Elements of Mathematics

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Preface

This is a latex version of subtle or important materials I encountered while studying in Peking university. I started this project in the fall of my third undergraduate year, noticing that I have a poor memory and consistently forget what I have already learned thus struggle to check details. So it came to me that I can compile all the proofs of theorems I cannot recall that is hard and subtle yet appearing over and over again. But finally it turns out I want to make it as comprehensive as possible. That's it.

Notice: This is hardly a *readable* book, I use it as a dictionary. It only contains materials that I'm interested in and many proofs are still missing. And maybe I will or maybe I won't complete them.

It should be made clear that I took proofs from many different places, so it should not be considered anything in this book originated from me. Until I get a full extensive reference of this note, I have few rights to the texts.

It's true that there is already a great online book StackProject that covers considerably many of this note, but it's TOO long, I'm far from finishing reading it yet. So I just reordered the materials that I learned and keep track of it in my own way. I write it much shorter and omitted easy proofs.

I truly hope this note can contribute to my study and help anyone who read it, but it comes with no warranty, please use at your own risk.

Table of Contents

Chapter I

Algebra

I.1 Set Theory

1 Cardinal & Ordinal

Def. (1.1.1). A *cardinal number* is an equivalence class, where equivalence and ordering is given by injectives and surjectives. it is used to describe the 'size' of a set. It can only be compared, not operated.

An *ordinal* is an equivalence class of isomorphic well-ordered transitive (i.e. every element is a subset of itself) sets. Notice that two ordinal can have the same cardinality. The ordering of ordinal is by inclusion. Ordinal numbers can apply arithmetics.

The least ordinal having cardinality α is called the initial ordinal of α. The axiom of choice asserts that every cardinal has a initial ordinal.

Prop. (1.1.2) (Bernstein's Theorem). If there is an injective from A to B and an from B to A, then there is a bijection from A to B. Thus the ordering of the cardinal is well-defined.

Lemma (1.1.3). The ordering of ordinals is an total ordering. Every element of an ordinal is an ordinal, and if an ordinal $\beta \subset \alpha$, then $\beta \in \alpha$. Cf.[Set Theory Jech P108].

Def. (1.1.4). The *cofinality* of an ordinal α is the smallest ordinal δ that is the order type of a cofinal subset of α.

The *cofinality* of or a poset (i.e partially ordered set) α is the is the smallest cardinality δ of a cofinal subset of α.

Ordinal Arithmetic

Cantor Normal Form

Prop. (1.1.5) (Transfinite Induction/Recurtion).

2 Filters & Ultrafilters

3 Axiomatic Set Theory

I.2 Linear Algebra

1 Similarity

Prop. (2.1.1). A matrix that $J^2+1=0$ is similar to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^n$. (Use cyclic decomposition).

Prop. (2.1.2) (Jordan Form). For a matrix over a algebraically closed field, it is similar to a matrix of blocks $\lambda_i I + N, Nx_i = x_i + 1$. A

For a real matrix, it is similar to a matrix of blocks of the above form together with $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ on the diagonal and $I_{2\times 2}$ on the upper side.

2 Conguence

Prop. (2.2.1). A symmetric matrix A is orthogonally diagonalizable. Similarly, a skew-symmetric matrix is orthogonally diagonalizable and an (skew)hermitian matrix is unitarily diagonalizable.

Proof: For any real matrix A and any vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle.$$

Now assume that A is symmetric, and $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of A corresponding to distinct eigenvalues λ and μ. Then

$$\lambda\langle \mathbf{x}, \mathbf{y}\rangle = \langle \lambda\mathbf{x}, \mathbf{y}\rangle = \langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x}, \mu\mathbf{y}\rangle = \mu\langle \mathbf{x}, \mathbf{y}\rangle.$$

Therefore, $(\lambda-\mu)\langle \mathbf{x}, \mathbf{y}\rangle = 0$. Since $\lambda - \mu \neq 0$, then $\langle \mathbf{x}, \mathbf{y}\rangle = 0$, i.e., $\mathbf{x} \perp \mathbf{y}$.

Now find an orthonormal basis for each eigenspace; since the eigenspaces are mutually orthogonal, these vectors together give an orthonormal subset of $\mathbb{R}^n$. □

Prop. (2.2.2) (Normal operator). More generally, a normal operator over $\mathbb{C}$ is unitary diagonalizable using resolution of identity(3.7.3) because the spectrum are discrete thus the point projection is orthogonal.

Prop. (2.2.3). Over $\mathbb{R}$, a skew-symmetric matrix are orthogonally congruent to $\mathrm{diag}\Big\{\begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}\Big\}_i$.

Proof: Choose a α, β and choose their orthogonal complement. □

Cor. (2.2.4). For a matrix that $J^2+1=0$, by (2.1.1),there is a unique inner product s.t. J is orthogonal and then it is orthogonally congruent to $\langle\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\rangle_n$. (Use cyclic decomposition).

so this J is equivalent to a complex structure, homeomorphic to $O(n)/U(\frac{n}{2})$.

3 Determinant

Prop. (2.3.1) (Sylvester's determinant identity). If A and B are matrices of sizes $m \times n$ and $n \times m$, then

$$\det(I_m + AB) = \det(I_n + BA)$$

Proof:

$$\begin{bmatrix} 1 & \mathrm{A} \\ \mathrm{B} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathrm{B} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 - \mathrm{BA} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{A} \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \mathrm{A} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 - \mathrm{AB} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathrm{B} & 1 \end{bmatrix}$$

□

Prop. (2.3.2). The determinant of a symplectic matrix $\in Sp(n)$ has determinant 1.

Proof: A symplectic matrix preserves the symplectic structure thus the symplectic form ω, hence ω^n which is $n!$ times the volume form. □

Prop. (2.3.3). $GL_n(\mathbb{C})$ can be embedded into $GL_{2n}(\mathbb{R})$, with determinant $|\det|^2$. And in this way, $U(n)$ is mapped into $O(2n)$. Also, $O(n)$ embeds into $U(n)$ diagonally.

Proof:

$$X + iY \mapsto \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \sim \begin{bmatrix} X & Y \\ iX - Y & X + iY \end{bmatrix} \sim \begin{bmatrix} X - iY & Y \\ 0 & X + iY \end{bmatrix}$$

□

Prop. (2.3.4). There is a polynomial Pf s.t. $\det M = \mathrm{Pf}(M)^2$ for a skew-symmetric matrix. This is because a skew symmetric is equal to $A^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^k A$ for A an orthogonal matrix (2.2.1), so it has determinant $(\det A)^2$ and A and depends polynomially on the entries of M.

Cor. (2.3.5).

$$\mathrm{Pf}(A^t M A) = \det A \cdot \mathrm{Pf}(M).$$

Because we only need to consider the sign and it is determined by letting $A = \mathrm{id}$.

4 Spectral Theory

See also 7

Prop. (2.4.1). a family of commuting diagonalizable operator can be simultaneously diagonalized.

Proof: □

Prop. (2.4.2). in an algebraically closed field, diagonalizable $\iff$ normal. And the eigenvectors are orthogonal to each other.

Proof: □

5 Decompositions

Prop. (2.5.1) (Polar Decomposition). $GL_n(\mathbb{R})$ can be decomposed as $P \cdot O(n)$, where P is a positive symmetric matrix and $O(n)$ the orthogonal matrix. a positive symmetric matrix can be diagonalized, so $GL_n(\mathbb{R})$ have $O(n)$ as deformation kernel.

Similarly, $Sp(2n)$ can be decomposed as $P \cdot U(n)$, because $O(2n) \cap Sp(2n) = U(n)$. And it has $U(n)$ as deformation kernel.

Prop. (2.5.2) (Bruhat Decomposition).

$$GL_n[K] = BWB$$

其中 W 为置换矩阵，B 为上三角矩阵，且分解是不交并。

Proof: Cf.[群与表示 王立中] □

Prop. (2.5.3) (Iwasawa Decomposition).

I.3 Abstract Algebra

1 Group Theory

Prop. (3.1.1) (Nielsen-Schreier). A subgroup of a free group is a free group. Moreover, a subgroup of index m in a free group on n generators is a free group on $1+m(n-1)$ generators.

Proof: A free group is the fundamental group of a wedge sum of circles, and a cover of it is a connected 1-graph. Now the graph has a maximal tree and module the tree gets us a wedge sum of circle. The second statement follows by two ways of counting Euler number χ. □

Finite Groups

Prop. (3.1.2) (Sylow Theorem).

Prop. (3.1.3) (Schur Zassenhaus). An exact sequence of groups $0 \to A \to E \to G \to 0$ must split when $|A|$ and $|G|$ are relatively prime.

2 Field Theory

Prop. (3.2.1). A separable extension or an extension having finitely many middle fields has a primitive element.

Proof: □

Brauer Groups

Prop. (3.2.2). The **Brauer group** $\mathrm{Br}(K)$ is defined as the profinite cohomology $H^2(G(K_s/K), K_s^*)$. For a Galois extension L/K, $\mathrm{Br}(L/K)$ is defined as $H^(G(L/K), L^*)$. Then by(2.2.6) we have

$$\varinjlim \mathrm{Br}(L/K) = \mathrm{Br}(K).$$

3 Transcendental extension

Prop. (3.3.1). Let K be an extension of a field k, a **transcendental base** is an algebraically independent set that any element is algebraic over it. Then the number of elements in any algebraically independent set $\leq$ the number of elements in any transcendental base. In particular, given any algebraically independent set $S \subset T$ a set over which K is algebraic, S can be extended to a transcendental base.

Proof: Let $X = \{x_1, \ldots, x_m\}$ transcendental base of minimal number, $S = \{w_1, \ldots, w_n\}$ an algebraically independent set. If $n > m$, we proceed by changing one element a time using induction and prove that K is algebraic over $\{w_1, \ldots, w_r, x_{r+1}, \ldots, x_m\}$, contradiction.

Because w_{r+1} is algebraic over $\{w_1, \ldots, w_r, x_{r+1}, \ldots, x_n\}$, we have a minimal polynomial

$$f = \sum g_j(w_{r+1}, w_1, \ldots, w_r, x_{r+2}, \ldots, x_m) x_{r+1}^j$$

s.t. $f(w_{r+1}, w_1, \ldots, x_m) = 0$ (after possibly renumbering x_i, this x must exists because S is itself algebraically independent). So x_r is algebraic over $\{w_1, \ldots, w_{r+1}, x_{r+2}, \ldots, x_m\}$, hence K is independent over it, too. □

4 Separability

Basic Reference is [Matsumura Ch10].

Def. (3.4.1). A field extension K/k is called separably generated iff it K is a separable algebraic extension of a purely transcendental field L/k. An algebra A/k is called separable iff $A \otimes_k k'$ is reduce for any k'/k algebraic.

Prop. (3.4.2). Let K/k by f.g. field ext, then K/k is separable algebra $\iff$ K/k is separably generated $\iff$ $K \otimes_k k^{1/p}$ is reduced, Cf.[Masumura P195].

5 Galois Theory

Prop. (3.5.1) (Primitive element). a finite extension E/k is primitive iff there are only finitely middle fields. And if E/k is separable, this is satisfied.

Proof: If k is finite, this is simple. Assume k infinite, for any two elements α, β, consider $k(\alpha + c_i\beta)$, if there is only finitely many middle fields, there exists two that is equal, so $k(\alpha, \beta) = k(\gamma)$. Proceeding inductively, E is primitive.

Conversely, if $k(\alpha) = E$, every middle field corresponds to a divisor of the irreducible polynomial of α. This map is injective, because for any g_F, degree of α over F is the same over the degree over the coefficient field of g_F, so it must be equal to F.

If E/k is separable, Let

$$P(X) = \prod_{i \neq j} (\sigma_i\alpha + X\sigma_i\beta - \sigma_j\alpha - X\sigma_j\beta)$$

for different embedding σ_i, σ_j of $E(\alpha, \beta)$ into k^{al}. Then it is not identically zero, thus there exists c that $\sigma_i(\alpha + c\beta)$ is all distinct, thus generate $K(\alpha, \beta)$. □

Prop. (3.5.2) ((Artin) Galois Main Theorem). Let G be a finite group of automorphisms of K. Then K/K^G is Galois of Galois group G.

Proof: For every element x, set $\{\sigma_1 x, \ldots, \sigma_r x\}$ be distinct conjugates, then $f(X) = \prod_i^r = (X - \sigma_i x)$ shows that K is separable and normal over K^G. And primitive element theorem shows that $[K : K^G] \leq |G|$, so it must equals G. □

Prop. (3.5.3) (Infinite Galois Theorem). The middle fields correspond to the closed subgp of $G(L/K)$.

Proof: The highlight is that $G(L/L^H) = H$ for a closed subgp H of $G(L/K)$. If σ fixes L^H but is not in H, because for every finite field M, $H \cdot G(L/M)$ corresponds to $M/(M \cap L^H)$, so $\sigma G(L/M) \cap H \neq \varnothing$. So σ is in the closure of H thus in H. □

Prop. (3.5.4) (Normal Basis Theorem). for a finite Galois extension, normal basis exists.

Proof: Finite case:

The Galois group is cyclic, and the linear independent of characters shows that the minimal polynomial of σ is n-dimensional thus equals $X^n - 1$. regard L as a $K[X]$ module thus by (5.2.5) is a direct sum of modules of the form $K[X]/(f(x))$, $f(x)|X^n - 1$ and the minimal polynomial for the action of X is $X^n - 1$. So it is isomorphic to $K(X)/(X^n - 1)$.

Infinite Case:
Let

$$f([X_\sigma]) = \det(t_{\sigma_i,\sigma_j}), \qquad t_{\sigma,\tau} = X_{\sigma^{-1}\tau}$$

□

We see $f \neq 0$ by substituting 1 for X_{id} and 0 otherwise. So it won't vanish for all x if we substitute $X_\sigma = \sigma(x)$ because $[\sigma(x)]$ is pairwisely different. Thus there exists w s.t.

$$\det(\sigma^{-1}\tau(w)) \neq 0.$$

Now if

$$\sum a_\tau \tau(w) = 0, \qquad a_\sigma \in K,$$

act by σ for all σ, we get $[\sigma^{-1}\tau(w)][a_\sigma] = 0$, thus $[a_\sigma] = 0$.

Prop. (3.5.5) (Kummer Theory). There exists an inclusion preserving isomorphism between the lattice of Kummer extensions L of K and the lattice of subgroups of L containing K^n:

$$L \mapsto \Delta = (L^\times)^n \cap K^\times, \qquad \Delta \mapsto K(\sqrt[n]{\Delta}).$$

And $\Delta/(K^\times)^n$ is isomorphic to $\chi(G_{L|K})$.

Proof: Cf.[Neukirch CFT P116]. □

I.4 Representation Theory

1 Linear Representation

Prop. (4.1.1) (Schur's lemma). If π is a countable dimensional $\mathbb{C}$-representation, then $\mathrm{End}(V) \cong \mathbb{C}$.

Proof: Notice we only have to find an eigenvalue of ϕ, but otherwise $\{(\phi - a)^{-1}$ is uncountable and linearly independent over $\mathbb{C}$, so $\dim(\mathbb{C}(\phi))$ is uncountable, contradiction. □

Def. (4.1.2). The induced and coinduced representation is that of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} -$ and $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], -)$.

If $[G:H]$ is finite, then induced is the same as coinduced.

Proof: Choose a left coset representation of H, then check $x \otimes a \to f : hx^{-1} \mapsto ha$ is an isomorphism Cf.[Weibel P172]. □

2 Locally Compact Groups

Prop. (4.2.1) (Brauer-Nesbitt). For a finite group G, if two finite dimensional semisimple representations over a field has the same char poly for every element g of G, then they are isomorphic.

Proof: Just use the irreducible representations are orthogonal and that they have the same and for char p, we can use divide by p and the char poly becomes p-th power and we can do this forever, contradiction. □

<u>**Compact Groups**</u>

Prop. (4.2.2) (Peter Weyl). For a compact group G, $\{\phi_{ij}(g); \phi(g) = (\phi_{ij}(g)), \phi$ an irreducible character$\}$ is a basis for the Hilbert space $L_2(G)$. Cf.[连续群 Pontryagin 第五章 § 33].

3 Locally Profinite Groups

Def. (4.3.1). A locally profinite group is a topological group that every open neighbourhood of id contains a compact open subgroup of G.

Def. (4.3.2). A representation of a topological gp on a discrete vector space is called **smooth** iff $G \times V \to V$ is continuous and **admissible** iff V^K is finite dimensional for every compact open subset K of G.

Prop. (4.3.3). A smooth irreducible representation is admissible. In fact, this is true for general connected reductive group.

Proof:

□

I.5 Commutative Algebra(Matsumura)

1 Basics

Prop. (5.1.1) (Induced & Coinduced). Given a ring homomorphism $R \to$S.

- $f^*M = N_R$, the restriction.
- $f_!M = M \otimes_R S$ is the induced module, it is left adjoint to restriction.
- $f_*M = \mathrm{Hom}_R(S, M)$ is the coinduced module, it is right adjoint to restriction. (It is a S-mod by $s(f)(t) = f(ts)$.)

Localization

Prop. (5.1.2) (Localization is exact). S^{-1} is an exact functor from $R-$mod to $R-$mod. Because it is a filtered colimit,(1.5.2).

Cor. (5.1.3).

$$(R/I)_{\bar{P}} \cong R_P/IR_P$$

in particular,

$$k(R/P) \cong R_P/PR_P$$

Prop. (5.1.4). Taking direct limits commutes with tensor product. (element chasing).

Prop. (5.1.5). Let k be a field, then the power series $k[[X_1, \ldots, X_n]]$ is a UFD.

Proof: Cf.[Algebra Lang P209]. □

Prop. (5.1.6). If finitely many primes cover an ideal, then one of them cover it.

Proof: Assume otherwise, use induction. For two primes, use $x + y$, for r primes, choose $x \notin p_i, i < r$, then $x \in p_r$, and choose $y \in JI_1 \ldots I_{r-1}$ and $y \notin y_r$, then $x + y$ suffice. □

Artinian Ring

Prop. (5.1.7). A ring A is Artinian iff the length of A as a A-mod is finite. This is equivalent to it is noetherian of dimension 0. Cf,[Matsumura P14].

So a f.d algebra over a field is Artinian. Artinian ring has finitely many primes.

Valuation Ring

Prop. (5.1.8). In a field K, the valuation ring is the maximum elements in the dominating ordering of local rings, where B dominate A iff $A \subset B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$. Cf.[Atiyah].

Prop. (5.1.9). The integral closure of a subring in a field k is the intersection of valuation rings containing A.

Noetherian

Prop. (5.1.10). Subring, quotient ring, finitely generated ring localization power series are Noetherian, graded algebra of a A by an ideal I is Noetherian.

Prop. (5.1.11). When A is Noetherian and is quipped with I-adic topology and I is f.g., then there is surjective ring map $A[[X]] \to A^*$ the completion, hence the completion is Noetherian.

2 Projective & Injective

Prop. (5.2.1). A module over a ring is projective iff it is a direct summand of a free module.

Prop. (5.2.2) (Baer's Criterion). A right $R-$module I is injective iff for every right ideal J of R, every map $J \to I$ can be extended to a map $R \to I$. (Direct from (1.5.8)).

Cor. (5.2.3). A module over a PID is injective iff it is divisible.

Prop. (5.2.4). The category of R-mod has enough injectives by(1.5.3), and it has enough projectives trivially.

Prop. (5.2.5) (Classification of Modules over PID).

1) PID is UFD thus Noetherian.
2) Submodule of a free module over a PID is free.
3) Finitely generated torsion-free module over a PID is free.
4) Finitely generated module over a PID has a primary decomposition $M = \bigoplus_i R/(q_i)$,where (q_i) is primary ideals.

So pojective $\iff$ free $\iff$ torsion-free.

Proof: Cf.[Lang P45] □

Prop. (5.2.6). Any projective module of finite type over $K[X_1, \dots, X_k]$ is free. (Highly nontrivial).

3 Associated Primes

Prop. (5.3.1). the associated prime $\mathrm{Ass}(M)$ of a A-module M is the set of $p = \mathrm{Ann}(m)$. $\mathrm{Ass}(M) \subset \mathrm{Supp}(M)$ and there minimal elements are the same. So $Ass(A/I)$ contains all the minimal primes over I. I is called **unmixed** if all of $\mathrm{Ass}(M)$ is minimal and of the same height. Cf.[Mat P50].

Flat ring extension

Def. (5.3.2). The following are equivalent for $f : A \to B$.

- f is **faithfully flat**, i.e. $-\otimes B$ is exact and $N \otimes B \neq 0$ for $n \neq 0$.
- f is flat and $Spec(B) \to Spec(A)$ is surjective.

Prop. (5.3.3). (Faithfaly)Flatness is stable under restriction, base change and localization. and it is a local property.

4 Integral Extension

Prop. (5.4.1). Let A a subring of $B, A \to B$ integral. Then:

1. If A is local and p is the maximal ideal of A, then the prime ideals of B lying over p is precisely the maximal ideal of B.
2. There is no inclusion relation between the prime ideals of B lying over a fixed prime ideal of A.

3. The Spec map is surjective.
4. The going-up holds.

Proof:
1. Since for two ring one integral over another, one is a field iff the other is a field.
2. Maximal ideal cannot contain each other.
3. Since $A_p \neq 0$, $B_p \neq 0$, so it has a maximal ideal.
4. Localize and use 3.

□

Prop. (5.4.2). Let A a subring of $B, A \to B$ integral noetherian. Then:
1.$\dim(A) = \dim(B)$
2.$\mathrm{ht(P)} = \mathrm{ht(P \cap A)}$
3.if going up holds, then $\mathrm{ht(J)} = \mathrm{ht(J \cap A)}$ for any ideal J.

Proof: 1.By the preceding lemma, there is no inclusion relation between prime over a fixed prim, so $\dim(B) \leq \dim(A)$. On the other hands, going-up holds, so $\dim(B) \geq \dim(A)$.
2.follows from (5.6.4)(1) since $\mathrm{ht(P/(P \cap A)B)} = 0$ by the preceding lemma.
3.by 2 and surjectiveness of Spec for integral extension. □

5 Graded Ring & Completion

Cf.[Matsumura Ch11].

Def. (5.5.1) (Hilbert Polynomial). Let A be an Artinian ring and $B = A[X_i]$. For a graded B-module $\oplus M_n$, we have $l(M_n)$ is a polynomial of n for n big, called the Hilbert Polynomial.

Completion

Prop. (5.5.2) (Artin-Rees). For A Noetherian and I an ideal, let $N \subset M$ be finite A-module, then

$$I^n M \cap N = I^{n-r}(I^r M \cap N)$$

hence the I-adic topology on M induce the I-adic topology on N.

Cor. (5.5.3) (Intersection Theorem). Notation as above, let $N = \cap^\infty I^n M$, then $IN = N$. So if $I \subset \mathrm{rad}(A)$, Nakayama tells us $N = 0$. This can be used to use induction to prove some theorem.

Cor. (5.5.4) (Krull). For A Noetherian, if $I \subset \mathrm{rad}(A)$ or A is a domain, then $\cap^\infty I^n = 0$.

Prop. (5.5.5). Let the topology on a A-module be defined by countable filtration of submodules, then iff M is complete, then M/N is complete in the quotient topology.

Proof: Write $x_{i+1} - x_i = y_i + z_i$ with $y_n \in M_n$ and $z_n \in N$, then the image of the limit of $\sum y_i$ is the limit of $\overline{x_i}$. □

Def. (5.5.6). The **completion** of a topological A-module is a functor $\varphi : M \to M'$ that are left adjoint to the forgetful functor from the category of complete Hausdorff A-modules. It is defined as composition of the Hausdorffization functor followed by $\lim M/M_n$ with the topology like that of profinite groups. The completion is right exact. For left exactness, otice the limit process is exact, so only the Hasudorffization can go astray.

Prop. (5.5.7). The completion of a submodule $N \subset M$ is the closure of $\varphi(N)$ (By direct construction). The completion of M/N is M^*/N^* because it is right exact.

Cor. (5.5.8). If N is open in M then $M/N \cong M^*/N^*$ because M/N is discrete hence complete Hausdorff.

Prop. (5.5.9). When N is finite, $0 \to N^* \to M^* \to (M/N)^* \to 0$ is exact, because the Hasfdorffization of N embeds in that of M by Artin-Rees.

Prop. (5.5.10). When A is Noetherian and M is finite A-module, then the natural map $M \otimes_A A^* \to M^*$ is an isomorphism (use M is finite presentation and tensor & completion is right exact), and five lemma.

Cor. (5.5.11). When A is Noetherian, A^*/A is flat (because flatness is check for finite module), and when A is complete Hausdorff, any finite module M is complete Hausdorff and hence any its submodule is complete thus closed in it. Hence the the completion of a submodule $N \subset M$ is $\varphi(N)A^*$ in $M^* = MA^*$. In fact this implies complete Hausdorff adic-ring is Zariski.

Prop. (5.5.12). A Noetherian I-adic ring is called **Zariski ring** if it satisfies the following equivalent conditions:

- Every finite module is Hausdorff in the I-adic topology.
- Every submodule in a finite module is closed in the I-adic topology.
- Every ideal is closed.
- $I \subset \mathrm{rad} A$.
- A^*/A is f.f.

Hence every complete Hausdorff ring is Zariski.

Proof: $1 \to 2$:apply it to the submodule M/N.

$3 \to 4$:If $I \not\subseteq m$, then $I^n + m = A$, thus $\overline{M} = A$, contradiction.

$4 \to 1$: by intersection theorem(5.5.3).

$4 \to 5$: for any maximal ideal m, $I \subset m$ so it is open, thus $A^*/mA^* = A/m \neq 0$ by(5.5.8) thus f.f. by(5.14.8).

$5 \to 1$:by(5.14.9), for any m maximal, there is a maximal ideal m' lying over m, so $IA^* \subset m^*$ by(5.5.11), thus $I \subset m$, hence $I \subset \mathrm{rad} A$. $\square$

Cor. (5.5.13). For a Zariski ring A, maximal ideal is open, thus $A/m \cong A^*/mA^*$ by(5.5.8), thus $\mathrm{Spec}\, A^* \to \mathrm{Spec}\, A$ is bijection on closed pt.

6 Dimension

Def. (5.6.1). A ring is called **universally catenary** if all its f.g. algebra is catenary, i.e. the dimension behave well. Dedekind domain, e.g. field is universally catenary, so f.g. domain over fields is catenary.

Prop. (5.6.2) (Noetherian Normalization Theorem). For a f.g. algebra over a field k, then there exists a purely transcendental field extension L/k of degree $\dim A$ that A/L is integral.

Prop. (5.6.3). For a Noetherian Local ring A, the Hilbert polynomial of A w.r.t $\mathfrak{m}$ has degree $\dim A$.

Prop. (5.6.4) (Dimension Extension Formula). Let $A \to B$ Noetherian, let $p = P \cap A$,then:

- $\mathrm{ht(P)} \leq \mathrm{ht(p)} + \mathrm{ht(P/pB)}$,in other words $\dim(B_P) \leq \dim(A_p) + \dim(B_P \otimes k(p))$. Where $k(p) = A_p/pA_p$ and $B \otimes k(p) = B_p/pB_p$.
- equality holds if going-down holds. For example, if it is flat.
- if Spec map is surjective and going-down holds, then we have i)$\dim B \geq \dim A$, and ii)$\mathrm{ht(I)} = \mathrm{ht(IB)}$ for ideal I of A.
- if going-up holds, then $\dim B \geq \dim A$. e.g. B integral over A Cf. (5.4.2)

Proof: Cf.[Commutative Algebra Matsumura (13.B)] □

Prop. (5.6.5) (Normalization Theorem). If A is a f.g. algebra over a field. then there are r alg. independent elements y_i that A is integral over $k[y_i]$ and $r = \dim A$. Hence $\dim A = \mathrm{tr.deg}\ A$ because integral extension has the same dimension.

Cor. (5.6.6) (Krull's Height Theorem). In a Noetherian domain, the height of the minimal prime of an ideal generated by n elements is at most n.

7 Dedekind Domain

Def. (5.7.1). A Dedekind domain is an integrally closed Noetherian domain of dimension 1.

Prop. (5.7.2). The integral closure of a Dedekind domain in a finite extension fields of its quotient fields is again a Dedekind domain.

8 Primary Ring

Def. (5.8.1). A primary ring is a unital ring with only one maximal ideal. Notice that this implys the ring is local. e.g. R/p where p is a primary ideal associated with a maximal ideal m, is primary.

Prop. (5.8.2). An Artinian ring is a direct sum of noetherian primary rings and the decomposition is unique.

Proof:

Take a primary decomposition to notice that 0 is a product of maximal ideals, (because of artinian). Then take

$$R_i = \prod_{j\neq i} \mathfrak{m}^{e_i}$$

then:

$$R \cong \bigoplus R/R_i, \qquad R_i \cong R/\mathfrak{m}^{e_i}$$

Notice R_i and $\mathfrak{m}^{e_i}$ coprime and nonintersecting, so take every decomposition of $x = x_i + y_i$ and prove $x = \sum x_i$. The map $R \to R : x \to R/R_i$ has kernel $\sum_{j\neq i} R_i \cong \mathfrak{m}^{e_i}$ by induction. Uniqueness:

Lemma (5.8.3). In a primary ring, there is no nontrivial idempotent element. Because e and $1 - e$ will all belong to the same maximal ideal m.

the decomposition gives a way to decompose 1 to sum of idempotent elements and is determines by it. $1 = \sum e_i = \sum f_i$, so $e_j = \sum e_j f_i$. But e_i cannot decompose, so $e_j = e_j f_{i(j)}$, $\exists\ i(j)$. the following is easy to show these two decomposition is the same. □

9 Jacobson Ring

Def. (5.9.1). A commutative ring is called Jacobson if every prime ideal is intersection of maximal ideals. In particular, the Jacobson radical equals the nilradical.

Def. (5.9.2). The **Jacobson radical** rad$R$ of R is: $J = \{r \in R | 1 + rs \text{ is a unit } \forall \text{s} \in \text{R}\}$.

The **nilradical** is the intersection of all primes.

Proof: One way is trivial and for the other if r is not in a maximal ideal M, then $(r) + M = (1)$, so contradiction. □

Prop. (5.9.3) (Generalized Nullstellensatz). If R is Jacobson and S is a finitely generated R-algebra, then S is Jacobson and the maximal ideal of S intersect with R a maximal ideal and the quotient ring extension is finite, (hence algebraic). In particular, a f.g. algebra over a field is Jacobson.

Proof: Cf.[Commutative Algebra Eisenbud P132] □

10 Depth & C.M. Ring

Prop. (5.10.1).

$$\operatorname{depth}_I(M) = \min\{i | \operatorname{Ext}^i_A(A/I, M) \neq 0\}$$

where $IM \neq M$ and $\operatorname{depth}_I(M)$ is the length of the maximal M-regular sequence in I.

Def. (5.10.2). For A Noetherian local, a A-module is called **Cohen-Macaulay** if $\operatorname{depth}(M) = \dim M$. A ring is called C.M. if all its localization at primes are C.M local. A ring is C.M. iff for all ideals, the associated primes of A/I all have the same height as I, i.e. unmixed.

11 Normal Ring & Regular Local Ring

Serre Conditions R_k & S_k

Def. (5.11.1). A ring is called R_k iff for all prime p of height $\leq k$, A_p is regular.

A ring is called S_k iff $\mathrm{depth}(A_p) \geq \min(k, \mathrm{ht}(p))$ for all prime p.

A module M is called S_k iff $\mathrm{depth}(M_p) \geq \min(k, \dim(\mathrm{Supp}(M_p)))$ for all prime p.

Prop. (5.11.2).

- M is S_1 iff M has no associated embedded primes. Cf.[StackProject 031Q].
- A Noetherian ring is reduced iff it is R_0 and S_1. Cf.[StackProject 031R].
- (Serre Criterion)A Noetherian ring is normal iff it is R_1 and S_2.
- A ring is C.M. iff it is $S_{\mathbb{N}}$.

Normal Ring

Def. (5.11.3). A ring is called **normal** iff all its localization at primes are integrally closed domain. So a normal domain is just the integrally closed domain. It is called **completely normal** iff all almost normal elements are in A, i.e. $\{u|\exists a, au^n \in A\forall n\} \in A$. For Noetherian ring, these notion are the same.

Prop. (5.11.4). A is completely normal $\Rightarrow$ $A[X]$ and $A[[X]]$ is completely normal. A is a normal ring then $A[X]$ is a normal ring. Hence it works well for A Noetherian. (Induction on the coefficient, Cf.[Matsumura P116]).

Prop. (5.11.5). Principal ideals in a Noetherian normal domain is unmixed and $A = \cap_{\mathrm{ht}\ p=1} A_p$. Cf.[Matsumura P124].

Prop. (5.11.6). A Noetherian domain is UFD iff all minimal primes are principal.

Regular Ring

Def. (5.11.7). A Noetherian local ring is called **regular** iff it $\mathrm{rank}_k\ \mathfrak{m}/\mathfrak{m}^2 = \dim A$. AThis is equivalent to gr $A \cong k[X_1, \ldots, X_d]$ by(5.6.3). Hence a regular local ring is normal Cf.[Matsumura P121]. A ring is called **regular** iff all its localization at primes are regular local.

Prop. (5.11.8). If A is regular, then $A[X_1, \ldots, X_n]$ is regular, and $A[[X_1, \ldots, X_n]]$ is regular, Cf.[Matsumura P176].

A regular local ring of dim 1 is the same as a principal DVR. A regular local ring is C.M.

Prop. (5.11.9). A Noetherian local ring of dim 1 is normal iff it is regular. i.e. integrally closed iff principal. Cf.[Matsumura P124].

Prop. (5.11.10) (Auslander-Buchsbaum). A regular local ring is UFD. A priori it is an integral domain.

Prop. (5.11.11). Localization of a regular local ring at primes are regular local. Cf.[Matsumura P139].

12 Differentials

Def. (5.12.1). Let $B \to A$ a ring map, $\mathrm{Der}_B(A, M)$ is defined as the R-mod map $A \to M$ that satisfies Leibniz rule and vanish on B. Then the **Kahler Differential** $\Omega_{A/B}$ is defined as a A-module that $\mathrm{Der}_B(A, M) \cong \mathrm{Hom}_A(\Omega_{B/A}, M)$.

Prop. (5.12.2). One construction is by the free group generated by elements of A module some relations.

It can also be constructed as follows: there are two ring maps λ_i from A to $A \otimes_B A$, and one map ε from $A \otimes_B A$ to A. Let $I = \ker \varepsilon$ as a A module by λ_1, then $I/I^2 \cong \Omega_{A/B}$ by (5.12.7) with $R = B$, $S' = S \otimes_R S$.

Cor. (5.12.3) (Functoriality). From the first construction, we can see directly that for a family of morphisms $R_i \to S_i$,

$$\Omega_{\lim_{\to} S_i / \lim_{\to} R_i} = \lim_{\to} \Omega_{S_i/R_i}.$$

In particular, we have:

$$T^{-1}\Omega_{B/A} = \Omega_{T^{-1}B/A}, \quad \Omega_{S^{-1}B/S^{-1}A} = S^{-1}\Omega_{B/A}.$$

Moreover, we have $\Omega_{S/R} \otimes_R R' = \Omega_{S \otimes_R R'/R'}$ by universal property.

Lemma (5.12.4). For a map of couples $(R \to S) \to (R' \to S')$, if $S \to S'$ is surjective, then so is the corresponding $\Omega_{S/R} \to \Omega_{S'/R'}$ and the kernel is generated by da, where $a \in S$ maps to the image of R'. Cf.[StackProject 00RR].

Prop. (5.12.5). For a exact sequence of rings:$0 \to A \to B \to C \to 0$, there is an exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules. $C \otimes_B \Omega_{B/A} \to \Omega_{C/A}$ has a left inverse (thus splits) iff any derivation B/A to a C-module can be extended to a C/A derivation. (This is by universal property).

Cor. (5.12.6). We have $\Omega_{A[X_1,\ldots,X_n]/A} = A[X_1, \ldots, X_n]\{dX_1, \ldots, dX_n\}$ (use the differential operator and universal property). thus if $B = A[X_1, \ldots, X_n]$, then $\Omega_{B/k} = \Omega_{A/k} \otimes_A B \oplus B[dX_1 \ldots, dX_n]$ because any any derivative of A/k can be extended to derivative of B/k by acting on the coefficients.

Prop. (5.12.7). If $S' = S/I$, then there is an exact sequence:

$$I/I^2 \to \Omega_{S/R} \otimes_S S' \to \Omega_{S'/R} \to 0.$$

Where $f \in I$ is mapped to $df \otimes 1$ and when $S \to S'$ has a right inverse, then the left hand side has a left inverse (thus splits). The essence is (5.12.4),Cf.[StackProject 00RU].

Cor. (5.12.8). If S/I is a field k that embeds in S, then $I/I^2 \cong \Omega_{S/k} \otimes_S k$.

Prop. (5.12.9). Let $k \subset K \subset L$ be fields, and L/K f.g., then

$$\dim_L \Omega_{L/k} \geq \dim_K \Omega_{K/k} + tr.\deg(L/K).$$

Equality holds if L/K is separably generated, i.e. separable over a transcendental basis. If $K = k$, then the equality hold iff L/k is separably generated. In particular, when L/k separable field extension, $\Omega_{L/k} = 0$, e.g. when k is perfect.

Proof: Consider extension by one element at a time, Cf.[Matsumura P190]. □

Prop. (5.12.10). Let B be a Noetherian local ring containing its residue field k and k is perfect, then $\Omega_{B/k}$ is a free B-module of rank $\dim B$ iff B is regular.

Proof: One way is by(5.12.8). Conversely, if B is regular, then it is integral(5.11.10), so $\Omega_{B/k} \otimes_B K = \Omega_{K/k}$(5.12.3) is of K-dimension $tr.\deg K/k = \dim B$, where K is the quotient field of B, and $\Omega_{B/k} \otimes k \cong m/m^2$ is of k-dimension $\dim B$ once again. These two facts shows that $\Omega_{B/k}$ is free B-module of rank $\dim B$ (first B is generated by $\dim B$ elements by Nakayama and the kernel R of $A^r \to \Omega_{B/k}$ vanishes tensoring K, thus vanish because it is torsion-free). □

13 Nagata Ring

Prop. (5.13.1). For a f.g. algebra A over a field, the integral closure of A in a finite algebraic extension of K is a f.g A-mod, in particular the integral closure of A. Cf.[Hartshorne P20].

14 Flatness

Prop. (5.14.1). Flatness need only be checked for finite modules, and it is equivalent to $\mathrm{Tor}_1(M, A/I) = 0$ for any f.g. ideal I. This is because tensor product commutes with colimit.

Prop. (5.14.2). If M is flat then $\mathrm{Tor}_i^A(M, N) = 0$ for all $i > 0$, because we have: if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

M_2, M_3 flat, then M_1 is flat (Use 9 entry sequence). So $\mathrm{Tor}_{n+1}(M_3, N) = \mathrm{Tor}_n(M_1, N) = 0$.

Thus we have the class of flat modules is adapted $-\otimes N$ for all N (free is flat).

Prop. (5.14.3). Flatness is transitive, stable under base change (by definition) and $S^{-1}A$ are A-flat because localization is exact.

Prop. (5.14.4). If B is A flat, then

$$\mathrm{Tor}_i^A(M, N) \otimes B = \mathrm{Tor}^B(M_{(B)}, N_{(B)}), \quad \mathrm{Ext}_i^A(M, N) \otimes B = \mathrm{Ext}^B(M_{(B)}, N_{(B)}).$$

Prop. (5.14.5). For a ring map $A \to B$ and a B-module M, we have M is B-flat iff M_m is $A_{\varphi^{-1}(m)}$-flat for all maximal ideal m of B.

Prop. (5.14.6) (Gororov-Lazard). Any flat $A-$module is isomorphic to a direct limit of free modules of finite type.

Prop. (5.14.7). A f.g. module M over a local Noetherian ring A is flat iff it is free. In particular, modules over a field is all flat.

Faithfully Flat

Prop. (5.14.8). M is f.f. $\iff$ it is flat and for any $N \neq 0$, $N \otimes M \neq 0$ $\iff$ M is flat and for any maximal ideal m of A, $mM \neq M$.

