7.2. Integrales Trigonométricas

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$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x \quad \div \cos^2 x$

$1 + \cot^2 x = \csc^2 x \quad \div \sin^2 x$

Integrales de la forma $\int \sin^n x * \cos^m x \, dx$

$\frac{d}{dx}(\sin x) = \cos x$

$u = \sin x$

$du = \cos x \, dx$

$\frac{d}{dx}(\cos x) = -\sin x$

$u = \cos x$

$du = -\sin x \, dx$

Primer Caso = Se necesita una par y una impar

Evalúe $\int \cos^5 x \, dx = \int \cos^4 x \, (\cos x \, dx)$

Reescribir $\cos^4 x = (\cos^2 x)^2 = (1 - \sin^2 x)^2$

$\cos^2 x = 1 - \sin^2 x$

$\therefore \int \cos^5 x \, dx = \int (1 - \sin^2 x)^2 (\cos x \, dx)$

$u = \sin x \qquad du = \cos x \, dx$

$= \int (1 - u^2)^2 \, du$

$= \int (1 - 2u^2 + u^4) \, du = u - \frac{2}{3} u^3 + \frac{1}{5} u^5 + C$

$\therefore = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C \; \square$

Aparte algún término $\sin x$ o $\cos x$.

a. Potencias impares de seno o coseno.

Evalué

esto es un problema — Preferimos potencias pares

$$\int \cos^3 x \sin^6 x \, dx =$$

$$\int \cos^2 x \sin^6 x \cos x \, dx \quad \text{ó} \quad \int \cos^3 x \sin^5 x \sin x \, dx$$

$$= \int \cos^2 x \sin^6 x (\cos x) dx = \int (1-\sin^2 x) \sin^6 x \, (\cos x \, dx)$$

$\cos^2 x = 1 - \sin^2 x$

$u = \sin x$

$du = \cos x \, dx$

$$= \int (1-u^2) u^6 \, du$$

$$= \int u^6 - u^8 \, du$$

$$= \frac{1}{7} u^7 - \frac{1}{8} u^9 + C$$

$$= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C$$

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b. $\int \cos^5 x \sin^3 x \, dx =$

$$\int \cos^4 x \sin^3 x \cos x \, dx \quad \text{ó}$$

$$\int \cos^5 x \sin^2 x \sin x \, dx$$

$$= \int \cos^5 x (\sin^2 x) \sin x \, dx$$

$$= \int \cos^5 x (1 - \cos^2 x) \sin x \, dx$$

$u = \cos x \qquad du = -\sin x \, dx$

$$= - \int u^5 (1 - u^2) \, du$$

$$= -\frac{1}{6} u^6 + \frac{u^8}{8} + C$$

$$\therefore -\frac{1}{6} \cos^6 x + \frac{1}{8} \cos^8 x + C$$

b) Potencias pares de seno y coseno

$$\int \cos^2 x\, dx = \int (1-\sin^2 x)\, dx = \underline{\frac{x}{2} + \frac{1}{4}\sin 2x + C}$$

$$1 = \cos^2 x + \sin^2 x \quad (1)$$

$$+\ \cos(x+x) = \cos^2 x - \sin^2 x \quad (2)$$

suma(1 y 2) $1 + \cos(2x) = 2\cos^2 x \Rightarrow \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

Ejercicio potencias pares:

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

a. $$\int_{-\pi}^{\pi} \sin^2 x\, dx = 2\int_0^{\pi} \sin^2 x\, dx = \frac{2}{2}\int_0^{\pi} (1-\cos 2x)\, dx = x - \frac{1}{2}\sin 2x$$

si fuera impar sería 0

$u = 2x \quad du = 2dx$

$$= x - \frac{1}{2}\sin^2 x\Big]_0^{\pi} = \pi - \cancel{\frac{1}{2}\sin 2\pi} - 0 + \cancel{\frac{\sin 0}{2}}$$

$$= \pi$$

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b. $$\int \sin^2 x \cos^2 x\, dx$$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$= \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)\, dx$$

diferencia de cuadrados

$$\frac{1}{4}\int (1-\cos^2 2x)\, dx = \frac{1}{4}\int (1-\cos^2 2x)\, dx$$

$$\cos^2(2x) = \frac{1}{2}(1+\cos 4x)$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \int \frac{1}{8} + \frac{1}{8}\cos 4x\, dx = \frac{1}{8}x + \underline{\frac{1}{8\cdot 4}\sin 4x + C}$$

