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CHAPTER 8 CONCEPT CHECK ANSWERS

1. (a) How is the length of a curve defined?

We can approximate a curve C by a polygon with vertices P_i along C. The length L of C is defined to be the limit of the lengths of these inscribed polygons:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

(b) Write an expression for the length of a smooth curve given by $y = f(x)$, $a \leqslant x \leqslant b$.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

(c) What if x is given as a function of y?

If $x = g(y)$, $c \leqslant y \leqslant d$, then $L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$.

2. (a) Write an expression for the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis.

$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

(b) What if x is given as a function of y?

If $x = g(y)$, $c \leqslant y \leqslant d$, then $S = \int_c^d 2\pi y \sqrt{1 + [g'(y)]^2}\, dy$.

(c) What if the curve is rotated about the y-axis?

$$S = \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2}\, dx$$

or $$S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2}\, dy$$

3. Describe how we can find the hydrostatic force against a vertical wall submersed in a fluid.

We divide the wall into horizontal strips of equal height Δx and approximate each by a rectangle with horizontal length $f(x_i)$ at depth x_i. If δ is the weight density of the fluid, then the hydrostatic force is

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} \delta x_i f(x_i)\, \Delta x = \int_a^b \delta x f(x)\, dx$$

4. (a) What is the physical significance of the center of mass of a thin plate?

The center of mass is the point at which the plate balances horizontally.

(b) If the plate lies between $y = f(x)$ and $y = 0$, where $a \leqslant x \leqslant b$, write expressions for the coordinates of the center of mass.

$$\bar{x} = \frac{1}{A} \int_a^b x f(x)\, dx \qquad \text{and} \qquad \bar{y} = \frac{1}{A} \int_a^b \tfrac{1}{2}[f(x)]^2\, dx$$

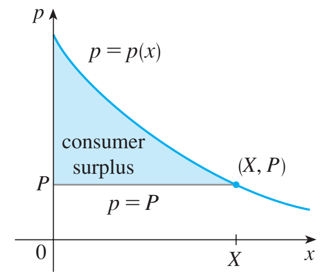
where $A = \int_a^b f(x)\, dx$.

5. What does the Theorem of Pappus say?

If a plane region $\mathcal{R}$ that lies entirely on one side of a line ℓ in its plane is rotated about ℓ, then the volume of the resulting solid is the product of the area of $\mathcal{R}$ and the distance traveled by the centroid of $\mathcal{R}$.

6. Given a demand function $p(x)$, explain what is meant by the consumer surplus when the amount of a commodity currently available is X and the current selling price is P. Illustrate with a sketch.

The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P [when they were willing to purchase it at price $p(x)$], corresponding to an amount demanded of X.



7. (a) What is the cardiac output of the heart?

It is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta.

(b) Explain how the cardiac output can be measured by the dye dilution method.

An amount A of dye is injected into part of the heart and its concentration $c(t)$ leaving the heart is measured over a time interval $[0, T]$ until the dye has cleared. The cardiac output is given by $A/\int_0^T c(t)\, dt$.

8. What is a probability density function? What properties does such a function have?

Given a random variable X, its probability density function f is a function such that $\int_a^b f(x)\, dx$ gives the probability that X lies between a and b. The function f has the properties that $f(x) \geqslant 0$ for all x, and $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

9. Suppose $f(x)$ is the probability density function for the weight of a female college student, where x is measured in pounds.

(a) What is the meaning of the integral $\int_0^{130} f(x)\, dx$?

It represents the probability that a randomly chosen female college student weighs less than 130 pounds.

(b) Write an expression for the mean of this density function.

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^{\infty} x f(x)\, dx$$

[since $f(x) = 0$ for $x < 0$]

(c) How can we find the median of this density function?

The median of f is the number m such that

$$\int_m^{\infty} f(x)\, dx = \frac{1}{2}$$

10. What is a normal distribution? What is the significance of the standard deviation?

A normal distribution corresponds to a random variable X that has a probability density function with a bell-shaped graph and equation given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

where μ is the mean and the positive constant σ is the standard deviation. σ measures how spread out the values of X are.

CHAPTER 9 CONCEPT CHECK ANSWERS

1. (a) What is a differential equation?

It is an equation that contains an unknown function and one or more of its derivatives.

(b) What is the order of a differential equation?

It is the order of the highest derivative that occurs in the equation.

(c) What is an initial condition?

It is a condition of the form $y(t_0) = y_0$.

2. What can you say about the solutions of the equation $y' = x^2 + y^2$ just by looking at the differential equation?

The equation tells us that the slope of a solution curve at any point (x, y) is $x^2 + y^2$. Note that $x^2 + y^2$ is always positive except at the origin, where $y' = x^2 + y^2 = 0$. Thus there is a horizontal tangent at $(0, 0)$ but nowhere else and the solution curves are increasing everywhere.

3. What is a direction field for the differential equation $y' = F(x, y)$?

A direction field (or slope field) for the differential equation $y' = F(x, y)$ is a two-dimensional graph consisting of short line segments with slope $F(x, y)$ at point (x, y).

4. Explain how Euler's method works.

Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field. Stop after a short time, look at the slope at the new location, and proceed in that direction. Keep stopping and changing direction according to the direction field until the approximation is complete.

5. What is a separable differential equation? How do you solve it?

It is a differential equation in which the expression for dy/dx can be factored as a function of x times a function of y, that is, $dy/dx = g(x)f(y)$. We can solve the equation by rewriting it as $[1/f(y)]\,dy = g(x)\,dx$, integrating both sides, and solving for y.

6. What is a first-order linear differential equation? How do you solve it?

A first-order linear differential equation is a differential equation that can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval. To solve such an equation, we multiply both sides by the integrating factor $I(x) = e^{\int P(x)\,dx}$ to put it in the form $\big(I(x)\,y\big)' = I(x)\,Q(x)$. We then integrate both sides and solve for y.

7. (a) Write a differential equation that expresses the law of natural growth. What does it say in terms of relative growth rate?

If $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then $\dfrac{dP}{dt} = kP$.

In this case the relative growth rate, $\dfrac{1}{P}\dfrac{dP}{dt}$, is constant.

(b) Under what circumstances is this an appropriate model for population growth?

It is an appropriate model under ideal conditions: unlimited environment, adequate nutrition, absence of predators and disease.

(c) What are the solutions of this equation?

If $P(0) = P_0$, the initial value, then the solutions are $P(t) = P_0e^{kt}$.

8. (a) Write the logistic differential equation.

The logistic differential equation is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where M is the carrying capacity.

(b) Under what circumstances is this an appropriate model for population growth?

It is an appropriate model for population growth if the population grows at a rate proportional to the size of the population in the beginning, but eventually levels off and approaches its carrying capacity because of limited resources.

9. (a) Write Lotka-Volterra equations to model populations of food-fish (F) and sharks (S).

$$\frac{dF}{dt} = kF - aFS \qquad \text{and} \qquad \frac{dS}{dt} = -rS + bFS$$

(b) What do these equations say about each population in the absence of the other?

