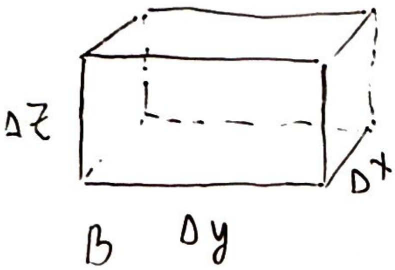
15.7 Integrales Triples.

Caja Rectangular $\quad B = [a,b] \times [c,d] \times [r,s].$



$V = \Delta x \, \Delta y \, \Delta z$ $\quad$ volumen caja

$$\iiint_B f(x,y,z)\,dV = \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx$$

Intercanbian el orden, 6 permutaciones posibles

$dx\,dy\,dz \qquad dx\,dz\,dy \qquad dy\,dx\,dz \qquad dy\,dz\,dx$

$dz\,dx\,dy \qquad dz\,dy\,dx.$

Ejercicio 1: Evalúe $\qquad$ sólido

a $\iiint_B (xy + 3z^2)\,dV \qquad B = [0,2] \times [0,1] \times [0,3]$

$$I_a = \int_0^1 \int_0^3 \int_0^2 (xy + 3z^2)\,dx\,dz\,dy.$$

$$I_a = \int_0^1 \int_0^3 \left. \frac{x^2 y}{2} + 3z^2 x \right]_{x=0}^{2} dz\,dy. = \int_0^1 \int_0^3 (2y + 6z^2)\,dz\,dy$$

$$I_a = \int_0^1 \left. 2yz + 2z^3 \right]_{z=0}^{z=3} dy = \int_0^1 (6y + 54)\,dy.$$

$$I_a = \left. 3y^2 + 54y \right]_{y=0}^{y=1} = 3 + 54 = 57.$$

b. $\iiint_B e^{x+y+z}\,dV \qquad B=[0,\ln 2]\times[0,\ln 3]\times[0,\ln 4].$

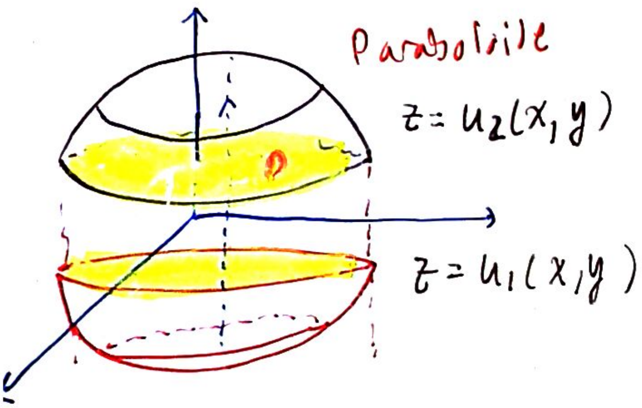
$$I_b = \int_0^{\ln 2}\int_0^{\ln 3}\int_0^{\ln 4} e^{x+y+z}\,dz\,dy\,dx. \qquad \text{son } e^{x+y+z}.$$

$$I_b = \int_0^{\ln 2}\int_0^{\ln 3} e^{x+y} e^{z}\Big]_{z=0}^{z=\ln 4} dy\,dx \qquad e^{\ln 4}-e^{0} = 4-1=3.$$

$$I_b = 3\int_0^{\ln 2} e^{x+y}\Big]_{y=0}^{y=\ln 3} dx \qquad e^{x+\ln 3}-e^{x} = 3e^{x}-e^{x}=2e^{x}$$

$$I_b = 3\cdot 2\int_0^{\ln 2} e^{x}\,dx = 3\cdot 2\, e^{x}\Big]_{x=0}^{x=\ln 2} = 3\cdot 2(2-1) = 3\cdot 2\cdot 1$$

Integrales triples sobre un sólido general.



$(x,y)\in D$

$$u_1(x,y)\le z\le u_2(x,y)$$

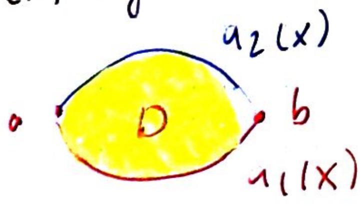
D es la región de proyección del sólido E sobre el plano xy.

$$\iiint_E f\,dV = \iint_D\left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right)dA.$$

D como una región tipo I, tipo II o en polares.

