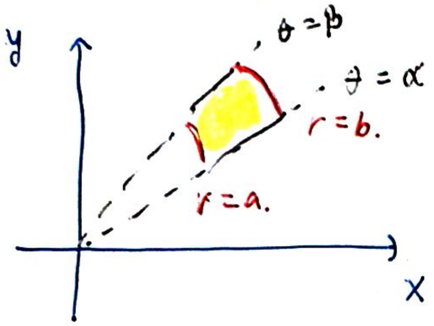
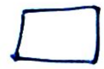
# 15.4 Integrales Dobles en Coordenadas Polares.




$R: \alpha \leq \theta \leq \beta. \qquad a \leq r \leq b.$

Rectángulo Polar $R$.

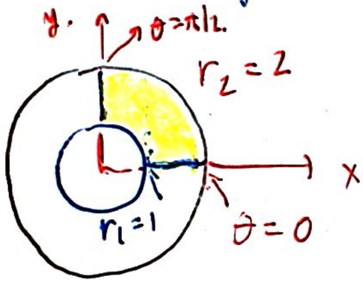
$dA = r\,dr\,d\theta. \qquad dA = dx\,dy$



$$\iint_{R} f(x,y)\,dA = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Ejercicio 1: Evalúe $\displaystyle\iint_R \sqrt{x^2+y^2}\,dA.$

$R$ es el cuarto de anillo en el 1er cuadrante con radio interno 1 y radio externo 2.




$R: \boxed{1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi/2}$

Simplifique

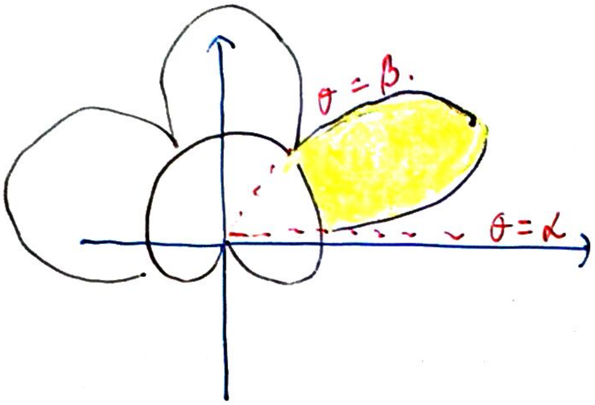
$$\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= r\sqrt{\cos^2\theta+\sin^2\theta} = r.$$

$$I_1 = \iint_R \underbrace{\sqrt{x^2+y^2}}_{r}\;\underbrace{dA.}_{r\,dr\,d\theta.} = \int_0^{\pi/2}\left(\int_1^2 r^2\,dr\right)d\theta$$

$$I_1 = \int_0^{\pi/2} \frac{r^3}{3}\Big]_{r=1}^{r=2} d\theta. = \int_0^{\pi/2} \frac{7}{3}\,d\theta. = \frac{7}{3}\,\theta\Big]_{\theta=0}^{\theta=\pi/2} = \frac{7\pi}{6}.$$

# Región Polar.

$R: \alpha \leq \theta \leq \beta. \qquad r_1(\theta) \leq \theta \leq r_2(\theta))$




$r_1 = 2 + \sin\theta, \quad r_2 = \sin 2\theta.$

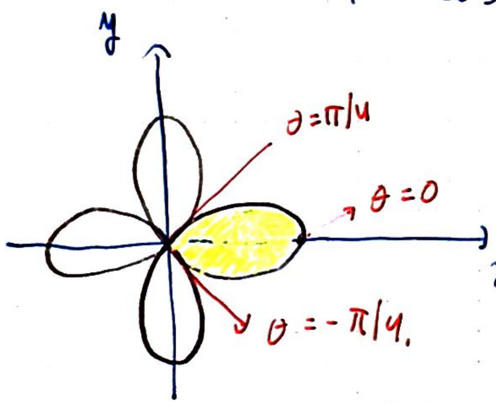
$$\iint_R f\,dA = \int_\alpha^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Área región Polar: $A = \iint_R dA.$

$$A = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} r\,dr\,d\theta. = \int_\alpha^\beta \frac{1}{2} r^2 \Big|_{r_1(\theta)}^{r_2(\theta)} d\theta.$$

$$A = \frac{1}{2}\int_\alpha^\beta \left[r_2^2(\theta) - r_1^2(\theta)\right] d\theta$$ misma fórmula secc 10.4.

