Fiberwise). For a ring map $R \to S$ and q is a prime of S over p. Then S/R is smooth at q iff S/R is of f.p. and S_q/R_p is flat and $S \otimes k(p)/k(p)$ is smooth.

Proof: One way is because smooth is flat, f.p. and stable under base change. Conversely, Cf.[StackProject 00TF]. □

Cor. (6.5.19) (Smooth Points and Flat Base Change). If $R \to S$ is of f.p. and $R \to R'$ is flat. Then the set of primes in $S' = S \otimes_R R'$ that has a nbhd that is smooth over R' is the inverse image of set of primes in S that has a nbhd that is smooth over R.

Proof: One direction is because smooth is stable under base change. Conversely, the local ring map is f.f., so $H_1(NL_{S'/R',q}) = H_1((NL_{S/R}\otimes_S S')_q) = H_1(NL_{S/R,p}\otimes_{S_p} S'_q)$. Then the result follows as S'_q/S_p is f.f. and triviality and finite projective descents for f.f. map(6.3.1). □

Cor. (6.5.20) (Smooth and Field Extension). For a f.g. algebra S over k and a field extension K/k, $S\otimes_k K$ is smooth over K iff S is smooth over k.

Proof: Because field extension is f.f. hence $\operatorname{Spec} S'\to\operatorname{Spec} S$ is surjective(6.2.12). □

Smooth over Fields

Prop. (6.5.21). A smooth k-algebra is a local complete intersection.

Proof: Cf.[StackProject 00T5]. □

Lemma (6.5.22). Let S be f.g. over a alg.closed field k and $\mathfrak{m}$ a maximal ideal, then the following are equivalent:

- S_m is regular.
- $\dim_k \Omega_{S/k}\otimes k\leq \dim S_{\mathfrak{m}}$
- $\dim_k \Omega_{S/k}\otimes k = \dim S_{\mathfrak{m}}$
- S/k is smooth in a nbhd of $\mathfrak{m}$.

Proof: Cf.[StackProject 00TS]. □

Prop. (6.5.23) (Smooth Differential Criterion). For a ring S f.g. over a field, then S is smooth in a nbhd of q iff $\dim_{k(q)}\Omega_{S/k}\otimes k(q)\leq \dim S_q$.

And in this case, equality hold, and S_q is regular.

Proof: Cf.[StackProject 00TT]. □

Cor. (6.5.24). Because regular is checked at closed pts(5.11.9), a f.g. algebra over a field k is smooth iff it is regular.

Prop. (6.5.25) (Smooth and Regular). Let S be f.g. over a field k, if $k(q)/k$ is separable(e.g. char 0) for q a prime of S, then S is smooth in a nbhd of p iff S_q is regular.

Proof: Cf.[StackProject 00TV]. □

Prop. (6.5.26). An injective morphism of domains is smooth at (0) iff the quotient field map is separable.

Proof: Cf.[StackProject 07ND]. □

6 Unramified

Formally Unramified

Def. (6.6.1). A ring map $R\to S$ is called **formally unramified** if for every R-ring A and an ideal I of A that $I^2=0$, a map $S\to A/I$ has at most one extension to a map $S\to A$.

Formally unramified is equivalent to $\Omega_{S/R}=0$. So it is stable under composition by Jacobi-Zariski sequence(1.2.4).

Proof: Let $J = \mathrm{Ker}(S \otimes_R S \to S)$, let $A_{univ} = S \otimes_R S/J^2$, then $J/J^2 \cong \Omega_{S/R}$(1.2.2), so we have two natural map from S to A_{univ}, they differ by the universal differential $S \to \Omega_{S/R}$. If S/R is unramified, then $ds = 0$ for all $s \in S$, so $\Omega_{S/R} = 0$.

Conversely, if there is a A and A/J that there are two liftings τ_1, τ_2, then we let $A_{univ} \to A$ defined by $s_1 \otimes s_2 \to \tau_1(s_1)\tau_2(s_2)$, this is well-defined, and because $A_{univ} \cong S$, this map descends to S, so $\tau_1(s_1 s_2) = \tau_2(s_1 s_2)$. □

Prop. (6.6.2) (Formally Unramified Stalkwise). Formally unramified is stalkwise both on the source and target(5.1.15).

Prop. (6.6.3). Colimits of formally unramified rings over R is formally unramified. (Trivial as one renders on the diagram in the definition of formally unramified).

Unramified Map

Def. (6.6.4). A ring map is called **unramified** iff it is formally unramified and f.g..

A ring map is called G-**unramified** iff it is formally unramified and of f.p.. In particular, an étale map is G-unramified.

These two notions are stable under composition and base change. These two notions are local on the source and target. $R \to R_f$ is G-unramified.(6.6.1)(5.1.15)

Prop. (6.6.5). $R \to R/I$ is unramified, and if I is f.g., then it is G-unramified. (Trivial).

Prop. (6.6.6) (Stalkwise and Fiberwise). If $R \to S$ is of f.t(f.p.), then it is unramified(G-unramified) at a prime q of S iff $(\Omega_{S/R})_q = 0$ iff $\Omega_{S/R} \otimes_S k(q) = 0$ iff $(\Omega_{S\otimes k(p)/k(p)})_q = 0$ iff $\Omega_{S\otimes k(p)/k(p)} \otimes k(q) = 0$.

Proof: By Nakayama, two pair of them are equivalent, and if $\Omega_{S/R,q} = 0$, then $\Omega_{S/R,g} = 0$ for some $g \notin q$(because support of finite module is open), so $R \to S_g$ is (G-)unramified. And notice in fact $\Omega_{S/R} \otimes_S k(q) = \Omega_{S\otimes k(p)/k(p)} \otimes_{k(p)} k(q)$. □

Prop. (6.6.7) (Equivalent Definition of Unramifiedness). A f.g. ring map $R \to S$ is unramified at a prime q of S over p iff $pS_q = qS_q$ and $k(q)/k(p)$ is finite separable.

Proof: Suppose $R \to S_g$ is unramified, then $S \otimes k(p)$ is unramified over $k(p)$, hence by(6.5.23), it is also smooth, so it is étale, and(6.7.7) gives the result.

For the converse, Cf[StackProject 02FM]. □

Prop. (6.6.8). A ring map is unramified iff it is locally a quotient of a standard étale map.

Proof: Cf.[StackProject 0395]. □

Prop. (6.6.9). Any G-unramified map is a base change of a G-unramified map over a ring R_0 f.g. over $\mathbb{Z}$. And similarly any unramified map is a quotient of a base change of a G-unramified map over a ring R_0 f.g. over $\mathbb{Z}$.

Proof: Let $S = R[X_1, \ldots, X_n]/(g_1, \ldots, g_c)$, then we have $dX_i = \sum a_{ij} dg_j + a_{ijk} g_j dX_k$, so we let R_0 be generated by g_i, a_{ij}, a_{ijk}, so $S_0 = R_0[X_1, \ldots, X_n]/(g_1, \ldots, g_c)$ is G-unramified. □

Prop. (6.6.10). If $R \to S$ is unramified, then $S \times_R S \to S$ is isomorphic to $S \otimes_R S \to (S \otimes_R S)_e$ for some idempotent $e \in S \otimes_R S$.

Proof: The kernel I satisfies $I/I^2 = 0$, and I is f.g. because S is f.g, so we can use(5.5.1). □

7 Étale

Formally Étale

Def. (6.7.1). A ring map $R \to S$ is called **formally étale** iff it is formally smooth and formally unramified.

Prop. (6.7.2). Colimits of formally étale rings over R is formally étale. (The lifting are compatible because of uniqueness).

Prop. (6.7.3). $R \to S^{-1}R$ is formally étale.

Proof: It suffice to prove that if $\varphi(s)$ is invertible modulo I, then $\varphi(s)$ is invertible, but this is true because I is nilpotent. □

Étale Map

Def. (6.7.4). A ring map $R \to S$ is called **étale** if the naive cotangent complex is exact, i.e. $I/I^2 \cong \Omega_{P/R} \otimes_P S$.

In particular, étale is equivalent to smooth+formally unramified($\Omega_{R/S} = 0$).

Cor. (6.7.5) (Properties of Étale).
1. Étale map is stable under base change and composition.
2. Étale map is local on the source and target. In particular, $R \to R_f$ is étale.
3. If $R \to S$ is of f.p. and $R \to R'$ is flat. Then the set of primes in $S' = S \otimes_R R'$ that has a nbhd that is étale over R' is the inverse image of set of primes in S that has a nbhd that is étale over R. (The same as(6.5.19)).
4. Étale map is syntomic, hence flat.
5. Any Étale map is a base change of an étale map over a ring R_0 f.g. over $\mathbb{Z}$. (Cf.[StackProject 00U2]).

Prop. (6.7.6) (Étale over Fields). An algebra over a field k is étale iff it is a finite product of finite separable extensions of k.

Proof: Cf.[StackProject 00U3]. □

Cor. (6.7.7). If $R \to S$ is étale at a nbhd of a prime q of S over p, then $pS_q = qS_q$, and $k(q)/k(p)$ is finite separable.

Proof: We can replace S by S_g so S_g/R is étale. Then $S \otimes k(p)/k(p)$ is étale, that is S_p/pS_p is a finite product of finite separable fields, so $S_q/pS_q = (S_p/pS_p)_q =$ some separable closed field. □

Lemma (6.7.8). If $R \to S$ is an étale map and q is a prime of S over p, then S/R is étale in a nbhd of q if
- $R \to S$ is of f.p.
- $R_p \to S_q$ is flat.
- $pS_q = qS_q$.
- $k(q)/k(p)$ is a finite separable field extension.

Proof: Cf.[StackProject 00U6]. □

Prop. (6.7.9) (Equivalent Definition of Étale). A ring map $R \to S$ is **étale** iff it is flat, of f.p. and $\Omega_{S/R}$ vanishes.

Proof: One way is by definition, and the converse is by(6.7.8) and(6.6.7). □

Prop. (6.7.10). A ring map of f.p. is formally étale iff it is étale. (Because in this case, formally smooth is equivalent to smooth(6.5.12).)

Prop. (6.7.11). Any étale map is standard smooth.

Proof: $I/I^2 \cong \Omega_{P/R} \otimes_P S$, so I/I^2 is free, so by(5.12.11), there is a presentation of S that $f_1, \ldots, f_c$ freely generate I/I^2, then obviously $c = n$ and $J(\frac{f_1,\ldots,f_c}{X_1,\ldots,X_n})$ is invertible in S, i.e. S is standard smooth. □

Prop. (6.7.12). If S/R and S'/R are étale, then any R-algebra map $S \to S'$ is étale.

Proof: $S \to S'$ is of f.p. by(5.12.9), the rest Cf.[StackProject 00U7]. □

Def. (6.7.13). A ring map $R \to R' = R[X]_g/(f)$ is called **standard étale** iff f is monic and the derivative f' is invertible in R'.

Standard étale is stable under base change and principal localization, but not stable under composition.

Prop. (6.7.14) (Étale and Standard Étale). A ring map is étale iff it is locally standard étale.

Proof: For a standard étale algebra $R[X]_g/(f) = R[X, Y]/(f, gY - 1)$ which is standard smooth and $\Omega_{R'/R} = 0$(1.2.6), so it is étale. To prove if it is locally standard étale then it is étale, Cf.[StackProject 00UE]. □

Prop. (6.7.15). Giving any ring R and a prime p, if there is a finite separable extension $L/k(p)$, then there is a standard étale map $R \to R'$ that for some q', $k(q') \cong L$ over k.

Proof: $L = k(p)[\alpha]$ by primitive element theorem, so the minimal polynomial of α is separable, and if we change α to $c\alpha$ for some $c \in k(p)$, we can assume f can be lifted to a $f \in R[X]$. Now $f'(\alpha)$ is invertible in L, so there is a map from $R[X]_{f'}/(f)$ to L, whose kernel gives the desired prime q. □

Étale over Fields

Prop. (6.7.16) (Étale and Unramified over Fields). A f.g. algebra is étale over field k iff it is G-unramified over it, by(6.5.23).

8 Separability

Basic reference is [Weibel Chap P309] and [StackProject 10.41].

Def. (6.8.1). A f.d simisimple algebra R over a field k is called **separable** iff for every field extension l/k, $R \otimes_k l$ is semisimple.

Prop. (6.8.2).

9 Jacobson Ring

Def. (6.9.1). The **Jacobson radical** $J = \mathrm{rad}(R)$ is the intersection of all maximal primes of R. $J = \{r \in R | 1 + rs \text{ is a unit } \forall s \in R\}$.

The **nilradical** is the intersection of all primes. It consists of nilpotent elements.

Proof: Jacobson: One way is trivial and for the other if r is not in a maximal ideal $\mathfrak{m}$, then $(r) + \mathfrak{m} = (1)$, so contradiction.

Nilpotent: Every nilpotent element is contained in every prime, and if a is not nilpotent, then A_a is nonzero, hence there is a maximal ideal, i.e. there is a prime of A not containing a. □

Def. (6.9.2). A commutative ring is called **Jacobson** if every prime ideal is an intersection of maximal ideals. In particular, the Jacobson radical equals the nilradical. This is equivalent to every radical ideal is an intersection of maximal primes.

Prop. (6.9.3). R is Jacobson iff $\mathrm{Spec}\, R$ is Jacobson space(1.12.8). In particular, the closed pts are dense in any closed subsets (Hilbert's Nullstellensatz satisfied).

Proof: We need to show that a locally closed subset contains a closed pt, we assume this set is of the form $V(I) \cap D(f)$, I is radical, then $f \notin I$, then by the condition, there is a $I \subset \mathfrak{m}$ that $f \notin \mathfrak{m}$, thus the result.

Conversely, for a radical ideal, let $J = \cap_{I \subset \mathfrak{m}} \mathfrak{m}$, then J is radical and $V(J)$ is the closure of $V(I) \cap X_0$, $V(I) = V(J)$, and because they are both radical, $I = J$. □

Cor. (6.9.4). If R is Jacobson, then R/I is Jacobson and R_f is Jacobson. And maximal ideals of R_f are maximal in R. (Immediate from(6.9.3) and(1.12.10)).

Prop. (6.9.5) (Generalized Nullstellensatz). If R is Jacobson and S is a finitely generated R-algebra, then S is Jacobson and the maximal ideal of S intersect with R a maximal ideal, and the quotient ring extension is finite, (hence algebraic).

In particular, a f.g. algebra over a ring of dimension 0, (e.g. Artinian ring or field) is Jacobson.

Proof: Cf.[StackProject 00GB]. □

10 Japanese& Nagata Rings

Def. (6.10.1). Let R be a domain with quotient field K, then R is called N**-1** iff the integral closure of R in K is a finite R-module.

R is called N-2 or **Japanese** iff for any finite field extension L/K, its integral closure in L is a finite R-module.

A ring R is called **universally Japanese** if for any finite type domain S/R, S is Japanese.

A ring R is called **Nagata** if it is Noetherian and for any prime p, R/p is Japanese.

Prop. (6.10.2). A f.g. algebra A over a field is Nagata.

Proof: Cf.[Hartshorne P20]. □

Cor. (6.10.3). The normalization of a f.g. integral domain over a field is f.g. over A.

11 Separably Generated Field Extension

Basic reference is [Matsumura Ch10].

Def. (6.11.1). A field extension K/k is called **separably generated** iff it K is a separable algebraic extension of a purely transcendental field L/k. An algebra A/k is called separable iff $A \otimes_k k'$ is reduce for any k'/k algebraic.

Prop. (6.11.2). Let K/k by f.g. field ext, then K/k is separable algebra $\iff$ K/k is separably generated $\iff$ $K \otimes_k k^{1/p}$ is reduced, Cf.[Masumura P195].

12 Dedekind Domain

Def. (6.12.1). A Dedekind domain is an integrally closed Noetherian domain of dimension 1.

Prop. (6.12.2). The integral closure of a Dedekind domain in a finite extension fields of its quotient fields is again a Dedekind domain.

13 Henselian Local Ring

Basic References are [StackProject Chap10.148].

Def. (6.13.1). A local ring $(R, \mathfrak{m}, k)$ is called **Henselian** iff for every $f \in R[X]$ and $a_0 \in k$ that $\overline{f}(a_0) = 0$ and $\overline{f}'(a_0) \neq 0$, then there is a root α of f lifting a_0. It is called **strict Henselian** if moreover its residue field is separably closed.

14 Local Algebra

Prop. (6.14.1). If A is a Noetherian local integral domain with residue field k and quotient field K, if M is a f.g. A-module that $\dim_k M \otimes_A k = \dim_K M \otimes_A K = r$, then M is free of rank r.

In other words, if the rank of M at the generic point and closed pt of B are the same, then M is free.

Proof: First M is generated by r elements by Nakayama and the kernel R of $A^r \to M$ vanishes when tensoring K, thus vanish because it is torsion-free. $\square$

15 Dualizing Module

I.7 Homological Algebra

1 Category

Exactness

Prop. (7.1.1). In an Abelian category, the functor $X \mapsto \mathrm{Hom}(X, Y)$ and $X \mapsto \mathrm{Hom}(Y, X)$ is both left exact. Note that left and right is seen on the image.

Adjointness

Prop. (7.1.2). A right adjoint functor is left exact and it preserves injectives if its left adjoint is exact.

A left adjoint functor is right exact and it preserves projectives if its right adjoint is exact.

Prop. (7.1.3). Any presheaf on a small category is a colimit of representable sheaves h_X. (Consider all $h_X \to \mathcal{F}$ and take colimit, prove it is isomorphism).

Prop. (7.1.4). The sheaf Γ functor is right adjoint to the constant sheaf functor over arbitrary site.

Prop. (7.1.5). The inclusion functor is right adjoint to the shifification functor over arbitrary site.

Prop. (7.1.6). The forgetful functor is right adjoint to the Shifification functor, and shifification is exact, so it preserves injectives.

Prop. (7.1.7). The stalk functor is left adjoint to the skyscraper sheaf operator.

Prop. (7.1.8). The valuation at k'th coordinate is left adjoint to the functor $k_*(A)(i) = \prod_{\mathrm{Hom}\, i,k} A$ and is exact. So k_* preserves injectives.

Kan Extension

Prop. (7.1.9) (Yoneda Lemma). $X \mapsto (Y \mapsto \mathrm{Hom}(Y, X))$ is a fully faithful embedding from $\mathcal{C}$ to $\hat{\mathcal{C}} = \mathrm{Func}(\mathcal{C}^\circ, \mathrm{Set})$. Thus if a $X \to Y$ induces isomorphism $\mathrm{Hom}(W, X) \to \mathrm{Hom}(W, Y)$ for every W, then $X \cong Y$.

So we can regard $\mathcal{C}$ as a fully faithful subcategory of $\widehat{\mathcal{C}}$.

Proof: In fact, there is a bijection $\mathrm{Hom}(h_X, F) \cong F(X)$ that maps a u to $u(X)(\mathrm{id}_X)$. We can define the inverse map as $x \in F(X) \mapsto (s \in \mathrm{Hom}(Y, X) \mapsto s^*(x) \in F(Y)) \in \mathrm{Hom}(h_X, F)$. □

Prop. (7.1.10). A presheaf of sets in C, i.e. $C^{op} \to \mathcal{S}et$ is a colimit of presentable sheaves of C. More precisely, there is an isomorphism

$$\mathcal{F} \cong \varinjlim_{h_X \to \mathcal{F}} h_X.$$

From this we see that any morphism $\widehat{\mathcal{C}} \to D$ is determined by its restriction on $\mathcal{C}$.

Proof: For any presheaf $\mathcal{G}$, there is a morphism $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\varinjlim_{h_X \to \mathcal{F}} h_X, \mathcal{G})$, i.e. a set of sections $f_s \in \mathcal{G}(X)$ for every $h_X \xrightarrow{s} \mathcal{F}$, that if $t \circ u = s$, then $u^*(f_t) = f_s$. Conversely, by Yoneda lemma, this just says that there is a morphism of presheaves $\mathcal{F} \to \mathcal{G} : F(X) \to G(X) : s \mapsto f_s$. □

Cor. (7.1.11) (Kan Extension). For a cocomplete category $\mathcal{D}$, there is a natural bijection between functor $\widehat{\mathcal{C}} \to \mathcal{D}$ that commutes with colimits and functors $\mathcal{C} \to \mathcal{D}$ by Yoneda embedding.

Proof: For this, we only have to notice the functor $\mathcal{D} \to \widehat{\mathcal{C}} : D \to \mathrm{Hom}(FX, D)$ is right adjoint to $F : \widehat{\mathcal{C}} \to \mathcal{D}$ when F is defined by colimit as in(7.1.10). $\square$

Cor. (7.1.12). Any contravariant functor $F : \widehat{\mathcal{C}} \to \mathit{Set}$ that take colimits to limits, F is representable. (Just use G in the last proof, F is representable by $G(\mathrm{pt})$.

Prop. (7.1.13) (Ends and Coends). Cf.[MacLane].

Prop. (7.1.14) (Category Equivalence). A Functor $\mathfrak{C} \to \mathfrak{D}$ is an equivalence if and only if it's fully faithful and essentially surjective.

Proof: There exist an object $G(X) \in \mathfrak{C}$ and an isomorphism $\xi_X : FG(X) \to X$ for every $X \in \mathfrak{D}$. Because F is fully faithful, there exists a unique morphism $G(f) : G(X) \to G(Y)$ such that $F(G(f)) = \xi_Y^{-1} \circ f \circ \xi_X$ for every morphism $f : X \to Y$ in $\mathfrak{D}$. Thus we obtain a functor $G : \mathfrak{C} \to \mathfrak{D}$ as well as a natural isomorphism $\xi : F \circ G \to \mathrm{Id}_{\mathfrak{D}}$. Moreover, the isomorphism $\xi_{F(Z)} : FGF(Z) \to F(Z)$ decides an isomorphism $\eta_Z : GF(Z) \to Z$ for every $Z \in \mathfrak{C}$.This yields a natural isomorphism $\eta : G \circ F \to \mathrm{Id}_{\mathfrak{C}}$. $\square$

Abelian Category

Prop. (7.1.15) (Axioms for Abelian Category).

- **A1**:$\mathrm{Hom}(X, Y)$ is an Abelian group.
- **A2**:There exists a zero object.
- **A3**:There exists a canonical sum&product with projections, and the sum induce the Abelian structure of $\mathrm{Hom}(X, Y)$.

(Satisfying this three is called a additive category.)

- **A4**:Coimage equals image.

Remark (7.1.16). WARNING: An additive category that epimorphism+monomorphism is isomorphism need not be an Abelian category. Cf.[https://mathoverflow.net/questions/41722/is-every-balanced-pre-abelian-category-abelian] for a counter-example.

Prop. (7.1.17). The $\mathrm{Hom}(X, -)$ operator is left exact in Abelian category by definition.

Prop. (7.1.18). Axiom A3 asserts the good existence of product and sum of objects as we wanted, and it can be used to prove that monomorphism and epimorphism are stable under pushout and pullback.

But this uses $A4$ strongly, Cf.[MacLane Categories for working mathematicians P203]. (For epimorphism, first prove $0 \to X \times_U Y \to X \times Y \to U \to 0$ is exact when $X \to U$ is epi).

Prop. (7.1.19). equalizer and finite product derives finite limit, thus finite limits and finite colimits exists in Abelian categories.

Prop. (7.1.20) (Mitchell's embedding theorem). If $\mathcal{A}$ is a small category, then there exists a unital ring R, not necessary commutative and a fully faithful and exact functor $\mathcal{A} \to R-$mod that preserves kernel and cokernel. WARNING: it may not preserve sum and product, let alone limits and colimits.

Prop. (7.1.21). If $\mathcal{C}, \mathcal{A}$ are categories and $\mathcal{A}$ is Abelian, then $\mathcal{H}om(\mathcal{C}, \mathcal{A})$ is an Abelian category. In particular, $Ch(\mathcal{A})$ is Abelian.

Serre Subcategory

Def. (7.1.22). A **Serre subcategory** of an Abelian category is a non-empty full subcategory $\mathcal{C}$ that if

$$A \to B \to C$$

is exact and $A, C \in Ob(\mathcal{C})$, then $B \in Ob(C)$.

A **weak Serre subcategory** of an Abelian category is a non-empty full subcategory $\mathcal{C}$ that if

$$A \to B \to C \to D \to E$$

is exact and $A, B, D, E \in \mathcal{C}$, then $C \in \mathcal{C}$.

Prop. (7.1.23). For an exact functor between Abelian categories, the objects that mapped to 0 forms a Serre subcategory. And any Serre subcategory is the kernel of a essentially surjective map.

Proof: The idea is to localize at all the morphisms that has kernel and cokernel in $\mathcal{C}$. Cf.[StackProject 02MS]. □

Prop. (7.1.24). For a Serre subcategory $\mathcal{B}$ of an Abelian category $\mathcal{A}$, the set of all complexes that has cohomology group in $\mathcal{B}$ is a strictly full triangulated subcategory of $\mathcal{D}(\mathcal{A})$. Cf.[StackProject 06UQ].

Others

Def. (7.1.25). In an Abelian category, an injection $A \to B$ is called **essential** iff every non-zero subobject of B intersects A. A surjection is called **essential** iff every proper subobject of A is not mapped to B.

Grothendieck Abelian Category

Prop. (7.1.26) (Axioms for Grothendieck Abelian Category).

- **AB3**: It is an Abelian category and arbitrary direct sums exists. (Thus colimits over small categories exists.)
- **AB5**: Filtered colimits over small categories are exact. This is equivalent to { for any family of subobjects $\{A_i\}$ of A to B indexed by inclusion can induce a morphism $\sum A_i \to B$ (internal sum)}?
- **GEN**: It has a generator, that is, an object U s.t. for any proper subobject $N \subsetneq M$, there is a map $U \to M$ that doesn't factor through N.

Prop. (7.1.27). The presheaf category $\mathcal{A}^C$ is a Grothendieck Abelian category if $\mathcal{A}$ is Grothendieck Abelian.

Proof: For the presheaf, the only problem is the existence of generator, for that, just construct a family of presheaves and sum them. Take $Z_X = i_X(U)$, where U is the generator of $\mathcal{A}$ then $F(X) = \mathrm{Hom}(Z_X, F)$ by adjointness(1.2.4). So they are a family of generators. □

Prop. (7.1.28) (Injectives). In a Grothendieck Abelian category with generator U, an object is injective iff it is extendable over subobjects of U. (AB5 assures we can extend by Zorn's lemma. Then use GEN, Cf.[StackProject 079G]). If it is a family of objects, it suffice to extend over each one of them.

Prop. (7.1.29). Grothendieck Abelian category has a functorial injective embedding, Cf.[StackProject 079H].

Prop. (7.1.30). A contravariant functor from a Grothendieck category to $\mathcal{S}ets$ is representable iff it takes colimits to limits.

Proof: $M \oplus M \to M$ with induce a map $F(M) \times F(M) \to F(M)$ thus $F(M)$ is a semigroup, and the inverse of id_Min $\mathrm{Hom}(M, M)$ maps to a $F(M) \to F(M)$ which is the inverse, Thus in fact F is a left adjoint functor to Ab.

Let U be a generator, $A = \sum_{s \in F(U)} U$, let $s_{univ} = (s) \in F(A) = \prod_{s \in F(U)} F(U)$. let A' be the largest objects that s_{univ} restrics to 0 in A', let $\overline{s}_{univ}$ be in $F(A/A')$ that maps to s_{univ} in $F(A)$ (because F is left exact). Then we claim $(A/A', \overline{s}_{univ})$ represents F. Cf.[StackProject 07D7]. □

Cor. (7.1.31). Grothendieck Category satisfies AB3*. (because $F = \prod_i \mathrm{Hom}(-, M_i)$ commutes with colimits).

Examples of Grothendieck Category

Prop. (7.1.32). The category of R-modules is a Grothendieck Abelian category with generator R because in R-mod category, taking filtered colimits is exact. (Diagram chasing).

Prop. (7.1.33). The category of Abelian presheaves and Abelian sheaves on a site is a Grothendieck category.

Proof: For the presheaf, C.f(7.1.27). For the sheaf, it follows from (7.1.35). □

Remark (7.1.34). The category of Abelian sheaves doesn't satisfy AB4*, i.e. not every limit of epimorphisms is epimorphism.

Proof: Consider the constant sheaf $\oplus B(\frac{p}{q}, \frac{1}{n})$ on $[0, 1]$. □

Prop. (7.1.35). The category of sheaf of $\mathcal{O}_X$-modules on a ringed site is a Grothendieck Abelian category. Moreover, injectives are flabby.

Proof: For a family of generators, take $j_!\mathcal{O}_U$ as the representative for $\Gamma(U, -)$, which is the sheaf associated to the sheaf Z_U in the proof of(7.1.27). Use $j_!\mathcal{O}_U$, we can see injectives are flabby, (because $j_!\mathcal{O}_U \to j_!\mathcal{O}_V$ is a monomorphism for $V \subset U$). □

Prop. (7.1.36). The category of Qco sheaves on a scheme is Grothendieck category, and we have a **coherentor** left adjoint to the forgetful functor.

Proof: Qco: First by(2.1.23), Qco is an Abelian category, and on affine open set, the colimit is an Qco sheaf, thus the limit exists in Qco and equals that of limits in the category of sheaves, thus filtered colimits is exact because $\mathcal{O}_X$-Mod is Grothendieck(7.1.35). The generator exists, Cf.[StackProject 077P].

The coherentor exists by the fact that $h_{\mathcal{F}}$ commutes with colimits and by the property of Grothendieck category(7.1.30). □

Lemma (7.1.37) (Gabber). Let X be a scheme, then there exists a cardinal κ that every Qco sheaf is a colimit of its κ-generated Qco subsheaves. Cf.[StackProject 077N].

Morita Equivalence

Basic References are [Morita Equivalence] and [Fuller Rings and Categories of Modules].

Def. (7.1.38). Two ring R, S are called **Morita equivalent** if the category of mod-R is equivalent to the category of mod-S.

Prop. (7.1.39). For an Abelian category $\mathcal{A}$ satisfying AB3 (i.e arbitrary sum exists), An object P of $\mathcal{A}$ is a **progenerator** if the functor $h' : X \mapsto \mathrm{Hom}_{\mathcal{A}}(P, X)$ is exact and and strict: $h'(X) = 0 \to X = 0$. Then h' determines an equivalence from $\mathcal{A}$ to mod-R, where $R = \mathrm{Hom}_{\mathcal{A}}(P, P)$.

Similarly, if $\mathcal{A}$ is an Abelian Noetherian category and P is a progenerator, then R is Noetherian and $\mathcal{A}$ is equivalent to the category of finitely generated R-categories.

Proof: Essentially surjective: construct using direct limit and cokernel.

Notice that $h'(X) \cong h'(X') \to X \cong X'$ by strictness and A4 axiom. So let $X = \mathrm{Coker}(P^{\oplus I}, P^{\oplus J})$,

$$\begin{aligned}
\mathrm{Hom}(h'(X), h'(Y)) &= \mathrm{Hom}(\mathrm{Coker}(h'(P^{\oplus J}), h'(P^{\oplus I}), h'(Y)) \\
&= \mathrm{Ker}(\mathrm{Hom}(h'(P^{\oplus J}), h'(Y)) \to \mathrm{Hom}(h'(P^{\oplus I}), h'(Y))) \\
&= \mathrm{Ker}(h'(Y^{\Pi I}) \to h'(Y^{\Pi J})) \\
&= \mathrm{Hom}(X, Y)
\end{aligned}$$

□

Prop. (7.1.40). In the case when A is the category mod-R, P is a generator $\iff$ $h' : X \mapsto \mathrm{Hom}_R(P, X)$ is faithful $\iff$ every M is a quotient of direct sums of P. And a **progenerator** is a f.g. projective generator.

Prop. (7.1.41). Let P be a (A, B)-bimodule, iff P is a progenerator as a right B module, then it is a progenerator as a left A module.

Prop. (7.1.42). Let P be a progenrator as a

Prop. (7.1.43) (Morita). The following are equivalent:

- categories A-mod and B-mod are equivalent.
- categories mod-A and mod-B are equivalent.
- There exist a finitely generated progenerator P of mod-A that $B \cong \mathrm{End}_A P$.

Proof: $2 \to 3$:A is a progenerator in mod-A, thus when $A \sim B$, F : mod-$A \to$ mod-B, $A \cong \mathrm{End}_A A = \mathrm{End}_B F(A)$, and $F(A)$ is a left A module as well as a progenerator of B. Thus there is a (A, B)-bimodule P that $A \cong \mathrm{End}_B P$, and a (B, A)-bimodule Q that $B \cong \mathrm{End}_A Q$. □

Prop. (7.1.44). There can be defined another Morita invariance that $R \sim S$ iff there are (R, S)-bimodule P and (S, R)-bimodule Q that $P \otimes_S Q \cong R$ as a (R, R)-bimodule and $Q \otimes_R P \cong S$ as a (S, S)-bimodule. This will immediately generate equivalence between R-mod and S-mod as well as equivalence between mod-R and mod-S by tensoring. And P and Q are projective modules respectively, because equivalence is a kind of adjoint.

Prop. (7.1.45) (Properties Preserved under Morita Invariance). Cf.[RIngs and Categories of Modules P54].

Fiber Product

Prop. (7.1.46). We have $(X\times_E Y)\times_S(Z\times_F W)=(X\times_S Z)\times_{E\times_S F}(Y\times_S W)$.

Prop. (7.1.47). The diagonal map $X\to X\times_Y X$ is an isomorphism iff $X\to Y$ is monomorphism. (This is equivalent to $\mathrm{pr}_1=\mathrm{pr}_2$).

Others

Prop. (7.1.48) (Eckmann-Hilton argument). If $\circ$ and $\otimes$ is two unital binary operator that commutes:$(a\otimes b)\circ(c\otimes d)=(a\circ c)\otimes(b\circ d)$, then they are equal and in fact commutative and associative. Cf.[Wiki].

Prop. (7.1.49). The group objects in the category of groups is abelian groups.

Proof: By Eckmann-Hilton argument, the category multiplication is the same as the group multiplication, so the unit is obviously the sam unit, thus the inverse. So the commutativity of m with inverse implies that it is abelian. □

Prop. (7.1.50). One should notice that the group object structure in any category $(m,id,i,X$ definition) is equivalent to a group structure on $\mathrm{Hom}(Y,X)$ that are preserve under composition with morphisms.

2 Cohomology of Complexes

Remark (7.2.1). Remember the translation operator $K[n]$ makes the complex lower n dimensions.

Def. (7.2.2). A **universal δ functor** between Abelian categories is one that any natural transformation from T^0 to another δ-functor will generate a δ-map. A **effaceable δ** functor is one that for any $n>0$ and any object A, there is an injection $A\to B$ that $T^n(A)\to T^n(B)=0$.

Prop. (7.2.3) (Grothendieck). A δ-functor is universal if it is effaceable.

Proof: We construct by induction on n. choose a $0\to A\to B\to C\to 0$ such that $T^{n+1}(A)\to T^{n+1}(B)=0$ then there is an isomorphism $T^{n+1}(A)\cong\mathrm{Coker}(T^n(B)\to T^n(C))$, and so we can construct the map on T^{n+1} induces by

$$\mathrm{Coker}(T^n(B)\to T^n(C))\to\mathrm{Coker}(G^n(B)\to G^n(C))\to G^{n+1}(A).$$

This can be verified to be a δ map. □

Def. (7.2.4) (Cone & Cylinder). The **mapping cone** of $f:K^\bullet\to L^\bullet$ is the complex $C(f)^\bullet$ that:

$$C(f)=K[1]^\bullet\oplus L^\bullet,\quad d(k^{i+1},l^i)=(-d_Kk^{i+1},f(k^{i+1})+d_Ll^i)$$

The **mapping cylinder** of $f:K^\bullet\to L^\bullet$ is the complex $\mathrm{Cyl}(f)$ that:

$$\mathrm{Cyl}(f)=K^\bullet\oplus K[1]^\bullet\oplus L^\bullet,\quad d(k^i,k^{i+1},l^i)=(d_Kk^i-k^{i+1},-d_Kk^{i+1},f(k^{i+1})+d_Ll^i)$$

It is a shame I haven't see clearly the similarity of this with the topological cone and cylinder, should study it further.

Prop. (7.2.5) (Distinguished Triangle of $K^*(\mathcal{A})$). For any morphism $K^\bullet \to L^\bullet$, there exists a termwise-spliting exact sequence of Complexes commuting in $K(\mathcal{A})$.

$$\begin{tikzcd} & K^\bullet \arrow[r] \arrow[d, equal] & L^\bullet \arrow[d, "\alpha"] & & \\ 0 \arrow[r] & K^\bullet \arrow[r] & \mathrm{Cyl}(f) \arrow[r] \arrow[d, "\beta"] & C(f) \arrow[r] \arrow[d, equal] & 0 \\ & 0 \arrow[r] & L^\bullet \arrow[r] & C(f) \arrow[r] & K^\bullet[1] \arrow[r] & 0 \end{tikzcd}$$

where $\beta\alpha = \mathrm{id}$ and $\alpha\beta \sim \mathrm{id}$. And $K^\bullet \to L^\bullet \to C(f) \to K^\bullet[1]$ is called a distinguished triangle. Any exact triple of complexes in $\mathrm{Kom}(\mathcal{A})$ is quasi-isomorphic to a distinguished triangle. In fact, we can define the distinguished triangle in $K(\mathcal{A})$ as that induced by a split exact sequence, Cf.[StackProject 014L].

Notice all this can imitate the similar parallel construction in the topology category.

Proof: Cf.[Gelfand P157] □

Cor. (7.2.6). A distinguished triangle will induce a long exact sequence, for this, just need to verify that the δ-homomorphism coincide with the morphism that $C(f) \to K^\bullet[1]$ induces.

Cor. (7.2.7). A morphism $f : K \to L$ is quasi-iso iff $C(f)$ is acyclic. It is homotopic to 0 iff f can be extended to a morphism $C(f) \to L$.

Prop. (7.2.8) (Five lemma). In an Abelian category, if there is a diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d, "s"] & * \arrow[r] \arrow[d, "g"] & * \arrow[r] \arrow[d, "f"] & * \arrow[r] \arrow[d, "h"] & * \arrow[d, "i"] \\ * \arrow[r] & * \arrow[r] & * \arrow[r] & * \arrow[r] & * \end{tikzcd}$$

Where the rows are exact and g, h are isomorphisms. If i is injective, then f is surjective; if s is surjective, then f is injective.

Proof: Rotate the diagram counterclockwise 90°. Then use the two different filtration both converge(8.5.7). □

Prop. (7.2.9) (Snake lemma). In an Abelian category, if there is a diagram

$$\begin{tikzcd} & * \arrow[r, "i"] \arrow[d, "f"] & * \arrow[r] \arrow[d, "g"] & * \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & * \arrow[r] & * \arrow[r, "s"] & * & \end{tikzcd}$$

where the rows are exact, then there is a long exact sequence

$$\mathrm{Ker}\, f \to \mathrm{Ker}\, g \to \mathrm{Ker}\, h \to \mathrm{Coker}\, f \to \mathrm{Coker}\, g \to \mathrm{Coker}\, h$$

And if i is injective, then the first one is injective; if s is surjective, then the last one is surjective.

Proof: Rotate the diagram counterclockwise 90°. Then use the two different filtration both converge(8.5.7). □

Prop. (7.2.10). For a 3×3 diagram of complexes, the connection homomorphism satisfies an anti-commutative diagram:

$$\begin{tikzcd} H^{q-1}(Z'') \arrow[r, "\delta"] \arrow[d, "\delta"] & H^q(X'') \arrow[d, "-\delta"] \\ H^q(Z) \arrow[r, "\delta"] & H^{q+1}(X) \end{tikzcd}$$

by(8.1.4) as the category $K(\mathcal{A})$ is triangulated.

Prop. (7.2.11) (Universal Coefficient Theorem). Should be somewhere in [Weibel].

Def. (7.2.12) (Herbrand Quotient). For a complex of R-modules cyclic of order 2, we define the **additive Herbrand quotient** as $\mathrm{length}_R(H^0) - \mathrm{length}_R(H^1)$, when both are definable and the **multiplicative Herbrand quotient** as $|H^0|/|H^1|$ when they are both finite.

Prop. (7.2.13). For an exact sequence $0 \to M \to N \to K \to$ of complexes of cyclic order 2, we have $h(N) = h(M) + h(K)$ and $h^*(N) = h^*(M)h^*(K)$ in the sense that if two of them are definable, then so is the third. This is an easy consequence of long exact sequence.

Prop. (7.2.14). If each term of this complex has finite length, then $h(M) = 0$. If each term is finite, then $h^*(M) = 0$. This is an consequence of isomorphism theorem. So we have, if a a morphism of complexes has kernel and cokernel finite, then it induce an isomorphism on h or h^*.

3 Injectives & Projectives

Def. (7.3.1). An **injective object** in a Abelian category is a I s.t. $\mathrm{Hom}(-, I)$ is an exact functor, equivalently, maps to I can be extended along injections.

A **projective object** in a Abelian category is a I s.t. $\mathrm{Hom}(I, -)$ is an exact functor, equivalently, maps to I can be pulled back along surjections.

Prop. (7.3.2). Product of injective elements are injective, coproducts of projective elements are projective.

Prop. (7.3.3). In an Abelian category, the direct summand of a projective object is projective. (The summand has definition in an Abelian category).

Prop. (7.3.4). If a functor f between Abelian categories is left adjoint to an exact functor, then it preserves injectives. Dually for projectives.

Prop. (7.3.5). If A is an Abelian category, the chain complex category $Ch(A)$ is abelian by(7.1.21). A chain complex P is projective iff it is a split exact complex of projective objects. The same is true by dual argument for injectives.

Proof: If K is projective, use the surjection $C(\mathrm{id}_K) \to K[1]$, there is a homotopy between id_K and 0. Thus we have $x = dhx + hdx$. And if $dhx = hdy$, then $dhdy = 0$, thus $dy = 0$, so $K = dhK \oplus hdK$ and thus $K[n] = B_n \oplus B_{n+1}$. Thus K is a direct product of $0 \to B \to B \to 0$. And this one is projective if B is projective. □

Injective Resolutions

Prop. (7.3.6) (Horseshoe Lemma). For a exact sequence $0 \to X_1 \to X \to X_2 \to 0$ and a injective resolution of X_1 and X_2, there is a injective resolution of X commuting with them. (Choose them one-by-one, in fact, $I_n = I_n^1 \oplus I_n^2$ using the injectivity of I_n^1. Snake lemma told us that the cokernel is an exact sequence, use that to define the next one.

Prop. (7.3.7). For two lifting of morphisms $X_1 \to Y_1$ and $X_2 \to Y_2$, there is a lifting of the morphism $X \to Y$ compatible with that. Cf.[Weibel P2.4.6].

Prop. (7.3.8) (Cartan-Eilenberg Resolution). If $\mathcal{I}_\mathcal{B}$ is sufficiently large, for any K in $K^+(\mathcal{B})$ there is a functorial Cartan-Eilenberg resolution, that is, It induces simultaneous injective resolutions of K^n, Z^n, B^n and H^n. Moreover, the resolution for $B^i \to Z^i \to H^i$ and $Z^i \to K^i \to B^{i+1}$ splits.

This is achieved by the functoriality of resolutions, it is natural and induces a functor from $K^+(\mathcal{B})$ to $K^{++}(\mathcal{I}_\mathcal{B})$. Cf., [Gelfand P210].[Weibel P146].

For a CE resolution, the spectral sequence can be applied, one side gets us: $K \to \mathrm{Tot}(L)$ is a quasi-isomorphism, i.e. $\underline{\mathrm{Tot}(L)\text{ is a injective resolution of }K}$. so $RG(K) = G(\mathrm{Tot}L)$ in $D(C)$

Prop. (7.3.9) (Functorial Injective Resolution). If $\mathcal{A}$ has an functorial injective embedding, then $K^+(\mathcal{A})$ has a functorial injective resolution (just construct one row by one row and use spectral sequence to show it is a quasi-isomorphism). This resolution functor induces a functor from $K^+(\mathcal{A})$ to $K^+(\mathcal{I})$. In particular, this applies to Grothendieck category(7.1.29).

4 K-injective

Prop. (7.4.1) (K-injective). For an Abelian category, a complex $I^\bullet$ in $K(\mathcal{A})$ is called a **K-injective** object iff it satisfies the following equivalent conditions:

- $\mathrm{Hom}_{K(\mathcal{A})}(S^\bullet, I^\bullet) = 0$ for any acyclic $S^\bullet$ in $K(\mathcal{A})$.
- $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet)$ for quasi-iso $M^\bullet \to N^\bullet$.
- $\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, I^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, I^\bullet)$ for every $X^\bullet$.
  In particular, a quasi-iso between two K-injective objects is a homotopy equivalence.

Proof: $1 \to 2$ is by(7.2.6); $2 \to 3$ use(8.2.2), for $3 \to 1$, notice an acyclic is quasi-iso to 0. □

Prop. (7.4.2). Objects in $K^+(\mathcal{I})$ are all K -injectives. thus the injective resolution is unique in K^+. Dually $K^-(\mathcal{P})$ are all K-projectives.

Proof: Use the first definition of K-injectives. Use induction, we construct the first homotopy, because $I^\bullet$ is bounded below, we see the map h^n factors through $\mathrm{Coker}\, d^{n-1} = \mathrm{Im}\ d^n$ because $S^\bullet$ is acyclic, so by injectivity, it can be extended to $S^{n+1} \to I^n$. □

Prop. (7.4.3). If a functor f between Abelian categories is left adjoint to an exact functor, then it preserves K-injectives (use definition1).

Prop. (7.4.4) (Functorial K-injective Resolution). If $\mathcal{A}$ is a Grothendieck category, then $K(\mathcal{A})$ has a functorial K-injective resolution $M^\bullet \to I^\bullet$, moreover, $I^\bullet$ consists of injective objects, Cf.[StackProject 079P].

5 Ring Category Case

Prop. (7.5.1). If A is Noetherian and $C^\bullet$ is a complex of A-modules bounded above that every cohomology group H^i is a finite A-module, then there is a complex $L^\bullet$ of finite free A-modules, that $g : L^\bullet \to C^\bullet$ is a quasi-isomorphism.

Moreover, if C^i are all flat A-modules, then $L^\bullet \otimes_A M \to C^\bullet \otimes_A M$ is quasi-isomorphism for every M.

Proof: $C^\bullet$ is bounded above so we choose $L^n = 0$, and use induction to construct L^n that $H^i(L) \to H^i(C)$ is isomorphism for $i > n+1$ and surjection for $i = n+1$. For this, choose a generator $x_1, \ldots, x_r$ of $H^n(C)$ in $Z^n(C)$, and let $y_{r+1}, \ldots, y_s$ be a generator of $g^{-1}(B^{n+1}(C))$(Noetherian used), and let $g(y_i) = dx_i$ for $x_i \in C^n$.

Now let L^n be freely generated by $e_1, \ldots, e_s$ and $de_i = 0$ for $i \leq r$ and $de_i = y_i$ for $i > r$, and let $g : L^n \to C^n$ be $ge_i = x_i$. Then it can be verified to be a quasi-isomorphism.

If C^i are all flat, we check isomorphism for all f.g. modules M, because $\otimes$ and cohomology all commutes with direct limits. Use induction, for n large, both are 0, and if we write $0 \to R \to F \to M \to 0$, for F finite free, then there is a commutative diagram of long exact sequences, and for F, H^i are obviously isomorphism, so I can use five lemma. □

I.8 Derived Category

Basic references are [Gelfand Homological Algebra], should consult [StackProject Ch13].

1 Triangulated Cateogory

Def. (8.1.1). A **triangulated category** is an additive category with a T: additive automorphism and an isomorphism class of distinguished triangles satisfying the following axiams:

TR1) $X \xrightarrow{id} X \to 0 \to X[1]$ is distinguished. Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle.

TR2) A morphism $X \to Y$ will generate a helix and a triangle is distinguished iff the helix it generate is all distinguished.

TR3) Any two consecutive morphisms of two distinguished class can be extended to a morphism of distinguished class.

TR4) Any diagram of the type "upper cap" can be completed to a octahedron diagram.

Def. (8.1.2). A functor from a triangulated category to an Abelian category is called **(co)homological** iff it maps a distinguished triangle to an exact sequence.

Conversely, A functor from an Abelian category to a triangulated category is called **δ-functor** iff it functorially maps an exact sequence to a distinguished triangle.

A functor between two triangulated category is called **exact** iff it preserves $-[1]$ and maps distinguished triangle to a distinguished triangle.

Prop. (8.1.3). For a distinguished category, $\mathrm{Hom}(-, C)$ and $\mathrm{Hom}(C, -)$ is (co)homological. In particular, composition of consecutive maps in a distinguished triangle is 0, (Easily from TR1 and TR3).

Thus the extension of TR3 of two isomorphisms is an isomorphism (by 5-lemma, $\mathrm{Hom}(C, X) \to \mathrm{Hom}(C, X')$ is an isomorphism, then use Yoneda). Hence the completion in TR2 is unique by TR3.

Prop. (8.1.4). In a triangulated category $\mathcal{D}$, any commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

can be extended to a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \arrow[d] \\ X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & X''[1] \arrow[d] \\ X[1] \arrow[r] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2] \end{tikzcd}$$

where the lower right is anti-commutative. Cf.[StackProject 05R0].

Prop. (8.1.5) ($K^*(\mathcal{A})$ is Triangulated). For Abelian category A, the categories $K^*(A)$ with distinguished triangles(7.2.6) is triangulated, and they are all subcategories of $K(\mathcal{A})$. This is hard to verify, but it solves every problem. Cf[Gelfand P246][StackProject 014S]. And an additive functor will induce exact functor between K^* because distinguished is split.

Localization of Triangulated Category

Def. (8.1.6). A class of morphisms S in a category is called **localizing** if:

- S is closed under composition and has identity.
- for every $s \in S$ and f, there is a $t \in S$ and g, s.t. $f \circ t = s \circ g$(resp. $t \circ f = g \circ s$).
- the existence of a $s \in S$ s.t. $sf = sg$ is equivalent to the existence of $t \in S$ s.t. $ft = gt$.

This will generate a roof-dominating equivalence and make sure it is an equivalence relation.

2 Derived Cateogry

Def. (8.2.1). The **derived category** $D(\mathcal{A})$ of an Abelian category $\mathcal{A}$ represents the universal property that any functor to a category $\mathcal{A} \to \mathcal{C}$ s.t. quasi-isomorphisms is mapped to isomorphisms uniquely factors through $D(\mathcal{A})$.

It can be defined as the localization of quasi-isomorphisms, but the class of quasi-isomorphisms is not localizing. But one can prove the quasi-isomorphisms in $K(\mathcal{A})$ is localizing and the localization by quasi-isomorphisms of $K(\mathcal{A})$ is equivalent to $D(\mathcal{A})$. Cf.[Gelfand P159]

Notice that equivalent roofs induce the same map on homology, so the cohomology functor can be regarded defined on $D(\mathcal{A})$.

$$\mathcal{A} \to K(\mathcal{A}) \to K(\mathcal{A})[S^{-1}] = D(\mathcal{A}) \xrightarrow{H^*} \mathcal{A}.$$

Prop. (8.2.2). The category $D^*(\mathcal{A})$ are the localized category of $K^*(\mathcal{A})$ at the class of quasi-isomorphisms respectively. The isomorphisms in $D^*(\mathcal{A})$ is of the form $t \circ s^{-1}$. (look at the homology map they induced).

Prop. (8.2.3). If $\mathcal{B}$ is a full subcategory that $S \cap \mathcal{B}$ is a localizing category of $\mathcal{B}$ and any $s \in S$ can be 'denominated' in one given side (any one is OK) into $\mathcal{B}$, then $\mathcal{B}[S \cap \mathcal{B}^{-1}]$ is a full subcategory of $\mathcal{C}[S^{-1}]$. The proof is easy, use left roof or right roof.

Prop. (8.2.4). K is a triangulated category and a localizing class S compatible with T, i.e. $s \in S \iff T(s) \in S$ and the extension in $TR3$ of f, g in S is in S. Then the localizing category $K[S^{-1}]$ is triangulated.

Cor. (8.2.5) (Derived Category is Triangulated). $D(\mathcal{A})$ is a triangulated category. The distinguished triangle is just the obvious one, and for a distinguished triangle, the long exact sequence exists,(7.2.5). In other words, H^0 is a cohomological functor for $D(\mathcal{A})$.

Derived Colimit

Def. (8.2.6). A **derived colimt** for a complex K_n in a triangulated category $\mathcal{D}$ is a K that

$$\oplus K_n \xrightarrow{(1-d)} \oplus K_n \to K \to \oplus K_n[1]$$

This exists when $\oplus K_n$ exists by TR1 and then it is unique by TR3. And the derived colimit is natural.

Dually for the definition of **derived limit**.

3 Acyclic Elements and Derived Functors

Def. (8.3.1). For a left exact F, a class R of elements is called **adapted to** F if it is sufficiently large and F maps acyclic objects in $Kom^+(\mathcal{R})$ to acyclic objects.

Injectives are F-acyclic for all left exact F because id : $I^\bullet \to I^\bullet$ is homotopic to 0, Cf(7.4.2).

Prop. (8.3.2). When RF exists, an object X is called F-acyclic iff $R^iF(X) = 0$ for all $i > 0$. Then: there is an adapted class of F iff the class of F-acyclic objects Z is sufficiently large.

If this is the case, then adapted class of F are exactly sufficiently large subclass of Z, and Z contains all injectives, Cf.[Gelfand P195].

Prop. (8.3.3) (Acyclic Criterion). If a class T of elements in an Abelian category of enough injectives is:

- sufficiently large.
- If $A \oplus A' \in T$ implies $A \in T$. (This implies all injectives are in T).
- Cokernel of elements of T is in T and $0 \to F(A) \to F(A') \to F(\mathrm{Coker}) \to 0$ is exact. (To use induction).

Then T is adapted to F.

Prop. (8.3.4). For a class of objects $\mathcal{R}$ in $\mathcal{A}$ stable under finite direct sum and are adapted to a left exact functor F, i.e. $Kom^+(\mathcal{R})$ is F-acyclic and every object in $\mathcal{A}$ is a subobject of an object from $\mathcal{R}$. Just need to verify the condition of(8.2.3). Similarly for the opposite category.

And in this case $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ is equivalent to $D^+(\mathcal{A})$.

Proof: The hard part is to prove every complex in $K^+(\mathcal{A})$ is quasi-isomorphic to a complex in $K^+(\mathcal{R})$, for this, use direct construction. Cf.[Gelfand P187]. □

Prop. (8.3.5). By(7.4.2), $K^+(\mathcal{I})$ is a saturated subcategory of $D+(\mathcal{A})$. And if $\mathcal{A}$ has enough injectives, this is an equivalence of category. (We only need to verify that the localization of $K^+(\mathcal{I})$ is itself, using the last proposition). In particular, this applies to Grothendieck categories. Cf.[Gelfand P179].

Prop. (8.3.6). By(7.4.2), if $\mathcal{A}$ contains sufficiently many injectives, then injective objects are adapted to any left exact functor F. (Because id on acyclic injective complexes is homotopic to 0 by the lemma).

Def. (8.3.7) (Derived functor). The **right derived functor** $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ for an additive functor F between Abelian categories is defined by the following universal property:

RF is exact and there is a natural tranformation

$$\varepsilon_F : Q_{\mathcal{B}}\ K^+F \to RF\ Q_{\mathcal{A}}.$$

and any other exact $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and a similar transformation must factor through ε_F uniquely. Thus this RF is unique up to natural isomorphism.

If a left exact functor F between Abelian categories has an adapted class, then by preceding proposition, $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ is equivalent to $D^+(\mathcal{A})$, then the derived functor exists, it is just F^+ on $K^+(\mathcal{R})$, Cf.[Gelfand P188].

Notice there is a more general derived functor that use inductive limits in $\hat{\mathcal{A}}$ that it maps $D^*(\mathcal{A})$ to $\mathrm{Ind}(D^*(\mathcal{B}))$, and if it has image in the subcategory of representable objects, then it coincide with RF. Similarly for right exact functor F. (This is easy to check)Cf.[Gelfand P198].

Yet there is another way to just look at the derived functors, it is the hypercohomology of the Cartan-Eilenberg resolution of the complex(8.3.11).

Prop. (8.3.8). $\mathcal{A},\mathcal{B},\mathcal{C}$ be Abelian categories with enough injectives in $\mathcal{A},\mathcal{B}$ and $F:\mathcal{A}\to\mathcal{B}, G:\mathcal{B}\to\mathcal{C}$ are left exact functors. If $F(\mathcal{I})\subset R_{\mathcal{B}}$ for $\mathcal{I}$ injective, then $R(G\circ F)\to RG\circ RF$ is an isomorphism . (Because the definition of RF is just F on $K^+(\mathcal{I})$).

Prop. (8.3.9). The derived functors form a universal δ-functor (when it exists).

Proof: It is a δ functor by(8.2.5), it is universal by(7.2.3). □

Prop. (8.3.10). Derived functor commutes with filtered colimits, when B is an Grothendieck category, this is by $AB5$.

Prop. (8.3.11) (Hypercohomology). we can define the **hypercohomology** of a left exact functor as $H^n(\mathrm{Tot}^{\Pi}F)$ if $\mathcal{B}$ satisfies AB3*.

Dually we can define the **hyperhomology** if $\mathcal{A}$ satisfies AB3* and AB4* and $\mathcal{B}$ satisfies AB3.

For complexes in $K^+(\mathcal{A})$, there is no restriction and everything is smooth.

When the Abelian category $\mathcal{A}$ satisfies AB3* and AB4*, i.e. the direct product is exact, then Tot^{Π} of the Cartan-Eilenberg resolution of any complex is a quasi-isomorphism to it by the dual of(8.5.10). (Take horizontal filtration, AB4* assures it collapse).

4 (Co)Homological Dimension

Prop. (8.4.1). If $\mathcal{A}$ has enough projectives, then the projective dimension of an object X is the length of projective resolutions. (Use resolution and long sequence).

Prop. (8.4.2) (Hilbert Theorem). For an Abelian category $\mathcal{A}$, the category $\mathcal{A}[T]$ is an Abelian category. If $\mathcal{A}$ has enough projectives and have infinite direct sum, then $\mathrm{dhp}_{\mathcal{A}[T]}(X,t)\leq\mathrm{dhp}_{\mathcal{A}}(X)+1$ and equality with $t=0$.

Cor. (8.4.3). The Categories Ab and $K[X]$-mod have homological dimension 1. $K[X_i,\ldots,X_k]$ has homological dimension k.

5 Spectral Sequence

Reference for this section is [Weibel Ch5]. All the definition below is dual for homology and cohomology, just rotate the diagram 180 degree.

We work in an Abelian category.

Def. (8.5.1). A convergent **Spectral Sequence** is a three-dimensional arrange of entries $E_r^{p,q}$ that:

1. $d_r:E_r^{p,q}\to E_r^{p+r,q-r+1}$ that $d_rd_r=0$.
2. $H^{p,q}(E_r^{p,q})\cong E_{r+1}^{p,q}$. And $E_r^{p,q}$ has a direct limit $E_\infty^{p,q}$.
3. There is a complex E^n and a decreasing bounded filtration F^pE^n on each E^n and $E_\infty^{p,q}\cong F^pE^{p+q}/F^{p+1}E^{p+q}$.

For a morphism of spectral sequences, if it defines an isomorphism for some r, then by five-lemma, it defines isomorphisms afterward, so it defines an isomorphism on E_∞^{pq}.

Def. (8.5.2). We say a (co)homology filtration is bounded below $F_{n_s}E_n = 0$ for some n_s, bounded above $F_{n_s}E_n = E_n$ for some n_s. It is exhaustive iff $\cap F_i E_n = E_n$. The spectral sequence is called regular iff $d^r_{pq} = 0$ for sufficiently large r.

Def. (8.5.3) (Spectral Sequence of a Filtered Complex). For a complex $K^\bullet$ and a filtration F^pK^n on K^n, we have a natural spectral sequence

$$E_1^{pq} = H^{p+q}(F^pE^{p+q}/F^{p+1}E^{p+q}), \quad E^n = H^n(K^\bullet), \quad F^pE^n = H^n(F^pK^\bullet).$$

For a morphism of filtered complexes that are isomorphism for some r, induction on the exact sequence $0 \to F^pE^n \to F^{p+1}E^n \to E_\infty^{p,n-p}$ and use five-lemma shows it induces isomorphism on H^*E.

Prop. (8.5.4) (Comparison Theorem). For a morphism between two convergent spectral sequences, if it is an isomorphism for some r, then it induce isomorphism on the infinite homology, because there are exact sequence

$$0 \to F^{p+1}H^n \to F^pH^n \to E_\infty^{p,n-p} \to 0$$

we can use five lemma and induction.

Prop. (8.5.5) (Classical Convergence). For homology, if the filtration is bounded below and exhaustive for all n, we have a convergence to E_n. Cf[Gelfand P203] for cohomological case and [Weibel P133] for homological case.

Prop. (8.5.6) (Complete convergence). For homology, if the filtration is complete, exhaustive, bounded above, and the spectral sequence is regular, then the spectral sequence converges to E_n.

There are two examples, the stupid filtration and the canonical filtration, the canonical filtration is natural and factors through $D(\mathcal{A})$.

Prop. (8.5.7) (Spectral Sequence of a Double Complex). A double complex has two natural filtration of the total complex, they defines two spectral sequence, one has

$$E_{2,x}^{p,q} = H_x^p(H_y^{\bullet,q}(L^{\bullet,\bullet}))$$

and the other has

$$E_{2,y}^{p,q} = H_y^p(H_x^{q,\bullet}(L^{\bullet\bullet})).$$

Cf.[Gelfand P209]. In fact under reflection, there is only one spectral sequence. For the horizontal filtration, the differential goes vertical first, for the vertical filtration, the differential goes horizontal first. The differential goes one way, the convergence goes reversely.

If both the filtration is finite and bounded, in particular if E is in the first quadrant, then they both converges to $H^n(E)$, this will generate important consequences.

Cor. (8.5.8). If a double complex in the first quadrant has its all column acyclic (3rd-quadrant pointing), then the total complex is acyclic. Thus a morphism of double complex inducing quasi-isomorphism on each column induces a quasi-isomorphism on the total complex.

If a double complex has $H_p(C_{*,q}) = 0, \forall p > 0, q$, then

$$H_n(\mathrm{Tot}C_{*,*}) = H_n(\mathrm{Coker}(C_{1,*} \to C_{0,*})$$

Prop. (8.5.9) (Horizontal Filtration). For a second-quadrant free homology double complex, the filtration is bounded below and exhaustive for $\mathrm{Tot}^{\oplus}$, so the classical convergence applies.

For a fourth-quadrant free homology double complex, the filtration is complete and exhaustive and regular ? for Tot^{Π}, so the complete spectral sequence applies. Cf.[Weibel P142].

Prop. (8.5.10) (Vertical Filtration). For a fourth-quadrant free homology double complex, the filtration is bounded below and exhaustive for $\mathrm{Tot}^{\oplus}$, so the classical convergence applies.

For a second-quadrant free homology double complex, the filtration is complete and exhaustive and regular ? for Tot^{Π}, so the complete spectral sequence applies. Cf.[Weibel P142].

Cor. (8.5.11) (Grothendieck Spectral Sequence). If $\mathcal{A},\mathcal{B},\mathcal{C}$ be Abelian categories and $\mathcal{A},\mathcal{B}$ has enough injectives, and $F:\mathcal{A}\to\mathcal{B}, G:\mathcal{B}\to\mathcal{C}$ are left exact functors. If $\mathcal{R}_{\mathcal{A}}$ is adapted to F, $\mathcal{R}_{\mathcal{B}}$ is adapted to G and $F(I_{\mathcal{A}})\subset\mathcal{R}_{\mathcal{B}}$, then for any $X\in K^+(\mathcal{A})$, there is a spectral sequence with $E_2^{p,q}=R^pG(R^qF(X))$ (to upper left) that converges to $E^n=R^n(G\circ F)(X)$. And this spectral sequence is functorial in X.

In particular, this applies to F is a right adjoint and its left adjoint is exact, then we may choose $\mathcal{R}_{\mathcal{A}}=\mathcal{I}_A$ and $\mathcal{R}_{\mathcal{A}}A=\mathcal{I}_{\mathcal{A}}$.

Proof: Let $K=F(I_X)=RF(X)$, and choose the CE resolution of K(7.3.8), because the resolutions for $B^i\to Z^i\to H^i$ and $Z^i\to K^i\to B^{i+1}$ split and G is additive, we have

$$H_x^{q,\bullet}(G(L^{\bullet,\bullet}))=G(H_x^{q,\bullet}(L^{\bullet,\bullet}))=RG(H^q(K))$$

So

$$E_{2,y}^{p,q}=H_y^p(H_x^{q,\bullet}(L^{\bullet\bullet}))=R^pG(H^q(K))=R^pG(R^qF(X))$$

and

$$E^\bullet=RG(\mathrm{Tot}(L))=G(\mathrm{Tot}(L))=RG(K)=RG\circ RF(X)=R(G\circ F)(X)(8.3.8).$$

□

Cor. (8.5.12). The low degree parts read:

$$0\to R^1G(F(A))\to R^1(G\circ F)(A)\to G(R^1F(A))\to R^2(G(F(A))\to R^2(G\circ F)(A).$$

(Check definition). More generally, if $R^pG(R^qF(A))=0, 0<q<n$, then

$$R^mG(F(A))\cong R^m(G\circ F)(A)\quad m<n$$

And

$$0\to R^nG(F(A))\to R^n(G\circ F)(A)\to G(R^nF(A))\to R^{n+1}(G(F(A))\to R^{n+1}(G\circ F)(A).$$

The Grothendieck spectral sequence is tremendously important.

Cor. (8.5.13). For chain complex K in $K^+(\mathcal{A})$ and a left exact functor F, the CE resolution will generate two spectral sequences: $E_{2,x}^{p,q}=H_x^p(R^qF(A_\bullet))$ and the other has $E_{2,y}^{p,q}=R^pF(H^q(A))$ that converges to the hypercohomology $\mathbb{R}^{p+q}F(K)$. Dually for derived homology.

6 Tor Hom and Ext

Prop. (8.6.1). $\mathcal{A}$ is categorically equivalent to the subcategory of $D(\mathcal{A})$ that has only H^0 nonzero. If we define $\mathrm{Ext}^i_{\mathcal{A}}(X,Y)$ as $\mathrm{Hom}_{D(\mathcal{A})}(X[0],Y[i])$, then it is equivalent to the i-term extension of Y by X, and it is an abelian group. We have a

$$\mathrm{Ext}^i_{\mathcal{A}}(X,Y)\times \mathrm{Ext}^i_{\mathcal{A}}(Y,Z)\to \mathrm{Ext}^{i+j}_{\mathcal{A}}(X,Z)$$

by composition or equivalently the conjunction of extensions.

Proof: Cf.[Gelfand P167] □

Cor. (8.6.2). The definition of $\mathrm{Ext}^n(X,Y)$ is equivalent to the usual definition as the derived functor of $\mathrm{Hom}(X,-)$. Because by(7.4.2) when we use a projective resolution or an injective resolution, then it is equivalent to hom in $K(\mathcal{A})$(7.4.1), which is exactly the homology group of the Hom.

Prop. (8.6.3). In an Abelian category with enough injectives, the extension $\mathrm{Ext}^1(N,M)$ is equivalent with the Abelian group of extensions with Baer sum as addition.

Proof: We choose a projective resolution $0\to K\to P\to N\to 0$, so $\mathrm{Hom}(K,M)\to \mathrm{Ext}^1(N,M)$ is surjective, so choose a lifting and the pushout $0\to M\to L\to N\to 0$ with be the corresponding extension, Now the Baer sum is easy to define and verify. □

Prop. (8.6.4). In an Abelian category with enough injectives, we have the $\mathrm{Ext}^i(-,G)$ forms a long exact sequence, by injective resolution.

Ring Category Case

Prop. (8.6.5). In the category of rings, $\mathrm{Tor}(A,B)=\mathrm{Tor}(B,A)$. This can be seen using spectral sequence of the double complex of flat resolutions of A and B. Similarly, we have two definitions of $\mathrm{Ext}^i(M,N)$ are compatible.

Prop. (8.6.6) (Base Change). For a ring extension $R\to S$, using projective resolution and spectral sequence, there is a first quadrant homology spectral sequence:

$$E^2_{pq}=\mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A,S),B)\Rightarrow \mathrm{Tor}^R_{p+q}(A,B).$$

Similarly, for Ext,

$$E_2^{pq}=\mathrm{Ext}^p_S(A,\mathrm{Ext}^q_R(S,B))\Rightarrow \mathrm{Ext}^{p+q}_R(A,B).$$

Prop. (8.6.7) (Universal Coefficient Theorem). Let P be a free R-module so $d(P_n)$ are all flat, then $Z(P_n)$ are also flat and

$$0\to d(P_{n+1})\to Z_n\to H_n\to 0$$

is a free resolution. we have an exact sequence:

$$0\to H_n(P)\otimes_R M\to H_n(P\otimes_R M)\to \mathrm{Tor}^R_1(H_{n-1}(P),M).$$

$$0\to \mathrm{Ext}^1_R(H_{n-1}(P),M)\to H^n(\mathrm{Hom}_R(P,M))\to Hom(H_n(P),M)\to 0$$

and these exact sequences non-canonically split because Z_n is a direct summand of P_n, thus $Z_n\otimes M$ is a direct summand of $P_n\otimes M$ and a fortiori $Z_n(P_n\otimes M)$. so $H_n(P)\otimes M$ is a direct summand of $H_n(P\otimes_R M)$.

Rhom and Rtensor

Prop. (8.6.8). The tensor product of complexes in R-mod is an exact functor of triangulated categories $K(R)$, because the distinguished triangles in $K(R)$ are termwise-split exact sequence(7.2.5).

Def. (8.6.9) (K-flat). A complex $K^\bullet$ in an Abelian category is called K-flat if for any acyclic complex $M^\bullet$, the total complex $\mathrm{Tot}(M^\bullet \otimes K^\bullet)$ is acyclic. This is equivalent to tensoring with $K^\bullet$ maps quasiiso to quasiiso, because quasiiso is equivalent to the cone is acyclic and tensoring is exact.

Prop. (8.6.10). Any complex of R-modules has a K-flat resolution, moreover, each term is a flat module. Cf.[StackProject 06Y4].

Prop. (8.6.11). In an Abelian category $\mathcal{A}$ with enough projectives, we can define a **general tensor product**

$$\otimes^L : D^-(\mathcal{A}) \times D^-(\mathcal{A}) \to D^-(\mathcal{A})$$

that is, for complexes $F^\bullet$ and $G^\bullet$, there are K-projective resolutions P and Q of F, G by duality of the CE resolution. Thus we define $F \otimes^L G = P \otimes Q$ as the total complex of the double complex. In fact, only one resolution will suffice.

This does descend to D^- because homotopy induce a homotopy in the the double complex and two K-projectives are quasi-isomorphic and quasi-isomorphisms induce isomorphism on E_1 of the spectral sequence associated to the double complex (used flatness) thus on the homology of total complex by comparison.

Prop. (8.6.12). In an Abelian category $\mathcal{A}$ with enough injectives or enough injectives, we can define a **general Hom**

$$R\operatorname{Hom} : (D^-(\mathcal{A}))^{op} \times D(\mathcal{A}) \to D(\mathcal{A}), \quad (D(\mathcal{A}))^{op} \times D^+(\mathcal{A}) \to D(\mathcal{A})$$

that is, for complexes $\mathcal{F}$ and $\mathcal{G}$, there are K-projective resolutions P of $F^\bullet$ or K-injective resolution Q of G by (duality) of the CE resolution. Thus we define $R\operatorname{Hom}^n(X^\bullet, Y^\bullet) = R\operatorname{Hom}(P, Q)$ where

$$R\operatorname{Hom}_n(P, Q) = \prod \operatorname{Hom}(P_i, Q_{n+i})$$

With the differential giving by $d_n(\{f_k\})_i = \{df_i + (-1)^n f_{i+1} d\}$.

This does descend to D^+ because homotopy induce a homotopy in the the double complex and two K-projectives(injectives) are quasi-isomorphic and quasi-isomorphisms induce isomorphism on E_1 of the spectral sequence associated to the double complex (used projectiveness or injectiveness) thus on the homology of total complex by comparison and also use the second definition of(7.4.1).

For any $X^\bullet \in K(\mathcal{A}), Y^\bullet \in K^+(\mathcal{A})$, we define $\operatorname{Ext}^n_{\mathcal{A}}(X, Y) = H^i(R\operatorname{Hom}(X, Y))$, this is seen to be equal to $\operatorname{Hom}_{K(\mathcal{A})}(X^\bullet, Q^\bullet[n]) = \operatorname{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet[n])$. Similarly for $P^\bullet$.

Inverse Limit

Prop. (8.6.13). The derived functor of lim from $K^+(\mathcal{A}) \to \mathcal{A}$ is $\operatorname{Coker}(a_i) \to (a_i - a_{i+1})$ for $\mathcal{A}$ Abelian, has enough injectives and satisfies $AB4^*$ (R-mod). $\lim^1$ vanishes for a complex that satisfies Mittag-Leffler conditions.

Proof: If A satisfies the M-L condition, the essential image $\{B_i\}$ is surjective so acyclic and $\{A_i/B_i\}$ is acyclic because the inverse image can be defined as a finite sum. So the long exact sequence gives A is acyclic.

The δ-functor is defined by the snake lemma and $AB4^*$ and we have to prove it is effaceable. For this, we use(7.1.8) to see that $E = \prod_k k_* A_k$ exists in $\mathcal{A}^C$ is injective and $A \to E$ is an injection. In this case E is a product of towers $\cdots \to A_k \to A_k \to 0 \to 0 \to \cdots$, hence surjective by $AB4^*$ so is acyclic. $\square$

For applications, Cf.[Weibel P82].

I.9 Lie Algebra

Note:[Lie Algebras of Finite and Affine Type Carter] is far more better than [Hymphreys].

1 Main Theorems

Prop. (9.1.1) (Engel). If all elements of L are ad-nilpotent, then L is nilpotent.

Proof: only need to show that If a subalgebra of $GL(n)$ consists of nilpotent elements, then there is a common 0-eigenvector. Use Induction, choose a maximal subalgebra of L, then it must be of codimension 1,$L = K + Fz$. Thus the 0-eigenvector for K is a nonzero subspace, and a 0-eigenvector for z will suffice. □

Prop. (9.1.2) (Lie's theorem). Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a solvable lie algebra. Then there exists a vector $v \in V$ which is a common eigen vector for all $X \in \mathfrak{g}$.

Proof: Idea is to prove by induction on dimension of $\mathfrak{g}$.

Produce a codimension 1 ideal $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g}$ be generated (as a vector space) by $\mathfrak{h}$ and Y.Being a subalgebra of solvable algebra $\mathfrak{g}$, $\mathfrak{h}$ is itself a solvable lie algebra. Apply induction step on $\mathfrak{h}$ and choose $v \in V$ such that v is an eigenvector for all $X \in \mathfrak{h}$.

The idea is to consider set W all common eigenvectors of elements of $\mathfrak{h}$ and produce an eigenvector of Y from this W. Let

$$W = \{v \in V | X(v) = \lambda(X)v \ \forall\ X \in \mathfrak{h} \text{ for a fixed } \lambda(X) \in \mathbb{C}\}.$$

Suppose W is an invariant subspace of Y, we then have restriction map $Y : W \to W$. As we are in complex vector space (algebraically closed) there exists an eigenvector for Y in W say w_0. Thus, w_0 is common eigenvector for all elements of $\mathfrak{g}$.

It remains to show that W is an invariant subspace of Y i.e., $Y(w) \in W$ for all $w \in W$ i.e., given $X \in \mathfrak{h}$, we need to have $X(Y(w)) = \lambda(X)Y(w)$.

Let $w \in W$, we have

$$\begin{aligned} X(Y(w)) &= Y(X(w)) + [X,Y](w) \\ &= Y(\lambda(X)w) + \lambda\left([X,Y]\right)w \\ &= \lambda(X)Y(w) + \lambda\left([X,Y]\right)w \end{aligned}$$

This is almost the same as what we want but with an extra term $\lambda\left([X,Y]\right)w$. Suppose we prove $\lambda\left([X,Y]\right) = 0$ for all $X \in \mathfrak{h}$ then we are done.

Then considers subspace U spanned by elements $\{w, Y(w), Y^2(w), \cdots\}$ and then says that U is invariant subspace of each element of $\mathfrak{h}$ and (assuming n is the smallest integer such that $Y^{n+1}w$ is in the subsapce generated by $\{w, Y(w), \cdots, Y^n(w)\}$) representation of an element Z of $\mathfrak{h}$ with the basis $\{w, Y(w), \cdots, Y^n(w)\}$ is an upper triangular matrix with $\lambda(Z)$ in the diagonal. So, $tr(Z) = n\lambda(Z)$.

So, $tr([X,Y]) = n\lambda([X,Y])$. As $[X,Y] = XY - YX$, we have $\mathrm{tr}([X,Y]) = \mathrm{tr}(XY) - \mathrm{tr}(YX) = 0$. Thus, $\lambda([X,Y]) = 0$ and we are done. □

Def. (9.1.3). A lie algebra is called semisimple if the maximal solvable ideal (Rad L)= 0.

Prop. (9.1.4) (Weyl). Representation of a semisimple lie algebra is completely reducible.

Proof: Cf.[Humphreys P28]. □

Prop. (9.1.5) (Cartan's Criteria for Solvability). If $\mathfrak{g}$ is a Lie algebra $\subset \mathfrak{gl}_n$, then

$$\mathfrak{g} \text{ is solvable } \iff Tr(xy) = 0,\ \forall x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}].$$

Note that a Lie algebra is solvable if the adjoint representation is solvable because the kernel is abelian. Cf.[Humphreys P20]

Prop. (9.1.6) (Cartan Criteria for Semisimplicity). A lie algebra is semisimple $\iff$ the Killing form is non-degenerate. Cf.[Humphreys P22].

Proof: Just show that the kernel of the Killing form is a solvable ideal and that $\mathrm{ad}x \cdot \mathrm{ad}y$ is nilpotent for x in an abelian ideal. □

Prop. (9.1.7). If L is semisimple, then every derivative of L is inner.

Proof: Cf.[Humphreys P23]. □

Prop. (9.1.8) (Abstract Jordan Decomposition). Let L be a semisimple lie algebra and $\phi : L \to GL(v)$ be a representation. If $x = s + n$ is the abstract Jordan decomposition of x, then $\phi(x) = \phi(s) + \phi(n)$ is the usual Jordan decomposition of $\phi(x)$.

Proof: In fact, we only need to prove that if L is a semisimple algebra $\subset \mathfrak{gl}(V)$, then L contains the semisimple and nilpotent element of all its element. Because the image of L is semisimple and the usual Jordan decomposition must be its abstract decomposition. The last assertion is due to the fact that if z is semisimple(nilpotent), then $\mathrm{ad}_{\mathfrak{gl}_n} z$ is semisimple(nilpotent), thus so do $\mathrm{ad}_L z$.

Cf.[Humphreys P27] for the following proof. □

Prop. (9.1.9) (Baker-Campbell-Hausdorff cor).

$$\exp(X)\exp(Y) = \exp(X + Y + 1/2[X, Y] + 1/12[X, [X, Y]] - 1/12[Y, [Y, X]] + \text{higher order terms})$$

Cf.[Hall Lie algebras GTM222 P76].

2 Complex Semisimple Lie Algebra

Following [Serre].

3 Reductive Lie Algebra

Prop. (9.3.1). A lie algebra is called reductive if $\mathrm{Rad}(L) = Z(L)$.

1. If L is reductive, then L is completely reducible ad L-module.
2. $L = [LL] \oplus Z(L)$.
3. If $L \subset GL(V)$ acting irreducibly on V, then L is reductive with $\dim \mathrm{Rad}(L) \le 1$. In particular, If $L \in SL(V)$ and $\mathrm{char}F \neq 0$, it must be semisimple. This can be used to prove that all classical algebras are semisimple. And the diagonal matrix will be toral and finding a set of simple roots will suffice to prove that every calssical lie algebra is simple.
4. If L is a completely reducible ad L-module, then L is reductive.
5. If L is reductive, then all finite dimensional representations of L in which $Z(L)$ is represented by semisimple endomorphism are completely reducible.

6. If $[LL]$ is semisimple, then L is reductive.

Proof: (1): Because $L/Z(L)$ is a semisimple lie algebra and $Z(L)$ is mapped to the kernel.

(2): Let $L = M \oplus Z(L)$ as a ad-L module, then $[LL] \subset [MM] \subset M$, but $[LL]$ maps onto $L/Z(L)$ because a semisimple is a sum of simple algebra. So $[LL] = M$.

(3): Cf.[Humphreys P102].

(4): In this way L decompose into $Z(L)$ and simple algebras, so it is reductive.

(5): First simultaneously diagonalize $Z(L)$, then the subspace corresponding to different characters are stable under L. Then decompose w.r.t. $[LL]$ with get the result. (6): Note that the element in $\mathrm{Rad}(L)$ will all be central. □

Prop. (9.3.2). Let L be a simple lie algebra, then any two symmetric associative bilinear forms on L is proportional. Because any of this form corresponds to a L-morphism from L to L^*. In particular, when $L \subset \mathfrak{gl}_n$, the usual trace is proportional to the Killing form.

4 Real Lie Algebra

Def. (9.4.1). A **compact real form** is a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ s.t. $\mathfrak{g}$ is the complexification of $\mathfrak{l}$ and $\mathfrak{l}$ is the lie algebra of a compact simply-connected Lie group.

Prop. (9.4.2). A real Lie algebra is compact iff there exists a inner product s.t.

$$([X,Y],Z) + (X,[Y,Z]) = 0,$$

iff the Killing form is negative definite.

Proof: One direction is easy, just use the average method to find a G-invariant inner product and then take derivative. For the other direction, the identity shows that a complement of an ideal is an ideal so $\mathfrak{g}$ is decomposed into simple lie groups and reduce to the case that $\mathfrak{g}$ is simple. The ideal is to show that $\mathfrak{g} \cong \mathrm{ad}(\mathfrak{g})$ is the whole outer derivative group $\partial(\mathfrak{g})$(the following lemma). So $\mathfrak{g}$ equals to the identity component of $\mathrm{Aut}(\mathfrak{g})$ which is a closed subgroup thus closed but it is also a subgroup of the compact group $O(\mathfrak{g})$ thus it is compact. □

Lemma (9.4.3). If a real semisimple Lie algebra X has a invariant inner product, then every outer derivative is inner.(In fact, this is true by Cartan Criterion for semisimplicity (9.1.7).

Proof: since $\mathrm{ad}(X)$ is skew-symmetric, it's diagonalizable and its eigenvalue is pure imaginary, so the Killing form of X is negative definite. Now choose the complement $\mathfrak{a}$ of $\mathrm{ad}(X)$ in $\partial(X)$, then $\mathfrak{a} \cap X = 0$. Thus for $D \in \mathfrak{a}$, $\mathrm{ad}(D(g)) = [D, \mathrm{ad}(g)] = 0$ for all g in X, so $D = 0$, thus $\mathrm{ad}(X) = \partial(X)$. □

Prop. (9.4.4). -

1. The complexification of the Lie algebra of a connected compact Lie group is reductive.
2. A complex Lie algebra is semisimple iff it is isomorphic to the complexification of the Lie algebra of a simply-connected compact Lie group. i.e. every complex semisimple Lie algebra has a compact real form.

Proof: 1: Because a connected compact Lie group is completely reducible so the does the Lie algebra and so does the complexification. So it is reductive by (9.3.1)4.

2: Cf.[Varadarajan Lie Groups Lie algebras and Their Representations]. The ideal is to find a real form whose corresponding simply-connected group is compact. □

Prop. (9.4.5). If $\mathfrak{g}$ is the Lie algebra of a matrix Lie group G, then:

1. every Cartan subalgebra comes from a maximal commutative subalgebra of a compact real form and any two Cartan subalgebras are conjugate under the Ad-action of G.
2. any two compact real form is conjugate under the Ad-action of G.
3. any two maximal commutative subalgebra of a compact real form is conjugate under the Ad-action of the corresponding compact compact subgroup.

Prop. (9.4.6). A real Lie algebra is semisimple iff its complexification is semisimple. Cf.[Varadarajan].

Cor. (9.4.7). The real Lie algebra of a compact simply-connected group is semisimple.

Note: For the classification of real semisimple Lie algebras, Cf.[李群讲义项武义 §6]

Prop. (9.4.8). If a complex representation of a Lie group admits an invariant bilinear form, then it is non-degenerate and unique. In fact, this is equivalent to a G-map from V to V^*. Thus there is unique invariant inner product in a compact real form by the preceding proposition.

5 Universal Enveloping Algebra

Prop. (9.5.1) (Chevalley). The center of the universal enveloping algebra is isomorphic to the polynomial ring over $\mathbb{C}$ of l elements, where L is a semisimple lie algebra of rank l. In particular, The center for $\mathfrak{sl}_2$ is the algebra generated by the Casimir element $1/2h^2 + ef + fe$.

Proof: Because there is a commutative diagram of isomorphisms of algebras:

$$\begin{array}{ccc} S(L)^G & \xrightarrow{\alpha} & P(L)^G \\ \big\downarrow{\scriptstyle\eta} & & \big\downarrow{\scriptstyle\phi} \\ S(H)^W & \xrightarrow{\beta} & P(H)^W \end{array}$$

Where P is the polynomial ring$\cong S(L^*)$, the horizontal is Killing isomorphisms and vertical is the restriction maps. Cf.[Carter Theorem 13.32].

The twisted Harish-Chandra map gives an isomorphism of algebras $Z(L) \to S(H)^W$ (It just maps $z \in Z(L)$ to its pure H part and transform every indeterminants h_i to $h_i - 1$). e.g. $z = h^2 + 2h + 1 + 4fe \in Z(\mathfrak{sl}_2)$ is maped to h^2 in $S(H)$. And $P(H)^W$ is isomorphic to a polynomial ring in l generators over $\mathbb{C}$. $\square$

6 Lie Algebra Cohomology

Prop. (9.6.1) (Chevalley-Eilenberg resolution).

I.10 Quantum Groups

1 Clifford Algebra

Prop. (10.1.1). Let $Cl_{r,s}$ denote the real Clifford algebra of signature $r-s$, then

$$Cl_{1,0}\cong\mathbb{C},\quad Cl_{0,1}\cong\mathbb{R}\oplus\mathbb{R},\quad Cl_{2,0}\cong\mathbb{H}\subset M(2,\mathbb{C}),\quad Cl_{0,2}\cong R(2)=M(2,\mathbb{R}),$$

And we have

$$Cl_{n+2,0}\cong Cl_{0,n}\otimes Cl_{2,0},\quad Cl_{0,n+2}\cong Cl_{n,0}\otimes Cl_{0,2}.$$

by the mapping $e_i\to e_i\otimes e_1'e_2'$, $e_{n+j}\to 1\otimes e_j'$.

So we have

$$Cl_{n+8,0}\cong Cl_n\otimes\mathbb{R}(16),\quad \mathbb{C}l_{n+2,0}=Cl_{n+2,0}\otimes\mathbb{C}=\mathbb{C}l_{n,0}\otimes\mathbb{C}(2).$$

because $\mathbb{H}\otimes\mathbb{C}=\mathbb{C}(2)$, and

$$\begin{bmatrix} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\\ Cl_{n,0} & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{H}\oplus\mathbb{H} & \mathbb{H}(2) & \mathbb{C}(4) & \mathbb{R}(8) & \mathbb{R}(8)\oplus\mathbb{R}(8)\\ \mathbb{C}l_{n,0} & \mathbb{C} & \mathbb{C}\oplus\mathbb{C} & & & & & & \end{bmatrix}$$

The Clifford algebra is a $\mathbb{Z}_2$-graded algebra, $Cl=Cl^0\otimes Cl^1$ and $Cl_{n-1}\cong Cl_n^0$ by the mapping $e_i\to e_i\otimes e_{n+1}$. This is in fact the decomposition of the chirality operator $\Gamma=(-1)^{\lfloor\frac{n+1}{2}\rfloor}e_1e_2\ldots e_n,\Gamma^2=1$.

Prop. (10.1.2). For n even, $\mathbb{C}(V)$ is naturally isomorphic to $\mathrm{End}_{\mathbb{C}}(\Lambda^*W)$, where $W=\{\frac{1}{\sqrt{2}}(e_{2i-1}-ie_{2i})\}$. This isomorphism is not obvious and restrict to a Spinor representation of $\mathrm{Spin}(n)$ and $\rho(\Gamma)^2=1$ induce two representations of $Cl(n)^0$, in particular $\mathrm{Spin}(n)$, called the **(half Spinor representations)**. This has a unique extension to representation of Spin^c. $\wedge^*W$ comes with a Hermitian metric which is preserved by the action of $\mathrm{Pin}(n)$ (check). So the image is $\mathrm{SO}(n)$ is in $SO(\wedge^*W)$. Cf.[Jost Geometric analysis P72].

Def. (10.1.3). denote $\mathrm{Pin}(n)$ as the group in Cl_n generated by v_i of norm 1. Because $v_i\cdot v_i=-1$, it is a group. And denote $\mathrm{Spin}(n)$ as the subgroup of $\mathrm{Pin}(n)$ generated by even number of v_is.

So the conjugation action $-Ad=v(-)v$ =reflection w.r.t v, maps $\mathrm{Pin}(n)$ to $O(n)$ and $\mathrm{Spin}(n)$ to $SO(n)$.

Prop. (10.1.4). The kernel of this mapping is $\{\pm\}$ when n is even. This is a double covering of $SO(n)$ and $O(n)$, it is nontrivial because $\{\pm\}$ is connected by $(\cos te_1+\sin te_2)(\cos te_1-\sin te_2)$.

Proof: Let $\alpha=e_i\beta+\gamma$, then $\beta,\gamma\in Cl^0$ and so $\alpha=ce_1\ldots e_n+d$, and c can happen only when n is odd. □

Prop. (10.1.5). As in (7.3.1) $SU(2)$ is a universal covering of $SO(3)$ and so does $\mathrm{Spin}(3)$(10.1.4), so $SU(2)\cong\mathrm{Spin}(3)$.

Prop. (10.1.6). $\mathrm{Spin}(4)\cong SU(2)\times SU(2)$ because of the action of $\mathbb{H}\times\mathbb{H}$ on $\mathbb{H}:x\to ux\overline{v}$. This map is a two cover of $SO(4)$.

Prop. (10.1.7). $\mathrm{Spin}(5)=\mathrm{Sp}(2)$ and $\mathrm{Spin}(6)=SU(4)$.

2 Quiver Hecke Algebra

Chapter II

Number Theory & Arithmetic Geometry

II.1 p-adic Analysis

1 General Valuation Theory

Prop. (1.1.1). Two valuation on a field is equivalent iff $|x|_1 < 1 \Rightarrow |x|_2 < 1$ and is equivalent to $|x|_1 = |x|_2^s$ for some $s > 0$.

Proof: Cf.[Neukirch Algebraic Number Theory P117]. □

Cor. (1.1.2) (Weak Approximation).

Proof: Cf.[Neukirch Algebraic Number Theory P117]. □

Prop. (1.1.3) (Gelfand). Any field with an Archimedean valuation K is a subfield of $\mathbb{C}$.

Proof: We consider its completion. when it contains $\mathbb{C}$, this is a corollary of**??**, otherwise, we consider $K \otimes \mathbb{C}$, then it is a finite dimensional module over K thus also complete. □

Prop. (1.1.4) (Ostrowski). Any non-trivial value on $\mathbb{Q}$ is equivalent to v_p or $|\ |$. Thus any complete Archimedean field is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ by(1.1.3).

Proof: □

Lemma (1.1.5) (Continuity of Roots). For a separable polynomial f over a valued alg.closed field $\overline{K}$, there is a ε that every polynomial g that are closed enough to f, the roots of g is closed to roots of f respectively.

Proof: This is easy to see by decomposition as each root of f is close to a root of g. f, g have the same degree so the roots correspond to each other. □

Prop. (1.1.6) (Fundamental Inequality). if (K, v) is a valued field and L/K be a field extension of degree n, if w_i are the valuations of L above v, then

$$\sum e(w_i/v)(f(w_i/v) \leq [L : K].$$

The equality holds when v is discrete and L/K is separable.

Proof: Cf.[Clark note Theorem4]. □

Completeness

Prop. (1.1.7) (Cauchy Sequence of Non-Archimedean field). For a sequence $\sum a_i$ in a non-Archimedean field, it is a Cauchy sequence iff $\lim |a_i| = 0$.

In particular, convergent sequence are all absolutely convergent and for a Cauchy sequence not converging to 0, the valuations of the terms stabilize.

Proof: One way is easy, the other way, notice $|\sum_{v=i}^{j} a_i| \leq \max_{i,i+1,\ldots j} |a_v| < \varepsilon$. □

Prop. (1.1.8) (Completion of a Field). The completion of a non-Archimedean field is preferred to choose the definition of Cauchy sequence, so we see by(1.1.7) that $v(\hat{K}) = v(K)$.

Prop. (1.1.9). For a complete field K and any finite vector space L, L has only one norm up to equivalence and it is complete.

Proof: Cf.[Formal and Rigid Geometry P230]. □

Prop. (1.1.10). A complete valuation on a field can extend uniquely to a valuation on its alg.closure. And in the finite case, it is $|\alpha| = |N(\alpha)|^{\frac{1}{d}}$. This is an immediate consequence of(1.1.21), and $|\alpha| \leq 1$ iff it is integral over valuation ring R of K.

Proof: Cf.[Formal and Rigid Geometry P232]. □

Prop. (1.1.11). Any infinite separable algebraic extension of a complete field is not complete.

Proof: We use Krasner's lemma. By Ostrowski theorem, we can assume it is non-Achimedian, otherwise it cannot by infinite dimensional. Choose an infinite linearly independent basis of decreasing value rapidly enough, then we can see the field generated by the limit contains all the partial sums, contradiction. □

Prop. (1.1.12). If K is alg.closed valued field, then its completion is also alg.closed.

Proof: Let $L = (\hat{K})^{alg}$, then we can extend to a valuation on L, now let f be a monic polynomial with coefficients in $\hat{K}$, we show its root $\alpha \in L$ can be approximated by elements in K, now let g monic in $K[X]$ be an approximation of f that $|g(\alpha)| \leq \varepsilon^n$, then there is a root β of g that $|\alpha - \beta| < \varepsilon$, and $\beta \in K$ by alg.closedness. □

Prop. (1.1.13). If F is a complete valued field, then F^{sep} is dense in F^{alg}.

Proof: Assume F is non-Archimedean, then for $y \in F^{alg}$, there is a n that $y^{p^n} = \alpha \in F^{sep}$. We may assume $|\alpha| \leq 1$, then let π be an element that $|\pi| < 1$, then if y_i is a root of the separable polynomial $Y^{p^n} - \pi^i Y - \alpha = 0$, then $(y - y_i)^{p^n} = \pi^i y_i$. So $|y - y_i| \to 0$. □

Banach Algebra

Def. (1.1.14). For K complete valued field, a complete normed(valued) K-vector space is called a **Banach space**. Open mapping theorem and closed graph theorem are applicable in this case.

A K-algebra with a complete K-algebra norm is called a **Banach algebra**.

Prop. (1.1.15). If K is complete, then each normed K-module is weakly-Cartesian, Cf.[Non-Archimedean Analysis P92].

Cor. (1.1.16). If K is complete, any two valuation on a finite K-vector space are equivalent. Cf.[Non-Archimedean Analysis P93].

Henselian Value Field

Def. (1.1.17). A valued field K is called **Henselian** iff the valuation ring is a Henselian local ring.

Prop. (1.1.18). K is Henselian iff the valuation of K has a unique extension to any finite extension L/K. Cf.[Algebraic Number Theory Neukirch P144].

Cor. (1.1.19). Thus for a normal extension, x and $\sigma(x)$ has the same valuation. Hence any polynomial in $K[X]$ has a decomposition into polynomials where all their roots has the same valuation.

Prop. (1.1.20) (Hensel's Lemma Generalized). Let K be a complete valued non-Archimedean field and $\mathcal{O}_K$ be the valuation ring. If $P, Q, R \in \mathcal{O}_K[X]$ and $0 \le \lambda < 1$ that $\deg P = m+n, \deg Q = n, \deg R = m$, and

$$\deg(P-QR) \le m+n-1, \quad |P-QR|_G \le \lambda |\operatorname{res}(Q,R)|^2$$

Where $|-|_G$ is the induced Gauss norm on $K[X]$. Then there exist polynomials U, V that

$$|U|_G, |V|_G \le \lambda |\operatorname{res}(Q,R)|^2, \deg U \le n-1, \deg V \le m-1$$

and $P = (Q+U)(R+V)$.

Proof: If $\rho = |\operatorname{res}(Q,R)| = 0$, then $P = QR$. Otherwise, the map $\theta_{Q,R} : W_m \oplus W_n \to W_{m+n}$ is invertible(3.2.5). Then we let $\varphi(U,V) = \theta_{Q,R}^{-1}(P-QR-UV)$, then If $U, V \in B(0, \lambda\rho)$, then $|\varphi(U,V)|_G \le \lambda\rho$. And it can be proved φ is a contraction map from $B(0,\lambda\rho)^2$ to itself with contraction factor λ, so it has a fixed point (U,V) by(1.6.3). So $QU + RV = P - QR - UV$. □

Cor. (1.1.21) (Hensel's Lemma). Let K be a complete valued non-Archimedean field and A be the valuation ring. If $P(X) \in A[X]$ and α_0 is an element of A s.t. $|P(\alpha_0)/P'(\alpha_0)^2| = \varepsilon < 1$, then there exists a $\alpha \in A$ that $P(\alpha) = 0$ and $|\alpha - \alpha_0| \le |P(\alpha_0)/P'(\alpha_0)|$.

The usual form is when $|P'(\alpha_0)| = 1$, in which case we can pass to the residue field. Equivalently, the valuation ring of a complete non-Archimedean field is a Henselian local ring.

Proof: Let $\lambda = |P(\alpha_0)/P'(\alpha_0)|$ and $\operatorname{res} = |P'(\alpha_0)$. Notice If $P(X) = Q(X)(X-\alpha_0) + P(\alpha)$, then $\operatorname{res}(Q(X), X-\alpha_0) = Q(\alpha_0) = P'(\alpha_0)$(3.2.7). □

Prop. (1.1.22) (Krasner's Lemma). For a Henselian non-Archimedean field K, the if $\alpha, \beta \in \overline{K}$ that $|\alpha - \beta| < |\alpha - \sigma(\alpha)|$ for all σ, then $K(\alpha,\beta)/K(\beta)$ is purely inseparable. So when α is separable over K, $K(\alpha) \in K(\beta)$.

Proof: It suffice to prove that for all field morphism $\tau : K(\alpha,\beta) \to \overline{K}$ fixing $K(\beta)$, $\tau(\alpha) = \alpha$. This is because $|\tau(\alpha) - \beta| = |\alpha - \beta| < |\alpha - \sigma(\alpha)|$, thus $|\tau(\alpha) - \alpha| \le \max\{|\tau(\alpha) - \beta|, |\beta - \alpha|\} < |\alpha - \sigma(\alpha)|$. □

Cor. (1.1.23). If f is a separable irreducible polynomial and α is a root, then for g closed enough to f, there is a root β of g that $K(\beta) = K(\alpha)$. (Immediate consequence of(1.1.5)).

Cor. (1.1.24). Let K be a non-Archimedean valued field with completion $\hat{K}$, then any finite separable extension $\mathcal{L}/\hat{K}$ is of the form $L\hat{K}$. (Because of Primitive element theorem).

Prop. (1.1.25) (Kaplansky-Schilling). A field which is Henselian w.r.t two inequivalent valuation is separably closed, and separably closed field is Henselian w.r.t any valuation.

2 (Non-Archimedean)Valuation Theory

The difference between This subsection and that of Functional AnalysisIV.3 is that here all the valuation is non-Archimedean and there we mainly deal about complete Archimedean valuation over $\mathbb{C}$.

As far as I know, all properties proved in Functional Analysis independent of complex analysis is applicable to the non-Archimedean case, and in fact, the goal of this section is to build an analytic theory parallel to complex analysis.

References are [Non-Archimedean Analysis].

Normed Rings

Def. (1.2.1). A **semi-normed group** is a group with a non-Archimedean valuation, it is called a **normed group** iff the valuation has kernel 0, which is equivalent to Hausdorff.

A normed group is totally connected, because open balls at 0 are subgroups hence closed.

Def. (1.2.2) (Normed Ring). A **(semi-)normed ring** is a (semi-)normed additive group that

- $|1| = 1$. or the valuation is trivial.
- $|ab| \leq |a||b|$.

A **valued ring** is a normed ring with $|ab| = |a||b|$. It is called **degenerate** if all non-zero valuation value≥ 1.

Prop. (1.2.3). A valuation on a ring is non-Archimedean iff $\{|n|\}$ is bounded. Thus any valuation on a field with finite characteristic is non-Archimedian.

Prop. (1.2.4). In a normed ring, every triangle is an acute isosceles triangle. (This is because the biggest is smaller than the maximal of the other two, thus the biggest two are equal). Hence we have, for a circle $B(O, r)$, any interior point P is a center of circle, because $OP < r$.

Def. (1.2.5). A normed ring R is call a B**-ring** if elements of valuation 1 is invertible, it is called **bald** if there is a ε that no elements has valuation in $(1-\varepsilon, 1)$.

Prop. (1.2.6). If K is a normed field with valuation ring R, the smallest subring containing a zero sequence $a_0, a_1, \ldots$ is bald.

Proof: Cf.[Formal and Rigid Geometry P25]. □

Normed Modules

Def. (1.2.7) (Normed Module). A module M over a normed ring A is called **normed module** iff it is a normed additive group and $|ax| \leq |a||x|$ for $a \in A, x \in M$. If A is valued and the equality always holds, we call it **faithfully normed** or **valued module**.

If A is a valued field, any normed module is valued.

Prop. (1.2.8) (Normed Algebra). A normed algebra is an A algebra B with $A \to B$ bounded of norm 1.

Prop. (1.2.9). For two valued module over A, if A is non-degenerate, a morphisms is bounded iff it is continuous. This is because we can multiply by elements of A to reduce to a nbhd of 0.

This applies to the case when A contains a field where the valuation is non-trivial, because we can use(1.2.7).

Def. (1.2.10) (Completed Tensor Product). For two normed modules over a normed ring R, there is a complete normed R-module $M\hat{\otimes}N$ called the **completed tensor product**, satisfying the following universal properties: $M \times N \to M\hat{\otimes}N$ is bounded by 1, and for any complete normed R-module T and a R-map $M \times N \to T$ bounded by a, then it factor through a R-map $M\hat{\otimes}N \to T$ bounded by a.

It satisfies many universal properties as you can imagine.

Proof: Cf.[Formal and Rigid Geometry P238]. □

Cor. (1.2.11). By(1.2.9), when A is non-degenerate, then the amalgamated sum is just the fibered pushout when restricted to the category of complete valued module over A with continuous maps as morphisms, because it satisfies the universal property.

Prop. (1.2.12). For two normed R-algebras there is an operation of **amalgamated sum** which satisfies universal properties similar to(1.2.10). In fact, it is just the completed tensor product when seen as modules.

Proof: Cf.[Formal and Rigid Geometry P242]. □

Extensions of Norms and Valuations

Def. (1.2.13). If L/K is a finite extension of valued field of degree n, then v extends uniquely to $w(\alpha) = \frac{1}{n}v(N_{L/K}(\alpha)$, now we define the **ramification degree** as $(w(L^*) : v(K^*))$, and the **inertia degree** as the degree of the residue field extension.

3 Non-Archimedean Functional Analysis

4 p-adic Analysis

Basic References are [p-adic Analysis Robert].

Prop. (1.4.1). For $b \in \mathbb{Z}_p$, we can define a power series in $\mathbb{Z}_p[[T]]$ as the limit of $(1 + a)^{b_n}$ for $b_n \to b$ in $\mathbb{Z}_p$. So for $a \in \mathbb{C}_p$ with $v(a) > 0$, there can be defined an element $(1 + a)^b \in \mathbb{C}_p$, and we have $(1 + a)^b = \sum C_b^k a^k$.

Holomorphic functions

Def. (1.4.2). For a p-adic field L, denote by $\mathcal{L}_L$ the set of Laurent series with coefficients in $\mathcal{L}$, then the set of valuations that a Laurent series converges $Conv(f)$ is an interval of $[-\infty, +\infty]$. Let $\mathcal{A}(I)$ denote the set of elements in L of valuation in I.

If f is bounded at r_1, r_2, then it is convergent on (r_1, r_2).

Def. (1.4.3). Denote

$\mathcal{L}_L[r_1, r_2] = \{f|f \text{ is convergent on } [r_1, r_2]\}$.

$\mathcal{L}_L(r_1, r_2] = \{f|f \text{ is convergent on } (r_1, r_2]\}$.

$\mathcal{L}_L]r_1, r_2] = \{f|f \text{ is convergent on } (r_1, r_2] \text{ and bounded at } r_1\}$.

$\mathcal{B}_L(I)$ is the subset of bounded functions. These are all rings under addition and multiplication. And if we define $v^{(r)}(f)$ as the minimum of $v(a_n) + nr$, then it is a valuation on these rings.

Proof: Cf.[Foundations of Theory of (φ, Γ)-modules over the Robba Ring P31]. □

Def. (1.4.4). If we set for $\mathcal{L}_L]r_1, r_2]$ the the valuation $v^{[r_1,r_2]}(f) = \min\{v^{(r_1)}(f), v^{(r_2)}(f)\}$, then this is a valuation on it.

Prop. (1.4.5). $\mathcal{L}_L(\{r\})$ is complete under valuation $v^{(r)}$. Similarly the valuation $v^{[r_1,r_2]}(f)$ makes $\mathcal{L}_L[r_1, r_2]$ a Banach space unless $r_1 = r_2 = \infty$.

Proof: We let $r = 0$. For a Cauchy sequence of Laurent series, we see that each coefficient is a Cauchy sequence, hence converge to some element in L, so it converge term-wise to a Laurent series f, so it converse to f in $v^{(r)}$. □

Cor. (1.4.6). We consider $\mathcal{L}_L(0, r]$, then it has a countable sequence of norms $v^{1/n,r}$, which makes it a locally convex space, and the last proposition shows that these valuations are complete, and a Cauchy sequence must converge to the term-wise limit, so $\mathcal{L}_L(0, r]$ is a complete Fréchet space in the Fréchet topology.

Cor. (1.4.7). The same method shows that $\mathcal{L}_L(I)$ is a Fréchet space for any interval I.

Def. (1.4.8) (Robba Ring and Overconvergent Elements). We define $\mathcal{E}$ as the Laurent sequences that are bounded at 0 and $\varinjlim_{n\to-\infty} v(a_n) = \infty$, and we define the **overconvergent elements** $\mathcal{E}^\dagger$ and **Robba ring** $\mathcal{R}$ as

$$\mathcal{E}^\dagger = \bigcup_{r>0} \mathcal{L}_L]0, r], \quad \mathcal{R} = \bigcup_{r>0} \mathcal{L}_L(0, r], \quad \mathcal{E}^\dagger \subset \mathcal{R}$$

and equip them with the final topology w.r.t. the Fréchet topologies on $\mathcal{L}_L(0, r]$. And denote by $\mathcal{E}^+ = \mathcal{E}^\dagger \cap L[[T]]$ and $\mathcal{R}^+ = \mathcal{R} \cap L[[T]]$.

For more properties of Robba ring, See [Foundations of Theory of (φ, Γ)-modules over the Robba Ring Chap4].

Def. (1.4.9) (Newton Polygon). For a non-Archimedean valued field K and a polynomial or power series $P(X) = a_0 + a_1X + \cdots + a_dX^d \in K[X]$, we denote by **Newton polygon** as the lower convex hull of the set of points $(0, v(a_0)), (1, v(a_1)), \ldots, (d, v(a_d))$.

Prop. (1.4.10). Foe a non-Archimedean field K the number of roots of P in $\overline{K}$ with valuation λ equals the horizontal width of the segment of Newton polynomial of P of slope $-\lambda$.

Proof: We may assume P is monic, then its coefficients are elementary polynomials of roots of P. And the conclusion follows as K is non-Archimedean. □

For Newton polynomial of power series, see[Berger Galois Representations Chap3] and Reference [Zeros of Power Series over complete Valued Field Lazard].