In the absence of sharks, an ample food supply would support exponential growth of the fish population, that is, $dF/dt = kF$, where k is a positive constant. In the absence of fish, we assume that the shark population would decline at a rate proportional to itself, that is $dS/dt = -rS$, where r is a positive constant.

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CHAPTER 10 CONCEPT CHECK ANSWERS

1. (a) What is a parametric curve?

A parametric curve is a set of points of the form $(x, y) = (f(t), g(t))$, where f and g are functions of a variable t, the parameter.

(b) How do you sketch a parametric curve?

Sketching a parametric curve, like sketching the graph of a function, is difficult to do in general. We can plot points on the curve by finding $f(t)$ and $g(t)$ for various values of t, either by hand or with a calculator or computer. Sometimes, when f and g are given by formulas, we can eliminate t from the equations $x = f(t)$ and $y = g(t)$ to get a Cartesian equation relating x and y. It may be easier to graph that equation than to work with the original formulas for x and y in terms of t.

2. (a) How do you find the slope of a tangent to a parametric curve?

You can find dy/dx as a function of t by calculating

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \text{if } dx/dt \neq 0$$

(b) How do you find the area under a parametric curve?

If the curve is traced out once by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then the area is

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$$

[or $\int_\beta^\alpha g(t)f'(t)\,dt$ if the leftmost point is $(f(\beta), g(\beta))$ rather than $(f(\alpha), g(\alpha))$].

3. Write an expression for each of the following:

(a) The length of a parametric curve

If the curve is traced out once by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then the length is

$$L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$$

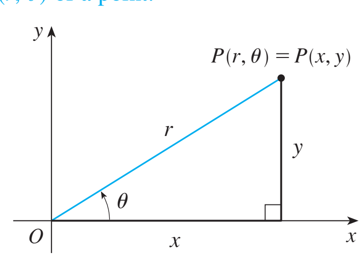
$$= \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

(b) The area of the surface obtained by rotating a parametric curve about the x-axis

$$S = \int_\alpha^\beta 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$$

$$= \int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

4. (a) Use a diagram to explain the meaning of the polar coordinates (r, θ) of a point.



(b) Write equations that express the Cartesian coordinates (x, y) of a point in terms of the polar coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta$$

(c) What equations would you use to find the polar coordinates of a point if you knew the Cartesian coordinates?

To find a polar representation (r, θ) with $r \geqslant 0$ and $0 \leqslant \theta < 2\pi$, first calculate $r = \sqrt{x^2 + y^2}$. Then θ is specified by $\tan\theta = y/x$. Be sure to choose θ so that (r, θ) lies in the correct quadrant.

5. (a) How do you find the slope of a tangent line to a polar curve?

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{d}{d\theta}(y)}{\dfrac{d}{d\theta}(x)} = \frac{\dfrac{d}{d\theta}(r\sin\theta)}{\dfrac{d}{d\theta}(r\cos\theta)}$$

$$= \frac{\left(\dfrac{dr}{d\theta}\right)\sin\theta + r\cos\theta}{\left(\dfrac{dr}{d\theta}\right)\cos\theta - r\sin\theta} \qquad \text{where } r = f(\theta)$$

(b) How do you find the area of a region bounded by a polar curve?

$$A = \int_a^b \tfrac{1}{2}r^2\,d\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta$$

(c) How do you find the length of a polar curve?

$$L = \int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta$$

$$= \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$$

$$= \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$$

6. (a) Give a geometric definition of a parabola.

A parabola is a set of points in a plane whose distances from a fixed point F (the focus) and a fixed line l (the directrix) are equal.

(b) Write an equation of a parabola with focus $(0, p)$ and directrix $y = -p$. What if the focus is $(p, 0)$ and the directrix is $x = -p$?

In the first case an equation is $x^2 = 4py$ and in the second case, $y^2 = 4px$.

7. (a) Give a definition of an ellipse in terms of foci.

An ellipse is a set of points in a plane the sum of whose distances from two fixed points (the foci) is a constant.

(b) Write an equation for the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a \geqslant b > 0$ and $c^2 = a^2 - b^2$.

(continued)

8. (a) Give a definition of a hyperbola in terms of foci.

A hyperbola is a set of points in a plane the difference of whose distances from two fixed points (the foci) is a constant. This difference should be interpreted as the larger distance minus the smaller distance.

(b) Write an equation for the hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$.

(c) Write equations for the asymptotes of the hyperbola in part (b).

$$y = \pm \frac{b}{a} x$$

9. (a) What is the eccentricity of a conic section?

If a conic section has focus F and corresponding directrix l, then the eccentricity e is the fixed ratio $|PF|/|Pl|$ for points P of the conic section.

(b) What can you say about the eccentricity if the conic section is an ellipse? A hyperbola? A parabola?

$e < 1$ for an ellipse; $e > 1$ for a hyperbola; $e = 1$ for a parabola

(c) Write a polar equation for a conic section with eccentricity e and directrix $x = d$. What if the directrix is $x = -d$? $y = d$? $y = -d$?

$$\text{directrix } x = d\text{:} \quad r = \frac{ed}{1 + e\cos\theta}$$

$$x = -d\text{:} \quad r = \frac{ed}{1 - e\cos\theta}$$

$$y = d\text{:} \quad r = \frac{ed}{1 + e\sin\theta}$$

$$y = -d\text{:} \quad r = \frac{ed}{1 - e\sin\theta}$$

CHAPTER 11 CONCEPT CHECK ANSWERS

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1. (a) What is a convergent sequence?

A convergent sequence $\{a_n\}$ is an ordered list of numbers where $\lim_{n\to\infty} a_n$ exists.

(b) What is a convergent series?

A series $\Sigma\, a_n$ is the *sum* of a sequence of numbers. It is convergent if the partial sums $s_n = \sum_{i=1}^{n} a_n$ approach a finite value, that is, $\lim_{n\to\infty} s_n$ exists as a real number.

(c) What does $\lim_{n\to\infty} a_n = 3$ mean?

The terms of the sequence $\{a_n\}$ approach 3 as n becomes large.

(d) What does $\Sigma_{n=1}^{\infty} a_n = 3$ mean?

By adding sufficiently many terms of the series, we can make the partial sums as close to 3 as we like.

2. (a) What is a bounded sequence?

A sequence $\{a_n\}$ is bounded if there are numbers m and M such that $m \leqslant a_n \leqslant M$ for all $n \geqslant 1$.

(b) What is a monotonic sequence?

A sequence is monotonic if it is either increasing or decreasing for all $n \geqslant 1$.

(c) What can you say about a bounded monotonic sequence?

Every bounded, monotonic sequence is convergent.

3. (a) What is a geometric series? Under what circumstances is it convergent? What is its sum?

A geometric series is of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

It is convergent if $|r| < 1$ and its sum is $\dfrac{a}{1-r}$.

(b) What is a p-series? Under what circumstances is it convergent?

A p-series is of the form $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$. It is convergent if $p > 1$.

4. Suppose $\Sigma\, a_n = 3$ and s_n is the nth partial sum of the series. What is $\lim_{n\to\infty} a_n$? What is $\lim_{n\to\infty} s_n$?

If $\Sigma\, a_n = 3$, then $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} s_n = 3$.