Proof: $3 \to 2$: any nonzero module has a submodule A/I, and thus $(A/I)M = M/IM \neq 0$.

$2 \to 1$:first show S is a complex if $S \otimes M$ is exact, then $H^*(S) \otimes M = H^*(S \otimes M)$ by flatness, thus $H^*(S) = 0$. □

Prop. (5.14.9). A ring map $A \to B$ is f.f. $\iff$ it is flat and $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective $\iff$ it is surjective on the closed pt.

Proof: $3 \to 1$ by definition(5.14.8) and $2 \to 3$ trivial, so only proves $1 \to 2$:Cf.[Matsumura P28]. □

I.6 Homological Algebra

1 Cohomology

Prop. (6.1.1) (Grothendieck). A δ-functor is universal if it is effaceable.

2 Simplicial Method

Prop. (6.2.1) (Eilenberg-Zilber). The three kinds of geometrization of a bisimplicial set is the same: geometrization the diagonal simplicial set, the twice geometrization of left(resp. right) simplicial set.

3 Derived Category

Def. (6.3.1). A class of morphisms S in a category is called localizing if:

- S is closed under composition and has identity.
- for every $s \in S$ and f, there is a $t \in S$ and g, s.t. $f \circ t = s \circ g$(resp. $t \circ f = g \circ s$).
- the existence of a $s \in S$ s.t. $sf = sg$ is equivalent to the existence of $t \in S$ s.t. $ft = gt$.

This will generate a roof-dominating equivalence and make sure it is an equivalence relation.

Prop. (6.3.2). The isomorphisms in $D^*(A)$ is of the form $t \circ s^{-1}$. (look at the homology map they induced).

Prop. (6.3.3). If $\mathcal{B}$ is a full subcategory that $S \cap \mathcal{B}$ is a localizing category of $\mathcal{B}$ and any $s \in S$ can be 'denominated' in one given side (any one is OK) into $\mathcal{B}$, then $\mathcal{B}[S \cap \mathcal{B}^{-1}]$ is a full subcategory of $\mathcal{C}[S^{-1}]$. The proof is easy, use left roof or right roof.

Remark (6.3.4). Remember the translation operator $K[n]$ makes the complex lower n dimensions.

Prop. (6.3.5) (Distinguished Triangle). For any morphism $K^\bullet \to L^\bullet$, there exists a exact sequence of Complexes

$$0 \to K^\bullet \to \mathrm{Cyl}(f) \to C(f) \to 0$$

commuting with (in $K(\mathcal{A})$)

$$0 \to K^\bullet \to \quad L^\bullet \quad \to C(f) \to K^\bullet[1].$$

And $K^\bullet \to L^\bullet \to C(f) \to K^\bullet[1]$ is called a distinguished triangle. And exact triple of complexes in $\mathrm{Kom}(\mathcal{A})$ is quasi-isomorphic to an distinguished triangle.

A distinguished triangle will induce a long exact sequence, for this, just need to verify that the δ-homomorphism coincide with the morphism that $C(f) \to K^\bullet[1]$ induces.

Notice all this can imitate the similar parallel construction in the topology category.

Proof: Cf.[Gelfand P157] □

Def. (6.3.6). A **triangulated category** is an additive category with a T: additive automorphism and an isomorphism class of distinguished triangles satisfying the following axiams:

TR1) $X \xrightarrow{id} X \to 0 \to X[1]$ is distinguished. Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle.

TR2) A triangle is distinguished iff the helix it generate is distinguished.

TR3) Any two consecutive morphisms of two distinguished class can be extended to a morphism of distinguished class.

TR4) Any diagram of the type "upper cap" can be completed to a octahedron diagram.

Prop. (6.3.7). For a distinguished triangle and an object, two long exact sequence exists. In particular, composition of consecutive maps in a distinguished triangle is 0.

Thus the extension of TR3 of two isomorphisms is an isomorphism by 5-lemma. And so the completion in TR2 is unique.

Prop. (6.3.8). For Abelian category A, the categories $K^*(A)$ is triangulated. This is hard to verify, but it solves every problem. Cf[Gelfand P246].

Prop. (6.3.9). K is a triangulated category and a localizing class S compatible with T, i.e. $s \in S \iff T(s) \in S$ and the extension in $TR3$ of f, g in S is in S. Then the localizing category $K[S^{-1}]$ is triangulated.

Cor. (6.3.10). $D(\mathcal{A})$ is a triangulated category. And for a distinguished triangle, the long exact sequence exists. The distinguished triangle is just the obvious one.

Def. (6.3.11). The **derived category** of an Abelian category $D(\mathcal{A})$ represents the universal property that any functor to a category $\mathcal{A} \to \mathcal{C}$ s.t. quasi-isomorphisms is mapped to isomorphisms uniquely factors through $D(\mathcal{A})$.

It can be defined as the localization of quasi-isomorphisms, but the class of quasi-isomorphisms is not localizing. But one can prove the quasi-isomorphisms in $K(\mathcal{A})$ is localizing and the localization of quasi-isomorphisms in $K(\mathcal{A})$ is equivalent to $D(\mathcal{A})$. Cf.[Gelfand P159]

Notice that equivalent roofs induce the same map on homology, so the cohomology functor can be regarded defined on $D(\mathcal{A})$.

$$\mathcal{A} \to K(\mathcal{A}) \to K(\mathcal{A})[S^{-1}] = D(\mathcal{A}) \xrightarrow{H^*} \mathcal{A}.$$

Prop. (6.3.12).

4 Acyclic Elements and Derived Functors

Def. (6.4.1). For a left exact F, a class R of elements is called **adapted to** F if it is sufficiently large and $R^n F(X) = 0$, for $n > 0$ and $X \in R$, this is equivalent to that $Kom^+(\mathcal{R})$ is F-acyclic.

Injectives are F-acyclic for all left exact F because $\mathrm{id} : I^\bullet \to I^\bullet$ is homotopic to 0, Cf(6.4.4).

Prop. (6.4.2) (Acyclic Criterion). If a class T of elements in an Abelian category of enough injectives is:

- sufficiently large.
- If $A \oplus A' \in T$ implies $A \in T$. (This implies all injectives are in T).
- Cokernel of elements of T is in T and $0 \to F(A) \to F(A') \to F(\mathrm{Coker}) \to 0$ is exact. (To use induction).

Then T is adapted to F.

Prop. (6.4.3). For a class of objects $\mathcal{R}$ in $\mathcal{A}$ stable under finite direct sum and are adapted to a left exact functor F, i.e. $Kom^+(\mathcal{R})$ is F-acyclic and every object in $\mathcal{A}$ is a subobject of an object from $\mathcal{R}$. Just need to verify the condition of(6.3.3). Similarly for the opposite category.

And in this case $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ is equivalent to $D^+(\mathcal{A})$.

Proof: The hard part is to prove every complex in $K^+(\mathcal{A})$ is quasi-isomorphic to a complex in $K^+(\mathcal{R})$, for this, use direct construction. Cf.[Gelfand P187]. □

Lemma (6.4.4). If s is a quasi-isomorphism between an object from $K^+(\mathcal{I})$ to an object from $K^+(\mathcal{A})$, then there exists a reverse map to compose to id_I. Cf.[Gelfand P180]

Prop. (6.4.5).

Cor. (6.4.6). $K^+(\mathcal{I})$ is a saturated subcategory of $D+(\mathcal{A})$. And if $\mathcal{A}$ has enough injectives, this is an equivalence of category. (We only need to verify that the localization of $K^+(\mathcal{I})$ is itself, using the last proposition). In particular, this applies to Grothendieck categories. Cf.[Gelfand P179].

Cor. (6.4.7). If $\mathcal{A}$ contains sufficiently many injectives, then injective objects are adapted to any left exact functor F. (Because id on acyclic injective complexes is homotopic to 0 by the lemma).

Prop. (6.4.8). In an Abelian category, the direct summand of a projective object is projective. (The summand has definition in an Abelian category).

Def. (6.4.9) (Derived functor). If a left exact functor between Abelian categories has an adapted class, then by preceding proposition, $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ is equivalent to $D^+(\mathcal{A})$, then we can use to define the **derived functor** $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ satisfying the following universal property:

RF is exact, i.e.respect the distinguished structure, and there is a natural isomorphism

$$\varepsilon_F : Q_{\mathcal{B}}\ K^+F \to RF\ Q_{\mathcal{A}}.$$

Moreover, any other exact $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and a similar transformation must factor through ε_F uniquely. Thus this RF is unique up to natural isomorphism. It is just F^+ on $K^+(\mathcal{R})$.

Notice there is a more general derived functor that use inductive limits in $\hat{A}$ that it maps $D^*(\mathcal{A})$ to $\mathrm{Ind}(D^*(\mathcal{B}))$, and if it has image in the subcategory of representable objects, then it coincide with RF. Similarly for right exact functor F. (This is easy to check)Cf.[Gelfand P198].

Yet there is another way to just look at the derived functors, it is the hypercohomology of the Cartan-Eilenberg resolution of the complex.**??**

Prop. (6.4.10). $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Abelian categories and $F : \mathcal{A} \to \mathcal{B}, G : \mathcal{B} \to \mathcal{C}$ are left exact functors. If $F(R_{\mathcal{A}}) \subset R_{\mathcal{B}}$, where $R_{\mathcal{A}}$ is the adapted class of F, then $R(G \circ F)$ and $RG \circ RF$ is natural isomorphic. (The definition of RF is just F on $K^+(R_{\mathcal{A}})$).

Prop. (6.4.11). The derived functor forms a universal δ-functor.

Cartan-Eilenberg Resolutions

Prop. (6.4.12) (Horseshoe Lemma). For a exact sequence $0 \to X_1 \to X \to X_2 \to 0$ and a injective resolution of X_1 and X_2, there is a injective resolution of X commuting with them. (Choose them one-by-one, in fact, $I_n = I_n^1 \oplus I_n^2$. Snake lemma told us that the cokernel is an exact sequence, use that to define the next one.

Prop. (6.4.13). For two lifting of morphisms $X_1 \to Y_1$ and $X_2 \to Y_2$, there is a lifting of the morphism $X \to Y$ compatible with that. Cf.[Weibel P2.4.6].

Prop. (6.4.14) (Cartan-Eilenberg Resolution). If $\mathcal{I}_\mathcal{B}$ is sufficiently large, for any K in $K^+(\mathcal{B})$ there is a functorial Cartan-Eilenberg resolution, that is, It induces simultaneous injective resolutions of K^n, Z^n, B^n and H^n. Moreover, the resolution for $B^i \to Z^i \to H^i$ and $Z^i \to K^i \to B^{i+1}$ splits.

This is achieved by the functoriality of resolutions, it is natural and induces a functor from $K^+(\mathcal{B})$ to $K^{++}(\mathcal{I}_\mathcal{B})$. Cf., [Gelfand P210].[Weibel P146].

For a CE resolution, the spectral sequence can be applied, one side gets us: $K \to \mathrm{Tot}(L)$ is a quasi-isomorphism, i.e. $\underline{\mathrm{Tot}(L) \text{ is a injective resolution of } K}$. so $RG(K) = G(\mathrm{Tot}L)$ in $D(C)$

Prop. (6.4.15) (Hypercohomology). we can define the **hypercohomology** of a left exact functor as $H^n(\mathrm{Tot}^\Pi F)$ if $\mathcal{B}$ satisfies AB3*.

Dually we can define the **hyperhomology** if $\mathcal{A}$ satisfies AB3* and AB4* and $\mathcal{B}$ satisfies AB3.

For complexes in $K^+(\mathcal{A})$, there is no restriction and everything is smooth.

When the Abelian category $\mathcal{A}$ satisfies AB3* and AB4*, i.e. the direct product is exact, then Tot^Π of the Cartan-Eilenberg resolution of any complex is a quasi-isomorphism to it by the dual of(6.6.9). (Take horizontal filtration, AB4* assures it collapse).

Prop. (6.4.16) (K-injective). Using(6.4.4), we can show when $Y^\bullet \in \mathrm{Ob}\,\mathrm{Kom}^+(\mathcal{I})$ or $X^\bullet \in \mathrm{Ob}\,\mathrm{Kom}^-(\mathcal{P})$,

$$\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, Y^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet)$$

is an isomorphism, thus they are K **-injectives**, i.e. $\mathrm{Hom}_{K(\mathcal{A}}(M^\bullet, I^\bullet) = 0$ for any acyclic $M^\bullet$ in $K(\mathcal{A})$. Thus the injective resolution is unique in K^+.

Cor. (6.4.17). Then we get that the definition of $\mathrm{Ext}^n(X, Y)$ as $\mathrm{Hom}_{D(\mathcal{A}}(X[0], Y[-n])$ is equivalent to the usual definition. And it also corresponds to the derived functor of $\mathrm{Hom}(X, -)$.

Lemma (6.4.18). The K-injective resolution is natural in $K(\mathcal{A})$. Cf.[Stack Project].

Ext & Tor

Prop. (6.4.19). $\mathcal{A}$ is categorically equivalent to the subcategory of $D(\mathcal{A})$ that has only H^0 nonzero. If we define $\mathrm{Ext}^i_\mathcal{A}(X, Y)$ as $\mathrm{Hom}_{D(\mathcal{A})}(X[0], Y[i])$, then it is equivalent to the i-term extension of Y by X, and it is an abelian group. We have a

$$\mathrm{Ext}^i_\mathcal{A}(X, Y) \times \mathrm{Ext}^i_\mathcal{A}(Y, Z) \to \mathrm{Ext}^{i+j}_\mathcal{A}(X, Z)$$

by composition or equivalently the conjunction of extensions.

Proof: Cf.[Gelfand P167] □

Prop. (6.4.20). In an Abelian category, the extension $\mathrm{Ext}^1(N, M)$ is equivalent with the extensions with Baer sum.

Proof: We choose a projective resolution $0 \to K \to P \to N \to 0$, so $\mathrm{Hom}(K, M) \to \mathrm{Ext}^1(N, M)$ is surjective, so choose a lifting and the pushout $0 \to M \to L \to N \to 0$ with be the corresponding extension, Now the Baer sum is easy to define and verify. □

5 Homological Dimension

Prop. (6.5.1). If $\mathcal{A}$ has enough projectives, then the projective dimension of an object X is the length of projective resolutions. (Use resolution and long sequence).

Prop. (6.5.2) (Hilbert Theorem). For an Abelian category $\mathcal{A}$, the category $\mathcal{A}[T]$ is an Abelian category. If $\mathcal{A}$ has enough projectives and have infinite direct sum, then $\mathrm{dhp}_{\mathcal{A}[T]}(X, t) \leq \mathrm{dhp}_{\mathcal{A}}(X) + 1$ and equality with $t = 0$.

Cor. (6.5.3). The Categories Ab and $K[X]$-mod have homological dimension 1. $K[X_i, \ldots, X_k]$ has homological dimension k.

6 Spectral Sequence

Reference for this section is [Weibel Ch5]. All the definition below is dual for homology and cohomology, just rotate the diagram 180 degree.

Def. (6.6.1). A convergent **Spectral Sequence** is a three-dimensional arrange of entries $E_r^{p,q}$ that:

1. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ that $d_r d_r = 0$.
2. $H^{p,q}(E_r^{p,q}) \cong E_{r+1}^{p,q}$. And $E_r^{p,q}$ has a direct limit $E_\infty^{p,q}$.
3. There is a complex E^n and a decreasing bounded filtration $F^p E^n$ on each E^n and $E_\infty^{p,q} \cong F^p E^{p+q} / F^{p+1} E^{p+q}$.

For a morphism of spectral sequences, if it defines an isomorphism for some r, then by five-lemma, it defines isomorphisms afterward, so it defines an isomorphism on E_∞^{pq}.

Def. (6.6.2). We say a (co)homology filtration is bounded below $F_{n_s} E_n = 0$ for some n_s, bounded above $F_{n_s} E_n = E_n$ for some n_s. It is exhaustive iff $\cap F_i E_n = E_n$. The spectral sequence is called regular iff $d_{pq}^r = 0$ for sufficiently large r.

Def. (6.6.3) (Spectral Sequence of a Filtered Complex). For a complex $K^\bullet$ and a filtration $F^p K^n$ on K^n, we have a natural spectral sequence

$$E_1^{pq} = H^{p+q}(F^p E^{p+q} / F^{p+1} E^{p+q}), \quad E^n = H^n(K^\bullet), \quad F^p E^n = H^n(F^p K^\bullet).$$

For a morphism of filtered complexes that are isomorphism for some r, induction on the exact sequence $0 \to F^p E^n \to F^{p+1} E^n \to E_\infty^{p,n-p}$ and use five-lemma shows it induces isomorphism on $H^* E$.

Prop. (6.6.4) (Classical Convergence). For homology, if the filtration is bounded below and exhaustive for all n, we have a convergence to E_n. Cf[Gelfand P203] for cohomological case and [Weibel P133] for homological case.

Prop. (6.6.5) (Complete convergence). For homology, if the filtration is complete, exhaustive, bounded above, and the spectral sequence is regular, then the spectral sequence converges to E_n.

There are two examples, the stupid filtration and the canonical filtration, the canonical filtration is natural and factors through $D(\mathcal{A})$.

Prop. (6.6.6) (Spectral Sequence of a Double Complex). A double complex has two natural filtration of the total complex, they defines two spectral sequence, one has

$$E_{2,x}^{p,q} = H_x^p(H_y^{\bullet,q}(L^{\bullet,\bullet}))$$

and the other has

$$E_{2,y}^{p,q} = H_y^p(H_x^{q,\bullet}(L^{\bullet\bullet})).$$

Cf.[Gelfand P209]. In fact under reflection, there is only one spectral sequence. For the horizontal filtration, the differential goes vertical first, for the vertical filtration, the differential goes horizontal first. The differential goes one way, the convergence goes reversely.

If both the filtration is finite and bounded, in particular if E is in the first quadrant, then they both converges to $H^n(E)$, this will generate important consequences.

Cor. (6.6.7). If a double complex in the first quadrant has its all column acyclic (3rd-quadrant pointing), then the total complex is acyclic. Thus a morphism of double complex inducing quasi-isomorphism on each column induces a quasi-isomorphism on the total complex.

If a double complex has $H_p(C_{*,q}) = 0, \forall p > 0, q$, then

$$H_n(\mathrm{Tot}C_{*,*}) = H_n(\mathrm{Coker}(C_{1,*} \to C_{0,*})$$

Prop. (6.6.8) (Horizontal Filtration). For a second-quadrant free homology double complex, the filtration is bounded below and exhaustive for $\mathrm{Tot}^{\oplus}$, so the classical convergence applies.

For a fourth-quadrant free homology double complex, the filtration is complete and exhaustive and regular ? for Tot^{Π}, so the complete spectral sequence applies. Cf.[Weibel P142].

Prop. (6.6.9) (Vertical Filtration). For a fourth-quadrant free homology double complex, the filtration is bounded below and exhaustive for $\mathrm{Tot}^{\oplus}$, so the classical convergence applies.

For a second-quadrant free homology double complex, the filtration is complete and exhaustive and regular ? for Tot^{Π}, so the complete spectral sequence applies. Cf.[Weibel P142].

Cor. (6.6.10) (Five lemma).

Cor. (6.6.11) (Snake lemma).

Cor. (6.6.12) (Grothendieck Spectral Sequence). $\mathcal{A},\mathcal{B},\mathcal{C}$ be Abelian categories and $F:\mathcal{A}\to\mathcal{B}, G:\mathcal{B}\to\mathcal{C}$ are left exact functors. If $R_{\mathcal{A}} = \mathcal{I}_{\mathcal{A}}$, $R_{\mathcal{B}} = \mathcal{I}_{\mathcal{B}}$, and $F(I_{\mathcal{A}}) \subset I_{\mathcal{B}}$, then there is a spectral sequence with $E_2^{p,q} = R^pG(R^qF(X))$ (to upper left) that converges to $E^n = R^n(G\circ F)(X)$. And this spectral sequence is functorial in X.

In particular, this applies to F is a right adjoint and its left adjoint is exact.

Proof: Let $K = F(I_X) = RF(X)$, and choose the CE resolution of K, because the resolutions for $B^i \to Z^i \to H^i$ and $Z^i \to K^i \to B^{i+1}$ split and G is additive, we have

$$H_x^{p,\bullet}(G(L^{\bullet,\bullet})) = G(H_x^{p,\bullet}(L^{\bullet,\bullet})) = RG(H^p(K))$$

So

$$E_{2,y}^{p,q} = H_y^q(H_x^{p,\bullet}(L^{\bullet\bullet})) = R^pG(H^q(K)) = R^pG(R^qF(X))$$

and

$$E^\bullet = RG(\mathrm{Tot}(L)) = G(\mathrm{Tot}(L)) = RG(K) = RG \circ RF(X) = R(G \circ F)(X)(6.4.10).$$

□

Cor. (6.6.13). The low degree parts read:

$$0 \to R^1G(F(A)) \to R^1(G \circ F)(A) \to G(R^1F(A)) \to R^2(G(F(A)) \to R^2(G \circ F)(A).$$

(Check definition). More generally, if $R^pG(R^qF(A)) = 0, 0 < q < n$, then

$$R^mG(F(A)) \cong R^m(G \circ F)(A) \quad m < n$$

And

$$0 \to R^nG(F(A)) \to R^n(G \circ F)(A) \to G(R^nF(A)) \to R^{n+1}(G(F(A)) \to R^{n+1}(G \circ F)(A).$$

The Grothendieck spectral sequence is tremendously important.

Cor. (6.6.14). For chain complex K in $K^+(\mathcal{A})$ and a left exact functor F, the CE resolution will generate two spectral sequences: $E_{2,x}^{p,q} = H_x^p(R^qF(A_\bullet))$ and the other has $E_{2,y}^{p,q} = R^pF(H^q(A))$ that converges to the hypercohomology $\mathbb{R}^{p+q}F(K)$. Dually for derived homology.

7 Tor and Ext

Prop. (6.7.1). $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A)$. This can be seen using spectral sequence of the double complex of flat resolutions of A and B.

Prop. (6.7.2) (Base Change). Using projective resolution and spectral sequence, we have a first quadrant homology spectral sequence:

$$E_{pq}^2 = \mathrm{Tor}_p^S(\mathrm{Tor}_q^R(A, S), B) \Rightarrow \mathrm{Tor}_{p+q}^R(A, B).$$

Similarly, for Ext,

$$E_2^{pq} = \mathrm{Ext}_S^p(A, \mathrm{Ext}_R^q(S, B)) \Rightarrow \mathrm{Ext}_R^{p+q}(A, B).$$

Prop. (6.7.3) (Universal Coefficient Theorem). Let P be a free R-module so $d(P_n)$ are all flat, then $Z(P_n)$ are also flat and

$$0 \to d(P_{n+1}) \to Z_n \to H_n \to 0$$

is a free resolution. we have an exact sequence:

$$0 \to H_n(P) \otimes_R M \to H_n(P \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(P), M).$$

$$0 \to \mathrm{Ext}_R^1(H_{n-1}(P), M) \to H^n(\mathrm{Hom}_R(P, M)) \to Hom(H_n(P), M) \to 0$$

and these exact sequences non-canonically split because Z_n is a direct summand of P_n, thus $Z_n \otimes M$ is a direct summand of $P_n \otimes M$ and a fortiori $Z_n(P_n \otimes M)$. so $H_n(P) \otimes M$ is a direct summand of $H_n(P \otimes_R M)$.

Inverse Limit

Prop. (6.7.4). The derived functor of lim from $K^+(\mathcal{A}) \to \mathcal{A}$ is Coker $(a_i) \to (a_i - a_{i+1})$ for $\mathcal{A}$ Abelian, has enough injectives and satisfies $AB4^*$ (R-mod). $\lim^1$ vanishes for a complex that satisfies Mittag-Leffler conditions.

Proof: If A satisfies the M-L condition, the essential image $\{B_i\}$ is surjective so acyclic and $\{A_i/B_i\}$ is acyclic because the inverse image can be defined as a finite sum. So the long exact sequence gives A is acyclic.

The δ-functor is defined by the snake lemma and $AB4^*$ and we have to prove it is effaceable. For this, we use(1.2.8) to see that $E = \prod_k k_* A_k$ exists in $\mathcal{A}^C$ is injective and $A \to E$ is an injection. In this case E is a product of towers $\cdots \to A_k \to A_k \to 0 \to 0 \to \cdots$, hence surjective by $AB4^*$ so is acyclic. □

For applications, Cf.[Weibel P82].

I.7 Lie Algebra

Note:[Lie Algebras of Finite and Affine Type Carter] is far more better than [Hymphreys].

1 Main Theorems

Prop. (7.1.1) (Engel). If all elements of L are ad-nilpotent, then L is nilpotent.

Proof: only need to show that If a subalgebra of $GL(n)$ consists of nilpotent elements, then there is a common 0-eigenvector. Use Induction, choose a maximal subalgebra of L, then it must be of codimension 1,$L = K + Fz$. Thus the 0-eigenvector for K is a nonzero subspace, and a 0-eigenvector for z will suffice. □

Prop. (7.1.2) (Lie's theorem). Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a solvable lie algebra. Then there exists a vector $v \in V$ which is a common eigen vector for all $X \in \mathfrak{g}$.

Proof: Idea is to prove by induction on dimension of $\mathfrak{g}$.

Produce a codimension 1 ideal $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g}$ be generated (as a vector space) by $\mathfrak{h}$ and Y.Being a subalgebra of solvable algebra $\mathfrak{g}$, $\mathfrak{h}$ is itself a solvable lie algebra. Apply induction step on $\mathfrak{h}$ and choose $v \in V$ such that v is an eigenvector for all $X \in \mathfrak{h}$.

The idea is to consider set W all common eigenvectors of elements of $\mathfrak{h}$ and produce an eigenvector of Y from this W. Let

$$W = \{v \in V | X(v) = \lambda(X)v \ \forall \ X \in \mathfrak{h} \text{ for a fixed } \lambda(X) \in \mathbb{C}\}.$$

Suppose W is an invariant subspace of Y, we then have restriction map $Y : W \to W$. As we are in complex vector space (algebraically closed) there exists an eigenvector for Y in W say w_0. Thus, w_0 is common eigenvector for all elements of $\mathfrak{g}$.

It remains to show that W is an invariant subspace of Y i.e., $Y(w) \in W$ for all $w \in W$ i.e., given $X \in \mathfrak{h}$, we need to have $X(Y(w)) = \lambda(X)Y(w)$.

Let $w \in W$, we have

$$\begin{aligned} X(Y(w)) &= Y(X(w)) + [X, Y](w) \\ &= Y(\lambda(X)w) + \lambda\left([X, Y]\right) w \\ &= \lambda(X)Y(w) + \lambda\left([X, Y]\right) w \end{aligned}$$

This is almost the same as what we want but with an extra term $\lambda\left([X, Y]\right) w$. Suppose we prove $\lambda\left([X, Y]\right) = 0$ for all $X \in \mathfrak{h}$ then we are done.

Then considers subspace U spanned by elements $\{w, Y(w), Y^2(w), \cdots\}$ and then says that U is invariant subspace of each element of $\mathfrak{h}$ and (assuming n is the smallest integer such that $Y^{n+1}w$ is in the subsapce generated by $\{w, Y(w), \cdots, Y^n(w)\}$) representation of an element Z of $\mathfrak{h}$ with the basis $\{w, Y(w), \cdots, Y^n(w)\}$ is an upper triangular matrix with $\lambda(Z)$ in the diagonal. So, $tr(Z) = n\lambda(Z)$.

So, $tr([X, Y]) = n\lambda([X, Y])$. As $[X, Y] = XY - YX$, we have $\mathrm{tr}([X, Y]) = \mathrm{tr}(XY) - \mathrm{tr}(YX) = 0$. Thus, $\lambda([X, Y]) = 0$ and we are done. □

Def. (7.1.3). A lie algebra is called semisimple if the maximal solvable ideal (Rad L)= 0.

Prop. (7.1.4) (Weyl). Representation of a semisimple lie algebra is completely reducible.

Proof: Cf.[Humphreys P28]. □

Prop. (7.1.5) (Cartan's Criteria for Solvability). If $\mathfrak{g}$ is a Lie algebra $\subset \mathfrak{gl}_n$, then

$$\mathfrak{g} \text{ is solvable } \iff Tr(xy) = 0,\ \forall x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}].$$

Note that a Lie algebra is solvable if the adjoint representation is solvable because the kernel is abelian. Cf.[Humphreys P20]

Prop. (7.1.6) (Cartan Criteria for Semisimplicity). A lie algebra is semisimple $\iff$ the Killing form is non-degenerate. Cf.[Humphreys P22].

Proof: Just show that the kernel of the Killing form is a solvable ideal and that $\mathrm{ad}x\cdot\mathrm{ad}y$ is nilpotent for x in an abelian ideal. □

Prop. (7.1.7). If L is semisimple, then every derivative of L is inner.

Proof: Cf.[Humphreys P23]. □

Prop. (7.1.8) (Abstract Jordan Decomposition). Let L be a semisimple lie algebra and $\phi : L \to GL(v)$ be a representation. If $x = s + n$ is the abstract Jordan decomposition of x, then $\phi(x) = \phi(s) + \phi(n)$ is the usual Jordan decomposition of $\phi(x)$.

Proof: In fact, we only need to prove that if L is a semisimple algebra $\subset \mathfrak{gl}(V)$, then L contains the semisimple and nilpotent element of all its element. Because the image of L is semisimple and the usual Jordan decomposition must be its abstract decomposition. The last assertion is due to the fact that if z is semisimple(nilpotent), then $\mathrm{ad}_{\mathfrak{gl}_n} z$ is semisimple(nilpotent), thus so do $\mathrm{ad}_L z$.

Cf.[Humphreys P27] for the following proof. □

Prop. (7.1.9) (Baker-Campbell-Hausdorff cor).

$$\exp(X)\exp(Y) = \exp(X+Y+1/2[X,Y]+1/12[X,[X,Y]]-1/12[Y,[Y,X]]+\text{higher order terms})$$

Cf.[Hall Lie algebras GTM222 P76].

2 Reductive Lie Algebra

Prop. (7.2.1). A lie algebra is called reductive if $\mathrm{Rad}(L) = Z(L)$.

1. If L is reductive, then L is completely reducible ad L-module.
2. $L = [LL] \oplus Z(L)$.
3. If $L \subset GL(V)$ acting irreducibly on V, then L is reductive with $\dim \mathrm{Rad}(L) \leq 1$. In particular, If $L \in SL(V)$ and $\mathrm{char} F \neq 0$, it must be semisimple. This can be used to prove that all classical algebras are semisimple. And the diagonal matrix will be toral and finding a set of simple roots will suffice to prove that every calssical lie algebra is simple.
4. If L is a completely reducible ad L-module, then L is reductive.
5. If L is reductive, then all finite dimensional representations of L in which $Z(L)$ is represented by semisimple endomorphism are completely reducible.

6. If $[LL]$ is semisimple, then L is reductive.

Proof: (1): Because $L/Z(L)$ is a semisimple lie algebra and $Z(L)$ is mapped to the kernel.

(2): Let $L = M \oplus Z(L)$ as a ad-L module, then $[LL] \subset [MM] \subset M$, but $[LL]$ maps onto $L/Z(L)$ because a semisimple is a sum of simple algebra. So $[LL] = M$.

(3): Cf.[Humphreys P102].

(4): In this way L decompose into $Z(L)$ and simple algebras, so it is reductive.

(5): First simultaneously diagonalize $Z(L)$, then the subspace corresponding to different characters are stable under L. Then decompose w.r.t. $[LL]$ with get the result. (6): Note that the element in $\mathrm{Rad}(L)$ will all be central. $\square$

Prop. (7.2.2). Let L be a simple lie algebra, then any two symmetric associative bilinear forms on L is proportional. Because any of this form corresponds to a L-morphism from L to L^*. In particular, when $L \subset \mathfrak{gl}_n$, the usual trace is proportional to the Killing form.

3 Real Lie Algebra

Def. (7.3.1). A **compact real form** is a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ s.t. $\mathfrak{g}$ is the complexification of $\mathfrak{l}$ and $\mathfrak{l}$ is the lie algebra of a compact simply-connected Lie group.

Prop. (7.3.2). A real Lie algebra is compact iff there exists a inner product s.t.

$$([X, Y], Z) + (X, [Y, Z]) = 0,$$

iff the Killing form is negative definite.

Proof: One direction is easy, just use the average method to find a G-invariant inner product and then take derivative. For the other direction, the identity shows that a complement of an ideal is an ideal so $\mathfrak{g}$ is decomposed into simple lie groups and reduce to the case that $\mathfrak{g}$ is simple. The ideal is to show that $\mathfrak{g} \cong \mathrm{ad}(\mathfrak{g})$ is the whole outer derivative group $\partial(\mathfrak{g})$(the following lemma). So $\mathfrak{g}$ equals to the identity component of $\mathrm{Aut}(\mathfrak{g})$ which is a closed subgroup thus closed but it is also a subgroup of the compact group $O(\mathfrak{g})$ thus it is compact. $\square$

Lemma (7.3.3). If a real semisimple Lie algebra X has a invariant inner product, then every outer derivative is inner.(In fact, this is true by Cartan Criterion for semisimplicity (7.1.7).

Proof: since $\mathrm{ad}(X)$ is skew-symmetric, it's diagonalizable and its eigenvalue is pure imaginary, so the Killing form of X is negative definite. Now choose the complement $\mathfrak{a}$ of $\mathrm{ad}(X)$ in $\partial(X)$, then $\mathfrak{a} \cap X = 0$. Thus for $D \in \mathfrak{a}$, $\mathrm{ad}(D(g)) = [D, \mathrm{ad}(g)] = 0$ for all g in X, so $D = 0$, thus $\mathrm{ad}(X) = \partial(X)$. $\square$

Prop. (7.3.4). -

1. The complexification of the Lie algebra of a connected compact Lie group is reductive.
2. A complex Lie algebra is semisimple iff it is isomorphic to the complexification of the Lie algebra of a simply-connected compact Lie group. i.e. every complex semisimple Lie algebra has a compact real form.

Proof: 1: Because a connected compact Lie group is completely reducible so the does the Lie algebra and so does the complexification. So it is reductive by (7.2.1)4.

2: Cf.[Varadarajan Lie Groups Lie algebras and Their Representations]. The ideal is to find a real form whose corresponding simply-connected group is compact. $\square$

Prop. (7.3.5). If $\mathfrak{g}$ is the Lie algebra of a matrix Lie group G, then:

1. every Cartan subalgebra comes from a maximal commutative subalgebra of a compact real form and any two Cartan subalgebras are conjugate under the Ad-action of G.
2. any two compact real form is conjugate under the Ad-action of G.
3. any two maximal commutative subalgebra of a compact real form is conjugate under the Ad-action of the corresponding compact compact subgroup.

Prop. (7.3.6). A real Lie algebra is semisimple iff its complexification is semisimple. Cf.[Varadarajan].

Cor. (7.3.7). The real Lie algebra of a compact simply-connected group is semisimple.

Note: For the classification of real semisimple Lie algebras, Cf.[李群讲义项武义 §6]

Prop. (7.3.8). If a complex representation of a Lie group admits an invariant bilinear form, then it is non-degenerate and unique. In fact, this is equivalent to a G-map from V to V^*. Thus there is unique invariant inner product in a compact real form by the preceding proposition.

4 Universal Enveloping Algebra

Prop. (7.4.1) (Chevalley). The center of the universal enveloping algebra is isomorphic to the polynomial ring over $\mathbb{C}$ of l elements, where L is a semisimple lie algebra of rank l. In particular, The center for $\mathfrak{sl}_2$ is the algebra generated by the Casimir element $1/2h^2+ef+fe$.

Proof: Because there is a commutative diagram of isomorphisms of algebras:

$$\begin{array}{ccc} S(L)^G & \xrightarrow{\alpha} & P(L)^G \\ \big\downarrow{\scriptstyle\eta} & & \big\downarrow{\scriptstyle\phi} \\ S(H)^W & \xrightarrow{\beta} & P(H)^W \end{array}$$

Where P is the polynomial ring$\cong S(L^*)$, the horizontal is Killing isomorphisms and vertical is the restriction maps. Cf.[Carter Theorem 13.32].

The twisted Harish-Chandra map gives an isomorphism of algebras $Z(L)\to S(H)^W$ (It just maps $z\in Z(L)$ to its pure H part and transform every indeterminants h_i to h_i-1). e.g. $z=h^2+2h+1+4fe\in Z(\mathfrak{sl}_2)$ is maped to h^2 in $S(H)$. And $P(H)^W$ is isomorphic to a polynomial ring in l generators over $\mathbb{C}$. $\square$

5 Lie Algebra Cohomology

I.8 Quantum Groups

1 Clifford Algebra

Prop. (8.1.1). Let $Cl_{r,s}$ denote the real Clifford algebra of signature $r-s$, then

$$Cl_{1,0} \cong \mathbb{C}, \quad Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}, \quad Cl_{2,0} \cong \mathbb{H} \subset M(2,\mathbb{C}), \quad Cl_{0,2} \cong R(2) = M(2,\mathbb{R}),$$

And we have

$$Cl_{n+2,0} \cong Cl_{0,n} \otimes Cl_{2,0}, \quad Cl_{0,n+2} \cong Cl_{n,0} \otimes Cl_{0,2}.$$

by the mapping $e_i \to e_i \otimes e_1' e_2'$, $e_{n+j} \to 1 \otimes e_j'$.

So we have

$$Cl_{n+8,0} \cong Cl_n \otimes \mathbb{R}(16), \quad \mathbb{C}l_{n+2,0} = Cl_{n+2,0} \otimes \mathbb{C} = \mathbb{C}l_{n,0} \otimes \mathbb{C}(2).$$

because $\mathbb{H} \otimes \mathbb{C} = \mathbb{C}(2)$, and

$$\begin{bmatrix} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ Cl_{n,0} & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{H}\oplus\mathbb{H} & \mathbb{H}(2) & \mathbb{C}(4) & \mathbb{R}(8) & \mathbb{R}(8)\oplus\mathbb{R}(8) \\ \mathbb{C}l_{n,0} & \mathbb{C} & \mathbb{C}\oplus\mathbb{C} & & & & & & \end{bmatrix}$$

The Clifford algebra is a $\mathbb{Z}_2$-graded algebra, $Cl = Cl^0 \otimes Cl^1$ and $Cl_{n-1} \cong Cl_n^0$ by the mapping $e_i \to e_i \otimes e_{n+1}$. This is in fact the decomposition of the chirality operator $\Gamma = (-1)^{\lfloor \frac{n+1}{2} \rfloor} e_1 e_2 \ldots e_n, \Gamma^2 = 1$.

Prop. (8.1.2). For n even, $\mathbb{C}(V)$ is naturally isomorphic to $\mathrm{End}_{\mathbb{C}}(\Lambda^* W)$, where $W = \{\frac{1}{\sqrt{2}}(e_{2i-1} - ie_{2i})\}$. This isomorphism is not obvious and restrict to a Spinor representation of $\mathrm{Spin}(n)$ and $\rho(\Gamma)^2 = 1$ induce two representations of $Cl(n)^0$, in particular $\mathrm{Spin}(n)$, called the **(half Spinor representations)**. This has a unique extension to representation of Spin^c. $\wedge^* W$ comes with a Hermitian metric which is preserved by the action of $\mathrm{Pin}(n)$ (check). So the image is $\mathrm{SO}(n)$ is in $SO(\wedge^* W)$. Cf.[Jost Geometric analysis P72].

Def. (8.1.3). denote $\mathrm{Pin}(n)$ as the group in Cl_n generated by v_i of norm 1. Because $v_i \cdot v_i = -1$, it is a group. And denote $\mathrm{Spin}(n)$ as the subgroup of $\mathrm{Pin}(n)$ generated by even number of v_is.

So the conjugation action $-Ad = v(-)v$ =reflection w.r.t v, maps $\mathrm{Pin}(n)$ to $O(n)$ and $\mathrm{Spin}(n)$ to $SO(n)$.

Prop. (8.1.4). The kernel of this mapping is $\{\pm\}$ when n is even. This is a double covering of $SO(n)$ and $O(n)$, it is nontrivial because $\{\pm\}$ is connected by $(\cos te_1 + \sin te_2)(\cos te_1 - \sin te_2)$.

