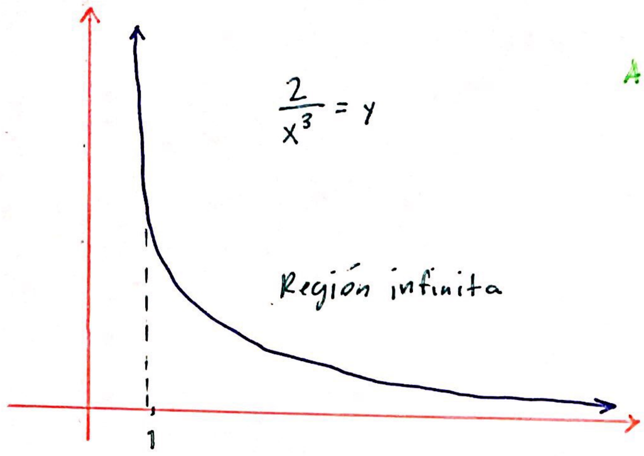
7.8. Integrales impropias

2019-09-03

Considere la región bajo la curva $y = \frac{2}{x^3}$ encima del eje-x y a la derecha de la recta $x = 1$



$$A = \int_1^t 2x^{-3}\,dx =$$

$$A = \frac{2}{-2}\,x^{-2}\Big]_1^t$$

$$A = -1 \cdot t^{-2} + 1 \cdot 1^{-2} = 1 - \frac{1}{t^2}$$

$$\lim_{t\to\infty} A = \lim\left(1 - \frac{1}{t^2}\right) = 1$$

$$\boxed{\int_1^t \frac{2}{x^3}\,dx = 1}$$

Límites básicos

a) $\lim_{x\to\infty}\left(\frac{1}{x^r}\right) = 0 \quad \left[\frac{1}{\infty}\right]$

b) $\lim_{x\to\infty}\left(e^x\right) = \infty \quad \left[e^{\infty}\right]$

c) $\lim_{x\to 0^+}\left(\ln x\right) = -\infty$

d) $\lim_{x\to\infty}\left(x^r\right) = \infty$

e) $\lim_{x\to-\infty}\left(e^x\right) = 0$

f) $\lim_{x\to\infty}\left(\ln x\right) = \infty$

Integrales impropias:

tipo 1: Intervalos infinitos $\pm\infty$

tipo 2: Funciones discontinuas (AVs, en $x = \pm a$)

Integrales Impropias tipo 1:

- $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

- $$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$$

- $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

- Convergente: se acerca a un número, el límite existe

Divergente: el límite no existe.

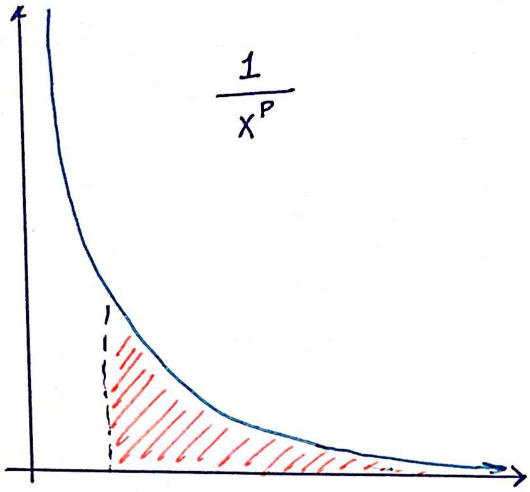
- Ejercicio: Evalúe

a) $$\int_1^\infty x^{-1/2}\,dx = 2x^{1/2}\Big]_1^\infty = \lim_{x\to\infty}\left(2\sqrt{x} - 2\right) = 2\sqrt{\infty} - 2 = \infty$$

Es una integral divergente.

b) $$\int_1^\infty \frac{1}{x}\,dx = \ln x\Big]_1^\infty = \lim_{t\to\infty}\left(\ln x - 0\right) = \infty$$

también es divergente.



$$\int_1^\infty \frac{1}{x^P}\,dx = \text{no necesariamente existe}$$

$$\int_1^\infty \frac{1}{x^P}\,dx = \begin{cases} P \le 1 & \text{Diverge} \\ P > 1 & \text{converge} \end{cases}$$

