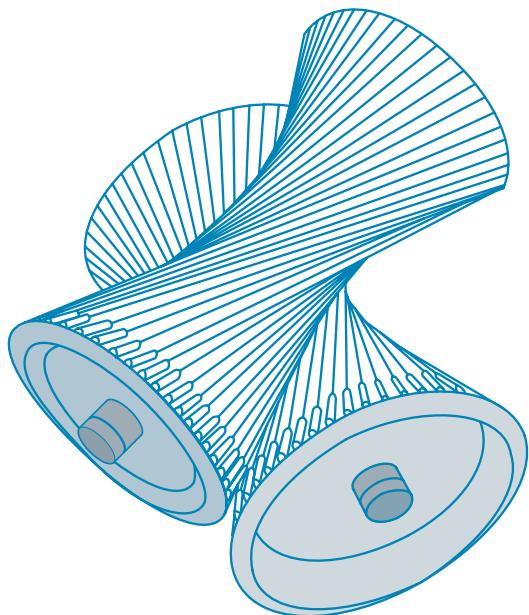


# 12

# Vectors and the Geometry of Space

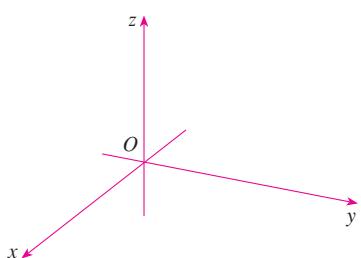
Each of these gears has the shape of a hyperboloid, a type of surface we will study in Section 12.6. The shape allows the gears to transmit motion between skew (neither parallel nor intersecting) axes.



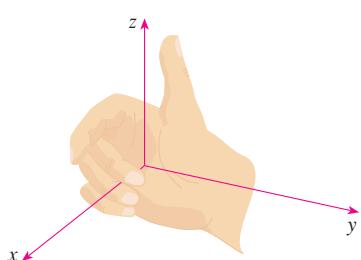
**IN THIS CHAPTER WE INTRODUCE** vectors and coordinate systems for three-dimensional space. This will be the setting for our study of the calculus of functions of two variables in Chapter 14 because the graph of such a function is a surface in space. In this chapter we will see that vectors provide particularly simple descriptions of lines and planes in space.

## 12.1 Three-Dimensional Coordinate Systems

### ■ 3D Space



**FIGURE 1**  
Coordinate axes

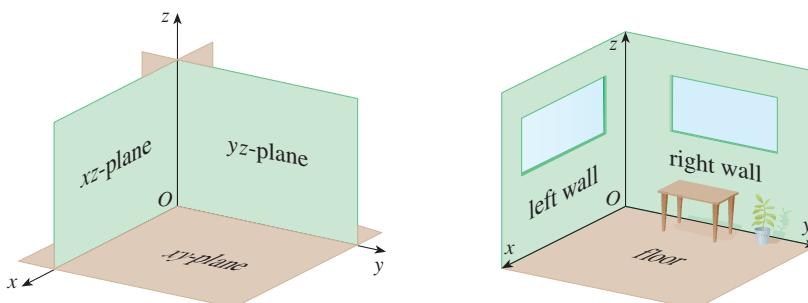


**FIGURE 2**  
Right-hand rule

To locate a point in a plane, we need two numbers. We know that any point in the plane can be represented as an ordered pair  $(a, b)$  of real numbers, where  $a$  is the  $x$ -coordinate and  $b$  is the  $y$ -coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple  $(a, b, c)$  of real numbers.

In order to represent points in space, we first choose a fixed point  $O$  (the origin) and three directed lines through  $O$  that are perpendicular to each other, called the **coordinate axes** and labeled the  $x$ -axis,  $y$ -axis, and  $z$ -axis. Usually we think of the  $x$ - and  $y$ -axes as being horizontal and the  $z$ -axis as being vertical, and we draw the orientation of the axes as in Figure 1. The direction of the  $z$ -axis is determined by the **right-hand rule** as illustrated in Figure 2: If you curl the fingers of your right hand around the  $z$ -axis in the direction of a  $90^\circ$  counterclockwise rotation from the positive  $x$ -axis to the positive  $y$ -axis, then your thumb points in the positive direction of the  $z$ -axis.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a). The  $xy$ -plane is the plane that contains the  $x$ - and  $y$ -axes; the  $yz$ -plane contains the  $y$ - and  $z$ -axes; the  $xz$ -plane contains the  $x$ - and  $z$ -axes. These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.



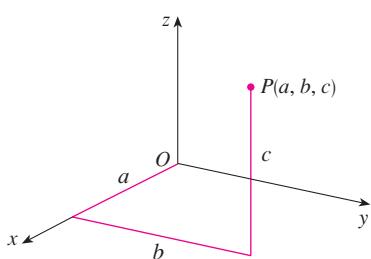
**FIGURE 3**

(a) Coordinate planes

(b)

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [see Figure 3(b)]. Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the  $xz$ -plane, the wall on your right is in the  $yz$ -plane, and the floor is in the  $xy$ -plane. The  $x$ -axis runs along the intersection of the floor and the left wall. The  $y$ -axis runs along the intersection of the floor and the right wall. The  $z$ -axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point  $O$ .

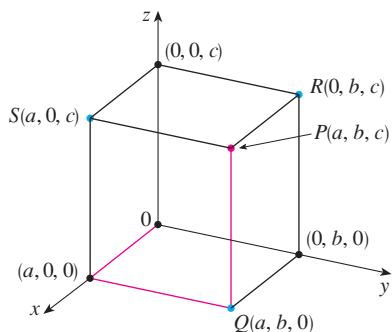
Now if  $P$  is any point in space, let  $a$  be the (directed) distance from the  $yz$ -plane to  $P$ , let  $b$  be the distance from the  $xz$ -plane to  $P$ , and let  $c$  be the distance from the  $xy$ -plane to  $P$ . We represent the point  $P$  by the ordered triple  $(a, b, c)$  of real numbers and we call  $a$ ,  $b$ , and  $c$  the **coordinates** of  $P$ ;  $a$  is the  $x$ -coordinate,  $b$  is the  $y$ -coordinate, and  $c$  is the  $z$ -coordinate. Thus, to locate the point  $(a, b, c)$ , we can start at the origin  $O$  and move  $a$  units along the  $x$ -axis, then  $b$  units parallel to the  $y$ -axis, and then  $c$  units parallel to the  $z$ -axis as in Figure 4.



