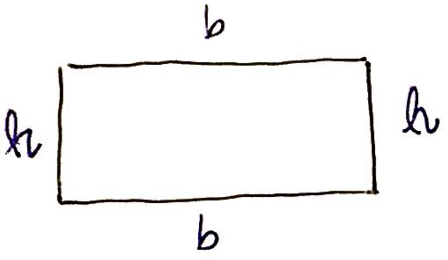
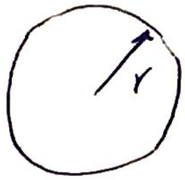
8.1 Longitud de una curva.

Geometría: longitud o perímetro



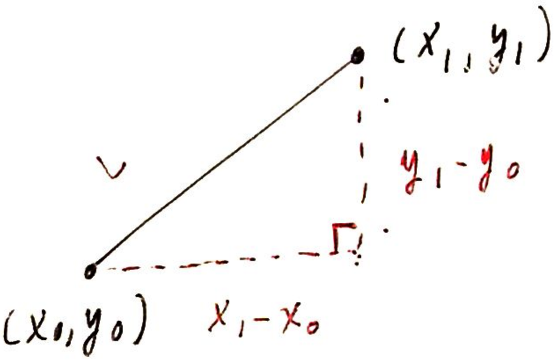


Rectángulo $L = 2b + 2h$

Circunferencia

$L = 2\pi r.$

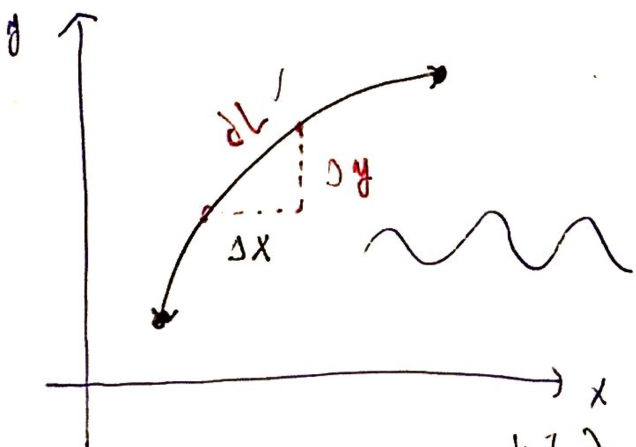
Segmento de recta



$$L = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Longitud de una curva: longitud de arco.



curva $\mathcal{C}$: $y = f(x)$

$a \leq x \leq b.$

región $\mathcal{R}$ sólido S.

parte infinitesimal del arco.

$$dL = \sqrt{(dx)^2 + (dy)^2}$$

$$dx^2 + dy^2 = dx^2\left(1 + \frac{dy^2}{dx^2}\right)$$

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Longitud L: integre dL en $a \leq x \leq b$.

Longitud de Arco:

$$\boxed{L = \int_a^b \sqrt{1+[y'(x)]^2}\,dx.} \qquad \frac{dy}{dx} = y'(x)$$

dado.

Utilice sólo esta fórmula. (P 109.)

Ejemplo: Halle la longitud de la curva $\mathcal{C}: 0 \le x \le \frac{8}{9}$

$$y = 1 + 2x^{3/2}.$$

$$y'(x) = \frac{6}{2}x^{1/2} = 3x^{1/2}$$

$$1 + [y'(x)]^2 = 1 + (2x^{1/2})^2 = 1 + 9x.$$

Longitud de Arco:

$$L = \int_0^{8/9} \sqrt{1+(y')^2}\,dx$$

$$L = \int_0^{8/9} \sqrt{1+9x}\,\underbrace{dx}_{du/9.}$$

$u = 1 + 9x$

$du = 9\,dx$

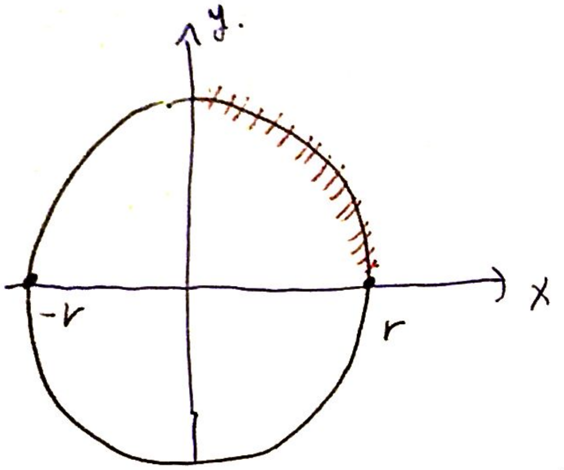
$u(8/9) = 1 + 8 = 9$

$u(0) = 1 + 0 = 1$

$$L = \int_1^9 u^{1/2}\,\frac{du}{9} = \frac{2}{3}\cdot\frac{1}{9}\,u^{3/2}\Big]_1^9$$

$$L = \frac{2}{27}\left((3^2)^{3/2} - 1^{3/2}\right) = \frac{2}{27}(3^3 - 1) = \frac{2\cdot 26}{27.}$$

Ejercicio 1: Encuentre la longitud (o perimetro) de una circunferencia de radio r.



