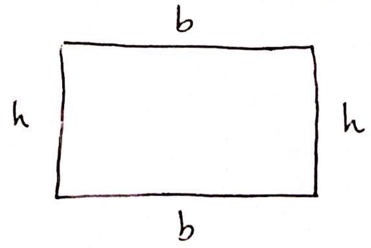
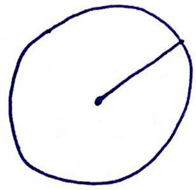
8.1 Longitud del arco

2019-09-24

Con geometría podemos encontrar la longitud ó perímetro.



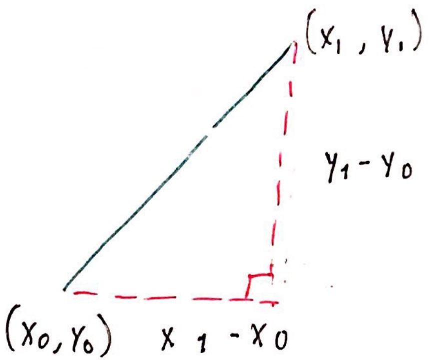


Circumferencia:

$L = 2\pi r$

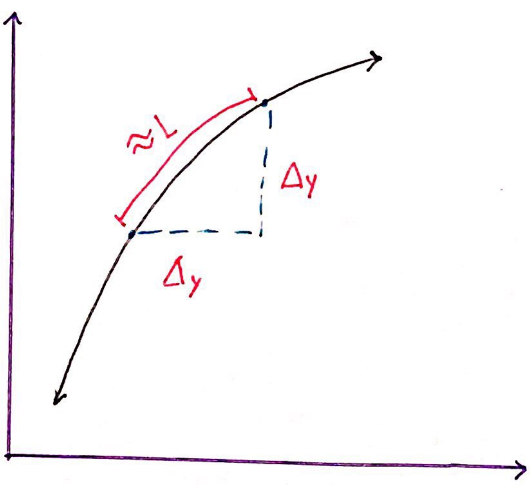
Rectángulo = $L = 2b + 2h$

- El segmento de recta:



$$L = \sqrt{(x_1 - x_0)^2 + \sqrt{y_1 - y_0}}$$

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$



curva $\mathcal{C}$: $y = f(x) \quad a \leq x \leq b$

Parte infinitesimal del arco:

$$dL = \sqrt{(dx)^2 + (dy)^2}$$

factorizar:

$$dx^2 + dy^2 = dx^2\left(1 + \frac{dy^2}{dx^2}\right) dx$$

$$\boxed{\text{Longitud de arco: } \quad \mathcal{L} = \int_a^b \sqrt{1 + [y'(x)]^2}\, dx}$$

$$\frac{dy}{dx} = y'(x)$$

Ej: Halle la longitud de la curva $\mathcal{C}$: $0 \leq x \leq \frac{8}{9}$

$$y = 1 + 2x^{3/2}$$

$$y'(x) = \frac{6}{2} x^{1/2} = 3x^{1/2} \qquad 1 + [y'(x)]^2 = 1 + 9x$$

$$\therefore L = \int_0^{8/9} \sqrt{1 + 9x}\, dx = \frac{1}{9}\int_0^{8/9} \sqrt{u}\, du = \frac{1}{9}\left[u^{\frac{3}{2}}\right]_0^{8/9}$$

