7.2 Integrales Trigonométricas.

Algunas integrales requieren el uso de identidades trigonométricas.

$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x$

$\div \cos^2 x$ $\quad \tan^2 x + 1 = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$\div \sin^2 x$ $\quad 1 + \cot^2 x = \csc^2 x \Rightarrow \cot^2 x = \csc^2 x - 1$

I. Forma $\int \sin^n x \cos^m x \, dx$.

II. Forma $\int \tan^n x \sec^m x \, dx$

III. Forma $\int \cot^n x \sec^m x \, dx$

Ia) Potencias Impares de Seno y Coseno.

$\int \sin^3 x \, dx. = \int \sin^4 x \sin x \, dx$

$u = \cos x$
$du = -\sin x \, dx$

$\sin^4 x = (\sin^2 x)^2 = (1 - \cos^2 x)^2$

Sustitución.
$u = \cos x$
$du = -\sin x \, dx$

$= \int (1 - \cos^2 x)^2 \sin x \, dx$

$= \int (1 - u^2)^2 (-du)$

$\cos^3 x = \cos x^3$

$= -\int (1 - 2u^2 + u^4) \, du.$

$= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C.$

$$= \boxed{-\cos x + \frac{2}{3}(\cos x)^3 - \frac{1}{5}\cos^5 x + C.}$$

Ejercicio 1: Evalúe Pág. 46.

a. $\int \cos^3 x \sin^6 x \, dx$

$\int \cos^2 x \sin^6 x \cos x \, dx$ ó $\int \cos^3 x \sin^5 x \sin x \, dx$

$\cos^2 x = 1 - \sin^2 x$ $\quad$ $u = \sin x$ $\quad du = \cos x \, dx$

$$\int \cos^2 x \sin^6 x \cos x \, dx = \int (1-\sin^2 x) \sin^6 x \, (\cos x \, dx)$$
$$= \int (1-u^2) u^6 \, du.$$
$$= \int (u^6 - u^8) \, du.$$
$$= \frac{1}{7} u^7 - \frac{1}{9} u^9 + C.$$
$$= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C.$$

b. $\int \cos^3 x \sin^3 x \, dx$

$\int \cos^2 x \sin^3 x \cos x \, dx$ ó $\int \cos^3 x \sin^2 x \sin x \, dx$

$\cos^2 x = 1 - \sin^2 x$ $\quad$ $\sin^2 x = 1 - \cos^2 x$

$\int (1-\sin^2 x) \sin^3 x \cos x \, dx$

$u = \sin x \quad du = \cos x \, dx$

$\int (1-u^2) u^3 \, du.$

$\int (u^3 - u^5) \, du$

$\frac{1}{4} u^4 - \frac{1}{6} u^6 + C.$

$\frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C.$

$\int \cos^3 x (1-\cos^2 x) \sin x \, dx$

$u = \cos x \quad du = -\sin x \, dx$

$-\int u^3 (1-u^2) \, du.$

$\int -u^3 + u^5 \, du$

$-\frac{1}{4} u^4 + \frac{1}{6} u^6 + C.$

$-\frac{1}{4} \cos^4 x + \frac{1}{6} \cos^6 x + C.$

Siempre y cuando haya un término impar de seno y coseno.

1b) Potencias Pares de seno y Coseno.

$$\int \cos^2 x\, dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \boxed{\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C.}$$

Utilice las identidades de doble ángulo.

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\cos(x+x) = -\sin x \sin x + \cos x \cos x$$
$$\cos 2x = -\sin^2 x + \cos^2 x.$$
$$1 = \sin^2 x + \cos^2 x$$

$$1+\cos(2x) = 2\cos^2 x.$$

Ejercicio 2: Evalúe. $(1-\cos 2x)(1+\cos 2x)$

a. $$\int \sin^2 x \cos^2 x\, dx = \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)\, dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x)\, dx$$

Vuelva a utilizarla

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x) \qquad = \frac{1}{4}\int \left(\underbrace{1-\frac{1}{2}}_{1/2} + \frac{1}{2}\cos(4x)\right) dx$$

$$= \frac{1}{4}\left(\frac{x}{2} + \frac{1}{2\cdot 4}\sin(4x) + C\right)$$

$$= \frac{x}{8} + \frac{1}{32}\sin(4x) + \frac{C}{4}$$

b. $\int_{-\pi/4}^{\pi/4} \sin^2 x\, dx = 2\int_0^{\pi/4} \sin^2 x\, dx$

PAR.