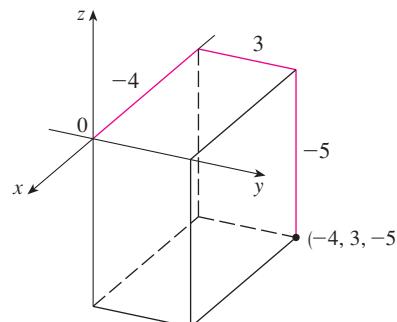
**FIGURE 4**

The point  $P(a, b, c)$  determines a rectangular box as in Figure 5. If we drop a perpendicular from  $P$  to the  $xy$ -plane, we get a point  $Q$  with coordinates  $(a, b, 0)$  called the **projection** of  $P$  onto the  $xy$ -plane. Similarly,  $R(0, b, c)$  and  $S(a, 0, c)$  are the projections of  $P$  onto the  $yz$ -plane and  $xz$ -plane, respectively.

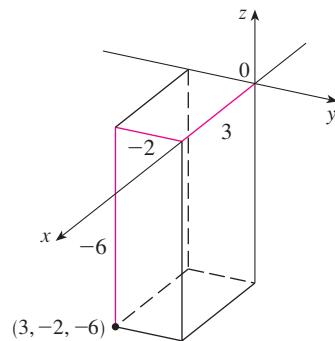
As numerical illustrations, the points  $(-4, 3, -5)$  and  $(3, -2, -6)$  are plotted in Figure 6.



## FIGURE 5



## FIGURE 6



The Cartesian product  $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$  is the set of all ordered triples of real numbers and is denoted by  $\mathbb{R}^3$ . We have given a one-to-one correspondence between points  $P$  in space and ordered triples  $(a, b, c)$  in  $\mathbb{R}^3$ . It is called a **three-dimensional rectangular coordinate system**. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

## Surfaces

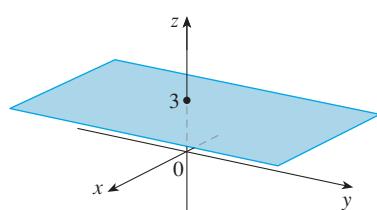
In two-dimensional analytic geometry, the graph of an equation involving  $x$  and  $y$  is a curve in  $\mathbb{R}^2$ . In three-dimensional analytic geometry, an equation in  $x$ ,  $y$ , and  $z$  represents a *surface* in  $\mathbb{R}^3$ .

**EXAMPLE 1** What surfaces in  $\mathbb{R}^3$  are represented by the following equations?

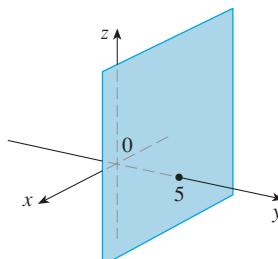
- (a)  $z = 3$  (b)  $y = 5$

## SOLUTION

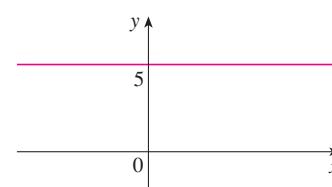
- (a) The equation  $z = 3$  represents the set  $\{(x, y, z) \mid z = 3\}$ , which is the set of all points in  $\mathbb{R}^3$  whose  $z$ -coordinate is 3 ( $x$  and  $y$  can each be any value). This is the horizontal plane that is parallel to the  $xy$ -plane and three units above it as in Figure 7(a).



(a)  $z = 3$ , a plane in  $\mathbb{R}^3$



(b)  $y = 5$ , a plane in  $\mathbb{R}^3$



(c)  $y = 5$ , a line in  $\mathbb{R}^2$

## FIGURE 7

- (b) The equation  $y = 5$  represents the set of all points in  $\mathbb{R}^3$  whose  $y$ -coordinate is 5. This is the vertical plane that is parallel to the  $xz$ -plane and five units to the right of it as in Figure 7(b).

**NOTE** When an equation is given, we must understand from the context whether it represents a curve in  $\mathbb{R}^2$  or a surface in  $\mathbb{R}^3$ . In Example 1,  $y = 5$  represents a plane in  $\mathbb{R}^3$ , but of course  $y = 5$  can also represent a line in  $\mathbb{R}^2$  if we are dealing with two-dimensional analytic geometry. See Figure 7(b) and (c).

In general, if  $k$  is a constant, then  $x = k$  represents a plane parallel to the  $yz$ -plane,  $y = k$  is a plane parallel to the  $xz$ -plane, and  $z = k$  is a plane parallel to the  $xy$ -plane. In Figure 5, the faces of the rectangular box are formed by the three coordinate planes  $x = 0$  (the  $yz$ -plane),  $y = 0$  (the  $xz$ -plane), and  $z = 0$  (the  $xy$ -plane), and the planes  $x = a$ ,  $y = b$ , and  $z = c$ .

### EXAMPLE 2

- (a) Which points  $(x, y, z)$  satisfy the equations

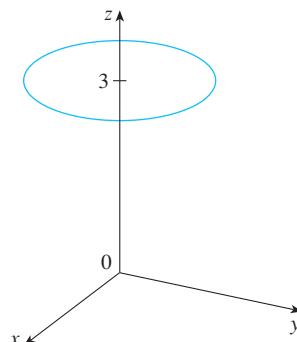
$$x^2 + y^2 = 1 \quad \text{and} \quad z = 3$$

- (b) What does the equation  $x^2 + y^2 = 1$  represent as a surface in  $\mathbb{R}^3$ ?

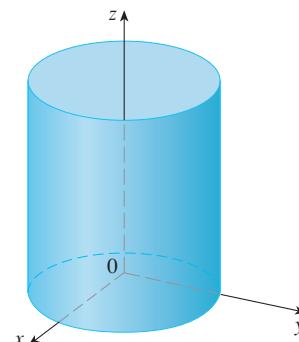
#### SOLUTION

(a) Because  $z = 3$ , the points lie in the horizontal plane  $z = 3$  from Example 1(a). Because  $x^2 + y^2 = 1$ , the points lie on the circle with radius 1 and center on the  $z$ -axis. See Figure 8.

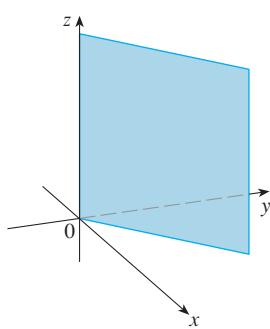
(b) Given that  $x^2 + y^2 = 1$ , with no restrictions on  $z$ , we see that the point  $(x, y, z)$  could lie on a circle in any horizontal plane  $z = k$ . So the surface  $x^2 + y^2 = 1$  in  $\mathbb{R}^3$  consists of all possible horizontal circles  $x^2 + y^2 = 1$ ,  $z = k$ , and is therefore the circular cylinder with radius 1 whose axis is the  $z$ -axis. See Figure 9.