5. State the following.

(a) The Test for Divergence

If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(b) The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$.

- If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) The Comparison Test

Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

- If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.
- If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

(d) The Limit Comparison Test

Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If $\lim\limits_{n\to\infty} a_n/b_n = c$, where c is a finite number and $c > 0$, then either both series converge or both diverge.

(e) The Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

where $b_n > 0$ satisfies (i) $b_{n+1} \leqslant b_n$ for all n and (ii) $\lim\limits_{n\to\infty} b_n = 0$, then the series is convergent.

(f) The Ratio Test

- If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive.

(g) The Root Test

- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

6. (a) What is an absolutely convergent series?

A series $\Sigma\, a_n$ is called absolutely convergent if the series of absolute values $\Sigma\, |a_n|$ is convergent.

(b) What can you say about such a series?

If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

(c) What is a conditionally convergent series?

A series $\Sigma\, a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

(continued)

7. (a) If a series is convergent by the Integral Test, how do you estimate its sum?

The sum s can be estimated by the inequality

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant s \leqslant s_n + \int_{n}^{\infty} f(x)\,dx$$

where s_n is the nth partial sum.

(b) If a series is convergent by the Comparison Test, how do you estimate its sum?

We first estimate the remainder for the comparison series. This gives an upper bound for the remainder of the original series (as in Example 11.4.5).

(c) If a series is convergent by the Alternating Series Test, how do you estimate its sum?

We can use a partial sum s_n of an alternating series as an approximation to the total sum. The size of the error is guaranteed to be no more than $|a_{n+1}|$, the absolute value of the first neglected term.

8. (a) Write the general form of a power series.

A power series centered at a is

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

(b) What is the radius of convergence of a power series?

Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$,

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(c) What is the interval of convergence of a power series?

The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (b), the interval of convergence is (i) the single point $\{a\}$, (ii) $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ that can contain neither, either, or both of the endpoints.

9. Suppose $f(x)$ is the sum of a power series with radius of convergence R.

(a) How do you differentiate f? What is the radius of convergence of the series for f'?

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, then $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ with radius of convergence R.

(b) How do you integrate f? What is the radius of convergence of the series for $\int f(x)\,dx$?

$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$ with radius of convergence R.

10. (a) Write an expression for the nth-degree Taylor polynomial of f centered at a.

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

(b) Write an expression for the Taylor series of f centered at a.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(c) Write an expression for the Maclaurin series of f.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \qquad [a = 0 \text{ in part (b)}]$$

(d) How do you show that $f(x)$ is equal to the sum of its Taylor series?

If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth-degree Taylor polynomial of f and $R_n(x)$ is the remainder of the Taylor series, then we must show that

$$\lim_{n\to\infty} R_n(x) = 0$$

(e) State Taylor's Inequality.

If $|f^{(n+1)}(x)| \leqslant M$ for $|x-a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leqslant d$$

11. Write the Maclaurin series and the interval of convergence for each of the following functions.

(a) $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad R = 1$

(b) $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}, \quad R = \infty$

(c) $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}, \quad R = \infty$

(d) $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}, \quad R = \infty$

(e) $\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}, \quad R = 1$

(f) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}, \quad R = 1$

12. Write the binomial series expansion of $(1+x)^k$. What is the radius of convergence of this series?

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

The radius of convergence for the binomial series is 1.

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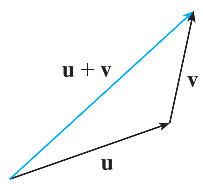
CHAPTER 12 CONCEPT CHECK ANSWERS

1. What is the difference between a vector and a scalar?

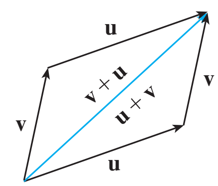
A scalar is a real number, whereas a vector is a quantity that has both a real-valued magnitude and a direction.

2. How do you add two vectors geometrically? How do you add them algebraically?

To add two vectors geometrically, we can use either the Triangle Law or the Parallelogram Law:



Triangle Law



Parallelogram Law

Algebraically, we add the corresponding components of the vectors.

3. If $\mathbf{a}$ is a vector and c is a scalar, how is $c\mathbf{a}$ related to $\mathbf{a}$ geometrically? How do you find $c\mathbf{a}$ algebraically?

For $c > 0$, $c\mathbf{a}$ is a vector with the same direction as $\mathbf{a}$ and length c times the length of $\mathbf{a}$. If $c < 0$, $c\mathbf{a}$ points in the direction opposite to $\mathbf{a}$ and has length $|c|$ times the length of $\mathbf{a}$. Algebraically, to find $c\mathbf{a}$ we multiply each component of $\mathbf{a}$ by c.

4. How do you find the vector from one point to another?

The vector from point $A(x_1, y_1, z_1)$ to point $B(x_2, y_2, z_2)$ is given by

$$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

5. How do you find the dot product $\mathbf{a} \cdot \mathbf{b}$ of two vectors if you know their lengths and the angle between them? What if you know their components?

If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$$

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

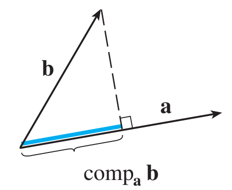
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

6. How are dot products useful?

The dot product can be used to find the angle between two vectors. In particular, it can be used to determine whether two vectors are orthogonal. We can also use the dot product to find the scalar projection of one vector onto another. Additionally, if a constant force moves an object, the work done is the dot product of the force and displacement vectors.

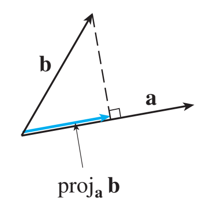
7. Write expressions for the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$. Illustrate with diagrams.

Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\quad \operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$



Vector projection of $\mathbf{b}$ onto $\mathbf{a}$:

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$$



8. How do you find the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors if you know their lengths and the angle between them? What if you know their components?

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ $(0 \leqslant \theta \leqslant \pi)$, then $\mathbf{a} \times \mathbf{b}$ is the vector with length $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ and direction orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, as given by the right-hand rule. If

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \qquad \text{and} \qquad \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

9. How are cross products useful?

The cross product can be used to create a vector orthogonal to two given vectors and it can be used to compute the area of a parallelogram determined by two vectors. Two nonzero vectors are parallel if and only if their cross product is $\mathbf{0}$. In addition, if a force acts on a rigid body, then the torque vector is the cross product of the position and force vectors.

(continued)

10. (a) How do you find the area of the parallelogram determined by **a** and **b**?

The area of the parallelogram determined by **a** and **b** is the length of the cross product: $|\mathbf{a} \times \mathbf{b}|$.

(b) How do you find the volume of the parallelepiped determined by **a**, **b**, and **c**?

The volume of the parallelepiped determined by **a**, **b**, and **c** is the magnitude of their scalar triple product: $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

11. How do you find a vector perpendicular to a plane?

If an equation of the plane is known, it can be written in the form $ax + by + cz + d = 0$. A normal vector, which is perpendicular to the plane, is $\langle a, b, c \rangle$ (or any nonzero scalar multiple of $\langle a, b, c \rangle$). If an equation is not known, we can use points on the plane to find two nonparallel vectors that lie in the plane. The cross product of these vectors is a vector perpendicular to the plane.

