Parcial 3 Lab 12 y 13.

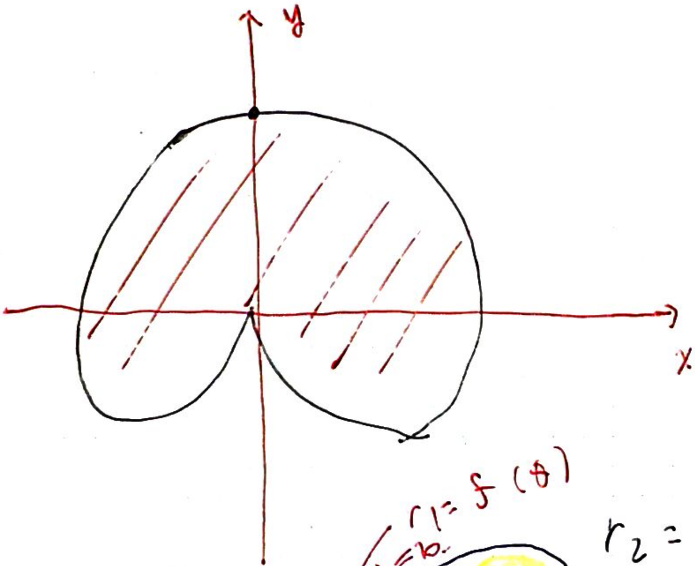
Cortos 10, 11 y 12 (Jueves)

Jueves Corto (1h).

Lunes Sim 3.

Martes Par 3. 10 AM.

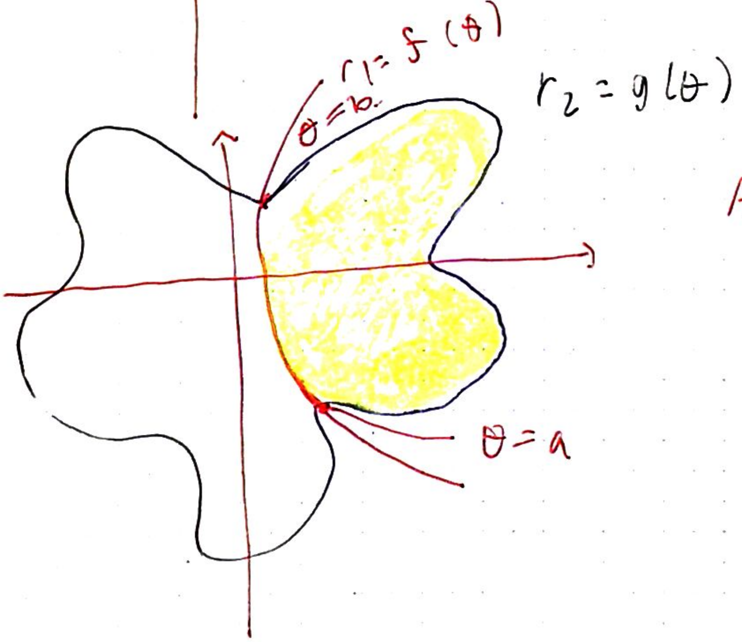
A'rea entre curvas Polares.



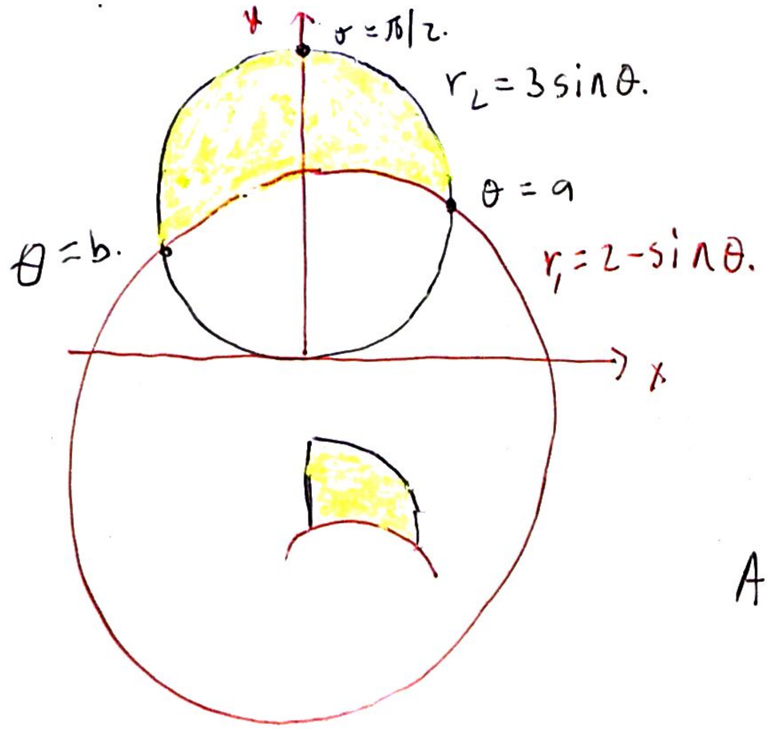
$$A = \frac{1}{2}\int_a^b r^2 \, d\theta.$$

$$\frac{1}{2} r^2 \theta.$$

$\theta = 2\pi$ $\quad 2\pi r^2$.