Proof: Let $\alpha = e_i\beta + \gamma$, then $\beta, \gamma \in Cl^0$ and so $\alpha = ce_1 \ldots e_n + d$, and c can happen only when n is odd. □

Prop. (8.1.5). As in (9.3.1) $SU(2)$ is a universal covering of $SO(3)$ and so does $\mathrm{Spin}(3)$(8.1.4), so $SU(2) \cong \mathrm{Spin}(3)$.

Prop. (8.1.6). $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$ because of the action of $\mathbb{H} \times \mathbb{H}$ on $\mathbb{H} : x \to ux\overline{v}$. This map is a two cover of $SO(4)$.

Prop. (8.1.7). $\mathrm{Spin}(5) = \mathrm{Sp}(2)$ and $\mathrm{Spin}(6) = SU(4)$.

Chapter II

Number Theory & Arithmetic Geometry

II.1 Algebraic Number Theory

1 Basics

Prop. (1.1.1).

$$\mathrm{G}(\mathbb{Q}[\mu_n]/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*.$$

Proof: We choose a prime p prime to n and show that μ_n^p is conjugate to μ_n.

Let $X^n - 1 = f(X)h(X)$ with f(X) minimal polynomial of μ_n.If $f(\mu_n^p) \neq 0$, then $h(\mu_n^p) = 0$,thus $h(X^p) = f(X)g(X)$.So module p, $X^n - 1$ has a multi root, which is wrong. □

Prop. (1.1.2).

$$\mathrm{Gal}(F_{p^n}/F_p) = \mathbb{Z}/n\mathbb{Z}.$$

Proof: Generated by Frobenius. □

Prop. (1.1.3). The ring of integers in the cyclotomic field is generated by the roots of identity.

Proof: First consider the case n a prime power. Because $d(1, \zeta, \cdots, \zeta^{d-1}) = \pm l^s$, $l^s\mathcal{O} \subset \mathbb{Z}[\zeta] \subset \mathcal{O}$. Because p totally splits, $\mathcal{O} = \mathbb{Z}[\zeta] + \pi\mathcal{O}$, thus $\mathcal{O} = \mathbb{Z}[\zeta] + \pi^t\mathcal{O}$. Choose $t = s\phi(n)$ yields $\mathbb{Z}[\zeta] = \mathcal{O}$.

Then for different p, the fields are disjoint and the discriminant are pairwisely coprime, thus by (2.11) in Neukirch, the products of the integral basis form an integral basis. □

Prop. (1.1.4) (Krasner's Lemma). In a separable extension, if $|\beta - \alpha| < |\alpha_i - \alpha|$ for any conjugate α_i of α, then $K(\alpha) \subset K(\beta)$.

Prop. (1.1.5) (Ostrowski). Any non-trivial value on $\mathbb{Q}$ is equivalent to v_p or $|\ |$. Thus any complete Archimedean field is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

Prop. (1.1.6) (Hilbert's Multiplicative Theorem 90). $H^1(G(L/K), L^*)$ for Galois extension L/K.

2 Ramification Theory

Prop. (1.2.1). If a prime $\mathfrak{p}$ splits completely in two separable extension LM of $\mathbb{K}$, then it also splits completely in the composite LM.

Proof: We use the language of valuation. The extension of a valuation v of K corresponds to the set of equivalent classes of algebra map from L to $\overline{K_v}$ module conjugacy over K_v. So We only need to show that two different maps of LM are not conjugate over K_v. But the restrict of them to L or M is different, thus non-conjugate over K_v. □

Cor. (1.2.2). A prime splits completely in a separable extension L if it splits completely in the Galois closure N of L.

Proof: This is because the Galois closure is the composite of the conjugates of L.

But it also can be proven directly : Set $H = \mathrm{Gal}(N/L)$,$\mathcal{P}$ a prime of N over $\mathfrak{p}$, then

$$H\backslash G/G_{\mathcal{P}} \longrightarrow \{\text{Primes of } L \text{ over } \mathfrak{p}\}, \quad H\sigma G_{\mathcal{P}} \mapsto \sigma\mathcal{P} \cap L$$

is a bijection. So it splits completely in $L \iff G_{\mathcal{P}}$ is trivial $\iff$ it splits completely in N by counting numbers. □

Prop. (1.2.3). A prime p splits in $\mathbb{Z}[\xi_n]$ iff $p \equiv 1(\mathrm{mod}\ n)$.

Proof: First, if it splits, then $f = 1$, Because the ring of integers is $\mathbb{Z}[\xi_n]$, so $X^n - 1$ splits in $\mathbb{F}_p$(1.1.3), thus $p \equiv 1(\mathrm{mod}\ n)$. And if $p \equiv 1(\mathrm{mod}\ n)$, it is unramified and $X^n - 1$ splits in $\mathbb{F}_p$, so $f = 1$. □

Prop. (1.2.4). The profinite group $\mathbb{Q}_p^{\mathrm{tame}}$ is $\widehat{\mathbb{Z}} \ltimes \Delta_p$. Which is the profinite group generated by the relationship $\sigma\tau\sigma^{-1} = \tau^p$, where σ is a lift of Frobenius. Which means that it is the limit of finite quotients of the group $\langle\sigma\tau\sigma^{-1} = \tau^p\rangle$.

Proof: Cf.[Local Fields Clark]. □

Prop. (1.2.5) (Hasse-Arf Theorem). For a complete discrete valuation field K and an abelian extension L of K, the jump in the upper numbering of higher ramification group G^v must happen at integers.

Proof: Cf.[Local Fields, Serre] □

3 Completion

Prop. (1.3.1). Any infinite separable algebraic extension of a complete field is not complete.

Proof: We use Krasner's lemma. By Ostrowski theorem, we can assume it is non-Achimedian. Choose an infinite linearly independent basis of decreasing value rapidly enough, then we can see the field generated by the limit contains all the partial sums, contradiction. □

II.2 Galois Cohomology

1 Group Cohomology

Def. (2.1.1). The **group cohomology** $H^n(G,A)$ is the derived functor of the left exact functor $H^0(G,A)=A^G=\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},A)$, so $H^n(G,A)=\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},A)$.

The **group homology** $H_n(G,A)$ is the derived functor of the right exact functor $H_0(G,A)=A_G=\mathbb{Z}\otimes_{\mathbb{Z}[G]}A$, so $H_n(G,A)=\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},A)$。

A^H is left exact from G-mod to G/H-mod because it is adjoint to the inclusion functor: $\mathrm{Hom}_G(X,A)=\mathrm{Hom}_{G/H}(X,A^H)$. And it preserves injectives because inclusion is exact**??**. Dually for A_H.

Prop. (2.1.2).

$$H^{-2}(G,\mathbb{Z})=G^{ab},\quad H^{-1}(G,\mathbb{Z})=0,\quad H^0(G,\mathbb{Z})=\mathbb{Z}/|G|\mathbb{Z},\quad H^1(G,\mathbb{Z})=0,\quad H^2(G,\mathbb{Z})=\chi(G).$$

Lemma (2.1.3) (Shapiro).

$$H_*(G,\mathrm{ind}_H^G(A))\cong H_*(H,A),\quad H^*(G,\mathrm{Coind}_H^G(A))\cong H^*(H,A)$$

by adjointness property of (co)induced.

And in the finite case, this is also true for Tate cohomology immediately form the standard resolution.

Prop. (2.1.4) (Dimension Shifting). There are fundamental split exact sequence $0\to J_G\to\mathbb{Z}[G]\to\mathbb{Z}\to 0$ and $0\to\mathbb{Z}\to\mathbb{Z}[G]\to I_G$, $A_G=A/J_GA$. This can be used to tensor with A and define natural dimension shifting of cohomology δ.

Prop. (2.1.5) (Serre-Hochschild Spectral Sequence). By Grothendieck Spectral sequence, the relation $A^G=(A^H)^{G/H}$ gives us a spectral sequence E that

$$E_2^{p,q}=H^p(G/H,H^q(H,A))\Longrightarrow E^n=H^n(G,A).$$

The lower parts give us:

$$0\to H^1(G/N,A^N)\xrightarrow{inf}H^1(G,A)\xrightarrow{res}H^1(N,A)^{G/N}\xrightarrow{transfer}H^2(G/N,A^N)\xrightarrow{inf}H^2(G,A).$$

dually for homology group.

Cor. (2.1.6). If $G=F/R$, F is free, then use the homology spectral sequence, $H_2(G,\mathbb{Z})\cong\frac{R\cap[F,F]}{[F,R]}$. Cf.[Weibel P198].

Prop. (2.1.7) (Tate Cohomology). Neukirch Constructed a standard resolution of the $\mathbb{Z}[G]$-module $\mathbb{Z}$, Cf.[Neukirch CFT P13]:

$$\cdots\longleftarrow X_{-2}\longleftarrow X_{-1}\xleftarrow{\mu\circ\varepsilon}X_0\longleftarrow X_1\longleftarrow\cdots$$

that $X_q=X_{-q-1}$ are $\mathbb{Z}$-module generated by q-cells $(\sigma_1,\ldots,\sigma_q)$, $X_0=X_{-1}=\mathbb{Z}[G]$.

It then can be verified that for G finite, Hom from this resolution gives out the Tate cohomology

$$H_T^n(G,A) = \begin{cases} H^n(G,A) & n \geq 1 \\ A^G/N_G A & n = 0 \\ {}_{N_G}A/I_G A & n = -1 \\ H_{-1-n}(G,A) & n \leq -2 \end{cases}$$

and H_T^n is a long exact sequence.

Prop. (2.1.8). For cyclic group, the Tate cohomology is 2-cyclic. For this, we only have to give an isomorphism of $H^{-1}(G,A)$ and $H^1(G,A)$ and use dimension shifting. Cf.[Neukirch CFT P51].

Prop. (2.1.9) (Tate's Theorem). Assume A is a G-module that $H^1(G,A) = 0$ and $H^2(g,A)$ is cyclic of order $|g|$, then for a generator a of $H^2(G,A)$, $a\cup : H^q(G,\mathbb{Z}) \to H^{q+2}(G,A)$ is an isomorphism.

In particular, this gives:

Cor. (2.1.10) (Hilbert's Additive Theorem 90). Form the normal basis theorem, we get for finite Galois extension L/K, $H^*(G,L) = H_*(G,L)$ for $* \neq 0$ and $H_T^*(G,L) = 0$. Thus the same hold for any Galois extensions.

MIscellaneous

Prop. (2.1.11) (H^2 and Extension). For a G-module A, there is a correspondence of equivalence classes of extension of G over A that are compatible with the G action and $H^2(G,A)$.

Proof: Cf.[Weibel P183]. In fact there are interpretations of $H^3(G,A)$ as $0 \to A \to N \to E \to G$ under some equivalences. □

2 Profinite Groups and Cohomology

Basic references are [Weibel] and [Shatz Profinite Groups, Arithmetic and Geometry].

Profinite Groups

Prop. (2.2.1). A profinite space is the same thing as a totally disconnected, compact Hausdorff topological space. A profinite group is the same thing as a totally disconnected, compact Hausdorff topological group.

Cor. (2.2.2). A closed subspace of a profinite space is profinite. A closed subgroup of a profinite group is profinite, and the quotient group is profinite.

Prop. (2.2.3). The category of profinite groups is Pontryagin dual to the category of torsion abelian group. (not that hard to verify).

Prop. (2.2.4). A profinite group $G \cong \lim G/U_i$, where U_i are open normal subgroups. (Only need to check it is injective and has a dense image).

Cohomology of Profinite Groups

Prop. (2.2.5). The category C_G of continuous discrete G-mod is a full Abelian category of the category of G-mod, and the forgetful functor is left adjoint to the functor $B \to \cap B^{U_i}$ where U_i range through all the open subgroup of G. So it preserves injectives and C_G has enough injectives.

Prop. (2.2.6). The profinite cohomology is the derived functor of $A \to A^G$ in the Abelian category C_G. And

$$H^*(G, A) = \varinjlim H^*(G/U, A^U)$$

Proof: It suffice to prove this is a universal δ-functor because $C^n(G, A) = \lim_{\to} C^n(G/U, A^U)$ and direct limit commutes with cohomology. It is effaceable because I^U is injective G/U-module. □

Prop. (2.2.7) (Serre-Hochschild Spectral sequence). Same as the finite case(2.1.5) also applies to profinite cohomology.

Brauer Groups

Prop. (2.2.8). The **Brauer group** $\mathrm{Br}(K)$ is defined as the profinite cohomology $H^2(G(K_s/K), K_s^*)$. For a Galois extension L/K, $\mathrm{Br}(L/K)$ is defined as $H^(G(L/K), L^*)$. Then by(2.2.6) we have

$$\varinjlim \mathrm{Br}(L/K) = \mathrm{Br}(K).$$

And by Hochschild-Serre spectral sequence and by Hilbert's multiplicative theorem90:$H^1(H, K_s^*) = 0$, we have the low term:

$$0 \to \mathrm{Br}(L/K) \xrightarrow{inf} \mathrm{Br}(K) \xrightarrow{res} \mathrm{Br}(L)^G \to H^3(G(L/K), L^*) \to H^3(K, K_s^*).$$

So $\mathrm{Br}(L/K)$ is the kernel of $\mathrm{Br}(K) \to \mathrm{Br}(L)$.

Cor. (2.2.9). $\mathrm{Br}(\mathbb{F}_q)$=0 for finite fields, because the finite Galois extension are cyclic and unramified.

In fact, the Brauer group has semisimple algebraic interpretations.

Duality

Prop. (2.2.10). For a Galois extension $\mathrm{Gal}(L/K) = G, H^1(G, GL_n(L)) = 1$.

3 Abstract Class Field Theory

4 Class Field Theory

Prop. (2.4.1) (Brauer Group). For a finite Galois extension L/K, there is an isomorphism $H^2(G_{L/K}, L^*) \cong Br(L/K)$. Cf.[Milne].

5 Iwasawa Theory

Iwasawa Module

II.3 Langlands Program

1 Local Langlands Correspondence

The basic object of LLC are the Weil group and its representations.
A representation ρ of W_K is called F**-semisimple** iff $\rho(\mathrm{Frob})$ is diagonalizable.

Prop. (3.1.1). A

Thm. (3.1.2) (LLC for $GL_n(\mathbb{C})$). The set of
irreducible smooth, admissible representations of $GL_n(K)$
corresponds to
n-dimensional F-semisimple Weil-Deligne representations of W_K.

Cor. (3.1.3) (LLC for $GL_1(\mathbb{C})$).

Local class field theory told us that W_K^{ab} is isometric to K^*, And notice by Schur's lemma, any smooth representation of K^* is 1-dimensional and factors through some U_k.

And a Weil-Deligne representation is now a continuous $W_K^{ab} \to C^*$. but it must factor through some U_K, so these two are equivalent.

most l-adic representation of G_K comes from étale cohomology.

<u>LLC for $GL_2(\mathbb{C})$</u>

II.4 Witt Theory (Local Fields Serre)

1 Witt Vectors

A ring φ lifting the Frobenius, i.e. $\varphi(x) = x^p + p\delta(x)$. It generate a $\delta-$ring structure.

This is a category and the right adjoint to the forgetful functor is $W(A) = \mathrm{Hom}(\Delta, A)$. Where Δ is the free ring $\mathbb{Z}[e, \delta, \delta^2, \ldots]$. and it add and product in the way of Leibniz rule. There is another description of Δ:

Prop. (4.1.1). Let θ_i be polynomials in δ with integer coefficients that

$$\varphi^n = \theta_0^{p^n} + p\theta_1^{p^{n-1}} + \cdots + p^n\theta_n$$

In fact

$$\mathbb{Z}[\theta_0, \theta_i, \ldots, \theta_n] = \mathbb{Z}[e, \delta, \delta^2, \ldots, \delta^n]$$

Proof: Use equation $\varphi \circ \varphi^n = \varphi^n \circ \varphi$ and module $p^n\mathbb{Z}[\theta_0, \theta_1, \ldots, \theta_n]$. □

So there is a map

$$Z[\varphi] \to \Delta$$

inducing an morphism of rings:

$$W(A) \to \prod_{\mathbb{Z}} A$$

that maps

$$(f(\delta^n)) \mapsto (f(\varphi^n))$$

Where the right hand side is the normal addition and multiplication, the left side is the usual coordinate of Witt vector, and $f(\theta_n)$ is called ghost component.

This is embedding if A is p-torsion free, and isomorphism iff $\frac{1}{p} \in A$.

Prop. (4.1.2). Notice in Serre book, he presented the Witt vectors in $(f(\theta_n))$ coordinates. In this coordinate, if k is a perfect ring and we let

$$T(A) = \sum a_i^{p^{-i}} p^i,$$

then T is an ring homomorphism from $W(k)$ to the strict p-ring with residue ring k.

Cor. (4.1.3). For example, $W(\mathbb{F}_p^n)$ is the unramified extension of $\mathbb{Z}_p$ of degree n. And $W(\overline{F})$ is the completion of the maximal unramified extension of $W(F)$.

Prop. (4.1.4). $\mathcal{O}_{\mathcal{E}} = W(K^{\frac{1}{p^\infty}})$ is a complete ring with maximal ideal $p\mathcal{O}_{\mathcal{E}}$. And $\mathcal{O}_{\mathcal{E}}[\frac{1}{p}] = \mathcal{E}$ is complete ring of character p. And the same construction of $\overline{K^{\frac{1}{p^\infty}}}$ yields the completion of maximal unramified extension of $\mathcal{O}_{\mathcal{E}}$, and the Galois group is the same as G_K.

II.5 Abelian Variety(Mumford)

An Abelian variety A is a smooth projective variety with a group structure.

Prop. (5.0.1). For a field K of characteristic p, then $A(K^{\text{sep}}$ is an Abelian group and its l^n torsion is isomorphic to $(\mathbb{Z}/l^n\mathbb{Z})^{2g}$ and its p^n torsion is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^r$.

Prop. (5.0.2). There is an isomorphism

$$H_t^m(\Lambda_{K^{\text{sep}}}, \mathbb{Q}_l) \cong \bigwedge_{\mathbb{Q}_l}^m (V_l(A))^*.$$

Cf.[Grothendieck Monodromy theorem].

II.6 Shimura Variety

II.7 Étale Cohomology

II.8 p-adic Hodge Theory

1 Adic Space

Prop. (8.1.1). A space is called **spectral** iff it is

2 Perfectoid space

3 l-adic representations

Prop. (8.3.1). Every continuous representation of G_K on a $\mathbb{Q}_l$ vector space (Continuous group morphism to $GL_n(\mathbb{Q}_l)$) has a $\mathbb{Z}_l$ lattice stable under the action. (notice that the stablizer of the standard lattice is $GL_n(\mathbb{Z}_l)$ which is open and so the inverse image has a finite coset. And the image of the wild ramification group is finite because it is in $GL_n(\mathbb{F}_l)$.

So the functor $\rho \to \rho \otimes \mathbb{Q}_l$ from $\mathrm{Rep}_{\mathbb{Z}_l}(G_K)$ to the Tannakian natural category $\mathrm{Rep}_{\mathbb{Q}_l}(G_K)$ is essentially surjective.

Prop. (8.3.2) (Grothendieck Monodromy theorem). For a local field K, the étale representation and the Tate module are all potentially semisimple. i.e. semisimple for a finite extension.

Chapter III

Geometry

III.1 Topology

1 Connected Component

Prop. (1.1.1). Let X be a topological space, $x \in X$, C is a connected component of x, i.e. a maximal connected subset containing x. Define A to be the intersection of all the open-and-closed sets that contain x (also called the pseudo-component sometimes). Then $A = C$, if X is normal.

Proof: Assume A splits into two components B, D. Since A is closed, B and D are both closed, because X is normal there are disjoint open neighborhoods U and V around B and D, respectively. The open sets U and V cover the intersection of all clopen neighborhoods of A, so cause X is compact, there must exist a finite number of clopen sets around A, say $A_1, \ldots, A_n$ such that $U \cup V$ covers $K = \bigcap_1^n A_i$.

Note that K is clopen. We can assume that $x \in U$. It is not difficult to see that $K \cap U$ is clopen and does not contain all of A, contradicting the definition of A. □

Prop. (1.1.2). A noetherian topological space has only finitely many connected components.

Proof: Let $\mathcal{C}$ be the family of closed subset that has infinitely many component, then there is a minimal element, but it is not connected, one of the component has infinitely many component and be smaller. □

2 Covering Space

Prop. (1.2.1). For a connected and locally connected space, it has a universal cover, and the fundamental group acts on it continuously and properly. (Define the universal cover as the homotopy equivalence class of lines starting from a base point).

Prop. (1.2.2). if X and Y are Hausdorff spaces, $f : X \to Y$ is a local homeomorphism, X is compact, and Y is connected, then f a covering map.

Proof: First, f is surjective (using the connectedness), and that for each $y \in Y$, $f^{-1}(y)$ is finite. Because X is compact, there exists a finite open cover of X by $\{U_i\}$ such that $f(U_i)$ is open and $f|_{U_i} : U_i \to f(U_i)$ is a homeomorphism. For $y \in Y$, let $\{x_1, \ldots, x_n\} = f^{-1}(y)$ (the

x_i all being different points). Choose pairwise disjoint neighborhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$, respectively (using the Hausdorff property).

By shrinking the U_i further, we may assume that each one is mapped homeomorphically onto some neighborhood V_i of y.

Now let $C = X \setminus (U_1 \cup \cdots \cup U_n)$ and set

$$V = (V_1 \cap \cdots \cap V_n) \setminus f(C)$$

V should be an evenly covered nbhd of y. □

Prop. (1.2.3). If $\pi : \tilde{B} \to B$ is a local onto homeomorphism with the property of lifting arcs. Let $\tilde{B}$ be arcwise connected and B simply connected, then π is a homomorphism.

Proof: only need to prove injective. If p_1 and p_2 map to the same point, then they can be connected, and the image is a loop thus contractable, contradiction. □

Cor. (1.2.4). If $\tilde{B}$ is locally arcwise connected and B is locally simply connected, then π is a covering map.(choose the connected component)

Prop. (1.2.5). a simply connected manifold is orientable. (Use the orientable double cover).

3 Paracompactness

Prop. (1.3.1). If X is regular, then TFAE:

1. Each open cover of X has an open locally finite refinement.
2. Each open cover of X has a locally finite refinement.
3. Each open cover of X has a closed locally finite refinement.
4. Each open cover of X is even. i.e. for any cover, there is an open nbhd V of diagonal of $X \times X$ such that $\forall x$, $V[x] = \{y | (x, y) \in V\}$ refines the cover.
5. Each open cover of X has an open σ-discrete refinement.
6. Each open cover of X has an open σ-locally finite refinement.

If this is satisfied, then X is called **paracompact**.

Proof: $6 \to 2$:Just minus every open set the part of open sets that appeared in families that ordered before it. $2 + 4 \to 1$:Use the lemma below, we can transform the cover $\mathcal{A}$ into $V[\mathcal{A}] \cap U_A$ which is an open locally finite cover
Cf.[General Topology Kelley] □

Lemma (1.3.2). If X satiesfies 4, let U be a nbhd of diagonal of $X \times X$, then their exists a symmetric nbhd of diagonal s.t. $V \circ V \subset U$, where $U \circ V = \{(x, z) | (x, y) \in U, (y, z) \in V, \exists y\}$.

Proof: $\forall x$ in X, there is a nbhd s.t. $W[x] \times W[x] \subset U$, this is an open cover, so there is a nbhd R of diagonal s.t. $R[x]$ refines it. Hense $R[x] \times R[x] \subset U$. Let $V = R \cap R^{-1}$, $V \circ V$ is the union of sets $V[x] \times V[x]$, so $V \circ V \subset U$. □

Lemma (1.3.3). In the preceding proposition, if X satisfies 4, Let $\mathcal{A}$ be a locally finite(resp. discrete i.e. intersect only one) family of subsets of X, then use the last lemma, there is a nbhd V of diagonal of $X \times X$ such that $V[A] = \{y | (x, y) \in V, \exists x \in A\}$ is locally finite(resp. discrete).

Proof: Choose for every pt a nbhd satisfy the property, then it is an open cover. Choose a diagonal nbhd U for the property 4, then choose coordinate symmetric nbhd V of diagonal s.t. $V \circ V \subset U$. If $V[x]$ intersect $V[A]$, then $V \circ V[x]$ intersect A. Done. □

Prop. (1.3.4). A regular paracompact space is normal.

Proof: The family consisting of two closed is locally discrete, by preceding lemma,there exists a V s.t.$V[A]$,$V[B]$ open and non-intersecting. □

Prop. (1.3.5). For a connected, Hausdorff, locally euclidian space, paracompact, second countable and a compact exhaustion is equivalent.

Proof: Cf.[Paracompactness and second countable]. □

Prop. (1.3.6). A metric space is paracompact.

Prop. (1.3.7). A compact Hausdorff space is paracompact.

Prop. (1.3.8) (Partition of unity). In a paracompact space, given any open cover, there exists a partition of unity $\{\rho_i\}$ that ρ_i has compact support and supp$\rho_i \subset U_i$.

4 Normal (T4)

Prop. (1.4.1) (Urysohn lemma). Let X be normal, A and B two closed subset of X, then there exists a continuous map from X to $[0,1]$ that maps A to 0 and B to 1.

Proof: Use the countability of rational numbers to construct a family of U_q s.t.

$$p < q \Rightarrow \bar{U}_p \subset U_q$$

Then choose $f(x) = \inf\{p \in \mathbb{Q} | x \in U_p\}$, then this f meets the requirement. □

Prop. (1.4.2) (Tietze extension). If X is normal and Y is a closed subspace, then any continuous function f on Y can be extended to a continuous function on X.

5 Compact-Open Topology

Prop. (1.5.1). The **compact-open topology** on X^Y is the topology generated by subbasis of $(K, U) =$ all f that maps K to U, for K compact and U open. When Y is compact and X a metric space, this coincides with the uniform topology.

Prop. (1.5.2).

- $X^Y \times Y \to X$ is continuous if Y is locally compact.
- $\mathrm{Map}(Y \times X, Z) \cong \mathrm{Map}(Z, X^Y)$.

6 Baire Space

Prop. (1.6.1) (Baire Category Theorem). Every complete metric space & locally compact Hausdorff space is a Baire space, i.e. not countable union of subsets whose closure have no interior point.

Proof: Choose consecutively open subsets that doesn't intersect $\overline{E_n}$ to find a limit point. □

7 Uniform Space

8 Spaces from Algebraic Geometry

Noetherian Space

Prop. (1.8.1). A Noetherian space is quasi-compact and its closed or open subspace is Noetherian hence quasi-compact.

Sober Space

Def. (1.8.2). A space X is called sober if every irreducible closed subset has a unique generic point.

Prop. (1.8.3). The underlying space of a scheme is sober.

Proof: First prove this for affine scheme, notice that closed irreducible subsets correspond to prime ideal. Then notice the generic point for $Z \cap U$ is the generic point for Z. □

Prop. (1.8.4) (Soberization). There is a left adjoint t to the forgetful functor from the Sober spaces. $t(X)$ consists of irreducible closed subsets of X, and use $t(Y)$ for Y closed as closed subsets. for a map $f : X \to Z$ to a sober space Z, the extension maps the generic point of an irreducible Y to the generic point of the closure of $f(Y)$.

Spectral Space

Prop. (1.8.5). A space is called **spectral** iff it is compact, T_0, sober and has a basis consisting of compact open sets.

A spectral space is exactly the underlying space of spectrum of a ring.

III.2 Riemann Geometry

Basic references are [Riemannian Geometry Do Carmo] and [Geometric Analysis Jost].

1 $\mathbb{R}^3$-Geometry

Different Coordinates

Prop. (2.1.1). In a polar coordinate,

$$g_{11} = 1, g_{12} = 0, g_{22} = |\frac{\partial f}{\partial \theta}|^2, \quad K = -\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}}$$

And $\sqrt{g_{22}} \sim \rho$. (Use the formula relating Jacobi Field with curvature)

Moving Frame Method

Prop. (2.1.2) (Theorema Egregium).

$$R_{1212} = K(g_{11}g_{22} - g_{12}^2)$$

Which is a special case of the definition of curvature.

Prop. (2.1.3) (Gauss-Bonnet). Let M be a compact oriented 2-dimensional Riemannian manifold, then

$$\chi(M) = \frac{1}{2\pi}\int_M K \text{ Vol}.$$

2 Hodge Theory

Def. (2.2.1) (Hodge Star Operator). given a volume-form ω on a vector space, the Hodge star operator $*-$ is an operator from $\bigwedge^k V \to \bigwedge^{n-k} V$ such that:

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \omega.$$

On a closed oriented Riemannian manifold, given a volume form ω, the star operator satisfies:

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \omega = \int_M \alpha \wedge *\beta.$$

For a operator d on $\Omega^* M$, we define the adjoint $d^* = (-1)^{n(p+1)+1} * d *$, which satisfies the property by calculation:

$$(d\alpha, \beta) = (\alpha, d\beta).$$

The laplacian $\Delta = d^* d$.

3 Connections

Def. (2.3.1). An affine connection on a vector bundle E is a map $D : \Gamma(E) \to \Gamma(E)\otimes\Gamma(T^*M)$ that satisfies differential-like properties, it can be written as $D = d+\omega$, with $\omega \in \Omega^1(\mathrm{End}(E))$.

Prop. (2.3.2) (Transformation Law). In two coordinates $\overline{e} = ea$ for $a : U \to GL(r, \mathbb{R})$, $d_A = d + \omega, d + \overline{\omega}$ respectively, $\Omega = d\omega + \omega \wedge \omega$. Then:

$$\overline{\omega} = a^{-1}\omega a + a^{-1}da, \quad \overline{\Omega} = a^{-1}\Omega a$$

Moreover, giving any locally compatible $d + \omega$,$\omega \in \Omega^1(\mathfrak{g})$ in the sense above, then for any G-associated bundle E, where G has lie algebra $\mathfrak{g}$, there is a connection that locally looks like $d + \omega$, (where $\mathfrak{g}$ embeds into $\mathfrak{gl}(E)$).

Prop. (2.3.3) (Second Bianchi's Identity). A affine connection on E looks like $d_A = d + \omega$, where $\omega \in \Omega^1(\mathrm{End}\, E)$. And $F_A = d_A \circ d_A \in \Omega^2(\mathrm{End}(E))$ satisfies

$$d_A F_A = dF_A + [\omega, F_A] = 0.$$

Cf.[Jost P111].

Def. (2.3.4). The torsion tensor of a connection ∇ on TM is defined as $T(X, Y) = \nabla_X Y - \nabla_Y X-[X, Y]$. The connection is called **torsion-free** if $T = 0$. This is equivalent to $\Gamma^k_{i,j} = \Gamma^k_{j,i}$. A connection is called **metric** if it preserves metric. i.e. $\nabla g = 0$.

Prop. (2.3.5) (Flat coordinate). A connection on TM assumes near every point a flat coordinate, i.e. $\nabla(\frac{\partial}{\partial x^i}) = 0$, iff it is flat and torsion-free.

Proof: One side is easy because its Christoffels vanish. On the other side, use integrability theorems (6.3.2). Cf.[Jost P115]. □

Prop. (2.3.6).

$$\Delta\langle\varphi, \varphi\rangle = 2(\langle D^*D\varphi, \varphi\rangle - \langle D\varphi, D\varphi\rangle).$$

Def. (2.3.7). The Christoffel symbol: $\nabla_{X_i} X_j = \sum_k \Gamma^k_{ij} X_k$.

The geodesic equation: $\dfrac{D}{dt}(\dfrac{d\gamma}{dt}) = \ddot{x}_k + \sum_{i,j} \Gamma^k_{ij}\dot{x}_i\dot{x}_j = 0 \quad \forall k.$

- Geodesic flow: the flow on TM whose trajectories are $t \mapsto (\gamma(t), \gamma'(t))$, where γ is a geodesic on M.
- **(The smoothness of geodesics)** for every point p, there exists a nbhd V and a C^∞ mapping
$$\gamma : (-\delta, \delta) \times V \times B(0, \epsilon) \to M,$$
s.t. $\gamma(t, q, v)$ is the geodesic passing through p with velocity v.

Def. (2.3.8). The **curvature** of a (affine) connection d_A is

$$F_A = d_A \circ d_A \in \Omega^2(\mathrm{End}(E)).$$

It induces a curvature tensor

$$F_A(Z)(X, Y) = R(X, Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]}Z.$$

(The third component is to assure it only depends on X_p and Y_p).

The connection is called **flat** if $F_A = 0$.

Prop. (2.3.9).
$$d_{gA}(s) = gd_A(g^{-1}(s))$$

So for any connection d_A and any point x_0, there is a gauge transformation that makes $d_A = d$ at x_0.

Proof: Just need to have $s(x_0) = \mathrm{id}$, $ds(x_0) = -A(x_0)$. this is possible because $A \in \Omega^1(\mathrm{Ad}E)$ which is the fiber of the frame bundle, use exp. □

Prop. (2.3.10). For a flat connection, there is a bundle isomorphism (Gauge transform) that transforms d_A into natural d.

Proof: Because $d_{gA}(s) = gd_A(g^{-1}(s))$,$d_{gA} = d - dg \cdot g^{-1} + g \cdot \omega \cdot g^{-1}$. Solve this PDE direvtly. (Cf.[Topics in Geometry Xie Yi week3]). □

Cor. (2.3.11). For a flat connection, the parallel transportation only depends on the homotopy type of the loop, thus gives an action of $\pi(X)$ on $SO(T_p(X))$ (or $SU(T_p(X))$. (because it is locally constant). And in this way, connections module gauge equivalence (preserving matric) equals representation of $\pi(X)$ module conjugations. The reverse map is giving by principal bundle.

Prop. (2.3.12). The connection action $d_A = d + \omega$ on a vector bundle E induces connection on relevant bundles. the action on dual bundle is by

$$d_A(s^*) = ds^* + \omega^t(s^*) = ds^* + s^* \circ \omega.$$

And the connection on $\mathrm{End}\, E$ by

$$d_A(\alpha) = d\alpha + [\omega, \alpha]$$

And they act on $\Omega^*(E)$ by Leibniz rule thus the formula looks the same. (Note that the convention is the matrix and composition act their way, and assume ω are always at left, so for example, $[\alpha, \alpha] = 2\alpha \wedge \alpha$).

Levi-Civita Connection

Def. (2.3.13) (Levi-Civita Connection). The Levi-Civita connection is the unique connection on M that is metric and torsion-free:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \qquad \nabla_X Y - \nabla_Y X = [X, Y].$$

It satisfies:

$$\langle Z, \nabla_Y X\rangle = 1/2\{X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [X, Y], Y\rangle - \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle\}.$$

Then
$$\Gamma_{ij}^m = 1/2 \sum_k \{g_{jk,i} + g_{ki,j} - g_{ij,k}\} g^{km}$$

Thus geodesic is a solution that only depends on the metric (2.3.7), so a local isometry preserves geodesics.

Prop. (2.3.14). Now the Lie derivative has the form:

$$L_X(S)(Y_1,\dots,Y_p)=\nabla_X(S)(Y_1,\dots,Y_p)+\sum_{i=1}^{p}S(Y_1,\dots,Y_{i-1},\nabla_{Y_i}X,\dots,Y_p).$$

The exterior derivative d and its adjoint d^* has the form:

$$d\omega(Y_i)=\sum(-1)^p\nabla_{Y_i}\omega(\check{Y_1}),\quad d^*\omega(Y_i)=-\sum\nabla_{e_j}\omega(e_j,Y_i)$$

where e_i is an orthonormal basis. Cf.[Jost P140].

- $$\frac{D}{\partial u}\frac{\partial s}{\partial v}=\frac{D}{\partial v}\frac{\partial s}{\partial u}.$$
- **(Totally normal nbhd)** For any point p, there exists a nbhd W and a number $\delta>0$ s.t. for every $q\in W$, $\exp_q$ is a diffeomorphism on $B_\delta(0)$ and $\exp_q(B_\delta(0))\supset W$. Thus, fine cover exists in every smooth manifold.
- (**Geodesic Frame**) In a neighborhood of every point p, there exists n vector fields, orthonormal at each point, and $\nabla_{E_i}E_j(p)=0$.(Choose normal nbhd and parallel a orthonormal basis to every point. (WARNING: this is not a flat coordinate, it only helps when dealing with point-wise properties).
- (Gauss Lemma) In a normal nbhd, the vectors orthogonal to geodesics is mapped under $(d\exp_p)_v$ to vectors orthogonal to geodesics.
- a locally minimizing piecewise differentiable curve is a geodesic.(Choose normal nbhd and use polar coordinate).

Def. (2.3.15). Killing field is which generates an infinitesimal isometry. X is killing $\iff L_X(g)=0\iff\langle\nabla_YX,Z\rangle+\langle\nabla_ZX,Y\rangle=0$ for all Y,Z (Killing equation).

A Killing field is a Jacobi field along geodesics. (by Calculation).

- The singularities of a Killing field is a submanifold and will generate a vector field along a geodesic sphere of the orthogonal component.
- gradient: $\langle \mathrm{grad}f(p),X\rangle=X(f)(p)$.
- divergence: $\mathrm{div}X(p)=$trace of the linear map $Y(p)\to\nabla_YX(p)=\sum_i\langle\nabla_{E_i}X,E_i\rangle$. It measures the variation of the volume and it depends only on the point.
- Hessian: Hess$f$ is a self-adjoint operator that $(\mathrm{Hess}f)Y=\nabla_Y\mathrm{grad}f$ as well as a symmetric form $(\mathrm{Hess}f)(X,Y)=\langle(\mathrm{Hess}f)X,Y\rangle$.
- Laplace: $\Delta f=\mathrm{div}\ \mathrm{grad}f=\mathrm{trace}\ \mathrm{Hess}f=\sum_i E_i(E_i(f))$.
- in a geodesic frame,

$$\mathrm{grad}f(p)=\sum_{i=1}^{n}(E_i(f))E_i$$

$$\mathrm{div}X(p)=\sum_{i=1}^{n}E_i(f_i)(p),\mathrm{where}X=\sum_i f_iE_i.$$

$$\Delta f=\sum_i E_i(E_i(f))(p).$$

- $\Delta(f\cdot g) = f\Delta g + g\Delta f + 2\langle \text{grad} f, \text{grad} g\rangle$.
- $d(i(X)m) = (\text{div}X)m$. where m is the volume form.

Cor. (2.3.16) (Hopf theorem). If f is a differentiable function on a compact orientable manifold with $\Delta f \geq 0$, then f is constant.

- The curvature tensor is determined by its sectional curvature, thus if M is isotropic at a point p (The sectional curvature depends only on the point), then $R(X,Y,W,Z) = K_0(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)$
- **Ricci curvature** $\text{Ric}_p(x) = \frac{1}{n-1}\sum\langle R(x,z_i)x, z_i\rangle$, for x a unit vector, where z_i is an orthonormal basis orthogonal to x. $\text{Ric}(x) = \text{Ric}(x,x)$, where $\text{Ric}(x,y)$ is the symmetric form of $\frac{1}{n}$ of trace of the map $z \to R(x,z)y$.
- **scalar curvature** $K(p) = 1/n\sum \text{Ric}_p(z_i)$, where z_i is an orthonormal basis.
- $$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})V. \quad \text{(obvious because } \frac{\partial}{\partial_s}, \frac{\partial}{\partial_t} \text{ commutes)}$$
- sectional curvature $K(X,Y) = \langle R(X,Y)X, Y\rangle$.
- curvature tensor only depends on the point and
$$R(X,Y,Z,W) = R(Z,W,X,Y), \quad R(X,Y,Z,W) = R(X,Y,W,Z).$$

Prop. (2.3.17) (Bianchi Identities). The covariant differential $\nabla R(Y_i, Z) = Z(R(Y_i)) - \sum_i R(\nabla_Z Y_i, Y_j)$.
(Bianchi Identity)$\sum_{(X,Y,Z)} R(X,Y)Z = 0$.
(Second Bianchi Identity)$\sum_{(Z,W,T)} \nabla R(X,Y,Z,W,T) = 0$.

Cor. (2.3.18) (Schur's Theorem). Let M be a manifold of dimension $n \geq 3$, suppose M is isotropic , then M has constant curvature. (Use the second Bianchi Identity and geodesic frame).