12. How do you find the angle between two intersecting planes?

The angle between two intersecting planes is defined as the acute angle θ between their normal vectors. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

13. Write a vector equation, parametric equations, and symmetric equations for a line.

A vector equation for a line that is parallel to a vector **v** and that passes through a point with position vector $\mathbf{r}_0$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. Parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the vector $\langle a, b, c \rangle$ are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

while symmetric equations are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

14. Write a vector equation and a scalar equation for a plane.

A vector equation of a plane that passes through a point with position vector $\mathbf{r}_0$ and that has normal vector **n** (meaning **n** is orthogonal to the plane) is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ or, equivalently, $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$.

A scalar equation of a plane through a point (x_0, y_0, z_0) with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

15. (a) How do you tell if two vectors are parallel?

Two (nonzero) vectors are parallel if and only if one is a scalar multiple of the other. In addition, two nonzero vectors are parallel if and only if their cross product is **0**.

(b) How do you tell if two vectors are perpendicular?

Two vectors are perpendicular if and only if their dot product is 0.

(c) How do you tell if two planes are parallel?

Two planes are parallel if and only if their normal vectors are parallel.

16. (a) Describe a method for determining whether three points P, Q, and R lie on the same line.

Determine the vectors $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PR} = \mathbf{b}$. If there is a scalar t such that $\mathbf{a} = t\mathbf{b}$, then the vectors are parallel and the points must all lie on the same line.

Alternatively, if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$, then $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, so P, Q, and R are collinear.

An algebraic method is to determine an equation of the line joining two of the points, and then check whether or not the third point satisfies this equation.

(b) Describe a method for determining whether four points P, Q, R, and S lie in the same plane.

Find the vectors $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$, $\overrightarrow{PS} = \mathbf{c}$. Then $\mathbf{a} \times \mathbf{b}$ is normal to the plane formed by P, Q, and R, and so S lies on this plane if $\mathbf{a} \times \mathbf{b}$ and **c** are orthogonal, that is, if $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$.

Alternatively, we can check if the volume of the parallelepiped determined by **a**, **b**, and **c** is 0 (see Example 12.4.5).

An algebraic method is to find an equation of the plane determined by three of the points, and then check whether or not the fourth point satisfies this equation.

17. (a) How do you find the distance from a point to a line?

Let P be a point not on the line L that passes through the points Q and R and let $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QP}$. The distance from the point P to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

(b) How do you find the distance from a point to a plane?

Let $P_0(x_0, y_0, z_0)$ be any point in the plane $ax + by + cz + d = 0$ and let $P_1(x_1, y_1, z_1)$ be a point not in the plane. If $\mathbf{b} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$, then the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of **b** onto the plane's normal vector $\mathbf{n} = \langle a, b, c \rangle$:

$$D = |\text{comp}_{\mathbf{n}}\,\mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(c) How do you find the distance between two lines?

Two skew lines L_1 and L_2 can be viewed as lying on two parallel planes, each with normal vector $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the direction vectors of L_1 and L_2. After choosing one point on L_1 and determining the equation of the plane containing L_2, we can proceed as in part (b). (See Example 12.5.10.)

(continued)

CHAPTER 12 CONCEPT CHECK ANSWERS *(continued)*

18. What are the traces of a surface? How do you find them?

The traces of a surface are the curves of intersection of the surface with planes parallel to the coordinate planes. We can find the trace in the plane $x = k$ (parallel to the yz-plane) by setting $x = k$ and determining the curve represented by the resulting equation. Traces in the planes $y = k$ (parallel to the xz-plane) and $z = k$ (parallel to the xy-plane) are found similarly.

19. Write equations in standard form of the six types of quadric surfaces.

Equations for the quadric surfaces symmetric with respect to the z-axis are as follows.

Ellipsoid: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Cone: $$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Elliptic paraboloid: $$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Hyperboloid of one sheet: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Hyperboloid of two sheets: $$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Hyperbolic paraboloid: $$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

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CHAPTER 13 CONCEPT CHECK ANSWERS

1. What is a vector function? How do you find its derivative and its integral?

A vector function is a function whose domain is a set of real numbers and whose range is a set of vectors. To find the derivative or integral, we can differentiate or integrate each component function of the vector function.

2. What is the connection between vector functions and space curves?

A continuous vector function $\mathbf{r}$ defines a space curve that is traced out by the tip of the moving position vector $\mathbf{r}(t)$.

3. How do you find the tangent vector to a smooth curve at a point? How do you find the tangent line? The unit tangent vector?

The tangent vector to a smooth curve at a point P with position vector $\mathbf{r}(t)$ is the vector $\mathbf{r}'(t)$. The tangent line at P is the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

4. If $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function, write the rules for differentiating the following vector functions.

(a) $\mathbf{u}(t) + \mathbf{v}(t)$

$$\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

(b) $c\mathbf{u}(t)$

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

(c) $f(t)\,\mathbf{u}(t)$

$$\frac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$$

(d) $\mathbf{u}(t) \cdot \mathbf{v}(t)$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

(e) $\mathbf{u}(t) \times \mathbf{v}(t)$

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

(f) $\mathbf{u}(f(t))$

$$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$$

5. How do you find the length of a space curve given by a vector function $\mathbf{r}(t)$?

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \leqslant t \leqslant b$, and the curve is traversed exactly once as t increases from a to b, then the length is

$$L = \int_a^b |\mathbf{r}'(t)|\,dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

6. (a) What is the definition of curvature?

The curvature of a curve is $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$ where $\mathbf{T}$ is the unit tangent vector.

(b) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{T}'(t)$.

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

(c) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

(d) Write a formula for the curvature of a plane curve with equation $y = f(x)$.

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

7. (a) Write formulas for the unit normal and binormal vectors of a smooth space curve $\mathbf{r}(t)$.

Unit normal vector: $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

Binormal vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

(b) What is the normal plane of a curve at a point? What is the osculating plane? What is the osculating circle?

The normal plane of a curve at a point P is the plane determined by the normal and binormal vectors $\mathbf{N}$ and $\mathbf{B}$ at P. The tangent vector $\mathbf{T}$ is orthogonal to the normal plane.

The osculating plane at P is the plane determined by the vectors $\mathbf{T}$ and $\mathbf{N}$. It is the plane that comes closest to containing the part of the curve near P.

The osculating circle at P is the circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which $\mathbf{N}$ points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature). It is the circle that best describes how C behaves near P; it shares the same tangent, normal, and curvature at P.

(continued)

8. (a) How do you find the velocity, speed, and acceleration of a particle that moves along a space curve?

If $\mathbf{r}(t)$ is the position vector of the particle on the space curve, the velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t)$, the speed is given by $|\mathbf{v}(t)|$, and the acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

(b) Write the acceleration in terms of its tangential and normal components.

$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, where $a_T = v'$ and $a_N = \kappa v^2$ ($v = |\mathbf{v}|$ is speed and κ is the curvature).

