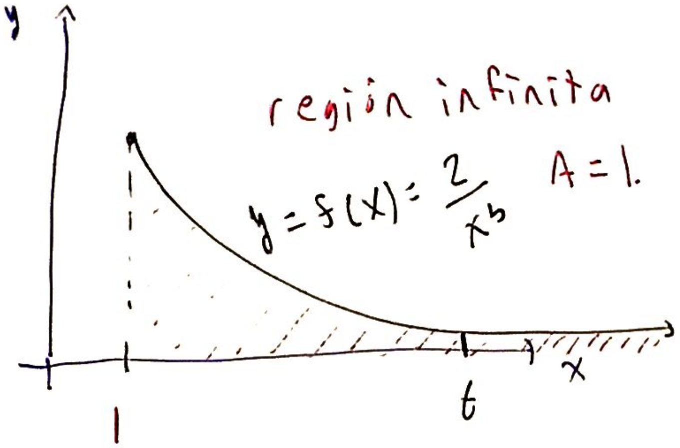
# 7.8 Integrales Impropias.

Considere la región bajo la curva $y = \frac{2}{x^3}$, encima del eje-$x$ y a la derecha de la recta $x = 1$.




$$A = \int_1^t 2x^{-3}\,dx.$$

$$A = \frac{2}{-2} x^{-2} \Big]_1^t$$

$$A = -\frac{1}{t^2} + \frac{2}{2}\frac{1}{1^2} = 1 - \frac{1}{t^2}.$$

A medida que $t$ aumenta. $\frac{1}{t^2} \to 0$ $\qquad \lim_{t\to\infty} A = \lim_{t\to\infty} 1 - \frac{1}{t^2} = 1$

Concluyendo $\int_1^\infty \frac{2}{x^3}\,dx = 1.$

Integral Impropia Tipo 1: (Pág 73.)

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

CONVERGENTE si el límite existe.

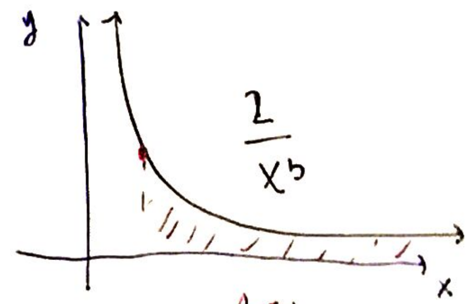
DIVERGENTE. si el límite no existe.

Ejercicio 1: Evalúe $\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$ y determine si la integral converge.

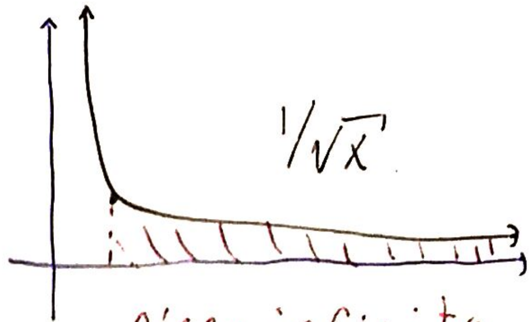
$$\int_1^{\infty} x^{-1/2}\, dx = 2x^{1/2}\Big]_1^{\infty} = \lim_{x\to\infty} 2x^{1/2} - 2 = +\infty.$$

$\sqrt{\infty} \to \infty.$

DIVERGENTE




$a=1$




Área infinita.

$f(x) \to 0$ a medida que $x \to \infty$ $\quad \int_a^{\infty} f(x)\, dx$ es posible que no exista.

Ej 2: Análisis de la integral $\int_1^{\infty} \frac{1}{x^p}\, dx$ (Pág. 79).

$$p=1 \quad \int_1^{\infty} \frac{1}{x}\, dx = \ln x \Big|_1^{\infty} = \lim_{x\to\infty} \underbrace{\ln x}_{+\infty} - \overset{0}{\ln(1)} = +\infty.$$

DIVERGE.

$$p=0.99 \quad \int_1^{\infty} x^{-0.99}\, dx = \frac{x^{0.01}}{0.01}\Big|_1^{\infty} = \lim_{x\to\infty} x^{0.01} - \frac{1}{0.01} = +\infty.$$

DIVERGE.