Prop. (1.4.11). If $I =]0, +\infty]$ and $f(X) \in \mathcal{H}(I)$, then the number of zeros of $f(X)$ in $\mathcal{A}(I)$ equals the length of the segment of $NP(f)$ whose slope is $-s$, and these roots gives a $P_s(X) \in K[X]$ that $f(X) = P_s(X)G(X)$, $G(X) \in \mathcal{H}(I)$.

Proof: Cf.[Zeros of Power Series over complete Valued Field Lazard]. □

Cor. (1.4.12). If $f(X) \in \mathcal{H}(I)$, then $f(X) \in \mathcal{B}_L(I)$ iff it has f.m. zeros in $\mathcal{A}(I)$.

Proof: Let $r = \inf I$ and $s = \sup I$. First notice that $f \in \mathcal{L}_L(I)$ is in $\mathcal{B}(I)$ iff $v(a_n)+nr$ is bounded from below as $n \to +\infty$ and $v(a_n) + ns$ is bounded below as $n \to -\infty$. And from the graph of $NP(f)$, this is equivalent to f has f.m. zeros in $\mathcal{A}(I)$. □

Prop. (1.4.13). $\mathcal{H}(I)$ is a Bezout domain.

II.2 Algebraic Number Theory

References are [Algebraic Number Theory Neukirch], should also include notes of Pete.L.Clark. [Neukirch Chap2.8, 2.9, 3.1, 3.2] should be added quickly.

1 Ramification Theory

Prop. (2.1.1). If a prime $\mathfrak{p}$ splits completely in two separable extension LM of K, then it also splits completely in the composite LM.

Proof: We use the language of valuation. The extension of a valuation v of K corresponds to the set of equivalent classes of algebra map from L to $\overline{K_v}$ module conjugacy over K_v. So We only need to show that two different maps of LM are not conjugate over K_v. But the restrict of them to L or M is different, thus not conjugate over K_v by the assumption. □

Cor. (2.1.2). A prime splits completely in a separable extension L if it splits completely in the Galois closure N of L.

Proof: This is because the Galois closure is the composite of the conjugates of L.

But it also can be proven directly : Set $H = \mathrm{Gal}(N/L)$,$\mathcal{P}$ a prime of N over $\mathfrak{p}$, then

$$H\backslash G/G_{\mathcal{P}} \longrightarrow \{\text{Primes of } L \text{ over } \mathfrak{p}\}, \quad H\sigma G_{\mathcal{P}} \mapsto \sigma\mathcal{P} \cap L$$

is a bijection. So it splits completely in $L \iff G_{\mathcal{P}}$ is trivial $\iff$ it splits completely in N by counting numbers. □

Prop. (2.1.3). A prime p splits in $\mathbb{Z}[\xi_n]$ iff $p \equiv 1 (\mathrm{mod}\ n)$.

Proof: First, if it splits, then $f = 1$, Because the ring of integers is $\mathbb{Z}[\xi_n]$, so $X^n - 1$ splits in $\mathbb{F}_p$(2.3.2), thus $p \equiv 1(\mathrm{mod}\ n)$. And if $p \equiv 1(\mathrm{mod}\ n)$, it is unramified and $X^n - 1$ splits in $\mathbb{F}_p$, so $f = 1$. □

Unramified Extension

Def. (2.1.4). For K a Henselian non-Archimedean valued field, L/K a finite extension is called **unramified** iff the residue field extension λ/k is separable and $[L : K] = [\lambda : k]$. Any algebraic extension is called **unramified** iff any finite extension is unramified.

This is compatible because unramified extensions form a distinguished class. So we can talk about the **maximal unramified extension** T of K.

Proof: It is faithfully transitive because the field extension degree is transitive, and for base change, as[the residue field is separable, we let $\lambda = k[\overline{\alpha}]$, and choose a lift $\alpha \in \mathcal{O}_L$, the minipoly of α is $f(X) \in \mathcal{O}_K[X]$. Then we have

$$[\lambda : k] \leq \deg \overline{f} = \deg f = [K(\alpha) : K] \leq [L : K] = [\lambda : k]$$

So $L = K(\alpha)$ and $\overline{f}$ is the minipoly of $\overline{\alpha}$. Then $L' = K'(\alpha)$, and let $g(X)$ be the minipoly of α over K', then $\overline{g}$ is a factor of $\overline{f}$ so separable, hence irreducible by Hensel's lemma. Noe:

$$[\lambda' : k'] \leq [L' : K'] = \deg g = \deg \overline{g} = [k'(\alpha) : k'] \leq [\lambda' : k].$$

So $[\lambda' : k'] = [L' : K']$. □

Prop. (2.1.5). The residue field of the maximal unramified extension T/K is $\overline{k}$, and the value group is the same as K.

Proof: The first assertion is because for any separable polynomial, it has a lift which is irreducible has a root lifting $\overline{\alpha}$, contradicting the maximality. For the second, look at finite subextensions, then it results from the fundamental inequality(1.1.6). □

Tamely Ramified Extension

Def. (2.1.6). For K a Henselian non-Archimedean valued field, L/K a finite extension is called **tamely ramified** iff the residue field extension is separable and $([L:T],p)=1$, where T is the maximal unramified subextension.

Prop. (2.1.7). A finite extension L/K is tamely unramified iff the extension is generated by radicals: $L=T(\sqrt[m]{a_i})$, where $a_i\in L$, (WARNING: make sure if $a_i\in K$ or not?).

Proof: Cf.[Algebraic Number Theory Neukirch P155]. □

Prop. (2.1.8). Tamely unramified extensions form a distinguished class, so we can talk about the maximal tamely unramified extensions.

Proof: Cf.[Algebraic Number Theory Neukirch P156]. □

Prop. (2.1.9). The value field of tamely ramified extensions. Cf.[Neukirch P157].

Ramification Groups

Prop. (2.1.10). For an extension valuation $w|v$, the **decomposition group** is $G_w(L/K)=\{\sigma\in G(L/K)|w\circ\sigma=w\}$. The **decomposition group** Z_w is the fixed field of G_w.

When w is non-Archimedean, we further define:

The **inertia group** is $I_w(L/K)=\{\sigma\in G_w(L/K)|\sigma(x)\equiv x \mod \mathfrak{P}\}$. The **inertia field** T_w is the fixed field of I_w.

The **ramification group** is $R_w(L/K)=\{\sigma\in G_w(L/K)|\sigma(x)/x\equiv 1 \mod \mathfrak{P}\}$. The **ramification field** V_w is the fixed field of R_w.

Prop. (2.1.11). For a local field, the ramification degree e equals the order of inertia group $|I_{L/K}|$.

Prop. (2.1.12). When w is non-Archimedean, the residue field extension λ/k is normal and there is an exact sequence

$$1\to I_w\to G_w\to G(\lambda/k)\to 1.$$

Proof: Cf.[Neukirch P172]. □

Prop. (2.1.13). T_w/Z_w is the maximal unramified subextension of L/Z_w.

Proof: Cf.[Neukirch P173]. □

Prop. (2.1.14). V_w/Z_w is the maximal tamely ramified subextension of L/Z_w.

Proof: Cf.[Neukirch P175]. □

Higher Ramification Groups

Def. (2.1.15). For L/K be a finite Galois extension of CDVR, we define the s-th **ramification group** $G_s(L/K)=\{\sigma\in G|v_L(\sigma(x)-x)\geq s+1 \text{ for all } a\in\mathcal{O}_L\}$.

Then we have $G=G_{-1}\supset G_0\supset G_1\subset\ldots$. And G_0 is the inertia group.

When K has finite quotient field, then G_1 is the ramification group(one way is trivial, for the other, we use Teichmüller representatives, then R_w preserves all them, and $\sigma(x)-x\equiv 0 \mod \mathfrak{P}^2$ is true for π, so it is true for all). In this case, we have

$$G_s(L/K)=\{\sigma\in G_0|\frac{\sigma(\pi_L)}{\pi_L}\in U_L^s\},\ \text{for } s\geq 0.$$

So there are injective morphism $G_s/G_{s+1}\to U_L^s/U_L^{s+1}:\sigma\mapsto\sigma(\pi_L)/\pi_L$ for $s\geq 0$.(This is independent of π_L chosen because units are mapped mod U_L^{s+1}).

Prop. (2.1.16). For local fields L/K, if σ is in the inertia group, then

$$v_L(\frac{\sigma(x)}{x}-1)\geq v_L(\frac{\sigma(\pi_L)}{\pi_L}-1)+\delta_{v_L(x),0}$$

for any $x\in\mathcal{O}_L$ and a uniformizer π_L. Equality holds when $v_L(x)=1$.

Proof: if L has residue field $\mathbb{F}_q$, then any element of $\mathcal{L}$ can be written as $\sum\xi_n\pi_L^n$, where ξ_n are all $q-1$-th roots of unity. And because σ is inertia group, all $q-1$-th roots of unity are preserved, so $\sigma(\xi_n\pi_L^n)-\xi_n\pi_L^n=\xi_n\pi_L(\frac{\sigma(\pi_L)}{\pi_L}-1)(\sigma(\pi_L)^{n-1}+\sigma(\pi_L)^{n-2}\pi_L+\ldots+\pi_L^{n-1})$ has valuation$\geq$ $v(\frac{\sigma(\pi_L)}{\pi_L}-1)+n$. Thus the result. □

Prop. (2.1.17). For a finite extension of CDVR, if the residue field extension λ/k is separable, then there exists a $x\in\mathcal{O}_L$ that $\mathcal{O}_K[x]=\mathcal{O}_L$.

Proof: If $\overline{x}$ is an element of λ that generate λ over k, by primitive element theorem, then let $\overline{f}$ be the minipoly of $\overline{x}$, then let f,x be lifting of them, then $f(x)$ is a uniformizer, otherwise, we now $f'(x)$ has valuation 0, so $f(x+\pi_L)$ is a uniformizer. Now we see that $x^if(x)^j$ is a basis of $\mathcal{O}_L$ over $\mathcal{O}_K$, □

Now in the sequel, we assume that the residue field extension is separable.

Lemma (2.1.18). We define $i_{L/K}(\sigma)=v_L(\sigma x-x)$, where x is the generator of $\mathcal{O}_L/\mathcal{O}_K$.

If $L/L'/K$ are Galois extensions that e is the ramification index of L/L'. Then

$$i_{L'/K}(\sigma')=\frac{1}{e}\sum_{\sigma|_{L'}=\sigma'}i_{L/K}(\sigma).$$

Proof: Cf.[Neukirch Algebraic Number Theory P178]. □

Def. (2.1.19) (Upper Numbering). We define the **Herbrand function** $\varphi_{L/K}(u)=\int_0^u\frac{\mathrm{dx}}{(G_0:G_x)}$. It maps $\{x\geq 1\}$ to itself and is strictly increasing.

If $m\leq s<m+1$, then it is just $\varphi_{L/K}(s)=\frac{1}{g_0}(g_1+g_2+\ldots+g_m+(s-m)g_{m+1})$, where $g_i=|G_i|$. By a double counting, it is

$$\varphi_{L/K}(s)=\frac{1}{g_0}\sum_{\sigma\in G}\min\{i_{L/K}(\sigma),s+1\}-1.$$

The derivative of $\varphi_{L/K}$ is $\varphi'_{L/K}(s)=\dfrac{|G_s|}{g_0}$.

Let $\psi_{L/K}$ be the inverse function of We define $G^t=G_{\psi_{L/K}(t)}$, this is called the **upper numbering**.

Lemma (2.1.20). For $L/L'/K$ Galois extensions, one has $G_s(L/K)H/H=G_t(L'/K)$, where $t=\varphi_{L/L'}(s)$. Equivalently, $G_s/H_s=(G/H)_{\varphi_{L/L'}(s)}$.

Proof: For $\sigma'\in G(L'/K)$, we choose a inverse image $\sigma\in G(L/K)$ of maximal $i_{L/K}(\sigma)$, then $i_{L'/K}(\sigma')-1=\varphi_{L/L'}(i_{L/K}(\sigma)-1)$. To prove this, let $i_{L/K}(\sigma)=m$, then we see $i_{L/K}(\sigma\tau)=\min\{i_{L/K}(\tau),m\}$, so by(2.1.18), $i_{L'/K}(\sigma')=\frac{1}{e}\sum_{\tau\in H}\min\{i_{L/K}(\tau),m\}$. And $e=|H_0|$ by(2.1.11). So the assertion follows from(2.1.19).

Now σ' is in the image of G_s is equivalent to $i_{L/K}(\sigma)-1\geq s\iff\varphi_{L/L'}(i_{L/K}(\sigma)-1)\geq\varphi_{L/L'}(s)$, which by what proved is equivalent to $\sigma'\in G_t(L'/K)$. □

Cor. (2.1.21). For $L/L'/K$ Galois extensions, $\varphi_{L/K}=\varphi_{L'/K}\circ\varphi_{L/L'}$, hence similar formula holds for ψ.

Proof: By the proposition and multiplicity of ramification index e, we get

$$\frac{1}{e_{L/K}}|G_s|=\frac{1}{e_{L'/K}}|(G/H)_t|\frac{1}{e_{L/L'}}|H_s|.$$

where $t=\varphi_{L/L'}(s)$, which is equivalent to the derivative $\varphi'_{L/K}(s)=\varphi'_{L'/K}(t)\varphi'_{L/L'}(s)=(\varphi_{L'/K}\circ\varphi_{L/L'})'(s)$, and they are equal at 0, so the conclusion follows. □

Prop. (2.1.22) (Herbrand's Theorem). For $L/L'/K$ Galois extensions, $G^t(L'/K)$ is the image of $G^t(L/K)$ under the quotient.

Proof: Let $r=\varphi_{L'/K}(t)$, by the above lemme and corollary,

$$G^tH/H=G_{\varphi_{L/K}(t)}H/H=G'_{\varphi_{L/L'}(\psi_{L/K}(t))}=G'_{\varphi_{L/L'}(\psi_{L/L'}(r))}=G_r(L'/K)=G^t(L'/K)$$

□

Prop. (2.1.23) (Hasse-Arf). For an Abelian extension of CDVRs L/K that the residue field extension is separable, the jump in the upper numbering of higher ramification group G^v must happen at integers. (Note: The proof in the case where K is a local field is much easier by Lubin-Tate group, See(4.1.33).

Proof: The theorem is just saying that if $G_s\neq G_{s+1}$ for s integer, then $\varphi_{L/K}(s)$ is an integer.

This follows follows from the following lemma, because if G is not totally ramified, then we can change it to the Galois field of G_0, this didn't change anything by the definition of(2.1.19), and the fact $\varphi(0)=0$. And when $G^v\neq G^{v+}$, then we consider splitting G/G^{v+} into product of cyclic groups, thus there is one cyclic group H that the projection of G^v into H is not trivial. Now H is a Galois group of some L'/K, and Herbrand's theorem shows that $H^v\neq H^{v+}$, hence v is an integer by the following lemma. □

Lemma (2.1.24). For a cyclic totally ramified extension of CDVRs L/K that the residue field extension is separable, if μ is the maximal integer that $G_\mu\neq 1$, then $\varphi_{L/K}(G_\mu)$ is an integer.

Proof: Cf.[Serre Local Fields P94]. □

Remark (2.1.25). An example: If $F_n = \mathbb{Q}_p(\zeta_{p^n})$, then

$$G(F_n/\mathbb{Q}_p)_s = G(F_n/F_t) \quad \text{for } p^t - 1 \leq s < p^{t+1} - 2.$$

(This is because $\zeta_{p^n} - 1$ is a uniformizer of F_n). Thus $G(F_n/\mathbb{Q}_p)^i = G(F_n/F_i)$.

Different and Discriminant

Def. (2.1.26). Let L/K be a finite separable field extension with separable residue field extensions, and $\mathcal{O}_K$ is a Dedekind domain with integral closure $\mathcal{O}_L$ in L, there is a **trace form** on L: $(x, y) \to \mathrm{tr}(xy)$.

We define the **dual module** for a fractional ideal I as $\check{I} = \{x \in L | \,\mathrm{tr}(xI) \in \mathcal{O}_K\}$. This is truly a fractional ideal because if $\alpha_i \in \mathcal{O}_L$ is a basis of L/K, and let $d = \det(\mathrm{tr}(\alpha_i\alpha_j))$, then for any $a \in I \cap \mathcal{O}_L$, $ad\check{I} \in \mathcal{O}_L$, because if $x = \sum x_i\alpha_i \in \check{I}$, then $\sum ax_i \,\mathrm{tr}(\alpha_i\alpha_j) = \mathrm{tr}(xa\alpha_j) \in \mathcal{O}_K$, so solve the equation shows $dax_i \in \mathcal{O}_K$.

The **different** of K/L is defined to be $\mathcal{D}_{L/K} = \check{\mathcal{O}}_L^{-1}$.

Prop. (2.1.27). Different is compatible with composition, localization and completion.

Proof: Cf.[Neukirch Algebraic Number Theory P195]. □

Prop. (2.1.28). If $\mathcal{O}_L = \mathcal{O}_K[\alpha]$, then $\mathcal{D}_{L/K} = (f'(\alpha))$, where f is the minipoly of α.

Proof: Let $f = a_0 + a_1X + \ldots + a_nX^n$ and $f(X)/(x - \alpha) = b_0 + b_1X + \ldots + b_{n-1}X^{n-1}$, and denote the roots of f be α_i, then

$$\sum \frac{f(X)}{X - \alpha_i}\frac{\alpha_i^r}{f'(\alpha_i)} = X^r$$

for all r by Lagrange interpolation. This is equivalent to

$$\mathrm{tr}(\frac{\alpha_i^r b_j}{f'(\alpha_i)}) = \delta_{ij}$$

So $\mathcal{D}_{L/K} = f'(\alpha)^{-1}(b_0\mathcal{O}_K + b_1\mathcal{O}_K + \ldots + b_{n-1}\mathcal{O}_K)$. Now the result follows if $(b_0\mathcal{O}_K + b_1\mathcal{O}_K + \ldots + b_{n-1}\mathcal{O}_K) = \mathcal{O}_L$, which is easy to see if we write b_i as polynomials of α. □

Cor. (2.1.29). If L/K is finite extension of local fields, then

$$v_L(\mathcal{D}_{L/K}) = \sum_{\sigma\in G, \sigma\neq 1} i_{L/K}(\sigma) = \int_{-1}^{\infty}(|G(L/K)_t| - 1)dt.$$

Notation as in(2.1.18).

Prop. (2.1.30). If L/K is a finite extension and if I is an ideal of $\mathcal{O}_L$, then $v_K(\mathrm{tr}_{L/K}(I)) = \lfloor v_K(I \cdot \mathcal{D}_{L/K})\rfloor$.

Proof: By definition, $\mathrm{tr}_{L/K}(x\mathcal{O}_L) \subset \mathcal{O}_K$ iff $x \in \mathcal{D}_{L/K}^{-1}$, thus $\mathrm{tr}_{L/K}(I) \subset J$ iff $I \subset \mathcal{D}_{L/K}^{-1}J$, i.e. $\mathrm{tr}_{L/K}(I)$ is the smallest ideal J of $\mathcal{O}_K$ that contains $I \cdot \mathcal{D}_{L/K}$, thus the result. □

2 Local Fields

The Group Structure of Local Fields

Prop. (2.2.1). A **local field** is a complete discrete valuation field with finite residue field. For a local field, $\mathcal{O}_K^*$ thus also K^* is locally compact Hausdorff.

Prop. (2.2.2). For $m > 0$, there is an isomorphism $(-)^m : U^n \cong U^{n+v(m)}$ when n is sufficiently large.

Proof: Let $m = u\pi^{v(m)}$. For surjectivity, we need to find x, that $1 + a\pi^{n+v(m)} = -(1 + x\pi^n)^m$. i.e.

$$-a + ux + \pi^{n-v(m)} f(x) = 0.$$

This has a solution x by Hensel's lemma. □

Cor. (2.2.3). $(K^*)^m$ is an open subgroup of K^*, and $\bigcap_m (K^*)^m = 1$. (Because if $a \in \bigcap_m (K^*)^m = 1$, then a is a unit, thus $a \in \bigcap_m (U)^m = 1$, thus $a \in U^n$ for every n thus $a = 1$.

Prop. (2.2.4). $[K^* : (K^*)^m]$ can be calculated, Cf.[Neukirch CFT P81].

Prop. (2.2.5) (p-adic Logarithm). Cf.[Neukirch P137].

Prop. (2.2.6) (Multiplicative Group Structure). For a local field K,
- If char $K = 0$, then $\mathcal{O}_K^* \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}_p^d$, where $d = [K : \mathbb{Q}_p]$.
- If char $K = p$, then $\mathcal{O}_K^* \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}_p^{\mathbb{N}}$.

Proof: Cf.[Neukirch P140]. □

Prop. (2.2.7). The inertia field, ramification field and the tower.

Ramification of Cyclotomic Fields

Prop. (2.2.8) (Unramified case). For K a finite extension of $\mathbb{Q}_p$ of residue field $\mathbb{F}_q$, we consider $L = K(\zeta_n)/K$, where $(n, p) = 1$. Then it is unramified of degree f where f is the minimal number that $q^f \equiv 1 \mod n$. And $\mathcal{O}_L = \mathcal{O}_K[\zeta_n]$.

Proof: ζ is a root of $\Phi_n | X^n - 1$, which is separable in k, so Φ and $\overline{\Phi}$ are both irreducible of the same degree by Hensel's lemma, so it is unramified, and λ is the minimal extension of $\mathbb{F}_q$ that contains the n-th roots and are generated by it, thus the result by the theory of finite fields.

For the last assertion, notice it is unramified so $\mathcal{O}_L = \mathcal{O}_K[\zeta_n] + p\mathcal{O}_L$ hence the result by Nakayama. □

Cor. (2.2.9). The maximal unramified extension of K is generated by adjoining all n-th roots where $(n, p) = 1$. This is because there is an inclusion relation and their residue field $\overline{\mathbb{F}}_p$ is already generated by roots of unity.

Prop. (2.2.10) (Totally Ramified case). Consider $\mathbb{Q}_p$(other local fields behave different), we have the $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is totally ramified of degree $\varphi(p^n)$ the Galois group is $(\mathbb{Z}/p^n\mathbb{Z})^*$. The ring of valuation of $\mathbb{Q}(\zeta_{p^n})$ is $\mathbb{Z}_p[\zeta_{p^n}]$ and $1 - \zeta$ is a uniformizer.

Proof: ζ is a root of the polynomial $\Phi = X^{p^{n-1}(p-1)} + X^{p^{n-2}(p-1)} + \ldots + 1 = 0$, which equals $\frac{X^{p^n}-1}{X^{p^{n-1}}-1} \equiv (X-1)^{p^{n-1}(p-1)} \mod p$ and $\Phi(1) = p$, so $\Phi(X+1)$ is a Eisenstein polynomial, hence irreducible. So $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is totally ramified of degree $p^{n-1}(p-1)$ and $N(1-zeta) = \prod(1-\sigma(\zeta)) = \Phi(1) = p$, so it is a uniformizer. The ring of integer is generated by a uniformizerby(1.1.6) as the extension is totally ramified. □

Prop. (2.2.11) (Infinite Cyclotomic Field). For a p-adic number field K, let $K_n = K(\zeta_{p^n})$ and $K_\infty = \cup K_n$ and $F = \mathbb{Q}_p$. Let χ be the cyclotomic character, then $\chi(G_K)$ is an open subgroup of $\mathbb{Z}_p^*$, thus contains a U_n for some n. Thus there is an isomorphism of groups: $\chi^{-1}(U_n) \cap G_K/\chi^{-1}(U_n + 1) \cap G_K \cong U_n/U_{n+1}$ which has order p, for n large.

So K_{n+1}/K_n is totally ramified of degree p, because $K_n = K \cdot F_n$, and its value group extension is of degree p, too.

And $|\{K_n : F_n\}|$ is decreasing and eventually equals to $[K_\infty : F_\infty]$. This is because its order equals $\chi^{-1}(U_n)/\chi^{-1}(U_n) \cap G_K \cong \chi^{-1}(U_n)G_K/G_K$, which is eventually $\mathrm{Ker}(\chi)G_K/G_K$, because $U_n \subset \chi(G_K)$.

Cor. (2.2.12). For n large, if x_i is a set of basis of $\mathcal{O}_{K_n}$ over $\mathcal{O}_{F_n}$, then they form a basis of K_N over F_N for all $N \geq n$. This is because it generate K_N over F_N and $[K_N : F_N] = [K_n : F_n]$.

Prop. (2.2.13). $p^n v_p(\mathcal{D}_{K_n/F_n})$ is bounded and eventually constant. In particular $v_p(\mathcal{D}_{K_n/F_n})$ converges to 0.

Proof: Cf.[Galois representation Berger P20]. □

Cor. (2.2.14). If L/K is a finite extension, then $\mathrm{tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty}) = \mathfrak{m}_{K_\infty}$.

Proof: By(2.1.30) and the fact $G(L_\infty/K_\infty) \cong G(L_n/K_n)$ for n large by(2.2.11), we have $\mathrm{tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_n}) = \mathfrak{m}_{K_n}^{c_n}$, where $c_n = \lfloor v_{K_n}(\mathfrak{m}_{L_n}\mathcal{D}_{L_n/K_n})\rfloor$. By the above proposition, c_n is bounded by a c. But if $x \in \mathfrak{m}_{K_\infty}$, $x \in \mathfrak{m}_{K_n}^c$ for n large, so $x \in \mathrm{tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty})$. □

Lemma (2.2.15). For any $\delta > 0$, when n is large, if $x \in \mathcal{O}_{K_{n+1}}$ and $g \in G(K_{n+1}/K_n)$, $v_p(g(x)-x) \geq \frac{1}{p-1} - \delta$. In particular, $v(N_{K_{n+1}/K_n}(x) - x^p) \geq \frac{1}{p-1} - \delta$.

Proof: Choose a basis e_i of $\mathcal{O}_{K_n}/\mathcal{O}_{F_n}$, then e_i^* is a basis for $\mathcal{D}_{K_n/F_n}$, and if $x_i = \mathrm{tr}_{K_{n+1}/F_{n+1}}(xe_i)$, then $x_i \in \mathcal{O}_{F_{n+1}}$ and $x = \sum x_i e_i$, by(2.2.12), and we have by(2.1.25), $v(g(x_i) - x_i) \geq 1/(p-1)$, so when n is large, by(2.2.13), $v(x_i) \geq -\delta$, so the require is satisfied. □

Prop. (2.2.16). if $\delta > 0$ and I is the ideal of elements of valuation$\geq 1/(p-1) - \delta$, then for n large, there is a map $x \mapsto x^p : \mathcal{O}_{K_{n+1}}/I \cap \mathcal{O}_{K_{n+1}} \to \mathcal{O}_{K_n}/I \cap \mathcal{O}_{K_n}$, and it is surjective.

Proof: For n large, choose a uniformizer π_{n+1} of K_{n+1}, then $\pi_n = N_{K_{n+1}/K_n}(\pi_{n+1})$ is the uniformizer of K_n because it is totally ramified(2.2.11), so any element $x \in \mathcal{O}_{K_{n+1}}$ can be written as $\sum \pi_{n+1}^i [x_i]$, where $x_i \in k_{K_{n+1}} = k_\infty$. Then $x^p \equiv \sum \pi_{n+1}^{pi}[x_i]^p \equiv \sum \pi_n^i [x_i^p] \mod I$ by the above proposition. And the surjection is verbatim. □

Def. (2.2.17) (Tate's Normalized Trace). The function $R_n(x) = p^{-k}\,\mathrm{tr}_{F_{n+k}/F_n}(x)$ is compatible with k and defines a F_n-linear projection from F_∞ to F_n, and it commutes with G_F action, called the **Tate's normalized trace**.

it's easily verified that $R_n(\mathcal{O}_{F_{n+k}}) \subset \mathcal{O}_{F_n}$, thus $R_n(\pi_n^j \mathcal{O}_{F_{n+k}}) \subset \pi_n^j \mathcal{O}_{F_n}$. So we have $v(R_n(x)) > v(x) - v(\pi_n)$. So R_n extends by continuity to a map $R_n : \hat{F}_\infty \to F_n$. If $x \in F_\infty$, then $R_n(x) = x$ for n large, thus $R_n(x) \to x$ for any $x \in \hat{F}_\infty$.

Now for a finite extension $K/\mathbb{Q}_p$, for n large, if e_i is a set of basis of $\mathcal{O}_{K_n}/\mathcal{O}_{\mathcal{F}_n}$, then for any $x \in \mathcal{O}_{K_N}$, $x = \sum x_i e_i^*$, where $x_i = \mathrm{tr}_{K_\infty/F_\infty}(xe_i) \in \mathcal{O}_{F_N}$, as in the proof of(2.2.15). So know we define $R_n(x) = \sum R_n(x_i)e_i^*$. Notice this is defined only for n large, and is independent of x_i chosen, and by the following lemma, it is continuous and extends to a K_n-linear projection from $\hat{K}_\infty$ to K_n.

Lemma (2.2.18). for any $\delta > 0$, when n is large, $v(R_n(x)) \geq v(x) - \delta$.

Proof: We have $v(x_i) > v(x) - v(\pi_N)$ by F_N-linearity, and $v(R_n(x_i)) > v(x_i) - v(\pi_n)$ as in(2.2.17), and $v(e_i^*) \geq -\delta$ when n is large, by(2.2.13). Thus the result. □

Prop. (2.2.19). There is a decomposition of $\hat{K}_\infty = X_n \oplus X_n$, where $X_n = \mathrm{Ker}\, R_n$. If $\delta > 0$, then for n large, $\alpha \in \mathbb{Z}_p^*$ and γ_n that $\chi(\gamma_n)$ is a topological generator Γ_{F_n}, $1 - \alpha\gamma_n : X_n \to X_n$ (because γ commutes with R_n) is invertible and $v_p((1-\alpha\gamma_n)^{-1}x) \geq v_p(x) - 1/(p-1) - \delta$, unless $\alpha = -1$ and $p = 2$, in which case it is only invertible on X_{n+1}.

Proof: As usual, x_i is a basis of $\mathcal{O}_{K_n/F_n}$, then $x = \sum x_i e_i^*$, $x_i = \mathrm{tr}_{K_\infty/F_\infty}(xe_i) \in \hat{F}_\infty$, and $R_n(x) = 0$. Then $(1 - \alpha\gamma_n)$ acts on x_i, so it reduce to the case $K = \mathbb{Q}_p$, if one notices(2.2.18) and(2.2.13).

Injectivity: If $\alpha = 1$, this is Ax-Sen-Tate. In other situations, $(1 - \alpha\gamma_n)(R_{n+k}(x)) = 0$ for all $k \geq 0$, so $R_{n+k}(x) = \alpha^{p^k}\gamma_n^{p^k}(R_{n+k}(X)) = \alpha^{p^k}R_{n+k}(X)$, so $R_{n+k}(x) = 0$, hence $x = 0$.

Surjectivity: Cf.[Galois representation Berger P23]. □

3 Global Fields

Prop. (2.3.1). $\mathrm{G}(\mathbb{Q}[\mu_n]/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$.

Proof: We choose a prime p prime to n and show that μ_n^p is conjugate to μ_n.

Let $X^n - 1 = f(X)h(X)$ with f(X) minimal polynomial of μ_n.If $f(\mu_n^p) \neq 0$, then $h(\mu_n^p) = 0$,thus $h(X^p) = f(X)g(X)$.So module p, $X^n - 1$ has a multi root, which is impossible. □

Prop. (2.3.2). The ring of integers in a cyclotomic field is generated by the roots of identity.

Proof: First consider the case n a prime power. Because $d(1, \zeta, \cdots, \zeta^{d-1}) = \pm l^s$, $l^s\mathcal{O} \subset \mathbb{Z}[\zeta] \subset \mathcal{O}$. Because p totally splits, $\mathcal{O} = \mathbb{Z}[\zeta] + \pi\mathcal{O}$, thus $\mathcal{O} = \mathbb{Z}[\zeta] + \pi^t\mathcal{O}$. Choose $t = s\phi(n)$ yields $\mathbb{Z}[\zeta] = \mathcal{O}$.

Then for different p, the fields are disjoint and the discriminant are pairwisely coprime, thus by (2.11) in Neukirch, the products of the integral basis form an integral basis. □

Prop. (2.3.3). fractional ideal.

Prop. (2.3.4) (Unit Theorem). If S is a finite set of primes containing all the infinite primes, the group K^S of elements of K^* that has only prime divisors in S, is a f.g. group of rank $|S| - 1$.

Prop. (2.3.5) (Class Number). The **ideal class group** is defined as the group of ideals in K quotients J_K the principal ideals, it has finite order, class the **class number** of K.

Prop. (2.3.6) (Hermite's theorem). There exists only f.m. number fields with bounded discriminant. Cf.[Neukirch Algebraic Number Theory P206].

Prop. (2.3.7) (Minkowski's theorem). The discriminant of a number field different from $\mathbb{Q}$ is not ± 1. Cf.[Neukirch Algebraic Number Theory P207].

Cor. (2.3.8). The field $\mathbb{Q}$ doesn't contain any unramified extensions.

Prop. (2.3.9) (Strong Approximation Theorem).

4 Adele and Idele

Def. (2.4.1) (Notations). We fix some notation.
S is a finite set of primes.
The **Idele** is a subset (a_p) of $\prod_p K_p^*$ that a_p is a unit for a.e. p.
The **ideal class group** $C_K = I_K/K^*$.
The group $I_K^S = \prod_{\mathfrak{p}\in S} K_\mathfrak{p}^* \times \prod_{p\notin S} U_\mathfrak{p}$ is called the **group of S-ideles** of K.
$K^S = K^* \cap I_K^S$ is the set of **S-units** of K.

Prop. (2.4.2). For a field extension L/K, $I_K \subset I_L$, and $I_L^G = I_K$, this is be the diagonal inclusion to all the primes above a given prime, and the action is by $(\sigma\mathfrak{a})_\mathfrak{P} = \sigma\mathfrak{a}_{\sigma^{-1}\mathfrak{P}}$. This induces an inclusion $C_K \subset C_L$ and $C_L^G = C_K$. The last assertion uses long exact sequence and $H^1(G, L^*) = 0$.

Prop. (2.4.3). I_K is locally compact in the restricted product topology, and K^* is a discrete subgroup of I_K, thus C_K is also Hausdorff locally compact.

Proof: Cf.[Neukirch P157]. □

Prop. (2.4.4). There is an absolute valuation on I_K and it vanish on K^*, thus induce a valuation on C_K. Then he kernel C_K^0 is compact and $C_K = C_K^0 \times \mathbb{R}_+^*$. Cf.[Neukirch P159].

Prop. (2.4.5). We let I_K^S be the group of ideles that has unit as components at all primes except S. Then we have a canonical isomorphism

$$I_K/J_K^{S_\infty} \cong J_K, \quad I_K/I_K^{S_\infty} \cdot K^* \cong J_K/P_K.$$

The proof is easy, just cut out the infinite prime part of $\mathfrak{a}$.

Prop. (2.4.6). If S is sufficiently large(containing a S_0) then $I_K = I_K^S \cdot K^*$ hence $C_K = I_K^S \cdot K^*/K^*$.

Proof: The ideal class group is finite, hence we can find a finite set of representative for it. Only finite set of primes are involved in it, thus we let S contain all these primes and infinite primes, then for any $\mathfrak{a}$, $\prod_{\mathfrak{p}\nmid\infty} a_\mathfrak{p} = A_i \cdot (x)$, and $A_i \in I_K^S$, hence $\mathfrak{a} \in I_K^S \cdot K^*$. □

II.3 Profinite Cohomology

Basic Reference is Neukirch's Wonderful book [Neukirch Class Field Theory 2015] and the giant book [Neukirch Cohomology of Number Fields]. More should be added to the discussion of CFT.

1 Group Cohomology

We usually consider finite group G, at least it should be discrete.

Def. (3.1.1). The **group cohomology** $H^n(G,A)$ is the derived functor of the left exact functor $H^0(G,A)=A^G=\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},A)$, so $H^n(G,A)=\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},A)$.

The **group homology** $H_n(G,A)$ is the derived functor of the right exact functor $H_0(G,A)=A_G=\mathbb{Z}\otimes_{\mathbb{Z}[G]}A$, so $H_n(G,A)=\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},A)$。

A^H is left exact from G-mod to G/H-mod because it is right adjoint to the inclusion functor: $\mathrm{Hom}_G(X,A)=\mathrm{Hom}_{G/H}(X,A^H)$ and it preserves injectives **??**. Dually for A_H.

Prop. (3.1.2) (Serre-Hochschild Spectral Sequence). By Grothendieck Spectral sequence, the relation $A^G=(A^H)^{G/H}$ gives us a spectral sequence E that

$$E_2^{p,q}=H^p(G/H,H^q(H,A))\Longrightarrow E^n=H^n(G,A).$$

The lower parts give us:

$$0\to H^1(G/H,A^H)\xrightarrow{inf}H^1(G,A)\xrightarrow{res}H^1(H,A)^{G/H}\xrightarrow{transgression}H^2(G/H,A^H)\xrightarrow{inf}H^2(G,A).$$

dually for homology group.

Moreover if $H^k(H,A)=0$ for $k=1,\ldots,n-1$, then the rows are blank, thus the above lower part can change to dimension n.

Cor. (3.1.3) (Hopf). If $G=F/R$, F is free, then use the homology spectral sequence, $H_2(G,\mathbb{Z})\cong\frac{R\cap[F,F]}{[F,R]}$. Cf.[Weibel P198].

Prop. (3.1.4). For $G=\mathbb{Z}$, we have a free resolution $0\to Z[t,t^{-1}]\xrightarrow{1-t}Z[t,t^{-1}]\to Z\to 0$. In particular, thus $H_n(\mathbb{Z},\mathbb{Z})=\mathbb{Z}$ iff $n=0,1$ and vanish otherwise.

Prop. (3.1.5) (Tate Cohomology). Neukirch Constructed a standard resolution of the $\mathbb{Z}[G]$-module $\mathbb{Z}$, Cf.[Neukirch CFT P13]:

$$\cdots\longleftarrow X_{-2}\longleftarrow X_{-1}\xleftarrow{\mu\circ\varepsilon}X_0\longleftarrow X_1\longleftarrow\cdots$$

that $X_q=X_{-q-1}$ are $\mathbb{Z}$-module generated by q-cells $(\sigma_1,\ldots,\sigma_q)$, $X_0=X_{-1}=\mathbb{Z}[G]$.

It then can be verified that for G finite, Hom from this resolution gives out the Tate cohomology

$$H_T^n(G,A)=\begin{cases}H^n(G,A) & n\geq 1\\ A^G/N_GA & n=0\\ {}_{N_G}A/I_GA & n=-1\\ H_{-1-n}(G,A) & n\leq -2\end{cases}$$

and H_T^n is a long exact sequence.

In particular, the Hom complex looks like:

$$\cdots \to A_{-2} \xrightarrow{\partial_{-1}} A_{-1} \xrightarrow{\partial_0} A_0 \xrightarrow{\partial_1} A_1 \xrightarrow{\partial_2} A_2 \to \cdots$$

where $A_{-1} = A_0 = A$ and $\partial_0 x = N_G x$, $(\partial_1 x)(\sigma) = \sigma x - x$,
$\partial_2(x)(\sigma_1, \sigma_2) = \sigma_1 x(\sigma_2) - x(\sigma_1\sigma_2) + x(\sigma_1)$.

From now on, consider only Tate cohomology.

Prop. (3.1.6).

$$H^{-2}(G, \mathbb{Z}) = G^{ab}, \quad H^{-1}(G, \mathbb{Z}) = 0, \quad H^0(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z}, \quad H^1(G, \mathbb{Z}) = 0, \quad H^2(G, \mathbb{Z}) = \chi(G).$$

Proof: H^0 is trivial and $H^1(G, \mathbb{Z}) = H^0(G, \mathbb{Q}/\mathbb{Z}) = 0$, $H^2(G, \mathbb{Z}) = H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. $H^{-1}(G, \mathbb{Z}) = {}_{N_G}\mathbb{Z}/I_G A = 0$.

For $H^{-2}(G, \mathbb{Z})$, use the dimension shifting $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$, $= H^{-1}(G, I_G) = I_G/I_G^2$. And $G^{ab} \cong I_G/I_G^2$ by $\sigma \mapsto \sigma - 1$. □

Prop. (3.1.7). $H^n(\mathbb{Z}/n\mathbb{Z}, A) = A^G/NA$ for n even and $H^n(\mathbb{Z}/n\mathbb{Z}, A) = {}_N A/(\sigma - 1)A$ for n odd.

Prop. (3.1.8). For a finite group G, $|G| \cdot H^n(G, A) = 0$ for any G-module A. (True for H^0 and use dimension shifting). In particular, a divisible G-module A has trivial cohomology).

Prop. (3.1.9).

<u>**Operations**</u>

Prop. (3.1.10) (Dimension Shifting). There are fundamental split exact sequence $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \to \mathbb{Z}[G] \to J_G \to 0$, thus $A_G = A/I_G A$. This can be used to tensor with A and define natural dimension shifting of cohomology δ.

Def. (3.1.11). The **inflation** is defined for $p \geq 0$ by composing with $G \to G/H$.

The **restriction** is the map $H^q(G, A) \to H^q(H, A)$ that is id when $q = 0$ and commutes with δ.

The **corestriciton** is the map $H^q(H, A) \to H^q(G, A)$ that maps a to $N_{G/H} a$ when $q = 0$ and commutes with δ.

Prop. (3.1.12). $cor \circ \mathrm{res} = [G : H]$ for a subgroup H. (check at degree 0 and use dimension shifting).

Prop. (3.1.13). For an isomorphism (σ^*, σ) of a group and its cochain map in the sense that $\sigma^*(g)(\sigma(a)) = g(a)$, we have an isomorphism of Conjugation acts trivially on the group cohomology, because it does on H^0 because $H^0 = A^G$ fixed by G, and it commutes with dimension shifting. (Warning, if you count directly $a(\sigma\tau\sigma^{-1}) - \sigma a(\tau)$, you won't get 0, but a 1-coboundary).

Prop. (3.1.14) (Cup Product). The cup product is defined by $C^p(X, A) \times C^q(X, B) \to C^{p+q}(X, A \otimes B)$:

$$(a \smile b)(\sigma_1, \ldots, \sigma_{p+q}) = a(\sigma_1, \ldots, \sigma_p) \otimes \sigma_1 \ldots \sigma_p b(\sigma_{p+1}, \ldots, \sigma_{p+q}).$$

It satisfies $\partial(a \smile b) = \partial(a) \smile b + (-1)^p a \smile \partial(b)$, thus defines a:

$$\smile : H^p(G, A) \times H^q(G, B) \to H^{p+q}(G, A \otimes B)$$

for $p, q \geq 0$. And in negative dimension this is also definable but not computable, Cf.[Neukirch Cohomology of Number Fields P42] or [Neukirch Class Field Theory 2015 P45].

- $a \smile b = a \otimes b$ for $a \in H^0(G, A), b \in H^0(G, B)$.
- $\delta(a \smile b) = \delta a \smile b, \delta(a \smile b) = (-1)^p(a \smile \delta b)$ for $a \in H^p(G, A)$.
- $\smile$ is associative and skew-symmetric (follows from dimension shifting and the last one.

Prop. (3.1.15) (Duality and Cup Product). Let $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ and $0 \to B' \xrightarrow{u} B \xrightarrow{v} B'' \to 0$ be exact and there is a pairing $\varphi : A \times B \to C$ that $\varphi(A' \times A') = 0$ hence induce a compatible pairing on $A' \times B''$ and $A'' \times B'$, then we have

$$\delta(\alpha) \smile \beta + (-1)^p \alpha \smile \delta(\beta) = 0$$

for $\alpha \in H^p(G, A'')$ and $\beta \in H^q(G, B'')$.

Proof: Use the definition of δ, let a, b be the preimage of α, β in A and B, and $ia' = \partial a, ub' = \partial b$, then $\delta(\alpha) \smile \beta + (-1)^p \alpha \smile \delta(\beta) = a' \smile vb + (-1)^p ja \smile b' = \partial a \smile b + (-1)^p a \smile \partial b = \partial(a \smile b)$ is a boundary. □

Prop. (3.1.16).

$$\mathrm{res}(a \smile \beta) = \mathrm{res}(a) \smile \mathrm{res}(b), \quad \mathrm{cor}(\mathrm{res} a \smile b) = a \smile \mathrm{cor} b$$

Cf.[Neukirch CFT P48].

Prop. (3.1.17). Let $\sigma \in G^{ab} = H^{-2}(G, \mathbb{Z})$ and $a_1 \in H^1(G, A), a_2 \in H^2(G, A)$, then

$$a_1 \smile \sigma = a_1(\sigma), \quad a_2(\sigma) = \sum_\tau a_2(\tau, \sigma).$$

Cf.[Neukirch CFT P50,P51].

Prop. (3.1.18). For cyclic group, the Tate cohomology is 2-cyclic.

Proof: There is an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[G] \xrightarrow{\sigma - 1} \mathbb{Z}[G] \to \mathbb{Z} \to 0$, and this defines an isomorphism $\delta^2 : H^0(G, \mathbb{Z}) \cong H^2(G, \mathbb{Z})$. And this is also true for any A when tensored with it. The isomorphism is $a \mapsto \delta^2 a = \delta^2(1) \smile a$. □

Prop. (3.1.19) (Duality). The cup product induces an isomorphism $H^i(G, A^\vee) \cong (H^{-i-1}(G, A))^\vee$, i.e, $H^n(G, A^\vee)$ and $H_n(G, A)$ are dual to each other when $n > 0$, where $A^\vee = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

Proof: We only need to verify $A^{*G}/N_G A^* \cong ({}_{N_G}A/I_G A)^*$ and use dimension shifting. Should use the injectivity of $\mathbb{Q}/\mathbb{Z}$ and the compatibility of cup product with dual. □

Cor. (3.1.20). When A is $\mathbb{Z}$-free, the cup product also induce an isomorphism $H^i(G, \mathrm{Hom}(A, \mathbb{Z})) \cong H^{-i}(G, A)^\vee$.

Prop. (3.1.21) (Theorem of Cohomological Triviality). For a G-module A, if there is a q s.t. $H^q(g, A) = H^{q+1}(g, A) = 0$ for all subgroups of G, then $H^p(g, A) = 0$ for any p and subgroup g. Cf.[Neukirch CFT P57].

Prop. (3.1.22) (Tate's Theorem). Assume A is a G-module that $H^1(G,A)=0$ and $H^2(g,A)$ is cyclic of order $|g|$ for any subgroup g of G, then for a generator a of $H^2(G,A)$, there is an isomorphism

$$a\smile: H^q(G,\mathbb{Z})\cong H^{q+2}(G,A).$$

Cf.[Neikirch CFT P79].

Cor. (3.1.23). In particular, by dimension shifting, if A is a G-module that $H^1(G,A)=0$ and $H^2(g,A)$ is cyclic of order $|g|$ for any subgroup g of G this gives an isomorphism:

$$a\smile: H^q(G,\mathbb{Z})\cong H^{q+2}(G,A).$$

for a generator a of $H^2(G,A)$, because cup product commutes with dimension shifting.

Miscellaneous

Prop. (3.1.24) ((Schreier)H^2 and Extensions). For a G-module A, there is a correspondence of equivalence classes of extension of G over A that are compatible with the G action and $H^2(G,A)$.

Proof: Cf.[Weibel P183]. In fact there are also interpretations of $H^3(G,A)$ as $0\to A\to N\to E\to G$ under some equivalences. □

Prop. (3.1.25). When G is a cyclic group and A is a G-module, let $f=\sigma-1$, $g=1+\sigma+\ldots+\sigma^{n-1}$, then we can form a cyclic complex of order 2 and compute the Herbrand quotient(7.2.12). In this case, $g_{f,g}$ is just $|H^0(G,A)|/|H^{-1}(G,A)|$. And by(7.2.14), if a G-morphism $A\to B$ has finite kernel and cokernel, then they have the same Herbrand quotient.

2 Profinite Groups

Basic references are [Neukirch Cohomology of Number Fields], [Serre Galois Cohomology] [Profinite Groups Zalesskii]and [Shatz Profinite Groups, Arithmetic and Geometry].

Def. (3.2.1). A **profinite group** is defined as an inverse limit of finite discrete groups.

Lemma (3.2.2). For a compact totally disconnected group G, any nbhd U of e contains a normal open subgroup.

Proof: U contains a precompact nbhd of e, then by(1.10.5), U contains an open subgroup V, so by(1.10.2), there is a nbhd V' of e that $xV'x^{-1}\subset V$ for all $x\in G$, this says $\cap x^{-1}Vx$ is open, so it is an open normal subgroup. □

Prop. (3.2.3) (Profinite Compact and Totally Disconnected). A profinite group is the same thing as a totally disconnected, compact Hausdorff topological group. In particular, $G\cong\varprojlim G/N$ for all open normal subgroups of G.

Proof: One way is because $\lim G_i$ is a closed subgroup of $\prod G_i$ which by Tychonoff's theorem is compact.

Conversely, by(3.2.2), G has a basis of e consisting of normal open subgroups, and by(1.10.3), the intersection of open normal subgroups is $\{e\}$. For any open normal subgroup N of G, G/N is compact discrete hence finite, the map $G\to\varprojlim G/N$ is continuous and has dense image, but G is compact and the right is Hausdorff, so the image is closed, hence it is surjective. It is injective because $\cap N=\{e\}$. Hence $G\cong\varprojlim G/N$. □

Cor. (3.2.4). A closed subgroup of a profinite group is profinite, and a quotient group is profinite.

A direct product of profinite groups are profinite, and so the inverse limit profinite groups are profinite, as it is a closed subgroup of a direct product.

Proof: The closed subgroup is totally disconnected by(1.1.5).

To show the quotient group is totally disconnected, by(1.10.3), it suffice to prove H is intersection of compact open nbhds in G/H. If $x \notin H$, then there is an open subgroup U disjoint from xH by(1.10.4), so it is closed hence compact. So UH is a compact nbhd of H in G/H that doesn't contains xH, hence the result. □

Cor. (3.2.5). A closed subgroup of a profinite group is a intersection of open normal subgroups of G containing it, as G/H is profinite and as in the proof of(3.2.3), H is the intersection of open normal subgroups of G/H.

Prop. (3.2.6). The category of profinite Abelian groups is Pontryagin dual to the category of torsion abelian group. (not that hard to verify).

Pro-p-Groups

Def. (3.2.7). To consider indexes of closed subgroups of a profinite group, the notion of surnatural numbers are needed. A **surnatural number** is a formal product $\prod_p p^{n_p}$, $n_p \in \mathbb{N} \cup \{0, \infty\}$.

For a closed subgroup H of a profinite group G, $[G : H]$ is defined to be the least common multiple of $[G/U : H/H \cap U]$ where U goes over all open normal subgroups of G. This also equals the least common multiple of $[G : V]$ for V open containing H(because for any such V, there is an open normal subgroup U that $HU \subset V$(1.10.2)).

Prop. (3.2.8). The index is compatible with composition and quotient: $[G : K] = [G : H][H : K]$ and $[G : H] = [G/K : H/K]$ for K closed normal in G.

$[G : H]$ is finite iff H is open. For a decreasing family of closed subgroups H_i of G, $[G : \cap H_i]$ equals the least common multiple of $[G : H_i]$.

Proof: $[G/U : K/K \cap U] = [G/U : H/H \cap U][H/H \cap U : K/K \cap U]|[G : H][H : K]$, giving one way of inequality. For the converse, Cf.[Etale Cohomology Fulei P150]. The quotient case is trivial.

If $[G : H]$ is finite, then For the final assertion, notice for a open subgroup V, $G - V$ is compact, so $\cap H_i \subset V$ iff $\cap H_i \subset V$ for some i. □

Def. (3.2.9). A profinite group G is called a **pro-p-group** iff $[G : 1]$ is a power of p. This is equivalent to G is an inverse limit of finite p-groups($G = \lim G/N$).

Given a profinite group, a closed subgroup H is called **Sylow p-subgroup** of G if H is pro-p and $[G : H]$ is prime to p.

Prop. (3.2.10). Any pro-p subgroup H of G is contained in a Sylow p-subgroup of G, and any two Sylow p-subgroups are conjugate. And a surjective morphism of profinite groups maps a pro-p group to a pro-p group.

Proof: For any open normal subgroup U of G, let I_U be the sets of all Sylow groups of G/U containing $H/H \cap U$, then the map $G/VG/U$ maps I_V to I_U, and I_U is finite nonempty by Sylow theory. So the inverse limit of I_U is nonempty, and let (P_U) be such an element, and $P = \varprojlim_U P_U$, then P is a pro-p subgroup of G, and $[G : P]$ equals the least common multiple of $[G/U : P_U]$, which is prime to p, so it is a Sylow p-group. Similarly, for two Sylow-p subgroup, we consider A_U the set of all $x \in G/U$ that $x^{-1}(PU/U)x = P'U/U$, then there is a inverse element x, and $x^{-1}Px = P'$.

If $G' = G/N$, then $[G/N : PN/N] = [G : PN] | [G : P]$ is prime to p, and $[PN/N : 1] = [P : P \cap N] | [P : 1]$ is a power of p, so PN/N is Sylow-p in G'. $\square$

Prop. (3.2.11). For a pro-p group G, any nonzero simple p-torsion G-module is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with trivial G-action.

Proof: The action of G on A factors through a finite quotient group which is a p-group, by**??**, $A^G \neq 0$, so $A = A^G$, then A must be $\mathbb{Z}/p\mathbb{Z}$. $\square$

3 Locally Profinite Groups

Def. (3.3.1). A **locally profinite group** is a topological group that is Hausdorff, locally compact and totally disconnected. A profinite group is locally profinite, and any compact open subgroup of a locally profinite group is profinite.

A p-adic number field F is a locally profinite group, so does $GL_n(F)$.

Cor. (3.3.2). A closed subgroup of a locally profinite group is locally profinite, and a quotient group is locally profinite.

Proof: The proof is very similar to that of(3.2.4), as the result of(1.1.3) remains true, because any connected nbhd of e is contained in any compact open subgroup. $\square$

Prop. (3.3.3). A representation of a locally profinite group G on a complex vector space is called **smooth** iff it is continuous w.r.t the discrete topology.

The category T_G of smooth representations is a full Abelian subcategory of the category of G-modules, and there is a right adjoint to the forgetful functor:

$$V^\infty = \bigcup_{H \subset G \text{ compact open}} V^K$$

So it preserves injectives and T_G has enough injectives.

4 Cohomology of Profinite Groups

Prop. (3.4.1) (Abelian Sheaves on T_G). If G is profinite, the category of Abelian sheaves on the canonical topology T_G of G-sets is equivalent to the the category of G-modules, by Yoneda functor. The inverse map is $F \mapsto \varinjlim F(G/H)$.

Proof: The task is to prove $F \cong h_{\varinjlim F(G/H)}$. Cf.[Tamme P29].

The inverse of the Yoneda functor is the functor $F \mapsto F(G)$ as a left G-set where $gs = F(\cdot g)s$. The task is to show that $F \cong h_{F(G)}$. For this, for any U we consider the covering $\{G \xrightarrow{\varphi_u} U$ where $\varphi_U(g) = gu$. Sheaf condition says

$$F(U) \to \prod_{u \in U} F(G) \rightrightarrows F(G \times_U G)$$

is exact, in other words, $F(U) \cong \mathrm{Hom}_G(U, F(G))$. $\square$

Prop. (3.4.2) (Profinite Cohomology). The profinite cohomology is the derived functor of $A \to A^G$ in the Abelian category C_G(It has enough injectives by(3.3.3)). And

$$H^*(G, A) \cong H^*(C(G, A)) \cong \varinjlim H^*(G/U, A^U)$$

where $C(G, A)$ is the set of continuous cochain complex of morphisms from G to A. Moreover, for the same reason, when $G = \varprojlim G_i$, and $A = \varinjlim A_i$, then

$$H^*(G, A) \cong \varinjlim H^*(G_i, A_i).$$

Proof: The second is an isomorphism because $C^n(G, A) = \operatorname{colim} C^n(G/U, A^U)$ and direct limit is exact.

For the first, the H^0 obviously coincide, so it suffice to prove $H^*(C(G, A))$ form a universal δ-functor. It is effaceable because I^U is injective G/U-module.

For the last one, we need to check $C^n(G, A) = \varinjlim C^n(G_i, A_i)$. Notice G has the profinite topology, thus must factor through some G_i, and the right through some A_i because the image of a morphism from G^n to A has finite image. Thus the result follows. □

Prop. (3.4.3). *cor* $\circ$ res $= [G : H]$ for a subgroup H is also true for profinite cohomology(3.1.12), if H is an open subgroup of G. This is because of(3.4.2).

Prop. (3.4.4). If H is a closed subgroup of a profinite group G that $[G : H]$ is relatively prime to p, then for any G-module A and i, the restriction map $H^i(G, A) \to H^i(H, A)$ is injective on the p-primary part of $H^i(G, A)$.

Proof: $H^i(H, A) = \varinjlim_U H^i(U, A)$ for open subgroups U containing H, by(3.4.2), and $H^i(G, A) \to H^i(U, A)$ is injective on the p-primary part by(3.4.3), so it is injective. □

Lemma (3.4.5) (Shapiro).

$$H_*(G, \operatorname{ind}_H^G(A)) \cong H_*(H, A), \quad H^*(G, \operatorname{Coind}_H^G(A)) \cong H^*(H, A)$$

because (co)induced is adjoint to exact functors, so it preserves injectives(projectives) and it is exact because $\mathbb{Z}[G]$ is free $\mathbb{Z}[H]$-module.

And in the finite case, this is also true for Tate cohomology using dimension shifting.

Prop. (3.4.6) (Serre-Hochschild Spectral sequence). Same as the finite case(3.1.2) also applies to profinite cohomology with H closed normal in G.

Cohomological Dimensions

Def. (3.4.7). The ***p*-cohomological dimension** $cd_p(G)$ of a profinite group G is defined as the smallest integer n that the p-primary part of $H^i(G, A)$ vanish for any torsion G-module A. The **strict *p*-cohomological dimension** $scd_p(G)$ of a profinite group G is defined as the smallest integer n that the p-primary part of $H^i(G, A)$ vanish for any G-module A.

The **cohomological dimension** $cd(G)$ is defined to be $\sup_p(cd_p(G))$. The **strict cohomological dimension** $scd(G)$ is defined to be $\sup_p(scd_p(G))$.

Prop. (3.4.8). For a profinite group G, the following are equivalent:
- $cd_p(G) \leq n$.

- $H^i(G,A)=0$ for any $i>n$ and any p-torsion G-module A.
- $H^{n+1}(G,A)$ for any simple p-torsion G-module A.

And if G is pro-p, then it suffice to check $H^{n+1}(G,\mathbb{Z}/p\mathbb{Z})=0$

Proof: For any torsion G-module A, $A=\oplus_p A(p)$, so $H^i(G,A(p))$ is the P-primary part of $H^i(G,A)$, so $1,2$ are equivalent. For $3\to 1$: use the fact cohomology commutes with colimits(3.4.2), reduce to the case of A finite, and then use use the quotient tower.

The last assertion is by(3.2.11). □

Prop. (3.4.9). $cd_p(G)\leq scd_p(G)\leq cd_p(G)+1$.

Proof: Let $A_p=\mathrm{Ker}(p:A\to A)$. There are exact sequences $0\to A_p\to A\xrightarrow{p} pA\to 0$ and $0\to pA\to A\to A/pA\to 0$. A_p and A/pA are p-torsion G-modules, so if $i>cd_p(G)+1$, then $H^i(G,A_p)$ and $H^{i-1}(G,A/pA)$ vanish. so $H^i(G,A)\xrightarrow{p} H^i(G<pA)$ and $H^i(G,pA)\to H^i(G,A)$ are injections, so their composition $H^i(G,A)\xrightarrow{p} H^i(G,A)$ is injective, showing $(H^i(G,A))_p=0$, so $scd_p(G)\leq cd_p(G)+1$. □

Prop. (3.4.10). For a closed subgroup H of a profinite group G, $cd_p(H)\leq cd_p(G)$ and $scd_p(H)\leq scd_p(G)$, and if $[G:H]$ is relatively prime to p, then equality holds.

Proof: The first is because of Shapiro's lemma(3.4.5). For the equality, use(3.4.4). □

Cor. (3.4.11). $cd_p(G)=cd_p(G_p)=cd(G_p)$, $scd_p(G)=scd_p(G_p)=scd(G_p)$.

Prop. (3.4.12). If H is a closed normal subgroup of G, then $cd_p(G)\leq cd_p(H)+cd_p(G/H)$, by Hochschild-Serre spectral sequence.

Prop. (3.4.13). If K is a field of char p, then $cd_p(G(K_s/K))=0$.

If $H^2(G(K_s/L),K_s^*)=0$ for all L/K separable, then $cd(G(K_s/K))\leq 1$. In particular $H^i(G(K_s/K),K_s^*)=0$ for $i\geq 1$.

Proof: Let G_p be the Sylow p-subgroup of $G(K_s/K)$ and $M=K_s^{G_p}$. There is an exact sequence $0\to\mu_p\to K_s\xrightarrow{x^p-x} K_s\to 0$, and combined with the fact that $H^i(G_p,K_s)=H^i(G(K_s/M),K_s)=0$ for $i\geq 1$(3.5.1), so $H^i(G_p,\mathbb{Z}/p\mathbb{Z})=0$ for $i\geq 2$. Thus by(3.4.8) and(3.4.11), $cd_p(G(K_s/K))\leq 1$.

For the second assertion, similarly, for $l\neq p$, consider the kernel of x^l, μ_l of l-th roots of unity in K_s, and $H^2(G_l,\mu_l(K_s))=\varinjlim_L H^2(G(K_s/L),\mu_l(K_s))=0$, so $cd_l(G(K_s/K))\leq 1$. Then $cd(G(K_s/K))\leq 1$, and $scd(G(K_s/K))\leq 2$, so $H^i(G(K_s/K),K_s^*)=0$ for $i\geq 1$. □

Prop. (3.4.14). For L/K field extension, $cd_p(G(L_s/L))\leq cd_p(G(K_s/K))+tr.deg(L/K)$.

Proof: Cf.[Etale Cohomology Fulei P169]. □

Cor. (3.4.15). If k is separably closed and K be a function field over k, then $cd(G(K_s/K))\leq 1$.

And if K is of char $p>0$, $H^2(G(K_s/K),K_s^*)$ is a p-torsion group.

Proof: Th first one is clear, for the second, for any $l\neq p$, use the exact sequence $\mu_l(K_s)\to K_s^*\xrightarrow{x\to x^l} K_s^*\to 0$, then $H^2(G(K_s/K),\mu_l(K_s))=0$, and $H^2(G(K_s/K),K_s^*)\xrightarrow{l} H^2(G(K_s/K),K_s^*)$ is injective. l is arbitrary, so $H^2(G(K_s/K),K_s^*)$ must be a p-torsion group. □

5 Galois Cohomology

References are [Neukirch Chap6]. Should include [Galois Cohomology Serre].

This subsection is not included in the following subsection about Galois/Profinite Cohomology because the G-groups may not be Abelian and it may not be endowed with the discrete topology.

Prop. (3.5.1) (Hilbert's Additive Satz 90). For L/K a Galois extension, $H^n(G(L/K), L) = 0$ for $n > 0$, where L is equipped with the discrete topology.

Proof: Form the normal basis theorem**??**, for finite Galois extension L/K, L is an induced module over K, thus $H^*(G, L) = H_*(G, L) = 0$ for $* \neq 0$ and $H_T^*(G, L) = 0$ by(3.4.5).

Hence the same is true, for arbitrary Galois extension, when L is equipped with the discrete topology, the same as in the proof of(3.5.7). □

Prop. (3.5.2) (Hilbert's Multiplicative Satz 90). $H^1(G_{L/K}, L^*) = 0$ for Galois extension L/K, where L is equipped with the discrete topology,(follows from(3.5.7)).

Non-Abelian Cohomology

Def. (3.5.3) (Non-Abelian Cohomology). Let G, M be topological groups, with a continuous action of G on M, then we define $H^0(G, M) = M^G$.

We define $Z^1(G, M)$=continuous maps $x : G \to M$ that

$$\sigma_1(x(\sigma_2)(x(\sigma_1\sigma_2))^{-1}x(\sigma_1) = 1, \quad \text{i.e.} \quad x(gh) = x(g)g(x(h))$$

If $x \in Z^1(G, M)$, then $x_m : \sigma \to m^{-1}x(\sigma)\sigma(m) \in Z^1(G, M)$ too. This defines an equivalence relation on $Z^1(G, M)$, the equivalence classes are called $H^1(G, M)$. This is compatible with the commutative case.

Prop. (3.5.4). Restriction map and inflation map is definable for H^0 and H^1, and $H^1(H, M)$ is a G/H-set where G acts on $H^1(H, M)$ by $g(c)(h) = g(c(g^{-1}hg))$.

Prop. (3.5.5). There is an exact sequence of pointed sets:

$$0 \to H^1(G/H, M^H) \xrightarrow{inf} H^1(G, M) \xrightarrow{res} H^1(H, M)^{G/H}.$$

Proof: First $\text{res}(H^1(G, M)) \subset H^1(H, M)^{G/H}$ because $g(c)(h) = c(g)^{-1}c(h)h(c(g))$ is checked so $g(c)$ is cohomologous to c.

$\text{res} \circ inf = 0$ is easy, if $\text{res}(c) = 0$, then c is trivial on H, hence $c(gh) = c(g)$ and $h(c(g)) = c(hg) = c(g \cdot g^{-1}hg) = c(g)$, so c is inflated from $H^1(G/H, M^H)$.

For the injectivity of inf. If $c(\overline{g}) = g^{-1}g(a)$, then $a \in M^H$, so it is a coboundary in $H^1(G/H, M^H)$. □

Prop. (3.5.6). Let $1 \to A \to B \to C \to 1$ be an exact sequence of G-groups, then there is a long exact sequence of pointed sets

$$1 \to A^G \to B^G \to C^G \xrightarrow{\delta} H^1(G, A) \to H^1(G, B) \to H^1(G, C) \xrightarrow{\Delta} H^2(G, A)$$

the last term is defined only when A is in the center of G.

Where δ is defined as follows: for $c \in C^G$, let b be an inverse image of c in B, then $a_\sigma = b^{-1}\sigma(b) \in A$, and it defines a cocycle in $H^1(G, A)$, different choice differ by a coboundary, so it is well-defined.

Δ is defines as: for c_σ a cocycle in $H^1(G, C)$, choose b_s inverse images of c_s, then $a_{\sigma,\tau} = b_\sigma\sigma(b_\tau)b_{\sigma\tau}^{-1}$ is a cocycle in $H^2(G, A)$.

Proof: The verification of well-definedness of Δ is checked at [Serre Local Fields P124].

For the exactness at C^G, the definition of δ shows that $\delta(c)=1$ iff there is an inverse image b that $b^{-1}\sigma(b)=1$ for all σ.

For the exactness at $H^1(G,A)$, $a_\sigma=b^{-1}\sigma(b)$ if a_σ is in the image of δ. Conversely, the image of b in C is in C^G, so it is in the image of δ.

For the exactness at $H^1(G,B)$, one way is clear, and for the other, if $\pi(b_\sigma)=c^{-1}\sigma(c)$, then if t is an inverse image of c, then $tb_\sigma\sigma(t)^{-1}$ is a cocycle in A cohomologous to b_σ.

For the exactness at $H^1(G,C)$, one way is clear, and if b_s is an inverse image of b_s and $a_{\sigma,\tau}=b_\sigma\sigma(b_\tau)b_{\sigma\tau}^{-1}$ is a coboundary, then it is $a_\sigma\sigma(a_\tau)a_{\sigma\tau}^{-1}$, so we change b to $a_\sigma^{-1}b_\sigma$, as A is in the center of B, this lifts c to a cocycle in B. □

Prop. (3.5.7). For L/K a Galois extension, $H^1(G(L/K),GL_n(L))=1$, where L is equipped with the discrete topology.

Proof: We prove any cocycle is a coboundary, for this, notice any cocycle factor through a finite quotient, and the images of it is contained in a finite extension of K, hence it reduce to the case of L/K finite.

For some $a\in H^1(G,GL_n(L))$, for a vector $x\in L^n$, let $P(x)=\sum a(\sigma)\sigma(x)$, then $\{P(x)\}$ generate L^n, because if f is a linear functional that vanish on it, then

$$0=f(P(\lambda x))=\sum f(a(\sigma)\sigma x)\sigma\lambda.$$

But automorphisms are linearly independent over L, hence $f(a(\sigma)\sigma(x))=0$ for all σ, so $f=0$ as $a(\sigma)\in GL_n(L)$

Now let $\{P(x_i)\}$ generate L^n, then let T be the matrix with x_i as rows, then $P=\sum a(\sigma)\sigma(T)$ is invertible. Now $a(\sigma)=P\cdot\sigma(P)^{-1}$ is a cocycle. □

Cor. (3.5.8). $H^1(G(L/K),SL_n(L))=1$. This is seen from the exact sequence $1\to SL_n(L)\to GL_n(L)\to L^*\to 1$.

Continuous Cochain Complex

In this subsubsection cohomology of G-modules with non-discrete topology is studied. References are [Cohomology of Number Fields Neukirch Chap 2.7].

Prop. (3.5.9). $H^*_{cts}(G,-)$ forms a long exact sequence for any $0\to A\to B\to C\to 0$ of continuous G-modules.

Prop. (3.5.10). If A is a compact G-module which is an inverse limit of finite discrete G-modules A_n, then if $H^i(G,A_n)$ is finite for all n, then

$$H^{i+1}_{cts}(G,A)=\varprojlim_n H^{i+1}(G,A_n).$$

Proof: Cf.[Cohomology of Number Fields Neukirch P142]. □

Prop. (3.5.11). Let π be a topologically nilpotent element of A which is complete in the π-adic topology and π is not a zero-divisor, let $R=A/\pi A$ equipped with discrete topology. Let G be a group which acts continuously on A and fix π, then if $H^1(G,R)$ is trivial, then $H^1(G,A)$ is trivial, and if moreover $H^1(G,GL_n(R))$ is trivial, then $H^1(G,GL_n(A))$ is trivial.

Proof: Cf.[Galois Representations Berger P15]. □

Prop. (3.5.12) (Cyclic Case). if G is a topological cyclic group $\overline{\langle g\rangle}$, then the map $H^1(G,M)\to M/(1-g)$ is well-defined and injective. And when M is profinite, p-adically complete, then the map is also surjective.

Proof: The surjection: there is only one choice: $c(g^i)=(1+g+\ldots+g^{i-1})(m)$. And we need to verify that it is continuous. The case of p-adic can be deduced from profinite case, because $c(\gamma)\in p^{-k}M$ for some k, and $p^{-k}M$ is then profinite. For any finite quotient N of M, there is a k that $kM=0$, and a n that $g^n=\mathrm{id}$ on N, so $c(g^{rkn})=0$ on N, which shows c is continuous. □

Interpretation of H^1

Prop. (3.5.13) (Semilinear Actions). For a topological group G and a G-ring R, now giving a free R module X of degree d with a G semilinear action of G(i.e. $\sigma(rm)=\sigma(r)\sigma(m)$), then the matrix of the G action defines a $[X]\in H^1(G,GL_d(R))$ and change of basis defines exactly the equivalence relations on $H^1(G,GL_d(R))$(3.5.3).

So there is a bijection of $H^1(G,GL_d(R))$ with the set of isomorphisms classes of semilinear representations of G on free R-modules of rank d.

Def. (3.5.14). A G-set X is a discrete set with a continuous G-action on X. Let A be a G-group, an A**-torsor** is a right A-action that is simply transitive and semi-linear in G.

Prop. (3.5.15) (H^1 and Torsors). We have a canonical bijection of pointed sets: $H^1(G,A)\cong TORS(A)$.

Proof: Let X be an A-torsor, choose $x\in X$, then $\sigma(x)=xa_\sigma$ for $a_\sigma\in A$. Now that $\sigma\mapsto a(\sigma)$ is checked to be a cocycle, and change of x changes to $\sigma\mapsto b^{-1}a_\sigma\sigma(b)$. Conversely, for an $a\in H^1(G,A)$, we let $X=A$ be a right A-module, and let $\sigma'(x)=a_\sigma\sigma(x)$, i.e. regarding coming from $x=1$, then this is a inverse map. □

Prop. (3.5.16) (Extension of Rings).

Prop. (3.5.17). There is an isomorphism of pointed sets $H^1(G,O(\varphi_L))\cong E_\varphi(L/K)$. Cf.[Neukirch Cohomology of Number Fields P346].

Prop. (3.5.18). There is an isomorphism of pointed sets $H^1(G,PSL_n(L))\cong BS_n(L/K)$, where $BS_n(L/K)$ is the isomorphism classes of Brauer-Severi varieties of dimension $n-1$ that splits in L. Cf.[Neukirch Cohomology of Number Fields P348].

6 Iwasawa Modules

II.4 Cohomology of Number Fields

1 Class Field Theory

Abstract Class Field Theory

Def. (4.1.1). A formation consists of a profinite group G regarded as a Galois group $G(K)$ and a G-module A. It is called a **field formation** iff for any normal extension L/K, $G(L/K, A^L) = 0$.

For a field extension, by(3.1.2), inf is an injection on H^2. We denote $H^2(K)$ as the profinite cohomology group $H^2(G, A) = \mathrm{Br}(K)$. Inflation should be thought of as inclusions.

It is called a **class formation** if moreover for every normal extension L/K, there is an canonical isomorphism

$$\mathrm{inv}_{L/K} : H^2(L/K) \to \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$$

that is compatible with inflation and restriction in the sense that:

- If $N/L/K$ with N/K and L/K normal, then $\mathrm{inv}_{L/K} = \mathrm{inv}_{N/K}|_{H^2(L/K)}$ via inflation.
- If $N/L/K$ with N/L and N/K normal, then $\mathrm{inv}_{N/L} \circ \mathrm{res}_L = [L:K] \cdot \mathrm{inv}_{N/K}$.

The element of H^2 that is mapped to $\frac{1}{[L:K]} + \mathbb{Z}$ is called the **fundamental class** $u_{L/K}$.

Prop. (4.1.2). inv also commutes with cor and conjugation:

$$\mathrm{inv}_{N/K}(\mathrm{cor}_K c) = \mathrm{inv}_{N/L} c, \quad \mathrm{inv}_{\sigma N/\sigma K}(\sigma^* c) = \mathrm{inv}(c).$$

The first is because inv commutes with res thus res is surjective, thus there is a c' that $c = \mathrm{res} c'$. Because of cor res=$[L:k]$, we have $\mathrm{cor}_K(c) = c'^{[L:K]}$. Thus $\mathrm{inv}_{N/K}(\mathrm{cor}_K c) = [L:K]\mathrm{inv}_{N/K}(c') = \mathrm{inv}_{N/L}(\mathrm{res}_L c') = \mathrm{inv}_{N/L}(c)$.

For the conjugation, Cf.[Neukirch CFT P69].

Cor. (4.1.3). From this we easily get that

$$u_{L/K} = (u_{N/K})^{[N:L]}, \quad \mathrm{res}_L(u_{N/K}) = u_{L/K}$$

$$\mathrm{cor}_K(u_{N/L}) = (u_{N/K})^{[L:K]}, \quad \sigma^*(u_{N/K}) = u_{\sigma N/\sigma K}.$$

Prop. (4.1.4) (Main Theorem). Tate's theorem(3.1.22) tells us for a class formation, for L/K normal extension, there is an isomorphism

$$u_{L/K} \smile : H^q(G_{L/K}, \mathbb{Z}) \cong H^{q+2}(L/K).$$

Especially, for $q = -2$, there is a canonical isomorphism $G^{ab}_{L/K} \cong A_K/N_{L/K}A_L$ that its inverse is called **reciprocity isomorphism** and $A_K \to G^{ab}_{L/K}$ is called **norm residue symbol**$(-, L/K)$. This norm residue symbol also induce a **universal residue symbol** $(-, K)$ on the limit G^{ab}_K, i.e. the maximal Abelian extension of K.

Lemma (4.1.5). Let L/K be a normal extension, $a \in A_K$ and $\chi \in \chi(G^{ab}_{L/K}) = H^1(G_{L/K}, \mathbb{Q}/\mathbb{Z})$ is a character, then

$$\chi((a, L/K)) = \mathrm{inv}_{L/K}(a \smile \delta\chi) \in \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}.$$

Proof: Cf.[Neukirch CFT P71]. □

Prop. (4.1.6) (Properties of Inv). There are commutative diagrams:

$$
\begin{array}{ccc}
A_K & \longrightarrow & G^{ab}_{N/K} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi} \\
A_K & \longrightarrow & G^{ab}_{L/K}
\end{array}
\qquad
\begin{array}{ccc}
A_K & \longrightarrow & G^{ab}_{N/K} \\
{\scriptstyle N_{L/K}}\uparrow\downarrow{\scriptstyle i} & & {\scriptstyle k}\uparrow\downarrow{\scriptstyle \mathrm{Ver}} \\
A_L & \longrightarrow & G^{ab}_{N/L}
\end{array}
\qquad
\begin{array}{ccc}
A_K & \longrightarrow & G^{ab}_{N/K} \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma^*} \\
A_{\sigma K} & \longrightarrow & G^{ab}_{\sigma L/\sigma K}
\end{array}
$$

Where Ver is the transfer map defined in**??**.

Proof: Cf.[Neukirch CFT P72]. □

Prop. (4.1.7). For a finite normal extension L/K, $N_{L/K}A_L = N_{L^{ab}/K}A_{L^{ab}}$. This is because the quotient both correspond to $G^{ab}_{L/K}$. So <u>class field theory doesn't tell about non-Abelian extension</u>.

Prop. (4.1.8) (Norm Group and Abelian Extension). The map $L \mapsto I_L = N_{L/K}A_L$ defines a inclusion reversing isomorphism between the lattice of Abelian extension L of K and the lattice of norm groups of A_K, i.e.:

$$I_{L_1L_2} = I_{L_1} \cap I_{L_2}, \quad I_{L_1\cap L_2} = I_{L_1} \cdot I_{L_2}.$$

And any group that contains a norm group is a norm group.

Proof: By the first commutative diagram of inv, if $(a, L_i/K) = 0$, then $(a, L_1L_2/K)$ is trivial on $G_{L_i/K}$, thus trivial on $G_{L_1L_2/K}$, thus $a \in I_{L_1L_2}$. so $I_{L_1} \cap I_{L_2} \subset I_{L_1L_2}$, the other side is easy. the second is because $|I_{L_1\cap L_2}/I_{L_1}| = |G_{L_1/L_1\cap L_2}| = |G_{L_1L_2/L_2}| = |I_{L_1}I_{L_2}/I_{L_1}|$. Also we deduce $I_{L_1} \subset I_{L_2} \iff L_2 \subset L_1$, thus by canonical isomorphism, groups containing $N_{L/K}A_L$ are one-to-one correspondence with middle fields of L/K by counting numbers. □

Remark (4.1.9). This shows the philosophy of CFT, i.e. the property of Abelian extensions of a field is can be read from its multiplicative group structure. Of course, determining and characterizing these norm groups requires some work.

<u>Local Class Field Theory</u>

The strategy is to first establish CFT for unramified extensions, then show that unramified extensions already cover $H^2(\overline{K}/K)$.

Lemma (4.1.10). Let L/K be an unramified extension, then $H^q(G_{L/K}, U_L) = 0$ for all q.

Proof: Cf.[Neukirch P83]. □

Prop. (4.1.11). The unramified extensions of K forms a class formation.

Proof: We first define the inv map: use the exact sequence $1 \to U_L \to L^* \xrightarrow{v_L} \mathbb{Z} \to 0$, using the lemma(4.1.10), we have

$$H^2(G_{L/K}, L^*) \xrightarrow{v_K} H^2(G_{L/K}, \mathbb{Z}) \xrightarrow{\delta^{-1}} H^1(G_{L/K}, \mathbb{Q}/\mathbb{Z}) = \chi(G_{L/K}).$$

And there is an isomorphism $\chi(G/K) \xrightarrow{\varphi} \frac{1}{[L;K]}\mathbb{Z}/\mathbb{Z}$, where φ is the Frobenius which generate $G_{L/K}$, and $\varphi(\chi) = \chi(\varphi)$.

To verify this is a class formation, we should verify(4.1.1), Cf.[Neukirch P85]. □

Prop. (4.1.12). If L/K is unramified, then $(a, L/K) = \varphi_{L/K}^{v_K(a)}$, Cf.[Neukirch CFT P86]. The same holds for L replaces by T, in which case

$$1 \to U_K \to K^* \to G_{T/K} \to 0$$

is exact. Cf[Neukirch P88].

Proof: We use(4.1.5), then $\chi(a, L/K) = \mathrm{inv}_{L/K}(\overline{a} \smile \delta\chi) = \varphi \circ \delta^{-1} \circ v_K(\overline{a} \smile \delta\chi) = \varphi(\delta^{-1}(v_K(a)\delta\chi) = \varphi(v_K(a)\chi) = v_K(a)\chi(\varphi_{L/K}) = \chi(\varphi_{L/K}^{v_K(a)})$, for any χ. The second assertion follows from the last prop(4.1.13). □

Cor. (4.1.13). The norm group of an unramified extension of degree f is

$$U_K \times \{\pi^{fn}\}(n = 0, 1, \ldots).$$

(This follows from the proposition as the degree f is the order of the Frobenius map).

Now we pass to ramified extensions.

Lemma (4.1.14). If L/K is normal, then $|H^2(L/K)|\Big|[L : K]$.

Proof: Cf.[Neukirch CFT P89]. Should use the fact that $G_{L/K}$ is solvable and Herbrand quotient. □

Lemma (4.1.15). If L/K is a normal extension and L'/K is another unramified extension of the same degree, then $H^2(L/K) = H^2(L'/K) \subset Br(K)$.

Proof: In view of(4.1.14) and(4.1.11), we only need to prove $H^2(L'/K) \subset H^2(L/K)$. For this, we let $N = LL'$, then there is an exact sequence(3.1.2)

$$1 \to H^2(L/K) \to H^2(N/K) \xrightarrow{res_L} H^2(N/L)$$

then we only need to prove $res_L(c) = 0$, and this follows from $\mathrm{inv}_{N/L}(res_L c) = 0$. This will follow, if we have

$$\mathrm{inv}_{N/L}(res_L c) = [L : K] \cdot \mathrm{inv}_{L'/K} c.$$

This follows from the lemma below(4.1.16). □

Lemma (4.1.16). For two subextensions L/K, L'/K in M/L normal with L'/K unramified extension, $N = LL'$, for $c \in H^2(L'/K)$,

$$\mathrm{inv}_{N/L}(\mathrm{res}_L c) = [L : K] \cdot \mathrm{inv}_{L'/K} c.$$

Proof: Cf[Neukirch CFT P90]. □

Prop. (4.1.17). (G_K, K^*) forms a class formation.

Proof: This almost follows from that of unramified extensions(4.1.11). We verify axioms(4.1.1) that inf is natural and commutes with res. It is natural because it is natural on unramified extensions, it commutes with res because we can assume $c \in H^2(L'/K)$ unramified and use(4.1.16). □

Cor. (4.1.18) (Main Theorem of Local Class Field Theory). Let L/K be a normal extension, then the homomorphism

$$u_{L/K}\smile: H^q(G_{L/K},\mathbb{Z})\cong H^{q+2}(L/K)$$

is an isomorphism.

Cor. (4.1.19). $H^3(L/K)=1, H^4(L/K)=\chi(G_{L/K})$, by(3.1.6).

Cor. (4.1.20). For a $\mathfrak{p}$-adic number field K, $Br(K)\cong\mathbb{Q}/\mathbb{Z}$.

Prop. (4.1.21). By(4.1.6), there is commutative diagrams

$$\begin{tikzcd} K^* \arrow[r] \arrow[d, "\mathrm{id}"] & G_{N/K}^{ab} \arrow[d, "\pi"] \\ K^* \arrow[r] & G_{L/K}^{ab} \end{tikzcd} \quad \begin{tikzcd} K^* \arrow[r] \arrow[d, shift left, "i"] & G_{N/K}^{ab} \arrow[d, shift left, "\mathrm{Ver}"] \\ L^* \arrow[r] \arrow[u, shift left, "N_{L/K}"] & G_{N/L}^{ab} \arrow[u, shift left, "k"] \end{tikzcd} \quad \begin{tikzcd} K^* \arrow[r] \arrow[d, "\sigma"] & G_{N/K}^{ab} \arrow[d, "\sigma^*"] \\ \sigma K^* \arrow[r] & G_{\sigma L/\sigma K}^{ab} \end{tikzcd}$$

Prop. (4.1.22). For an Abelian extension L/K, the higher principal units U_K^n are mapped under the higher ramification groups of $G_{L/K}$ under the upper numbering. ?

Prop. (4.1.23). Getting things together, we get a universal norm residue map

$$K^*\xrightarrow{(-,K)}G_K^{ab}$$

It is injective because $(K^*)^n$ are all norm groups by(4.1.25), so the kernel is there intersection with is 1 by(2.2.3). It image is called the **Weil group**.

Now we want to characterize the norm groups of K^*.

Prop. (4.1.24) (Norm Group and Abelian Extension). The map $L\mapsto I_L=N_{L/K}A_L$ defines a inclusion reversing isomorphism between the lattice of Abelian extension L of K and the lattice of norm groups of A_K, i.e.:

$$I_{L_1L_2}=I_{L_1}\cap I_{L_2},\quad I_{L_1\cap L_2}=I_{L_1}\cdot I_{L_2}.$$

And any group that contains a norm group is a norm group. This follows from(4.1.8) and (4.1.17).

Prop. (4.1.25). The norm groups are precisely the open(closed) subgroups of finite index in K^*. In fact finite index are itself open because it contains $(K^*)^n$ which is open.

Proof: One part follows from(4.1.24) and the fact that $(K^*)^m$ is open(2.2.3). For the converse, we only need to prove $(K^*)^m$ is a norm group. This uses Kummer theory and Cf.[Nuekirch CFT P96]. □

Prop. (4.1.26) (Norm Groups of Local Fields). The norm groups of K^* are exactly the groups containing $U_K^n\times(\pi^f)$ for some n,f.

Proof: $U_K^n\times(\pi^f)$ is a norm group because it is closed of finite index. Conversely, any norm group contains some U_K^n because it is open and contains some (π^f) because it is of finite index. □

Lubin-Tate Formal Group

This is a continuation of2.

Prop. (4.1.27). There is an isomorphism of $\mathcal{O}$-modules $\Lambda_{f,n} \cong \mathcal{O}/\pi^n\mathcal{O}$, Cf.[Neukirch CFT P101]. Thus the automorphism of $\Lambda_{f,n}$ is all of the form u_f for units, isomorphic to U_K/U_K^n.

So we can define a **Tate module** $TG = \varprojlim \mathrm{Ker}[\pi_K^n]$, it is a free $\mathcal{O}_K$-module of rank 1.

Def. (4.1.28). As TG is a free $\mathcal{O}_G$-module of dimension 1, and G_K acts on TG, there can be attached a **Lubin-Tate character** $\chi_K : G_K \to \mathcal{O}_K^*$ by $g(\alpha) = [\chi_K(g)](\alpha)$, this depends on π_K, but its restriction on I_K doesn't depend on π_K, and is just the local CFT isomorphism composed with $x \to x^{-1}$.

Proof: $[\chi_K(g)]$ is, by definition, the morphism that is id on K^{ur} and g on L_π. So it equals g on all K^{ab} iff g is id on K^{ur}, that is, $g \in I_K$. So if $g \in I_K$, by local CFT, $(\chi(g))^{-1}$ corresponds to g, uniquely. □

Prop. (4.1.29). $G_{\pi,n} \cong \mathcal{O}_K^*/U_K^n$, thus we have $G_\pi \cong \mathcal{O}_K^*$. $L_{\pi,n}/K$ is Abelian totally ramified of degree $p^{n-1}(p-1)$ generated by a Eisenstein polynomial with constant coefficient π so π is in the norm group.

Proof: For this, first note Galois action induce an isomorphism on $\Lambda_{f,n}$, thus correspond to an element of U_K/U_K^n by(4.1.27), this is an injection because $\Lambda_{f,n}$ generate $L_{\pi,n}$. Then we use the canonical polynomial $f(Z) = \pi Z + Z^q$, $f^n = f^{n-1}\varphi(n)$, where $\varphi(n)$ is a Eisenstein polynomial, thus $L_{\pi,n}/K$ is totally ramifies with $|G_{\pi,n}| = q^{n-1}(q-1) = |U_K/U_K^n|$, thus the result. □

Prop. (4.1.30) (Explicit Local Norm Residue Symbol). Now we can write the universal residue symbol little bit more explicitly. For $a = u\pi^m$, (a, K) acts by φ^m on T and generated by the action $(u^{-1})_f$ on $\Lambda_{f,n}$ on $L_{\pi,n}$. Cf.[Neukirch CFT P106].

Thus the norm group of $L_{\pi,n}$ is just U^n by(4.1.29).

Cor. (4.1.31). The norm groups of the totally ramified Abelian extension is precisely the groups that contains some $U_K^n \times (\pi)$ for some uniformizer π. And every totally ramified Abelian extension L/K is contained in some $L_{\pi,n}$.

Proof: For any totally ramified extension, choose a uniformizer, then its norm is a uniformizer π of K. And $N_{L/K}$ is open (as it contains $(K^*)^m$**??**.) Thus it contains some U^n. The rest follows from local CFT(4.1.24). □

Cor. (4.1.32) (Maximal Abelian Extension of Local Fields). Let $L_\pi = \cup L_{\pi,n} = K(\Lambda_f)$, where $\Lambda_f = \cup \Lambda_{f,n}$, then $T \cdot L_\pi$ is the maximal extension of Abelian extension of K. Hence $G_K^{ab} = G_{T,K} \times G_\pi$. This follows immediately from(4.1.26).

Cor. (4.1.33) (Hasse-Arf). We can prove Hasse-Arf(2.1.23) in the case where K is a local field. This is because we already know the maximal Abelian extension, and $G(K^{ab}/T) \cong G(L_\pi/K) \cong \mathbb{Z}_p$ for which we know the Galois action well(4.1.27)(4.1.29), so $i(\sigma) = v(\sigma(\alpha_n) - \alpha_n) = v([\sigma - 1](\alpha))$, which jumps at U_K^n(the same pattern as $K = \mathbb{Q}_p$(2.1.25), thus the result.

Remark (4.1.34). There is a concrete example. When $K = \mathbb{Q}_p$, we can choose $f(Z) = (1+Z)^p - 1$, thus $L_{\pi,n}$ is just $\mathbb{Q}_p(\xi_{p^n})$. And we have $r_f = (1+Z)^r - 1$, thus we have

$$(a, \mathbb{Q}_p(\xi_{p^n})/\mathbb{Q}_p)\zeta = \zeta^r$$

where $a = up^m$, and $r \equiv u^{-1} \bmod p^n$.

Global Class Field Theory

For Basic Notations regarding Idele and Adele, See(2.4.1).

The **Ideal class group** $C_K = I_K/K^*$ is a the main object of global class field theory. We will denote $H^q(G_{L/K}, C_L)$ by $H^q(L/K)$. $H^2(G_{L/K}, I_L)$ is the secondary object.

Prop. (4.1.35). Let $\mathfrak{P}$ be a prime of L lying over $\mathfrak{p}$, then $H^q(G, I_L^{\mathfrak{p}}) \cong H^q(G_{\mathfrak{P}}, L_{\mathfrak{P}}^*)$. If $\mathfrak{p}$ is a finite unramified prime of L, then $H^q(G, U_L^{\mathfrak{p}}) = 1$ for all q.

Proof: Notice $I_L^{\mathfrak{p}} = \prod_{\sigma \in G/G_{\mathfrak{P}}} L_{\sigma\mathfrak{P}}^* = \prod_{\sigma \in G/G_{\mathfrak{P}}} \sigma L_{\mathfrak{P}}^*$ is an induced module, so by(3.4.5), we have $H^q(G_{\mathfrak{P}}, L_{\mathfrak{P}}^*)$, and similarly for $U_{\mathfrak{p}}$, which vanish by(4.1.10). □

Cor. (4.1.36).

$$H^q(G, I_L^S) = \bigoplus_{p \in S} H^q(G_{\mathfrak{P}/\mathfrak{p}}, L_{\mathfrak{P}}^*), \quad H^q(G, I_L) = \bigoplus_{p} H^q(G_{\mathfrak{P}}, L_{\mathfrak{P}}^*).$$

And the isomorphism is natural, by restriction to components.

Proof: For this, just notice $I_L = \cup_S I_L^S$, then use the last proposition, notice group cohomology commutes with colimits(3.4.2). □

Cor. (4.1.37). $H^1(G, I_L) = H^3(G, I_L) = 0$, by(4.1.19).

Cor. (4.1.38). An idele $\mathfrak{a} \in I_K$ is the norm of an idele $\mathfrak{b}$ in I_L if each component $\mathfrak{a}_{\mathfrak{p}}$ is the norm of an element $b_{\mathfrak{P}} \in L_{\mathfrak{P}}^*$.

Prop. (4.1.39). The decomposition commutes with inf, res and cor. Cf.[Neukirch CFT P125].

The strategy is to first establish CFT for cyclic extensions, then show they cover all $H^2(\overline{K}/K)$.

Lemma (4.1.40). For a cyclic extension L/K of order p, C_L is a Herbrand module with Herbrand quotient $h(C_L) = p$.

Proof:

□

Prop. (4.1.41) (First Fundamental Inequality). $(C_K : N_G C_L) \geq p$

Prop. (4.1.42). If K contains p-th roots of unity and L/K is a cyclic extension of order p, then $(C_K : N_G C_L) \leq p$.

Cor. (4.1.43) (Second Fundamental Inequality). If L/K is a cyclic extension of order p, then $(C_K : N_G C_L) = p$.

Cor. (4.1.44) (Hass Norm Theorem). For a cyclic extension L/K, an element $x \in K^*$ is a norm iff it is locally a norm everywhere.

Proof: Use the long exact sequence for $1 \to L^* \to I_L \to C_L \to 1$, we see that $H^0(G, L^*) \to H^0(G, I_L)$ is an injection, which is

$$0 \to K^*/N_{L/K}L^* \to \bigoplus_{p} K_{\mathfrak{p}}^*/N_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} L_{\mathfrak{P}}^*.$$

In fact, by(4.1.37), we say that this is equivalent to $H^1(G_{L/K}, C_L) = 1$, which is equivalent to second fundamental inequality. □

Prop. (4.1.45). For L/K normal extension, $|H^2(G,C_L)|\Big|[L:K]$.

Proof: Cf.[Neukirch P137]. □

Prop. (4.1.46). Let K be a finite algebraic number field, then

$$Br(K)=\bigcup_{L/K\text{ cyclic}}H^2(G_{L/K},L^*),\quad H^2(G_{\overline{K}/K},I_{\overline{K}})=\bigcup_{L/K\text{ cyclic}}H^2(G_{L/K},I_L).$$

Proof: Cf,[Neukirch P127]. □

Next we construct the Invariant map, first for $H^2(G_{L/K},I_L)$, then for $H^2(G_{L/K},C_K)$.

Def. (4.1.47). We define for $c=(c_\mathfrak{p})\in H^2(G_{L/K},I_L)$ by

$$\mathrm{inv}_{L/K}c=\sum_{\mathfrak{p}}\mathrm{inv}_{L_\mathfrak{P}/K_\mathfrak{p}}c_\mathfrak{p}.$$

For an Abelian extension L/K, we define for $\mathfrak{a}\in I_K$:

$$(\mathfrak{a},L/K)=\prod_{\mathfrak{p}}(a_\mathfrak{p},L_\mathfrak{P}/K_\mathfrak{p})\in G_{L/K}.$$

Prop. (4.1.48). If $c\in H^2(G_{L/K},L^*)$, then $\mathrm{inv}_{L/K}c=0$. Cf.[Neukirch P141].

Cor. (4.1.49). Now we can define the inv map for C_K when . By the exact sequence $1\to L^*\to I_L\to C_K\to 1$, we have

$$1\to H^2(G_{G/K},L^*)\to H^2(G_{L/K},I_L)\to H^2(G_{L/K},C_L)\to H^3(G_{L/K},L^*)$$

The last one is 1 if L/K is cyclic, thus tby this proposition, inv is defined for $H^2(G_{L/K},C_L)$.

Prop. (4.1.50) (Hasse's Main Theorem). For every finite algebraic number field K, there is a canonical exact sequence

$$1\to Br(K)\to\bigoplus_{\mathfrak{p}}Br(K_\mathfrak{p})\xrightarrow{\mathrm{inv}_K}\mathbb{Q}/\mathbb{Z}\to 0$$

Proof: Cf.[Neukirch P146]. □

Prop. (4.1.51). If L/K is normal and L'/K is cyclic and they have the same degree, then $H^2(L'/K)=H^2(L/K)\subset H^2(\overline{K}/K)$.

Cor. (4.1.52). $H^2(\overline{K}/K)=\cup_{L/K\text{ cyclic}}H^2(L/K)$, thus the homomorphism $H^2(G_K,I_{\overline{K}})\to H^2(\overline{K}/K)$ is surjective by(4.1.49).

Prop. (4.1.53). The inv map is defined for $H^2(\overline{K}/K)$, and $inv_{L/K}:H^2(L/K)\to\frac{1}{[L;K]}\mathbb{Z}/\mathbb{Z}$ is an isomorphism for every normal extension L/K.

Prop. (4.1.54) (Main Theorem). The formation $(G_K,C_{\overline{K}})$ is a class formation with the inv map.

Cor. (4.1.55) (Artin's Reciprocity Law). The cup product with the fundamental class in $H^2(L/K)$ defines an isomorphism **reciprocity map**

$$G_{L/K}^{ab} \cong H^{-2}(G_{L/K}, \mathbb{Z}) \to H^0(L/K) = C_K/N_{L/K}C_L.$$

And the reverse map is called the **norm residue symbol**

$$1 \to N_{L/K}C_L \to C_K \xrightarrow{(-,L/K)} G_{L/K}^{ab} \to 1$$

Remark (4.1.56). WARNING: we have already defined a norm residue map in(4.1.47), they are compatible with that derived from CFT mechanism. i.e. local global correspondence and vanish on K^*.

Proof: Cf.[Neukirch CFT P154]. □

Prop. (4.1.57). Properties of Norm Residue symbol.

Cor. (4.1.58). By(4.1.8), the map $L \mapsto I_L = N_{L/K}C_L$ defines a inclusion reversing isomorphism between the lattice of Abelian extension L of K and the lattice of norm groups of C_K, i.e.:

$$I_{L_1L_2} = I_{L_1} \cap I_{L_2}, \quad I_{L_1 \cap L_2} = I_{L_1} \cdot I_{L_2}.$$

And any group that contains a norm group is a norm group.

Prop. (4.1.59). Let L/K be an Abelian extension, then $(\mathfrak{a}, L/K) = \prod_{\mathfrak{p}}(a_{\mathfrak{p}}, L_{\mathfrak{P}}/K_{\mathfrak{p}})$

Proof: Cf.[Neukirch P154]. □

Prop. (4.1.60) (Existence Theorem). The norm groups of C_K are precisely the (open)closed subgroups of finite index.

Proof: Cf.[Neukirch P162]. □

Now we want to further characterize the norm groups of C_K in an arithmetic way.

Def. (4.1.61) (Notations). A **modulus** $\mathfrak{m}$ is a $\prod_{\mathfrak{p}} \mathfrak{p}_{\mathfrak{p}}^n$ that $n_{\mathfrak{p}} = 0$ a.e.. $\{\mathfrak{m}\}$ is the set of primes in $\mathfrak{m}$.

$I_K^{\mathfrak{m}} = \{\mathfrak{a} \in I_K | \mathfrak{a} \equiv 1 \bmod \mathfrak{m}\}$.

The **congruence subgroup mod** $\mathfrak{m}$ $C_K^{\mathfrak{m}} = I^{\mathfrak{m}} \cdot K^*/K^* \subset C_K$.

$C_K^{\mathfrak{m}}$ is a norm group by(4.1.63), the Abelian class field L/K associated with $C_K^{\mathfrak{m}}$ is called the **ray class field mod** $\mathfrak{m}$, so its Galois group is isomorphic to $C_K/C_K^{\mathfrak{m}}$.

Prop. (4.1.62). For a field K, if S is a finite set of primes that contains all the infinite primes and all the primes lying above the primes dividing n, and $I_K = I_K^S \cdot K^*$, then $C_K^n \cdot U_K^S$ is a norm group. If K contains the n-th roots of unity, then it corresponds to the Kummer extension $T = K(\sqrt[n]{K^S}/K)$.

Prop. (4.1.63). The norm groups $\mathcal{N}_{L/K}$ of C_K is precisely the groups containing some congruence subgroup $C_K^{\mathfrak{m}}$. Such $\mathfrak{m}$ are called a **modulus of definition for** L/K.

Proof: Cf.[Neukirch P164]. □

Prop. (4.1.64). Getting things together, we get a **universal norm residue symbol** $C_K \xrightarrow{(-,K)} G_K^{ab}$, and its kernel is $D_K = \cap_L N_{L/K} C_L$.

Then we have $D_K = \cap C_K^n$ and it is the connected component of $1 \in C_K$ and $C_K/D_K \to G_K^{ab}$ is an isomorphism.

Proof: Cf.[Neukirch P167]. and [Class Field Theory Artin Tate Chap9]. □

Prop. (4.1.65). When $K = \mathbb{Q}$ and $\mathfrak{m} = m \cdot p_\infty$, then the ray class field mod $\mathfrak{m}$ is $\mathbb{Q}(\zeta_m)$.

Proof: Cf.[Neukirch P165]. □

Cor. (4.1.66) (Kronecker Theorem). Every Abelian extension of $\mathbb{Q}$ is a subfield of $Q(\zeta_m)$ for some cyclotomic field.

Remark (4.1.67). The ray class field mod 1 is important, it is the **Hibert class field** of K, its Galois group is isomorphic to $C_K/C_K^1 \cong I_K/I_K^{S_\infty} \cdot K^* \cong J_K/P_K$ by(2.4.5). Its degree is equal to the ideal class number h of K.

Next we investigate the relation of CFT with the decomposition of primes in extension fields.

Prop. (4.1.68). If L/K is an Abelian extension, then $N_{L/K} C_L \cap K_\mathfrak{p}^* = N_{L_\mathfrak{P}/K_\mathfrak{p}} L_\mathfrak{P}^*$.

Proof: For the non-trivial part, notice if $\mathfrak{a} \in N_\mathfrak{P} L_\mathfrak{P}^*$ is a norm times a $a \in K^*$, then it is a norm at all primes except $\mathfrak{p}$, thus it is also norm at $\mathfrak{p}$ by the multiplicative definition of the inv map(4.1.47). □

Cor. (4.1.69). Let L/K be Abelian and $\mathcal{N} = N_{L/K} C_L$ be the norm group, then $\mathfrak{p}$ is unramified in L iff $U_\mathfrak{p} \subset \mathcal{N}$ and $\mathfrak{p}$ splits completely in L iff $K_\mathfrak{p}^* \subset \mathcal{N}_{L/K}$.

Cor. (4.1.70) (Conductor). We can define the **conductor** $\mathfrak{f}$ of L/K as the gcd of all $\mathfrak{m}$ that $C_K^\mathfrak{m} \in \mathcal{N}_{L/K}$. Then all primes not in $\mathfrak{f}$ are unramified and in particular, all primes not in $\mathfrak{m}$ are unramified in $C^\mathfrak{m}$.

Prop. (4.1.71) (Ramification and Norm Group). Let L/K is an Abelian extension of degree n and $\mathfrak{p}$ is an unramified prime ideal of K and π is a uniformizer, then if f is the smallest number that $(\ldots, 1, \pi^f, 1, \ldots) \in N_{L/K} C_L$, then $\mathfrak{p}$ factors in the extension L into $r = n/f$ distinct primes of degree f.

Proof: The degree the extension of $\mathfrak{p}$ is just the order of the Frobenius automorphism of $G_{\mathfrak{P}/\mathfrak{p}}$, which is just the order in $G_{L/K} \cong C_K/N_{L/K} C_L$. The Frobenius of $\mathfrak{p}$ correspond exactly to $(\ldots, 1, \pi, 1, \ldots)$ by(4.1.12), so the result follows. □

Prop. (4.1.72). The Hilbert class field is the maximal unramified extension of K.

Prop. (4.1.73) (Principal Ideal Theorem). In the Hilbert class field over K, every ideal $\mathfrak{a}$ of K becomes a principal ideal.

Proof: Cf.[Neukirch P171]. It should use a Finite Group theory theorem(3.1.10) □

Next we interpret the conclusions of GCFT in the language of ideals.

Def. (4.1.74) (Notations). $J^{\mathfrak{m}}$ is the group of all ideals relatively prime to $\mathfrak{m}$.

The **ray mod $\mathfrak{m}$** $P^{\mathfrak{m}}$ is the group of all principal ideals (a) that $a \equiv 1 \bmod \mathfrak{m}$.

All subgroups of $J^{\mathfrak{m}}/P^{\mathfrak{m}}$ are called **ideal groups defined mod $\mathfrak{m}$**.

If L/K is an Abelian extension with a modulus of definition $\mathfrak{m}$, then $H^{\mathfrak{m}} = N_{L/K}J_L^{\mathfrak{m}} \cdot P^{\mathfrak{m}}$ is called the **ideal group defined mod $\mathfrak{m}$**.

Def. (4.1.75). We have a homomorphism $J^{\mathfrak{m}} \to G_{L/K}$ called the **Artin symbol** $(\frac{L/K}{})$. On primes $\mathfrak{p}$, it maps a prime $\mathfrak{p}$ which is unramified by(4.1.70) to its local Frobenius automorphism in $G_{\mathfrak{P}/\mathfrak{p}} \subset G_{L/K}$, which doesn't depend on $\mathfrak{P}$ because it is Abelian.

Lemma (4.1.76). When $\mathfrak{m}$ is a modulus of definition, the restriction to finite part defines isomorphism $C/C^{\mathfrak{m}} \cong J^{\mathfrak{m}}/P^{\mathfrak{m}}$ and $N_{L/K}C_L/C^{\mathfrak{m}} \cong H^{\mathfrak{m}}/P^{\mathfrak{m}}$.

Proof: Cf.[Neukirch CFT P176]. □

Prop. (4.1.77) (classical Artin Reciprocity Law). If L/K is an Abelian extension and $\mathfrak{m}$ is a modulus of definition, then there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & N_{L/K}C_L \arrow[r] \arrow[d] & C_K \arrow[r, "{(-,L/K)}"] \arrow[d] & G_{L/K} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & H^{\mathfrak{m}}/P^{\mathfrak{m}} \arrow[r] & J^{\mathfrak{m}}/P^{\mathfrak{m}} \arrow[r, "(\frac{L/K}{})"] & G_{L/K} \arrow[r] & 1
\end{tikzcd}$$

Thus the second row is exact by(4.1.55), and $G_{L/K} \cong J^{\mathfrak{m}}/H^{\mathfrak{m}}$.

Prop. (4.1.78) (Ramification and Ideal Group). Let L/K is an Abelian extension of degree n with a modulus of definition $\mathfrak{m}$(e.g. the conductor) and $\mathfrak{p}$ doesn't divid $\mathfrak{m}$. then if f is the smallest number that $\mathfrak{p}^f \in H^{\mathfrak{m}}$, then $\mathfrak{p}$ factors in the extension L into $r = n/f$ distinct primes of degree f.

Proof: The degree the extension of $\mathfrak{p}$ is just the order of the Frobenius automorphism of $G_{\mathfrak{P}/\mathfrak{p}}$, which is just the order in $G_{L/K} \cong J^{\mathfrak{m}}/H^{\mathfrak{m}}$. The Frobenius of $\mathfrak{p}$ correspond exactly to $\mathfrak{p}$ by(4.1.12), so the result follows. □

2 Cohomology of Local Fields

Def. (4.2.1) (Notations). For an algebraic extension K/Q_p, we let G_K be $G(\overline{\mathbb{Q}_p}/K)$.

For a finite extension $K/\mathbb{Q}_p$, K_∞ is defined to be K adding all the p^n-th roots of unities.

H_K is defined to be $G(\overline{\mathbb{Q}_p}/K_\infty)$, $\Gamma_K = G_K/H_K$.

The **cyclotomic character** χ is defined to be the multiplicative map $\chi : G_K \to \mathbb{Z}_p^*$ that $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ for every $\sigma \in G_K$ and ζ a p^n-th root of unity. The kernel of χ is H_K, and it identifies $\Gamma_{\mathbb{Q}_p}$ as $\mathbb{Z}_p^*$ and Γ_K as an open subgroup of $\mathbb{Z}_p^*$.