- $B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$. It is bilinear and symmetric.
- $H_\eta(x,y) = \langle B(x,y), \eta\rangle$. Thus $B(x,y) = \sum H_i(x,y)E_i$ for an orthonormal frame E_i in $\mathfrak{X}(U)^\perp$.
- $S_\eta(x) = -(\overline{\nabla}_x \eta)^T$. It satisfies:$\langle S_\eta(x), y\rangle = H_\eta(x,y) = \langle B(x,y), \eta\rangle$. It is self-adjoint. When codimension 1, it is the derivative of the Gauss mapping.
- (**Gauss Formula**): let x,y be orthonormal tangent vector. Then:
$$K(x,y) - \overline{K}(x,y) = \langle B(x,x), B(y,y)\rangle - |B(x,y)|^2.$$
- An immersion is called **geodesic** at p if the second fundamental form S_η is zero for all η, (which means $\nabla_X Y$ has no normal component). It is called **minimal** if the trace of S_η is zero.
- An immersion is called umbilic if there exists a normal unit field η s.t. $\langle B(X,Y), \eta\rangle(p) = \lambda(p)\langle X,Y\rangle$.
- If the ambient space has constant sectional curvature and the immersed manifold is totally umbilic, then λ is constant.

- mean curvature tensor of immersion $f = 1/n \sum_i (\text{tr } S_i)E_i = 1/n \text{ tr } B$. It is zero if f is minimal.
- normal connection $\nabla^{\perp}_X \eta = (\overline{\nabla}_X \eta)^N = \overline{\nabla}_X \eta + S_\eta(X)$.
- (Gauss equation)
$$\langle \overline{R}(X,Y)Z,T\rangle = \langle R(X,Y)Z,T\rangle - \langle B(Y,T), B(X,Z)\rangle + \langle B(X,T), B(Y,Z)\rangle.$$
- (Ricci equation)
$$\langle \overline{R}(X,Y)\eta,\zeta\rangle - \langle R^{\perp}(X,Y)\eta,\zeta\rangle = \langle [S_\eta, S_\zeta]X, Y\rangle.$$
- (Codazzo equation)
$$\langle \overline{R}(X,Y)Z,\eta\rangle = (\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta). \qquad \text{(Lie bracket)}$$

4 Complete manifold

Prop. (2.4.1) (Hopf-Rinow theorem). The following is equivalent definition of **completeness**.

1. $\exp_p$ is defined for all of $T_p(M)$.
2. The closed and bounded sets of M are compact.
3. M is complete as a metric space.
4. M is $\sigma-$compact and if $q_n \notin K_n$, $d(p, q_n) \to \infty$.
5. The length of any divergent (compact escaping) curve is unbounded.

and if M is complete, then for any q, there exists a minimizing geodesic. And any compact submanifold of a complete manifold is complete.

Prop. (2.4.2) (Hadamard theorem). M a complete simply connected Riemann manifold of sectional curvature ≤ 0, then $\exp_p : T_pM \to M$ is an isomorphism of M to $\mathbb{R}^n$.(negative sectional curvature to show exp is a local isomorphism, complete to show it is a covering map)

- For any two manifold of the same constant curvature and any two orthogonal basis, there is a local isometry (It is locally isotropic).
- Any complete manifold with a sectional curvature is like $\tilde{M}/\Gamma$, where $\tilde{M}$ is $\mathbf{H}^n, \mathbf{R}^n$ or $\mathbf{S}^n$.
- Every compact orientable surface of genus $p > 1$ can be provided with a metric of constant negative curvature.

5 Jacobi Field and Comparison Theorems

- Jacobi field equation along a geodesic γ: $D^2 J(t) + R(\dot{\gamma(t)}, J(t))\dot{\gamma}(t) = 0$. It is defined by its initial condition $J(0)$ and $J^{'}(0)$. It can be used to detect the sectional curvature、the critical point of $\exp_p$ and calculate variation of energy.
- The Jacobi field along a point with initial velocity 0 all has the form $J(t) = (d\exp_p)_{t\dot{\gamma}(0)}(tJ^{'}(0))$. Corollary: the Jacobi transport from p to q is an isomorphism iff p and q is not conjugate.

- for general Jacobi field,

$$\langle J(t), \dot\gamma(t)\rangle = \langle J^{'}(0), \dot\gamma(0)\rangle t + \langle J(0), \dot\gamma(0)\rangle.$$

- If J is a Jacobi field $J(t) = (d\exp_p)_{tv}(tw), |v| = |w| = 1$, then

$$|J(t)| = t - \frac{1}{6}K_p(v,w)t^3 + o(t^3).$$

- Energy

$$E(s) = \int_0^a |\frac{\partial f}{\partial t}(s,t)|^2 dt.$$

- A minimizing geodesic must minimize energy.
- (**First Variation of Energy**)

$$1/2E^{'}(0) = -\int_0^a \langle V(t), D\dot c(t)\rangle dt + \langle V(a), \dot c(a)\rangle - \langle V(0), \dot c(0)\rangle.$$

  A piecewise differentiable curve is a geodesic iff every proper variation has first derivative 0.

- (**Second Variation of Energy**) If γ is a geodesic,

$$1/2E^{''}(0) = \int_0^a \{\langle DV(t), DV(t)\rangle - \langle R(\dot\gamma, V)\dot\gamma, V\rangle\}dt + \langle D_sV(a), \dot\gamma(a)\rangle - \langle D_sV(0), \dot\gamma(0)\rangle.$$

- a variation is equivalent to a vector field along the curve, and a variation that $f_s(t)$ are all piecewise geodesics corresponds to a piecewise Jacobi field(Choose a normal partition).

Prop. (2.5.1) (Rauch Comparison theorem). Let M and $\tilde M$ be manifolds, $\dim\tilde M \geq \dim M$. If J and $\tilde J$ be two Jacobi fields along geodesics γ and $\tilde\gamma$ that $|J^{'}(0)| = |\tilde J^{'}(0)|$. If $\tilde\gamma$ has no conjugate point or focal point free and $\tilde K(\tilde x, \dot{\tilde\gamma}(t)) \geq K(x, \dot\gamma)$ for any vector $x, \tilde x$, then $|\tilde J| \leq |J|$.

Cor. (2.5.2). If the sectional curvature of M satisfies: $0 < L \leq K \leq H$, then the distance between any two conjugate points satisfies: $\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}$.

Cor. (2.5.3) (Bishop theorem). Let M be complete manifold such that $\mathrm{Ric}_M \geq H$, let $\tilde M(H)$ be a complete simply connected manifold of constant sectional curvature H, then $\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(B_r(\tilde p))$. ?

Prop. (2.5.4). If two manifold M and $M^{'}$ satisfy $K \leq K^{'}$, then in a normal nbhd of a point p in M and a nbhd of $p^{'}$ that exp is nonsingular, the transformation of a curve c shortens length.

Note that this is not Toponogov thoerem, because if you try to map from a large curvature manifold to a small curvature, then you cannot guarantee that the mapped curve is the shortest.

Cor. (2.5.5). In a complete simply connected manifold of non-positive curvature,

$$A^2 + B^2 - 2AB\cos\gamma \leq C^2$$

thus $\alpha + \beta + \gamma \leq \pi$.

Prop. (2.5.6) (Moore theorem). Let $\overline{M}$ be a complete simply connected manifold of sectinoal curvature $\overline{K} \leq -b \leq 0$, M a compact manifold of sectional curvature satisfying $K - \overline{K} \leq b$. If $\dim\overline{M} < \dim M$, M cannot be immersed into $\overline{M}$.(use Hadamard theorem to choose the furthest geodesic and calculate the second variation of energy and use Gauss formula).

Cor. (2.5.7). Let $\overline{M}$ be a complete simply connected manifold of sectinoal curvature $\overline{K} \leq 0$, M a compact manifold of sectional curvature satisfying $K \leq \overline{K}$. If $\dim\overline{M} < \dim M$, M cannot immerse into $\overline{M}$.

Remark (2.5.8). There exist complete surfaces with $K \leq 0$ in $\mathbb{R}^3$, but the hyperbolic surface cannot be immersed into $\mathbb{R}^3$(Hilbert Theorem).

Lemma (2.5.9) (Klingenberg). (P236)Let M be a complete manifold of sectional curvature $K \geq K_0$, let γ_0, γ_1 be two homotopic geodesics from p to q, then there exists a middle curve γ_s s.t.

$$l(\gamma_0) + l(\gamma_1) \geq \frac{2\pi}{\sqrt{K_0}}.$$

Prop. (2.5.10) (Klingenberg). Let M be a simply connected compact manifold of dimension $n \geq 3$ such that $\frac{1}{4} < K \leq 1$, then $i(M)$(The infimum of distance to the cut locus)$\geq \pi$.

Cor. (2.5.11). If M is a compact orientable manifold of even dimension satisfying $0 < K \leq 1$, then $i(M) \geq \pi$.

Prop. (2.5.12) (1/4-pinch Sphere Theorem). Let M be a compact simply connected manifold satisfying $0 < 1/4K_{\max} < K \leq K_{\max}$, then M is homeomorphic to a sphere.

(Use Klingenberg Theorem, this is a special case of diameter geodesic sphere theorem). Cf.(2.5.22).

It can be shown that in this case, this sphere is even diffeomorphic to S^n using Ricci flow.

Remark (2.5.13). $0 < 1/4K_{\max} < K$ cannot be changed to $\geq$. In fact, the Funibi-Study metric on CP^n has sectional curvature $1 \geq K \geq 4$. Cf. **??**

Hess$\rho(X, Y)$ where ρ is the distance to a fixed point, is important.

Prop. (2.5.14). Hess$\rho(X, Y)$ is positive definite on the tangent space of the geodesic sphere within the injective radius, and its principal value is $|\frac{J'}{J}|$ for a Jacobi field in that direction. And it is zero on the normal direction.

So there would be a Riccati comparison theorem on the eigenvalue of $\Pi_2 : \lambda' \leq -K - \lambda^2$,Hess$(\rho)$ is bounded.

Proof: Notice that

$$\mathrm{Hess}\rho(X, Y) = (\nabla_X \mathrm{grad}\rho, Y) = XY\rho - (\nabla_X Y)\rho$$

so if choose a normal geodesic γ of initial vector X, then

$$\mathrm{Hess}\rho(X, X) = X\langle\dot{\gamma}, d\rho\rangle - (\nabla_X\dot{\gamma})\rho = X\langle\dot{\gamma}, d\rho\rangle = \langle\dot{\gamma}, d\langle\dot{\gamma}, d\rho\rangle\rangle = E''(0)$$
$$= I_q(X, X) = ((\nabla_{\dot{\gamma}}X)(q), X(q)) = \frac{\langle J', J\rangle}{|J|^2}$$

□

Prop. (2.5.15) (Toponogov). Let$M$ be a complete manifold with $K \geq H$.

If a hinge satisfies γ_1 is minimal and $\gamma_2 \geq \frac{\pi}{\sqrt{H}}$ if $H > 0$., then on M^H the same hinge has smaller distance of endpoints than this hinge

Proof: Cf.[Cheeger Comparison Theorems in Riemannian Geometry P42]. And there is another triangle version: For a minimal geodesic triangle, the comparison triangle has smaller angles. NOTE this theorem cannot be derived from Rauch Comparison Theorem. □

Critical Point for Distance Function

Prop. (2.5.16). The critical point for distance function on a complete manifold is that for every direction v, there is a minimal geodesic γ s.t. $\langle \gamma'(l), v\rangle \leq \frac{\pi}{2}$.

The set of regular point is open and there exists a smooth gradient like vecter field (i.e. acute angle with every minimal geodesic) on this open subset .

Prop. (2.5.17) (Berger's Lemma). A maximal point for the distance function is a critical point.

Proof: If not, choose a convergent point v of the minimal geodesics with endpoint in a curve of that direction, then exp near v will generate a Jacobi field with endpoint Jacobi is the sam of that direction. So the distance will increase by $\cos\theta$ along that direction, contradiction. □

Prop. (2.5.18) (Soul Lemma). Let M is a Riemannian manifold and A is a closed submanifold. If $dist(A, -)$ has no critical point on $D(A, R)\backslash A$, then $B(A, R)$ is diffeomorphic to the normal bundle of $A \to M$.

Proof: A has a normal exp radius ϵ, and we can vary the gradient-like vector field to be identical to the normal vector near A, and use Morse lemma (the flow) to get a diffeomorphism. □

Cor. (2.5.19) (Disk Theorem). If A is a point then M is diffeomorphic to a disk.

Lemma (2.5.20) (Generalized Schoenflies Theorem). Easy to do, just use the fact that exp is continuous to find a boundary sphere depending continuously on the direction (both p and q).

Prop. (2.5.21) (Sphere Theorem). If M is a closed manifold and has a distance function with only one critical point (the furthest one), then M is homeomorphic to a twisted ball.

Proof: There exists a ϵ and r that $B(q, \epsilon)$ and $B(p, r)$ covering M, (Use the convergent point argument). Then use the generalized Schoenflies theorem. □

Prop. (2.5.22) (Diameter Sphere Theorem). If a closed manifold M satisfies sec$M \geq K > 0$, and diam$(M) > \frac{\pi}{2\sqrt{K}}$, then M is homeomorphic to S^n.

Proof: First, if there are two maximal distance point, then use Toponogov to show contradiction. Second, at other points x,

$$\angle pxq > \frac{\pi}{2}$$

(Regular domain) because of Toponogov and The formula

$$\cos\tilde{\alpha} = \frac{\cos l - \cos l_1 \cos l_2}{\sin l_1 \sin l_2}.$$

So the geodesic direction $\overrightarrow{xq}$ will serve as a geodesic-like vector field (might need paracompactness). $\square$

Prop. (2.5.23) (Critical Principle). In a complete manifold M of sectional curvature $> K$, if q is a critical point of p, then for any point x with $d(p,x) > d(p,q)$ and any minimal geodesic from p to x, the $\angle xpq$ is smaller than the $\cosh_K^{-1}(\frac{d(p,x)}{d(p,q)})$.

Proof: Use Toponogov for the hinge xpq. Then notice that there is a different minimal geodesic from $p \to q$ that makes the $\angle pqx < \pi/2$ by the definition of critical point, thus there is another Toponogov inequality, this two inequality contradicts. $\square$

Cor. (2.5.24). For a complete open manifold whose K are lower bounded, then it is homeomorphic to the interior of a manifold with boundary. (Use Soul lemma, otherwise there will be a sequence of critical point whose angles are big).

Prop. (2.5.25). ray construction and Line construction?

Cor. (2.5.26) (Splitting Theorem). The universal cover of a compact RIemannian manifold with non-negative Ricci curvature splits isometrically as a product $\widetilde{M} = N \times \mathbb{R}^k$ where N is a compact manifold manifold.

Prop. (2.5.27) (Soul Theorem). If M is an open manifold with $K \geq 0$, then there is a totally geodesic submanifold S that M is diffeomorphic to the normal bundle over S.

Proof: Use the ray construction to get a totally convex compact subset, hence it is a manifold or with boundary , if it has boundary, then find to set of maximal distance to the distance to boundary, the distance to the boundary is a convex function, so it is a smaller totally geodesic manifold. So a S without boundary must exist and this constitutes a stratification, all the level set is strongly convex. Thus all point outside S is not critical, hence the soul lemma applies. Cf.[GeJian Comparison theorems in Riemannian Geometry Lecture7]. $\square$

Prop. (2.5.28) (Perelman). There is a distance non-increasing contraction unto the soul, and it must be just the projection along the normal bundle. Moreover, for any geodesic on the soul and a parallel vector field in the normal bundle along it, it spans a flat surface (by Rauch comparison).

Cor. (2.5.29) (Soul Conjecture). For an open(non-compact) complete manifold M with $K \geq 0$, if it has a point p s.t. sectional curvature at p are all positive, then M is diffeomorphic to R^n. (It's enough to show that its soul is a point, otherwise for any point, it must has a direction that is flat, $K = 0$).

Prop. (2.5.30) (Finiteness Theorem). Cf.[Jost P234].

Prop. (2.5.31) (π_2-Finiteness Theorem).

6 Curvature and Topology

Prop. (2.6.1) (Liouville Theorem). Any conformal mapping for an open subset of $\mathbb{R}^n, n > 2$ is restriction of a composition of isometry, dilations and/or inversions, at most once.

Prop. (2.6.2) (Bonnet-Mayer). M a complete manifold of Ricci curvature $\mathrm{Ric}_p(v) \geq \frac{1}{r^2}$, Then M is compact and have diameter $\leq \pi r$.

Cor. (2.6.3). *M is a complete manifold of Ricci curvature $\geq \delta > 0$, then the universal cover is compact thus $\pi_z(M)$ is finite.*

Prop. (2.6.4) (First Betti Number Theorem). There is a number $f(n, \lambda, D)$, $f(n, 0, D) = n, f(n, \lambda, D) = 0$ for $\lambda > 0$ that for a manifold of diameter $\leq D$ and Ricci curvature $\geq \lambda$, $b_1(M) \leq f(n, \lambda, D)$.

Prop. (2.6.5) (Synge). f is an isometry of a compact oriented manifold M^n of positive sectional curvature, f alter orientation by $(-1)^n$, then f has a fixed pt.

Cor. (2.6.6). *M a compact manifold of positive sectional curvature, then*

1. If M is orientable and n is even, then M is simply connected. So If M is compact and even dimension, then $\pi(M) = 1$ or $\mathbb{Z}_2$.
2. If n is odd, then M is orientable.

(Use the universal cover and covering transformation.)

Morse Index

Prop. (2.6.7) (Index Lemma). Among the piecewise differentiable vector fields along a geodesic without conjugate point or without focal point, with initial value 0 and fixed end value, the Jacobi field attain minimum of the index form:

$$I_a(V, V) = \int_0^a \{\langle DV(t), DV(t)\rangle - \langle R(\dot{\gamma}, V)\dot{\gamma}, V\rangle\}dt.$$

Cor. (2.6.8). *$I_l(J, J) = \langle J, J^{'}\rangle(l)$ for a Jacobi field.*

Prop. (2.6.9). a focal point is a critical value of $exp^{\perp}$. For an embedded manifold, the focal point equals $x + 1/t\eta$, where η is a vertical vector and t is a principal value of S_eta.

Prop. (2.6.10) (Morse Index theorem). The index of the the index form $I_a(V, W)$ on the space of vector fields 0 at the endpoints, equal to the number of points conjugate to $\gamma(0)$ in $[0, a)$.

Cor. (2.6.11). *If γ is minimizing, γ has no conjugate points on $(0, a)$, γ has a conjugate point, it is not minimizing.*

Prop. (2.6.12) (Cartan). in any nontrivial homotopy class in a compact manifold , there exists a closed geodesic.

Prop. (2.6.13) (Morse). If M is complete with non-negative sectional curvature, then $\pi_1(M)$ have no finite non-trivial cyclic group and $\pi_k(M) = 0$.

Proof: because universal cover of M is contractible, so the higher homotopy group vanish and $H^k(M) = H^k(\pi_1(M))$, so if a subgroup is finite cyclic, its homology is periodic, contradiction. □

Prop. (2.6.14) (Preissman). For a compact manifold with $K < 0$, any nontrivial abelian subgroup of π_1 is infinite cyclic.

Prop. (2.6.15). If M is compact and $K < 0$, $\pi_1(M)$ is not abelian.

Assuming M complete,

- The cut point of p along γ is the maximum $\gamma(t)$ s.t. $d(p, \gamma(t)) = t$. It is either the first conjugate point of p or the intersection of two minimizing geodesics.
- Conversely, if a point is a conjugate point of p or is intersection of two geodesics of equal length, then there is a cut point before it. So, if intersection of two minimizing geodesics happens, it must happen before the occurrence of conjugate point.
- thus the cut point relation is reflexive, and if $q \in M \setminus C_m(p)$, then there exists a unique minimizing geodesic joining p and q.
- $M \setminus C_m(p)$ is homeomorphic to an open ball through exp.
- the distance of p to the cut locus is continuous, thus $C_m(p)$ is closed.
- If M is complete and there is a p which has a cut point for every geodesic, then M is compact.
- for q the closest of $C_m(p)$ to p, either there exists a minimizing geodesic and q is conjugate to p or there is to minimizing geodesic connecting at q.

Prop. (2.6.16). The index of a geodesic will decrease when transferred to a manifold of smaller sectional curvature K.

Prop. (2.6.17). In a complete manifold, if there is a sequence of points $\{p_i\}$ converging to a point p, choose for each point a minimal geodesic, then a subsequence of them will converge to a minimal geodesic to p.

Proof: The convergence is by smoothness and of exp and Hadamard. The minimality is by comparing distance. $\square$

III.3 Geometric Analysis

1 Simplifications

Prop. (3.1.1). For every vector field $X$and every point $X(p) \neq 0$, there exists a coordinate nbhd $(x_1, \dots, x_{n-1}, t)$ such that $X = \frac{\partial}{\partial t}$.

2 Differential Forms

Lemma (3.2.1).

$$[X, Y] = \frac{\partial}{\partial t}(d(\phi_{-t})Y)|_{t=0}$$

Proof: For any function f,set $g(t,q) = \frac{f(\phi_t(q)) - f(q)}{t}$,$g(0,q) = Xf(q)$. Then g is differentiable (because $g(t,q) = \int_0^1 Xf(\phi_{ts}(p))ds$, and:

$$\begin{aligned}
\lim_{t\to 0} d(\phi_{-t})Yf(p) &= \lim \frac{Yf(p) - Y(f\phi_{-t})(\phi_t(p))}{t} \\
&= \lim \frac{Yf(p) - Yf(\phi_t p) - Y(tg(-t, \phi_t(p))}{t} \\
&= ((XY - YX)f)(p) \\
&= [X, Y]f(p)
\end{aligned}$$

□

Prop. (3.2.2) (Lie formula).

$$L_X(g(Y, Z)) = L_X(g)(Y, Z) + g(L_X Y, Z) + g(Y, L_X Z).$$

Prop. (3.2.3) (Derivative formula).

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

.

Prop. (3.2.4) (Cartan's magic formula).

$$L_X \omega = \iota_X(d\omega) + d(\iota_X \omega)$$

$$\iota([X, Y]) = [L_X, \iota_Y]$$

Proof: Notice that four of them are derivatives (check because $\iota_X(w \wedge v) = \iota_X w \wedge v + (-1)^{|w|} w \wedge \iota_X v$). So by induction, we only has to verify them on dimension 0 and 1. □

Prop. (3.2.5) (Stoke's theorem).

$$\oint_\Omega d\omega = \oint_{\partial\Omega} i^* \omega.$$

In a 3-dimensional Riemanian manifold, If we set:

$$df = \omega^1_{\mathrm{grad}_f}, \quad d\omega^1_A = \omega^2_{\mathrm{curl}_A}, \quad d\omega^2_A = (\nabla A)\omega^3,$$

Then:

$$f(y) - f(x) = \int_l \mathrm{grad} f \cdot dl.$$

$$\int_l A \cdot dl = \oint_S \mathrm{curl} A \cdot dn.$$

$$\oint_U \nabla \cdot F dV = \oint_{\partial U} F \cdot n dS.$$

Prop. (3.2.6). Lie bracket commutes with derivative.$[df(X), df(Y)] = df([X,Y])$. (Use $XY - YX$ to see).

Prop. (3.2.7) (Frobenius Theorem). If X is an involutive distribution on a manifold M, then there is a unique maximal integration manifold passing through it. Where a distribution is involutive if it is closed under Lie bracket.

Proof: The key to the proof is to prove that involutive is equivalent to integrable, i.e. flat locally as $\{\frac{\partial}{\partial x_i}\}$ for some local coordinate. Cf.[李群讲义 项武义 P226] □

Cor. (3.2.8). X, Y in a Lie algebra commute iff their corresponding vector fields commute.

3 Transversality

Prop. (3.3.1) (Parametric Transversality Theorem). Suppose N and M are smooth manifolds, $X \subset M$ is an embedded submanifold, and F_s is a smooth family of maps from N to M. If the map $F : N \times S \to M$ is transverse to X, then for almost every s, the map $Fs : N \to M$ is transverse to X. Cf.[Smooth Manifold Lee T6.35].

Proof: □

Prop. (3.3.2) (Transversality Homotopy Theorem). Suppose N and M are smooth manifolds and $X \subset M$ is an embedded submanifold. Every smooth map $f : N \to M$ is homotopic to a smooth map $g : N \to M$ that is transverse to X. Cf.[Smooth Manifold Lee T6.36].

Proof: embed M into a R^k and take a tubular neighbourhood, then we can construct a $N \times S^k$ transversal to M. □

Cor. (3.3.3). For a vector bundle over a compact manifold, there exists a global section transversal to the zero section, in particular, if $\dim E > M$, then it has no zero.

Proof: choose a finite trivializing cover that there closure is compact and choose a compact subcover, find finitely many sections to assure $C^N \times X \to E$ is transversal, and use parametric trnasversality theorem to prove there is a section that is transversal. □

Cor. (3.3.4). There is a vector field on compact manifold of only isolated zeros. And a vector bundle over a k dimensional curve splits to components of dimension no bigger than k. Determined by its Chern class.

4 Flow

Prop. (3.4.1) (Isotopy Extension Theorem). Let M be a manifold and A be a compact subset. Then an isotopy $F : A \times I \to M$ can be extended to an diffeotopy of M.

Proof: Consider $F(A \times I) \subset M \times I$ is a compact set, and $TM \times I \to M \times I$ is a vector bundle. The time lines generate a section $F(A \times I) \to TM \times I$, so (7.1.2) guarantees an extension $M \times I \to TM \times I$, and because manifolds are locally compact, this section can be chosen to be compactly supported, then the flow it generates is a diffeotopy. □

5 Differential Topology

Prop. (3.5.1) (transversality).

Prop. (3.5.2) (Sard Theorem). The set of critical values is of measure zero in the image manifold.

Prop. (3.5.3) (Hopf Index theorem). In a compact manifold, any vector field V with isolated zeros has sum of its index equal to $\chi(M)$. Where the index of a singularity is the mapping degree of V on a surrounding sphere.

6 Young-Mills Euqation & Seiberg-Witten Equation

Def. (3.6.1) (Yong-Mills). The Young-Mills functional on connections on a compact oriented space:

$$YM(A)^2 = ||F_A||^2 = -\int_X \mathrm{tr}(F_A \wedge F_A)$$

it is a critical point when $d_A \star F_A = 0$ and $d_A F_A = 0$.

Prop. (3.6.2) (2-dim Case). $\star F \in \Omega^0(su(E))$ is parallel thus its characteristic spaces is orthogonal and a stable under parallel transport. So an irreducible YM $SU(n)$-connection must by flat, thus correspond to irreducible $SU(n)$ representation of $\pi_1(X)$.

Prop. (3.6.3) (4-dim Case). $** = (-1)^{2*2} = \mathrm{id}$ on $\Omega^2(E)$ on E a $SU(n)$-bundle, so $\Omega^2(E) = \Omega^+ \oplus \Omega^-$. We have

$$||F_A^+||^2 + ||F_A^-||^2 \geq ||F_A^-||^2 - ||F_A^+||^2 = \int_X \mathrm{tr}(F_A \wedge F_A) = 8\pi^2 c_2(E)$$

Cf.[谢毅 Lecture5]. So it attains minimum at the connection that $*F_A = \pm F_A$ and $d_A F_A = 0$.((Anti)self-dual((anti)instanton)) depending on the sign of $c_2(E)$.

Prop. (3.6.4) (Anti-Instanton Connection on Complex Line Bundle). For a $U(1)$-bundle, $d_A F_A = dF_A$, so F_A is harmonic, thus $c_1(L) = [\frac{-1}{2\pi i} F_A] \in H^2(X, \mathbb{Z}) \cap \mathcal{H}^2_-(X, \mathbb{R})$, In fact, this is equivalent to the existence of a anti-self-dual connection on this bundle.

If this is the case, then we have the ASD-connections module Gauge equivalence is isomorphic to $H^1(X, \mathbb{R})/H^1(X, \mathbb{Z}) = T^{b_1(X)}$.

Proof: Because a gauge is just a $X \to S^1$, and its connected component thus equals $[X, S^1] = H^1(X, \mathbb{Z})$ (MacLane space), and its identity is just the map that is homotopic to id. and $d(gA) = dA - g^{-1}dg = dA - idu$, for $g = exp(iu)$, so $\Omega^1/\mathcal{G} = H^1(X, \mathbb{R}/H^!(X, \mathbb{Z}) = T^{b_1(X)}$. □

Lemma (3.6.5) (Weizenbock Formula).

$$\int |\mathcal{D}_A\varphi|^2 = \int |\nabla_A\varphi|^2 = \frac{1}{4}R|\varphi|^2 + \frac{1}{2}\langle F_A^+\varphi, \varphi\rangle.$$

Prop. (3.6.6) (Seiberg-Witten). The Seiberg-Witten equation functional for a unitary connection A on the determinant bundle of a Spin^c structure of M and a section of $\mathcal{S}^+$ is:

$$\begin{aligned} SW(\varphi, A) &= \int \Big(|\nabla_A\varphi|^2 + |F_A^+|^2 + \frac{R}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^2\Big)Vol. \\ &= \int \Big(|\mathcal{D}_A\varphi|^2 + |F_A^+ - \frac{1}{4}\langle e_je_k\varphi, \varphi\rangle e^j \wedge e^k|^2\Big)Vol \end{aligned}$$

So the Seiberg-Witten equation is the lowest topological possible value of the Seiberg-Witten functional. It writes:

$$\mathcal{D}_A\varphi = 0, \quad F_A^+ = \frac{1}{4}\langle e_je_k\varphi, \varphi\rangle e^j \wedge e^k.$$

Cf.[Jost Chapter 7].

Cor. (3.6.7). If a compact oriented Spin^c manifold M has nonnegative scalar curvature, then the only possible solution is $\varphi = F_A^+ = 0$. (See from the equivalence of forms of Seiberg-Witten functional.)

7 Spin Structure

Prop. (3.7.1) (Spin Structure Obstruction). For a oriented real bundle, its transformation map can be chosen to be in $SO(n)$, and constitute a Cech Cohomology $H^1(X, SO(n))$, and by exact sequence of

$$0 \to \pm 1 \to \mathrm{Spin}(n) \to SO(n),$$

this can be lifted to a $H^1(X, \mathrm{Spin}(n))$ iff its image w in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ is 0. and then its inverse image will be parametrized by $H^1(X, \mathbb{Z}/2\mathbb{Z})$ (By the non-commutative spectral sequence of Čech).

We have $w = w_2$, the Whitney class, (Just need to reduce to sk_2X and in this case, check they both equivalent to the bundle can be lifted). Cf.[XieYi 几何学专题]. Or we can use the Postnikov system of $BO(n)$(5.5.2).

Proof: First prove that if $E \oplus R^n$ is spin, then E is spin, and then pull $H^2(X, \mathbb{Z}/2\mathbb{Z})$ into $H^2(\mathrm{sk}_2(X), \mathbb{Z}/2\mathbb{Z})$, this in a injection, and the homology is natural, so we only have to prove this for $\mathrm{sk}_2(X)$. But E on $\mathrm{sk}_2(X)$ can decompose into a E' of dimension on more than 2, and for this, we see E is Spin iff it is the square of another bundle, so w and w_2 are the same. □

Prop. (3.7.2). For a Spin bundle E, the Spin-principal bundle with the Spinor representation(8.1.2) will generate a bundle S called the **Spinor bundle**. And the Ad action of Spin(n) on $Cl_{n,0}$ will generate a **Clifford bundle** $Cl(E)$. The Spin(n) actions are compatible, so the Clifford bundle can act on the spinor bundle. bundle. The act of the chirality operator on the Spinor bundle will generate two half spinor bundles $S^\pm$. Then TM will maps $S^\pm \to S^\mp$ for n even,(because of anti-commutative with Γ).

Prop. (3.7.3) (Spin^c-structure). The group Spin^c is the covering space of $SO(n) \times S^1$ (n>2) that corresponds to the group of elements mod 0 mod 2 in $\mathbb{Z}_2 \times \mathbb{Z}$, i.e. $\mathrm{Spin}(n) \times S^1/\{\pm 1\}$.

For example, $\mathrm{Spin}^c(4) = \{(A_1, A_2) \in U(2) \times U(2) | \det A_1 = \det A_2\}$, and $\mathrm{Spin}^C(3) = U(2)$.

Then a $SO(n)$ bundle can be lift to be a Spin^c-bundle if the line bundle determined by S^1 is determine the same w_2 as it, i.e. $w_2 = c_1(L)$ mod 2, This is equivalent to w_2 is in the image of $H^(X, \mathbb{Z})$, and this is equivalent to the Bockstein image of it is zero.

Use a variant of Wu's formula: $w_2(TM)[\alpha] = \alpha \cdot \alpha (\text{mod } 2)$ for M orientable of dimension 4, we have any orientable manifold of dimension 4 has a Spin^c-structure. Cf.[XieYi 几何学专题 Homework3].

There is a connection on the Clifford bundle and on the Spinor bundle induced by the Levi-Civita connection of M(2.3.2). This is compatible with the Clifford action. and it is also metric because the connection 1-form is in $\mathfrak{so}(n)$ because the action of $SO(n)$ preserves metric.

8 Chern-Weil Theory

Prop. (3.8.1) (Chern-Weil). For any connection on E, the map from invariant polynomial ring to $H^*(X) : P \mapsto [P(\Omega)]$ is a ring homomorphism independent on the connection.

The invariant polynomial ring is generated by coefficients f_k of the $\det(1+t\Omega)$ polynomial and also generated by the $\mathrm{tr}(\Omega^k)$ polynomials. Cf.[Loring Tu Appendix].

For a complex line bundle of degree r over a complex manifold,

$$\det(1 - \frac{1}{2\pi i} F_A) = 1 + c_1 + \ldots + c_n$$

gives out the Chern class, because it satisfies the axioms of Chern class (7.2.1).

For a real line bundle of degree r,

$$\det(1 - \frac{1}{2\pi i} F_A) = 1 + p_1 + \ldots + p_{\lfloor \frac{r}{2} \rfloor}$$

gives out the Pontrjagin class, where $p_k \in H^{4k}(X)$. (Notice the Ω can be chosen to be skew-symmetric thus for odd k, so the classes $\mathrm{tr}(\Omega^k) \in H^{2k}(X)$ vanish).

For an oriented real bundle of degree $2r$, the ω and thus Ω can be chosen to be skew-symmetric and the transformation matrix in $SO(2r)$, then

$$\mathrm{Pf}(\frac{1}{2\pi}\Omega) \in H^{2r}(X)$$

is well-defined and closed and gives the Euler class $e(E)$ (recall $e(E)^2 = p_r(E)$). (Use $\mathrm{Pf}^2 = \det$ to get that $[\frac{\partial \mathrm{Pf}}{\partial \Omega_{ij}}]^t$ commutes with Ω, then calculate $d\mathrm{Pf}(\Omega) = 0$).

There are relations between c_i and $\mathrm{tr}(F_A^k)$, they can be derived by considering diagonal elements.

Cor. (3.8.2).

$$c_1(E) = c_1(\wedge^{\dim E} E).$$

Direct from the formula.

Cor. (3.8.3) (Whitney Product Formula).

$$c(E \oplus F) = c(E)c(F), \quad p(E \oplus F) = p(E)p(F)$$

Directly form the product connection on $E \oplus F$.

Prop. (3.8.4) (Chern Character). The Chern character

$$ch(E) = [\operatorname{tr}\exp(\frac{i}{2\pi}F_A)]$$

satisfies $ch(E \oplus F) = ch(E) + ch(F)$ and $ch(E \otimes F) = ch(E)ch(F)$. So it defines a ring homomorphism from $K(X)$ to $H^*(X)$.

Prop. (3.8.5) (Chern-Gauss-Bonnet). For a $2n$-dimensional orientable manifold M,

$$\int_M e(TM) = \chi(M).$$

Prop. (3.8.6) (Hirzebruch Signature Formula). On a $4n$-dimensional orientable manifold M, the Poincare duality defines a bilinear pairing $H^{2n}(M) \times H^{2n}(M) \to \mathbb{R}$, its signature $\sigma(M)$ is given by:

$$\sigma(M) = \int_M L_n(p_1, \ldots, p_n).$$

Where L_n is the degree n part of the Taylor expansion of $\prod_{i=1}^r \dfrac{\sqrt{p_i}}{\tanh p_i}$.

Cor. (3.8.7). For a $4n$-dimensional M which is a boundary of a manifold, its signature is 0.

Proof: By Stokes theorem, if M is a boundary of a manifold, then all its Pontryagin numbers, i.e. $\int_M \prod p_i^{n_i}, \sum n_i = n$, vanish. □

Prop. (3.8.8) (Riemann-Roch). for a n-dimensional complex line bundle E over a Riemann Surface M, let

$$\chi(M, E) = \sum_{q=0}^n (-1)^q \dim H^q(M, E), \quad \deg L = \int_M c_1(E).$$

then

$$\chi(M, L) = \deg L - g + 1.$$

Prop. (3.8.9) (Hirzebruch-Riemann-Roch). For a n-dimensional complex line bundle E over a complex manifold M,

$$\chi(M, E) = \int_M [\mathrm{ch}(E)\mathrm{td}(TM)]_n.$$

Where $\chi(M, E)$ is defined as in (3.8.8), ch is the Chern character and $\mathrm{td}(TM)$ is the Todd polynomial, i.e. Taylor expansion of $\prod_{i=1}^r \dfrac{t_i}{1 - e^{-t_i}}$ applied to $c_i(TM)$.

Prop. (3.8.10). For a vector bundle and a flat connection d_A on a manifold, i.e. $d_A^2 = 0$, we have a deRham like cohomology, and there is a sheaf of flat sections.

$$H^*(X, A) = H^*(X, E).$$

III.4 Morse Theory & Floer Homology

1 Morse Theory(Milnor)

Prop. (4.1.1) (Morse Lemma). In a non-degenerate critical point of f, there is a coordinate that

$$f = f(p) + x_1^2 + \cdots + x_{n-\lambda}^2 - y_1^2 - \cdots - y_\lambda^2.$$

Proof: Just extract the first order part out and reform the bilinear form one-by-one. Cf.[Milnor Morse Theory lemma 2.2]. □

Prop. (4.1.2). If f is a smooth function that $f^{-1}([a,b])$ is compact and have no critical points, then M^a is a deformation retracts of M^b using $\mathrm{grad} f/|\mathrm{grad} f|^2$.

Prop. (4.1.3) (Morse Main Lemma). If f is a smooth function with p a non-degenerate critical point and λ downward pointing direction. If for some $f^{-1}([c-\epsilon, c+\epsilon])$ is compact, then $M^{c+\epsilon}$ is homotopic to $M^{c-\epsilon}$ gluing a λ dimensional cell.

Proof: Cf.[Milnor Prop3.2]. □

Prop. (4.1.4). For an embedded manifold and almost all point p, the distance to p is a morse function. (Use Sard theorem and degenerate $\iff$ p is a focal point.

Cor. (4.1.5). smooth manifold has CW type; on a compact manifold any vector field with discrete singular points has its index sum equal to $\chi(M)$ (Hopf-Rinow), and there exists one.

Prop. (4.1.6). for $\Omega(p,q)^c$ the path space of energy $< c$, the piecewise geodesic path space B (piece fixed), the energy function is smooth and B^a is compact and is the deformation contraction of $\mathrm{int}\Omega^a$ for $a < c$. E has the same critical point and same index and nullity on B and Ω^c. (Just geodesicrize any path in Ω.

So for two point not conjugate in B^a, Ω^a has a finite CW complex type and a λ-dimensional cell for every geodesic of index λ in B^a.

Prop. (4.1.7) (Morse Main Theorem). If p and q are not conjugate along any geodesic, then $\Omega(p,q)$ has a countable CW complex type and has a λ-cell for every geodesic of index λ.

If M has nonnegative Ricci curvature, then M has only finite cell for every dimension.

Proof: Cf.[Milnor Morse Theory Prop17.3]. □

Cor. (4.1.8). The path space homotopy type only depend on the homotopy type of M (use the two homotopy to id to get a composition of homotopy of the two path space), so one can get the information of path space of M by looking at the homotopy type of M.