$$A = \frac{1}{2}\int_a^b r_2^2 - r_1^2 \, d\theta.$$



Ejemplo: p. 163.

Encuentre el área de la región fuera del limaçon $r_1=2-\sin\theta$ y adentro del círculo $r_2=3\sin\theta$.

$$A=\frac{1}{2}\int_a^b r_2^2-r_1^2\,d\theta.$$

$r_2>r_1$ en $a\le\theta\le b$ el círculo está más alejado del origen.

P.I's: $r_1=r_2$ $\quad 3\sin\theta=2-\sin\theta.$

$4\sin\theta=2 \Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3},\frac{5\pi}{6}.$

$$A=\frac{1}{2}\int_{\pi/3}^{5\pi/6}(3\sin\theta)^2-(2-\sin\theta)^2\,d\theta.$$

$$A=\frac{2}{2}\int_{\pi/3}^{\pi/2}9\sin^2\theta-(4-4\sin\theta+\sin^2\theta)\,d\theta.$$

$$A=\int_{\pi/3}^{\pi/2}\left(\underline{8\sin^2\theta}+4\sin\theta-4\right)d\theta.$$ $\quad \frac{1}{2}-\frac{1}{2}\cos2\theta=\sin^2\theta$

$$A=\int_{\pi/3}^{\pi/2}(4-4\cos2\theta+4\sin\theta-4)\,d\theta.$$

$$A=\int_{\pi/3}^{\pi/2}-4\cos2\theta+4\sin\theta\,d\theta.$$

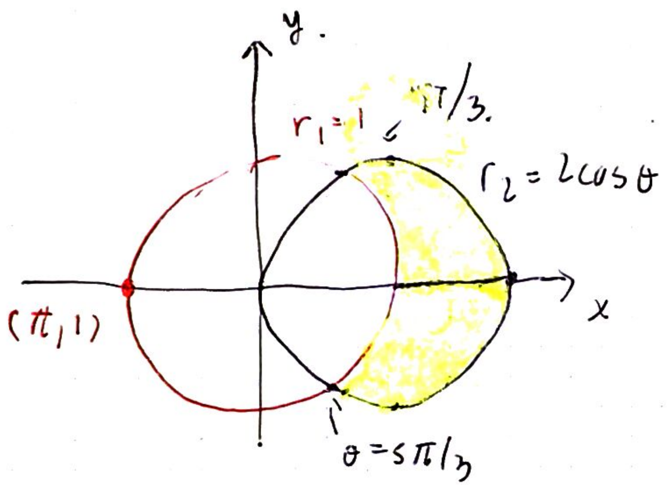
$$A = -\frac{4}{2}\sin 2\theta \Big]_{\pi/3}^{\pi/2} - 4\cos\theta \Big]_{\pi/3}^{\pi/2}$$

$\int \cos 2\theta = \frac{\sin 2\theta}{2}$

$$A = -2\cancel{\sin \pi} + 2\sin\frac{2\pi}{3} - 4\cancel{\cos\frac{\pi}{2}} + 4\cos\frac{\pi}{3}$$

$$A = 2\frac{\sqrt{3}}{2} + 4\frac{\sqrt{3}}{2} = \sqrt{3} + 2\sqrt{3} = 3\sqrt{3}!$$

Ejercicio 3: Encuentre el área que está adentro del círculo $r_2 = 2\cos\theta$ y fuera del círculo $r_1 = 1$.



r_2 está más alejada del origen que r_1

$$A = \frac{1}{2}\int_a^b r_2^2 - r_1^2 \, d\theta.$$

P.I's. $r_2 = r_1$

$2\cos\theta = 1 \Rightarrow \cos\theta = 1/2$

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (4\cos^2\theta - 1^2)\, d\theta.$$

$$A = \frac{2}{2}\int_0^{\pi/3} (4\cos^2\theta - 1)\, d\theta.$$

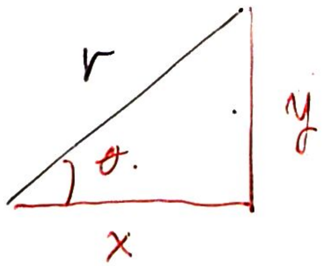
$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta.$

$$A = \int_0^{\pi/3} (2\cos 2\theta + 2 - 1)\, d\theta. = \sin 2\theta + \theta \Big]_0^{\pi/3}$$

$$A = \sin\frac{2\pi}{3} + \frac{\pi}{3} - 0 = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$

Longitud de Arco Coordenadas Polares.