Prop. (4.2.2). The profinite group $\mathbb{Q}_p^{\text{tame}}$ is $\widehat{\mathbb{Z}} \ltimes \Delta_p$. Which is the profinite group generated by the relationship $\sigma\tau\sigma^{-1} = \tau^p$, where σ is a lift of Frobenius. Which means that it is the limit of finite quotients of the group $\langle \sigma\tau\sigma^{-1} = \tau^p \rangle$.

Proof: Cf.[Local Fields Clark]. □

3 Cohomology of Global Fields

4 Iwasawa Theory

II.5 Langlands Program

1 Tate's thesis (LLC for GL_1)

2 Local Langlands Correspondence

The basic object of LLC are the Weil group and its representations.
A representation ρ of W_K is called F**-semisimple** iff ρ(Frob) is diagonalizable.

Thm. (5.2.1) (LLC for GL_n). The set of
irreducible smooth, admissible representations of $GL_n(K)$
corresponds to
n-dimensional F-semisimple Weil-Deligne representations of W_K.

Cor. (5.2.2) (LLC for GL_1).
Local class field theory told us that W_K^{ab} is isometric to K^*, And notice by Schur's lemma, any smooth representation of K^* is 1-dimensional and factors through some U_k.

And a Weil-Deligne representation is now a continuous $W_K^{ab} \to C^*$. but it must factor through some U_K, so these two are equivalent.

most l-adic representation of G_K comes from étale cohomology.

LLC for $GL_2(\mathbb{C})$

3 Global Langlands Correspondence

II.6 Modular Forms

1 Fermat's Last Theorem

II.7 Abelian Variety(Mumford)

1 Algebraic Aspects

Def. (7.1.1). An **Abelian variety** A over a field k is a group scheme over k that is proper and geometrically integral.

Prop. (7.1.2). Abelian variety is projective, by(9.1.15) and(3.3.3).

Prop. (7.1.3). Abelian Variety is smooth.

Proof: When k is perfect, this is by(9.1.13) or(9.1.14). When k is not perfect, we choose its perfect closure, then use descent for smoothness. □

Prop. (7.1.4). For a field K of characteristic p, then $A(K^{sep})$ is an Abelian group and its l^n torsion is isomorphic to $(\mathbb{Z}/l^n\mathbb{Z})^{2g}$ and its p^n torsion is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^r$.

Prop. (7.1.5). There is an isomorphism

$$H_t^m(\Lambda_{K^{\text{sep}}}, \mathbb{Q}_l) \cong \bigwedge_{\mathbb{Q}_l}^m (V_l(A))^*.$$

Cf.[Grothendieck Monodromy theorem].

2 Elliptic Curves

II.8 Shimura Variety

II.9 Rigid Analytic Geometry

Basic references are [Formal and Rigid Geometry Siegfried Bosch] and [Non-Archimedean Analysis Remmert].

1 Adic Space

2 Affinoid Spaces

Tate Algebra

Def. (9.2.1). For a complete non-Archimedean field K with residue field k, we define the **Tate algebra** $T_n = K\langle x_1, \ldots, x_n\rangle$ to be the subalgebra of $K[[x_1, \ldots, x_n]]$ that $\varinjlim_{|v|\to\infty} |a_v| = 0$. It is endowed with the norm $|f| = \max |a_v|$.

The norm satisfies $|fg| = |f||g|$ and $|f + g| \leq |f| + |g|$.

There is a **reduction map** from T_n to $k[x_1, \ldots, x_n]$, it is surjective.

Proof: T_n is an algebra because the values of coefficients of f is bounded. $|fg| \leq |f||g|$ is easy, to show $|fg| \geq |f||g|$, we assume $|f| = |g| = 1$, then their reduction in $K[x_1, \ldots, x_n]$ is non-zero, thus $\overline{fg}$ is non-zero, which shows $|fg| \geq 1$. □

Prop. (9.2.2) (Maximum Principle). A formal power series f converges in $B^n(\overline{K})$ iff it is in T_n.

And when it is in T_n, $|f(x)|$ attains a maximum$= |f|$ in $B^n(\overline{K})$.

Proof: If it converges at $(1, \ldots, 1)$, then $\varinjlim_{|v|\to\infty} |a_v| = 0$ by(1.1.7). Conversely, for any point in $B^n(\overline{K})$, it can be considered in a finite extension field of K, thus complete, hence we can apply(1.1.7) again.

For the second assertion, we assume $|f| = 1$, then consider its reduction to $k[x_1, \ldots, x_n]$, then there is a $\overline{x}$ in the alg.closure of k that $\overline{f}(\overline{x}) \neq 0$. Now $\overline{k}$ can be seen as the residue field of $\overline{K}$. Then the lifting of $\overline{x}$ to a $x \in \overline{K}$ has valuation 1 and $|f(x)| = 1$. □

Prop. (9.2.3). T_n is a Banach algebra.

Proof: Cf.[Formal and Rigid Geometry P14]. □

Cor. (9.2.4). An element f of norm 1 of T_n is invertible in T_n iff its reduction in $k[x_1, \ldots, x_n]$ is a unit. Elements of other norms can be reduced to the case of norm 1. (One way is trivial, the other is because $|f - f(0)| < 1$, hence $f = f(0)(1 + g)$, this is invertible by power expansion.

Prop. (9.2.5) (Noether Normalization). For any proper ideal $\mathfrak{a}$ of T_n, There is a d and a finite injection $T_d \to T_n/\mathfrak{a}$.

Proof: Cf.[Formal and Rigid Geometry P19]. □

Cor. (9.2.6). The residue field of a maximal ideal of T_n is a finite extension field of K, because $T_n/\mathfrak{m}$ has dimension 0, thus $K \to T_n/\mathfrak{m}$ finite injective.

Cor. (9.2.7). The map from $B^n(\overline{K})$ to the set of maximal ideals of T_n are surjective.

Proof: Cf.[Formal and Rigid Geometry P19]. □

Cor. (9.2.8) (Main Theorem). T_n is Noetherian, UFD, Jacobson of Krull dimension n.

Prop. (9.2.9). For an ideal $\mathfrak{a} \in T_n$, there are $a_1, \ldots, a_r$ which generate $\mathfrak{a}$ that $|a_i| = 1$, and any elements in f has a representation of the form $\sum f_i a_i$ with $|f_i| \leq |f|$.

The same assertion holds for submodules of T_n^k. Cf.[Formal and Rigid Geometry P29].

Should include [Formal and Rigid Geometry P28 Cor9].

Proof: Cf.[Formal and Rigid Geometry P27]. □

Cor. (9.2.10). Each ideal of T_n is complete hence closed in T_n. (This is because for a Cauchy sequence, $|f_i|$ converges to 0(1.1.7), thus its its coordinates in(9.2.9) converges in T_n, thus the ideal is complete).

Cor. (9.2.11). For any ideal $\mathfrak{a}$ of T_n, the distance from an element to $\mathfrak{a}$ attains minimum.

Proof: Cf.[Formal and Rigid Geometry P28]. □

Affinoid Algebras

Def. (9.2.12). Algebras of the form $T_n/\mathfrak{a}$ are called **affinoid algebras**. an affinoid algebra has a natural semi-norm by $|f|_{sup} = \sup |f|_{\mathfrak{m}}$ in $A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ of A by(9.2.6).

Proof: We need to show the sup is finite, for this, we use(9.2.11) to see $|f| = |g|$ for some g in the induced norm, so for any maximal ideal $\mathfrak{m}$ of A, the inverse is a maximal ideal $\mathfrak{n}$ in T_n by finiteness, thus $|f|_{\mathfrak{m}} = |g|_{\mathfrak{n}} \leq |g|_{sup}$.

To finish the proof, notice on T_n, $|\cdot|_{sup}$ and $|\cdot|$ equal, by(9.2.2) and(9.2.7). □

Prop. (9.2.13). For $T_d \to A$ a finite injection, assume A is a torsion-free T_d-module, then for any $f \in A$, there is a unique monic polynomial P of f over T_d.

In this case, $|f|_{sup} = \sup |a_i|_{sup}^{1/i}$ where a_i are coefficients of P.

Proof: Because A is torsion-free, we reduce to the quotient field of T_n, then f has a minimal monic polynomial, and T_n is UFD, hence Gauss lemma shows that this polynomial has coefficients in T_d. Hence $T_n[f] = T_n[X]/(p)$.

For the second, notice first for finite extension the Spec map is surjective, thus we may assume $A = T_n[f] = T_n[X]/(p)$, and for a maximal ideal $\mathfrak{m}$ of T_n, let $T_n/\mathfrak{m} = k$, then $A/(\mathfrak{m}) = k[X]/(\overline{p})$, then maximal ideals of $A/(\mathfrak{m})$ corresponds to roots α_i of $\overline{p}$ in $\overline{k}$, so $\sup_{\mathfrak{n} \text{ over } \mathfrak{m}} |f|_{\mathfrak{n}} = \sup |\alpha_i| = \max |a_i|_{\mathfrak{m}}^{1/i}$, so the result follows. □

Cor. (9.2.14). We have $|f|_{sup}$ in the multiplicative group $\sqrt[N]{|K|}$ for some N and all $f \in A$, because the minimal polynomial has bounded degree.

Cor. (9.2.15) (Maximal Principle). $|f|_{sup} = |f|_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$.

Proof: Since A is Noetherian(9.2.8), it has f.m. minimal primes, hence $|f|_{sup} = |f|_{sup}$in A/p_i for some minimal prime of A. Hence we reduce to the case of(9.2.13), hence the conclusion follows from(9.2.2) and the proof of(9.2.13). □

Prop. (9.2.16). Any morphism between two affinoid algebras is continuous w.r.t any residue norms. In particular, all residue norms on an affinoid algebra are equivalent. Hence they induce the same topology.

Cor. (9.2.17). For an affinoid algebra A, the restricted power series in A:

$$A\langle X_i\rangle = \{\sum a_v X^v | \varinjlim_{|v|\to\infty} a_v = 0\}$$

is an affinoid algebra, this is independent of the residue norm chosen.

Prop. (9.2.18) (Fibererd-Pushouts). When R, A_1, A_2 are all affinoid algebras, the amalgamated sum is also a affinoid algebra. In other words, the category of affinoid algebras admits amalgamated sums(fibered pushouts by(1.2.11)).

Proof: Cf.[Formal and Rigid Geometry P245]. □

Affinoid K-Spaces

Def. (9.2.19). In view of the above proposition, we can now view A as the function ring on the space $\mathrm{Sp}\,A$ of maximal ideals of A with the usual Zariski topology called the **affinoid K-space associated to A**. A morphism of affinoid algebras induce a map on their $\mathrm{Sp}\,A$. This is because residue fields of maximal ideals are finite over K. So we *define* the category of affinoid spaces as the opposite category of affinoid algebras.

Prop. (9.2.20). The category of affinoid spaces admits fiber products, because of(9.2.18).

Prop. (9.2.21). By the properties of a Jacobson space(1.12.11), the affinoid K-space has good properties w.r.t. closed, open hence irreducible compared to $\mathrm{Spec}\,A$. Cf.[Formal and Rigid Geometry P41].

Prop. (9.2.22). The affinoid K-space has another topology, called the **canonical topology**, generated by $X(f,\varepsilon) = \{x | f(x) < \varepsilon\}$ as a basis. And we can show they in fact are generated by $X(f) = X(f,1)$ as a subbasis.

Proof: For the last assertion, notice $f(x)$ assume value in $|\overline{K}|$, which is dense in $\mathbb{R}_+$, so we can assume $\varepsilon \in |\overline{K}|$, hence $\varepsilon^n = |c|$, where $c \in K$, so $X(f,\varepsilon) = X(f^n, c) = X(c^{-1}f^n)$. □

Prop. (9.2.23). $\{x | f(x) = \varepsilon\}$ is open in $\mathrm{Sp}\,A$, hence inverse of open or closed intervals are open.

Proof: We let $f(x) = \varepsilon$ and $k = A/\mathfrak{m}_x$, let the minipoly of f in $A/\mathfrak{m}_x$ be P of degree n, and let $g = P(f)$, then $g(x) = 0$, and if $|g(y)| < \varepsilon^n$, then $|f(y)| = \varepsilon$, otherwise $|f(y) - \alpha_i| \geq |\alpha_i| = \varepsilon$ for every root α_i of P, hence $|P(f(y))| \geq \varepsilon^n$, contradiction. □

Cor. (9.2.24). By the proof, we have, $X(f_1, \ldots, f_r)$, $f_i \in \mathfrak{m}_x$ forms a basis of x in $\mathrm{Sp}\,A$.

Prop. (9.2.25). For an affinoid K-space X, a subset U is called a **affinoid subdomain** of X if there is an closest affinoid space map $X' \to X$ with image in U, i.e. any other these maps factor through it.

Prop. (9.2.26). For an affinoid subdomain $i : X' \to X$,
- i is injective and $\mathrm{Im}\, i = U$.
- i^* induce an isomorphism $A/\mathfrak{m}_{i(x)}^k \cong A'/\mathfrak{m}_x^k$.
- $\mathfrak{m}_x = \mathfrak{m}_{i(x)} A'$.

Proof: Consider a point $y \in U$, there is a commutative diagram $\begin{tikzcd} A \arrow[r, "i^*"] \arrow[d] & A' \arrow[d] \arrow[dl, "\alpha"] \\ A/\mathfrak{m}_y^n \arrow[r, "\sigma"] & A'/\mathfrak{m}_y^n A' \end{tikzcd}$. Then there is a map $\alpha : A' \to A/\mathfrak{m}_y^n$ that makes the upper diagram commutative by universal property, and the lower is commutative by universal properties again. Then we see σ is surjective and we notice the kernel of the projection is $\mathfrak{m}_y A'$ is in the kernel of α, thus σ is injective.

Now the case $n = 1$ shows $\mathfrak{y}A'$ is maximal, hence i is surjective and the inverse image is just one point. □

Prop. (9.2.27). There are three special affinoid subdomain of X: **Weierstrass domain** $X(f_1, \ldots, f_r)$, **Laurent domain** $X(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1})$, **rational domain** $X(\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}) = \{x | f_i(x) \leq f_0(x)\}$ for $(f_0, \ldots, f_r) = (1)$. They are all open by(9.2.23).

Weierstrass domain are Laurent, and Laurent domain are rational, this is because intersection of rational domains are rational.

Proof: The Weierstrass domain corresponds to $A \to A\langle X_1, \ldots, X_r\rangle/(X_i - f_i)$.

The Laurent domain corresponds to $A\langle X_1, \ldots, X_{r+s}\rangle/(X_i - f_i, 1 - X_{r+j}g_j)$.

The rational domain corresponds to $A\langle X_1, \ldots, X_r\rangle/(f_i - f_0X_i)$. □

Lemma (9.2.28) (Pullback & Composition of Affinoid Domain). The pullback(hence intersections) of affinoid domains is affinoid domain, and specialness are preserved, because fiber product exist in the category of affinoid K-spaces(9.2.20).

The affinoid domain of affinoid domain are affinoid, and Weierstrassness and rationalness are preserved(while Laurentness not). rational domain is a Weierstrass domain of a rational domain.

Proof: A rational domain f_0 is a unit in U, hence its inverse has a bounded value, then $|cf_0| \geq 1$ for some $c \in K^*$. Hence U is Weirersrass in $X((cf_0)^{-1})$.

For the transversality, Cf.[Formal and Rigid Geometry P56]. □

Lemma (9.2.29). Every affinoid subdomain of X is open and has the restriction topology of X.

Proof: Cf.[Formal and Rigid Geometry P60]. □

Prop. (9.2.30) (Gerritzen-Grauert). Any affinoid subdomain is a finite union of rational subdomains. Cf.[Formal and Rigid Geometry P77].

Def. (9.2.31). We define a **weak Grothendieck category** on an affinoid space X by coverings defined by the finite cover by affinoid subdomains, called **affinoid covering**. This is truly a topology by(9.2.28).

The **Strong Grothendieck category** on an affinoid space X is defined by: elements are union of affinoid subdomains U that for any morphism from an affinoid space $Z \to U \subset X$, the pullback covering has a finite subcover by affinoid subdomains. A covering is defined by the same property. This is truly a topology is verified routinely.

The weak one is a temporary notion, we are interested in the strong one.

Prop. (9.2.32). Morphisms of affinoid spaces are continuous in weak Grothendieck topology by(9.2.28). It is also continuous in the strong Grothendieck topology, as one look at the finiteness assumption.

Def. (9.2.33). There is a presheaf of affinoid algebras defined on the weak Grothendieck category, and stalks are defined routinely.

Then the stalk $\mathcal{O}_{X,x}$ are local ring with maximal ideal $\mathfrak{m}_x\mathcal{O}_{X,x}$. And let $X = \operatorname{Sp} A$, then the stalk map factor thorough $A_\mathfrak{m}$ by an injection and

$$A/\mathfrak{m}^n \cong A_{\mathfrak{m}_x}/\mathfrak{m}_x^n A_{\mathfrak{m}_x} \cong \mathcal{O}_{X,x}/\mathfrak{m}_x^n\mathcal{O}_{X,x}$$

so it induce and isomorphism between there $\mathfrak{m}_x$-adic completions.

Proof: Cf.[Formal and Rigid Geometry P66]. □

Cor. (9.2.34). $f \in A$ vanish iff it vanish at every stalk, this is because $A \to \prod_\mathfrak{m} A_\mathfrak{m} \to \prod \mathcal{O}_{X,x}$ is injective.

Cor. (9.2.35). For a subdomain of an affinoid space X, the corresponding algebra map is flat.

Proof: Cf.[Formal and Rigid Geometry P68]. □

Prop. (9.2.36). The stalk $\mathcal{O}_{X,x}$ is Noetherian.

Proof: FIrst it is $\mathfrak{m}$-adically separated, because by(9.2.33), for a $f \in \cap \mathfrak{m}^n\mathcal{O}_{X,x}$, we can choose an affinoid subdomain $\operatorname{Sp} A$ that $f \in A$, then $f \in \mathfrak{m}^n A$, so by Krull's intersection theorem(5.8.5), we have $f = 0$ in $A_\mathfrak{m}$.

In the same way, we see that any f.g. ideal $\mathfrak{a}$ of $\mathcal{O}_{X,x}$ is $\mathfrak{m}$-adically closed, this is because we can assume it is generated by an ideal in the affinoid algebra of a nbhd.

Now we pass a chain of f.g. ideals to their completion, then that chain is stationary because $\hat{\mathcal{O}}_{X,x} = \hat{A}_{\mathfrak{m}_x}$ is Noetherian(5.1.7). And now this chain is also stationary because ideals are closed in $\mathfrak{m}$-adic topology. □

Prop. (9.2.37). For n functions $f_1, \ldots, f_n$ that has no common zeros, the rational subdomains $U_i = (\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i})$ is an affinoid covering, called the **rational covering**. For n functions $f_1, \ldots, f_n$, there is a **Laurent covering** $X(\prod f_i^{\varepsilon_i})$.

Every affinoid covering has a refinement of rational covering. For every rational covering, there is a Laurent covering $\{V_i\}$ that restriction on each V_i is rational covering generated by units. Every rational covering generated by units has a refinement of Laurent covering.

Proof: Cf.[Formal and Rigid Geometry P84]. □

Prop. (9.2.38) (Tate's Acyclicity Theorem). The presheaf of affinoid functions on an affinoid space $X = \operatorname{Sp} A$ is a sheaf w.r.t the weak Grothendieck category. In fact, for any A module M, the presheaf $\widetilde{M} = M \otimes_A \mathcal{O}_X$ is a sheaf w.r.t. the weak Grothendieck topology, called the **quasi-coherent** sheaf on X.

In fact, for any finite cover of affinoid subdomains, the Čech cohomology group vanish for $q \neq 0$.

Proof: Cf.[Formal and Rigid Geometry P87,90]. First reduce to the case of Laurent covering,

The last assertion follows from the first because we can choose a free resolution of M, then use dimension shifting, notice the covering is finite. In the process, the flatness of the algebra map(9.2.35) is used to deduce the long exact sequence. □

Prop. (9.2.39). If X is an affinoid K-space, any sheaf w.r.t to the weak topology has a unique extension to a sheaf w.r.t to the strong topology by(1.2.5).

In particular, this applies to the case $\mathcal{O}_X$ by(9.2.38), the resulting sheaf is called the **sheaf of rigid analytic functions** on X.

Prop. (9.2.40). Let X be affinoid space and $f \in \mathcal{O}_X(X)$, then the following sets $U = \{x||f(x)| > 1\}$, $U' = \{x||f(x)| < 1\}$, $U'' = \{x||f(x)| > 0\}$. Any finite union of sets of these type are admissible open and finite cover of finite union of sets of these type are admissible. Cf.[Formal and Rigid Geometry P96].

Cor. (9.2.41). The last type is Zariski open, thus strong topology is finer than Zariski topology.

3 Rigid Analytic Spaces

Def. (9.3.1). A **G-ringed K-space** is a pair $(X, \mathcal{O}_X)$ where X is a G-topological space and $\mathcal{O}_X$ is a sheaf of K-algebras. It is called **local G-ringed K-space** if the stalks are all local rings. Their morphisms are defined routinely.

Prop. (9.3.2). An affinoid K-space in the strong topology with the structure sheaf is a local G-ringed K-space, And a morphism induce a local G-ringed morphism. And it is easy to see all morphisms comes from these.

Moreover, an affinoid K-space is a complete G-ringed K-space(i.e. rigid)(1.1.4).

Proof: It is a G-space by(9.2.38)(9.2.39), morphisms by(9.2.32), it is local because of(9.2.33), notice the shape of the stalks shows the morphism is local. □

Def. (9.3.3). A **rigid (analytic) space** is a complete local G-ringed K-space that it has a covering of affinoid K-spaces.

It follows easily that an admissible open of X is again rigid.

Prop. (9.3.4). Glue operation for rigid analytic spaces are legitimate, the proof is the same as(1.5.10).

Cor. (9.3.5). If X is rigid and Y is affinoid, then $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_X)$. This follows from(9.3.2) and glue.

Prop. (9.3.6). We can define the connected components of X as the equivalence class of elements that can be reached using connected admissible open subsets of X. Then the connected components are admissible and forms an admissible cover of X.

Proof: We use completeness, notice that we can choose a covering that all subdomains are connected, because an affinoid subdomain is connected iff it is Zariski connected, because of Tate's acyclicity. And $\mathrm{Sp}\, A$ has f.m. connected components because $\mathrm{Spec}\, A$ has f.m. irreducible components.

Now we see that a connected subdomain either non-intersect a connected components or contained in it, hence by completeness(1.1.4), this is admissible and the cover is admissible. □

Prop. (9.3.7). Fiber products exist in the category of rigid analytic space. This is fiber product of affinoid spaces are affinoid so we can glue them by universal property, and the resulting space is truly rigid.

Def. (9.3.8). For a K-scheme X of locally f.t., there is a rigid K-space $X^{rig} \to X$ which is a morphism of local G-ringed space that is closest to X w.r.t this property, called the **rigid analytification** of X.

The underlying map of $X^{rig} \to X$ identifies points of X^{rig} with the closed pts of X, and the analytification defines a functor from the category of K-schemes of locally f.t. to the category of rigid analytic spaces, called the **GAGA** functor.

Proof: Cf.[Formal and Rigid Geometry P109]. □

4 Coherent Sheaves on Rigid Spaces

Prop. (9.4.1). The category of $\mathcal{O}_X$-modules on a rigid K-space is a Grothendieck category. The proof is routine, notice shifification is left exact because $\check{H}^0$ does and right exact by adjointness.

Prop. (9.4.2). The quasi-coherent presheaf construction as in(9.2.38) is an exact fully faithful functor, this is because the restriction morphisms are flat by(9.2.35).

Def. (9.4.3). Now we can define the right derived functor for Γ and more general f_p, these are left exact by(1.2.4). And routinely $R^p f_*\mathcal{F} = (f_*\mathcal{H}^p(\mathcal{F}))^\sharp$ by Grothendieck spectral sequence. And the Cech to Derived spectral sequence is applied

In particular, if we have $H^q(U_{i_0 i_1 \ldots i_r}, \mathcal{F}) = 0, q > 0$, then $H^p(\{U_i \to U\}, \mathcal{F}) = H^p(U, \mathcal{F})$(4.2.7). And it is enough to have$\check{H}^q(U_{i_0 i_1 \ldots i_r}, \mathcal{F}) = 0, q > 0$ by(4.4.3).

Cor. (9.4.4). A Qco sheaf on an affinoid space has vanishing higher sheaf cohomology by Tate's acyclicity(9.2.38).

Def. (9.4.5). For an $\mathcal{O}_X$-module $\mathcal{F}$ on a rigid space X, **finite type, of finite presentation, coherent** are defined w.r.t the strong topology similar to the case of ringed space.

Prop. (9.4.6). An $\mathcal{O}_X$-module on an affinoid K-space X is coherent iff it is associated to a finite A-module. Cf.[Formal and Rigid Geometry P119].

Def. (9.4.7). A morphism is called a **closed immersion** if there is a covering by affinoid subdomains that it restricts to a closed immersion of affinoid spaces. The **(quasi-)separatedness, quasi-compactness** are defined as usual.

Separated morphism is quasi-separated because closed immersion is quasi-compact**?**.

Prop. (9.4.8). For a morphism of schemes locally of f.t. over K, it is proper iff its rigid analytification is proper. Cf.[U. Köpf, Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen. Schriftenr. Math. Inst. Univ. Münster, 2. Serie, Heft 7 (1974) Satz 2.16].

Prop. (9.4.9) (Proper Mapping theorem Kiehl). The higher direct images of a proper map of rigid analytic spaces takes coherent sheaves to coherent sheaves.

Prop. (9.4.10). For a scheme X locally of f.t. over K, an $\mathcal{O}_X$-module $\mathcal{F}$ on X gives rise to an $\mathcal{O}_{X^{rig}}$-module on X^{rig}, and it is coherent iff $\mathcal{F}$ is coherent.

Prop. (9.4.11). For a proper scheme over K, $H^q(X, \mathcal{F}) \cong H^q(X^{rig}, \mathcal{F}^{rig})$ for $\mathcal{F}$ coherent.

Prop. (9.4.12). When X is proper, coherent sheaves on X^{rig} corresponds to coherent sheaves on X. This gives an analog of Chow's theorem when applied to $X = \mathbb{P}^n_K$ and $\mathcal{F}'$ is a sheaf of ideal in $\mathcal{O}_{X^{rig}}$.

5 Perfectoid space

II.10 p-adic Hodge Theory

1 Witt Theory (Local Fields Serre)

Complete discrete Valuation Ring

Structure of complete discrete valuation ring will be studied in this subsubsection.

Prop. (10.1.1) (0-Type). Let A be a local ring which is complete Hausdorff in the topology defined by the decreasing chain of ideals $\mathfrak{m} = \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots$ such that $\mathfrak{a}_m\mathfrak{a}_n \subset \mathfrak{a}_{m+n}$. Now if $A/\mathfrak{m}$ is field of characteristic 0, then A contains a field mapped isomorphically onto k.

Proof: $\mathbb{Z} \to A \to k$ is injective, so $\mathbb{Z}$ is units in A, thus $\mathbb{Q} \in A$, hence A contains a field. Now we show using Zorn's lemma that the maximal field S of A is mapped onto k.

First $\overline{K}$ is algebraic over $\overline{S}$, this is because if there is a transcendental element $\overline{a}$, then the inverse image a is transcendental over S and $S[a] \cap \mathfrak{a}_1 = 0$, so $S(a) \subset A$. Now for any a that is algebraic over $\overline{S}$, the minipoly has no multiple roots, so it has a lifting by Hensel's lemma, thus the result. □

Def. (10.1.2) (Strict p-Ring). A **p-ring** A is a ring which is complete Hausdorff in the topology defined by the decreasing chain of ideals $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots$ such that $\mathfrak{a}_m\mathfrak{a}_n \subset \mathfrak{a}_{m+n}$ that $k = A/\mathfrak{a}_1$ is a perfect ring of characteristic p.

It is called a **strict p-ring** if moreover $\mathfrak{a}_n = p^n A$ and p is not a zero-divisor of A.

Prop. (10.1.3). For a p-ring, there exists a unique system of representatives $k \to A$ that $f(\lambda^p) = f(\lambda)^p$, called the **Teichmüller lift**.

For this representative, it is also multiplicative, and if A has char p, then it is also additive. And an element is in the image of f iff it is a p^n-th power for any n.

Proof: For any $\lambda \in k$, the $\lambda^{p^{-n}}$ is unique in k, and if we consider U_n the set of all x^{p^n} where x is a lift of $\lambda^{p^{-n}}$, then U_n is a descending set. Moreover, the diameter converges to 0, because $a \equiv b \mod \mathfrak{a}_1$ implies $a^{p^n} \equiv b^{p^n} \mod \mathfrak{a}_{n+1}$ as $p \in \mathfrak{a}_1$. So it converses to a unique point $f(\lambda)$ in A. And we see that any other f' maps λ to a p^n-th root hence in U_n for any n, hence it map be equal to $f(\lambda)$. The rest is easy. □

Cor. (10.1.4) (Equal Characteristic case). If A is a complete discrete valuation ring with residue field k. If k and A have the same characteristic and k is perfect, then $A \cong k[[T]]$.

Cor. (10.1.5). When A is a strict p-ring, elements of A can be written uniquely as $\sum f(\alpha_n)p^n$.

Def. (10.1.6) ($(0,p)$-type case). When A is a complete DVR with residue field k and quotient field K. If $\mathrm{char} K = 0$ and $\mathrm{char} k = p$, then p goes to 0 in k, so $e = v(p) \geq 1$, called the **absolute ramification index** of A. It is called **absolutely unramified** iff $e = 1$.

Remark (10.1.7) (Canonical Strict p-Ring). The canonical strict p-ring is the ring $\hat{S} = \hat{\mathbb{Z}}[X_\alpha^{p^{-n}}]$. Its residue ring is $\mathbb{F}_p[X_\alpha^{p^{-n}}]$ which is perfect. X_i are all Teichmüller lifts, as they has all p^n roots.

Now we consider the $* = + - \times$ in $\hat{S}$. then there are elements $Q_i^* \in \mathbb{F}_p[X_\alpha^{p^{-n}}, Y_\alpha^{p^{-n}}]$ that $x * y = \sum f(Q_i^*)p^i$ where f is the Teichmüller lift.

Prop. (10.1.8) (Universal Law of p-Rings). For any p-ring A with residue ring k, the calculation in A is dominated by Q_i^* defined in(10.1.7), i.e.

$$\left(\sum f(\alpha_i)p^i\right) * \left(\sum f(\beta_i)p^i\right) = \sum f(\gamma_i)p^i$$

where $\gamma_i = Q_i^*(\alpha_0, \alpha_1, \ldots, \beta_0, \beta_1, \ldots)$.

Proof: There is a map θ from $\hat{S} = \hat{\mathbb{Z}}[X_i^{p^{-n}}, Y_i^{p^{-n}}]$ to A induced by $f(\alpha_i), f(\beta_i)$ as they all has p^{-n}-th roots. Then notice θ induce a $\overline{\theta}$ on residue ring and these two θ commutes with Teichmüller lift, as seen by the definition of the latter. Then the theorem follows immediately. □

Cor. (10.1.9). For two p-ring A, A' that A is strict, then any map φ of their residue ring induces a unique ring homomorphism $A \to A'$. In particular, two strict p-ring with the same residue ring is canonically isomorphic.

Proof: We have already seen that ring homomorphism commutes with Teichmüller lift. Now we define

$$g(a) = \sum g(f(\alpha_i))p^i = \sum f(\varphi(\alpha_i))p^i$$

and this is the unique choice. It is a ring homomorphism by universal law of(10.1.8). □

Prop. (10.1.10) (Witt Vectors of Perfect Rings). For any perfect ring k of char p, there exists uniquely a strict p-ring $W(k)$ that has residue ring k, called the **ring of Witt vectors** with coefficients in k. W is a faithful functor from perfect rings to strict p-rings by(10.1.9).

Proof: For a canonical ring $\mathbb{F}_p[X_\alpha^{p^{-n}}]$, $\hat{\mathbb{Z}}[X_\alpha^{p^{-n}}]$ is a strict p-ring. Now arbitrary perfect p-ring is a quotient of $\mathbb{F}_p[X_\alpha^{p^{-n}}]$, so we can construct its strict p-ring $W(k)$ as the quotient of $\hat{\mathbb{Z}}[X_\alpha^{p^{-n}}]$. Uniqueness is by(10.1.9).

Notice it is nothing mysterious, it is just the set of all formal sum $\sum f(x_i)p^i$ under the operation defined in(10.1.7). See also(10.1.13). □

Witt Vectors

A ring homomorphism φ lifting the Frobenius, i.e. $\varphi(x) = x^p + p\delta(x)$. It generate a $\delta-$**ring structure**.

δ-rings form a category and the right adjoint to the forgetful functor is $W(A) = \mathrm{Hom}(\Delta, A)$. Where Δ is the free ring $\mathbb{Z}[e, \delta, \delta^2, \ldots]$. The ring structure of $W(A)$ is that: as φ is a ring homomorphism, there are sum and product formulae for $\delta^n(x+y)$ and $\delta^n(xy)$ in forms of $\delta^n(x)$ and $\delta^n(y)$. So that is the ring structure of $W(A)$.

There is another description of Δ:

Prop. (10.1.11) (Ghost Component). Let θ_i be polynomials in δ with integer coefficients that

$$\varphi^n = \theta_0^{p^n} + p\theta_1^{p^{n-1}} + \cdots + p^n\theta_n = W_n(\theta_0, \ldots, \theta_n)$$

In fact $\mathbb{Z}[\theta_0, \theta_i, \ldots, \theta_n] = \mathbb{Z}[e, \delta, \delta^2, \ldots, \delta^n]$.

Proof: Use equation $\varphi \circ \varphi^n = \varphi^n \circ \varphi$ and module $p^n\mathbb{Z}[\theta_0, \theta_1, \ldots, \theta_n]$. □

So there is a map $Z[\varphi] \to \Delta$ inducing an morphism of rings: $W(A) \to \prod_{\mathbb{Z}} A$ that maps

$$(f(\delta^n)) \mapsto (f(\varphi^n))$$

Where the right hand side is the usual addition and multiplication, the left side is the usual coordinate of Witt vector, and $f(\theta_n)$ is called **ghost component**.

This is embedding if A is p-torsion free, and isomorphism iff $\frac{1}{p} \in A$, because θ_n can be presented by φ^n.

Lemma (10.1.12) (Formula for p-Rings). For $* = +$ or $\times$, there are integral polynomials $S_*(X_i, Y_i)$ that

$$\left(\sum f(\alpha_i)p^i\right) * \left(\sum f(\beta_i)p^i\right) = \sum f(\gamma_i)p^i$$

where $\gamma_i = Q_i^*(\alpha_0, \alpha_1, \ldots, \beta_0, \beta_1, \ldots)$. And for $+$, when reduced to $\mathbb{F}_p$, Q_i^+ are polynomials in $X_i^{p^{-n}}, Y_i^{p^{-n}}$ for $i \leq n$ and homogenous of degree 1. And

$$Q_i^+ = (X_n + Y_n) + (X_{n-1}^{p^{-1}} + Y_{n-1}^{p^{-1}})R_{n,n-1} + \ldots + (X_0^{p^{-n}} + Y_0^{p^{-n}})R_{n,0}.$$

Proof: We solve S_n by induction. Notice for any lift $\hat{S}_i$ of S_i,

$$f(S_i) \equiv \hat{S}_i(X^{1/p^{n-i}}, Y^{1/p^{n-i}})^{p^{n-i}} \mod p^{n-i+1}$$

so we mod p^{n+1} to solve S_n:

$$S_n \equiv 1/p^n\Big(X_0 + Y_0 + \ldots + p^n X_n + p^n Y_n - \hat{S}_0(X^{1/p^n}, Y^{1/p^n})^{p^n} - \ldots - p^{n-1}\hat{S}_{n-1}(X^{1/p}, Y^{1/p})^p\Big)$$

The rest follows by induciton. □

Prop. (10.1.13). Notice in Serre book, he presented the Witt vectors in $(f(\theta_n))$ coordinates. In this coordinate, if k is a perfect ring and we let

$$T(\{a_i\}) = \sum f(a_i)^{p^{-i}} p^i,$$

then T is a ring isomorphism from $W(k)$ to the strict p-ring with residue ring k.

Proof: We need to prove this is a ring homomorphism. That on $W(A)$ is to make φ a ring homomorphism, and that on the right is usual. It suffice to prove for the canonical strict p-ring, as seen by the universal law(10.1.8).

For this, we let $(\sum X_i^{p^{-i}} p^i) * (\sum Y_i^{p^{-i}} p^i) = \sum f(\psi_i(X_i, Y_i))^{p^{-i}} p^i$, and $W_n(a_i) * W_n(b_i) = W_n(\varphi_i)$, where $\psi_i \in \mathbb{F}_p[X_i, Y_i]$ and $\varphi_i \in \mathbb{Z}[X_i, Y_i]$, they both exist, the latter because of(10.1.11).

Then we mod p^{n+1}, and let $X_i = X_i^{p^n}$, $Y_i = Y_i^{p^n}$, so

$$W_n(\varphi_i) = W_n(X_i) * W_n(Y_i) \equiv \sum_{i \leq n} f(\psi_i(X_i^{p^n}, Y_i^{p^n}))^{p^{-i}} p^i \equiv W_n(\psi_i) \mod p^{n+1}$$

Now induction, $\varphi_i \equiv \psi_i \mod p$, then $p^n\varphi_n \equiv p^n\psi_n \mod p^{n+1}$ so this is true for n, too. □

Cor. (10.1.14). For example, $W(\mathbb{F}_p^n)$ is the unramified extension of $\mathbb{Z}_p$ of degree n. And $W(\overline{F})$ is the completion of the maximal unramified extension of $W(F)$.

Def. (10.1.15) (Witt Vectors over Valued Rings). If a perfect ring R itself has a complete valuation v, then we can endow $W(R)$ with a finer topology: we let $w_k(x) = \inf_{i \leq k} v(x_i)$, where $x = \sum p^i f(x_i)$. Now $w_k(x+y) \geq \inf(w_k(x), w_k(y))$ by(10.1.12). The **weak topology** of $W(R)$ is defined by the semi-valuations w_k.

Prop. (10.1.16). If $a, b \in \mathcal{O}_R) + p^{n+1}W(R)$, then

$$p^n v(a_n - b_n) \geq w_n(a-b) \geq \inf_{k \leq n} p^{-k} v(a_{n-k} - b_{n-k}).$$

So we see that a sequence is Cauchy in $W(R)$ if each coordinate is Cauchy in R, so $W(R)$ is complete in the weak topology.

Proof: Firstly the last proposition follows from the first because we can always multiply by a $f(\alpha)$ to make the first n coordinate in $\mathcal{O}_R$.

The first is nearly an immediate consequence of(10.1.12). □

Prop. (10.1.17). $\mathcal{O}_{\mathcal{E}} = W(K^{\frac{1}{p^\infty}})$ is a complete ring with maximal ideal $p\mathcal{O}_{\mathcal{E}}$. And $\mathcal{O}_{\mathcal{E}}[\frac{1}{p}] = \mathcal{E}$ is complete ring of character p. And the same construction of $\overline{K^{\frac{1}{p^\infty}}}$ yields the completion of maximal unramified extension of $\mathcal{O}_{\mathcal{E}}$, and the Galois group is the same as G_K.

2 Galois Representations

Lemma (10.2.1). If $P(X) \in \overline{F}[X]$ is a monic polynomial of degree n, all of its roots satisfied $val_p(\alpha) \geq c$ for some constant c. We let $q = p^k$ if $n = p^k d, d \neq 1$ or $n = p^{k+1}$.

Then the derivative $P^{(q)}(X)$ has a root β with $val_p(\beta) \geq c$ or in case $n = p^{k+1}$, $val_p(\beta) \geq c - \frac{1}{p^k(p-1)}$.

Proof: Let $P = X^n + a_{n-1}X^{n-1} + \ldots + a_0$, then $val_p(a_i) \geq (n-i)c$. And

$$1/q! \cdot P^{(q)}(X) = \sum_{i=0}^{n-1} C^q_{n-i} a_{n-i} X^{-i-q}.$$

So at lest one root β satisfies

$$val_p(\beta) \geq \frac{1}{n-q}((n-q)c - val_p(C^q_n)) = c - \frac{1}{p^k(p-1)}.$$

□

Lemma (10.2.2). If F is a complete valued field and $\alpha \in \overline{F}$, let $\Delta_K(\alpha) = \inf_{g \in G_K} val_p(g(\alpha) - \alpha)$, then there exists $\delta \in K$ that $val_p(\alpha - \delta) \geq \Delta_K(\alpha) - p/(p-1)^2$.

Proof: We strengthen the assertion and use induction on $n = [F(\alpha) : F]$ to prove that there is a δ that $val_p(\alpha - \delta) \geq \Delta_K(\alpha) - \sum_{k=2}^m \frac{1}{p^k(p-1)}$, where p^{m+1} is the largest power of p that $\leq n$.

$n = 1$ is sure, let the minipoly be $P(X)$. By lemma(10.2.1), there is a root β of $P^{(q)}$ that $val_p(\beta - \alpha) \geq \Delta_K(\alpha)$ or minus a factor. Then for any σ, $val_p(\sigma(\beta) - \beta) \geq val_p(\sigma(\alpha) - \alpha)$ or minus a factor. Then $\Delta(\beta) \geq \Delta(\alpha)$ or minus a factor. Now $[F(\beta) : F] < n$, so we can use induction hypothesis to get the result. □

Prop. (10.2.3) (Ax-Sen-Tate). If F is a complete p-adic field and if $K \in F^{alg}$, then $\widehat{F^{alg}}^{G_K} = \widehat{K}$. Thus $\hat{L}^{G_{L/K}} = \hat{F}$ for any alg.ext L/K.

Proof: Any $\alpha \in \hat{\overline{F}}$ can be written as $\sum \alpha_n$ with $\alpha_n \in \overline{F}$. Then $\Delta_K(\alpha_n) \to \infty$, and α_n can be approximated by $\delta_n \in K$ by lamma(10.2.2), thus $\alpha \in \widetilde{K}$. □

Cohomology of G_K action on $\mathbb{C}_p$

K is assumed to be a p-adic number field.

Lemma (10.2.4). Giving an $\sigma \in G(K/\mathbb{Q}_p)$, if $x, y \in \mathfrak{m}_{\mathbb{C}_p}$ that $x \equiv y \mod \pi_K^n$, then $[\pi_K]^\sigma(x) \equiv [\pi_K]^\sigma(y) \mod \pi_K^{n+1}$, where f^σ is given by action of σ on the coefficients.

Proof: This is because the coefficients of $[\pi_K]^\sigma$ are divisible by π_K except for degree q, where it is $x^q - y^q = (x-y)(x^{q-1} + x^{q-2}y + \ldots + y^{q-1})$ which is divisible by π_K^{n+1} because the residue field of K is of order q. □

Prop. (10.2.5). If we let the action of $\sigma \in G(K/\mathbb{Q}_p)$ on the residue field giving by $\overline{\sigma} : k_K \to \overline{\mathbb{F}}_p : x \mapsto x^{q_\sigma}$, where $q_\sigma = p^{n_\sigma}$ is a p-power, given an element $\eta = (\eta_0, \eta_1, \ldots) \in TG$, we have $\eta^{q_\sigma} \equiv [\pi_K]^\sigma(\eta_{n+1}^{q_\sigma}) \mod \pi_K$, hence the above lemma(10.2.4) shows that $[\pi_K^n]^\sigma \eta_n^{q_\sigma} \equiv [\pi_K^{n+1}]^\sigma(\eta_{n+1}^{q_\sigma}) \mod \pi_K^{n+1}$, so $[\pi_K^n]^\sigma(\eta_n^{q_\sigma})$ is a Cauchy sequence, converging to an element μ_σ(don't care about η).

If $g \in G_K$, then $g(\eta_n) = [\chi_K(g)](\eta_n)$, hence take q_σ-th power, $g(\eta_n^\sigma) \equiv [\chi_K(g)]^\sigma(\eta_n^{q_\sigma}) \mod \pi_K$, then

$$[\chi_K(g)]^\sigma[\pi_K^n]^\sigma(\eta_n^{q_\sigma}) \equiv [\pi_K^n]^\sigma g(\eta_n^{q_\sigma}) = g([\pi_K^n]^\sigma \eta_n^{q_\sigma}) \mod \pi_K.$$

hence by limiting, $g(\mu_\sigma) = [\chi_K(g)]^\sigma(\mu_\sigma)$.

Lemma (10.2.6).

$$v_p(\mu_\sigma) = \begin{cases} \frac{q_\sigma}{e_K(q-1)} + \frac{1}{e_K} & n(\sigma) \neq 0 \\ \frac{1}{e_K(q-1)} + v_p(\sigma(\pi_K) - \pi_K) & n(\sigma) = 0 \end{cases}$$

Proof: By(9.2.17), we know the Newton polygon of $[\pi_K^n]^\sigma$. When $n(\sigma) \neq 0$, $v(\eta_1^{q_\sigma}) = \frac{q_\sigma}{e_K(q-1)} > \frac{1}{e_K(q-1)}$, so the valuation of $[\pi_K]^\sigma(\eta_1^{q_\sigma})$ equals the valuation of its degree 1 term, which is $v(\pi_K \eta_1^{q_\sigma}) = \frac{q_\sigma}{e_K(q-1)} + \frac{1}{e_K}$. Now we have by(10.2.5), we have $[\pi_K]^\sigma \eta^{q_\sigma} \equiv [\pi_K^2]^\sigma(\eta_2^{q_\sigma}) \mod \pi_K^2$, and $\frac{q_\sigma}{e_K(q-1)} + \frac{1}{e_K} < 2/e_K$, so valuation already stable at degree 1, and $v(\mu_\sigma) = v([\pi_K]^\sigma(\eta_1^{q_\sigma}))$.

If $q_\sigma = 1$, it's more delicate, because degree 1 and degree q term has the same minimal valuation, so they may jump to higher valuations. Notice $[\pi_K^n](\eta_n) = 0$, so $[\pi_K^n]^\sigma(\eta_n) = ([\pi_K^n]^\sigma - [\pi_K^n])(\eta_n)$. And we have by(2.1.16), for $x \in \mathcal{O}_K$, $v(\sigma(x) - x) \geq v(x) + v(\frac{\sigma(\pi_K)}{\pi_K} - 1) + \delta_{v(x),0} v(\pi_K)$, with equality when $v_p(x) = q/e_K$. So by the Newton polygon, the minimum valuation of the coefficient of $[\pi_K^n]^\sigma - [\pi_K^n]$ appear at degree p^{n-1} and possibly p^n. The valuation of η_n is too small($\frac{1}{e_K p^{n-1}(p-1)}$) that we don't need to consider other degrees but can assure that degree p^{n-1} is of minimum valuation, which is $v(\eta_n^{p^{n-1}}) + v(\sigma(\pi_L) - \pi_L) = \frac{1}{e_K(q-1)} + v_p(\sigma(\pi_K) - \pi_K)$. □

Prop. (10.2.7). For any $\sigma \in G(K/\mathbb{Q}_p) \backslash \{\mathrm{id}\}$, there is an element $\alpha_\sigma \in \mathbb{C}_p^*$ that $\sigma \circ \chi_K(g) = g(\alpha_\sigma)/\alpha_\sigma$ for all $g \in G_K$, where χ_K is the Lubin-Tate character.

Proof: We let $\alpha_\sigma = \log_{\mathcal{F}_\pi}^\sigma(\mu_\sigma)$, by(10.2.6), $1/e_K < \mu_\sigma < \infty$, so by the Newton polygon analysis of $\log_{F_\pi}$(9.2.18), α_σ has the same valuation of μ_σ, in particular, $\alpha_\sigma \neq 0$. Then

$$g(\alpha_\sigma) = \log_{\mathcal{F}_\pi}^\sigma(g(\mu_\sigma)) = (\log_{\mathcal{F}} \circ [\chi_K(g)])^\sigma(\mu_\sigma) = (\chi_K(g) \cdot \log_{\mathcal{F}_\pi})^\sigma(\mu_\sigma) = \sigma(\chi_K(g)) \cdot \alpha_\sigma.$$

□

Cor. (10.2.8). $\log_p(\sigma(\chi_K(g))) = g(\log(\alpha_\sigma)) - \log_p(\alpha_\sigma)$.

Def. (10.2.9). Let $\psi : G_K \to \Gamma_K \to \mathbb{Z}_p^*$ be a character factoring through Γ_K. Then we can form arepresentation $\mathbb{C}_p(\psi)$ of G_K on $\mathbb{C}_p$ that $\rho(\sigma)(x) = \psi(\sigma)\sigma(x)$. This is an action because G_K acts trivial on $\mathbb{Z}_p^*$.

If $\psi^k = \mathrm{id}$ for some k, then it is trivial on Γ_K^k. Γ_K is an open subgroup of $\mathbb{Z}_p$, so Γ_K^n is of finite index in Γ_K by(2.2.3), hence also does its inverse image in G_K. So ψ comes from a finite extension L/K.

Prop. (10.2.10). $H^0(G_K, \mathbb{C}_p(\psi)) = K$ if ψ is of finite order, and vanish if ψ is of infinite order.

Proof: Finite case: ψ factor through some G_L, so ψ corresponds to a continuous cocycle w.r.t the discrete topology of $\mathbb{C}_p$. So by(3.5.2) there is a $a \in \mathbb{C}_p^*$ that $\psi(\sigma) = \sigma(a)/a$, so $\mathbb{C}_p(\psi) \cong \mathbb{C}_p : x \mapsto ax$. And the result follows from Ax-Sen-Tate, as $K = \hat{K}$.

Infinite case: $H^0(G_K, \mathbb{C}_p(\psi)) \subset H^0(H_K, \mathbb{C}_p(\psi)) = \hat{K}_\infty(\psi)$ by Ax-Sen-Tate and the fact ψ is trivial on H_K. Then for the normalized trace R_n, which commutes with G_K, $g(R_n(x)) = \psi^{-1}(g)R_n(x)$. But $G(K_n/K)$ is finite, so $R_n(x) = \psi^{-N}(g)R_n(x)$ for any g. So $R_n(x) = 0$, otherwise ψ is of finite order. Now $R_n(x) \to x$, so $x = 0$. □

Prop. (10.2.11). Now we compute $H^1(G_K, \mathbb{C}_p(\psi))$. There is a inf-res exact sequence

$$0 \to H^1(\Gamma_K, \hat{K}_\infty(\psi)) \to H^1(G_K, \mathbb{C}_p(\psi)) \to H^1(H_K, \mathbb{C}_p(\psi))$$

Then $H^1(H_K, \mathbb{C}_p(\psi)) = 0$. The first two vanish iff ψ is of infinite order, and is a K-vector space of dimension 1 if ψ is of finite order.

Proof: For the first assertion, ψ is trivial on H_K, so $\mathbb{C}_p(\psi) \cong \mathbb{C}_p$ as H_K-representation, so it suffice to show for $\psi = \mathrm{id}$. Let f be a cocycle, as H_K is compact, $f(H_K) \in p^{-k}\mathcal{O}_{\mathbb{C}_p}$ for some integer k. So the lemma below(10.2.12) shows that we can move f cohomologouly to higher valuation, i.e. $f(g) = \sum x_i - g(\sum x_i)$, so f is a coboundary.

For the second assertion, we assume $\Gamma_K \neq \mathbb{Z}_p^*$, for this case, see remark(10.2.13) below.

let γ be a topological generator of $\Gamma_K = 1 + p^k\mathbb{Z}_2^*$, $k \geq 0$, because $\mathbb{Z}_p^*$ are all topological cyclic groups except for $\mathbb{Z}_2^* \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}_2$, and γ_n be a topological generator of Γ_{F_n} which is also a power of γ. By(3.5.12) we know $H^1(\Gamma_K, \hat{K}_\infty(\psi)) = \hat{K}_\infty(\psi)/1 - \gamma$.

For n large, we have a decomposition $\hat{K}_\infty(\psi) = K_n(\psi) \oplus X_n(\psi)$ by(2.2.19), and $1 - \gamma_n$ is invertible on $X_n(\psi)$. Now $1 - \gamma_n = (1 - \gamma)(1 + \gamma + \ldots + \gamma^{k-1})$, so $1 - \gamma$ is also invertible in $X_n(\psi)$. And on $K_n(\psi)$, if ψ is of infinite order, then $1 - \gamma$ is injective, otherwise $x = \psi(\gamma)^N\gamma^N(x) = \psi(\gamma)^N x$. So it is also surjective because it is a K-linear mapping of K_n. So $\hat{K}_\infty(\psi)/1 - \gamma = 0$. If ψ is of finite order then $K_n(\psi) \cong K_n$ as Γ_K-module when n is large enough that γ factors through Γ_{K_n}, by(3.5.1). So $K_n/1 - \gamma = K_n/\operatorname{Ker}(\mathrm{tr}_{K_n/K}) = K$. □

Lemma (10.2.12). If $f : H_K \to p^n\mathcal{O}_{\mathbb{C}_p}$ is a continuous cocycle, then there exists a $x \in p^{n-1}\mathcal{O}_{\mathbb{C}_p}$ that the cohomologous cocycle $g \mapsto f(g) - (x - g(x))$ has values in $p^{n+1}\mathcal{O}_{\mathbb{C}_p}$.

Proof: $p^{n+2}\mathcal{O}_{\mathbb{C}_p}$ is open in $p^n\mathcal{O}_{\mathbb{C}_p}$, so there is a finite extension L/K that $f(H_L) \in p^{n+2}\mathcal{O}_{\mathbb{C}_p}$. By(2.2.14), there is a z that $\mathrm{tr}_{L_\infty/K_\infty}(z) = p$, so there is a $y \in p^{-1}\mathcal{O}_{L_\infty}$ that $\mathrm{tr}_{L_\infty/K_\infty}(y) = 1$.

Now for a set of representatives Q of H_K/H_L, denote $x_Q = \sum_{h\in Q} h(y)f(h)$, then for $g \in H_K$, $g(Q)$ is also a set of representative, and $g(x_Q) = \sum_{h\in Q} gh(y)gf(h) = \sum_{h\in Q} gh(y)(f(gh) - f(g)) = x_{g(Q)} - f(g)$, as $\mathrm{tr}(y) = 1$. So $f(g) - (x_Q - g(x_Q)) = x_{g(Q)} - x_Q$. The RHS is in $p^{n+1}\mathcal{O}_{\mathbb{C}_p}$, because: if we let $gh_i = h_{g(i)}a_i$, where $a_i \in H_L$, then $x_{g(Q)} - x_Q = \sum h_{g(i)}(y)f(h_{g(i)}a_i) - \sum h_{g(i)}(y)f(h_{g(i)}) = \sum h_{g(i)}(y)h_{g(i)}(f(a_i))$, which is in p^{n+1} because $h_{g(i)}(y) \in p^{-1}\mathcal{O}_{\mathbb{C}_p}$ and $f(a_i) \in p^{n+2}\mathcal{O}_{\mathbb{C}_p}$ by the choice of L. □

Remark (10.2.13). In case $\Gamma_K = \mathbb{Z}_2^*$,

$$0 \to H^1(\{\pm 1\}, K(\psi)) \to H^1(\mathbb{Z}_2^*, \hat{K}_\infty(\psi)) \to H^1(1+2\mathbb{Z}_2^*, \mathbb{C}_p(\psi))$$

$H^1(\{\pm 1\}, K(\psi)) = 0$ whether $\psi(-1) = 1$ or -1. And by the same proof as above, possibly replace X_n with X_{n+1}, to remedy the singularity of $p = 2$, $H^1(1+2\mathbb{Z}_2^*, \mathbb{C}_p(\psi)) = K$, with generator $[g \mapsto \frac{\chi(g)-1}{\gamma-1}(a)]$ for some a. This cocycle extends to a cocycle of $\mathbb{Z}_2^*$, so the map is surjective.

Prop. (10.2.14). The 1-dimensional K-vector space $H^1(G_K, \mathbb{C}_p)$ is generated by the cocycle $[g \mapsto \log_p \chi(g)]$.

Proof: By the proof of(10.2.11), we know that $H^1(\Gamma_K, K_n) \xrightarrow{f} H^1(G_K, \mathbb{C}_p)$ is an isomorphism. for $\alpha \in K$, if $\chi(g) = \gamma^k$, then $f(\alpha)(g) = (1+\gamma+\ldots+\gamma^{k-1})(\alpha) = k\alpha = \alpha \cdot \log_p(\chi(g))/\log_p(\gamma)$. So by continuity, f is a multiple of $[g \mapsto \log_p(\chi(g))]$. □

Lemma (10.2.15). And $f \in \mathrm{Hom}(I_K^{ab}, \mathbb{Q}_p)$ is of the form $f(g) = \mathrm{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g))$ for some $\beta_f \in K$.

Proof: By(4.1.28), χ_K is a canonical isomorphism $I_K^{ab} \cong \mathcal{O}_K^*$. Any $f \in \mathrm{Hom}(\mathcal{O}_K^*, \mathbb{Q}_p)$ is of the form $f(y) = \mathrm{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p(y))$ for some $\beta_f \in K$, because: by(2.2.5), when n is large, $\log_p$ is a bijection between U_K^n and $\pi_K^n \mathcal{O}_K$.

$\pi_K^n \mathcal{O}_K \to \mathbb{Q}_p$ can be extended to a map $K \to \mathbb{Q}_p$ as $\mathbb{Q}_p$ is divisible. Now trace is a invertible bilinear form on K, so the assertion is true on U_K^n for some n, and because U_K^n is of finite index in $\mathcal{O}_K^*$ and $\mathbb{Q}_p$ is of char 0, this is true for all $\mathcal{O}_K^*$. □

Prop. (10.2.16). The map $H^1(G_K, \mathbb{Q}_p) \to H^1(G_K, \mathbb{C}_p)$ is given as follows: as $f \in H^1(G_K, \mathbb{Q}_p)$ must factor through G_K^{ab}, if the restriction of f to I_K^{ab} corresponds to β_f, then f maps to $\beta_f[g \mapsto \log_p \chi(g)]$.

Proof: $f(g) = \mathrm{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g))$ on I_K, but this map extends to map on G_K. So $f(g) = \mathrm{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g)) + c(g)$ for a unramified map c on G_K.

Now by(3.5.11), $H^1(G, \hat{\mathbb{Q}}_p^{ur}/\mathbb{Q}_p)$ vanish because $H^1(G, \overline{\mathbb{F}}_p)$ vanish(3.5.1), so there is a $z \in \hat{\mathbb{Q}}_p^{ur}$ that $c(g) = g(z) - z$. And

$$\mathrm{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g)) = \sum_\sigma \sigma(\beta_f \log_p \chi_K(g)) = \beta_f \,\mathrm{tr}_{K/\mathbb{Q}_p}(\log_p \chi_K(g)) + \sum_\sigma (\sigma(\beta_f) - \beta_f)\sigma(\log_p \chi_K(g)).$$

Notice(10.2.7) gives a β_σ that $\sigma(\log_p \chi_K(g)) = g(\beta_\sigma) - \beta_\sigma$, and $\mathrm{tr}_{K/\mathbb{Q}_p}(\log_p \chi_K(g)) = \log_p \chi(g)$ because $(N_{K/\mathbb{Q}_p}\chi_K(g))^{-1} = (\chi(g))^{-1}$, as they both correspond via local CFT to the element in G_K^{ab} which acts by g on L_π and id on K^{ur}. Thus the result. □

Cor. (10.2.17). If $\eta : G_K \to \mathbb{Z}_p^*$ is a character and there is $y \in \mathbb{C}_p^*$ that $\eta(g) = g(y)/y$, then there exists a finite Abelian extension L of K that $\eta|_{G_L}$ is unramified, i.e. η is **potentially unramified**.

Proof: Apply $\log_p$, then the image of $f = \log_p \eta$ in $H^1(G_K, \mathbb{C}_p)$ is trivial, so the above proposition shows $\beta_f = 0$, so $\log_p \eta$ is trivial on I_K, so I_K is mapped by η into the μ_p, so $\eta(I_K^{ab} \cap (G_K^{ab})^{p-1}) = 1$. $G_K^{ab} \cong \hat{\mathbb{Z}} \times \mathcal{O}_K^*$, so $(G^{ab})^{p-1}$ is open hence of finite index in G_K^{ab}, so correspond to a finite Abelian extension L. □

Prop. (10.2.18). If $G_K \to GL_d(\mathbb{Q}_p)$ is such $\rho(g) = g(M)M^{-1}$ for $M \in GL_d(\mathbb{C}_p)$, then ρ is potentially unramified.

Proof: Cf.[Sen Continuous Cohomology and p-adic Galois representations]. □

3 (φ, Γ)-modules

Basic References are [Fontaine90: Représentations p-adiques des corps locaux],[Fontaine94a: Le corps des périodes p-adiques] and [Fonatine94b: Repésentations p-adiques semi-stables] but I cannot understand French so [Foundations of Theory of (φ, Γ)-modules over the Robba Ring] is used and I'm basically following [Berger Galois representations and (φ, Γ)-modules].

Def. (10.3.1) (φ-module). Let M be a A-module and $\sigma : A \to A$ is a ring map. Then an additive map $\varphi : M \to M$ is called **σ-semi-linear** iff $\varphi(am) = \sigma(a)\varphi(m)$ for $a \in A$. A **φ-module** over (A, σ) is just a M with a σ-semi-linear φ.

Giving a A-module M and a $\varphi : M \to M$, there is a map $\Phi : A_\sigma \otimes_A M = M_\sigma \to M : \lambda \otimes m \to \lambda\varphi(m)$, which is a A-module map iff φ is σ-semi-linear.

Prop. (10.3.2). We define a ring $A_\sigma[\varphi]$ as the free group $A[X]$ modulo the relation $Xa = \sigma(a)X$ and ring relations in A, then it is a ring. Then a φ-module over (A, σ) is equivalent to a left $A_\sigma[\varphi]$-module.

Cor. (10.3.3). Thus we know that the category of φ-modules is a Grothendieck Abelian category $\Phi\mathcal{M}$, and moreover, the kernel as $A_\sigma[\varphi]$-module is the same as the kernel as a A-module.

Def. (10.3.4). If A is Noetherian, then a φ-module M is called **étale** iff it is f.g and the corresponding $\Phi : M_\sigma \to M$ in(10.3.1) is a bijection. The subcategory of étale φ-modules is denoted by $\Phi\mathcal{M}^{ét}$.

In case when σ is a bijection, Φ is a bijection iff φ is a bijection.

Proof: Note that in this case $M_\sigma \to M$ is a bijection by $\lambda \otimes m \to \sigma^{-1}(\lambda)m$, so the rest is easy. □

Prop. (10.3.5). If A is Noetherian and A_σ is flat, then $\Phi\mathcal{M}^{ét}$ is Abelian category.

Proof: 0 is the zero object, the canonical sum&product are clearly étale. And we need to check the kernel and cokernel are etale. But we have an exact sequence $0 \to \mathrm{Ker} \to M \to N \to \mathrm{Coker} \to 0$ so we tensor with A_σ to get a morphism of sequences that $M_\sigma \to M, N_\sigma \to N$ are both bijective, so by 5-lemma, it is bijection at kernel and cokernel, so they are étale. □

Def. (10.3.6). If there is a map $\alpha : (A_1, \sigma_1) \to (A_2, \sigma_2)$ that commutes with σs, then we have a **pullback** from $\Phi\mathcal{M}_1$ to $\Phi\mathcal{M}_2$: $\alpha^*(M) = A_2 \otimes_{A_1} M$, and φ is given by $\varphi(a \otimes m) = \sigma_2(a) \otimes \varphi(m)$.

Def. (10.3.7) ((φ, Γ)-module). If A has a action of a group Γ that commutes with σ, then a **(φ, Γ)-module** is a φ-module with a semi-linear action of Γ that commutes with φ.

If A, Γ has Hausdorff and complete topologies and the action is continuous, and A is Hausdorff and A_σ flat, then a (φ, Γ)-module M is called **étale** iff it is a étale φ-module and it has a topology that the action of Γ is continuous on M.

(φ, Γ)-modules forms an Abelian category with tensor products, Cf.[Fon90 3.3.2].

Dieudonné-Manin Theory

Def. (10.3.8). We consider a perfect field k and $K = W(k)[1/p]$, K is equipped with the natural σ lifting the Frobenius. We consider φ-modules for $\sigma = \sigma^a$, where $a \in \mathbb{Z}\backslash\{0\}$. We don't care about this a much.

Then a $K - \varphi$-module D is called **effective** if there is a (complete)$W(k)$-lattice M of D that $\varphi(M) \subset M$. In this case, let a_n be the maximum integer that $\varphi^n(M) \subset p^{a_n}M$, then we have $a_{m+n} \geq a_m + a_n$, thus by(2.1.1), we have a_n/n converges to $\sup a_n/n = \lambda$. λ doesn't depend on M because of the cofinality of lattices.

Lemma (10.3.9). Let M be a lattice of D that $\varphi^{h+1}(M) \subset p^{-1}M$, where h is the dimension of D, then D is effective.

Proof: Let $M_j = M + \varphi(M) + \ldots + \varphi^j(M)$, then $M_j/M \subset p^{-1}M/M$, which is a k-vector space of dimension h, then $M_j = M_{j+1}$ for some j, hence M_j is stable under φ. □

Prop. (10.3.10). $\lambda \geq 0$ iff D is effective. And $\lambda = s/r$, where $1 \leq r \leq h$.

Proof: If D is effective, then $a_n \geq 0$, conversely, if $a_n \geq 1$, then $M' = M + \varphi(M) + \ldots + \varphi^{n-1}(M)$ is stable under φ, so D is effective.

For the second assertion, we first notice, if $\lambda > 0$, then φ is nilpotent on M/pM, which is a k-vector space of dimension h, then $\varphi^h = 0$ on M/pM, so $\lambda \geq 1/h$.

Now we find s, r that $|r\lambda - s| \leq 1/(h+1)$, and $\widetilde{\varphi} = p^{-s}\varphi^r$ has $|\widetilde{\lambda}| \leq 1/(h+1)$, so(10.3.9) shows that $\widetilde{\varphi}$ is effective, hence $\widetilde{\varphi} \geq 0$, and by what we have proved, $\widetilde{\varphi} = 0$, hence it is $\lambda = s/r$. □

Lemma (10.3.11). For a φ-stable $W(k)$-lattice of D, one has $M = M_0 \oplus M_{>0}$, where φ is bijection on M_0 and topologically nilpotent on $M_{>0}$.

Proof: We consider M/p^nM, then by(3.4.1) under slight modification, we have a decomposition, thus it has a decomposition. And the decompositions for different n are compatible, so it gives a decomposition of M it self. □

Def. (10.3.12). A φ-module is called **pure of slope** $\lambda = s/r \in \mathbb{Q}$ if D admits a lattice M on which $p^{-s}\varphi^r$ is a bijection. This is independent of M because λ is independent of M.

Prop. (10.3.13) (Dieudonné-Manin). If M is a φ-module, then D is a finite sum of modules pure of slopes λ_i.

Proof: We use the $\widetilde{\varphi}$ as in(10.3.10), we see that M has a decomposition $M_0 \oplus M_{>0}$ by(10.3.11), and $M_0 \neq 0$ by definition. Then we use induction to get the result. □

Prop. (10.3.14). If k is a separably closed field and V is a φ-module with $a \geq 1$, then V has a basis of elements fixed by φ, and $1 - \varphi$ is a surjection.

If A is a ring with $A/pA = k$ a separably close field and V is a φ-module with $a \geq 1$, then V has a basis of elements fixed by φ, and $1 - \varphi$ is a surjection.

Proof: We choose a $e_0 \in V$, and set $e_i = \varphi^i(e_0)$, and suppose $e_d = a_0e_0 + \ldots + a_{d-1}e_{d-1}$, then if we consider the equation $\varphi(b_0e_0 + \ldots + b_{d-1}e_{d-1}) = b_0e_0 + \ldots + b_{d-1}e_{d-1}$, then we need to assure b_{d-1} is a zero of

$$x = a_0^{q^{d-1}}x^{q^d} + a_1^{q^{d-2}}x^{q^{d-1}} + \ldots + a_{d-1}x^q$$

which is separable, so it has a non-zero solution in k, so φ has a fixed point v. By induction, we have $V/k \cdot v$ admits a basis fixed by φ. We know that $1 - \varphi : k \cdot v \to k \cdot v : x \mapsto (x - x^q)$ is surjective, so we can adjust the coefficient of v to get a basis of V fixed by φ. And meanwhile we proved $1 - \varphi$ is surjective.

The second assertion follows from successive approximation, as $x^p - x - a$ always has a root in k. □

Def. (10.3.15). When k is pure and separably closed(i.e. k is alg.closed), for $\lambda = s/r$, we define a φ-module over $K = W(k)[1/p]$ $E_\lambda = \oplus_{i+0}^{r-1} Ke_i$ that $\varphi(e_i) = e_{i+1}$, and $\varphi(e_{r+1}) = p^s e_0$. In this case, E_λ is irreducible.

Proof: If D is a $W(k)$-lattice stable under φ, then we may assume it is pure of slope d/h by(10.3.13), and then we find an element $y = \sum y_i e_i$ fixed by $p^{-d}\varphi^h$, then $p^{sh}\varphi^{rh}(y_i) = p^{rd}y_i$, which by valuation is only possible when $sh = rd$, so $h \geq r$, so D generate E_λ. □

Prop. (10.3.16) (Dieudonné-Manin). If k is alg.closed, then any φ-module over K has a unique decomposition as sums of E_{λ_i}.

Proof: By(10.3.13) we assume D is pure, then by(10.3.14) we find a basis y_i that $\varphi^r(y) = p^s y$, then there is a map $E_\lambda \to D$. Since E_λ is irreducible, this is injective, and we consider all y_i until $E_\lambda^m \to V$ is surjective, then it is an isomorphism(this is like the case of simple modules). □

4 l-adic representations

Prop. (10.4.1). Every continuous representation of G_K on a $\mathbb{Q}_l$ vector space (Continuous group morphism to $GL_n(\mathbb{Q}_l)$ has a $\mathbb{Z}_l$ lattice stable under the action.

So the functor $\rho \to \rho \otimes \mathbb{Q}_l$ from $\mathrm{Rep}_{\mathbb{Z}_l}(G_K)$ to the Tannakian natural category $\mathrm{Rep}_{\mathbb{Q}_l}(G_K)$ is essentially surjective.

Proof: Notice that the stablizer of the standard lattice is $GL_n(\mathbb{Z}_l)$ which is open and so the inverse image has a finite coset. And the image of the wild ramification group is finite because it is in $GL_n(\mathbb{F}_l)$. □

Prop. (10.4.2) (Grothendieck Monodromy theorem). For a local field K, the étale representation and the Tate module are all potentially semisimple. i.e. semisimple for a finite extension.

II.11 Logarithmic Geometry

II.12 Diophantine Geometry

1 Artin's Conjecture

Basic references are [Serre Galois Cohomology Chap2.4.5].

Def. (12.1.1). A field K is called C_1 or for any homogenous polynomial $F(X_1,\ldots,X_n)$ of degreed with coefficient in K that $d^k<n$ has a non-zero solution in K^n.

C_0 fields are just alg.closed fields, C_1 fields are also called **quasi-algebraically closed**.

Prop. (12.1.2). Any field L algebraic over a quasi-alg.closed field is quasi-alg.closed.

Proof: For a homogenous polynomial $F(X_1,\ldots,X_n)$, its coefficient lies in a finite extension of K contained in L, so we may assume L/K is finite. Then choose a basis $\{e_1,\ldots,e_m\}$ of L over K, then consider the function $f(x_{11},\ldots,x_{1m},\ldots,x_{n1},\ldots,x_{nm})=N_{L/K}(F(x_11e_1+\ldots+x_{1m}e_m,\ldots,x_{n1}e_1+\ldots+x_{nm}e_m)$, which is a homogenous polynomial of degree nm with coefficient in K, because it has values all in K. So it has a nonzero solution in K^{nm} by(6.1.19), Krull's height theorem and k is alg.closed. □

Prop. (12.1.3) (Chevalley-Warning). Any finite field is quasi-algebraically closed.

Proof: □

Prop. (12.1.4) (Tsen). Algebraic function fields of dimension 1 over an alg.closed field K is quasi-alge.closed.

Proof: By(12.1.2), it suffice to consider the case $K=k(t)$ purely transcendental. for a polynomial F with coefficient in $k(t)$, we can assume it has coefficient in $k[t]$, then let δ be their maximal degree. If substituted with $X_i=\sum_{i=0}^{N}a_ijt^j$, the function becomes a system of $\delta+dN+1$ homogenous equation with $n(N+1)$ unknowns a_{ij}, since $d<n$, $\delta+dN+1<n(N+1)$ for N large. In this case, □

Prop. (12.1.5). If K is quasi-alg.closed, then $H^2(G(K_s/K),K_s^*)=0$.

Proof: Cf.[Etale Cohomology Fulei 5.7.15]. □

Cor. (12.1.6). By this and(12.1.2), the condition of(3.4.13) are satisfied. So if K is quasi-alg.closed, then $cd(G(K_s/K))\le 1$ and $H^i(G(K_s/K),K_s^*)=0$ for $i\ge 1$.

Prop. (12.1.7) (Ax-Kochen). For any d, there is a N_d that if $p>N_d$, then any homogenous polynomial $f(X_1,\ldots,X_n)$ of degree d with coefficient in $\mathbb{Q}_p$ that $d^k<n$ has a non-zero solution in $\mathbb{Q}_p^n$. The proof uses Model theory.

Chapter III

Geometry

III.1 Topology

Basic references are [Topology Munkres].

1 Basics

Def. (1.1.1). A function from X to $\mathbb{R}\cup\{-\infty,\infty\}$ is called **upper semicontinuous** iff $f^{-1}(<a)$ are all open. It is called **lower semicontinuous** iff $f^{-1}(>a)$ are all open.

Connected Component

Prop. (1.1.2). Let X be a topological space, $x\in X$, C is a connected component of x, i.e. a maximal connected subset containing x. Define A to be the intersection of all the open-and-closed sets that contain x (also called the pseudo-component sometimes). Then $A=C$, if X is normal.

Proof: Assume A splits into two components B,D. Since A is closed, B and D are both closed, because X is normal there are disjoint open neighborhoods U and V around B and D, respectively. The open sets U and V cover the intersection of all clopen neighborhoods of A, so cause X is compact, there must exist a finite number of clopen sets around A, say $A_1,\ldots,A_n$ such that $U\cup V$ covers $K=\bigcap_1^n A_i$.

Note that K is clopen. We can assume that $x\in U$. It is not difficult to see that $K\cap U$ is clopen and does not contain all of A, contradicting the definition of A. $\square$

Cor. (1.1.3). For a compact Hausdorff topological space X and a point $x\in X$, the connected component of X containing x is the intersection of all compact open neighborhoods of x, because X is normal(1.5.1).

Def. (1.1.4). A space is called **totally disconnected** iff any connected subset of X contains only one point.

Prop. (1.1.5). A subspace of a totally disconnected space is totally disconnected, because totally disconnected is equivalent to the only connected subsets are pt sets.

Compactness

Prop. (1.1.6) (Tychonoff). An arbitrary direct product of compact topological groups is compact.

Def. (1.1.7). $f: X \to Y$, X is compact and Y is Hausdorff, then for a descending chain Y_i of closed subsets of X,

$$f(\bigcap_n Y_n) = \bigcap_n f(Y_n).$$

Proof: The left side is compact, so if $x \notin f(\bigcap_n Y_n)$, there is a closed subsets $x \in T$ that $T \cap f(\bigcap_n Y_n)$=0, so $f^{-1}(T) \bigcap_n Y_n = 0$, so $f^{-1}(T) \cap Y_n = 0$ for some n, hence $x \notin f(Y_n)$. □

Prop. (1.1.8). A locally compact second countable space X is σ-compact.

Proof: For every point, there is an open nbhd that the closure is compact. Then we find a basis B_i in this nbhd, then its closure are also compact. Then we have countable compact subsets that cover X. □

Compact-Open Topology

Prop. (1.1.9). The **compact-open topology** on X^Y is the topology generated by subbasis of (K, U) = all f that maps K to U, for K compact and U open. When Y is compact and X a metric space, this coincides with the uniform topology.

Prop. (1.1.10).
- $X^Y \times Y \to X$ is continuous if Y is locally compact.
- $\mathrm{Map}(Y \times X, Z) \cong \mathrm{Map}(Z, X^Y)$.

Profinite Space

Def. (1.1.11). A space is called **profinite** if it is a limit of discrete topological spaces.

Prop. (1.1.12). A profinite space is the same thing as a totally disconnected, compact Hausdorff topological space.

Cor. (1.1.13). A closed subspace of a profinite space is profinite, by criterion(1.10.3).

2 Covering Space

Prop. (1.2.1). For a connected and locally connected space, it has a universal cover, and the fundamental group acts on it continuously and properly. (Define the universal cover as the homotopy equivalence class of lines starting from a base point).

Prop. (1.2.2). if X and Y are Hausdorff spaces, $f: X \to Y$ is a local homeomorphism, X is compact, and Y is connected, then f a covering map.

Proof: First, f is surjective (using the connectedness), and that for each $y \in Y$, $f^{-1}(y)$ is finite. Because X is compact, there exists a finite open cover of X by $\{U_i\}$ such that $f(U_i)$ is open and $f|_{U_i}: U_i \to f(U_i)$ is a homeomorphism. For $y \in Y$, let $\{x_1, \ldots, x_n\} = f^{-1}(y)$ (the x_i all being different points). Choose pairwise disjoint neighborhoods $U_1, \dots, U_n$ of $x_1, \dots, x_n$, respectively (using the Hausdorff property).

By shrinking the U_i further, we may assume that each one is mapped homeomorphically onto some neighborhood V_i of y.

Now let $C = X \setminus (U_1 \cup \cdots \cup U_n)$ and set

$$V = (V_1 \cap \cdots \cap V_n) \setminus f(C)$$

V should be an evenly covered nbhd of y. □

Prop. (1.2.3). If $\pi : \tilde{B} \to B$ is a local onto homeomorphism with the property of lifting arcs. Let $\tilde{B}$ be arcwise connected and B simply connected, then π is a homomorphism.

Proof: only need to prove injective. If p_1 and p_2 map to the same point, then they can be connected, and the image is a loop thus contractable, contradiction. □

Cor. (1.2.4). If $\tilde{B}$ is locally arcwise connected and B is locally simply connected, then π is a covering map.(choose the connected component)

Prop. (1.2.5). a simply connected manifold is orientable. (Use the orientable double cover).

3 Paracompactness

Prop. (1.3.1). If X is regular, then TFAE:
1. Each open cover of X has an open locally finite refinement.
2. Each open cover of X has a locally finite refinement.
3. Each open cover of X has a closed locally finite refinement.
4. Each open cover of X is even. i.e. for any cover, there is an open nbhd V of diagonal of $X \times X$ such that $\forall x$, $V[x] = \{y|(x, y) \in V\}$ refines the cover.
5. Each open cover of X has an open σ-discrete refinement.
6. Each open cover of X has an open σ-locally finite refinement.

If this is satisfied, then X is called **paracompact**.

Proof: $6 \to 2$:Just minus every open set the part of open sets that appeared in families that ordered before it. $2 + 4 \to 1$:Use the lemma below, we can transform the cover $\mathcal{A}$ into $V[\mathcal{A}] \cap U_A$ which is an open locally finite cover
Cf.[General Topology Kelley] □

Lemma (1.3.2). If X satiesfies 4, let U be a nbhd of diagonal of $X \times X$, then their exists a symmetric nbhd of diagonal s.t. $V \circ V \subset U$, where $U \circ V = \{(x, z)|(x, y) \in U, (y, z) \in V, \exists y\}$.

Proof: $\forall x$ in X, there is a nbhd s.t. $W[x] \times W[x] \subset U$, this is an open cover, so there is a nbhd R of diagonal s.t. $R[x]$ refines it. Hense $R[x] \times R[x] \subset U$. Let $V = R \cap R^{-1}$, $V \circ V$ is the union of sets $V[x] \times V[x]$, so $V \circ V \subset U$. □

Lemma (1.3.3). In the preceding proposition, if X satisfies 4, Let $\mathcal{A}$ be a locally finite(resp. discrete i.e. intersect only one) family of subsets of X, then use the last lemma, there is a nbhd V of diagonal of $X \times X$ such that $V[A] = \{y|(x, y) \in V, \exists x \in A\}$ is locally finite(resp. discrete).

Proof: Choose for every pt a nbhd satisfy the property, then it is an open cover. Choose a diagonal nbhd U for the property 4, then choose coordinate symmetric nbhd V of diagonal s.t. $V \circ V \subset U$. If $V[x]$ intersect $V[A]$, then $V \circ V[x]$ intersect A. Done. □

Prop. (1.3.4). A regular paracompact space is normal.

Proof: The family consisting of two closed is locally discrete, by preceding lemma,there exists a V s.t.$V[A]$,$V[B]$ open and non-intersecting. □

Prop. (1.3.5). For a connected Hausdorff locally euclidian space, the condition of paracompact, second countable and a compact exhaustion is equivalent.

Proof: Cf.[Paracompactness and second countable]. □

Prop. (1.3.6). A metric space is paracompact.

Prop. (1.3.7). A compact Hausdorff space is paracompact.

Prop. (1.3.8) (Partition of unity). In a paracompact space, given any open cover, there exists a partition of unity $\{\rho_i\}$ that ρ_i has compact support and supp$\rho_i \subset U_i$.

4 Separation Axioms

Regular

Completely Regular

5 Normal (T4)

Prop. (1.5.1). A compact Hausdorff space is normal.

Prop. (1.5.2) (Urysohn lemma). Let X be normal, A and B two closed subset of X, then there exists a continuous map from X to $[0, 1]$ that maps A to 0 and B to 1.

Proof: Use the countability of rational numbers to construct a family of U_q s.t.

$$p < q \Rightarrow \bar{U}_p \subset U_q$$

Then choose $f(x) = \inf\{p \in \mathbb{Q} | x \in U_p\}$, then this f meets the requirement. □

Prop. (1.5.3) (Tietze extension). If X is normal and Y is a closed subspace, then any continuous function f on Y can be extended to a continuous function on X.

6 Complete Metric Space

Prop. (1.6.1) (Hausdorff). In a complete space, a subset M is sequentially compact iff it is totally bounded. (Use the diagonal method).

In a metric space, a subset M is sequentially compact iff its closure is compact. Hence in Fréchet space, a closed subset is compact iff it is totally bounded.

Cor. (1.6.2) (Arzela-Ascoli). For M compact, $F \subset C(M)$ is a sequentially compact subset iff it is uniformly bounded and equicontinuous.

Prop. (1.6.3) (Fixed point theorem). If X is a complete metric space and $f : X \to X$ satisfies $d(f(x), f(y)) \leq \lambda d(x, y)$ for some $0 \leq \lambda < 1$, then f has a unique fixed point in X. If X is moreover compact, then and f that $d(f(x), f(y)) < d(x, y)$ will have a unique fixed point.

Proof: $x + f(x) + f^2(x) + \cdots$ is the fixed point. And uniqueness is easy. For compact case, notice the image $\operatorname{Im} f^n$ is a descending chain, it must stable to some T. If $x, y \in Y$ attains the diameter of Y, and let $x = f(X), y = f(Y)$, where $X, Y \in T$, then $d(x, y) < d(X, Y) \leq d(x, y)$, contradiction. □

7 Baire Space

Prop. (1.7.1) (Baire Category Theorem). Every complete metric space & locally compact Hausdorff space is a Baire space, i.e. not countable union of subsets whose closure have no interior point.

Proof: Choose consecutively compact open subsets that doesn't intersect $\overline{E_n}$ to find a limit point. □

8 Uniform Space

9 Manifold

Def. (1.9.1). A **manifold** of dimension n is a Hausdorff topological space that is locally subsets of R^n and it is second countable. By(1.3.5), the last condition is equivalent to say it is paracompact.

10 Topological Groups

Prop. (1.10.1). For a topological group G, the following are equivalent:

- e is a closed pt.
- G is T_1.
- G is Hausdorff(T_2).
- G is regular.

Proof: □

Prop. (1.10.2). For a compact subset K and a nbhd U of e in a topological group, there exists a nbhd V of e that $xVx^{-1} \subset U$ for any $x \in K$.

Proof: For any x, there exists a nbhd W_x of x and a nbhd V_x of e that $txt^{-1} \in U$ for any $t \in W_x$ and $y \in V_x$. Let f.m. W_{x_i} cover K, then $V = \cap V_{x_i}$ satisfies the condition. □

Prop. (1.10.3). A compact topological group is totally disconnected iff the intersection of all compact open nbhds of e equals $\{e\}$.

Proof: If it is totally disconnected, then $\{1\}$ is closed, so G is Hausdorff(1.10.1), so by(1.1.3), the assertion is true. Conversely, if the intersection of all compact open nbhds of e equals $\{e\}$, then $\{1\}$ is closed because G is a group. □

Prop. (1.10.4). A compact open nbhd of e in a Hausdorff topological group contains an open subgroup of G.

Proof: Cf.[Etale Cohomology Fulei P147] □

Prop. (1.10.5). A precompact nbhd of a e in a totally disconnected topological group contains a compact open subgroup.

Proof: Cf.[Etale Cohomology Fulei P147]. □

11 Hausdorff Geometry

Def. (1.11.1). The **Hausdorff distance** for two subset $Y_1, Y_2 \in X$ is the

$$d_X^H(Y_1, Y_2) = \inf\{\varepsilon | Y_2 \subset B(Y_1, \varepsilon), Y_1 \subset B(Y_2, \varepsilon)\}.$$

The **Gromov-Hausdorff metric** for two metric space is

$$d^{GH}(X_1, X_2) = \inf\{d_Z^H(i_1(X_1), i_2(X_2))\}$$

where i_1, i_2 are isometry of X_1, X_2 into a metric space Z.

This metric makes the set of all compact metric space into a complete Hausdorff space $\mathcal{MET}$.

Def. (1.11.2). A map from X to Y is called a ε **approximation** iff $B(f(X), \varepsilon) = Y$ and $|d(x, y) - d(f(x), f(y))| \leq \varepsilon$.

We have: if there is a ε approximation, then $d^{GH}(X, Y) \leq 3\varepsilon$, and if $d^{GH}(X, Y) \leq \varepsilon$, there is a 3ε approximation.

Prop. (1.11.3). The set of isometries of

12 Spaces from Algebraic Geometry

Noetherian Space

Prop. (1.12.1). A Noetherian space is quasi-compact and all subsets of it in the induced topology is Noetherian hence quasi-compact.

Proof: Let $T \subset X$, for a chain of closed subsets $Z_i \cap T$ of T, $Z_1, Z_1 \cap Z_2, \ldots$ stabilize in X, hence the chain stabilize in T. □

Prop. (1.12.2). A Noetherian space has only f.m. irreducible component, hence it has only f.m. connected components.

Proof: Let $\mathcal{C}$ be the family of closed subset that has infinitely many component, then there is a minimal object, but it is not irreducible, one of the component has infinitely many components and be smaller. □

Specialization & Generalization

Constructible Set

Def. (1.12.3). A set of X is called **retrocompact** if the inclusion map is quasi-compact.

Def. (1.12.4). A set of X is called **constructible** if it is a finite union of sets of the form $U \cap V^c$ where U, V are open and retrocompact in X. In the case when X is Noetherian, by(1.12.1), all subsets are retrocompact hence constructible sets are just union of locally closed subsets of X.

A set of X is called **locally constructible** if it is locally constructible.

Prop. (1.12.5). A locally constructible set is constructible on every quasi-compact subset.

Irreducible

Def. (1.12.6). A space is irreducible iff there are no two nonempty nonintersecting open subsets. Thus an open subset of an irreducible set is dense and irreducible.

Prop. (1.12.7). If Y is irreducible in X, then $\overline{Y}$ is also irreducible.

Proof: Any two nonempty open sets of $\overline{Y}$ must intersect Y thus must intersect. □

Jacobson Space

Def. (1.12.8). Let X be a space and X_0 the set of closed pts of X, then X is called **Jacobson** iff $\overline{Z \cap X_0} = Z$ for every closed subset Z of X. This is equivalent to every non-empty locally closed subset of X contains a closed pt.

Thus there is a correspondence between closed subsets of X_0 and closed subsets of X, so they have the same Krull dimension.

Prop. (1.12.9). Being Jacobson is local. And for an open covering, we have $X_0 = \cup U_{i,0}$.

Proof: Cf.[StackProject 005W]. □

Cor. (1.12.10). If X is Jacobson, then any locally constructible sets of X is Jacobson. And its closed pts are closed in X.

Proof: By the proposition, we only have to prove for constructible sets. For $T = \cup T_i$ where T_i is locally closed, then a locally closed set in T has a non-empty intersection $T \cap T_i$ which is also locally closed for some i.

Hence is has a closed pt in X hence in T, so T is Jacobson. The second assertion is implicit in the proof. □

Prop. (1.12.11). If X is Jacobson, then an open set U of X is compact iff $U \cap X_0$ is compact, hence an open set U is retrocompact iff $U \cap X_0$ is retrocompact.

Hence the constructible sets of X correspond to the constructible sets of X_0.

And Irreducible closed subsets correspond to irreducible subsets of X_0

Krull Dimension

Def. (1.12.12). The **Krull dimension** of a topological space is the length of the longest chain of closed irreducible subsets.

The **local dimension** $\dim_x(X) = \min\{\dim U | x \in U \subset X \text{ open in X}\}$.

Prop. (1.12.13). If $Y \subset X$, then $\dim Y \leq \dim X$, because the closure of any chain of Y is a chain of X by(1.12.7).

For an open covering of X, $\dim X = \sup \dim U_i$, because for any chain of closed irreducible subsets, if U_i contains the minimal one, then $\dim U_i =$ length of this chain.

Prop. (1.12.14). $\dim X = \sup \dim_x(X)$.

Proof: The right is smaller than the left, and for any chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of irreducible closed subset of X, if I choose a point $x \in Z_0$, then $\dim_x(X) \geq n$. □

Prop. (1.12.15). In case $X = \operatorname{Spec} A$ for a Noetherian ring A, $\dim X = \sup \dim A_p$, because A is of finite

Catenary space

Def. (1.12.16). A space X is called **catenary** iff for any inclusion of irreducible closed subsets of X, their codimension is finite and every maximal chain of irreducible closed subsets has the same dimension. This is equivalent to $\operatorname{codim}(X,Y)+\operatorname{codim}(Y,Z)=\operatorname{codim}(X,Z)$.

Prop. (1.12.17). Catenary is a local property. Cf.[StackProject 02I2].

Sober Space

Def. (1.12.18). A space X is called sober if every irreducible closed subset has a unique generic point.

Prop. (1.12.19). The underlying space of a scheme is sober.

Proof: First prove this for affine scheme, notice that closed irreducible subsets correspond to prime ideal. Then notice the generic point for $Z\cap U$ is the generic point for Z. □

Prop. (1.12.20) (Soberization). There is a left adjoint t to the forgetful functor from the Sober spaces. $t(X)$ consists of irreducible closed subsets of X, and use $t(Y)$ for Y closed as closed subsets. for a map $f:X\to Z$ to a sober space Z, the extension maps the generic point of an irreducible Y to the generic point of the closure of $f(Y)$.

Def. (1.12.21). A Noetherian Sober space is called a Zariski space.

Dimension Function

The dimension function is usually considered when the space is sober.

Def. (1.12.22). On a topological space, we consider the specialization relation, a **dimension function** δ on X is one that if y is a specialization of x, then $\delta(y)<\delta(x)$, and if it is a direct specialization, then $\delta(y)=\delta(x)-1$.

13 Spectral Space

References are [StackProject 5.23] and [Adic Spaces].

Prop. (1.13.1). A space is called **spectral** iff it is quasi-compact, sober and the intersection of two affine open is affine open, and the affine opens form a basis for the topology. A morphism between two spectral spaces is called **spectral** iff it is quasi-compact.

A spectral space is exactly the underlying space of spectrum of a ring.

Def. (1.13.2). The constructible topology on a spectral space X is generated by the U,U^c, where U is a quasi-compact open. It is the coarsest topology that every constructible open are both open and closed.

Prop. (1.13.3). A set closed in the constructible topology in a spectral space stable under specilization is closed.

Proof: Cf.[StackProject 0903]. □

III.2 Riemannian Geometry

Basic references are [Riemannian Geometry Do Carmo], [Geometric Analysis Jost] and [Differentia Geometry Loring Tu].

1 $\mathbb{R}^3$-Geometry

Different Coordinates

Prop. (2.1.1). In a polar coordinate,

$$g_{11} = 1, g_{12} = 0, g_{22} = |\frac{\partial f}{\partial \theta}|^2, \quad K = -\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}}$$

And $\sqrt{g_{22}} \sim \rho$. (Use the formula relating Jacobi Field with curvature)

Moving Frame Method

Prop. (2.1.2) (Theorema Egregium).

$$R_{1212} = K(g_{11}g_{22} - g_{12}^2)$$

Which is a special case of the definition of curvature.

Prop. (2.1.3) (Gauss-Bonnet). Let M be a compact oriented 2-dimensional Riemannian manifold, then

$$\chi(M) = \frac{1}{2\pi}\int_M K \text{ Vol}.$$

Proof: Should be an direct corollary of(3.8.6). □

Topology and Geometry

Prop. (2.1.4). Every compact orientable surface of genus $p > 1$ can be provided with a metric of constant negative curvature.

Remark (2.1.5) (Hilbert Theorem). There exist complete surfaces with $K \leq 0$ in $\mathbb{R}^3$, but the hyperbolic surface cannot be immersed into $\mathbb{R}^3$.

2 Basics

Prop. (2.2.1). If the metric tensor on the tangent space is g in a coordinate, then it is g^{-1} in the cotangent space. (Follows from**??**).

3 Connections

Def. (2.3.1). An affine connection on a vector bundle E is a map $D : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M)$ that satisfies differential-like properties, it can be written as $D = d + \omega$, with $\omega \in \Omega^1(\text{End}(E))$.

Prop. (2.3.2) (Transformation Law). In two coordinates $\overline{e} = ea$ for $a : U \to GL(r, \mathbb{R})$, $d_A = d + \omega, d + \overline{\omega}$, then $\overline{\omega} = a^{-1}\omega a + a^{-1}da$.

Moreover, giving any locally compatible $d + \omega$,$\omega \in \Omega^1(\mathfrak{g})$ in the sense above, then for any G-associated bundle E, where G has lie algebra $\mathfrak{g}$, there is a connection that locally looks like $d + \omega$, (where $\mathfrak{g}$ embeds into $\mathfrak{gl}(E)$).

Cor. (2.3.3) (Local Nature of Connection). From the description of connection given above, it's easy to say if the is a local connection that satisfies these transformation laws, then it generate a global connection. So by partition of unity(1.3.8), connection exists in any vector bundle over a manifold.

Cor. (2.3.4) (Simplification). $d_{gA}(s) = gd_A(g^{-1}(s))$, So for any connection d_A and any point x_0, there is a gauge transformation that makes $d_A = d$ at x_0.

Proof: Just need to have $s(x_0) = \mathrm{id}$, $ds(x_0) = -A(x_0)$. this is possible because $A \in \Omega^1(\mathrm{Ad}E)$ which is the fiber of the frame bundle, use exp. □

Prop. (2.3.5) (Induced connections). The connection action $d_A = d+\omega$ on a vector bundle E induces connection on many relevant bundles. the action on dual bundle is by

$$d_A(s^*) = ds^* - \omega^t(s^*) = ds^* - s^* \circ \omega.$$

And the connection on $\mathrm{End}\, E$ by

$$d_A(\alpha) = d\alpha + [\omega, \alpha] = [\nabla, \alpha]$$

And they act on $\Omega^*(E)$ by Leibniz rule thus the formula looks the same. (Note that the convention is section write on the left of the differential forms, so for example, $[\omega, \omega] = 2\omega \wedge \omega$).

Proof: Cf.[Jost P110]. □

Cor. (2.3.6). For a line bundle L, for a connection on it with curvature Ω, the induced on the dual line bundle L^* has connection $-\Omega$. (because $\Omega = d\omega$ and $\omega' = -\omega$).

Prop. (2.3.7) (Second Bianchi's Identity). A affine connection on E looks locally like $d_A = d + \omega$, where $\omega \in \Omega^1(\mathrm{End}\, E)$. And $F_A = d_A \circ d_A \in \Omega^2(\mathrm{End}(E))$ satisfies

$$d_A F_A = dF_A + [\omega, F_A] = 0.$$

Proof: Notice $dF_A = dd\omega + d(\omega \wedge \omega) = d\omega \wedge \omega - \omega \wedge d\omega$, and $\omega(d\omega + \omega \wedge \omega) - (d\omega + \omega \wedge \omega)\omega = \omega \wedge d\omega - d\omega \wedge \omega$. □

Def. (2.3.8). The **Christoffel symbol** of a connection is defined by the equation: $\nabla_{X_i} X_j = \sum_k \Gamma^k_{ij} X_k$.

The **geodesic equations** is $\dfrac{D}{dt}(\dfrac{d\gamma}{dt}) = \ddot{x}_k + \sum_{i,j} \Gamma^k_{ij}\dot{x}_i\dot{x}_j = 0 \quad \forall k$.

Def. (2.3.9). The **torsion tensor** of a connection ∇ on TM is defined as $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. The connection is called **torsion-free** if $T = 0$. This is equivalent to $\Gamma^k_{i,j} = \Gamma^k_{j,i}$.

A connection is called **metric** if it preserves metric. i.e. $\nabla g = 0$.

Proof: T is a tensor because it is skew=symmetric, and

$$T(fX,Y) = f\nabla_X Y - f\nabla_Y X - df(Y)X - (f[X,Y] - df(Y)X) = fT(X,Y),$$

where(3.2.5) is used. □

Prop. (2.3.10). If ∇ is torsion-free connection on TM, then its induced connection on T^*M satisfies

$$(d\alpha)(v_1,\ldots,v_k) = \sum(-1)^i (D_{v_i}\alpha)(v_1,\ldots,\hat{v}_i,\ldots,v_k).$$

Proof: □

Def. (2.3.11). The **curvature** of a (affine) connection d_A is $F_A = d_A \circ d_A \in \Omega^2(\mathrm{End}(E))$. It induces a curvature tensor

$$F_A(Z)(X,Y) = R(X,Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]}Z.$$

Locally $F_A = d\omega + \omega\wedge\omega$, and in two coordinates $\overline{e} = ea$ for $a : U \to GL(r,\mathbb{R})$,$\overline{F_A} = a^{-1}F_A a$. The connection is called **flat** if $F_A = 0$.

Proof: To verify the equation, check first the left side is pointwise, and the third component of the right side assures it is pointwise, too, thus we can check for a local coordinate vector field, then because $\nabla s = \sum_i \nabla_i s dx_i$,

$$\nabla^2 s = \nabla(\sum_i \nabla_i s dx_i) = \sum_{ij} \nabla_j\nabla_i s dx_j dx_i = \sum_{i<j}(\nabla_i\nabla_j - \nabla_j\nabla_i) s dx_i \wedge dx_j$$

□

Prop. (2.3.12) (Flat coordinate). A connection on TM assumes near every point a flat coordinate, i.e. $\nabla(\partial/\partial x^i) = 0$, iff it is flat and torsion-free.

Proof: One side is easy because its Christoffels vanish. On the other side, use integrability theorems (6.6.2). Cf.[Jost P115]. □

Prop. (2.3.13).

$$\Delta\langle\varphi,\varphi\rangle = 2(\langle D^*D\varphi,\varphi\rangle - \langle D\varphi, D\varphi\rangle).$$

Proof: Cf.[Jost P118]. □

Prop. (2.3.14). For a flat connection, there is a bundle isomorphism (Gauge transform) that transforms d_A into natural d.

Proof: Because $d_{gA}(s) = gd_A(g^{-1}(s))$,$d_{gA} = d - dg\cdot g^{-1} + g\cdot\omega\cdot g^{-1}$. Solve this PDE direvtly. (Cf.[Topics in Geometry Xie Yi week3]). □

Cor. (2.3.15). For a flat connection, by(2.3.14), the parallel transportation only depends on the homotopy type of the loop, thus gives an action of $\pi(X)$ on $SO(T_p(X))$ (or $SU(T_p(X))$. (because it is locally constant).

In this way, connections module gauge equivalence (preserving matric) equals representation of $\pi(X)$ module conjugations. The reverse map is giving by principal bundle.

Proof: □

Levi-Civita Connection

Def. (2.3.16) (Levi-Civita Connection). The Levi-Civita connection is the unique connection on M that is metric and torsion-free:

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \qquad \nabla_X Y - \nabla_Y X = [X,Y].$$

It satisfies:

$$\langle Z, \nabla_Y X\rangle = 1/2\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle\}.$$

Then

$$\Gamma_{ij}^m = 1/2\sum_k \{g_{jk,i} + g_{ki,j} - g_{ij,k}\}g^{km}$$

Thus geodesic is a solution that only depends on the metric (2.3.8), so a local isometry preserves geodesics.

Prop. (2.3.17). Now the Lie derivative has the form:

$$L_X(S)(Y_1,\dots,Y_p) = \nabla_X(S)(Y_1,\dots,Y_p) + \sum_{i=1}^p S(Y_1,\dots,Y_{i-1},\nabla_{Y_i}X,\dots,Y_p).$$

The exterior derivative d and its adjoint d^* has the form:

$$d\omega(Y_i) = \sum(-1)^p \nabla_{Y_i}\omega(\check{Y_1}), \quad d^*\omega(Y_i) = -\sum \nabla_{e_j}\omega(e_j, Y_i)$$

where e_i is an orthonormal basis. Cf.[Jost P140].

- $$\frac{D}{\partial u}\frac{\partial s}{\partial v} = \frac{D}{\partial v}\frac{\partial s}{\partial u}.$$

Prop. (2.3.18) (Totally normal nbhd). For any point p, there exists a nbhd W and a number $\delta > 0$ s.t. for every $q \in W$, $\exp_q$ is a diffeomorphism on $B_\delta(0)$ and $\exp_q(B_\delta(0)) \supset W$. Thus, fine cover exists in every smooth manifold, because Riemannian metric exists on these manifolds.

Proof: □

- (**Geodesic Frame**) In a neighborhood of every point p, there exists n vector fields, orthonormal at each point, and $\nabla_{E_i}E_j(p) = 0$.(Choose normal nbhd and parallel a orthonormal basis to every point. (WARNING: this is not a flat coordinate, it only helps when dealing with point-wise properties).
- (Gauss Lemma) In a normal nbhd, the vectors orthogonal to geodesics is mapped under $(d\exp_p)_v$ to vectors orthogonal to geodesics.
- a locally minimizing piecewise differentiable curve is a geodesic.(Choose normal nbhd and use polar coordinate).

Def. (2.3.19). Killing field is which generates an infinitesimal isometry. X is killing $\iff L_X(g) = 0 \iff \langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ for all Y, Z (Killing equation).

A Killing field is a Jacobi field along geodesics. (by Calculation).

- The singularities of a Killing field is a submanifold and will generate a vector field along a geodesic sphere of the orthogonal component.
- gradient: $\langle \mathrm{grad} f(p), X\rangle = X(f)(p)$.
- divergence: $\mathrm{div}X(p)$ =trace of the linear map $Y(p) \to \nabla_Y X(p) = \sum_i \langle \nabla_{E_i} X, E_i\rangle$. It measures the variation of the volume and it depends only on the point.
- Hessian: $\mathrm{Hess}f$ is a self-adjoint operator that $(\mathrm{Hess}f)Y = \nabla_Y \mathrm{grad}f$ as well as a symmetric form $(\mathrm{Hess}f)(X,Y) = \langle (\mathrm{Hess}f)X, Y\rangle$.
- Laplace: $\Delta f = \mathrm{div}\ \mathrm{grad}f = \mathrm{trace}\ \mathrm{Hess}f = \sum_i E_i(E_i(f))$.
- in a geodesic frame,
$$\mathrm{grad}f(p) = \sum_{i=1}^n (E_i(f))E_i$$
$$\mathrm{div}X(p) = \sum_{i=1}^n E_i(f_i)(p), \mathrm{where} X = \sum_i f_i E_i.$$
$$\Delta f = \sum_i E_i(E_i(f))(p).$$
- $\Delta(f \cdot g) = f\Delta g + g\Delta f + 2\langle \mathrm{grad}f, \mathrm{grad}g\rangle$.
- $d(i(X)m) = (\mathrm{div}X)m$. where m is the volume form.

Cor. (2.3.20) (Hopf theorem). If f is a differentiable function on a compact orientable manifold with $\Delta f \geq 0$, then f is constant.

- The curvature tensor is determined by its sectional curvature, thus if M is isotropic at a point p (The sectional curvature depends only on the point), then $R(X,Y,W,Z) = K_0(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)$
- **Ricci curvature** $\mathrm{Ric}_p(x) = \frac{1}{n-1}\sum\langle R(x,z_i)x, z_i\rangle$, for x a unit vector, where z_i is an orthonormal basis orthogonal to x. $\mathrm{Ric}(x) = \mathrm{Ric}(x,x)$, where $\mathrm{Ric}(x,y)$ is the symmetric form of $\frac{1}{n}$ of trace of the map $z \to R(x,z)y$.
- **scalar curvature** $K(p) = 1/n\sum \mathrm{Ric}_p(z_i)$, where z_i is an orthonormal basis.
- $$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})V. \quad \text{(obvious because } \frac{\partial}{\partial_s}, \frac{\partial}{\partial_t} \text{ commutes)}$$
- sectional curvature $K(X,Y) = \langle R(X,Y)X, Y\rangle$.
- curvature tensor only depends on the point and
$$R(X,Y,Z,W) = R(Z,W,X,Y), \quad R(X,Y,Z,W) = R(X,Y,W,Z).$$

Prop. (2.3.21) (Bianchi Identities). The covariant differential $\nabla R(Y_i, Z) = Z(R(Y_i)) - \sum_i R(\nabla_Z Y_i, Y_j)$.
(Bianchi Identity)$\sum_{(X,Y,Z)} R(X,Y)Z = 0$.
(Second Bianchi Identity)$\sum_{(Z,W,T)} \nabla R(X,Y,Z,W,T) = 0$.

Cor. (2.3.22) (Schur's Theorem). Let M be a manifold of dimension $n \geq 3$, suppose M is isotropic , then M has constant curvature. (Use the second Bianchi Identity and geodesic frame).

- $B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$. It is bilinear and symmetric.
- $H_\eta(x,y) = \langle B(x,y), \eta\rangle$. Thus $B(x,y) = \sum H_i(x,y)E_i$ for an orthonormal frame E_i in $\mathfrak{X}(U)^\perp$.
- $S_\eta(x) = -(\overline{\nabla}_x \eta)^T$. It satisfies:$\langle S_\eta(x), y\rangle = H_\eta(x,y) = \langle B(x,y), \eta\rangle$. It is self-adjoint. When codimension 1, it is the derivative of the Gauss mapping.
- (**Gauss Formula**): let x, y be orthonormal tangent vector. Then:

$$K(x,y) - \overline{K}(x,y) = \langle B(x,x), B(y,y)\rangle - |B(x,y)|^2.$$

- An immersion is called **geodesic** at p if the second fundamental form S_η is zero for all η, (which means $\nabla_X Y$ has no normal component). It is called **minimal** if the trace of S_η is zero.
- An immersion is called umbilic if there exists a normal unit field η s.t. $\langle B(X,Y), \eta\rangle(p) = \lambda(p)\langle X, Y\rangle$.
- If the ambient space has constant sectional curvature and the immersed manifold is totally umbilic, then λ is constant.
- mean curvature tensor of immersion $f = 1/n \sum_i (\mathrm{tr}\ S_i)E_i = 1/n\ \mathrm{tr}\ B$. It is zero if f is minimal.
- normal connection $\nabla^\perp_X \eta = (\overline{\nabla}_X \eta)^N = \overline{\nabla}_X \eta + S_\eta(X)$.
- (Gauss equation)

$$\langle \overline{R}(X,Y)Z, T\rangle = \langle R(X,Y)Z, T\rangle - \langle B(Y,T), B(X,Z)\rangle + \langle B(X,T), B(Y,Z)\rangle.$$

- (Ricci equation)

$$\langle \overline{R}(X,Y)\eta, \zeta\rangle - \langle R^\perp(X,Y)\eta, \zeta\rangle = \langle [S_\eta, S_\zeta]X, Y\rangle.$$

- (Codazzo equation)

$$\langle \overline{R}(X,Y)Z, \eta\rangle = (\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta). \qquad \text{(Lie bracket)}$$

Parallel Transportation

Def. (2.3.23) (Parallel Transportation).