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$= \int_0^{\pi/4} (1-\cos 2x)\, dx$$

$$= x - \frac{1}{2}\sin 2x \Big]_0^{\pi/4}$$

$$= \frac{\pi}{4} - \frac{1}{2}\overset{1}{\cancel{\sin}}\frac{\pi}{2} - 0 + \frac{1}{2}\overset{0}{\cancel{\sin}}\,0$$

$$= \boxed{\frac{\pi}{4} - \frac{1}{2}.}$$

II. Integrales de la forma $\int \sec^m x \tan^n x\, dx$

$\frac{d}{dx}\tan x = \sec^2 x\, dx$ $\qquad$ $\frac{d}{dx}\sec x = \sec x \tan x$

$\sec^2 x = \tan^2 x + 1$ $\qquad$ $\tan^2 x = \sec^2 x - 1$

Aparte $\sec^2 x$ ó $\sec x \tan x$

Utilice las identidades: $\sec^2 x = \tan^2 x + 1$

ó $\tan^2 x = \sec^2 x - 1$

Ejercicio 3: Evalúe las sigs. integrales.

a. $\int \tan^6 x \sec^4 x\, dx = \int \tan^6 x\, \sec^2 x\,(\sec^2 x\, dx)$

$\int \tan^6 x \sec^2 x\ \sec^2 x\, dx$ ✓ $\quad$ ¿ $\int \tan^5 x \sec^5 x\ \overline{\sec x \tan x\, dx}$ ✗

Use $\sec^2 x = \tan^2 x + 1$

$\int \tan^6 x\,(\tan^2 x + 1)(\sec^2 x\,dx)$

$u = \tan x \qquad du = \sec^2 x\,dx$

$\int u^6 (u^2+1)\,du = \int u^8 + u^6\,du$

$= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C.$

$= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C.$

b. $\int \tan^5 x \sec^6 x\,dx$

$\int \tan^5 x\,\underbrace{\sec^4 x}\,(\sec^2 x\,dx)$ ¿ $\int \tan^4 x \sec^5 x(\sec x \tan x\,dx)$

$(\sec^2 x)^2 = (1+\tan^2 x)^2$ ✓ $\qquad (\tan^2 x)^2 = (\sec^2 x - 1)^2$

$\int \underbrace{\tan}_{u=\tan x}{}^5 x\,(1+\tan^2 x)^2\,\underbrace{(\sec^2 x\,dx)}_{du = \sec^2 x\,dx}$

$\int u^5(1+u^2)^2\,du = \int u^5(1+2u^2+u^4)\,du$

$= \int (u^5 + 2u^7 + u^9)\,du.$

$= \frac{1}{6}u^6 + \frac{2}{8}u^8 + \frac{1}{10}u^{10} + C.$

$= \boxed{\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^8 x + \frac{1}{10}\tan^{10} x + C.}$

c. $\int \tan^3 x \sec^3 x\,dx = \int \tan^2 x \sec^2 x(\sec x \tan x\,dx)$

$\tan^2 x = \sec^2 x - 1$

$= \int (\sec^2 x - 1)\sec^2 x(\sec x \tan x\,dx)$

$u = \sec x$

$du = \sec x \tan x\,dx$

$= \int (u^2-1)u^2\,du.$

$$= \int (u^4 - u^2)\, du.$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C.$$

$$= \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.}$$

$\cdot\, -\ln a = \ln(a^{-1})$

Casos especiales $\quad \int \sec^n x\, dx \quad \int \tan^n x\, dx$

a. $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{du}{u} = -\ln|u| + C.$

$u = \cos x \quad du = -\sin x\, dx$

$= -\ln|\cos x| + C.$

$= \ln|\sec x| + C.$

b. $\int \sec x\, dx = \int \frac{\sec x \overbrace{(\sec x + \tan x)}^{1}}{\tan x + \sec x}\, dx$

$\int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}\, dx$

$u = \tan x + \sec x \qquad du = (\sec^2 x + \sec x \tan x)\, dx$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \boxed{\ln|\tan x + \sec x| + C.}$$

c. $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \boxed{\tan x - x + C.}$

d. $\int \sec^2 x\, dx = \boxed{\tan x + C.}$

e. $\int \tan^3 x\, dx = \int \tan x(\tan^2 x)\, dx \qquad \int f - g\, dx$

$= \int \tan x(\sec^2 x - 1)\, dx \qquad \int f\, dx - \int g\, dx$

$$= \int \underbrace{\tan x}_{u} \underbrace{\sec^2 x\, dx}_{du} - \int \tan x\, dx \qquad \frac{\sin}{\cos}$$

$w = \cos x$

$$= \frac{1}{2} u^2 \qquad + \ln|w| + c$$

$$= \frac{1}{2} \tan^2 x \qquad + \ln|\cos x| + C.$$

Integral de $\sec^3 x$

$$\int \sec x \sec^2 x\, dx = \sec x \tan x - \int \underbrace{\tan^2 x} \sec x\, dx$$

IPP.

$u = \sec x$ $\qquad du = \sec^2 x\, dx$ $\qquad \tan^2 x = \sec^2 x - 1$

$du = \sec x \tan x\, dx$ $\qquad v = \tan x$

$$\int \sec^3 x\, dx = \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx$$

$$\underline{\int \sec^3 x\, dx} = \sec x \tan x - \underline{\int \sec^3 x\, dx} + \int \sec x\, dx$$

cíclica.

$$2 \int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

$$2 \int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| + C.$$

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

$$= \frac{1}{2} \text{derivada}(\sec) + \frac{1}{2} \text{integral}(\sec)$$