Reglas Límites $\quad \lim_{x\to\infty} x^r = +\infty$, $r>0$. $\quad \lim_{x\to\infty} \frac{1}{x^r} = 0.$

$$p=1.01 \quad \int_1^{\infty} x^{-1.01}\, dx = \frac{x^{-0.01}}{-0.01}\Big|_1^{\infty} = \lim_{x\to\infty} \frac{1}{-0.01}\,\frac{1}{x^{0.01}} + \frac{1}{0.01} = \frac{1}{0.01}$$

$0 + \frac{1}{0.01}$

CONVERGE

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{DIVERGE} & \text{si } p \le 1 \\ \text{CONVERGE} & \text{si } p > 1. \end{cases}$$

Ejercicio 3: Evalúe.

a. $$\int_{-\infty}^{0} e^{-x^2} x\,dx = \int_{-\infty}^{0} e^u \frac{du}{-2} = -\frac{1}{2} e^u \Big]_{-\infty}^{0} \qquad e^{-\infty} = \frac{1}{e^\infty} \to 0$$

$u = -x^2 \qquad u(0) = 0^2$

$du = -2x\,dx \qquad u(-\infty) = -\infty.$

$$= -\frac{1}{2} e^0 + \cancel{\lim_{u\to-\infty} \frac{1}{2} e^u}^{\,0}$$

$$= -\frac{1}{2}\cdot 1 = -\frac{1}{2}.$$

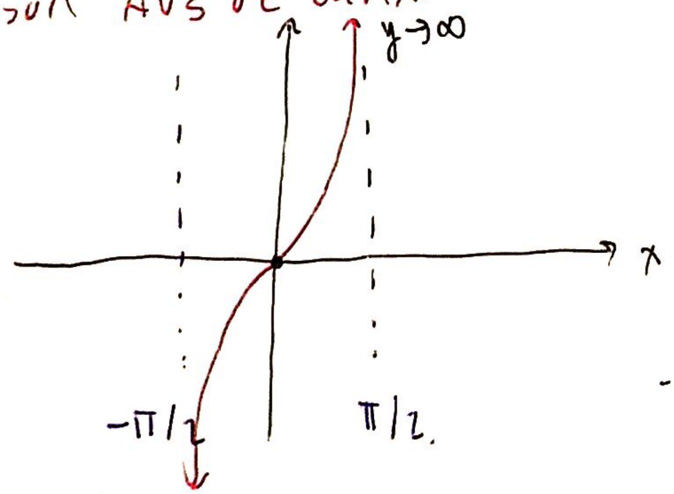
$$\int_{-\infty}^{0} e^{-x^2} x\,dx = -\frac{1}{2} \quad \text{converge.}$$

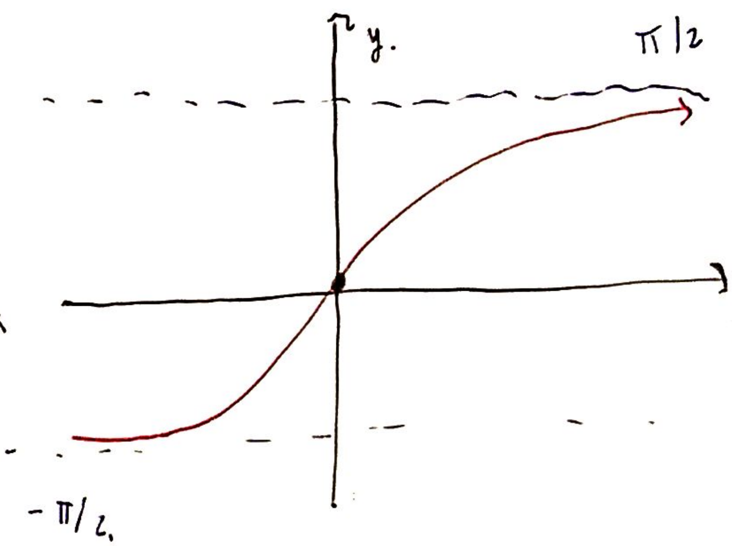
b. $$\int_{-\infty}^{\infty} \frac{2}{1+x^2}\,dx = 2\tan^{-1}(x)\Big]_{-\infty}^{\infty} = \pi + \frac{2\pi}{2} = 2\pi. \quad \text{CONVERGE}$$