Def. (2.3.24). The **holonomy group** $Hol_x(g)$ of a Riemannian manifold M w.r.t to the Levi-Civita connection is defined to be the subgroup of $O(T_x(M))$ induced by the parallel transportation along a loop. If M is connected, For different points, holonomy groups are conjugate, so holonomy group is defined up to conjugation.

Prop. (2.3.25) (Berger). in fact, the groups that can be realized as a holonomy group of a simply connected complete Riemannian manifold can be classified, Cf.[Complex geometry Daniel P214].

Def. (2.3.26). The **Geodesic flow** for a connection on TM is the flow on TM whose trajectories are $t \mapsto (\gamma(t), \gamma'(t))$, where γ is a geodesic on M.

Prop. (2.3.27) (The smoothness of geodesics). For every point p, there exists a nbhd V and a C^∞ mapping

$$\gamma : (-\delta, \delta) \times V \times B(0,\epsilon) \to M,$$

s.t. $\gamma(t,q,v)$ is the geodesic passing through p with velocity v.

Complete manifold

Prop. (2.3.28) (Hopf-Rinow theorem). The following is equivalent definition of **completeness**.

1. $\exp_p$ is defined for all of $T_p(M)$.
2. The closed and bounded sets of M are compact.
3. M is complete as a metric space.
4. M is $\sigma-$compact and if $q_n \notin K_n$, $d(p, q_n) \to \infty$.
5. The length of any divergent (compact escaping) curve is unbounded.

and if M is complete, then for any q, there exists a minimizing geodesic. And any compact submanifold of a complete manifold is complete.

- For any two manifold of the same constant curvature and any two orthogonal basis, there is a local isometry (It is locally isotropic).
- Any complete manifold with a sectional curvature is like $\tilde{M}/\Gamma$, where $\tilde{M}$ is $\mathbf{H}^n, \mathbf{R}^n$ or $\mathbf{S}^n$.

Prop. (2.3.29) (Cartan). in any nontrivial homotopy class in a compact manifold , there exists a closed geodesic.

4 Jacobi Field and Comparison Theorems

- Jacobi field equation along a geodesic γ: $D^2 J(t) + R(\dot{\gamma(t)}, J(t))\dot{\gamma}(t) = 0$. It is defined by its initial condition $J(0)$ and $J^{'}(0)$. It can be used to detect the sectional curvature、the critical point of $\exp_p$ and calculate variation of energy.
- The Jacobi field along a point with initial velocity 0 all has the form $J(t) = (d\exp_p)_{t\dot{\gamma}(0)}(tJ^{'}(0))$. Corollary: the Jacobi transport from p to q is an isomorphism iff p and q is not conjugate.
- for general Jacobi field,

$$\langle J(t), \dot{\gamma}(t)\rangle = \langle J^{'}(0), \dot{\gamma}(0)\rangle t + \langle J(0), \dot{\gamma}(0)\rangle.$$

- If J is a Jacobi field $J(t) = (d\exp_p)_{tv}(tw), |v| = |w| = 1$, then

$$|J(t)| = t - \frac{1}{6}K_p(v, w)t^3 + o(t^3).$$

- Energy

$$E(s) = \int_0^a |\frac{\partial f}{\partial t}(s, t)|^2 dt.$$

- A minimizing geodesic must minimize energy.
- (**First Variation of Energy**)

$$1/2E^{'}(0) = -\int_0^a \langle V(t), D\dot{c}(t)\rangle dt + \langle V(a), \dot{c}(a)\rangle - \langle V(0), \dot{c}(0)\rangle.$$

A piecewise differentiable curve is a geodesic iff every proper variation has first derivative 0.

- (**Second Variation of Energy**) If γ is a geodesic,

$$1/2E^{''}(0) = \int_0^a \{\langle DV(t), DV(t)\rangle - \langle R(\dot\gamma, V)\dot\gamma, V\rangle\}dt + \langle D_s V(a), \dot\gamma(a)\rangle - \langle D_s V(0), \dot\gamma(0)\rangle.$$

- a variation is equivalent to a vector field along the curve, and a variation that $f_s(t)$ are all piecewise geodesics corresponds to a piecewise Jacobi field(Choose a normal partition).

Prop. (2.4.1) (Rauch Comparison theorem). Let M and $\tilde{M}$ be manifolds, dim$\tilde{M} \geq$ dim M. If J and $\tilde{J}$ be two normal Jacobi fields along geodesics γ and $\tilde\gamma$ that $|J(0)| = |J'(0)| = 0$ and $|J^{'}(0)| = |\tilde{J}^{'}(0)|$. If $\tilde\gamma$ has no conjugate point or focal point free and $\tilde{K}(\tilde{x}, \dot{\tilde\gamma}(t)) \geq K(x, \dot\gamma)$ for any vector $x, \tilde{x}$, then $|\tilde{J}| \leq |J|$.

Cor. (2.4.2) (Injectivity Radius Estimate). If the sectional curvature of M satisfies: $0 < L \leq K \leq H$, then the distance between any two conjugate points satisfies: $\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}$.

Prop. (2.4.3). If two manifold M and $M^{'}$ satisfy $K \leq K^{'}$, then in a normal nbhd of a point p in M and a nbhd of $p^{'}$ that exp is nonsingular, the transformation of a curve c shortens length.

Note that this is not Toponogov thoerem, because if you try to map from a large curvature manifold to a small curvature, then you cannot guarantee that the mapped curve is the shortest.

Cor. (2.4.4). In a complete simply connected manifold of non-positive curvature,

$$A^2 + B^2 - 2AB\cos\gamma \leq C^2$$

thus $\alpha + \beta + \gamma \leq \pi$.

Prop. (2.4.5) (Moore theorem). Let $\overline{M}$ be a complete simply connected manifold of sectinoal curvature $\overline{K} \leq -b \leq 0$, M a compact manifold of sectional curvature satisfying $K - \overline{K} \leq b$. If dim$\overline{M}$ < dim$M$, M cannot be immersed into $\overline{M}$.(use Hadamard theorem to choose the furthest geodesic and calculate the second variation of energy and use Gauss formula).

Cor. (2.4.6). Let $\overline{M}$ be a complete simply connected manifold of sectinoal curvature $\overline{K} \leq 0$, M a compact manifold of sectional curvature satisfying $K \leq \overline{K}$. If dim$\overline{M}$ < dim$M$, M cannot immerse into $\overline{M}$.

Lemma (2.4.7) (Klingenberg). (P236)Let M be a complete manifold of sectional curvature $K \geq K_0$, let γ_0, γ_1 be two homotopic geodesics from p to q, then there exists a middle curve γ_s s.t.

$$l(\gamma_0) + l(\gamma_1) \geq \frac{2\pi}{\sqrt{K_0}}.$$

Prop. (2.4.8) (Klingenberg). Let M be a simply connected compact manifold of dimension $n \geq 3$ such that $\frac{1}{4} < K \leq 1$, then $i(M)$(The infimum of distance to the cut locus)$\geq \pi$.

Cor. (2.4.9). If M is a compact orientable manifold of even dimension satisfying $0 < K \leq 1$, then $i(M) \geq \pi$.

Prop. (2.4.10) (1/4-pinch Sphere Theorem). Let M be a compact simply connected manifold satisfying $0 < 1/4K_{\max} < K \leq K_{\max}$, then M is homeomorphic to a sphere.

(Use Klingenberg Theorem, this is a special case of diameter geodesic sphere theorem). Cf.(2.4.20).

It can be shown that in this case, this sphere is even diffeomorphic to S^n using Ricci flow.

Remark (2.4.11). $0 < 1/4K_{\max} < K$ cannot be changed to $\geq$. In fact, the Funibi-Study metric on CP^n has sectional curvature $1 \geq K \geq 4$. Cf. **??**

Hess$\rho(X,Y)$ where ρ is the distance to a fixed point, is important.

Prop. (2.4.12). Hess$\rho(X,Y)$ is positive definite on the tangent space of the geodesic sphere within the injective radius, and its principal value is $|\frac{J'}{J}|$ for a Jacobi field in that direction. And it is zero on the normal direction.

So there would be a Riccati comparison theorem on the eigenvalue of $\Pi_2 : \lambda' \leq -K - \lambda^2$,Hess$(\rho)$ is bounded.

Proof: Notice that

$$\mathrm{Hess}\rho(X,Y) = (\nabla_X \mathrm{grad}\rho, Y) = XY\rho - (\nabla_X Y)\rho$$

so if choose a normal geodesic γ of initial vector X, then

$$\mathrm{Hess}\rho(X,X) = X\langle\dot\gamma, d\rho\rangle - (\nabla_X\dot\gamma)\rho = X\langle\dot\gamma, d\rho\rangle = \langle\dot\gamma, d\langle\dot\gamma, d\rho\rangle\rangle = E^{''}(0)$$

$$= I_q(X,X) = ((\nabla_{\dot\gamma}X)(q), X(q)) = \frac{\langle J', J\rangle}{|J|^2}$$

□

Prop. (2.4.13) (Toponogov). Let$M$ be a complete manifold with $K \geq H$.

If a hinge satisfies γ_1 is minimal and $\gamma_2 \geq \frac{\pi}{\sqrt{H}}$ if $H > 0.$, then on M^H the same hinge has smaller distance of endpoints than this hinge

Proof: Cf.[Cheeger Comparison Theorems in Riemannian Geometry P42]. And there is another triangle version: For a minimal geodesic triangle, the comparison triangle has smaller angles. NOTE this theorem cannot be derived from Rauch Comparison Theorem. □

Critical Point for Distance Function

Prop. (2.4.14). The critical point for distance function on a complete manifold is that for every direction v, there is a minimal geodesic γ s.t. $\langle\gamma'(l), v\rangle \leq \frac{\pi}{2}$.

The set of regular point is open and there exists a smooth gradient like vector field (i.e. acute angle with every minimal geodesic) on this open subset .

Prop. (2.4.15) (Berger's Lemma). A maximal point for the distance function is a critical point.

Proof: If not, choose a convergent point v of the minimal geodesics with endpoint in a curve of that direction, then exp near v will generate a Jacobi field with endpoint Jacobi is the sam of that direction. So the distance will increase by $\cos\theta$ along that direction, contradiction. □

Prop. (2.4.16) (Soul Lemma). Let M is a Riemannian manifold and A is a closed submanifold. If $dist(A,-)$ has no critical point on $D(A,R)\backslash A$, then $B(A,R)$ is diffeomorphic to the normal bundle of $A \to M$.

Proof: A has a normal exp radius ϵ, and we can vary the gradient-like vector field to be identical to the normal vector near A, and use Morse lemma (the flow) to get a diffeomorphism. □

Cor. (2.4.17) (Disk Theorem). If A is a point then M is diffeomorphic to a disk.

Lemma (2.4.18) (Generalized Schoenflies Theorem). Easy to do, just use the fact that exp is continuous to find a boundary sphere depending continuously on the direction (both p and q).

Prop. (2.4.19) (Sphere Theorem). If M is a closed manifold and has a distance function with only one critical point (the furthest one), then M is homeomorphic to a twisted ball.

Proof: There exists a ϵ and r that $B(q,\epsilon)$ and $B(p,r)$ covering M, (Use the convergent point argument). Then use the generalized Schoenflies theorem. □

Prop. (2.4.20) (Diameter Sphere Theorem). If a closed manifold M satisfies $\sec M \geq K > 0$, and $\mathrm{diam}(M) > \frac{\pi}{2\sqrt{K}}$, then M is homeomorphic to S^n.

Proof: First, if there are two maximal distance point, then use Toponogov to show contradiction. Second, at other points x,

$$\angle pxq > \frac{\pi}{2}$$

(Regular domain) because of Toponogov and The formula

$$\cos\tilde{\alpha} = \frac{\cos l - \cos l_1 \cos l_2}{\sin l_1 \sin l_2}.$$

So the geodesic direction $\overrightarrow{xq}$ will serve as a geodesic-like vector field (might need paracompactness). □

Prop. (2.4.21) (Critical Principle). In a complete manifold M of sectional curvature $> K$, if q is a critical point of p, then for any point x with $d(p,x) > d(p,q)$ and any minimal geodesic from p to x, the $\angle xpq$ is smaller than the $\cosh_K^{-1}(\frac{d(p,x)}{d(p,q)})$.

Proof: Use Toponogov for the hinge xpq. Then notice that there is a different minimal geodesic from $p \to q$ that makes the $\angle pqx < \pi/2$ by the definition of critical point, thus there is another Toponogov inequality, this two inequality contradicts. □

Cor. (2.4.22). For a complete open manifold whose K are lower bounded, then it is homeomorphic to the interior of a manifold with boundary. (Use Soul lemma, otherwise there will be a sequence of critical point whose angles are big).

Prop. (2.4.23). ray construction and Line construction**?**

Prop. (2.4.24) (Soul Theorem). If M is an open manifold with $K \geq 0$, then there is a totally geodesic submanifold S that M is diffeomorphic to the normal bundle over S.

Proof: Use the ray construction to get a totally convex compact subset, hence it is a manifold or with boundary , if it has boundary, then find to set of maximal distance to the distance to boundary, the distance to the boundary is a convex function, so it is a smaller totally geodesic manifold. So a S without boundary must exist and this constitutes a stratification, all the level set is strongly convex. Thus all point outside S is not critical, hence the soul lemma applies. Cf.[GeJian Comparison theorems in Riemannian Geometry Lecture7]. □

Prop. (2.4.25) (Perelman). There is a distance non-increasing contraction unto the soul, and it must be just the projection along the normal bundle. Moreover, for any geodesic on the soul and a parallel vector field in the normal bundle along it, it spans a flat surface (by Rauch comparison).

Cor. (2.4.26) (Soul Conjecture). For an open(non-compact) complete manifold M with $K \geq 0$, if it has a point p s.t. sectional curvature at p are all positive, then M is diffeomorphic to R^n. (It's enough to show that its soul is a point, otherwise for any point, it must has a direction that is flat, $K = 0$).

5 Curvature and Topology

<u>**Sectional Curvature**</u>

Prop. (2.5.1) (Hadamard theorem). M a complete simply connected Riemann manifold of sectional curvature ≤ 0, then $\exp_p : T_pM \to M$ is an isomorphism of M to $\mathbb{R}^n$.(negative sectional curvature to show exp is a local isomorphism, complete to show it is a covering map)

Prop. (2.5.2) (Liouville Theorem). Any conformal mapping for an open subset of $\mathbb{R}^n, n > 2$ is restriction of a composition of isometry, dilations and/or inversions, at most once.

Prop. (2.5.3) (Synge). f is an isometry of a compact oriented manifold M^n of positive sectional curvature, f alter orientation by $(-1)^n$, then f has a fixed pt.

Cor. (2.5.4). M a compact manifold of positive sectional curvature, then

1. If M is orientable and n is even, then M is simply connected. So If M is compact and even dimension, then $\pi(M) = 1$ or $\mathbb{Z}_2$.
2. If n is odd, then M is orientable.

(Use the universal cover and covering transformation.)

Remark (2.5.5) (Hopf Conjecture). If M is a compact Riemannian manifold of even dimension that $K > 0$, then it has positive Euler characteristic.

<u>**Morse Index**</u>

Prop. (2.5.6) (Index Lemma). Among the piecewise differentiable vector fields along a geodesic without conjugate point or without focal point, with initial value 0 and fixed end value, the Jacobi field attain minimum of the index form:

$$I_a(V, V) = \int_0^a \{\langle DV(t), DV(t)\rangle - \langle R(\dot{\gamma}, V)\dot{\gamma}, V\rangle\}dt.$$

Cor. (2.5.7). $I_l(J, J) = \langle J, J'\rangle(l)$ for a Jacobi field.

Prop. (2.5.8). a focal point is a critical value of $exp^\perp$. For an embedded manifold, the focal point equals $x + 1/t\eta$, where η is a vertical vector and t is a principal value of S_eta.

Prop. (2.5.9) (Morse Index theorem). The index of the the index form $I_a(V, W)$ on the space of vector fields 0 at the endpoints, equal to the number of points conjugate to $\gamma(0)$ in $[0, a)$.

Cor. (2.5.10). If γ is minimizing, γ has no conjugate points on $(0, a)$, γ has a conjugate point, it is not minimizing.

Prop. (2.5.11) (Morse). If M is complete with non-negative sectional curvature, then $\pi_1(M)$ have no finite non-trivial cyclic group and $\pi_k(M) = 0$.

Proof: because universal cover of M is contractible, so the higher homotopy group vanish and $H^k(M) = H^k(\pi_1(M))$, so if a subgroup is finite cyclic, its homology is periodic, contradiction. □

Prop. (2.5.12) (Preissman). For a compact manifold with $K < 0$, any nontrivial abelian subgroup of π_1 is infinite cyclic.

Prop. (2.5.13). If M is compact and $K < 0$, $\pi_1(M)$ is not abelian.

Assuming M complete,

- The cut point of p along γ is the maximum $\gamma(t)$ s.t. $d(p, \gamma(t)) = t$. It is either the first conjugate point of p or the intersection of two minimizing geodesics.
- Conversely, if a point is a conjugate point of p or is intersection of two geodesics of equal length, then there is a cut point before it. So, if intersection of two minimizing geodesics happens, it must happen before the occurrence of conjugate point.
- thus the cut point relation is reflexive, and if $q \in M \setminus C_m(p)$, then there exists a unique minimizing geodesic joining p and q.
- $M \setminus C_m(p)$ is homeomorphic to an open ball through exp.
- the distance of p to the cut locus is continuous, thus $C_m(p)$ is closed.
- If M is complete and there is a p which has a cut point for every geodesic, then M is compact.
- for q the closest of $C_m(p)$ to p, either there exists a minimizing geodesic and q is conjugate to p or there is to minimizing geodesic connecting at q.

Prop. (2.5.14). The index of a geodesic will decrease when transferred to a manifold of smaller sectional curvature K.

Prop. (2.5.15). In a complete manifold, if there is a sequence of points $\{p_i\}$ converging to a point p, choose for each point a minimal geodesic, then a subsequence of them will converge to a minimal geodesic to p.

Proof: The convergence is by smoothness and of exp and Hadamard. The minimality is by comparing distance. □

Ricci Curvature

Prop. (2.5.16) (Ricci Comparison). Volume comparison, Laplacian Comparison, Mean Curvature comparison. Cf.[葛健 Week13].

Prop. (2.5.17) (Bishop-Gromov). Let M be an open manifold with $\mathrm{Ric} \geq H$, let $\tilde{M}(H)$ be a complete simply connected manifold of constant sectional curvature H, then

$$\mathrm{Vol}(B_r(x)) \leq \mathrm{Vol}(B_r(\tilde{p})), \quad \frac{\mathrm{Vol}(B_R(x))}{\mathrm{Vol}(B_r(x))} \leq \frac{\mathrm{Vol}(B_R(\tilde{p}))}{\mathrm{Vol}(B_r(\tilde{p}))}.$$

Cf.[葛健 Week13].

Prop. (2.5.18) (Bonnet-Myer). M a complete manifold of Ricci curvature $\mathrm{Ric}_p(v) \geq \frac{1}{r^2}$, Then M is compact and have diameter $\leq \pi r$.

And if the identity is achieved, $M \cong \mathbb{S}^n$.

Proof: Use Laplacian comparison $\Delta r \leq (n-1)\cot r$. Cf.[葛健 week13]. □

Cor. (2.5.19). M is a complete manifold of Ricci curvature $\geq \delta > 0$, then the universal cover is compact thus $\pi_1(M)$ is finite. This cam be seen as a obstruction for a compact manifold to have positive Ricci curvature.

Cor. (2.5.20) (Calabi-Yau). For an open manifold with non-negative Ricci curvature, for any point, $\mathrm{Vol}(B(p,r)) \geq c_p r$.

Prop. (2.5.21) (Milnor). Let M be an open manifold of non-negative Ricci curvature of dimension n, then any f.g. subgroup of $\pi_1(M)$ has polynomial growth $\leq n$. Milnor conjectured that $\pi_1(M)$ in fact is f.g..

Prop. (2.5.22) (First Betti Number Theorem). There is a number $f(n,\lambda,D)$, $f(n,0,D) = n, f(n,\lambda,D) = 0$ for $\lambda > 0$ that for a manifold of diameter $\leq D$ and Ricci curvature $\geq \lambda$, $b_1(M) \leq f(n,\lambda,D)$.

Cor. (2.5.23) (Splitting Theorem). The universal cover of a compact RIemannian manifold with non-negative Ricci curvature splits isometrically as a product $\widetilde{M} = N \times \mathbb{R}^k$ where N is a compact manifold manifold.

Scalar Curvature

III.3 Geometric Analysis

1 Simplifications

Prop. (3.1.1). For every vector field $X$and every point $X(p) \neq 0$, there exists a coordinate nbhd $(x_1, \ldots, x_{n-1}, t)$ such that $X = \frac{\partial}{\partial t}$.

2 Differential Forms

Prop. (3.2.1) (Frobenius Theorem). If X is an involutive distribution on a manifold M, then there is a unique maximal integration manifold passing through it. Where a distribution is involutive if it is closed under Lie bracket.

Proof: The key to the proof is to prove that involutive is equivalent to integrable, i.e. flat locally as $\{\frac{\partial}{\partial x_i}\}$ for some local coordinate. Cf.[李群讲义 项武义 P226] □

Cor. (3.2.2). X, Y in a Lie algebra commute iff their corresponding vector fields commute.

Interior and Exterior Derivatives

Lie Derivatives

Def. (3.2.3). The **Lie bracket** of two vector fields X, Y is defined to be $[X, Y](f) = (XY - YX)f$, then if $X = \sum a_i \partial/\partial x_i$, $Y = \sum b_i \partial/\partial x_i$, then $[X, Y] = \sum (X(b_i) - Y(a_i))\partial/\partial x_i$.

Lemma (3.2.4). $[X, Y] = \frac{\partial}{\partial t}(d(\phi_{-t})Y)|_{t=0}$.

Proof: For any function f,set $g(t, q) = \frac{f(\phi_t(q)) - f(q)}{t}$,$g(0, q) = Xf(q)$. Then g is differentiable (because $g(t, q) = \int_0^1 Xf(\phi_{ts}(p))ds$, and:

$$\begin{aligned}
\lim_{t\to 0} d(\phi_{-t})Yf(p) &= \lim \frac{Yf(p) - Y(f\phi_{-t})(\phi_t(p))}{t} \\
&= \lim \frac{Yf(p) - Yf(\phi_t p) - Y(tg(-t, \phi_t(p))}{t} \\
&= ((XY - YX)f)(p) \\
&= [X, Y]f(p)
\end{aligned}$$

□

Prop. (3.2.5). $[fu, v] = f[u, v] - df(u)v$.

Prop. (3.2.6). Lie bracket commutes with derivative.$[df(X), df(Y)] = df([X, Y])$. (Use $XY - YX$ to see).

Prop. (3.2.7) (Lie formula).

$$L_X(g(Y, Z)) = L_X(g)(Y, Z) + g(L_X Y, Z) + g(Y, L_X Z).$$

Prop. (3.2.8) (Derivative formula).

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

.

Prop. (3.2.9) (Cartan's magic formula).

$$L_X\omega = \iota_X(d\omega) + d(\iota_X\omega)$$

$$\iota([X,Y]) = [L_X, \iota_Y]$$

Proof: Notice that four of them are derivatives (check because $\iota_X(w\wedge v) = \iota_X w\wedge v + (-1)^{|w|}w\wedge \iota_X v$). So by induction, we only has to verify them on dimension 0 and 1. □

Prop. (3.2.10) (Stoke's theorem).

$$\oint_\Omega d\omega = \oint_{\partial\Omega} i^*\omega.$$

In a 3-dimensional Riemanian manifold, If we set:

$$df = \omega^1_{\mathrm{grad}f},\quad d\omega^1_A = \omega^2_{\mathrm{curl}A},\quad d\omega^2_A = (\nabla A)\omega^3,$$

Then:

$$f(y) - f(x) = \int_l \mathrm{grad}f\cdot dl.$$

$$\int_l A\cdot dl = \oint_S \mathrm{curl}A\cdot dn.$$

$$\oint_U \nabla\cdot F dV = \oint_{\partial U} F\cdot n dS.$$

Hodge Star

Def. (3.2.11) (Hodge Star Operator). given a volume-form ω on a vector space, the Hodge star operator $*-$ is an operator from $\bigwedge^k V\to\bigwedge^{n-k}V$ such that:

$$\alpha\wedge(*\beta) = \langle\alpha,\beta\rangle\omega.$$

On a closed oriented Riemannian manifold, given a volume form ω, the star operator satisfies:

$$(\alpha,\beta) = \int_M\langle\alpha,\beta\rangle\omega = \int_M \alpha\wedge *\beta.$$

And $** = (-1)^{p(n-p)}$ on $\Omega^p M$.

Def. (3.2.12). For a operator d on $\Omega^* M$, we define the adjoint $d^* = (-1)^{n(p+1)+1} * d*$ on Ω^p, which satisfies the adjoint property by calculation:

$$(d^*\alpha,\beta) = (\alpha, d\beta).$$

The laplacian $\Delta = d^*d + dd^*$. It can be verified that Δ commutes with $*$ and d.

3 Transversality

Prop. (3.3.1) (Parametric Transversality Theorem). Suppose N and M are smooth manifolds, $X \subset M$ is an embedded submanifold, and F_s is a smooth family of maps from N to M. If the map $F : N \times S \to M$ is transverse to X, then for almost every s, the map $F_s : N \to M$ is transverse to X. Cf.[Smooth Manifold Lee T6.35].

Proof: □

Prop. (3.3.2) (Transversality Homotopy Theorem). Suppose N and M are smooth manifolds and $X \subset M$ is an embedded submanifold. Every smooth map $f : N \to M$ is homotopic to a smooth map $g : N \to M$ that is transverse to X. Cf.[Smooth Manifold Lee T6.36].

Proof: embed M into a R^k and take a tubular neighbourhood, then we can construct a $N \times S^k$ transversal to M. □

Cor. (3.3.3). For a vector bundle over a compact manifold, there exists a global section transversal to the zero section, in particular, if $\dim E > M$, then it has no zero.

Proof: choose a finite trivializing cover that there closure is compact and choose a compact subcover, find finitely many sections to assure $C^N \times X \to E$ is transversal, and use parametric trnasversality theorem to prove there is a section that is transversal. □

Cor. (3.3.4). There is a vector field on compact manifold of only isolated zeros. And a vector bundle over a k dimensional curve splits to components of dimension no bigger than k. Determined by its Chern class.

4 Flow

Prop. (3.4.1) (Isotopy Extension Theorem). Let M be a manifold and A be a compact subset. Then an isotopy $F : A \times I \to M$ can be extended to an diffeotopy of M.

Proof: Consider $F(A\times I) \subset M \times I$ is a compact set, and $TM \times I \to M \times I$ is a vector bundle. The time lines generate a section $F(A\times I) \to TM\times I$, so (5.1.2) guarantees an extension $M\times I \to TM\times I$, and because manifolds are locally compact, this section can be chosen to be compactly supported, then the flow it generates is a diffeotopy. □

5 Differential Topology

Prop. (3.5.1) (Sard Theorem). The set of critical values is of measure zero in the image manifold.

Prop. (3.5.2) (Hopf Index theorem). In a compact manifold, any vector field V with isolated zeros has sum of its index equal to $\chi(M)$. Where the index of a singularity is the mapping degree of V on a surrounding sphere.

6 Spin Structure

Prop. (3.6.1) (Spin Structure Obstruction). For a oriented real bundle, its transformation map can be chosen to be in $SO(n)$, and constitute a Cech Cohomology $H^1(X, SO(n))$, and by exact sequence of

$$0 \to \pm 1 \to \mathrm{Spin}(n) \to SO(n),$$

this can be lifted to a $H^1(X, \mathrm{Spin}(n))$ iff its image w in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ is 0. and then its inverse image will be parametrized by $H^1(X, \mathbb{Z}/2\mathbb{Z})$ (By the non-commutative spectral sequence of Čech).

We have $w = w_2$, the Whitney class, (Just need to reduce to sk_2X and in this case, check they both equivalent to the bundle can be lifted). Cf.[XieYi 几何学专题]. Or we can use the Postnikov system of $BO(n)$(4.5.2).

Proof: First prove that if $E \oplus R^n$ is spin, then E is spin, and then pull $H^2(X, \mathbb{Z}/2\mathbb{Z})$ into $H^2(\mathrm{sk}_2(X), \mathbb{Z}/2\mathbb{Z})$, this in a injection, and the homology is natural, so we only have to prove this for $\mathrm{sk}_2(X)$. But E on $\mathrm{sk}_2(X)$ can decompose into a E' of dimension on more than 2, and for this, we see E is Spin iff it is the square of another bundle, so w and w_2 are the same. □

Prop. (3.6.2). For a Spin bundle E, the Spin-principal bundle with the Spinor representation(10.1.2) will generate a bundle S called the **Spinor bundle**. And the Ad action of $\mathrm{Spin}(n)$ on $Cl_{n,0}$ will generate a **Clifford bundle** $Cl(E)$. The $\mathrm{Spin}(n)$ actions are compatible, so the Clifford bundle can act on the spinor bundle. bundle. The act of the chirality operator on the Spinor bundle will generate two half spinor bundles $S^{\pm}$. Then TM will maps $S^{\pm} \to S^{\mp}$ for n even,(because of anti-commutative with Γ).

Prop. (3.6.3) (Spin^c-structure). The group Spin^c is the covering space of $SO(n) \times S^1$ (n>2) that corresponds to the group of elements mod 0 mod 2 in $\mathbb{Z}_2 \times \mathbb{Z}$, i.e. $\mathrm{Spin}(n) \times S^1/\{\pm 1\}$.

For example, $\mathrm{Spin}^c(4) = \{(A_1, A_2) \in U(2) \times U(2) | \det A_1 = \det A_2\}$, and $\mathrm{Spin}^C(3) = U(2)$.

Then a $SO(n)$ bundle can be lift to be a Spin^c-bundle if the line bundle determined by S^1 is determine the same w_2 as it, i.e. $w_2 = c_1(L)$ mod 2, This is equivalent to w_2 is in the image of $H^2(X, \mathbb{Z})$, and this is equivalent to the Bockstein image of it is zero.

Use a variant of Wu's formula: $w_2(TM)[\alpha] = \alpha \cdot \alpha(\text{mod } 2)$ for M orientable of dimension 4, we have any orientable manifold of dimension 4 has a Spin^c-structure. Cf.[XieYi 几何学专题 Homework3].

There is a connection on the Clifford bundle and on the Spinor bundle induced by the Levi-Civita connection of M(2.3.2). This is compatible with the Clifford action. and it is also metric because the connection 1-form is in $\mathfrak{so}(n)$ because the action of $SO(n)$ preserves metric.

7 Young-Mills Euqation & Seiberg-Witten Equation

Def. (3.7.1) (Yong-Mills). The Young-Mills functional on connections A on a bundle E on a compact oriented space:

$$YM(A)^2 = ||F_A||^2 = -\int_X \mathrm{tr}(F_A \wedge *F_A)$$

it is a critical point when $d_A \star F_A = 0$ and $d_A F_A = 0$.

Prop. (3.7.2) (2-dim Case). $\star F \in \Omega^0(su(E))$ is parallel thus its characteristic spaces is orthogonal and a stable under parallel transport. So an irreducible YM $SU(2)$-connection must by flat, thus correspond to irreducible $SU(2)$ representation of $\pi_1(X)$.

Prop. (3.7.3) (4-dim Case). $** = (-1)^{2*2} = \mathrm{id}$ on $\Omega^2(E)$ on E a $SU(n)$-bundle, so $\Omega^2(E) = \Omega^+ \oplus \Omega^-$. We have

$$||F_A^+||^2 + ||F_A^-||^2 \geq ||F_A^-||^2 - ||F_A^+||^2 = \int_X \mathrm{tr}(F_A \wedge F_A) = 8\pi^2 c_2(E)$$

Cf.[谢毅 Lecture5]. So it attains minimum at the connection that $*F_A = \pm F_A$ and $d_A F_A = 0$.((Anti)self-dual((anti)instanton)) depending on the sign of $c_2(E)$.

Prop. (3.7.4) (Anti-Instanton Connection on Complex Line Bundle). For a $U(1)$-bundle, $d_A F_A = dF_A$, so F_A is harmonic, thus $c_1(L) = [\frac{-1}{2\pi i} F_A] \in H^2(X,\mathbb{Z}) \cap \mathcal{H}^2_-(X,\mathbb{R})$, In fact, this is equivalent to the existence of a anti-self-dual connection on this bundle.

If this is the case, then we have the ASD-connections module Gauge equivalence is isomorphic to $H^1(X,\mathbb{R})/H^1(X,\mathbb{Z}) = T^{b_1(X)}$.

Proof: Because a gauge is just a $X \to S^1$, and its connected component thus equals $[X, S^1] = H^1(X,\mathbb{Z})$ (MacLane space), and its identity is just the map that is homotopic to id. and $d(gA) = dA - g^{-1}dg = dA - idu$, for $g = exp(iu)$, so $\Omega^1/\mathcal{G} = H^1(X, \mathbb{R}/H^!(X,\mathbb{Z}) = T^{b_1(X)}$. □

Lemma (3.7.5) (Weizenbock Formula). On a Riemannian manifold M, the Laplace operator has the form:

$$\Delta = -\nabla^2_{e_i e_i} - \xi^i \wedge \iota(e_j) R(e_i, e_j)$$

where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$.

$$\int |\mathcal{D}_A \varphi|^2 = \int |\nabla_A \varphi|^2 + \frac{1}{4} R|\varphi|^2 + \frac{1}{2}\langle F_A^+ \varphi, \varphi\rangle.$$

If M is a spin manifold, then the Dirac operator D satisfies:

$$D^2 = -\nabla^2_{e_i e_i} + \frac{1}{4} R$$

where R is the scalar curvature form on M. If M is a $Spin^c$ manifold with a $Spin^c$ connection ∇_A, then the Dirac operator satisfies

$$D_A^2 = -\nabla^2_{A, e_i e_i} + \frac{1}{4} R + \frac{1}{2} F_A$$

Cf.[Geometric Analysis Jost P143,153].

Prop. (3.7.6) (Seiberg-Witten). The Seiberg-Witten equation functional for a unitary connection A on the determinant bundle of a Spin^c structure of M and a section of $\mathcal{S}^+$ is:

$$\begin{aligned} SW(\varphi, A) &= \int \Big(|\nabla_A \varphi|^2 + |F_A^+|^2 + \frac{R}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^2\Big) Vol. \\ &= \int \Big(|\mathcal{D}_A \varphi|^2 + |F_A^+ - \frac{1}{4}\langle e_j e_k \varphi, \varphi\rangle e^j \wedge e^k|^2\Big) Vol \end{aligned}$$

So the Seiberg-Witten equation is the lowest topological possible value of the Seiberg-Witten functional. It writes:

$$\mathcal{D}_A \varphi = 0, \quad F_A^+ = \frac{1}{4}\langle e_j e_k \varphi, \varphi\rangle e^j \wedge e^k.$$

Cf.[Jost Chapter 7].

Cor. (3.7.7). If a compact oriented Spin^c manifold M has nonnegative scalar curvature, then the only possible solution is $\varphi = F_A^+ = 0$. (See from the equivalence of forms of Seiberg-Witten functional.)

8 Chern-Weil Theory

Prop. (3.8.1) (Chern-Weil). An **Invariant polynomial** of a the entries of $M_n(k)$ is one that is invariant under the conjugation action(3.8.2).

For any connection on E, the **Chern-Weil** map CW from invariant polynomial ring to $H^*(X)$: $P \mapsto [P(\Omega)]$ is a ring homomorphism independent on the connection A.

There are relations between c_i and $\mathrm{tr}(\Omega^k)$, they can be derived formally by considering diagonal elements.

Proof: To prove $P(\Omega)$ is closed, notice by(3.8.2), it suffice to show $\mathrm{tr}(\Omega^k)$ is closed. By(2.3.7), $d\,\mathrm{tr}(\Omega^k) = \mathrm{tr}(\omega \wedge \Omega^k - \Omega^k \wedge \omega) = 0$, which is zero because Ω is of even dimension.

For the independence of connections, use(4.1.14). For two connection ∇_i, $\nabla = t\nabla_0 + (1-t)\nabla_1$(you can smooth it) is a connection on the vector bundle $\pi^* E$ on $M \times I$, and the section 0 and 1 induces the connection ∇_0 and ∇_1. Thus s_0^* and s_1^* are the same map, thus $CW_M(p) = s_i^* CW_{M\times I}(p)$ are all the same map. □

Cor. (3.8.2). For a complex line bundle of degree r over a complex manifold,

$$\det(1 - \frac{1}{2\pi i} F_A) = 1 + c_1 + \ldots + c_r$$

gives out the **Chern class**, because it satisfies the axioms of Chern class (5.4.1). In other words, $c_k = \mathrm{tr}((-\frac{1}{2\pi i}F_A)^k)$.

For a real line bundle of degree r,

$$\det(1 - \frac{1}{2\pi i} F_A) = 1 + p_1 + \ldots + p_{\lfloor \frac{r}{2} \rfloor}$$

gives out the **Pontryagin class**, where $p_k \in H^{4k}(X)$. (Notice the ω thus Ω can be chosen to be skew-symmetric, thus for odd k the classes $\mathrm{tr}(\Omega^k) \in H^{2k}(X)$ vanish).

For an oriented real bundle of degree $2r$, the ω and thus Ω can be chosen to be skew-symmetric and the transformation matrix in $SO(2r)$, then

$$\mathrm{Pf}(\frac{1}{2\pi}\Omega) \in H^{2r}(X)$$

is well-defined and closed and gives the **Euler class** $e(E)$ (recall $e(E)^2 = p_r(E)$). (Use $\mathrm{Pf}^2 = \det$ to get that $[\dfrac{\partial \mathrm{Pf}}{\partial \Omega_{ij}}]^t$ commutes with Ω, then calculate $d\mathrm{Pf}(\Omega) = 0$).

Proof: In fact, the construction is natural w.r.t the connection because connection can be pulled back and summed. Then the only task is the normality, which is direct calculation on $\mathbb{CP}^1$. □

Cor. (3.8.3).

$$c_1(E) = c_1(\wedge^{\dim E} E).$$

Direct from the formula.

Cor. (3.8.4) (Whitney Product Formula).

$$c(E \oplus F) = c(E)c(F), \quad p(E \oplus F) = p(E)p(F)$$

Directly form the product connection on $E \oplus F$.

Prop. (3.8.5) (Chern Character). The Chern character

$$ch(E) = [\operatorname{tr}\exp(\frac{i}{2\pi}F_A)]$$

satisfies $ch(E \oplus F) = ch(E) + ch(F)$ and $ch(E \otimes F) = ch(E)ch(F)$ by simple calculation. So it defines a ring homomorphism from $K(X)$ to $H^*(X)$.

Prop. (3.8.6) (Chern-Gauss-Bonnet). For a $2n$-dimensional orientable manifold M,

$$\int_M e(TM) = \chi(M).$$

Prop. (3.8.7). For a vector bundle and a flat connection d_A on a manifold, i.e. $d_A^2 = 0$, we have a deRham like cohomology, and there is a sheaf of flat sections.

$$H^*(X, A) = H^*(X, E).$$

9 Index Theorems(Atiyah-Singer)

References are [Heat equation and the Index Theorem Atiyah] and [Index Theorem].

Prop. (3.9.1) (Gilkey). For a natural transformation ω from the functor $p : M \to$ the Riemannian structure on M to the functor $q : M \to k$-forms on M, if it is homogenous of weight 0 w.r.t to metric g(i.e. $\omega(\lambda^2 g) = \omega(g)$) and in local coordinates it has the coefficients of $\omega(g)$ generated by g_{ij} and $\det g^{-1}$ and their derivatives , then is is a polynomial of Pontryagin classes of the given dimension. (not only up to homology).

Proof: Cf.[Heat equation and the Index Theorem Atiyah P284]. □

Prop. (3.9.2) (Gilkey Generalized). For a natural transformation ω from the functor $p : M \to$ Riemannian structures on M with a Hermitian bundle E with a Hermitian connection and the functor $q : M \to k$-forms on M, if it is homogenous of weight $(0, 0)$ w.r.t to metric g, h and the Hermitian structure(i.e. $\omega(\lambda^2 g, \mu^2\xi) = \omega(g, \xi)$) and in local coordinates it has the coefficients of $\omega(g, \xi)$ generated by $g_{ij}, h_{ij}, \det h^{-1}, \det g^{-1}$ and Γ^{ij}_k(the connection form) and their derivatives, then is is a polynomial of Pontryagin classes and Chern classes of E of the given dimension. (not only up to homology).

Proof: Cf.[Heat equation and the Index Theorem Atiyah P290]. □

Cor. (3.9.3). For a natural transformation ω from the functor $p : M \to$ Hermitian bundle E on M with a Hermitian connection and the functor $q : M \to k$-forms on M, if it is homogenous of weight 0 w.r.t to metric h and the Hermitian structure(i.e. $\omega(\mu^2\xi) = \omega(\xi)$) and in local coordinates it has the form $\omega(g, \xi)$ generated by $h_{ij}, \det h^{-1}$ and Γ^{ij}_k(the connection form) and their derivatives, then is is a polynomial of Chern classes of E of the given dimension. Because when composed with the forgetful functor, it gives a transformation as above. And it is obviously independent of g.

Prop. (3.9.4) (Hodge). For any differential operator A from a vector bundle E to a vector bundle F, we form two operators AA^* and A^*A, then they are both self adjoint elliptic operators, let these corresponding eigenspace be $\Gamma_\lambda(E)$ and $\Gamma_\lambda(F)$, then A and A^* define an isomorphism between $\Gamma_\lambda(E)$ and $\Gamma_\lambda(F)$.

Proof: □

Prop. (3.9.5) (Hirzebruch Signature Formula). On a $4n$-dimensional orientable manifold M, the Poincare duality defines a bilinear pairing $H^{2n}(M)\times H^{2n}(M)\to\mathbb{R}$, its signature $\sigma(M)$ is given by:

$$\sigma(M)=\int_M L_n(p_1,\ldots,p_n).$$

Where L_n is the degree n part of the Taylor expansion of $\prod_{i=1}^r \dfrac{\sqrt{x_i}}{\tanh\sqrt{x_i}}$ in terms of the symmetric polynomial.

Proof: We consider the operator $\tau:\alpha\mapsto i^{l+p(p-1)}*\alpha$, $\tau^2=1$, thus Γ^* is decomposed into two eigenspaces of τ. We define the **signature operator** A as the restriction of $\Delta=d-\tau d\tau$ to Γ_+. Δ anti commutes with τ thus maps Ω_+ to Ω_-, then we have $\operatorname{Ker}A=\operatorname{Ker}\Delta\cap\Omega_+$, which is the positive harmonic forms H_+. So

$$\mathrm{Ind}A=\dim H_+-\dim H_-.$$

And we notice the positive and negative harmonic forms neutralize each other unless on the $2n$-forms, so only need to consider them. In fact, if we consider $4n+2$ manifolds, then τ is pure imaginary and the conjugation neutralize even the $2n+1$ forms, so there are no signature.

Now the inner product $\alpha\to\int\alpha\wedge*\alpha$ is positive definite for a real form α, so this index of A is just the signature of the intersection form defined by cup product. □

Cor. (3.9.6). For a $4n$-dimensional M which is a boundary of a manifold, its signature is 0.

Proof: By Stokes theorem, if M is a boundary of a manifold, then all its Pontryagin numbers, i.e. $\int_M\prod p_i^{n_i},\sum n_i=n$, vanish. □

Prop. (3.9.7) (Generalized Hirzebruch Signature Formula). Let M be a $2l$ dimensional smooth manifold and E be a Hermitian bundle over M, then The index of the generalized signature operator is giving by

$$\mathrm{Ind}A_\eta=2^l\cdot ch(E)L(p_1,\ldots,p_l).$$

where $L(M)(p_i)=\prod\dfrac{x_i/2}{\tanh x_i/2}$.

Prop. (3.9.8) (Hirzebruch-Riemann-Roch). For a n-dimensional complex line bundle L over a compact Kähler manifold M,

$$\chi(M,L)=\int_M[\mathrm{ch}(E)\mathrm{td}(T^{1,0}M)]_n.$$

Where $\chi(M,L)=\sum_{q=0}^n(-1)^q\dim H^q(M,E)$, ch is the Chern character(3.8.5) and $\mathrm{td}(T^{1,0}M)$ is the Todd polynomial, i.e. Taylor expansion of $\prod_{i=1}^r\dfrac{t_i}{1-e^{-t_i}}$ in terms of the symmetric polynomial, applied to $c_i(T^{1,0}M)$.

Cor. (3.9.9) (Riemann-Roch). For a n-dimensional complex vector bundle E over a Riemann Surface M, let $\deg E=\int_M c_1(E)$, then

$$\chi(M,E)=H^0(M,E)-\dim H^1(M,E)=\deg E+\mathrm{rk}(E)(1-g).$$

Cf.[Index Theorem P115].

Hodge Theory

Prop. (3.9.10) (Hodge). By(6.8.11), if we investigate the Laplace operator Δ_d on a compact orientable Riemannian manifold, we get that

$$\Omega^i = \mathcal{H}^i \oplus \operatorname{Im}\Delta_d = \mathcal{H}^i \oplus \operatorname{Im} d \oplus \operatorname{Im} d^*.$$

Thus H^i can be uniquely represented by elements of $\mathcal{H}^i$.

Proof: It suffice to prove Δ_d is self-adjoint elliptic.

$\operatorname{Im}\Delta_d \subset \operatorname{Im} d \oplus \operatorname{Im} d^*$, and the result follows if we show $\mathcal{H}^i, \operatorname{Im} d, \operatorname{Im} d^*$ are orthogonal. In fact, let ω be harmonic, then $(\omega, d^*\xi) = (d\omega, \xi) = 0$, $(\omega, d\eta) = (d^*\omega, \eta) = 0$, $(d\eta, d^*\xi) = (dd\eta, \xi) = 0$. □

Cor. (3.9.11) (Poincare Duality for deRham Cohomology). If M is a n-dimensional oriented Riemannian manifold, then

$$H^p_{dR}(M) \cong H^{n-p}_{dR}(M)$$

Induced by $*$, because $** = \pm 1$ and $*$ commutes with Δ_d(3.2.12), so it induce an isomorphism $\mathcal{H}^p \cong \mathcal{H}^{n-p}$.

Moreover, $*$ in fact induces a perfect pairing:

$$H^k_{dR}(M) \times H^{n-k}(M) \to \mathbb{R}$$

induced by the map

$$* : \mathcal{H}^k(M) \times H^{n-k}(M) \to \mathbb{R} : (\alpha, \beta) \mapsto \int_M \alpha \wedge *\beta$$

As $\int_M \alpha \wedge *\alpha = ||\alpha||^2 \neq 0$.

Prop. (3.9.12). On a compact complex manifold, the formal adjoint of $\overline{\partial}$ is $*\partial*$. (By direct calculation). Also $d^* = (-1)^{n(p+1)+1} * d* = - * d*$.

Prop. (3.9.13) (Hodge). Given a compact Hermitian complex manifold (X, J, g) and a holomorphic line bundle E over it, there is a Hermitian metric on $A^{p,q}E$, and an operator $\overline{\partial}$ on it. Then $\overline{\partial}$ has a formal adjoint $\overline{\partial}^*$, and $\Delta_{\overline{\partial}_E}$ can be defined. Let $\mathcal{H}^{p,q}_E$ be the kernel of $\Delta_{\overline{\partial}}$ on $A^{p,q}E$, called the **E-valued (p, q)-forms**, then there is a orthonormal decomposition

$$A^{p,q}E = \mathcal{H}^{p,q}_E \oplus \operatorname{Im}\Delta_{\overline{\partial}_E} = \mathcal{H}^{p,q}_E \oplus \operatorname{Im}\overline{\partial}_E \oplus \operatorname{Im}\overline{\partial}^*_E$$

And thus $\mathcal{H}^{p,q}(X, E) \cong H^{p,q}_{\overline{\partial}}(X, E)$.

Proof: It suffice to prove $\Delta_{\overline{\partial}_E}$ is self-adjoint elliptic. The rest is verbatim as the proof of(3.9.10). □

Cor. (3.9.14) (Hodge). In case $E = \mathcal{O}_X$, $\mathcal{H}^{p,q}(X) \cong H^{p,q}_{\overline{\partial}}(X)$.

Cor. (3.9.15) (Kodaira-Serre Duality). For a Hermitian line bundle over a compact Hermitian complex manifold X, from Hodge theorem and(10.5.2), we get

$$H^p(X, \Omega^q(E)) \cong H^{n-p}(X, \Omega^{n-q}(E^*))$$

induced by $\overline{*}_E$ and $\overline{*}_{E^*}$.

III.4 Algebraic Topology

1 Homology and Cohomology

Def. (4.1.1). The singular homology of a topological space with coefficients R is the cohomology groups of the Moore complex of $R[\mathrm{Sing}X]$(1.1.4).

Prop. (4.1.2) (Homotopy Axiom for Singular Cohomology). For two homotopic map between two topological space, they induce the same map on singular (co)homology.

Proof: For singular homology, the combinatorial 'pillariazation' can be constructed that $f-g=k^{n-1}\circ d+d\circ k^n$. □

Prop. (4.1.3). The cellular (co)homology coincides with the singular (co)homology for CW-complex.

Prop. (4.1.4) (Morse Inequality). for any field F,

$$\sum^{k}(-1)^i\dim H_i(X,F)\leq\sum^{k}(-1)^i c_i,$$

where c_i is the number of I-dimensional cells. (Use the dimension counting of the long exact sequence).

Prop. (4.1.5) (Universal Coefficient Theorem). See (7.2.11).

Cor. (4.1.6). A map between topological spaces that induce isomorphism on arbitrary homology group induce isomorphisms on cohomology groups.

Prop. (4.1.7) (Poincare Duality). For X a closed manifold, if X is oriented or $\mathrm{char}k=2$, then there is an isomorphism

$$H_i(X,k)\cong H^{n-i}(X,k)$$

which follows immediately from(5.4.15) and(4.5.12). (Should also attain the compact cohomology case if know the relation of compact sheaf cohomology better).

Cor. (4.1.8).

$$H^*(\mathbb{RP}^n,\mathbb{Z}_2)=\mathbb{Z}_2[X]/X^n,\quad H^*(\mathbb{CP}^n,\mathbb{Z})=\mathbb{Z}[X]/X^n$$

Proof: Use induction and Poincare duality to find that $\alpha*\alpha^{n-1}=\alpha^n$. □

Prop. (4.1.9) (Alexander Duality).

Prop. (4.1.10) (Thom isomorphism). Cf.[姜伯驹同调论].

Prop. (4.1.11) (Gysin Sequence). Cf.[姜伯驹同调论].

Prop. (4.1.12) (Lefschetz Fixed Point Theorem).

deRham Cohomology

Prop. (4.1.13) (De Rham). For a smooth manifold and an Abelian group G,

$$H^*_{dR}(X,G) \cong H^*(X,G)$$

Where the right is constant sheaf cohomology.(4.5.11).

Prop. (4.1.14) (Homotopy Axiom for deRham Cohomology). For two homotopic map between two smooth manifold, they induce the same map on deRham Cohomology.

Proof: We only have to prove the case of $M \times \mathbb{R} \to M$, where any constant section map induces an isomorphism $H^*_{dR}(M \times I) \cong H^*_{dR}(M)$. Because any homotopy is a morphism $M \times I \to N$ where f and g are the sections 0 and 1.

For the zero section, we define $K : a + bdt \mapsto \int_0^t b$. This is the desired homotopy, Cf.[Differential Forms in Algebraic Topology Bott Tu]. □

Cohomology of Fiber Bundles

Def. (4.1.15). A **Serre fibration** is the right lifting class of $D^n \to D^n \times I$ for every n. This is equivalent to: for any homotopy of ∂D^n and a image D^n, there is a homotopy of D^n.

Prop. (4.1.16) (Leray-Hirsch). For a fiber bundle $F \to E \to B$ and a ring R s.t. $H^n(F,R)$ is f.g free for all n, and there exist classes c_j of $H^*(E)$ that constitute a basis for each fiber F, then

$$H^*(B,R) \otimes H^*(F,R) \to H^*(E,R)$$

is an isomorphism of $H^*(B,R)$-modules.

Cor. (4.1.17).
- $H^*(U(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_1, x_3, \ldots, x_{2n-1}]$.
- $H^*(SU(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3, \ldots, x_{2n-1}]$.
- $H^*(Sp(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3, x_7, \ldots, x_{4n-1}]$.

Prop. (4.1.18). $H^*(G_n(\mathbb{K}^\infty);\mathbb{Z})$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is generated by the symmetric polynomials, where for $\mathbb{R}$ the coefficient is $\mathbb{Z}_2$.

Proof: Use the flag variety and first calculate for ∞. Then use Poincare duality to show it is mapped onto the symmetric polynomials. Cf.[Hatcher P435]. □

Prop. (4.1.19) (Leray-Serre). For a Serre fibration, especially fiber bundle, $F \to E \to B$, that B is simply connected, then there is a spectral sequence

$$E_2^{pq} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E) \quad E_2^{pq} = H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)$$

Cor. (4.1.20) (Wang Sequence). When $B = S^n$, there is a long exact sequence:

$$\cdots \to H_q(F) \to H_q(E) \to H_{q-n}(F) \to H_{q-1}(F) \to H_{q-1}(E) \to \cdots$$

Cor. (4.1.21) (Gysin Sequence). When $F = S^n$, there is a long exact sequence:

$$\cdots \to H_{p-n}(B) \to H_p(E) \to H_p(B) \to H_{p-n-1}(B) \to H_{p-1}(E) \to \cdots$$

Cup Product and Cohomology Operators

Prop. (4.1.22). The cup product will restrict to a relative version:

$$H^*(X,A)\times H^*(X,B)\to H^*(X,A\cup B),$$

This implies that if X is a union of n contractible open set, then the cup product of n-elements vanish. In particular, the cup product in a suspension vanishes.

Prop. (4.1.23) (Steenrod Powers). The total Steenrod squares Sq is a map from $H^n(X,\mathbb{Z}_2)\to H^{n+*}(X,\mathbb{Z}_2)$ that:

- it is natural and stable under suspension.
- it is additive.
- $\mathrm{Sq}(\alpha\cup\beta)=\mathrm{Sq}(\alpha)\cup\mathrm{Sq}(\beta)$.
- $Sq^i(\alpha)=\alpha^2$ if $i=|\alpha|$, and 0 if $i>|\alpha|$.

The total Steenrod Powers P is a similar map from $H^n(X,\mathbb{Z}_p)\to H^{n+*}(X,\mathbb{Z}_p)$ that $P^i(\alpha)=\alpha^p$ if $2i=|\alpha|$ and 0 if $2i>\alpha$.

The algebra of powers is generated respectively by elements Sq^{2^k}, and for p it is generated by β and the elements P^{p^k}. (Because of Adem relations) Cf.[Hatcher P497].

2 Fundamental Groups

Prop. (4.2.1). The fundamental group of a topological group is abelian.

Proof: This is because π_1 preserves products, so takes group objects to group objects. And the group objects in the category of groups is the abelian groups (7.1.49) □

Prop. (4.2.2) (Van Kampen). If X is a union of path-connected subsets A_α all containing x_0 that $A_\alpha\cap A_\beta$ and $A_\alpha\cap A_\beta\cap A_\gamma$ are all path-connected, then $*\pi_1(A_\alpha)/\sim$ where $\sim$ is generated by $i_*(\pi_1(A_\alpha\cap A_\beta))\in\pi_1(A_\alpha)\sim i_*(\pi_1(A_\alpha\cap A_\beta))\in A_\beta$ for every α,β, Cf.[Hatcher P52].

3 CW Complex

Prop. (4.3.1). If (X,A) is a CW pair, then $X\times\{0\}\cup A\times I$ is a deformation retract of $X\times I$, thus (X,A) has the **homotopy extension property** because we can perform infinite induction on dimension.

Prop. (4.3.2). The loop space ΩX for X a CW complex has CW complex type. In particular, if it has noly finitely many cells for a given dimension, then so does ΩX. Milnor proved this.

Prop. (4.3.3). The homotopy group defines a long exact sequence for triples (X,A,B), in particular for $B=\mathrm{pt}$.

Prop. (4.3.4) (Compression Theorem). If (X,A) is a CW pair that (Y,B) be a pair that $\pi_n(Y,B,y_0)$=0, for any n, then every map (X,A) to $(Y<B)$ is homotopic rel$A$ to a map $X\to B$. (Use extension property to extend one dimension a time). This shows that the homotopy doesn't depend on higher dimensions, (but might on lower one).

Cor. (4.3.5) (Whitehead Combinatorial Homotopy I). If M and K is dominated by CW complexes, then any weak homotopy equivalence is an homotopy equivalence. If the map is an inclusion, then it is a deformation retract. In particular, if M is manifold, then it is dominated by its tubular nbhd, so this theorem is applied.

Proof: For inclusion, use compression, and in general use mapping cylinder and cellular approximation. □

Cor. (4.3.6). If $\pi_n(X) = 0$ for all n and a CW complex X, then X is contractible.

Def. (4.3.7). A morphism is called a **weak homotopy equivalence** iff it induces isomorphism on homotopy groups on every dimension.

Prop. (4.3.8). A weak homotopy equivalence induce isomorphism on all homology and cohomology. And also $[K, A] \cong [K, B]$ and $\langle K, A\rangle = \langle K, B\rangle$ for every finite CW complex K.

Proof: Pass to the mapping cylinder, the homotopy case follows easily from the compression lemma(4.3.4), and the cohomology follows from universal coefficient theorem(4.1.5).

We may use (reduced) mapping cylinder to assume $A \to B$ is an injection, then compression shows surjectivity, and the relative case for homotopy also show injectivity. □

Prop. (4.3.9) (Cellular Approximation Theorem). Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. This makes calculation of homotopy easy. (It suffice to show a map cannot be surjective on a higher dim cell, Cf.[Hatcher P349].

Moreover, Any map of pairs of CW complexes can be deformed to a cellular map. (first deform the small complex, then deform the big by dimension.

Cor. (4.3.10). The cellular approximation makes the computation of homotopy theoretically easier, but the difficulty comes from the complexity of the homotopy group of the sphere. If a CW complex has only cells of dim> n, then it's homotopy group vanishes for $i < n$. In particular, $\pi_n(S^k) = 0$ for $n < k$.

Prop. (4.3.11) (CW Approximations). If A is CW, then there is a n-connected CW models (Z, A) to (X, A), i.e. $\pi_{\leq n}(Z, A) = 0$ and $Z \to X$ induce isomorphism on $\pi_{>n}$ and injection for π_n, moreover it can be constructed from A by attaching cells of dimension greater than n. Cf.[Hatcher P353].

Thus there exists a CW approximation for any space A, thus there exists a CW approximation for any pair (X, X_0), (first approximate X_0 and use the mapping cylinder to get a embedding) that is, induce isomorphism on $\pi_n X$ and $\pi_n X_0$ and on relative homotopy group.

Use long exact sequence, compression and mapping cylinder, we can prove the approximations preserve (co)homology and mapping classes.

And this approximation is unique up to homotopy equivalence rel$A$, (use relative mapping cylinder and use compression). They act like injective resolutions. Cf.[Hatcher P55].

Cor. (4.3.12). For any n-connected CW pair (X, A), there exist a homotopic $(Z, A) \cong (X, A)$ rel A that $Z\backslash A$ has only cells of dimension greater than n.

Proof: Choose the n-connected approximation as above. The map induce and isomorphism on $\pi_{>n}$ by definition and on $\pi_{<n}$ because $\pi_i(A) \to \pi_i(Z)$ and $\pi_i(A) \to \pi_i(X)$ are isomorphisms. And on π_n, it is injective by definition and surjective because $\pi_n(A) \to \pi_n(Z) \to \pi_n(X)$ is isomorphism. Hence we know that the collapsed mapping cylinder at is weak-homotopy equivalent to Z, thus it deforms into Z by(4.3.5), thus $Z \to X$ rel A by(4.4.1). □

Cor. (4.3.13) (Whitehead theorem). A f between two simply connected CW complexes that induce isomorphism on homology groups is a homotopy equivalence. (using mapping cylinder, we can assume it's an inclusion, and $\pi_1(Y, X) = 0$, so the theorem shows that $\pi_n(Y, X) = 0$, and use Whitehead(4.3.5)).

Prop. (4.3.14). A closed manifold or the interior of a manifold with boundary has a homotopy type of a CW complex of finite type.

Remark (4.3.15). The use of mapping cylinder and relative mapping cylinder is important.

4 Homotopy

Prop. (4.4.1). A map $X \to Y$ is a homotopy equivalence iff the mapping cylinder deformation retracts onto X.

Prop. (4.4.2). The universal cover have the same homotopy group $\pi_{>1}$, by lifting property.

Prop. (4.4.3) (Excision Theorem). If A, B are CW-complexes, then if $(A, A \cap B)$ are m-connected and $(B, A \cap B)$ are n-connected, then $\pi_i(A, A \cap B) \to \pi_i(A \cup B, A)$ is isomorphism for $i < m + n$, and surjective for $i = m + n$. Cf.[Hatcher P360].

Moreover, if (X, A) is r-connected and A is s-connected, then $\pi_i(X, A) \to \pi_i(X/A)$ is isomorphism for $i \leq r + s$ and surjection for $i = r + s + 1$.

Cor. (4.4.4). For $n > 1$, $\pi_n(\bigvee_\alpha S^\alpha)$ is free Abelian with $\pi_n(S^n)$ as generators. This is because $(\prod_\alpha S^n, \bigvee_\alpha S^n)$ is $(2n-1)$-connected thus use excision, because $\pi_n \prod_\alpha S^n$ is easy to calculate.

Cor. (4.4.5) (Freudenthal Theorem). For $i < 2n - 1$, $\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$ and . (Can also be derived considering antipodal point point of S^n by (4.6.9)) and surjective for $i = 2n - 1$. In general, this holds when X is $(n-1)$-connected. Thus we have $\pi_n(S^n) = \mathbb{Z}$.

Proof: Use the suspension, $\pi_i(X) \cong \pi_{i+1}(C_+X, X) \to \pi_{i+1}(SX, C_-X) \cong \pi_{i+1}(SX)$.
for $n = 1$ for the homotopy of sphere, we can use Hopf bundle. □

Prop. (4.4.6) (Generalized Hurewicz theorem). If (X, A) is a $(n-1)$-connected pair of spaces, $n \geq 2$, then the Hurewicz map induces isomorphism

$$\pi_n(X, A)/(\pi_1(A)\text{action}) \cong H_k(X, A),$$

and $H_k(X, A) = 0, k < n$. And on π_{n+1}, the Hurwicz map is surjective for $n > 1$.Cf.[Hatcher P390Ex23] for surjectiveness.

Prop. (4.4.7) (Fiber Bundle). For a fiber bundle $S \to M \to N$, there is a long exact sequence of homotopy groups:

$$\cdots \to \pi_i(N) \to \pi_{i-1}(S) \to \pi_{i-1}(M) \to \pi_{i-1}(N) \to \cdots$$

Because it has lifting property.

Prop. (4.4.8). $\pi_{i+1}(M) \cong \pi_i(\Omega(M))$, where Ω is the loop space. More generally,

$$\langle \Sigma X, K \rangle = \langle X, \Omega K \rangle.$$

Prop. (4.4.9). The homotopic direct limit of a family of homotopy equivalence is a homotopy equivalence. Cf.[Morse Theory Milnor].

Prop. (4.4.10). for $i \leq 2m$, $\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i-1}U(m)$, and

$$\pi_{i-1}U(m) \cong \pi_{i-1}U(m+1) \cong \cdots .$$

and for $j \neq 1$, $\pi_j U(m) \cong \pi_j SU(m)$.

Similarly, $\pi_i \Omega_1(2m) \cong \pi_{i+1}O(2m)$ for $i \leq n-4$. (4.6.10),Cf[Morse Theory Milnor Prop23.4].

Cor. (4.4.11) (Bott Periodicity theorem for Unitary Groups). The stable homotopy group $\pi_i U$ has period 2. $\pi_{2k+1}U \cong 0$ and $\pi_{2k}U \cong \mathbb{Z}$.

Proof: Use the last proposition and long exact sequence to show that for $1 \leq i \leq 2m$,

$$\pi_{i-1}U = \pi_{i-1}U(m) \cong \pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i+1}SU(2m) \cong \pi_{i+1}U.$$

Notice that $U(m) \to U(2m)/U(m) \to G_m(\mathbb{C}^{2m})$ □

Prop. (4.4.12) (Bott Periodicity for O). For the infinite dimensional orthogonal space O, $\Omega_8(16r) \cong O(r), \Omega_4(8r) \cong Sp(2r)$. So $\Omega_8 \cong O$ and $\Omega_4 O \cong Sp$. Thus by (4.4.8),

$$\pi_i(O) = \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z} \ldots, \quad \pi_i(Sp) = 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z} \ldots$$

respectively. (Use (4.6.11)) Cf.[Morse Theory Prop24.7].

Prop. (4.4.13). Homotopy Fibers.

5 Obstruction Theory & General Cohomology Theory

Towers

Prop. (4.5.1) (Towers). There are Whitehead Towers and Postnikov Towers for a CW complex X.

$$\cdots \to Z_2 \to Z_1 \to Z_0 \to X \to \cdots \to X_2 \to X_1 \to X_0$$

Z_n annihilate $\pi_{\leq n}(X)$, X_n remains only $\pi_{\leq n}(X)$. The towers can be chosen to be fibrations, with fibers $K(\pi_n X, n)$ by(1.4.16).

Prop. (4.5.2). There is a Postnikov towers of :

$$B\mathrm{String}(n) \to B\mathrm{Spin}(n) \to BSO(n) \to BO(n)$$

with corresponding obstructions $w_1(X), w_2(X)$ and $p_1(X)/2$.

Prop. (4.5.3) (Obstructions). If a connected abelian CW complex X ($\pi_1(X)$ abelian and action on higher homotopy trivial) and (W, A) satisfies $H^{n+1}(W, A; \pi_n X) =$ for all n, then $A \to X$ can extend to a map $M \to X$.

Proof: Cf.[Hatcher P417]. □

Cor. (4.5.4). A map between Abelian CW complexes that induce isomorphisms on homology is a homotopy equivalence.

Proof: Notice that $\pi_1(X)$ acts trivially on $\pi_1(Y, X)$ and use Hurewicz. □

<u>**Eilenberg-Maclane Space**</u>

Prop. (4.5.5) (Generalized Cohomology). If K_n is an Ω-spectrum, i.e. $K_n \cong \Omega K_{n+1}$ weak equivalence, then the functors $X \mapsto h^n(X) = \langle X, K_n \rangle$ define a reduces cohomology theory on the category of basepointed CW complexes, i.e. it satisfies the long exact sequence for $A \to X \to X/A$ and wedge axiom. Cf.[Hatcher P397].

Proof: Use(4.4.8) and there is a Cofibration sequence:

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \dots$$

□

Def. (4.5.6). For a discrete Abelian group G, an **Eilenberg-Maclane spaces** $K(G,n)$ is a space having only one nontrivial homotopy group $\pi_n(K(G,n)) = G$.

It can be constructed by $K(A,0) = A$, $K(A,n+1) = B(K(A,n))$(5.3.4). Note $K(G,1)$ is constructed the same as by(1.3.31).

Alternatively, it can also be constructed by first use(4.4.4) and then use higher cells to kill higher homotopies.

Prop. (4.5.7). The homotopy typw of a CW complex $K(G,n)$ is unique, thus $\Omega(K(G,n)) \cong K(G,n-1)$ hence $H^n(X,A) \cong [X, K(A,n)]$(4.5.5) and this isomorphism is generated by a distinguished class of $H^n(K(G,n),G)$.

Proof: Cf.[Hatcher P366]. □

Prop. (4.5.8). $K(\mathbb{Z},1) = S^1 = U(1)$, $K(\mathbb{Z},2) = \mathbb{CP}^\infty$, Because $S^\infty \to \mathbb{CP}^\infty$ is a contractable covering.

6 Morse Theory & Floer Homology

<u>**Morse Theory(Milnor)**</u>

Prop. (4.6.1) (Morse Lemma). In a non-degenerate critical point of f, there is a coordinate that

$$f = f(p) + x_1^2 + \cdots + x_{n-\lambda}^2 - y_1^2 - \cdots - y_\lambda^2.$$

Proof: Just extract the first order part out and reform the bilinear form one-by-one. Cf.[Milnor Morse Theory lemma 2.2]. □

Prop. (4.6.2). If f is a smooth function that $f^{-1}([a,b])$ is compact and have no critical points, then M^a is a deformation retracts of M^b using $\mathrm{grad} f/|\mathrm{grad} f|^2$.

Prop. (4.6.3) (Morse Main Lemma). If f is a smooth function with p a non-degenerate critical point and λ downward pointing direction. If for some $f^{-1}([c-\epsilon, c+\epsilon])$ is compact, then $M^{c+\epsilon}$ is homotopic to $M^{c-\epsilon}$ gluing a λ dimensional cell.

Proof: Cf.[Milnor Prop3.2]. □

Prop. (4.6.4). For an embedded manifold and almost all point p, the distance to p is a morse function. (Use Sard theorem and degenerate $\iff$ p is a focal point.

Cor. (4.6.5). smooth manifold has CW type; on a compact manifold any vector field with discrete singular points has its index sum equal to $\chi(M)$ (Hopf-Rinow), and there exists one.

Prop. (4.6.6). for $\Omega(p,q)^c$ the path space of energy $< c$, the piecewise geodesic path space B (piece fixed), the energy function is smooth and B^a is compact and is the deformation contraction of $\mathrm{int}\Omega^a$ for $a < c$. E has the same critical point and same index and nullity on B and Ω^c. (Just geodesicrize any path in Ω.

So for two point not conjugate in B^a, Ω^a has a finite CW complex type and a λ-dimensional cell for every geodesic of index λ in B^a.

Prop. (4.6.7) (Morse Main Theorem). If p and q are not conjugate along any geodesic, then $\Omega(p,q)$ has a countable CW complex type and has a λ-cell for every geodesic of index λ.

If M has nonnegative Ricci curvature, then M has only finite cell for every dimension.

Proof: Cf.[Milnor Morse Theory Prop17.3]. □

Cor. (4.6.8). The path space homotopy type only depend on the homotopy type of M (use the two homotopy to id to get a composition of homotopy of the two path space), so one can get the information of path space of M by looking at the homotopy type of M.

Prop. (4.6.9) (Minimal Geodesics). If p, q in a complete manifold M has distance $\sqrt{d}$ and the minimal geodesics form a topological manifold, and if all non-minimal geodesic has index $\geq \lambda$, then for $0 \leq i < \lambda$, $\pi_i(\Omega, \Omega^d) = 0$.

Lemma (4.6.10). In $SU(2m)$, the minimal geodesic from I to $-I$ is homeomorphic to Grassmannian $G_m(\mathbb{C}^{2m})$ and and non-minimal geodesic has index $\geq 2m+2$.

Similarly, The space of minimal geodesic from I to $-I$ in $O(2m)$ is homeomorphic to the space of complex structures in $\mathbb{R}^{2m}$, and any non-minimal geodesic has index $\geq 2m-2$.

Proof: Cf.[Milnor Morse Theory Lemma23.1 Lemma24.4]. □

Lemma (4.6.11). Ω_{k+1} is homotopic to the space of minimal geodesics in Ω_k from J to $-J$. (The same way, calculate the index of geodesics from J to $-J$ and use (4.6.9)). Cf.[Milnor Morse Theory Prop24.5] for definition of Ω_{k+1}.

III.5 Vector Bundle & K-Theory

1 Fundamentals

Prop. (5.1.1). A vector bundle can have its transform map $\in O(n)$ (or $U(n)$) by constructing a riemannian metric on it. And for every local trivialization, we choose the metric on it compatible with the given metric, thus the transform map is $\in O(n)$ (or $U(n)$).

Prop. (5.1.2) (Tietze extension general). For a Hausdorff paracompact(hence normal) space X and a paracompact subspace Y, every section on Y can be extended to a section on X. (For every point of Y, find a local trivialization and an even smaller open set. Use Tietze extension to extend locally to this nbhd, then use partition of unity to unify all).

Prop. (5.1.3) (Homotopy Invariance of Vector Bundles). For a continuous family of maps from a paracompact Hausdorff space Y to a Hausdorff paracompact space X, then the pullback bundle is isomorphic.

Proof: Consider the space $Y \times I$ and the pullback bundle E, then for every t_0, consider a new bundle $\mathrm{Hom}(E, \pi_1^* E_{t_0})$, then Y has a section id, this section by the last proposition can be extended, so it spans the vector space for nearby t (because of paracompactness), thus is an isomorphism because it is a locally invertible vector bundle homomorphism. □

Prop. (5.1.4) (Splitting Principle). For a vector bundle $E \to X$, there is a space $Y \to X$ that p^* is injective on $H^*(-, \mathbb{Z})$ and p^*E splits as a sum of line bundles. This proposition is useful when proving theorems about characteristic classes.

Proof: It suffice to find a Y that p^*E has a subbundle, then choose its orthogonal part, and use induction. For this, choose $Y = P(E)$, then Y has a tautological bundle, which is a subbundle of p^*E, and Y is fibered over X with fiber $\mathbb{P}^n$, and we want to use Leray-Hirsch, so check the fact $H^*(\mathbb{P}^n)$ is free and generated by the first Chern class, by(5.4.1) and(4.4.14). And Chern class is functorial, so the powers of Chern class of f^*E will generate the cohomology ring of any stalks. □

2 Thom isomorphism

Prop. (5.2.1) (Thom Class). For a vector bundle, we can compactify its bundles to get a (D^n, S^n)-bundle, if there is a Thom class that induce a generator $H^n(D^n, S^n)$ on every fiber. Then the relative Leray-Hirsch will give that c induces an isomorphism $H^i(B, R) \to H^{i+n}(E, E', R)$. For $\mathbb{Z}_2$ coefficient there exists a Thom class, and for orientable bundle there exists a $\mathbb{Z}$-Thom class. Notice that fiber bundle over a simply connected base is orientable.

Prop. (5.2.2). Similarly, for a orientable fiber bundle $S^{n-1} \to E \to B$, make it a $D^n \to E' \to B$ bundle, then E' is homotopy equivalent to B so there is a Gysin sequence

$$\to H^{i-n}(B) \xrightarrow{*e} H^i(B) \to H^i(E) \to H^{i-n+1}(B) \to$$

Where the Euler class e is chosen to commute with the Thom isomorphism.

3 Principal Bundles

Basic reference is [Principal Bundles and Classifying Space].

Def. (5.3.1). A **principal bundle** or G-bundle is a bundle P with G-fibers that the transition function is right G-map, i.e. left multiplication by some $g_{\alpha\beta}$. a associated bundle of a representation $G \to \mathrm{End}(V)$ is the total space of $P \times V$ module the equivalence $[gg_0, v] = [g, g_0 v]$. The corresponding transition function is just the left action by $g_{\alpha\beta}$.

Prop. (5.3.2) (Homogenous Space). If G is a Lie group and H is a closed subgroup, then the quotient $H\backslash G$ can be given a structure of a G-homogenous space and $G \to H\backslash G$ is a principal H-bundle.

Prop. (5.3.3). The projection $S^{2n+1} \to \mathbb{C}P^n$ is a principal S^1-bundle.

Classifying Space

Def. (5.3.4). The **classifying space** for a topological group G is a CW complex BG with a weakly contractable universal cover EG that EG is a G-fiber bundle on BG.

$\pi_{n+1}(BG) = \pi_n(G)$ by(4.4.7).

Prop. (5.3.5). $[X, BG] \cong G$-bundles on X. And BG is Abelian if G is Abelian. Thus the classifying space BG is unique up to homotopy equivalence because they all represent the functor from the CW homotopy category to the set of G-bundles on it.

Proof: Cf.[Principal Bundles and Classifying Space P13]. □

Cor. (5.3.6). If $H \to G$ is a homomorphism of topological groups, then any H-principal bundle can be made into a G-bundle by right tensor G. Thus there is a map $BH \to BG$ by Yoneda lemma. In other words, there is a **classifying functor** θ from the category of topological space to the category of homotopy class of CW complexes.

Prop. (5.3.7) (Examples).

- $B(\mathbb{Z}/2) = \mathbb{RP}^\infty$, and $B(\mathbb{Z}/n\mathbb{Z}) = S^\infty/(\mathbb{Z}/n)$.
- $BSU(2) = \mathbb{HP}^\infty$.
- $B(\mathbb{Z}^{2g})$ = torus of genus g because torus has the upper half plane as universal cover, this can be seen observing only has to satisfy the sum of inner angle is π.
- $BO(n), BU(n), BSp(n)$ are respectively the Grassmannian of n-planes in the infinite dimensional real, complex and quaternion vector spaces, because we have

$$O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty).$$

  and similarly for $\mathbb{C}$ and $\mathbb{H}$, and $V_n(\mathbb{R}^\infty)$ is contractible by linear homotopy and Schmidt orthogonalization. In particular, $B(\mathbb{Z}/2) = \mathbb{RP}^\infty$ and $BS^1 = \mathbb{CP}^\infty$.

Def. (5.3.8). A subgroup of a topological group G is called **admissible** if $G \to G/H$ is a H-bundle.

Prop. (5.3.9). If H is an admissible subgroup of G, then there is a homotopy fiber sequence $G/H \to BH \to BG$.

Proof: Cf.[Principal Bundles and Classifying Space P22]. □

Cor. (5.3.10). There is a homotopy equivalence $\Omega BK \cong K$ and $B\Omega K \cong K$.

Prop. (5.3.11). If H is an admissible normal subgroup of G, then there is a homotopy fiber sequence $BH \to BG \to B(G/H)$.

Cor. (5.3.12).
- there are fiber bundles $S^0 \to BSO(n) \to BO(n)$ and similarly for $\mathbb{C}$ and $\mathbb{H}$.
- there are fiber bundles $S^n \to BO(n) \to BO(n+1)$.
- there are fiber bundles $U(n)/T^n \to (\mathbb{CP}^\infty)^n \to BU(n)$, and where $U(n)/T^n$ is the variety of complete flags in $\mathbb{C}^n$.
- for a discrete group $H \subset G$, $BH \to BG$ is a covering map.
- there are fiber bundles $BSO(n) \to BO(n) \to \mathbb{RP}^\infty$ and similarly for $\mathbb{C}$ and $\mathbb{H}$.
- there are fiber bundles $\mathbb{RP}^\infty \to BSpin(n) \to BSO(n)$.

Prop. (5.3.13). $H_*(BG, \mathbb{Z}) \cong H_*(G, \mathbb{Z})$ and $H^*(BG, \mathbb{Z}) \cong H^*(G, \mathbb{Z})$.

Proof: Because EG is weakly contractible, $S_*(EG)$ is a free $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$ and $S_*(EG)_G$ is identified with $S_*(BG)$. The rest is easy. □

4 Characteristic Classes

References are [Cohomology of Classifying Space Toda] and [Characteristic Classes Milnor].

Def. (5.4.1). Axioms for **Chern classes** for complex bundles:
- $c(E) = 1 + c_1(E) + \ldots + c_n(E) \in H^*(X, \mathbb{Z})$, $n = \deg(E)$.
- $f^*(c(E)) = c(f^*(E))$.
- $c(E \oplus F) = c(E)c(F)$.
- On the tautological bundle over $\mathbb{CP}^1, c(\eta) = 1 + c_1(\eta)$ and $\int_{\mathbb{CP}^k} c_1(\eta) = -1$. There is an affine connection definition of Chern class.

Prop. (5.4.2). There exists uniquely a natural transformation $c : Vect_{\mathbb{C}}(X) \to H^*(X, \mathbb{Z})$ satisfying these axioms. (For this, it suffice to calculate the cohomology ring of $BGL_n(\mathbb{C})$, Cf.[Cohomology of Classifying Space Toda].

Prop. (5.4.3) (First Chern Class Map). A complex line bundle can be seen as an element of $H^1(X, \underline{\mathbb{C}}^*)$, by(4.2.6), by the exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathbb{C} \xrightarrow{exp(1\pi i-)} \mathbb{C}^* \to 0$$

($\mathbb{C}$ is sheaf of smooth functions from X to $\mathbb{C}$)which gives a map $H^1(X, \mathbb{C}^*) \to H^2(X, \mathbb{Z})$, called the **first Chern class map**. It is called so because it gives the first Chern class of this complex line bundle. It is also an isomorphism because $\mathbb{C}$ is fine sheaf so acyclic.

Proof: Only have to prove they are equal in $H^2(X,\mathbb{C})$. We choose a totally convex covering U_i of X by(2.3.18), then it is a fine cover, so by(4.2.7) the Čech cohomology and sheaf cohomology equal.

Use the Chern-Weil map definition of the Chern class, a connection on a line bundle satisfies $\nabla e_\alpha = \omega_\alpha e_\alpha$, and if $e_\beta = e_\alpha g_{\alpha\beta}$, then $\omega_\beta = g_{\alpha\beta}^{-1}\omega_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} = \omega_\alpha + d(\log g_{\alpha\beta})$. So $\Omega_\alpha = d\omega_\alpha$ locally, and the first Chern class is giving by Ω_α in $H^2(X,\mathbb{C})$.

Then we need to understand the deRham isomorphism. For the exact sequence $0 \to \underline{\mathbb{C}} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \ldots$, it has a splitting: $0 \to \underline{\mathbb{C}} \to \mathcal{K}^1 \to 0$ and $0 \to \mathcal{K}^1 \to 0 \to \mathcal{A}^1 \to \mathcal{K}^2 \to 0$, this gives

$$0 \to H^1(X,\mathcal{K}^1) \xrightarrow{\delta} H^2(X,\underline{\mathbb{C}}) \to 0, \quad \mathcal{A}^1(X) \to \mathcal{K}^2(X) \xrightarrow{\delta} H^1(X,\mathcal{K}^1) \to 0.$$

because $\mathcal{A}^k$ are fine sheaves. The composite of them is just the de Rham isomorphism(Here we are identifying $H^2(X,\underline{\mathbb{C}})$ to $H^2(X,\mathbb{C})$ by(4.5.12)). Tracking the lifting, we notice Ω is mapped to the cocycle $\{\log g_{\alpha\beta} + \log g_{\beta\gamma} - \log g_{\alpha-\beta}\}$, which is exactly the image of the first Chern class map. □

Cor. (5.4.4). Complex line bundles are characterized by the first Chern class up to smooth isomorphism, because $H^1(X,\mathbb{C}^*) \to H^2(X,\mathbb{Z})$ is an isomorphism.

Def. (5.4.5). Axioms for **Stiefel-Whitney classes** for real bundles:

- $w(E) = 1 + w_1(E) + \ldots + w_n(E) \in H^*(X,\mathbb{Z}/2\mathbb{Z})$, $n = \deg(E)$.
- $f^*(w(E)) = w(f^*(E))$.
- $w(E \oplus F) = w(E)w(F)$.
- On the tautological bundle over $\mathbb{RP}^1$, $w(\eta) = 1 + w_1(\eta)$ and $\int_{\mathbb{CP}^k} c_1(\eta) = -1$.

Def. (5.4.6). The Pontryagin class is defined as $p_k(E) = (-1)^k c_k(E_\mathbb{C}) \in H^{4k}(X,\mathbb{Z})$.

Def. (5.4.7). Axioms for **Euler classes** for orientable real bundles:

- if E has non-vanishing section, then $e(E) = 0$.
- $f^*(w(E)) = w(f^*(E))$.
- $w(E \oplus F) = w(E)w(F)$.
- for the opposite orientation $\overline{E}$, $e(\overline{E}) = -e(E)$.

Definition via Classifying Space

Prop. (5.4.8).

$$H^*(BO(n),\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2, \ldots, w_n].$$

$$H^*(BSO(n),\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_2, \ldots, w_n].$$

Cf.[Cohomology of Classifying Space Toda P82].

Prop. (5.4.9).

$$H^*(BU(n),R) = R[c_1, c_2, \ldots, c_n].$$

$$H^*(BSU(n),R) = R[c_2, \ldots, c_n].$$

Cf.[Cohomology of Classifying Space Toda P81].

Prop. (5.4.10). For R of characteristic $\neq 2$,

$$H^*(BSO(2n+1),R) = R[p_1,p_2,\dots,p_n].$$

$$H^*(BSO(2n),R) = R[p_1,p_2,\dots,p_n,e], e^2 = p_n.$$

Cf.[Cohomology of Classifying Space Toda P81].

Prop. (5.4.11). There are maps $t: SO(n) \to U(n)$, and it will induce a map of classifying spaces, and induce

$$p_k = (-1)^k Bt^*(c_{2k}).$$

There are maps $O(n) \xrightarrow{i} U(n) \xrightarrow{j} SO(2n)$, and it will induce a map of classifying spaces, and induce

$$Bi^*(c_k) = w_k^2, \quad Bj^*(w_{2k}) = c_k.$$

There are maps $k: U(n) \to SO(m)$ $m = 2n$ or $2n+1$, then for a field R of characteristic $\neq 2$,

$$Bk^*(p_k) = \sum_{i+j=k} (-1)^i c_i c_j, \quad Bk^*(e) = c_n.$$

Cf.[Cohomology of Classifying Space Toda P81].

Applications

Prop. (5.4.12). Note that $K(\mathbb{Z},2) = \mathbb{CP}^\infty = B(K(\mathbb{Z},1)) = BS^1$(5.3.7) thus it is also the classifying space of $U(1)$, thus we have $H^2(X,\mathbb{Z}) \cong [X, K(\mathbb{Z},2)] \cong$complex line bundles on X. Similarly, we have $H^1(X,\mathbb{Z}/2\mathbb{Z}) \cong$ real line bundles on X.

III.6 Symplectic Geometry

Cf.[Methods in Classical Mechanics Arnold Chapter8],[辛几何讲义范辉军].

1 Basics

Symplectic Forms

Def. (6.1.1). A **symplectic form** ω is a closed 2-form that is non-degenerate on any point. A smooth manifold with a symplectic form is called a **symplectic manifold**. A symplectic manifold must be even dimensional and orientable.

Prop. (6.1.2). A hamiltonian phase flow preserves the symplectic form. $g^{t*}\omega = \omega$.

Proof: by Cartan's magic formula,

$$\frac{d}{dt}(g^t)^*\omega = L_X\omega = \iota_X(d\omega) + d(\iota_X\omega) = d(\iota_X\omega)$$

because ω is closed. And by definition, $d(\iota_X\omega)(\eta) = \omega(J\mathrm{d}H, \eta) = \langle dH, \eta\rangle$, so $d(\iota_X\omega) = dH$, Thus the theorem. □

For the following Cf.[辛几何讲义范辉军 lecture3].

Prop. (6.1.3) (Moser's Stability). If ω_t is a smooth family of cohomologous forms on a closed manifold M, then there exists an isotopy Ψ_t s.t.

$$\Psi_t^*(\omega_t) = \omega_0.$$

Prop. (6.1.4) (Relative Moser Stability). If M is a closed manifold and S is a compact submanifold, then if two closed 2-form equals on S, then there is an open neighborhood N_0, N_1 of S and a diffeomorphism $\Psi : N_0 \to N_1$ that

$$\Psi|S = \mathrm{id},\ \ \Psi^*\omega_1 = \omega_0.$$

Cor. (6.1.5) (Darboux's Theorem). Every symplectic form ω on M is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.

Proof: Choose $S = \mathrm{pt}$ and uses relative Moser stability. □

Prop. (6.1.6). For a compact symplectic manifold M, its even dimensional cohomology groups doesn't vanish, because ω^k are nontrivial.

Proof: This is because ω^n is a volume form on M that never vanish, so it gives M an orientation and $\int_M \omega^n \neq 0$. If ω^k is exact, then ω^n is exact, so $\int_M \omega^n = 0$ by Stokes', contradiction. □

III.7 Lie Groups & Symmetric spaces

1 Main Theorems

Prop. (7.1.1). For a Lie group G, for any lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there exists uniquely a connected Lie subgroup H s.t. $\mathfrak{h}$ is the lie algebra of H.

Proof: By (3.2.1), there is a maximal connected manifold H corresponding to $\mathfrak{h}$, we only need to show that it is a group. But the left invariance of $\mathfrak{h}$ shows that $HH \subset H$ because H is maximal. □

Cor. (7.1.2). If G_1 is a simply connected Lie group and G_2 is a connected Lie group, then any Lie algebra homomorphism $\tilde{h} : \mathfrak{g}_1 \to \mathfrak{g}_2$ can be lifted to a Lie group homomorphism.

Proof: use the image of $\tilde{h}$: $\Gamma(\tilde{h}) \subset \mathfrak{g}_1 \times \mathfrak{g}_2$, the prop shows that there is a Lie group in $G_1 \times G_2$ for $\Gamma(\tilde{h})$. It is isomorphic to G_1 because the Lie algebra is the same and both are connected, thus a covering map and G_1 is simply connected. □

Prop. (7.1.3) (Closed Subgroup Theorem). If H is a closed subgroup of a Lie group G, then there exists uniquely a differential structure s.t. H is a Lie subgroup of G. Cf.[Helgason Symmetric Spaces].

Prop. (7.1.4) (Ado). Any finite dimensional Lie algebra can be embedded in some $\mathfrak{gl}(n, \mathbb{C})$.

Cor. (7.1.5). From the preceding propositions, it follows that the category of finite dimensional Lie algebras is equivalent to the category of simply connected Lie groups.

2 Generals

Prop. (7.2.1). A connected Lie group is second countable.

Proof: This follows from the fact that a Lie group is a manifold hence locally compact and it is a union of their products. □

Prop. (7.2.2). A continuous homomorphism between Lie groups is smooth.

Proof: use exp coordinates. □

Prop. (7.2.3). Any connected Lie group has a compact subgroup as deformation contraction.

Prop. (7.2.4).

$$SU(2) = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}, \alpha, \beta \in \mathbb{C}$$

is isomorphic to the group of unit quaternions and diffeomorphic to S^3.

3 Classical Groups

For more classical groups, Cf.[Classical Groups Baker].

Fundamental Groups

Prop. (7.3.1).

- $SU(n-1) \to SU(n) \to S^{2n-1}$ gives us $SU(n)$ is simply connected.

$$\pi_1(Sp(2n)) = \pi_1(U(n)) = \mathbb{Z}$$

  and the determinant induces an isomorphism onto $\pi_1(S^1)$. In fact, this is used to define the Maslov index.

- $SO(n-1) \to SO(n) \to S^{n-1}$ gives us $\pi_1(SO(n)) \cong \pi_1(SO(3))$. And $SU(2)$ as the unit sphere in $\mathbb{H}$ maps to $SO(3)$ via the conjugation:$\mathrm{Ad}(z): w \mapsto zw\overline{z}$ has kernel ± 1, so $SO(3)$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$.

Generals

Prop. (7.3.2). As in (7.3.1) $SU(2)$ is a universal covering of $SO(3)$ and so does Spin(3)(10.1.4), so $SU(2) \cong \mathrm{Spin}(3)$.

Prop. (7.3.3). The symplectic group $Sp(2n,\mathbb{R}) = Sp(2n,\mathbb{C}) \cap U(n,n,\mathbb{C}) = Sp(n,\mathbb{H})$. And

$$Sp(2n) \cap O(2n) = Sp(2n) \cap GL(n,\mathbb{C}) = O(2n) \cap GL(n,\mathbb{C}) = U(n) = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}, \quad X + iY \in U(n).$$

Exponential Map

Prop. (7.3.4). The exponential map for $GL_n(\mathbb{C})$ and $U(n)$ is surjective and the image of the exponential map for $GL_n(\mathbb{R})$ is $GL_n(\mathbb{R})^2$.

Proof: Use Jordan Decomposition (Real). For complex case, it is unitary diagonalizable. □

4 Analysis

Lemma (7.4.1). Bi-invariant metric exists in a compact manifold.

Proof: Because the Haar measure on a compact metric is bi-invariant. Choose a Riemann metric and set

$$\langle V, W\rangle = \int_{G\times G} \langle L_{\sigma *}R_{\tau *}(V), L_{\sigma *}R_{\tau *}(W)\rangle d\mu(\sigma)d\mu(\tau).$$

Note that L_* and R_* commute. □

Prop. (7.4.2). If G is a Lie group with a bi-invariant metric, then

$$2\nabla_X Y = [X,Y], \quad \langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle,$$

$$\nabla_X Y = 1/2[X,Y], \quad R(X,Y)Z = 1/4[[X,Y],Z], \quad K(\sigma) = 1/4\Big|[X,Y]\Big|^2.$$

So its curvature is non-negative, and all 1-parameter subgroups are geodesic.

Prop. (7.4.3). A bi-invariant Lie group with $\mathfrak{g}$ having trivial center is compact and $\pi_1(G)$ finite.

Proof: From Myer Theorem because the Ricci curvature has a positive lower bound.
Cf.[Morse Theory Milnor Prop20.5]. □

Prop. (7.4.4) (Structure theorem for bi-invariant Lie group). A simply connective Lie group with a bi-invariant metric is equal to $G^{'} \times R^k$, $G^{'}$ compact.

Proof: Because the orthogonal complement of the center of $\mathfrak{g}$ is a Lie algebra, G is like $G^{'} \times R^k$, and a simply connected abelian Lie group is R^k ?. □

5 Symmetric space

Prop. (7.5.1). A **symmetric space** is that for every point p, there is a isometry reversing the geodesics passing p. A manifold is called **locally symmetric** if $\nabla R = 0$. Locally symmetric is equivalent to the fact that every local reversing map is an isometry. A symmetric space is complete because two folding is an extension of geodesic.

Prop. (7.5.2). A Lie group with a bi-invariant metric is a symmetric space.

Prop. (7.5.3). The conjugate points in a symmetric space is easy to calculate, they are $\exp(\frac{\pi k}{\sqrt{e_i}}V)$, counting multiplicity, where e_i is the eigenvalue of the sself-adjoint operator $K_V(W) = R(V, W)V$ at p.

III.8 Other Geometries

1 Hyperbolic Geometry

Prop. (8.1.1). Isometries of hyperbolic ball are all given by Mobius transformations, because the distance to three non-colinear point can localize a point. Cf.[双曲几何 刘毅].

2 Metric Geometry

Def. (8.2.1) (Hausdorff dimension). $\dim^H(X)$.

Def. (8.2.2). The **Hausdorff distance** for two subset $Y_1, Y_2 \in X$ is the

$$d_X^H(Y_1, Y_2) = \inf\{\varepsilon | Y_2 \subset B(Y_1, \varepsilon), Y_1 \subset B(Y_2, \varepsilon)\}.$$

The **Gromov-Hausdorff metric** for two metric space is

$$d^{GH}(X_1, X_2) = \inf\{d_Z^H(i_1(X_1), i_2(X_2))\}$$

where i_1, i_2 are isometry of X_1, X_2 into a metric space Z.

This metric makes the set of all compact metric space into a complete Hausdorff space $\mathcal{MET}$.

Def. (8.2.3). A map from X to Y is called a ε**-approximation** iff $B(f(X), \varepsilon) = Y$ and $|d(x, y) - d(f(x), f(y))| \le \varepsilon$.

We have: if there is a ε approximation, then $d^{GH}(X, Y) \le 3\varepsilon$, and if $d^{GH}(X, Y) \le \varepsilon$, there is a 3ε approximation.

Prop. (8.2.4). Fix a function $N : (0, 1) \to \mathbb{N}$, the space $\mathcal{MET}(D, N)$ of complete metric space of diameter bounded by D and for every ε, there is a ε-net with no more then $N(\varepsilon)$ points. Then it is a compact subspace of $\mathcal{MET}$.

Proof: We show it is totally bounded and closed. It is totally bounded because the space of discrete space of no more than $N(\varepsilon)$ is compact and it ε approximate $\mathcal{MET}(D, N)$ by definition. Thus we have it is totally bounded. And □

Prop. (8.2.5) (Gromov Compactness Theorem). Denote the space $\mathcal{RIC}^D_{*,-1}(n)$ of manifold with Ricci curvature bounded below by -1 and diameter bounded above by D, then it is a precompact subset of $\mathcal{MET}$.

Proof: By Bishop-Gromov(2.5.17), there is a $N(\varepsilon)$ that M can only have $N(\varepsilon)$ many balls of radius ε, because M has bounded diameter (Packing argument). So $\mathcal{RIC}^D_{*,-1}(n) \subset \mathcal{MET}(D, 2N)$ is precompact. □

Prop. (8.2.6). Any metric space X in the closure of $\mathcal{RIC}^D_{*,-1}(n)$ has Hausdorff dimension $\dim^H(X) \le n$.

Prop. (8.2.7) (Gromov). If a sequence of manifold $\{M_i\}$ in $\mathcal{M}^{D,k}_{V,-k}(n)$, then they has a limit point $X \in \mathcal{MET}$. Then X is a C^∞ manifold and there is a $C^{1,\alpha}$-metric for every $\alpha < 1$. And M_i are all diffeomorphic to X for large X.

In particular, this implies that there are only finitely many diffeomorphic classes.

Prop. (8.2.8) (Peterson). $\mathcal{M}^D_{*,v,k}(n)$ has only finitely many homotopy classes.

3 Spectral Geometry

Chapter IV

Analysis

IV.1 Real Analysis

Basic references are [Folland Real Analysis].

1 Basics

Prop. (1.1.1). A function f is real analytic on an open set iff there is a extension to a complex analytic function to an open set. And this is equivalent to: For every compact subset, there is a constant C that for every positive integer k, $|\frac{d^k f}{dx^k}(x)| \leq C^{k+1}k!$.

Proof: Use Lagrange residue(中值定理) to show that it will converge to f. □

Prop. (1.1.2) (convergences). There are three different kinds of convergences.

Prop. (1.1.3) (Dominant Convergence Theorem).

Prop. (1.1.4). For a pair of Hilbert basis $\{e_i\}$ of $L^2(M)$ and $\{f_j\}$ of $L^2(N)$, $\{e_i \otimes f_j\}$ gives a basis for $L^2(M \times N)$. (Use Fubini).

Prop. (1.1.5) (Fubini-Tonelli). For two σ-finite measure space, if $f \in L^+(X \times Y)$, then $f_x \in L^+(Y)$ and $f^y \in L^+(X)$, and $\int_{X\times Y} f dxdy = \int_Y \int_X f dxdy = \int_X \int_Y f dydx$.

If $f \in L^1(X \times Y)$, then $f_x \in L^1(Y)$ and $f^y \in L^1(X)$, a.e. and the product formula is definable and holds.

Proof: Cf. [Folland P67]. □

2 Approximations

Prop. (1.2.1). The polynomial functions are dense in $C[-1, 1]$.

Proof: We only have to prove that $|x|$ can be approximated, because then all piecewise linear function can. For this, Taylor expand $\sqrt{1 + (x^2 - 1)}$. (or we can use Stone-Weierstrass). □

Prop. (1.2.2) (Stone-Weierstrass Approximation). If a unital C^*-algebra of continuous functions on a compact Hausdorff space separates points, then it is dense in $C(X)$.

Proof: This is a consequence of Bishop theorem(3.5.13) □

Prop. (1.2.3). for $1 \leq p < +\infty$, $C(X)$ are dense in $L^p(X)$ for a Radon measure, but not for $p = \infty$.

Proof: Use finite stair approximation and then inner regular approximation and then Tietz extension. □

Prop. (1.2.4) (Approximate Identity). A family of $L^\infty(\mathbb{T})$ functions $\{\Phi_N\}$ are called an approximate identity if:

1. $\int_0^1 \Phi_N(x)dx = 1$.
2. $\sup\int_0^1 |\Phi_N(x)|dx < \infty$.
3. For any $\delta > 0$, $\int_{|x|>\delta} |\Phi_N(x)|dx \to 0$ as $N \to +\infty$.

For any approximate identity, if $f \in C(\mathbb{T})$ or $L^p(\mathbb{T})$ for $1 \leq p < +\infty$, then $\Phi_N * f \to f$.

Proof: Use uniform continuity and also use continuous approximation (1.2.3). □

Cor. (1.2.5). for $1 \leq p < +\infty$,trigonometric polynomials are dense in $L^p(\mathbb{T})$ and $C(\mathbb{T})$, but not for $p = \infty$. So $e^{2\pi inx}$ forms an orthogonal basis in $L^2(\mathbb{T})$.

Thus, the Parseval's identity holds.

Proof: Just use the fact that Fejer kernels are an approximate identity. □

Prop. (1.2.6). For a integrable function u that has compact support, $u_\delta = j_\delta * u$ is a smooth function of compact support that $||u_\delta - u||_{C^k} \to 0$ when $u \in C^k$. Where j_δ is the scaling of a smooth function of compact support. So Smooth function of compact support are dense in C_0^k.

Prop. (1.2.7). $D(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$.

Proof: Use the fact that C_0 are dense in L^p by(1.2.6). And $f_\delta \to f$ in L^p norm for $f \in C_0$. So we can use the three-part argument applied to $D_\alpha u$ to get $D_\alpha(u_\delta) \to D_\alpha u$ in L^p norm for $|\alpha| \leq m$. Thus the result. □

3 Convolution

Prop. (1.3.1). Convolution with a smooth function makes the function smooth, in particular, $\frac{\partial}{\partial x}(f * g) = \frac{\partial f}{\partial x} * g$.

Prop. (1.3.2) (Young's Inequality). $||f * g||_r \leq ||f||_p ||g||_q$ for all $1 \leq r, p, q \leq \infty$ and

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

In particular, $||K * f||_p \leq ||K||_1 ||f||_p$.

Proof: By Riesz representation, it suffices to show that: for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$,

$$\int\int f(x)g(y-x)h(y) \leq ||f||_p ||g||_q ||h||_r.$$

write the LHS as

$$\int\int (f^p(x)g(y-x)^q)^{1-\frac{1}{r}} (f^p(x)h^r(y))^{1-\frac{1}{q}} (g^q(y-x)h^r(y))^{1-\frac{1}{p}}$$

and use Holder inequality. □

4 Measures

Def. (1.4.1). A Brel measure μ is called **inner regular** iff $\mu(E) = \inf\{\mu(K)|K \subset E\text{compact}\}$ for every Borel set E. It is called **outer regular** iff $\mu(E) = \sup\{\mu(U)|E \subset U\text{open}\}$.

A **Radon measure** is a Borel measure that is finite on compact set, outer regular on Borel sets, inner regular on open sets.

Prop. (1.4.2) (Radon-Nikodym). If two σ-finite measures v, μ on a measurable space satisfies v is absolutely continuous w.r.t μ, then there is a μ-integrable function f such that

$$dv = f d\mu.$$

Cor. (1.4.3). Special case of the Freudenthal spectral theorem (3.8.15).

Prop. (1.4.4) (Riesz Representation Theorem). on $C_c(X)$ for a LCH space X,

If I is a positive linear functional, there is a unique regular (both inner and outer) Radon measure μ on X such that $I(f) = \int f d\mu$. Moreover,

$$\mu(U) = \sup\{I(f) : f < U\} \text{ for } U \text{ open},$$

$$\mu(K) = \inf\{I(f) : f > \chi(K)\} \text{ for } K \text{ compact}.$$

If I is a continuous linear functional, there is a unique regular countably additive complex Borel measure μ on X that $I(f) = \int f d\mu$.

So if X is compact, $M(X)$ the space of Borel measures on X is the dual spcae of $C(X)$.

Proof: Cf.[Real Analysis Folland P212]. □

IV.2 Complex Analysis

1 Topology

Prop. (2.1.1). A first differentiable conformal map in $\mathbb{C}$ is holomorphic or anti-holomorphic. Cf.[Ahlfors P74]. In higher dimension, conformal is equivalent to $\langle df_p(v_1), df_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle$.

Prop. (2.1.2). The roots of a polynomial depends continuously on the coefficients. (Use Rouch Principle).

2 Basics

Prop. (2.2.1). For the equation $(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})f=0$ or $(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})u=0$, the function is elliptic, so by(6.8.4), the solution of the equation is automatically smooth, so no smoothness condition need to be added.

Prop. (2.2.2) (Uniqueness). If the zeros of a holomorphic function f has a convergent point in the domain of definition, then $f=0$.

Def. (2.2.3). We introduce the following notation:

$$\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}),\quad \frac{\partial}{\partial \overline{z}}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}),\quad dz=dx+idy\quad d\overline{z}=dx-idy.$$

Then dz is dual to $\frac{\partial}{\partial z}$ and $d\overline{z}$ is dual to $\frac{\partial}{\partial \overline{z}}$. And for any function f,

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial \overline{z}}d\overline{z}.$$

3 Theorems

Prop. (2.3.1) (Uniformization Theorem). Any connected Riemann Surface is the quotient by a discrete subgroup of $\mathbb{C},\mathbb{H}$ or $\mathbb{P}^1$.

Proof: □

Prop. (2.3.2) (Runge's Theorem). Let K be a compact subset of $\overline{\mathbb{C}}$ and let f be a function which is holomorphic on an open set containing K. If A is a set containing at least one complex number from every bounded connected component of $\overline{\mathbb{C}}\backslash K$, then there exists a sequence of rational functions which converges uniformly to f on K and all the poles of the functions are in A.

Proof: □

Prop. (2.3.3) (Mergelyan's theorem). If K is compact in $\mathbb{C}$ and f is a continuous function on K that is holomorphic in $\mathrm{int}(K)$, then f can be uniformly approximated by polynomials.

Prop. (2.3.4) (Montel's Theorem). Sets of holomorphic functions bounded in the topology of $H(\Omega)$, inter convex uniform convergence, is sequentially compact.

Proof: □

4 Multi-Variable case

Basics

Should cover the part from [Complex Analytic and Differential Geometry Demailly], [Principle of Algebraic Geometry Griffith/Harris] and [Complex Geometry Daniel].

Def. (2.4.1). A function is called **holomorphic** in several variables iff it is holomorphic for each indeterminate.

Prop. (2.4.2) (Hartog's Extension Theorem). If K is a compact subset in an open domain Ω of $\mathbb{C}^n(n \geq 2)$ and $\Omega - K$ is connected, then any holomorphic function on $\Omega - K$ extends to a holomorphic function on Ω.

IV.3 Functional Analysis

Reference: [Rudin Functional Analysis].

1 Topological Vector Space

Def. (3.1.1). A **topological vector space**(TVS) over a valued field k is a k-vector space that the addition and scalar multiplication is continuous.

Def. (3.1.2). there are different topology in the space of operators on a Hilbert space.
Norm operator topology: $||H_i - H|| \to 0$.
Strong operator topology: $\forall u, ||(H_i - H)u|| \to 0$.
Weak operator topology: $\forall u, v, \langle H_i(u), v\rangle \to \langle H(u), v\rangle$

Def. (3.1.3). A space is called a F**-space** if its toopology is induced by a complete invariant metric. F-space is of second Baire category by(1.7.1)

A locally convex F-space is called a **Fréchet space**.

A TVS is said to satisfy **Heine-Borel** iff every closed and bounded subset of X is compact.

Def. (3.1.4). A **sublinear functional** is a function p that $p(x+y) \leq p(x)+p(y)$ and $p(\lambda x) = \lambda p(x)$ for $\lambda > 0$.

A **seminorm** is a non-negative function p that $p(x + y) \leq p(x) + p(y)$ and $p(\alpha x) = |\alpha|p(x)$ for all complex α.

Prop. (3.1.5) (Minkowski Functional). The set of seminorms/sublinear functionals correspond 1-to-1 with convex balanced absorbing open sets containing 0 through Minkowski functional. and it is uniformly continuous iff 0 is an interior point.

Prop. (3.1.6). A separating family of seminorms is equivalent to a convex balanced local basis at 0. And it generate a metric making the space a Fréchet space. Cf.[Rudin P27].

Prop. (3.1.7). In a locally bounded space, if E is totally bounded, then $\mathrm{co}(E)$ is totally bounded. Thus in a Fréchet space, the closed convex closure of a compact set is compact.

Prop. (3.1.8). There is only one topology on a finite dimensional space and it is complete. A TVS is locally compact iff it is f.d. Cf.[Rudin P17].

Prop. (3.1.9). In Fréchet space, a closed subset is compact iff it is totally bounded by(1.6.1).

Prop. (3.1.10). If a subspace of a TVS is a F-space, then it is closed in it. Cf.[Rudin P21].

Cor. (3.1.11). A f.d subspace in a TVS is closed.

