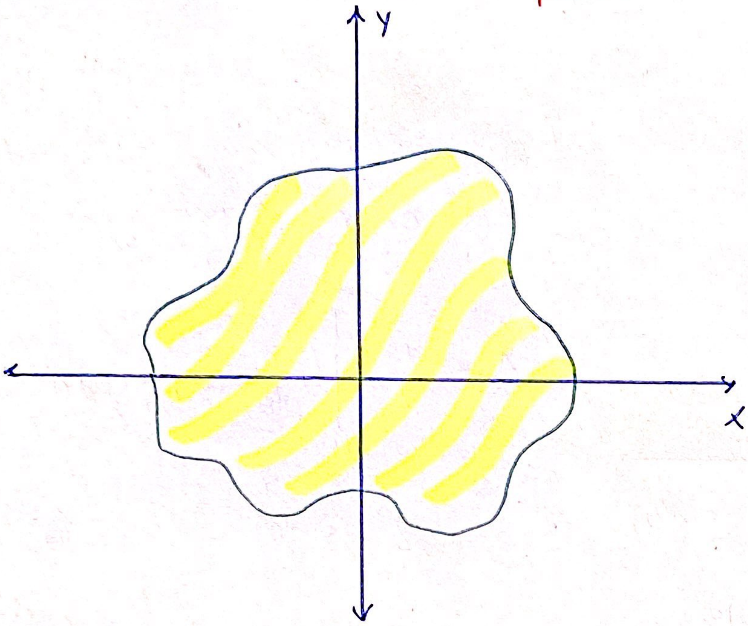
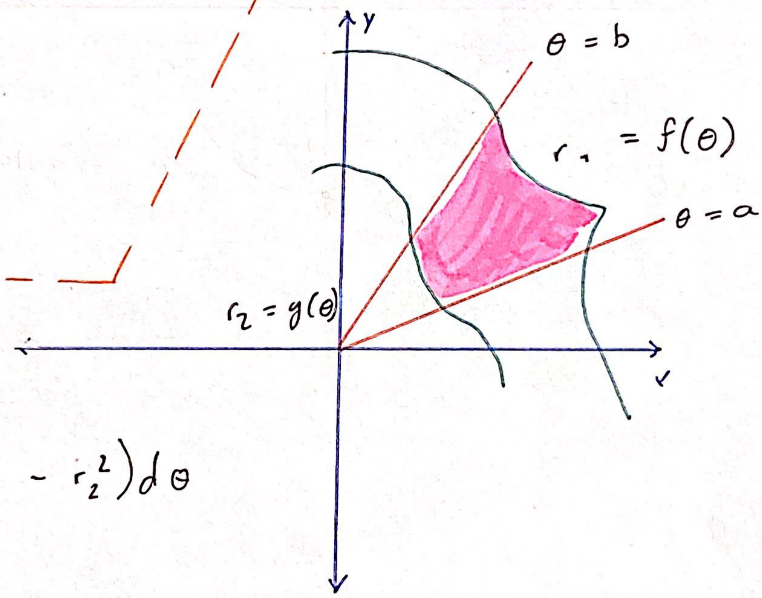
2019-11-06