Prop. (4.1.9) (Minimal Geodesics). If p, q in a complete manifold M has distance $\sqrt{d}$ and the minimal geodesics form a topological manifold, and if all non-minimal geodesic has index $\geq \lambda$, then for $0 \leq i < \lambda$, $\pi_i(\Omega, \Omega^d) = 0$.

Lemma (4.1.10). In $SU(2m)$, the minimal geodesic from I to $-I$ is homeomorphic to Grassmannian $G_m(\mathbb{C}^{2m})$ and and non-minimal geodesic has index $\geq 2m+2$.

Similarly, The space of minimal geodesic from I to $-I$ in $O(2m)$ is homeomorphic to the space of complex structures in $\mathbb{R}^{2m}$, and any non-minimal geodesic has index $\geq 2m-2$.

Proof: Cf.[Milnor Morse Theory Lemma23.1 Lemma24.4]. □

Lemma (4.1.11). Ω_{k+1} is homotopic to the space of minimal geodesics in Ω_k from J to $-J$. (The same way, calculate the index of geodesics from J to $-J$ and use (4.1.9)). Cf.[Milnor Morse Theory Prop24.5] for definition of Ω_{k+1}.

III.5 Algebraic Topology

1 Homology and Cohomology

Prop. (5.1.1). The fundamental group of a topological group is abelian.

Proof: This is because π_1 preserves products, so takes group objects to group objects. And the group objects in the category of groups is the abelian groups (1.8.2) □

Prop. (5.1.2) (de Rham). The de Rham cohomological group $H^*_{dR}(X)$ is isomorphic to the singular cohomological group $H^*(X,\mathbb{R})$.

Proof: First, $H^*_{dR}(X) \cong H^*(X,R)$ for the constant sheaf cohomology by(3.2.19), and prove □

Prop. (5.1.3). For two homotopic map between two topological space (Fine enough), they induce the same map on singular (co)homology and de Rham cohomology.

Proof: For singular homology, the combinatorial 'pillariazation' can be constructed that $f-g=k^{n-1}\circ d+d\circ k^n$. And for de Rham cohomology, then a similar k^n can be constructed. Cf.[Gelfand Homological Algebra P52]. □

Prop. (5.1.4). The cellular (co)homology coincides with the singular (co)homology for CW-complex.

Prop. (5.1.5) (Morse Inequality). for any field F,

$$\sum^k(-1)^i\dim H_i(X,F)\le\sum^k(-1)^i c_i,$$

where c_i is the number of I-dimensional cells. (Use the dimension counting of the long exact sequence).

Prop. (5.1.6) (Poincare Duality).

Cor. (5.1.7).

$$H^*(\mathbb{RP}^n,\mathbb{Z}_2)=\mathbb{Z}_2[X]/X^n,\quad H^*(\mathbb{CP}^n,\mathbb{Z})=\mathbb{Z}[X]/X^n$$

Proof: Use induction and Poincare duality to find that $\alpha*\alpha^{n-1}=\alpha^n$. □

Prop. (5.1.8) (Alexander Duality).

Prop. (5.1.9) (Thom isomorphism). Cf.[姜伯驹同调论].

Prop. (5.1.10) (Gysin Sequence). Cf.[姜伯驹同调论].

Prop. (5.1.11) (Lefschetz Fixed Point Theorem).

Cohomology of Fiber Bundles

Prop. (5.1.12) (Leray-Hirsch). For a fiber bundle and a ring R s.t. $H^n(F,R)$ is f.g free for all n, and there exist classes c_j that constitute a basis for each fiber F, then

$$H^*(B,R)\otimes H^*(F,R)\to H^*(E,R)$$

is an isomorphism of $H^*(B,R)$-modules.

Cor. (5.1.13).

- $H^*(U(n);\mathbb{Z})=\Lambda_{\mathbb{Z}}[x_1,x_3,\ldots,x_{2n-1}]$.
- $H^*(SU(n);\mathbb{Z})=\Lambda_{\mathbb{Z}}[x_3,\ldots,x_{2n-1}]$.
- $H^*(Sp(n);\mathbb{Z})=\Lambda_{\mathbb{Z}}[x_3,x_7,\ldots,x_{4n-1}]$.

Prop. (5.1.14). $H^*(G_n(\mathbb{K}^\infty);\mathbb{Z})$ where $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$ is generated by the symmetric polynomials, where for $\mathbb{R}$ the coefficient is $\mathbb{Z}_2$.

Proof: Use the flag variety and first calculate for ∞. Then use Poincare duality to show it is mapped onto the symmetric polynomials. Cf.[Hatcher P435]. $\square$

Prop. (5.1.15) (Leray-Serre). For a Serre fibration, especially fiber bundle, $F\to E\to B$, that B is simply connected, then there is a spectral sequence

$$E_2^{pq}=H_p(B,H_1(F))\Rightarrow H_{p+q}(E).$$

Cor. (5.1.16) (Wang Sequence). When $B=S^n$, there is a long exact sequence:

$$\cdots\to H_q(F)\to H_q(E)\to H_{q-n}(F)\to H_{q-1}(F)\to H_{q-1}(E)\to\cdots$$

Cor. (5.1.17) (Gysin Sequence). When $F=S^n$, there is a long exact sequence:

$$\cdots\to H_{p-n}(B)\to H_p(E)\to H_p(B)\to H_{p-n-1}(B)\to H_{p-1}(E)\to\cdots$$

Cup Product and Cohomology Operators

Prop. (5.1.18). The cup product will restrict to a relative version:

$$H^*(X,A)\times H^*(X,B)\to H^*(X,A\cup B),$$

This implies that if X is a union of n contractible open set, then the cup product of n-elements vanish. In particular, the cup product in a suspension vanishes.

Prop. (5.1.19) (Steenrod Powers). The total Steenrod squares Sq is a map from $H^n(X,\mathbb{Z}_2)\to H^{n+*}(X,\mathbb{Z}_2)$ that:

- it is natural and stable under suspension.
- it is additive.
- $\mathrm{Sq}(\alpha\cup\beta)=\mathrm{Sq}(\alpha)\cup\mathrm{Sq}(\beta)$.
- $Sq^i(\alpha)=\alpha^2$ if $i=|\alpha|$, and 0 if $i>|\alpha|$.

The total Steenrod Powers P is a similar map from $H^n(X,\mathbb{Z}_p)\to H^{n+*}(X,\mathbb{Z}_p)$ that $P^i(\alpha)=\alpha^p$ if $2i=|\alpha|$ and 0 if $2i>\alpha$.

The algebra of powers is generated respectively by elements Sq^{2^k}, and for p it is generated by β and the elements P^{p^k}. (Because of Adem relations) Cf.[Hatcher P497].

2 Fundamental Groups

Prop. (5.2.1) (Van Kampen).

3 CW Complex

Prop. (5.3.1). If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, thus (X, A) has the **homotopy extension property** because we can perform infinite induction on dimension.

Prop. (5.3.2). The loop space ΩX for X a CW complex has CW complex type. In particular, if it has noly finitely many cells for a given dimension, then so does ΩX. Milnor proved this.

Prop. (5.3.3). The homotopy group defines a long exact sequence for triples (X, A, B), in particular for $B = \mathrm{pt}$.

Prop. (5.3.4) (Compression Theorem). If (X, A) is a CW pair that (Y, B) be a pair that $\pi_n(Y, B, y_0)$=0, for any n, then every map (X, A) to $(Y < B)$ is homotopic rel$A$ to a map $X \to B$. (Use extension property to extend by dimension). This shows that the homotopy doesn't depend on higher dimensions, (but might on lower one).

Cor. (5.3.5) (Whitehead Combinatorial Homotopy I). If M and K is dominated by CW complexes, then any map $M \to K$ inducing homotopy group isomorphisms is an homotopic equivalence. If the map is inclusion, then it is a deformation retract. In particular, if M is manifold, then it is dominated by its tubular nbhd, so this theorem is applied.

Proof: For inclusion, use compression, and in general use mapping cylinder and cellular approximation. □

Cor. (5.3.6). If $\pi_n(X) = 0$ for all n and a CW complex X, then X is contractible.

Prop. (5.3.7) (Cellular Approximation Theorem). Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. This makes calculation of homotopy easy. (It suffice to show a map cannot be surjective on a higher dim cell, Cf.[Hatcher P349].

Moreover, Any map of pairs of CW complexes can be deformed to a cellular map. (first deform the small complex, then deform the big by dimension.

Cor. (5.3.8). If a CW complex has only cells of dim> n, then it's homotopy group vanishes for $i < n$. In particular, $\pi_n(S^k) = 0$ for $n < k$.

Prop. (5.3.9) (CW Approximations). There exists a CW approximation for any pair (X, A), that is, induce isomorphism on X and X_0 thus on relative homotopy group. Cf.[Hatcher P353].

If A is CW, then there is a n-connected CW models (Z, A) to (X, A), i.e. $\pi_{\leq n}(Z, A) = 0$ and $Z \to X$ induce isomorphism on $\pi_{>n}$ and injection for π_n. And this approximation is unique up to homotopy equivalence rel$A$, (use relative mapping cylinder and use compression). They act like injective resolutions.

Use Long exact sequence compression and mapping cylinder, we can prove the approximations preserve (co)homology and mapping classes.

Cor. (5.3.10). For any n-connected CW pair (X, A), there exist a homotopic (Z, A) that $Z\backslash A$ has only cells of dimension n.

Cor. (5.3.11) (Whitehead theorem). A f between two simply connected CW complexes that induce isomorphism on homology groups is a homotopy equivalence. (using mapping cylinder, we can assume it's an inclusion, and $\pi_1(Y, X) = 0$, so the theorem shows that $\pi_n(Y, X) = 0$, and use Whitehead).

Prop. (5.3.12). A closed manifold or the interior of a manifold with boundary has a homotopy type of a CW complex of finite type.

Remark (5.3.13). The use of mapping cylinder and relative mapping cylinder is important.

4 Homotopy

Prop. (5.4.1). The universal cover have the same homotopy group $\pi_{>1}$, by lifting property.

Prop. (5.4.2) (Excision Theorem). If A, B are CW-complexes, then if $(A, A \cap B)$ are m-connected and $(B, A\cap B)$ are n-connected, then $\pi_i(A, A\cap B) \to \pi_(A\cup B, A)$ is isomorphism for $i < m + n$, and surjective for $i = m + n$. Cf.[Hatcher P360].

Moreover, if (X, A) is r-connected and A is s-connected, then $\pi_i(X, A) \to \pi_i(X/A)$ is isomorphism for $i \le r + s$ and surjection for $i = r + s + 1$.

Cor. (5.4.3) (Freudenthal Theorem). For $i \le 2n - 2$, $\pi_i(S^n) \cong \pi_{i+1}(S^{n+1}) = \mathbb{Z}$. (Can also be derived considering antipodal point point of S^n by (4.1.9)) and surjective for $i = 2n-1$. In general, $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n + 1$.

Proof: Use the suspension, for $n = 1$, we can use Hopf bundle. □

Prop. (5.4.4) (Generalized Hurewicz theorem). If (X, A) is a $(n-1)$-connected pair of spaces, $n \ge 2$, then the Hurewicz map induces isomorphism $\pi_n(X, A)/(\pi_1(A) - id) \to H_k(X, A)$, and $H_k(X, A) = 0, k < n$. And for on π_{n+1}, the Hurwicz map is surjective for $n > 1$.Cf.[Hatcher P390Ex23] for surjectiveness.

Prop. (5.4.5). For a fiber bundle $S \to M \to N$, there is a long exact sequence of homotopy groups:

$$\cdots \to \pi_i(N) \to \pi_{i-1}(S) \to \pi_{i-1}(M) \to \pi_{i-1}(N) \to \cdots$$

Because it has lifting property.

Prop. (5.4.6). $\pi_{i+1}(M) \cong \pi_i(\Omega(M))$, where Ω is the loop space. More generally,

$$\langle \Sigma X, K\rangle = \langle X, \Omega K\rangle.$$

Prop. (5.4.7). If K_n is an Ω-spectrum, then the functors $X \mapsto h^n(X) = \langle X, K_n\rangle$ define a reduces cohomology theory on the category of basepointed CW complexes, i.e. it satisfies the long exact sequence for $A \to X \to X/A$ and wedge axiom. Cf.[Hatcher P397].

Proof: Use(5.4.6) and there is a Cofibration sequence:

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \dots$$

□

Prop. (5.4.8). Every map can be decomposed as a homotopy equivalence followed by a fibration, by the construction of homotopy fibers. Cf.[Hatcher P407].

Prop. (5.4.9). The homotopic direct limit of a family of homotopy equivalence is a homotopy equivalence. Cf.[Morse Theory Milnor].

Prop. (5.4.10). for $i \leq 2m$, $\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i-1}U(m)$, and

$$\pi_{i-1}U(m) \cong \pi_{i-1}U(m+1) \cong \cdots .$$

and for $j \neq 1$, $\pi_j U(m) \cong \pi_j SU(m)$.

Similarly, $\pi_i \Omega_1(2m) \cong \pi_{i+1}O(2m)$ for $i \leq n-4$. (4.1.10),Cf[Morse Theory Milnor Prop23.4].

Cor. (5.4.11) (Bott Periodicity theorem for Unitary Groups). The stable homotopy group $\pi_i U$ has period 2. $\pi_{2k+1}U \cong 0$ and $\pi_{2k}U \cong \mathbb{Z}$.

Proof: Use the last proposition and long exact sequence to show that for $1 \leq i \leq 2m$,

$$\pi_{i-1}U = \pi_{i-1}U(m) \cong \pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i+1}SU(2m) \cong \pi_{i+1}U.$$

Notice that $U(m) \to U(2m)/U(m) \to G_m(\mathbb{C}^{2m})$ □

Prop. (5.4.12) (Bott Periodicity for O). For the infinite dimensional orthogonal space O, $\Omega_8(16r) \cong O(r), \Omega_4(8r) \cong Sp(2r)$. So $\Omega_8 \cong O$ and $\Omega_4 O \cong Sp$. Thus by (5.4.6),

$$\pi_i(O) = \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z} \ldots, \quad \pi_i(Sp) = 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z} \ldots$$

respectively. (Use (4.1.11)) Cf.[Morse Theory Prop24.7].

5 Obstruction Theory & Classifying Space

Towers

Prop. (5.5.1) (Towers). There are Whitehead Towers and Postnikov Towers for a CW complex X.

$$\cdots \to Z_2 \to Z_1 \to Z_0 \to X \to \cdots \to X_2 \to X_1 \to X_0$$

Z_n annihilate $\pi_{\leq n}(X)$, X_n remains only $\pi_{\leq n}(X)$. The towers can be chosen to be fibrations, with fibers $K(\pi_n X, n)$ by(5.4.8).

Prop. (5.5.2). There is a Postnikov towers of :

$$B\mathrm{String}(n) \to B\mathrm{Spin}(n) \to BSO(n) \to BO(n)$$

with corresponding obstructions $w_1(X), w_2(X)$ and $p_1(X)/2$.

Prop. (5.5.3) (Obstructions). If a connected abelian CW complex X($\pi_1(X)$ abelian and action on higher homotopy trivial) and (W, A) satisfies $H^{n+1}(W, A; \pi_n X) =$ for all n, then $A \to X$ can extend to a map $M \to X$.

Proof: Cf.[Hatcher P417]. □

Cor. (5.5.4). A map between Abelian CW complexes that induce isomorphisms on homology is a homotopy equivalence.

Proof: Notice that $\pi_1(X)$ acts trivially on $\pi_1(Y,X)$ and use Hurewicz. □

Def. (5.5.5). The **classifying space** for a topological group is a space BG with a weakly contractable universal cover EG that EG is a G-fiber bundle on BG. $[X, BG] \cong G$-bundles on X. And BG is Abelian if G is Abelian.

For an discrete Abelian group A, we denote $K(A,0) = A$, $K(A,n+1) = B(K(A,n))$. Then $H^n(X,A) = [X, K(A,n)]$. Notice that $\pi_{n+1}(BG) = \pi_n(G)$. It is called **Eilenberg-Maclane spaces**, i.e. it has only a nontrivial homotopy group $\pi_n(K(A,n)) = A$. The compression theorem shows $K(G,n)$ is unique up to homotopy.

EM complexes can also be built by a set of $n, n+1$ cells and use cells of dim$\geq n+2$ to kill other homotopy.

Prop. (5.5.6) (Examples).

- $K(\mathbb{Z},1) = S^1 = U(1)$, $K(\mathbb{Z},2) = \mathbb{CP}^\infty$, Because $S^\infty \to \mathbb{CP}^\infty$ is a contractable covering and use the fiber sequence.
- $B(\mathbb{Z}/2) = \mathbb{RP}^\infty$, and $B(\mathbb{Z}/n\mathbb{Z}) = S^\infty/(\mathbb{Z}/n)$.
- $BSU(2) = \mathbb{HP}^\infty$.
- $B(\mathbb{Z}^{2g})$ = torus of genus g because torus has the upper half plane as universal cover, this can be seen observing only has to satisfy the sum of inner angle is π.
- $BO(n), BU(n), BSp(n)$ are respectively the Grassmannian of n-planes in the infinite dimensional real, complex and quaternion vector spaces, because we have

$$O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty).$$

  and similarly for $\mathbb{C}$ and $\mathbb{H}$, and $V_n(\mathbb{R}^\infty)$ is contractible by linear homotopy and Schmidt orthogonalization.
- there are fiber bundles

$$S^0 \to BSO(n) \to BO(n)$$

  and similarly others, because they both have the flag variety $V_n(R^\infty)$ as EG.

Prop. (5.5.7). $H_*(BG,\mathbb{Z}) \cong H_*(G,\mathbb{Z})$ and $H^*(BG,\mathbb{Z}) \cong H^*(G,\mathbb{Z})$.

Proof: Because EG is weakly contractible, $S_*(EG)$ is a free $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$ and $S_*(EG)_G$ is identified with $S_*(BG)$. The following is easy. □

III.6 Differential Geometry

1 Different Coordinates

Prop. (6.1.1). In a polar coordinate,

$$g_{11} = 1, g_{12} = 0, g_{22} = |\frac{\partial f}{\partial \theta}|^2, \quad K = -\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}}$$

And $\sqrt{g_{22}} \sim \rho$. (Use the formula relating Jacobi Field with curvature)

2 Moving Frame Method

Prop. (6.2.1) (Theorema Egregium).

III.7 Vector Bundle & K-Theory

1 Fundamentals

Prop. (7.1.1). A vector bundle can have its transform map $\in O(n)$ (or $U(n)$) by constructing a riemannian metric on it. And for every local trivialization, we choose the metric on it compatible with the given metric, thus the transform map is $\in O(n)$ (or $U(n)$).

Prop. (7.1.2) (Tietze extension general). For a Hausdorff paracompact(hence normal) space X and a paracompact subspace Y, every section on Y can be extended to a section on X. (For every point of Y, find a local trivialization and an even smaller open set. Use Tietze extension to extend localy to this nbhd, then use partition of unity to unify all).

Prop. (7.1.3). For a continuous family of maps from a paracompact Hausdorff space Y to a Hausdorff paracompact space X, then the pullback bundle is isomorphic.

Proof: Consider the space $Y \times I$ and the pullback bundle E, then for every t_0, consider a new bundle $\mathrm{Hom}(E, \pi_1^* E_{t_0})$, then Y has a section id, this section by the last proposition can be extended, so it spans the vector space for nearby t (because of paracompactness), thus is an isomorphism because it is a locally invertible vector bundle homomorphism. □

2 Chern Class

Prop. (7.2.1). Axioms for Chern classs for complex bundles:

- $c(E) = 1 + c_1(E) + \ldots + c_n(E)$, $n = \deg(E)$.
- $f^*(c(E)) = c(f^*(E))$.
- $c(E \oplus F) = c(E)c(F)$.
- On the tautological bundle over $\mathbb{CP}^k$, $c(\eta) = 1 + c_1(\eta)$ and $\int_{\mathbb{CP}^k} c_1(\eta) = -1$. There is a Affine connection definition of Chern class.

Proof: □

Prop. (7.2.2). For a complex line bundle, the first Chern class characterize them. We have by the Affine definition that $c_1(E) = c_1(\wedge^{\dim E} E)$, so $c_1(E) = 0 \iff \wedge^{\dim E} E$ is trivial $\iff E$ is orientable.

3 Thom isomorphism

Prop. (7.3.1) (Thom Class). For a vector bundle, we can compactify its bundles to get a (D^n, S^n)-bundle, if there is a Thom class that induce a generator $H^n(D^n, S^n)$ on every fiber. Then the relative Leray-Hirsch will give that c induces an isomorphism $H^i(B, R) \to H^{i+n}(E, E', R)$. For $\mathbb{Z}_2$ coefficient there exists a Thom class, and for orientable bundle there exists a $\mathbb{Z}$-Thom class. Notice that fiber bundle over a simply connected base is orientable.

Prop. (7.3.2). Similarly, for a orientable fiber bundle $S^{n-1} \to E \to B$, make it a $D^n \to E' \to B$ bundle, then E' is homotopy equivalent to B so there is a Gysin sequence

$$\to H^{i-n}(B) \xrightarrow{*e} H^i(B) \to H^i(E) \to H^{i-n+1}(B) \to$$

Where the Euler class e is chosen to commute with the Thom isomorphism.

4 Principal Bundles

Def. (7.4.1). A principal bundle is a bundle P with G-fibers that the transition function is right G-map, i.e. left multiplication by some $g_{\alpha\beta}$. a associated bundle of a representation $G \to \mathrm{End}(V)$ is the total space of $P \times V$ module the equivalence $[gg_0, v] = [g, g_0 v]$. The corresponding transition function is just the left action by $g_{\alpha\beta}$.

Prop. (7.4.2) (Homogenous Space). If G is a Lie group and H is a closed subgroup, then the quotient $H\backslash G$ can be given a structure of a G-homogenous space and $G \to H\backslash G$ is a principal H-bundle.

Prop. (7.4.3). The projection $S^{2n+1} \to \mathbb{C}P^n$ is a principal S^1-bundle.

III.8 Symplectic Geometry

1 Basics

Cf.[Methods in Classical Mechanics Arnold Chapter8],[辛几何讲义范辉军].

Prop. (8.1.1). A hamiltonian phase flow preserves the symplectic form. $g^{t*}\omega = \omega$.

Proof: by Cartan's magic formula,

$$\frac{d}{dt}(g^t)^*\omega = L_X\omega = \iota_X(d\omega) + d(\iota_X\omega) = d(\iota_X\omega)$$

because ω is closed. And by definition, $d(\iota_X\omega)(\eta) = \omega(J\mathrm{d}H, \eta) = \langle dH, \eta\rangle$, so $d(\iota_X\omega) = dH$, Thus the theorem. □

For the following Cf.[辛几何讲义范辉军 lecture3].

Prop. (8.1.2) (Moser's Stability). If ω_t is a smooth family of cohomologous forms on a closed manifold M, then there exists an isotopy Ψ_t s.t.

$$\Psi_t^*(\omega_t) = \omega_0.$$

Prop. (8.1.3) (Relative Moser Stability). If M is a closed manifold and S is a compact submanifold, then if two closed 2-form equals on S, then there is an open neighborhood N_0, N_1 of S and a diffeomorphism $\Psi : N_0 \to N_1$ that

$$\Psi|S = \mathrm{id}, \;\; \Psi^*\omega_1 = \omega_0.$$

Cor. (8.1.4) (Darboux's Theorem). Every symplectic form ω on M is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.

Proof: Choose $S = \mathrm{pt}$ and uses relative Moser stability. □

III.9 Lie Groups & Symmetric spaces

1 Main Theorems

Prop. (9.1.1). For a Lie group G, for any lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there exists uniquely a connected Lie subgroup H s.t. $\mathfrak{h}$ is the lie algebra of H.

Proof: By (3.2.7), there is a maximal connected manifold H corresponding to $\mathfrak{h}$, we only need to show that it is a group. But the left invariance of $\mathfrak{h}$ shows that $HH \subset H$ because H is maximal. □

Cor. (9.1.2). If G_1 is a simply connected Lie group and G_2 is a connected Lie group, then any Lie algebra homomorphism $\tilde{h} : \mathfrak{g}_1 \to \mathfrak{g}_2$ can be lifted to a Lie group homomorphism.

Proof: use the image of $\tilde{h}$: $\Gamma(\tilde{h}) \subset \mathfrak{g}_1 \times \mathfrak{g}_2$, the prop shows that there is a Lie group in $G_1 \times G_2$ for $\Gamma(\tilde{h})$. It is isomorphic to G_1 because the Lie algebra is the same and both are connected, thus a covering map and G_1 is simply connected. □

Prop. (9.1.3) (Closed Subgroup Theorem). If H is a closed subgroup of a Lie group G, then there exists uniquely a differential structure s.t. H is a Lie subgroup of G. Cf.[Helgason Symmetric Spaces].

Prop. (9.1.4) (Ado). Any finite dimensional Lie algebra can be embedded in some $\mathfrak{gl}(n, \mathbb{C})$.

Cor. (9.1.5). From the preceding propositions, it follows that the category of finite dimensional Lie algebras is equivalent to the category of simply connected Lie groups.

2 Generals

Prop. (9.2.1). A connected Lie group is second countable.

Proof: This follows from the fact that a Lie group is a manifold hence locally compact and it is a union of their products. □

Prop. (9.2.2). A continuous homomorphism between Lie groups is smooth.

Proof: use exp coordinates. □

Prop. (9.2.3). Any connected Lie group has a compact subgroup as deformation contraction.

Prop. (9.2.4).

$$SU(2) = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}, \alpha, \beta \in \mathbb{C}$$

is isomorphic to the group of unit quaternions and diffeomorphic to S^3.

3 Classical Groups

For more classical groups, Cf.[Classical Groups Baker].

Fundamental Groups

Prop. (9.3.1).

- $SU(n-1) \to SU(n) \to S^{2n-1}$ gives us $SU(n)$ is simply connected.

  $$\pi_1(Sp(2n)) = \pi_1(U(n)) = \mathbb{Z}$$

  and the determinant induces an isomorphism onto $\pi_1(S^1)$. In fact, this is used to define the Maslov index.

- $SO(n-1) \to SO(n) \to S^{n-1}$ gives us $\pi_1(SO(n)) \cong \pi_1(SO(3))$. And $SU(2)$ as the unit sphere in $\mathbb{H}$ maps to $SO(3)$ via the conjugation:$\mathrm{Ad}(z) : w \mapsto zw\overline{z}$ has kernel ± 1, so $SO(3)$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$.

Generals

Prop. (9.3.2). As in (9.3.1) $SU(2)$ is a universal covering of $SO(3)$ and so does $\mathrm{Spin}(3)$(8.1.4), so $SU(2) \cong \mathrm{Spin}(3)$.

Prop. (9.3.3). The symplectic group $Sp(2n, \mathbb{R}) = Sp(2n, \mathbb{C}) \cap U(n, n, \mathbb{C}) = Sp(n, \mathbb{H})$. And

$$Sp(2n)\cap O(2n) = Sp(2n)\cap GL(n, \mathbb{C}) = O(2n)\cap GL(n, \mathbb{C}) = U(n) = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}, \quad X+iY \in U(n).$$

Exponential Map

Prop. (9.3.4). The exponential map for $GL_n(\mathbb{C})$ and $U(n)$ is surjective and the image of the exponential map for $GL_n(\mathbb{R})$ is $GL_n(\mathbb{R})^2$.

Proof: Use Jordan Decomposition (Real). For complex case, it is unitary diagonalizable. □

4 Analysis

Lemma (9.4.1). Bi-invariant metric exists in a compact manifold.

Proof: Because the Haar measure on a compact metric is bi-invariant. Choose a Riemann metric and set

$$\langle V, W\rangle = \int_{G\times G} \langle L_{\sigma *}R_{\tau *}(V), L_{\sigma *}R_{\tau *}(W)\rangle d\mu(\sigma)d\mu(\tau).$$

Note that L_* and R_* commute. □

Prop. (9.4.2). If G is a Lie group with a bi-invariant metric, then

$$2\nabla_X Y = [X, Y], \quad \langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle,$$

$$\nabla_X Y = 1/2[X, Y], \quad R(X, Y)Z = 1/4[[X, Y], Z], \quad K(\sigma) = 1/4\Big|[X, Y]\Big|^2.$$

So its curvature is non-negative, and all 1-parameter subgroups are geodesic.

Prop. (9.4.3). A bi-invariant Lie group with $\mathfrak{g}$ having trivial center is compact and $\pi_1(G)$ finite.

Proof: From Myer Theorem because the Ricci curvature has a positive lower bound. □
Proof: Cf.[Morse Theory Milnor Prop20.5]. □

Prop. (9.4.4) (Structure theorem for bi-invariant Lie group). A simply connective Lie group with a bi-invariant metric is equal to $G^{'} \times R^k$, $G^{'}$ compact.

Proof: Because the orthogonal complement of the center of $\mathfrak{g}$ is a Lie algebra, G is like $G^{'} \times R^k$, and a simply connected abelian Lie group is R^k **?**. □

5 Symmetric space

Prop. (9.5.1). A **symmetric space** is that for every point p, there is a isometry reversing the geodesics passing p. A manifold is called **locally symmetric** if $\nabla R = 0$. Locally symmetric is equivalent to the fact that every local reversing map is an isometry. A symmetric space is complete because two folding is an extension of geodesic.

Prop. (9.5.2). A Lie group with a bi-invariant metric is a symmetric space.

Prop. (9.5.3). The conjugate points in a symmetric space is easy to calculate, they are $\exp(\frac{\pi k}{\sqrt{e_i}}V)$, counting multiplicity, where e_i is the eigenvalue of the sself-adjoint operator $K_V(W) = R(V,W)V$ at p.

III.10 Hyperbolic Geometry

Prop. (10.0.1). 双曲圆盘的保距同构都是由 Mobius 变换给出的。因为任何三点为半径做圆就可以确定出每一个点。Cf.[双曲几何 刘毅]

Chapter IV

Analysis

IV.1 Real Analysis

1 Basics

Prop. (1.1.1). A function f is real analytic on an open set iff there is a extension to a complex analytic function to an open set. And this is equivalent to: For every compact subset, there is a constant C that for every positive integer k, $|\frac{d^k f}{dx^k}(x)| \leq C^{k+1}k!$.

Proof: Use Lagrange residue(中值定理) to show that it will converge to f. □

Prop. (1.1.2) (convergences). There are three different kinds of convergences.

Prop. (1.1.3) (Bounded Convergence Theorem).

Prop. (1.1.4) (Riesz Representation Theorem). on $C_c(X)$ for a LCH space X,

If I is a positive linear functional, there is a unique regular (both inner and outer) Radon measure μ on X such that $I(f) = \int f d\mu$. Moreover,

$$\mu(U) = \sup\{I(f) : f < U\} \text{ for } U \text{ open},$$

$$\mu(K) = \inf\{I(f) : f > \chi(K)\} \text{ for } K \text{ compact}.$$

If I is a continuous linear functional, there is a unique regular countably additive complex Borel measure μ on X that $I(f) = \int f d\mu$.

So if X is compact, $M(X)$ the space of Borel measures on X is the dual spcae of $C(X)$.

Proof: Cf.[Real Analysis Folland]. □

Prop. (1.1.5). For a pair of Hilbert basis $\{e_i\}$ of $L^2(M)$ and $\{f_j\}$ of $L^2(N)$, $\{e_i \otimes f_j\}$ gives a basis for $L^2(M \times N)$. (Use Fubini).

2 Approximations

Prop. (1.2.1). The polynomial functions are dense in $C[-1, 1]$.

Proof: We only have to prove that $|x|$ can be approximated, because then all piecewise linear function can. For this, Taylor expand $\sqrt{1 + (x^2 - 1)}$. (or we can use Stone-Weierstrass). □

Prop. (1.2.2) (Stone-Weierstrass Approximation). If a unital C^*-algebra of continuous functions on a compact Hausdorff space separates points, then it is dense in $C(X)$.

Proof: This is a consequence of Bishop theorem(3.4.14) □

Prop. (1.2.3). for $1 \le p < +\infty$,continuous functions are dense in $L^p(\mathbb{T})$, but not for $p = \infty$.

Proof: 用有限阶梯函数逼近，再用内闭逼近，再用 Tietze 扩张。 □

Prop. (1.2.4) (Approximate Identity). A family of $L^\infty(\mathbb{T})$ functions $\{\Phi_N\}$ are called an approximate identity if:

1. $\int_0^1 \Phi_N(x)dx = 1$.
2. $\sup\int_0^1 |\Phi_N(x)|dx < \infty$.
3. For any $\delta > 0$, $\int_{|x|>\delta} |\Phi_N(x)|dx \to 0$ as $N \to +\infty$.

For any approximate identity, if $f \in C(\mathbb{T})$ or $L^p(\mathbb{T})$ for $1 \le p < +\infty$, then $\Phi_N * f \to f$.

Proof: Use uniform continuity and also use continuous approximation (1.2.3). □

Cor. (1.2.5). for $1 \le p < +\infty$,trigonometric polynomials are dense in $L^p(\mathbb{T})$ and $C(\mathbb{T})$, but not for $p = \infty$. So $e^{2\pi inx}$ forms an orthogonal basis in $L^2(\mathbb{T})$.

Thus, the Parseval's identity holds.

Proof: Just use the fact that fejer kernels are an approximate identity. □

3 Convolution

Prop. (1.3.1). Convolution with a smooth function makes the function smooth, in particular,$\frac{\partial}{\partial x}(f * g) = \frac{\partial f}{\partial x} * g$.

Prop. (1.3.2) (Young's Inequality). $||f * g||_r \le ||f||_p||g||_q$ for all $1 \le r, p, q \le \infty$ and

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

In particular, $||K * f||_p \le ||K||_1||f||_p$.

Proof: BY Riesz representation, it suffices to show that: for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$,

$$\int\int f(x)g(y-x)h(y) \le ||f||_p||g||_q||h||_r.$$

write the LHS as

$$\int\int (f^p(x)g(y-x)^q)^{1-\frac{1}{r}}(f^p(x)h^r(y))^{1-\frac{1}{q}}(g^q(y-x)h^r(y))^{1-\frac{1}{p}}$$

and use Holder inequality. □

4 Measures

Prop. (1.4.1) (Radon-Nikodym). If two σ-finite measures v, μ on a measurable space satisfies v is absolutely continuous w.r.t μ, then there is a μ-integrable function f such that

$$dv = fd\mu.$$

Cor. (1.4.2). Special case of the Freudenthal spectral theorem **??**.

IV.2 Complex Analysis

1 Topology

Prop. (2.1.1). A first differentiable conformal map in $\mathbb{C}$ is holomorphic or anti-holomorphic. Cf.[Ahlfors P74]. In higher dimension, conformal is equivalent to $\langle df_p(v_1), df_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle$.

Prop. (2.1.2). The roots of a polynomial depends continuously on the coefficients. (Use Rouch Principle).

2 Theorems

Prop. (2.2.1) (Uniformization Theorem). Any connected Riemann Surface is the quotient by a discrete subgroup of $\mathbb{C}, \mathbb{H}$ or $\mathbb{P}^1$.

Proof: □

Prop. (2.2.2) (Runge's Theorem). Let K be a compact subset of $\overline{\mathbb{C}}$ and let f be a function which is holomorphic on an open set containing K. If A is a set containing at least one complex number from every bounded connected component of $\overline{\mathbb{C}}\backslash K$, then there exists a sequence of rational functions which converges uniformly to f on K and all the poles of the functions are in A.

Proof: □

Prop. (2.2.3) (Mergelyan's theorem). If K is compact in $\mathbb{C}$ and f is a continuous function on K that is holomorphic in $\mathrm{int}(K)$, then f can be uniformly approximated by polynomials.

Prop. (2.2.4) (Montel's Theorem). Sets of holomorphic functions bounded in the topology of $H(\Omega)$, inter convex uniform convergence, is sequentially compact.

Proof: □

IV.3 Functional Analysis

1 Various Spaces and Duality

For a bounded connected open set Ω,

- The space $C^k(\Omega)$ is the Banach space of C^k functions u on $\overline{\Omega}$ with the norm

$$||u|| = \max_{|\alpha|\le k} \max_{x\in\overline{\Omega}} |\partial^\alpha u(x)|.$$

  It is complete because the limits of u^α is compatible.

- **Sobolev Space** $H^{m,p}(\Omega)$ is the completion of a subspace $C^k(\Omega)$ with the norm

$$||u||_{m,p} = (\sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha u(x)|^p \,\mathrm{dx}).$$

  And we denote $H^{m,p}(\Omega)$ by $H^m(\Omega)$.

- $C_0^k(\Omega)$ is the subspace of $C^k(\Omega)$ that have support in Ω. Its completion $H_0^m(\Omega)$ is a closed subspace of $H^m(\Omega)$.
- $C(\Omega)$ in the topology of compact convergence is a Fréchet space. It is not locally convex.
- $H(\Omega)$ the space of holomorphic functions in Ω is a closed subspace of $C(\Omega)$ thus is a Fréchet space. Montel's theorem says exactly that $H(\Omega)$ is Heine-Borel.
- $C^\infty(\Omega)$ in the topology defined by seminorms $p_N(f) = \max\{|D^\alpha f(x)| : x\in K_N, |\alpha|\le N\}$, is a metrizable Fréchet space thus locally convex and it has the Heine-Borel property by Arzela-Ascoli.
- $D(K)$ is the closed subspace of functions on Ω with support in K, thus a Fréchet space.

Prop. (3.1.1) (Dual Spaces).

- For a σ-finite measure μ on a measurable space X, for $1\le p<\infty$

$$L^p(X,\Omega,\mu)^* = L^q(X,\Omega,\mu).$$

- $C[0,1]^* = BV[0,1]$ and $C[X]^* = M(X)$, the space of complex measure on compact X with the norm of total variance, by Riesz representation theorem(1.1.4).

2 Topological Vector Space

Def. (3.2.1). there are different topology in the space of operators on a Hilbert space.
Norm operator topology: $||H_i - H|| \to 0$.
Strong operator topology: $\forall u, ||(H_i - H)u|| \to 0$.
Weak operator topology: $\forall u, v, \langle H_i(u), v\rangle \to \langle H(u), v\rangle$

Def. (3.2.2). A space is called a F**-space** if its toopology is induced by a complete invariant metric.

A locally convex F-space is called a **Fréchet space**.

Def. (3.2.3). A **sublinear functional** is a function p that $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for $\lambda > 0$.

A **seminorm** is a non-negative function p that $p(x+y) \leq p(x)+p(y)$ and $p(\alpha x) = |\alpha| p(x)$ for all complex α.

Prop. (3.2.4) (Minkowski Functional). The set of seminorms/sublinear functionals correspond 1-to-1 with convex balanced absorbing open sets containing 0 through Minkowski functional. and it is uniformly continuous iff 0 is an interior point.

Prop. (3.2.5). A separating family of seminorms is equivalent to a convex balanced local basis at 0. And it generate a metric making the space a Fréchet space. Cf.[Rudin P27].

Prop. (3.2.6) (Hausdorff). In a complete space, a subset M is sequentially compact iff it is totally bounded. (Use the diagonal method).

In a metric space, a subset M is sequentially compact iff its closure is compact. Hence in Fréchet space, a closed subset is compact iff it is totally bounded.

Cor. (3.2.7) (Arzela-Ascoli). $F \subset C(M)$ M compact is a sequentially compact subset iff it is uniformly bounded and equicontinuous.

Prop. (3.2.8). In a locally bounded space, if E is totally bounded, then $\mathrm{co}(E)$ is totally bounded. Thus in a Fréchet space, the closed convex closure of a compact set is compact.

Prop. (3.2.9). A topological vector space is locally compact iff it is f.d., and there is only one topology on a finite dimensional space and it is complete. Cf.[Rudin P17].

Cor. (3.2.10). A f.d subspace in a F-space is complete thus closed.

Prop. (3.2.11) (Schauder Fixed Point Theorem). If C is a closed convex subset in a normed space and $T : C \to C$ has sequentially complete image, e.g. C is compact, then T has a fixed point.