9. State Kepler's Laws.

- A planet revolves around the sun in an elliptical orbit with the sun at one focus.
- The line joining the sun to a planet sweeps out equal areas in equal times.
- The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

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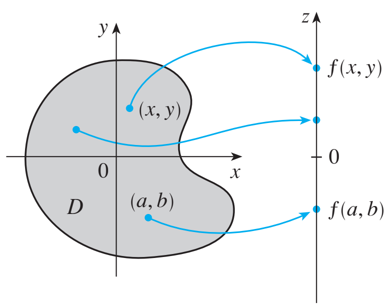
CHAPTER 14 CONCEPT CHECK ANSWERS

1. (a) What is a function of two variables?

A function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$.

(b) Describe three methods for visualizing a function of two variables.

One way to visualize a function of two variables is by graphing it, resulting in the surface $z = f(x, y)$. Another method is a contour map, consisting of level curves $f(x, y) = k$ (k a constant), which are horizontal traces of the graph of the function projected onto the xy-plane. Also, we can use an arrow diagram such as the one below.



2. What is a function of three variables? How can you visualize such a function?

A function f of three variables is a rule that assigns to each ordered triple (x, y, z) in its domain a unique real number $f(x, y, z)$. We can visualize a function of three variables by examining its level surfaces $f(x, y, z) = k$, where k is a constant.

3. What does

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

mean? How can you show that such a limit does not exist?

$\lim_{(x,y)\to(a,b)} f(x, y) = L$ means that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f. We can show that a limit at a point does not exist by finding two different paths approaching the point along which $f(x, y)$ has different limits.

4. (a) What does it mean to say that f is continuous at (a, b)?

A function f of two variables is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

(b) If f is continuous on $\mathbb{R}^2$, what can you say about its graph?

If f is continuous on $\mathbb{R}^2$, its graph will appear as a surface without holes or breaks.

5. (a) Write expressions for the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ as limits.

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

(b) How do you interpret $f_x(a, b)$ and $f_y(a, b)$ geometrically? How do you interpret them as rates of change?

If $f(a, b) = c$, then the point $P(a, b, c)$ lies on the surface S given by $z = f(x, y)$. We can interpret $f_x(a, b)$ as the slope of the tangent line at P to the curve of intersection of the vertical plane $y = b$ and S. In other words, if we restrict ourselves to the path along S through P that is parallel to the xz-plane, then $f_x(a, b)$ is the slope at P looking in the positive x-direction. Similarly, $f_y(a, b)$ is the slope of the tangent line at P to the curve of intersection of the vertical plane $x = a$ and S.

If $z = f(x, y)$, then $f_x(x, y)$ can be interpreted as the rate of change of z with respect to x when y is fixed. Thus $f_x(a, b)$ is the rate of change of z (with respect to x) when y is fixed at b and x is allowed to vary from a. Similarly, $f_y(a, b)$ is the rate of change of z (with respect to y) when x is fixed at a and y is allowed to vary from b.

(c) If $f(x, y)$ is given by a formula, how do you calculate f_x and f_y?

To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

6. What does Clairaut's Theorem say?

If f is a function of two variables that is defined on a disk D containing the point (a, b) and the functions f_{xy} and f_{yx} are both continuous on D, then Clairaut's Theorem states that $f_{xy}(a, b) = f_{yx}(a, b)$.

7. How do you find a tangent plane to each of the following types of surfaces?

(a) A graph of a function of two variables, $z = f(x, y)$

If f has continuous partial derivatives, an equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(b) A level surface of a function of three variables, $F(x, y, z) = k$

The tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

(continued)

8. Define the linearization of f at (a, b). What is the corresponding linear approximation? What is the geometric interpretation of the linear approximation?

The linearization of f at (a, b) is the linear function whose graph is the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear approximation of f at (a, b) is

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Geometrically, the linear approximation says that function values $f(x, y)$ can be approximated by values $L(x, y)$ from the tangent plane to the graph of f at $(a, b, f(a, b))$ when (x, y) is near (a, b).

9. (a) What does it mean to say that f is differentiable at (a, b)?

If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. In other words, a differentiable function is one for which the linear approximation as stated above is a good approximation when (x, y) is near (a, b).

(b) How do you usually verify that f is differentiable?

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

10. If $z = f(x, y)$, what are the differentials dx, dy, and dz?

The differentials dx and dy are independent variables that can be given any values. If f is differentiable, the differential dz is then defined by

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy$$

11. State the Chain Rule for the case where $z = f(x, y)$ and x and y are functions of one variable. What if x and y are functions of two variables?

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

12. If z is defined implicitly as a function of x and y by an equation of the form $F(x, y, z) = 0$, how do you find $\partial z/\partial x$ and $\partial z/\partial y$?

If F is differentiable and $\partial F/\partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

13. (a) Write an expression as a limit for the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$. How do you interpret it as a rate? How do you interpret it geometrically?

The directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

We can interpret it as the rate of change of f (with respect to distance) at (x_0, y_0) in the direction of $\mathbf{u}$.

Geometrically, if P is the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f and C is the curve of intersection of the graph of f with the vertical plane that passes through P in the direction of $\mathbf{u}$, then $D_{\mathbf{u}} f(x_0, y_0)$ is the slope of the tangent line to C at P.

(b) If f is differentiable, write an expression for $D_{\mathbf{u}} f(x_0, y_0)$ in terms of f_x and f_y.

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$

14. (a) Define the gradient vector ∇f for a function f of two or three variables.

If f is a function of two variables, then

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

For a function f of three variables,

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

(b) Express $D_{\mathbf{u}} f$ in terms of ∇f.

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

or $$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

(continued)

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(c) Explain the geometric significance of the gradient.

The gradient vector of f gives the direction of maximum rate of increase of f. On the graph of $z = f(x, y)$, ∇f points in the direction of steepest ascent. Also, the gradient vector is perpendicular to the level curves or level surfaces of a function.

15. What do the following statements mean?

(a) f has a local maximum at (a, b).

f has a local maximum at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b).

(b) f has an absolute maximum at (a, b).

f has an absolute maximum at (a, b) if $f(x, y) \leqslant f(a, b)$ for all points (x, y) in the domain of f.

(c) f has a local minimum at (a, b).

f has a local minimum at (a, b) if $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b).

(d) f has an absolute minimum at (a, b).

f has an absolute minimum at (a, b) if $f(x, y) \geqslant f(a, b)$ for all points (x, y) in the domain of f.

(e) f has a saddle point at (a, b).

f has a saddle point at (a, b) if $f(a, b)$ is a local maximum in one direction but a local minimum in another.

16. (a) If f has a local maximum at (a, b), what can you say about its partial derivatives at (a, b)?

If f has a local maximum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

(b) What is a critical point of f?

A critical point of f is a point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or one of these partial derivatives does not exist.

17. State the Second Derivatives Test.

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\,f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- If $D < 0$, then $f(a, b)$ is not a local maximum or minimum. The point (a, b) is a saddle point of f.

18. (a) What is a closed set in $\mathbb{R}^2$? What is a bounded set?

A closed set in $\mathbb{R}^2$ is one that contains all its boundary points. If one or more points on the boundary curve are omitted, the set is not closed.