$u = 1 + 4x$

$du = 9\,dx$

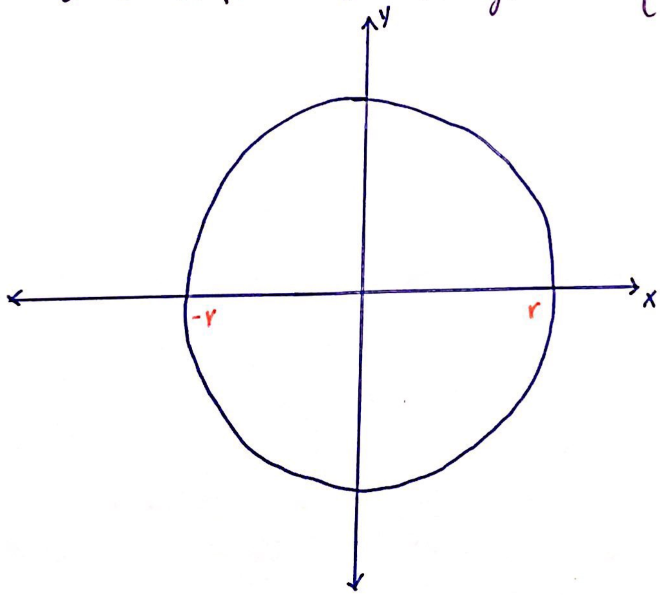
$\frac{du}{9} = dx$

$u(0) = 1 \qquad u(8/9) = 9$

$$= \frac{1}{9} \cdot \frac{3}{2}\left((\sqrt{u})^3 .\right)\Big]_1^9$$

$$= \frac{2}{27}\left((3^2)^{3/2} - 1^{3/2}\right) = \frac{2}{27}(3^3 - 1)$$

Ej: Encuentre la longitud (ó perímetro) de una circumferencia



$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$y = +\sqrt{r^2 - x^2}$$

límites : $(-r, r)$

$$L = 4\int_0^r \sqrt{1 + (y'(x))^2}\, dx =$$

$$L = 4\int_0^r \frac{\sqrt{r^2}}{\sqrt{r^2 - x^2}}\, dx = 4r \sin^{-1}(x)\Big]_0^r = \rightarrow$$

$$y' = \frac{1}{2}(r^2 - x^2)^{-1/2} \cdot (-2x)$$

$$(y')^2 = \left[\frac{x}{(r^2 - x^2)}\right]^2 = 1 + [y'(x)]^2 = \frac{1}{1} + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + 1}{r^2 - x^2}$$

$$= 4r\{\sin^{-1}(\underbrace{\tfrac{r}{r}}_{1})\} - \{\cancel{\sin^{-1}(0)}^{0}\} = 4r\{\sin^{-1}(1)\} = 4r\left(\tfrac{\pi}{2}\right)$$

se indefine en $x = r$ es impropia convergente

Ejercicio = p. 101. un cable teléfono cuelga entre dos postes con posiciones horizontales en $x = \pm 25$. El cable toma la forma de una catenaria con ecuación:

$$y = -5 + 25\cosh\left(\tfrac{x}{25}\right)$$

Hallar la longitud:

$$y'(x) = \cancel{25}\sinh\left(\tfrac{x}{25}\right)\cdot\frac{1}{\cancel{25}} = \left[\sinh\left(\tfrac{x}{25}\right)\right]^2 = [y'(x)]^2$$

$$\underbrace{1 + \sinh^2\left(\tfrac{x}{25}\right)}_{} =$$

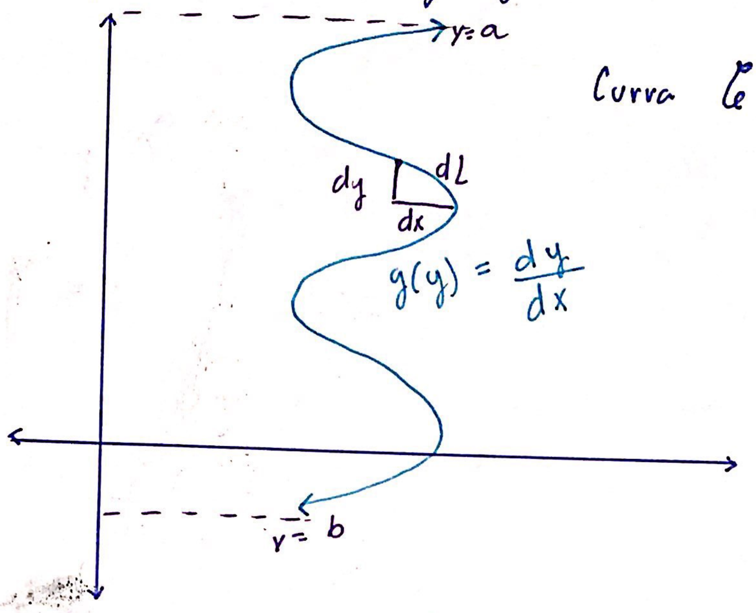
utilizar identidad pitagórica $1 = \sin^2 x + \cos^2 x$