Proof: Use a $1/n$-net and construct a contraction to their convex hull. Then use Brauwer fixed point theorem to find $Tx_n = x_n$, and choose a convergent point x to show $Tx = x$. □

Reflexive and Separable (张恭庆)

Prop. (3.2.12) (Banach). For a norms space X, if X^* is separable, then X is separable.

Proof: Choose a countable dense set in X^*, then there projection to the unit sphere $\{g_n\}$ is dense, and choose for each of them a x_n that $||x_n|| = 1$ and $g_n(x_n) > \frac{1}{2}$. Use Hahn to show span of $\{x_n\}$ is dense in X. □

Prop. (3.2.13) (Pettis). Closed subspace of a reflexive normed space is reflexive.

Prop. (3.2.14). If a subspace of normed space X is separable or reflexive, then its unit sphere in X^* is weak*-sequentially compact.

Closed Range Theorem

Prop. (3.2.15). Let T be continuous mapping between Banach spaces X and Y, then $T(X) = Y \iff ||T^*y^*|| \geq \delta||y^*|| \iff T^*$ is one-to-one and $R(T^*)$ is closed(By Banach theorem).

$R(T)$ is closed in Y iff $R(T^*)$ is closed in X^*.

Proof: Cf.[Rudin P100]. □

3 Completeness

Prop. (3.3.1) (Banach-Steinhaus). Γ is a collection of continuous linear mapping between two TVS, then if the set B of x that $\Gamma(x)$ is bounded is a second category set in X, then $B = X$ and Γ is equicontinuous (thus maps bounded sets to bounded sets).

Similarly, if for a compact convex set K in X, if a set Γ of continuous linear mapping is bounded for every $x \in K$, then Γ is equicontinuous on K.

Proof: For an open set of 0, choose a balanced nbhd U s.t. $\overline{U}+\overline{U} < W$, set $E = \cap_{\Lambda\in\Gamma}\lambda^{-1}(\overline{U})$, then $B \subset \bigcup_{i=1}^{\infty} nE$, so by Baire, E has a interior point thus has a nbhd V s.t. $\Gamma(V) \subset \overline{U}+\overline{U} \subset W$. □

Cor. (3.3.2) (Uniform Boundedness Theorem). If a set Γ of continuous linear mapping from a F-space X to Y satisfies $\Gamma(x)$ is bounded for every x, then Γ is equicontinuous.

Prop. (3.3.3) (Open Mapping theorem). If a continuous linear mapping T from a F-space X to Y is surjective and $R(T)$ is of second category, then it is a surjective open mapping and Y is a F-space.

Proof: We can show that $T(B(0, \frac{1}{2^n})$ are all of second category, so $\overline{T(B(0, \frac{1}{2^n}))}$ is open, thus for a $y \in T(B(0,1)$ we can consecutively choose $x_n \in B(0, \frac{1}{2^n})$ s.t. $y - \sum_{i\leq n} T(x_i) \in \overline{T(B(0, \frac{1}{2^n}))}$. So by completeness of X, $y \in T(B(0.1)$, thus it is open. □

Cor. (3.3.4) (Banach). If a continuous T between F-spaces is a bijection, then it has a continuous inverse.

Cor. (3.3.5). If a F-space is complete w.r.t two different topologies and one is stronger than the other, then they are equivalent.

Cor. (3.3.6) (Closed Graph Theorem). If T is a closed linear mapping between two F-spaces, i.e. the graph of it is closed, then it is continuous. (Because the metric induced by the graph is stronger than the original one). This is very useful when proving some map is continuous.

Prop. (3.3.7). Any symmetric operator on a Hilbert space is continuous. (Because $x_n \to 0$ implies $Tx_n \to 0$ weakly, so we can use closed graph theorem.

Cor. (3.3.8). If the image of a continuous linear mapping T between F-spaces has finite codimensional image, then the image is closed. (Construct $\mathbb{C}^n \oplus X/N(T) \to Y$, by Banach it is a homeomorphism.

4 Convexity

Hahn-Banach

Prop. (3.4.1) (Real Hahn). For a sublinear functional p on a real linear space X and a subspace X_0, if a functional f satisfies $f(x) < p(x)$ on X_0, then it can be extended to a functional on X with the same condition. (Use Zorn's lemma)

Prop. (3.4.2) (Complex Hahn). For a seminorm p (i.e. it can attain 0) on a complex linear space X and a subspace X_0, if a functional f satisfies $f(x) < p(x)$ on X_0, then it can be extended to a functional on X with the same condition.

Proof: Let $g(x) = \operatorname{Re} f(x)$ and extend it and set $f(x) = g(x) - ig(ix)$. □

Prop. (3.4.3) (Hahn). In a normed space X, a bounded linear functional on a subspace X_0 can extend to a bounded functional on X with the same norm.

Cor. (3.4.4). For every $x \neq 0$, there is a continuous functional f of norm 1 that $f(x) = ||x||$. So continuous functionals can separate points. Thus the inclusion from X to X^{**} is an isometry into and the conjugation $T*$ form $L(X, Y)$ to $L(Y^*, X^*)$ is an isometry into a closed subspace.

Prop. (3.4.5) (Geometric Hahn).

- If E_1 and E_2 are two convex set that $E_1 \cap E_2 = \emptyset$ and E_1 is open, then there is a continuous linear functional that separate them, i.e $\operatorname{Re} f(E_1) < \operatorname{Re} f(E_2)$. (use the continuous sublinear functional for $E_1 - E_2$).
- In a locally convex TVS, if E_1 is convex compact and E_2 is convex closed, then there is a real functional that separate them. Thus for a set E and a point x, $x \in \overline{\operatorname{span} E} \iff$ for all f that $f(E) = 0$, $f(x) = 0$.

Cor. (3.4.6). If a sequence $\{x_n\}$ weak converge to x in a normed space, then convex combination of $\{x_n\}$ strongly converge to x, i.e. $x \in \overline{\operatorname{co}}(\{x_n\})$. The weak closure of a convex set in a locally convex space equals the original closure.

Prop. (3.4.7). For a bounded operator T,

$$\overline{R(T)} = N(T^*)^\perp, \text{ Thus } \overline{R(T^*)} = N(T)^\perp$$

Prop. (3.4.8). For a finite dimensional space in a Banach space, the projection exists. (construct the dual functional for a basis and extends it to a functional using Hahn.

Prop. (3.4.9) (Banach-Alaoglu theorem). For a nbhd V of 0, the set

$$K = \{f |\ |fx| \leq 1,\ \forall x \in V\}$$

is weak*-compact.

Proof: The point is that the weak*-topology coincides with the pointwise convergence topology. And that topology is embedded in a compact space(Tychonoff) and K is a closed subspace of that space. □

Prop. (3.4.10). In a locally convex space, bounded $\iff$ weakly bounded. Cf.[Rudin Prop3.18].

Prop. (3.4.11). For a commuting family of continuous affine maps from K to K where K is a compact convex set in a TVS, then there is a fixed point.

Proof: Consider the semigroup generated by these maps together with their average, they have a common image, and for this image, consider $p = \frac{1}{n}(I + T + t^2 + \ldots + T^{n-1})(x)$, then $p - Tp = \frac{1}{n}(x - T^n x) \in \frac{1}{n}(K - K)$, thus $p = Tp$ for all T. □

Cor. (3.4.12) (Invariant Hahn). For a commuting family Γ of operators on a normed space and Y a invariant space, then for any Γ-invariant continuous functional y^* has a Γ-invariant Hahn extension. (Consider the action of T^* on all the Hahn extension of f, use Alaoglu).

Krein-Milman theorem

Prop. (3.4.13). For a compact convex set in a TVS that is weak-Hausdorff, then $K = \overline{\text{co}}(\text{Extreme}(K))$.

If K is a compact set in a locally convex space, then $K \subset \overline{\text{co}}(K) = \overline{\text{co}}(E(K))$.

Proof: Show that every extreme set contains a extreme point, and use the geometric Hahn, because the extreme value point for any functional on K is a extreme set. □

Prop. (3.4.14). If K is a compact set in a locally convex space X and if $\overline{\text{co}}(K)$ is also compact, e.g in a Fréchet space, then every extreme point of $\overline{\text{co}}(K)$ lies in K.

Cor. (3.4.15). There is a Bishop theorem that derive Stone-Weierstrass theorem, proved using Krein-Milman.

5 Sobolev Space

6 Banach Algebra

Def. (3.6.1). For a bounded operator from on X complete, then a λ is called a:

- **point spectrum** $\rho(A)$ if $(\lambda I - A)^{-1}$ doesn't exists;
- **continuous spectrum** if $R(\lambda I - A) \neq X$ but $\overline{R(\lambda I - A)} = X$.
- **residue point** if $\overline{R(\lambda I - A)} \neq X$.
- **regular point** if $(\lambda I - A)^{-1}$ exists and is continuous, i.e. $R(\lambda I - A) = X$;

denote $\sigma(A) = \mathbb{C}\backslash\rho(A)$ the spectrum of A.

Prop. (3.6.2). $\mathbb{C}\backslash\sigma(T)$ is an open set and $\lambda \to (\lambda I - T)^{-1}$ is a holomorphic function on $\mathbb{C}\backslash\sigma(T)$.

Thus for every bounded operator T, $\sigma(T)$ is not empty, otherwise this holomorphic function is bounded.

Cor. (3.6.3) (Gelfand-Mazur). If in a Banach space where all the nonzero element is invertible, then it is isomorphic to $\mathbb{C}$.

Prop. (3.6.4). Notice $(I-T)$ is invertible for $||T||<1$ and the derivative can be calculated by definition.

Cor. (3.6.5). The spectrum of an element of a Banach algebra is continuous.

Prop. (3.6.6). In a Banach algebra A, $e-xy$ is invertible iff $e-yx$ is invertible, thus $\sigma(xy)\cup\{0\}=\sigma(yx)\cup\{0\}$.

Proof: Use power expansion to get an expression and prove it is the inverse. □

Prop. (3.6.7) (Gelfand). The spectrum radius

$$r_\sigma(A)=\lim_{n\to\infty}||A^n||^{\frac{1}{n}}=\inf||A^n||^{\frac{1}{n}}.$$

So $\sigma(A)$ is compact.

Proof: Use Hadamard radius formula and for the other side, use the fact that $|f(\frac{A^n}{(r_A+\varepsilon)^{n+1}})|$ is bounded, so by uniform boundedness theorem, $\frac{||A||^n}{(r_A+\varepsilon)^{n+1}}<M$ for all n. And $\lambda\in\sigma(A)$ implies $\lambda\in\sigma(A^n)$ thus the limit is well-defined. □

Cor. (3.6.8). For Banach algebra B and its closed subalgebra A, $\sigma_A(x)$ is obtained from $\sigma_B(x)$ by filling some holes. So when $\sigma_B(x)$ doesn't separate $\overline{\mathbb{C}}$ or $\sigma(A)$ has empty interior, then $\sigma_A(x)=\sigma_B(x)$. Cf.[Rudin P256].

Symbolic Calculus

Prop. (3.6.9). For a Banach algebra A. For a domain Ω in $\mathbb{C}$, define A_Ω as the set of x that $\sigma(x)\in\Omega$, it is an open set by(3.6.5), then:

$$f\mapsto\tilde{f}(x)=\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda e-x)^{-1}d\lambda$$

for any contour Γ that surrounds $\sigma(x)$, is a continuous algebra isomorphism of $H(\Omega)$ into the set of A-valued functions on A_Ω with the compact-open topology.

We have $\widetilde{g\circ f}=\tilde{g}\circ\tilde{f}$.

Proof: The nontrivial part is that this map is multiplicative, but for this we can use Runge's theorem to approximate any function on $\sigma(x)$. □

This theorem makes it possible to implant complex analysis to the study of Banach Algebra.

Cor. (3.6.10). $\exp(x)$ is defined on A and is continuous. If $\sigma(x)$ doesn't separate 0 from ∞, then $\log(x)$ is defined but might not be continuous.

Prop. (3.6.11) (Spectral Mapping Theorem). $\tilde{f}(x)$ is invertible in A iff $f(\lambda)\neq 0$ on $\sigma(x)$. Thus we have $\sigma(\tilde{f}(x))=f(\sigma(x))$.

Prop. (3.6.12). If f doesn't vanish identically on any component of Ω, then $f(\sigma_p(T))=\sigma_p(\tilde{f}(T))$. Cf.[Rudin P266].

Commutative Banach Algebra

Prop. (3.6.13). For A a commutative Banach algebra, the set of maximal ideals has codimension 1 corresponds to kernels of complex homomorphisms to $\mathbb{C}$. (Consider the quotient space and use Gelfand-Mazur). Note that a complex homomorphism is all continuous because $\lambda e - x$ maps to nonzero.

$\lambda \in \sigma(x)$ iff there is a complex homomorphism h s.t. $h(x) = \lambda$. (Because x is invertible iff it is not contained in any proper ideal of A.

Prop. (3.6.14) (Gelfand Transform). The set Δ of maximal ideals of a commutative Banach algebra is a compact Hausdorff space w.r.t to the weak*-topology and the Gelfand transform: $x \mapsto \hat{x}(h) = h(x)$ is a map of A into $C(\Delta)$. And the range of $\hat{x}$ equals $\sigma(x)$, so $||\hat{x}|| = \rho(x) \le ||x||$.(Use Alaoglu).

Prop. (3.6.15). For $A = C(X)$ where X is compact Hausdorff, Δ is homeomorphic to X. (otherwise it has finite $f_i \neq 0$, then $\sum |f_i|^2$ is positive thus invertible but maps to 0). In fact, for a space X, $\Delta(C(X))$ is the stone-$\check{C}$ech compactification of X.

Prop. (3.6.16). For $A = L^\infty(m)$, the spectrum of f is just the essential range of f.

Lemma (3.6.17). If $\hat{A} \subset C(\Delta)$ with a chosen topology that makes it compact, and A separate points, then the topology of it is the same of the weak*-topology. (Compact to Hausdorff).

Prop. (3.6.18). The algebra $L^1(\mathbb{R}^n) + \delta$ with the multiplication by convolution has the spectrum $R^n \cap \{\infty\}$. (Use $L^{p*} = L^q$ and see when will it be homomorphism).

B^*-Algebra and Hilbert space

Prop. (3.6.19). A closed convex subset in a Hilbert space has a unique element that attains the minimum norm.

Proof: Choose a sequence approaching the infimum and use parallelogram identity to show it is a Cauchy sequence. □

Cor. (3.6.20). The orthogonal complement of a closed subspace of a Hilbert space exists. and the projection on to a closed subspace exists. This is a good trait of Hilbert space.

Prop. (3.6.21). Linear functionals on a Hilbert space is all of the form $x \mapsto (x, z)$ (Choose a orthogonal of the kernel).

Prop. (3.6.22). For a Hilbert space, the adjoint operation serves as an involution and makes $B(H)$ into a B^*-algebra, i.e. $||T^*T|| = ||T||^2$. (In fact, $||T|| = ||T^*|| = \sup\{(Tx, y)|||x||, ||y|| \le 1\}$).

Prop. (3.6.23). Any B^*-algebra is isomorphic to a closed subspace of $B(H)$ for some Hilbert space. Cf.[Rudin P338].

Prop. (3.6.24) (Gelfand-Naimark). A commutative B^*-algebra A is a Banach algebra with involution s.t. $||xx^*|| = ||x||^2$. Then the Gelfand transform is an isomorphism from A to $C(\Delta)$ with $||x|| = ||\hat{x}||_\infty$ and $\widehat{x^*} = \overline{\hat{x}}$.

Proof: First use $||xx^*|| = ||x||^2$ to prove that a hermitian element is mapped to real function, and use Stone-Weierstrass to show that the image is dense, then let $y = xx^*$ and $||y^{2^m}|| = ||y||^{2^m}$ to prove $||\hat{x}|| = ||x||$, so its image is closed. □

Now we want to use commutative algebra methods in the non-commutative case, there are two ways.

Prop. (3.6.25). For a commutative set of elements S in A, Γ the set of elements that commute with S, then $B = \Gamma(\Gamma(S))$ is commutative and contains S. And $\sigma_B(x) = \sigma_A(x)$ for $x \in B$.

Proof: Because $S \subset \Gamma(S)$, $\Gamma(\Gamma(S)) \subset \Gamma(S)$, thus $\Gamma(\Gamma(S))$ is commutative. And if $xy = yx$, then $x^{-1}y = yx^{-1}$. □

Cor. (3.6.26). In a Banach algebra, if x, y commutes, then

$$\sigma(x + y) \subset \sigma(x) + \sigma(y), \quad \sigma(xy) \subset \sigma(x)\sigma(y).$$

(because $\sigma(x)$ is just the range of $\hat{x}$ on Δ that x, y generated).

The second method applies to normal elements:

Prop. (3.6.27). In a Banach algebra with an involution, a set S is called **normal** if it is commutative and $S^* = S$. A maximal normal set B is a closed subalgebra and $\sigma_B(x) = \sigma_A(x)$.

Proof: Cf.[Rudin P294]. □

Cor. (3.6.28). In a B^*-algebra A,

- Hermitian elements have real spectra.
- If x is normal, then $\rho(x) = ||x||$.
- If $u, v \geq 0$, then $u + v \geq 0$, i.e. $\sigma(u + v) \subset [0, \infty)$.
- $yy^* \geq 0$. Thus $e + yy^*$ is invertible.

Proof: Cf.[Rudin P295]. □

Prop. (3.6.29). In a Banach algebra with an involution, a **positive functional** is such that $F(xx^*) \geq 0$. It has the following properties.

- $F(x^*) = \overline{F(x)}$ and $|F(xy^*)|^2 \leq F(xx^*)F(yy^*)$. (Use Swartz like trick).
- $|F(x)|^2 \leq F(e)F(xx^*) \leq F(e)^2\rho(xx^*)$, because $e = ee^*$. Thus $|F(x)| \leq F(e)\rho(x)$ for every normal x (By the last prop), so $||F|| = F(e)$ if A is commutative.

Cf.[Rudin P297].

Prop. (3.6.30). If A is a commutative Banach algebra with an involution that $h(x^*) = \overline{h(x)}$, then The map

$$\mu \to F(x) = \int_\Delta \hat{x} d\mu$$

is a one-to-one correspondence between the convex set of μ that $\mu(\Delta) \leq 1$ to the convex set K of positive functionals on A of norm ≤ 1, i.e. $F(e) \leq 1$, so maps the extreme points, i.e. the point mass to extremes points, thus the extreme points of K is exactly Δ. This can be used to prove **Bochner's theorem**.

Proof: Use the last prop to show that there is a functional on $C(\Delta)$ and use Riesz representation. It is positive and by Stone-Weierstrass, it is unique. □

7 Spectral Theory

Resolution of Identity

Def. (3.7.1). A **resolution of identity** on a Hilbert space H for a σ-algebra on a set Ω is a E that:

1. $E() = 0, E(\Omega) = 1$.
2. $E(\omega)$ is self-adjoint projection.
3. $E(\omega' \cap \omega) = E(\omega')E(\omega)$.
4. $E(\omega \cup \omega') = E(\omega) + E(\omega')$ for disjoint ω, ω'.
5. $E_{x,y}(\omega) = (E(\omega)x, y)$ is a complex measure on E.

Thus for any x, $\omega \to E(\omega)x$ is a countably additive H-valued measure.

This will generate an isometric*-isomorphism Ψ of the Banach algebra $L^\infty(E)$ onto a closed normal subalgebra A of $B(H)$. (Define on simple function first).

$$\Psi(f) = \int_\Omega f dE, \quad (\Psi(f)x, y) = \int_\Omega f dE_{x,y}$$

Prop. (3.7.2) (Spectral Decomposition). For any closed B^*-algebra A of $B(H)$, there is a unique resolution E of identity on the Borel subsets of Δ that the inverse of Gelfand transform extends to an isometric *-isomorphism Φ of the algebra $L^\infty(E)$ to a closed subalgebra B containing A. Cf.,[Rudin P322]. In fact, $B = \Gamma(\Gamma(A))$ is normal by Fuglede theorem(3.7.8).

Cor. (3.7.3) (Generalized Symbolic Calculus). If T is a commutative B^*-algebra that x, x^* topologically generate A, then the map

For a normal operator T and the closed normal B^*-algebra it generates, we have $\hat{x}$ maps $\Delta \cong \sigma(x)$ and the inverse of Gelfand transform (by Naimark) gets us a map $\Psi : C(\sigma(x)) \to A$ that $\Psi(z) = x$, $\Psi(\overline{z}) = x^*$. And this can be extended to a resolution of identity on the Borel set of $\sigma(T)$ that maps $L^\infty(m)$ to $B(H)$. $||\Psi(f)|| = ||f||_\infty$.

Cor. (3.7.4). If T is normal, then

1. $||T|| = \sup\{|(Tx, x)|\ ||x|| = 1\}$.
2. T is self-adjoint iff $\sigma(T)$ is real.
3. T is unitary iff $|\sigma(T)| = 1$.

Proof: For 1, use the fact that $||T|| = \rho(T) = ||z_0||$ for some $z_0 \in \rho(T)$, then use Urysohn to show $E(U) \neq 0$ for a open U near x, then there are x_0 that $E(U)x_0 = x_0$ and use $f = z - z_0$ to show that this x_0 get near $||T||$. □

Normal Operator on Hilbert Space

Prop. (3.7.5) (Normal Operators). An operator is normal iff $||Tx|| = ||T^*x||$. So we have $N(T) = N(T^*)$ thus $\sigma_p(T^*) = \overline{\sigma_p(T)}$. And different eigenspaces are orthogonal.

An operator is unitary iff $R(U) = H$ and $||Ux|| = ||x||$ for every x. (Because an operator is defined by its value (Tx, y).

Prop. (3.7.6). For a normal operator T on a Hilbert space, $N(T) = R(T)^\perp$, so T is invertible iff there is a δ that $||Tx|| = ||T^*x|| \geq \delta||x||$.

Prop. (3.7.7) (Polar Decomposition). A positive operator is self-adjoint and has positive spectrum, they have a positive square root. (use the last prop).

So polar decomposition exists in $B(H)$ and normal operator has commuting decomposition. Thus two similar normal operator are unitarily equivalent, (use Fuglede).

Prop. (3.7.8) (Fuglede). If N_1 and N_2 are normal operators and A is a bounded linear operator on a Hilbert space such that $N_1A = AN_2$, then $N_1^*A = AN_2^*$.

Proof: For any $S \in B(H)$, $\exp(S - S^*)$ is unitary thus $||\exp(S - S^*)|| = 1$, $\exp(N_1)A = A\exp(N_2)$. So we have

$$||\exp(\lambda N_1^*)T\exp(-\lambda N_2^*)|| \leq ||T||$$

because λN_i is normal. Thus by Liouville, compare the coefficients of λ, we get the result. □

Prop. (3.7.9). An operator $A \in B(H)$ has the same spectrum w.r.t all the closed *-algebras of $B(H)$.

Proof: Because AA^* is self-adjoint thus has real thus doesn't separate $\mathbb{C}$ thus it is invertible in any closed *-algebra of $B(H)$. so does $T^{-1} = T^*(TT^*)^{-1}$. □

Prop. (3.7.10). For T normal and E its spectral decomposition, then if $f \in C(\sigma(T))$ and $\omega_0 = f^{-1}(0)$, then $N(f(T)) = R(E(\omega_0))$.

Proof: $\chi_E f = 0$, and let $\omega_n = f^{-1}([1/(n-1), 1/n))$, then $E(\omega_n)x = 0(f(T)x = 0)$, so countable additivity shows that $E(\sigma\backslash\omega_0)x = 0$, so $E(\omega_0)x = x$. □

Cor. (3.7.11).

1. $N(T - \lambda I) = R(\{\lambda\})$.
2. every isolated point of $\sigma(T)$ is point spectra, because this point is open thus is $E(\{x\}) \neq 0$.
3. if $\sigma(T)$ is countable, then every $x \in H$ has a unique orthogonal decomposition $x = \sum E(\lambda_i)x$ and $T(E(\lambda_i)x) = \lambda_i E(\lambda_i)x$.

Normal Compact Operator

It is assumed to be an operator on a Hilbert space.

Prop. (3.7.12). A normal compact operator $T \in B(H)$ is compact iff $\sigma(T)$ has no limit point and $\dim N(T - \lambda I) < \infty$ for $\lambda \neq 0$. In particular, a compact operator is a limit of f.d. operators.

Cor. (3.7.13) (Spectral Theorem). A compact normal operator (in particular a normal operator on a f.d linear space) is unitarily diagonalizable. (Use resolution of identity(3.7.11)).

Cor. (3.7.14) (Hilbert-Schimidt). For a symmetric compact operator A on a Hilbert space H, there is a set of orthonormal basis that A is diagonal on it. And of course, its eigenvalue is real and converges to 0.

Prop. (3.7.15) (Stum-Liouville). The integral operator is a compact operator. This can be used via Green's function to solve for example eigenvalue problem for Liouville's equation:

$$(pu')' + qu = \lambda\sigma u.$$

Cf.(6.2.8).

Cor. (3.7.16). The Hermite functions $C_n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2}$, as the eigenvector of $\hat{H} = x^2 - \frac{d^2}{dx^2}$, forms a complete basis for $L^2(\mathbb{R})$. Because it is e^{-x^2} times the solution of the operator $(e^{-x^2}F')' - e^{-x^2}F$.

Prop. (3.7.17) (Freudenthal Spectral Theorem).

Compact Operator & Fredholm Operator(张恭庆)

Def. (3.7.18). An operator between Banach spaces is called **compact** if it maps bounded set to sequentially compact(Closure compact) set.

Prop. (3.7.19). The space of compact operator is a closed subspace of $L(X, Y)$. (Use Hausdorff theorem(3.2.6) to show a limit is totally bounded). Thus the limit of f.d. operators is compact.

If one of A or B is compact and the other is continuous, then AB is compact, because continuous maps bounded to bounded and compact to compact.

Prop. (3.7.20). T is compact $\iff$ T^* is compact.

Proof: We need only to show that $T^* y_n^*$ has a uniformly convergent subsequence on the unit sphere, and we use Arzela-Ascoli because $\overline{T(B(0,1))}$ is compact. For the other half, use reflexive. □

Prop. (3.7.21) (Riesz-Fredholm). For a compact operator A, let $T = I - A$. Then:

1. $R(T)$ is closed. So $R(T) = N(T^*)^\perp$.
2. $\sigma(T) = \sigma(T^*)$.
3. $\mathrm{codim} R(T) = \dim N(T) = \dim N(T^*) < \infty$. Equivalently, $\sigma(A)\backslash\{0\} = \sigma_p(A)\backslash\{0\}$.
4. $\sigma(A)$ has at most one convergent point 0 (it must attain 0 if X is a infinite-dimensional). Hence it at most countable spectrum.

Proof: For 3,4, it suffices to find a convergent series that cannot converge. □

Prop. (3.7.22) (Jordan Decomposition for Compact Operators). For a compact operator A and all the non-zero eigenvalues λ_i, we can find space

$$\bigoplus_{i=1}^{\infty} N((\lambda_i - A)^{p_i})$$

on which A has a Jordan decomposition.

Def. (3.7.23) (Fredholm Operator). A bounded operator between Banach space is called a Fredholm operator if $\dim N(T) < \infty$ and $\mathrm{codim} R(T) < \infty$. It necessarily has closed image. The index is defined as $\mathrm{ind}(T) = \dim N(T) - \mathrm{codim} R(T)$, thus for a compact operator A, $I - A$ has index 0.

Prop. (3.7.24) (Characterization of Fredholm Operator). If T is Fredholm from X to Y iff there exist a bounded S from Y to X that $S_1T = I - A_1, TS_2 = I - A_2$, where A_1, A_2 is compact. S_1 and S_2 can be chosen the same, so S is Fredholm as well.

Cor. (3.7.25). Fredholm operators constitute an open set in $L(X, Y)$, and it is closed under composition. and index is an open map on it. $\operatorname{ind}(T_1T_2) = \operatorname{ind}(T_1) + \operatorname{Ind}(T_2)$ (Direct calculation).

Proof: Use the fact that composition with a compact operator is compact and Notice $I - S$ is invertible for $||S|| < 1$. □

Cor. (3.7.26). If T is Fredholm and A is compact, then $T + A$ is compact, and $\operatorname{ind}(T + A) = \operatorname{ind}(T)$.

So the Fredholm operator is the set of operators 'invertible module compact ones'.

IV.4 Abstract Harmonic Analysis(Folland)

1 Locally Compact Groups

Prop. (4.1.1). Topological group is completely regular.

Proof: Use a sequence of neighbourhood of identity to construct a uniform metric on G. Then set $\phi(x) = min\{1, 2\sigma(a, x)\}$. Cf.[Abstract Harmonic Analysis Ross §8.4] □

Prop. (4.1.2). Locally compact group (Hausdorff) is normal. In particular, Dirac Sequence exists.

Proof: Notice that by choosing a precompact symmetric open neighbourhood U of identity, there exists a σ-compact clopen subgroup H. So H can σ-locally refine every open cover, thus G can, too. So by (1.3.1) G is paracompact. As a topological group, G is regular, thus G is normal by (1.3.4). □

2 Analysis on Locally compact groups

Prop. (4.2.1). The dual group G^* can be regarded as the spectrum of $L^1(G)$:

$$\xi \mapsto \Big(f \mapsto \int \overline{(x,\xi)} f(x)dx\Big),$$

and in this way, the Fourier transform is just the Gelfand transform from $L^1(G)$ to $C(\hat{G})$. Its range is a dense space of $C_0(\hat{G})$.

Prop. (4.2.2). There is another map from $M(\hat{G})$ to bounded continuous functions on G:

$$\mu \mapsto \Big(\phi_\mu : x \mapsto \int (x,\xi)d\mu(\xi)\Big).$$

This is a norm decreasing injection.

Prop. (4.2.3). $\widehat{(f * g)} = \widehat{f} \cdot \widehat{g}$, so if $f, g \in L^2(G)$, $\widehat{(fg)} = \widehat{f} * \widehat{g}$. Cf.[Folland Abstract Harmonic Analysis].

Def. (4.2.4). A function of **positive type** on a closed compact group G is a function $\phi \in L^\infty(G)$ that defines a positive linear functional on the B^*-algebra $L^1(G)$.

We set $P = P(G) =$ the set of continuous functions of positive type on G and $P_0(G) = \{\phi | ||\phi||_\infty \leq 1\}$. By Alaoglu, $P_0(G)$ is a weak*-compact set.

Prop. (4.2.5) (Bochner's Theorem). If $\phi \in P(G)$, there is a unique positive $\mu \in M(\widehat{G})$ s.t. $\phi = \phi_\mu$.

Proof: We have the map defined in(4.2.2) maps into $P_0(G)$ and it is weakly*continuous, so maps the compact convex set of positive measures that $\mu(\hat{G}) \leq 1$ to a compact convex set. And the image contains all the extreme point of P_0, i.e. characters of G and 0. So by Krein-Milman, this map is surjective. Cf. [Folland Abstract Harmonic Analysis Prop4.19]. □

Cor. (4.2.6) (Herglotz). A numerical sequence $\{a_n\}$ is positive iff there is a positive measure $\mu \in M(T)$ s.t. $a_n = \widehat{\mu}(n)$.

Prop. (4.2.7). The set of regular Borel probability measures on a compact X is weak*-compact in $C(X)^*$. (Use Alaoglu).

IV.5 Harmonic Analysis

1 Fourier Analysis on $\mathbb{R}^n$

Prop. (5.1.1). If $f \in L^1(\mathbb{T})$ is absolutely continuous, then $\widehat{(f')}(n) = 2\pi i n \cdot \widehat{f}(n)$.

Prop. (5.1.2). $f \in L^1(\mathbb{T})$ is determined by its Fourier coefficients.

IV.6 Differential Equations

1 ODE-Fundamentals

Prop. (6.1.1).

$$x^{(2)} = f(x)$$

It can be solved.

Proof:

$$x^{'}x^{(2)} = f(x)x^{'}$$

$$\frac{1}{2}(x^{'})^2 = \int^x f(t)dt$$

□

2 ODE-Theorems

Prop. (6.2.1) (Existence and Uniqueness of ODE of Lipschitz Type). If $F(t,x)$ defined on $[-h,.h] \times [\eta-\delta, \eta+\delta]$ is a function that is locally Lipschitz: that is, $\exists \delta, L$, s.t. if $|t| \leq h, |x_i - \eta| \leq \delta$, then

$$|F(t,x_1) - F(t,x_2)| \leq L|x_1 - x_2|.$$

Then the initial value problem:

$$(Tx)(t) = \eta + \int_0^t F(\tau, x(\tau))d\tau$$

has a unique solution on the interval $[-h,h]$ if $h < \min\{\delta/M, 1/L\}$, where M is the maximum of F on $[-h,.h] \times [\eta-\delta, \eta+\delta]$. Because T is a contraction.

Prop. (6.2.2) (Existence of ODE of continuous Type (Caratheodory)). If $F(t,x)$ defined on $[-h,.h] \times [\eta-\delta, \eta+\delta]$ is a continuous function, then

$$(Tx)(t) = \eta + \int_0^t F(\tau, x(\tau))d\tau$$

has a unique solution on the interval $[-h,h]$ if $h < \delta/M$, where M is the maximum of F on $[-h,.h] \times [\eta-\delta, \eta+\delta]$. (Use Schauder fixed point theorem and Arzela-Ascali).

Prop. (6.2.3) (Existence Theorem for Complex Differential Equations). Let $f(z,\mathbf{w})$ be a holomorphic vector function in a domain $D \subset \mathbb{C}^{n+1}$, then the initial value problem

$$\mathbf{w}' = f(z, \mathbf{w}), \quad w(z_0) = w_0$$

has exactly on holomorphic solution locally (Thus on a simply connected domain).

Cor. (6.2.4). So a holomorphic high-order ODE for a complex variable can be solves. And luckily it can be solved even $\overline{z}$ appears (just regard it as a constant).Δ

Proof: Cf.[Ordinary Differential Equations, P110]. □

Prop. (6.2.5). For the equation:

$$\frac{d\mathbf{y}}{d\mathbf{x}} = \mathbf{Ay},$$

One solution basis is:

$$\begin{cases} e^{\lambda_1 x}\mathbf{P}_1^{(1)}(x), \cdots, e^{\lambda_1 x}\mathbf{P}_{n_1}^{(1)}(x); \\ \cdots\cdots\cdots \\ e^{\lambda_s x}\mathbf{P}_1^{(d)}(x), \cdots, e^{\lambda_s x}\mathbf{P}_{n_s}^{(1)}(x); \end{cases}$$

Where

$$\mathbf{P}_j^{(i)}(x) = \mathbf{r}_{j0}^{(i)} + \frac{x}{1!}\mathbf{r}_{j1}^{(i)} + \cdots,$$

where $\mathbf{r}_{j0}^{(i)}$ is a basis of solution of $(\mathbf{A} - \lambda_i I)^n \mathbf{x} = 0$, and $\mathbf{r}_{k+1}^{(i)} = (\mathbf{A} - \lambda_i I)\mathbf{r}_k^{(i)}$.

Proof: Cf.[常微分方程丁同仁定理 6.6]. □

Cor. (6.2.6). For the equation:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0,$$

If the characteristic equation has s different roots $\lambda_1, \ldots, \lambda_s$ and corresponding multiplicities $n_1, \ldots, n_s$, then:

$$\begin{cases} e^{\lambda_1 x}, xe^{\lambda_1 x}, \cdots, x^{n_1-1}e^{\lambda_1 x}; \\ \cdots\cdots\cdots \\ e^{\lambda_s x}, xe^{\lambda_s x}, \cdots, x^{n_s-1}e^{\lambda_s x}; \end{cases}$$

is a solution basis.

Proof: Cf.[常微分方程丁同仁 P198]. □

Prop. (6.2.7) (Lyapunov). Consider the Lyapunov stability of an autonomous system of the form:

$$\frac{dx}{dt} = Ax + o(|x|),$$

Then:

1. If A has a eigenvalue whose real part is positive, then the trivial solution is not weak stable.
2. If all eigenvalues of A has negative real part, then the trivial solution is strong stable.

Prop. (6.2.8) (Stum-Liouville). The eigenvalue BVP problem of L-S equation:

$$Lu = (pu')' + qu = \lambda\sigma u, \quad a_1 u(a) + a_2 u'(a) = 0, b_1 u(b) + b_2 u'(b) = 0, \sigma(x) > 0.$$

can be solved by the method of Green's function. For the function:

$$G(x,s) = \begin{cases} Cu_1(x)u_2(s), & x < s \\ Cu_2(x)u_1(s), & x > s \end{cases}$$

for some C, where u_1 is a solution of the L-S equation with boundary value at a, and u_2 with boundary value at b that are linear independent (This happens when the homogenous equation has no solution). It satisfies: $LG(x,s) = \delta(x-s)$ and satisfies the boundary conditions.

Because L is self-adjoint, we have:

$$Gf(x) = \int f(s)G(x,s)ds, LG = \mathrm{id}, GL = \mathrm{id}$$

thus the eigenvalues of L is the reciprocal of the eigenvalues of G, and G is a compact self-adjoint operator on $L^2(\sigma, \mathbb{R})$, so by spectral theorem, the eigenvectors are countable and form an orthonormal basis.

And when the homogenous function do have a solution ϕ, then we have:$Lu = f$ has a solution iff $(f, \phi) = 0$. one way is simple and the other way is because we solve the initial problem of ODE and find that it automatically satisfies the boundary condition. Cf.[Stum Liouville Theory].

3 PDE

Direct Solution

Prop. (6.3.1) (Characteristic Line). Consider a 1-dimensional parabolic equation:

$$p_t + c(p,x,t)p_x = r(p,x,t)$$

Let $P(t) = p(X(t),t)$, this equation is equivalent to

$$P_t = r(X(t), t, P(t)), \quad X_t = c(X(t), t).$$

an ODE equation.

Prop. (6.3.2). A set of equations:

$$\frac{\partial}{\partial x^i}\mu = A_i \mu$$

where μ is a n-vector. It has a solution iff

$$[A_i, A_j] = \frac{\partial}{\partial x^i}A_i - \frac{\partial}{\partial x^i}A_j.$$

Cor. (6.3.3). This seems to be able to derive Frobenius integrability theorem, but I cannot figure it out.

Chapter V

Algebraic Geometry

V.1 Schemes

1 Sheaves

Prop. (1.1.1) (Glueing Sheaves). We have a space X and . Further, we are given a family of sheaves on a cover for this space $\{U_i\}_{i\in I}$. We are given isomorphisms $\phi_{ij}:\mathcal{F}_i|_{U_i\cap U_j}\longrightarrow \mathcal{F}_j|_{U_i\cap U_j}$ along with the so-called cocycle condition $\phi_{ik}=\phi_{jk}\circ\phi_{ij}\quad\text{on}\quad U_i\cap U_j\cap U_k$.

The task is to construct a sheaf on X compatible with these local sheaves. In the same way, we can glue schemes and also morphisms with a fixed target (compatible with the glueing).

Proof: For every open set $V\subset X$, we define the group of sections $\mathcal{F}(V)$ to be a set consisting of all tuples $(s_i)_{i\in I}$ required to obey the compatibility condition:

$$\phi_{ij}(s_i|_{V\cap U_i\cap U_j})=s_j|_{V\cap U_i\cap U_j}\qquad(*)$$

for all $i,j\in I$. The group addition on $\mathcal{F}(V)$ is the obvious one.

The $\mathcal{F}$ that I defined is guaranteed to be a sheaf, but we also need to satisfy ourselves that the restriction $\mathcal{F}|_{U_k}$ really is isomorphic to the $\mathcal{F}_k$ that we started with, for each $k\in I$. It is here that the cocycle condition is required.

It is easy to write down what the isomorphism $\psi:\mathcal{F}_k\overset{\cong}{\to}\mathcal{F}|_{U_k}$ ought to be. Given an open $V\subset U_k$ and given a section $s\in\mathcal{F}_k$, we would like to define its image under ψ to be

$$\psi(s)=(\phi_{ki}(s|_{V\cap U_i}))_{i\in I}$$

However, we need to be sure that the tuple $(\phi_{ki}(s|_{V\cap U_i}))_{i\in I}$ represents a well-defined element of $\mathcal{F}(V)$. In particular, we must verify that $(\phi_{ki}(s|_{V\cap U_i}))_{i\in I}$ obeys the condition $(*)$, which states that

$$\phi_{ij}\circ\phi_{ki}(s|_{V\cap U_i\cap U_j})=\phi_{kj}(s|_{V\cap U_i\cap U_j})$$

for any $i,j\in I$. This is true by virtue of the cocycle condition.