Área de una región acotada por dos curvas polares



$$A = \frac{1}{2}\int_a^b r^2 \, d\theta$$

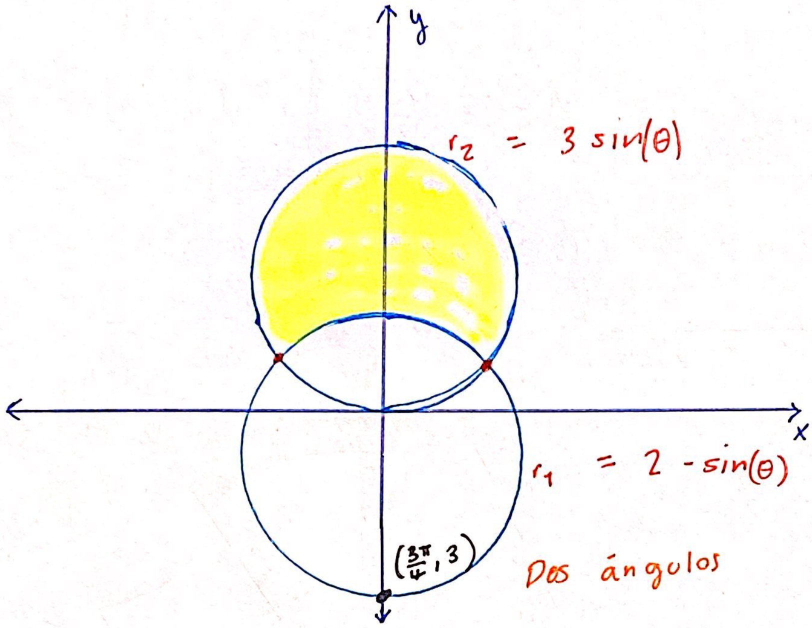


$$A = \frac{1}{2}\int_a^b \left(r_1^2 - r_2^2\right) d\theta$$

r_1 es la que está más alejada del origen

$R: \quad r_2 \leq r \leq r_1 \qquad a \leq \theta \leq b$

Ejemplo: Encuentre el área de la región fuera del limagón $r_1 = 2 - \sin\theta$ & entre el círculo



[r_2 está más alejada del orígen que r_1]

$$A = \frac{1}{2}\int_a^b (r_2^2 - r_1^2)\, d\theta$$

$$2 - \sin(\theta) = 3\sin(\theta)$$

$$2 = 4\sin(\theta)$$

$$\frac{1}{2} = \sin(\theta)$$

$$\theta = \frac{\pi}{6} \; ; \; \theta = \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 - (2 - \sin\theta)^2\, d\theta$$

Duplicar resultado

$$\ldots = \frac{2}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta - 1 + 2\sin\theta - \sin^2\theta \; d\theta$$

$$\ldots = \int_{\pi/6}^{\pi/2} (2\cdot\sin\theta - 4\cos(2\theta))\, d\theta = -2\cos\theta - 2\sin 2\theta \Big]_{\pi/6}^{\pi/2}$$

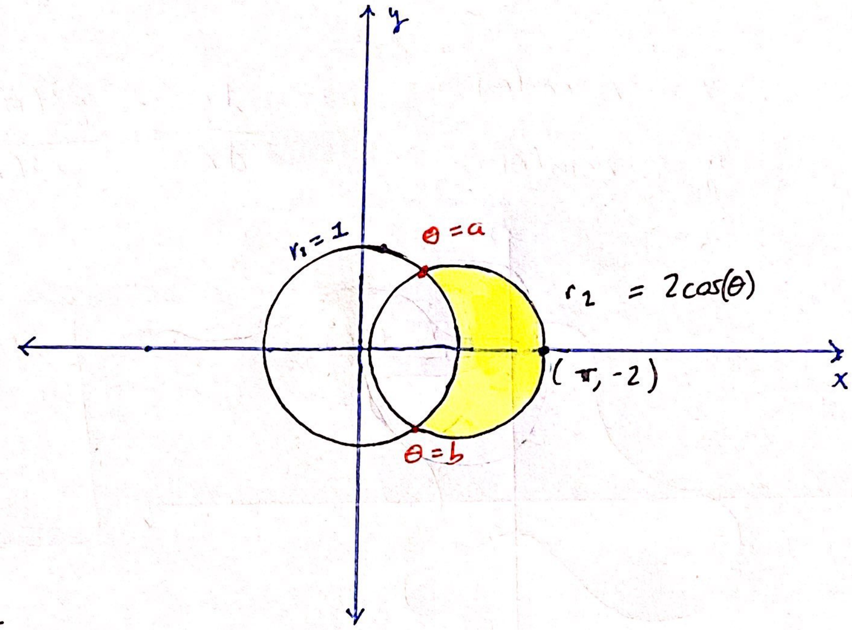
$$\ldots = -2\cos\left(\frac{\pi}{2}\right) + 2\cos\left(\frac{\pi}{6}\right) - \cancel{2\sin(\pi)}^{0} + 2\sin\left(\frac{\pi}{3}\right) = \ldots$$

$$\ldots = \frac{2\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} = \sqrt{3} + \sqrt{3} = \boxed{2\sqrt{3}}$$

Ejercicio 3: Considerar curvas $r_1 = 1$ & $r_2 = 2\cos(\theta)$.

a) Encontrar el área de la región que está adentro de $r_2 = 2\cos\theta$ & fuera de $r_1 = 1$

θ	r_2
$\emptyset$	2
$\frac{\pi}{2}$	$\emptyset$
π	-2
$\frac{\pi}{4}$	$\frac{2\sqrt{2}}{2} = \sqrt{2}$