- $P = 0.99$

$$\int_1^\infty \frac{1}{x^{0.99}}\,dx = \frac{x^{0.01}}{0.01}\Big]_1^\infty = \lim_{x\to\infty}\left(x^{0.01} - \frac{1}{0.01}\right) = +\infty$$

Diverge

- $P = 1.001$

$$\int_1^\infty x^{-1.001}\,dx = \frac{1}{x^{0.001}}\cdot\frac{1}{0.001}\Big]_1^\infty = \lim_{x\to\infty}\left[\frac{1000}{x^{0.001}} + \frac{1}{0.001}\right) =$$

$$= \frac{1000}{x^{0.001}}\Big]_\infty^1 = 1000 - \lim_{x\to\infty}\left(\frac{1000}{x^{0.001}}\right) = \underline{1000}$$

Diverge.

$$\int_{-\infty}^0 e^{-x^2}\cdot x\,dx = \int_{-\infty}^0 e^u\,\frac{du}{-2} = -\frac{1}{2}e^u\Big]_{-\infty}^0 = -\frac{1}{2}e^0 + \frac{1}{2}e^{-\infty} = \underline{-\frac{1}{2}}$$

Converge

$u = -x^2$ $\qquad u(0) = -0^2$

$-\frac{du}{2} = x\,dx$ $\qquad u(-\infty) = -(-\infty)^2 = -\infty$

b.) $\dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} \dfrac{dx}{1+x^2} = \dfrac{1}{2}\tan^{-1}(x)\Big]_{-\infty}^{\infty} = \left\{\dfrac{1}{2}\tan^{-1}(\infty)\right\} - \left\{\dfrac{1}{2}\tan^{-1}(-\infty)\right\} = \dfrac{\pi}{2}$ Diverge

$\tan x \Rightarrow$ ID : $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

R : $(-\infty, \infty)$

AV: $x = -\pi/2, \pi/2$

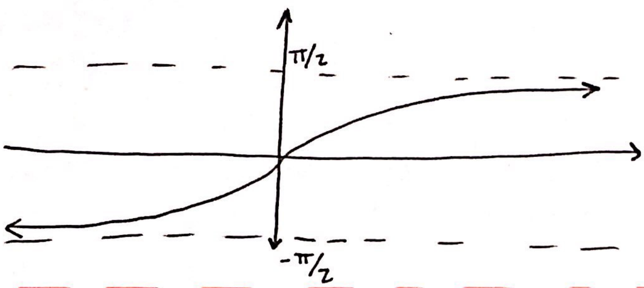
$= \tan^{-1} x$ ID $= (-\infty, \infty)$

R $= (-\pi/2, \pi/2)$

A.H $= \pm \pi/2$

$= \tan^{-1}(\infty) = \pi/2$

$= \tan^{-1}(-\infty) = -\pi/2$

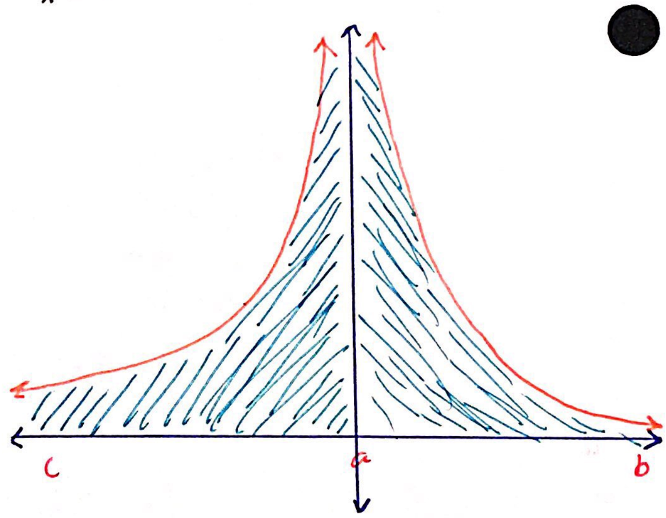


Integrales impropias: Tipo 2

Hay una asíntota vertical en $x = a$:

$$\int_b^a f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

$$\int_c^a f(x)\,dx = \lim_{t\to a^-} \int_c^t f(x)\,dx$$



A.V. $x=a$ $\displaystyle\int_c^b f(x)\,dx = \int_c^a f(x)\,dx + \int_a^b f(x)\,dx$