**FIGURE 8**  
The circle  $x^2 + y^2 = 1$ ,  $z = 3$



**FIGURE 9**  
The cylinder  $x^2 + y^2 = 1$



**FIGURE 10**  
The plane  $y = x$

### EXAMPLE 3

Describe and sketch the surface in  $\mathbb{R}^3$  represented by the equation  $y = x$ .

**SOLUTION** The equation represents the set of all points in  $\mathbb{R}^3$  whose  $x$ - and  $y$ -coordinates are equal, that is,  $\{(x, x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$ . This is a vertical plane that intersects the  $xy$ -plane in the line  $y = x$ ,  $z = 0$ . The portion of this plane that lies in the first octant is sketched in Figure 10.

### Distance and Spheres

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

**Distance Formula in Three Dimensions** The distance  $|P_1P_2|$  between the points  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

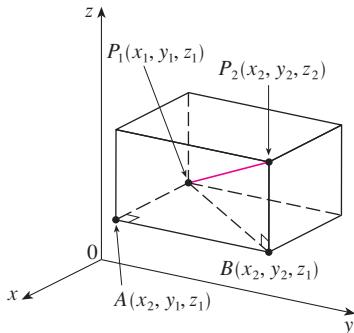


FIGURE 11

To see why this formula is true, we construct a rectangular box as in Figure 11, where  $P_1$  and  $P_2$  are opposite vertices and the faces of the box are parallel to the coordinate planes. If  $A(x_2, y_1, z_1)$  and  $B(x_2, y_2, z_1)$  are the vertices of the box indicated in the figure, then

$$|P_1A| = |x_2 - x_1| \quad |AB| = |y_2 - y_1| \quad |BP_2| = |z_2 - z_1|$$

Because triangles  $P_1BP_2$  and  $P_1AB$  are both right-angled, two applications of the Pythagorean Theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

and

$$|P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combining these equations, we get

$$\begin{aligned} |P_1P_2|^2 &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

$$\text{Therefore } |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

**EXAMPLE 4** The distance from the point  $P(2, -1, 7)$  to the point  $Q(1, -3, 5)$  is

$$|PQ| = \sqrt{(1 - 2)^2 + (-3 + 1)^2 + (5 - 7)^2} = \sqrt{1 + 4 + 4} = 3$$

**EXAMPLE 5** Find an equation of a sphere with radius  $r$  and center  $C(h, k, l)$ .

**SOLUTION** By definition, a sphere is the set of all points  $P(x, y, z)$  whose distance from  $C$  is  $r$ . (See Figure 12.) Thus  $P$  is on the sphere if and only if  $|PC| = r$ . Squaring both sides, we have  $|PC|^2 = r^2$  or

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

The result of Example 5 is worth remembering.

**Equation of a Sphere** An equation of a sphere with center  $C(h, k, l)$  and radius  $r$  is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin  $O$ , then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

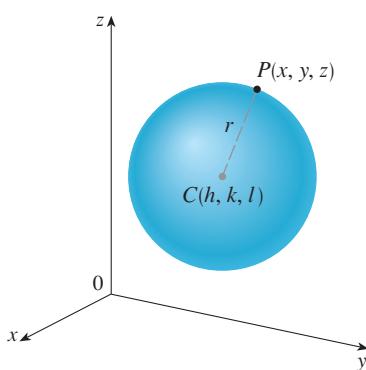


FIGURE 12

**EXAMPLE 6** Show that  $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$  is the equation of a sphere, and find its center and radius.

**SOLUTION** We can rewrite the given equation in the form of an equation of a sphere if we complete squares:

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 4 + 9 + 1$$

$$(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8$$

Comparing this equation with the standard form, we see that it is the equation of a sphere with center  $(-2, 3, -1)$  and radius  $\sqrt{8} = 2\sqrt{2}$ . ■

**EXAMPLE 7** What region in  $\mathbb{R}^3$  is represented by the following inequalities?

$$1 \leq x^2 + y^2 + z^2 \leq 4 \quad z \leq 0$$

**SOLUTION** The inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 4$$

can be rewritten as

$$1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2$$

so they represent the points  $(x, y, z)$  whose distance from the origin is at least 1 and at most 2. But we are also given that  $z \leq 0$ , so the points lie on or below the  $xy$ -plane. Thus the given inequalities represent the region that lies between (or on) the spheres  $x^2 + y^2 + z^2 = 1$  and  $x^2 + y^2 + z^2 = 4$  and beneath (or on) the  $xy$ -plane. It is sketched in Figure 13. ■

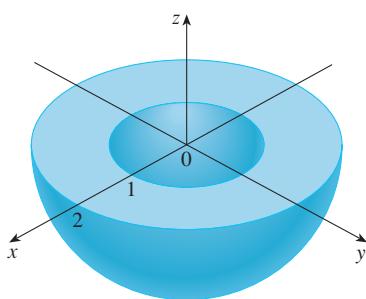


FIGURE 13

## 12.1 EXERCISES

- Suppose you start at the origin, move along the  $x$ -axis a distance of 4 units in the positive direction, and then move downward a distance of 3 units. What are the coordinates of your position?
- Sketch the points  $(1, 5, 3)$ ,  $(0, 2, -3)$ ,  $(-3, 0, 2)$ , and  $(2, -2, -1)$  on a single set of coordinate axes.
- Which of the points  $A(-4, 0, -1)$ ,  $B(3, 1, -5)$ , and  $C(2, 4, 6)$  is closest to the  $yz$ -plane? Which point lies in the  $xz$ -plane?
- What are the projections of the point  $(2, 3, 5)$  on the  $xy$ -,  $yz$ -, and  $xz$ -planes? Draw a rectangular box with the origin and  $(2, 3, 5)$  as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.
- What does the equation  $x = 4$  represent in  $\mathbb{R}^2$ ? What does it represent in  $\mathbb{R}^3$ ? Illustrate with sketches.
- What does the equation  $y = 3$  represent in  $\mathbb{R}^3$ ? What does  $z = 5$  represent? What does the pair of equations  $y = 3$ ,  $z = 5$  represent? In other words, describe the set of points  $(x, y, z)$  such that  $y = 3$  and  $z = 5$ . Illustrate with a sketch.
- Describe and sketch the surface in  $\mathbb{R}^3$  represented by the equation  $x + y = 2$ .
- Describe and sketch the surface in  $\mathbb{R}^3$  represented by the equation  $x^2 + z^2 = 9$ .
- Find the lengths of the sides of the triangle  $PQR$ . Is it a right triangle? Is it an isosceles triangle?
- $P(3, -2, -3)$ ,  $Q(7, 0, 1)$ ,  $R(1, 2, 1)$
- $P(2, -1, 0)$ ,  $Q(4, 1, 1)$ ,  $R(4, -5, 4)$
- Determine whether the points lie on a straight line.
  - $A(2, 4, 2)$ ,  $B(3, 7, -2)$ ,  $C(1, 3, 3)$
  - $D(0, -5, 5)$ ,  $E(1, -2, 4)$ ,  $F(3, 4, 2)$
- Find the distance from  $(4, -2, 6)$  to each of the following.
  - The  $xy$ -plane
  - The  $yz$ -plane
  - The  $xz$ -plane
  - The  $x$ -axis
  - The  $y$ -axis
  - The  $z$ -axis