Prop. (3.1.12) (Schauder Fixed Point Theorem). If C is a closed convex subset in a normed space and $T : C \to C$ has sequentially complete image, e.g. C is compact, then T has a fixed point.

Proof: Use a $1/n$-net and construct a contraction to their convex hull. Then use Brauwer fixed point theorem to find $Tx_n = x_n$, and choose a convergent point x to show $Tx = x$. □

2 Various Spaces and Duality

For a bounded connected open set Ω,

- **Sobolev Space** $W^{m,p}(\Omega)$ is the completion of a subspace of $C^\infty(\Omega)$ with the norm
$$||u||_{m,p} = (\sum_{|\alpha|\leq m}\int_\Omega |\partial^\alpha u(x)|^p \,\mathrm{dx})^{1/p}.$$
for $m > 0$. And we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. It is also a subspace of $L^p(\Omega)$ that satisfies this, without completion(5.3.1).
- $C_0^\infty(\Omega)$ is the subspace of $C^\infty(\Omega)$ that have compact support in Ω. Its completion $W_0^{m,p}(\Omega)$ is a closed subspace of $W^{m,p}(\Omega)$. And we denote $W_0^{m,2}(\Omega)$ by $H_0^m(\Omega)$ and the dual space of $H_0^m(\Omega)$ by $H^{-m}(\Omega)$.
- $C(\Omega)$ in the topology of compact convergence is a Fréchet space. It is not locally convex.
- $H(\Omega)$ the space of holomorphic functions in Ω is a closed subspace of $C(\Omega)$ thus is a Fréchet space. Montel's theorem says exactly that $H(\Omega)$ is Heine-Borel.
- $C^\infty(\Omega)$ in the topology defined by seminorms $p_N(f) = \max\{|D^\alpha f(x)| : x \in K_N, |\alpha| \leq N\}$, is a Fréchet space thus locally convex and it has the Heine-Borel property by Arzela-Ascoli.
- $D(K)$ is the closed subspace of smooth functions on Ω with support in K, thus a Fréchet space with Heine-Borel property.
- $D(\Omega)$ is the space of smooth functions with support in Ω. It has the topology generated by translation of basis consisting of convex balanced sets W that $W \cap D(K)$ is open for every compact K. This makes $D(\Omega)$ into a locally convex TVS, Cf.[Rudin P152]. It has Heine-Borel property(5.1.1).

Dual Spaces

- For a σ-finite measure μ on a measurable space X, for $1 \leq p < \infty$
$$L^p(X,\Omega,\mu)^* = L^q(X,\Omega,\mu).$$
- $C[0,1]^* = BV[0,1]$ and $C[X]^* = M(X)$, the space of complex measure on compact X with the norm of total variance, by Riesz representation theorem(1.4.4).

3 Dual Space

Prop. (3.3.1) (Closed Range Theorem). Let T be continuous mapping between Banach spaces X and Y, then $T(X) = Y \iff ||T^*y^*|| \geq \delta||y^*|| \iff T^*$ is one-to-one and $R(T^*)$ is closed(By Banach theorem).

$R(T)$ is closed in Y iff $R(T^*)$ is closed in X^*.

Proof: Cf.[Rudin P100]. □

Prop. (3.3.2). For a bounded operator T,
$$\overline{R(T)} = N(T^*)^\perp,\ \text{Thus}\ \overline{R(T^*)} = N(T)^\perp$$

Cf.[Rudin P99].

Weak Convergence

Prop. (3.3.3). In a normed space, iff $x_n \to x$ weakly, then $\underline{\lim}||x_n|| \geq ||x||$. (choose a functional that $||f|| = 1$ and $|f(x)| = 1$).

Prop. (3.3.4). In a Banach space, if $x_n \to x$ weakly, then $\{x_n\}$ is bounded, by Banach-Steinhaus.

Reflexive and Separable

Prop. (3.3.5) (Banach). For a norms space X, if X^* is separable, then X is separable.

Proof: Choose a countable dense set in X^*, then their projection to the unit sphere $\{g_n\}$ is dense, and choose for each of them a x_n that $||x_n|| = 1$ and $g_n(x_n) > \frac{1}{2}$. Use Hahn to show span of $\{x_n\}$ is dense in X. □

Prop. (3.3.6). For an exact sequence of normed spaces $0 \to X \to Y \to Z \to 0$, there is a exact sequence $0 \to Z^* \to Y^* \to X^* \to 0$. This is because Hahn extension.

Prop. (3.3.7) (Pettis). Closed subspace and quotient space of a reflexive normed space is reflexive. (Use the fact that $0 \to X \to Y \to Z \to 0$ induces an exact sequence $0 \to X^{**} \to Y^{**} \to Z^{**} \to 0$, and we use snake lemma.)

Prop. (3.3.8) (Eberlein-Smulian). Closed ball of a reflexive space is weak-sequencially compact. Thus a set is weak-sequencially compact iff it is bounded (the other side use Banach-Steinhaus).

Proof: Cf.[泛函分析张恭庆 P147]. □

Prop. (3.3.9). If a normed space X is separable, then the unit sphere of X^* is weak*-sequentially compact. Cf.[泛函分析张恭庆 P145].

Prop. (3.3.10). A closed convex set of a reflexive Banach space attains minimal norm.

Proof: By Hahn, a closed convex set is weakly closed. cut it with a ball, we can assume it is weak-sequencially compact(3.3.8), thus some $x_n \to x$ weakly. and use(3.3.3), we see x attains minimal norm. □

Prop. (3.3.11) (Banach-Alaoglu). For a nbhd V of 0 in a TVS,

$$K = \{f| \ |fx| \leq 1, \ \forall x \in V\}$$

is weak*-compact,(3.5.8).

4 Completeness

Def. (3.4.1). A F**-space** is a TVS induced by a translation-invariant metric that is complete.

Prop. (3.4.2) (Banach-Steinhaus). Γ is a collection of continuous linear mapping between two TVS, then if the set B of x that $\Gamma(x)$ is bounded is a second category set in X, then $B = X$ and Γ is equicontinuous (thus maps bounded sets to bounded sets).

Similarly, if for a compact convex set K in X, if a set Γ of continuous linear mapping is bounded for every $x \in K$, then Γ is equicontinuous on K.

Proof: For an open set of 0, choose a balanced nbhd U s.t. $\overline{U}+\overline{U}<W$, set $E=\cap_{\Lambda\in\Gamma}\lambda^{-1}(\overline{U})$, then $B\subset\bigcup_{i=1}^{\infty}nE$, so by Baire, E has a interior point thus has a nbhd V s.t. $\Gamma(V)\subset\overline{U}+\overline{U}\subset W$. □

Cor. (3.4.3) (Uniform Boundedness Theorem). If a set Γ of continuous linear mapping from a F-space X to Y satisfies $\Gamma(x)$ is bounded for every x, then Γ is equicontinuous.

Cor. (3.4.4). The dual space of a F-space is complete.

Prop. (3.4.5) (Open Mapping theorem). If a continuous linear mapping T from a F-space X to Y is surjective and $R(T)$ is of second category, then it is a surjective open mapping and Y is a F-space.

Proof: We can show that $T(B(0,\frac{1}{2^n})$ are all of second category, so $\overline{T(B(0,\frac{1}{2^n}))}$ is open, thus for a $y\in T(B(0,1)$ we can consecutively choose $x_n\in B(0,\frac{1}{2^n})$ s.t. $y-\sum_{i\le n}T(x_i)\in\overline{T(B(0,\frac{1}{2^n}))}$. So by completeness of X, $y\in T(B(0.1)$, thus it is open. □

Cor. (3.4.6) (Banach). If a continuous T between F-spaces is a bijection, then it has a continuous inverse.

Cor. (3.4.7). If a F-space is complete w.r.t two different topologies and one is stronger than the other, then they are equivalent.

Cor. (3.4.8). For every operator between F-spaces that has closed image, we have $X/N(T)\cong R(T)$.

Cor. (3.4.9) (Closed Graph Theorem). If T is a closed linear mapping between two F-spaces, i.e. the graph of it is closed, then it is continuous. (Because the metric induced by the graph is stronger than the original one). This is very useful when proving some map is continuous.

Prop. (3.4.10). Any symmetric operator on a Hilbert space is continuous. (Because $x_n\to 0$ implies $Tx_n\to 0$ weakly, so we can use closed graph theorem.

Cor. (3.4.11). If the image of a continuous linear mapping T between F-spaces has finite codimensional image, then the image is closed. (Construct $\mathbb{C}^n\oplus X/N(T)\to Y$, by Banach it is a homeomorphism.

Prop. (3.4.12) (Separate Continuous). If a bilinear map $B:X\times Y\to Z$ where X is a F-space is separately closed, then $B(x_n,y_n)$ converges to $B(x_0,y_0)$. (Use Banach-Steinhaus to prove $B_{y_n}(x)$ is equicontinuous, then use $B(x_n-x_0,y_n)+B(x_0,y_n-y_0)$.

5 Convexity

Hahn-Banach

Prop. (3.5.1) (Real Hahn). For a sublinear functional p on a real linear space X and a subspace X_0, if a functional f satisfies $f(x)<p(x)$ on X_0, then it can be extended to a functional on X with the same condition. (Use Zorn's lemma)

Prop. (3.5.2) (Complex Hahn). For a seminorm p (i.e. it can attain 0) on a complex linear space X and a subspace X_0, if a functional f satisfies $f(x)<p(x)$ on X_0, then it can be extended to a functional on X with the same condition.

Proof: Let $g(x) = \operatorname{Re} f(x)$ and extend it and set $f(x) = g(x) - ig(ix)$. □

Prop. (3.5.3) (Hahn). In a normed space X, a bounded linear functional on a subspace X_0 can extend to a bounded functional on X with the same norm.

Cor. (3.5.4). For every $x \neq 0$, there is a continuous functional f of norm 1 that $f(x) = ||x||$. So continuous functionals can separate points. Thus the inclusion from X to X^{**} is an isometry into and the conjugation $T*$ form $L(X, Y)$ to $L(Y^*, X^*)$ is an isometry into a closed subspace.

Prop. (3.5.5) (Geometric Hahn).

- If E_1 and E_2 are two convex set that $E_1 \cap E_2 = \emptyset$ and E_1 is open, then there is a continuous linear functional that separate them, i.e $\operatorname{Re} f(E_1) < \operatorname{Re} f(E_2)$. (use the continuous sublinear functional for $E_1 - E_2$).
- In a locally convex TVS, if E_1 is convex compact and E_2 is convex closed, then there is a real functional that separate them. Thus for a set E and a point x, $x \in \overline{\operatorname{span} E} \iff$ for all f that $f(E) = 0$, $f(x) = 0$.

Cor. (3.5.6). If a sequence $\{x_n\}$ weak converge to x in a normed space, then convex combination of $\{x_n\}$ strongly converge to x, i.e. $x \in \overline{\mathrm{co}}(\{x_n\})$. The weak closure of a convex set in a locally convex space equals the original closure.

Prop. (3.5.7). For a finite dimensional space in a Banach space, the projection exists. (construct the dual functional for a basis and extends it to a functional using Hahn.

Prop. (3.5.8) (Banach-Alaoglu). For a nbhd V of 0 in a TVS, the set

$$K = \{f |\ |fx| \leq 1,\ \forall x \in V\}$$

is weak*-compact.

Proof: The point is that the weak*-topology coincides with the pointwise convergence topology. And that topology is embedded in a compact space(Tychonoff) and K is a closed subspace of that space. Cf.[Rudin P68]. □

Prop. (3.5.9). In a locally convex space, bounded $\iff$ weakly bounded. Cf.[Rudin Prop3.18].

Prop. (3.5.10). For a commuting family of continuous affine maps from K to K where K is a compact convex set in a TVS, then there is a fixed point.

Proof: Consider the semigroup generated by these maps together with their average, they have a common image, and for this image, consider $p = \frac{1}{n}(I + T + t^2 + \ldots + T^{n-1})(x)$, then $p - Tp = \frac{1}{n}(x - T^n x) \in \frac{1}{n}(K - K)$, thus $p = Tp$ for all T. □

Cor. (3.5.11) (Invariant Hahn). For a commuting family Γ of operators on a normed space and Y a invariant space, then for any Γ-invariant continuous functional y^* has a Γ-invariant Hahn extension. (Consider the action of T^* on all the Hahn extension of f, use Alaoglu).

Krein-Milman theorem

Prop. (3.5.12). For a compact convex set in a TVS that is weak-Hausdorff, then $K = \overline{\text{co}}(\text{Extreme}(K))$.

If K is a compact set in a locally convex space, then $K \subset \overline{\text{co}}(K) = \overline{\text{co}}(E(K))$.

Proof: Show that every extreme set contains a extreme point, and use the geometric Hahn, because the extreme value point for any functional on K is a extreme set. □

Prop. (3.5.13). If K is a compact set in a locally convex space X and if $\overline{\text{co}}(K)$ is also compact, e.g in a Fréchet space, then every extreme point of $\overline{\text{co}}(K)$ lies in K.

Cor. (3.5.14). There is a Bishop theorem that derive Stone-Weierstrass theorem, proved using Krein-Milman.

6 Banach Algebra

Def. (3.6.1). For a bounded operator $A \in L(X)$ where X is Banach space, then a λ is called a:

- **point spectrum** $\rho(A)$ if $(\lambda I - A)^{-1}$ doesn't exists;
- **continuous spectrum** if $R(\lambda I - A) \neq X$ but $\overline{R(\lambda I - A)} = X$.
- **residue point** if $\overline{R(\lambda I - A)} \neq X$.
- **regular point** if $(\lambda I - A)^{-1}$ exists and is continuous, i.e. $R(\lambda I - A) = X$;

denote $\sigma(A) = \mathbb{C}\backslash\rho(A)$ the spectrum of A.

Prop. (3.6.2). $\mathbb{C}\backslash\sigma(T)$ is an open set and $\lambda \to (\lambda I - T)^{-1}$ is a holomorphic function on $\mathbb{C}\backslash\sigma(T)$.

Thus for every bounded operator T, $\sigma(T)$ is not empty, otherwise this holomorphic function is bounded.

Cor. (3.6.3) (Gelfand-Mazur). If in a Banach algebra where all the nonzero element is invertible, then it is isomorphic to $\mathbb{C}$.

Prop. (3.6.4). $\sigma(A) = \sigma(A^*)$. Cf.[张恭庆泛函分析 P218].

Prop. (3.6.5). Notice $(I - T)$ is invertible for $||T|| < 1$ and the derivative can be calculated by definition.

Cor. (3.6.6). The spectrum of an element of a Banach algebra is continuous.

Prop. (3.6.7). In a Banach algebra A, $e - xy$ is invertible iff $e - yx$ is invertible, thus $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.

Proof: Use power expansion to get an expression and prove it is the inverse. □

Prop. (3.6.8) (Gelfand). The spectrum radius

$$r_\sigma(A) = \lim_{n\to\infty} ||A^n||^{\frac{1}{n}} = \inf ||A^n||^{\frac{1}{n}}.$$

So $\sigma(A)$ is compact.

Proof: Use Hadamard radius formula and for the other side, use the fact that $|f(\frac{A^n}{(r_A+\varepsilon)^{n+1}})|$ is bounded, so by uniform boundedness theorem, $\frac{||A||^n}{(r_A+\varepsilon)^{n+1}} < M$ for all n. And $\lambda \in \sigma(A)$ implies $\lambda \in \sigma(A^n)$ thus the limit is well-defined. □

Cor. (3.6.9). For Banach algebra B and its closed subalgebra A, $\sigma_A(x)$ is obtained from $\sigma_B(x)$ by filling some holes. So when $\sigma_B(x)$ doesn't separate $\overline{\mathbb{C}}$ or $\sigma(A)$ has empty interior, then $\sigma_A(x) = \sigma_B(x)$. Cf.[Rudin P256].

Symbolic Calculus

Prop. (3.6.10). For a Banach algebra A. For a domain Ω in $\mathbb{C}$, define A_Ω as the set of x that $\sigma(x) \in \Omega$, it is an open set by(3.6.6), then:

$$f \mapsto \tilde{f}(x) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda e - x)^{-1} d\lambda$$

for any contour Γ that surrounds $\sigma(x)$, is a continuous algebra isomorphism of $H(\Omega)$ into the set of A-valued functions on A_Ω with the compact-open topology.

We have $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$.

Proof: The nontrivial part is that this map is multiplicative, but for this we can use Runge's theorem to approximate any function on $\sigma(x)$. □

This theorem makes it possible to implant complex analysis to the study of Banach Algebra.

Cor. (3.6.11). $\exp(x)$ is defined on A and is continuous. If $\sigma(x)$ doesn't separate 0 from ∞, then $\log(x)$ is defined but might not be continuous.

Prop. (3.6.12) (Spectral Mapping Theorem). $\tilde{f}(x)$ is invertible in A iff $f(\lambda) \neq 0$ on $\sigma(x)$. Thus we have $\sigma(\tilde{f}(x)) = f(\sigma(x))$.

Prop. (3.6.13). If f doesn't vanish identically on any component of Ω, then $f(\sigma_p(T)) = \sigma_p(\tilde{f}(T))$. Cf.[Rudin P266].

Commutative Banach Algebra

Prop. (3.6.14). For A a commutative Banach algebra, the set of maximal ideals has codimension 1 corresponds to kernels of complex homomorphisms to $\mathbb{C}$. (Consider the quotient space and use Gelfand-Mazur). Note that a complex homomorphism is all continuous because $\lambda e - x$ maps to nonzero.

$\lambda \in \sigma(x)$ iff there is a complex homomorphism h s.t. $h(x) = \lambda$. (Because x is invertible iff it is not contained in any proper ideal of A.

Prop. (3.6.15) (Gelfand Transform). The set Δ of maximal ideals of a commutative Banach algebra is a compact Hausdorff space w.r.t to the weak*-topology and the Gelfand transform: $x \mapsto \hat{x}(h) = h(x)$ is a map of A into $C(\Delta)$. And the range of $\hat{x}$ equals $\sigma(x)$, so $||\hat{x}|| = \rho(x) \leq ||x||$.(Use Alaoglu).

Prop. (3.6.16). For $A = C(X)$ where X is compact Hausdorff, Δ is homeomorphic to X. (otherwise it has finite $f_i \neq 0$, then $\sum |f_i|^2$ is positive thus invertible but maps to 0). In fact, for a space X, $\Delta(C(X))$ is the stone-Čech compactification of X.

Prop. (3.6.17). For $A = L^\infty(m)$, the spectrum of f is just the essential range of f.

Lemma (3.6.18). If $\hat{A} \subset C(\Delta)$ with a chosen topology that makes it compact, and A separate points, then the topology of it is the same of the weak*-topology. (Compact to Hausdorff).

Prop. (3.6.19). The algebra $L^1(\mathbb{R}^n) + \delta$ with the multiplication by convolution has the spectrum $R^n \cap \{\infty\}$. (Use $L^{p*} = L^q$ and see when will it be homomorphism).

7 B^*-Algebra and Hilbert space

Prop. (3.7.1). A closed convex subset in a Hilbert space has a unique element that attains the minimum norm, because it is reflexive(3.3.10).

Cor. (3.7.2). The orthogonal complement of a closed subspace of a Hilbert space exists. and the projection on to a closed subspace exists. This is a good trait of Hilbert space.

Prop. (3.7.3). Linear functionals on a Hilbert space is all of the form $x \mapsto (x, z)$ (Choose an orthogonal of the kernel).

Prop. (3.7.4). For a Hilbert space, the adjoint operation serves as an involution and makes $B(H)$ into a B^*-algebra, i.e. $||T^*T|| = ||T||^2$. (In fact, $||T|| = ||T^*|| = \sup\{(Tx, y)|||x||, ||y|| \leq 1\}$).

B^*-algebra

Prop. (3.7.5). A **B^*-algebra** is a Banach algebra with an involution s.t. $||xx^*|| = ||x||^2$. Any B^*-algebra is isomorphic to a closed subspace of $B(H)$ for some Hilbert space. Cf.[Rudin P338].

Prop. (3.7.6) (Gelfand-Naimark). For a commutative B^*-algebra, the Gelfand transform is an isomorphism from A to $C(\Delta)$ with $||x|| = ||\hat{x}||_\infty$ and $\widehat{x^*} = \overline{\hat{x}}$.

Proof: First use $||xx^*|| = ||x||^2$ to prove that a hermitian element is mapped to real function, and use Stone-Weierstrass to show that the image is dense, then let $y = xx^*$ and $||y^{2^m}|| = ||y||^{2^m}$ to prove $||\hat{x}|| = ||x||$, so its image is closed. □

Now we want to use commutative algebra methods in the non-commutative case, there are two ways.

Prop. (3.7.7). For a commutative set of elements S in A, Γ the set of elements that commute with S, then $B = \Gamma(\Gamma(S))$ is commutative and contains S. And $\sigma_B(x) = \sigma_A(x)$ for $x \in B$.

Proof: Because $S \subset \Gamma(S)$, $\Gamma(\Gamma(S)) \subset \Gamma(S)$, thus $\Gamma(\Gamma(S))$ is commutative. And if $xy = yx$, then $x^{-1}y = yx^{-1}$. □

Cor. (3.7.8). In a Banach algebra, if x, y commutes, then

$$\sigma(x + y) \subset \sigma(x) + \sigma(y), \quad \sigma(xy) \subset \sigma(x)\sigma(y).$$

(because $\sigma(x)$ is just the range of $\hat{x}$ on Δ that x, y generated).

The second method applies to normal elements:

Prop. (3.7.9). In a Banach algebra with an involution, a set S is called **normal** if it is commutative and $S^* = S$. A maximal normal set B is a closed subalgebra and $\sigma_B(x) = \sigma_A(x)$.

Proof: Cf.[Rudin P294]. □

Cor. (3.7.10). In a B^*-algebra A,

- Hermitian elements have real spectra.
- If x is normal, then $\rho(x) = ||x||$.
- If $u, v \geq 0$, then $u + v \geq 0$, i.e. $\sigma(u + v) \subset [0, \infty)$.

- $yy^* \geq 0$. Thus $e + yy^*$ is invertible.

Proof: Cf.[Rudin P295]. □

Prop. (3.7.11). In a Banach algebra with an involution, a **positive functional** is such that $F(xx^*) \geq 0$. It has the following properties.
- $F(x^*) = \overline{F(x)}$ and $|F(xy^*)|^2 \leq F(xx^*)F(yy^*)$. (Use Swartz like trick).
- $|F(x)|^2 \leq F(e)F(xx^*) \leq F(e)^2\rho(xx^*)$, because $e = ee^*$. Thus $|F(x)| \leq F(e)\rho(x)$ for every normal x (By the last prop), so $||F|| = F(e)$ if A is commutative.

Cf.[Rudin P297].

Prop. (3.7.12). If A is a commutative Banach algebra with an involution that $h(x^*) = \overline{h(x)}$, then The map

$$\mu \to F(x) = \int_\Delta \hat{x} d\mu$$

is a one-to-one correspondence between the convex set of μ that $\mu(\Delta) \leq 1$ to the convex set K of positive functionals on A of norm ≤ 1, i.e. $F(e) \leq 1$, so maps the extreme points, i.e. the point mass to extremes points, thus the extreme points of K is exactly Δ. This can be used to prove **Bochner's theorem**.

Proof: Use the last prop to show that there is a functional on $C(\Delta)$ and use Riesz representation. It is positive and by Stone-Weierstrass, it is unique. □

8 Spectral Theory on Hilbert Spaces

Resolution of Identity

Def. (3.8.1). A **resolution of identity** on a Hilbert space H for a σ-algebra on a set Ω is a E that:
1. $E() = 0, E(\Omega) = 1$.
2. $E(\omega)$ is self-adjoint projection.
3. $E(\omega' \cap \omega) = E(\omega')E(\omega)$.
4. $E(\omega \cup \omega') = E(\omega) + E(\omega')$ for disjoint ω, ω'.
5. $E_{x,y}(\omega) = (E(\omega)x, y)$ is a complex measure on E.

Thus for any x, $\omega \to E(\omega)x$ is a countably additive H-valued measure.

This will generate an isometric*-isomorphism Ψ of the Banach algebra $L^\infty(E)$ onto a closed normal subalgebra A of $B(H)$. (Define on simple function first).

$$\Psi(f) = \int_\Omega f dE, \quad (\Psi(f)x, y) = \int_\Omega f dE_{x,y}$$

Prop. (3.8.2) (Spectral Decomposition). For any closed B^*-algebra A of $B(H)$, there is a unique resolution E of identity on the Borel subsets of Δ that the inverse of Gelfand transform extends to an isometric *-isomorphism Φ of the algebra $L^\infty(E)$ to a closed subalgebra B containing A. Cf.[Rudin P322]. In fact, $B = \Gamma(\Gamma(A))$ is normal by Fuglede theorem(3.8.8).

Cor. (3.8.3) (Generalized Symbolic Calculus). If T is a commutative B^*-algebra that x, x^* topologically generate A, then the map

For a normal operator T and the closed normal B^*-algebra it generates, we have $\hat{x}$ maps $\Delta \cong \sigma(x)$ and the inverse of Gelfand transform (by Naimark) gets us a map $\Psi : C(\sigma(x)) \to A$ that $\Psi(z) = x$, $\Psi(\overline{z}) = x^*$. And this can be extended to a resolution of identity on the Borel set of $\sigma(T)$ that maps $L^\infty(m)$ to $B(H)$. $||\Psi(f)|| = ||f||_\infty$.

Cor. (3.8.4). If T is normal, then

1. $||T|| = \sup\{|(Tx, x)| \; ||x|| = 1\}$.
2. T is self-adjoint iff $\sigma(T)$ is real.
3. T is unitary iff $|\sigma(T)| = 1$.

Proof: For 1, use the fact that $||T|| = \rho(T) = ||z_0||$ for some $z_0 \in \rho(T)$, then use Urysohn to show $E(U) \neq 0$ for a open U near x, then there are x_0 that $E(U)x_0 = x_0$ and use $f = z - z_0$ to show that this x_0 get near $||T||$. $\square$

<u>**Normal Operator on Hilbert Space**</u>

Prop. (3.8.5) (Normal Operators). An operator is normal iff $||Tx|| = ||T^*x||$. So we have $N(T) = N(T^*)$ thus $\sigma_p(T^*) = \overline{\sigma_p(T)}$. And different eigenspaces are orthogonal.

An operator is unitary iff $R(U) = H$ and $||Ux|| = ||x||$ for every x. (Because an operator is defined by its value (Tx, y).

Prop. (3.8.6). For a normal operator T on a Hilbert space, $N(T) = R(T)^\perp$, so T is invertible iff there is a δ that $||Tx|| = ||T^*x|| \geq \delta||x||$.

Prop. (3.8.7) (Polar Decomposition). A positive operator is self-adjoint and has positive spectrum, they have a positive square root, by(3.8.6).

So polar decomposition exists in $B(H)$ and normal operator has commuting decomposition. Thus two similar normal operator are unitarily equivalent, (use Fuglede).

Prop. (3.8.8) (Fuglede). If N_1 and N_2 are normal operators and A is a bounded linear operator on a Hilbert space such that $N_1A = AN_2$, then $N_1^*A = AN_2^*$.

Proof: For any $S \in B(H)$, $\exp(S - S^*)$ is unitary thus $||\exp(S - S^*)|| = 1$, $\exp(N_1)A = A\exp(N_2)$. So we have

$$||\exp(\lambda N_1^*)T\exp(-\lambda N_2^*)|| \leq ||T||$$

because λN_i is normal. Thus by Liouville, compare the coefficients of λ, we get the result. $\square$

Prop. (3.8.9). An operator $A \in B(H)$ has the same spectrum w.r.t all the closed *-algebras of $B(H)$.

Proof: Because TT^* is self-adjoint thus has real spectrum so doesn't separate $\mathbb{C}$ thus it is invertible in any closed B^*-algebra of $B(H)$(3.6.9). so does $T^{-1} = T^*(TT^*)^{-1}$. $\square$

Prop. (3.8.10). For T normal and E its spectral decomposition, then if $f \in C(\sigma(T))$ and $\omega_0 = f^{-1}(0)$, then $N(f(T)) = R(E(\omega_0))$.

Proof: $\chi_E f = 0$, and let $\omega_n = f^{-1}([1/(n-1), 1/n))$, then $E(\omega_n)x = 0(f(T)x = 0)$, so countable additivity shows that $E(\sigma\backslash\omega_0)x = 0$, so $E(\omega_0)x = x$. $\square$

Cor. (3.8.11).

1. $N(T-\lambda I) = R(\{\lambda\})$.
2. every isolated point of $\sigma(T)$ is point spectra, because this point is open thus is $E(\{x\}) \neq 0$.
3. if $\sigma(T)$ is countable, then every $x \in H$ has a unique orthogonal decomposition $x = \sum E(\lambda_i)x$ and $T(E(\lambda_i)x) = \lambda_i E(\lambda_i)x$.

Normal Compact Operator

It is assumed to be an operator on a Hilbert space.

Prop. (3.8.12). A normal operator $T \in B(H)$ is compact iff $\sigma(T)$ has no limit point except 0 and $\dim N(T-\lambda I) < \infty$ for $\lambda \neq 0$. In particular, a normal compact operator is a limit of f.d. operators.

Cor. (3.8.13) (Spectral Theorem). A compact normal operator (in particular a normal operator on a f.d linear space) is unitarily diagonalizable. (Use resolution of identity(3.8.11)).

Cor. (3.8.14) (Hilbert-Schimidt). For a symmetric compact operator A on a Hilbert space H, there is a set of orthonormal basis that A is diagonal on it. And of course, its eigenvalue is real and converges to 0.

Prop. (3.8.15) (Freudenthal Spectral Theorem).

Trace Class and Hilbert-Schmidt Operator

9 Compact Operator & Fredholm Operator

Def. (3.9.1). An operator between Banach spaces is called **compact** if it maps bounded set to sequentially compact(Closure compact) set. It is necessarily continuous because the norm function is continuous thus $||Tx||$ is bounded on the unit ball.

Prop. (3.9.2). Examples of compact operators conclude

- $Lu(x) = \int_X K(x,y)u(y)dy$ for X compact and $K \in C(X \times X)$. This is a compact operator on $C(X)$ by Arzela-Ascali.
- $Lu(x) = \int_\Omega K(x,y)u(y)dy$ for $K(x,y) \in L^2(\Omega)$. This is a compact operator on $L^2(\Omega)$, because we only need to show this is totally continuous(3.9.4). For this, we use(3.3.4) and dominant convergence.

Prop. (3.9.3). The space of compact operator is a closed subspace of $L(X,Y)$. (Use Hausdorff theorem(1.6.1) to show a limit is totally bounded). Thus the limit of f.d. operators is compact.

If one of A or B is compact and the other is continuous, then AB is compact, because continuous maps bounded to bounded and compact to compact.

Prop. (3.9.4). Let $x_n \to x$ weakly, if T is compact, then $Tx_n \to T_n$ strongly. The converse is true when X is reflexive. In particular, this applies to Hilbert space.

Proof: $x_n \to x$ weakly, so $Tx_n \to Tx$ weakly because T is continuous, and $\{x_n\}$ is bounded(3.3.4), so by T compact, there is a $Ax_n \to z$ strongly, thus $z = Ax$. The converse is easy by(3.3.8). $\square$

Prop. (3.9.5). T is compact $\iff$ T^* is compact.

Proof: We need only to show that $T^*y_n^*$ has a uniformly convergent subsequence on the unit sphere, but fo this it suffice to prove y_n^* is sequencially compact in $C(\overline{T(B(0,1))})$. And we use Arzela-Ascoli because $\overline{T(B(0,1))}$ is compact. For the other half, use the double dual space. □

Prop. (3.9.6) (Riesz-Fredholm). For a compact operator $A \in L(X)$, let $T = I - A$. Then:

1. $0 \in \sigma(A)$ if X is not f.d.
2. A is Fredholm of index 0. Equivalently, $\sigma(A)\backslash\{0\} = \sigma_p(A)\backslash\{0\}$(because either T not injective or T is surjective).
3. $\sigma(A)$ has at most one convergent point 0 (it must attain 0 if X is a infinite-dimensional). Hence it at most countable spectrum.

Proof: 1: If 0 is a regular value, then T is invertible, thus $T^{-1}T = \mathrm{id}$ is compact, thus X has f.d.(3.1.8).

For 2,3, it suffices to find a convergent series that cannot converge. Cf.[泛函分析张恭庆 P216,P223].

First prove this for $N(T) = 0$. In this case, if $R(T) \neq X$, then $R(T^1) \supset R(T^2) \supset \ldots$ and are all closed. Thus we can find $y_n \in R(T^n)\backslash R(T^{n+1})$ that $||y_n|| = 1$ and $\mathrm{dist}(y_n, T^{n+1}) \geq 1/2$, Now observe $|Ay_n - Ay_{n+p}| = |y_n - Ty_n + Ty_{n+p} - y_{n+p}| \geq 1/2$ because $-Ty_n + Ty_{n+p} - y_{n+p} \in R^{n+1}$. Thus contradicting the fact that A is compact. □

Prop. (3.9.7) (Jordan Decomposition for Compact Operators). For a compact operator A and all the non-zero eigenvalues λ_i, we can find a subspace

$$\bigoplus_{i=1}^{\infty} N((\lambda_i - A)^{p_i})$$

on which A has a Jordan decomposition.

Proof: By**??**, we only have to prove there is a $R(T^m) \oplus N(T^n) = X$. But the same proof as in(3.9.6) for the case $N(T) = 0$ shows that $p < \infty$, and(3.9.6) show that $R(T^p) = R(T^{p+1})$ also and $p = q$. Thus**??** agains shows that $R(T^p) \oplus N(T^q) = X$. □

Def. (3.9.8). A bounded operator between Banach space is called a **Fredholm operator** if $\dim N(T) < \infty$ and $\mathrm{codim} R(T) < \infty$. It necessarily has closed image(3.4.11), so $R(T) = N(T^*)^\perp$(3.3.2). The index is defined as $\mathrm{ind}(T) = \dim N(T) - \mathrm{codim} R(T)$, thus for a compact operator A, $I - A$ has index 0.

Prop. (3.9.9). For a Fredholm operator, we have

$$X = N(T) \oplus R(T) \quad Y = Y/R(T) \oplus R(T)$$

and $X/N(T) \cong R(T)$. because $R(T)$ and $N(T)$ is closed.

Prop. (3.9.10) (Characterization of Fredholm Operator). T is Fredholm from X to Y iff there exist a bounded S from Y to X that $S_1T = I - A_1, TS_2 = I - A_2$, where A_1, A_2 is compact. S_1 and S_2 can be chosen the same, so S is Fredholm as well.

So the Fredholm operator is the set of operators 'invertible module compact ones'.

Proof: Because $R(T)$ is closed, we have $X/N(T) \cong R(T)$ by Banach, and we have a projection of Y onto $R(T)$ by(3.9.9). Thus we composed them to get a $S : Y \to X$. And ST and TS are both 1 minus projection.

For the converse, use the fact that composition with a compact operator is compact. □

Cor. (3.9.11). Fredholm operators constitute an open set in $L(X, Y)$, and it is closed under composition. and index is an open map on it. $\mathrm{ind}(T_1T_2) = \mathrm{ind}(T_1) + \mathrm{Ind}(T_2)$.

Proof: We use snake lemma, there is a diagram with

$$U \to V \to \mathrm{Coker} \to 0$$

$$0 \to W \to W \to 0$$

Then we get

$$0 \to \mathrm{Ker}\, T_2 \to \mathrm{Ker}\, T_1T_2 \to \mathrm{Ker}\, T_2 \to \mathrm{Coker}\, T_2 \to \mathrm{Coker}\, T_1T_2 \to \mathrm{Coker}\, T_1 \to 0.$$

(The left one is splinted), which gives the desired results.

For the rest, notice $1 + S$ is invertible for $||S|| < 1$. □

Cor. (3.9.12). If T is Fredholm and A is compact, then $T+A$ is Fredholm, and $\mathrm{ind}(T+A) = \mathrm{ind}(T)$.

Proof: It is Fredholm by(3.9.10), and we notice $S(T + A)$ and ST are both 1 minus compact operators, thus(3.9.11) and(3.9.6) gives the result. □

IV.4 Abstract Harmonic Analysis(Folland)

1 Locally Compact Groups

Prop. (4.1.1). Topological group is completely regular.

Proof: Use a sequence of neighbourhood of identity to construct a uniform metric on G. Then set $\phi(x) = \min\{1, 2\sigma(a, x)\}$. Cf.[Abstract Harmonic Analysis Ross §8.4] □

Prop. (4.1.2). Locally compact group (Hausdorff) is normal. In particular, Dirac Sequence exists.

Proof: Notice that by choosing a precompact symmetric open neighbourhood U of identity, there exists a σ-compact clopen subgroup H. So H can σ-locally refine every open cover, thus G can, too. So by (1.3.1) G is paracompact. As a topological group, G is regular, thus G is normal by (1.3.4). □

2 Analysis on Locally compact groups

Prop. (4.2.1). The dual group G^* can be regarded as the spectrum of $L^1(G)$:

$$\xi \mapsto \Big(f \mapsto \int \overline{(x,\xi)} f(x)dx\Big),$$

and in this way, the Fourier transform is just the Gelfand transform from $L^1(G)$ to $C(\hat{G})$. Its range is a dense space of $C_0(\hat{G})$.

Prop. (4.2.2). There is another map from $M(\hat{G})$ to bounded continuous functions on G:

$$\mu \mapsto \Big(\phi_\mu : x \mapsto \int (x,\xi)d\mu(\xi)\Big).$$

This is a norm decreasing injection.

Prop. (4.2.3). $\widehat{(f*g)} = \widehat{f}\cdot\widehat{g}$, so if $f, g \in L^2(G)$, $\widehat{(fg)} = \widehat{f} * \widehat{g}$. Cf.[Folland Abstract Harmonic Analysis].

Def. (4.2.4). A function of **positive type** on a closed compact group G is a function $\phi \in L^\infty(G)$ that defines a positive linear functional on the B^*-algebra $L^1(G)$.

We set $P = P(G) =$ the set of continuous functions of positive type on G and $P_0(G) = \{\phi | ||\phi||_\infty \leq 1\}$. By Alaoglu, $P_0(G)$ is a weak*-compact set.

Prop. (4.2.5) (Bochner's Theorem). If $\phi \in P(G)$, there is a unique positive $\mu \in M(\widehat{G})$ s.t. $\phi = \phi_\mu$.

Proof: We have the map defined in(4.2.2) maps into $P_0(G)$ and it is weakly*continuous, so maps the compact convex set of positive measures that $\mu(\hat{G}) \leq 1$ to a compact convex set. And the image contains all the extreme point of P_0, i.e. characters of G and 0. So by Krein-Milman, this map is surjective. Cf. [Folland Abstract Harmonic Analysis Prop4.19]. □

Cor. (4.2.6) (Herglotz). A numerical sequence $\{a_n\}$ is positive iff there is a positive measure $\mu \in M(T)$ s.t. $a_n = \widehat{\mu}(n)$.

Prop. (4.2.7). The set of regular Borel probability measures on a compact X is weak*-compact in $C(X)^*$. (Use Alaoglu).

3 Locally Compact Abelian Group

Prop. (4.3.1) (Pontryagin Duality). For a locally compact Abelian group G, $G \to G^{\vee\vee}$ is an isomorphism of topological groups. Cf.[Folland Abstract Harmonic Analysis P110].

IV.5 Harmonic Analysis

1 Distributions

Def. (5.1.1). The space $D(\Omega)$ of **test functions** has the induced topology coincides with that of $D(K)$, and any bounded subsets are in some $D(K)$, thus it is complete and has Heine-Borel because $D(K)$ does.

The space of continuous linear functionals of $D(\Omega)$ is called the space of **distributions** $D'(\Omega)$. It is equivalence to the restriction to every $D(K)$ is continuous, Cf.[Rudin P155]. The **order** of a distribution Λ is the minimal N that $|\Lambda\varphi| \leq C_K||\varphi||_N$ on every K , it might be ∞.

Def. (5.1.2). The **differentiation** of a distribution Λ is defined as $D^\alpha\Lambda(\varphi) = (-1)^{|\alpha|}\Lambda(D^\alpha\varphi)$. The multiplication by a smooth function f is defined by $f\Lambda(\varphi) = \Lambda(f\varphi)$. Then

$$D^\alpha(f\Lambda) = \sum_{\beta\leq\alpha} C_{\alpha\beta}(D^{\alpha-\beta}f)(D^\beta\Lambda).$$

Support of a Distribution

Def. (5.1.3). The **support** $Supp(\Lambda)$ of a distribution is the complement of the open sets U that $\Lambda(f) = 0$ for any f with support in U.

If $Supp(\Lambda)$ is compact, then Λ has finite order and $|\Lambda\varphi| \leq C||\varphi||_N$ for some N, and Λ extends uniquely to a continuous linear functional on $C^\infty(\Omega)$.

Proof: This is because its support is compact so we can choose a smooth ψ that $= 1$ on $Supp\varphi$ and has support in $W \subset \Omega$. Then by (5.1.1), there is a C that $|\Lambda(\psi\varphi)| < C||\psi\varphi||_N$, and Leibniz rule will give us the result. □

Prop. (5.1.4). If the support of a Λ is a pt p (thus has finite order m), then it is a linear combination of $D^\alpha\delta_p$,$|\alpha| \leq m$. (use approximate identity and show the kernel of Λ is contained in the kernel of $D^\alpha\delta_p$. Cf.[Rudin P165].

Prop. (5.1.5). For any distribution Λ, there exist continuous functions g_α in $C^\infty(\Omega)$ that each compact K intersects support of f.m g_α and $\Lambda = \sum D^\alpha g_\alpha$. When Λ has finite order, we can use only f.m g_α.

Proof: use partition of unity. Then for a compact K, find a compact-open W, then find a bump function between $K \subset W$, thus reduce to the case of $D_{\overline{W}}$. For the rest, Cf.[Rudin P169]. □

Convolution on $\mathbb{R}^n$

Denote $D = D(\mathbb{R}^n), D' = D'(\mathbb{R}^n)$.

Def. (5.1.6). The **translation** of a distribution u is defined as $(\tau_x u)(\varphi) = u(\tau_{-x}\varphi)$, where $\tau_x\varphi(y) = \varphi(y - x)$.

The **convolution** with a distribution u is defined as $(u * \varphi)(x) = u(\tau_x\check{\varphi})$, where $\check{\varphi}(y) = \varphi(-y)$.

Prop. (5.1.7) (Special Case of (5.1.10)). For $u \in D', \varphi \in D, \psi \in D$,

- $\tau_x(u * \varphi) = (\tau_x u) * \varphi = u * (\tau_x\varphi)$.
- $u * \varphi \in C^\infty$ and $D^\alpha(u * \varphi) = (D^\alpha u) * \varphi = u * (D^\alpha\varphi)$.

- $u * (\varphi * \psi) = (u * \varphi) * \psi$.

If u has compact support, then(5.1.3) shows that u can extend to C^∞, thus convolution is defined for $\varphi \in C^\infty$ and the first two formulae still hold, and when $\psi \in D$,

$$u * \psi \in D, \quad u * (\varphi * \psi) = (u * \varphi) * \psi = (u * \psi) * \varphi$$

Proof: Cf.[Rudin P171], [Rudin P174]. □

Cor. (5.1.8). $L : \varphi \mapsto u * \varphi$ is a continuous linear map into C^∞ that commutes with τ_x. (It is continuous because of of closed graph theorem(3.4.9), $\lim(u * \varphi_i)(x) = \lim u(\tau_x \check{\varphi}) = u(\tau_x \check{\varphi})$). And any these map comes from a $u*$:let $u = (L\check{\varphi})(0)$.

Cor. (5.1.9). When $u, v \in D'$ and one of them has compact support, then similar to(5.1.8), $L\varphi = u*(v*\varphi)$ is a continuous linear map that commutes with τ_x, so there is a unique **convolution distribution** $u * v$ that $(u * v) * \varphi = u * (v * \varphi)$. This convolution is compatible with the previous one when $v \in D$.

Prop. (5.1.10) (Convolution of Distributions). For $u, v, w \in D'$,

- if one of u, v has compact support, then $u * v = v * u$, and $Supp(u * v) \subset Supp(u) + Supp(v)$.
- if two of three of u, v, w has compact support, then $(u * v) * w = u * (v * w)$.
- $D^\alpha u = (D^\alpha \delta) * u$.
- if one of u, v has compact support, then $D^\alpha(u * v) = (D^\alpha u) * v = u * (D^\alpha v)$.

Proof: Cf.[Rudin P177]. □

Def. (5.1.11). A **approximate identity** here is a $h \in D$ that $h_k(x) = k^n h(kx)$. Then we will have $\lim \varphi * h_j = \varphi$ for $\varphi \in D$, $\lim u * h_j = u$ in D'.

2 Fourier Analysis on $\mathbb{R}^n$

Def. (5.2.1). We denote the normalized notation $\mathbb{R}^n$ as $dm = (2\pi)^{-n/2}dx$ and $D_\alpha = 1/i^{|\alpha|}D^\alpha$, this will simplify notations. The **Fourier transform** here of a function $f \in L^1(\mathbb{R}^n)$ is the function $\hat{f}$ that $\hat{f}(t) = \int_{\mathbb{R}^n} f e_{-t} dm_n = (f * e_t)(0)$.

See(5.2.13) for general Fourier transform.

Prop. (5.2.2). For $f \in L^1(\mathbb{R})$,

$$\widehat{\tau_x f} = e_{-x}\hat{f}, \quad \widehat{e_{-x} f} = \tau_x \hat{f},$$

$$\widehat{f * g} = \hat{f}\hat{g}, \quad \widehat{f(x/\lambda)}(t) = \lambda^n \hat{f}(\lambda t).$$

(Note $||f * g||_1 \le ||f||_1 ||g||_1$).

Def. (5.2.3). The class of **Shwarts functions** $\mathcal{S}$ is defined as smooth functions on $\mathbb{R}^n$ that

$$\sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |(D_\alpha f)(x)| < \infty.$$

Lemma (5.2.4). Let $f = e^{-1/2|x|^2}$, then $f \in \mathcal{S}$, $\hat{f} = f$ and $f(0) = \int \hat{f}$. (reduce to the 1 dimensional case, in which case, $f' + xy = 0$, and $\hat{f}$ also satisfies this).

Lemma (5.2.5). For $f, g \in L^1$, Fubini gives us $\int \hat{f} g = \int f \hat{g}$.

Prop. (5.2.6) (Classical Fourier Transform).

- $\mathcal{S}$ is a Fréchet space in the topology defined by these norms.
- multiplication by $g \in \mathcal{S}$ and derivations are continuous linear map from S to S (direct calculation).
- $\widehat{P(D)f}(t) = P(t)\hat{f}(t)$ and $\widehat{Pf} = P(-D)\hat{f}$.
- The Fourier transform is a continuous linear one-to-one automorphism of S, and $\Psi^2 g = \check{g}$.

Proof: 3: use(5.1.10) for the first one, and for the second one, should use definition of derivative and dominated convergence.

4:$\Psi f \in \mathcal{S}$ by 3, and it is continuous by closed graph theorem. By(5.2.5) and(5.2.2), $\int \hat{f}(t)g(t/\lambda) = \int f(t/\lambda)\hat{g}(y)$. If $\hat{f}, \hat{g} \in L^1$, dominant convergence shows $g(0)\int \hat{f} = f(0)\int \hat{g}$. So we only need one f that $f(0) = \int \hat{f}$, $f = e^{-1/2|x|^2}$ will suffice(5.2.4). Hence $g(0) = \int \hat{g}$ for every such g, and the conclusion follows by translation(5.2.2), and(5.2.8) also follows.

□

Cor. (5.2.7). If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$, and $||\hat{f}||_\infty \leq ||f||_1$, because $\mathcal{S}$ is dense in $L^1(\mathbb{R}^n)$.

Prop. (5.2.8) (Inversion Theorem). If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then $\check{f} = \Psi^2 f$ a.e.

Proof: In(5.2.5), let $g \in \mathcal{S}$ and substitute $g = \Psi g$ and use Fubini, we get $\check{f} - \Psi^2 f$ is orthogonal to every $\mathcal{S}$, then every continuous function with compact support by(1.2.6). Thus they equal a.e. □

Cor. (5.2.9). If $f, g \in \mathcal{S}$, then $\widehat{fg} = \hat{f} * \hat{g}$ (apply Fourier one time and use(5.2.2)), and thus $f * g \in \mathcal{S}$.

Prop. (5.2.10) (Fourier-Plancherel). If $f, g \in \mathcal{S}$, then

$$\int f\overline{g} = \int \overline{g}(x)\hat{f}(t)e^{ixt} = \int \hat{f}(t)\int \overline{g}(x)e^{ixt} = \int \hat{f}\overline{\hat{g}}$$

by inversion formula. And $\mathcal{S}$ is dense in L^2, thus it extends to a linear isometry of $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. This coincides with the Fourier transform on $L^1 \cap L^2$.

Prop. (5.2.11). D injects into $\mathcal{S}$ and is dense.(Notice they both are complete, but the subspace topology are different)(Use scaling, Cf.[Rudin Functional Analysis P189]). So we call a distribution **tempered** iff it comes from a continuous functional of $\mathcal{S}$.

From(5.1.3), we know any distribution with compact support is tempered. By Holder, every $f \in L^p(\mathbb{R}^n), p \geq 1$ is tempered distribution, and every polynomial or functions of polynomial growth are tempered distribution.

$$D \subset \mathcal{S} \subset L^2 = (L^2)^\vee \subset \mathcal{S}' \subset D'.$$

$\mathcal{S}, \mathcal{S}'$ is complete(3.4.4).

Prop. (5.2.12). A $f \in \mathcal{S}'$ iff $f = \sum_{|\alpha|\leq m} D_\alpha(u_\alpha(1+|x|^2)^{m/2})$ for some m, where $u_\alpha \in L^2(\mathbb{R}^n)$.

Proof: In fact,

$$||\varphi||'_m = (\sum_{|\alpha|\le m}\int (1+|x|^2)^m |D_\alpha \varphi|^2 \,\mathrm{dx})^2$$

is an equivalent set of norms of $\mathcal{S}'$, Cf.[泛函分析张恭庆 P182]. And each of them defines a Hilbert space. So by Riesz we get the result. □

Prop. (5.2.13) (Generalized Fourier Transform). For a tempered distribution $u \in \mathcal{S}'$, we define the **Fourier transformation** as the tempered distribution $\hat{u}(\varphi) = u(\hat{\varphi})$. It is easily verified that it is compatible with previously defined Fourier transform when seen as tempered distributions by**??** In particular, this is defined for compactly supported distribution, $L^p(\mathbb{R}^n), p \ge 1$ and smooth functions of polynomial growth(5.2.11).

Prop. (5.2.14). $\widehat{P(D)u} = P\hat{u}$ and $\widehat{Pu} = P(-D)\hat{u}$. And The Fourier transformation is a continuous linear isometry of $\mathcal{S}'$ in the weak* topology.

Cor. (5.2.15). $\hat{1} = \delta$, thus $\hat{P} = P(-D)\delta$ and $P(\hat{D})\delta = P$. Now(5.1.4) tells us a distribution is the Fourier transform of a polynomial iff it has support in the origin.

Prop. (5.2.16) (Convolution of Tempered Distributions). Let $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$, then

- $u * \varphi \in C^\infty$ of polynomial growth and $D^\alpha(u*\varphi) = (D^\alpha u)*\varphi = u*(D^\alpha \varphi)$.
- $u*(\varphi*\psi) = (u*\varphi)*\psi$.
- $\widehat{u*\varphi} = \hat{\varphi}\hat{u}, \hat{u}*\hat{\varphi} = \widehat{\varphi u}$.
- If P is a polynomial and $g \in \mathcal{S}$, then $D^\alpha u, Pu$ and gu are all tempered.

Cf.[Rudin Functional Analysis P195] for the first 3.

Paley-Wiener Theory

Prop. (5.2.17). For $\varphi \in D(\mathbb{R}^n)$ that has support in rB, the You-Know-How defined $\hat{\varphi}(z)$ is an entire function of several variable and satisfies:

$$|\varphi'(z)| \le \gamma_N (1+|z|)^{-N} e^{r|\operatorname{Im} z|}.$$

For $N \ge 0$. Conversely, any such function correspond to a $\varphi \in D(\mathbb{R}^n)$ that has support in rB.

Proof: Cf.[Rudin P198]. □

Prop. (5.2.18) (Fourier-Laplace transformation). For $u \in D'(\mathbb{R}^n)$ that has support in rB, of order N, the $\hat{u}(z) = u(e_{-z})$ is an entire function of several variable and satisfies:

$$|f(z)| \le \gamma(1+|z|)^N e^{r|\operatorname{Im} z|}.$$

Conversely, any such function correspond to a $u \in D'(\mathbb{R}^n)$ that has support in rB.

Proof: Cf.[Rudin P199]. □

3 Sobolev Space

Def. (5.3.1). For $1 \leq p < \infty$, the **Sobolev space** $W^{m,p}(\Omega)$ is the space of functions u that $D^\alpha u \in L^p(\Omega)$ for every $|\alpha| \leq m$. The **Sobolev space** $W_0^{m,p}(\Omega)$ is the completion of the subspace $C_0^\infty(\Omega)$.

Prop. (5.3.2) (Meyers-Serrin). The Sobolev space $W^{m,p}(\Omega)$ is the completion of $u \in C^\infty(\Omega)$ that $D^\alpha u \in L^p(\Omega)$ for every $|\alpha| \leq m$.

Proof: Choose a countable partition of unity ψ_k, then as in the proof of(1.2.7), we can choose δ_k small enough and $||\psi u - (\psi u)_{\delta_k}|| < \varepsilon/2^k$ and $\varphi = \sum(\psi u)_{\delta_k}$ is definable. □

Prop. (5.3.3). We denote $H^m(\Omega) = W^{m,2}(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$ and $H^{-m}(\Omega) = (H_0^m(\Omega))^*$ when m is an integer. Notice derivative is not applicable for $H^{-m}(\Omega)$ unless $\Omega = \mathbb{R}^n$.

When $\Omega = \mathbb{R}^n$, $D(\mathbb{R}^n)$ is dense in $W^{m,p}$(1.2.7), thus $W_0^{m,p} = W^{m,p}$. Define the **Sobolev space**

$$H^s = \{u|(1+|y|^2)^{s/2}|\hat{u}| \in L^2\}$$

H^s is a Hilbert space and $H^s \subset \mathcal{S}'$ for every s (use Holder to show $\hat{u} \in \mathcal{S}'$). H^m coincides with previously defined H^m when m is a positive integer thus also negative-integer. A linear operator on $H = \cup H^s$ is said to have **order** t if it maps every H^s continuously into H^{s-t}.

Proof: By Plancherel,

$$||\varphi||_m' = (\sum_{|\alpha|\leq m} ||D_\alpha u||_2^2)^{1/2} \quad \text{and} \quad (\int (1+|x|^2)^m |\hat{u}|^2)^{1/2}$$

are equivalence norms on H^m. □

Prop. (5.3.4) (Poincare Inequality). For $u \in C_0^m(\Omega)$ and Ω bounded, its $W^{m,p}$ norm is controlled by its mth order derivative L^p norm.

Prop. (5.3.5). When $t < s$, $H^s \subset H^t$. And H^s are isometric to H^t by $\hat{v} = (1+|y|^2)^{t/2}\hat{u}$ and is of order t. D^α is of order $|\alpha|$. If $f \in \mathcal{S}$, then $u \to fu$ is an operator of order 0, Cf.[Rudin P217].

Every distribution of compact support is in some H^s(5.1.3), in particular $D(\Omega)$.

Prop. (5.3.6) (Sobolev Embedding Theorem). On a manifold of dimension n which is compact with Lipschitz boundary or complete of positive injective radius and bounded sectional curvature,

- if $k > l$ be integers and
$$\frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{l}{n}$$
then $W^{k,p}(\mathrm{int}(M)) \subset W^{l,q}(M)$ continuously. Cf.[Evans P290].
- if
$$\frac{1}{p} - \frac{k}{n} = -\frac{r+\alpha}{n}$$
then $W^{k,p}(\mathrm{int}(M)) \subset C^{r,\alpha}(M)$ continuously.

Cor. (5.3.7) (Gagliardo–Nirenberg–Sobolev). On a manifold of dimension n which is compact with Lipschitz boundary or a complete of positive injective radius and bounded sectional curvature, if $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ (Sobolev conjugate), then $W^{1,p}(\mathrm{int}(M)) \subset L^{p^*}(M)$ continuously.

Cor. (5.3.8). On a manifold of dimension n which is compact with Lipschitz boundary or a complete of positive injective radius and bounded sectional curvature, if $m > n/2$, then $W^{m,2}(\mathrm{int}(M)) \subset C(\overline{\Omega})(M)$ continuously. And the functions in $W_0^{m,2}$ are continuous and vanish at the boundary, by C_0 approximation.

Proof: The $\mathbb{R}^n$ case can be directly proved: because we have the equivalent norm(5.3.3), $\hat{u} \in L^2$ thus $u \in L^2$, and

$$\int |\hat{u}| \leq (\int (1+|x|^2)^m |\hat{u}|^2)^{1/2} \cdot \Big(\int 1/(1+|x|^2)^m\Big)^{1/2}.$$

We have $\hat{u} \in L^1$, thus inversion formula applies that u is continuous and $||u||_\infty \leq ||\hat{u}||_1 \leq C||u||_{H^m}$. □

Cor. (5.3.9). $\cap_s H^s = C^\infty(M)$.

Prop. (5.3.10) (Rellich-Kondrechov). On a compact manifold with C^1 boundary of dimension n, if $k > l$ and

$$\frac{1}{p} - \frac{k}{n} < \frac{1}{q} - \frac{l}{n}$$

then $W^{k,p} \subset W^{l,q}$ completely continuously. Cf.[Evans P290].

Proof: Cf.[Distributions and Operators P199]. □

Cor. (5.3.11). On a bounded extension domain of $\mathbb{R}^n$, $W^{1,p} \subset L^p$ completely continuously.

Proof: We prove the $p = 2$ case. For a sequence u_m in $W^{1,2}$, we have $||u_m - u_p||_2 = ||U_m - U_p||_2 = ||\hat{U}_m - \hat{U}_p||_2$. By(3.3.8), there is a subsequence that $\hat{U}_m$ pointwise converge. Notice they are uniformly bounded, Now apply two region argument, for $|x| < r$, use Lebesgue dominant convergence, and for $|x| > r$, use $\int (1+|x|^2)|\hat{U}_m - \hat{U}_p|^2$ is bounded to conclude $||u_m - u_p||_2 \to 0$. □

Prop. (5.3.12). $u \in D'(\Omega)$ is a locally $H^s \iff \psi u \in H^s$ for every $\psi \in D(\Omega) \iff D_\alpha u$ is locally L^2 for every $|\alpha| \leq s$.

Thus every smooth function is locally H^s for every s.

Proof: $1 \to 2$ use partition of unity, $2 \to 1$ easy, and $2, 3$ are all equivalent to $D_\alpha(\psi u) \in L^2$ for every $\psi \in D(\Omega)$. by Leibniz+Plancherel or (5.3.5). □

Prop. (5.3.13). If $r > p + n/2$, then if a function f on Ω has all the distribution derivative $D_i^k f$ locally $L^2, = g_{is}$, for $0 \leq k \leq r$, then $f \in C^p(\Omega)$ a.e.

Cor. (5.3.14). If $u \in D'(\Omega)$ is locally H^s, then $u \in C^{s-n/2}(\Omega)$. Thus $\cap$locally $H^s = C^\infty(\Omega)$.

Holder Space

Def. (5.3.15). Holder space $C^{k,\alpha}(\Omega)$ is the subspace of $C^k(\Omega)$ with the norm

$$||f||_{C^{k,\alpha}} = ||f||_{C^k} + \max_{|\beta|=k} \sup_{x\neq y\in\Omega} \frac{|D^\beta f(x) - D^\beta f(y)|}{||x-y||^\alpha}.$$

4 Fourier Analysis on $\mathbb{T}^n$

Prop. (5.4.1). If $f \in L^1(\mathbb{T})$ is absolutely continuous, then $\widehat{(f')}(n) = 2\pi i n \cdot \hat{f}(n)$.

Prop. (5.4.2). $f \in L^1(\mathbb{T})$ is determined by its Fourier coefficients.

IV.6 Differential Operators

1 ODE-Fundamentals

Prop. (6.1.1).

$$x^{(2)} = f(x)$$

It can be solved.

Proof:

$$x' x^{(2)} = f(x)x'$$

$$\frac{1}{2}(x')^2 = \int^x f(t)dt$$

□

Prop. (6.1.2) (Wronsky).

2 ODE-Theorems

Prop. (6.2.1) (Existence and Uniqueness of ODE of Lipschitz Type). If $F(t,x)$ defined on $[-h,.h]\times[\eta-\delta,\eta+\delta]$ is a function that is locally Lipschitz: that is, $\exists\delta, L$, s.t. if $|t|\leq h, |x_i-\eta|\leq\delta$, then

$$|F(t,x_1)-F(t,x_2)|\leq L|x_1-x_2|.$$

Then the initial value problem:

$$(Tx)(t)=\eta+\int_0^t F(\tau,x(\tau))d\tau$$

has a unique solution on the interval $[-h,h]$ if $h<\min\{\delta/M,1/L\}$, where M is the maximum of F on $[-h,.h]\times[\eta-\delta,\eta+\delta]$. Because T is a contraction.

Prop. (6.2.2) (Existence of ODE of continuous Type (Caratheodory)). If $F(t,x)$ defined on $[-h,.h]\times[\eta-\delta,\eta+\delta]$ is a continuous function, then

$$(Tx)(t)=\eta+\int_0^t F(\tau,x(\tau))d\tau$$

has a unique solution on the interval $[-h,h]$ if $h<\delta/M$, where M is the maximum of F on $[-h,.h]\times[\eta-\delta,\eta+\delta]$. (Use Schauder fixed point theorem and Arzela-Ascali).

Prop. (6.2.3) (Existence Theorem for Complex Differential Equations). Let $f(z,\mathbf{w})$ be a holomorphic vector function in a domain $D\subset\mathbb{C}^{n+1}$, then the initial value problem

$$\mathbf{w}'=f(z,\mathbf{w}),\quad w(z_0)=w_0$$

has exactly on holomorphic solution locally (Thus on a simply connected domain).

Cor. (6.2.4). So a holomorphic high-order ODE for a complex variable can be solves. And luckily it can be solved even $\overline{z}$ appears (just regard it as a constant).Δ

Proof: Cf.[Ordinary Differential Equations, P110]. □

Prop. (6.2.5). For the equation:

$$\frac{d\mathbf{y}}{d\mathbf{x}} = \mathbf{A}\mathbf{y},$$

One solution basis is:

$$\begin{cases} e^{\lambda_1 x}\mathbf{P}_1^{(1)}(x), \cdots, e^{\lambda_1 x}\mathbf{P}_{n_1}^{(1)}(x); \\ \cdots\cdots\cdots \\ e^{\lambda_s x}\mathbf{P}_1^{(d)}(x), \cdots, e^{\lambda_s x}\mathbf{P}_{n_s}^{(1)}(x); \end{cases}$$

Where

$$\mathbf{P}_j^{(i)}(x) = \mathbf{r}_{j0}^{(i)} + \frac{x}{1!}\mathbf{r}_{j1}^{(i)} + \cdots,$$

where $\mathbf{r}_{j0}^{(i)}$ is a basis of solution of $(\mathbf{A} - \lambda_i I)^n \mathbf{x} = 0$, and $\mathbf{r}_{k+1}^{(i)} = (\mathbf{A} - \lambda_i I)\mathbf{r}_k^{(i)}$.

Proof: Cf.[常微分方程丁同仁定理 6.6]. □

Cor. (6.2.6). For the equation:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0,$$

If the characteristic equation has s different roots $\lambda_1, \ldots, \lambda_s$ and corresponding multiplicities $n_1, \ldots, n_s$, then:

$$\begin{cases} e^{\lambda_1 x}, xe^{\lambda_1 x}, \cdots, x^{n_1 - 1}e^{\lambda_1 x}; \\ \cdots\cdots\cdots \\ e^{\lambda_s x}, xe^{\lambda_s x}, \cdots, x^{n_s - 1}e^{\lambda_s x}; \end{cases}$$

is a solution basis.

Proof: Cf.[常微分方程丁同仁 P198]. □

Prop. (6.2.7) (Lyapunov). Consider the Lyapunov stability of an autonomous system of the form:

$$\frac{dx}{dt} = Ax + o(|x|),$$

Then:

1. If A has a eigenvalue whose real part is positive, then the trivial solution is not weak stable.
2. If all eigenvalues of A has negative real part, then the trivial solution is strong stable.

Stum-Liouville

Prop. (6.2.8) (Stum-Liouville). The eigenvalue BVP problem of L-S equation:

$$Lu = (pu')' + qu = \lambda u, \quad a_1 u(a) + a_2 u'(a) = 0, b_1 u(b) + b_2 u'(b) = 0, \sigma(x) > 0.$$

can be solved by the method of Green's function. For the function:

$$G(x,s) = \begin{cases} Cu_1(x)u_2(s), & x < s \\ Cu_2(x)u_1(s), & x > s \end{cases}$$

for some C, where u_1 is a solution of the L-S equation with boundary value at a, and u_2 with boundary value at b that are linear independent (This happens when the homogenous equation has no solution). It satisfies: $LG(x,s)=\delta(x-s)$ and satisfies the boundary conditions.

Because L is self-adjoint, we have:

$$Gf(x)=\int f(s)G(x,s)ds, LG=\mathrm{id}, GL=\mathrm{id}$$

thus the eigenvalues of L is the reciprocal of the eigenvalues of G, and G is a compact self-adjoint operator on $L^2(\sigma,\mathbb{R})$, so by spectral theorem, the eigenvectors are countable and form an orthonormal basis.

And when the homogenous problem do have a solution ϕ, then we have:$Lu=f$ has a solution iff $(f,\phi)=0$. one way is simple and the other way is because we solve the initial problem of ODE and find that it automatically satisfies the boundary condition. Cf.[Stum Liouville Theory].

Prop. (6.2.9). More generally, if there the boundary is mixed of $u(a),U'(a),u(b),u'(b)$, the solution of

$$Lu=(pu')'+qu=0, B_1(u)=\alpha, B_2(u)=\beta.$$

has a unique solution for any α,β iff the homogenous equation has only non-trivial solution. (Because the solution space is of 2 dimensional.

Prop. (6.2.10) (Stum Seperation Theorem).

Prop. (6.2.11) (Stum Comparison Theorem). If $y''+K_i(x)y=0$ are equations. If $y_i(0)=0$ and $|y_1'(0)|=|y_2'(0)|$, then if $K_1(x)\geq K_2(x)$, then $y_1(x)\geq y_2(x)$ until $y_2(x)$ is zero. (directly from(2.4.1)).

3 Linear PDE

Def. (6.3.1). For a linear PDE with constant coefficients $P(D)u=v$, the **fundamental solution** is a distribution $E\in D'(\mathbb{R}^n)$ that $P(D)E=\delta$. This is important because if v is a distribution with compact support, $P(D)(E*v)=(P(D)E)*v=\delta*v=v$(5.1.10), so $u=E*v$ is a distribution solution.

Prop. (6.3.2). When $v\in D'(\mathbb{R}^n)$ has compact support, $P(D)u=v$ has a solution u with compact support iff $Pg=\hat{v}$ has a solution g entire. In this case, $g=\hat{u}$ for some distribution u, and u has support in the convex hull of the support of v.

Proof: Use(5.2.18), and some bound relation between g and Pg. Cf.[Rudin Functional Analysis P212]. □

Prop. (6.3.3). The fundamental solution always exist when for PDE of constant coefficients.

Proof: For a $\varphi\in D(\mathbb{R}^n)$, there is at most one ψ that $\psi=P(D)\varphi$ because $\hat{\psi}=P\hat{\varphi}$ and they are entire function. Thus the task is to verify the functional $u:P(D)\varphi\to\varphi(0)$ is continuous and extend to a distribution $u\in D'(\mathbb{R}^n)$. Cf.[Rudin Functional Analysis P215]. □

4 Differential Operator on Manifolds

Prop. (6.4.1) (Index Theorem P109). has a nice definition of symbol of a differential operator on a manifold as a map form $\mathrm{Sym}^m T^*M\otimes\mathbb{C}\to\mathrm{Hom}(E,F)$.

5 Pseudo-Differential Operator

Def. (6.5.1). Denote the **Japanese bracket** $[x] = (1+|x|^2)^{1/2} \sim 1+|x|$.

Motivated by the formula $(\widehat{P\hat{f}})^\vee = P(D)f$ for $f \in \mathcal{S}$ and polynomial P of ξ with coefficients smooth functions of x**??** we define the **symbol class** $S^{\mu,\beta}$ as the space of smooth functions $a : \mathbb{R}^{2n} \to \mathbb{C}$ that

$$|D_{x,\alpha} D_{\xi,\beta} a(x,\xi)| \leq C_{\alpha,\beta} [x]^\mu [\xi]^{m-|\beta|}$$

and denote $S^m = S^{0,m}$.

We denote the **symbol class** $\mathcal{A}^v$ as the space of smooth functions $a : \mathbb{R}^{2n} \to C$ that $|D_\alpha a| \leq C_\alpha [x+\xi]^v$ for any α. So $S^{\mu,m} \subset \mathcal{A}^{|\mu|+|m|}$

And we define the **pseudo-differential operator of symbol** a:

$$(a(x,D)u)(x) = \int_\xi e^{ix\xi} a(x,\xi)\hat{u}$$

Moreover, we can define the **amplitude function** $p(x,y,\xi)$ and define

$$Pu(x) = \int e^{i(x-y)\xi} p(x,y,\xi) u(y) dy.$$

Def. (6.5.2). We define the space S^d of **polyhomogenous symbols of degree** d as the set of all symbols in $S^d_{0,1}$ that there exists a set of p_{d-l} homogenous in ξ of degree $d-l$ that $p = \sum p_{d-l}$ modulo an operator in $S^{-\infty}$. Note that when p_{d-l} is homogenous of degree $d-l$, then it is automatically in $S^{d-l}_{0,1}$.

Def. (6.5.3). A ψdo operator a is called **elliptic** if $\sigma(a) \in S^m$ and $\sigma(a) \geq [\xi]^{-m}$ for ξ big enough.

Prop. (6.5.4) (Peetre's Inequality). For all $v \in \mathbb{R}$, there is a constant C that

$$[X+Y]^v < C[X]^v[Y]^v.$$

Proof: For $v > 0$, just as normal. For $v < 0$, use $X = (X+Y) + (-Y)$ applied to $-v$. □

Prop. (6.5.5). The mapping $a(x,\xi) \times u(x) \mapsto a(x,D)u$ is continuous from $\mathcal{A}^v \times \mathcal{S} \to \mathcal{S}$, thus also continuous from $S^{\mu,m} \times \mathcal{S} \to \mathcal{S}$. Cf.[Pseudo Differential Operator P28].

Lemma (6.5.6) (Schur Test). For a function K on $\mathbb{R}^{2n}$ and $u \in L^p(\mathbb{R}^n)$, let $||K||_1 = \sup_x \int |K(x,y)| dy$ and $||K(x,y)||_2 = \sup_y \int |K(x,y)| dx$. Let $Au(x) = \int K(x,y) u(y) dy$, then

$$||Au||_{l^p} \leq ||K||_1^{1-1/p} ||K||_2^{1/p} ||u||_{L^p}.$$

by Holder.

Prop. (6.5.7) (Calderón-Vaillancourt). There is a constant C, N_{CV} that for $u \in \mathcal{A}^0$ and $\varphi \in \mathcal{S}$,

$$||Op(u)\varphi||_{L^2} \leq C \max_{|\alpha|+|\beta| \leq N_{CV}} ||\partial_x^\alpha D_{\beta,\xi} u||_{L^\infty} ||\varphi||_{L^2}.$$

This in particular applies to $u \in S^0$.

Proof: Cf.[Calderon-Vaillancourt]. □

Cor. (6.5.8). S^m maps H^s to H^{s-m}. Because by symbolic calculus(6.5.10), we have

$$Op([\xi]^{s-m}) Op(u) Op([\xi]^{-s}) = Op(b) \in S^0,$$

thus $Op(u) = Op([\xi]^{m-s}) Op(b) Op([\xi]^s)$ maps H^s into H^{s-m}.

Symbolic Calculus

Def. (6.5.9) (Semiclassical Operator). For $a \in S^{\mu,m}$ and $h \in (0,1]$, we denote $a_h(x,\xi) = a(x,h\xi)$, it is also in $S^{\mu,m}$.

Prop. (6.5.10) (Composition). If $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$, there is a pseudo-differential operator $(a\#b)(h) \in S^{\mu_1+\mu_2,m_1+m_2}$ for every $h \in (0,1]$ that

$$Op(a_h)Op(b_h) = Op((a\#b)(h)_h)$$

and for all $J > 0$, $(a\#b)(h)$ can be written as

$$a\#b(h) = \sum_{j<J} h^j \Big(\sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b \Big) + h^J r_J^\#(a,b,h)$$

where $r_J^\#(a,b,h) \in S^{\mu_1+\mu_2,m_1+m_2-J}$ and it is bilinear of a,b and equicontinuous independently of h.

Proof: Cf.[Pseudo Differential Operator P36]. □

Prop. (6.5.11) (Adjoint). If $a \in S^{\mu,m}$ and $u,v \in \mathcal{S}$, there is a pseudo-differential operator $a*(h)$ for every $h \in (0,1]$ that

$$(u, Op(a_h)v) = (Op(a^*(h)_h)u, v)$$

in the L^2 norm and for all $J > 0$, $a^*(h)$ can be written as

$$a^*(h) = \sum_{j<J} h^j (\sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{a}) + h^J r_J^*(a,h)$$

where $r_J^*(a,h) \in S^{\mu,m-J}$ and it is anti-linear of a and equicontinuous independently of h.

Proof: Cf.[Pseudo Differential Operator P30]. □

Def. (6.5.12). For $u \in \mathcal{S}'$, we define the action of $a(x,\xi)$ on u by

$$(Op(a_h)u)(\overline{\varphi}) = u(\overline{Op(a^*(h)_h)\varphi}).$$

This is compatible with the definition on $\mathcal{S}$.

6 General PDE

Direct Solution

Prop. (6.6.1) (Characteristic Line). Consider a 1-dimensional parabolic equation:

$$p_t + c(p,x,t)p_x = r(p,x,t)$$

Let $P(t) = p(X(t),t)$, this equation is equivalent to

$$P_t = r(X(t),t,P(t)), \quad X_t = c(X(t),t).$$

an ODE equation.

Prop. (6.6.2). A set of equations:

$$\frac{\partial}{\partial x^i}\mu = A_i\mu$$

where μ is a n-vector. It has a solution iff

$$[A_i, A_j] = \frac{\partial}{\partial x^i}A_i - \frac{\partial}{\partial x^i}A_j.$$

Proof: □

Cor. (6.6.3). This seems to be able to derive Frobenius integrability theorem, but I cannot figure it out.

7 Analysis on Manifolds

Prop. (6.7.1) (Peetre's Theorem). For a linear operator from $C^\infty(M)$ to $C^\infty(M)$ that $Supp(Lu) \subset Supp(u)$ where M is a compact manifold, then on every compact subset of a coordinate chart L looks like a differential operator of finite order.

Proof: The first thing is to prove on a chart Ω, L is continuous on $C_0^\infty(\Omega)$. In fact, it suffice to show it is continuous from $C_0^\infty(\Omega)$ to $C_0^0(\Omega)$ because we can apply to $D_\alpha L$. For this, Cf.[Pseudo Differential Operator P86].

Then we have $|Lu| \leq C\max_{|\alpha\leq m}\sup_K|D_\alpha\varphi|$ for every $\varphi \in C_0(K)$. And the functional $\varphi \to (L\varphi)(x)$ is a distribution supported on x, thus by(5.1.4), it is of the form

$$Lu(x) = \sum_{|\alpha|\leq m} a_\alpha(x)D_\alpha\varphi(x).$$

We need to show a_α is smooth, which we choose a bump function χ to show a_0 is smooth and then choose $x_i\chi$ applied to $L\varphi - a_0\varphi$ to show a_i is smooth, etc. □

Prop. (6.7.2). The property of ψdo of order d is preserved under diffeomorphism, Cf.[Distributions and Operators P195], giving us the possibility to define ψdo differential operator on manifolds, and the principal symbol variate in this way that it forms a function from the cotangent bundle to the $M_n(\mathbb{C})$. And the Sobolev space is defined by the property that all of its restrictions on a atlas are Sobolev, using the partition of unity.

Prop. (6.7.3). All the norms of different are commensurable up to constant factor w.r.t. each other, so it doesn't quite matter with different norms.

Prop. (6.7.4). The parametrix exists for an elliptic operator on manifolds. Cf.[Distributions and Operators P207].

8 Elliptic Operator

Prop. (6.8.1). Elliptic operator is a Fredholm operator. And the kernel and cokernel are smooth functions, so it is also a Fredholm operator on $C^\infty(\Omega)$.

Proof: It suffice to find a left and right inverse modulo compact operators, and in fact we find it module $S^{-\infty}$. Since $S^{-\infty}$ are all compact operators, i.e. it has a parametrix. Cf.[Distributions and Operators, P184]. □

Prop. (6.8.2) (Garding Inequality). For an elliptic operator of order d on $\Gamma(E)$,

$$||f||_{H^s} \leq C(||f||_{H^{s-d}} + ||Pf||_{H^{s-d}})$$

Proof: □

Cor. (6.8.3) (Elliptic Regularity Theorem). The inverse image of a smooth function under an elliptic operator is a smooth function, because the intersection of $H^s(E)$ is $C^\infty(E)$.

Cor. (6.8.4) (Elliptic Regularity Theorem). For $L = \sum_{|\alpha|\leq N} f_\alpha D_\alpha$, where $f_\alpha \in C^\infty(\Omega)$ and the equation $Lu = v$ for distributions u and $v \in D'(\Omega)$, when v is locally H^s, u is locally H^{s+N}. Thus if $v \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$ by(5.3.12)(5.3.14).

Proof: We prove the case when L has leading coefficients constant. For every $\varphi \in D(\Omega)$ that is 1 on some open ball B, φu has compact support thus in some H^t< and then we use a sublemma that says if ψ is 1 on the support of φ, then if ψu is in H^t, where $t \leq s + N - 1$, then $\varphi u \in H^{t+1}$. In this way, we can shrink the nbhd to reach H^{s+N}. The proof of the lemma is in [Rudin Functional Analysis P220]. □

Prop. (6.8.5). The formal adjoint of an elliptic operator is an elliptic operator.

Proof: □

Cor. (6.8.6). The index of an elliptic operator, regarded as an operator form $L_s \to L_{s-d}$ doesn't depend on s, because all the kernel of P and P^* are smooth.

Prop. (6.8.7). For an elliptic operator, It has a inverse, the Green function which is a compact operator, so it has countable eigenfunctions consisting of smooth functions on L^2 with eigenvalues converging to ∞. Moreover, the eigenvalues satisfy $|\lambda_n| \geq Cn^\delta$ for some δ, C.

Proof: We prove for P self-adjoint. Use(6.8.1), $\operatorname{Ker} P$ is all smooth, so there is a map $P(H^{-2d}) \to P(H^{-d})$ which is bijective thus an isomorphism by Banach. So the inverse of this isomorphism composed with the Sobolev embedding $H^{-d} \to L^2$ is a compact operator G. we notice that this map has the same eigenfunctions as P, thus the result from that of compact operators.

For the second assertion, if suffice to prove $\dim N(\lambda) \leq C\lambda^M$. Using Garding inequality and Sobolev embedding, we have for $f \in N(\lambda)$, $||f||_{C^0} \leq C(1+\lambda^l)||f||_{L^2}$ for large l. So if we choose an orthonormal basis f_i, then $|a_i f_i(x)| \leq C(1+\lambda^l)|\sqrt{\sum |a_i|^2}$. Let $a_i = f_i(x)$ and integrate over M, we get the desired result. □

Cor. (6.8.8). For a self-adjoint elliptic operator P which is not a constant, $L^2(E)$ has a basis consisting of eigenfunctions of P.

Cor. (6.8.9) (Stum-Liouville). This can be used to solve for example eigenvalue problem for Liouville's equation:

$$(pu')' + qu = \lambda\sigma u.$$

where p and σ are positive. Cf.(6.2.8).

Cor. (6.8.10). The Hermite functions $C_n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2}$, as the eigenvector of $\hat{H} = x^2 - \frac{d^2}{dx^2}$, forms a complete basis for $L^2(\mathbb{R})$. Because it is e^{-x^2} times the solution of the operator $(e^{-x^2}F')' - e^{-x^2}F$.

Prop. (6.8.11). For a formally self-adjoint elliptic operator P of degree d on E, $\Gamma(E) = \operatorname{Im} P \oplus P(\Gamma(E))$.

Proof: We know that $L^2(E) = P(H^d E) \oplus \operatorname{Ker} P$, and $\operatorname{Ker} P$ are all smooth by(6.8.3), so $\Gamma(E) = \operatorname{Ker} P \oplus P(H^d E) \cap \Gamma(E)$. Now use Garding's inequality(6.8.2), the intersection is just $P(\Gamma(E))$, thus the result. □

Prop. (6.8.12) (Asymptotic Heat Equation). In this case we have the series

$$h_t(A^*A) = \sum_\lambda e^{-\lambda t} \dim \Gamma_\lambda(E)$$

converges and h_t has an asymptotic expansion

$$h_t = \sum_{k \geq -n} t^{k/2m} U_k(A^*A)$$

where $n = \dim M$ and $U_k = \int_M \mu_k$ for a differential form on M. Cf.[Heat Equation and the Index Theorem P297].

By the proposition above, the eigenspaces of eigenvalue non-zero neutralize, so $\operatorname{Ind} A = h_t(A^*A) - h_t(AA^*)$, so

$$\operatorname{Ind} A = U_0(A^*A) - U_0(AA^*) = \int_M \mu_0(A^*A) - \mu_0(AA^*).$$

The proof consists of the following propositions,

Prop. (6.8.13). Using the fact that an elliptic operator has a countable basis, for an elliptic operator P, when $t > 0$, we let $K(t, x, y) = \sum_n e^{-t\lambda_n} \Phi_n(x) \overline{\Phi}_n(y)$, then

$$e^{-tP} f(x) = \int K(t, x, y) f(y) dy.$$

$K(t, x, y)$ is smooth. and the trace of e^{-tA^*A} is exactly $h_t(A^*A)$ as in the last proposition. And the trace is just $\int_M K(t, x, x)$, as can be easily seen.

Proof: Use Garding inequality and(6.8.7), we can show $||K||_{C^k}$ is bounded. □

Chapter V

Algebraic Geometry

V.1 Sites

References are [StackProject].

1 Sites

Def. (1.1.1). A **site** is given by a category $\mathcal{C}$ and a set $Cov(\mathcal{C})$ of families of morphisms with fixed target, called the **coverings** of $\mathcal{C}$ that:

- An isomorphism is a covering.
- Coverings of covering is a covering.
- Base change of a covering is a covering.

Sometimes A site is called called a topology, the difference is that the morphism of site is reverse of a morphism of topology.

Def. (1.1.2). A **morphism of topologies** $\mathcal{D} \to \mathcal{C}$ is a morphism that preserves covering and base change by covering morphisms. A **morphism of sites** $\mathcal{C} \to \mathcal{D}$ is a morphism u of topologies $\mathcal{D} \to \mathcal{C}$ that u_s(1.2.5) is exact.

This exact condition is easy to be satisfied, by(1.2.7).

Def. (1.1.3). A G**-topological space** is a set X with a family of subsets of X that they form a Grothendieck topology w.r.t inclusions and that covering are all set-theoretic coverings(but not conversely). These subsets are called **admissible opens** of X and covers are called **admissible covers**.

Def. (1.1.4) (Completeness). The completeness of a G-topological space X:

- G0: $\varnothing$ and X are admissible open.
- G1: Let $\{U_i \to U\}$ be an admissible cover, then a subset $V \subset U$ is admissible if $V \cap U_i$ are all admissible.
- G2: Let $\{U_i \to U\}$ be a cover of admissible opens for U admissible, then the cover is admissible if it has an admissible cover as a refinement.

Def. (1.1.5). An object U in a site is called **quasi-compact** if for each covering of U, f.m. of them still forms a covering of U. The topology T is called **Noetherian** if each object of T is quasi-compact.

Given a site T, we can define a new site T^f whose coverings are coverings of T that are finite. Then this is truly a site and it is Noetherian.

Topoi

Def. (1.1.6). A **topos** is the category of sheaves over a site $\mathcal{C}$. For sites $\mathcal{C}, \mathcal{D}$, a morphism of topoi consists of two natural adjoint morphism $f_* : Sch(\mathcal{C}) \to Sch(\mathcal{D})$ and $f^{-1} : Sch(\mathcal{D}) \to Sch(\mathcal{C})$ that f_* is right adjoint to f^{-1} and f^{-1} is exact.

2 Sheaves on Sites

Def. (1.2.1). An epimorphism $\{U_i \to V\}$ in a category is called a **family of effective epimorphisms** if

$$\mathrm{Hom}(V, Z) \to \prod \mathrm{Hom}(U_i, Z) \rightrightarrows \prod \mathrm{Hom}(U_i \times_V U_j, Z)$$

is exact for each Z. Similarly for a **family of universal effective epimorphisms**.

Prop. (1.2.2). The set of all families of universal effective epimorphisms in a category forms a Grothendieck topology, called the **canonical topology**. It is the finest topology that all representable presheaves are sheaves.

Proof: We only need to verify that family of universal effective epimorphisms is closed under composition. For this, first prove epimorphism, then use epimorphism to prove effectiveness. Universal follows routinely. Cf.[Tamme]. $\square$

Prop. (1.2.3) (Sheafification). The operator F^+ is the presheaf that

$$F^+(U) = \varinjlim \mathrm{Ker}(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)) = \check{H}^0(U, F)$$

It is a separated presheaf, i.e. $0 \to F(U) \to \prod_i F(U_i)$ and when F is separated, $F \to F^+$ is injective and F^+ is a sheaf. (The problem of separated is that the cover may not be identical in $U_i \times_U U_j$ but only on a cover of it).

The sheafification F^{++} is exact and it is left adjoint to the forgetful functor.

So the forgetful functor is left exact and it preserves injectives. Thus the sheaf cokernel is the shifification of the presheaf kernel, the sheaf kernel is the presheaf kernel.

Proof: The separatedness is simple. For sheaf condition, an element of $F^+(U_i)$ is represented by a covering $\{V_{ij} \to U_i\}$, and there restriction to $U_i \times_U U_j$ coincide by separatedness hence the covering $\{V_{ij} \to U\}$ is an element of $F^+(U)$.

Sh is left exact because $-^+$ is left exact from PAb to PAb by(4.2.2) checked on every element U. It is right exact trivially, hence it is exact. $\square$

transfer of sheaves under morphisms

Def. (1.2.4) (Pullback & Pushforward of Presheaves). Given a morphism of topologies $T \to T'$, which should be regarded as an inverse map. There are maps

$$f^p F'(U) = F'(f(U)) : \mathcal{P}' \to \mathcal{P}, \quad f_p(F)(U') = \varinjlim_{U_i | U \to f(U_i)} F(U_i) : \mathcal{P} \to \mathcal{P}'$$

Then f_p is left adjoint to f^p, and f^p is exact, checked easily.

Def. (1.2.5) (Pullback & Pushforward of Sheaves). Given a morphism of topologies $T \to T'$, which should be regarded as an inverse map. There are maps

$$f^s = f^p \circ i : \mathcal{S}' \to \mathcal{S}, \quad f_s = \sharp \circ f_p \circ i : \mathcal{S} \to \mathcal{S}'.$$

f_s is left adjoint to f^s, by adjointness of f_p, f^p and $\sharp, \iota$.

This is dual to the case of usual topology space.

Prop. (1.2.6). If a sheaf F on T is represented by $Z \in T$, then $f_p F$ is represented by $f(Z) \in T'$.

Proof: Cf.[Tamme P44]. □

Prop. (1.2.7) (When is f_s Exact). If $f : T \to T'$ is a morphism of topologies that has final objects and finite fiber products and f respects final objects and finite fiber produces, then $f_s : \mathcal{S} \to S'$ is exact.

Proof: It suffices to show the left exactness of f_p. By definition, $f_p(U') = \varinjlim_{\mathcal{I}_{U'}^{op}} F_{U'}$, where $\mathcal{I}_{U'}$ is the category of all (U, φ) that $U' \to F(U)$ and $F_{U'}$ is the covariant functor $(U, \varphi) \to F(U)$. Now T has fiber products and products, then $\mathcal{I}_{U'}^{op}$ is seen to be cofiltered, so this limit process is exact form $\mathrm{Hom}(\mathcal{I}_{U'}^{op}, \mathcal{A}b)$ to $\mathcal{A}b$ and $F \to F_{U'}$ is clearly exact. □

Prop. (1.2.8). For a topology T and an object Z of T, there is a category T/Z as objects over T, and $i : T/Z \to T$ is continuous. Then i^s is exact.

Proof: $R^q i^s(F) = (i^p(\mathcal{H}^q(F)))^\sharp$(4.2.8), and $(\mathcal{H}^q(F))^\sharp = 0$(4.2.6), so it suffices to show i^p and $\sharp$ commutes. But i^s and $^+$ commutes obviously. □

Prop. (1.2.9) (Sheaf Condition is Local). To check sheaf condition for sheaf, it suffice to show that for any covering, there is a refinement covering of it that sheaf condition hold, because by the definition of sheafification functor, $F^+ = F$, so F is a sheaf.

Cor. (1.2.10). For two topology on a category that $\mathcal{I}'$ is finer than $\mathcal{I}$, then any $\mathcal{I}'$-sheaf is a $\mathcal{I}$-sheaf. And if any covering in $\mathcal{I}'$ can be refined by a covering in $\mathcal{I}$, then $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories. In particular, if T is Noetherian, $\mathcal{S}(T)$ and $\mathcal{S}(T^f)$ are equivalent.

Prop. (1.2.11) (Comparison lemma). Let T' be a fully subcategory of T, $i : T' \to T$ is a morphism of topologies, if any cover of T between objects of T' is a cover of T', and each object U of T as a covering $\{U_i \to U\}$ with objects $U_i \in T'$, then i^s, i_s forms an equivalence between sheaves on T and sheaves on T'.

Proof: $i^s i_s G(U) = (i_p G)^\sharp(U)$ for $U \in T'$, we show $G(U) \cong i_p(G)(U) cong (i_p G)^\sharp(U)$. For the first, by definition of i_p, and use the fact (U, id_U) is the initial object. For the second, notice the colimit definition of $^+$, then $(i_p G)^+(U) \cong i_p G(U) = G(U)$.

Now i^s is exact, because $^+$ commutes with i^p commutes for presheaves on T, and $R^q i^s(F) = (i^p \mathcal{H}^q(F))^\sharp$ and $(\mathcal{H}(F))^\sharp = 0$(4.2.6).

To prove $i_s i^s F \cong F$, notice $i_s i^s F \cong (i_p i^s F)^\sharp$, and there are commutative diagram $i_p i^s F(U) \to (i_p i^s F)^\sharp(U) \to F(U)$. If we let $U \in T'$, then the first part of the proof applies and shows $i_p i^s \mathcal{F} \cong (i_p i^s F)^\sharp(U)$, and $F(U) = i^s F(U) \cong i_p i^s F(U)$, so $i_s i^s F(U) \cong F(U)$.

Now for any $U \in T$, we can choose a covering $\{U_i \to U\}$ with U_i in T', and then choose a covering $\{U_{ij}^k \to U_i \times_U U_j\}$, then there is a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & i_s i^s F(U) & \longrightarrow & \prod_i i_s i^s F(U_i) & \longrightarrow & \prod_{ijk} i_s i^s F(U_{ij}^k) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(U) & \longrightarrow & \prod_i F(U_i) & \longrightarrow & \prod_{ijk} F(U_{ij}^k) \end{array}$$

The last two vertical map are isomorphisms, so we can use five lemma. □

Cor. (1.2.12). Let T' be a fully subcategory of T, $i : T' \to T$ is a morphism of topologies, and each object U of T' and a covering $\{U_i \to U\}$ in T has a refinement $\{U'_j \to U\}$ in T'. Then $G \cong i^s i_s G$ for any sheaf G on T' and $i^s : \mathcal{S} \to \mathcal{S}'$ is exact. (This is implicit in the proof of the last proposition).

In particular, this applies to the case $T' = T/Z \to T$, the localization category, in with case $i^s F(Z') = F(Z)$ is called the **restriction sheaf**.

3 Sheaves on Topological Spaces

Prop. (1.3.1) (Stalks). Taking stalks is a left adjoint to the skyscraper sheaf from $\mathcal{A}b$ to Ab thus preserves cokernel, moreover it is exact.

Epimorphism and monomorphism can be checked on stalks, so also can be checked on affine opens. Cf.[Hartshorne P63].

Prop. (1.3.2). If a sheaf has only one non-vanishing stalk, then it is a skyscraper stalk. (Because the restriction to that point for every open set is an isomorphism).

Def. (1.3.3).

- the pushforward $f_p F$, $f_p F(U) = F(f^{-1}(U))$ sends presheaf to presheaf.
- the direct image $f_* \mathcal{F}$, $f_* \mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ sends sheaf to sheaf.
- the inverse image $f^{-1}\mathcal{F}$, $f^{-1}\mathcal{F}(U) = \mathcal{F}(f(U))^\sharp$.
- the extending by zero sheaf: for an open subset U, $j_!(F)$ is shifification of presheaf that $G(V) = F(V)$ when $V \subset U$ and 0 otherwise.
- the inverse image with compact support $i^!$ for an inclusion of closed subset is $i^!(F)(U = V \cap Y) = \{s \in \Gamma(V, X) | supp(s) \in Y\}$.

f^{-1} is left adjoint to f_* by(2.1.5) because Ab are just $\mathbb{Z}$-modules. And f^{-1} is exact(**??**

$j_!$ is left adjoint to the functor f^{-1} for an inclusion of open subset $j : U \subset X$. $i^!$ is right adjoint to f_* for an inclusion of closed subset $i : Y \to X$, in particular f_* is exact when f is a closed inclusion.

4 Sites over Schemes

Prop. (1.4.1). Fiber products exist in the category of schemes.

Proof: Cf.[Hartshorne P88]. You should use glueing(1.5.10). □

Zariski Topology

Def. (1.4.2). The **Zariski topology** has the the covering of a scheme T as classes of open immersions $\{T_i \to T\}$ that their images cover T.

The **big Zariski site** Sch_{Zar}/S has the objects as all schemes over S.

The **small Zariski site** S_{Zar} has the objects as all open subschemes over S.

The **restricted Zariski site** S_{Zarfp} has the objects as all schemes that are qcqs open subschemes of S.

The **big affine Zariski site** Aff_{Zar}/S has the objects as all schemes affine over S.

These are all topologies because open immersions satisfies base change trick(3.2.36).

In particular when the cover has only one element and is affine, the descent datum is equivalent to compatible isomorphisms

$$\varphi_{13} : N \otimes_A B \otimes_A B \xrightarrow{\varphi_{12}} B \otimes_A M \otimes_B \xrightarrow{\varphi_{23}} B \otimes_A B \otimes_A M.$$

Prop. (1.4.3). A sheaf w.r.t the small Zariski topology is equivalent to a sheaf on S, trivially, so the sheaf cohomology on Aff_{Zar}/S is equivalent to usual sheaf cohomology on S.

Prop. (1.4.4). If X is qs, then $\widetilde{X_{Zar}} \to \widetilde{X_{Zarfp}}$ is an equivalence by i_s and i^s, the same proof as(1.4.9).

Étale Topology

Def. (1.4.5). The **étale topology** has the covering of a scheme T as classes of étale morphisms that their images cover T.

The **big étale site** $Sch_{étale}/S$ has the objects as all schemes over S.

The **small étale site** $S_{étale}$ has the objects as all schemes that are étale over S.

The **restricted étale site** $S_{étfp}$ has the objects as all schemes that are étale and qcqs over S.

The **big affine étale site** $Aff_{étale}/S$ has the objects as all schemes affine over S.

These are truly topologies because étale is stable under base change and composition.

Prop. (1.4.6). Zariski covering is étale, because open immersions are étale.

Prop. (1.4.7). An étale covering of a qc scheme can be refined a finite affine étale covering, this is because étale map are open(3.7.3). Thus so does all above coverings.

Prop. (1.4.8). The restricted étale site of a qc scheme X is Noetherian, because étale map is open, and any object in $X_{étfp}$ is qc.

Prop. (1.4.9). If X is qs, then $\widetilde{X_{ét}} \to \widetilde{X_{étfp}}$ is an equivalence by i_s and i^s.

Proof: Want to use(1.2.11), one condition is satisfied by(1.4.7), so it suffice to check any étale scheme X'/X has a covering of qcqs étale schemes over X. For any point $p \in X'$, there is an affine nbhd U' that maps to an affine nbhd U' of X and the ring map is f.p., so $U' \to U$ is étale and f.p, and $U \to X$ is open immersion and qs, it is qc because X is qs and U is qc(3.2.25). □

Prop. (1.4.10) (Cohomology Big and Small Sites). The inclusion of small sites to the big sites has no infection on the sheaf cohomology, by(4.2.12). This is applicable to all topologies τ considered here.

Syntomic Topology

Def. (1.4.11). The syntomic topology has the covering of a scheme T as classes of syntomic morphisms that their images cover T.

fppf Topology

Def. (1.4.12). The fppf topology has the covering of a scheme T as classes of flat locally of finite presentation morphisms that their images cover T. (f.f.+locally of f.p.).

The **big Zariski site** Sch_{fppf}/S has the objects as all schemes over S.

The **big affine Zariski site** Aff_{fppf}/S has the objects as all schemes affine over S.

They are all topologies because flatness and finite presentation satisfies base change trick by(3.4.2) and(3.9.3).

Prop. (1.4.13). A syntomic covering is fppf by definition(6.4.1).

Prop. (1.4.14). A fppf covering of an affine scheme can be refined a finite affine fppf covering, because fppf map are open(3.4.7).

fpqc Topology

Def. (1.4.15). The **fpqc topology** has the covering of a scheme T as classes of flat morphisms s.t. their images cover T and for any affine open $U \subset T$, the restriction on T can be refined by a finite affine cover of open affine subschemes of the covering(f.f.+qc). It is a topology by(3.4.2) and(3.2.24).

When the covering consists of affine schemes, it is called **standard fpqc covering**.

Prop. (1.4.16). Fppf coverings are fpqc. (Use(3.4.7), we see that fppf covering consists of open morphisms, thus it is qc because affine scheme is quasi-compact.)

Prop. (1.4.17). A covering consisting of flat morphisms refined by a fpqc covering is a fpqc covering.

Hence being fpqc is local on the target, because a Zariski cover is a fpqc covering.

If U is a covering consisting of flat morphisms that there is a fpqc covering V that $U \times V \to V$ is a fpqc covering, then U is fpqc, because $U \times V$ does and it refines U.

Remark (1.4.18). Defining fpqc sites has inescapable set-theoretic difficulties, thus we don't consider fpqc sites and fpqc cohomologies. Cf.[StackProject 0BBK].

Prop. (1.4.19). A presheaf is a sheaf w.r.t the fpqc topology iff it is a sheaf w.r.t the Zariski topology and satisfies sheaf property w.r.t the single covering $V \to U$ f.f. between affine schemes.

Proof: For any covering $\{X_i \to X\}$, choose an affine open cover U_i of X, then by definition, the pullback cover on U_i can use refined by a finite affine cover $U_{ik} \to U_i$, so The composition covering of $\{U_{ij} \to U_i\}$ and $\{U_i \to X\}$ refines $\{X_i \to X\}$. And sheaf condition for $\{U_{ij} \to U_i\}$ is the same as sheaf condition for $\{\coprod U_{ij} \to U_i\}$, thus the result, by(1.2.9). □

Prop. (1.4.20). The coverings in X_{fpqc} are families of universal effective epimorphisms, in the category of X-schemes.

Proof: Cf.[StackProject 023Q]. □

Cor. (1.4.21). For $f : Y \to X$ a morphism of schemes, if $Z \in X_\tau$ for the above topologies τ, then $f^*(\mathrm{Hom}_X(-, Z)) \cong \mathrm{Hom}(-, Z \otimes_X Y)$, in other words, the inverse sheaf of a representable sheaf is representable.

Proof: By definition, $f^*(\mathrm{Hom}_X(-, Z))$ is the sheaf associated to the presheaf $f_p(\mathrm{Hom}_X(-, Z))$, which by(1.2.6) is just the presheaf represented by $Z \otimes_X Y$, but by the proposition, it is already a sheaf. □

Prop. (1.4.22). Let M be a Qco sheaf of $\mathcal{O}_X$-modules, then the functor $X' \to \Gamma(X', M \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})$ is an Abelian sheaf on $X_{\acute{e}t}$. To verify this, use(1.4.19), and it suffice to check sheaf condition for étale surjection of affine schemes, in which case the morphism is f.f., and the result follows from fpqc lemma(1.5.3).

5 Descent

Basic References are [StackProject Chap34, 10.158].

General Principal

Prop. (1.5.1). A property of schemes is called **local** in a topology if for any covering $\{U_i \to S\}$, S has P iff U_i has P. A property of morphisms is called **local** in a topology if for any covering $\{U_i \to S\}$, $X \to S$ has P iff $X \times_S U_i \to U_i$ has P.

For these, we only need to show that it is local in the Zariski topology and check for a standard topology(i.e. affine covering of affine scheme). This is because any covering of an affine scheme can be refined by an affine covering(1.4.15) and(1.4.16).

Def. (1.5.2). A **descent datum** for qco sheaves for a covering in a site is just a family of local Qco sheaves that satisfies the cocycle condition on their intersections.

fpqc Descent

Prop. (1.5.3) (fpqc-Poincare Lemma). If a ring map $A \to B$, either has a section $B \to A$, or it is faithfully flat, then the Amitsur complex $s(N)$ for the canonical descent datum (with augmentation):

$$0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M \to \dots$$

with Čech-like maps, is exact.

Proof: Nullhomotopy can be constructed, the f.f. case can be reduced to the first case by tensoring B to consider$B \to B \otimes_A B$, Cf.[Sheaf Cohomology notes P23]. □

Lemma (1.5.4) (Affine fpqc Descent). When $A \to B$ is f.f., there is an equivalence of categories:

$$\{M \in A - mod\} \leftrightarrow \{(N, \varphi) \text{ descent datum}\}$$

giving by $M \to B \otimes_A M$ with the canonical descent datum. and $M \to B \otimes_A H^0(s(N))$.

Proof: Cf.[StackProject 023N]. □

Remark (1.5.5). In fact, a descent datum is always effective iff $A \to B$ is universally injective. Cf.[StackProject]. And f.f. extension is u.i.(6.2.18).

Prop. (1.5.6) (fpqc Descent). If S is a scheme and $\{U_i \to S\}$ is a fpqc covering, then the category of descent datum for the covering is equivalent to the category of qco sheaves on S.

Proof: Faithfulness: if a, b are two morphism of Qco sheaves over S, then for $s \in S$, $s = \varphi_i(u)$ for some $\varphi_i : U_i \to S$, and φ_i is flat, so the stalk map is f.f. by(6.2.14), so $a = b$ on every stalk of S.

Fully faithfulness: if we have a morphism $\{\varphi_i\}$ of descent datums of qco sheaves, then for any affine subset V of S, we get a morphism of the pullback descent datum. Then by(1.5.4) above, we get a morphism on V. These morphism are compatible on their intersection by the faithfulness just proved, so they gives a morphism on S.

Essentially surjectivity: for a descent datum, pull it back to any affine subscheme V_i of S, then there is a qco sheaf on V_i by (1.5.4) above, and there is a canonical isomorphism of their restriction on the intersection, by fully faithfulness just proved, so it gives a qco sheaf on S by Zariski descent(1.5.10) and the fact qco is local by(2.1.22). □

Cor. (1.5.7). For any Qco sheaf $\mathcal{F}$ on S, the functor $(Sch/S)^{op} \to Ab : T \to \Gamma(T, f^*\mathcal{F})$ is a sheaf in the fpqc topology, hence also in the fppf, étale Zariski topology.

Prop. (1.5.8). For any Qco sheaf $\mathcal{F}$ on a separated scheme X. If T is a Grothendieck topology on Sch/S containing the Zariski topology and every cover is refined by a fpqc cover by a finite collection of affine schemes, then $H^p(T, X, \mathcal{F}) = H^p(X, \mathcal{F})$. Same as the proof of(4.4.2), with the Zariski-Poincare lemma replaced by the fpqc-Poincare lemma.

Prop. (1.5.9) (fpqc Descent for morphisms). For a faithfully flat morphism f that is qc, the following property holds for a morphism iff it holds for its base change along f.

1. isomorphism/monomorphism.
2. (quasi-)separated.
3. quai-compact.
4. (locally)of f.t.
5. (locally)of f.p.
6. proper
7. (quasi-)affine.
8. (quasi-)finite.
9. flat.
10. smooth, unramified, étale.
11. (closed/open)immersion.

Proof: Cf.[EGAIV-2, Proposition 2.7.1]. and [StackProject 34.20]. □

Galois Descent

Galois descent is a special case of fpqc descent.

Zariski Descent

Prop. (1.5.10) (Zariski Descent). Any descent datum for the Zariski topology is effective. In fact, Qco is no needed. In the same way, we can glue schemes and also morphisms with a fixed target (compatible with the glueing).

Proof: For every open set $V \subset X$, we define the group of sections $\mathcal{F}(V)$ to be a set consisting of all tuples $(s_i)_{i \in I}$ required to obey the compatibility condition:

$$\phi_{ij}(s_i|_{V \cap U_i \cap U_j}) = s_j|_{V \cap U_i \cap U_j} \qquad (*)$$

for all $i, j \in I$. The group addition on $\mathcal{F}(V)$ is the obvious one.

The $\mathcal{F}$ that I defined is guaranteed to be a sheaf, but we also need to satisfy ourselves that the restriction $\mathcal{F}|_{U_k}$ really is isomorphic to the $\mathcal{F}_k$ that we started with, for each $k \in I$. It is here that the cocycle condition is required.

It is easy to write down what the isomorphism $\psi : \mathcal{F}_k \overset{\cong}{\to} \mathcal{F}|_{U_k}$ ought to be. Given an open $V \subset U_k$ and given a section $s \in \mathcal{F}_k$, we would like to define its image under ψ to be

$$\psi(s) = (\phi_{ki}(s|_{V \cap U_i}))_{i \in I}$$

However, we need to be sure that the tuple $(\phi_{ki}(s|_{V \cap U_i}))_{i \in I}$ represents a well-defined element of $\mathcal{F}(V)$. In particular, we must verify that $(\phi_{ki}(s|_{V \cap U_i}))_{i \in I}$ obeys the condition $(*)$, which states that

$$\phi_{ij} \circ \phi_{ki}(s|_{V \cap U_i \cap U_j}) = \phi_{kj}(s|_{V \cap U_i \cap U_j})$$

for any $i, j \in I$. This is true by virtue of the cocycle condition.

This map is obviously injection and it is surjection by virtue of $(*)$. $\square$

V.2 Schemes

1 Ringed Spaces & $\mathcal{O}_X$-Modules

Def. (2.1.1). A **ringed space** X is a topological space together with a sheaf of rings $\mathcal{O}_X$. There morphisms are topological maps and a reverse ring map. A $\mathcal{O}_X$**-module** is an Abelian sheaf with a ring module structure compatible with restriction maps. A ringed space is called **local ringed space** iff kts stalks are all local rings.

Prop. (2.1.2). Glueing sheaves is available for ringed spaces, similar to(1.5.10).

Transfer of Modules

Def. (2.1.3).

- the direct image modules: $f_*\mathcal{F}$, $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ sends $\mathcal{O}_X$-module to $\mathcal{O}_Y$-module.
- the pullback of modules:$f^*(\mathcal{F}) = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.
- The **tensor** of two modules is the sheaf associated to the tensor of two presheaves.
- for a closed immersion $Y \to X$, there is $i^! : Qco(X) \to Qco(Y)$ that is right adjoint to i_*: $i^!\mathcal{G} = i^*((\mathcal{H}_Z(\mathcal{G}))')$, where $\mathcal{H}_Z(\mathcal{G})$ is the sheaf of sections annihilated by $\mathcal{I}$ and $\mathcal{F}'$ is the maximal Qco sheaf of $\mathcal{F}$.
- For f proper between locally Noetherian scheme, there is a inverse sheaf $f^!\mathcal{G} = \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})$, which maps $Qco(Y)$ to $Qco(X)$ by(2.1.25) and(4.4.12). When f is affine, in particular when it is finite, then $f^!$ is right adjoint to f_* on Qco(4.4.13).