$a\le x\le b.$

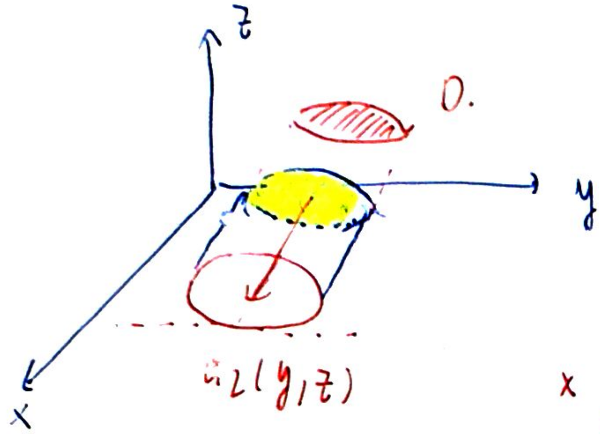
$a_1(x)\le y\le a_2(x)$



$$\iiint_E f\,dV = \int_a^b\int_{a_1(x)}^{a_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Sólido Tipo I, con región D tipo I.

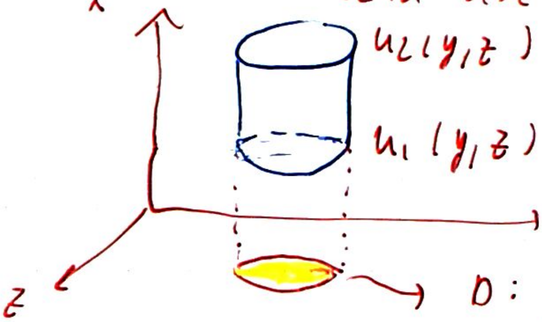
Sólido tipo II $\quad u_1(y,z) \le x \le u_2(y,z)$.



$(y,z) \in D$

$$\iiint_E f\,dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f\,dx \right) dA.$$

atrás hacia adelante.



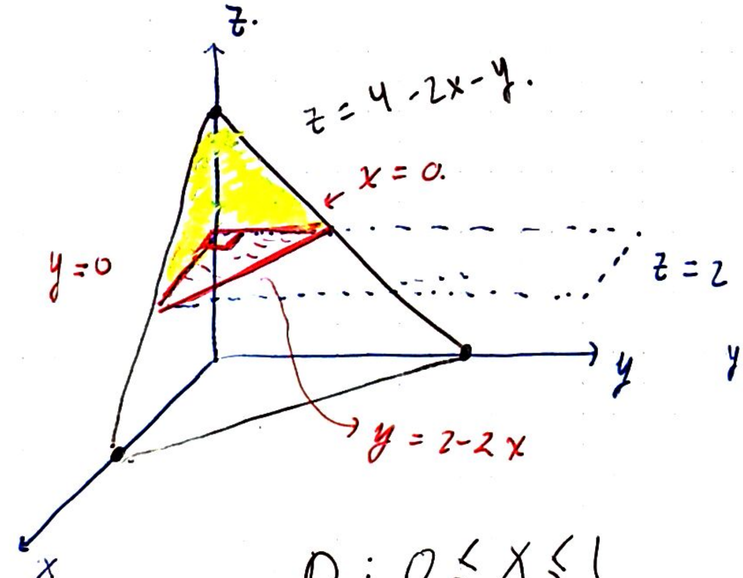
$D: \; c \le y \le d$

$f(y) \le x \le g(y)$

$$\text{Área} = \iint_D dA$$

Ejercicio 2: Evalúe $\displaystyle\iiint_E 6x\,dV$.