This map is obviously injection and it is surjection by virtue of $(*)$.

This method can also be used to glue schemes. □

Prop. (1.1.2) (Stalks). Taking stalks is a right adjoint thus preserves kernel on PAb, and it is exact on Ab. And the epimorphism and monomorphism can be checked on stalks, so also can be checked on affine opens. Cf.[Hartshorne P63].

Prop. (1.1.3). The category of presheaves and sheaves on a site is a Grothendieck category.

Proof: For the presheaf, the only problem is the existence of generator, for that, just construct a family of presheaves and sum them. Take $Z_U(V) = \oplus_{\mathrm{Mor}(V,U)}\mathbb{Z}$, then $F(U) = \mathrm{Hom}(Z_U, F)$(Similar to Yoneda Lemma). So they are a family of generators.

For the sheaf, the shifification is exact(1.1.6), it follows that coimage is the image. colimits in ShA is the shifification of the colimits in PShA. $\sharp$ is exact so filtered limits is exact, and Z_U^{++} is a family of generators, Z_U^{++} represents the functor $\Gamma(U, -)$. □

Prop. (1.1.4). The category of sheaves of sets on the site of the G-sets S_G is equivalent to S_G.

Prop. (1.1.5). Fiber products exist in the category of schemes.

Proof: Cf.[Hartshorne P88]. You should use (1.1.1). □

<u>**transfer of sheaves under morphisms**</u>

Prop. (1.1.6) (Sheafification). The operator F^+ is the sheaf that

$$F^+(U) = \varinjlim \ker(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)) = \check{H}^0(U, F)$$

takes presheaf to a separated presheaf, i.e. $0 \to F(U) \to \prod_i F(U_i)$ and a separated presheaf to a sheaf. (The problem of separated is that the cover may not be identical in $U_i \times_U U_j$ but only on a cover of it.

The sheafification F^{++} is exact and it is left adjoint to the forgetful functor, so the forgetful functor is left exact and it preserves injectives, Thus the sheaf cokernel is the shifification of the presheaf kernel, the sheaf kernel is the presheaf kernel.

Proof: Sh is left exact because F^+ is left exact Cf.(3.2.1). □

Def. (1.1.7).

- the pushforward f_pF, $f_pF(U) = F(f^{-1}(U))$ sends presheaf to presheaf.
- the direct image $f_*\mathcal{F}$, $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ sends sheaf to sheaf.
- the inverse image $f^{-1}\mathcal{F}$, $f^{-1}\mathcal{F}(U) = \mathcal{F}(f(U))^\sharp$.
- the extending by zero sheaf: for an open subset U, $j_!(F)$ is shifification of presheaf that $G(V) = F(V)$ when $V \subset U$ and 0 otherwise. We have an exact sequence of sheaves:$0 \to j_!(F|_U) \to F \to i_*(F|_Z) \to 0$(check on stalks).

2 Spec and Schemes

Prop. (1.2.1). On Spec(A), $\mathcal{O}(D(f)) = A_f$. Cf.[Hartshorne P71]. We can also define it this way and check the sheaf condition.

Prop. (1.2.2). The closure of a subset T of Spec$(A) = V(\cap p, p \in T)$.

Prop. (1.2.3). The Spec operator from $CRing^*$ to Scheme is right adjoint to $X \to \Gamma(X, \mathcal{O}_X)$,

$$\mathrm{Hom}_{Sch}(X, \mathrm{Spec}(A)) \cong \mathrm{Hom}_{Ring}(A, \Gamma(X, \mathcal{O}_X)).$$

Notice the category of schemes is a full subcategory of the category of locally ringed spaces.

Proof: First prove this for $X = \mathrm{Spec}(B)$. Cf.[Hartshorne P73]. Then choose affine cover of X and glue them ($\mathcal{H}om$ is a sheaf). Should notice this is the special case of Global spec with $S = \mathrm{Spec}(\mathbb{Z})$. $\square$

Prop. (1.2.4) (Global Spec). There is a S-scheme $f : \mathbf{Spec}_S\,\mathcal{A} \to S$ for every Qco sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ on S that $f^{-1}(U) \cong \mathrm{Spec}\,\mathcal{A}(U)$. This construction is right adjoint to the direct image map:

$$\mathrm{Hom}_{\mathcal{O}_S-alg}(\mathcal{A}, \pi_*\mathcal{O}_X) \cong \mathrm{Hom}_{Sch/S}(X, \mathbf{Spec}_S\,\mathcal{A}).$$

and defines an equivalence of affine morphisms over S and Qco $\mathcal{O}_S$-algebras. Moreover, this defines an equivalence of the category of $\mathcal{A}$-modules and the category of $\mathcal{O}_{\mathbf{Spec}\,\mathcal{A}}$-modules.

Proof: It suffice to prove for affine opens in S and glue. For this, use the adjointness of $\sim$ and Γ and adjointness for Spec. $\square$

Prop. (1.2.5). A A-point for $\mathrm{Spec}(A)$ a point, is a morphism $\mathrm{Spec}(A) \to X$. For $A = K$, this correspond to points of X with $k(x) \subset K$, for $A = k[\varepsilon]/\varepsilon^2$, this correspond to a rational point x and an element in the dual of the $k(x)$-space m_x/m_x^2, i.e. the Zariski tangent space. (notice the local map).

Prop. (1.2.6) (Fiber Products). Fiber products exist in the category of schemes. Cf.[Hartshorne P87].

One should use universal properties of fiber products to get subschemes of the fiber product.

Dimensions

Prop. (1.2.7). For any scheme, $\dim \mathcal{O}_x = \mathrm{codim}(\overline{\{x\}}, X)$.

Prop. (1.2.8). For an integral scheme of finite type over a field, $\dim X = \dim \mathcal{O}_p = \dim U = tr.dK(X)/k$ for any closed point p and open subscheme U. (Use closed point are dense(2.2.17) and k is universal catenary to prove it is true for some U and all the closed point in it, so other U's because X is irreducible.

3 $\mathcal{O}_X$-modules

Transfer of Modules

Def. (1.3.1).

- the pushforward f_pF, $f_pF(U) = F(f^{-1}(U))$ sends presheaf to presheaf.
- the direct image of modules $f_*\mathcal{F}$, $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ sends $\mathcal{O}_X$-module to $\mathcal{O}_Y$-module.
- the pullback of modules:$f^*(\mathcal{F}) = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.

Prop. (1.3.2). f^{-1} or f^* is left adjoint to f_* (sheave is just $\underline{\mathbb{Z}}$-module):

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

And f^{-1} is exact (Check on stalks).

Coherent Sheaves

Prop. (1.3.3). For any A-module M, there is a sheaf of modules $\widetilde{M}$ on $\operatorname{Spec} A$ similar to(1.2.1). This is left adjoint to Γ

$$\operatorname{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F})$$

and defines a fully faithful functor from the category of A-modules to the category of $\mathcal{O}_{\operatorname{Spec} A}$-modules (because $\Gamma(X, \widetilde{M}) = M$)= the category of quasi-coherent sheaves over $\operatorname{Spec} A$ because Qco is affine local.

This is also an equivalence between f.g. A-modules and coherent sheaves over $\operatorname{Spec} A$..

Def. (1.3.4). A sheave on a ringed space is called **quasi-coherent** if it is locally $\widetilde{M_i}$. It is called **coherent** if X is of f.t. and finitely presented. When X is a locally Noetherian scheme, this is equivalent to M_is are f.g. A_i-module. When talking about coherent sheaves over scheme, I tacitly assume the scheme is locally Noetherian.

(Quasi-)coherent is an affine local (check(2.1.2) Cf.[Hartshorne P112]), hence quasi-coherent sheaves over affine scheme is just $\widetilde{M}$.

Prop. (1.3.5).

- The (Q)co modules forms an Abelian category (local on the affine open, check on stalks, localization is exact).
- Extensions of (quasi-)coherent sheaves are (quasi-)coherent checked on affine open by five-lemma. (because quasi-coherent is cyclic for Γ).
- Tensor product of two (Q)co sheaf is (Q)co, and locally free if they are locally free (because $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ as tensor product commutes with pullbacks)
- pullback of (quasi-)coherent sheaves are (quasi-)coherent. (Local on the affine open, check all the distinguished open sets). Note in the coherent case, both scheme should be locally Noetherian.
- If f is qcqs, then the pushforward of of a quasi-coherent sheaf is quasi-coherent(3.2.12).

Prop. (1.3.6). On a locally Noetherian scheme, any Qco sheaf is sum of coherent sheaves, so any coherent sheaf an be extended to a global coherent sheaf. (First prove for affine opens).

Prop. (1.3.7). The category of coherent sheaves over a ringed space is an Abelian category.

Prop. (1.3.8). The category of $\mathcal{O}_X$-modules on a scheme is a Grothendieck Abelian category. Injectives are flabby.

Proof: For a family of generators, take $j_!\mathcal{O}_U$ as the representative for $\Gamma(U, -)$. Use $j_!\mathcal{O}_U$, we can see injectives are flabby. □

Support of Modules

Prop. (1.3.9). The support of a Qco sheaf of f.t over a scheme is closed, e.g. coherent sheaf. This have many consequences applied to kernel and cokernel, for example, a coherent sheaf is locally free iff all its stalk is free (choose a presentation and see kernel and cokernel).

$\operatorname{Supp}(f^*(\mathcal{F})) = f^{-1}(\mathcal{F})$, (should use Nakayama).

Def. (1.3.10). A f.t. Qco sheaf on a scheme has a minimal closed scheme on its support, it is generated locally by the Qco ideal $\mathrm{Ann}_A(M)$(2.2.27). And there is a f.t. Qco sheaf $\mathcal{G}$ that $i_*(\mathcal{G}) = \mathcal{F}$. Cf.[StackProject 01QY].

4 Projective Space

Def. (1.4.1) (Projective Scheme). For a graded ring S, we have a scheme $\mathrm{Proj}(S)$ that consists of homogenous primes of S minus S_+ and the affine cover is $D(f) = \{p | f \nsubseteq p\}$, and $\mathcal{O}(D(f)) = \mathrm{Spec}\, S_{(f)}$, where $S_{(T)}$ is the degree zero part of $T^{-1}S$. It has $\mathcal{O}_p = S_{(p)}$.

Proof: Define the sheaf using stalks, then we only have to check that $\mathrm{Spec}\, S_{(f)} \cong$ homogenous $p \in S_f$ by natural intersection of ideals φ. and $S_{(p)} \cong (S_{(f)})_{\varphi(p)}$ for $p \in D(f)$.

We check that for $S_{(f)} \subset S_f$, $p \to p \cap S_{(f)}$ and $p' \to pS$ is natural and inverse to each other. $S_{(f)} \to S_{(p)}$ maps $\varphi(p)$ to invertible, and any $x/a \in S_{(p)}$ can be written as $\frac{xa^{\deg f - 1}/f^{\deg a}}{a^{\deg f}/f^{\deg a}}$. $\square$

Prop. (1.4.2).

$$\mathrm{Proj}^n_{\mathbb{Z}} \times \mathrm{Spec}\, A = \mathrm{Proj}^n_A.$$

(Choose the canonical affine open sets to see).

Prop. (1.4.3). For two graded ring with the same $S_0 = A$, $\mathrm{Proj}(S \times_A T) \cong X \times_A Y$, where $(S \times_A T)_n = S_n \times_A T_n$ (natural morphism from left to right).

Prop. (1.4.4). For a graded S-module, there is a Qco-sheaf $\widetilde{M}$ on Proj$S$, that $\widetilde{M}_p = M_{(p)}$ and $\widetilde{M}|_{D^+(f)} \cong \widetilde{M_{(f)}}$. the construction is as in(1.4.1).

Def. (1.4.5) (Relative Proj). The relative $\mathrm{Proj}\, \mathcal{S}$ over locally Noetherian Y of a Qco graded $\mathcal{O}_Y$-algebra $\mathcal{S}$ f.g. over $\mathcal{S}_0$ by coherent $\mathcal{S}_1$ is the glueing of locally $\mathrm{Proj}\, S$. $\mathrm{Proj}\, \mathcal{S} \to Y$ is locally projective thus proper. It is equipped with invertible sheaf $\mathcal{O}(1)$ by glueing.

Prop. (1.4.6) (Twisting of Proj). With notation as in (1.4.5), Let $S' = S * \mathcal{L} : S'_d = S_d \otimes \mathcal{L}^d$, then $\varphi : \mathrm{Proj}\, S' \to \mathrm{Proj}\, S$ is an isomorphism that induces

$$\mathcal{O}'(1) \cong \varphi^* \mathcal{O}(1) \otimes \pi'^* \mathcal{L}.$$

Prop. (1.4.7). If Y is Noetherian and admits an ample invertible sheaf, then by definition, we have $S_1 \otimes \mathcal{L}^n$ is base point free for some n, thus we have a morphism $\mathrm{Proj}\, S * \mathcal{L}^n \to \mathbb{P}^N_Y$, so $P = \mathrm{Proj}\, S$ is H-quasi-projective with $\mathcal{O}(1) = \mathcal{O}_P(1) \otimes \pi^* \mathcal{L}^n$.

Serre Twisting

Def. (1.4.8). Define $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) = \mathbb{Z}[\widetilde{X_0, \ldots, X_n}](1)$, this is an invertible sheaf. The invertible **Serre twisting sheaf** $\mathcal{O}(1)$ on $\mathbb{P}^r_Y$ is the pullback of that of $\mathbb{P}^n_{\mathbb{Z}}$ and an invertible **Serre twisting sheaf** of the relative $X = \mathrm{Proj}\, \mathcal{S}$ over Y is locally the pullback of that of $\mathbb{P}^r_Y$. Giving a Serre twisting sheaf of X over Y, the **Serre twisting sheaf** of $\mathcal{F}$ over X is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Prop. (1.4.9). For X projective over Spec(A), (i.e. $X = \mathrm{Proj}(S)$(2.2.26)), $\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ and many other properties involving the Serre twisting, all this boil down to the fact that $(M \otimes_S N)_{(f)} = M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ for $f \in S_1$.

and by virtue of(1.4.4), when $X = \mathrm{Proj}(S)$ projective, we have:

- $\widetilde{M}(n) \cong \widetilde{M(n)}$.
- For a graded ring map $S \to T$, we have the corresponding Proj map $f : U \to T$ that $f^*(\widetilde{M}) \cong (\widetilde{M \otimes_S T})|_U$ and $f_*(\widetilde{N}|_U) \cong \widetilde{N_S}$. That's to say, $f^*(\widetilde{M}(n)) = f^*(\widetilde{M})(n)$ and $f_*(\widetilde{M}(n)) = f_*(\widetilde{M})(n)$.

Cor. (1.4.10). $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$ for any scheme X projective over Y.

Vector Bundles

Def. (1.4.11). A locally free module on schemes can induce a symmetric vector bundle $S(E)$, and the section sheaf recovers $E^\vee$. This defines a reverse equivalence of locally free sheaves and vector bundles on X.

When E is Qco, we can define the **associated projective space bundle** $\mathbb{P}(E)$ as $\operatorname{Proj} S(E)$. It is equipped with a Serre twisting sheaf $\mathcal{O}(1)$. There is a surjective morphism $\pi^*(\mathcal{E}) \to \mathcal{O}(1)$ (local check).

Prop. (1.4.12). Let $g : Y \to X$ by a scheme over X, a morphism $Y \to \mathbb{P}(\mathcal{E})$ over X is equivalent to an invertible sheaf $\mathcal{L}$ and a surjective map $g^*\mathcal{E} \to \mathcal{L}$.

In particular, giving a morphism $X \to \mathbb{P}^n_A$ is essentially equivalent to a base point free invertible sheaf with n generators on X.

Proof: If there is a morphism, it will pullback $\pi^*(\mathcal{E}) \to \mathcal{O}(1)$ into $g^*\mathcal{E} \to \mathcal{L}$. For the coverse, construct locally and glue, we have the natural morphisms $A[x_1/x_i, \ldots, x_n/x_i] \to \mathcal{O}_{X_{s_i}} : x_j/x_i \to s_j/s_i$ in a homogenous sense. It is natural hence glue together. For the module, maps $x_i \to s_i$. □

Cor. (1.4.13). All automorphisms of $\mathbb{P}^n_k$ is linear.

Proof: The Picard group of $\mathbb{P}^n_k$ is $\mathbb{Z}$ and is generated by $\mathcal{O}(1)$(1.6.13), so the automorphism will map $\mathcal{O}(1)$ to $\mathcal{O}(\pm 1)$ and $\mathcal{O}(-1)$ has no global section(1.5.1). And the globals section is n-dimensional and determines the morphism by the prop. □

Blow Up

Prop. (1.4.14). On a locally Noetherian scheme, the **blowing up** $\widetilde{X}_I$ along a closed scheme (Corresponding to a coherent sheaf) is defined as $\operatorname{Proj}(\oplus I^d)$. It has the universal property that any morphism $Z \to X$ that pulls back Y to an effective Cartier divisor uniquely factors through $\widetilde{X}_I$.

Proof: Notice first an effective devisor is equivalent to an invertible sheaf of ideal. And any morphism $Z \to X$ pulls back I to the image of $f^{-1}I \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to f^{-1}I \cdot \mathcal{O}_Z$. This is just $\mathcal{O}(1)$ of $\widetilde{X}_I$ so invertible.

For the construction, the local uniqueness implies the existence. Notice locally $\widetilde{X}_I$ is projective over X. Now because the $Z \to X$ pulls back I to an invertible sheaf and it is generated by $f^{-1}(a_i)$, we use(2.3.9) to get another $Z \to \operatorname{Proj}^n_X$ and it factors through the closed subscheme $\widetilde{X}_I$. If there is another morphism g, then $f^{-1}I \cdot \mathcal{O}_Z = g^{-1}(\pi^{-1}I \cdot \mathcal{O}_{\widetilde{X}_I}) \cdot \mathcal{O}_Z = g^{-1}(\mathcal{O}_{\widetilde{X}_I}) \cdot \mathcal{O}_Z$ surjective, and a surjective morphism between two invertible sheave is an isomorphism, and they are both ideal sheaves, thus is the same, so this morphism is unique as it is determined by the map on $\mathcal{O}_X$(2.3.9). □

Cor. (1.4.15). If the sheaf of ideal is itself invertible, then the blowing up is an isomorphism by the universal property. In particular, on the open set $U = X - Y$, $I_U \cong \mathcal{O}_U$, so $\pi^{-1}(U) \cong U$.

Cor. (1.4.16). $\pi : \widetilde{X}_I \to X$ is birational, proper thus surjective. If X is a (complete)variety, then so does $\widetilde{X}_I$.

Prop. (1.4.17) (Strict Transformation). Sam notation as before, for any locally Noetherian scheme $Z \to X$, we have the pullback sheaf $J = i^{-1}(I) \cdot \mathcal{O}_Z$ on Z, so $\widetilde{Z}_J \to X$ factors through $\widetilde{X}_I$. This a pullback diagram. (Recall the definition of fiber product, we only need to check for Z,X affine and glue. For this, check $B \otimes_A (\oplus I^d) \to \oplus (IB)^d$ defines the fiber map).

Prop. (1.4.18). If X is H-(quasi-)projective, then so does $\widetilde{X}_I$ and π is H-projective(1.4.7). And any birational projective morphism from another variety Z to X comes from a blowing-up.

Proof: Cf.[Hartshorne P166]. □

5 Invertible Sheaves

Prop. (1.5.1). Let $\mathcal{L}$ be an invertible sheaf over qcqs X, let $\Gamma_*(\mathcal{F}) = \oplus \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$, then $\mathcal{F}(X_f) \cong \Gamma_*(\mathcal{F})_{(f)}$.

Proof: This is nearly the same as the proof that $(\operatorname{Spec} A)_f = \operatorname{Spec} A_f$, Cf.[StackProject 01PW]. □

Cor. (1.5.2). when $X = \operatorname{Proj}(S)$ projective and $\mathcal{F}$ Qco, $\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$. In particular, Γ_* for projective space $\mathbb{P}^n_A$ equals $A[x_1, \ldots, x_n]$.

Ample & Very Ample Sheaves

Def. (1.5.3). A **very ample** invertible sheaf on X/Y quasi-projective over Y is the pullback along some immersion of $\mathcal{O}(1)$ of $\operatorname{Proj}(\mathcal{E})$ for some Qco module $\mathcal{E}$ over Y, Cf.(1.4.8). It is called H-**very ample** iff $\mathcal{E}$ is trivial. Notice when X is proper, this immersion must be closed by(2.3.3).

On a quasi-compact scheme X, an invertible sheaf $\mathcal{L}$ is called **ample** iff there is a n and sections $s_i \in \Gamma(X, \mathcal{L}^n)$ that X_{s_i} is an affine cover of X. On a Noetherian scheme X, an invertible sheaf $\mathcal{L}$ is called H-**ample** iff for any coherent sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for large n.

Prop. (1.5.4). An invertible sheaf $\mathcal{L}$ is ample iff $\mathcal{L}^m$ is ample.

Prop. (1.5.5). When X is Noetherian, H-ample $\iff$ ample.

Proof: Cf.[StackProject 01Q3], the left to right: For any point, choose a open affine U that $\mathcal{L}$ is free, then the sheaf of ideal for $X - U$ is coherent because X is Noetherian so $I_Y \otimes \mathcal{L}^n$ is generated by global sections thus some s that $p \in supp(s)$. So as U is affine, $X_s \subset U$ is affine. Then use finiteness argument. □

Prop. (1.5.6). Let X/S be locally of f.t , then for any ample invertible sheaf $\mathcal{L}$ over X, every $\mathcal{L}^m$ for m large is H-very ample.

Proof: As in the proof of(1.5.5), we see that there are f.m affine opens X_{s_i} that cover X refining a inverse image of affine cover of S, we can make them the same degree then by f.t., there are f.m generators $\{c_{ij}\}$(1.5.1). So consider the projective space $A[x_i, c_{ij}]$, X is closed immersed into an open subscheme of P^N_S. Cf.[01VS]. □

Prop. (1.5.7). When X/S is of f.t. and S is affine, H-very ample is equivalent to very ample. And $\mathcal{L}$ ample $\iff$ $\mathcal{L}$ relative ample(not defined yet) $\iff$ $\mathcal{L}^n$ is $(H-)$very ample for some(all large)n, Cf.[StackProject 01VT].

Cor. (1.5.8) (Serre). When X/S is of f.t. and S is Noetherian affine, (H-)very-ample implies (H-)ample.

6 Divisors

Weil Divisors

We only consider divisors on a Noetherian integral separated scheme regular in codimension 1. Cartier divisor and Picard Group are more general.

Prop. (1.6.1). If X is a Noetherian integral separated scheme regular in codimension 1, then so does $X \times \operatorname{Spec}\mathbb{Z}[T]$ and $X \times \mathbb{P}^n_{\mathbb{Z}}$(local check), and $\mathrm{Cl}(X \times \operatorname{Spec}\mathbb{Z}[T]) = \mathrm{Cl}(X)$ and $\mathrm{Cl}X \times \mathbb{P}^n_{\mathbb{Z}} = \mathbb{Z} \times \mathrm{Cl}(X)$. Cf.[Hartshorne P134].

Prop. (1.6.2). For A a Noetherian domain, it is a UFD iff $X = \operatorname{Spec}(A)$ is normal and $\mathrm{Cl}(X) = 0$.

Proof: We only have to show minimal primes of A is principal iff minimal primes of A is a principal divisor. This is done by (5.11.5) and (5.11.6). □

Cor. (1.6.3). The divisors on $\mathbb{P}^n_k$ is is locally defined by a function, this is because the affine opens are UFD.

Prop. (1.6.4). A hypersurface of degree d in P^n_k is equivalent to dH, where H is the surface $x_0 = 0$. This is because irreducible hypersurface of P^n_k correspond to a homogeneous prime ideal of height 1 which is principal. So $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$.

Prop. (1.6.5). An element $\notin k$ in the function fields of a projective non-singular curve over an alg.closed k defines a inclusion $k(f) \subset K(X)$ thus a morphism from X to P^1_k(5.0.8), and $(f) = \varphi^*(\{0\} - \{\infty\})$.

Prop. (1.6.6). For a finite morphism f between two non-singular curves over alg.closed field, e.g. dominant morphism between complete non-singular curves, $\deg f^*D = \deg f \cdot \deg D$. Cf.[Hartshorne P138].

Prop. (1.6.7). A **complete linear system** on a nonsingular projective variety is the set of effective divisors linearly equivalent to D_0. When X is non-singular over a alg.closed field, the equivalent divisors correspond to projective space of $\Gamma(X, L(D_0))$, (should use X is normal and so

Cartier Divisors

Def. (1.6.8). A **Cartier divisor** on an integral scheme is an element in $\Gamma(X, K^*/\mathcal{O}_X)$. An **effective Cartier divisor** is a Cartier divisor that is locally defined as $\{(U_i, f_i)\}$ where $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$, it is equivalent to a closed subscheme locally defined by a single element.

Prop. (1.6.9). For an integral separated Noetherian scheme that is locally factorial, the Cartier divisor is the same thing as Weil divisor. This in particular applies to non-singular curves. Cf.[Hartshorne P141].

Prop. (1.6.10). For a 1-dimensional integral scheme proper over k and a function $f \in K(X)^*$,

$$\sum_{x \text{ closed}} [k(x):k]\mathrm{ord}_{\mathcal{O}_x}(f) = 0.$$

Cf.[StackProject 02RU].

Picard Group

Prop. (1.6.11) (Cartier-Pic). If X is an integral scheme, the homomorphism $\mathrm{CaCl}(X) \to \mathrm{Pic}(X) : D \to \mathcal{L}(D)$ is an isomorphism. (It is always an injective). (It suffice to show any invertible sheaf can embed into the constant sheaf, tensor with K and restrict to the stalk of the generic point, i.e. there is a compatible choice of isomorphisms to $K(X)$).

Cor. (1.6.12) (Cl-Pic). For an integral separated Noetherian scheme that is locally factorial, $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$(1.6.9).

Remark (1.6.13). Take $\mathbb{P}^n_k$ for example, the hyperplane $x_0 = 0$ defines a Cartier divisor (x_0/x_i) on U_i, thus it define the subsheaf of $\underline{K^*}$ generated by (x_i/x_0) on U_i, thus it is isomorphic to the Serre sheaf $\mathcal{O}(1)$ by multiplication by x_0. The Picard group of $\mathbb{P}^n_k$ are generated by $\mathcal{O}(1)$(1.6.4).

7 Differentials

Def. (1.7.1). The diagonal map $\Delta : X \to X \otimes_Y X$ is an immersion hence an isomorphism onto the image. So we use the locally sheaf of ideal $\mathcal{I}$ corresponding to $\Delta(X)$ to get the **Sheaf of differentials** $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ on X. It is a $\mathcal{O}_X$-module on X.

It is a Qco sheaf because pullback of Qco is Qco, and when $X \to Y$ is locally of f.t. and Y is locally Noetherian, X and $X \otimes_Y X$ is also locally Noetherian thus $\Omega_{X/Y}$ is coherent.

By(5.12.2)(5.12.3) $\Omega_{X/Y}$ can also be constructed by locally $\widetilde{\Omega_{B/A}}$ and glue because it is functorial. And we see form this that it is compatible with base change of schemes. Form this we see the stalk of $\Omega_{X/Y}$ at p is $\Omega_{X_p/Y_{f(p)}}$.

Prop. (1.7.2). Let $f : X \to Y$ and $g : Y \to Z$, then there is an exact sequence of sheaves on X:

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y}.$$

Immediate from(5.12.5).

Prop. (1.7.3). Let $f : Z \to X$ be closed immersion and $g : X \to Y$, then there is an exact sequence of sheaves on Z:

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X,Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

Immediate from(5.12.7).

Prop. (1.7.4). For an irreducible scheme of f.t. over an alg.closed field k, then $\Omega_{X/k}$ is a locally free sheaf of rank $n = \dim X$ iff X is a nonsingular.

By the same method, we can show that an integral scheme of f.t. over k perfect has an open dense subset U that is regular.

Proof: It suffice to consider closed point by**??**, the alg.closed,irreducible and f.t. conditions are here to use(5.12.10), and a coherent sheaf is locally free iff its stalks are free(1.3.9). It is regular because regular local rings are integral hence reduced.

For the second assertion, we consider the stalk of $\Omega_{X/k}$ at the generic point, it is $\Omega_{K/k}$, which is free by(5.12.9). So by(1.3.9) again there is an open dense nbhd of the generic point that Ω is free hence all the points in it are regular. □

8 Limit of Schemes

Def. (1.8.1). For a locally Noetherian scheme and a Qco sheaf of ideal I on it corresponding to a closed scheme Y, there is a **Formal completion of** X **along** I defined the ringed space with the glue of locally the functorial completion of A along I on the topological space Y,(5.5.6)[Hartshorne P194]. In fact, any coherent sheaf on X can be completed along Y.

A Locally ringed space $\widetilde{X}$ is called Locally Noetherian formal scheme if it is locally a formal complete of some X along I. A sheaf of $O_{\widetilde{X}}$-modules is called coherent iff it is locally the complete of a sheaf of coherent module.

V.2 Properties of Schemes(Hartshorne)

Basic References are [Algebraic Geometry Hartshorne] and [Hartshorne Solution 田翊].

1 Basic Scheme Properties

Affine Local

Lemma (2.1.1) (Nike's Trick). In a scheme X and $x \subset \operatorname{Spec}(A) \cap \operatorname{Spec}(B)$, x has an open nbhd in $\operatorname{Spec}(A) \cap \operatorname{Spec}(B)$ that are distinguished in both $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$.

Prop. (2.1.2) (Affine Communication Theorem). A property P of affine open subsets is called **affine local** if: $\operatorname{Spec}(A)$ has $P \Rightarrow$ all $\operatorname{Spec}(A_f)$ has P, and any cover of $\operatorname{Spec}(A_{f_i})$ has $P \Rightarrow \operatorname{Spec}(A)$ has P.

Then suppose $X = \bigcup_i \operatorname{Spec}(A_i)$ that A_i has P, then any affine open of X has P. Notice a stalk-wise property is obviously affine-local. If X has property $\widetilde{P}$, then all the open subscheme of X has property $\widetilde{P}$.

Cor. (2.1.3). List of affine local properties:

- (Locally)Noetherian. Cf,[Hartshorne P83].
- Reducedness. (Stalk-wise)

Proof: Reducedness: if there is an affine cover that is reduced, then the stalks will be like R_P is reduced if R is reduces. And if the stalks are all reduced, then a nilpotent element will be 0 in every local set, thus 0 because $\mathcal{O}$ is a sheaf. □

Connectedness

Prop. (2.1.4). $\operatorname{Spec}(A)$ is not connected $\iff A = A_1 \times A_2 \iff A$ has no nontrivial idempotent element.

Proof: This is all equivalent to the fact that there exists $e + f = 1, ef = 0$. □

Prop. (2.1.5). For geometrically connected. Cf[StackProject 32.7]

Irreducible

Prop. (2.1.6). A scheme is irreducible iff for every affine open U, $X(U)$ is irreducible iff X has an irreducible affine open cover that pairwise intersects.

Cor. (2.1.7). The fiber product of irreducible schemes is irreducible.

Def. (2.1.8). A scheme over k is called **geometrically irreducible** if $X \times_k K$ is irreducible for every field extension, it suffice to check for $K = k^{sep}$, Cf.[StackProject 32.8].

Reduced

Def. (2.1.9). Call a scheme is called **reduced** if $\Gamma(U, \mathcal{O}_X)$ is reduced for every open set U. Reduced is a stalk-wise property(2.1.3).

Prop. (2.1.10). A scheme over k is called **geometrically reduced** if $X \times_k K$ is irreducible for every field extension, it suffice to check for $K = k^{per}$, Cf.[StackProject 32.6].

Prop. (2.1.11). There is a $X_{red} \to X$ associated tot every scheme, it is $\mathbf{Spec}(\mathcal{O}_X/\mathcal{N})$ where $\mathcal{N}$ is the sheaf of nilpotent elements. This construction is right adjoint to the forgetful functor by the adjoint property of **Spec**(1.2.4). $X_{red} \to X$ is an closed immersion.

It's useful to change to X_{red} when the proposition only involve topology because X_{red} has the same topology as X.

Prop. (2.1.12). There is a reduced induced scheme structure on a closed subset Y of a scheme X, it is the **Spec** of the $\mathcal{O}_X$-algebra of $\mathcal{O}_X(U)/\cap p_i, (i \in Y)$. It has the universal property.

Cor. (2.1.13). Any map morphism from a reduced scheme X to Y that factors through the closed subscheme of the closure of its image. (By virtue of reducedness).

Integral

Def. (2.1.14). A scheme X is called integral if $X(U)$ is all integral. This is equivalent to reduced and irreducible. So a scheme is integral iff there is an integral open affine cover that are pairwise-intersect(2.1.6). Cf.[Hartshorne P82].

Cor. (2.1.15). The projective space over an integral scheme is integral. (Check the affine covers are dense). The projective space $P^n_{\mathbb{Z}}$ is integral.

Prop. (2.1.16). For geometrically integral, Cf.[32.9].

Noetherian

Def. (2.1.17). A scheme is called locally Noetherian if it can be covered by open affine schemes of noetherian rings. It is called **Noetherian** if moreover it is quasi-compact.

Prop. (2.1.18). (Locally)Noetherian is affine local, i.e. X is locally Noetherian if any affine open of X is spec of a Noetherian ring(2.1.3).

Normal & Regular

Def. (2.1.19). A scheme is called **normal** if all its stalk is normal domain, so all its affine sections are normal ring. It is called **regular** iff all its stalk is regular local ring, i.e. all affine opens are regular ring. Regular only have to be checked at close pt because of(5.11.11).

Prop. (2.1.20). For an integral scheme X, there is a $X_{nom} \to X$ which is $\mathbf{Spec}(\mathcal{O}_{X,nom})$, any dominant morphism f from a normal integral scheme to X will factor through X_{nom}. (Use the adjointness for **Spec** and notice f maps generic to generic.

Prop. (2.1.21). For geometrically normal, Cf.[StackProject 32.10].

Prop. (2.1.22). For a curve, normal is equivalent to regular. This is because for a Noetherian local domain of dim 1, principal $\iff$ normal $\iff$ regular $\iff$ DVR.

Cor. (2.1.23). A Noetherian Normal scheme is regular in codimension 1.

2 Basic Morphism Properties

Local Property

Our fundamental tool is (2.1.2).

Prop. (2.2.1). List of properties local on the target:

- isomorphism, injective, surjective, open, closed

Prop. (2.2.2). All the property besides the projectiveness is local on the target.

Prop. (2.2.3). List of properties affine local on the source:

- Finite type. (A_{f_i} f.g$\Rightarrow A$ f.g)

Valuation Criterion

Prop. (2.2.4). The valuation criterion for $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$ where R is a valuation ring: For a quasi-compact morphism,

- it is separated iff there is at most one lifting.
- it is universally closed iff there is at least one lifting.
- it is proper iff it is finite type(auto quasi-compact) and lifting exists uniquely (More useful).

Cf.[StackProject].

Base Change Trick

Prop. (2.2.5). If A property P of morphisms satisfy:

- Closed immersion has P.
- Stable under base change and composition.

Then

- it is stable under product.
- $g \circ f$ has $P + g$ separated $\Rightarrow f$ has P.
- it is stable under f_{red}. (Notice $X_{red} \to X$ is closed immersion).
  Cf.[Hartshorne Ex2.4.8].

Closed Map

Prop. (2.2.6). Let $A \to B$ noetherian. Then going-up holds $\iff$ Spec map is closed.

Proof: going-up is equivalent to $f^*(V(q)) = V(f^*(q))$, $\forall q$ prime. Use primary decomposition of $\sqrt{I}$, $V(I) = \bigcup V(q_i)$. □

Prop. (2.2.7) (Universal Closed). Universal closedness is local on the basis and and satisfies the base change trick(2.2.5).

Prop. (2.2.8). If g is surjective, then $f \circ g$ is universally closed iff f is universally closed (because surjective is S.u.B).

Prop. (2.2.9). In a normal scheme. A constructible set is stable under specialization iff it is close, and dual argument.

Prop. (2.2.10). The image of a quasi-compact morphism is closed iff it is stable under base change. Cf.[Hartshorne P98].

Cor. (2.2.11). Spec map is closed iff the ring map satisfies going-up.

Affine Map

Prop. (2.2.12). X is affine if there is a finite set of elements $f_i \in \Gamma(X, \mathcal{O}_X)$ that generate the unit ideal and X_{f_i} is affine.

Proof: First prove that $X_{f_i} \cap X_{f_j} = X_{f_i f_j}$ is affine because affine intersect X_{f_i} is affine. Second, prove $\Gamma(X_f, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)_f$, finally glue them to get a map $X \to \mathrm{Spec}(A)$ and use the fact isomorphism is local on the target. Cf.[Hartshorne Ex 2.2.17]. X is affine scheme if $X \to \mathrm{Spec}(\Gamma(X))$ is affine. □

Cor. (2.2.13). Affineness is affine local on the target and satisfies the base change trick(2.2.5). Hence open and closed morphisms are affine.

Quasi-Compact

Def. (2.2.14). A morphism $f : X \to S$ is called quasi-compact if the inverse image of affine open is quasi-compact.

Quasi-compactness is local on the target and satisfies the base change trick(2.2.5).

Dominant

Prop. (2.2.15). A quasi-compact morphism of schemes:$X \to S$ is dominant if every generic point of irreducible components of S is in the image of f.(Use quasi-compactness to reduce to the affien case). In particular, if X, S is affine, dominant is equivalent to image of f contains minimal primes and equivalent to the kernel is in the nilradical. (Because the closure of image $= V(\mathrm{ker})$.

Finite Type

Def. (2.2.16). A morphism $f : X \to S$ is called of **locally finite type** if for there exists an affine open cover $\{\mathrm{Spec}(B_i)\}$ of S that $f^{-1}(U_i)$ has an affine open cover of spec of finite generated B_i-algebras. It is called finite type if moreover it is quasi-compact.

(Locally)Finite type is affine local on the target (check(2.1.2)), affine local on the source, and satisfies the base change trick(2.2.5).

Prop. (2.2.17). For a scheme locally of finite type over a field, the closed points of X are dense in X.

Proof: It suffice to show that a point is closed iff $k(x)/k$ is finite iff $k(x)/k$ is algebraic.

The first is by generalized Nullstellensatz**??**, and if $k(x)/k$ is algebraic, then A/m_x is algebraic ring extension over a field thus a field, so x is closed in any affine nbhd, so it is closed in X. □

Prop. (2.2.18) (Chevalley). A morphism of finite type between Noetherian schemes maps constructible subset to constructible subset. (reduce to affine integral dominant case and use a commutative algebra proposition. Cf.[Hartshorne P94].

Finite & Integral Map

Def. (2.2.19). A morphism $f: X \to S$ is called **finite** if it is affine and the inverse image of an affine cover is finite module.

Finiteness is affine local on the target Cf.[StackProject 02JL] and satisfies the base change trick(2.2.5).

A morphism $f: X \to S$ is called **quasi-finite** if it is of finite-type and the inverse of a point is a discrete hence finite set.

A morphism $f: X \to S$ is called **integral** if it is affine and the inverse image an affine cover is integral ring extension.

Integral is affine local on the target Cf.[StackProject 02JK] and satisfies the base change trick(2.2.5).

Prop. (2.2.20). A locally f.t. integral morphism is finite.

Prop. (2.2.21) (Chevalley). Finite $\iff$ quasi-finite+proper.?

Proof: The fiber of $f: X \to S$ is $\mathrm{Spec}(k(y) \otimes_A B)$, which is Artinian (5.1.7), so it has finitely many primes. Finite morphism is proper because it is affine and integral(2.2.22).

For the converse, one should use Zariski's Main Theorem. □

Prop. (2.2.22). Integral is equivalent to u.c. and affine. cf.[StackProject 01WM].

