Cálculo con ecuaciones paramétricas.

a. Primera Derivada $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$

b. Tangentes Horizontales: $y'(t) = 0$ $x'(t) \neq 0.$

Tangentes Verticales: $x'(t) = 0$ $y'(t) \neq 0.$

c. Segunda Derivada. P. 139.

Dada la curva $\mathcal{C}$: $x = f(t)$ $y = g(t)$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{d}{dt}(y)}{x'(t)} \qquad \frac{d^2y}{dx^2} = \frac{d(dy/dx)}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{x'(t)}. \quad , \quad \frac{d^3y}{dx^3} = \frac{\frac{d}{dt}(d^2y/dx^2)}{x'(t)}$$

$x = f(t)$ $y = dy/dx$

Ejercicio 4: Encuentre la primera, segunda y tercera derivada de las sigs. curvas paramétricas.

a. $x = 3t^2$ $y = t^3 + 3t^6.$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2 + 18t^5}{6t.} = 0.5t + 3t^4$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{x'(t)} = \frac{0.5 + 12t^3}{6t.} = \frac{1}{12t} + 2t^2$$

$$\frac{d^3y}{dx^3} = \frac{\frac{d}{dt}\left(d^2y/dx^2\right)}{x'(t)} = \frac{1}{6t}\left(\frac{-1}{12t^2} + 4t\right)$$

b. $x = e^t \qquad y = te^t$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{1e^t + te^t}{e^t} = 1 + t.$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(dy/dx\right)}{x'(t)} = \frac{1}{e^t} = e^{-t} = \frac{1}{x}$$

$$\frac{d^3y}{dx^3} = \frac{\frac{d}{dt}\left(d^2y/dx^2\right)}{x'(t)} = \frac{-e^{-t}}{e^t} = \frac{-1}{e^{2t}}$$

c. $x = \cos\theta \qquad y = \cos 2\theta. \qquad -1 \leq x \leq 1$

$$\cos^2\theta = \frac{1}{2}\left(1 + \cos 2\theta\right)$$

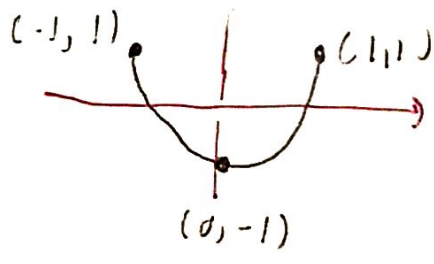
$$2x^2 = 1 + \cos 2\theta.$$

$\cos 2\theta = 2x^2 - 1$

$y = 2x^2 - 1$

$$\frac{dy}{dx} = 4x$$

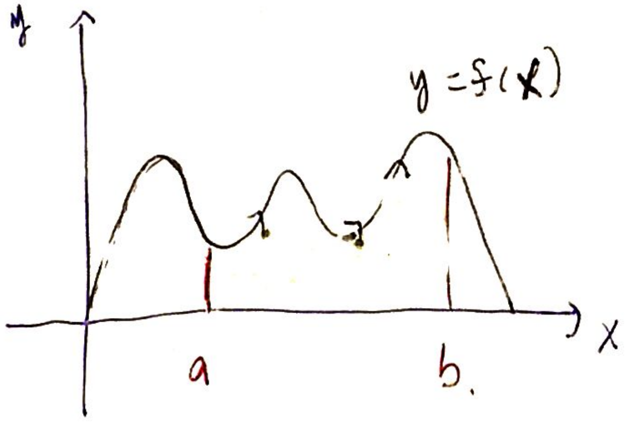
$$\frac{d^2y}{dx^2} = 4 \qquad \frac{d^3y}{dx^3} = 0$$



Paramétricas $\dfrac{dy}{dx} = \dfrac{-2\sin 2\theta}{-\sin\theta.} = \dfrac{4\sin\theta\cos\theta}{\sin\theta} = 4\cos\theta$

$$\frac{d^2y}{dx^2} = \frac{-4\sin\theta.}{-\sin\theta} = 4.$$

Área de una Región encerrada por una curva.



$$A = \int_a^b y\,dx.$$

$\left(A = \int_c^d x\,dy \right)$ no se usa mucho.

$\mathcal{C}$: $x = f(t)$ $\quad dx = f'(t)\,dt.$

$y = g(t).$

$t_1 \le t \le t_2.$

Reescriba

$$\boxed{A = \int_a^b y\,dx = \int_{t_1}^{t_2} g(t)\,f'(t)\,dt}$$

Ejercicio 6: p.141 Encuentre el área de la región dada.

