7.2 Integrales Trigonométricas.

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x \quad \div \cos^2 x.$

$1 + \cot^2 x = \csc^2 x \quad \div \sin^2 x.$

I. Integrales de la forma $\int \sin^n x \cos^m x \, dx$

$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$

$u = \sin x, \; du = \cos x \, dx \qquad u = \cos x, \quad du = -\sin x \, dx$

Evalúe $\int \cos^5 x \, dx = \int \cos^4 x \underbrace{(\cos x \, dx)}^{du}$

Reescriba $\cos^4 x = (\cos^2 x)^2 = (1-\sin^2 x)^2$

$\cos^2 x = 1 - \sin^2 x$

$\int \cos^5 x \, dx = \int (1-\sin^2 x)^2 \underbrace{(\cos x \, dx)}_{du = \cos x \, dx}$

$u = \sin x$

$= \int (1-u^2)^2 \, du$

$= \int (1 - 2u^2 + u^4) \, du$

$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C.$

$\sin^3 x \neq \sin x^3$

$(\sin x)^3$

$= \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.}$

Aparte algún término sinx ó cosx

a. Potencias impares de seno o coseno.

Ejercicio 1: Evalúe.

a. $\int \cos^3 x \sin^6 x \, dx = \int \cos^2 x \sin^6 x (\cos x \, dx)$

$\int \underbrace{\cos^2 x}_{1-\sin^2 x} \sin^6 x \cos x \, dx$ ó $\int \cos^3 x \overset{impar}{\sin^5 x} \sin x \, dx$

$1-\cos^2 x$ ✗

$\cos^2 x = 1 - \sin^2 x$

$u = \sin x$

$du = \cos x \, dx$

$(a+b+c)d$

$ad + bd + cd.$

$= \int (1-\sin^2 x) \sin^6 x (\cos x \, dx)$

$= \int (1-u^2) u^6 \, du$

$= \int (u^6 - u^8) \, du$

$= \frac{1}{7} u^7 - \frac{1}{9} u^9 + C.$

$= \boxed{\frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C.}$

b. $\int \cos^5 x \sin^3 x \, dx = \int \cos^5 x (\sin^2 x) \sin x \, dx$

$\int \underbrace{\cos^4 x}_{(1-\sin^2 x)^2} \sin^3 x \cos x \, dx$ ✓ ó $\int \cos^5 x \underbrace{\sin^2 x}_{1-\cos^2 x} \sin x \, dx$ ✓

$u = \cos x$

$du = -\sin x \, dx$

$= \int \cos^5 x (1-\cos^2 x) \sin x \, dx$

$= -\int u^5 (1-u^2) \, du = \int (-u^5 + u^7) \, du$

$= -\frac{1}{6} u^6 + \frac{1}{8} u^8 + C.$

$= \boxed{-\frac{1}{6} \cos^6 x + \frac{1}{8} \cos^8 x + C.}$

b) Potencias pares de seno y coseno

$$\int \cos^2 x\, dx = \int \frac{1}{2} + \frac{1}{2}\cos 2x\, dx = \boxed{\frac{x}{2} + \frac{1}{4}\sin 2x + C.}$$

$$1 = \cos^2 x + \sin^2 x. \quad (1)$$

$$\underline{\cos(x+x) = \cos^2 x - \sin^2 x \quad (2)}$$

$$1 + \cos(2x) = 2\cos^2 x \Rightarrow \boxed{\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x}$$

(1) + (2)

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Ejercicio 2: Evalúe.

a $$\int_{-\pi}^{\pi} \sin^2 x\, dx = 2\int_0^{\pi} \sin^2 x\, dx = \frac{2}{2}\int_0^{\pi} (1 - \cos 2x)\, dx$$

PAR.

$$\int_{-\pi}^{\pi} \sin x\, dx = 0 \quad \text{impar.}$$

360°

$$\int_0^{\pi} (1 - \cos 2x)\, dx = x - \frac{1}{2}\sin 2x \Big]_0^{\pi} = \pi - \frac{1}{2}\cancel{\sin 2\pi}^{\,0} - 0 + \frac{\sin 0}{2}$$

$u = 2x \quad du = 2dx$

$$\int_{-\pi}^{\pi} \sin^2 x\, dx = \pi.$$

$$\cos^2 \square = \frac{1}{2}(1 - \cos 2\square)$$

b. $$\int \sin^2 x \cos^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)\frac{1}{2}(1 + \cos 2x)\, dx$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

doble

$$= \frac{1}{4}\int (1 - \cos^2 2x)\, dx$$

$$\cos^2(2x) = \frac{1}{2}(1 + \cos 4x) \quad = \frac{1}{4}\int \left(1 - \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\int\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right)dx$$

$$= \int \frac{1}{8} + \frac{1}{8}\cos 4x \quad dx.$$

$\int af\,dx$
$= aF + C$

$$= \boxed{\frac{x}{8} + \frac{1}{8\cdot 4}\sin 4x + C.}$$

II Forma $\int \tan^m x \sec^n x \, dx$

$(\tan x)' = \sec^2 x$ aparte | $(\sec x)' = \sec x \tan x$ aparte.