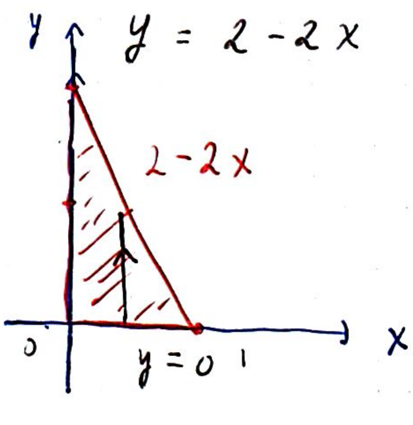
E se encuentra debajo del plano $z = 4-2x-y$, encima del plano $z=2$ y entre $x=0$, $y=0$. 1er octante.



$2 \le z \le 4-2x-y$.

Límites D.

$2 = 4 - 2x - y$.



$D: \; 0 \le x \le 1$

$\quad 0 \le y \le 2-2x$

sólido E: $0 \le x \le 1$, $0 \le y \le 2-2x$, $2 \le z \le 4-2x-y$.

$$I_2 = \iiint_E 6x\,dV = \int_0^1 \int_0^{2-2x} \int_2^{4-2x-y} 6x \,dz\,dy\,dx.$$

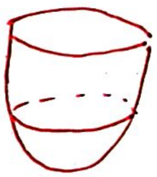
$$6x \int_2^{4-2x-y} dz = (2-2x-y)6x = 12x - 12x^2 - 6xy.$$

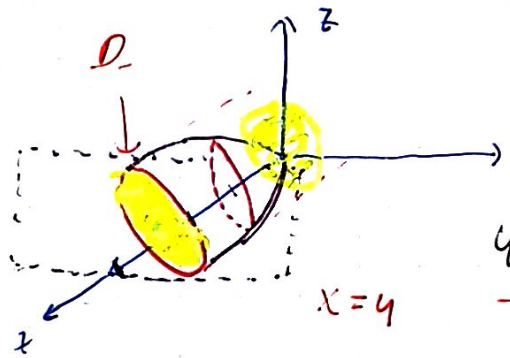
$$I_2 = \int_0^1 \int_0^{2-2x} (12x - 12x^2 - 6xy)\,dy\,dx$$

$$I_2 = \int_0^1 (12x - 12x^2)(2-2x) - 3x(2-2x)^2\,dx$$

Ejercicio 3: Evalúe $\iiint_E x\,dV$

E está entre el paraboloide $x = 4y^2 + 4z^2$ & $x = 4$





$4y^2 + 4z^2 \le x \le 4.$

D es la "intersección" entre $4y^2 + 4z^2$ & 4:

$4y^2 + 4z^2 \le 4$. D: $y^2 + z^2 \le 1$ disco radio 1

$$I_3 = \iiint_E x\,dV = \iint_D \left(\int_{4y^2+4z^2}^{4} x\,dx \right) dA$$

$4y^2 + 4z^2 \le x \le 4$

D: $0 \le y^2 + z^2 \le 1$

$$I_3 = \iint_D \frac{x^2}{2}\Big]_{x=4y^2+4z^2}^{x=4} dA. \qquad (4y^2+4z^2)^2 = 16(y^2+z^2)^2$$

$$I_3 = \iint_{D.} 8 - 8(y^2+z^2)^2 \, dA.$$

como D es un disco use coordenadas polares

$dA = r\,dr\,d\theta$, $0 \leq \theta \leq 2\pi.$, $0 \leq r \leq 1$

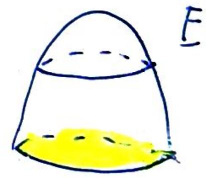
use $y^2+z^2 = r^2$, $y = r\sin\theta.$ $z = r\cos\theta.$

$$I_3 = \int_0^{2\pi}\int_0^1 (8 - 8r^4)\, r\,dr\,d\theta.$$

$$I_3 = \left(\int_0^{2\pi} d\theta\right)\int_0^1 (8r - 8r^5)\,dr$$

$$I_3 = 2\pi\left(4r^2 - \frac{8}{6}r^6\Big]_{r=0}^{r=1}\right) = 2\pi\left(4 - \frac{8}{6}\right) = \frac{16\pi}{3.}$$