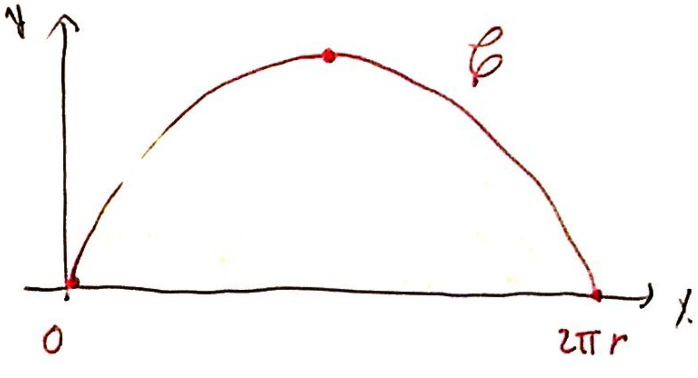
a. La región debajo de un arco de la cicloide

$x = r(\theta - \sin\theta)$

$y = r(1-\cos\theta)$

$0 \le \theta \le 2\pi.$

r es constante



$$A = \int_0^{2\pi} y\,dx = \int_0^{2\pi} r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta.$$

$dx = r(1-\cos\theta)\,d\theta.$

$$A = r^2 \int_0^{2\pi} (1-\cos\theta)^2\,d\theta.$$

$$A = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta.$$

Identidad Doble Ángulo: $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta.)$

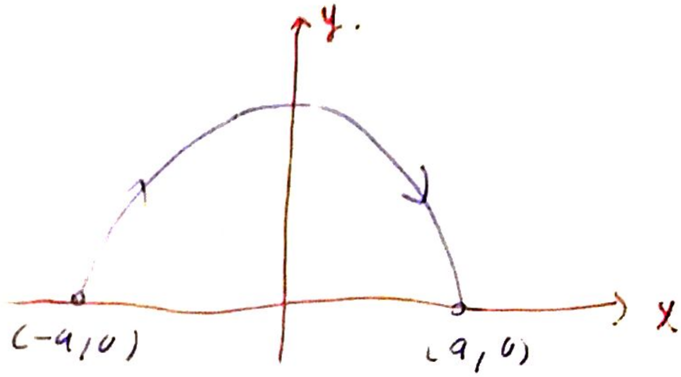
$$A = r^2 \int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta.$$

$$A = r^2 \left(\frac{3\theta}{2} - 2\sin\theta, + \frac{1}{2\cdot 2}\sin 2\theta \Big]_0^{2\pi} \right) \qquad \sin 0 = 0$$

$$A = r^2 \left(\frac{3\cdot 2\pi}{2} - 2\cancelto{0}{\sin 2\pi} + \frac{1}{4}\cancelto{0}{\sin 4\pi} - 0 \right)$$

$\boxed{A = 3\pi r^2.}$ 3 veces el área del círculo πr^2.

b. Media elipse.



$x = -a\cos\theta$
$y = b\sin\theta.$
$0 \leq \theta \leq \pi.$

$$A = \int_0^{\pi} y\, dx = \int_0^{\pi} b\cdot\sin\theta\; a\cdot\sin\theta\, d\theta. = ab\int_0^{\pi} \sin^2\theta\, d\theta.$$

$dx = a\cdot\sin\theta\, d\theta$

$$A = \frac{ab}{2}\int_0^{\pi} 1 - \cos 2\theta\; d\theta. = \frac{ab}{2}\left(\theta - \frac{1}{2}\sin 2\theta \Big]_0^{\pi}\right)$$

$$A = \frac{ab}{2}\left(\pi - \frac{1}{2}\cancelto{0}{\sin 2\pi} - 0\right) = \frac{\pi ab}{2} \quad \text{½ Elipse.}$$

Área toda la elipse. $A = \pi ab.$

de un círculo: $A = \pi a^2.$ $a = b.$

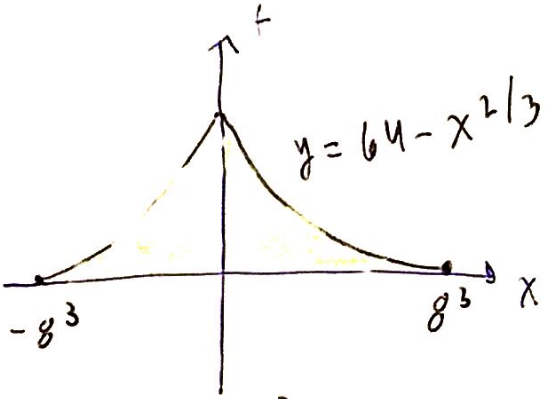
d. $\mathcal{G}$: $x = t^3$, $y = 64 - t^2$ y el eje-x.