Ejercicio 4: Evalúe. Indique donde es discontinua

a) $$\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^8 u^{-1/3}\,du = \frac{3}{2}u^{2/3}\Big]_{0^+}^{8} =$$

Discontinuidades: $x = 1$ denominador igual a $1/0$

$u = x - 1$ $\quad u(1) = 0$

$du = dx$ $\quad u(9) = 8$

$$= \frac{3}{2}(8^2)^{1/3} - \frac{3}{2}\lim_{u\to 0^+} u^{2/3} = \frac{3}{2}\sqrt[3]{64} - 0 = \frac{3}{2}\cdot 4 = 6$$

$\square$

b) discontinua en 0

$$\int_{-2}^{3} \frac{3}{x^4}\,dx = \underbrace{\int_{-2}^{0} 3x^{-4}\,dx}_{(1)} + \underbrace{\int_0^3 3x^{-4}\,dx}_{(2)} = \infty \text{ diverge}$$

$$(1) = \frac{3x^{-3}}{-3}\Big]_{-2}^{0^-} = \underbrace{\lim_{x\to 0^-}\left(-\frac{1}{x^3}\right)}_{\infty} + \frac{1}{-2^8} = +\infty$$

$$(2) = \int_0^3 3x^{-4}\,dx = -x^{-3}\Big]_0^3 = -\frac{1}{3^3} + \lim_{x\to 0^+}\frac{1}{x^3} = +\infty$$

c.) $$\int_0^1 \ln(x)\,dx = x\ln(x) - \int dx = x\ln x - x + C\Big]_{0^+}^{1} = \overbrace{1\cdot\ln(1)}^{0} - \overbrace{1 - \lim_{x\to 0} x\ln x}^{\infty}$$

$$= -1 - \lim_{x\to 0}(x\ln x)$$

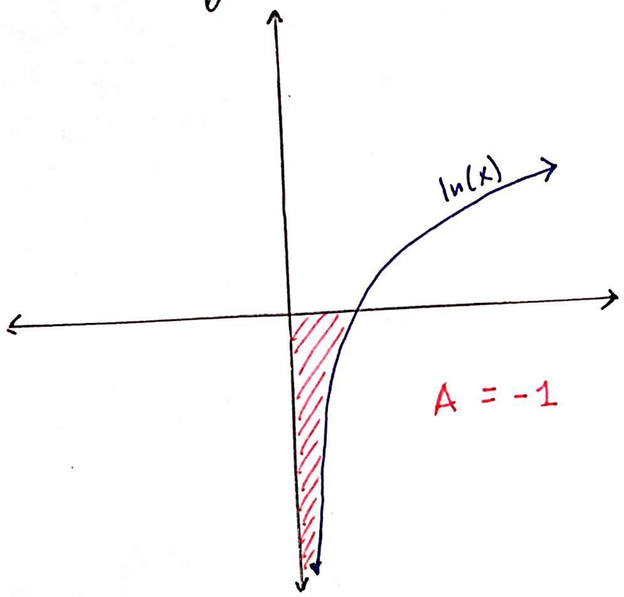
Regla de L'Hopital $\quad \lim_{x\to a}\frac{f(x)}{g(x)}$

$$\lim_{x\to 0^+}\left(x\ln(x)\right) = \lim_{x\to 0^+}\left(\frac{\ln(x)}{x^{-1}}\right) \overset{LH}{=} \lim_{x\to 0^+}\left(\frac{1}{x(-x^{-2})}\right) = \lim_{x\to 0^+}\left(\frac{1}{-x^{-1}}\right) = \dots$$

$$\dots = \lim_{x\to 0^+}\left(\frac{1}{-x^{-1}}\right) = \lim_{x\to 0^+}\left(-x\right) = \underline{0}$$

$$\therefore \int_0^1 \ln(x)\,dx = -1 - \lim_{x\to 0} x\ln(x) = -1 + 0$$

converge



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