15.9 Coordenadas Esféricas (ρ, θ, φ).

ρ distancia de P al origen O $\quad \rho = \sqrt{x^2+y^2+z^2}$

θ es el ángulo polar $\quad \tan\theta = y/x$

φ es el azimut $\quad \cos\varphi = z/\rho$.

$0 \leq \varphi \leq \pi$

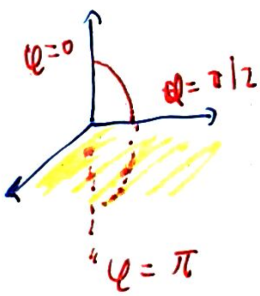
Cambio de Coordenadas.

$x = \rho\sin\varphi\cos\theta$. $\quad \rho^2 = x^2+y^2+z^2$

$y = \rho\sin\varphi\sin\theta$. $\quad \tan\theta = y/x$

$z = \rho\cos\varphi$ $\quad \cos\varphi = \dfrac{z}{\rho}$.

$r = x^2+y^2 = \rho\sin\varphi$.



$z>0$ Polo Norte $\varphi = 0$ $\quad$ Hemisferio Norte $0 < \varphi < \pi/2$

Ecuador (plano xy) $\varphi = \pi/2$

$z<0$ Polo Sur $\varphi = \pi$. $\quad$ Hemisferio sur $\pi/2 < \varphi < \pi$

$(0,0,-K)$

Evaluación de Integrales Triples. $\iiint_E f(x,y,z)\,dU$.

Sólido E sea una "cuña" esférica

segmento de una esfera, segmento de un cono, plano horizontal

$E: \alpha \leq \theta \leq \beta \quad c \leq \varphi \leq d \quad r_1 \leq \rho \leq r_2$.

$$I = \int_\alpha^\beta \int_c^d \int_{r_1}^{r_2} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \underbrace{\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}_{dV.}$$

Volumen de un sólido.

$$V = \iiint_E dV = \int_\alpha^\beta \int_c^d \int_{r_1(\theta,\varphi)}^{r_2(\theta,\varphi)} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

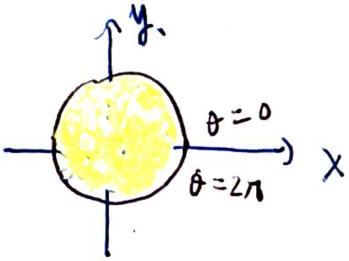
Masa: $m = \iiint_E \delta(x,y,z)\, dV$ densidad volumétrica $\delta(x,y,z)$.

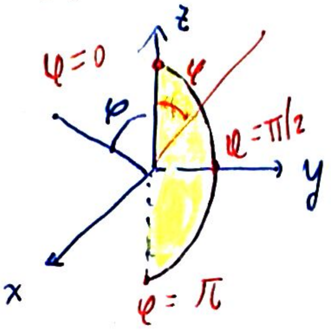
Ejercicio 1: Evalúe $\iiint_E (x^2+y^2)\, dV$

E está entre las esferas $x^2+y^2+z^2=4$ & $x^2+y^2+z^2=9$.

$\rho_1^2 = 4$ & $\rho_2^2 = 9$

$$\boxed{\begin{array}{l} 2 \le \rho \le 3. \\ 0 \le \theta \le 2\pi. \\ 0 \le \varphi \le \pi \end{array}}$$





$$x^2+y^2 = \rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi.$$

$$\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta) = \rho^2\sin^2\varphi.$$

$$\iiint_E (x^2+y^2)\, dV = \int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^2\sin^2\varphi \; \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta.$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin^3\varphi\, d\varphi\right)\left(\int_2^3 \rho^4\, d\rho\right)$$

$$\int_0^{2\pi} d\theta = 2\pi \qquad \int_2^3 \rho^4\, d\rho = \frac{1}{5}\rho^5\Big]_2^3 = \frac{1}{5}(3^5-2^5)$$

$$= 211/5.$$

$$\int \sin^2\varphi \sin\varphi \, d\varphi = \int (1-\cos^2\varphi)\sin\varphi \, d\varphi.$$