$$= 2\underbrace{\lim_{x\to\infty} \tan^{-1}(x)}_{\pi/2} - 2\underbrace{\lim_{x\to-\infty} \tan^{-1}(x)}_{-\pi/2}.$$

AHs de $\tan^{-1}(x)$

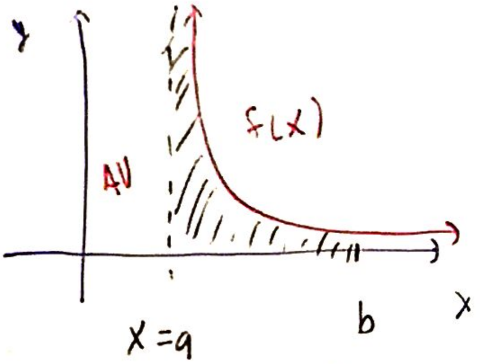
son AVs de $\tan x$.



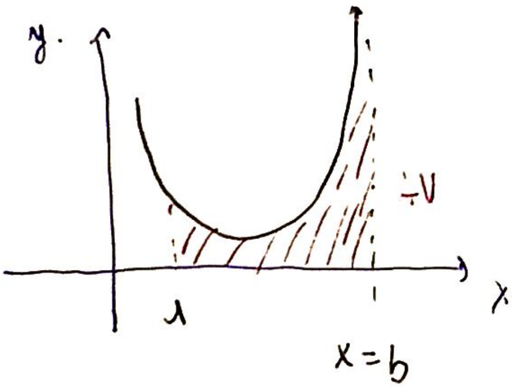
# Integrales impropias Tipo 2.

Hay una A.V en $x = a$.



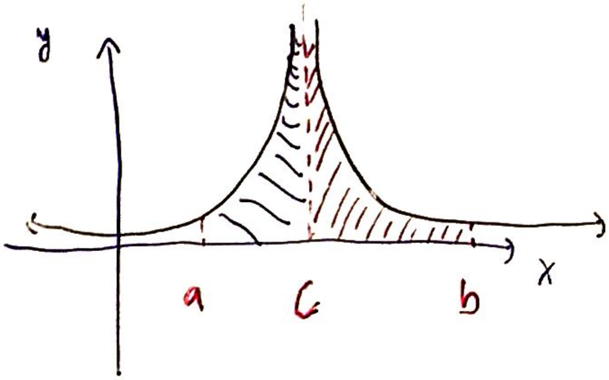

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx.$$

existe, es CONVERGENTE.




$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

AV en $x = c$ y está en medio del intervalo.




$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

2 integrales impropias.

Ejercicio 4: Evalúe las sigs. integrales. Indique donde son discontinuas.

a. $$\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^8 u^{-1/3}\,du. = \frac{3}{2} u^{2/3} \Big]_{0^+}^{8} = \frac{3}{2}(8)^{2/3} - \lim_{u \to 0^+} \frac{3}{2} u^{2/3}$$

$u = x - 1$ $\quad u(9) = 8$

$du = dx$ $\quad u(1) = 0$

indefinida en $x = 0$

$\frac{1}{0}$ indefinido.

$$\frac{3}{2}(64)^{1/3} - \frac{3}{2}0^{2/3} = \frac{3}{2}\cdot 4 - 0 = 6. \quad \text{converge.}$$

b. $$\int_{-2}^{3} \frac{3}{x^4}\,dx = \int_{-2}^{0} 3x^{-4}\,dx + \int_{0}^{3} 3x^{-4}\,dx$$

$\frac{1}{0}$ indefinida en $x=0$.

$$\int_{-2}^{0} 3x^{-4}\,dx = -x^{-3}\Big]_{-2}^{0^-} = \underbrace{\lim_{x\to 0^-} -\frac{1}{x^3}}_{+\infty.} + \frac{1}{(-2)^3}$$

No EXISTE.