$x^2 + y^2 = r^2$ Ec. Circunferencia.

$y^2 = r^2 - x^2$

$y = +(r^2 - x^2)^{1/2}$, ✓ Semi circunferencia superior.

$-r \leq x \leq r.$

$$L = 4\int_0^r \sqrt{1 + [y'(x)]^2}\, dx.$$

$$y'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$[y'(x)]^2 = \frac{x^2}{r^2 - x^2}.$$

$$1 + [y'(x)]^2 = \frac{1}{1} + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$

$$L = 4\int_0^r \sqrt{\frac{r^2}{r^2 - x^2}}\, dx = 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} \qquad \sin^{-1}\left(\frac{x}{r}\right)$$

$$L = 4r \sin^{-1}\left(\frac{x}{r}\right)\Big]_0^r = 4r\left(\overset{\pi/2}{\cancel{\sin^{-1}(1)}} - \overset{0}{\cancel{\sin^{-1}(0)}}\right)$$

$$= 4r\,\frac{\pi}{2} = 2\pi r.$$

Indefine en $x = r$, integral impropia convergente.

Ejercicio 2: p 101. Un cable telefónico cuelga entre dos postes con posiciones horizontales en $x = \pm 25$.

El cable toma la forma de una catenaria con ecuación

$$y = -5 + 25\cosh\left(\frac{x}{25}\right). \qquad \sqrt{1+(y')^2}$$

Halle la longitud del cable.

$$y'(x) = 25\sinh\left(\frac{x}{25}\right)\frac{1}{25} = \sinh\left(\frac{x}{25}\right)$$

$$1+[y'(x)]^2 = 1+\sinh^2\left(\frac{x}{25}\right) = \cosh^2\left(\frac{x}{25}\right)$$

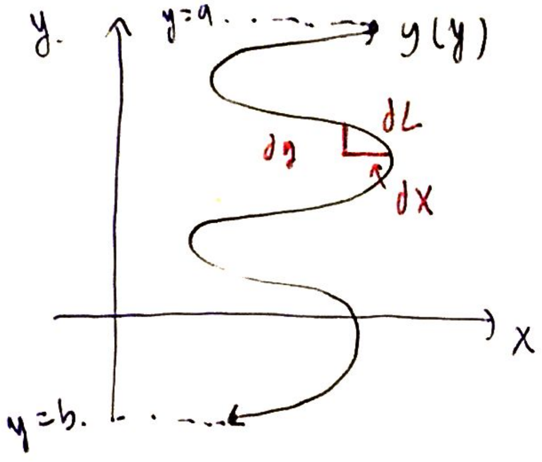
$$L = \int_{-25}^{25}\sqrt{1+(y')^2}\,dx = \int_{-25}^{25}\sqrt{\cosh^2\left(\frac{x}{25}\right)}\,dx$$

$$L = \int_{-25}^{25}\cosh\left(\frac{x}{25}\right)dx = 2\int_0^{25}\cosh\left(\frac{x}{25}\right)dx$$

$$L = 2\cdot 25\sinh\left(\frac{x}{25}\right)\Big]_0^{25} = 50(\sinh 1 - \sinh 0)$$

$$= 50\sinh 1 \approx 58.76.$$

Integración en el eje-y



curva $\mathcal{C}$: $a \le y \le b$, $x = g(y)$

$$dL = \sqrt{(dx)^2 + (dy)^2}$$

$$dL = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\ dy.$$

$$L = \int_a^b \sqrt{[g'(y)]^2 + 1}\ dy.$$

Ejercicio 3: Encuentre la longitud de arco para las curvas dadas. (P. 112).