Ejercicio 2: Encuentre el área de un pétalo de la rosa $r = \cos 2\theta.$




$A = \iint_R dA. \qquad 0 \leq r \leq \cos 2\theta.$ ✓

$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}.$ ✓

$\cos 2\theta = 0. \qquad 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{-\pi}{4}.$

$$A = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta. = \int_{-\pi/4}^{\pi/4} \frac{r^2}{2}\Big|_0^{\cos 2\theta} d\theta = \int_{-\pi/4}^{\pi/4} \frac{\cos^2(2\theta)}{2} d\theta.$$

Difícil intercambiar el orden.

cos · y $\cos^2\theta$ son funciones pares. $\int \sin = -\cos x$

$$A = \int_0^{\pi/4} \cos^2(2\theta)\,d\theta. = \frac{1}{2}\int_0^{\pi/4}(1+\cos 4\theta)\,d\theta. \qquad \int \cos x = \sin x$$

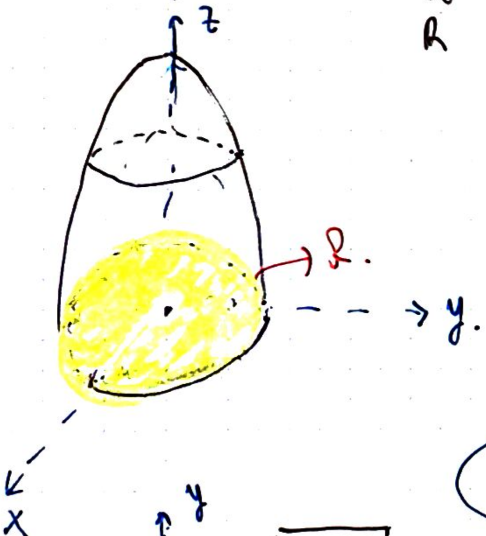
$$\cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta) \quad ó \quad \sin^2 2\theta = \frac{1}{2}(1-\cos 4\theta)$$

$$A = \frac{1}{2}\left(\theta + \frac{1}{4}\sin 4\theta \Big]_{\theta=0}^{\theta=\pi/4}\right) = \frac{1}{2}\left(\frac{\pi}{4} + \cancel{\frac{1}{4}\sin\pi}^{0} - \cancel{0}^{0} - \cancel{\sin 0}^{0}\right)$$

$$\boxed{A = \pi/8.}$$

Ejercicio 3: Encuentre el volumen del sólido entre el paraboloide $z = 18 - 2x^2 - 2y^2$ y el plano $xy$.

Volumen: $V = \iint_R f(x,y)\,dA.$ $\quad f(x,y) \geq 0$ en $R$




Encuentre las ecs. que describen $R$

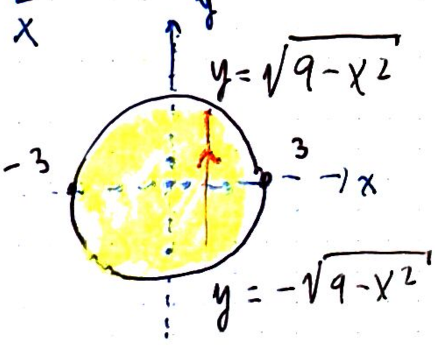
es la proyección de $z$ sobre el plano $xy$.

$z \geq 0. \quad 18 - 2x^2 - 2y^2 \geq 0$

$18 \geq 2x^2 + 2y^2$

$0 \leq x^2 + y^2 \leq 9.$ región de proyección

círculo de radio 3.




$R: -3 \leq x \leq 3 \quad -\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}$

cartesianas.