$u = \cos\varphi \qquad -du = \sin\varphi \, d\varphi$

$$= -\int (1-u^2)\, du$$

$$= -u + \frac{u^3}{3} + C$$

$$\boxed{\int \sin^3\varphi \, d\varphi = \frac{1}{3}\cos^3\varphi - \cos\varphi + C}$$

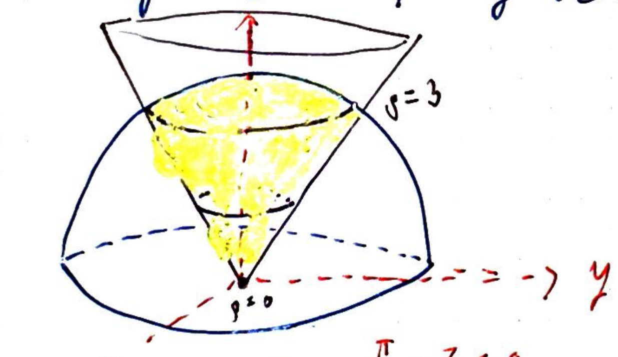
$$\iiint_E (x^2+y^2)\, dV = 2\pi \cdot \frac{211}{5}\left(\frac{1}{3}\cos^3\varphi - \cos\varphi \Big]_{\varphi=0}^{\varphi=\pi}\right).$$

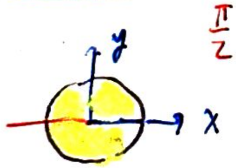
$2 - \frac{2}{3} = \frac{4}{3}.$

$$= \frac{422\pi}{5}\left(-\frac{1}{3} + 1 - \frac{1}{3} + 1\right) = \frac{422\pi}{5}\,\frac{2}{3}.$$

$$= \frac{844\pi}{15} \approx 176.767.$$

Ejercicio 2: Encuentre el volumen del sólido que se encuentra arriba del plano xy, dentro de la esfera $x^2+y^2+z^2=9$ y dentro del semicono $z=\sqrt{x^2+y^2}$





El sólido en amarillo es un segmento de la esfera $\rho=3$.

$0 \le \rho \le 3.$ "No hay cascarones"

$0 \le \theta \le 2\pi.$ "secciones transversales circulares"

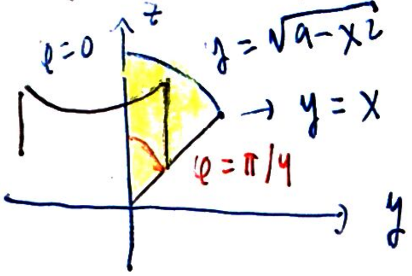
$0 \le \varphi \le \pi/4.$

No está el hemisferio sur

- Reescriba el cono en coordenadas esféricas.

$\underline{\rho\cos\varphi} = r = \underline{\rho\sin\varphi}. \Rightarrow \tan\varphi = 1 \Rightarrow \varphi = \frac{\pi}{4}$

$x^2+y^2 = r^2 = \rho^2\sin^2\varphi.$



$E: 0\leq\rho\leq 3,\ 0\leq\theta\leq 2\pi,\ 0\leq\varphi\leq\frac{\pi}{4}.$

$V = \iiint_E dV$

$x^2 = 9-x^2$

$2x^2 = 9$

$x = \sqrt{4.5}$

$V = \int_0^3\int_0^{2\pi}\int_0^{\pi/4}\rho^2\sin\varphi\cdot d\varphi\, d\theta\cdot d\rho.$

$V = 2\pi\int_0^a r h\, dx = 2\pi\int_0^{\sqrt{4.5}} x\left(\sqrt{9-x^2}-x\right)dx.$

$V = \int_0^{2\pi} d\theta\int_0^{\pi/4}\sin\varphi\, d\varphi\int_0^3\rho^2 d\rho.$

$V = 2\pi\left(-\cos\varphi\Big]_0^{\pi/4}\right)\left(\frac{1}{3}\rho^3\Big]_0^3\right)$

$V = 2\pi\left(-\frac{\sqrt{2}}{2}+\frac{2}{2}\right)\frac{27}{3.} = 9\pi(2-\sqrt{2}).$

Volumen Esfera $\rho=3$ $\quad V = \frac{4}{3}\rho^3\pi = 36\pi.$

semiesfera $\quad V = 18\pi.$

Ejercicio 3: Evalúe la integral dada.