DIVERGE.

DIVERGE $\to \pm\infty$.

CONVERGE $\to$ número, límite existe.

c. $$\int_{0}^{1} \ln x\,dx$$

$\ln 1 = 0$

indefinida en $x=0$.

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C.$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$\int_{0}^{1} \ln x\,dx = x\ln x - x\Big]_{0^+}^{1} = 1\cdot\cancel{\ln(1)}^{0} - 1 - \lim_{x\to 0^+} x\ln x - \cancel{x}^{0}$$

$$= -1 - \lim_{x\to 0^+} \cancel{x\ln x}^{0.} \qquad 0\cdot\infty$$

Regla de L'Hospital $$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \underset{LH}{=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}}$$

$$\lim_{x\to 0^+} -x = -0$$

$$\boxed{\int_0^1 \ln x\,dx = -1 \quad \text{CONVERGE}}$$

$0 \cdot \infty \neq 0$ $\quad \lim\limits_{\substack{x \to \infty \\ 0 \cdot \infty.}} \frac{1}{x} e^x = \lim\limits_{\substack{x \to \infty \\ \infty/\infty}} \frac{e^x}{x} \underset{LH}{\overset{\sim}{=}} \lim\limits_{x \to \infty} \frac{e^x}{1} = +\infty.$

Problema 2: $a(t) = 10t \cos\left(\frac{t}{2}\right)$

$v(t) = \int 10t \cos\left(\frac{t}{2}\right) dt. = 20t \sin\frac{t}{2} - \int 20 \sin\frac{t}{2}\, dt.$

$u = 10 \cdot t \qquad dv = \cos\left(\frac{t}{2}\right) dt. \qquad 20t\sin\frac{t}{2} + 40\cos\frac{t}{2} + C.$

$du = 10\, dt. \qquad v = 2\sin\left(\frac{t}{2}\right)$

a) Reposo $v(0) = 0.$ $\quad v(0) = 0 + 40 + C = 0$

$C = -40.$

b) $s(t) = \int \left(20t\sin\frac{t}{2} + 40\cos\frac{t}{2} - 40\right) dt.$

$s(0) = 5.$

Problema 6: $\int_1^e \frac{24 \ln^2 x}{(\ln^6 x + 1)^2} \frac{dx}{x} = \int_0^1 \frac{24u^2}{(u^6+1)^2}\, du.$

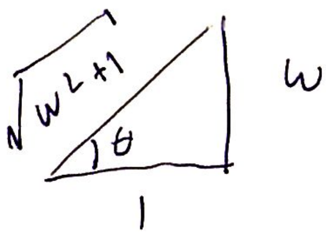
$u = \ln x \qquad u = \ln^3 x \qquad \ln^3 x = \tan\theta.$

$u^3 = \tan\theta.$

$du = \frac{dx}{x} \qquad u(e) = 1, \quad u(1) = 0$

$w = u^3 \qquad w(1) = 1 \qquad \frac{24}{3} \int_0^1 \frac{1}{(w^2+1)^{4/2}} \frac{dw}{3.}$

$dw = 3u^2 du \qquad w(0) = 0$




$w = \tan\theta.$

$dw = \sec^2\theta . d\theta.$

$\sqrt{w^2+1} = \sec\theta.$

$$\frac{24}{3}\int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{\sec^4\theta} = 8\int_0^{\pi/4} \cos^2\theta \, d\theta$$

$$= 4\int_0^{\pi/4} (1+\cos 2\theta) \, d\theta$$

$$= 4\left( \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} \right)$$

$$= \frac{4\pi}{4} + 2\sin\frac{\pi}{2} - 0 - 0$$

$$= \pi + 2.$$