13. Find an equation of the sphere with center  $(-3, 2, 5)$  and radius 4. What is the intersection of this sphere with the  $yz$ -plane?
14. Find an equation of the sphere with center  $(2, -6, 4)$  and radius 5. Describe its intersection with each of the coordinate planes.
15. Find an equation of the sphere that passes through the point  $(4, 3, -1)$  and has center  $(3, 8, 1)$ .
16. Find an equation of the sphere that passes through the origin and whose center is  $(1, 2, 3)$ .

17–20 Show that the equation represents a sphere, and find its center and radius.

17.  $x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$

18.  $x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$

19.  $2x^2 + 2y^2 + 2z^2 = 8x - 24z + 1$

20.  $3x^2 + 3y^2 + 3z^2 = 10 + 6y + 12z$

21. (a) Prove that the midpoint of the line segment from  $P_1(x_1, y_1, z_1)$  to  $P_2(x_2, y_2, z_2)$  is

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

- (b) Find the lengths of the medians of the triangle with vertices  $A(1, 2, 3)$ ,  $B(-2, 0, 5)$ , and  $C(4, 1, 5)$ . (A *median* of a triangle is a line segment that joins a vertex to the midpoint of the opposite side.)

22. Find an equation of a sphere if one of its diameters has endpoints  $(5, 4, 3)$  and  $(1, 6, -9)$ .
23. Find equations of the spheres with center  $(2, -3, 6)$  that touch (a) the  $xy$ -plane, (b) the  $yz$ -plane, (c) the  $xz$ -plane.
24. Find an equation of the largest sphere with center  $(5, 4, 9)$  that is contained in the first octant.

25–38 Describe in words the region of  $\mathbb{R}^3$  represented by the equation(s) or inequality.

25.  $x = 5$

26.  $y = -2$

27.  $y < 8$

28.  $z \geq -1$

29.  $0 \leq z \leq 6$

30.  $y^2 = 4$

31.  $x^2 + y^2 = 4$ ,  $z = -1$

32.  $x^2 + y^2 = 4$

33.  $x^2 + y^2 + z^2 = 4$

34.  $x^2 + y^2 + z^2 \leq 4$

35.  $1 \leq x^2 + y^2 + z^2 \leq 5$

36.  $x = z$

37.  $x^2 + z^2 \leq 9$

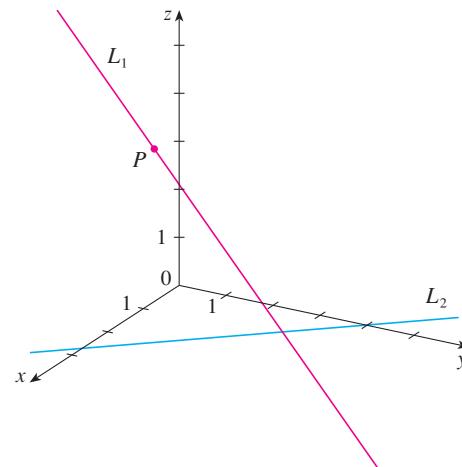
38.  $x^2 + y^2 + z^2 > 2z$

39–42 Write inequalities to describe the region.

39. The region between the  $yz$ -plane and the vertical plane  $x = 5$

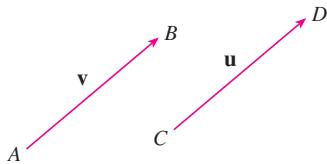
40. The solid cylinder that lies on or below the plane  $z = 8$  and on or above the disk in the  $xy$ -plane with center the origin and radius 2
41. The region consisting of all points between (but not on) the spheres of radius  $r$  and  $R$  centered at the origin, where  $r < R$
42. The solid upper hemisphere of the sphere of radius 2 centered at the origin

43. The figure shows a line  $L_1$  in space and a second line  $L_2$ , which is the projection of  $L_1$  onto the  $xy$ -plane. (In other words, the points on  $L_2$  are directly beneath, or above, the points on  $L_1$ .)
- (a) Find the coordinates of the point  $P$  on the line  $L_1$ .
- (b) Locate on the diagram the points  $A$ ,  $B$ , and  $C$ , where the line  $L_1$  intersects the  $xy$ -plane, the  $yz$ -plane, and the  $xz$ -plane, respectively.

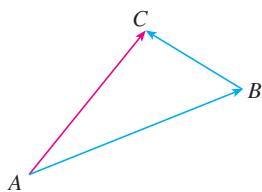


44. Consider the points  $P$  such that the distance from  $P$  to  $A(-1, 5, 3)$  is twice the distance from  $P$  to  $B(6, 2, -2)$ . Show that the set of all such points is a sphere, and find its center and radius.
45. Find an equation of the set of all points equidistant from the points  $A(-1, 5, 3)$  and  $B(6, 2, -2)$ . Describe the set.
46. Find the volume of the solid that lies inside both of the spheres  $x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$  and  $x^2 + y^2 + z^2 = 4$
47. Find the distance between the spheres  $x^2 + y^2 + z^2 = 4$  and  $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$ .
48. Describe and sketch a solid with the following properties. When illuminated by rays parallel to the  $z$ -axis, its shadow is a circular disk. If the rays are parallel to the  $y$ -axis, its shadow is a square. If the rays are parallel to the  $x$ -axis, its shadow is an isosceles triangle.

## 12.2 Vectors



**FIGURE 1**  
Equivalent vectors



**FIGURE 2**

The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We denote a vector by printing a letter in boldface (**v**) or by putting an arrow above the letter ( $\vec{v}$ ).