A bounded set is one that is contained within some disk. In other words, it is finite in extent.

(b) State the Extreme Value Theorem for functions of two variables.

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

(c) How do you find the values that the Extreme Value Theorem guarantees?

- Find the values of f at the critical points of f in D.
- Find the extreme values of f on the boundary of D.
- The largest of the values from the above steps is the absolute maximum value; the smallest of these values is the absolute minimum value.

19. Explain how the method of Lagrange multipliers works in finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. What if there is a second constraint $h(x, y, z) = c$?

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$], we first find all values of x, y, z, and λ where $\nabla f(x, y, z) = \lambda\,\nabla g(x, y, z)$ and $g(x, y, z) = k$. (Thus we are finding the points from the constraint where the gradient vectors ∇f and ∇g are parallel.) Evaluate f at all the resulting points (x, y, z); the largest of these values is the maximum value of f, and the smallest is the minimum value of f.

If there is a second constraint $h(x, y, z) = c$, then we find all values of x, y, z, λ, and μ such that

$$\nabla f(x, y, z) = \lambda\,\nabla g(x, y, z) + \mu\,\nabla h(x, y, z)$$

Again we find the extreme values of f by evaluating f at the resulting points (x, y, z).

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CHAPTER 15 CONCEPT CHECK ANSWERS

1. Suppose f is a continuous function defined on a rectangle $R = [a, b] \times [c, d]$.

(a) Write an expression for a double Riemann sum of f. If $f(x, y) \geqslant 0$, what does the sum represent?

A double Riemann sum of f is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

where ΔA is the area of each subrectangle and (x_{ij}^*, y_{ij}^*) is a sample point in each subrectangle. If $f(x, y) \geqslant 0$, this sum represents an approximation to the volume of the solid that lies above the rectangle R and below the graph of f.

(b) Write the definition of $\iint_R f(x, y)\, dA$ as a limit.

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

(c) What is the geometric interpretation of $\iint_R f(x, y)\, dA$ if $f(x, y) \geqslant 0$? What if f takes on both positive and negative values?

If $f(x, y) \geqslant 0$, $\iint_R f(x, y)\, dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x, y)$. If f takes on both positive and negative values, then $\iint_R f(x, y)\, dA$ is $V_1 - V_2$, where V_1 is the volume above R and below the surface $z = f(x, y)$, and V_2 is the volume below R and above the surface.

(d) How do you evaluate $\iint_R f(x, y)\, dA$?

We usually evaluate $\iint_R f(x, y)\, dA$ as an iterated integral according to Fubini's Theorem:

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

(e) What does the Midpoint Rule for double integrals say?

The Midpoint Rule for double integrals says that we approximate the double integral $\iint_R f(x, y)\, dA$ by the double Riemann sum $\sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$, where the sample points $(\bar{x}_i, \bar{y}_j)$ are the centers of the subrectangles.

(f) Write an expression for the average value of f.

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where $A(R)$ is the area of R.

2. (a) How do you define $\iint_D f(x, y)\, dA$ if D is a bounded region that is not a rectangle?

Since D is bounded, it can be enclosed in a rectangular region R. We define a new function F with domain R by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Then we define

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA$$

(b) What is a type I region? How do you evaluate $\iint_D f(x, y)\, dA$ if D is a type I region?

A region D is of type I if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

(c) What is a type II region? How do you evaluate $\iint_D f(x, y)\, dA$ if D is a type II region?

A region D is of type II if it lies between the graphs of two continuous functions of y, that is,

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous on $[c, d]$. Then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

(d) What properties do double integrals have?

- $$\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$
- $$\iint_D c f(x, y)\, dA = c \iint_D f(x, y)\, dA$$
  where c is a constant
- If $f(x, y) \geqslant g(x, y)$ for all (x, y) in D, then
  $$\iint_D f(x, y)\, dA \geqslant \iint_D g(x, y)\, dA$$
- If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then
  $$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$
- $\iint_D 1\, dA = A(D)$, the area of D.
- If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then
  $$mA(D) \leqslant \iint_D f(x, y)\, dA \leqslant MA(D)$$

(continued)

3. How do you change from rectangular coordinates to polar coordinates in a double integral? Why would you want to make the change?

We may want to change from rectangular to polar coordinates in a double integral if the region D of integration is more easily described in polar coordinates:

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

To evaluate $\iint_R f(x, y)\, dA$, we replace x by $r\cos\theta$, y by $r\sin\theta$, and dA by $r\, dr\, d\theta$ (and use appropriate limits of integration):

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

4. If a lamina occupies a plane region D and has density function $\rho(x, y)$, write expressions for each of the following in terms of double integrals.

(a) The mass: $m = \iint_D \rho(x, y)\, dA$

(b) The moments about the axes:

$$M_x = \iint_D y\rho(x, y)\, dA \qquad M_y = \iint_D x\rho(x, y)\, dA$$

(c) The center of mass:

$$(\bar{x}, \bar{y}), \qquad \text{where } \bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

(d) The moments of inertia about the axes and the origin:

$$I_x = \iint_D y^2 \rho(x, y)\, dA$$

$$I_y = \iint_D x^2 \rho(x, y)\, dA$$

$$I_0 = \iint_D (x^2 + y^2)\rho(x, y)\, dA$$

5. Let f be a joint density function of a pair of continuous random variables X and Y.

(a) Write a double integral for the probability that X lies between a and b and Y lies between c and d.

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d) = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

(b) What properties does f possess?

$$f(x, y) \geqslant 0 \qquad \iint_{\mathbb{R}^2} f(x, y)\, dA = 1$$

(c) What are the expected values of X and Y?

The expected value of X is $\mu_1 = \iint_{\mathbb{R}^2} x f(x, y)\, dA$

The expected value of Y is $\mu_2 = \iint_{\mathbb{R}^2} y f(x, y)\, dA$

6. Write an expression for the area of a surface with equation $z = f(x, y)$, $(x, y) \in D$.

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

(assuming that f_x and f_y are continuous).

7. (a) Write the definition of the triple integral of f over a rectangular box B.

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

where ΔV is the volume of each sub-box and $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is a sample point in each sub-box.

(b) How do you evaluate $\iiint_B f(x, y, z)\, dV$?

We usually evaluate $\iiint_B f(x, y, z)\, dV$ as an iterated integral according to Fubini's Theorem for Triple Integrals: If f is continuous on $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Note that there are five other orders of integration that we can use.

(c) How do you define $\iiint_E f(x, y, z)\, dV$ if E is a bounded solid region that is not a box?

Since E is bounded, it can be enclosed in a box B as described in part (b). We define a new function F with domain B by

$$F(x, y, z) = \begin{cases} f(x, y, z) & \text{if } (x, y, z) \text{ is in } E \\ 0 & \text{if } (x, y, z) \text{ is in } B \text{ but not in } E \end{cases}$$

Then we define

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

(continued)

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(d) What is a type 1 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?

A region E is of type 1 if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\right] dA$$

(e) What is a type 2 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?

A type 2 region is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

where D is the projection of E onto the yz-plane. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx\right] dA$$

(f) What is a type 3 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?