Una función polar $r = f(\theta)$ tiene las sigs. ecs. paramétricas cartesianas.

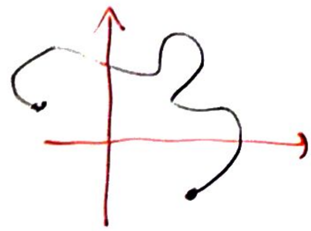


$$\boxed{\begin{aligned} y &= r\sin\theta \\ x &= r\cos\theta \end{aligned}}$$

θ parámetro

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)}$$

Longitud Curva $\quad L = \displaystyle\int_a^b \sqrt{(x')^2 + (y')^2}\, d\theta$



r no es constante $\quad r(\theta)$

$$x'(\theta) = r'(\theta)\cos\theta - r\sin\theta$$
$$y'(\theta) = r'\sin\theta + r\cos\theta.$$

$\cos^2\theta + \sin^2\theta = 1$

$$(x')^2 = (r')^2\cos^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta.$$
$$(y')^2 = (r')^2\sin^2\theta + 2rr'\cos\theta\sin\theta + r^2\cos^2\theta$$

$$(r')^2 \qquad\qquad 0 \qquad\qquad r^2.$$

$$\boxed{L = \int_a^b \sqrt{(r')^2 + r^2}\, d\theta.}$$

Fórmula que se puede utilizar.

Última Páginas Ejercicio 4: Encuentre la longitud exacta de las siguientes curvas.

a. $r = 2\cos\theta$; $0 \leq \theta \leq \pi$. Círculo de radio 1

$2\pi(1)$

$r = 2\cos\theta$ $\qquad r^2 = 4\cos^2\theta.$

$r'(\theta) = -2\sin\theta$ $\qquad (r')^2 = 4\sin^2\theta.$

$r^2 + (r')^2 = 4(\cos^2\theta + \sin^2\theta) = 4.$

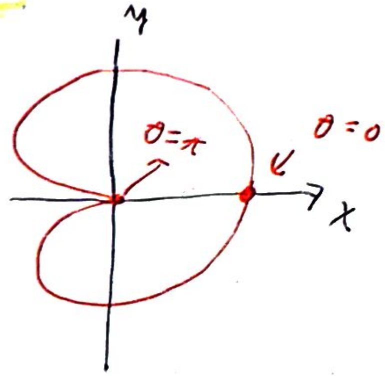
$r(0) = 2$

$r(\pi) = -2.$

$(0,2)$ y $(\pi, -2)$ son el mismo punto.

$$L = \int_0^\pi \sqrt{r^2 + (r')^2}\, d\theta$$

$$L = \int_0^\pi \sqrt{4}\, d\theta = 2\int_0^\pi d\theta. = 2\theta\Big]_0^\pi = 2\pi$$



b. $r = 1 + \cos\theta$ $\qquad 0 \leq \theta \leq \pi.$

Medio cardioide.

$$L = \int_0^\pi \sqrt{r^2 + (r')^2}\, d\theta.$$

$r = 1 + \cos\theta$ $\qquad r' = -\sin\theta.$

$r^2 = (1+\cos\theta)^2 = 1 + 2\cos\theta + \cos^2\theta$, $(r')^2 = \sin^2\theta.$

$$r^2 + (r')^2 = 1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}$$

$$r^2 + (r')^2 = 2 + 2\cos\theta.$$

$$L = \int_0^{\pi} \sqrt{2+2\cos\theta}\, d\theta. \qquad \text{cicloide } \int_0^{2\pi} \sqrt{2+2\sin\theta}\, d\theta.$$

$$\cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta) \qquad 4\cos^2\frac{\theta}{2} = 2+2\cos\theta.$$

$$4\sin^2\frac{\theta}{2} = 2-2\cos\theta.$$

$$L = \int_0^{\pi} \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta = 2\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta.$$

$$L = 2\cdot 2\sin\left(\frac{\theta}{2}\right)\Big]_0^{\pi} = 4\left(\sin\frac{\pi}{2} - \sin 0\right) = 4.$$

Longitud de todo el cardioide es $2L = 8$.