For instance, suppose a particle moves along a line segment from point  $A$  to point  $B$ . The corresponding **displacement vector**  $\mathbf{v}$ , shown in Figure 1, has **initial point**  $A$  (the tail) and **terminal point**  $B$  (the tip) and we indicate this by writing  $\mathbf{v} = \overrightarrow{AB}$ . Notice that the vector  $\mathbf{u} = \overrightarrow{CD}$  has the same length and the same direction as  $\mathbf{v}$  even though it is in a different position. We say that  $\mathbf{u}$  and  $\mathbf{v}$  are **equivalent** (or **equal**) and we write  $\mathbf{u} = \mathbf{v}$ . The **zero vector**, denoted by  $\mathbf{0}$ , has length 0. It is the only vector with no specific direction.

### Combining Vectors

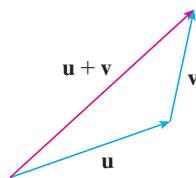
Suppose a particle moves from  $A$  to  $B$ , so its displacement vector is  $\overrightarrow{AB}$ . Then the particle changes direction and moves from  $B$  to  $C$ , with displacement vector  $\overrightarrow{BC}$  as in Figure 2. The combined effect of these displacements is that the particle has moved from  $A$  to  $C$ . The resulting displacement vector  $\overrightarrow{AC}$  is called the **sum** of  $\overrightarrow{AB}$  and  $\overrightarrow{BC}$  and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

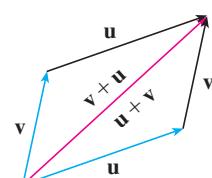
In general, if we start with vectors  $\mathbf{u}$  and  $\mathbf{v}$ , we first move  $\mathbf{v}$  so that its tail coincides with the tip of  $\mathbf{u}$  and define the sum of  $\mathbf{u}$  and  $\mathbf{v}$  as follows.

**Definition of Vector Addition** If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors positioned so the initial point of  $\mathbf{v}$  is at the terminal point of  $\mathbf{u}$ , then the **sum**  $\mathbf{u} + \mathbf{v}$  is the vector from the initial point of  $\mathbf{u}$  to the terminal point of  $\mathbf{v}$ .

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.



**FIGURE 3**  
The Triangle Law



**FIGURE 4**  
The Parallelogram Law

In Figure 4 we start with the same vectors  $\mathbf{u}$  and  $\mathbf{v}$  as in Figure 3 and draw another copy of  $\mathbf{v}$  with the same initial point as  $\mathbf{u}$ . Completing the parallelogram, we see that  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ . This also gives another way to construct the sum: if we place  $\mathbf{u}$  and  $\mathbf{v}$  so they start at the same point, then  $\mathbf{u} + \mathbf{v}$  lies along the diagonal of the parallelogram with  $\mathbf{u}$  and  $\mathbf{v}$  as sides. (This is called the **Parallelogram Law**.)

**EXAMPLE 1** Draw the sum of the vectors  $\mathbf{a}$  and  $\mathbf{b}$  shown in Figure 5.

**SOLUTION** First we move  $\mathbf{b}$  and place its tail at the tip of  $\mathbf{a}$ , being careful to draw a copy of  $\mathbf{b}$  that has the same length and direction. Then we draw the vector  $\mathbf{a} + \mathbf{b}$  [see



**FIGURE 5**

Figure 6(a)] starting at the initial point of  $\mathbf{a}$  and ending at the terminal point of the copy of  $\mathbf{b}$ .

Alternatively, we could place  $\mathbf{b}$  so it starts where  $\mathbf{a}$  starts and construct  $\mathbf{a} + \mathbf{b}$  by the Parallellogram Law as in Figure 6(b).

**TEC** Visual 12.2 shows how the Triangle and Parallellogram Laws work for various vectors  $\mathbf{a}$  and  $\mathbf{b}$ .

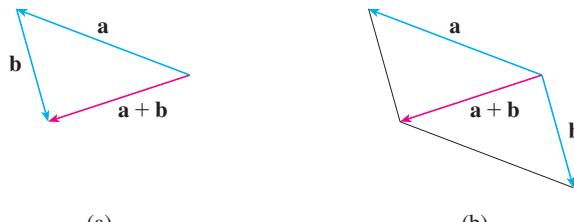


FIGURE 6

(a)

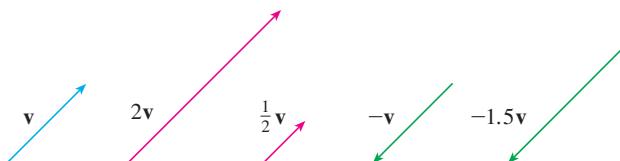
(b)

It is possible to multiply a vector by a real number  $c$ . (In this context we call the real number  $c$  a **scalar** to distinguish it from a vector.) For instance, we want  $2\mathbf{v}$  to be the same vector as  $\mathbf{v} + \mathbf{v}$ , which has the same direction as  $\mathbf{v}$  but is twice as long. In general, we multiply a vector by a scalar as follows.

**Definition of Scalar Multiplication** If  $c$  is a scalar and  $\mathbf{v}$  is a vector, then the **scalar multiple**  $c\mathbf{v}$  is the vector whose length is  $|c|$  times the length of  $\mathbf{v}$  and whose direction is the same as  $\mathbf{v}$  if  $c > 0$  and is opposite to  $\mathbf{v}$  if  $c < 0$ . If  $c = 0$  or  $\mathbf{v} = \mathbf{0}$ , then  $c\mathbf{v} = \mathbf{0}$ .

This definition is illustrated in Figure 7. We see that real numbers work like scaling factors here; that's why we call them scalars. Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another. In particular, the vector  $-\mathbf{v} = (-1)\mathbf{v}$  has the same length as  $\mathbf{v}$  but points in the opposite direction. We call it the **negative** of  $\mathbf{v}$ .

FIGURE 7  
Scalar multiples of  $\mathbf{v}$



By the **difference**  $\mathbf{u} - \mathbf{v}$  of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

So we can construct  $\mathbf{u} - \mathbf{v}$  by first drawing the negative of  $\mathbf{v}$ ,  $-\mathbf{v}$ , and then adding it to  $\mathbf{u}$  by the Parallellogram Law as in Figure 8(a). Alternatively, since  $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$ , the vector  $\mathbf{u} - \mathbf{v}$ , when added to  $\mathbf{v}$ , gives  $\mathbf{u}$ . So we could construct  $\mathbf{u} - \mathbf{v}$  as in Figure 8(b) by means of the Triangle Law. Notice that if  $\mathbf{u}$  and  $\mathbf{v}$  both start from the same initial point, then  $\mathbf{u} - \mathbf{v}$  connects the tip of  $\mathbf{v}$  to the tip of  $\mathbf{u}$ .