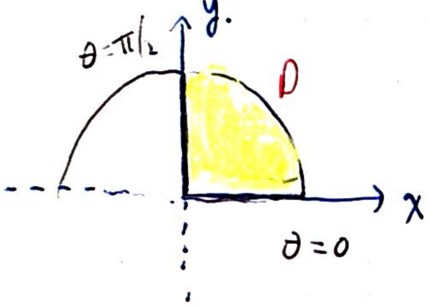
$$I_1 = \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{1-x^2-y^2}} \underline{xy}\, dz \right) dy\, dx.$$

Sólido D $\quad 0 \leq x \leq 1$, $\quad 0 \leq y \leq \sqrt{1-x^2}$, $\quad 0 \leq z \leq \sqrt{1-x^2-y^2}$

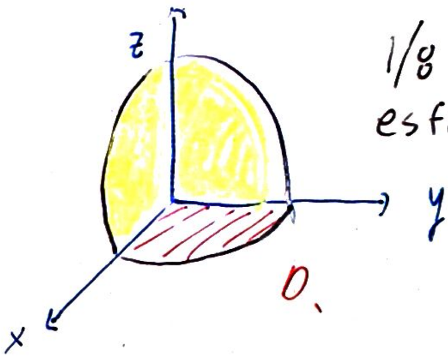
semicircunferencia superior. semiesfera superior hemisferio norte.

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2 \qquad z^2+x^2+y^2 = 1 \qquad z = \pm\sqrt{K^2-x^2-y^2}$$



$$\boxed{\begin{matrix} 0 \leq \theta \leq \pi/2. \\ 0 \leq \rho \leq 1 \\ 0 \leq \varphi \leq \pi/2. \\ E \end{matrix}}$$



1/8 esfera.

$$xy = \rho \sin\varphi \cos\theta \, \rho \sin\varphi \sin\theta. = \rho^2 \sin^2\varphi \cos\theta \sin\theta$$

$$I_1 = \iiint_E xy\, dV = \int_0^{\pi/2} \int_0^1 \int_0^{\pi/2} \rho^2 \sin^2\varphi \underline{\cos\theta \sin\theta} \cdot \rho^2 \sin\varphi\, d\varphi\, d\rho\, d\theta.$$

$$I_1 = \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta. \int_0^1 \rho^4 d\rho. \int_0^{\pi/2} \sin^3\varphi\, d\varphi.$$

$$I_1 = \frac{\sin^2\theta}{2}\Big]_0^{\pi/2} \frac{\rho^5}{5}\Big]_0^1 \left(\frac{1}{3}\cos^3\varphi - \cos\varphi \Big]_0^{\pi/2} \right)$$

$$I_1 = \frac{1}{2} \, \frac{1}{5.} \left(-\frac{1}{3} + 1 \right) = \frac{1}{2}\,\frac{1}{5}\,\frac{2}{3} = \frac{1}{15.}$$

$$I_2 = \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2+y^2)\,dz\,dx\,dy.$$

E: $-a \leq y \leq a.$ $\quad -\sqrt{a^2-y^2} \leq x \leq \sqrt{a^2-y^2}$

$$-\sqrt{a^2-x^2-y^2} \leq z \leq \sqrt{a^2-x^2-y^2}$$

La totalidad de una esfera de radio a.