$R.$

Polares: $0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi.$

Volumen $V = \iint_R (18 - 2x^2 - 2y^2)\, dA.$

Se recomienda usar polares $dA = r\,dr\,d\theta.$

$$f(r\cos\theta, r\sin\theta) = 18 - \underbrace{2r^2\cos^2\theta - 2r^2\sin^2\theta}.$$

$$= 18 - 2r^2.$$

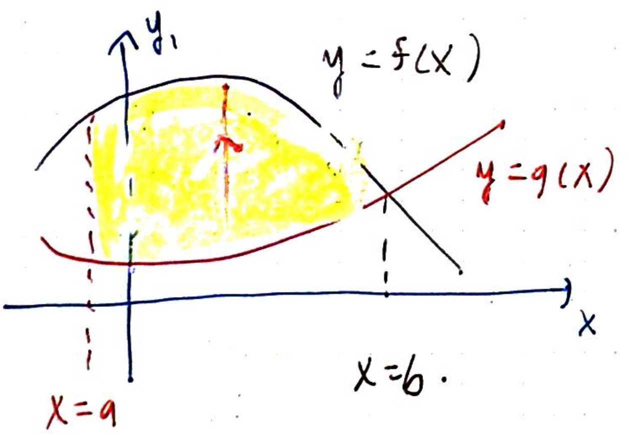
$$V = \iint_R f\, dA. = \int_0^3 \int_0^{2\pi} (18 - 2r^2)\, r\, d\theta\, dr = \int_0^{2\pi} \int_0^3 (18 - 2r^2)\, r\, dr\, d\theta.$$

$$V = \int_0^3 (18r - 2r^3)\, \theta \Big]_{\theta=0}^{\theta=2\pi} dr = \pi \int_0^3 (36r - 4r^3)\, dr.$$

$$V = \pi\left( 18r^2 - r^4 \Big]_{r=0}^{r=3} \right) = \pi\left( \underbrace{18 \cdot 9}_{162} - 81 \right) = 81\pi.$$

$$V = \Big( \underbrace{\int_0^{2\pi} d\theta}_{2\pi,} \Big)\Big( \int_0^3 18r - 2r^3\, dr \Big) = 2\pi\left( 9r^2 - \frac{r^4}{2} \Big]_0^3 \right)$$

## Integrales Dobles en Coordenadas Generales.



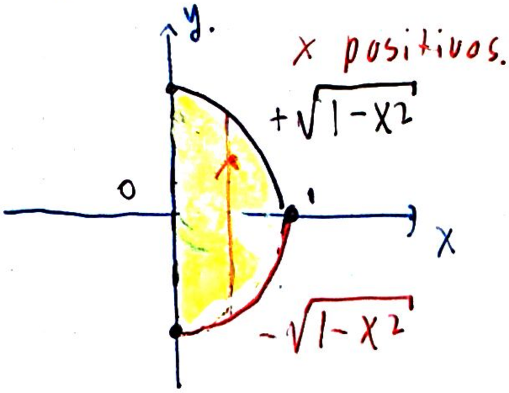

$a \le x \le b. \quad g(x) \le y \le f(x)$

$$\iint_R Z\, dA = \int_a^b \int_{g(x)}^{f(x)} z(x, y)\, dy\, dx$$

Región Tipo I.

Ejercicio 4: Considere la región encerrada por $x=0$ & $x=\sqrt{1-y^2} \rightarrow$ ¿ ¿ )

a. Bosqueje la región y descríbala como una tipo I, tipo II y como un rectángulo polar.

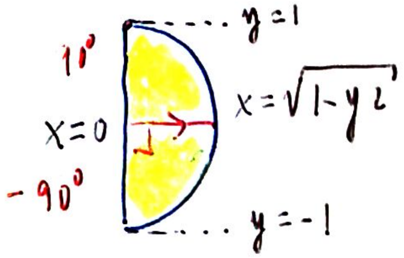



$$x^2 = 1-y^2 \Rightarrow x^2+y^2=1$$

Tipo I: $f(x) \leq y \leq g(x)$

$$y = \pm\sqrt{1-x^2}$$

$$\boxed{0 \leq x \leq 1 \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}}$$




Tipo II: $f(y) \leq x \leq g(y)$.