FIGURE 8  
Drawing  $\mathbf{u} - \mathbf{v}$

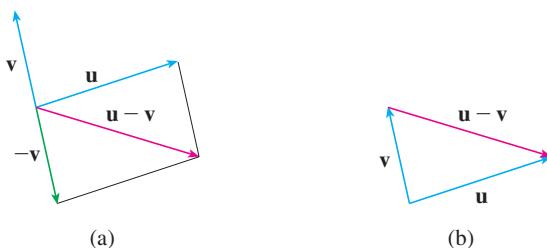




FIGURE 9

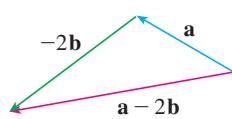


FIGURE 10

**EXAMPLE 2** If  $\mathbf{a}$  and  $\mathbf{b}$  are the vectors shown in Figure 9, draw  $\mathbf{a} - 2\mathbf{b}$ .

**SOLUTION** We first draw the vector  $-2\mathbf{b}$  pointing in the direction opposite to  $\mathbf{b}$  and twice as long. We place it with its tail at the tip of  $\mathbf{a}$  and then use the Triangle Law to draw  $\mathbf{a} + (-2\mathbf{b})$  as in Figure 10. ■

### ■ Components

For some purposes it's best to introduce a coordinate system and treat vectors algebraically. If we place the initial point of a vector  $\mathbf{a}$  at the origin of a rectangular coordinate system, then the terminal point of  $\mathbf{a}$  has coordinates of the form  $(a_1, a_2)$  or  $(a_1, a_2, a_3)$ , depending on whether our coordinate system is two- or three-dimensional (see Figure 11). These coordinates are called the **components** of  $\mathbf{a}$  and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \quad \text{or} \quad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation  $\langle a_1, a_2 \rangle$  for the ordered pair that refers to a vector so as not to confuse it with the ordered pair  $(a_1, a_2)$  that refers to a point in the plane.

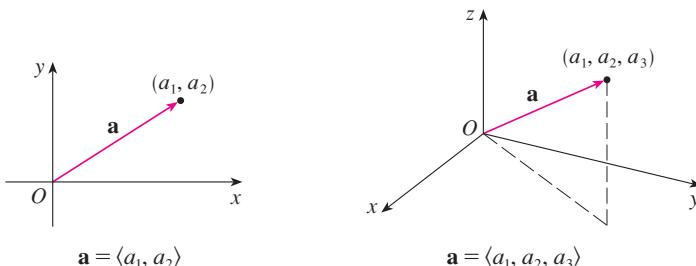


FIGURE 11

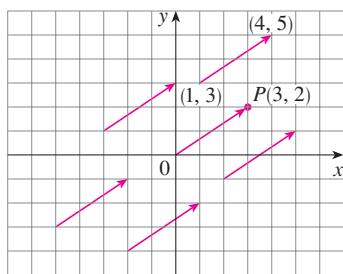


FIGURE 12

Representations of  $\mathbf{a} = \langle 3, 2 \rangle$

For instance, the vectors shown in Figure 12 are all equivalent to the vector  $\overrightarrow{OP} = \langle 3, 2 \rangle$  whose terminal point is  $P(3, 2)$ . What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward. We can think of all these geometric vectors as **representations** of the algebraic vector  $\mathbf{a} = \langle 3, 2 \rangle$ . The particular representation  $\overrightarrow{OP}$  from the origin to the point  $P(3, 2)$  is called the **position vector** of the point  $P$ .

In three dimensions, the vector  $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$  is the **position vector** of the point  $P(a_1, a_2, a_3)$ . (See Figure 13.) Let's consider any other representation  $\overrightarrow{AB}$  of  $\mathbf{a}$ , where the initial point is  $A(x_1, y_1, z_1)$  and the terminal point is  $B(x_2, y_2, z_2)$ . Then we must have  $x_1 + a_1 = x_2$ ,  $y_1 + a_2 = y_2$ , and  $z_1 + a_3 = z_2$  and so  $a_1 = x_2 - x_1$ ,  $a_2 = y_2 - y_1$ , and  $a_3 = z_2 - z_1$ . Thus we have the following result.

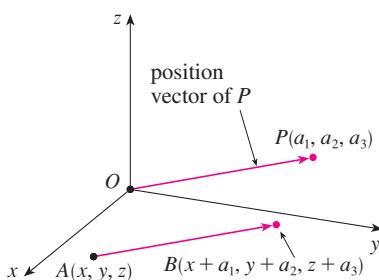


FIGURE 13

Representations of  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

**1** Given the points  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$ , the vector  $\mathbf{a}$  with representation  $\overrightarrow{AB}$  is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**EXAMPLE 3** Find the vector represented by the directed line segment with initial point  $A(2, -3, 4)$  and terminal point  $B(-2, 1, 1)$ .

**SOLUTION** By (1), the vector corresponding to  $\overrightarrow{AB}$  is

$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle = \langle -4, 4, -3 \rangle$$

The **magnitude** or **length** of the vector  $\mathbf{v}$  is the length of any of its representations and is denoted by the symbol  $|\mathbf{v}|$  or  $\|\mathbf{v}\|$ . By using the distance formula to compute the length of a segment  $OP$ , we obtain the following formulas.

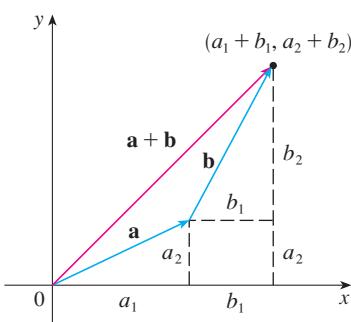


FIGURE 14

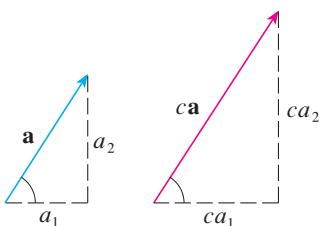


FIGURE 15

The length of the two-dimensional vector  $\mathbf{a} = \langle a_1, a_2 \rangle$  is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

How do we add vectors algebraically? Figure 14 shows that if  $\mathbf{a} = \langle a_1, a_2 \rangle$  and  $\mathbf{b} = \langle b_1, b_2 \rangle$ , then the sum is  $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$ , at least for the case where the components are positive. In other words, *to add algebraic vectors we add corresponding components*. Similarly, *to subtract vectors we subtract corresponding components*. From the similar triangles in Figure 15 we see that the components of  $c\mathbf{a}$  are  $ca_1$  and  $ca_2$ . So *to multiply a vector by a scalar we multiply each component by that scalar*.