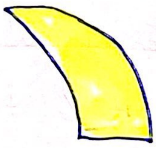
$x = r\cos(\theta)$

$= -2\cos\pi = 2$

Pts de intersección

$2\cos(\theta) = 1 \Rightarrow \cos(\theta) = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}\ ;\ -\frac{\pi}{3}\ ;\ \frac{5\pi}{2}$



$$A = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(r_2^2 - r_2^2\right) d\theta$$

Duplicamos & cambiamos

$$A = \int_0^{\frac{\pi}{3}} \left(1 \cdot \cos^2\theta - 1\right) d\theta = \theta + \sin(2\theta)\Big]_0^{\frac{\pi}{3}} = \ldots$$

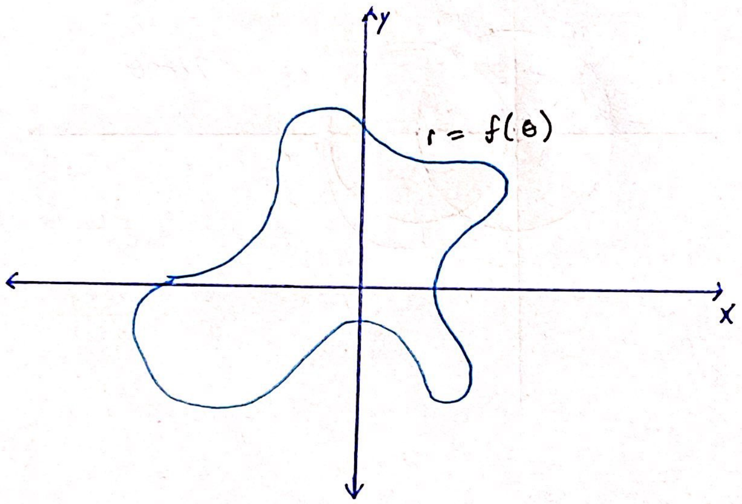
$$\ldots = \left[\left(\frac{\pi}{3} + \underbrace{\sin\left(2\frac{\pi}{3}\right)}_{\frac{\sqrt{3}}{2}}\right) - (\emptyset + \emptyset)\right] = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

Longitud de arco de curvas polares

La curva polar $r = f(\theta)$ se escribe en términos de sus ecuaciones paramétricas

$$x = r\cos(\theta) \qquad \frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)}$$

$$y = r\sin(\theta)$$



Longitud de arco paramétrica

$$\boxed{L = \int_a^b \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}\, d\theta}$$

$$x'(\theta) = r'\cos\theta - r\sin\theta$$

$$y'(\theta) = r'\sin\theta + r\cos\theta$$

$$[x'(\theta)]^2 = (r')^2\cos^2(\theta) - \cancel{2r\cdot r'\cos\theta\sin\theta}^{\,0} + r^2\sin^2\theta$$

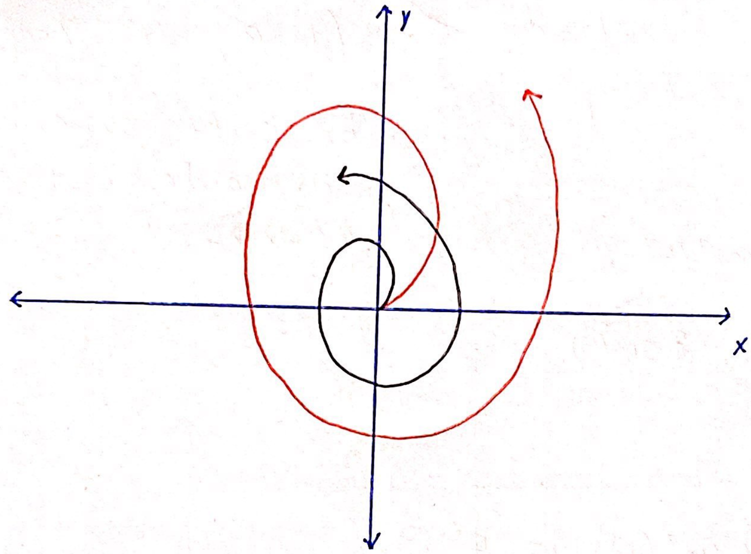
$$+ \quad [y'(\theta)]^2 = (r')^2\sin^2\theta + \cancel{2r\cdot r'\cos\sin\theta} + r^2\cos^2\theta$$