$$\cosh^2(x/25)$$

$$L = 2\int_0^{25}\sqrt{\cosh^2(x/25)}\,dx = 2\int_0^{25}\cosh(x/25)\,dx = 2\sinh(x/25)\cdot 25\Big]_0^{25} =$$

$$= 50\sin^{-1}(x/25)\Big]_0^{25} = \{50\sin^{-1}(1)\} - \{\cancel{50\sin^{-1}(0)}^{0}\} = 50\sin^{-1}(1)$$

Integración en el eje-y:



Curva $\mathcal{C} = a \leq y \leq b$

$$dL = \sqrt{(dx)^2 + (dy)^2}$$

$$dL = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\ dy$$

$$L = \int_a^b \sqrt{[g'(x)]^2 + 1}\ dy$$

<u>Ejercicio 3</u>: encuentre la longitud de arco para las curvas dadas.

a.) $\mathcal{C}: \ x = \dfrac{y^3}{6} + \dfrac{1}{2y} \qquad 1 \leq y \leq 2$

$$x'(y) = \frac{3y^2}{6} + \frac{1}{2y^2} = \frac{1}{2}y^2 - \frac{1}{2y^2} = \frac{1}{2}\left(y^2 - \frac{1}{y^{-2}}\right)$$

$$[x'(y)]^2 = \frac{1}{4}\left(y^4 - 2\cancel{y^2}\cdot\frac{1}{\cancel{y^2}} + \frac{1}{y^4}\right)$$

$$1 + [x'(y)]^2 = \frac{1}{4}\left(4 + y^2 - 2 + y^{-4}\right) + 1$$

$$= \frac{1}{4}\left(4 + y^4 - 2 + y^{-4}\right)$$

$$= \frac{1}{4}\underbrace{\left(y^4 + 2y + y^{-4}\right)}_{\left(\sqrt{y^4} + \sqrt{\frac{1}{y^4}}\right)^2} = \frac{1}{4}\left(y^2 + y^{-2}\right)^2$$

factorización

$$L = \int_1^2 \sqrt{(y^2+y^{-2})^2}\, dy = \frac{1}{2}\int_1^2 (y^2+y^{-2})\, dy = \frac{1}{2}\left(\frac{1}{3}y^3 + \frac{1}{-1}y^{-1}\right)\Big]_1^2$$

$$= \frac{1}{2}\left(\left\{\frac{8}{3} - \frac{1}{2}\right\} - \left\{\frac{1}{3} - 1\right\}\right) = \frac{17}{12}$$

b.) $\mathcal{C}_2 : \; y = \ln(\csc\theta) \qquad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$

$$L = \int_{\pi/6}^{\pi/2} \sqrt{1 + [y'(\theta)]^2}$$

$$y' = \frac{1}{\cancel{\csc\theta}} \cdot -\cancel{\csc\theta} \cdot \cot\theta = -\cot\theta$$

$$1 + [y'(x)]^2 = \underbrace{1 + \cot^2\theta}_{\text{Identidad pitagórica}} = \csc\theta$$

$$L = \int_{\pi/6}^{\pi/2} \csc\theta \, d\theta = -\ln|\csc\theta + \cot\theta| \Big]_{\pi/6}^{\pi/2} = -\ln(1) + \ln(2+\sqrt{3})$$