$u = \tan x$ | $u = \sec x$

$\sec^2 x = \tan^2 x + 1$ | $\tan^2 x = \sec^2 x - 1$

Ejercicio 3: Evalúe   Pág. 48.

a. $\int \tan^5 x \sec^4 x \, dx$

$\int \tan^5 x \underbrace{\sec^2 x}_{\tan^2 x + 1} (\sec^2 x\, dx)$; $\int \tan^4 x \sec^3 x (\tan x \sec x \, dx)$

$u = \tan x$ ✓ | $u = \sec x$ ✓

$(\tan^2 x)^2 = (\sec^2 x - 1)^2$

$\int \tan^5 x \sec^2 x (\sec^2 x\, dx)$

$\int \tan^5 x (\tan^2 x + 1)(\sec^2 x \, dx)$

$u = \tan^2 x$ ×
$du = 2\tan x \sec^2 x \, dx$ ×

$u = \tan x$ $du = \sec^2 x \, dx$

$$\int u^5(u^2+1)\,du = \int (u^7 + u^5)\,du = \frac{u^8}{8} + \frac{u^6}{6} + C.$$

$$= \boxed{\frac{1}{8}\tan^8 x + \frac{1}{6}\tan^6 x + C.}$$

Sharon: Resumen Identidades y luego sustitución.

b. Alejandro $\int \tan^5 x \sec^5 x \, dx$

$\int \tan^4 x \sec^4 x (\sec x \tan x \, dx)$ ✓

$\int \tan^5 x \sec^3 x (\sec^2 x \, dx)$ X

$\int (\tan^2 x)^2 \sec^4 x (\sec x \tan x \, dx)$ $\tan^2 x = \sec^2 x - 1$

$\int (\sec^2 x - 1)^2 \sec^4 x (\sec x \tan x \, dx)$

$u = \sec x$ $du = \sec x \tan x \, dx$.

$\int (u^2 - 1)^2 u^4 \, du = \int (u^4 - 2u^2 + 1) u^4 \, du$.

$\int (u^8 - 2u^6 + u^4) \, du = \frac{1}{9} u^9 - \frac{2}{7} u^7 + \frac{1}{5} u^5 + C$.

$= \frac{1}{9} \sec^9 x - \frac{2}{7} \sec^7 x + \frac{1}{5} \sec^5 x + C$.

c. $\int \tan^4 x \sec^4 x \, dx. = \int \tan^4 x \sec^2 x \sec^2 x \, dx$

sólo tan's.

$\sec^2 x = \tan^2 x + 1$ $= \int \tan^4 x (\tan^2 x + 1)(\sec^2 x \, dx)$

$u = \tan x$ $du = \sec^2 x \, dx$ $= \int u^4 (u^2 + 1) \, du$.

$= \int u^6 + u^4 \, du = \frac{1}{7} u^7 + \frac{1}{5} u^5 + C$

$= \frac{1}{7} \tan^7 x + \frac{1}{5} \tan^5 x + C$.

Casos especiales $\int \tan^m x\,dx \quad \int \sec^n x\,dx$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\ln|u| + C.$$

$$= -\ln|\cos x| + C.$$

$u = \cos x \qquad du = -\sin x\,dx$

$$\int \sec x\,dx = \int \sec x \frac{(\sec x + \tan x)^*}{\tan x + \sec x^*}\,dx$$ Brillante.

$$= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}\,dx$$

$u = \tan x + \sec x \qquad du = (\sec^2 x + \sec x \tan x)\,dx$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\tan x + \sec x| + C.$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C.$$

$$\int \sec^2 x\,dx = \tan x + C.$$

$\tan^2 x = \sec^2 x - 1$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C.$$

$$\int \tan^3 x\,dx = \int \tan^2 x \tan x\,dx$$

$$= \int (\sec^2 x - 1)\tan x\,dx$$

$u = \tan x$
$du = \sec^2 x\,dx$

$$= \int (\sec^2 x \tan x - \tan x)\,dx$$

$$= \int \underbrace{\tan x}_{u} \cdot \underbrace{\sec^2 x\,dx}_{du} - \int \tan x\,dx$$

$$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

Más Difícil $\int \sec^3 x \, dx$.

$$\int \sec x \sec^2 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx$$

IPP $u = \sec x$ $\qquad dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx$ $\qquad v = \tan x$

$$\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx$$

$$= \int (\sec^3 x - \sec x) \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| + C.$$

$$\int \sec^3 x \, dx = \frac{1}{2} \text{derivada}(\sec) + \frac{1}{2} \text{integral}(\sec)$$

$$\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.$$