$\int a\, f\, dx$

$= aF + C$

Forma $\int \tan^m x \sec^n x\, dx$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\sec x = \sec x \tan x$

$u = \tan x$

$w = \sec x$

$\sec^2 x = \tan^2 x + 1$

$\tan^2 x = \sec^2 x + 1$

Ejercicio 3 Evalúe Pg. 48

1. $\int \tan^5 x \sec^4 x\, dx$

$\int \tan^5 x \sec^2 x (\sec^2 x\, dx)$ ó $\int \tan^4 x \sec^3 x (\tan x \sec x\, dx)$

$u = \tan x \quad \tan^2 x + 1$

$w = \sec x$

$(\tan^2 x)^2 = (\sec^2 x - 1)^2$

$\int \tan^5 x \sec^2 x (\sec^2 x\, dx)$

$\int \tan^5 x (\tan^2 x + 1)(\sec^2 x\, dx)$

$u = \tan^2 x$

$du = 2 \tan x \sec^2 x$

$\int u^5 (u^2 + 1)\, du = \int (u^7 + u^5)\, du = \frac{u^8}{8} + \frac{u^6}{6} + C$

$= \frac{1}{8} \tan^8 x + \frac{1}{6} \tan^6 x + C$

b. $\int \tan^5 x \sec^5 x \, dx =$

$\int \tan^4 x \sec^4 x (\sec x \tan x) dx$ ó $\int \tan^5 x \sec^3 x (\sec^2 x \, dx)$

$\int (\tan^2 x)^2 \sec^4 x (\sec x \tan x \, dx)$ $\qquad \tan^2 x = \sec^2 x - 1$

$\int (\sec^2 x - 1)^2 \sec^4 x (\sec x \tan x \, dx)$

$u = \sec x \qquad du = \sec x \tan x \, dx$

$\int (u^2-1)^2 u^4 \, du = \int (u^4 - 2u^2 + 1) u^4 \, du$

$= \int u^8 - 2u^6 + u^4$

$= \frac{u^9}{9} - \frac{2u^7}{7} + \frac{u^5}{5} + C$

$\frac{1}{9}\sec^9 x - \frac{2}{7}\sec^7 x + \frac{1}{5}\sec^5 x + C$

c. $\int \tan^4 x \sec^4 x \, dx$

$= \int \tan^4 x \underbrace{\sec^2 x}_{\tan x^?} \sec^2 x \, dx$

$\sec^2 x = \tan^2 x + 1$

$= \int \tan^4 x (\tan^2 x + 1) \sec^2 x \, dx$

$u = \tan x \qquad du = \sec^2 x$

$= \int u^4 (u^2 + 1) \, du$

$= \frac{1}{7}u^7 + \frac{1}{5}u^5 + C = \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$

Casos especiales $\int \tan^m x\, dx \qquad \int \sec^n x$

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln|u| + C$$

$w = \cos x$
$dw = -\sin x$

$$= -\ln(\cos x) + C$$

$$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} dx$$

$$= \int \frac{du}{u} = \ln|u| + C$$
$$= \ln|\tan x + \sec x| + C$$

$u = \tan x + \sec x$
$du = \sec^2 x + \sec x \tan x \; dx$

$$\int \csc x\, dx = -\ln|\csc + \cot x + C$$

$$\int \sec^2 x\, dx = \tan x + C$$

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$$

$$\int \tan^3 x\, dx = \int \tan^2 x \tan x\, dx = \int (\sec^2 x - 1) \tan x\, dx$$

$u = \tan x \quad du = \sec^2 x$

$$= \int \sec^2 x \tan x - \tan x\, dx$$

$$= \int \underset{u}{\tan x}\, \underset{du}{\sec^2 x\, dx} - \int \tan x\, dx$$

$$= \frac{1}{2} \tan^2 x + \ln|\cos x| + C$$

$$\int \sec^3 x \, dx = \int \sec^2 x \sec x \, dx$$

IPP $\quad u = \sec x \qquad dv = \sec^2 x \, dx$

$\quad du = \sec x \tan x \qquad v = \tan x$

$$= \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$= \int \tan^2 x \sec x \, dx = \int \sec^2 x - 1 \sec x \, dx$$

$$= \int \sec^3 x - \sec x \, dx$$

$$= \int \sec^3 x \, dx - \int \sec x \, dx$$

$$= \frac{\sec x \tan x + \ln|\sec x + \tan x| + C}{2}$$