$$\boxed{-1 \leq y \leq 1 \qquad 0 \leq x \leq \sqrt{1-y^2}}$$

Polares: $\boxed{0 \leq r \leq 1 \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.}$

b. Evalúe $\iint_D (3x-6y)\, dA.$ $\qquad dA = r\,dr\,d\theta.$

$$I_4 = \int_{-\pi/2}^{\pi/2} \int_0^1 \left(3r^2\cos\theta - 6r^2\sin\theta\right) dr\, d\theta.$$

$$I_4 = \int_{-\pi/2}^{\pi/2} r^3\cos\theta - 2r^3\sin\theta \Big]_{r=0}^{r=1} d\theta. = \int_{-\pi/2}^{\pi/2} \cos\theta - 2\sin\theta \; d\theta.$$

$$I_y = \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta - 2\cancel{\int_{-\pi/2}^{\pi/2} \sin\theta \, d\theta}^{\,0}$$

par. impar

$$I_y = 2\int_0^{\pi/2} \cos\theta \, d\theta. = 2\sin\theta\Big]_{\theta=0}^{\theta=\pi/2} = 2$$

El problema es más complicado de trabajar en cartesianas.

$$I_y = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}} (3x-6y)\,dx\,dy. = \int_{-1}^{1} \frac{3}{2}x^2 - 6yx\Big]_{x=0}^{x=\sqrt{1-y^2}} dy.$$

$$I_y = \int_{-1}^{1} \frac{3}{2}(1-y^2) - 6y\sqrt{1-y^2}\; dy.$$

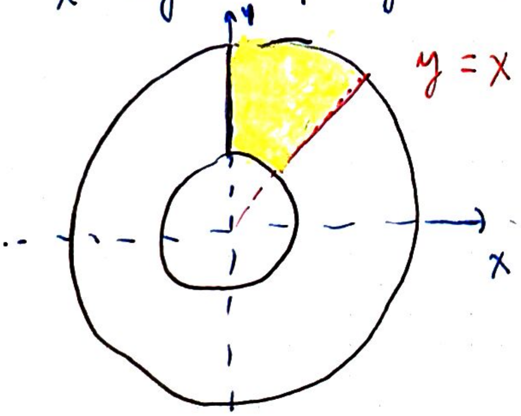
$$I_y = \frac{3}{2}\int_{-1}^{1}\underbrace{(1-y^2)}_{\text{PAR}}dy - 6\cancel{\int_{-1}^{1} \underbrace{y}_{\text{impar}}\underbrace{\sqrt{1-y^2}}_{\text{par.}}\,dy}^{\,0.}$$

misma respuesta

$$I_y = 3\int_0^1 (1-y^2)\,dy = 3y - y^3\Big]_{y=0}^{y=1} = 3-1 = 2.$$

Ejercicio 5: Evalúe $\iint_R y \sin(x^2+y^2)\,dA.$

$R$ es la región entre las circunferencias $x^2+y^2=1$ & $x^2+y^2=9$ y las rectas $x=0$ & $y=x.$




$1 \le r \le 3.$ $\quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}.$

$r\cos\theta = 0. \Rightarrow \theta = \cos^{-1}(0) = \pi/2$

$r\sin\theta = r\cos\theta.$

$\tan\theta = 1 \Rightarrow \theta = \pi/4.$

$x^2 + y^2 = r^2$ $\qquad dA = r\,dr\,d\theta.$

$$\iint_R 4\sin(x^2+y^2)\,dA. = \int_{\pi/4}^{\pi/2}\int_1^3 4\sin(r^2)\,r\,dr\,d\theta.$$

$$\int_{\pi/4}^{\pi/2} d\theta = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4.}$$

$$2\int_1^3 \sin(\underbrace{r^2}_{u})(\underbrace{2r\,dr}_{du.}) = -2\cos(r^2)\Big]_1^3 = -2\cos 9 + 2\cos 1$$

$$\iint_R 4\sin(x^2+y^2)\,dA = \frac{\pi}{4.}\left(2\cos 1 - 2\cos 9\right)$$