$$(x'(\theta))^2 + (y'(\theta)) = (r'(\theta))^2 + 0 + r^2$$

$$\boxed{\therefore L = \int_a^b \sqrt{r^2 + (r'(\theta))^2}\, d\theta}$$

para polares

c) La espiral $r = \theta^2$ $\quad 0 \le \theta \le \sqrt{\pi}$



$$\mathcal{L} = \int_0^{\sqrt{\pi}} \sqrt{r^2 + (r'(\theta))^2}\, d\theta$$

$$r = \theta^2 \Rightarrow r = \theta^4$$

$$r' = 2\theta \Rightarrow (r')^2 = 4\theta^2$$

$$\mathcal{L} = \int_0^{\sqrt{\pi}} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \dots$$

$$\dots = \int_0^{\sqrt{\pi}} \sqrt{\theta^2 + 4}\;\theta\, d\theta = \dots$$

$$\dots = \int_{u(0)}^{u(\sqrt{\pi})} u^{\frac{1}{2}} \frac{du}{2} = \frac{1}{3} u^{\frac{3}{2}} \Big]_{\pi}^{\pi+4} =$$

$u = \theta^2 + 4$

$u(\sqrt{\pi}) = \pi + 4$

$du = 2\theta\, d\theta$

$u(0) = \pi$

$$= \boxed{\frac{1}{3}\left[(\pi+4)^{\frac{3}{2}} - \left(\pi^{\frac{3}{2}}\right) \right]}$$

Derivadas paramétricas & polares, Áreas, Longitud de arco.

Jueves súper corto

Ej 4: Encuentre la longitud de arco de los sigs. problemas:

a) Circunferencia radio 1 centrada en $(1,0)$; $r = 2\cos\theta$; $0 \le \theta \le \pi$

b) Parte superior del cardioide $r = 1 + \cos\theta$; $0 \le \theta \le 2\pi$.

$$r = 2\cos\theta \implies r^2 = 4\cos^2(\theta)$$

$$r'(\theta) = -2\sin\theta \implies (r'(\theta))^2 = 4\sin^2(\theta)$$

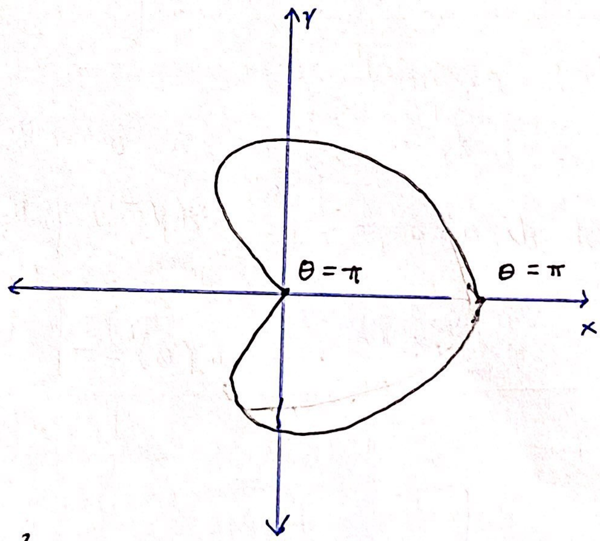
Simplificar:

$$r^2 + (r'(\theta))^2 = 4\underbrace{(\cos^2(\theta) + \sin^2(\theta))}_{1} = 4$$

Longitud de arco:

$$L = \int_0^{\pi} \sqrt{r^2 + (r'(\theta))^2}\, d\theta$$

$$= \int_0^{\pi} 2\, d\theta = 2\theta\Big]_0^{\pi} = \boxed{2\pi}$$



$$r^2 = (1 + \cos\theta)^2$$
$$= 1 + 2\cos(\theta) + \cos^2\theta$$

$$r'(\theta) = -\sin(\theta) \Rightarrow (r'(\theta)) = \sin^2\theta$$

$$r^2(\theta) + (r'(\theta))^2 = 2 + 2\cos(\theta)$$

$$L = \int_0^{\pi} \sqrt{2 + 2\cos(\theta)}\, d\theta = \int_0^{\pi} \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta = 4\sin\left(\frac{\theta}{2}\right)\Big]_0^{\pi} = 4 - 0 = \boxed{4}$$

recordar: $\cos^2\frac{\theta}{2} = \frac{1}{2} + \frac{1}{2}\cos(\theta)$