a. $\mathcal{C}_1$: $x = \frac{y^3}{6} + \frac{1}{2y}$ $\quad 1 \le y \le 2.$

$y^2 y^{-2} = 1$

$$x'(y) = \frac{3y^2}{6} - \frac{1}{2y^2} = \frac{1}{2}y^2 - \frac{1}{2}y^{-2} = \frac{1}{2}(y^2 - y^{-2})$$

$$[x'(y)]^2 = \frac{1}{4}(y^2 - y^{-2})^2 = \frac{1}{4}(y^4 - 2 + y^{-4})$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$1 + [x'(y)]^2 = 1 + \frac{1}{4}(y^4 - 2 + y^{-4})$$

$$= \frac{1}{4}(4 + y^4 - 2 + y^{-4})$$

$$= \frac{1}{4}(y^4 + 2 + y^{-4}) = \frac{1}{4}\overbrace{(y^2 + y^{-2})^2}.$$

$$a^2 + 2a + 1 = (a+1)(a+1).$$

$$L = \int_1^2 \sqrt{\frac{1}{4}(y^2+y^{-2})^2}\, dy. = \sqrt{\frac{1}{4}} \int_1^2 (y^2+y^{-2})\, dy.$$

$$L = \frac{1}{2}\left(\frac{y^3}{3} - \frac{1}{y}\Big]_1^2\right) = \frac{1}{2}\left(\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + \frac{1}{1}\right) = \frac{17}{6\cdot 2}.$$

b. $\mathcal{C}_2$ $\quad y = \ln(\csc\theta)$, $\quad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}.$

$$L = \int_{\pi/6}^{\pi/2} \sqrt{1+(y')^2}\, d\theta.$$

$$y' = \frac{-\csc\theta \cot\theta}{\csc\theta} = -\cot\theta.$$

$$1+(y')^2 = 1+\cot^2\theta = \csc^2\theta.$$

$$L = \int_{\pi/6}^{\pi/2} \sqrt{\csc^2\theta}\, d\theta. = \int_{\pi/6}^{\pi/2} \csc\theta\, d\theta.$$

$$L = -\ln|\csc\theta + \cot\theta|\Big]_{\pi/6}^{\pi/2}.$$

$$L = -\ln|\underset{1}{\csc \pi/2} + \underset{0}{\cot \pi/2}| + \ln|\underset{2}{\csc \pi/6} + \underset{\sqrt{3}}{\cot \pi/6}|.$$

$$\frac{1}{\sin\pi/2} = 1 \qquad \frac{\cos\pi/2}{\sin\pi/2} = 0 \qquad \frac{1}{\sin\pi/6} = 2. \qquad \frac{\cos\pi/6}{\sin\pi/6.} = \frac{\sqrt{3}/2}{1/2.}$$

$$L = -\ln(1) + \ln(2+\sqrt{3})$$

$$L = \ln(2+\sqrt{3}).$$

función longitud de arco: $a \le x \le t$.

$$s(t) = \int_a^t \sqrt{1+[y']^2}\,dx$$

$y = \ln(\sin t)$, $\frac{\pi}{2} \le t \le x$.

$$L = \int_{\pi/2}^{x} \sqrt{1+(y')^2}\,dt.$$

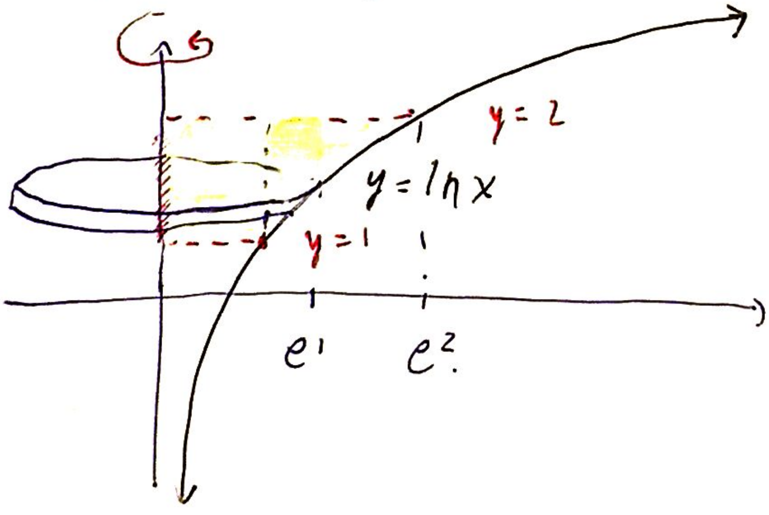
$$y' = \frac{-\cos t}{\sin t} = -\cot(t) \qquad 1+(y')^2 = 1+\cot^2(t)$$
$$\csc^2(t).$$

$$L = \int_{\pi/2}^{x} \sqrt{\csc^2(t)}\,dt = \int_{\pi/2}^{x} \csc t\,dt.$$

$$L = -\ln|\csc t + \cot t|\Big]_{\pi/2}^{x} = -\ln|\csc x + \cot x| + \ln(1).$$

Laboratorio 8. Problema 2:

Región $1 \le y \le 2$, $y = \ln x$ & $x = 0$.



$y = \ln x$ DISCOS.

$$V = \pi \int_1^2 x^2\,dy.$$

$$x = e^y$$

$$V = \pi \int_1^2 e^{2y}\,dy.$$