Aplicaciones: $\iiint f\,dV \to m^4$



dV volumen de una caja infinitesimal

Volumen: $V = \iiint_E dV.$ Área $A = \iint_D dA$

Densidad volúmetrica $\rho = \dfrac{m}{V} \Rightarrow m = \rho V.$

Si la densidad no es constante $\rho(x,y,z)$

$dm = \rho(x,y,z)\,dV.$

masa: $m = \iiint_E \rho(x,y,z)\,dV$

Ejercicio 4: Considere el sólido $E: 0 \le x \le 2,\ 0 \le y \le x^3,\ 0 \le z \le y$

plano xy.

a. Encuentre la masa del objeto si la densidad $\rho = xy^2$.

$$m = \iiint_E xy^2 \, dV = \int_0^2 \int_0^{x^3} \int_0^y xy^2 \, dz \, dy \, dx.$$

$$xy^2 z \Big]_{z=0}^{y} = xy^2 \cdot y = xy^3.$$

$$m = \int_0^2 \int_0^{x^3} xy^3 \, dy \, dx \quad , \quad \frac{xy^4}{4}\Big]_{y=0}^{y=x^3} = \frac{x \cdot x^{12}}{4} = \frac{x^{13}}{4}.$$

$$m = \int_0^2 \frac{x^{13}}{4} \, dx = \frac{x^{14}}{4 \cdot 14}\Big]_{x=0}^{x=2} = \frac{2^{14}}{8 \cdot 7} = \frac{2^{11}}{7}.$$

b. Encuentre el volumen del objeto $V = \iiint_E dV = \iint_D \overbrace{f(x,y)}^{y.} \, dA.$

$$V = \iiint_E dV = \int_0^2 \int_0^{x^3} \int_0^{y} dz \, dy \, dx = \int_0^2 \int_0^{x^3} y \, dy \, dx$$

$$V = \int_0^2 \frac{1}{2} y^2 \Big]_{y=0}^{y=x^3} dx = \frac{1}{2} \int_0^2 x^6 \, dx = \frac{1}{2 \cdot 7} x^7 \Big]_{x=0}^{x=2.}$$

$$V = \frac{2^7}{2 \cdot 7} = \frac{2^6}{7} = \frac{64}{7}.$$

2a $\int_0^1 \int_0^{\pi/2} y \cos(xy)\, dy\, dx$ cuite IPP.

2b) $\int_0^2 \int_{y^2}^4 y^2 \sqrt{x} \sin x \, dx\, dy$ = número.

$u = xy \quad du = y\,dx$

2a $\int_0^{\pi/2} \int_0^1 y \cos(xy)\, dx\, dy. = \int_0^{\pi/2} \frac{y}{y} \sin(xy) \Big]_{x=0}^{x=1} dy.$

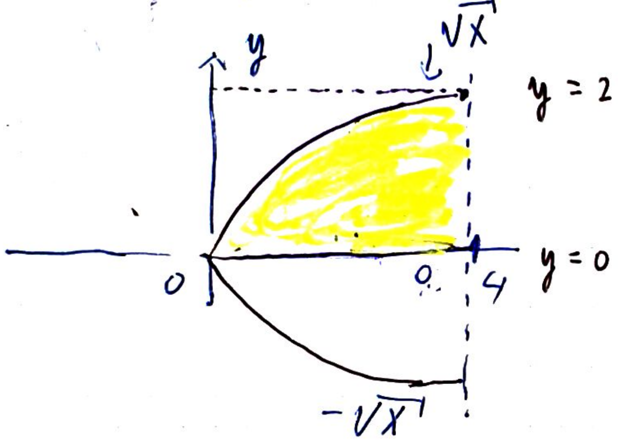
$I_{2a} = \int_0^{\pi/2} \sin y \, dy = \cos y \Big]_{\pi/2}^{0} = 1$

$x = y^2.$

2b) $\int_0^2 \int_{y^2}^4 y^2 x^{1/2} \sin x \, dx\, dy.$

$y^2 \leq x \leq 4$

$0 \leq y \leq 2.$



$0 \leq y \leq \sqrt{x}$

$0 \leq x \leq 4$

$\int_0^4 \int_0^{\sqrt{x}} y^2 x^{1/2} \sin x \, dy\, dx.$

$\int_0^4 \frac{1}{3} x^{3/2} x^{1/2} \sin x \;\, dx$