Proof: For one way, it suffice to show it is locally closed(5.4.1). □

Immersions

Def. (2.2.23). An **immersion** is a closed immersion followed by an open immersion. A open immersion followed by a closed immersion can be written as a closed immersion followed by an open immersion, but not reversely. The reverse happens if the immersion is quasi-compact or the source is reduced (use the reduced induced structure) Cf.[StackProject 01QV].

Prop. (2.2.24). Open and closed immersions are affine local on the target.

Prop. (2.2.25). Immersion, closed immersion, open immersion are stable under base change. Closed immersion satisfies the base change trick(2.2.5).

Proof: For closed immersion, check locally, for open immersion, notice that $U \times_W V \to X \times_S Y$ is open immersion. □

Prop. (2.2.26). The closed subscheme of Spec(A) corresponds to the quotients A/I. The closed scheme of $X = \mathrm{Proj}(A[x_1, \ldots, x_n])$ corresponds to the saturated homogenous ideal I_Y, (i.e. if there is an n that cor any $i, x_i^n s \in I_Y \Rightarrow x \in I_Y$).

So projective scheme over Spec(S_0) corresponds to Proj(S), where S are f.g. over S_0 by S_1 saturated in the sense above.

Proof: A closed immersion is proper, thus the kernel I_Y of the structural map is a Qco-module(1.3.5), so it must be an ideal on every affine open, because Qco is affine local. Then we should use that $\Gamma_*(\mathcal{O}_X) = A[x_1, \ldots, x_n]$ and $\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$ for $\mathcal{F}$ Qco over X, then $\gamma_*(I_Y)$ will suffice. Cf.[Hartshorne P118,P119]. □

Cor. (2.2.27). The closed subscheme of a scheme corresponds to Qco $\mathcal{O}_X$-ideals.

Proof: The closed immersion is qcqs, so it maps $\mathcal{O}_Y$ to $i_*(\mathcal{O}_Y)$ Qco, thus the kernel of $\mathcal{O}_X \to i_*(\mathcal{O}_Y)$ is Qco. Conversely, for a Qco sheaf of ideals, $Y = \mathbf{Spec}_X(\mathcal{O}_X/I)$ for the Qco $\mathcal{O}_X$-algebra $\mathcal{O}_X/I$. □

Separatedness

Def. (2.2.28). A map $f : X \to Y$ is called **separated** if the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. It is called **quasi-separated** if the diagonal is quasi-compact, this is equivalent to intersection of two affine opens mapped into and affine open is quasi-compact.

In fact Δ is always an immersion because maps between affine scheme is separated so $\Delta(X)$ is closed in $\cup U_{ij} \otimes_{V_i} U_{ij}$ where U, V are affine open, hence it suffice to check the image is closed.

(Quasi-)Separateness is local on the target because closed immersion and quasi-compact is local on the target. ($f^{-1}(U) \times_U f^{-1}(U)$ form a basis).

Prop. (2.2.29). the class of separated map satisfies the base change trick(2.2.5) because closed immersion do. So X separated $\Rightarrow X \to Y$ separated.

Prop. (2.2.30). monomorphism is separated because the diagonal map is isomorphism(1.7.2), so <u>immersions are separated</u> as they are monomorphisms in the category of schemes (because of surjectiveness on the stalk).

Prop. (2.2.31). A morphism is separated iff for any two affine open that mapped to an affine open, their intersection is affine and $\mathcal{O}(U) \otimes_{\mathcal{O}(W)} \mathcal{O}(V) \to O(U \cap V)$ is surjective. (One way is easy, the other way Cf.[StackProject 01KP]).

Cor. (2.2.32). A projective scheme $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is separated.

Prop. (2.2.33). If X is Noetherian, then $X \to Y$ is quasi-compact (any open set is quasi-compact).

Prop. (2.2.34). Affine morphism is separated (Check closed immersion directly).

3 Proper & Projective Morphism

Prop. (2.3.1). A morphism that is separated, finite-type and universally closed is called proper.

proper is local on the target, because all these three properties do.

Prop. (2.3.2). The class of proper morphisms satisfies the base change trick(2.2.5)(Valuation Criterion). (Closed immersion is proper because it is f.t. and is affine so separated(2.2.28), and it is universally closed because immersions are stable under base change(2.2.25)).

Prop. (2.3.3) (Image of Proper Map). If $X \to Y$ is morphism between separated schemes f.t over S, then if X is proper, then the image is closed (base change trick) and is proper in its induced reduced structure(2.2.8).

Cor. (2.3.4). A morphism from a connected proper scheme to an Noetherian affine scheme $\operatorname{Spec} A$ is constant.

Proof: Because the image is proper and use(3.2.15), so A is a finite module over $\operatorname{Spec} k$ thus Artinian so has finitely many point. So it is discrete. □

Projective Morphism

Def. (2.3.5). A **projective** morphism $X \to Y$ is a closed immersion $X \to \operatorname{Proj}(\mathcal{E})$ for some Qco f.t. module $\mathcal{E}$. A H**-projective** $\to Y$ is a closed immersion $X \to \mathbb{P}^n_Y$. A H-quasi-projective morphism is a H-projective morphism composed with an open immersion. Some proposition about projective is written before the language of Hartshorne so I may not have changed them to the more general projective notion yet.

Prop. (2.3.6). H-(Quasi-)Projectiveness satisfies the base change trick(2.2.5). (because Segre embedding is closed). Disjoint union of finite projective morphisms is projective (embed into the Segre embedding).

Cor. (2.3.7). Projective morphism is locally projective and locally projective is proper[StackProject 01WC], because closed immersion is proper and $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is u.c. by valuation criterion. Cf.[Hartshorne P103]. And a quasi-projective morphism is of f.t. and separated(2.2.30).

A quasi-projective is proper iff it is separated (base change trick).

Prop. (2.3.8). Projective scheme over $\operatorname{Spec} A$ is of the form $\operatorname{Proj} S$ where $S_0 = A$ and S is f.g over S_0 by S_1(2.2.26).

Prop. (2.3.9). H-quasi-projective morphism over an affine scheme correspond to a very ample invertible sheaf(1.4.12).

Prop. (2.3.10) (Chow's Lemma). Let $X \to S$ be separated of f.t over a Noetherian S, then there is a birational, proper, surjective $X' \to X$ that X' is quasi-projective.

X is proper iff X' can be projective. And if X is integral(irreducible,reduced), x' can be chosen to be so.

Proof: Basic idea: reduce the the irreducible case, and use f.t. to generate a local quasi-projectives, then the closure of the image of $U \to X \times_S P_1 \times_S \ldots \times_S P_n$ will suffice. □

4 More Properties of Schemes

5 Zariski's Main Theorem

6 Flatness & Smoothness

Def. (2.6.1). For a morphism $f : X \to Y$, a $\mathcal{O}_X$-module M is called **flat** over Y iff its stalk is flat as a $\mathcal{O}_{Y,f(x)}$-module. f is called **flat** iff $\mathcal{O}_X$ is flat.

Prop. (2.6.2). Flatness is local on the target, it is stable under base change, composition. A coherent $\mathcal{O}_X$ module is flat over X iff it is locally free.(5.14.7)(5.14.3). A Qco sheaf is flat w.r.t to a affine morphism iff it is flat over affine opens.

Prop. (2.6.3). A finite morphism $f : X \to S$ with S locally Noetherian is flat iff $f_*(\mathcal{O}_X)$ is locally free, Cf.[StackProject 02KB].

Smoothness

Def. (2.6.4). A morphism between schemes of f.t. over k is called **smooth** of relative dimension n iff f is flat and every fiber of f is geometrically regular of dimension n. (geometrically regular$\Rightarrow$ regular?)

Prop. (2.6.5). Smooth morphism is stable under base change and composition.

Prop. (2.6.6). Smooth over a field $k \iff$ geometrically regular by(5.14.7).

7 Étale Morphism

V.3 Cohomology

1 Sites of Schemes

Zariski Site

Def. (3.1.1). The Zariski topology has the the covering of a scheme T as classes of open immersions $\{T_i \to T\}$ that their images cover T.

Étale Site

Def. (3.1.2). The étale topology has the covering of a scheme T as classes of étale morphisms that their images cover T.

fppf Site

Def. (3.1.3). The fppf topology has the covering of a scheme T as classes of flat locally of finite presentation morphisms that their images cover T.

fpqc Site

Def. (3.1.4). The fppf topology has the covering of a scheme T as classes of flat morphisms s.t. that their images cover T and for any affine open $U \subset T$, f.m of them can cover U.

2 Sheaf cohomology

Cohomology on Site

Prop. (3.2.1) (Čech-Cohomology). For any U and a cover in a site, the corresponding Čech cohomology usually defined is a derived functor on the category of presheaves on a site, and if we take limit for coverings, $F \to \check{H}^0(U, F)$ is a left exact functor from presheaves to sheaves, the derived functors are just the limits $\check{H}^q(U, F)$.

Proof: Cf.[Tamme P34]. It is left exact because direct product, kernel are left exact. To check the refinement colimit is exact, we sow that the refinement is independent of the refinement map chosen, in this way, this is obviously a filtered colimit which is exact. For two refinement map, they defined a pullback map from $U_{\varphi(\beta)} \otimes_X U_{\psi(\beta)}$ and by definition of kernel, this is the same. □

Prop. (3.2.2) (Non-Abelian Čech). For a exact sequence of sheaves $1 \to A \to B \to C \to 1$, where A is in the center of B, then there is a exact sequence:

$$1 \to H^0(U, A) \to H^0(U, B) \to H^0(U, C) \to H^1(U, A) \to H^1(U, B) \to H^1(U, C) \to H^2(U, A)$$

which is by direct calculation, the last one is the Čech composed with the injection to sheaf cohomology(3.2.4). Use the same method.

Prop. (3.2.3) (Sheaf-Cohomology-Presheaf). The forgetful functor is right adjoint to the exact shifi functor, the Grothendieck spectral sequence applies to the exact functor $\Gamma(U, -)$ from $\mathcal{P}$ to Ab shows its right derived functor is

$$\mathcal{H}^p(F) = R^p\iota(F) : U \to H^p(U, F).$$

Prop. (3.2.4) (Čech to Sheaf). The Grothendieck spectral sequence applied to

$$\Gamma(U,-) = H^0(\{U_i \to U\},-) \circ \iota = \check{H}^0(U,-) \circ \iota$$

gives us:

$$H^p(\{U_i \to U\}, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(U,F).$$
$$\check{H}^p(U, \mathcal{H}^q(F)) \to H^{p+q}(U,F).$$

Cor. (3.2.5). The Grothendieck spectral sequence applied to forgetful functor and exact $\sharp$ functor shows that $\mathcal{H}^p(F)^{++} = \mathcal{H}^p(F)^\sharp = 0$ for $p > 0$, so

$$\mathcal{H}^p(F)^+(U) = \check{H}^0(U, \mathcal{H}^p(F)) = 0 \quad p > 0.$$

Thus the low degree of Čech to sheaf says:

$$0 \to \check{H}^1(U,F) \to H^1(U,F) \to 0 \to \check{H}^2(U,F) \to H^2(U,F).$$

Prop. (3.2.6) (Leray Spectral Sequence). There is a spectral sequence with

$$E_2^{pq} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}\mathcal{F}.$$

Cohomology of Sheaves

Prop. (3.2.7) (Grothendieck). The cohomology of a sheaf over a Noetherian topological space of dimension n vanish for $k > n$. Cf.[Hartshorne P208].

Cohomology of Modules

Lemma (3.2.8) (Zariski-Poincare). A quasi-coherent sheaf on a affine scheme X is Čech-acyclic. (Direct calculation). Cf.[Sheaf Cohomology notes Lemma5].

Prop. (3.2.9). For a quasi-coherent sheaf on a separated scheme, we have $H^p(X,F) = \check{H}^p(X,F) = H^p(\{U_i \to X\}, F)$. for U_i any open affine cover.

Proof: The intersection of affine opens is affine open, we only have to show that $H^p(U,F) = 0$, then we can use Čech to derived1. For this, we use Čech to sheaf2 together with Zariski-Poincare and $\check{H}^0(U, \mathcal{H}^q(F)) = 0$(3.2.5). □

Prop. (3.2.10) (Serre). An open subset U of X is Coh-acyclic iff $\mathcal{O}_{X|U}$ is isomorphic to an affine scheme as a ringed space. Cf.[Hartshorne P215].

Prop. (3.2.11) (Cartan). The class of Coh-Acyclic subsets of an analytic space is exactly the Stein manifold.

Prop. (3.2.12). For sheaves on schemes, the trivial spectral sequence for $f_* \circ \iota$ shows that $R^p f_* F = (f_p \mathcal{H}^p(F))^\sharp$. Thus if f is qcqs then $R^n f_*$ maps Qco to Qco, $f_p \mathcal{H}^p(F) = \check{H}^p(f^{-1}(U), F)$ is a sheaf, so $R^p f_* F = \check{H}^p(f^{-1}(U), F)$.

Cor. (3.2.13). f_* is exact for Qco for affine morphisms.

Prop. (3.2.14). If $f : X \to Y$ is proper and Y is locally Noetherian, then $R^n f_*$ maps coherent to coherent. Cf.[StackProject 02O6].

Cor. (3.2.15). If X is proper over an affine variety, its global section is a f.g. A-module.

Acyclic Sheaves

Def. (3.2.16). An Abelian sheaf on a site is called **flask** if it satisfies the following equivalent conditions:

- It is acyclic for the forgetful functor ι,
- It is acyclic for any $\check{H}^0(\{U_i \to U\}, -)$
- It is acyclic for $\Gamma(U, -)$. (Use the two Čech to derived SpecSeqs).

So it is adapted to ι.

An Abelian sheaf on a site is called **flasque** iff it is acyclic for all $\mathrm{Mor}(S, -)$ for any S a sheaf of sets. It is obviously flask.

An Abelian sheaf on a topological space X is called

flabby iff for any open $U, \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective;

soft iff X is paracompact Hausdorff and $\forall$ closed $V, \mathcal{F}(X) \to \mathcal{F}(V)$ is surjective. A flabby sheaf is soft.

fine if it is on a paracompact Hausdorff space and the sheaf of rings $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$ is soft.

Fine, flabby and soft, are local properties. (Use Zorn's lemma to construct one-by-one).

Prop. (3.2.17). For a sheaf of *rings* over a paracompact Hausdorff space X, the following are equivalent,

1. it is a soft sheaf.
2. for any disjoint closed sets V, W, there is a section of X that is 0 on V, and 1 on W.
3. it possesses a partition of unity.
4. it is a fine sheaf.

Note any soft sheaf possesses a partition of unity.

Proof: $1 \iff 2$ is easy and $1 \to 3$ is the to choose a closed locally finite subcover and use Zorn's lemma to construct one-by-one. For $3 \to 1$, notice a closed section can extend to a slightly larger nbhd.

Because for a sheaf of rings $\mathcal{F}$, a partition of unity is equivalent to a partition of unity $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$, so 34 are equivalent because 13 are equivalent. □

Note that a fine sheaf possesses a decomposition of section because the previous proposition applies to $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$, and a partition of unity in $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$ yields a decomposition of section in $\mathcal{F}$.

Thus a fine sheaf is soft. (extend to a small nbhd and use partition of unity).

The sheaf of modules over a soft sheaf of rings is soft.

The continuous function sheaf on a paracompact Hausdorff space or the smooth function sheaf on a smooth manifold is fine, thus any smooth module is fine (Use bump function).

Prop. (3.2.18). Flabby sheaf is adapted to ι, i.e. adapted to any $\Gamma(U, -)$, soft sheaf, e.g. fine sheaf is adapted to $\Gamma(X, -)$.

Proof: Just need to verify(6.4.2). The product of skyscraper sheaves of stalks is flabby, so it is sufficiently large.

For a exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of sheaves, if $\mathcal{F}$ is flabby, then H is just the presheaf cokernel. (It reduces to $\check{H}^1(\{U_i \to U\}, F) = 0$, and this is done by Zorn's lemma). Thus if $\mathcal{F}$ is flabby, $\mathcal{G}$ is flabby iff $\mathcal{H}$ is flabby.

Similar for softness, since flabby is soft and the others are the same as before. □

Smooth Sheaves

Prop. (3.2.19) (De Rham). For a smooth manifold and an Abelian group G,

$$H^*_{dR}(X,G) \cong H^*(X,G)$$

Where the right is constant sheaf cohomology. (Use the fact that smooth sheaf is fine so adapted to sheaf cohomology, and Poincare-lemma).

Prop. (3.2.20) (Dolbeault). For a complex bundle on a complex manifold,

$$H^{p,q}(X,\mathcal{E}) \cong H^q(M,\Omega^p \otimes_{\mathcal{O}_X} \mathcal{E}),$$

where the left is Dolbeault cohomology and the right is sheaf cohomology. (Use the fact that smooth sheaf is fine so adapted to sheaf cohomology, and $\overline{\partial}$-Poincare lemma).

Moreover, there is a spectral sequence of

$$E_1^{p,q} = H^{p,q}_{\overline{\partial}}(X) \Rightarrow E^n = H^n_{dR}(X,\mathbb{R}\times_{\mathbb{R}}\mathbb{C}).$$

Prop. (3.2.21) (Singular). For a locally contractible topological space,

$$H^p_{sing}(X,R) \cong H^p(X,R_X).$$

Proof: Shifification of the singular cochain complex is a flabby presheaf resolution of R_X because it is locally contractible, check on stalks. Then we only have to prove $C^\bullet(X) \to (C/V)^\bullet(X)$ is quasi-isomorphism, where V is the presheaf of locally vanishing cochain. It suffice to prove $V^\bullet(X)$ is exact.

For any i-cocycle φ, for any $i-1$-complex σ, use barycentric subdivision, we can construct a c_σ whose boundary is σ and other simplexes on which ϕ vanishes, so we have the coboundary of $\eta:\sigma \to \varphi(c_\sigma)$ is φ. $\square$

3 Étale Cohomology

fpqc Topology

Prop. (3.3.1). If a ring map $A \to B$, either has a section $B \to A$, or it is faithfully flat, then the Amitsur complex:

$$M \to M\otimes_A B \to M\otimes_A B\otimes_A B \to \dots$$

with Čech like maps, is exact. Cf.[Sheaf Cohomology notes].

4 Crystalline Cohomology

V.4 Derived Category of Schemes

V.5 Curves

Varieties

Prop. (5.0.1). the underlying space of a scheme is sober, Cf.(1.8.3).

Prop. (5.0.2). For k alg.closed, the soberization functor t induce a fully faithful functor from $\mathrm{Var}(k) \to \mathrm{Sch}(k)$ that maps to quasi-projective integral schemes over k. It maps projective varieties to projective integral schemes. And this functor preserves fiber products ?.

Proof: We assign the irreducible closed subsets space $t(X)$ and show that this embeds X in $t(X)$, and for an affine variety $(V, \mathcal{O}_V)$, the regular function sheaf is isomorphic to the pullback sheaf on $t(V) = \mathrm{Spec}(A)$.

By definition $t(X)$ is quasi-projective, and for a closed subscheme of $\mathbb{P}^n_k$, the closed pt of any closed subscheme are dense so $t(V)$ is homeomorphic to X. And because they are both reduced, they are isomorphic. So it is essentially surjective.

It is fully faithful because the closed point are equivalent to $k(x) = k$ and is dense in a f.t scheme over k so it maps closed pt to closed pt. □

Prop. (5.0.3). The soberization of a variety X is regular at a closed point iff the local defining functions has rank $n - \dim X$.

Proof: Consider the space of closed point of X, they correspond to classical points because k is alg.closed. Let $a_p = (x_1 - a_1, \ldots, x_n - a_n)$ and b be the locally defining ideal. Then the differential defines an isomorphism of vector space $a_p/a_p^2 \cong k^n$, and the local ring at p is $m/m^2 \cong (a_p/b)/(a_p/b)^2 \cong a_p/(b + a_p^2)$. The rank of the defining functions is $b + a_p^2/a_p^2$. Counting dimension gives us the result. (Use (1.2.8) also). □

Def. (5.0.4) (Abstract Variety). An **abstract variety** is an integral separated scheme of finite type over an alg. closed field k. A variety is an abstract variety because quasi-projective is f.t. and separated(2.3.7). It is called **complete** if it is also proper.

A curve over k is an abstract variety of dimension 1. It is called **non-singular** iff all the local rings are regular local.

Cor. (5.0.5). An abstract variety is birational to an integral quasi-projective scheme. A complete variety is birational to an integral projective scheme by Chow's lemma(2.3.10)(2.3.3).

Prop. (5.0.6). By valuation criterion, for a complete variety, every valuation of the function fields of K/k dominate a unique point of X. So the points of X correspond to valuations of K/k (valuation ring is the maximal local ring).

Prop. (5.0.7) (Nagata's Theorem). Any abstract variety can be embedded as an open subset of a complete variety.

Curves

Prop. (5.0.8). The following categories are equivalent.

- The category of varieties (curves) over k with dominant rational morphisms.
- The dual category of f.g. field extensions over k (of dimension 1).

- The category of non-singular projective curves and dominant morphisms.

Proof: 1,2 are equivalent by [Hartshorne P25]. We construct functor from 2 to 3. The solution is construct a rational map of the abstract non-singular curve C_K Cf.[Hartshorne P42], then extend it to the whole C_K by the following lemma. □

Lemma (5.0.9). Rational map from a non-singular curve to a projective variety can be extended to a morphism. Cf.[StackProject 0BX7].

Prop. (5.0.10). Every proper integral curve over field k is projective. Cf.[StackProject 02A6].

Def. (5.0.11). A projective variety over a field k is called **degree** d if $\mathcal{O}_X(1)$ has degree d.

Prop. (5.0.12). A projective non-singular curve of degree d in $\mathbb{P}^n_k$ isomorphic to where $d \leq n$ not contained in any $\mathbb{P}^{n-1}_k$ is isomorphic to the n-tuple embedding, and $d = n$.

This has easy generalization to surfaces and higher dimensions.

Proof: (1.6.13)shows $\mathcal{O}_X(1) \cong \mathcal{O}(d)$ over $\mathbb{P}^1_k$, and the restriction of global sections is injective. So the global section is an isomorphism, and it defines the embedding up to a linear automorphism. □

Prop. (5.0.13) (Riemann-Roch). Let D be a curve on a complete curve of genus g, then

$$l(D) - l(K - D) = \deg D + 1 - g.$$

Cf.[Hartshorne P295].

Cor. (5.0.14). Let $\deg(\mathcal{F}) = \chi(F) - r\chi(\mathcal{O}_X)$, where $\chi(F) = \sum(-1)^i \dim_k H^i(X < \mathcal{F})$ and $r = \dim_K F_\eta$, then $\deg L(D) = \deg D$.

Prop. (5.0.15). a non-singular curve in $P^2(k)$ where char$k \neq 0$ is projectively isomorphic to $xy - z^2$ if it has a rational point. (Use Riemann-Roch to show that $O(p)$ has a nontrivial section which gives a isomorphism to P^1). And in fact the assertion can be checked directly.

Surfaces

Prop. (5.0.16). Any birational transformation of non-singular surfaces will be factorized into f.m blowing-ups and blowing-downs of points.

V.6 Group Schemes

Finite Flat Group Schemes

Def. (6.0.1). An **Affine Group Scheme** is a representable covariant functor $G : R-\mathcal{A}lg \to \mathcal{G}rps$. It is called algebraic iff it is of f.t..

In order for an object to represent a functor to $\mathcal{G}rps$ rather than $\mathcal{S}ets$, we suffice to have:

- comultiplication: $\Delta : A \to A \otimes_R A$.
- counit: $A \to R$.
- antipode:$A \to A$

that satisfy the supposed identities.

We have the left(right)translation for an elements in $G(R)$, equivalently, a natural transformation on G, and base change $(G \otimes_R R')(T'_{R'}) = G(T'_R)$

Lemma (6.0.2). A bialgebra over a field k is direct limit of bialgebras of f.t. over k.

Prop. (6.0.3). Affine group schemes over a field is reduced. And it is smooth over k. Cf.[Jacob Stix P5].

Prop. (6.0.4). There is a Cartier duality on the category of finite flat affine commutative group schemes over $\operatorname{Spec} R$. This is because a finite flat module is locally free(2.6.3) , thus $A^{\vee\vee} = A$ for a R-algebra A.

p-divisible Groups

V.7 Complex Geometry

Prop. (7.0.1). the Funibi-Study metric on CP^n has sectional curvature $1 \geq K \geq 4$. Cf.[Do Carmo P188].

Chapter VI

Higher Algebra

VI.1 Category

1 Exactness

Prop. (1.1.1). In an Abelian category, the functor $X \mapsto \mathrm{Hom}(X,Y)$ and $X \mapsto \mathrm{Hom}(Y,X)$ is both left exact. Note that left and right is seen on the image.

2 Adjointness

Prop. (1.2.1). A right adjoint functor is left exact and it preserves injectives if its left adjoint is exact.

A left adjoint functor is right exact and it preserves projectives if its right adjoint is exact.

Prop. (1.2.2). Any presheaf on a small category is a colimit of representable sheaves h_X. (Consider all $h_X \to \mathcal{F}$ and take colimit, prove it is isomorphism).

Prop. (1.2.3) (Kan Extension). For a cocomplete category $\mathcal{D}$, there is a natural bijection between left adjoint functor $\widehat{\mathcal{C}} \to \mathcal{D}$ and functors $\mathcal{C} \to \mathcal{D}$ by Yoneda embedding.

Prop. (1.2.4). The sheaf Γ functor is right adjoint to the constant sheaf functor over arbitrary site.

Prop. (1.2.5). The inclusion functor is right adjoint to the shifification functor over arbitrary site.

Prop. (1.2.6). The forgetful functor is right adjoint to the Shifification functor, and shifification is exact, so it preserves injectives.

Prop. (1.2.7). The stalk functor is left adjoint to the skyscraper sheaf operator.

Prop. (1.2.8). The valuation at k'th coordinate is left adjoint to the functor $k_*(A)(i) = \prod_{\mathrm{Hom}\, i,k} A$ and is exact. So k_* preserves injectives.

3 Injective & Projective

4 Abelian Category

Prop. (1.4.1) (Axioms for Abelian Category).

- **A1**:$\mathrm{Hom}(X,Y)$ is an Abelian group.
- **A2**:There exists a zero object.
- **A3**:There exists a canonical sum and product with projections.

(Satisfying this three is called a additive category.)

- **A4**:Coimage equals image. This is equivalent to mono+epi $\iff$ isomorphism, just notice that the cokernel of 0 is an isomorphism.

Prop. (1.4.2). The $\mathrm{Hom}(X,-)$ operator is left exact in Abelian category, because kernel is a kind of limit.

Prop. (1.4.3). Axiom A3 asserts the good existence of product and sum of objects as we wanted, and it can be used to prove that monomorphism and epimorphism are stable under pushout and pullback. (For epimorphism, first prove $0 \to X \times_U Y \to U \to 0$ is exact when $X \to U$ is epi).

Prop. (1.4.4). equalizer and finite product derives finite limit, thus finite limits and finite colimits exists in Abelian categories.

Prop. (1.4.5) (Mitchell's embedding theorem). If $\mathcal{A}$ is a small category, then there exists a unital ring R, not necessary commutative and a fully faithful and exact functor $\mathcal{A} \to R-$mod that preserves kernel and cokernel. WARNING: it may not preserve sum and product, let alone limits and colimits.

Prop. (1.4.6) (Examples). If A is an Abelian category, then A^C is an Abelian category.

5 Grothendieck Abelian Category

Prop. (1.5.1) (Axioms for Grothendieck Abelian Category).

- **AB3**: It is an Abelian category and arbitrary direct sums exists. (Thus colimits over small categories exists.)
- **AB5**: Filtered colimits over small categories are exact. This is equivalent to $\{$ for any family of subobjects $\{A_i\}$ of A to B indexed by inclusion can induce a morphism $\sum A_i \to B$ (internal sum)$\}$ Cf.[Tamme].
- **GEN**: It has a generator, that is, an object U s.t. for any proper subobject $N \subsetneq M$, there is a map $U \to M$ that doesn't factor through N.

Lemma (1.5.2). In R-mod category, taking filtered colimits is exact. (Diagram chasing).

Prop. (1.5.3). The category of R-modules is a Grothendieck Abelian category with generator R.

Prop. (1.5.4). The category $\mathcal{A}^C$ is a Grothendieck category iff $\mathcal{A}$ is. Cf.[Tamme P8].

Cor. (1.5.5). The category of Abelian presheaves on a site is a Grothendieck topology.

The category of Abelian sheaves on a site is a Grothendieck topology. Cf.(1.1.3)

Prop. (1.5.6). The category of $\mathcal{O}_X$-modules on a scheme is a Grothendieck Abelian category.

Prop. (1.5.7). The category of abelian sheaves satisfies AB4 and AB3*, but doesn't satisfy AB4*, i.e. not every limit of epimorphisms is epimorphism.

Proof: Consider the zero extension constant sheaf on $B(0, \frac{1}{n})$. □

Prop. (1.5.8) (Injectives). In a Grothendieck Abelian category with generator U, an object is injective iff it is extendable over subobjects of U. Cf.[Sheaf Cohomology Notes1.2]

Prop. (1.5.9). Grothendieck Abelian category has a functorial injective embedding. Cf.[StackProject 079H].

6 Category Equivalence

Prop. (1.6.1). A Functor $\mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it's fully faithful and essentially surjective.Hom

Proof: There exist an object $G(X) \in \mathcal{C}$ and an isomorphism $\xi_X : FG(X) \to X$ for every $X \in \mathcal{D}$. Because F is fully faithful, there exists a unique morphism $G(f) : G(X) \to G(Y)$ such that $F(G(f)) = \xi_Y^{-1} \circ f \circ \xi_X$ for every morphism $f : X \to Y$ in $\mathcal{D}$. Thus we obtain a functor $G : \mathcal{C} \to \mathcal{D}$ as well as a natural isomorphism $\xi : F \circ G \to \mathrm{Id}_{\mathcal{D}}$. Moreover, the isomorphism $\xi_{F(Z)} : FGF(Z) \to F(Z)$ decides an isomorphism $\eta_Z : GF(Z) \to Z$ for every $Z \in \mathcal{C}$.This yields a natural isomorphism $\eta : G \circ F \to \mathrm{Id}_{\mathcal{C}}$. □

Prop. (1.6.2) (Yoneda Lemma). $X \mapsto (Y \mapsto \mathrm{Hom}(Y, X))$ is a fully faithful embedding from $\mathcal{C}$ to $\hat{\mathcal{C}} = \mathrm{Func}(\mathcal{C}^\circ, \mathrm{Set})$.

Prop. (1.6.3). For an Abelian category $\mathcal{A}$ satisfying AB3 (i.e arbitrary sum exists), An object P of $\mathcal{A}$ is called a projective generator if the functor $h' : X \mapsto \mathrm{Hom}_{\mathcal{A}}(P, X)$ is exact and and strict: $h'(X) = 0 \to X = 0$. Then h' determines an equivalence from $\mathcal{A}$ to mod-R, where $R = \mathrm{Hom}_{\mathcal{A}}(P, P)$.

Similarly, if $\mathcal{A}$ is an Abelian Noetherian category and P is a projective generator, then R is Noetherian and $\mathcal{A}$ is equivalent to the category of finitely generated R-categories.

Proof: Essentially surjective: construct using direct limit and cokernel.

Notice that $h'(X) \cong h'(X') \to X \cong X'$ by strictness and A4 axiom. So let $X = \mathrm{Coker}(P^{\oplus I}, P^{\oplus J})$,

$$\begin{aligned}\mathrm{Hom}(h'(X), h'(Y)) &= \mathrm{Hom}(\mathrm{Coker}(h'(P^{\oplus J}), h'(P^{\oplus I}), h'(Y)) \\ &= \ker(\mathrm{Hom}(h'(P^{\oplus J}), h'(Y)) \to \mathrm{Hom}(h'(P^{\oplus I}), h'(Y))) \\ &= \ker(h'(Y^{\Pi I}) \to h'(Y^{\Pi J})) \\ &= \mathrm{Hom}(X, Y)\end{aligned}$$

□

Cor. (1.6.4) (Morita Equivalence). The following are equivalent:

1. categories A-mod and B-mod are equivalent.
2. categories mod-A and mod-B are equivalent.
3. There exist a finitely generated projective generator P of mod-A that $B \cong \mathrm{End}_A P$.

7 Fiber Product

Prop. (1.7.1). We have $(X \times_E Y) \times_S (Z \times_F W) = (X \times_S Z) \times_{E \times_S F} (Y \times_S W)$.

Prop. (1.7.2). The diagonal map $X \to X \times_Y X$ is an isomorphism iff $X \to Y$ is monomorphism. (This is equivalent to $\mathrm{pr}_1 = \mathrm{pr}_2$).

8 General Category

Prop. (1.8.1) (Eckmann-Hilton argument). If $\circ$ and $\otimes$ is two unital binary operator that commutes:$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$, then they are equal and in fact commutative and associative. Cf.[Wiki].

Prop. (1.8.2). The group objects in the category of groups is abelian groups.

Proof: By Eckmann-Hilton argument, the category multiplication is the same as the group multiplication, so the unit is obviously the sam unit, thus the inverse. So the commutativity of m with inverse implies that it is abelian. □

Prop. (1.8.3). One should notice that the group object structure in any category $(m, id, i, X$ definition) is equivalent to a group structure on $\mathrm{Hom}(Y, X)$ that are preserve under composition with morphisms.

VI.2 Higher Category

1 Kan Complex

Prop. (2.1.1). The fact that any simplicial set X is a colimit of $\Delta[n]$ (1.2.3) is important in proving properties of constructions of simplicial set.

Prop. (2.1.2). A simplicial group is a Kan complex. In particular, simplicial abelian group and simplicial R-module are Kan complexes.

Proof: Cf.[Simplicial Homology Theory Jardine P12] □

2 Simplicial Set

Prop. (2.2.1). The Nerve functor N is a fully faithful functor from the category of small categories to the category of simplicial sets.

Prop. (2.2.2). A natural transformation will induce homotopic nerve map. thus a pair of adjoint functors will induce a simplicial homotopy between their nerve.

3 ∞-Algebras

Prop. (2.3.1).

- The category of functors from the $E_\infty = Fin_*$ (pointed finite sets) to Cat that

$$X([n]) \xrightarrow{\prod_{i=2}^n X(\rho^i)} \prod_{i=1}^n X([1]) \quad n \geq 0$$

  is equivalent to the category of symmetric unital monoidal categories $(X([1]))$.

- The category of functors from the Δ^{op} to Cat that

$$X([n]) \xrightarrow{\prod_{i=2}^n X(\rho^i)} \prod_{i=1}^n X([1]) \quad n \geq 0$$

  is equivalent to the category of symmetric unital monoidal categories $(X([1]))$. And it is symmetric iff it factors through Cut:$\Delta^{op} \to Fin_*$.

VI.3 Simplicial Homotopy Theory

1 Cyclic Homology Theory(欧阳恩林)

Dold-Kan Correspondence

Prop. (3.1.1). The normalized Moore complex gives an equivalence between the category of simplicial abelian group and the category of chain complex of abelian groups. And $NA_*, A_*, (A/DA_*)$ are all homotopically equivalent, $A_* \cong NA_* \oplus DA_*$. Cf.[Wiki hyperref] https://web.archive.org/web/20160913201635/http://people.fas.harvard.edu/~amathew/doldkan.pdf.

Proof: □

2 Homotopy Algebra

3 Topological Cyclic Homology(Scholze)

VI.4 Derived Algebraic Geometry

Chapter VII

Theoretical Physics

VII.1 Hamiltonian Mechanics

1 TBA

Prop. (1.1.1). Yang-Mills Field.

VII.2 Fluid Dynamics

VII.3 Quantum Mechanics

1 Schrodinger Equation

Prop. (3.1.1) (Axioms). The Schrodinger equation can be derived from the Dirac-von Neumann axioms:

The state of particals is a countable dimensional Hilbert space, and

- The observables of a quantum system are defined to be the (possibly unbounded) self-adjoint operators A on $\mathbb{H}$.
- The state φ of the quantum system is a unit vector of $\mathbb{H}$, up to scalar multiples.
- The expectation value of an observable A for a system in a state φ is given by the inner product $\langle\varphi, A\varphi\rangle$.
- (Unitarity) the time evolution of a quantum state according to the Schrodinger equation is mathematically represented by a unitary operator $U(t)$ (depends only on the state an relative time)(one-parameter subgroup).

Now that $\varphi(t) = \hat{U}(t)\varphi(t_0)$, so $\hat{U}(t)\varphi(t_0) = e^{-i\hat{\mathcal{H}}t}, \hat{\mathcal{H}}$ hermitian.

So now take derivative w.r.t t, we get $i\frac{d\varphi}{dt} = \hat{\mathcal{H}}\varphi$. By quantum correspondence principle, it is possible to derive the expression of $\hat{\mathcal{H}}$ by classical methods.

Prop. (3.1.2). The solution of a Schrodinger equation for a non Relativistic particle is assumed to be a Schwartz function (Vanish fast enough at infinity). The coefficients is assumed smooth enough to guarantee at least uniqueness and existence locally.

Prop. (3.1.3). The wave function on the (p,t) coordinates is the Fourier Transform of the wave function on the (x,t) coordinates, because the eigenstate of the p-operator $i\hbar\frac{\partial}{\partial x}$ is e^{ikx}, the coefficients of which is the value (probability) of the wave function of the (p,t) coordinates.

Prop. (3.1.4) (Schrodinger Uncertainty Principle). Set $\sigma_A = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$, then:

$$\sigma_A^2 2\sigma_B^2 \geq \left|\frac{1}{2}\langle\hat{A}\hat{B} + \hat{B}\hat{A}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right|^2 + \left|\frac{1}{2i}\langle[\hat{A},]\hat{B}]\rangle\right|^2.$$

(Derived from definition and Schwarz inequality, Cf.[Wiki]).

Cor. (3.1.5) (Heisenberg Uncertainty Principle).

$$\sigma_x\sigma_p \geq \frac{\hbar}{2}$$

Proof:

$$[x, i\hbar\frac{\partial}{\partial x}] = i\hbar$$

. □

Prop. (3.1.6) (Spectral Decomposition). In Quantum physics, one need to use spectral decomposition of the Hamiltonian operator. But at most cases, there are only countably many eigenstate and the eigenvalue has a lower bound and tends to infinity. In this case, $(\hat{H} + A)^{-1}$ is a compact operator thus by spectral theorem(3.7.13) the eigenstate of $\hat{H}$ forms a set of complete basis.

2 Calculations

Prop. (3.2.1) (Virial Theorem). For a system that $V(r) \sim r^n$, the average kinetic energy and the average potential energy has the relation :

$$2\langle T\rangle = n\langle V\rangle.$$

3 Spin

VII.4 Quantum Field Theory

VII.5 General Relativity

1 Basics

Prop. (5.1.1) (Maxwell's Equation). Normal Maxwell's equation reads:

$$\begin{cases} \mathrm{div}E = q & \text{(Coulomb's law)} \\ \mathrm{div}H = 0 & \text{(Gaussian law)} \\ \mathrm{curl}E = -\frac{1}{c}\frac{\partial H}{\partial t} & \text{(Faraday's law)} \\ \mathrm{curl}H = j + \frac{1}{c}\frac{\partial E}{\partial t} & \text{(Ampère-Maxwell law)} \end{cases}$$

where E is the magnetic field, H is the electric field, q the charge density, j the electric current.

In Minkowski space, we define the electromagnetic 2-form

$$F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta,$$

where $F_{i0} = E_i, \quad Fij = H_k$, and electric current J, $J^i = -j^i, J^0 = q$.

Maxwell's equation can be re-written as:

$$d^* F = J \qquad dF = 0.$$

Where $d^* = *d*$.

Proof: The Minkowski space is flat, the equivalence can be seen by direct calculation. □

VII.6 String Theory

Chapter VIII

Unknown

VIII.1 TBA

- regularity theorem for elliptic operator.
- facts about linear algebra.
- a right Kan fibration which is a weak equivalence is a trivial fibration.
- smooth irreducible representations of Weil group is admissible.
- fundamental class relation with Weil group
- conductor of a Weil representation is an integer
- $\mathcal{O}_\mathfrak{p} \cong \mathbb{Z}_p^d$.