Prop. (2.1.4). Tensoring is strongly left adjoint to $\mathcal{H}om(\mathcal{F}, -)$:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{H} \otimes \mathcal{F}, \mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})).$$

(Recall the definition of tensor sheaf).

Prop. (2.1.5). f^* is left adjoint to f_* by(3.4.3): $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$. In fact

$$f_*(\mathcal{H}om(f^*\mathcal{G}, \mathcal{F})) = \mathcal{H}om(\mathcal{G}, f_*\mathcal{G}).$$

Cor. (2.1.6). The f^* may not be exact. f^{-1} is exact, but we tensored with $\mathcal{O}_X$, it is exact when f is flat.

Prop. (2.1.7). Tensor commutes with pullbacks, in particular with taking stalks. So tensoring with a locally free sheaf is exact.

Proof: We have

$$\mathrm{Hom}(f^*\mathcal{F} \otimes f^*\mathcal{G}, \mathcal{H}) = \mathrm{Hom}(\mathcal{F}, f_*\,\mathcal{H}om(f^*\mathcal{G}, \mathcal{H})) = \mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, f_*\mathcal{H})) = \mathrm{Hom}(f^*(\mathcal{F} \otimes \mathcal{G}), \mathcal{H}).$$

□

Prop. (2.1.8). On a ringed space X, for a qc open subset U, $(\oplus\mathcal{F}_i)(U) = \oplus\mathcal{F}_i(U)$. This uses the compactness of U.

Prop. (2.1.9). For a closed immersion f, f_* on $\mathcal{O}_X$-mod is fully faithful, with image those killed by $\mathcal{I}$, where $\mathcal{I}$ is the structural kernel, Cf.[StackProject 08KS].

Modules of Finite Type

Def. (2.1.10). An $\mathcal{O}_X$-module is called of **finite type** iff locally it is a cokernel of a finite free sheaf.

Prop. (2.1.11). if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}_X$-modules and $\mathcal{F}_1, \mathcal{F}_3$ are of f.t., then $\mathcal{F}_2$ is of finite type.

Proof: Choose a finite generators of $\mathcal{F}_3$ on some nbhd, and shrink it to get inverse image of them in $\mathcal{F}_2$, and choose a smaller nbhd that $\mathcal{F}_1$ is f.g., then on this nbhd, $\mathcal{F}_2$ is f.g. □

Prop. (2.1.12). If $\mathcal{G} \to \mathcal{F}$ is surjective on the stalk for a x and $\mathcal{F}$ is of f.t., then it is surjective on a nbhd of x. Thus the support of a f.t. sheaf is closed (look at $0 \to \mathcal{F}$).

(Quasi-)Coherent Sheaves

Def. (2.1.13). A sheaf of module $\mathcal{F}$ is called **quasi-coherent** iff locally it is a cokernel of free modules.

Prop. (2.1.14). The pullback of a qco module is a qco, because f^* is right adjoint.

And for a ringed space $(X, \mathcal{O}_X)$ and a $R = \Gamma(X, \mathcal{O}_X)$-module M, we have a coherent sheaf $\mathcal{F}_M$ on X, defined as $\pi^*(M)$, where M is seen as a qco sheaf on (pt, R). It is the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U) \otimes M$.

This construction is a functor from the category of R-module to the category of Qco $\mathcal{O}_X$-modules, and it commutes with colimits because f^* does. And it is left adjoint to Γ by(2.1.5):

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{G})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{G})$$

Def. (2.1.15). A sheaf of module $\mathcal{F}$ is called **of finite presentation** iff locally it is a cokernel of finite free modules. A finitely presented sheaf of modules is Qco, and the pullback of a f.p. sheaf is f.p, by the right adjointness of f^*.

Prop. (2.1.16). If $\mathcal{G} \to \mathcal{F}$ is a surjection and $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is of f.t., then the kernel is of finite type. (Use a restriction to smaller nbhd technique, this has the same proof as**??**).

Def. (2.1.17). On a ringed space X, a **coherent module** is a module that is of f.t. and on any open set U and for any set of elements of $\Gamma(U, \mathcal{F})$, the kernel of $\oplus \mathcal{O}_U \to \mathcal{F}|_U$ is of f.t.. A coherent sheaf is of finite presentation, by a restriction to nbhd technique.

A coherent sheaf is of finite presentation and Qco, Cf.[StackProject 01BW].

Prop. (2.1.18). Any f.t. subsheaf of a coherent sheaf is coherent, by definition. Any kernel of a f.t. sheaf to a coherent sheaf is of f.t. And the category of coherent sheaves is a weak Serre subcategory of $\mathcal{O}_X$-modules. Thus if $\mathcal{O}_X$ is coherent, then a sheaf is coherent iff it is f.p.

Proof: Cf.[StackProject 01BY]. □

Cor. (2.1.19). If $\mathcal{G} \to \mathcal{F}$ is injective on some point x, $\mathcal{G}$ is of f.t. and $\mathcal{F}$ is coherent, then it is injective on a nbhd of x.

Proof: By the proposition, the kernel is of f.t., so its support is closed by(2.1.12). □

(Quasi-)Coherent Sheaves over Schemes

Lemma (2.1.20). On an affine scheme $\operatorname{Spec} A$, there is a sheaf $\widetilde{M}$, that is M_f on $\operatorname{Spec} A_f$. To check it is a sheaf, we only need to check to affine coverings, and this is by(1.5.3).

Prop. (2.1.21) (Quasi-Coherent Sheaves). For any A-module M, there is a sheaf of modules $\mathcal{F}_M$ on $\operatorname{Spec} A$ by(2.1.14). This is left adjoint to Γ and defines a functor from the category of A-modules to the category of $\mathcal{O}_{\operatorname{Spec} A}$-modules.

And in fact, this is an equivalence to the category of quasi-coherent sheaves over $\operatorname{Spec} A$ because Qco is locally like $\widetilde{M_i}$, by the fact that localization is exact, and(1.5.4) shows that locally of the form $\widetilde{M_i}$ must be globally of the form $\widetilde{M}$ and $\Gamma(X, \widetilde{M}) = M$.

This is also an equivalence between f.g. A-modules and coherent sheaves over $\operatorname{Spec} A$, by fpqc descent(1.5.4).

Def. (2.1.22) (Coherent Sheaves). When X is a locally Noetherian scheme, coherence is equivalent to M_is are f.g. A_i-module. When talking about coherent sheaves over schemes, I tacitly assume the scheme is locally Noetherian.

(Quasi-)coherent is an affine local by(2.1.21) and(5.1.16).

Prop. (2.1.23).

- $(Q)co(X)$ forms a weak Serre subcategory of Mod-$\mathcal{O}_X$.
- Tensor product of two (Q)co sheaf is (Q)co, and locally free if they are locally free (because $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ as tensor product commutes with pullbacks)
- pullback of (qco)coherent sheaves are (qco)coherent. (Local on the affine opens, check $f^*(\widetilde{M}) \cong \widetilde{M \otimes_A B}$. Note in the coherent case, both scheme should be locally Noetherian.

Proof: For the weak Serre subcategory, Coherent case is by(2.1.18). For Qco, just need to verify the kernel, cokernel and extension, by the fact that localization is exact. For the extension of Qco, use(4.4.4) that the global section is exact, so there is a morphism of exact sequences $\Gamma\widetilde{(X, \mathcal{F})} \to \mathcal{F}$, and five lemma gives the result. □

Prop. (2.1.24). If f is qcqs, then the pushforward of of a Qco sheaf is Qco. (Used in(4.3.7)).

Proof: The question is local so we let Y be affine, and then X is qcqs, so we cover it with affine opens U_i and their intersections are U_{ijk}. Then we see by sheaf property

$$0 \to f_*\mathcal{F} \to \oplus_i f_*(\mathcal{F}|_{U_i}) \to \oplus_{ijk} f_*(\mathcal{F}|_{U_{ijk}})$$

The last two are Qco because two are maps between affine schemes, so the first is Qco. □

Prop. (2.1.25). f_* for f proper maps coherent sheaf to a coherent sheaf.

Prop. (2.1.26) (Extensions of Coherent Sheaves). On a locally Noetherian scheme, any Qco sheaf is sum of coherent sheaves, so any coherent sheaf on an open subset can be extended to a global coherent sheaf.

Proof: First prove for affine opens, this is true, then we extend by Zorn lemma. The last is because for any section $s \in \Gamma(U)$, we can extend it to a global section of the pushforward sheaf. □

Prop. (2.1.27) (Deligne). On a Noetherian scheme X, let $\mathcal{F}$ be a Qco sheaf, $\mathcal{G}$ be a coherent sheaf and $\mathcal{I}$ be a Qco sheaf of ideals correspongding to Z, $U = X - Z$, then we have

$$\varinjlim \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular,

$$\varinjlim \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \cong \Gamma(U, \mathcal{F}).$$

Proof: Cf.[StackProject 01YB]. □

Prop. (2.1.28) (Kleinmann). If X is a Noetherian integral separated locally factorial scheme, then every coherent sheaf on X is a quotient of a finite locally free sheaf.

Proof: Cf.[Hartshorne P238]. □

Prop. (2.1.29) (Support of Modules). The support of a Qco sheaf of f.t over a scheme is closed(because the support of a single section is closed in every affine open), e.g. coherent sheaf.

This have many consequences applied to kernel and cokernel, for example, a coherent sheaf is locally free iff all its stalk is free (choose a presentation and see kernel and cokernel).

For a flat morphism f, $\mathrm{Supp}(f^*(\mathcal{F})) = f^{-1}(\mathrm{Supp}\,\mathcal{F})$, by(5.1.12).

Proof: because for a set of generators x_i of M, $Ann(\mathcal{F}) = \cup Ann(x_i)$, and $Ann(x_i)$ is closed. □

Cor. (2.1.30) (Semicontinuity). For a Qco sheaf $\mathcal{F}$ of f.t., $\varphi(y) = \dim_{k(y)(\mathcal{F}\otimes k(y))}$ is an upper semicontinuous function on the scheme.

Proof: By Nakayama, $\varphi(y)$ is equal to the minimal number of generators of the $\mathcal{O}_y$-module $\mathcal{F}_y$. But these generators extends to a nbhd of y, so $\varphi \le n$ on this nbhd. □

Prop. (2.1.31). For X a scheme and any $\mathcal{O}_X$-module $\mathcal{F}$, there is a Qco submodule of $\mathcal{F}$ maximal among all Qco submodules of $\mathcal{F}$. This is because the colimit of Qco sheaves are Qco.

Prop. (2.1.32). A f.t. Qco sheaf on a scheme has a minimal closed scheme on its support, it is generated locally by the Qco ideal $\mathrm{Ann}_A(M)$(3.2.37). And there is a f.t. Qco sheaf $\mathcal{G}$ that $i_*(\mathcal{G}) = \mathcal{F}$. Cf.[StackProject 01QY].

Devissage of Coherent Sheaves

Lemma (2.1.33). Let $\mathcal{F}$ be a coherent sheaf on a Noetherian scheme X, let I be a sheaf of ideals that correspond to Z, then $\mathrm{Supp}(\mathcal{F}) \subset Z$ iff $\mathcal{I}^n\mathcal{F} = 0$ for some n. (This follows easily from Noetherian and(5.6.5)).

Lemma (2.1.34). If we have a coherent sheaf $\mathcal{F}$ on a Noetherian scheme X, that $\mathrm{Supp}(\mathcal{F}) = Z_1 \cup Z_2$, then we have an exact sequence of coherent sheaves $0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$ that $\mathrm{Supp}(\mathcal{G}_i) \subset Z_i$.

Proof: Let I be the reduced ideal sheaf of Z_1, we use the exact sequence $0 \to \mathcal{I}^n\mathcal{F} \to \mathcal{F} \to \mathrm{Coker} \to 0$, by(2.1.33), we can choose n that $\mathrm{Supp}(\mathcal{I}^n\mathcal{F}) \subset Z_2$, thus the result. □

Prop. (2.1.35). Let $\mathcal{F}$ be a coherent sheaf on a Noetherian scheme X, then there is a filtration of coherent sheaves that the quotients are pushforward of ideal sheaves on integral subschemes of X. This is analogous to the filtration in the module case.

Proof: We consider the set of these counterexamples and their Supp, then use Noetherian induction, the minimal one if not irreducible, then from(2.1.34) we find a filtration for it. Then let the ideal of sheaf be $\mathcal{I}$, then $\mathcal{I}^n\mathcal{F} = 0$, then we should use [StackProject 01YE] to finish to proof. Cf.[StackProject 01YF]. □

Prop. (2.1.36). Let P be a property of coherent sheaves on X Noetherian that

- for an exact sequence of sheaves:$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$, if $\mathcal{F}_i$ has P, then $\mathcal{F}$ has P.
- If $\mathcal{F}^{\oplus r}$ has P, then $\mathcal{F}$ has P.
- For every integral closed subscheme Z of X with generic point ξ, there is a coherent sheaf $\mathcal{G}$ that
  1. $\operatorname{Supp}\mathcal{G} \subset Z$.
  2. $\mathcal{G}_\xi$ is annihilated by m_ξ.
  3. For every sheaf of ideal $\mathcal{I}$ on X that $\mathcal{I}_\xi = \mathcal{O}_{X,\xi}$, there is a sheaf $\mathcal{G}' \subset \mathcal{I}\mathcal{G}$ that $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and has P.

Then we have P holds for every coherent sheaf on X.

Proof: Use Noetherian induction, the minimal counterexample should have Supp irreducible by(2.1.34) and then we use[StackProject 01YL]. Note this has nothing to do with reducedness. □

2 Spec and Schemes

Def. (2.2.1). The category of schemes is a fully faithful category of the category of ringed spaces that locally isomorphic to $\operatorname{Spec} A$.

Prop. (2.2.2). On $\operatorname{Spec}(A)$, $\mathcal{O}(D(f)) = A_f$. Cf.[Hartshorne P71]. We can also define it this way and check the sheaf condition.

Cor. (2.2.3). For an qcqs scheme X and a Qco module $\mathcal{F}$, $(\Gamma(X, \mathcal{F}))_s \cong \Gamma(X_s, \mathcal{F})$.

Proof: This is the canonical map $f : X \to \operatorname{Spec}\Gamma(X)$ is qcqs, (Notice qc is local on the target). Then $f_*\mathcal{F}$ is Qco on $\operatorname{Spec}\Gamma(X)$ thus the result. □

Prop. (2.2.4). The closure of a subset T of $\operatorname{Spec}(A) = V(\cap p, p \in T)$.

Prop. (2.2.5). The Spec operator from $CRing^*$ to Scheme is right adjoint to $X \to \Gamma(X, \mathcal{O}_X)$,

$$\operatorname{Hom}_{Sch}(X, \operatorname{Spec}(A)) \cong \operatorname{Hom}_{Ring}(A, \Gamma(X, \mathcal{O}_X)).$$

Notice the category of schemes is a full subcategory of the category of locally ringed spaces.

Proof: First prove this for $X = \operatorname{Spec}(B)$. Cf.[Hartshorne P73]. Then choose affine cover of X and glue them ($\mathcal{H}om$ is a sheaf). Should notice this is the special case of Global spec with $S = \operatorname{Spec}(\mathbb{Z})$. □

Prop. (2.2.6) (Global Spec). There is a S-scheme $f : \mathbf{Spec}_S\,\mathcal{A} \to S$ for every Qco sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ on S that $f^{-1}(U) \cong \operatorname{Spec}\mathcal{A}(U)$. This construction is right adjoint to the direct image map:

$$\operatorname{Hom}_{\mathcal{O}_S-alg}(\mathcal{A}, \pi_*\mathcal{O}_X) \cong \operatorname{Hom}_{Sch/S}(X, \mathbf{Spec}_S\,\mathcal{A}).$$

and defines an equivalence of affine morphisms over S and Qco $\mathcal{O}_S$-algebras. Moreover, this defines an equivalence of the category of $\mathcal{A}$-modules and the category of $\mathcal{O}_{\mathbf{Spec}\,\mathcal{A}}$-modules.

Proof: It suffice to prove for affine opens in S and glue. For this, use the adjointness of $\tilde{}$ and Γ and adjointness for Spec. □

Prop. (2.2.7). A A-point for Spec(A) a point, is a morphism Spec$(A) \to X$. For $A = K$, this correspond to points of X with $k(x) \subset K$, for $A = k[\varepsilon]/\varepsilon^2$, this correspond to a rational point x and an element in the dual of the $k(x)$-space m_x/m_x^2, i.e. the Zariski tangent space. (notice the local map).

Prop. (2.2.8) (Fiber Products). Fiber products exist in the category of schemes. Cf.[Hartshorne P87].

One should use universal properties of fiber products to get subschemes of the fiber product.

Dimensions

Prop. (2.2.9). For any scheme, $\dim \mathcal{O}_x = \operatorname{codim}(\overline{\{x\}}, X)$.

Prop. (2.2.10). For an integral scheme of finite type over a field,

$$\dim X = \dim \mathcal{O}_{p,X} = \dim U = tr.\deg K(X)/k$$

for any closed point p and any open subscheme U. (Use closed point are dense(3.2.27) and k is universal catenary to prove it is true for some U and all the closed point in it, so other U's because X is irreducible).

Lemma (2.2.11). For a Noetherian local ring $(A, \mathfrak{m})$, Spec $A - \mathfrak{m}$ is affine iff $\dim A \le 1$.

Proof: if $\dim A = 0$, this is true, if $\dim A = 1$, let $f \in m$ not in any other minimal primes of A, then Spec $A - m =$ Spec A_f.

Conversely, Cf.[StackProject 0BCR]. □

Prop. (2.2.12). Let X be a locally Noetherian scheme, if $U \subset X$ is an open subscheme that $U \to X$ is affine, then every irreducible complements of $X - U$ has codimension≤ 1. And if U is dense, then equality must hold.

Proof: Cf.[StackProject 0BCU]. □

Associated Points

Basic References are [StackProject Chap30].

Def. (2.2.13). For a scheme X and a Qco sheaf $\mathcal{F}$ on X, a point is called **associated to $\mathcal{F}$** iff $\mathfrak{m}_x$ is associated to $\mathcal{F}_x$, which is equivalent to $\mathfrak{m}_x$ are all zero-divisors in M by(5.6.9). When $\mathcal{F} = \mathcal{O}_X$, x is called an **associated point of X**.

Prop. (2.2.14). If X is locally Noetherian, then an associated prime is equivalent to it is an associated prime of $\Gamma(X, \mathcal{O}_X)$ of $\Gamma(U, \mathcal{F})$ for a nbhd U of x.

Proof: Cf.[StackProject 02OK]. □

Prop. (2.2.15). Same results of associated points are parallel to the discussion of associated primes:

- relations of $Ass(\mathcal{F})$ w.r.t exact sequences(5.6.6).

- $Ass(\mathcal{F}) \subset \mathrm{Supp}(\mathcal{F})$(5.6.7).
- When X is locally Noetherian and $\mathcal{F}$ is coherent, for a quasi-compact open set U of X, the number of associated points in U is finite(5.6.7).
- When X is locally Noetherian, $\mathcal{F} = 0$ iff $Ass(\mathcal{F})$ is empty(5.6.7).
- When X is locally Noetherian, If $Ass(\mathcal{F}) \subset$ an open subset U, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is injective(5.6.11).
- If X is locally Noetherian, then the minimal elements(under specialization) of $\mathrm{Supp}(\mathcal{F})$ are associated points of $\mathcal{F}$. in particular, any generic point of an irreducible component of X is an associated points of X.
- If X is locally Noetherian, then if a map $\varphi : \mathcal{F} \to \mathcal{G}$ that is injective at all the stalks of $Ass(\mathcal{F})$, then φ is injective.

3 Projective Space

Def. (2.3.1) (Projective Scheme). For a graded ring S, we have a scheme $\mathrm{Proj}(S)$ that consists of homogenous primes of S minus S_+ and the affine cover is $D(f) = \{p | f \nsubseteq p\}$, and $\mathcal{O}(D(f)) = \mathrm{Spec}\, S_{(f)}$, where $S_{(T)}$ is the degree zero part of $T^{-1}S$. It has $\mathcal{O}_p = S_{(p)}$.

Proof: Define the sheaf using stalks, then we only have to check that $\mathrm{Spec}\, S_{(f)} \cong$ homogenous $p \in S_f$ by natural intersection of ideals φ. and $S_{(p)} \cong (S_{(f)})_{\varphi(p)}$ for $p \in D(f)$.

We check that for $S_{(f)} \subset S_f$, $p \to p \cap S_{(f)}$ and $p' \to pS$ is natural and inverse to each other. $S_{(f)} \to S_{(p)}$ maps $\varphi(p)$ to invertible, and any $x/a \in S_{(p)}$ can be written as $\frac{xa^{\deg f - 1}/f^{\deg a}}{a^{\deg f}/f^{\deg a}}$. □

Prop. (2.3.2).

$$\mathrm{Proj}^n_{\mathbb{Z}} \times \mathrm{Spec}\, A = \mathrm{Proj}^n_A.$$

(Choose the canonical affine open sets to see).

Prop. (2.3.3). For two graded ring with the same $S_0 = A$, $\mathrm{Proj}(S \times_A T) \cong X \times_A Y$, where $(S \times_A T)_n = S_n \times_A T_n$ (natural morphism from left to right).

Prop. (2.3.4). For a graded S-module, there is a Qco-sheaf $\widetilde{M}$ on $\mathrm{Proj} S$, that $\widetilde{M}_p = M_{(p)}$ and $\widetilde{M}|_{D^+(f)} \cong \widetilde{M_{(f)}}$. the construction is as in(2.3.1).

Def. (2.3.5) (Relative Proj). The relative $\mathrm{Proj}\,\mathcal{S}$ over locally Noetherian Y of a Qco graded $\mathcal{O}_Y$-algebra $\mathcal{S}$ f.g. over $\mathcal{S}_0$ by coherent $\mathcal{S}_1$ is the glueing of locally $\mathrm{Proj}\, S$. $\mathrm{Proj}\,\mathcal{S} \to Y$ is locally projective thus proper. It is equipped with invertible sheaf $\mathcal{O}(1)$ by glueing.

Prop. (2.3.6) (Closed Subscheme of Projective Scheme). The closed scheme of $X = \mathbb{P}^n_A$ corresponds to the saturated homogenous ideal $\mathcal{I}_Y$, (i.e. for any s, if there is an n that for any $i, x_i^n s \in \mathcal{I}_Y$, then $s \in I_Y$).

So projective scheme over $\mathrm{Spec}\, S_0$ corresponds to $\mathrm{Proj}\, S$, where S are f.g. over S_0 by S_1 saturated in the sense above.

Proof: A closed immersion is proper, thus the kernel $\mathcal{I}_Y$ of the structural map is a Qco(2.1.23), so it must be an ideal on every affine open, because Qco is affine local. Then we should use(2.4.5), $\Gamma_*(\mathcal{I}_Y)$ will suffice. Cf.[Hartshorne Ex2.5.10]. □

Prop. (2.3.7). The global section of a projective space $\operatorname{Proj} S \to \operatorname{Spec} S_0$ is just S_0, this is by(2.4.5).

Prop. (2.3.8). A quasi-projective scheme X over a field k of dimension r can be covered by $r+1$ open affine subsets. This is because there are r hyperplane that intersect X non-empty. This can happen by choosing a hyperplane non-intersecting the generic point of X, otherwise we choose many hyperplane, then their intersection is empty.

Serre Twisting

Def. (2.3.9). Define $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) = \mathbb{Z}[X_0,\widetilde{\ldots,}X_n](1)$, this is an invertible sheaf. The invertible **Serre twisting sheaf** $\mathcal{O}(1)$ on $\mathbb{P}^r_Y$ is the pullback of that of $\mathbb{P}^n_{\mathbb{Z}}$ and an invertible **Serre twisting sheaf** of the relative $X = \operatorname{Proj} \mathcal{S}$ over Y is locally the pullback of that of $\mathbb{P}^r_Y$. Giving a Serre twisting sheaf of X over Y, the **Serre twisting sheaf** of $\mathcal{F}$ over X is the sheaf $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_X(n)$.

Prop. (2.3.10). For X projective over $\operatorname{Spec}(A)$, (i.e. $X = \operatorname{Proj}(S)$(2.3.6)), $\widetilde{M\otimes_S N}\cong \widetilde{M}\otimes_{\mathcal{O}_X}\widetilde{N}$ and many other properties involving the Serre twisting, all this boil down to the fact that $(M\otimes_S N)_{(f)} = M_{(f)}\otimes_{S_{(f)}} N_{(f)}$ for $f\in S_1$.

and by virtue of(2.3.4), when $X = \operatorname{Proj}(S)$ projective, we have:

- $\widetilde{M}(n)\cong \widetilde{M(n)}$.
- For a graded ring map $S\to T$, we have the corresponding Proj map $f: U\to T$ that $f^*(\widetilde{M})\cong (\widetilde{M\otimes_S T})|_U$ and $f_*(\widetilde{N}|_U)\cong \widetilde{N_S}$. That's to say, $f^*(\widetilde{M}(n)) = f^*(\widetilde{M})(n)$ and $f_*(\widetilde{M}(n)) = f_*(\widetilde{M})(n)$.

Cor. (2.3.11). $\mathcal{O}_X(n)\otimes\mathcal{O}_X(m)\cong\mathcal{O}_X(n+m)$ for any scheme X projective over Y.

Prop. (2.3.12) (Twisting of Proj). With notation as in (2.3.5), Let $S' = S * \mathcal{L} : S'_d = S_d\otimes\mathcal{L}^d$, then $\varphi: \operatorname{Proj} S'\to\operatorname{Proj} S$ is an isomorphism that induces

$$\mathcal{O}'(1)\cong\varphi^*\mathcal{O}(1)\otimes\pi'^*\mathcal{L}.$$

Prop. (2.3.13). If Y is Noetherian and admits an ample invertible sheaf, then by definition, we have $S_1\otimes\mathcal{L}^n$ is base point free for some n, thus we have a morphism $\operatorname{Proj} S*\mathcal{L}^n\to\mathbb{P}^N_Y$, so $P = \operatorname{Proj} S$ is H-quasi-projective with $\mathcal{O}(1) = \mathcal{O}_P(1)\otimes\pi^*\mathcal{L}^n$.

Locally Free sheaves

Prop. (2.3.14). Pullback and pushforward of locally free sheaves are locally free.

Prop. (2.3.15). For a finite locally free sheaf $\mathcal{E}$ on X,

- $\mathcal{E}^{\vee\vee}\cong\mathcal{E}$.
- $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})\cong\mathcal{E}^\vee\otimes_{\mathcal{O}_X}\mathcal{F}$.
- $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})\otimes\mathcal{H}\cong\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{H})$ if $\mathcal{F}$ or $\mathcal{H}$ is finite locally free.
- $\mathcal{H}om(\mathcal{E}\otimes\mathcal{F},\mathcal{G})\cong\mathcal{H}om(\mathcal{F},\mathcal{E}^\vee\otimes\mathcal{G})$, by the first and(2.1.4).

Proof: We define the map, and verify locally. □

Prop. (2.3.16) (Wedge Product). For a exact sequence of locally free sheaves: $0 \to F' \to F \to F'' \to 0$,

$$\wedge\mathcal{F}' \otimes \wedge\mathcal{F}'' \cong \wedge\mathcal{F}.$$

Let $\mathcal{F}$ be a locally free sheaf of rank n, then there is a perfect pairing $\wedge^r\mathcal{F} \otimes \wedge^{n-r}\mathcal{F} \to \wedge\mathcal{F}$ which is a perfect pairing.

Prop. (2.3.17). For a exact sequence of locally free sheaves:$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to 0$, where L is a line bundle, there is an exact sequence

$$0 \to \wedge^r(\mathcal{F}') \to \wedge^r(\mathcal{F}) \to \wedge^{r-1}(\mathcal{F}') \otimes \mathcal{L} \to 0$$

This is a special case of [Hartshorne Ex2.5.16c].

Proof: □

Prop. (2.3.18). The map $(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ is an isomorphism in the case when $\mathcal{F}$ is a locally free or when it is of finite presentation, by(5.12.6).

Def. (2.3.19). A locally free module on schemes can induce a symmetric vector bundle $S(\mathcal{E})$, and the section sheaf recovers $E^\vee$. This defines a reverse equivalence of locally free sheaves and vector bundles on X.

When $\mathcal{E}$ is Qco, we can define the **associated projective space bundle** $\mathbb{P}(\mathcal{E})$ as $\mathrm{Proj}\, S(\mathcal{E})$. It is equipped with a Serre twisting sheaf $\mathcal{O}(1)$. There is a surjective morphism $\pi^*(\mathcal{E}) \to \mathcal{O}(1)$ (local check).

Prop. (2.3.20). Let $g : Y \to X$ by a scheme over X, a morphism $Y \to \mathbb{P}(\mathcal{E})$ over X is equivalent to an invertible sheaf $\mathcal{L}$ and a surjective map $g^*\mathcal{E} \to \mathcal{L}$.

In particular, giving a morphism $X \to \mathbb{P}^n_A$ is essentially equivalent to a base point free invertible sheaf with n generators on X.

Proof: If there is a morphism, it will pullback $\pi^*(\mathcal{E}) \to \mathcal{O}(1)$ into $g^*\mathcal{E} \to \mathcal{L}$. For the coverse, construct locally and glue, we have the natural morphisms $A[x_1/x_i, \ldots, x_n/x_i] \to \mathcal{O}_{X_{s_i}} : x_j/x_i \to s_j/s_i$ in a homogenous sense. It is natural hence glue together. For the module, maps $x_i \to s_i$. □

Cor. (2.3.21). All automorphisms of $\mathbb{P}^n_k$ is linear.

Proof: The Picard group of $\mathbb{P}^n_k$ is $\mathbb{Z}$ and is generated by $\mathcal{O}(1)$(5.1.14), so the automorphism will map $\mathcal{O}(1)$ to $\mathcal{O}(\pm 1)$ and $\mathcal{O}(-1)$ has no global section(2.4.4). And the globals section is n-dimensional and determines the morphism by the prop. □

4 Invertible Sheaves

Def. (2.4.1). An invertible sheaf on a ringed space is a sheaf that $\mathcal{L} \otimes -$ is an equivalence of categories. A locally free sheaf of rank 1 is inveritble and when X is local ringed space, the converse is also true.

Proof: Cf.[StackProject 0B8M]. □

Prop. (2.4.2). For any ringed space X, the $\mathrm{Pic}(X) \cong H^1(Z, \mathcal{O}_X^*)$. This is by choosing a locally trivial opens of X and the first Čech cohomology equals sheaf cohomology(4.2.6).

Prop. (2.4.3). Giving a morphism $X \to \mathbb{P}^n_A$ is essentially equivalent to a base point free invertible sheaf with n generators on X. This follows from(2.3.20).

Prop. (2.4.4) (Global Section). Let $\mathcal{L}$ be an invertible sheaf over qcqs scheme X, for a Qco module $\mathcal{F}$ let the **global section functor** $\Gamma_*(\mathcal{F}) = \oplus\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$, then

$$\Gamma_*(\mathcal{F})_{(f)} \cong \mathcal{F}(X_f).$$

where $s \in \Gamma(X, \mathcal{L})$. In particular that if there is a section f of $\mathcal{F}$ on X_s, then for some n, $f \otimes s^n$ is a global section of $\mathcal{F} \otimes \mathcal{L}^n$.

Proof: This is nearly the same as the proof that $(\operatorname{Spec} A)_f = \operatorname{Spec} A_f$, Cf.[StackProject 01PW]. $\square$

Cor. (2.4.5). when $X = \operatorname{Proj} S$ projective over $\operatorname{Spec} S_0$ and $\mathcal{F}$ Qco, $\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$, where $\Gamma_*(\mathcal{F}) = \oplus_{n\in\mathbb{Z}}\Gamma(X, \mathcal{F}(n))$, which is a graded S-module. In particular, Γ_* for projective space $\mathbb{P}^n_A$ equals $A[x_1, \ldots, x_n]$.

Ample Invertible Sheaves

Def. (2.4.6). On a quasi-compact scheme X, an invertible sheaf $\mathcal{L}$ is called **ample** iff there is a n and sections $s_i \in \Gamma(X, \mathcal{L}^n)$ that X_{s_i} is an affine cover of X.

For a qc morphism $f : X \to Y$, an invertible sheaf on X is called **f-ample** iff it is ample restricted to every open subscheme $f^{-1}(V)$, where V are affine open in Y.

On a locally Noetherian scheme X, an invertible sheaf $\mathcal{L}$ is called **H-ample** iff for any coherent sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for large n.

Prop. (2.4.7). An invertible sheaf $\mathcal{L}$ is (f-)ample iff $\mathcal{L}^m$ is (f-)ample.

Prop. (2.4.8). When X is Noetherian, H-ample $\iff$ ample.

Proof: Cf.[StackProject 01Q3], the left to right: For any point, choose a open affine U that $\mathcal{L}$ is free, then the sheaf of ideal for $X - U$ is coherent because X is Noetherian so $I_Y \otimes \mathcal{L}^n$ is generated by global sections thus some s that $p \in \mathit{supp}(s)$. So as U is affine, $X_s \subset U$ is affine. Then use finiteness argument. $\square$

Prop. (2.4.9). When There is a f-ample sheaf for $f : X \to Y$ qc, then f is separated.

Proof: Being separated is local on the target, so we assume Y is affine, then this follows from [StackProject 01PY]. $\square$

Lemma (2.4.10). For an invertible sheaf $\mathcal{L}$ on a qc scheme X, if for each Qco sheaf of ideals $\mathcal{I} \in \mathcal{O}_X$, there is a n that $H^1(X, \mathcal{I} \otimes \mathcal{L}^n) = 0$, then $\mathcal{L}$ is ample.

Proof: For any closed pt P, choose an open affine nbhd U that $\mathcal{L}$ is trivial, let $Y = X - U$, by the exact sequence $0 \to \mathcal{I}_{Y\cup\{p\}} \to \mathcal{I}_Y \to k(P) \to 0$, we have

$$0 \to \mathcal{I}_{Y\cup\{p\}} \otimes \mathcal{L}^n \to \mathcal{I}_Y\mathcal{L}^n \to k(P) \otimes \mathcal{L}^n \to 0.$$

Thus by assumption we have a surjective map $\Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^n) \to \Gamma(X, k(P) \otimes \mathcal{L}^n)$. Now $k(P) \otimes \mathcal{L}^n$ is A/m_P, so we let $s \in \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^n)$ maps to a section in $\Gamma(X, k(P) \otimes \mathcal{L}^n)$ that restricts to $1 \in A/m_P$, then $P \in \operatorname{Supp} s \subset U$ is affine. So we find an affine X_s for every closed pt of X, these will cover X. $\square$

Prop. (2.4.11) (Serre's Cohomological Criterion of Ample). If X is proper over a Noetherian affine scheme, $\mathcal{L}$ is an invertible sheaf, then the following is equivalent.

- $\mathcal{L}$ is ample
- For each coherent sheaf $\mathcal{F}$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for n large enough.
- For each Qco sheaf of ideals $\mathcal{I} \in \mathcal{O}_X$, there is a n that $H^1(X, \mathcal{I} \otimes \mathcal{L}^n) = 0$

(Notice in this case H-ample $\iff$ ample).

Proof: $1 \to 2$: Because $\mathcal{L}^m$ is H-very ample for some m, thus X is projective, then we use Serre theorem(4.4.29).

$3 \to 1$: (2.4.10). □

Prop. (2.4.12). $f : X \to Y$, let $\mathcal{L}$ be f-ample on X and $\mathcal{M}$ ample on Y, then $\mathcal{L} \otimes f^*\mathcal{M}^n$ is ample for n large.

Proof: Cf.[StackProject 0892]. □

Cor. (2.4.13). If $f : X \to Y$ is quasi-affine, then the pullback of an ample invertible sheaf is ample. This is because quasi-affine $\iff$ $\mathcal{O}_X$ is f-ample.

Prop. (2.4.14). If $f : Y \to X$ is finite and surjective morphism between schemes proper over a Noetherian affine scheme, then for an invertible sheaf $\mathcal{L}$ on X, $\mathcal{L}$ is ample iff $f^*\mathcal{L}$ is ample.

Proof: One way follows from(2.4.13), For the other we use Serre criterion(2.4.11) and devissage(2.1.36). We only verify 3: By(3.2.31), there exists such coherent sheaf $f_*\mathcal{F}$ for any integral subscheme, and for a any Qco sheaf of ideals $\mathcal{I}$, $\mathcal{I}f_*\mathcal{F} = f_*(f^{-1}\mathcal{I}\mathcal{F})$ because f is affine, thus

$$H^p(X, \mathcal{I}f_*\mathcal{F}) = H^p(X, f_*(f^{-1}\mathcal{I}\mathcal{F} \otimes \mathcal{L}^n)) = H^p(Y, f^{-1}\mathcal{I}\mathcal{F} \otimes \mathcal{L}^n)$$

by projection formula, and f is affine. This vanish for n large. □

Prop. (2.4.15). If $i : Z \to X$ is a closed immersion that induce homeomorphism on topology between Noetherian schemes, then $\mathcal{L}$ is ample iff $i^*\mathcal{L}$ is ample.

In particular, this applies to $X_{red} \to X$.

Proof: Cf.[StackProject 09MS]. □

Prop. (2.4.16). Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let n_0 be an integer. If $H^p(X, \mathcal{L}^{-n}) = 0$ for $n \geq n_0$ and $p > 0$, then X is affine.

Proof: Cf.[StackProject 0EBD]. □

Very Ample Invertible Sheaves

Def. (2.4.17). A **very ample** invertible sheaf on X/Y quasi-projective over Y is the pullback along some immersion of $\mathcal{O}(1)$ of $\mathrm{Proj}(\mathcal{E})$ for some Qco module $\mathcal{E}$ over Y, Cf.(2.3.9). It is called **H-very ample** iff $\mathcal{E}$ is trivial. Notice when X is proper, this immersion must be closed by(3.3.3).

When S is affine and X/S is of f.t., then very ample is equivalent to H-very ample.

Proof: Cf.[StackProject 02NP]. □

Prop. (2.4.18). Let X/S be locally of f.t., then for any ample invertible sheaf $\mathcal{L}$ over X, every $\mathcal{L}^m$ for m large is H-very ample.

Proof: As in the proof of(2.4.8), we see that there are f.m affine opens X_{s_i} that cover X refining a inverse image of affine cover of S, we can make them the same degree then by f.t., there are f.m generators $\{c_{ij}\}$(2.4.4). So consider the projective space $A[x_i, c_{ij}]$, X is closed immersed into an open subscheme of P_S^N. Cf.[StackProject 01VS]. □

Prop. (2.4.19). If X/S is qc, then f-very ample implies f-ample.

Proof: Cf.[StackProject 01VN]. □

Prop. (2.4.20) (Serre). When $f : X \to S$ is of f.t. and S is affine, $\mathcal{L}$ is $(H\text{-})$ample $\iff$ $\mathcal{L}$ is f-relative ample $\iff$ $\mathcal{L}^n$ is $(H\text{-})$very ample for some(all large)n. (All these follow from propositions above).

Prop. (2.4.21). A proper scheme that has a (H)-very ample invertible sheaf is projective, because the image of a proper scheme is proper.

Prop. (2.4.22). When X is Noetherian and has an H-ample invertible sheaf, any coherent sheaf is a quotient of a a finite direct sum of $\mathcal{O}(-n)$.

Proof: This is because X is qc and $\mathcal{F}(n)$ is globally generated for some n. So for any pt p we find f.m. section that generate the stalk, then by coherence, there is a nbhd that generate the stalk, and the compactness shows that there is f.m that generate the stalk, thus $\mathcal{O}_X^N \to \mathcal{F}(n)$ surjective, then we tensor it with $\mathcal{O}_X(-n)$. □

Linear System

Prop. (2.4.23). A **complete linear system** on a regular projective variety is the set of effective divisors linearly equivalent to D_0.

When X is non-singular over a alg.closed field, the equivalent divisors correspond to projective space of $\Gamma(X, \mathcal{L}(D_0))$,

Proof: Any divisor equivalent to D_0 defines a global section on $\mathcal{L}(D_0)$. And $\Gamma(X, \mathcal{O}_X^*) = k^*$ by(2.3.7). □

Prop. (2.4.24). To give a morphism from X to $\mathbb{P}_k^n$ is equivalent to give a linear system without base point on X. Cf.[Hartshorne P150]

5 Differentials

Def. (2.5.1). The diagonal map $\Delta : X \to X \times_Y X$ is an immersion hence an isomorphism onto the image. So we use the locally sheaf of ideal $\mathcal{I}$ corresponding to $\Delta(X)$ to get the **Sheaf of differentials** $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ on X. It is a $\mathcal{O}_X$-module on X.

It is a Qco sheaf because pullback of Qco is Qco, and when $X \to Y$ is locally of f.t. and Y is locally Noetherian, X and $X \otimes_Y X$ is also locally Noetherian thus $\Omega_{X/Y}$ is coherent.

By(1.2.2)(1.2.3) $\Omega_{X/Y}$ can also be constructed by locally $\widetilde{\Omega_{B/A}}$ and glue because it is functorial. And we see form this that it is compatible with base change of schemes. From this we see the stalk of $\Omega_{X/Y}$ at p is $\Omega_{X_p/Y_{f(p)}}$.

Prop. (2.5.2) (Jacobi-Zariski Sequence). Let $f: X \to Y$ and $g: Y \to Z$, then there is an exact sequence of sheaves on X:

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Immediate from(1.2.5).

Prop. (2.5.3). Let $f: Z \to X$ be closed immersion and $g: X \to Y$, then there is an exact sequence of sheaves on Z:

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

Immediate from(1.2.4).

Prop. (2.5.4). For an irreducible algebraic separated scheme X over a perfect field k, $\Omega_{X/k}$ is a locally free sheaf of rank $n = \dim X$ iff X is a regular. Plus the condition of separatedness, X will be a regular variety.

By the same method, we can show that an integral scheme of f.t. over k perfect has an open dense subset U that is regular.

Proof: It suffice to consider closed point by**??**, the alg.closed,irreducible and f.t. conditions are here to use(1.2.9), and a coherent sheaf is locally free iff its stalks are free(2.1.29).

For the second assertion, we consider the stalk of $\Omega_{X/k}$ at the generic point, it is $\Omega_{K/k}$, which is free by(1.2.8). So by(2.1.29) again there is an open dense nbhd of the generic point that Ω is free hence all the points in it are regular. □

Prop. (2.5.5). If $X = \mathbb{P}^n_A$ over $Y = \operatorname{Spec} A$, then there is an exact sequence

$$0 \to \Omega_{X/Y} \to (\mathcal{O}_X(-1))^{n+1} \to \mathcal{O}_X \to 0.$$

This is because locally the kernel is generated by $(e_j - (x_j/x_i)e_i)/e_i = d((x_j/x_i))$.

When A is a field k, this sequence is locally free by(6.1.11), so taking dual we get:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to \mathcal{T}_X \to 0.$$

Taking highest exterior power we get $\omega_X \cong \mathcal{O}_X(-n-1)$.

6 Limit of Schemes

Def. (2.6.1). For a locally Noetherian scheme and a Qco sheaf of ideal I on it corresponding to a closed scheme Y, there is a **Formal completion of X along I** defined the ringed space with the glue of locally the functorial completion of A along I on the topological space Y,(5.8.8)[Hartshorne P194]. In fact, any coherent sheaf on X can be completed along Y.

A Locally ringed space $\widetilde{X}$ is called Locally Noetherian formal scheme if it is locally a formal complete of some X along I. A sheaf of $O_{\widetilde{X}}$-modules is called coherent iff it is locally the completion of a sheaf of coherent module.

7 Others

Geometry of schemes

Def. (2.7.1). The **Zariski Tangent space** of a scheme at x is defined to be the dual vector space of $\mathfrak{m}_x/\mathfrak{m}_x^2$. Giving a tangent vector of X is equivalent to giving a morphism $\operatorname{Spec} D \to X$, where $D = k(x)[t]/t^2$ is the set of dual numbers over $k(x)$.

Def. (2.7.2). Let X be a scheme algebraic over field k, an **infinitesimal deformation** of X is a scheme X', flat over D, that $X' \otimes_D k = X$. By definition(2.7.1), a global deformation(flat family) gives rise to an infinitesimal deformation of X.

V.3 Properties of Schemes(Hartshorne)

Basic References are [Algebraic Geometry Hartshorne] and [Hartshorne Solution 田翊].

1 Basic Scheme Properties

Affine Local Properties of Schemes

Lemma (3.1.1) (Nike's Trick). In a scheme X and $x \subset \mathrm{Spec}(A) \cap \mathrm{Spec}(B)$, x has an open nbhd in $\mathrm{Spec}(A) \cap \mathrm{Spec}(B)$ that are distinguished in both $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$.

Proof: □

Prop. (3.1.2) (Affine Communication Theorem). A property P of affine open subsets is called **affine local** if: $\mathrm{Spec}(A)$ has $P \Rightarrow$ all $\mathrm{Spec}(A_f)$ has P, and any cover of $\mathrm{Spec}(A_{f_i})$ has $P \Rightarrow \mathrm{Spec}(A)$ has P. Notice a stalk-wise property is obviously affine-local.

Now if we call X has $\widetilde{P}$ if $X = \bigcup_i \mathrm{Spec}(A_i)$ that A_i has P. Then the following is equivalent:

- X has $\widetilde{P}$.
- all open subscheme of X has $\widetilde{P}$.
- X has an open cover that has $\widetilde{P}$.

Proof: □

When proving locality of morphism properties, one usually resort to1.

Cor. (3.1.3) (List of affine local properties). (not complete)

- (Locally)Noetherian. Cf,[Hartshorne P83].
- Reducedness. (Stalk-wise)

Proof: Reducedness: if there is an affine cover that is reduced, then the stalks will be like R_P is reduced if R is reduces. And if the stalks are all reduced, then a nilpotent element will be 0 in every local set, thus 0 because $\mathcal{O}$ is a sheaf. □

Connectedness

Prop. (3.1.4). $\mathrm{Spec}(A)$ is not connected $\iff A = A_1 \times A_2 \iff A$ has no nontrivial idempotent element.

Proof: This is all equivalent to the fact that there exists $e + f = 1, ef = 0$. □

Prop. (3.1.5). For geometrically connected. Cf[StackProject 32.7]

Irreducible

Prop. (3.1.6). A scheme is irreducible iff for every affine open U, $X(U)$ is irreducible iff X has an irreducible affine open cover that pairwise intersects.

Cor. (3.1.7). The fiber product of irreducible schemes is irreducible.

Reduced

Def. (3.1.8). Call a scheme is called **reduced** if $\Gamma(U, \mathcal{O}_X)$ is reduced for every open set U. Reduced is a stalk-wise property(3.1.3).

Prop. (3.1.9). There is a $X_{red} \to X$ associated tot every scheme, it is $\mathbf{Spec}(\mathcal{O}_X/\mathcal{N})$ where $\mathcal{N}$ is the sheaf of nilpotent elements. This construction is right adjoint to the forgetful functor by the adjoint property of **Spec**(2.2.6). $X_{red} \to X$ is an closed immersion.

It's useful to change to X_{red} when the proposition only involve topology because X_{red} has the same topology as X. A map can induce a map on their reduced structure.

Prop. (3.1.10). There is a reduced induced scheme structure on a closed subset Y of a scheme X, it is the **Spec** of the $\mathcal{O}_X$-algebra of $\mathcal{O}_X(U)/\cap p_i, (i \in Y)$. It has the universal property.

Cor. (3.1.11). Any map morphism from a reduced scheme X to Y factors through the closed subscheme of the closure of its image. (By virtue of reducedness).

Integral

Def. (3.1.12). A scheme X is called integral if $X(U)$ is all integral. This is equivalent to reduced and irreducible. So a scheme is integral iff there is an integral open affine cover that are pairwise-intersect(3.1.6). Cf.[Hartshorne P82].

Cor. (3.1.13). The projective space over an integral scheme is integral. (Check the affine covers are dense). The projective space $P^n_{\mathbb{Z}}$ is integral.

Noetherian

Def. (3.1.14). A scheme is called locally Noetherian if it can be covered by open affine schemes of noetherian rings. It is called **Noetherian** if moreover it is quasi-compact.

Prop. (3.1.15). (Locally)Noetherian is affine local, i.e. X is locally Noetherian if any affine open of X is spec of a Noetherian ring(3.1.3).

Cohen-Macaulay

Def. (3.1.16). A scheme is called C.M. iff all its stalks is C.M. local.

Normal & Regular

Def. (3.1.17). A scheme is called **normal** if all its stalk is normal domain, so all its affine sections are normal ring. It is called **regular** iff all its stalk is regular local ring, i.e. all affine opens are regular rings. Regular only have to be checked at close pt because of(5.11.9).

Prop. (3.1.18). For an integral scheme X, there is a $X_{nom} \to X$ which is $\mathbf{Spec}(\mathcal{O}_{X,nom})$, any dominant morphism f from a normal integral scheme to X will factor through X_{nom}. (Use the adjointness for **Spec** and notice f maps generic to generic.

Prop. (3.1.19). For a curve, normal is equivalent to regular. This is because for a Noetherian local domain of dim 1, principal $\iff$ normal $\iff$ regular $\iff$ DVR.

Cor. (3.1.20). A Noetherian Normal scheme is regular in codimension 1.

Prop. (3.1.21). A Noetherian connected regular scheme is irreducible, since it has f.m. closed components and they cannot intersect, because at the intersection pt, an affine nbhd has multiple minimal primes, thus the local ring also has multiple stalk, thus not integral, not regular.

Prop. (3.1.22). For geometrically normal, Cf.[StackProject 32.10].

Geometrical properties

Def. (3.1.23). A scheme over k is called **geometrically reduced** if $X \times_k K$ is irreducible for every field extension, it suffice to check for $K = k^{per}$, Cf.[StackProject 32.6].

Def. (3.1.24). A scheme X is called **geometrically integral/reduced/separated/irreducible**... over a field k iff for any field extension k'/k, $X_{k'}$ is integral/reduced/separated/....

Prop. (3.1.25). Use of faithfully flat descent.

Prop. (3.1.26). Geometrically irreducible is enough to check for $K = k^{sep}$, Cf.[StackProject 32.8].

Prop. (3.1.27). If X is geometrically reduced, connected and proper over a field k, then $\Gamma(X, \mathcal{O}_X) = k$. In particular, this is true for a complete variety over alg.closed field k.

Proof: Cf.[StackProject 0BUG]. □

2 Basic Morphism Properties

Base Change Trick

Prop. (3.2.1). If A property P of morphisms satisfy:
- Closed immersion has P.
- Stable under base change and composition.

Then
- it is stable under product.
- $g \circ f$ has $P + g$ separated $\Rightarrow f$ has P.
- it is stable under f_{red}. (Notice $X_{red} \to X$ is closed immersion).

Cf.[Hartshorne Ex2.4.8].

Prop. (3.2.2). Lists of properties satisfying the base change trick(not complete):
1. Universal closed morphism.
2. Affine morphism.
3. Quasi-affine morphism.
4. closed immersions.
5. Quasi-compact morphism.
6. (Quasi-)Separatedness.
7. (Locally) of Finite Presentation.

8. Unramified.

Proof:
1.
2. Trivial.
3.
4. For closed immersion, check locally, for open immersion, notice that $U \times_W V \to X \times_S Y$ is open immersion.
5. Trivial.
6. For $X \to Y \to Z$, $X \to X \times_Y \times X \to X \times_Z X$, the second one is a base change of $Y \to Y \times_Z Y$(3.2.41). And the diagonal of base change is the base change of diagonal, so this follows from that of closed immersion and qc.
7. By(5.12.8).
8.

□

Local Properties of Morphisms

Our fundamental tool is (3.1.2).

Prop. (3.2.3) (List of properties affine local on the target). (All the property besides the H-projectiveness is local on the target).
1. All properties defined by a ring map property local on the target(5.1.16) .
2. Isomorphism, injective, surjective, open, closed.
3. Quasi-compactness.
4. (Open/Closed)immersions.
5. (Quasi-)Separateness.
6. Finite morphism.
7. Integral morphism.

Proof:
1. Trivial.
2. Only isomorphism need proving, Cf.[Hartshorne Ex 2.2.17].
3. Because affine open is compact and $(\operatorname{Spec} A)_f$ is also compact.
4. Because open and closed are local on the target and check closedness on stalks.
5. Use criterion(3.2.45).
6. Cf.[StackProject 02JL].
7. Cf.[StackProject 02JK].

□

Prop. (3.2.4) (List of properties affine local on the source). (not complete)

1. All properties defined by a ring map property local on the source(5.1.16) .
2. Openness.
3. (Locally)Finite presentation.

Proof:
1. Trivial.
2. Trivial.
3. By(5.12.8).

□

Valuation Criterion

Prop. (3.2.5). The valuation criterion for $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$ where R is a valuation ring: For a quasi-compact morphism,
- it is separated iff there is at most one lifting.
- it is universally closed iff there is at least one lifting.
- it is proper iff it is finite type(auto quasi-compact) and lifting exists uniquely (More useful).

Cf.[StackProject].

Injective Morphism

Prop. (3.2.6). For a morphism of schemes, the following are equivalent:
- It is universally injective.
- It is injective and the residue field extension are all purely inseparable.
- The diagonal map is surjective.
- For any field K, $\mathrm{Hom}(\mathrm{Spec}\, K, X) \to \mathrm{Hom}(\mathrm{Spec}\, K, S)$ is injective.

Proof: Cf.[StackProject 01S4]. □

Closed Map

Prop. (3.2.7). Let $A \to B$ noetherian. Then going-up holds $\iff$ Spec map is closed.

Proof: going-up is equivalent to $f^*(V(q)) = V(f^*(q))$, $\forall q$ prime. Use primary decomposition of $\sqrt{I}$, $V(I) = \bigcup V(q_i)$. □

Prop. (3.2.8) (Universal Closed). Universal closedness is local on the basis and satisfies the base change trick(3.2.2).

Prop. (3.2.9). If g is surjective, then $f \circ g$ is universally closed iff f is universally closed (because surjective is S.u.B).

Prop. (3.2.10). The image of a quasi-compact morphism is closed iff it is stable under specialization. And it is a closed map iff specialization lifts along f.

Proof: For the first, the question is local, so reduce to Y affine, and then X is qc$=\cup U_i$, then we can replace X by an affine $\coprod U_i$, then reduce to the affine case(5.5.4).

For the second, for any closed subset of X with its induced reduced structure, the restriction to it is still qc and specialization lifts, so we prove the image is closed. Now the image is stable under specialization, so the result follows from the first assertion. □

Affine Map

Lemma (3.2.11). Isomorphism is local on the target(3.2.3)

Prop. (3.2.12). X is affine if there is a finite set of elements $f_i \in \Gamma(X, \mathcal{O}_X)$ that generate the unit ideal and X_{f_i} is affine.

Proof: First prove that $X_{f_i} \cap X_{f_j} = X_{f_i f_j}$ is affine because affine intersect X_{f_i} is affine. Second, prove $\Gamma(X_f, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)_f$, finally glue them to get a map $X \to \mathrm{Spec}(A)$ and use(3.2.11). X is affine scheme if $X \to \mathrm{Spec}(\Gamma(X))$ is affine. □

Cor. (3.2.13). Affineness is affine local on the target and satisfies the base change trick(3.2.2).

Prop. (3.2.14) (Serre Criterion of Affiness). For a qc scheme $(X, \mathcal{O}_X)$, it is isomorphic to an affine scheme as a ringed space $\iff$ X is $(Co)h$-acyclic $\iff$ $H^1(X, \mathcal{I}) = 0$ for every Qco sheaf of ideals $\mathcal{I}$.

Proof: The case of affine scheme is proven by(4.4.1) and(4.4.2). The converse: For every point p, choose an open affine nbhd U, let $Y = X - U$, by the exact sequence

$$0 \to \mathcal{I}_{Y \cup \{p\}} \to \mathcal{I}_Y \to k(P) \to 0,$$

we have a surjective map $\Gamma(X, \mathcal{I}_Y) \to \Gamma(X, k(P))$ thus there is a $f \in A = \Gamma(X, \mathcal{O}_X)$ that $P \in X_f \subset U$ is affine. So using(3.2.12), we only have to show that for f.m f_i, they generate $\Gamma(X, \mathcal{O}_X)$. This is by considering the kernel F of $\mathcal{O}_X^r \to \mathcal{O}_X : (a_1, \ldots, a_r) \to \sum f_i a_i$, and there is a filtration on F, the quotient of which are all coherent sheaves because kernel and cokernel are Qco, and there by induction and hypothesis, $H^1(X, F) = 0$, thus the result. □

Cor. (3.2.15). If X is qcqs, then if $H^1(X, \mathcal{I}) = 0$ for every Qco sheaf of ideals $\mathcal{I}$ of f.t., then X is an affine scheme. (Because by(4.2.11), it we can use colimit to show that $H^1(X, \mathcal{I}) = 0$ for Qco sheaf of ideals).

Cor. (3.2.16). For a Noetherian scheme X, X is affine iff X_{red} is affine.

Proof: The canonical exact sequence(4.3.2) reads: $0 \to \mathcal{N}\mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$, so iff X_{red} is affine, then we have $H^i(\mathcal{F}) \cong H^i(\mathcal{N}\mathcal{F})$, and notice $\mathcal{N}^k = 0$ for some k. □

Cor. (3.2.17). For a Noetherian reduced scheme X, X is affine iff each irreducible component is affine. (The same as the above, notice that $\prod p_i = 0$, for the minimal primes of A). (The reducedness can be dropped by the last proposition).

Quasi-affine

Def. (3.2.18). A scheme is called **quasi-affine** iff it is quasi-compact and isomorphic to an open subscheme of an affine scheme. A morphism is called **quasi-affine** iff the inverse of any affine scheme is quasi-affine.

Prop. (3.2.19). Quasi-affine morphism is separated and qc by(3.2.45).

Prop. (3.2.20). Quasi-affine is local on the target and satisfies the base change trick. Cf.[StackProject 01SN].

Prop. (3.2.21). A scheme is quasi-affine iff $\mathcal{O}_X$ is ample. Cf.[StackProject 01QE].

Cor. (3.2.22). A morphism f is quasi-affine iff $\mathcal{O}_X$ is f-ample.

Dominant

Prop. (3.2.23). A quasi-compact morphism of schemes:$X \to S$ is dominant if every generic point of irreducible components of S is in the image of f.(Use quasi-compactness to reduce to the affien case). In particular, if X, S is affine, dominant is equivalent to image of f contains minimal primes and equivalent to the kernel is in the nilradical. (Because the closure of image $= V(\mathrm{Ker})$).

Quasi-Compact

Def. (3.2.24). A morphism $f : X \to S$ is called quasi-compact if the inverse image of affine open is quasi-compact.

Quasi-compactness is local on the target and satisfies the base change trick(3.2.2).

Prop. (3.2.25). Let $f : X \to Y$, $g : Y \to Z$. If $g \circ f$ is quasi-compact and g is qs, then f is qc.

Proof: Factor it through $X \to X \times_Z Y \to Y$. The second map is a base change of $X \to Z$ hence qc, the first map is a section of $X \times_Z Y \to X$, which is a base change of $Y \to Z$, hence qs, so by(3.2.43), the first map is also qc. □

Finite Type

Def. (3.2.26). A morphism $f : X \to S$ is called of **locally finite type** if for there exists an affine open cover $\{\mathrm{Spec}(B_i)\}$ of S that $f^{-1}(U_i)$ has an affine open cover of spec of finite generated B_i-algebras. It is called **finite type** if moreover it is quasi-compact.

A scheme over a field k is called **(locally)algebraic** iff it is (locally) of finite type over $\mathrm{Spec}\, k$.

(Locally)Finite type is affine local on the target and on the source, and satisfies the base change trick(3.2.2).

Prop. (3.2.27) (Locally Finite Type over Field is Jacobson). For a scheme locally of finite type over a field k, the set of closed points X_0 is dense in every closed subset of X, Because it is a Jacobson space by(1.12.9) and(6.9.5).

Moreover, the residue field at a closed stalks is finite over k by(6.9.5).

Prop. (3.2.28) (Chevalley). A qc morphism locally of f.p. maps locally constructible subset to locally constructible subset.

Proof: We prove $f(E) \cap U_i$ is constructible for every U_i affine open in X. The inverse image of U_i is qc, hence a locally constructible set is constructible by(1.12.5). So we reduce to the affine case(5.12.12). □

Finite & Integral Map

Def. (3.2.29). A morphism $f : X \to S$ is called **finite** if it is affine and the inverse image of an affine cover is finite module.

Finiteness is affine local on the target and satisfies the base change trick(3.2.2).

A morphism $f : X \to S$ is called **quasi-finite** if it is of finite-type and the inverse of a point is a discrete hence finite set.

A morphism $f : X \to S$ is called **integral** if it is affine and the inverse image an affine cover is integral ring extension.

Integral is affine local on the target and satisfies the base change trick(3.2.2).

Prop. (3.2.30). A locally f.t. integral morphism is finite.

Lemma (3.2.31). For $f : Y \to X$ finite surjective and X locally Noetherian, for every integral subscheme Z of X with generic point ξ, there is a coherent sheaf $\mathcal{F}$ on Y that the support of $f_*\mathcal{F}$ is Z and $(f_*Z)_\xi$ is annihilated by m_ξ.

Proof: W consider an inverse image of $\xi = \xi'$, and let $Z' = \overline{\{\xi'\}}$ with the induced reduced structure, then let $\mathcal{F} = i_*\mathcal{O}_{Z'}$ on Y, $\mathcal{F}$ is coherent, then we need to show that $(f_*\mathcal{F})_\xi$ is annihilated by m_ξ. This is because it factors through Z.**?** Cf[StackProject 01YO]. □

Prop. (3.2.32) (Chevalley). If $f : Y \to X$ is finite surjective, Y is affine, then X is affine. Cf.[StackProject 01ZT].

Proof: We prove the Noetherian case.

We use(2.1.36) and(3.2.15). In fact we prove $H^1(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$. We check the conditions of(2.1.36), the sheaf $\mathcal{G}$ exists by(3.2.31), just let $\mathcal{G} = f_*\mathcal{F}$. Then for any Qco sheaf of ideals $\mathcal{I}$, we have $\mathcal{I}\mathcal{G} = f_*(f^{-1}\mathcal{I}\mathcal{F})$, because f is affine, and(4.4.23) shows that $\mathcal{I}\mathcal{G}$ satisfies the condition. □

Prop. (3.2.33). Integral map is equivalent to u.c. and affine. cf.[StackProject 01WM].

Proof: For one way, it suffice to show it is locally closed(5.7.1). □

Immersions

Def. (3.2.34). An **immersion** is a closed immersion followed by an open immersion. A open immersion followed by a closed immersion can be written as a closed immersion followed by an open immersion, but not reversely. The reverse happens if the immersion is quasi-compact or the source is reduced (use the reduced induced structure) Cf.[StackProject 01QV].

Prop. (3.2.35). Open and closed immersions are affine local on the target(3.2.3).

Prop. (3.2.36). Closed immersion satisfies the base change trick(3.2.2). Open immersion are stable under base change and composition.

Prop. (3.2.37). The closed subscheme of a scheme corresponds to Qco $\mathcal{O}_X$-ideals. Hence the closed subscheme of $\operatorname{Spec} A$ corresponds to the quotients A/I.

Proof: The closed immersion is qcqs, so it maps $\mathcal{O}_Y$ to $i_*(\mathcal{O}_Y)$ Qco(2.1.24), thus the kernel of $\mathcal{O}_X \to i_*(\mathcal{O}_Y)$ is Qco. Conversely, for a Qco sheaf of ideals, $Y = \mathbf{Spec}_X(\mathcal{O}_X/I)$ for the Qco $\mathcal{O}_X$-algebra $\mathcal{O}_X/I$. □

Def. (3.2.38). For a morphism $f : X \to Y$, there is a closed scheme called **scheme-theoretic image** that is the smallest subscheme of Y that f factors through Z. This is by considering the kernel of the structural map, and the kernel has a maximal Qco sheaf of ideal $\mathcal{I}$(2.1.31).

For an immersion of schemes, the scheme-theoretic closure of the immersion is called the **scheme-theoretic closure**.

Prop. (3.2.39). When f is qc, this set-theoretic image behaves well and is the closure of the image because the kernel of the structural map is Qco. And when X is reduced, it is then the reduced induced structure of the closure of image of f.

Proof: Cf.[StackProject 01R8]. □

Separatedness

Def. (3.2.40). A map $f : X \to Y$ is called **separated** if the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. It is called **quasi-separated** if the diagonal is quasi-compact.

In fact Δ is always an immersion because maps between affine scheme is separated so $\Delta(X)$ is closed in $\cup U_{ij} \otimes_{V_i} U_{ij}$ where U, V are affine open, hence it suffice to check the image is closed.

Lemma (3.2.41). For $X \to T$ and $Y \to T$ and $T \to S$, $X \times_T Y \to X \times_S Y$ is a base change of $T \to T \times_S T$.

Cor. (3.2.42). For $X \to S$ and $Y \to S$, the map $X = X \times_Y Y \to X \times_S Y$ is an immersion. It is closed immersion if $Y \to S$ is separated, and it is qc if $Y \to S$ is quasi-separated.

Cor. (3.2.43). If $s : S \to X$ is a section of $f : X \to S$, the above corollary applies to this case, because $S = S \times_X X \to S \times_S X = X$.

Prop. (3.2.44). (Quasi-)Separateness is local on the target because closed immersion and quasi-compact is local on the target(3.2.3).

(Quasi-)Separatedness satisfies base change trick by(3.2.2).

Prop. (3.2.45). A morphism is quasi-separated iff for any two affine open that mapped to an affine open, their intersection is quasi-compact. This is because quasi-compact is local on the target.

A morphism is separated iff for any two affine open that mapped to an affine open, their intersection is affine and $\mathcal{O}(U) \otimes_{\mathcal{O}(W)} \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is surjective. This is because closed immersion is local on the target.

Cor. (3.2.46). A locally Noetherian scheme is quasi-separated.

Cor. (3.2.47). If $g \circ f$ is (quasi-)separated, then so is f.

Cor. (3.2.48). If X is (quasi-)separated, then $X \to Y$ is (quasi-)separated.

Prop. (3.2.49). monomorphism is separated because the diagonal map is isomorphism(7.1.47), so immersions are separated as they are monomorphisms in the category of schemes (because of surjectiveness on the stalk).

Prop. (3.2.50). Affine morphism is separated (Check closed immersion directly).

3 Proper & Projective

Prop. (3.3.1). A morphism that is separated, finite-type and universally closed is called proper.
proper is local on the target, because all these three properties do.

Prop. (3.3.2). The class of proper morphisms satisfies the base change trick(3.2.1)(Valuation Criterion). (Closed immersion is proper because it is f.t. and is affine so separated(3.2.40), and it is universally closed because immersions are stable under base change(3.2.36)).

Prop. (3.3.3) (Image of Proper Map). If $X \to Y$ is morphism between separated schemes f.t over S, then if X is proper, then the image is closed (base change trick) and is proper in its scheme-theoretic structure(3.2.9). Notice proper is qc and use(3.2.38).

Cor. (3.3.4). A morphism from a connected proper scheme to an Noetherian affine scheme $\operatorname{Spec} A$ is constant.

Proof: Because the image is proper and use(4.4.28), so A is a finite module over $\operatorname{Spec} k$ thus Artinian so has finitely many point. So it is discrete. $\square$

<u>**Projective Morphism**</u>

Def. (3.3.5). A **projective** morphism $X \to Y$ is a closed immersion $X \to \operatorname{Proj}(\mathcal{E})$ for some Qco f.t. module $\mathcal{E}$. A **H-projective** $\to Y$ is a closed immersion $X \to \mathbb{P}^n_Y$. A H-quasi-projective morphism is a H-projective morphism composed with an open immersion. Some proposition about projective is written before the language of Hartshorne so I may not have changed them to the more general projective notion yet.

Prop. (3.3.6). H-(Quasi-)Projectiveness satisfies the base change trick(3.2.1). (because Segre embedding is closed). Disjoint union of f.m. projective morphisms is projective (embed into the Segre embedding).

Cor. (3.3.7). Projective morphism is locally projective and locally projective is proper[StackProject 01WC], because closed immersion is proper and $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is u.c. by valuation criterion. Cf.[Hartshorne P103]. And a quasi-projective morphism is of f.t. and separated(3.2.49).

Prop. (3.3.8). Projective scheme over $\operatorname{Spec} A$ is of the form $\operatorname{Proj} S$ where $S_0 = A$ and S is f.g over S_0 by S_1(2.3.6).

Prop. (3.3.9) (Chow's Lemma). Let $X \to S$ be separated of f.t over a Noetherian S, then there is a birational, proper, surjective $X' \to X$ that X' is quasi-projective.

X is proper iff X' can be projective. And if X is integral(irreducible,reduced), X' can be chosen to be so.

Proof: Basic idea: reduce the the irreducible case, and use f.t. to generate a local quasi-projectives, then the closure of the image of $U \to X \times_S P_1 \times_S \ldots \times_S P_n$ will suffice. $\square$

4 Flatness

Def. (3.4.1). For a morphism $f : X \to Y$ of ringed spaces, a $\mathcal{O}_X$-module $\mathcal{F}$ is called **flat** over Y iff its stalk is flat as a $\mathcal{O}_{Y,f(x)}$-module. f is called **flat** iff $\mathcal{O}_X$ is flat, it is called **faithfully flat** iff moreover it is surjective.

For a Qco sheaf $\mathcal{F}$, this is equivalent to $\Gamma(U, \mathcal{F})$ is flat over A for every U that mapped to $\operatorname{Spec} A \subset Y$ by(6.2.21).

Prop. (3.4.2). Flatness is local on the target, it is stable under base change, composition. A coherent $\mathcal{O}_X$ module is flat over X iff it is locally free.(6.2.7)(6.2.17).

Prop. (3.4.3). For a morphism $f : X \to S$ locally of f.p., and a Qco sheaf on X that is locally of finite presentation, the set of points that $\mathcal{F}$ is flat over S is open.

Proof: Cf.[StackProject 0399]. □

Prop. (3.4.4). For a flat morphism of ringed space, f^* is exact, because it is f^{-1} followed by tensoring with $\mathcal{O}_X$, check on stalks.

Prop. (3.4.5). A finite morphism $f : X \to S$ with S locally Noetherian is flat iff $f_*(\mathcal{O}_X)$ is locally free, Cf.[StackProject 02KB].

Prop. (3.4.6). Generalization lifts along a f.f. morphism.

Proof: We can find an affine nbhd, then choose a nbhd of the inverse image, then a generalization in an affine open is a true generalization, so it reduce to the affine case. The rest follows from going-down(6.2.22). □

Prop. (3.4.7) (Flatness and Openness). A flat morphism locally of f.p. is (universally)open, hence it is qc.

And a qc f.f. morphism of schemes is submersive.

Proof: We need only consider they are both affine. Then the assertion follows from(6.2.23).

For the second, by(3.4.6), a subset whose inverse image is closed is stable under specialization(surjectiveness used), then the complement is closed by(3.2.10) □

Prop. (3.4.8) (Flat Points are Open). For a morphism $f : X \to Y$ of f.t. of Noetherian schemes, the set of points of X that f is flat is open in X.

Proof: Cf.[EGA3,11.1.1]. □

Prop. (3.4.9) (Flat Family and Hilbert Polynomial). For $\underline{X/T \text{ projective}}$, where T is an integral Noetherian scheme and $X \subset \mathbb{P}^n_T$. Then for each point T, X_t is a closed subscheme of $\mathbb{P}^n_{k(t)}$, so we can consider its Hilbert Polynomial P_t. Then X/T is flat iff P_t is independent of T.

Proof: $P_t(m) = \dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t}(m))$ for m large by(4.4.18). And we may let $X = \mathbb{P}^n_T$ and prove for any coherent sheaf $\mathcal{F}$. Moreover, we may let T be a affine local Noetherian, because flatness is local and we only need to compare Hilbert polynomial with the generic point. Now we prove a stronger assertion: The following are equivalent:

- $\mathcal{F}$ is flat over T.
- $H^0(X, \mathcal{F}(m))$ is a free A-module of finite rank, for m large.

- The Hilbert polynomial P_t of $\mathcal{F}_t$ on $X_t = \mathbb{P}^n_{k(t)}$ is independent of t.

1 $\to$ 2: Use the canonical cover and Čech cohomology, then we notice when m is large, $H^0(X, \mathcal{F}(m))$ is a kernel of the Čech resolution, so it is flat. And it is also finite by(4.4.28). Then it is free because it is flat by(6.2.7).

2 $\to$ 1: Let $M = \oplus_{m \geq m_0} H^0(X, \mathcal{F}(m))$, then $\widetilde{M} = \mathcal{F}$(2.4.5), notice that the truncation doesn't affect.

2 $\to$ 3: It suffice to prove that for any $t \in T$, when m is large,

$$H^0(X_t, \mathcal{F}_t(m)) \cong H^0(X, \mathcal{F}(m)) \otimes_A k(t).$$

For this, we may use(4.4.31) to pass to the localization and assume t is the closed pt of T. Then $A \to k(t)$ is surjective and we may let $A^q \to A \to k \to 0$, then by(4.4.30), we have $H^0(X_t, \mathcal{F}_t(m))$ is the cokernel of $H^0(X, \mathcal{F}(m))^q \to H^0(X, \mathcal{F}(m))$, but this cokernel is $H^0(X, \mathcal{F}(m)) \otimes_k$ because tensoring is right-adjoint, so we are done.

3 $\to$ 2: We have the rank of $H^0(X, \mathcal{F}(m))$ at the generic and closed point of T are the same(still use $H^0(X_t, \mathcal{F}_t(m)) \cong H^0(X, \mathcal{F}(m)) \otimes_A k(t)$.) Now(6.14.1) gives $H^0(X, \mathcal{F}(m))$ is free. It is f.g. automatically. □

Cor. (3.4.10). For a flat morphism to a connected scheme T, the dimension, degree, and arithmetic genus of the fibers are independent of t.

Proof: By(4.4.20)and (6.1.20). □

Def. (3.4.11). For a surjective map of varieties $f : X \to T$ over an alg.closed field k, its fiberes over closed points with induced reduced structure $X_{(t)}$ is called a **algebraic family of varieties parametrized by** T if

1. $f^{-1}(t)$ is irreducible of dimension $\dim X - \dim T$ for every closed point t.
2. If ζ is the generic point of $f^{-1}(t)$, then $F^\sharp \mathfrak{m}_t$ generates the maximal ideal $\mathfrak{m}_\zeta \subset \mathcal{O}_{\zeta,X}$.

Prop. (3.4.12). if $X_{(t)}$ is an algebraic family of normal varieties over an alg.closed field k parametrized by a nonsingular curve T, then it is a flat family of schemes.

Proof: By(6.2.14), $X \to T$ is flat. So what we need to do is to prove X_t is reduced so $X_t = X_{(t)}$. Let $A = \mathcal{O}_{x,X}$, let u_t be a uniformizer of $\mathcal{O}_{t,T}$, then A/tA is the local ring of x on X_t. By hypothesis X_t is irreducible so tA has a unique minimal prime p in A, and t generate the maximal ideal of A_p by hypothesis. The local ring of $X_{(t)}$ is A/p, so A/p is normal by hypothesis. Then the result follows from(5.11.8). □

Cor. (3.4.13) (Igusa). Let $X_{(t)}$ be an algebraic family of normal varieties in $\mathbb{P}^n_k$ for k alg.closed parametrized a variety T, then the Hilbert polynomials of $X_{(t)}$ are independent of t.

Proof: ? Why is X/T projective? Cf.[Hartshorne P265]. □

Relative Dimension

Def. (3.4.14). A morphism of schemes which is locally of f.t. is called of **relative dimension** n iff all fibers X_s are equidimensional of dimension n.

Prop. (3.4.15). If $f : X \to Y$ is a morphism of schemes locally of f.t., then $\dim_x(X_s) = \dim \mathcal{O}_{X_s,x} + tr.deg_{k(s)} k(x)$.

Proof: Cf.[StackProject 02FX]. □

Prop. (3.4.16). If $f: X \to Y, g: Y \to S$ are locally of f.t., then $\dim_x(X_s) \le \dim_x(X_y) + \dim_y(X_s)$. Moreover, equality holds if $\mathcal{O}_{X_s,x}/\mathcal{O}_{Y_s,y}$ is flat.

Proof: Cf.[StackProject 02JS]. □

Cor. (3.4.17). If $f: X \to Y, g: Y \to Z$ are of relative dimension m and n, and f is flat, then $g \circ f$ is of relative dimension $m + n$.

Prop. (3.4.18). For a morphism $X \to S$ locally of f.t, and its base change $X' \to S'$, then $\dim_x(X_s) = \dim_{x'}(X'_{s'})$.

Proof: Cf.[StackProject 02FY]. □

Cor. (3.4.19) (Relative Dimension and Base Change). The base change of a morphism locally of f.t. of relative dimension n is still of relative dimension n.

Prop. (3.4.20). For a morphism $f: X \to Y$ between locally Noetherian schemes which is flat and locally of f.t and of relative dimension n, then if $y = f(x)$, we have $\dim_x(X_y) = \dim_x(X) - \dim_y(Y)$.

Proof: Shrinking the nbhd, we may assume $\dim_x(X) = \dim X$ and $\dim_y(Y) = \dim Y$ and X, Y are affine. Now f is locally of f.p. and flat, so it is open(3.4.7). So we may assume f is surjective. Then $\dim \mathcal{O}_{X,a} = \dim \mathcal{O}_{Y,b} + \dim \mathcal{O}_{X_b,a} = \mathcal{O}_{Y,b} + n$ by(5.9.7), then taking supremum(5.9.2), the result follows. □

Cor. (3.4.21). For a morphism of schemes that is flat and of f.t., if Y is irreducible, then X is equidimensional of dimension $\dim Y + n$ iff X_y is equidimensional of dimension n for every $y \in Y$. This follows immediately from the proposition and(3.2.27).

Proof: The proof highly relies on(2.2.10).

$1 \to 2$: For $Z \subset X_y$ an irreducible component, choose a closed pt x of Z not contained in any other irreducible component, then

$$\dim_x Z = \dim_x X - \dim_y Y = \dim X - \dim \overline{\{x\}} - \dim Y + \dim \overline{\{y\}}.$$

The two closures are of the same dimension because by(6.9.5), their quotient field extension is finite.

$2 \to 1$: Now for an irreducible component of X, choose a closed pt x of Z not contained in any other irreducible component, then the result is immediate. □

5 Smoothness

Def. (3.5.1). A morphism $f: X \to Y$ of schemes is called **smooth** if there is an open affine cover $\{U_i\}$ of S and an open affine cover V_{ij} of $f^{-1}(\{U_i\})$ that the ring map is smooth. A **standard smooth morphism** is the Spec map of a standard smooth ring map.

Smoothness is local on the source and target(6.5.13). Smoothness is stable under base change and composition(6.5.13).

Lemma (3.5.2). For a smooth morphism $X \to S$, the morphism of differential $\Omega_{X/S}$ is locally free and $\dim_x \Omega_{X/S} = \dim_x(X_{f(x)})$(local dimension(1.12.12)).

Proof: We can assume that $X \to S$ is standard smooth, so by the proof in(6.5.12), $\Omega_{X/S}$ is free of dimension $n-c$, and also standard smooth is relative global complete intersection(6.5.10), so $U_{f(x)}$ is equidimensional of dimension $n-c$, thus the result. □

Prop. (3.5.3) (Fiberwise and Stalkwise). For a morphism $X \to S$ locally of f.p., the following are equivalent:

- It is smooth at a point x.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $X_{f(x)}/k(x)$ is smooth at x.(6.5.18). Moreover, using(6.5.20), we even only have to check for the geometric fibers.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $\Omega_{X/S,x}$ can be generated by at most $\dim_x(X_{f(x)})$ elements(3.5.2) and(6.5.23).
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $\Omega_{X_s/s,x} \otimes \Omega_{\mathcal{O}_{X_s,x}} k(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} k(x)$ can be generated by at most $\dim_x(X_{f(x)})$ elements, by Nakayama.

Prop. (3.5.4). Open immersion is smooth. Smooth morphism is syntomic hence flat. Smooth morphism is locally of f.p. Hence smooth morphism is universally open(3.4.7).

Smooth morphism is locally standard smooth(6.5.12).

Prop. (3.5.5). If $X \to Y$ is smooth and a morphism $Y \to S$, then there is an exact sequence of sheaves(2.5.2)(6.5.5):

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega S/Y \to 0$$

Prop. (3.5.6). If $Z \to X \to S$, Z/S is smooth and $Z \to X$ is an immersion, then there is an exact sequence of sheaves(2.5.3)(6.5.6):

$$0 \to \Omega_{C/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

Prop. (3.5.7). If $X \to Y \to S$, and $X \to Y$ is surjective, flat and locally of f.p., $X \to S$ is smooth, then $Y \to S$ is smooth.

Proof: Cf.[StackProject 05B5]. □

Def. (3.5.8). By(3.5.3)(3.5.2), A morphism is smooth of relative dimension n is equivalent to fppf+fibers equidimensional of dimension n and $\Omega_{X/S}$ is locally free of dimension n.

Prop. (3.5.9). A morphism between schemes algebraic over a field k is smooth of relative dimension n iff f is flat and every fiber of f is geometrically regular of dimension n. (geometrically regular⇒ regular by(6.3.1))

Prop. (3.5.10). Smooth over a field $k \iff$ geometrically regular by(6.2.7).

6 Unramified

More advanced materials to learn at [StackProject Chap40].

Def. (3.6.1). A morphism is called **(G-)unramified** iff there is an open affine cover U_i and an open affine cover of $f^{-1}(U_i)$ that the induced ring map is (G-)unramified. Equivalently, $\Omega_{X/S} = 0$ and it is locally of f.t.(f.p.).

(G-)unramifiedness is local on the source and target(3.2.3)(3.2.4). (G-)unramifiedness is stable under base change and composition(6.6.4). Moreover, Unramifiedness satisfied the base change trick.

Prop. (3.6.2). An unramified map is locally quasi-finite.

Proof: Cf.[StackProject 02V5]. □

Prop. (3.6.3) (Fiberwise). A morphism is (G-)unramified iff it is locally of f.t.(f.p.) and all the fibers X_s are disjoint unions of spectra of finite separable extensions of $k(p)$.

Proof: By(6.6.7), Notice $pS_q = qS_q$ is equivalent to every q is minimal in X_p, which is equivalent to X_p is discrete. □

Cor. (3.6.4) (Unramified over Fields). A scheme over a field k is unramified iff it is a disjoint union of spectra of finite separable extensions of k, because locally of f.p. is trivially satisfied.

Prop. (3.6.5). A morphism $X \to S$ is (G-)unramified iff it is of f.t.(f.p.) and the diagonal is an open immersion.

Proof: If it is unramified, then it is an open immersion by(6.6.10). Conversely, $\Omega_{X/S}$ is just the conormal sheaf of the diagonal map, so it is zero. □

Cor. (3.6.6). Let X, Y be schemes over S, if f, g are two maps from X to Y, then if Y/S is unramified and f, g are equal on a pt x of X(both on image and residue field), then there is a nbhd of x that f, g are equal.

Proof: This follows as $\Delta_{Y/S}$ is open immersion, so the set that f, g are equal is open in X. □

Prop. (3.6.7) (Stalkwise and Fiberwise). For a morphism locally of f.t.(f.p.), the following are equivalent:

- It is (G-)unramified at a point x,
- The fiber $X_{f(x)}$ is smooth over $k(f(x))$ at x.
- $\Omega_{X_{f(x)},x} = 0$.
- $\Omega_{X_s/s,x} \otimes \Omega_{\mathcal{O}_{X_s,x}} k(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} k(x) = 0$.(6.6.6).
- $\mathfrak{m}_x \mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)/k(s)$ separable.(6.6.7).

Prop. (3.6.8). If $X \to Y \to S$, X/S is unramified, then X/Y is unramified. And if X/S is G-unramified and Y/S is of f.t., then X/Y is G-unramified.(By(3.9.5)and(2.5.2)).

Cor. (3.6.9) (Unramified Points Base Change). If f is of f.t.(f.p.), then the set of points that f is unramified is stable under base change by the above proposition.

Noetherian Case

7 Étale

More advanced materials to learn at [StackProject Chap40].

Def. (3.7.1). A morphism $f : X \to Y$ of schemes is called **étale** if there is an open affine cover $\{U_i\}$ of S and an open affine cover V_{ij} of $f^{-1}(\{U_i\})$ that the ring map is étale. A **standard étale morphism** is the Spec map of a standard étale ring map.

étale is local on the source and target(6.7.5). Étale is stable under base change and composition(6.7.5).

Prop. (3.7.2). Étale at a point x is equivalent to smooth and unramified at a x(6.7.4).

étale at a point x is equivalent to flat and G-unramified at that point, by(6.7.9). So Étale over field is equivalent to G-unramified, because over a field it is obviously flat.

Étale at a point x is equivalent to locally standard étale at that point(6.7.14).

A morphism is étale iff it is smooth of dimension 0, by definition(3.5.8).

Étale is equivalent to smooth flat, locally of f.p. and formally unramified, by(6.7.9).

Cor. (3.7.3). Étale map is smooth, hence syntomic, flat.

Étale map is universally open because it is flat and lcoally of f.p.(3.4.7).

Prop. (3.7.4). If X, Y are étale over S, then any map $X \to Y$ is étale.(6.7.12).

Prop. (3.7.5) (Fiberwise). A morphism of schemes is étale iff it is flat, locally of f.p., and every fiber X_s is a disjoint union of spectra of finite separable field extensions of $k(s)$.

Proof: Follows from(3.7.2)(3.6.3) and(6.7.8). □

Cor. (3.7.6). A scheme is étale over a field k iff it is a disjoint union of spectra of finite separable field extensions.

Prop. (3.7.7). If $X \to Y$ is smooth at x, then there exist a nbhd of x that it factors through $U \xrightarrow{\pi} \mathbb{A}^d_V \to V$, where π is étale.

Proof: Any standard smooth morphism can be factorized as an étale map over a polynomial algebra, as easily seen. □

Prop. (3.7.8) (Stalkwise and Fiberwise). For a morphism locally of f.p., the following are equivalent:

- It is étale at a point x.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ if flat and $X_{f(x)}/k(x)$ is smooth at x.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ if flat and $X_{f(x)}/k(x)$ is unramified at x.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $\Omega_{X_{f(x)},x} = 0$.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $\Omega_{X_s/s,x} \otimes \Omega_{\mathcal{O}_{X_s,x}} k(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} k(x) = 0$.
- $\mathcal{O}_{x,X}/\mathcal{O}_{f(x),S}$ is flat and $\mathfrak{m}_x\mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)/k(s)$ separable.

By (3.6.7) and(3.5.3).

Prop. (3.7.9). If $X \to Y \to S$, and $X \to Y$ is surjective, flat and locally of f.p., $X \to S$ is étale, then $Y \to S$ is étale.

Proof: Cf.[StackProject 05B5]. □

Def. (3.7.10) (Étale Neighborhood).

Prop. (3.7.11). For a morphism $f : Y \to X$ of schemes étale over field k, then f is surjective iff $Y(k_s) \to X(k_s)$ is surjective.

Proof: If $Y \to X$ is surjective, then □

Noetherian Case

8 Zariski's Main Theorem

References are [StackProject Chap36.38].

Prop. (3.8.1) (Zariski's Main Theorem). For a morphism $X \to S$ that is quasi-finite and separated, if S is qcqs, Then there is a factorization $X \to T \to S$ that $X \to T$ is a qc open immersion and $T \to S$ is finite.

Proof: Cf.[StackProject 05K0]. □

Prop. (3.8.2) (Chevalley). Finite $\iff$ quasi-finite+proper.**?**

Proof: The fiber of $f : X \to S$ is $\mathrm{Spec}(k(y) \otimes_A B)$, which is Artinian (5.1.9), so it has finitely many primes. Finite morphism is proper because it is integral(3.2.33).

For the converse, one should use Zariski's Main Theorem. □

9 More Properties of Schemes

Universal Catenary Ring

Def. (3.9.1). A scheme S is called **universally catenary** iff S is locally Noetherian and every scheme locally of f.t. over S is catenary.

Universally catenary is a local property, this follows from(1.12.17).

Prop. (3.9.2). A locally Noetherian scheme is universally catenary iff all its stalks are universally catenary. Cf.[StackProject 02JA].

Morphism of Finite Presentation

Def. (3.9.3). A morphism between schemes is called **of locally finite presentation** iff for any point $x \in X$, there is an open affine mapped into an open affine that the ring map is of finite presentation. It is called **of finite presentation** iff moreover it is qcqs.

locally finite presentation is local on the source and target and it is stable under composition and base change but it doesn't satisfies the base change trick by(3.2.4)(3.2.3) and(3.2.2)

Prop. (3.9.4). When the target is locally Noetherian, (locally)finite type and (locally)finite presentation is equivalent.

Prop. (3.9.5). For $f : X \to Y$ over S, if X/S is locally of f.p. and Y/S is locally of f.t., then f is locally of f.p.. If moreover X is of f.t. and Y is qs, then f is of f.t..

Proof: The first follows from(5.12.9), the second needs to check qcqs. Qc follows from(3.2.25). □

Finite Locally Free Morphism

Def. (3.9.6). A morphism f is called **fintie locally free** of rank d iff $f_*\mathcal{O}_X$ is locally free of rank d.

Prop. (3.9.7). When $f : X \to Y$ and Y is locally Noetherian, then f is finite locally free iff it is finite and flat. Cf.[StackProject 02KB].

V.4 Cohomology

1 Acyclic Sheaves

Def. (4.1.1). An Abelian sheaf on a site is called **flask** if it satisfies the following equivalent conditions:

- It is acyclic for the forgetful functor ι,
- It is acyclic for any $\check{H}^0(\{U_i \to U\}, -)$
- It is acyclic for all $\Gamma(U, -)$.

Also the class of flask sheaves are adapted to ι.

An Abelian sheaf on a site is called **flasque** iff it is acyclic for all $\mathrm{Mor}(S, -)$ for any S a sheaf of sets, which is obviously flask.

It is called **flabby** iff for any $U \to X$, $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective;

Proof: $1 \iff 3$ is by(4.2.4), $3 \to 2$ use Čech to sheaf1(4.2.5).

$2 \to 1$: suffices to check(8.3.3) for ι, should use ι takes injective to injective, $\check{H}^0(\{U_i \to U\}, -)$ commutes with finite sum and the fact that $\check{H}^1 = H^1$ and long exact sequence. □

Prop. (4.1.2). Flabby sheaf is flask. By the way, injective sheaves in the $\mathcal{O}_X$-module category are flabby by(7.1.35).

Proof: Just need to verify(8.3.3). Injectives are flabby, so it is sufficiently large.

For an exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of sheaves, if $\mathcal{F}$ is flabby, then H is just the presheaf cokernel. (It reduces to $\check{H}^1(\{U_i \to U\}, F) = 0$, and this is done by Zorn's lemma). Thus if $\mathcal{F}$ is flabby, $\mathcal{G}$ is flabby iff $\mathcal{H}$ is flabby(by five lemma). □

Prop. (4.1.3). For a morphism os topologies $f : T \to T'$, if F' is a flask sheaf on T', then $f^s F'$ is also flask.

Proof: Notice $H^q(\{U_i \to U\}, f^s F) = H^q(\{f(U_i) \to f(U)\}, F)$. □

Prop. (4.1.4). Filtered colimits of flabby sheaves is flabby. (This is because filtered colimits is exact).

Filtered colimits of injective sheaves over a Noetherian topological space is injective. (Use Baer criterion, then notice every sub-object of $\mathbb{Z}_U$ is finitely generated because it has only f.m. connected component(1.12.2) so it maps to some F_α.

Cor. (4.1.5). For an injective Abelian presheaf F on T, $F(U)$ is injective Abelian group for every U, this is because the morphism $i : \mathrm{pt} \to T : \mathrm{pt} \mapsto U$ is exact($i_p A(V) = \oplus_{\mathrm{Hom}(V,U)} A$), hence i^p preserves injectives.

Prop. (4.1.6). Let I be an injective module over a Noetherian ring A, then the sheaf $\widetilde{I}$ on $\mathrm{Spec}\, A$ is flabby.

Proof: We have for a Qco module over $\mathrm{Spec}\, A$, $\Gamma(U, \widetilde{M}) \cong \varinjlim \mathrm{Hom}(I^n, M)$(2.1.27), so if we have two open set $X - V(a)$ and $X - V(b)$, and a, b radical, then the restriction map is induce by the inclusion $b \subset a$, and it is surjective because I is injective and filtered colimits is exact. □

Lemma (4.1.7). A constant sheaf on an irreducible topological space is flabby, thus flask.

Prop. (4.1.8). If $\mathcal{I}$ is an injective $\mathcal{O}_X$-mod, then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-mod for U open, this is because $-|_U$ is right adjoint to the exact $j_!$.

Prop. (4.1.9). If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then for a coherent locally free sheaf $\mathcal{L}$, $\mathcal{L}\otimes\mathcal{I}$ is also injective, because tensoring with $\mathcal{L}$ is adjoint to tensoring with $\mathcal{L}^\vee$(2.3.15), which is exact.

Def. (4.1.10). A sheaf of modules $\mathcal{F}$ is called **flat** iff $\mathcal{F}\otimes -$ is an exact functor. This is equivalent to the stalks are all exact, because tensor commutes with stalks and use skyscraper sheaf.

Locally free sheaves are flat. There are enough flat sheaves because $j_!\mathcal{O}_U$ is flat and any sheaf of module is a quotient of sums of these.

2 Cohomology on Site

More Materials to add from [StackProject Chap21].

Čech Cohomology

Def. (4.2.1). Let X be a site, we have a canonical complex of presheaves $K(U)_\bullet$ w.r.t. an open covering U that is

$$\cdots\to\bigoplus Z_{U_{i_0i_1i_2}}\to\bigoplus Z_{U_{i_0i_1}}\to Z_{U_{i_0}}\to 0.$$

And for any presheaf of $\mathcal{O}_X$-module $\mathcal{F}$, $\mathrm{Hom}_{\mathcal{O}_X}(K(U)_\bullet,\mathcal{F})$ gives out the Čech complex of $\mathcal{F}$. Hence we have: an injective sheaf is Čech acyclic.

This complex has homotopy 0 unless $i=0$. This is because we have a homotopy: choose a fixed i_0, for a $s\in\Gamma(X,U_{i_1\ldots i_n})$, we map it to $(hs)_{ii_1\ldots i_n}=\delta_{i,i_0}s$.

Prop. (4.2.2) (Čech-Cohomology). For any U and a cover in a site, the corresponding Čech cohomology is a derived functor on the category of presheaves on a site.

If we take colimit for coverings, $F\to\check{H}^0(U,F)$ is a left exact functor from presheaves to sets, the derived functors are just the limits $\check{H}^q(U,F)$.

Proof: It suffice to prove the Čech cohomology is universal, for this, we only need to prove the sheave defined in(4.2.1) is exact then cech cohomology group vanish for $\mathcal{F}$ injective. We check on every V, then the complex can be classified by its image in $\mathrm{Hom}(V,U)$, after that, if we denote $S(\varphi)=\oplus\,\mathrm{Hom}(V,U_i)$, then the complex is of the form

$$\bigoplus_{\varphi\in\mathrm{Hom}(V,U)}(\ldots\to\oplus_{S(\varphi)\times S(\varphi)}\mathbb{Z}\to\oplus_{S(\varphi)}\mathbb{Z})$$

which is easily to seen to be nullhomotopic.

To check the refinement colimit is exact, we sow that the refinement is independent of the refinement map chosen, in this way, this is obviously a filtered colimit which is exact. For two refinement map, there is a commutative diagram
$$\begin{tikzcd}\prod F(U_i) \arrow[r, "d^0"] \arrow[d, "f-g"] & \prod F(U_i\times_U U_j) \arrow[dl, "\Delta^1"] \\ \prod F(U_j') & \end{tikzcd}$$
, so it induce the same map on the kernel. □

Prop. (4.2.3) (Non-Abelian Čech). For a exact sequence of sheaves of groups $1 \to A \to B \to C \to 1$, where A is in the center of B, then there is a exact sequence:

$$1 \to H^0(U,A) \to H^0(U,B) \to H^0(U,C) \to H^1(U,A) \to H^1(U,B) \to H^1(U,C) \to H^2(U,A)$$

which is by direct calculation, the last one is the Čech composed with the injection to sheaf cohomology(4.2.5). Use the same method.

Sheaf Cohomology

Prop. (4.2.4) (Sheaf-Cohomology-Presheaf). The forgetful functor is right adjoint to the exact shifi functor, the Grothendieck spectral sequence applies to the exact functor $\Gamma(U,-)$ from $\mathcal{P}$ to Ab shows its right derived functor is

$$\mathcal{H}^p(F) = R^p\iota(F) : U \to H^p(U,F).$$

Prop. (4.2.5) (Čech to Sheaf). For any Abelian sheaf $\mathcal{F}$ on a site, the Grothendieck spectral sequence applied to $\Gamma(U,-) = H^0(\{U_i \to U\},-) \circ \iota = \check{H}^0(U,-) \circ \iota$ gives us:

$$H^p(\{U_i \to U\}, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(U,F).$$

$$\check{H}^p(U, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(U,F).$$

Cor. (4.2.6). The Grothendieck spectral sequence applied to forgetful functor and exact $\sharp$ functor shows that $\mathcal{H}^p(F)^{++} = \mathcal{H}^p(F)^\sharp = 0$ for $p > 0$, so

$$\mathcal{H}^p(F)^+(U) = \check{H}^0(U, \mathcal{H}^p(F)) = 0 \quad p > 0.$$

because $\mathcal{H}^p(F)^+$ is separated, See(1.2.3).

Thus the low degree of Čech to sheaf says:

$$0 \to \underline{\check{H}^1(U,F) \to H^1(U,F)} \to 0 \to \check{H}^2(U,F) \to H^2(U,F).$$

Cor. (4.2.7). If we have $H^q(U_{i_0 i_1 \ldots i_r}, \mathcal{F}) = 0, q > 0$, then $H^p(\{U_i \to U\}, \mathcal{F}) = H^p(U, \mathcal{F})$. (because $H^p(\{U_i \to U\}, \mathcal{H}^q(F))$ vanish for $q > 0$).

Prop. (4.2.8) (Higher Direct Image). For $f : T' \to T$ a morphism of topologies and $\mathcal{F}$ a sheaf on T, the trivial spectral sequence for $(\sharp \circ f^p) \circ \iota$(because $\sharp, f^p$ are exact) shows that $R^p f^s \mathcal{F} = (f^p \mathcal{H}^p(\mathcal{F}))^\sharp$. So flask sheaf thus flabby sheaf is acyclic for f^s.

Prop. (4.2.9) (Leray Spectral Sequence). For $T'' \xrightarrow{g} T \xrightarrow{f} T'$ of topologies, for a sheaf $\mathcal{F}'$ on T', there is a spectral sequence

$$E_2^{p,q} = R^p g^s(R^q f^s(F')) \Rightarrow R^{p+q}(fg)^s(\mathcal{F}'),$$

by Grothendieck spectral sequence applied to $g^s f^s = (fg)^s$. (Use(4.1.3) and Grothendieck spectral sequence).

Cor. (4.2.10) (Leray Spectral Sequence). Let $f : T \to T'$ be a morphism of topologies and $U \in T$, then for a sheaf F', there is a spectral sequence

$$E_2^{pq} = H^p(U, R^q f^s(\mathcal{F}') \Rightarrow H^{p+q}(f(U), \mathcal{F}').$$

Letting g be the morphism from pt to T that maps pt to U.

Prop. (4.2.11) (Filtered Colimits). $H^n(U,-)$ commutes with filtered colimits if T is Noetherian topology.

Proof: For $n=0$ the limit presheaf is already a sheaf, because for any finite cover, the Čech complex of the limit sheaf is the limit of Čech sheaves, and direct limit is exact.

And the limit sheaf of flask sheaves are flask, because flask need only be checked for finite covers at this case(because T and T^f have equivalent category of sheaves(1.1.5) and definition of flask(4.1.1)). Then the limit of exact Čech complexes is exact. So we can use the limit of the flask sheaf resolutions to calculate cohomology, thus the result. □

Comparison Lemma

Prop. (4.2.12) (Change to Subtopologies). Let T' be a fully subcategory of T, $i: T' \to T$ is a morphism of topologies, and each object U of T' and a covering $\{U_i \to U\}$ in T has a refinement $\{U'_j \to U\}$ in T'. Then

$$H^p(T'; U, i^s F) \cong H^p(T; U, F), \quad H^p(T'; U, F') \cong H^p(T; U, i_s F')$$

Proof: Use(1.2.12), for the first one, use Leray spectral sequence, and the fact i^s is exact. The second follows from the fist because $F' \cong i^s i_s F'$. □

3 Cohomology on Ringed Spaces

There are three basic objects, the derived functor for f_* as an Abelian sheaf, f_* as a $\mathcal{O}_X$-module, $\Gamma(U,-)$ as an Abelian sheaf. Notice that an Abelian group is just a $\mathbb{Z}$-module.

Prop. (4.3.1) (Grothendieck). The sheaf cohomology of a sheaf over a Noetherian topological space of dimension n vanish for $k > n$. Cf.[Hartshorne P208].

Proof: Use(4.3.2) and(4.4.23) and long exact sequence, we can reduce to the case of X irreducible. Then we induct on dimension. Notice first any sheaf is a filtered colimits of sheaf generated by f.m sections, thus we can use(4.2.11) to reduce to f.m sections case. And notice $\mathcal{F}_{\alpha'} \to \mathcal{F}_\alpha \to \mathcal{G}$, then G Is generated by at most $|\alpha| - |\alpha'|$ elements, so reduce to the one section case.

Now it is a quotient sheaf of $\mathbb{Z}$, look at the kernel R. If the kernel is $d\mathbb{Z}$ at the generic pt, then $R|_V \cong \mathbb{Z}$ on some nbhd, and $R|_V/\mathbb{Z}$ supports on a lower dimension set, then we only need to consider the pushout of constant sheaf $\mathbb{Z}_U$.

Now there is an exact sequence $0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0$(4.3.2), $\mathbb{Z}$ is flabby(4.1.7) so flask, and the conclusion follows by induction. □

Prop. (4.3.2). We have a canonical exact sequences of sheaves of modules:

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0$$

$$0 \to i_* i^!_Y \mathcal{F} \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to 0$$

(check on stalks), which is important to use reduction to calculate sheaf cohomology. The latter induces long exact sequences:

$$0 \to H^0_Y(X,\mathcal{F}) \to H^0(X,\mathcal{F}) \to H^0(U,\mathcal{F}|_U) \to H^1_Y(X,\mathcal{F}) \to \cdots$$

Prop. (4.3.3). For $f: X \to Y$, if $\mathcal{I}$ is an injective module on X, then $\check{H}^p(\{U_i \to U\}, f_*\mathcal{I}) = 0$ for every open cover for an open subset U(4.1.1). This is because Čech cohomology is a derived functor. (Notice $f_*\mathcal{I}$ may not be injective when f is not flat).

Prop. (4.3.4). $H^i(X, -)$ commutes with direct limits if X is a qcqs ringed space,Cf.[StackProject 01FF].

Cor. (4.3.5) (Mayer-Vietoris). For $X = U \cup V$, there is a long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \otimes H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \cdots$$

derived from the Čech to sheaf1 because it has only two column, just wrap out the definition.

Prop. (4.3.6). For a subscheme of $\mathbb{P}^2_k$ defined by a d-dimensional homogenous polynomial f that $f(1,0,0) \neq 0$, then using the two open affines $\{x_1 \neq 0\}$ and $\{x_2 \neq 0\}$, we see that $\dim H^0(X, \mathcal{O}_X) = 1$, $\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$.

Proof: We need to see that $\sum x_0^{a_i} x_1^p / x_2^q \equiv \sum x_0^{b_j} x_2^s / x_1^t \mod f$, where $a_i, b_j < d$. Just look at the degree of x_0. For H^1, notice $\{x_0^{a+b}/x_1^a x_2^b\}$ where $a + b < d$ forms a basis of H^1. □

Higher Direct Image

Prop. (4.3.7). For f a morphisms of ringed spaces, $R^p f_* \mathcal{F} = (f_p \mathcal{H}^p(\mathcal{F}))^\sharp$, by(4.2.8).

So flask sheaf thus flabby sheaf is acyclic for f_*. When $\mathcal{F}$ has $\mathcal{O}_X$ structure, injective $\mathcal{O}_X$-modules are flabby(4.1.2) thus acyclic as Abelian sheaf, so the higher direct image is the same in the category of $\mathcal{O}_X$-module as in the category SAb.

Prop. (4.3.8) (Projection Formula). Let $f: X \to Y$, and $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module, then we have

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong R^i f_*(\mathcal{F}) \otimes \mathcal{E}.$$

Proof: It suffice to prove for $i = 0$, because then we know that $f^*\mathcal{E}$ and $\mathcal{E}$ are locally free thus flat and preserves injectives(4.1.9) and then use Grothendieck spectral sequence.

For $i = 0$, there is a map from the right to the left, and there stalk are both $(f_*(\mathcal{F})_x)^{\text{rank } \mathcal{E}}$, so they are equal. □

Base Change

4 Cohomology on Schemes

Lemma (4.4.1) (Zariski-Poincare). A Qco sheaf on an affine scheme X is Čech-acyclic.

Proof: Because the principal affine covers are cofinal in the ordering of covers, we only need to consider principal affine covers. Let $R \to A = \prod R_{f_i}$, then it is f.f., so we can use(1.5.3), just notice the higher term is $\prod_{i_0, \ldots, i_n} R_{f_{i_0} \cdots f_{i_n}}$. □

Prop. (4.4.2) (Separated Čech Sheaf Coh Equal). For a Qco sheaf $\mathcal{F}$ on a separated scheme, we have $H^p(X, \mathcal{F}) = \check{H}^p(X, \mathcal{F}) = H^p(\{U_i \to X\}, F)$. for U_i any open affine cover.

Proof: By(4.2.7), the intersection of affine opens is affine open, we only have to show that $H^p(U, F) = 0$. Use induction on p, we can use Čech to sheaf1 $H^p(\{U_i \to U\}, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(U, F)$. The case $p \neq 0$ is by(4.4.1) and induction hypothesis. For $p = 0$, use (4.2.6) □

Remark (4.4.3). Notice the proof extends to any case that there is a family of open sets that are closed under intersection and Čech cohomology vanish.

Cor. (4.4.4) (Affine Qco Cohomology Vanish). For a Qco sheaf on an affine scheme, $H^i(X,\mathcal{F})$ vanish for $i>0$. For a Qco sheaf on a qcqs scheme X, the sheaf cohomology vanish for n large enough. (Use check to sheaf2).

Prop. (4.4.5) (Compatibility of Qco and $\mathcal{O}_X$-mod). We have in the category of Qco sheaves, injective objects are flabby sheaves, thus nearly calculating all derived functors are legitimate in the category of Qco sheaves(4.1.1).

Proof: We use the Deligne formula(2.1.27) and the definition of injective, just need to consider the sheaf of ideals of the corresponding induced reduced structure. □

Prop. (4.4.6) (Filtered Colimits). If X is qcqs, then sheaf cohomology on X commutes with filtered colimits. (Follows from(1.4.4) the same way(7.1.6) follows from(1.4.9)).

Prop. (4.4.7). In the category of $Qco(X)$, we have two Ext, for $\mathrm{Hom}(\mathcal{F},-)$ and $\mathcal{H}om(\mathcal{F},-)$.

We have

$$\mathcal{E}xt^i_X(\mathcal{F},\mathcal{G})|_U=\mathcal{E}xt^i_U(\mathcal{F}|_U,\mathcal{G}|_U),$$

because both gives a universal delta functor for $\mathcal{G}$. In particular, we have $\mathcal{H}om(\mathcal{F},-)$ is exact for $\mathcal{F}$ locally free.

Prop. (4.4.8). Ext and $\mathcal{E}xt$ are universal δ functors in $\mathcal{G}$ and a δ functors in $\mathcal{F}$ using injective resolution of $\mathcal{G}$. (Notice injective are acyclic for $\mathcal{E}xt$ because $\mathcal{I}|_U$ is also injective).