Elimine el parámetro ó $\int y\,dx$ $dx = x'(t)\,dt.$

$t = x^{1/3} \Rightarrow y = 8^2 - x^{2/3}.$

Intersecciones-x: $y = 0$: $x^{2/3} = 8^2$ $\left(\ ^{2/3}\right)^{3/2}.$

$x = (8^2)^{3/2} = \pm 8^3$



$$A = \int_{-8^3}^{8^3} y\,dx$$

$$A = \int_{-8^3}^{8^3} (8^2 - x^{2/3})\,dx$$

$$A = 2\int_0^{8^3} (8^2 - x^{2/3})\,dx = 2\left(8^2 x - \frac{3}{5}x^{5/3}\Big]_0^{8^3}\right)$$

$$A = 2\left(8^5 - \frac{3}{5}(8^3)^{5/3}\right) = 2\left(8^5 - \frac{3}{5}8^5\right) = 2\cdot 8^5\left(1 - \frac{3}{5}\right)$$

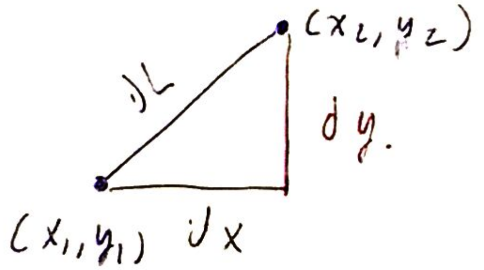
soln 2: $y = 64 - t^2 = 0 \Rightarrow t^2 = 64 \Rightarrow t = \pm 8$

$$A = \int_{-8}^{8} y\,x'(t)\,dt. = \int_{-8}^{8} (64 - t^2)\,3t^2\,dt.$$

$$A = 2\int_0^8 64\cdot 3t^2 - 3t^4\,dt = 2\left(8^2\cdot t^3 - \frac{3}{5}t^5\Big]_0^8\right)$$

$$A = 2\left(8^5 - \frac{3}{5}8^5\right)$$

e. Longitud de Arco.



$$dL = \sqrt{(dx)^2 + (dy)^2}$$

$$\mathcal{C}: \quad x = f(t) \qquad y = g(t)$$

$$dL = \sqrt{(dx)^2 + (dy)^2}\,\frac{dt}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Longitud de Arco de una curva $\mathcal{C}: x = f(t),\ y = g(t)$

en $a \le t \le b$. $\quad L = \displaystyle\int_a^b \sqrt{(x')^2 + (y')^2}\, dt.$

cartesiana: $\quad L = \displaystyle\int_a^b \sqrt{1 + (y')^2}\, dx$

Ejercicio 7: Encuentre la longitud exacta de la curva dada.

a. Longitud de una circunferencia de radio 4.

$x = 4\cos\theta, \quad y = -4\sin\theta \quad , \quad 0 \le \theta \le 2\pi.$

$x'(\theta) = -4\sin\theta \qquad y'(\theta) = -4\cos\theta. \qquad y' = \sqrt{16 - x^2}$

$(x')^2 + (y')^2 = 16\sin^2\theta + 16\cos^2\theta = 16.$

$$L = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\, d\theta = 4\int_0^{2\pi} d\theta = 4\theta\Big]_0^{2\pi} = 8\pi.$$

c. $x = e^t \cos t \qquad y = e^t \sin t \qquad 0 \leq t \leq \ln 2$

Use la Regla del Producto.

$x'(t) = e^t \cos t - e^t \sin t. \qquad e^t e^t = e^{2t}$

$y'(t) = e^t \sin t + e^t \cos t. \qquad \sin t \sin t = \sin^2 t$

$(x')^2 = e^{2t}\cos^2 t - 2e^{2t}\cos t \sin t + e^{2t}\sin^2 t$

$(y')^2 = e^{2t}\sin^2 t + 2e^{2t}\cos t \sin t + e^{2t}\cos^2 t.$

$(x')^2 + (y')^2 = e^{2t} + 0 + e^{2t} = 2e^{2t}.$

$$L = \int_0^{\ln 2} \sqrt{(x')^2 + (y')^2}\, dt. = \int_0^{\ln 2} (2e^{2t})^{1/2}\, dt.$$

$$L = 2^{1/2}\int_0^{\ln 2} e^t\, dt. = 2^{1/2} e^t \Big]_0^{\ln 2}$$

$$L = 2^{1/2}(e^{\ln 2} - e^0) = \sqrt{2}(2-1) = \sqrt{2}$$