$0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi,\ 0 \leq \rho \leq a.$

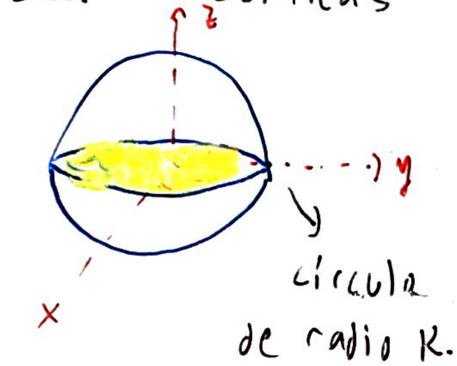
$$I_2 = \iiint_E (x^2+y^2)\,dV. = \left(\int_0^{2\pi} d\theta.\right)\int_0^a \int_0^{\pi} \rho^2\sin^2\varphi\ \rho^2 \sin\varphi\, d\varphi\, d\rho.$$

$r^2 = \rho^2\sin^2\varphi.$

$$I_2 = 2\pi \int_0^a \rho^4\, d\rho \int_0^{\pi} \sin^3\varphi\, d\varphi. = \frac{8\pi}{15}a^?$$

Volumen de una Esfera. de radio K.

Ecs. Cartesianas $\quad z = \pm\sqrt{K^2 - x^2 - y^2}$

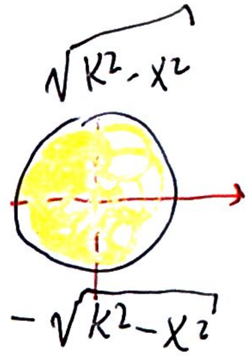


D: $\quad z = 0.$

$$K^2 - x^2 - y^2 = 0.$$

$$y = \pm\sqrt{K^2 - x^2}$$

$$-K \le x \le K$$



$$E: -K \le x \le K, \; -\sqrt{K^2-x^2} \le y \le \sqrt{K^2-x^2}$$

$$-\sqrt{K^2-x^2-y^2} \le y \le \sqrt{K^2-x^2-y^2}$$

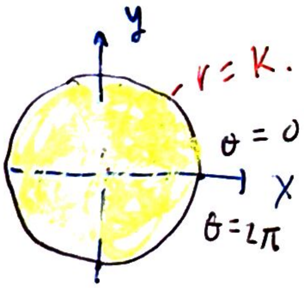
$$V = \iiint dV = \int_{-K}^{K} \int_{-\sqrt{K^2-x^2}}^{\sqrt{K^2-x^2}} \int_{-\sqrt{K^2-x^2-y^2}}^{\sqrt{K^2-x^2-y^2}} dz\, dy\, dx.$$

Cartesianas:

Ecs. de una esfera en coordenadas cilíndricas

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z.$$

$$z = \pm\sqrt{K^2 - r^2\cos^2\theta - r^2\sin^2\theta} = \pm\sqrt{K^2 - r^2}$$



$$E: 0 \le \theta \le 2\pi, \; 0 \le r \le K.$$

$$-\sqrt{K^2-r^2} \le z \le \sqrt{K^2-r^2}$$

$$V = \iiint dV = \int_0^{2\pi} \int_0^{K} \int_{-\sqrt{K^2-r^2}}^{\sqrt{K^2-r^2}} r\, dz\, dr\, d\theta.$$

$$V = \int_0^{2\pi} d\theta \int_0^{K} r \quad z \Big]_{-\sqrt{K^2-r^2}}^{\sqrt{K^2-r^2}} dr.$$

$$V = 2\pi \int_0^{K} \underbrace{2r (K^2-r^2)^{1/2}}_{u} \underbrace{dr.}_{-du.} \qquad \frac{2}{3} K^3.$$

$$V = -2\pi \frac{2}{3.} (K^2-r^2)^{3/2.} \Big]_{r=0}^{r=K.} = -0 + \frac{4\pi}{3} (K^2)^{3/2.}$$

$$V = \frac{4\pi}{3} K^3.$$

Esféricas $\quad 0 \leq \rho \leq K \quad 0 \leq \theta \leq 2\pi \quad 0 \leq \varphi \leq \pi$

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{K} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

$$V = 2\pi \int_0^{\pi} \sin\varphi \, d\varphi. \int_0^{K} \rho^2 \, d\rho.$$

$$V = 2\pi \left(\cos\varphi \Big]_{\pi}^{0} \right) \frac{\rho^3}{3} \Big]_0^{K.}$$

$$V = 2\pi (1+1) \frac{K^3}{3} = \frac{4\pi}{3} K^3.$$

$\rho = 2\cos\varphi$ $\qquad$ esfera de radio 1 centrada en $(0,0,1)$.