A type 3 region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

where D is the projection of E onto the xz-plane. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy\right] dA$$

8. Suppose a solid object occupies the region E and has density function $\rho(x, y, z)$. Write expressions for each of the following.

(a) The mass:

$$m = \iiint_E \rho(x, y, z)\, dV$$

(b) The moments about the coordinate planes:

$$M_{yz} = \iiint_E x\,\rho(x, y, z)\, dV$$

$$M_{xz} = \iiint_E y\,\rho(x, y, z)\, dV$$

$$M_{xy} = \iiint_E z\,\rho(x, y, z)\, dV$$

(c) The coordinates of the center of mass:

$$(\bar{x}, \bar{y}, \bar{z}), \text{ where } \bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m}$$

(d) The moments of inertia about the axes:

$$I_x = \iiint_E (y^2 + z^2)\,\rho(x, y, z)\, dV$$

$$I_y = \iiint_E (x^2 + z^2)\,\rho(x, y, z)\, dV$$

$$I_z = \iiint_E (x^2 + y^2)\,\rho(x, y, z)\, dV$$

(continued)

9. (a) How do you change from rectangular coordinates to cylindrical coordinates in a triple integral?

$$\iiint\limits_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

where

$$E = \{(r, \theta, z) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta),\ u_1(r\cos\theta, r\sin\theta) \leqslant z \leqslant u_2(r\cos\theta, r\sin\theta)\}$$

Thus we replace x by $r\cos\theta$, y by $r\sin\theta$, dV by $r\, dz\, dr\, d\theta$, and use appropriate limits of integration.

(b) How do you change from rectangular coordinates to spherical coordinates in a triple integral?

$$\iiint\limits_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

where $$E = \{(\rho, \theta, \phi) \mid \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d,\ g_1(\theta, \phi) \leqslant \rho \leqslant g_2(\theta, \phi)\}$$

Thus we replace x by $\rho\sin\phi\cos\theta$, y by $\rho\sin\phi\sin\theta$, z by $\rho\cos\phi$, dV by $\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$, and use appropriate limits of integration.

(c) In what situations would you change to cylindrical or spherical coordinates?

We may want to change from rectangular to cylindrical or spherical coordinates in a triple integral if the region E of integration is more easily described in cylindrical or spherical coordinates. Regions that involve symmetry about the z-axis are often simpler to describe using cylindrical coordinates, and regions that are symmetrical about the origin are often simpler in spherical coordinates. Also, sometimes the integrand is easier to integrate using cylindrical or spherical coordinates.

10. (a) If a transformation T is given by $x = g(u, v)$, $y = h(u, v)$, what is the Jacobian of T?

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

(b) How do you change variables in a double integral?

We change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing

$$dA = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

Thus, under the appropriate conditions,

$$\iint\limits_R f(x, y)\, dA = \iint\limits_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

where R is the image of S under the transformation.

(c) How do you change variables in a triple integral?

Similarly to the case of two variables in part (b),

$$\iiint\limits_R f(x, y, z)\, dV = \iiint\limits_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw$$

where $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

is the Jacobian.

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CHAPTER 16 CONCEPT CHECK ANSWERS

1. What is a vector field? Give three examples that have physical meaning.

A vector field is a function that assigns a vector to each point in its domain.

A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vector of the earth's gravitational field at a location in space.

2. (a) What is a conservative vector field?

A conservative vector field $\mathbf{F}$ is a vector field that is the gradient of some scalar function f, that is, $\mathbf{F} = \nabla f$.

(b) What is a potential function?

The function f in part (a) is called a potential function for $\mathbf{F}$.

3. (a) Write the definition of the line integral of a scalar function f along a smooth curve C with respect to arc length.

If C is given by the parametric equations $x = x(t)$, $y = y(t)$, $a \leqslant t \leqslant b$, we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width. The ith subinterval corresponds to a subarc of C with length Δs_i. Then

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

where (x_i^*, y_i^*) is any sample point in the ith subarc.

(b) How do you evaluate such a line integral?

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Similarly, if C is a smooth space curve, then

$$\int_C f(x, y, z)\, ds$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

(c) Write expressions for the mass and center of mass of a thin wire shaped like a curve C if the wire has linear density function $\rho(x, y)$.

The mass is $m = \int_C \rho(x, y)\, ds$.

The center of mass is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m} \int_C x\, \rho(x, y)\, ds$$

$$\bar{y} = \frac{1}{m} \int_C y\, \rho(x, y)\, ds$$

(d) Write the definitions of the line integrals along C of a scalar function f with respect to x, y, and z.

$$\int_C f(x, y, z)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta x_i$$

$$\int_C f(x, y, z)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta y_i$$

$$\int_C f(x, y, z)\, dz = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta z_i$$

(We have similar results when f is a function of two variables.)

(e) How do you evaluate these line integrals?

$$\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\, dt$$

$$\int_C f(x, y, z)\, dy = \int_a^b f(x(t), y(t), z(t))\, y'(t)\, dt$$

$$\int_C f(x, y, z)\, dz = \int_a^b f(x(t), y(t), z(t))\, z'(t)\, dt$$

4. (a) Define the line integral of a vector field $\mathbf{F}$ along a smooth curve C given by a vector function $\mathbf{r}(t)$.

If $\mathbf{F}$ is a continuous vector field and C is given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

(b) If $\mathbf{F}$ is a force field, what does this line integral represent?

It represents the work done by $\mathbf{F}$ in moving a particle along the curve C.

(c) If $\mathbf{F} = \langle P, Q, R \rangle$, what is the connection between the line integral of $\mathbf{F}$ and the line integrals of the component functions P, Q, and R?

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz$$

5. State the Fundamental Theorem for Line Integrals.

If C is a smooth curve given by $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, and f is a differentiable function whose gradient vector ∇f is continuous on C, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

6. (a) What does it mean to say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path?

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial points and the same terminal points.

(b) If you know that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, what can you say about $\mathbf{F}$?

We know that $\mathbf{F}$ is a conservative vector field, that is, there exists a function f such that $\nabla f = \mathbf{F}$.

(continued)

7. State Green's Theorem.

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

8. Write expressions for the area enclosed by a curve C in terms of line integrals around C.

$$A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

9. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$.

(a) Define curl $\mathbf{F}$.

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
$$= \nabla \times \mathbf{F}$$

(b) Define div $\mathbf{F}$.

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$$

(c) If $\mathbf{F}$ is a velocity field in fluid flow, what are the physical interpretations of curl $\mathbf{F}$ and div $\mathbf{F}$?

At a point in the fluid, the vector curl $\mathbf{F}$ aligns with the axis about which the fluid tends to rotate, and its length measures the speed of rotation; div $\mathbf{F}$ at a point measures the tendency of the fluid to flow away (diverge) from that point.

10. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, how do you determine whether $\mathbf{F}$ is conservative? What if $\mathbf{F}$ is a vector field on $\mathbb{R}^3$?

If P and Q have continuous first-order derivatives and $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, then $\mathbf{F}$ is conservative.

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative.

11. (a) What is a parametric surface? What are its grid curves?