Cor. (4.4.9). When X is locally Noetherian and $\mathcal{F}$ is coherent, we have

$$\mathcal{E}xt^i(\mathcal{F},\mathcal{G})_x\cong \mathrm{Ext}^i_{\mathcal{O}_x}(\mathcal{F}_x,\mathcal{G}_x).$$

Proof: Check local on an affine open, Use a finite locally free resolution and(2.3.18), notice the stalk function is exact. □

Cor. (4.4.10). If X is locally Noetherian, suppose that every coherent sheaf is a quotient of a locally free sheaf, we can define the **homological dimension** $hd(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ as the minimal length of locally free resolution of $\mathcal{F}$. Then $hd(\mathcal{F})\le n \iff \mathcal{E}xt^i(\mathcal{F},\mathcal{G})=0$ for every $\mathcal{G}$ and every $i>n$. And $hd(\mathcal{F})=\sup pd_{\mathcal{O}_{X,x}}\mathcal{F}_x$. This follows easily from(4.4.9).

Prop. (4.4.11). When $\mathcal{L}$ is a locally free sheaf, we have:

$$\mathcal{E}xt^i(\mathcal{F}\otimes\mathcal{L},\mathcal{G})\cong\mathcal{E}xt^i(\mathcal{F},\mathcal{L}^\vee\otimes\mathcal{G})\cong\mathcal{E}xt^i(\mathcal{F},\mathcal{G})\otimes\mathcal{L}^\vee$$

because there are maps between them(2.3.15), and $\mathcal{E}xt$ is local, so check locally. In particular,

$$\mathrm{Ext}^i(\mathcal{F}\otimes\mathcal{L},\mathcal{G})=\mathrm{Ext}^i(\mathcal{F},\mathcal{L}^\vee\otimes\mathcal{G}).$$

Prop. (4.4.12). On a Noetherian affine scheme, if M is f.g., then

$$\mathcal{E}xt^i(\widetilde{M},\widetilde{N})\cong\mathcal{E}xt^i(M,N).$$

So on a locally Noetherian scheme, when $\mathcal{F}$ is coherent and $\mathcal{G}$ Qco, $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})$ is Qco and if moreover $\mathcal{G}$ is coherent, then $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})$ is coherent (because this is true for Ext by free resolution).

Proof: Show that they are both universal effaceable. □

Prop. (4.4.13). For f proper between locally Noetherian scheme, there is a inverse sheaf $f^!\mathcal{G} = \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})$, which maps $Qco(Y)$ to $Qco(X)$ by(2.1.25) and(4.4.12). When f is affine, in particular when it is finite, then for $\mathcal{F}$ coherent and $\mathcal{G}$ Qco,

$$f_*\mathcal{H}om(\mathcal{F}, f^!\mathcal{G}) \cong \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{G})$$

and when X, Y is separated and X has the resolution property and f is flat, then

$$\mathrm{Ext}^i(\mathcal{F}, f^!\mathcal{G}) \cong \mathrm{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G})$$

is also an isomorphism.

Proof: The first on e is just local check, for the second one, just use Grothendieck spectral sequence and the fact $f_*\mathcal{O}_X$ is locally free thus $f^!$ is exact. □

Cohomology of Proper & Projective Spaces

Prop. (4.4.14). Let $X = \mathbb{P}^r_A$ we have:

- $H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all n.
- There is a perfect pairing
$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A.$$
- Of course for $i > r$, the cohomology vanish because X is separated. And when $n > 0$, $H^r(X, \mathcal{O}_X(n-r-1)) = 0$.

Proof: X is separated, we use Cech cohomology, the second one is easy, $(x_0x_1\dots x_r)^{-1}$ forms a basis of H^r.

For the first one, induction on r,Cf.[Hartshorne P225]. □

Prop. (4.4.15). Let $X = \mathbb{P}^n_k$ and $0 \le p, q \le n$, then $H^q(X, \Omega^p_X) = 0$ for $p \ne q$ and when $p = q$, $H^q(X, \Omega^p_X) = k$.

Proof: By(2.3.17) and(2.5.5), there is an exact sequence $0 \to \Omega^q \to \wedge^q\mathcal{O}(-1) \to \Omega^{q-1} \to 0$, and the middle has vanishing q-th cohomology by(4.4.14), thus we can induct and (4.4.14) gives the result. □

Def. (4.4.16) (Euler Characteristic). Let X be proper over a field k and $\mathcal{F}$ be coherent, then we define

$$\chi(\mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

It is an additive functor on $Coh(X)$.

Prop. (4.4.17). For a proper scheme X over a field k and $\mathcal{L}_i$ be invertible sheaves on X. Then for any coherent sheaf $\mathcal{F}$ on X,

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r})$$

is a polynomial in $(n_1, \dots, n_r)$ of total degree at most $\dim \operatorname{Supp} \mathcal{F}$.

Proof: Cf.[StackProject 0BEM]. □

Cor. (4.4.18) (Hilbert Polynomial). For a projective scheme over a field k and a coherent sheaf $\mathcal{F}$, there is a polynomial **Hilbert polynomial** P that $\chi(\mathcal{F}(n)) = P(n)$, and $\deg P \leq \dim \operatorname{Supp}(\mathcal{F})$.

This Hilbert polynomial is compatible with the definition in(6.1.20), because by(4.4.29), the higher cohomology group vanishes for n large, so $\chi(\mathcal{F}(n)) = \Gamma(\mathcal{F}(n)) = \Gamma_*(\mathcal{F})_n$.

Prop. (4.4.19). Let $f : Y \to X$ be morphism between schemes proper over field k and $\mathcal{F}$ a coherent sheaf, then we have

$$\chi(Y, \mathcal{F}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{F}).$$

This comes from the Leray spectral sequence.

Def. (4.4.20). Thr **arithmetic genus** of a proper scheme of dimension r over a field is defined as $p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1)$. In particular, when X is a curve over alg.closed field k, then $p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$.

Prop. (4.4.21) (Asymptotic Riemann-Roch). If X is a proper scheme over a field k of dimension d and $\mathcal{L}$ is an ample invertible sheaf, then $\dim \Gamma(X, \mathcal{L}^n) \sim cn^d$, Cf.[StackProject 0BJ8].

Prop. (4.4.22). Let X be H-projective over a Noetherian affine scheme and $\mathcal{F}$, $\mathcal{G}$ be coherent, then for n large,

$$\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \cong \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n)).$$

Proof: This true for $i = 0$, so let $i > 0$. When $\mathcal{F} = mclO_X$, this is true by(4.4.29), and hence true for $\mathcal{F}$ locally free(4.4.11), and for $\mathcal{F}$ general, choose a locally free surjective $\mathcal{E} \to \mathcal{F}$ with kernel $\mathcal{G}$, then for n large, there is an exact sequence

$$\operatorname{Hom}(\mathcal{E}, \mathcal{G}(n)) \to \operatorname{Hom}(\mathcal{R}, \mathcal{R}(n)) \to \operatorname{Ext}^1(\mathcal{F}, \mathcal{G}(n))$$

and $\operatorname{Ext}^i(\mathcal{R}, \mathcal{G}(n)) \cong \operatorname{Ext}^{i+1}(\mathcal{F}, \mathcal{G}(n))$. And similarly for $\mathcal{E}xt$. When proving $i = 1$, we need to use(4.4.30) to choose n even larger to get the corresponding global section exact sequence. □

Higher Direct Image

Cor. (4.4.23) (Sheaf Cohomology Commutes with Affine Map). For f affine and $\mathcal{F}$ Qco, we have $H^n(Y, \mathcal{F}) = H^n(X, i_*\mathcal{F})$ (because $R^i f_* \mathcal{F}(U) = 0$(by Serre criterion(3.2.14) and(4.3.7)) and then use(4.2.9)).

Prop. (4.4.24) (Higher Direct Image is Qco and Local). If f is qcqs then $R^n f_*$ maps Qco to Qco, $\underline{R^p f_* \mathcal{F}(U) = H^p(\widetilde{f^{-1}}(U), \mathcal{F})}$.

Proof: check local on affine open of Y, both side are δ-functors from $Qco(X)$ to $\widetilde{Mod_Y}$, injectives in $Qco(X)$ are flabby, thus both are effaceable. We only need to show $f_*\mathcal{F} = \widetilde{\Gamma(X}, \mathcal{F})$, and this is(2.1.24). Cf.[Hartshorne P251]. □

Prop. (4.4.25). For a qcqs morphism $f : X \to S$, if S is qc, there is a N that for every base change f' of f, we have $R^n f'_* \mathcal{F} = 0$ for every $\mathcal{F}$ Qco and $n \geq N$.

Proof: We check local on affine open and use(4.4.24), choose an finite affine cover of X, their intersection are all f.m. affine opens. Then local on a base change, the number of affine opens are the same. So when n is large enough, using Cech to Sheaf2, we have the cohomology vanish.(This uses the fact that the intersection of affine opens are separated and(4.4.2). □

Cor. (4.4.26). For a qcqs scheme X, the cohomology vanish for n large. And when X is separated and can be covered by r affine opens, then N can chosen to be r.

Prop. (4.4.27) (Proper Pushout of Coherent). If $f : X \to Y$ is proper and Y locally Noetherian, then $R^n f_*$ maps coherent to coherent. Cf.[StackProject 02O5].

Cor. (4.4.28). If X is proper over a Noetherian affine scheme, its global section is a f.g. A-module.

Prop. (4.4.29) (Serre). If $X \to Y$ is a projective morphism of Noetherian schemes and $\mathcal{F}$ be a coherent sheaf on X, then we have $R^i f_*(\mathcal{F}(n)) = 0$ for $i > 0$ and n large enough.

For this it suffice to prove the local case: If X is projective scheme over a Noetherian affine scheme, $H^i(X, \mathcal{F}(n)) = 0$.

Proof: Because $i_*\mathcal{F}$ is coherent on $\mathbb{P}^r_A$, we reduce to the case $X = \mathbb{P}^r_A$. The conclusion is true for $\mathcal{O}_X(n)$ by(4.4.14), and for general $\mathcal{F}$, we use descending induction on i, choose a $\oplus\mathcal{O}_X(n_i) \to \mathcal{F} \to 0$ with kernel $\mathcal{R}$(2.4.22), then

$$H^i(X, \oplus\mathcal{O}_X(n_i + n)) \to H^i(X, \mathcal{F}(n)) \to H^{i+1}(X, \mathcal{R}(n)),$$

and the left term vanish for n large(4.4.14) thus the result. □

Cor. (4.4.30). For any finite exact sequence of coherent sheaves on a H-projective scheme over a Noetherian affine scheme, if tensoring it with $\mathcal{O}(n)$ for large n, the resulting global section is exact.

Base Change

Prop. (4.4.31) (Flat Base Change). For a Cartesian diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

if g is flat and f is qcqs, then for every Qco sheaf $\mathcal{F}$ on X with base change $\mathcal{F}'$, there is an canonical isomorphism

$$g^* R^i f_* \mathcal{F} \cong R^i f'_* \mathcal{F}'$$

when S, S' is affine, this reads:

$$H^i(X', \mathcal{F}) \otimes_A B \cong H^i(X, \mathcal{F}').$$

Proof: First we show the existence canonically such a morphism. Choose an injective resolution $\mathcal{I}$ of $\mathcal{F}$ and an injective resolution J of $(g')^*\mathcal{F}$, then we $g_*\mathcal{J}$ is also injective because (g') is exact for flat morphism. Now the canonical map $\mathcal{F} \to (g')_*(g')^*\mathcal{F}$ gives rise to a morphism of complexes from $\mathcal{I} \to (g')_*\mathcal{J}$, thus giving a morphism of complexes $f_*\mathcal{I} \to f_* g'_* \mathcal{J} = g_* f'_* \mathcal{J}$, which gives by adjointness a morphism $g^* f_* \mathcal{I} \to f'_* \mathcal{J}$.

By(4.4.24), we only need to check the results affine opens, so let S, S' be affine open. If X is separated, then the cohomology equals Čech cohomology, and the Čech cohomology of $\mathcal{F}'$ is just the cohomology of the Čech complex tensored with B, so it commutes with taking cohomology because B is A-flat.

Now if X is only qs, then we choose an open affine cover(finite) $\{U_i\}$, then all the intersections of these U_i are separated. Now we use Čech-to-sheaf2 spectral sequence(4.2.5), then by what we proved for separated case, there is an isomorphism of spectral sequences E_2, so their limit are the same. □

Cor. (4.4.32). Let $X \to Y$ be qcqs and Y affine, then for any $y \in Y$, let X_y be the fiber, then $H^i(X_y, \mathcal{F}_y) \cong H^i(X, \mathcal{F} \otimes k(y))$.

Proof: The only problem is to reduce to the case that $Y' = \{y\}$ with the induced reduced structure, because then $\operatorname{Spec} k(y) \to Y$ is flat. All we care is the fiber over y, and $X' = X \times_Y Y'$ is a closed scheme of X, $\mathcal{F}$ pullbacks to $\mathcal{F} \otimes_A A/p_y$, so $H^i(X', \mathcal{F} \otimes_A A/p_y \otimes k(y)) \cong H^i(X, \mathcal{F} \otimes k(y))$. □

Semicontinuity Theorem

Prop. (4.4.33). If X is projective over a Noetherian affine scheme $\operatorname{Spec} A$, and $\mathcal{F}$ is a coherent sheaf on X which is flat over Y, then if we define $T^i(M) = H^i(X, \mathcal{F} \otimes_A M)$ as a functor from A-modules to A-modules, then they form a δ-functor as $\mathcal{F}$ is flat.

And there is a complex $L^\bullet$ of f.g. free A modules bounded above that $T^i(M) \cong h^i(L^\bullet \otimes_A M)$.

Proof: Firstly, the Ćech complex satisfies the requirement, but it is not finite free. But, its cohomology equals $H^i(\mathcal{F})$(4.4.2), so(7.5.1) can be used. □

Prop. (4.4.34). T^i is left exact iff $\operatorname{Coker} d^{i-1}$ is a projective A-module, iff it is representable by a finite A-module.

Proof: Denote $W^i = \operatorname{Coker} d^{i-1}$, then $\operatorname{Coker} d^{i-1} \otimes_A M = W^i \otimes M$, because tensoring is right exact. Thus $T^i(M) = \operatorname{Ker}(W^i \otimes M \to L^{i+1} \otimes M)$. Then for $M' \subset M$, there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^i(M) & \longrightarrow & W^i \otimes M' & \longrightarrow & L^{i+1} \otimes M' \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ 0 & \longrightarrow & T^i(M) & \longrightarrow & W^i \otimes M & \longrightarrow & L^{i+1} \otimes M \end{array}$$

γ is injective, so using spectral sequence, its clear α is injective iff β is injective, i.e. W^i is flat, which is equivalent to finite projective(5.2.4).

To prove T^i is representable, let $Q = \operatorname{Coker}(L^{i+1,*} \to W^{i,*})$, then Q is finite because W^i is finite(5.2.11), and $0 \to \operatorname{Hom}(Q, M) \to \operatorname{Hom}(W^{i,*}, M) \to \operatorname{Hom}(L^{i+1,*}, M)$, but by(5.2.12), the last two are just $W^i \otimes M$ and $L^{i+1} \otimes M$, $\operatorname{Hom}(Q, M) = T^i(M)$ by what has already be proved. □

Prop. (4.4.35). T^i is right exact iff the morphism $T^i(A) \otimes_A M \to T^i(M)$ are all isomorphism.

Proof: Because T^i and $\otimes$ commutes with direct limit, it suffices to prove for M finite. In this case, choose a finite presentation $A^r \to A^s \to M \to 0$, then there is a diagram

$$\begin{array}{ccccccc} T^i(A) \otimes A^r & \longrightarrow & T^i(A) \otimes A^s & \longrightarrow & T^i(A) \otimes M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ T^i(A^r) & \longrightarrow & T^i(A^s) & \longrightarrow & T^i(M) & & \end{array}$$

The first two vertical arrows are isomorphisms, so if T^i is right exact, so does the third vertical arrow. Conversely, if $T^i(A) \otimes_A M \to T^i(M)$ are all isomorphisms, then by a similar diagram, there $T^i(M) \to T^i(M')$ are surjective for $M \to M'$ surjective. □

Cor. (4.4.36). T^i is exact iff it is right exact and $T^i(A) = H^i(\mathcal{F})$ is a finite projective A-module.

Proof: Because in this case $T^i(M) \cong T_i(A) \otimes_A M$, so it is exact iff $T^i(A)$ is flat, and it is also finite, so it is equivalent to projective(5.2.4). □

Def. (4.4.37). For a prime p of A, we define $T^i_p(N) = H^i(L^\bullet_p \otimes N)$, then T^i is (left/right)exact iff T^i_p are all (left/right)exact(exact sequence is stalkwise(5.1.15).

Prop. (4.4.38). If T^i is (left/right)exact at a point y, then the same is true on a nbhd of y.

Proof: From(4.4.34), $(\operatorname{Coker} d^{i-1})_y$ is a finite projective A_p module, so it is free. Now $\operatorname{Coker} d^{i-1}$ is a coherent sheaf, so it is free at a nbhd of y, so the same is true on a nbhd of y. Now right exactness of T^i is equivalent to left exactness of T^{i+1}, and exact is left exact+right exact, so we are done. □

Prop. (4.4.39) (Semicontinuity of Cohomology). Let $X \to Y$ be a projective morphism of locally Noetherian schemes and $\mathcal{F}$ is a coherent sheaf on X, flat over Y, then for each i, $h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$ is an upper semicontinuous function on Y.

Proof: The question is local on Y, so we may assume Y is affine Noetherian. By(4.4.31), $H^i(y, \mathcal{F}) = \dim_{k(y)} T^i(k(y))$. And as in the proof of(4.4.34), $T^i(M) = \operatorname{Ker}(W^i \otimes M \to L^{i+1} \otimes M)$, and $W^i \to L^{i+1} \to W^{i+1} \to 0$ is exact, so $0 \to T^i(k(y)) \to W^i \otimes k(y) \to L^{i+1} \otimes k(y) \to W^{i+1} \otimes k(y) \to 0$, and counting dimension, $h^i(y, \mathcal{F}) = \dim W^i \otimes k(y) + \dim W^{i+1} \otimes k(y) - \dim L^{i+1} \otimes k(y)$. Notice the last term is constant as L^{i+1} is free A-module and the first two terms are upper semicontinuous by(2.1.30), thus $h^i(y, \mathcal{F})$ is upper-semicontinuous. □

Cor. (4.4.40) (Grauert). If Y is integral and $h^i(y, \mathcal{F})$ is constant on Y, then $R^i f_*(\mathcal{F})$ is locally free on Y and $R^i f_*(\mathcal{F}) \otimes k(y) \cong H^i(X_y, \mathcal{F}_y)$.

Proof: Following the above proof, we get $\dim W^i \otimes k(y)$ and $\dim W^{i+1} \otimes k(y)$ are all constant. This implies that W^i and W^{i+1} are all locally free, so T^i and T^{i+1} are both left exact, so T^i is exact. So $T^i(A)$ is finite projective by(4.4.36). So $R^i f_X(\mathcal{F})$, as equal to $widetilde T^i(A)$, is locally free. The last assertion follows from(4.4.32) and(4.4.35). □

Prop. (4.4.41). To check T^i is right exact, it suffice to check $T^i(A) \otimes k(y) \to T^i(k(y))$ is surjective.

Proof: Cf.[Hartshorne P289]. □

Theorem of Formal Functions

Basic References are [StackProject 29.20].

Prop. (4.4.42).

5 Topological Sheaves

Acyclic sheaves

Def. (4.5.1). An Abelian sheaf on a paracompact Hausdorff topological space X is called
soft iff is and $\forall$ closed $V, \mathcal{F}(X) \to \mathcal{F}(V)$ is surjective. A flabby sheaf is soft.
fine iff the sheaf of rings $\operatorname{Hom}(\mathcal{F}, \mathcal{F})$ is soft.
Fine and soft are local properties (Use Zorn's lemma to construct one-by-one).

Prop. (4.5.2). For a sheaf of *unital rings* over a paracompact Hausdorff space X, the following are equivalent,

1. it is a soft sheaf.
2. for any disjoint closed sets V, W, there is a section of X that is 0 on V, and 1 on W.
3. it possesses a partition of unity.
4. it is a fine sheaf.

Note any soft sheaf possesses a partition of unity.

Proof: $1 \iff 2$ is easy and $1 \to 3$ is the to choose a closed locally finite subcover and use Zorn's lemma to construct one-by-one. For $3 \to 1$, notice a closed section can extend to a slightly larger nbhd.

Because for a sheaf of rings $\mathcal{F}$, a partition of unity is equivalent to a partition of unity $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$, so 34 are equivalent because 13 are equivalent. □

Cor. (4.5.3).

- Note that a fine sheaf possesses a decomposition of section because the previous proposition applies to $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$, and a partition of unity in $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$ yields a decomposition of section in $\mathcal{F}$. Thus <u>a fine sheaf is soft</u>. (extend to a small nbhd and use partition of unity).
- The sheaf of modules over a soft sheaf of rings is soft, by partition of unity.
- The continuous function sheaf on a paracompact Hausdorff space or the smooth function sheaf on a smooth manifold is fine, thus any smooth module is fine (Use bump function).

Prop. (4.5.4). Soft sheaf, e.g. fine sheaf is adapted to $\Gamma(X, -)$. (Similar as in(4.1.2), notice flabby is soft and the others are the same as before).

Prop. (4.5.5). Let X be a locally compact space of finite compact dimension, when S is a soft sheaf, and one of S and $\mathcal{F}$ is flat, then $S \otimes_k \mathcal{F}$ is soft. Cf.[Cohomology of Sheaves Iversen P319].

Prop. (4.5.6). Over a locally compact space of finite dimension, any flat sheaf $\mathcal{F}$ on X has a resolution of soft flat sheaves, Cf.[Gelfand P232].

<u>**Comparison Theorems**</u>

Lemma (4.5.7) (Poincare Lemma). For a smooth manifold X of dimension n, there is an exact sequence

$$0 \to \mathbb{R}_X \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n \to 0$$

Lemma (4.5.8) ($\overline{\partial}$-Poincare Lemma). If X is a complex manifold of dimension n, there are exact sequences:

$$0 \to \Omega^p_{hol} \xrightarrow{\overline{\partial}} \Omega^{p,0} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \Omega^{p,n-p} \to 0$$

Proof: Cf.[Sheaf Cohomology P21]. □

Cor. (4.5.9). If X is a complex manifold of dimension n, there are exact sequences:

$$0 \to \underline{\mathbb{C}} \xrightarrow{d} \Omega^0_{hol} \xrightarrow{d} \Omega^1_{hol} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{hol} \to 0$$

Proof: This follows from the Poincare lemma(4.5.7) and ∂-Poincare lemma(4.5.8), by applying the Spectral sequence(I mean in the category of sheaves). □

Cor. (4.5.10) (Holomorphic Cohomology). For a complex manifold X,

$$H^p(X,\underline{\mathbb{C}}) = H^p(X,\Omega^\bullet_{hol}).$$

Prop. (4.5.11) (De Rham). For a smooth manifold and an Abelian group G,

$$H^*_{dR}(X,G) \cong H^*(X,\underline{G})$$

Where the right is constant sheaf cohomology. (Use the fact that smooth sheaf is fine so adapted to sheaf cohomology(4.5.4), and Poincare lemma(4.5.7)).

Prop. (4.5.12) (Singular). For a locally contractible topological space,

$$H^p_{sing}(X,G) \cong H^p(X,\underline{G}).$$

Proof: Shifification of the singular cochain complex is a flabby presheaf resolution of $\underline{G}$ because it is locally contractible, check on stalks. Then we only have to prove $C^\bullet(X) \to (C/V)^\bullet(X)$ is quasi-isomorphism, where V is the presheaf of locally vanishing cochain. It suffice to prove $V^\bullet(X)$ is exact.

For any i-cocycle φ, for any $i-1$-complex σ, use barycentric subdivision, we can construct a c_σ whose boundary is σ and other simplexes on which ϕ vanishes, so we have the coboundary of $\eta:\sigma \to \varphi(c_\sigma)$ is φ. □

Prop. (4.5.13) (Dolbeault). For a complex bundle on a complex manifold,

$$H^{p,q}_{\overline{\partial}}(X,\mathcal{E}) \cong H^q(M,\Omega^p_{hol}\otimes_{\mathcal{O}_X}\mathcal{E}),$$

where the left is Dolbeault cohomology and the right is sheaf cohomology. (Use the fact that smooth sheaf is fine so adapted to sheaf cohomology(4.5.4), and $\overline{\partial}$-Poincare lemma(4.5.8)).

Moreover, there is a spectral sequence of

$$E_1^{p,q} = H^{p,q}_{\overline{\partial}}(X) \Rightarrow E^n = H^n_{dR}(X,\mathbb{R})\times_{\mathbb{R}}\mathbb{C}.$$

Prop. (4.5.14) (Cartan). The class of Coh-Acyclic subsets of an analytic space is exactly the Stein manifold.

Cohomology with Proper Support

References are [Cohomology of Sheaves Iversen].

Prop. (4.5.15). For a morphism of locally compact spaces, we can define a **direct image of proper support**:

$$f_!(\mathcal{F})(U) = \{s\in\Gamma(f^{-1}(U),\mathcal{F})\,|\,\mathrm{Supp}(s)\to U \text{ proper}\}$$

This is a subsheaf of $f_*\mathcal{F}$ and it is left exact. we denote $\Gamma_c(X,\mathcal{F})$ as the group $f_!(\mathcal{F})$ where $f: X\to \mathrm{pt}$. And the stalk $f_!(\mathcal{F})_y = \Gamma_c(f^{-1}(y),\mathcal{F}|_{f^{-1}(y)})$ Cf.[Gelfand P224 P225].

Prop. (4.5.16). Soft sheaf is adapted to $f_!$ when X, Y are locally compact. Cf.[Gelfand P226]. So we can use soft resolution to define $R^i f_!$, in particular, when $Y = \mathrm{pt}$, we denote it by $H^i_c(X, \mathcal{F})$. Using(4.5.15), we get the stalk of $R^i f_!(\mathcal{F})$ at y is just $H^i_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$.

Def. (4.5.17). The **compact dimension** of a locally compact topological space is the smallest n that $H^i_c(X, \mathcal{F}) = 0$ for $i > n$. It is also the maximal length of minimal soft resolution.

$\dim_c \mathbb{R}^n = n$, and when Y is an open or closed subset of X, $\dim_c Y \leq \dim_c X$. $\dim_c$ is local in the sense if every point has a nbhd of dimension$\leq n$, then $\dim_c X \leq n$. Cf.[Iversen].

V.5 Topics in Schemes

1 Divisors

Weil Divisors

We only consider divisors on a Noetherian integral separated scheme regular in codimension 1. Cartier divisor and Picard Group are more general.

Def. (5.1.1). A **Weil divisor** is on a Noetherian integral separated scheme regular in codimension 1 is a linear combination of closed integral subschemes of codimension 1.

Prop. (5.1.2). If X is a Noetherian integral separated scheme regular in codimension 1, then so does $X \times \operatorname{Spec}\mathbb{Z}[T]$ and $X \times \mathbb{P}^n_{\mathbb{Z}}$(local check), and $\mathrm{Cl}(X \times \operatorname{Spec}\mathbb{Z}[T]) = \mathrm{Cl}(X)$ and $\mathrm{Cl}(X) \times \mathbb{P}^n_{\mathbb{Z}} = \mathbb{Z} \times \mathrm{Cl}(X)$. Cf.[Hartshorne P134].

Prop. (5.1.3). For A a Noetherian domain, it is a UFD iff $X = \operatorname{Spec}(A)$ is normal and $\mathrm{Cl}(X) = 0$.

Proof: We only have to show minimal primes of A is principal iff minimal primes of A is a principal divisor. This is done by (5.11.7) and (3.3.10). □

Cor. (5.1.4). The divisors on $\mathbb{P}^n_k$ is is locally defined by a function, this is because the affine opens are UFD.

Prop. (5.1.5) (Picard Group of $\mathbb{P}^n_k$). A hypersurface of degree d in P^n_k is equivalent to dH, where H is the surface $x_0 = 0$. This is because irreducible hypersurface of P^n_k correspond to a homogeneous prime ideal of height 1 which is principal. So $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$. Cf.[Hartshorne P132].

Cartier Divisors

Def. (5.1.6). A **Cartier divisor** on a scheme is an element in $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*_X)$. An **effective Cartier divisor** is a Cartier divisor that is locally defined as $\{(U_i, f_i)\}$ where $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$, it is equivalent to a closed subscheme locally defined by a single element.

Notice by definition, $\mathcal{K}$ is the localization w.r.t. non-zero-divisors, and f_i is invertible in $\mathcal{K}^*$ so f_i must be non-zero-divisors.

The **Cartier group** CaCl is the quotient of $\Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}^*_X)$.

Prop. (5.1.7) (Weil-Cartier). For an integral separated Noetherian scheme that is locally factorial, the (effective)Cartier divisor is the same thing as (effective)Weil divisor.

This in particular applies to non-singular curves.

Proof: Cf.[Hartshorne P141]. □

Cor. (5.1.8). If X is Noetherian and the diagonal map is affine, for a dense affine open U, if all the stalk of $X - U$ are UFD, then U is the complement of an effective Cartier divisor.

Proof: The irreducible complements of $X - U$ is finite and has codimension 1 by(2.2.12) because $U \to X$ is affine, and it is an effective Cartier divisor by(5.1.7)., so their sum will suffice. □

Picard Group

Def. (5.1.9). For any ringed space, the **Picard group** is the group of isomorphism classes of invertible sheaves on X, under the tensor operation.

The Picard group is seen via Cech cohomology isomorphic to $H^1(X, \mathcal{O}_X^*)$ by(4.2.6).

Def. (5.1.10). For a Cartier divisor on a scheme X, we can define $\mathcal{L}(D)$ the **sheaf associated to** D as the sub $\mathcal{O}_X$-module of $\mathcal{K}$ locally generated by (f_i^{-1}), where $D = (f_i)$ locally.

Prop. (5.1.11). For an integral separated Noetherian scheme X that is locally factorial, by(5.1.7), a Weil divisor is equivalent to a Cartier divisor, so giving an integral closed subscheme E of X, $\mathcal{L}(E)$ can be defined, and there is an exact sequence of sheaves on X:

$$0 \to \mathcal{L}(-E) \to \mathcal{O}_X \to i_*\mathcal{O}_E \to 0$$

Prop. (5.1.12) (Cartier-Pic). For an integral scheme X, the homomorphism $\mathrm{CaCl}(X) \to \mathrm{Pic}(X) : D \to \mathcal{L}(D)$ is an isomorphism. (It is alway injective, as it is in fact the δ-functor of the exact sequence of sheaves $0 \to \mathcal{O}_X^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}_X^* \to 0$.)

Proof: It suffices to show any invertible sheaf can embed into the constant sheaf, tensor with K and restrict to the stalk of the generic point, i.e. there is a compatible choice of homomorphisms into $K(X)$. $\square$

Cor. (5.1.13) (Cl-Pic). For an integral separated Noetherian scheme that is locally factorial, $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$(5.1.7).

Remark (5.1.14). Take $\mathbb{P}_k^n$ for example, the hyperplane $x_0 = 0$ defines a Cartier divisor (x_0/x_i) on U_i, thus it define the subsheaf of $\underline{K^*}$ generated by (x_i/x_0) on U_i, thus it is isomorphic to the Serre sheaf $\mathcal{O}(1)$ by multiplication by x_0. The Picard group of $\mathbb{P}_k^n$ are generated by $\mathcal{O}(1)$(5.1.5).

Prop. (5.1.15). For an integral normal projective scheme of dimension≥ 2 over an alg.closed field, then the support of an effective ample divisor is connected.

Proof: We may assume the divisor is very ample, denote $\mathcal{O}_X(1) = \mathcal{L}(D)$, let Y_q be the closed subscheme corresponding to the divisor qD, then there is an exact sequence

$$0 \to \mathcal{O}_X(-q) \to \mathcal{O}_X \to i_*\mathcal{O}_{Y_q} \to 0$$

(5.1.11), so for q large,(5.4.8) shows that $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_{Y_q})$ is surjective. But the former is k by(3.1.27) and the second contains k, thus the latter is also k, thus it is connected. $\square$

2 Blowing Up

Prop. (5.2.1). On a locally Noetherian scheme, the **blowing up** $\widetilde{X}_I$ along a closed scheme (Corresponding to a coherent sheaf) is defined as $\mathrm{Proj}(\oplus I^d)$. It has the universal property that any morphism $Z \to X$ that pulls back Y to an effective Cartier divisor uniquely factors through $\widetilde{X}_I$.

Proof: Notice first an effective devisor is equivalent to an invertible sheaf of ideal. And any morphism $Z \to X$ pulls back I to the image of $f^{-1}I \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to f^{-1}I \cdot \mathcal{O}_Z$. This is just $\mathcal{O}(1)$ on $\widetilde{X}_I$ so invertible.

For the construction, the local uniqueness will implie the existence. Notice locally $\widetilde{X}_I$ is projective over X. Now because the $Z \to X$ pulls back I to an invertible sheaf and it is generated by $f^{-1}(a_i)$, we use**??** to get another $Z \to \mathrm{Proj}^n_X$ and it factors through the closed subscheme $\widetilde{X}_I$. If there is another morphism g, then $f^{-1}I \cdot \mathcal{O}_Z = g^{-1}(\pi^{-1}I \cdot \mathcal{O}_{\widetilde{X}_I}) \cdot \mathcal{O}_Z = g^{-1}(\mathcal{O}_{\widetilde{X}_I}) \cdot \mathcal{O}_Z$ surjective, and a surjective morphism between two invertible sheave is an isomorphism, and they are both ideal sheaves, thus is the same, so this morphism is unique as it is determined by the map on $\mathcal{O}_X$**??**. □

Cor. (5.2.2). If the sheaf of ideal is itself invertible, then the blowing up is an isomorphism by the universal property. In particular, on the open set $U = X - Y$, $I_U \cong \mathcal{O}_U$, so $\pi^{-1}(U) \cong U$.

Cor. (5.2.3). $\pi : \widetilde{X}_I \to X$ is birational, proper thus surjective. If X is a (complete)variety, then so does $\widetilde{X}_I$.

Prop. (5.2.4) (Strict Transformation). Same notation as before, for any locally Noetherian scheme $Z \to X$, we have the pullback sheaf $J = i^{-1}(I) \cdot \mathcal{O}_Z$ on Z, so $\widetilde{Z}_J \to X$ factors through $\widetilde{X}_I$. This a pullback diagram. (Recall the definition of fiber product, we only need to check for Z,X affine and glue. For this, check $B \otimes_A (\oplus I^d) \to \oplus (IB)^d$ defines the fiber map).

Prop. (5.2.5). If X is H-(quasi-)projective, then so does $\widetilde{X}_I$ and π is H-projective(2.3.13). And any birational projective morphism from another variety Z to X comes from a blowing-up.

Proof: Cf.[Hartshorne P166]. □

Blowing up along a regular variety

Prop. (5.2.6). If X is a regular variety over k and Y is a regular closed subvariety defined by $\mathcal{I}$, then the blowing up along $\mathcal{I}$ is also regular, and the inverse image Y' of Y is locally principal in it. In fact, $Y' \to Y$ is isomorphic to $\mathbb{P}(\mathcal{I}/\mathcal{I}^2)$, the projective space associated to the locally free bundle $\mathcal{I}/\mathcal{I}^2$ on Y, and the normal sheaf $\mathcal{N}_{Y'/X'} \cong \mathcal{O}_{\mathbb{P}(\mathcal{I}/\mathcal{I}^2)}(-1)$.

Proof: (Imagine the blowing up of $\mathbb{A}^2$ along $\{0\}$). $X' \cong \mathrm{Proj} \oplus \mathcal{I}^d$ and $Y' \cong \mathrm{Proj} \oplus \mathcal{I}^d/\mathcal{I}^{d+1}$. Then since Y is regular, (5.10.12) tells us $\mathcal{I}$ is locally generated by a regular sequence and(5.10.11) tells $Y' = \mathbb{P}(\mathcal{I}/\mathcal{I}^2)$. Y' is regular by(5.11.10), and then(5.11.15) shows that X' is regular also. For the normal sheaf, the defining sheaf $\mathcal{I}' = \mathcal{O}_{X'}$ and then $\mathcal{I}'/\mathcal{I}'^2 = \mathcal{O}_Y(1)$, thus $\mathcal{N}_{Y'/X'} \cong \mathcal{O}_{\mathbb{P}(\mathcal{I}/\mathcal{I}^2)}(-1)$. □

Prop. (5.2.7). In a blowing up along a regular variety of codimension $r \geq 2$, There is an isomorphism $\mathrm{Pic}X' \cong \mathrm{Pic}X \oplus \mathbb{Z}$ induced by the Weil divisor exact sequence of $Y' \subset X'$. This is because $r \geq 2$ and (5.2.2).

We also have $\omega_{X'} = f^*\omega_X \otimes \mathcal{L}((r-1)Y')$ because $\mathcal{L}(Y') = \mathcal{O}(-1)$ and $\omega_Y \cong \omega_X \otimes \mathcal{L}(D) \otimes \mathcal{O}_Y$ by(6.1.13), so it suffice to prove $\omega_{Y'} \cong f^*\omega_X \otimes \mathcal{O}_{Y'}(-r)$. For this, notice for a closed pt of Y, the fiber is a $\mathbb{P}^{r-1}$ because $\mathcal{I}/\mathcal{I}^2$ is locally free of rank r by(6.1.11) and the functoriality of $\mathcal{O}(1)$.

3 Derived Category of Schemes

Def. (5.3.1). For a ringed space X,

We denote $K(\mathcal{O}_X)$ the complexes of $\mathcal{O}_X$ modules modulo quasi-isomorphisms and $D(\mathcal{O}_X)$ the derived category of Mod-$\mathcal{O}_X$.

We denote $K(Qco(X))$ the complexes of Qco $\mathcal{O}_X$-modules modulo quasi-isomorphisms and $D(Qco(X))$ the derived category of Qco $\mathcal{O}_X$-modules.

We denote $K(Coh(X))$ the complexes of coherent $\mathcal{O}_X$-modules modulo quasi-isomorphisms and $D(Coh(X))$ the derived category of coherent $\mathcal{O}_X$-modules.

Perfect Complex and Pseudo-Coherent Module

Def. (5.3.2). A complex of $\mathcal{O}_X$-modules is called **strictly perfect** if it is finite and every term is a direct summand of a finite free sheaf.

Prop. (5.3.3). Every mapping from a strictly perfect complex to an acyclic complex has a cover of open sets that on each open set the map is nullhomotopic.

Proof: This is true for a single direct summand of a finite free sheaf, and we can use induction to prove, Cf.[StackProject 08C7]. □

Cor. (5.3.4). The strictly perfect complex is fake "K-projective" object in $K(\mathcal{O}_X)$. Note it is note technically K-projective, but it has all the properties of K-projective when proven, noticing the fact it is irrelevant when taken shifification.

Def. (5.3.5). We say an object $K^\bullet$ in $K(\mathcal{O}_X)$ **perfect** if there is a an open cover that on each open set there is a quasi-iso $K_i^\bullet \to K^\bullet|_{U_i}$ with $K_i^\bullet$ strictly perfect.

This is equivalent to $K^\bullet$ is locally represented by perfect objects in $D(\mathcal{O}_X)$ by the fact that perfect object is fake K-projective.

Prop. (5.3.6). When X is local ringed space, perfectness is equivalent to the fact that it is locally a finite free $\mathcal{O}_{U_i}$-module. This is because direct summand of a finite free module is free, Cf.[StackProject 0BCI].

Resolution Property

Def. (5.3.7). A scheme X is said to have **resolution property** iff every Qco $\mathcal{O}_X$-module of f.t. is a quotient of a locally free sheaf.

Prop. (5.3.8). If X is Noetherian scheme and has an ample invertible sheaf, then X has the resolution property(2.4.22). In fact, every coherent sheaf is a quotient of a finite direct sum of $\mathcal{O}(-n)$.

Prop. (5.3.9). If X is qc regular scheme with an affine diagonal, then X has the resolution property, Cf.[StackProject 0F8A]. Conversely, if X is qcqs with the resolution property, then X has affine diagonal. Cf.[StackProject 0F8C].

Prop. (5.3.10) (Kleiman). If X is a qc irreducible and locally factorial scheme with affine diagonal map, then X has the resolution property.

Proof: By(5.1.8), we have an basis of the form X_s for $s \in \Gamma(X, \mathcal{L})$ for various invertible sheaves, then for any coherent sheaf, it is generated by f.m. sections in $\Gamma(U_i, \mathcal{F})$ and $U_i = X_s$ for $s \in \Gamma(X, \mathcal{L})$, and for each of them, we can use(2.4.4), we can extend these to global sections on $\Gamma(\mathcal{F} \otimes \mathcal{L}_i^{n_i})$ for n_i large. Then tensoring $\mathcal{L}_i^{-n}$, we find a $\oplus L_i^{-n_i} \to \mathcal{F}$ surjective. □

Prop. (5.3.11). When X has the resolution property, $\mathcal{E}xt^\bullet(-, \mathcal{G})$ is an universal δ-functor for every Qco $\mathcal{G}$, this is because locally free sheaf is adapted to $\mathcal{E}xt^\bullet(-, \mathcal{G})$ by(4.4.8), so we can calculate $\mathcal{E}xt(\mathcal{F}, \mathcal{G})$ using a finite locally free resolution of $\mathcal{F}$.

4 Duality for Schemes

Serre Duality Theorem

Def. (5.4.1). Let X be a proper scheme of dimension n over a field k, then a **dualizing sheaf** for X is a coherent sheaf ω_X together with a trace map $H^n(X,\omega_X)\to k$ that for every coherent sheaf $\mathcal{F}$,

$$\mathrm{Hom}(\mathcal{F},\omega_X)\times H^n(X,\mathcal{F})\to H^n(X,\omega_X)\to k$$

is a perfect pairing. In other words, ω_X represents the functor $\mathcal{F}\to (H^n(X,\mathcal{F}))^\vee$.

Prop. (5.4.2). If X is proper over a field k, then there exists uniquely a dualizing sheaf.

Proof: □

Lemma (5.4.3). For $X=\mathbb{P}^n_k$, the canonical sheaf ω_X is the dualizing sheaf. Moreover, we even have a perfect pairing

$$\mathrm{Ext}^i(\mathcal{F},\omega_X)\times H^{n-i}(X,\mathcal{F})\to k$$

Proof: In this case, $\omega_X=\mathcal{O}_X(-n-1)$. For $i=0$, when $\mathcal{F}=\mathcal{O}_X(n)$, then this follows from (4.4.14)and(4.4.11). And X has the resolution property(5.3.8), so we can write $\mathcal{F}$ as a quotient of two finite direct sum of $\mathcal{O}(-n)$. then the long exact sequence gives us the result as H^{n+1} vanish.

For $i>0$, both side are universal δ-functors, so we show they are both coeffaceable. write $\mathcal{F}$ as a quotient of two finite direct sum of $\mathcal{O}(-n)$ for n large, then $\mathrm{Ext}^i(\mathcal{O}(-n),\omega)=H^i(X,\omega(n))=0$ for $i>0$. And $H^{n-i}(X,\mathcal{O}_X(-n))=0$ by(4.4.14). □

Cor. (5.4.4). If X is a closed subscheme of $\mathbb{P}^n_k$ of codimension r, then X has a dualizing sheaf $\omega_X=\mathcal{E}xt^r_P(i_*\mathcal{O}_X,\omega_P)$.

Proof: We only have to prove that $\mathrm{Hom}_X(\mathcal{F},\omega_X)\cong\mathrm{Ext}^r_P(i_*\mathcal{F},\omega_P)$, then the above proposition will give the desired result together with the fact pushforward commutes with sheaf cohomology.

For this, we choose an injective resolution $\mathcal{I}^\bullet$ of ω_X and let $\mathcal{J}^\bullet=\mathcal{H}om_P(\mathcal{O}_X,\mathcal{I}^\bullet)$. Then $\mathcal{J}^\bullet$ are injective $\mathcal{O}_X$-modules because $\mathrm{Hom}_X(\mathcal{F},\mathcal{H}om_P(\mathcal{O}_X,\mathcal{I}^\bullet))=\mathrm{Hom}_P(\mathcal{F},\mathcal{I}^\bullet)$. And by the lemma(5.4.5) below, $\mathcal{J}^\bullet$ is exact up to $r=\operatorname{codim}X$, so it splits and $\omega_X=\operatorname{Coker}\mathcal{J}^n$ hence $\mathrm{Hom}(\mathcal{F},\omega_X)=\mathrm{Ext}^r_P(\mathcal{F},\omega_P)$. □

Lemma (5.4.5). Let X be a closed subscheme of $\mathbb{P}^n_k$ of codimension r, then $\mathcal{E}xt^i(\mathcal{O}_X,\omega_P)=0$ for $i<r$.

Proof: Since $\mathcal{F}^i=\mathcal{E}xt^i(\mathcal{O}_X,\omega_P)=0$ is a coherent sheaf, it suffice to show that $\Gamma(P,\mathcal{F}^i(q))=0$ for q large enough. But this equals $\mathrm{Ext}^i_P(\mathcal{O}_X,\omega_P(q))$, which is the dual of $H^{n-i}(P,\mathcal{O}_X(-q))=H^{n-i}(X,\mathcal{O}_X(-q))$ which vanish by Grothendieck vanishing theorem. □

Prop. (5.4.6). Let X be projective of dimension n over a field k and ω_X be the dualizing sheaf, then For $\mathcal{F}$ coherent, there is a natural map

$$\mathrm{Ext}^i(\mathcal{F},\omega_X)\to (H^{n-i}(X,\mathcal{F}))^\vee$$

Then the following are equivalent:

- For any $\mathcal{F}$ locally free on X, $H^i(X,\mathcal{F}(-q))=0$ for $i<n$ and q large.
- $H^i(X,\mathcal{O}_X(-q))=0$ for $i<n$ and q large.

- This is an isomorphism of δ-functors.
- X is C.M. and equidimensional.

Proof: Notice the left side is an universal δ-functor in $\mathcal{F}$ by(5.3.11), so the map exist, and

$2 \to 3$: This implies that the right is also universal by(5.3.8).

$3 \to 1$: For $\mathcal{F}$ locally free,

$$H^i(X, \mathcal{F}(-q)) = (\mathrm{Ext}^{n-i}(\mathcal{F}(-q), \omega_X))^\vee = (H^{n-i}(X, \mathcal{F})^\vee \otimes \omega_X(q))^\vee$$

which is 0 for q large.

$4 \to 1$: Embed X in $P = \mathbb{P}_k^N$, for $\mathcal{F}$ locally free, since X is catenary, equidimensional is equivalent to $\dim \mathcal{F}_x = n$ for all closed pt x, and C.M. says depth $\mathcal{F}_x = n$. Thus by(5.11.17), $pd_{\mathcal{O}_{P,x}} \mathcal{F}_x = N - n$. Thus $\mathcal{E}xt_P^k(\mathcal{F}, -)$ vanish for $k > N - n$ checked on stalks.

Now $H^i(X, \mathcal{F}(-q))$ is dual to $\mathrm{Ext}_P^{N-i}(\mathcal{F}, \omega_P(q))$ by the proof of(5.4.4), which is isomorphic to $\Gamma(P, \mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P(q)))$ for q large by(4.4.22), so it vanish when $i < n$ by what we proved.

$1 \to 4$: The same as the proof of $4 \to 1$, then for $i < n$,

$$\Gamma(P, \mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P(q))) = 0 = \Gamma(P, \mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P)(q))$$

for q large, so $\mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P) = 0$ as it is coherent. Then the stalk is $\mathrm{Ext}_{\mathcal{O}_{P,x}}^{N-i}(\mathcal{O}_{X,x}, \mathcal{O}_{P,x})$, so $pd_{\mathcal{O}_{P,x}} \mathcal{F}_x \leq N - n$ by(5.11.18), so depth $\mathcal{O}_{X,x} \geq n$, we must have equality, thus X is C.M. and equidimensional, as it suffice to check at closed pts. $\square$

Cor. (5.4.7). If X is C.M and equidimensional over alg.closed field k, e.g. it is a regular projective variety, then for any locally free sheaf $\mathcal{F}$, there is an isomorphism:

$$H^i(X, \mathcal{F}) \cong (H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X))^\vee.$$

Prop. (5.4.8) (Enriques-Severi-Zariski). Let X be a normal projective scheme that every irreducible component has dimension≥ 2, then for any locally free sheaf $\mathcal{F}$ on X, $H^1(X, \mathcal{F}(-q)) = 0$.

Proof: Just notice that $\dim \mathcal{F}_x \geq 2$, and Serre criterion shows depth $\mathcal{F}_x \geq 2$, the rest is the same as $4 \to 1$ in the proof of(5.4.6). $\square$

Prop. (5.4.9). When X is a closed subscheme of $P = \mathbb{P}_k^n$ which is a local complete intersection of dimension r, then $\omega_X = \omega_P \otimes \wedge(\mathcal{I}/\mathcal{I}^2)^{-1}$, which is an invertible sheaf on X. Notice $\mathcal{I}/\mathcal{I}^2$ is locally free by(6.1.25).

In particular, when X is regular over an alg. closed field k, then ω_X is just the canonical sheaf(6.1.13).

Proof: Cf.[Harthorne P245]. The basic idea is to use the free resolution of Koszul complex for the stalk of $\mathcal{O}_X$ to calculate $\omega_X = \mathcal{E}xt^r(\mathcal{O}_X, \omega_P)$. It depends on the regular sequence, and the transition of $(\mathcal{I}/\mathcal{I}^2)^{-1}$ neutralize this. $\square$

Cor. (5.4.10). For a projective regular variety over an alg.closed field k, $H^n(X, \omega_X) = k$, and when X is a curve, $H^1(X, \mathcal{O}_X)$ and $H^0(X, \omega_X)$ are dual to each other, thus the arithmetic genus equals the geometric genus.

Cor. (5.4.11). Let X be a regular projective variety of dimension n over a alg.closed field k, $\Omega = \Omega_{X/k}$ is locally free by(2.5.4), thus by(2.3.16), $\Omega^{n-p} \cong (\Omega^p)^\vee \otimes \omega$. Thus

$$H^q(X, \Omega^p) \cong (H^{n-q}(X, \Omega^{n-p}))^\vee.$$

by(5.4.7).

Topological Sheaves

Prop. (5.4.12) (Global Verdier Duality). If $f : X \to Y$ is a map between locally compact space with finite dimension, then there exists a functor $f^! : D^+(SAb_Y) \to D^+(SAb_X)$ that

$$R\operatorname{Hom}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong R\operatorname{Hom}(\mathcal{F}, f^!\mathcal{G}^\bullet).$$

In particular, $f^!$ is right adjoint to $Rf_!$. Cf.[Gelfand P228].

There is also a local form of Verdier duality, which implies the global version by taking global section, Cf.[Cohomology of Sheaves Iversen P330].

Prop. (5.4.13). When $X \to Y$ is an inclusion of open subset, $f_!$ is just $j_!$ defined in(1.3.3) and $f^!$ is the restriction. When it is an inclusion of closed subset of locally compact spaces, it is the direct image f_* and $f^!$ is the $j^!$ previously defined in(1.3.3). They are not barely defined on $D^+(SAb)$ but on SAb.

Prop. (5.4.14). We consider the case where $f : X \to \mathrm{pt}$, and let $G = \mathbb{Z}$, denote $f^!(\mathbb{Z})$ by $\mathcal{D}_X^\bullet$, called the **dualizing complex**, then there is a duality:

$$R\operatorname{Hom}(R\Gamma_c(X, \mathcal{F}^\bullet), \mathbb{Z}) \cong R\operatorname{Hom}(\mathcal{F}^\bullet, \mathcal{D}_X^\bullet).$$

for $\mathcal{F}^\bullet \in D^+(SAb_X)$.

Prop. (5.4.15). When X is a n dimensional topological manifold with boundary, then $\mathcal{D}_X^\bullet = \omega_X[n]$, where the sheaf ω_X is defined by

$$\Gamma(U, \omega_X) = \operatorname{Hom}_{Ab}(H_c^n(U, \mathbb{Z}), \mathbb{Z}).$$

Cf.[Gelfand P234]. If we replace $\mathbb{Z}$ by a field k, then w_X is the sheaf of k-orientations of $\mathrm{int}(X)$, thus the constant sheaf when X is oriented or $\mathrm{char} k = 2$?.

In particular, place k in dimension i then we get an isomorphism

$$\operatorname{Hom}_k(H_c^i(X, \mathcal{F}), k) = \operatorname{Ext}^{n-i}(\mathcal{F}, \omega_X)$$

(because k is a field thus injective). Gelfand even gives an interpretation of this pair in [Gelfand P236].

And if $\mathcal{F} = \omega_X$ and X oriented or $\mathrm{char} k = 2$, we have $\operatorname{Ext}^{n-i}(k_X, k_X[n]) = H^{n-i}(X, k_X)$ using the adjointness of constant sheaf, so we get the Poincare duality:

$$H_c^i(X, k_X)^\vee \cong H^{n-i}(X, k_X).$$

Prop. (5.4.16). compact cohomology commute with colimits, Cf.[Cohomology of Sheaves Iversen P173].

5 Deformation Theory

Def. (5.5.1). Let X be a scheme algebraic over a field k and $\mathcal{F}$ is a coherent sheaf on X, then a **infinitesimal extension** of X by the sheaf $\mathcal{F}$ is a scheme X' over k that has a sheaf of ideals $\mathcal{I}$ that $\mathcal{I}^2 = 0$ and $(X', \mathcal{O}_{X'}/\mathcal{I}) \cong (X, \mathcal{O}_X)$, and moreover, $\mathcal{I}$ with the $\mathcal{O}_X$-structure is isomorphic to $\mathcal{F}$.

There is a trivial extension, that is $(X', \mathcal{O}_{X'}) \cong (X, \mathcal{O}_X \oplus \mathcal{F})$, where the multiplication is $(a, f)(a', f') = (aa', af' + a'f)$.

Def. (5.5.2). Let X be a scheme algebraic over a field k, an **infinitesimal deformation** of X is a scheme X' flat over $D = k[t]/(t^2)$ that $X' \otimes_D k = X$.

If Y is a closed subscheme of X, then we define the **infinitesimal deformation of Y in X** to be a closed subscheme $Y' \subset X \otimes_k D$ which is flat over D and $Y' \otimes_D k = Y$.

A scheme algebraic over a field k is called **rigid** if it has no infinitesimal deformations.

Prop. (5.5.3). Let X be a nonsingular variety over an alg.closed field k, infinitesimal deformation of X is the same as an infinitesimal extension of X by the sheaf $\mathcal{O}_X$. Thus we get the set of infinitesimal deformations of X is parametrized by $H^1(X, \mathcal{T}_X)$, by(5.5.5)below.

Proof: For an infinitesimal deformation, tensoring $\mathcal{O}_{X'}$ with the exact sequence $0 \to k \xrightarrow{t} D \to k \to 0$, we get(by flatness)

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_{X'} \to \mathcal{O}_X \to 0,$$

, and conversely, an extension is locally free(because it is f.g. so flat over D is equivalent to free). $\square$

Prop. (5.5.4). If X is an affine regular scheme algebraic over an alg.closed field k, then any extension by coherent sheave is trivial.

Proof: For any infinitesimal extension, the morphism $X \to X'$ is a closed immersion and surjection, so X' is also affine by(3.2.32), $= \operatorname{Spec} A'$. Now the rest follows from(1.2.21). $\square$

Cor. (5.5.5) (Infinitesimal Extension and Cohomology). Let X be a nonsingular variety over an alg.closed field k, then the set of infinitesimal extensions by a coherent sheaf $\mathcal{F}$ is parametrized by $H^1(X, \mathcal{F} \otimes \mathcal{T}_X)$.

If Y is a closed subscheme of X, then the set of infinitesimal deformation of Y in X is parametrized by $H^0(Y, \mathcal{N}_{Y/X})$.

Proof: By the proposition, we know that an infinitesimal extension s locally isomorphic to $(U, \mathcal{O}_X(U) \otimes \mathcal{F}(U))$, by a section $\mathcal{F}(U) \to \mathcal{O}_{X'}(U)$.

But there is a twist, because there can because different sections. But the different sections different at a $\operatorname{Hom}_{\mathcal{O}_X(U)}(\Omega_{\mathcal{O}_X(U)/k}, \mathcal{F}(U)) = (\mathcal{T} \otimes \mathcal{F})(U)$. These forms a Čech cocycle for $\mathcal{F} \otimes \mathcal{T}_X$, and the converse is also true. Finally, use the fact that X is separated so Čech and sheaf cohomology coincide.

For the subscheme, $\square$

6 Intersection Theory

[Hartshorne Ex2.6.2] might be useful.

Basics

Def. (5.6.1). The setup is a universally catenary scheme with a dimension function.

Def. (5.6.2). For X/S locally of f.t., the function

$$\delta(x) = \delta(f(x)) + tr.deg_{k(f(x))} k(x)$$

is a dimension function on X. Cf.[StackProject 02JW].

Prop. (5.6.3). For $f : X \to Y$between schemes integral and locally of f.t. over S, if $\dim_\delta X = \dim_\delta Y$, then either $f(X)$ not dominant or the function field extension is finite. (Because the generic stalk has tr.deg 0). The extension degree d is called the **degree** of f.

Pushforward and Pullback

Rational Equivalence

Chern Class

Proper Intersection

Chow Ring

Prop. (5.6.4) (Bezout). The Chow ring of $\mathbb{P}^n_k$ is isomorphic to $\mathbb{Z}[x]/(x^{n+1})$. The degree of an irreducible closed variety corresponds to the coefficient of it.

V.6 Varieties

Basic references are [StackProject] and [Hartshorne].

1 Varieties

Classical variety

Prop. (6.1.1). the underlying space of a scheme is sober, Cf.(1.12.19).

Prop. (6.1.2). For k alg.closed, the soberization functor t induce a fully faithful functor from $\mathrm{Var}(k) \to \mathrm{Sch}(k)$ that maps to quasi-projective integral schemes over k. It maps projective varieties to projective integral schemes. And this functor preserves fiber products ?.

Proof: We assign the irreducible closed subsets space $t(X)$ and show that this embeds X in $t(X)$, and for an affine variety $(V, \mathcal{O}_V)$, the regular function sheaf is isomorphic to the pullback sheaf on $t(V) = \mathrm{Spec}(A)$.

By definition $t(X)$ is quasi-projective, and for a closed subscheme of $\mathbb{P}^n_k$, the closed pt of any closed subscheme are dense so $t(V)$ is homeomorphic to X. And because they are both reduced, they are isomorphic. So it is essentially surjective.

It is fully faithful because the closed point are equivalent to $k(x) = k$ and is dense in a f.t scheme over k so it maps closed pt to closed pt. □

Prop. (6.1.3). The soberization of a classical variety X is regular at a closed point iff the local defining functions has rank $n - \dim X$.

Proof: Consider the space of closed point of X, they correspond to classical points because k is alg.closed. Let $a_p = (x_1 - a_1, \ldots, x_n - a_n)$ and b be the locally defining ideal. Then the differential defines an isomorphism of vector space $a_p/a_p^2 \cong k^n$, and the local ring at p is $m/m^2 \cong (a_p/b)/(a_p/b)^2 \cong a_p/(b + a_p^2)$. The rank of the defining functions is $b + a_p^2/a_p^2$. Counting dimension gives us the result. (Use (2.2.10) also). □

Abstract Variety

Def. (6.1.4) (Abstract Variety). An **abstract variety** is an integral separated scheme of finite type over an alg. closed field k. A variety is an abstract variety because quasi-projective is f.t. and separated(3.3.7). It is called **complete** if it is also proper.

A curve over k is an abstract variety of dimension 1. It is called **non-singular** iff all the local rings are regular local.

Cor. (6.1.5). An abstract variety is birational to an integral quasi-projective scheme. A complete variety is birational to an integral projective scheme by Chow's lemma(3.3.9)(3.3.3).

Prop. (6.1.6). By valuation criterion, for a complete variety, every valuation of the function fields of K/k dominate a unique point of X. So the points of X correspond to valuations of K/k (valuation ring is the maximal local ring).

Prop. (6.1.7) (Extension of Morphism). Let X, Y be schemes over S, X is Noetherian and Y is proper. If there is a morphism from an open subset U of X to Y, and there is a point x in the closure of U with the stalk being a valuation ring, then the morphism can be extended to an open set containing x.

Proof: Cf.[StackProject 0BX7]. □

Prop. (6.1.8) (Nagata's Theorem). Any abstract variety can be embedded as an open subset of a complete variety.

Prop. (6.1.9). The product of two varieties over k alg.closed is also a variety.

Proof: The only problem is integral. By (3.1.12), it suffice to prove the affine case, this follows from (5.1.13). □

Prop. (6.1.10). The following categories are equivalent.

- The category of varieties (curves) over k with dominant rational morphisms.
- The dual category of f.g. field extensions over k (of trans.dimension 1).

Proof: Cf.[StackProject 0BXN]. □

Canonical Sheaves

Prop. (6.1.11) (Non-singular and Conormal Shraf). Let X be a regular variety over alg.closed k, then an irreducible closed subscheme Y of X is regular iff $\Omega_{Y/k}$ is locally free and(2.5.3) is exact on the left.

In this case, $\mathcal{I}$ is locally generated by r elements and $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf of rank r on Y by(6.1.25).

Proof: Cf.[Hartshorne P178]. □

Def. (6.1.12). For a nonsingular variety X over an alg.closed field k and Y a nonsingular subvariety of codimension r, by(6.1.11), $\Omega_{X/k}$ and $\Omega_{Y/k}$ are locally free, so we define:

- The **canonical sheaf** $\omega_X = \wedge^n\Omega_{X/k}$ on X.
- The **tangent sheaf** $\mathcal{T}_X = (\Omega_{X/k})^{-1}$ on X.
- The **conormal sheaf** $\mathcal{I}/\mathcal{I}^2$ on Y.
- The **normal sheaf** $\mathcal{N}_{Y/X} = (\mathcal{I}/\mathcal{I}^2)^{-1}$ on Y.

Prop. (6.1.13) (Adjunction Formulas). For a nonsingular variety X over an alg.closed field k and Y a nonsingular subvariety of codimension r, from(6.1.11), $0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$.

Taking the highest exterior power(2.3.16), we get:

$$\omega_Y = \omega_X \otimes \wedge^r \mathcal{N}_{Y/X} = \omega_X \otimes (\wedge \mathcal{I}/\mathcal{I}^2)^{-1}$$

In particular, if $r = 1$ then Y is a divisor D in X, the canonical sheaf

$$\omega_Y \cong \omega_X \otimes \mathcal{L}(D) \otimes \mathcal{O}_Y, \quad \omega_{\mathbb{P}^n_k/k} = \mathcal{O}(-n-1)\text{(2.5.5)}.$$

because $\mathcal{I}_Y \cong \mathcal{L}(-Y)$ in this case so $\mathcal{I}_Y/\mathcal{I}_Y^2 = L(-Y) \otimes \mathcal{O}_Y$.

Taking dual, we get:

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X \otimes \mathcal{O}_Y \to \mathcal{N}_{Y/X} \to 0$$

Prop. (6.1.14). For a regular proper variety over a field k, the **geometric genus** p_g is defined as the rank of the global section of the invertible canonical sheaf $\omega_X = \wedge^n \Omega_{X/k}$. It is a birational invariance. With the same methods, we can prove the rank of global sections of any other functorially defined bundles of Ω_X is birational invariance, e.g. Hodge numbers.

Proof: For any rational map $U \to Y$, there is a subset $V \in U$ and a local isomorphism V and $f(V)$, that will define an isomorphism of global sections. Because a nonzero section of an invertible sheaf cannot vanish on a dense open set $f(V)$, the morphism of global sections is injective into $\Gamma(U, \mathcal{O}_U)$. Now we find a U that $\mathrm{codim}(X - U) > 1$, then we can use(5.11.7) to get $\Gamma(U) = \Gamma(X)$, then $p_g(X) \geq p_g(X')$, and the converse is also true. For this, we use valuation criterion of properness, then for any codimension 1 point, the stalk is a DVR, thus we find a $\mathrm{Spec}\, \mathcal{O}_p \to X'$, this extends to a nbhd of p because X' is of f.t.. □

Cor. (6.1.15). By the exact sequence(2.5.5) for $\mathbb{P}^n_k$ and(2.3.15), we have $\omega_{\mathbb{P}^n_k} \cong \mathcal{O}(-n-1)$, so it has no global section, $p_g(\mathbb{P}^n_k) = 0$. Hence every rational variety over a field k, i.e. one that is birational to $\mathbb{P}^n_k$, has geometric genus 0.

Projective Variety

Prop. (6.1.16) (Bertini). Any regular projective variety over k alg.closed with f.m singular pt has a hyperplane that intersect it with a regular variety. These hyperplanes form an open dense subset of the complete linear system $|H|$ of $\mathbb{P}^n_k$, Cf.[Hartshorne P179].

Cor. (6.1.17). When $\dim X \geq 2$, this is even a regular variety by(5.1.15)and(3.1.21).

Prop. (6.1.18) (Affine Dimension Theorem). For two affine variety Y, Z of dimension r, s in $\mathbb{A}^n_k$ over fields, there intersection has every component dim $\geq r + s - n$.

Proof: The theorem follows from Krull's theorem(5.9.9) when $Y = H$, and for the general case, notice $Y \cap Z \cong (Y \times Z) \cap \Delta$ in $\mathbb{A}^n \times \mathbb{A}^n$. □

Cor. (6.1.19) (Projective Dimension Theorem). For two projective variety Y, Z of codimension r, s in $\mathbb{P}^n_k$ over fields, there intersection has every component of codimension$\leq r + s$.

Proof: First prove this for $Y = H$, then we can either induct or directly from the theorem above. For this, we just use Krull's theorem(5.9.9). □

Degree of Projective Varieties

Basic References are [Hartshorne I.7].

Def. (6.1.20) (Hilbert Polynomial). For a scheme projective over a field k of dimension r, we define the **Hilbert polynomial** P_Y as the Hilbert polynomial of its homogenous coordinate ring $\Gamma_*(Y)$. It has dimension r by(5.8.1).

The **degree** of Y is defined as the $r!$ times the leading coefficients of P_Y.

Prop. (6.1.21).

- The degree is a positive integer.
- If $Y = Y_1 \cup Y_2$ and $\dim Y_1 \cap Y_2 < r$, then $\deg Y = \deg Y_1 + \deg Y_2$.

- If H is a hypersurface whose ideal is generated by a homogeneous polynomial of degree d, then $\deg H = d$.

Proof: Cf.[Hartshorne P52]. □

Prop. (6.1.22). For a variety of degree k and a general linear space, the intersection has k points.

Complete Intersection

Def. (6.1.23). A closed subscheme of a nonsingular variety over a field k of codimension r is called **locally complete intersection** iff Y is locally generated by $r = \operatorname{codim}(Y, X)$ elements. Because regular is C.M, Y is C.M by(5.10.12). In particular, by(6.1.11), a regular variety is always a locally complete intersection.

Def. (6.1.24). A variety Y of codimension r in $\mathbb{P}^n_k$ is a **strict complete intersection** iff $\mathcal{I}_Y$ can be generated by r elements. It is called a **set-theoretic complete intersection** iff it can be written as an intersection of r hypersurfaces.

Prop. (6.1.25). A local complete intersection has its ideal sheaf $\mathcal{I}$, then $\mathcal{I}/\mathcal{I}^2$ locally free by(5.10.11).

Prop. (6.1.26). If Y is a complete intersection in $\mathbb{P}^n_k$ of hypersurfaces of degree $d_1, \ldots, d_r$, then $\omega_Y = \mathcal{O}_Y(\sum d_i - n - 1)$.

Proof: Use the exact sequence $0 \to \mathcal{O}_Z(n-d) \to \mathcal{O}_Z \to i_*\mathcal{O}_Y \to 0$ and(6.1.13). □

Prop. (6.1.27). For a complete intersection of dimension q, $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$. And the natural map $\Gamma(P, \mathcal{O}_P(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is a surjection for every n. In particular, Y is connected, and the arithmetic genus $p_a(Y) = \dim H^q(Y, \mathcal{O}_Y)$.

Proof: We use induction, the case $Y = P$ follows from(4.4.14), let $Y = Z \cap H$, where H has degree d, then

$$0 \to \mathcal{O}_Z(n-d) \to \mathcal{O}_Z \to i_*\mathcal{O}_Y \to 0$$

thus use long exact sequence. The rest is easy. □

Cor. (6.1.28). If Y is a nonsingular hypersurface of degree d in $\mathbb{P}^n$, then $p_g(Y) = C^n_{d-1}$. If Y is a non-singular curve which is an intersection of two non-singular hypersurface of degree d, e in $\mathbb{P}^3_k$, then $p_g(Y) = \frac{1}{2}de(d+e-4)+1$.

Proof: Use the long exact sequence to reduce to $\mathbb{P}^n_k$. Cf.[Hartshorne Ex2.8.4]. □

2 Curves

Basic references are [Hartshorne Chap4] and [StackProject Algebraic Curves].

Prop. (6.2.1). A Noetherian separated scheme of dimension 1 has an ample invertible sheaf.

Proof: First reduce to the case when X is reduced. This is because(6.2.22) shows this invertible sheaf is a pullback of a sheaf of X and(2.4.15) shows this sheaf is ample.

Second we reduce to the case X is integral. Cf.[StackProject 09NX]. □

Cor. (6.2.2) (Complete Curve is Projective). A proper scheme of dimension 1 over a field k is H-projective, by(6.2.1).

Prop. (6.2.3). A separated scheme of f.t. of dimension≤ 1 over a field k is a H-projective scheme $\overline{X}$ called the **completion** of X minus f.m. closed pts. And when X is reduced, the stalk are discrete valuation rings at these closed pts. Cf.[StackProject 0BXV,0BXW].

Cor. (6.2.4). A morphism of varieties $X \to Y$ with X a curve and Y proper over a field k factors through the completion $\overline{X}$ of X by(6.1.7).

Prop. (6.2.5). A curve over a field k is either affine or H-projective. Cf.[StackProject 0A27].

Cor. (6.2.6). Two birationally equivalent complete curve are isomorphic. Thus if a curve is birationally equivalent to another complete curve, then it is an open immersion, by(6.2.3).

Prop. (6.2.7) (Non-constant Morphism Finite). Let $f : X \to Y$ be a morphism of schemes over a field k that Y is separated and X is proper of dimension ≤ 1. If the image of every irreducible component of X is not a pt, then f is finite.

Proof: Cf.[StackProject 0CCL]. □

Prop. (6.2.8). For an Noetherian integral scheme of dimension 1, there is an isomorphism $\mathcal{K}/\mathcal{O}_X \to \sum_p i_*(\mathcal{K}/\mathcal{O}_p)$.

Proof: Check on stalks, this is because closed subsets are finite. □

Nonsingular Curves

Lemma (6.2.9) (Extension of Morphism). Rational map from a non-singular curve to a proper variety can be extended to a morphism. This is a consequence of(6.1.7).

Prop. (6.2.10) (Category of Non-singular Complete Curves). The category of non-singular complete curves over a field k with non-constant morphisms is the opposite category of f.g. field extenision of k of trans.deg 1.

Proof: First a non-constant morphism maps the generic pt to the generic pt, thus inducing a map of function fields, and a map of there function fields induce a birational map by(6.1.10), and this extends to a morphism by(6.2.9).

It's left to show that any these fields is a function fields, for this, Cf.[StackProject 0BY1]. □

Cor. (6.2.11). It follows from this that two birational equivalent non-singular proper curve over a field is isomorphic.

Cor. (6.2.12). Comparing this and(6.1.10), we see that every curve over k correspond to a unique non-singular proper curve over k with the same function field, which is called the **non-singular projective model**.

Prop. (6.2.13) (Flatness, Nonsingular Curve and Associated Points). $f : X \to Y$ with Y integral and regular of dimension 1. Then f is flat iff every associated prime of X is mapped to the generic point of Y.

In particular when X is reduced, this is equivalent to every irreducible component of X dominants Y, by(5.6.13).

Proof: If x is mapped to a closed pt of Y, then $\mathcal{O}_{y,Y}$ is a DVR, let t be a uniformizer, then t is not a zero-divisor, and $f^\sharp(t) \in \mathfrak{m}_x$ is also not a zero-divisor. So x is not an associated point.

Conversely, to show f is flat, if y is the generic point, then $\mathcal{O}_{y,Y}$ is a field, so it is flat. When y is a closed pt, $\mathcal{O}_{y,Y}$ is a DVR, so by(6.2.8), we need to show that it is torsion free. If it is not, then $f^\sharp(t)$ must be a zero-divisor for a uniformizer t of $\mathcal{O}_{y,Y}$. But then it is contained in some associated prime p of $\mathcal{O}_{x,X}$(5.6.8). Now p is mapped to y, which is a contradiction. □

Cor. (6.2.14). If $f : X \to Y$ is a dominant morphism from a variety to a nonsingular curve over k, then f is flat.

Cor. (6.2.15). Let Y be integral and regular of dimension 1 and P a closed pt. X is a closed subscheme in $\mathbb{P}^n_{Y-P}$ that is flat over $Y - P$, then there is a unique closed subscheme $\overline{X}$ closed in $\mathbb{P}^n_Y$ that is flat over Y and restrict to X on $\mathbb{P}^n_{Y-P}$.

Proof: Choose the scheme-theoretic closure of X in $\mathbb{P}^n_Y$. Cf[Hartshorne P258]. □

Cor. (6.2.16). Combining this with(6.2.7), we say that a morphism between two non-singular curves are finite flat.

Prop. (6.2.17). A projective non-singular curve of degree d in $\mathbb{P}^n_k$, where $d \le n$ not contained in any $\mathbb{P}^{n-1}_k$ is isomorphic to the n-tuple embedding, and $d = n$.

This has easy generalization to surfaces and higher dimensions.

Proof: (5.1.14)shows $\mathcal{O}_X(1) \cong \mathcal{O}(d)$ over $\mathbb{P}^1_k$, and the restriction of global sections is injective. So the global section is an isomorphism, and it defines the embedding up to a linear automorphism. □

Cor. (6.2.18). For a projective regular curve over an alg. closed field k, the arithmetic genus=geometric genus= $\dim H^1(X, \mathcal{O}_X)$ by Serre duality(5.4.10).

Divisors on Curves

Prop. (6.2.19). For a finite morphism f between two non-singular curves over alg.closed field, e.g. dominant morphism between complete non-singular curves, $\deg f^*D = \deg f \cdot \deg D$. This is because f is finite locally free(6.2.16), thus this follows from [StackProject 02RH].

Prop. (6.2.20). An element $\notin k$ in the function fields of a projective non-singular curve over an alg.closed k defines a inclusion $k(f) \subset K(X)$ thus a morphism from X to P^1_k(6.1.10), and $(f) = \varphi^*(\{0\} - \{\infty\})$.

Cor. (6.2.21).

Prop. (6.2.22). If $Z \to X$ is a closed immersion and $\dim X \le 1$, then Pic $X \to$ Pic Z is a surjection.

Proof: Use the exact sequence $0 \to (1 + \mathcal{I})\mathcal{O}^*_X \to \mathcal{O}^*_X \to i_*\mathcal{O}^*_Z \to 0$, $\dim X \le 1$ and the Grothendieck vanishing theorem gives the desired result, also notice i is affine. □

Prop. (6.2.23). For a 1-dimensional integral scheme proper over k and a function $f \in K(X)^*$,

$$\sum_{x \text{ closed}} [k(x) : k]\mathrm{ord}_{\mathcal{O}_x}(f) = 0.$$

Cf.[StackProject 02RU].

Residues

Prop. (6.2.24) (Riemann-Roch). Let D be a divisor on a complete curve of genus g, then

$$l(D) - l(K - D) = \deg D + 1 - g.$$

Cf.[Hartshorne P295].

Cor. (6.2.25). Let $\deg(\mathcal{F}) = \chi(F) - r\chi(\mathcal{O}_X)$, where $\chi(F) = \sum(-1)^i \dim_k H^i(X, \mathcal{F})$ and $r = \dim_K F_\eta$, then $\deg \mathcal{L}(D) = \deg D$.

Prop. (6.2.26). a non-singular curve in $\mathbb{P}^2_k$ where char $k \neq 0$ is projectively isomorphic to $xy - z^2$ if it has a rational point. (Use Riemann-Roch to show that $O(p)$ has a nontrivial section which gives a isomorphism to P^1). And in fact the assertion can be checked directly.

Prop. (6.2.27). Let X be a complete regular curve over an alg.closed field k, K be the function field, then for any closed pt P, there is a unique k-linear map $res_P : \Omega_{K/k} \to k$ with the following properties:

- $res_P(\tau) = 0$ for $\tau \in \Omega_P$, where Ω_P is the stalk of the canonical sheaf at P.
- $res_P(f^n df) = 0$ for $f \in K$ and $n \neq -1$.
- $res_P(f^{-1}df) = v_P(f)$, where v_P is the valuation associated to P.

Prop. (6.2.28) (Residue Theorem). For every $\tau \in \Omega_{K/k}$, we have $\sum res_P \tau = 0$.

Picard Schemes of Curves

Basic References are [StackProject Chap43].

3 Surfaces

Prop. (6.3.1). Any birational transformation of non-singular surfaces will be factorized into f.m blowing-ups and blowing-downs of points.

V.7 Étale Cohomology

Basic references are [Étale Cohomology Fulei], [StackProject] and [Etale Cohomology Tamme].

1 Basics(Tamme Level Stuff)

Prop. (7.1.1). Considering the inclusion $\varepsilon : X_{Zar} \to X_{ét}$ of topologies, for any Abelian sheaf F on $X_{ét}$, there is a Leray spectral sequence(4.2.9)

$$E_2^{pq} = H^p_{Zar}(X,, R^q\varepsilon^s(F)) \Rightarrow H^{p+1}_{ét}(X,F).$$

Def. (7.1.2). Denote $\widetilde{X_{ét}}$ as the category of sheaves on $X_{ét}$. For a morphism of schemes $X \to Y$, there is a morphism of topologies $f_{ét} : Y_{ét} \to X_{ét}$, and we define

$$f_* = (f_{ét})^s : \widetilde{X_{ét}} \to \widetilde{Y_{ét}}, \quad f^* = (f_{ét})_s : \widetilde{Y_{ét}} \to \widetilde{X_{ét}}$$

f^* is called the inverse image, it is exact because $f_{ét}$ preserves fiber product and final object(1.2.7). So it is a morphisms of sites $X_{ét} \to Y_{ét}$. $f^*G(X')$ equals the colimit over all $X' \to Y' \times_Y X$, equivalently, all $X' \to Y'$ over Y, by definition(1.2.5).

Cor. (7.1.3). For a $f : X' \to X$ étale, f^* induce a morphism of topoi, that $F/X'(Z') = f^*F(Z') = F(Z')$, and $H^q(X_{ét}; Z', F) \cong H^q(X'_{ét}; X', F/X')$, by(4.2.12) and(1.2.12).

Prop. (7.1.4). For any Abelian sheaf on $X_{ét}$ and any étale scheme Y'/Y, there is a Leray spectral sequence(4.2.10):

$$E_2^p = H^p(Y', R^q f_*(F)) \Rightarrow H^{p+q}(Y' \times_Y X, F)$$

Prop. (7.1.5). If $f : X \to Y, Y \to Z$ is a morphism of schemes, then for any sheaf on $X_{ét}$, there is a Leray spectral sequence(4.2.9)

$$E_2^{pq} = R^p g_*(R^q f_*(F)) \Rightarrow R^{p+q}(gf)_*(F)$$

Prop. (7.1.6) (Commutes with Colimits). If X is qcqs, then by(1.4.8)(1.4.9) and(4.2.11), $H^q_{ét}(X, -)$ commutes with filtered colimits.

Field Case

Prop. (7.1.7). The functor $f : X' \to X'(k_s)$ is an equivalence of topologies from the small étale site $(\mathrm{Spec}(k))_{ét}$ to the canonnical topology T_G on the category of G-sets, where $G = G(k_s/k)$.

In particular, any Abelian sheaf on $(\mathrm{Spec}(k))_{ét}$ is representable.

Proof: First f maps a family of morphisms of schemes to a covering iff this family is a covering itself. This is because both are defined by set-theoretical surjectivity, and this is by(3.7.11).

Next we need to show this is an equivalence of categories. f has a left adjoint g because $X' \to \mathrm{Hom}_G(U, X'(k_s))$ is representable for any G-set U, because any G-set is equivalent to disjoint sums of G/H, and both category has arbitrary sums, so it suffice to prove for G/H, but this is represented by $\mathrm{Spec}\, k'$, where k' is the fixed field of H.

To prove $fg \cong \mathrm{id}$ and $gf \cong id$, they commutes with direct sums, so the first one is true because $G/H \to fg(G/H)) = \mathrm{Spec}(k_s)(k)$ is an isomorphism, and the second follows from(3.7.6) as all étale schemes over field k is a disjoint union of spectra of finite separable field extensions of k. □

Cor. (7.1.8). By(3.4.1), $F \to \varinjlim F(\mathrm{Spec}\, k_s)$ is an equivalence between the category of Abelian sheaves on $(\mathrm{Spec}\, k_s)_{ét}$ to the category of continuous G-modules. So

$$H^q_{ét}(\mathrm{Spec}\, k_s, F) \cong H^q(G, \varinjlim F(\mathrm{Spec}\, k_s)).$$

Artin-Schreier Theory and Kummer Theory

Strict Henselization

Locally Constant Sheaves

Prop. (7.1.9). $\mu_{n,X}$ is étale over X iff n is prime to the characteristic of all local residue fields of X. (Only unramifiedness is concerned, and it is fiberwise(6.6.6). And we can compute the Kahler differential of $k[T]/(T^n-1)$ vanish iff $n=0$ in k.

In this case, μ_n is locally isomorphic to $(\mathbb{Z}/n\mathbb{Z})_X$, because for any affine open $U = \operatorname{Spec} A$, $U' = \operatorname{Spec} A[t]/(t^n-1) \to U$ is étale and surjective(6.2.13) and U' has all n-th roots of unity.

2 Étale Fundamental Group

Basic references are [StackProject Chap53] and [Fulei Chap3].

Relative Cohomology

Curve Case

Comparison Theorems

Proper Base Change

Prop. (7.2.1). By Leray spectral sequence(4.2.9), there are edge morphisms $R^pg_*(f_*F) \to R^p(gf)_*(F)$ and $R^p(gf)_*(F) \to g_*(R^pf_*(F))$. So if there is a cartesian diagram $\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y \end{array}$, there is a morphism $F \to v'_*v'^*F$, and

$$R^pf_*F \to R^pf_*(v'_*v'^*F) \to R^p(fv')_*(v'^*F) = R^p(vf')_*(v^*F) \to v_*(R^pf_*(v'^*F)).$$

Hence by adjointness a morphism

$$v^*(R^pf_*(F)) \to R^pf'_*(v'^*F))$$

called the **base change morphism**.

Duality

Finiteness Theorems

l-adic Cohomology

V.8 Crystalline Cohomology

1 Algebraic deRham Cohomology

Prop. (8.1.1). The algebraic deRham cohomology for a ring R is defined by Kahler differential similar to the geometric deRham cohomology, and it has the same homotopy invariance as in(4.1.14).

Prop. (8.1.2). There is a similar construction of connections on a f.g. projective R-module M and Weil-Chern theory parallel to that of 8 and 3.

But in this case, the trace map is defined only when M is f.g. projective, which is called the **Hattoris-Stallings trace**: If A is f.g. projective, the natural map $\mathrm{Hom}_R(A,R)\otimes_R A\to\mathrm{End}_A(P)$ is an isomorphism (Because locally it is an isomorphism**??**, and the inverse composed with $\mathrm{Hom}_R(A,R)\otimes_R A\to A$, we get the desired map.

Also, when M is f.g. projective, there is a **Levi-Cevita connection** induced by the $A\to\Omega^1_{A/R}$ because M is a direct summand of some A^n. This is verified to be independent of n, or one can more algeoly use the fact that projective module is locally free.

The Chern character is important, it defines a ring map from $K_0(R)$ to $H^{ev}_{dR}(A)$. In fact, this can be lifted to a morphism $K_0(A)\to HC_0^{per}(A)\to H^{ev}_{dR}(A)$, Cf.[阳恩林 循环同调 Dennis trace].

V.9 Group Schemes

1 Group Schemes

Basic References are [StackProject Chap38].

Def. (9.1.1). A **group scheme** is a representable contravariant functor $G : Sch/S \to \mathcal{G}rps$.

In order for an object to represent a functor to $\mathcal{G}rps$ rather than $\mathcal{S}ets$, we suffice to have:

- multiplication: $m : G \times G \to G$.
- unit: $S \to G$.
- inverse: $G \to G$.

that satisfy the supposed identities.

A **open/closed subgroup scheme** of a group scheme G/S is an open/closed subscheme of G/S that the restriction of multiplication $m : G \times G \to G$ on $H \times H$ factors through H.

We call a group scheme **smooth/flat/separated/**... iff G/S is **smooth/flat/separated/**....

We have the left(right)translation for an elements in $G(R)$, equivalently, a natural transformation on G, and base change $(G \otimes_R R')(T'_{R'}) = G(T'_R)$

Remark (9.1.2). We do not need to verify all the relations, whenever we have a natural group structure on all the set $\mathrm{Hom}(T, G)$, we immediately recover the map $m : G \times G \to G$ as $pr_1 pr_2$ in $G(G \times G)$, $inv : G \to G$ as id^{-1} in $G(G).u : S \to G$ as 1 in $G(S)$.

Lemma (9.1.3). A bialgebra over a field k is direct limit of bialgebras of f.t. over k.

Prop. (9.1.4). Affine group schemes over a field is reduced. And it is smooth over k. Cf.[Jacob Stix P5].

Prop. (9.1.5). Common group schemes include

- $\mathbb{G}_a = \mathbb{Z}[T]$
- $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$
- $\mu_n = \mathbb{Z}[T]/(T^n - 1)$
- $\mathbb{GL}_n = \mathbb{Z}[T_{ij}][1/\det]$
- $D(g) = \operatorname{Spec} Z[g]$ for a commutative group g.
- The **constant group scheme** $g_{\mathbb{Z}} = \coprod_{\sigma \in g} \operatorname{Spec} \mathbb{Z}$, where m maps the component of (g, g') to the component of gg'. It represents the functor : $T \to$ the group of locally constant functions $T \to G$.

Def. (9.1.6) (Character Group Scheme). A **character** of a group scheme G is a homomorphism of group sheaves of Sch/S from G to $\mathbb{G}_m$, it is equivalent to a non-vanishing section χ of G that $m^*\chi = pr_1\chi \cdot pr_2\chi$ multiplication as sections. This is a subgroup of $\mathbb{G}_m(G)$.

A **character group scheme** of G is one that represent the functor $T \to \mathrm{Hom}_{Gr/T}(G_T, \mathbb{G}_{m,T})$. This will induce a compatible pairing $G(T) \times G'(T) \to \mathbb{G}_m(T)$, which gives a map $\mathbb{G}_{mS} \to G \times G'$.

Prop. (9.1.7) ($g_{\mathbb{Z}}$ and $D(g)$). $\Gamma(g_S, \mathcal{O}_{X_S})$ = group homomorphism from g to $\Gamma(S, \mathcal{O}_S)$. So we see that a character group scheme(9.1.6) of g_S is equivalent to a group homomorphism $g \to \mathbb{G}_m(S)$, equivalent to $D(g)(S)$. So $D(g)_S$ is the character group scheme of g_S.

Conversely, g_S is also the character group of $D(g)_S$, because the the composition gives a pairing

$$D(X)(T) \times X(T) \to \mathbb{G}_{mT}$$

This gives an isomorphism from g_S to the character of $D(g)_S$, Cf.[Tate Finite Flat Group Scheme].

Prop. (9.1.8).

Prop. (9.1.9). For a group scheme G/S, there is a Cartesian diagram:

$$\begin{tikzcd} G \arrow[r, "\Delta_{X/S}"] \arrow[d] & G \times_S G \arrow[d, "(g,g') \mapsto m(i(g),g')"] \\ S \arrow[r, "e"] & G \end{tikzcd}$$

This can be seen by a testing scheme T.

Cor. (9.1.10). G/S is (quasi-)separated iff e is qc(closed immersion).

Prop. (9.1.11). If G/S is a flat group scheme, T/S is a scheme and $\psi : T \to G$ is a morphism. Then $T \times_S G \to T \times_S G : (t, g) \mapsto m(\psi(t), g)$ is flat. In particular, m is flat.

Proof: Notice $T \times_S G \to T \times_S G : (t, g) \to (t, m(\psi(t), g))$ is an isomorphism, and the desired morphism is this composed with the projection, which is base change of $T \to S$, so it is flat. □

(Algebraic) Group Schemes over Fields

Prop. (9.1.12). Any group scheme over a field of char 0 is reduced.

Proof: Cf.[StackProject 047O]. □

Prop. (9.1.13) (Cartier's Theorem). A locally algebraic group scheme over a field of char 0 is smooth.

Proof: Cf.[StackProject 045X]. □

Prop. (9.1.14). For a locally algebraic group scheme over a perfect field, if it is reduce, then it is smooth.

Proof: Cf.[StackProject 047P]. □

Prop. (9.1.15). An algebraic group scheme over a field k is quasi-projective.

Proof: Cf.[StackProject 0BF7]. □

Prop. (9.1.16). For a locally algebraic group scheme G over a field k, its center is a closed subgp scheme of G.

Proof: Cf.[StackProject 0BF8]. □

2 Formal Groups

Basic References are [Cartier Theory of Commutative Formal Groups Zink]

Def. (9.2.1). A **formal group law** of dimension n over a commutative ring R is a set of n power series $G=(G_1,\ldots,G_n)$ in $K[[X_1,\ldots,X_n,Y_1,\ldots,Y_n]]$ that

$$G(X,0)=G(0,X)=X,\quad G(G(X,Y),Z)=G(X,G(Y,Z)).$$

Note this immediately induce an inverse $\mathrm{inv}(X)$ that $G(X,\mathrm{inv}(X))=G(\mathrm{inv}(X),X)=0$. This can be constructed noticing $G(X,Y)=X+Y+o(X,Y)$.

A morphism of formal groups is a vector of power series $\varphi(X)$ that $\varphi(G(X,Y))=H(\varphi(X),\varphi(Y))$.

A **formal R-module** is a formal group G over R together with a ring homomorphism $R\to\mathrm{End}_R(G)$ that $[a](X)=aX+\ldots$.

Prop. (9.2.2) (Automorphisms). If $\alpha\in R^*$ and F_i are power series that the degree 1 term of (F_i) is invertible, then there are unique power series G_i that $G\circ F=\mathrm{id}$ and $F\circ G=\mathrm{id}$.

Proof: Use induction to find G that $F\circ G=id$. Then the degree 1 terms of G is also invertible, thus there are $G\circ H=\mathrm{id}$, now $F=H$ and the proof is finished. □

Prop. (9.2.3). $\mathbb{G}_a$ is the one-dimensional formal group with $\mathbb{G}_a(X,Y)=X+Y$, $\mathbb{G}_m$ is the one-dimensional formal group with $\mathbb{G}_m(X,Y)=X+Y+XY$. Over a $\mathbb{Q}$-algebra K, there is an isomorphism between $\mathbb{G}_a$ and $\mathbb{G}_m$ giving by $X\to\exp(X)-1$.

Def. (9.2.4). A continuous K-linear mapping $D:K[[X]]\to K[[X]]$ is called a **differential operator** of degree N iff

$$L_D:K[[X,Z]]\to K[[X]]:\sum p_\alpha(X)Z^\alpha\to\sum p_\alpha(X)D(X^\alpha)$$

vanish on J^{N+1}, where $J=(X_i-Z_i)$.

It can be shown D is of degree N if $fD-Df$ is of degree $N-1$ for all f, Cf.[Cartier Theory of Commutative Formal Groups Zink P20].

Prop. (9.2.5). There is a representation $G(X+Y)=\sum D_\alpha g(X)Y^\alpha$, and every D_α is a differential operator of degree $|\alpha|$. And D_α forms a basis for the differential operators.

Prop. (9.2.6) ($\mathbb{Q}$-Theorem). Any commutative connected formal group over Λ a $\mathbb{Q}$-algebra is a direct sum of $\hat{\mathbb{G}}_a$, Cf.[Cartier Theory of Commutative Formal Groups Zink P19].

1-dimensional Formal Groups

Def. (9.2.7). For a 1-dimensional formal group $\mathcal{F}$ over R, the **invariant differential** is a differential form $\omega=P(T)dT\in R[[T]]dT$ that $\omega\circ F(T,S)=\omega$. It is called **normalized** if $P(0)=1$.

There exists uniquely an invariant differential, it is given by $F_X(0,T)^{-1}dT$.

Proof: We need to check $F_X(0,F(T,S))^{-1}F_X(T,S)=F_X(0,T)^{-1}$, and this is just $F(U,F(T,S))=F(F(U,T),S)$ differentiated at U and let $U=0$.

Conversely, if ω is an invariant differential, then $P(F(T,S))F_X(T,S)=P(T)$, let $T=0$, then $P(S)=P(0)F_X(0,S)^{-1}$. □

Prop. (9.2.8). For a morphism $f : \mathcal{F} \to \mathcal{G}$ of 1-dimensional formal groups over R, $\omega_{\mathcal{G}} \circ f = f'(0)\omega_{\mathcal{F}}$.

Proof: We only need to show that $\omega_{\mathcal{G}} \circ f$ is an invariant differential for $\mathcal{F}$ and then compare their constant coefficients. For this, notice

$$\omega_{\mathcal{G}} \circ f(F(T,S)) = \omega_{\mathcal{G}}(G(f(T), f(S))) = \omega_{\mathcal{G}}(f(T)) = \omega_{\mathcal{G}} \circ f(T).$$

□

Def. (9.2.9). When R has characteristic 0, the **formal logarithm** $\log_{\mathcal{F}}$ for a 1-dimensional formal group is the integration of invariant differential $\int_0^T \omega_{\mathcal{F}} = T + c_1/2T^2 + \cdots$.

Then the **formal power exponential** is the the unique power series $\exp_{\mathcal{F}}$ that is the inverse of $\log_{\mathcal{F}}$. It exists uniquely by(9.2.2).

Prop. (9.2.10). For R char= 0 and an 1 dimensional formal group $\mathcal{F}$ over R, $\log_{\mathcal{F}} : \mathcal{F} \to \mathbb{G}_a$ is an isomorphism of formal groups over $R \otimes_{\mathbb{Z}} \mathbb{Q}$.

And if $\mathcal{F}$ is a formal R-module, then it is an isomorphism of R-modules, because from(9.2.8) that $\omega_{\mathcal{F}} \circ [a] = a\omega_{\mathcal{F}}$, thus $\log_{\mathcal{F}} \circ [a] = a \cdot \log_{\mathcal{F}}$.

Proof: From $\omega_{\mathcal{F}}(F(T,S)) = \omega_{\mathcal{F}}(T)$, we get that $\log_{\mathcal{F}}(F(T,S)) = \log_{\mathcal{F}}(S) + \log_{\mathcal{F}}(T)$. So it is a homomorphism. Now the inverse $\exp_{\mathcal{F}}$ is already given, so it is an isomorphism. □

Cor. (9.2.11). A 1-dimensional formal group over a ring R that has no torsion nilpotents is commutative.

Proof: We only prove for R torsion free. $F(T,S) = \exp_{\mathcal{F}}(\log_{\mathcal{F}}(T) + \log_{\mathcal{F}}(S))$. □

Lubin-Tate Formal Group

Def. (9.2.12). For a p-adic number field K with a uniformizer π_K with residue field $\mathbb{F}_q$, a **Lubin-Tate power series** for π_K is a $\varphi(X) \in \mathcal{O}_K[[X]]$ that $\varphi(X) \equiv \pi_K X \mod X^2$ and $\varphi(X) \equiv X^q \mod \pi_K$.

A **Lubin-Tate module** G over $\mathcal{O}_K$ is a formal $\mathcal{O}_K$-module that $[\pi_K](X)$ is a Lubin-Tate power series.

Prop. (9.2.13). Given a p-adic number field K with residue field $\mathbb{F}_q$, we consider the set ξ_π of all Lubin-Tate power series for π.

If $f, g \in \xi_\pi$ and $L(X) = \sum a_i X_i$ be a linear form, then there exists a unique power series $F(X)$ that $F(X) \equiv L(X)$ mod degree 2 and $f(F(X)) = F(g(X_1), \ldots, g(X_n))$.

Proof: Choose F consecutively, if $F_{r+1} = F_r + \Delta_r$, then must

$$\Delta \equiv \frac{f(F_r(X)) - F_r(g(X))}{\pi^{r+1} - \pi} \text{ mod degree } (r+2).$$

This has coefficient in $\mathcal{O}$ because $f \equiv g \equiv Z^q \mod \pi$. □

Cor. (9.2.14). If we let $f = g, L = X + Y$ to get F_f and $f, g, L = aX$ to get $a_{f,g}$, then

- $F_f(X,Y) = F_f(Y,X)$.
- $F_f(F_f(X,Y), Z) = F_f(X, F_f(Y,Z))$.
- $a_{f,g}(F_g(X,Y)) = F_f(a_{f,g}(X), a_{f,g}(Y))$.

- $a_f b_f(Z) = (ab)_f(Z)$.
- $(a+b)_f(Z) = F_f(a_f(Z), b_f(Z))$.
- $\pi_f(Z) = f(Z)$.

all follow from the unicity of the last proposition.

Cor. (9.2.15) (Existence of Lubin-Tate Module). We get a commutative formal $\mathcal{O}$-module F_f for every f. And this group can act on $\mathfrak{p}_L$ for an alg.ext L/K. The set of zeros $\Lambda_{f,n}$ of f^n in L, as the elements annihilated by π^n, is a submodule of $\mathfrak{p}_L^{(f)}$.

And $u_{g,f}$ for any unit $u \in \mathcal{O}$ defines an isomorphism between F_f and F_g, thus this formal group only depends on π, called F_π. Hence $L_{f,n} = K(\Lambda_{f,n})$ only depends on π, with Galois group $G_{\pi,n}$.

Prop. (9.2.16) (Different Uniformizers). Now consider different π, it is proven that F_π and $F_{\pi'}$ are isomorphic, but the coefficient in $\mathcal{O}_{\hat{T}}$ where T is the maximal unramified extension.

Thus $L_{\pi,n}$ and $L_{\pi',n}$ may not be isomorphic, but $T\cdot L_{\pi,n} = T\cdot L_{\pi',n}$ since $\hat{T}\cdot L_{\pi,n} = \hat{T}\cdot L_{\pi',n}$ and both of them is the algebraic closure of K in it.

Proof: Cf.[Neukirch CFT P105]. □

Lemma (9.2.17). The Newton polygon of $[\pi_K^n]/\pi_K^n$ has vertices $(1,0),(q,-1/e_K),(q^2,-2/e_K),\ldots$.

Proof: Notice $[\pi_K^n]$ has no infinite edge of negative slope because all its coefficient are in $\mathcal{O}_K$. Now look at its roots, it has a root 0, and $q-1$ roots of valuation $v_p(\pi_K)/(q-1)$, $q(q-1)$ roots of valuation $v_p(\pi_K)/q(q-1)$, and so on. So by factor out these roots, $[\pi_K^n]/\pi_K^n$ is left with a power series whose Newton polygon is a single line, which shows the desired result. □

Prop. (9.2.18). The formal logarithm of the Lubin-Tate formal group F_π satisfies:

$$\log_{\mathcal{F}_\pi}(T) = \varinjlim [\pi_{\mathcal{F}}^n]/\pi_{\mathcal{F}}^n.$$

Proof: By(9.2.10) we have

$$\log_{\mathcal{F}}(T) = \log_{\mathcal{F}}([\pi_{\mathcal{F}}^n])/\pi_{\mathcal{F}}^n = ([\pi_K^n] + a_2/2[\pi_K^n]^2 + \ldots)/\pi_K^n$$

and for any degree n, the coefficient of $[\pi_K^{2n}]/\pi_K^{2n}$ is bounded below by a $c(n)$, so $[\pi_K^{2n}]/\pi_K^n$ converges to 0, thus the result. □

Cor. (9.2.19). The Newton polygon of $\log_{\mathcal{F}}(T)$ has vertices $(1,0),(q,-1/e_K),(q^2,-2/e_K),\ldots$.

The discussion is continued at1.

3 Finite Flat Group Schemes

Def. (9.3.1) (Cartier Duality). There is a Cartier duality on the category of finite flat affine commutative group schemes over $\operatorname{Spec} R$. This is because a finite flat module is locally free(3.4.5), thus $A^{\vee\vee} = A$ for a R-algebra A.

Prop. (9.3.2). When $G = \operatorname{Spec} A$ over R, $A^\vee$ represent the character group scheme of G. Cf.[Jakob P10].

Prop. (9.3.3). Frobenius and Relative Frobenius.

Prop. (9.3.4). If G is a finite flat commutative group scheme over R of constant order n, then multiplication by n kills the group. Cf.[Jakob P12].

4 p-divisible Groups

Def. (9.4.1). Let Λ be a local complete Noetherian ring and A_Λ^f be the category of finite length Artinian Λ-algebra,

Then a **Λ-formal functor** is a functor $A_\Lambda^f \to \mathcal{S}ets$.

The **formal completion** of a functor $A_\Lambda \to \mathcal{S}ets$ is its restriction on A_Λ^f. We denote the formal completion of $\operatorname{Spec} A$ by $\mathrm{Spf}A$.

Then a **Λ-formal scheme** is a filtered colimits of functors $\varinjlim \mathrm{Spf}A_i$, or equivalently a profinite Λ-algebra $A = \varprojlim A_i$ with profintie topology.

A **Λ-formal group** is a Λ-formal scheme with values in groups.

A **formal Lie group** over Λ is a connected formally smooth Λ-formal group. It is necessarily isomorphic to $\mathcal{G} = \mathrm{Spf}\Lambda[[X_1, \ldots, X_n]]$ where $n = \dim \mathcal{G}$.

A p **divisible formal Lie group** is a commutative formal Lie group $\mathcal{G} = \mathrm{Spf}\Lambda[[X_1, \ldots, X_n]]$ that multiplication by $p : [p]^*$ is a finite flat morphism on $\Lambda[[X_1, \ldots, X_n]]$.

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V.10 Complex Geometry

Basic References are [Voisin], [Principle of Algebraic Geometry Griffith/Harris], more advanced stuffs to add from [Kähler Geometry] and [Complex Geometry Daniel]. Even more advanced is [Demailly Complex Analytic and Differential Geometry].

1 Complex Manifold

Def. (10.1.1). A **complex manifold** is an even dimensional manifold that the transformation matrix is holomorphic.

An **analytic subvariety** is a closed subset of a complex manifold that is locally defined by f.m. holomorphic functions. The **regular points** of an analytic subvatiety locally defined by k functions is the points that $\mathrm{rank}(\frac{\partial(f_1,\ldots,f_k)}{\partial(z_1,\ldots,z_n)})=k$.

Prop. (10.1.2) (Adjunction Formula). The normal sheaf of a submanifold $Y\subset X$ is defined the same as the case of nonsingular varieties(6.1.13), then the same is true of the adjunction formula:

$$\mathcal{K}_Y\cong\mathcal{K}_X\otimes\det\mathcal{N}_{Y/X}$$

In case Y is of codimension 1, $\mathcal{N}_{Y/X}\cong\mathcal{L}(Y)|_Y=\mathcal{O}_Y(Y)$.

Prop. (10.1.3) (Remmert's Theorem). A non-compact manifold admits a proper holomorphic embedding into $\mathbb{C}^N$ for some N iff it is a Stein manifold.

Prop. (10.1.4) (Hirzebruch-Riemann-Roch). By(3.9.8), for a n-dimensional complex line bundle L over a compact Kähler manifold M,

$$\chi(M,L)=\int_M[\mathrm{ch}(E)\mathrm{td}(T^{1,0}M)]_n.$$

Where $\chi(M,L)=\sum_{q=0}^n(-1)^q\dim H^q(M,E)$, ch is the Chern character(3.8.5) and $\mathrm{td}(T^{1,0}M)$ is the Todd polynomial, i.e. Taylor expansion of $\prod_{i=1}^r\dfrac{t_i}{1-e^{-t_i}}$ in terms of the symmetric polynomial, applied to $c_i(T^{1,0}M)$.

Cor. (10.1.5) (Riemann-Roch). By(3.9.9), for a complex vector bundle E over a Riemann Surface M, let $\deg E=\int_M c_1(E)$, then

$$\chi(M,E)=H^0(M,E)-\dim H^1(M,E)=\deg E+\mathrm{rk}(E)(1-g).$$

Cor. (10.1.6). For other examples of corollaries of Hirzebruch-Riemann-Roch theorem, Cf.[Complex Geometry P232].

Analytic Subvarieties

Def. (10.1.7) (Analytic subvariety). An **analytic subvariety** of a complex manifold is a subset that is locally defined by f.m. holomorphic functions.

Prop. (10.1.8) (Proper Mapping Theorem). If U,M are complex manifolds and $M\subset U$ is an analytic subvariety, then if $f:U\to N$ is a holomorphic mapping whose restriction on M is proper, then $f(M)$ is an analytic subvariety of N.

Proof: Cf.[Griffith/harris P395]. □

Almost Complex Structure

Def. (10.1.9). For M a real orientable manifold of dimension $2n$, an **almost complex structure** is a bundle map $J : TM \to TM$ satisfying $J^2 = -1$.

A complex manifold has an almost complex structure, just define $J(\partial/\partial x_i) = \partial/\partial y_i$ and $J(\partial/\partial y_i) = -\partial/\partial x_i$.

Def. (10.1.10). J will define a bundle map on $T^*M \to T^*M$, and it has two eigenvalues $\pm i$, denoted by $T^{*1.0}M$ and $T^{*0,1}M$. The **formal differential forms** $\wedge^k T^*M \cong \sum \wedge^{p,k-p}T^*M$. ∂ is defined to be $\pi_{p+1,q} \circ d$ on $\wedge^{p,q}T^*M$, and $\overline{\partial}$ is defined to be $\pi_{p,q+1} \circ d$.

Def. (10.1.11) (Integrability). An almost complex structure is called **integrable** iff it satisfies the following equivalent conditions:

- $d\alpha = \partial\alpha + \overline{\partial}\alpha$.
- $d\alpha = \partial\alpha + \overline{\partial}\alpha$ is true for $\alpha \in \mathcal{A}^{1,0}(X)$
- $[T^{0,1}X, T^{0,1}X] \subset T^{0,1}X$.
- $\overline{\partial}^2 f = 0$ for functions f.

Proof: 1 $\iff$ 3 is because by(3.2.8), if $u, v \in T^{0,1}X$,

$$d\alpha(u, v) = u(\alpha(v)) - v(\alpha(u)) - \alpha([u, v]) = -\alpha([u, v]).$$

3 $\iff$ 4 is because by(3.2.8), if $\alpha = \overline{\partial} f$ and $u, v \in T^{0,1}X$, then

$$\overline{\partial}^2 f(u, v) = u(\overline{\partial} f(v)) - v(\overline{\partial} f(u)) - \overline{\partial} f([u, v]) = u(df(v)) - v(df(u)) - \overline{\partial} f([u, v]) = \partial f([u, v])$$

□

Prop. (10.1.12) (Nirenberg-Newlander). Given an almost complex manifold (M, J), it is integrable iff it comes from a complex structure.

Proof: Cf.[Foundation of Differential Geometry Kobayashi Chap9.2]. □

Blowing-up

Blowing-up serves as a way to magnify local properties to global ones.

Remark (10.1.13). Cf.[Complex Geometry P98] for blowing up along an arbitrary subvariety.

Def. (10.1.14) (Blowing-up along Point). For a nbhd U of 0 in $\mathbb{C}^n$, we can define the **blowing-up** $\pi : \widetilde{U} \to U$: $\widetilde{U}$ is the subset of $U \times \mathbb{CP}^n$ consisting of $(z, [l])$ that $z \subset [l]$. Then $\pi^{-1}(U - \{0\}) \cong U - \{0\}$ holomorphically.

For a complex manifold M and a point x, then choose a local coordinate centered at x, then we can form the blowing-up, because it is holomorphism away from x, so it can glue with the rest of M and form a knew manifold $\widetilde{M}$, called the **blowing-up** of M along x.

Notice this is independent of the coordinate chosen, because if $f(U)$ is a new coordinate of U, then $\pi'^{,-1} f\pi : \widetilde{U} - \{x\} \to \widetilde{U}' - \{x\}$ is a holomorphism, and it can be extended to $\widetilde{U} \to \widetilde{U}'$ by setting $f(x, [l]) = (x, [(\frac{\partial f_i}{\partial z_j})(0)l])$.