If  $\mathbf{a} = \langle a_1, a_2 \rangle$  and  $\mathbf{b} = \langle b_1, b_2 \rangle$ , then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \quad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\begin{aligned} \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle &= \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \\ \langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle &= \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle \\ c\langle a_1, a_2, a_3 \rangle &= \langle ca_1, ca_2, ca_3 \rangle \end{aligned}$$

**EXAMPLE 4** If  $\mathbf{a} = \langle 4, 0, 3 \rangle$  and  $\mathbf{b} = \langle -2, 1, 5 \rangle$ , find  $|\mathbf{a}|$  and the vectors  $\mathbf{a} + \mathbf{b}$ ,  $\mathbf{a} - \mathbf{b}$ ,  $3\mathbf{b}$ , and  $2\mathbf{a} + 5\mathbf{b}$ .

**SOLUTION**  $|\mathbf{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle 4, 0, 3 \rangle + \langle -2, 1, 5 \rangle \\ &= \langle 4 + (-2), 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle \end{aligned}$$

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= \langle 4, 0, 3 \rangle - \langle -2, 1, 5 \rangle \\ &= \langle 4 - (-2), 0 - 1, 3 - 5 \rangle = \langle 6, -1, -2 \rangle \end{aligned}$$

$$3\mathbf{b} = 3\langle -2, 1, 5 \rangle = \langle 3(-2), 3(1), 3(5) \rangle = \langle -6, 3, 15 \rangle$$

$$\begin{aligned} 2\mathbf{a} + 5\mathbf{b} &= 2\langle 4, 0, 3 \rangle + 5\langle -2, 1, 5 \rangle \\ &= \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle \end{aligned}$$

Vectors in  $n$  dimensions are used to list various quantities in an organized way. For instance, the components of a six-dimensional vector

$$\mathbf{p} = \langle p_1, p_2, p_3, p_4, p_5, p_6 \rangle$$

might represent the prices of six different ingredients required to make a particular product. Four-dimensional vectors  $\langle x, y, z, t \rangle$  are used in relativity theory, where the first three components specify a position in space and the fourth represents time.

We denote by  $V_2$  the set of all two-dimensional vectors and by  $V_3$  the set of all three-dimensional vectors. More generally, we will later need to consider the set  $V_n$  of all  $n$ -dimensional vectors. An  $n$ -dimensional vector is an ordered  $n$ -tuple:

$$\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle$$

where  $a_1, a_2, \dots, a_n$  are real numbers that are called the components of  $\mathbf{a}$ . Addition and scalar multiplication are defined in terms of components just as for the cases  $n = 2$  and  $n = 3$ .

**Properties of Vectors** If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_n$  and  $c$  and  $d$  are scalars, then

- |   |  |
|---|--|
| 1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$      | 2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ |
| 3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$                   | 4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$   |
| 5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ | 6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$                                   |
| 7. $(cd)\mathbf{a} = c(d\mathbf{a})$                        | 8. $1\mathbf{a} = \mathbf{a}$  |

These eight properties of vectors can be readily verified either geometrically or algebraically. For instance, Property 1 can be seen from Figure 4 (it's equivalent to the Parallelogram Law) or as follows for the case  $n = 2$ :

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle \\ &= \langle b_1 + a_1, b_2 + a_2 \rangle = \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle \\ &= \mathbf{b} + \mathbf{a} \end{aligned}$$

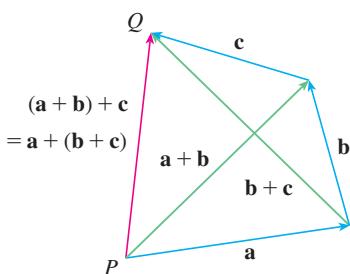


FIGURE 16

We can see why Property 2 (the associative law) is true by looking at Figure 16 and applying the Triangle Law several times: the vector  $\overrightarrow{PQ}$  is obtained either by first constructing  $\mathbf{a} + \mathbf{b}$  and then adding  $\mathbf{c}$  or by adding  $\mathbf{a}$  to the vector  $\mathbf{b} + \mathbf{c}$ .

Three vectors in  $V_3$  play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \quad \mathbf{j} = \langle 0, 1, 0 \rangle \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

These vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are called the **standard basis vectors**. They have length 1 and point in the directions of the positive  $x$ -,  $y$ -, and  $z$ -axes. Similarly, in two dimensions we define  $\mathbf{i} = \langle 1, 0 \rangle$  and  $\mathbf{j} = \langle 0, 1 \rangle$ . (See Figure 17.)

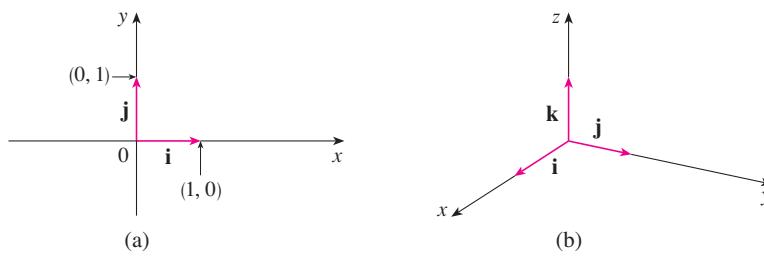


FIGURE 17

Standard basis vectors in  $V_2$  and  $V_3$

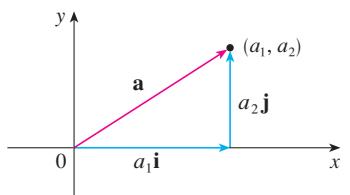
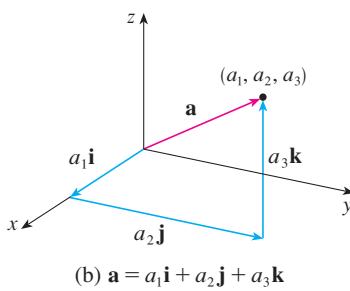
(a)  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ (b)  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ 

FIGURE 18

**Gibbs**

Josiah Willard Gibbs (1839–1903), a professor of mathematical physics at Yale College, published the first book on vectors, *Vector Analysis*, in 1881. More complicated objects, called quaternions, had earlier been invented by Hamilton as mathematical tools for describing space, but they weren't easy for scientists to use. Quaternions have a scalar part and a vector part. Gibbs's idea was to use the vector part separately. Maxwell and Heaviside had similar ideas, but Gibbs's approach has proved to be the most convenient way to study space.