A parametric surface S is a surface in $\mathbb{R}^3$ described by a vector function

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

of two parameters u and v. Equivalent parametric equations are

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

The grid curves of S are the curves that correspond to holding either u or v constant.

(b) Write an expression for the area of a parametric surface.

If S is a smooth parametric surface given by

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

where $(u, v) \in D$ and S is covered just once as (u, v) ranges throughout D, then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

(c) What is the area of a surface given by an equation $z = g(x, y)$?

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

12. (a) Write the definition of the surface integral of a scalar function f over a surface S.

We divide S into "patches" S_{ij}. Then

$$\iint_S f(x, y, z)\,dS = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

where ΔS_{ij} is the area of the patch S_{ij} and P_{ij}^* is a sample point from the patch. (S is divided into patches in such a way that ensures that $\Delta S_{ij} \to 0$ as $m, n \to \infty$.)

(b) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?

$$\iint_S f(x, y, z)\,dS = \iint_D f(\mathbf{r}(u, v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

where D is the parameter domain of S.

(c) What if S is given by an equation $z = g(x, y)$?

$$\iint_S f(x, y, z)\,dS$$
$$= \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$$

(d) If a thin sheet has the shape of a surface S, and the density at (x, y, z) is $\rho(x, y, z)$, write expressions for the mass and center of mass of the sheet.

The mass is

$$m = \iint_S \rho(x, y, z)\,dS$$

The center of mass is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\,\rho(x, y, z)\,dS$$
$$\bar{y} = \frac{1}{m}\iint_S y\,\rho(x, y, z)\,dS$$
$$\bar{z} = \frac{1}{m}\iint_S z\,\rho(x, y, z)\,dS$$

(continued)

Cut here and keep for reference

13. (a) What is an oriented surface? Give an example of a nonorientable surface.

An oriented surface S is one for which we can choose a unit normal vector $\mathbf{n}$ at every point so that $\mathbf{n}$ varies continuously over S. The choice of $\mathbf{n}$ provides S with an orientation.

A Möbius strip is a nonorientable surface. (It has only one side.)

(b) Define the surface integral (or flux) of a vector field $\mathbf{F}$ over an oriented surface S with unit normal vector $\mathbf{n}$.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

(c) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

We multiply by -1 if the opposite orientation of S is desired.

(d) What if S is given by an equation $z = g(x, y)$?

If $\mathbf{F} = \langle P, Q, R \rangle$,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

for the upward orientation of S; we multiply by -1 for the downward orientation.

14. State Stokes' Theorem.

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

15. State the Divergence Theorem.

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

16. In what ways are the Fundamental Theorem for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem similar?

In each theorem, we integrate a "derivative" over a region, and this integral is equal to an expression involving the values of the original function only on the *boundary* of the region.

CHAPTER 17 CONCEPT CHECK ANSWERS

1. (a) Write the general form of a second-order homogeneous linear differential equation with constant coefficients.

$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

(b) Write the auxiliary equation.

$$ar^2 + br + c = 0$$

(c) How do you use the roots of the auxiliary equation to solve the differential equation? Write the form of the solution for each of the three cases that can occur.

If the auxiliary equation has two distinct real roots r_1 and r_2, the general solution of the differential equation is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

If the roots are real and equal, the solution is

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

where r is the common root.

If the roots are complex, we can write $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, and the solution is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

2. (a) What is an initial-value problem for a second-order differential equation?

An initial-value problem consists of finding a solution y of the differential equation that also satisfies given conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$, where y_0 and y_1 are constants.

(b) What is a boundary-value problem for such an equation?

A boundary-value problem consists of finding a solution y of the differential equation that also satisfies given boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.

3. (a) Write the general form of a second-order nonhomogeneous linear differential equation with constant coefficients.

$ay'' + by' + cy = G(x)$, where a, b, and c are constants and G is a continuous function.

(b) What is the complementary equation? How does it help solve the original differential equation?

The complementary equation is the related homogeneous equation $ay'' + by' + cy = 0$. If we find the general solution y_c of the complementary equation and y_p is any particular solution of the nonhomogeneous differential equation, then the general solution of the original differential equation is $y(x) = y_p(x) + y_c(x)$.

(c) Explain how the method of undetermined coefficients works.

To determine a particular solution y_p of $ay'' + by' + cy = G(x)$, we make an initial guess that y_p is a general function of the same type as G. If $G(x)$ is a polynomial, choose y_p to be a general polynomial of the same degree. If $G(x)$ is of the form Ce^{kx}, choose $y_p(x) = Ae^{kx}$. If $G(x)$ is $C \cos kx$ or $C \sin kx$, choose $y_p(x) = A \cos kx + B \sin kx$. If $G(x)$ is a product of functions, choose y_p to be a product of functions of the same type. Some examples are:

$G(x)$	$y_p(x)$
x^2	$Ax^2 + Bx + C$
e^{2x}	Ae^{2x}
$\sin 2x$	$A \cos 2x + B \sin 2x$
xe^{-x}	$(Ax + B)e^{-x}$

We then substitute y_p, y_p', and y_p'' into the differential equation and determine the coefficients.

If y_p happens to be a solution of the complementary equation, then multiply the initial trial solution by x (or x^2 if necessary).

If $G(x)$ is a sum of functions, we find a particular solution for each function and then y_p is the sum of these.

The general solution of the differential equation is

$$y(x) = y_p(x) + y_c(x)$$

(d) Explain how the method of variation of parameters works.

We write the solution of the complementary equation $ay'' + by' + cy = 0$ as $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$, where y_1 and y_2 are linearly independent solutions. We then take $y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$ as a particular solution, where $u_1(x)$ and $u_2(x)$ are arbitrary functions. After computing y_p', we impose the condition that

$$u_1' y_1 + u_2' y_2 = 0 \qquad \mathbf{(1)}$$

and then compute y_p''. Substituting y_p, y_p', and y_p'' into the original differential equation gives

$$a(u_1' y_1' + u_2' y_2') = G \qquad \mathbf{(2)}$$

We then solve equations **(1)** and **(2)** for the unknown functions u_1' and u_2'. If we are able to integrate these functions, then a particular solution is $y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$ and the general solution is $y(x) = y_p(x) + y_c(x)$.

4. Discuss two applications of second-order linear differential equations.

The motion of an object with mass m at the end of a spring is an example of simple harmonic motion and is described by the second-order linear differential equation

$$m \frac{d^2x}{dt^2} + kx = 0$$

(continued)

where k is the spring constant and x is the distance the spring is stretched (or compressed) from its natural length. If there are external forces acting on the spring, then the differential equation is modified.

Second-order linear differential equations are also used to analyze electrical circuits involving an electromotive force, a resistor, an inductor, and a capacitor in series.

See the discussion in Section 17.3 for additional details.

5. How do you use power series to solve a differential equation?

We first assume that the differential equation has a power series solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Differentiating gives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n + 1) c_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n - 1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n + 2)(n + 1) c_{n+2} x^n$$

We substitute these expressions into the differential equation and equate the coefficients of x^n to find a recursion relation involving the constants c_n. Solving the recursion relation gives a formula for c_n and then

$$y = \sum_{n=0}^{\infty} c_n x^n$$

is the solution of the differential equation.