Prop. (10.1.15) (Exceptional Divisor). Let E be $\pi^{-1}(x)$ for a blowing-up, called the **exceptional divisor**. Often the line bundle $\mathcal{O}_{\widetilde{X}}(E)$ associated with it is called denoted by E.

There are canonical coordinates near E: let $\widetilde{U}_i$ be $\widetilde{U} - \{(l_i = 0)\}$, then endow $\widetilde{U}_i$ with the coordinate $z(i) = (l_j/l_i, \ldots, z_i, \ldots, l_n/l_i)$, it is holomorphic to $\mathbb{C}^n$. π in this coordinate is written as $(z(1), \ldots, z(n)) \mapsto (z(i)z(1), \ldots, z(i), \ldots, z(n)z(i))$.

The transition function can be written, it is

$$\varphi_j \circ \varphi_i^{-1}((z(i)_1, \ldots, z(i)_n)) = (\frac{z(i)_1}{z(i)_j}, \ldots, \frac{1}{z(i)_j}, \ldots, z(i)_i z(i)_j, \ldots, \frac{z(i)_n}{z(i)_j}).$$

Notice it is somewhat tricky because it has two different coordinates.

The defining function of E in this coordinate is $(z(i)) = (z_i)$. So the line bundle $\mathcal{O}_{\widetilde{X}}(E)$ has transition function $g_{ij} = z(i)/z(j)$, and it can be thought of as the line bundle that has line $[l]$ at the point $(z, [l]) \in \widetilde{U}$. So it is kindof tautological, in fact its restriction on $E \cong \mathbb{CP}^{n-1}$ is just the tautological line bundle.

Prop. (10.1.16). The canonical line bundle $\mathcal{K}_{\widetilde{X}} = \pi^*\mathcal{K}_X + (n-1)E$, where n is the dimension of X.

Proof: Away from E, the π is a holomorphism, so It suffices to compare the two transition function of the two canonical maps near E using the coordinates in(10.1.15), with the local section given by $dz_1 \wedge \ldots \wedge dz_n$ and $dz(i)_1 \wedge dz(i)_n$ respectively. On $\widetilde{U}_i$, locally $dz_1 \wedge \ldots \wedge dz_n$ is pulled by π^* to the trivial bundle on U', and by calculation, $dz(j)_1 \wedge dz(j)_n = z(i)_j^{n-1} dz(i)_1 \wedge dz(i)_n$, so $\mathcal{K}_{\widetilde{X}} - (n-2)E$ has a global section $z(i)_i^{n-1} dz(i)_1 \wedge dz(i)_n$, so it is also trivial on $\widetilde{U}$, so $\mathcal{K}_{\widetilde{X}} = \pi^*\mathcal{K}_X + (n-1)E$ is true. □

2 Deformation of Complex Structures

Cf.[Kähler Geometry] and [Complex Geometry Chap6], should be completed as soon as possible.

Calabi-Yau Manifolds

3 Coherent Sheaves and Analytic Spaces

Cf.[Demailly] and [GAGA Serre].

Def. (10.3.1). A subset U of $\mathbb{C}^n$ is called a **analytic** if it is locally defined by f.m. holomorphic functions. Hence it is locally closed in $\mathbb{C}^n$ and it is locally compact.

On an analytic space, there is a sheaf of holomorphic functions $\mathcal{H}_U$. So we can define **holomorphic map** φ as continuous functions that maps holomorphic germs to holomorphic germs, which is equivalent to the coordinates of φ are all holomorphic.

Def. (10.3.2). An **analytic space** is a Hausdorff space X with a structure sheaf $\mathcal{H}_X$ that is locally isomorphic to an analytic set. Morphisms are continuous maps that are locally holomorphic. Sub-analytic spaces are defined as usual.

An **analytic module** is just a module over the sheaf $\mathcal{H}_X$. For a sub-analytic space Y, we have a sheaf of ideals $\mathcal{A}_Y$ which is the sheaf of germs vanishing at Y, and $\mathcal{H}_X/\mathcal{A}_Y$ is a sheaf of X that is zero outside Y, and we identify it with $\mathcal{H}_Y$.

The products of analytic spaces can be defined, and it has the product topology, unlike the case of schemes.

Prop. (10.3.3). The structure sheaf of an analytic space is coherent, and the sheaf of ideals of a subanalytic space is coherent.

Proof: First prove for X is an open subset of $\mathbb{C}^n$, Cf.[GAGA Serre P4]. And by definition $\mathcal{A}_X$ is a $\mathcal{O}_X$-module of f.t., and it is also coherent?, so $\mathcal{H}_X$ is coherent. $\mathcal{A}_Y$ is coherent because it is a kernel of $\mathcal{H}_X \to \mathcal{H}_Y$. □

Prop. (10.3.4) (Analytification). For any algebraic variety over $\mathbb{C}$, any open affine subset is isomorphic to an analytic space of $\mathbb{C}^n$, hence can be given an analytic structure $X^{an} \to X$, this is because algebraic isomorphisms are analytic isomorphism, and X^{an} is Hausdorff because analytification preserves products and morphisms, and separability of X shows that $\Delta(X)$ is closed in $X \times X$, hence it is also closed in the analytification, hence X^{an} is Hausdorff.

X^{an} is locally compact and σ-compact, because X is qc hence covered by f.m. affine subsets hence second-countable and use(1.1.8). X^{an}/X is flat because completion of Noetherian rings are flat(5.8.13).

Remark (10.3.5). There is in fact a more general analytification for any scheme locally of finite type over $\mathbb{C}$. That is, we define it as the right adjoint to the forgetful functor from analytic spaces to local ringed spaces. Where an analytic space is a local ringed space that locally has immersions into $\mathbb{C}^n$. Should consult [Grothendieck EGA1-7].

Proof: Notice the schemes that have an analytification is stable under open subscheme, closed subscheme and products, and we can make a glue a large space from open subschemes by the unicity. So we only need to consider $\operatorname{Spec}\mathbb{C}[T]$, whose analytification is $\mathbb{C}$. □

4 Positive Current

5 Hermitian Vector Bundles

Def. (10.5.1). A **holomorphic vector bundle** is a vector bundle on a complex manifold that the transition function is holomorphic. A **Hermitian vector bundle** is a holomorphic vector bundle endowed with a Hermitian metric. Any holomorphic vector bundle has a Hermitian structure, by partition of unity method.

Prop. (10.5.2) (Hodge Star for Hermitian bundles). If E is a Hermitian vector bundle over a compact complex manifold of complex dimension n, we define a conjugate-linear operator $\overline{*} : A^{p,q}(X) \to A^{n-p,n-q}(X) : \eta \mapsto *\overline{\eta}$, and a conjugate-linear functor $\tau E \to E^*$ induced by the Hermitian metric on E.

Then we can define $\overline{*}_E : A^{p,q}(E) \to A^{n-p,n-q}E : \eta \otimes s \mapsto \overline{*}(\eta) \otimes \tau(s)$. It can be checked that

$$(\alpha, \beta) * 1 = \alpha \wedge *_E \beta,$$

$$\overline{\partial}_E^* = -\overline{*}_{E^*}\overline{\partial}_{E^*}\overline{*}_E, \quad \overline{*}_E \Delta_{\overline{\partial}_E} = \Delta_{\overline{\partial}_{E^*}} \overline{*}_{E^*}, \quad \overline{*}_{E^*}\overline{*}_E = (-1)^{p+1} \text{ on } \Omega^{p,q}(E).$$

Hermitian Metric

Def. (10.5.3). The complexified tangent bundle $T_{\mathbb{C}}M$ is defined as $TM \otimes_{\mathbb{R}} \mathbb{C}$, the **holomorphic tangent bundle** $T^{1,0}M$ and anti-holomorphic bundle $T^{0,1}M$ are defined to be the vectors generated resp. by $\partial/\partial z_i$ and $\partial/\partial \overline{z_i}$. The **holomorphic cotangent bundle** and anti-holomorphic cotangent bundle is defined to be the covectors generated by dz_i and $d\overline{z}_i$.

Def. (10.5.4). A metric on TM is called **Hermitian** iff it is J-invariant, that is $g(Ju, Jv) = g(u, v)$. If g is Hermitian, then it can be checked that $g(T^{0,1}, T^{0,1}) = 0 = g(T^{1,0}, T^{1,0})$. A metric extend by linearity to a bilinear form on $T_{\mathbb{C}}M$. And if we define $(Z, W) = g(Z, \overline{W})$, then it is a Hermitian metric on $T^{1,0}$(non-degenerate because g is, and g trivial on $T^{0,1}$ and $T^{1,0}$). Conversely, the same construction shows a Hermitian metric on $T^{1,0}$ is equivalent to a Hermitian metric on TM, that's what the name means.

Def. (10.5.5). Given a Hermitian metric g on TM, define the **Kahler form** ω_g as $\omega_g(u, v) = g(Ju, v)$. Then it is a real 2-form on M.

Notice $g(u, v) = \omega_g(u, Jv)$, so g can be constructed by ω_g, iff ω_g is positive(10.7.1).

Picard Group

Def. (10.5.6). The group of isomorphisms of holomorphic line bundles on a complex manifold X is denoted by $Pic_{\mathbb{C}}(X)$.

Prop. (10.5.7) (Picard Group). For a connected space X, there is an exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{f \to e^{1\pi i f}} \mathcal{O}_X^* \to 0$, and it induces a map $Pic_{\mathbb{C}}(X) = H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$, which is a just the first Chern class(same proof as in(5.4.3)).

WARNING: in this case it is not necessarily isomorphism, not as in the case of topological line bundles.

in particular, The image of the first Chern class is trivial in $H^2(X, O_X)$.

Def. (10.5.8). The dual of the universal line bundle on $\mathbb{CP}^n$ is called the **hyperplane line bundle**, denoted by H or $\mathcal{O}(1)$.

Prop. (10.5.9). $Pic_{\mathbb{C}}(\mathbb{CP}^n) \cong \mathbb{Z}$, with $\mathcal{O}(1)$ as a generator.

Proof: As $\mathbb{CP}^n$ is Kähler, use(10.6.29), then $H^{0,k}(X, \mathbb{C}) \cong H^k(X, \mathcal{O}_X) = H^k(X, \mathcal{K}_X \otimes \mathcal{O}(2)) = 0$ for $k \geq 1$ by Kodaira vanishing(10.7.8), and then $NS(X) = H^{1,1}(X) = H^2(X, \mathbb{Z}) = Z$ by Lefschetz $(1,1)$-form theorem(10.6.30). It remains to prove $c_1(\mathcal{O}(1))$ is the generator, for this, Cf.[Demailly P280]. $\square$

Prop. (10.5.10). Let S_d be the set of homogenous polynomials of degree d, then

$$H^0(\mathbb{CP}^n, \mathcal{O}(d)) = \begin{cases} S_d & d \geq 0 \\ 0 & d < 0 \end{cases}$$

Proof: This is because it is sections that satisfy $f_\alpha([z]) = (\frac{z_\beta}{z_\alpha})^k f_\beta([z])$, which says f_α glue together to give a holomorphic function homogenous of degree k on $\mathbb{C}^n - \{0\}$, which extends to a function on $\mathbb{C}^n$ by(2.4.2), then it is easy to see it is a homogenous polynomial using the power series expansion. $\square$

Def. (10.5.11) (Neron-Severi Group). For a compact complex manifold, the **Neron-Severi group** $NS(X)$ is the image of $Pic_{\mathbb{C}}(X) \to H^2(X, \mathbb{R})$. $rank_{\mathbb{R}}(NS(X))$ is called the **Picard number** of X.

There is a good description of $NS(X)$ in case X is Kahler, See Lefschetz theorem(10.6.30).

Chern Connection

Prop. (10.5.12) (Chern Connection). Given a Hermitian holomorphic bundle $E \to M$ on a complex manifold, there is a unique **Chern connection** ∇ on E, that ∇ is holomorphic(i.e. the connection matrix is holomorphic w.r.t a homomorphic frame), and it is compatible with the Hermitian metric.

Proof: Write out the requirement: if $H = h_{ij}$ is the matrix of the Hermitian metric, so H is Hermitian, and we need $dh_{ij} = (\nabla e_i, e_j) + (e_i, \nabla e_j) = \sum_k \omega_{ik} h_{kj} + \sum \overline{\omega_{jk}} h_{ik}.\omega$ is holomorphic, so must

$$\partial H = \theta H, \quad \overline{\partial} H = H \overline{\theta}^t.$$

But $H^t = \overline{H}$ so these two equations are equivalent and $\theta = \partial H H^{-1}$. □

Cor. (10.5.13). The curvature of the Chern connection is $\Omega = \overline{\partial}(\partial(h)h^{-1})$. In particular, it is a skew-symmetric matrix of $(1,1)$-forms. If it is of dimension 1, then $\Omega = \overline{\partial}\partial \log h$.

Proof: Ω is locally $d\omega + \omega \wedge \omega$, so if we choose a unitary basis, then ω is skew-symmetric by definition and $\omega \wedge \omega$ is also skew-symmetric, so Ω is skew-symmetric. The calculation is direct calculation. □

Prop. (10.5.14). The transformation matrix of a complex manifold is holomorphic, so it is possible to define globally $\overline{\partial}$ operator. And locally on a nbhd, ∂ is defined as $d - \overline{\partial}$.

Prop. (10.5.15) (Normal Coordinate). For a Hermitian vector bundle E over a complex manifold X, given any coordinate frame (z_j), there exists a holomorphic frame (e_λ) that

$$\langle e_\lambda, e_\mu \rangle = \delta_{\lambda,\mu} - \sum c_{jk\lambda\mu} z_j \overline{z}_k + O(|z|^3)$$

where $c_{ij\lambda\mu}$ is the coefficient of the Chern connection Ω. Such a coordinate is called the **normal coordinate frame** of E at x.

Proof: Cf.[Demailly P270]. □

Def. (10.5.16) (Dolbeault Cohomology). The **Dolbeault cohomology group** $H^{p,q}_{\overline{\partial}}(X, \mathcal{E})$ of a holomorphic vector bundle $\mathcal{E}$ over a complex manifold X is defined to be the q-th cohomology group of the complex

$$0 \to \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \Omega^{p,n-p} \to 0$$

and $H^{p,q}_{\overline{\partial}}$ is defined to be $H^{p,q}_{\overline{\partial}}(X, \mathbb{C}_X)$. By(4.5.13), $H^{p,q}_{\overline{\partial}}(X, \mathcal{E}) \cong H^q(M, \Omega^p_{hol} \otimes_{\mathcal{O}_X} \mathcal{E})$.

6 Kähler Geometry

Kähler Metric

Def. (10.6.1). The metric g is called **Kähler** iff ω_g is closed. In which case, it is called the **Kähler class** of g in $H^2_{dR}(M)$. A complex manifold with a Kähler metric is called a **Kähler manifold**.

If $g_{ij} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j})$, then $\omega_g = \sum_{ij} g_{ij} dz_i \wedge d\overline{z}_j$. Then the condition of ω_g being closed can in fact be written in derivatives of g.

Prop. (10.6.2). If g is Hermitian, then ω_g is real, non-degenerate and $\frac{1}{n!}\omega^n$ is a volume form on M. In particular, if ω is Kähler, then it is a symplectic form.

Proof: If $g = \sum \varphi_i \otimes \overline{\varphi}_i$, then $\omega = i \sum \varphi_i \wedge \overline{\varphi}_i$, so it is clear that $\overline{\omega} = \omega$. ω is non-degenerate as g is. The last assertion follows from(2.4.3). □

Cor. (10.6.3). If M is a compact Kähler manifold, then its even dimensional cohomology group doesn't vanish(6.1.6).

Remark (10.6.4). Notice there are notions like almost Hermitian and almost Kähler, similar to the definition of Hermitian and Kähler, but they are just defined using an almost complex structure on M. And a almost Kähler structure is Kähler iff $\nabla J = 0$, Cf.[Foundation of Differential Geometry Kobayashi].

Remark (10.6.5) (Examples of Kähler Manifolds).

- If $M = \mathbb{R}^{2n}$, $g = \sum dx_i \wedge dx_i + \sum dy_i \wedge dy_i$, then $\omega_g = \frac{i}{2} \sum dz_i \wedge d\overline{z}_i$ is Kähler.
- The metric $\omega_g = \sum dz_i \wedge dz_i$ on a complex tori $\mathbb{C}^n/\Lambda$ is Kähler.
- Any compact Riemann surface is Kähler, because $d\omega$ is a 3-form so vanish.
- if $M = B(0,1) \in \mathbb{C}^n$ and $\omega_g = i\partial\overline{\partial} \log \frac{1}{1-|z|^2}$, then it is Kähler.
- The product metric on the product space $M \times N$ of two Kähler manifold is Kähler.
- A submanifold of a Kähler manifold is Kähler, as the Kähler form is the pullback of the Kähler form of the large manifold.

Prop. (10.6.6) (Fubini-Study Metric). The **Fubini-Study metric** form on $\mathbb{CP}^n$ is defined locally to be $i\partial\overline{\partial}|s|^2$, for any local lifting of the projection $\mathbb{C}^n - \{0\} \to \mathbb{CP}^n$. This doesn't depend on the lifting, as $\partial\overline{\partial}(\log f + \log \overline{f}) = 0$, so they glue together to be a global form on $\mathbb{CP}^n$. It can be checked, ω is translation invariant and on the coordinate $(1, w_1, \ldots, w_n) \to (w_1, \ldots, w_n)$, $\omega|_{(0,\ldots,0)} = \sum dw_i \wedge dw_i$, so it is positive definite.

Cor. (10.6.7). Any projective manifold is Kähler.

Prop. (10.6.8). the Fubini-Study metric on $\mathbb{CP}^n$ has sectional curvature $1 \leq K \leq 4$.

Proof: Cf.[Do Carmo P188]. □

Prop. (10.6.9) (Kähler Normal Coordinate). For a Hermitian metric g on M, g is Kähler iff for any point p of M, there is a holomorphic coordinate centered at p, $\omega_g = \sum g_{ij} dz_i \wedge d\overline{z}_j$ satisfying $g_{ij}(p) = 0$ and $dg_{ij}(p) = 0$. This coordinate is called **Kähler normal coordinate**. (Notice this is different from Darboux theorem, because this coordinate should be holomorphic).

Proof: Cf.[Complex Geometry P210]. □

Curvature Tensor of Kähler Manifolds

Prop. (10.6.10). Let (M, J, g) be a Kähler manifold, then the complexification of the Levi-Civita connection of g restricts to the Chern connection on $T^{1,0}M$.

Proof: Cf.[Complex Geometry note 石亚龙 48] and [Complex geometry Daniel Chap4.A]. □

Prop. (10.6.11). For a Kähler manifold, $\nabla J = 0$.

Proof: The problem depends only on first derivative, so choosing a Kähler normal nbhd(10.6.9), we may choose J to be constant, so obviously $\nabla J(p) = 0$, P is arbitrary, so $\nabla J = 0$. □

Cor. (10.6.12). $\nabla(JX) = J\nabla X$, so $R(X,Y)JZ = JR(X,Y)Z$, thus

$$\langle R(JX,JY)Z,W\rangle = \langle R(Z,W)JX,JY\rangle = \langle R(Z,W)X,Y\rangle = \langle R(X,Y)Z,W\rangle,$$

so $R(JX,JY)Z = R(X,Y)Z$.

Prop. (10.6.13). The curvature tensor of the complexified Levi-Civita connection on a Kähler manifold can be calculated in terms of $\partial_i, \overline{\partial}_j$, Cf.[Complex Geometry note 石亚龙 50].

Kähler Identities

Let X be a compact complex Kähler manifold.

Def. (10.6.14). Introduce some operators:

- $d^c = i(\overline{\partial} - \partial)$, then $dd^c = 2i\partial\overline{\partial}$.
- The **Lefschetz operator** $L(\eta) = \omega \wedge \eta$. Λ is defined as the formal adjoint of L as $A^{p,q}$ is an inner space. In fact, $\Lambda = \pm * L *$.
- $h = (k-n)$ on $\mathcal{A}^k(X)$.

Prop. (10.6.15). $[L,\Lambda] = p+q-n$ on (p,q)-forms.

Proof: The problem doesn't depends on the derivatives, so using the Kähler normal coordinate(10.6.9), it suffice to prove for $\mathbb{C}^n$, for this, Cf.[Griffith/Harris P120] or [Complex Geometry P34]. □

Prop. (10.6.16) (Kähler Identities).

$$[\Lambda, \partial] = i\overline{\partial}^*, \quad [\Lambda, \overline{\partial}] = -i\partial^*.$$

Proof: The second one follows from the first because ω is a real form. For the first, notice only first derivation are involved, so by using the Kähler normal coordinate, it suffice to prove for $\mathbb{C}^n$, and this is by [Complex Geometry 石亚龙 P61]. □

Cor. (10.6.17).

$$[\Lambda, d^c] = d^*, \quad [\Lambda, d] = -d^{c*}.$$

Prop. (10.6.18). Δ_d commutes with both L and Λ.

Proof: L commutes with d because ω is closed, so taking adjoints, Λ commutes with d^*. Now by Kähler identities,

$$\Lambda\Delta_d = \Lambda(dd^* + d^*d) = -d^{c*}d^* + dd^*\Lambda - dd^{*c} + d^*d\Lambda = \Delta_d\Lambda.$$

So taking adjoints, Δ_d also commutes with L. □

Prop. (10.6.19). In the Kähler case, $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$.

Proof:

$$\Delta_d = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) = \Delta_\partial + \Delta_{\overline{\partial}} + (\partial\overline{\partial}^* + \overline{\partial}^*\partial) + (\overline{\partial}\partial^* + \partial^*\overline{\partial})$$

So it suffice to prove $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$(so $\overline{\partial}\partial^* + \partial^*\overline{\partial} = 0$ by conjugation), and $\Delta_\partial = \Delta_{\overline{\partial}}$. For the first, use Kähler identities, then

$$i(\partial\overline{\partial}^* + \overline{\partial}^*\partial) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = 0$$

For the second, using Kähler identities,

$$i\Delta_{\overline{\partial}} = \overline{\partial}[\Lambda, \partial] + [\Lambda, \partial]\overline{\partial} = \overline{\partial}\Lambda\partial + \partial\overline{\partial} - \Lambda\overline{\partial}\partial - \partial\Lambda\overline{\partial}$$

and the same is miraculous true for Δ_∂, so the result is true. □

Hodge Theory

Prop. (10.6.20) (Hodge Decomposition of compact Kähler Manifold). For a compact Kähler manifold X,

$$H^r_{dR}(X, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}_{\overline{\partial}}(X) \cong \bigoplus_{p+q=r} H^q(X, \Omega^p)$$

and $\overline{H^{p,q}_{\overline{\partial}}(X)} \cong H^{q,p}_{\overline{\partial}}(X)$. Moreover, this decomposition doesn't depends on the Kähler metric.

Proof: (10.6.19) shows that Δ_d maps $A^{p,q}$ to $A^{p,q}$, so $\mathcal{H}^{p+q}_d \cap A^{p,q} = H^{p,q}_{\overline{\partial}}(X)$. The last assertion is seen using the Δ_d definition.

If chosen two different Kähler metric g, g', there $\mathcal{H}^{p,q}(X, g) \cong H^{p,q}(X) \cong H^{p,q}(X, g')$. If α, α' be g, g' $\overline{\partial}$-harmonic respectively, so by definition $\alpha - \alpha' = \overline{\partial}\gamma$ for some γ, and they are both d-harmonic, so $d\overline{\partial}\gamma = 0$, and $\overline{\partial}\gamma$ is g-orthogonal to $\mathcal{H}^k(X, g)$ by Hodge decomposition for $\overline{\partial}$ with metric g, so by Hodge theorem for d with metric g, $\partial\gamma$ is d-exact, so $[\alpha] = [\alpha']$.

□

Cor. (10.6.21). Betti number $b_r = \sum_{p+q=r} h^{p,q}$, $h^{p,q} = h^{q,p}$. In particular, b_{2k+1} is always even.

Cor. (10.6.22) (Holomorphic Form on Kahler Manifold is Closed). $\mathcal{H}^{p,0}_{\overline{\partial}}(X) = H^0(X, \Omega^p)$. Now a $(p,0)$-form is automatically $\overline{\partial}^*$-closed, so it is $\overline{\partial}$-harmonic iff it is holomorphic. So we conclude any holomorphic p-form on a Kähler manifold is d-closed, even d-harmonic.

Lemma (10.6.23) ($\partial\overline{\partial}$-lemma). A closed differential form η on a compact Kähler manifold M is d-exact iff it is ∂-exact iff it is $\overline{\partial}$-exact iff it is $\partial\overline{\partial}$-exact.

Proof: Now $\Delta_d, \Delta_{\overline{\partial}}, \Delta_\partial$ are all the same, By Hodge theorem, it suffice to prove, if a form is orthogonal to $\mathcal{H}^{p,q}(X)$, then it is $\partial\overline{\partial}$-exact(this implies other exactness).

Noe η is d-closed hence ∂ and $\overline{\partial}$-closed, then $\eta = \partial\gamma$ for some γ, and then $\gamma = \overline{\partial}\beta + \overline{\partial}^*\beta' + \beta''$ for β'' harmonic. So $\eta = \partial\overline{\partial}\beta + \partial\overline{\partial}^*\beta'$, and then $\overline{\partial}\eta = \overline{\partial}\overline{\partial}^*\partial\beta = 0$, but then inner product with $\partial\beta$ shows $\overline{\partial}^*\partial\beta = 0$, so $\eta = \partial\overline{\partial}\beta$. □

Cor. (10.6.24) (Kodaira-Serre Duality). By(3.9.15), For a Hermitian line bundle over a compact Hermitian complex manifold X, from Hodge theorem and(10.5.2), we get

$$H^p(X, \Omega^q(E)) \cong H^{n-p}(X, \Omega^{n-q}(E^*))$$

induced by $\overline{*}_E$ and $\overline{*}_{E^*}$. Moreover, there is a perfect pairing

$$H^p(X, \Omega^q(E)) \times H^{n-p}(X, \Omega^{n-q}(E^*)) \to \mathbb{C}$$

induced by

$$\mathcal{H}^{p,q}(X,E)\times\mathcal{H}^{n-p,n-q}(X,E^*)\to\mathbb{C}:(\alpha,\beta)\mapsto\int_X\alpha\wedge\overline{*}_E\beta$$

In fact, $\int_X\alpha\wedge\overline{*}_E\alpha=||\alpha||^2\neq 0$.

Prop. (10.6.25). Holomorphic 1-forms on a compact complex surface is closed.?

Prop. (10.6.26) (Hard Lefschetz Theorem). For a compact Kähler manifold M, the map

$$L^k:H^{n-k}(M)\to H^{n+k}(M)$$

is an isomorphism,(notice it is defined because L commutes with d).

Define the **primitive cohomology class** $P^{n-k}(M)=\operatorname{Ker}L^{k+1}$ on H^{n-k}, then

$$H^m(M)=\oplus_k L^kP^{m-2k}(M).$$

Proof: Cf.[Griffith/Harris P122], using representation theory of $\mathfrak{sl}_2$. □

Prop. (10.6.27) (Hodge-Riemann Bilinear Relation). Let (X,ω) be a Kähler manifold, if $\alpha\neq 0\in H^{p,q}(X)$ is a primitive cohomology class, then

$$i^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}}\int_X\alpha\wedge\overline{\alpha}\wedge\omega^{n-p-q}>0$$

Proof: Cf.[Griffith/Harris] or [Complex Geometry Daniel P138]. □

Cor. (10.6.28). For a compact Kähler manifold of complex dimension $2m$,

$$\operatorname{sgn}(X)=\sum_{p,q=0}^m(-1)^p h^{p,q}(m)$$

Proof: Cf.[Complex Geometry Daniel P140]. □

Formality of Complex Kähler Geometry

Cf.[Complex Geometry Daniel Chap3.A].

Jacobian and Abanese Torus

Lemma (10.6.29). if X is compact Kahler, then the natural map $H^k(X,\mathbb{C})\to H^k(X,\mathcal{O}_X)$ is just the projection onto the $(0,k)$-part. In particular, the image is in $H^{0,k}(X)$.

Proof: By Hodge decomposition, the definition of Dolbeault cohomology and the commutative diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] & \mathcal{A}^0(X) \arrow[r, "d"] \arrow[d, "="] & \mathcal{A}^1(X) \arrow[r, "d"] \arrow[d, "\pi_{0,1}"] & \mathcal{A}^2(X)\dots \arrow[d, "\pi_{0,2}"] \\
\mathcal{O}_X \arrow[r] & \mathcal{A}^0(X) \arrow[r, "\overline{\partial}"] & \mathcal{A}^1(X) \arrow[r, "\overline{\partial}"] & \mathcal{A}^2(X)\dots
\end{tikzcd}$$

□

Prop. (10.6.30) (Lefschetz theorem on $(1,1)$**-forms).** By(10.5.7), we know that the image of $Pic_{\mathbb{C}}(X) \to H^2(X,\mathbb{Z})$ is trivial in $H^2(X,\mathcal{O}_X)$. If X is compact Kähler, there is Hodge decomposition(10.6.20) $H^2(X,\mathcal{O}_X) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$.

So if we define $H^{1,1}(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$, then the image of $Pic_{\mathbb{C}}(X)$ is contained in $H^{1,1}(X,\mathbb{Z})$ by(10.6.29), and it is also surjective, this is to say, $NS(X) = H^{1,1}(X)$

Proof: Because by the long exact sequence of(10.5.7) and(10.6.29) again, an $\alpha \in H^2(X,\mathbb{C})$ is in $H^{1,1}(X,\mathbb{Z})$ iff α is in the image of $Pic_{\mathbb{C}}(X) \to H^{1,1}(X,\mathbb{Z})$. □

Cor. (10.6.31). The image of $H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X)$ is a lattice. In particular, it is isomorphic to $\mathbb{Z}^{b_1(X)}$.

Proof: $H^1(X,\mathbb{Z})$ is a lattice in $H^1(X,\mathbb{R})$, and $H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O}_X) = H^{0,1}(X,\mathbb{C})$ is an isomorphism, because $H^{0,1}(X,\mathbb{C})$ are conjugate to $H^{1,0}(X,\mathbb{C})$ and $H^1(X,\mathbb{R})$ is real. □

Def. (10.6.32) (Jacobian). The **Jacobian** $\mathrm{Jac}(X)$ of a compact Kähler manifold X is defined to be $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$, so it is a complex torus of dimension $b_1(X)$ by(10.6.31), it is also the kernel of the first Chern class map by the long exact seqence(10.5.7), i.e.

$$0 \to \mathrm{Jac}(X) \to Pic(X) \xrightarrow{c} NS(X) \to 0$$

Def. (10.6.33) (Albanese). The **Albanese** $\mathrm{Alb}(X)$ of a compact Kähler manifold X is defined to be the complex torus $H^0(X,\Omega^1_X)^*/H_1(X,\mathbb{Z})$, where

$$H^1(X,\mathbb{Z}) \to H^0(X,\Omega_X)^* : [\gamma] \mapsto \Big(u \mapsto \int_\gamma u\Big)$$

(Notice this is well-defined because by(10.6.22) any $u \in H^0(X,\Omega^1_X)$ is closed).

Fix a base point x_0 of X, the **Albanese map** $\mathrm{Alb} : X \to \mathrm{Alb}(X)$ is defined to be

$$x \mapsto \Big(u \mapsto \int_{x_0}^x u\Big)$$

It is holomorphic and functorial in (X, x_0). It is just the so called **Abel-Jacobi map** in case when X is a Riemann surface.

7 Positive Vector Bundles & Vanishing Theorems

Positivity

Def. (10.7.1) (Positive Line Bundle). A 2-form ω on a Hermitian complex manifold M is called **positive** iff $\omega(u, Ju) \geq 0$ for $u \neq 0 \in TM$, which is equivalent to $-i\omega(v,\overline{v}) > 0$ for all $v \in T^{1,0}X$.

A holomorphic vector bundle is called **(Griffith-)positive** iff there exists a Hermitian metric on it that the curvature form Ω for the Chern connection(10.5.12) satisfies $h(\Omega(s),s)(v,\overline{v}) > 0$ for all $s \in E$ and $v \in T^{1,0}X$.

The pullback of a positive line bundle along an immersion is positive.

Prop. (10.7.2) (Positivity on Kähler Manifolds). On a compact Kähler manifold, being positive is a topological property for line bundles. It is equivalent to the first Chern class of L can be represented by a positive form in $H^2_{dR}(M)$.

Proof: $c_1(L) = [\frac{i}{2\pi}\Omega]$, so one direction is trivial, and if $c_1(L) = [\frac{i}{2\pi}\theta]$, choose an arbitrary Hermitian metric h on L, then by $\partial\overline{\partial}$-lemma(10.6.23), $\theta = \Omega + \overline{\partial}\partial\rho$ for some smooth function ρ. Then $e^{\rho}h$ has $\Omega = \theta$ by formula(10.5.13). □

Cor. (10.7.3). On a compact Kähler manifold, if L is positive, then for any other Hermitian line bundle L', $kL + L'$ is positive.

Prop. (10.7.4). The hyperplane line bundle $\mathcal{O}(1)$(10.5.8) is positive.

Proof: The hyperplane line bundle is dual to the tautological line bundle. The metric on the tautological line bundle is given by locally $g_i = \frac{1}{|z_i|^2}\sum|z_i|^2$. It is compatible with the transition map, and then by(10.5.13), the Chern curvature is

$$\overline{\partial}\partial(\frac{1}{|z_i|^2}\sum|z_i|^2) = \overline{\partial}\partial(\sum|z_i|^2).$$

So by(2.3.6)the curvature of the hyperplane line bundle times i is just the Fubini-Study metric form(10.6.6), so it is positive. □

Prop. (10.7.5). For $\widetilde{X} \to X$ the blowing-up of X at a point x, If L is a positive line bundle on X, then for any integer n, there exists a $k > 0$ that $\pi^*L^k - nE$ is a positive line bundle on $\widetilde{X}$, where E is the exceptional divisor.

Proof: Involves explicit metric calculation, Cf.[Kodaira Embedding Theorem P11] and [Complex Geometry P249].. □

Kodaira Vanishing Theorem and Applications

Prop. (10.7.6) (Nakano Identities). For a holomorphic vector bundle over a compact Kähler manifold (M, ω) with Hermitian metric h, introduce operators L and Λ as before. If we denote the $(1, 0)$ and $(0, 1)$-part of the Chern connection on E by D' and $D'' = \overline{\partial}$, then

$$[\Lambda, \overline{\partial}] = -iD'^*, \quad [\Lambda, D'] = i\overline{\partial}^*$$

Proof: The question is local, choose normal coordinate frame at x(10.5.15), then by the formula of Chern connection(10.5.13), $\nabla_E = d + A$, $A(x) = 0$, and $\nabla_{E^*} = d + B$, $B(x) = 0$. so

$$[\Lambda, \overline{\partial}_E] + iD'^* = [\Lambda, \partial] + i\partial^* + [\Lambda, A^{0,1}] + iB^{0,1}$$

where the usual Kähler identities(10.6.16) are used. Then it is zero when evaluated at x, Cf.[Demailly Complex Analytic and Differential Geometry P329]. □

Cor. (10.7.7) (Bochner-Kodaira-Nakano Identity).

$$\Delta_{\overline{\partial},E} - \Delta_{D',E} = i[\Omega, \Lambda]$$

Proof:

$$-i\Delta_{D',E} = D'[\Lambda, \overline{\partial}] + [\Lambda, \overline{\partial}]D' = D'\Lambda\overline{\partial} - D'\overline{\partial}\Lambda + \Lambda\overline{\partial}D' - \overline{\partial}\Lambda D'$$

and similar calculation for $i\Delta_{\overline{\partial},E}$, so

$$i\Delta_{\overline{\partial},E} - i\Delta_{D',E} = \Lambda(\overline{\partial}D' + D'\overline{\partial}) - (\overline{\partial}D' + D'\overline{\partial})\Lambda = -[\Omega, \Lambda].$$

□

Prop. (10.7.8) (Kodaira-Akizuki-Nakano Vanishing Theorem). If L is a positive line bundle on a compact Kähler manifold M, then

$$H^p(M, \Omega^q(L)) = 0$$

for $p+q > n$. In particular, $H^q(M, \mathcal{K}_X \otimes L) = 0$ for $q > 0$.

Proof: By Hodge theorem(3.9.13), it suffice to prove there are no harmonic (p,q)-forms$\in \mathcal{H}^{p,q}(X, L)$ on L.

As $i\Omega = \omega$ is positive, we may endow M with the metric ω, then by(10.7.7) and(10.6.15), $\Delta_{\overline{\partial}} - \Delta_{D'} = [L, \Lambda] = p+q-n$ on $A^{p,q}$.

So if $s \in \mathcal{H}^{p,q}(X, L)$, then $(\Delta_{\overline{\partial}} s - \Delta_{D'} s, s) = (p+q-n)||s||^2 \geq 0$, but $(\Delta_{\overline{\partial}} s - \Delta_{D'} s, s) = -(\Delta_{D'} s, s) = -||D' s||^2 - ||D'^* s||^2 \leq 0$, so $s = 0$. □

Cor. (10.7.9) (Serre's Theorem). Let L be a positive line bundle on a compact complex Kähler manifold X, then for any holomorphic vector bundle E, for m large, $H^q(X, L^m \otimes E) = 0$.

Proof: Same notation as in the proof of(10.7.8), choose Hermitian structure on E and L and their Chern connections by ∇_E, ∇_L, the corresponding Chern connection on $E \otimes L^m$ is denoted by ∇, and make sure $\frac{i}{2\pi} F_{\nabla_L}$ is the Kahler form ω, then for any harmonic form $\alpha \in \mathcal{H}^{p,q}(X, E \otimes L^m)$, by(10.7.7), $\frac{i}{2\pi}([\Lambda, F_\nabla](\alpha), \alpha) \geq 0$, but $\frac{i}{2\pi} F_\nabla = \frac{i}{2\pi} F_{\nabla_E} + m\omega$, so

$$0 \leq \frac{i}{2\pi}([\Lambda, F_{\nabla_E}](\alpha), \alpha) + m(n-p-q)||\alpha||^2$$

Notice $|([\Lambda, F_{\nabla_E}](\alpha), \alpha)|$ has a bound by Schwartz inequality, then if $p+q > n$ and m sufficiently large, α must by 0. In this case $\mathcal{H}^{p,q}(X, E \otimes L^m) = 0$, but $\mathcal{H}^{0,q}(X, \mathcal{K}_X \otimes E \otimes L^m) \subset \mathcal{H}^{n,q}(X, E \otimes L^m)$, so it is 0. Now we've proved $H^q(X, \mathcal{K}_X \otimes E \otimes L^m) = 0$ for any E if m is large. But E is arbitrary, so the conclusion is true. □

Cor. (10.7.10) (Grothendieck's Lemma). Every holomorphic line bundle E over $\mathbb{CP}^1$ is uniquely isomorphic to a finite direct sum of $\mathcal{O}(a_i)$.

Proof: If E has rank 1, this is the content of(10.5.9), so use induction on rank of E. Choose a maximal a that $\mathrm{Hom}(\mathcal{O}(a), E) = H^0(CP^1, E(-a)) \neq 0$. This a exists because Serre's Theorem(10.7.9) shows that $H^1(\mathbb{CP}^1, E(-a)) = 0$ for a sufficiently small, and Riemann-Roch(3.9.9) shows that $\chi(\mathbb{CP}^1, E(-a)) = \deg E + \mathrm{rk}(E)(1-a)$ is positive for a sufficiently small, so $H^0(CP^1, E(-a)) \neq 0$. Conversely, if a is sufficiently large, then $H^0(\mathbb{CP}^1, E(-a)) \cong H^1(\mathbb{CP}^1, E^*(a-2)) = 0$(Notice $\mathcal{K}_{\mathbb{CP}^n} = \mathcal{O}(-n-1)$).

So now there is an exact sequence of sheaves

$$0 \to \mathcal{O}(a) \xrightarrow{s} E \to E_1 \to 0$$

I claim E_1 is also a vector bundle, because s never vanishes, otherwise if it vanish at some x, then we can divide by a linear factor $s_x \in H^0(\mathbb{CP}^1, \mathcal{O}(1))$ to get a map $\mathcal{O}(a+1) \to E$, contradicting the maximality. So by induction $E_1 = \oplus \mathcal{O}(a_i)$, then I claim $a_i \leq a$, because otherwise $H^0(\mathbb{CP}^1, E_1(-a-1)) \neq 0$, and by the exact sequence $0 \to \mathcal{O}(-1) \to E(-a-1) \to E_1(-a-1) \to 0$, $H^0(\mathbb{CP}^1, E(-a-1)) \neq 0$, contradiction.

Then we want to show the above sequence splits, this is equivalent to

$$0 \to E_1^*(a) \to E^*(a) \to \mathcal{O} \to 0$$

splits, and his follows from the fact $H^1(\mathbb{CP}^1, E_1^*(a)) = H^1(\mathbb{CP}^1, \oplus \mathcal{O}(a-a_i)) = 0$, by Serre duality. So there is a section lifting $\mathcal{O} \to E^*(a)$, which splits the sequence. □

Prop. (10.7.11) (Weak Lefschetz Theorem). Let X be a compact Kähler manifold and Y be a submanifold that the line bundle $\mathcal{L}(Y)$ is positive, then the canonical restriction map $H^k(X,\mathbb{C}) \to H^k(Y,\mathbb{C})$ is isomorphism for $k \le n-2$ and injective for $k = n-1$.

Proof: In fact, using Hodge decomposition, it suffices to prove on the level of $H^q(X,\Omega^p_X)$. Tensoring the exact sequence
$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \rangle_* i_Y^* O_X \to 0$$
with Ω^p_X and taking the cohomology. By Serre duality and Kodaira vanishing(10.7.8), the map $H^q(X,\Omega^p_X) \to H^q(X,\Omega^p i_* i_Y^* O_X)$ is isomorphism for $p+q < n-1$ and injection for $p+q = n-1$.

Next consider the exact sequence $0 \to TY \to TX \to \mathcal{N}_{Y/X} \to 0$. By(2.3.17) there is an exact sequence
$$0 \to \wedge^p TY \to \wedge^p TX|_Y \to \wedge^{q-1} TY \mathcal{N}_{Y/X} \to 0$$
Taking dual and applying adjunction formula(10.1.2), it becomes:
$$0 \to \Omega_Y^{q-1} \otimes \mathcal{O}(-N) \to \Omega^q_X|Y \to \Omega^q_Y \to 0$$
Taking cohomology and use Serre duality and Kodaira vanishing as before, the result follows, and the composition is also true. □

Remark (10.7.12). There is a topological proof of weak Lefschetz theorem in [Bott On a theorem of Lefschetz].

Kodaira Embedding Theorem

Prop. (10.7.13) (Kodaira map). For a holomorphic line bundle L on a compact complex manifold M, if $s_0, \ldots, s_n$ be a basis of $H^0(X,L)$, we try to define a map from M to $\mathbb{CP}^n : x \to [s_0(x), \ldots, s_n(x)]$. This is independent of the change of coordinates because $g_{\alpha\beta}$ is invertible, and it is definable iff L is basepoint-free. This map is holomorphic where it is definable.

Def. (10.7.14). For a holomorphic vector bundle L on a compact complex manifold X, L is called
- **semi-ample** iff for m large, L^m is basepoint-free.
- **very ample** iff L is basepoint-free and the Kodaira map $\iota_L : X \to \mathbb{CP}^N$ is a holomorphic embedding.
- **ample** iff for m large, L^m is very ample.

Lemma (10.7.15) (Cohomological Method for Very Ampleness). For the above Kodaira map to be a holomorphic embedding, it suffice to show that the map is definable, injective and surjective on cotangent space. For these, it is equivalent to $H^0(X,L) \to L_x$ surjective, $H^0(X,L) \to L_x \oplus L_y$ surjective, and $L \otimes \mathcal{I}_x \to L_x \otimes T^{1,0*}(X)_x$ surjective. And they are true if
$$H^1(X, L\otimes \mathcal{I}_x) = 0, \quad H^1(X, L\otimes \mathcal{I}_{x,y}) = 0, \quad H^1(X, L\otimes \mathcal{I}_x^2) = 0.$$
respectively.

Proof: Basepoint-free at x is easily seen to be equivalent to $H^0(X,L) \to L_x$ surjective. And there is an exact sequence of sheaves:
$$0 \to L\otimes \mathcal{I}_x \to L \to L_x \to 0$$

where L_x means the skyscraper sheaf. So $H^1(X, L\otimes \mathcal{I}_x^2)=0$ induces the result.

Injective is easily seen to be equivalent to $H^0(X,L)\to L_x\oplus L_y$ surjective. And there is an exact sequence of sheaves:

$$0\to L\otimes \mathcal{I}_{x,y}\to L\to L_x\oplus L_y\to 0$$

where $\mathcal{I}_{x,y}$ is the sheave of functions vanishing at x and y, and $L_x\oplus L_y$ means the skyscraper sheaf. So $H^1(X,L\otimes\mathcal{I}_{x,y})=0$ induces the result.

For the surjection on cotangent spaces, given any point x, choose a basis $s_1,\ldots,s_n$ of sections in $H^0(X,L)$ vanishing at x, and by basepoint-free, there is a s_0 not vanishing at x, then on a coordinate, the Kodaira map is given by $x\to(s_1/s_0,\ldots,s_n/s_0)$, then it need to be checked $d_x(s_i/s_0)=d_x(x_i)/s_0$ span $T^{1,0*}(X)_x$. But there are exact sequences of sheaves:

$$0\to L\otimes\mathcal{I}_x^2\to L\otimes\mathcal{I}_x\xrightarrow{d_x} L_x\otimes T_x^{1,0*}\to 0$$

where d_x is given by $d_x(s\otimes f)=s(x)\otimes d_x(f)$(by the universal property of skyscraper sheaf, it suffice to give a map $(L\otimes\mathcal{I}_x\to L_x\otimes T_x^{1,0*})$, notice this is independent of the coordinate because $d_x(s_\alpha)=d_x(g_{\alpha\beta}s_\beta)=g_{\alpha\beta}d_x(s_\beta)$, as s_α vanishes at x, so this is truly a sheaf map, and its kernel is $L\otimes\mathcal{I}_x^2$. So $H^1(X,L\otimes\mathcal{I}_x^2)=0$ induces the result. □

Prop. (10.7.16). A holomorphic line bundle L on a compact Kähler manifold is ample iff it is positive.

Proof: If L is ample, then L^m is the pullback of the hyperplane bundle by the Kodaira map. The hyperplane line bundle is positive by(10.7.4), so L^m is positive with the induced metric, so L is also positive given the m-th roots of the induced metric (notice the metric of line bundle is just locally a number compatible with transition map).

Conversely, using(10.7.15), we want to find a L^k that $H^1(X,L^k\otimes\mathcal{I}_x)=0, H^1(X,L^k\otimes\mathcal{I}_{x,y})=0, H^1(X,L^k\otimes\mathcal{I}_x^2)=0$. First notice it suffice to prove for single points when k is sufficiently large, because the holomorphic embedding is an open property and X is compact so a sufficiently large k will suffice.

Consider the blowing-up $\widetilde{X}$ at a point x, there is a commutative diagram

$$\begin{tikzcd}
H^0(X,L^k) \arrow[r] \arrow[d, "\pi^*"] & L_x^k \arrow[d, "\cong"] \\
H^0(\widetilde{X},\pi^*L^k\otimes L_x^k) \arrow[r] & H^0(E,\mathcal{O}_E)\otimes L_x^k
\end{tikzcd}$$

The right vertical map is isomorphism as $E\cong\mathbb{CP}^n$, so $H^0(E,\mathcal{O}_E)=\mathbb{C}$. The left exact sequence is also isomorphism: it is injective because π is surjective, and it is surjective because: if $\dim X=1$, then $\pi=\mathrm{id}$ so trivially true, and if $\dim X\geq 2$, then because $\pi:\widetilde{X}-E\cong X-\{x\}$, any holomorphic function on $\widetilde{X}$ induces a holomorphic function on $X-\{x\}$ and by Hartog's theorem(2.4.2), it comes from a holomorphic function on X.

Now the second horizontal line is part of the cohomology exact sequence of(5.1.11)

$$0\to\pi^*L^k\otimes\mathcal{O}_{\widetilde{X}}(-E)\to\pi^*L^k\to\pi^*L^k|_E\to 0$$

So it is reduced to prove $H^1(\widetilde{X},\pi^*L^k\otimes\mathcal{O}(-E))=0$, but by(10.1.16), $\pi^*L^k-E=\pi^*L^k-E+\mathcal{K}_{\widetilde{X}}-\pi^*\mathcal{K}_X-(n-1)E=\mathcal{K}_{\widetilde{X}}+(\pi^*L^k-E)+\pi^*(L^k-\mathcal{K}_X)$, and by(10.7.5)(10.7.3) the last two are positive when k is large, so the conclusion follows from Kodaira vanishing(10.7.8).

The proof of $H^1(X, L^k \otimes \mathcal{I}_{x,y}) = 0$ is verbatim, just use blowing-up at two different points.

To prove $H^1(X, L^k \otimes \mathcal{I}_x^2) = 0$, consider the blowing-up $\widetilde{X}$ at a point x, notice there is a commutative diagram

$$\begin{tikzcd} H^0(X, L^k \otimes \mathcal{I}_x) \arrow[r, "d_x"] \arrow[d, "\pi^*"] & L_x^k \otimes T^{1,0*}X_x \arrow[d, "\cong"] \\ H^0(\widetilde{X}, \pi^* L^k - E) \arrow[r] & L_x^k \otimes H^0(E, -E) \end{tikzcd}$$

In fact this comes from the two commuting exact sequences twisted with $\pi^* L^k$:

$$\begin{tikzcd} 0 \arrow[r] & \pi^*\mathcal{I}_x^2 \arrow[r] \arrow[d] & \pi^*\mathcal{I}_x \arrow[r, "d_x"] \arrow[d] & \pi^* T^{1,0*}X_x \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}(-2E) \arrow[r] & \mathcal{O}(-E) \arrow[r] & \mathcal{O}_E(-E) \arrow[r] & 0 \end{tikzcd}$$

The second line is(5.1.11) and the fact a section vanishing at x lifts to a section vanishing at E thus equivalent to a section in the twisted sheaf $- \otimes \mathcal{O}(-E)$. These two exact sequences commutes because

Back to the commutative diagram, the above argument also shows that the first vertical map is isomorphism. To show the second vertical map is isomorphism, notice by(10.1.15) $\mathcal{O}(-E)$ is just the hyperplane line bundle on E, so $H^0(E, -E) \cong T^{1,0*}X_x$, we need to know the vertical map is the natural map $V^* \to H^0(\mathbb{P}(V), \mathcal{O}(1))$. This in fact need some careful calculation using coordinates in(10.1.15).**?**

Now the map d_x is surjective iff the second horizontal map is surjective, with is part of the cohomology exact sequence of

$$0 \to \pi^* L^k \otimes \mathcal{O}_{\widetilde{X}}(-2E) \to \pi^* L^k \mathcal{O}_{\widetilde{X}}(-E) \to \pi^* L^k|_E \to 0$$

So it is reduced to prove $H^1(\widetilde{X}, \pi^* L^k \otimes \mathcal{O}(-wE)) = 0$, which is by Kodaira vanishing theorem the same reason as before. □

Cor. (10.7.17) (Kodaira Embedding Theorem). If a compact complex manifold M has a positive line bundle, then it is projective.

Cor. (10.7.18). A compact Kähler manifold X is a projective submanifold iff it has a closed positive $(1,1)$-form ω whose cohomology class $[\omega]$ is rational(i.e. in $H^2(X, \mathbb{Q})$).

Proof: if ω is rational, then a multiple of it is integral, then there is a L that $c_1(L) = k[\omega]$ by Lefschetz theorem on $(1,1)$-forms(10.6.30), so L is positive by(10.7.2), so X is projective. Conversely, the Chern class of the pullback of the hyperplane line bundle is positive rational(10.7.4)(10.6.30). □

Cor. (10.7.19). if $\widetilde{X}$ is the blowing-up of a Kähler manifold X at a point x, then if X is projective, then $\widetilde{X}$ is also projective, because by(10.7.5) $\pi^* L^k - E$ is positive for k large.

Cor. (10.7.20). For a finite unbranched cover of compact Kähler manifolds $\widetilde{X} \to X$, $\widetilde{X}$ is projective iff X is projective.

Proof: A positive rational closed $(1,1)$-form on X pull backs to a positive rational closed $(1,1)$-form on $\widetilde{X}$, and it can even be pulled forward: $\omega' = \sum_{y\in\pi^{-1}(x)}(\pi^{-1})^*\omega(y)$, then it is also positive closed. It is rational because $\int_X \omega'\wedge\eta = \frac{1}{d}\int_{X'}\omega\wedge\pi^*\eta$, where $\widetilde{X}\to X$ is branched of degree d. □

Cor. (10.7.21). If X is projective, then the map $Div(X)\to Pic(X) : D\to\mathcal{L}(D)$ is surjective.

Proof: In fact, it suffice to show any line bundle E has a meromorphic section s, thus $L=\mathcal{L}(div(s))$. But X has a positive line bundle, so L^k+E and L^k are very ample thus clearly effective, with sections s_1 and s_2, so s_1/s_2 is a section of E. □

Cor. (10.7.22) (Riemann Bilinear Form). For a complex variety V/Λ, it is projective iff there is a **Riemann form** on V, Cf.[Complex Geometry Daniel P251].

Def. (10.7.23). For a Kähler manifold X, the **Kähler cone** K_X is defined to be the set of closed real positive $(1,1)$-forms. Then K_X is an open convex cone in $H^{1,1}(X)\cap H^2(X,\mathbb{R})$. Then(10.7.18) says X is projective iff $\mathcal{K}_X\cap H^2(X,\mathbb{Z})\neq 0$.

8 Riemann Surfaces

Basic references are [黎曼曲面 伍鸿熙].

Prop. (10.8.1). A compact Riemann surface is Kähler, so by Hodge decomposition(10.6.20),

$$H^1(M,\mathbb{C})\cong H^0(M,\Omega^1)\oplus\overline{H^0(M,\Omega^1)},$$

so the number of holomorphic 1-forms on M is equal to $b_1(M)/2$.

9 GAGA

Basic Reference are [GAGA Serre].

Remark (10.9.1) (GAGA Principle). Any global analytic object on an algebraic variety over $\mathbb{C}$ is algebraic.

Prop. (10.9.2) (Chow). Any analytic subvariety of $\mathbb{CP}^n$ is projective algebraic.

Proof: if Y is a hypersurface, then it is a prime divisor, so $\mathcal{L}(Y)$ is of the form $\mathcal{O}(d)$ for some $d>0$ by(10.5.9), and Y is a the divisor of a global holomorphic section of it, by by(10.5.10) this section is a homogenous polynomial of degree d, so Y is algebraic.

In general, if Y is of degree k, then it suffice to find for any $x\notin Y$, a homogenous polynomial that vanish on Y but not on x. The strategy is to find a $\mathbb{CP}^{k+1}$ to projection to it, and make sure $\pi(x)\notin\pi(Y)$ still, then use proper mapping theorem(10.1.8) to show $\pi(Y)$ is still an analytic subvariety, so it is defined by a polynomial, but the the inverse image is also defined by the same polynomial.? □

Prop. (10.9.3).

- Any meromorphic function on an algebraic variety $V\subset\mathbb{CP}^n$ is rational.
- Any meromorphic differential form on a smooth variety is algebraic.
- Any holomorphic map between smooth varieties can be given by rational maps.
- Any holomorphic vector bundle on a smooth variety is algebraic, i.e. transition function can be made rational.

Cf.[Griffith/Harris P168,170].

Complex Algebraic Varieties

Prop. (10.9.4). If X_t is an algebraic family of nonsingular projective varieties over $\mathbb{C}$ parametrized by a variety T, then the functions $h^i(X_t, \mathcal{O}_y)$ are constant for all i.

V.11 Algebraic Spaces and Stacks

Basic references are [Algebraic Spaces and Stacks Olsson], [Fibered Categories and Descent Theory Vistoli] and [Fibered Category to Algebraic Stacks Lamb].

Chapter VI

Higher Algebra

VI.1 Simplicial Homotopy Theory

References are [Jardine Simplicial Homotopy Theory].

1 Simplicial Category

Simplicial Set

Def. (1.1.1). The category of simplicial objects Δ consists of $[n]$ for each $n \geq 0$ and there maps are order-preserving maps. A **simplicial object** in A is a functor from $\Delta^{op} \to A$. A **cosimplicial object** in A is a functor from $\Delta \to A$. $\Delta[n]$ is the simplicial set $\Delta^n([m]) = \mathrm{Hom}([m],[n])$.

Prop. (1.1.2). The fact that any simplicial set X is a colimit of Δ^n (7.1.10) is important in proving properties of constructions of simplicial set.

Def. (1.1.3). The **nerve** of a category C is a simplicial category with $NC_n = \mathrm{Hom}([n], C)$, i.e. composable arrows of morphisms of length n. It is a fully faithful functor from the category of small categories to the category of simplicial sets.

Def. (1.1.4). The **geometrization** of a simplicial object is

$$|X| = \varinjlim_{\Delta[n]\to X} \Delta_n.$$

The **singular functor** maps a topological space X to a simplicial object $\mathrm{Sing}Y_n = \mathrm{Hom}(\Delta_n, Y)$. The geometrization functor is left adjoint to the singular functor (use colimit definition of X). This is just the Kan adjoint in(7.1.11).

Moreover, the geometrization as a functor from $\Delta_{Set} \to CGHaus$ preserves finite limits. Cf.[Jardine P9].

The three kinds of geometrization of a bisimplicial set is the same: geometrization the diagonal simplicial set, the twice geometrization of left(resp. right) simplicial set.

Def. (1.1.5). A morphism of simplicial set is called **Kan fibration** iff it has right lifting property w.r.t all $\Lambda^n_k \to \Delta^n$. So a morphism between topological spaces $X \to Y$ is a Serre fibration iff $S(X) \to S(Y)$ is a Kan fibration(1.1.4).

Def. (1.1.6). A **groupoid** is a category that every morphism is invertible. The nerve of a groupoid is a Kan fibration, because we only need to consider dimension< 3.

Prop. (1.1.7). A surjection of simplicial groups is a Kan fibration. In particular, simplicial abelian group and simplicial R-module are Kan complexes.

Proof: Cf.[Simplicial Homology Theory Jardine P12] □

Prop. (1.1.8). The bar resolution BG is a Kan fibration for every group G.

Prop. (1.1.9). A principal G fibration, i.e. $X \to X/G$ where X is a simplicial object of G-sets that G acts freely on X_n, is a Kan fibration.

Def. (1.1.10). A class of monomorphisms in Δ_{Set} is called **saturated** iff it contains all isomorphisms, closed under pushout, retraction, countable composition and arbitrary direct sum.

Def. (1.1.11). The saturated class generated by either of the following three class of monomorphisms is called **anodyne**:

1. $\Lambda^n_k \to \Delta[n]$, $0 \le k \le n$.
2. $(\Delta[1] \times \partial\Delta[n]) \cup (\{e\} \times \Delta[n]) \to \Delta[1] \times \Delta[n]$, $e = 0$ or 1.
3. $(\Delta[1] \times Y) \cup (\{e\} \times X) \to \Delta[1] \times X$, $e = 0$ or 1, for any $Y \subset X$.

Proof: 2 and 3 are equivalence because any inclusion comes from attaching cells(1.1.3). For 1 and 2, Cf.[Jardine P17]. □

Prop. (1.1.12). A natural transformation will induce homotopic nerve map. thus a pair of adjoint functors will induce a simplicial homotopy between their nerve.

Prop. (1.1.13) (Koszul Resolution). The Koszul Complete or the sequence r_i is the tensor complex $K[r; R] = K[r_1, R] \otimes_R \cdots \otimes_R K[r_n; R]$, where $K[x; R] = 0 \to R \xrightarrow{r} R \to 0$.Cf.[Weibel P111].

Prop. (1.1.14) (Chevalley-Eilenberg Resolution).

2 André-Quillen Cohomology

Basic references are [Andre-Quillen Cohomology of Commutative Algebras Iyenger]. See also [Quillen Cohomology of Commutative Rings] and [Quillen On the (Co-)homology of Commutative Rings].

Kahler Differentials

Def. (1.2.1). Let $S \to R$ a ring map, $\mathrm{Der}_S(R, M)$ is defined as the set of S-mod maps $R \to M$ that satisfies Leibniz rule and vanish on R. Then the **Kahler Differential** $\Omega_{R/S}$ is defined as a R-module that $\mathrm{Der}_S(R, M) \cong \mathrm{Hom}_S(\Omega_{R/S}, M)$. In particular, $\mathrm{Der}_S(R, R)$ is the R-dual of $\Omega_{R/S}$.

Prop. (1.2.2). One construction is by the free group generated by elements of R module some relations.

It can also be constructed as follows: there are two ring maps λ_i from S to $R \otimes_R S$, and one map ε from $S \otimes_R S$ to S. Let $I = \mathrm{Ker}\,\varepsilon$ as a R module by λ_1, then $I/I^2 \cong \Omega_{S/R}$ by (1.2.5) that

$$0 \to I/I^2 \to \Omega_{S\otimes_R S/S} \otimes_{S\otimes_R S} S \to \Omega_{S/S} \to 0.$$

So $I/I^2 \cong \Omega_{S/R} \otimes_S (S \otimes_R S) \otimes_{S\otimes_R S} S \cong \Omega_{S/R}$. And it can be verified that $a \otimes 1 - 1 \otimes a$ corresponds to da.

Cor. (1.2.3) (Functoriality). From the first construction, we can see directly that for a family of morphisms $R_i \to S_i$,

$$\Omega_{\operatorname{colim} S_i/\operatorname{colim} R_i} = \operatorname{colim} \Omega_{S_i/R_i}.$$

In particular, we have:

$$T^{-1}\Omega_{B/A} = \Omega_{T^{-1}B/A}, \quad \Omega_{S^{-1}B/S^{-1}A} = S^{-1}\Omega_{B/A}.$$

Moreover, we have

$$\Omega_{S/R} \otimes_R R' = \Omega_{S\otimes_R R'/R'}, \quad (S \otimes_R \Omega_{T/R}) \oplus (T \otimes_R \Omega_{S/R}) \cong \Omega_{S\otimes_R T/R}$$

by universal property.

Prop. (1.2.4) (Jacobi-Zariski Sequences). For a sequence of commutative rings:$A \to B \to C$, there is an exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules. It has a left inverse and splits iff any derivation B/A to a C-module can functorially be extended to a C/A derivation. This is true when C/B is smooth(6.5.5).

Proof: Taking Hom with an arbitrary C-module M, by universal property, we need to check the exactness of $0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$, which is easy. □

Prop. (1.2.5) (Second Exact Sequence). (This is a special case of(1.2.15)). If $S' = S/I$, then there is an exact sequence of R'-modules:

$$I/I^2 \to \Omega_{S/R} \otimes_S S' \to \Omega_{S'/R} \to 0.$$

Where $f \in I$ is mapped to $df \otimes 1$ and it has a left inverse and splits iff $S/I^2 \to S'$ has a right inverse.

And in fact $\Omega_{S/R} \otimes S' \cong \Omega_{(S/I^2)/R} \otimes S'$.

Proof: For a S/I-module M, we check:

$$0 \to \mathrm{Der}_R(S/I, M) \to \mathrm{Der}_R(S, M) \to \mathrm{Hom}_{S/I}(I/I^2, M)$$

To prove $\Omega_{S/R} \otimes S' \cong \Omega_{(S/I^2)/R} \otimes S'$, we apply Hom for a S'-module M.

So to prove the left exactness, we may assume $I^2 = 0$. If we have an inverse $\Omega_{S/R} \otimes_S S' \to I$. then it gives a derivation $D : A \to I$ that is identity on I, so $a - D(a)$ gives a R-ring map $S \to S$ that is trivial on I(because $I^2 = 0$). Hence it gives a $S/I \to S$ that is inverse to the projection.

For the converse, if $d : S/I \to S$ is a right inverse, then $a - d(\overline{a})$ is a derivation $S \to I$, which is identity on I, so it gives a inverse map $\Omega_{S/R} \otimes_S S' \to I$ by universal property. □

Cor. (1.2.6) (Examples).

- $\Omega_{A[X_1,\ldots,X_n]/A} = A[X_1, \ldots, X_n]\{dX_1, \ldots, dX_n\}$ (use the differential operator and universal property).
- If $S = A[X_i]/\{f_j\}$, then $\Omega_{S/A} = S[dX_i]/\{df_j\}$ by exact sequence2.
- $\Omega_{A[X_i]/k} = \Omega_{A/k} \otimes_A A[X_i] \oplus A[X_i]\{dX_1 \ldots, dX_n\}$ because any derivative of A/k can be extended to derivative of B/k by acting on the coefficients.
- (Standard Étale Algebra)For $A = R[x]_g/(f)$, where f' has image invertible in $A, \Omega_{A/R} = 0$.

- The differential for the inclusion $k[y^2, y^3] \to k[y]$ is $k[y]/(2y, 3y^2)\{dy\}$.

Cor. (1.2.7). If S/I is a field k that embeds in S, then $I/I^2 \cong \Omega_{S/k} \otimes_S k$.

Prop. (1.2.8). Let $k \subset K \subset L$ be fields, and L/K f.g., then

$$\dim_L \Omega_{L/k} \geq \dim_K \Omega_{K/k} + tr.\deg(L/K).$$

Equality holds if L/K is separably generated, i.e. separable over a transcendental basis. If $K = k$, then the equality hold iff L/k is separably generated. In particular, when L/k separable field extension, $\Omega_{L/k} = 0$, e.g. when k is perfect.

Proof: Consider extension by one element at a time, Cf.[Matsumura P190]. □

Prop. (1.2.9). Let B be a Noetherian local ring containing its residue field k and k is perfect, then $\Omega_{B/k}$ is a free B-module of rank $\dim B$ iff B is regular. [Hartshorne Ex.2.8.1] has a generalization of this fact.

Proof: One way is by(1.2.7). Conversely, if B is regular, then it is integral(5.11.13), so $\Omega_{B/k} \otimes_B K = \Omega_{K/k}$(1.2.3) is of K-dimension $tr.\deg K/k = \dim B$, where K is the quotient field of B, and $\Omega_{B/k} \otimes k \cong m/m^2$ is of k-dimension $\dim B$ once again. These two facts shows that $\Omega_{B/k}$ is free B-module of rank $\dim B$ by(6.14.1). □

Cotangent Complex

Def. (1.2.10). The **naive cotangent complex** of a ring map $R \to S$ is defined as the complex $NL_{S/R} = (I/I^2 \to \Omega_{R[S]/R} \otimes_{R[S]} S)$ as in(1.2.4), where I/I^2 is in degree 1 and $\Omega_{R[S]/R} \otimes_{R[S]} S$ in degree 0.

So it has homology $H^0 = \Omega_{S/R}$.

The naive cotangent complex is the canonical truncation of cotangent complex at degree 1.

For a ring map $R \to S$, if we choose another presentation $\alpha : P \to S \to 0$ where P is a polynomial algebra over R, then we denote $NL(\alpha) = NL_{P/S}$.

Def. (1.2.11). Definition of cotangent complex See[StackProject 08PN]. Should consult definition using model category.

Prop. (1.2.12). For a morphism of ring morphisms $(R \to S) \to (R' \to S')$, if there is a morphism of presentations, then we get by functoriality(1.2.3) a S-module morphism $\Omega_{P/R} \otimes_P S \to \Omega_{P'/R'} \otimes_{P'} S'$ and also a map $I/I^2 \to (I')/(I')^2$ for the kernel, so we get a morphism $NL(\alpha) \to NL(\alpha')$. The morphism constructed is compatible with composition.

In particular, we get a morphism $NL_{S/R} \to NL_{S'/R'}$.

Prop. (1.2.13). For a morphism of ring morphisms $(R \to S) \to (R' \to S')$, let α, α' be two presentations, then there exists morphism of presentations, and different morphisms induce homotopic maps $NL_{S/R} \to NL_{S'/R'}$.

Proof: Cf.[StackProject 00S1]. □

Cor. (1.2.14). If $A = R[X_i]$ be a polynomial algebras, then $NL_{A/R}$ is homotopic to $(0 \to \Omega_{B/A})$ because $A \to A$ is a presentation with zero kernel.

If $R \to A$ is surjective with kernel I, then $NL_{A/R}$ is homotopic to $(I/I^2 \to 0)$.

Prop. (1.2.15) (Jacobi-Zariski Sequence). Let $A \to B \to C$ be a ring map. Choose a presentation $\alpha : P \to B$ for B/A with kernel I, a presentation $\beta : Q \to C$ for C/B with kernel J, a presentation $\gamma : R \to C$ for the induced representation C/A with kernel K, then there is an exact sequence of complexes:

$$\begin{tikzcd} & I/I^2 \otimes_B C \arrow[r] \arrow[d] & K/K^2 \arrow[r] \arrow[d] & J/J^2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega_{P/A} \otimes_B C \arrow[r] & \Omega_{R/A} \otimes C \arrow[r] & \Omega_{Q/B} \otimes C \arrow[r] & 0 \end{tikzcd}$$

Applying snake lemma, we get

$$H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

Proof: Cf.[StackProject 00S2]. □

Prop. (1.2.16) (Flat Base Change). For a $R \to S$ and a flat R-ring R', for a presentation α of S/R, tensoring R' gives a presentation α' of S'. Then $NL(\alpha) \otimes_R R' = NL(\alpha) \otimes_S S' = NL(\alpha')$. This is because flatness implies kernel commutes with tensoring. In particular, $NL_{S/R} \otimes_R R' \to NL_{S'/R'}$ is a homotopy equivalence.

Prop. (1.2.17) (Colimit). NL commutes with colimit. (Because the kernel commutes with colimit).

Cor. (1.2.18) (Localization). Let $A \to B$ be a ring map, for a multiplicative set S of B, we have $NL_{B/A} \otimes_B S^{-1}B$ is quasi-isomorphic to $NL_{S^{-1}B/A}$.

Proof: Because it commutes with colimit, it suffice to prove for $S = f$, and this is the content of lemma(1.2.19) below. □

Lemma (1.2.19). If $A \to B$ is a ring map and $\alpha : P \to B$ is a presentation of B with kernel I, then $\beta : P[X] \to B_g : X \to 1/g$ is a presentation of B_g with kernel $J = I + (gX - 1)$. Then we have

- $J/J^2 = (I/I^2)_g \oplus B_g(fX - 1)$.
- $\Omega_{P[X]/A} \otimes_{P[X]} B_g = \Omega_{P/A} \otimes_P B_g \oplus B_g dX$.
- $NL(\beta) \cong NL(\alpha) \otimes_B B_g \oplus (B_g \xrightarrow{g} B_g)$.

Hence $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ is a homotopy equivalence.

Proof: Cf.[StackProject 08JZ]. □

<u>**Infinitesimal Deformation**</u>

Def. (1.2.20). An **infinitesimal deformation** of a f.g. k-algebra is defined as a algebra A' flat over $D = k[t]/(t^2)$ that $A' \otimes_D k = A$.

A f.g. k-algebra is called **rigid** if it has no infinitesimal deformations.

Lemma (1.2.21) (Infinitesimal Lifting Property). If A is a f.g. k-algebra that is regular, where k is alg.closed, then for any k-algebra homomorphism $0 \to I \to B' \to B \to 0$ that $I^2 = 0$, we can lift a map $A \to B$ to a map $A \to B'$.

Proof: Cf.[Hartshorne Ex2.8.6]. □

Cor. (1.2.22). Let A be a f.g. k-algebra, write A as a quotient of a polynomial ring over k with kernel J, then we get an exact sequence $J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0$ by(1.2.4), then we apply $\mathrm{Hom}_A(-, A)$ and let $T^1(A) = \mathrm{Coker}(\mathrm{Hom}_A(\Omega_{P/k} \otimes_A A, A) \to \mathrm{Hom}_A(J/J^2, A))$. Then $T^1(A)$ parametrize infinitesimal deformations of A.

Proof:

□

3 Cyclic Homology Theory(欧阳恩林)

Combinatorial Category

Def. (1.3.1). The **Segal category** Fin_* is the category of pointed finite sets. A morphism is called **inert** iff $|f^{-1}(\{i\})| = 1$ for all $i \neq *$. It is called **active** iff $f^{-1}(\{*\}) = \{*\}$.

A morphism can be uniquely factorized as a composition gh, where h is inert and g is active.

Prop. (1.3.2). There is a morphism $\mathrm{Cut} : \Delta^{op} \to \mathrm{Fin}_*$ where we interpret $[n] \in \mathrm{Fin}_*$ as the set of cut in $[n]$, and

$$\mathrm{Cut}(\alpha)(i) = \begin{cases} j & \text{if there are } j \text{ s.t. } \alpha(j-1) < i \leq \alpha(j) \\ 0 & \text{otherwise} \end{cases}$$

Prop. (1.3.3). The category of functors from the $E_\infty = \mathrm{Fin}_*$ to $\mathcal{C}at$ that

$$X([n]) \xrightarrow{\prod_{i=2}^n X(\rho^i)} \prod_{i=1}^n X([1]) \quad n \geq 0$$

and $X([0])$ is the final object, is equivalent to the category of symmetric unital monoidal categories with base category $(X([1]))$. (Because the commutativity of morphisms encodes the fact that the tensor action is symmetric).

Similarly, the category of functors from the Δ^{op} to $\mathcal{C}at$ that

$$X([n]) \xrightarrow{\prod_{i=2}^n X(\rho^i)} \prod_{i=1}^n X([1]) \quad n \geq 0$$

is equivalent to the category of symmetric unital monoidal categories $(X([1]))$. And it is symmetric iff it factors through Cut:$\Delta^{op} \to Fin_*$.

Def. (1.3.4). The **Conne cyclic category** Δ_C is a category containing Δ that $\mathrm{Aut}_{\Delta C}([n])$ is C_{n+1}. And every morphism $[n] \to [m]$ in Δ_C can be uniquely written as the form φg, where $\varphi \in \mathrm{Hom}_\Delta([n], [m])$ and $g \in \mathrm{Aut}_{\Delta_C}([n])$.

Δ_C^{op} is isomorphic to Δ_C Cf.[杨恩林循环同调 P31], thus Δ and Δ^{op} are all subcategories of Δ_C.

Def. (1.3.5). The category Δ_S is the category that $\mathrm{Aut}_{\Delta_S}([n]) \cong S^n$ and every morphism $[n] \to [m]$ in Δ_S can be uniquely written as the form φg, where $\varphi \in \mathrm{Hom}_\Delta([n], [m])$ and $g \in \mathrm{Aut}_{\Delta_S}([n])$.

Def. (1.3.6). For a category C, a **cyclic object** in C is a functor $\Delta_C^{op} \to C$.

For example, the functor that maps $[n]$ to C_{n+1} and the functor maps to the pull back of the order of the cyclic, is a cyclic object.

Simplicial Homology

Def. (1.3.7) (Moore Complex). Giving a simplicial object in an Abelian category, we can have a **Moore chain complex** with Čech-like differenticals. $\partial_n = \sum_1^n (-1)^i d_i$. And we have $\partial^2 = 0$.

Proof: Should use $d_i d_j = d_{j-1} d_i$ for $i < j$. □

Def. (1.3.8). The **normalization** of a Simplicial Abelian group M is the chain complex

$$NM : \cdots \to NM_n \xrightarrow{(-1)^n d_n} NM_{n-1} \to \cdots$$

where $NM_n = \bigcap_{i=0}^{n-1} \mathrm{Ker}(d_i) \in M_n$. This is a chain complex because $d_{n-1}d_n = d_{n-1}d_{n-1}$ is 0 on NM_n. In fact NM is preserved by all injections.

The **degenerate complex** of a Moore complex DM is the chain complex that $D_n = \sum_{i=0}^{n-1} s_i M_{n-1}$ is a sub chain complex of M by the relation of d_i, s_j.

Prop. (1.3.9). The simplicial homology of the Moore complex of the bar resolution BG of group homology with coeeficient in R is just the group homology $H_n(G, R)$ for the trivial module R. And it has the same homology with the geometrization $|BG|$.

Lemma (1.3.10). $A_* \cong NA_* \oplus DA_*$ as a complex, $NA_*, A_*, (A/DA_*)$ are all homotopically equivalent.

Proof: We define similarly $N_k A_*$ and $D_k A_*$ and induct on k, our conclusion is the case $k = n-1$. When $k = 0$, $\mathrm{Im}\, d_0 \oplus \mathrm{Ker}\, s_0 A_n = A_n$ because $d_0 s_0 = id_{n-1}$ thus $A_{n-1} \xrightarrow{s_0} A_n$ is a split injection.

There are two split exact rows by simplicial relations:

$$\begin{tikzcd}
0 \arrow[r] & N_{k-1}A_{n-1} \arrow[r, "s_k"] \arrow[d] & N_{k-1}A_n \arrow[r, "1-s_k d_k"] \arrow[d] & N_k A_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1}/D_{k-1}A_{n-1} \arrow[r, "s_k"] & A_n/D_{k-1}A_n \arrow[r] & A_n/D_k A_n \arrow[r] & 0
\end{tikzcd}$$

The first one split because it has a right section, the second one split because it has a left section. So by induction, $N_k A_n \to A_n/D_k A_n$ is an isomorphism, thus $N_k A_n \oplus D_k A_n = A_n$ because it splits.

For the homotopy equivalence, Cf.[Jardine P150]. □

Prop. (1.3.11) (Dold-Kan Correspondence). The normalized Moore complex NA_* gives an equivalence between

$$\text{simplicial Abelian group} \cong \text{chain complex of Abelian groups.}$$

Proof: We define a functor that maps a chain complex to a simplicial Abelian group as follows: $\sigma(C)_n = \bigoplus_{[n]\to[k]\text{surjects}} C_k$, and a morphism $\sigma_n \to \sigma_m$ for a morphism $[m] \to [n]$ is defined as follows: For $[n] \to [k]$ surjects, write $[m] \to [n] \to [k]$ as $[m] \to [r] \xrightarrow{\psi} [k]$ where $[m] \to [r]$ surjects and $[r] \to [k]$ injects, thus maps $a \in C_k$ in σC_n to $\psi^*(a) \in C_r$ in σC_m, where ψ^* is zero unless $\psi = d^n : \Delta[n-1] \to \Delta[n]$. This is natural and defines a simplicial Abelian group because of the unicity of the the canonical decomposition. There is a natural map from $\sigma(N\mathcal{A})$ to $\mathcal{A}$.

Now the task is to show that $\sigma(NA) \cong A$ and $N(\sigma C) \cong C$. We has $N(\sigma C)_n = C_n$ because $d^i C_n$ is 0 for $i \neq n$ and the other components are all degeneracies thus are not in $N(\sigma C)_n = C_n$ by(1.3.10).

Then we prove $\sigma(NA) \cong A$. It is a surjection by(1.3.10) and induction. For the injectivity, if $(a_\varphi) \neq 0$ is mapped to 0, a_{id_n} is 0 by(1.3.10). And we choose an ordering on the $\varphi : [n] \to [k]$ by dominating, and suppose ψ is a minimal one. Now choose a section ξ of ψ that ξ is the maximal section, thus $\varphi\xi$ cannot by id_k for any other φ. Now by induction we have $a_\psi = 0$, contradiction. □

Prop. (1.3.12). There is a functor form a R-algebra S to a trivial simplicial R-algebra $s(S)$, it is a fully faithful embedding and π_0 is left adjoint to it. (The action of A_n on $s(S)_n$ is $A_n \to A_{n-1} \to \cdots \to A_0 \to S$).

Hochschild Homology

Def. (1.3.13). For a R-algebra A and a (A, A)-bimodule L, there is a simplicial module $C(A, L)$ called the **Hochschild complex** of A with coefficient in M, with $M_n = L \otimes A^n$ that

$$d_i(m, a_1, \ldots, a_n) = \begin{cases} (m_0 a_1, a_2, \ldots, a_n) & i = 0 \\ (a_n m_0, a_1, \ldots, a_{n-1}) & i = n \\ (m_0, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) & \text{otherwise} \end{cases}$$

$$s_j(m, a_1, \ldots, a_n) = (m, a_1, \ldots, a_{j-i}, 1, a_{j+1}, \ldots, a_n)$$

When $L = A$, this is even a cyclic module, denoted by $C(A, A)$.

Def. (1.3.14). The homology group of the Moore complex associated to the Hochschild complex is called the **Hochschild homology** $H_n(A, M)$. And we denote the homology of $C(A, A)$ as $HH_*(A)$. $H_n(A, M)$ is a $Z(A)$ module by the action of $Z(A)$ on M and HH_* defines a functor $\mathcal{A}lg_R \to {}_R\mathrm{Mod}$.

Prop. (1.3.15). For a commutative ring R and a symmetric R-bimodule M, there is a spectral sequence

$$E^2_{pq} = \mathrm{Tor}^R_p(H_p(A, A), M) \Rightarrow H_{p+q}(A, M).$$

Cor. (1.3.16). For A commutative and a symmetric (A, A)-module M,$HH_0(A, A) = A^{ab}$ and $HH_1(A, A) \cong \Omega^1_{A/R}$ giving by $a \otimes x \mapsto adx$ by direct calculation. Thus we have $H_1(A, M) = M \otimes_A A^{ab}$ and $H_1(A, M) = M \otimes_A \Omega^1_{A/R}$. And if M is flat, $H_n(A, M) = M \otimes_A H_n(A, A)$.

Prop. (1.3.17) (Hochschild-Kostant-Rosenberg). The isomorphism $\Omega^1_{A/R} \cong HH_1(A)$ extends to a graded ring map

$$\Psi : \Omega^*_{A/k} \to H_*(A, A)$$

. If A/R be smooth algebra and R Noetherian, then Ψ is an isomorphism of graded algebra. Cf.[Weibel P322], [阳恩林循环同调 P133].

Def. (1.3.18) (Tsygan's Double Complex). For a cyclic objec M in an Abelian category, let t_* be the cyclic morphism and $\partial_n = \sum_{i=0}^n (-1)^i d_i$, $\partial'_n = \sum_{i=0}^{n-1} (-1)^i d_i$, $N_n = \sum_{k=0}^n ((-1)^n t_n)^k$, then there is a double complex $CC(M)$:

$$\begin{tikzcd} {} \arrow[d, "\partial"] & {} \arrow[d, "-\partial'"] & {} \arrow[d, "\partial"] & \\ M_1 & M_1 \arrow[l, "1-(-1)^1 t"'] & M_1 \arrow[l, "N"'] & {} \arrow[l, "1-(-1)^1 t"'] \\ M_0 \arrow[from=u, "\partial"] & M_0 \arrow[from=u, "-\partial'"] \arrow[l, "1-(-1)^0 t"'] & M_0 \arrow[from=u, "\partial"] \arrow[l, "N"'] & {} \arrow[l, "1-(-1)^0 t"'] \end{tikzcd}$$

That the column are 2-cyclic. Cf.[Weibel P337]. The first column is called the **Hochschild complex of** M: $C^h(M)$, the second column is called **acyclic complex of** M(1.3.19) $C^a(M)$. And we can even augment a cokernel column on the left, which is the complex of M modulo the cyclic action, called the **Conne complex** $C^\lambda(M)$.

We define the **Cyclic Homotopy Group** $HC_n(M) = H_n(\mathrm{Tot}CC(M))$ and when M is the cyclic module $C(A)$(1.3.13), denote $CC(C(A)) = CC(A)$, $HC_n(A) = HC_n(C(A))$.

Lemma (1.3.19). The second column is exact and $h = t_{n+1}s_n$ is a null-homotopy. Cf.[阳恩林循环同调 P122].

Lemma (1.3.20). Notice the rows are in fact a group homology $\mathrm{Hom}(\mathbb{Z}/(n+1)\mathbb{Z}, M_n)$, thus when $\mathbb{Q} \in R$, we have the rows are acyclic because the group homology is killed by $|G|$**??**, thus $HC_*(M) \cong H_*^\lambda(M)$ are isomorphisms by spectral sequence.

Prop. (1.3.21) (Conne SBI Sequence). For a cyclic module M, there is a long exact sequence

$$\cdots \to HH_n(M) \xrightarrow{I} HC_n(M) \xrightarrow{S} HC_{n-2}(M) \xrightarrow{B} HH_{n-1}(M) \to \cdots$$

Proof: shift the diagram 2 column right, then there is an exact sequence of double complexes and notice the second column is exact(1.3.19), thus we have the kernel is quasi-isomorphic to $C^h(M)$. So the sequence follows. □

Cor. (1.3.22). $HC_0(A) = HH_0(A) = A^{ab}$.

When A is commutative, $HC_1(A) = \mathrm{Coker}(HC_0(A) \xrightarrow{B} HH_1(A)) = \Omega^1_{A/R}/dA$ as a R module, because we can verify that $B(a) = a \otimes 1 - 1 \otimes A$.

Cor. (1.3.23). For a morphism of two cyclic objects, $HH_*(M) \cong HH_*(M')$ iff $HC_n(M) \cong HC_n(M')$. (Use five lemma).

Def. (1.3.24). A **mixed complex** (M, b, B) is a complex with $b : M_n \to M_{n-1}$ and $B : M_n \to M_{n+1}$ that makes M into a double chain complex. And there is a **Conne double complex** associated with this mixed complex. And similarly there is a same SBI sequence associated to the following diagram:

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
M_2 \arrow[d, "b"] & M_1 \arrow[l, "B"'] \arrow[d, "b"] & M_0 \arrow[l, "B"'] \\
C_1 \arrow[d, "b"] & C_0 \arrow[l, "B"'] & \\
C_0 & &
\end{tikzcd}$$

From a cyclic object M, we notice that the $2k$-th column is acyclic(1.3.19), thus there is a snake-like connection homomorphism B that makes M into a mixed complex BM. Cf.[Weibel P344]. And the Conne double complex will compute the same cyclic homology with previous defined cyclic homology, Cf.[Weible P345].

Notice for this B, B_* on homology is exactly the composition BI.

Prop. (1.3.25). Let R be a unital commutative ring and A is a commutative R-algebra and M is a A-module, then there is a natural morphism

$$M \otimes_A \Omega^n_{A/R} \xrightarrow{\varepsilon_n} H_n(A, M) \xrightarrow{\pi_n} M \otimes_A \Omega^n_{A/R}.$$

such that $\pi_n \circ \varepsilon_n = n!$.

We first define a map $\varepsilon_n : M \otimes \wedge^n A \to H_n(A, M)$ that

$$\varepsilon_n(m, a_1, \ldots, a_n) = \sum \mathrm{sgn}(\sigma)(m, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$$

then define $\varepsilon_n(m \otimes x da_1 \wedge \cdots \wedge da_n) = \varepsilon_n(mx, a_1, \ldots, a_n)$. And we verify that this map is well-defined and maps into $Z_n(C(A, M))$,Cf.[阳恩林循环同调 P99].

Then we define $\pi_n(m, a_1, \ldots, a_n) = m \otimes da_1 \wedge \cdots \wedge da_n$ and verify easily that this vanish on $B_n(C(A, M))$. And it is easy to verify $\pi_n \circ \varepsilon_n = n!$.

Prop. (1.3.26). When A is a unital R-algebra, there is a commutative diagram

$$\begin{tikzcd} \Omega^n_{A/R} \arrow[r, shift left, "(n+1)d"] \arrow[r, shift right, "d"'] \arrow[d, shift right, "\pi_n"'] & \Omega^{n+1}_{A/R} \arrow[d, shift right, "\pi_{n+1}"'] \\ HH_n(A) \arrow[u, shift right, "\varepsilon_n"'] \arrow[r, "B_*"] & HH_{n+1}(A) \arrow[u, shift right, "\varepsilon_{n+1}"'] \end{tikzcd}$$

Proof: We notice $B = (1 - (-1)^n t)sN$:

$$(m, a_1, \ldots, a_n) \mapsto \sum_{i=0}^{n} (-1)^{in}(1, a_i, \ldots, a_n, m, a_1, \ldots, a_{i-1}) - \sum_{i=0}^{n} (-1)^{in}(a_i, 1, a_{i+1}, \ldots, a_n, m, a_1, \ldots, a_{i-1}).$$

Cf.[阳恩林循环同调 P128]. □

Cor. (1.3.27). For a commutative unital R-algebra A, there is a functorial $\varepsilon_n : \Omega^n_{A/R}/d\Omega^{n-1}_{A/R} \to HC_n(A)$ making the following diagram commutative:

$$\begin{tikzcd} {} \arrow[r, "0"] & \Omega^{n-1}/d\Omega^{n-2} \arrow[r, "d"] \arrow[d, "\varepsilon_{n-1}"] & \Omega^n \arrow[r] \arrow[d, "\varepsilon_n"] & \Omega^n/d\Omega^{n-1} \arrow[r, "0"] \arrow[d, "\varepsilon_n"] & \Omega^{n-2}/d\Omega^{n-3} \arrow[r] \arrow[d, "\varepsilon_{n-2}"] & \cdots \\ {} \arrow[r] & HC_{n-1} \arrow[r, "B"] & HH_n \arrow[r, "I"] & HC_n \arrow[r, "S"] & HC_{n-2} \arrow[r, "B"] & \cdots \end{tikzcd}$$

which is induced by the cokernel. Cf.[阳恩林循环同调 P130]. When $\mathbb{Q} \in R$, ε_n is a split injection.

Prop. (1.3.28). When $\mathbb{Q} \in R$, $\frac{1}{n!}\pi_n$ induces a morphism of mixed complexes $(BA, \partial, B) \to (\Omega^*_{A/R}, 0, d)$ by(1.3.25), thus there is a natural map

$$HC_n(A) \to \Omega^n_{A/R}/d\Omega^{n-1}_{A/R} \bigoplus_{i>0} H^{n-2i}_{dR}(A).$$

Prop. (1.3.29) (Morita Invariance). $Tr : HH_*(M_r(A), M_r(M)) \cong HH_*(A, M)$ by the trace and inclusion functors. Cf.[阳恩林循环同调 Morita Invariance]. In particular, there is an isomorphism $HH_*(M_r(A)) \cong HH_*(A)$, thus also $HC_*(M_r(A)) \cong HC_*(A)$ by(1.3.21).

Prop. (1.3.30) (Karoubi). BG is a cyclic group, and then the cyclic homology group $HC_n(G, A) \cong \bigoplus_{k \geq 0} H_{n-2k}(G, A)$. Cf.[Weibel P339].

Simplicial Homotopy

Prop. (1.3.31). For a Kan fibration X, there can be defined a homotopy groups π_n that they agree with $\pi_i(|X|)$ thus also $\pi_i(S|X|)$, Cf.[Weibel P263]. Thus we see that $|BG|$ is truly the Eilenberg-Maclane spaces BG.

4 Model Category

References are [Simplicial Homotopy Theory Jardine] and [Model Category and SImplicial Methods Goerss].

Def. (1.4.1). A **model category** is a category C with three classes of morphisms:fibrations, cofibrations and weak equivalences that satisfy the following axioms.

- M0: C is closed under finite limits and colimits.
- M1: We have a lifting property with a cofibration i and fibration p when either of them is a weak equivalence.
- M2: Any map f can be factored as pi where i is cofibration and p is a fibration and assure any of them be a weak equivalence, i.e. trivial (co)fibration.
- M3: Fibration is stable under composition, base change and isomorphism is a fibration. Dually for cofibrations.
- M4: The base change of a trivial fibration is a weak equivalence. Dually for cofibration.
- M5: If two of f, g, fg is weak equivalence, then so is the third.

The definition is dual, i.e., if A is a model category, then so is A^{op}.

It is called a **closed model category** iff moreover it satiesfies

- M6: (co)fibration, weak equivalence is closed under retract.

It is called **simplicial model category** iff all $\mathrm{Hom}(X, Y)$ are simplicial sets and it satisfies:

- SM7: If $i : U \to V \in Cof$ and $p : X \to Y \in F$, then the induced map

$$\mathrm{Hom}(V, X) \xrightarrow{(i^*,p_*)} \mathrm{Hom}(U, X) \times_{\mathrm{Hom}(U,Y)} \mathrm{Hom}(V, Y)$$

  is a fibration, and trivial iff any of i, p is trivial.

Prop. (1.4.2). A model category satisfies M6 iff:
fibration= r(trivial cofibrations),
cofibration= l(trivial fibrations),
weak equivalence= uv, where $v \in l(F)$ and $u \in r(Cof)$.

Proof: If these are satisfied, M6 is easy: a retract of an isomorphism is an isomorphism, so $\gamma(f)$ is an isomorphism and(1.4.3) shows $f \in r(Cof)$ thus a weak-equivalence.

Conversely, notice for a diagram
$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Z \arrow[r, "u"'] \arrow[ur, "s"] & Y \end{tikzcd}$$
induce p as a retraction of u. straightforward for (co)fibrations and for $f = uv$, the same diagram proves u, v are all weak-equivalences. □

Prop. (1.4.3). Let p be a fibration in C_{cf}, then $p \in r(Cof)$ iff $\gamma(p)$ is an isomorphism, Cf.[Quillen 5.2]. So if conditions of(1.4.2) are satisfied (i.e. C is a closed model category), $\gamma(f)$ is an isomorphism iff f is a weak equivalence by the charaterization of weak-equivalence of(1.4.2).

Prop. (1.4.4). A **cylinder object** for an object A is a C with $X \coprod X \xrightarrow{i} C \xrightarrow{j} X$, where $i \in Cof$ and $j \in W$ and ji is the codiagonal map.

Dually, a **path object** for Y is a P with $Y \xrightarrow{q} P \xrightarrow{p} Y \times Y$ where $q \in W$ and $p \in F$ and pq is the diagonal map. They are named because $C = A \times I$ and $P = Y^I$ is the prototype and we will write this way often.

Two morphisms $f, g : X \to Y$ are called **left homotopic** iff there is a cylinder object $X \coprod X \to X \times I$ with $X \times I \to Y$ that induce $(f, g) : X \coprod X \to Y$. Dually for right homotopic.

Lemma (1.4.5). If A is Cof and $A \times I$ is a cylinder object for A, then $\partial_i : A \to A \times I$ are trivial fibrations. (Because it's pushout of Cof and $\sigma \circ \partial_i = \mathrm{id}_A$).

Cor. (1.4.6) (Covering Homotopy Theorem). If A is Cof and, then $\partial_i : A \to A \times I$ has left lifting property w.r.t. all fibrations.

Cor. (1.4.7) (Homotopy Extension Theorem). If B is fibrant, then $\sigma_i : B^I \to B$ has right lifting property w.r.t. all cofibrations.

Prop. (1.4.8). If A is Cof, the left homotopy is an equivalence relation on $\mathrm{Hom}(A, B)$.

Proof: For this, the only problem is transitivity, so we construct a glueing A'' as the pushout of $\partial_1 : A \to A \times I$ and $\partial_0' : A \to A \times I'$. $A'' \to A$ is a weak equivalence by M4,M5 and(1.4.5). $A \coprod A \to A''$ is a Cof because it is composition of two pushouts. So it is a cylinder object. □

Path object and cylinder object exists by M2.

Prop. (1.4.9). If A is Cof and $f, g \in \mathrm{Hom}(A, B)$, then

1. f, g are left homotopic iff they are right homotopic.
2. If f, g are right homotopic, then $s \to B^I$ can be chosen to be trivial Cof.
3. If f, g are right homotopic, then so does $uf \sim ug$ or $fv \sim gv$. Thus if A is Cof, hence there is a map: $\pi^r(A, B) \times \pi^r(B, C) \to \pi^r(A, C)$.
4. For a $X \to Y \in TF$, $\pi^l(A, X) \to \pi^l(A, Y)$ is a bijection.

And dual argument hold for B fibrant.

Proof:

1. Cf.[Quillen Homotopical Algebra 1.8].
2. factorize $B \to B^I$ to $B \to B^{I'} \to B^I$ where $B \to B^{I'} \in TCof$ and $B^{I'} to B^I \in W$, so $B^{I'}$ is also a cylinder object and the homotopy $A \to B^I$ can be lifted to $A \to B^{I'}$.
3. there is a diagram $\begin{array}{ccc} B & \xrightarrow{su} & C^I \\ \downarrow s & & \downarrow (d_0, d_1) \\ B^I & \xrightarrow{(d_0 u, d_1 u)} & C \times C \end{array}$ which have a lifting φ, then composed with $A \to B^I$ will give the desired homotopy.
4. the map is well-defined, it is surjective because of lifting property, and it is injective because $A \coprod A \to A \times I \in Cof$ so the homotopy can be lifted to X.

□

Def. (1.4.10). Let C_c, C_f, C_{cf} denote the full subcategory of cofibrant, fibrant and cofibrant-fibrant objects. And we define πC_c as the category module right homotopy equivalence between morphisms, dually for πC_f. Notice for C_{cf}, left homotopy is equivalent to right homotopy by(1.4.9), so πC_{cf} is full subcategory for both πC_c and πC_f.

Def. (1.4.11). The **localization** of a category is defined as usual, and the **homotopy category** hC for a model category C is the localization of C w.r.t. to the class of weak equivalences.

Lemma (1.4.12). A functor $C \to B$ that maps weak equivalence to isomorphisms will map all left homotopic or right homotopic morphisms to the same morphism (look at the definition of cylinder object). Thus it induces a functor $\gamma : \pi C_{cf} \to hC$, and similarly $\gamma_f : \pi C_f \to hC_f$ and $\gamma_c : \pi C_c \to hC_c$.

Prop. (1.4.13). $\pi C_{cf} \cong hC \cong hC_c \cong hC_f$. So hC_c injects into πC_c and is right adjoint to γ_c. hC_f injects into πC_f and is left adjoint to γ_c, Cf.[Quillen Homotopical Algebra 1.13].

<u>**Examples**</u>

Prop. (1.4.14) (Serre-Quillen). The category $\mathcal{T}op$ is a closed model category with Serre fibrations, weak homotopy equivalence and cofibrations defined as the left lifting class of trivial fibrations.

Cor. (1.4.15). $\partial D^n \to D^n$ is an cofibration, hence all inclusion of CW complexes are cofibration. All topologocal space are fibrant.

Proof: Use mapping cylinder, we can regard it as an injection and then use compression lemma.
?
□

Cor. (1.4.16). Every map can be decomposed as a homotopy equivalence followed by a fibration, by the construction of homotopy fibers. Cf.[Hatcher P407].

Prop. (1.4.17) (Derived Category Model). If $\mathcal{A}$ is an Abelian category with enough injectives, then $K^+(\mathcal{A})$ is a closed model category with Fibration= epimorphisms with Ker in $K^+(\mathcal{I})$, cofibration=monomorphisms, weak equivalence=quasi-isomorphisms.

Prop. (1.4.18). The category C of semi-simplicial sets is a closed model category with fibrations=Kan fibrations, cofibration= injective maps, weak equivalence= maps which induce homotopy equivalence on geometrizations.

Prop. (1.4.19) (Kan Model). The category of Simplicial sets $\mathcal{S}et_\Delta$ is a model category with cofibrations=monomorphisms and fibrations=Kan fibrations, weak equivalence= which induce homotopy equivalence of their geometrizations.

Proof: Cf.[Jardine P62]. □

Prop. (1.4.20). The singular functor and the geometrization functor defines an equivalence of categories between $h(\mathcal{S}et_\Delta)$ and $h(\mathcal{T}op)$. Cf.[Jardine P63].

Prop. (1.4.21) (Joyal). The

VI.2 Higher Topos Theory

1 ∞-Categories

Def. (2.1.1). An ∞**-category** is a simplicial set that has lifting property w.r.t all $\Lambda_i^n \to \Delta^n$, where $0 < i < n$.

2 ∞-Algebras

3 Topological Cyclic Homology(Scholze)

VI.3 K-Theory

1 Milnor K-Group

Def. (3.1.1). The Grothendieck group $K_0(A)$ for a ring A is the free group generated by f.g. projective module over A modulo exact sequences. Then we have $P \sim Q$ iff $P \oplus A^n \cong Q \oplus A^n$ for some n. This is a functor $\mathcal{R}ing \to \mathcal{A}b$.

2 Topological K-theory

K-group of Coherent sheaves

Def. (3.2.1). We denote by $K_0(Coh(X))$ the Grothendieck group of the category of coherent groups on a scheme X Noetherian, and $K_0(Vect(X))$ the subgroup generated by locally free sheaves. The tensor product defines a commutative ring structure on $K_0(Vect(X))$ because locally free sheaves are flat.

Prop. (3.2.2). For a Noetherian regular scheme of finite dimension, the inclusion $K_0(Vect(X)) \to K_0(Coh(X))$ is an isomorphism.

Proof: Cf.[StackProject 0FDI,0FDJ]. $\square$

VI.4 Derived Algebraic Geometry(Lurie)

VI.5 Condensed Mathematics(Scholze)

VI.6 HoTT

VI.7 Model Theory

1 Filters & Ultrafilters

Chapter VII

Theoretical Physics

VII.1 Hamiltonian Mechanics

1 Basics

Prop. (1.1.1) (Principle of Least Action). Typically for a physical system, we can find a functional $L(q,q',t)$ that the actual evolution of this system must be an extremal point of the configuration of the system:

$$S[1,t_1,t_2]=\int_{t_1}^{t_2}L(q(t),q'(t),t)dt.$$

Prop. (1.1.2) (Dimensional Analysis). In an equation raising from a physical problem, we can normalize all the indeterminants to get a non-dimensional one, giving the equation some kind of characteristic length.

2 TBA

Prop. (1.2.1). Yang-Mills Field.

VII.2 Fluid Dynamics

VII.3 Quantum Mechanics

1 Basics

Prop. (3.1.1) (Axioms). The Schrodinger equation can be derived from the Dirac-von Neumann axioms:

The state of particals is a countable dimensional Hilbert space, and

- The observables of a quantum system are defined to be the (possibly unbounded) self-adjoint operators A on $\mathbb{H}$.
- The state φ of the quantum system is a unit vector of $\mathbb{H}$, up to scalar multiples.
- The expectation value of an observable A for a system in a state φ is given by the inner product $\langle\varphi, A\varphi\rangle$.
- (Unitarity) the time evolution of a quantum state according to the Schrodinger equation is mathematically represented by a unitary operator $U(t)$ (depends only on the state an relative time)(one-parameter subgroup).

Now that $\varphi(t)=\hat{U}(t)\varphi(t_0)$, so $\hat{U}(t)\varphi(t_0)=e^{-i\hat{\mathcal{H}}t}$, $\hat{\mathcal{H}}$ hermitian.

So now take derivative w.r.t t, we get $i\frac{d\varphi}{dt}=\hat{\mathcal{H}}\varphi$. By quantum correspondence principle, it is possible to derive the expression of $\hat{\mathcal{H}}$ by classical methods.

Prop. (3.1.2). The solution of a Schrodinger equation for a non Relativistic particle is assumed to be a Schwartz function (Vanish fast enough at infinity). The coefficients is assumed smooth enough to guarantee at least uniqueness and existence locally.

Prop. (3.1.3). The wave function on the (p,t) coordinates is the Fourier Transform of the wave function on the (x,t) coordinates, because the eigenstate of the p-operator $i\hbar\frac{\partial}{\partial x}$ is e^{ikx}, the coefficients of which is the value (probability) of the wave function of the (p,t) coordinates.

Prop. (3.1.4) (Schrodinger Uncertainty Principle). Set $\sigma_A=\sqrt{\langle\hat{A}^2\rangle-\langle\hat{A}\rangle^2}$, then:

$$\sigma_A^2 2\sigma_B^2 \geq \left|\frac{1}{2}\langle\hat{A}\hat{B}+\hat{B}\hat{A}\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle\right|^2+\left|\frac{1}{2i}\langle[\hat{A},]\hat{B}]\rangle\right|^2.$$

(Derived from definition and Schwarz inequality, Cf.[Wiki]).

Cor. (3.1.5) (Heisenberg Uncertainty Principle).

$$\sigma_x\sigma_p\geq\frac{\hbar}{2}$$

Proof:

$$[x, i\hbar\frac{\partial}{\partial x}]=i\hbar.$$

□

Prop. (3.1.6) (Spectral Decomposition). In Quantum physics, one need to use spectral decomposition of the Hamiltonian operator. But at most cases, there are only countably many eigenstate and the eigenvalue has a lower bound and tends to infinity. In this case, $(\hat{H}+A)^{-1}$ is a compact operator thus by spectral theorem(3.8.13) the eigenstate of $\hat{H}$ forms a set of complete basis.

Calculations

Prop. (3.1.7) (Virial Theorem). For a system that $V(r)\sim r^n$, the average kinetic energy and the average potential energy has the relation :

$$2\langle T\rangle = n\langle V\rangle.$$

Spin

VII.4 Quantum Field Theory

VII.5 General Relativity

1 Basics

Prop. (5.1.1) (Maxwell's Equation). Normal Maxwell's equation reads:

$$\begin{cases} \mathrm{div}E = q & \text{(Coulomb's law)} \\ \mathrm{div}H = 0 & \text{(Gaussian law)} \\ \mathrm{curl}E = -\frac{1}{c}\frac{\partial H}{\partial t} & \text{(Faraday's law)} \\ \mathrm{curl}H = j + \frac{1}{c}\frac{\partial E}{\partial t} & \text{(Ampère-Maxwell law)} \end{cases}$$

where E is the magnetic field, H is the electric field, q the charge density, j the electric current.

In Minkowski space, we define the electromagnetic 2-form

$$F = \frac{1}{2}F_{\alpha\beta}dx^\alpha \wedge dx^\beta,$$

where $F_{i0} = E_i, \quad Fij = H_k$, and electric current J, $J^i = -j^i, J^0 = q$.

Maxwell's equation can be re-written as:

$$d^*F = J \qquad dF = 0.$$

Where $d^* = *d*$.

Proof: The Minkowski space is flat, the equivalence can be seen by direct calculation. $\square$

VII.6 String Theory

Chapter VIII

Others

VIII.1 TBA

- a right Kan fibration which is a weak equivalence is a trivial fibration.
- smooth irreducible representations of Weil group is admissible.
- fundamental class relation with Weil group
- conductor of a Weil representation is an integer

VIII.2 Elementary Mathematics

1 Algebra

Prop. (2.1.1). If a_n is a series of real numbers that $a_{m+n} \geq a_m + a_n$, then a_m/m converges to $\lambda = \sup a_n/n$.

Proof: Let $\lambda - a_n/n < \varepsilon$, then for any N large, $N = kn + m$ for $m < n$, so $a_N \geq ka_n + a_m$, so $\liminf a_N/N \geq a_n/n$ for N large. Thus the result follows. $\square$

2 Number Theory

Prop. (2.2.1). $v_p(n!) = \frac{n-c(n)}{p-1}$, where $c(n)$ is the sum of the presentation of n in the p-adic base.

Cor. (2.2.2). $v_p(C^b_{a+b})$ equals the number of carries when adding a and b in base p.

VIII.3 Music Principle

General Principle

Prop. (3.0.1). The Scales and the months that has 31 days can correspond in this way:

$$\begin{array}{ccccccc} F & G & A & B & C & D & E \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 5 & 7 & 8 & 10 & 12 \end{array}$$

C corresponds to 8, which is the luckiest number among them.

Prop. (3.0.2). A X**-junior** 3**-chord** is three numbers $X, X+4, X+7 \pmod{12}$.
A X**-minor** 3**-chord** is three numbers $X, X+3, X+7 \pmod{12}$.
A X**-plus** 3**-chord** is three numbers $X, X+4, X+8 \pmod{12}$.
A X**-minus** 3**-chord** is three numbers $X, X+3, X+6 \pmod{12}$.
When swiping a chord on guitar, the chord should begin with X.

Prop. (3.0.3). The basic chords is the $C(D)E(F)G, A(B)C(D)E$ etc, in the C-tone, which can be parallel transported to other tones.

Guitar

Prop. (3.0.4) (Finding Scales). On a standard tuned acoustic guitar,

- The n-th fret of the 6-th string and the n-th fret of the 1-st string have the same scale.
- The n-th fret of the 6-th sting and the $n+2$-th fret of the 4-th string have the same scale; The n-th fret of the 5-th fret and the $n+2$-th fret of the 3-rd string have the same scale.
- The n-th fret of the 4-th string and the $n+3$-th fret of the 2-nd string have the same scale; the n-th fret of the 3-rd string and the $n+2$-th fret of the 1-st string have the same scale.
- The n-th fret of the 2-nd sting and the $n+3$-th fret of the 4-th sting have the same scale.

These should be enough to quickly locate where the scale goes.