J. Un pétalo de la rosa $r = \cos 2\theta$, $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

$$r = \cos(2\theta) \qquad r^2 = \cos^2(2\theta)$$

$$r' = -2\sin(2\theta) \qquad (r')^2 = 4\sin^2(2\theta)$$

$$r^2 + (r')^2 = \cos^2(2\theta) + 4\sin^2(2\theta) = 1 + 3\sin^2(2\theta)$$

$$L = \int_{-\pi/4}^{\pi/4} \sqrt{1+3\sin^2 2\theta}\, d\theta.$$

No se puede integrar de manera, sólo de manera aproximada.

$$L = 2\int_0^{\pi/4} \sqrt{1+3\sin^2 2\theta}\, d\theta.$$

c. La espiral $r = \theta^2$ en $0 \leq \theta \leq \sqrt{\pi}$.

$$r = \theta^2 \qquad r^2 = \theta^4.$$

$$r' = 2\theta \qquad (r')^2 = 4\theta^2.$$

$$L = \int_0^{\sqrt{\pi}} \sqrt{r^2 + (r')^2}\, d\theta = \int_0^{\sqrt{\pi}} \sqrt{\theta^4 + 4\theta^2}\, d\theta. \qquad \theta^2. \ \text{F.C.}$$

$$L = \int_0^{\sqrt{\pi}} \sqrt{\underbrace{\theta^2 + 4}_{u}}\ \theta\, d\theta. \qquad u = \theta^2 + 4$$

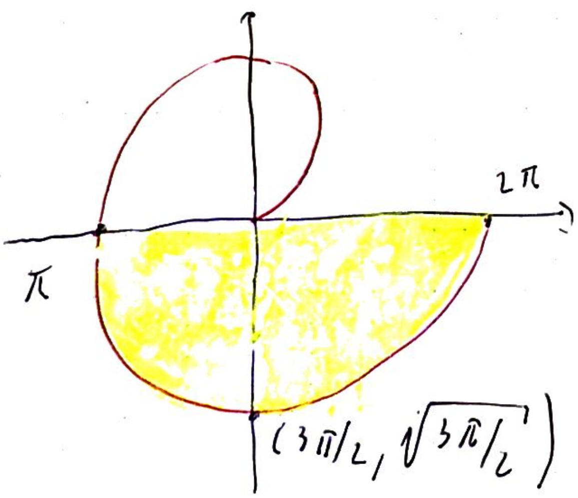
$$du = 2\theta\, d\theta.$$

$$L = \int u^{1/2} \frac{du}{2} = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} = \frac{1}{3}(\theta^2 + 4)^{3/2} \Big]_0^{\sqrt{\pi}}$$

$$L = \frac{1}{3}(\pi + 4)^{3/2} - \frac{1}{3} \underbrace{4^{3/2}}_{8}. = \frac{1}{3}\left[(\pi + 4)^{3/2} - 8\right].$$

$$\int_a^b y\, dx$$

WA 10.4 Prob. $r = \sqrt{\theta}$



$$\boxed{A = \frac{1}{2} \int_a^b r^2\, d\theta}$$

$$A = \frac{1}{2} \int_{\pi}^{2\pi} \theta\, d\theta.$$

$$A = \frac{1}{4} \theta^2 \Big]_{\pi}^{2\pi}$$

$$A = \frac{1}{4}(4\pi^2 - \pi^2) = \frac{3\pi^2}{4}.$$

Prob 1b) Recta Tangente a $r = \frac{1}{\theta}$ en $\theta = \pi$.

Ec. Recta Tangente $y = y(\pi) + m(\pi)(x - x(\pi))$

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)}$$

$y = r\sin\theta = \theta^{-1}\sin\theta.$

$x = r\cos\theta = \theta^{-1}\cos\theta.$

R.P.

$y'(\theta) = -\theta^{-2}\sin\theta + \theta^{-1}\cos\theta.$

$x'(\theta) = -\theta^{-2}\cos\theta - \theta^{-1}\sin\theta.$

$\sin\pi = 0$

$\cos(\pi) = -1$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{-\frac{\sin\pi}{\pi^2} + \frac{\cos\pi}{\pi}}{-\frac{\cos\pi}{\pi^2} - \frac{\cancel{\sin\pi}}{\cancel{\pi}}} = \frac{\frac{-1}{\pi}}{\frac{+1}{\pi^2}} * \frac{\pi^2}{\pi^2} = -\pi$$

$x(\pi) = \frac{1}{\pi}\cos\pi = -\frac{1}{\pi}$ $\qquad y(\pi) = \frac{1}{\pi}\sin\pi = 0$

Ec. Recta Tangente: $y = -\pi\left(x + \frac{1}{\pi}\right)$

$$\boxed{y = -\pi x - 1}$$