$$L = \ln(2+\sqrt{3})$$

ángulos especiales:

$$\frac{1}{\sin\left(\frac{\pi}{2}\right)} = 1 \qquad \frac{\cos(\pi/2)}{\sin(\pi/2)} = 0$$

$$\frac{1}{\sin(\pi/6)} = 2 \qquad \frac{\cos(\pi/6)}{\sin(\pi/6)} = \frac{\sqrt{3}\cdot\cancel{2}}{\cancel{2}} = \sqrt{3}$$

Función longitud de arco:

$a \leq x \leq t$

$$s(t) = \int_a^t \sqrt{1 + [y'(x)]^2}$$

$y = \ln(\sin t)$, $\pi/2 \leq t \leq x$

$$L = \int_{\pi/2}^{x} \sqrt{1 + y'(x)^2}\, dx = \int_{\pi/2}^{x} \csc x \, dx = -\ln|\csc t + \cot t| \Big]_{\pi/2}^{x}$$

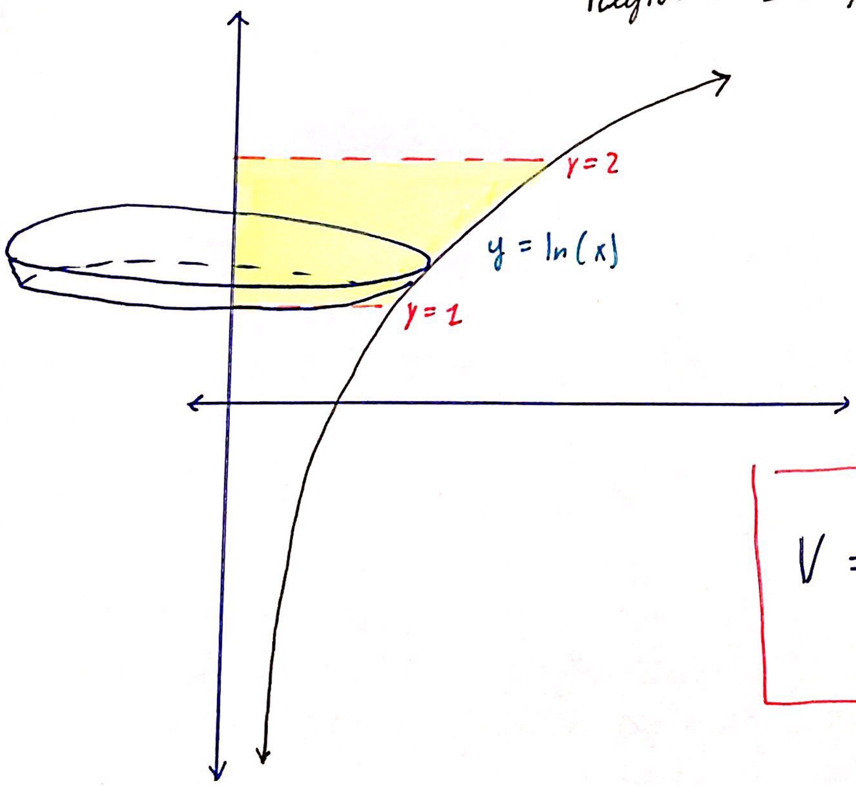
$$= -\ln|\csc x + \cot x| + \ln(1)$$

$y' = -\cot x$

$(y')^2 = \cot^2 x$ $\quad \underbrace{1 + \cot^2\theta}_{\csc\theta}$

Laboratorio 8 : 2 :

Región : $1 \leq y \leq 2$, $y = \ln(x)$ & $x = 0$



$y = \ln(x)$ Discos

$$V = \pi \int_1^2 x^2 \, dy$$

$x = e^y$

$$\boxed{V = \pi \int_1^2 e^{2y} \, dy}$$