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ , then we can write

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle \\ \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\end{aligned}$$

Thus any vector in  $V_3$  can be expressed in terms of  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ . For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

See Figure 18 for the geometric interpretation of Equations 3 and 2 and compare with Figure 17.

**EXAMPLE 5** If  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$  and  $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$ , express the vector  $2\mathbf{a} + 3\mathbf{b}$  in terms of  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ .

**SOLUTION** Using Properties 1, 2, 5, 6, and 7 of vectors, we have

$$\begin{aligned}2\mathbf{a} + 3\mathbf{b} &= 2(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3(4\mathbf{i} + 7\mathbf{k}) \\ &= 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} + 12\mathbf{i} + 21\mathbf{k} = 14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}\end{aligned}$$

A **unit vector** is a vector whose length is 1. For instance,  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are all unit vectors. In general, if  $\mathbf{a} \neq \mathbf{0}$ , then the unit vector that has the same direction as  $\mathbf{a}$  is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

In order to verify this, we let  $c = 1/|\mathbf{a}|$ . Then  $\mathbf{u} = c\mathbf{a}$  and  $c$  is a positive scalar, so  $\mathbf{u}$  has the same direction as  $\mathbf{a}$ . Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$$

**EXAMPLE 6** Find the unit vector in the direction of the vector  $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ .

**SOLUTION** The given vector has length

$$|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

so, by Equation 4, the unit vector with the same direction is

$$\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

**Applications**

Vectors are useful in many aspects of physics and engineering. In Chapter 13 we will see how they describe the velocity and acceleration of objects moving in space. Here we look at forces.

A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

**EXAMPLE 7** A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces)  $\mathbf{T}_1$  and  $\mathbf{T}_2$  in both wires and the magnitudes of the tensions.

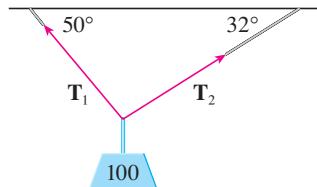


FIGURE 19

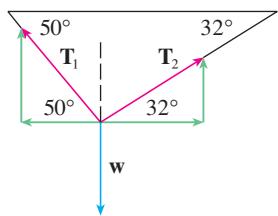


FIGURE 20

**SOLUTION** We first express  $\mathbf{T}_1$  and  $\mathbf{T}_2$  in terms of their horizontal and vertical components. From Figure 20 we see that

5

$$\mathbf{T}_1 = -|\mathbf{T}_1| \cos 50^\circ \mathbf{i} + |\mathbf{T}_1| \sin 50^\circ \mathbf{j}$$

6

$$\mathbf{T}_2 = |\mathbf{T}_2| \cos 32^\circ \mathbf{i} + |\mathbf{T}_2| \sin 32^\circ \mathbf{j}$$

The resultant  $\mathbf{T}_1 + \mathbf{T}_2$  of the tensions counterbalances the weight  $\mathbf{w} = -100 \mathbf{j}$  and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100 \mathbf{j}$$

Thus

$$(-|\mathbf{T}_1| \cos 50^\circ + |\mathbf{T}_2| \cos 32^\circ) \mathbf{i} + (|\mathbf{T}_1| \sin 50^\circ + |\mathbf{T}_2| \sin 32^\circ) \mathbf{j} = 100 \mathbf{j}$$

Equating components, we get

$$-|\mathbf{T}_1| \cos 50^\circ + |\mathbf{T}_2| \cos 32^\circ = 0$$

$$|\mathbf{T}_1| \sin 50^\circ + |\mathbf{T}_2| \sin 32^\circ = 100$$

Solving the first of these equations for  $|\mathbf{T}_2|$  and substituting into the second, we get

$$\begin{aligned} |\mathbf{T}_1| \sin 50^\circ + \frac{|\mathbf{T}_1| \cos 50^\circ}{\cos 32^\circ} \sin 32^\circ &= 100 \\ |\mathbf{T}_1| \left( \sin 50^\circ + \cos 50^\circ \frac{\sin 32^\circ}{\cos 32^\circ} \right) &= 100 \end{aligned}$$

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 85.64 \text{ lb}$$

and

$$|\mathbf{T}_2| = \frac{|\mathbf{T}_1| \cos 50^\circ}{\cos 32^\circ} \approx 64.91 \text{ lb}$$

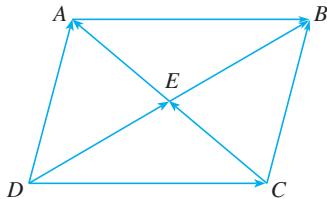
Substituting these values in (5) and (6), we obtain the tension vectors

$$\mathbf{T}_1 \approx -55.05 \mathbf{i} + 65.60 \mathbf{j}$$

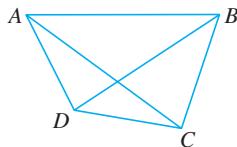
$$\mathbf{T}_2 \approx 55.05 \mathbf{i} + 34.40 \mathbf{j}$$

## 12.2 EXERCISES

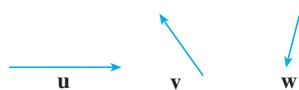
- Are the following quantities vectors or scalars? Explain.
  - The cost of a theater ticket
  - The current in a river
  - The initial flight path from Houston to Dallas
  - The population of the world
- What is the relationship between the point  $(4, 7)$  and the vector  $\langle 4, 7 \rangle$ ? Illustrate with a sketch.
- Name all the equal vectors in the parallelogram shown.



- Write each combination of vectors as a single vector.
  - $\vec{AB} + \vec{BC}$
  - $\vec{CD} + \vec{DB}$
  - $\vec{DB} - \vec{AB}$
  - $\vec{DC} + \vec{CA} + \vec{AB}$



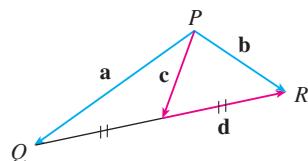
- Copy the vectors in the figure and use them to draw the following vectors.
  - $\mathbf{u} + \mathbf{v}$
  - $\mathbf{u} + \mathbf{w}$
  - $\mathbf{v} + \mathbf{w}$
  - $\mathbf{u} - \mathbf{v}$
  - $\mathbf{v} + \mathbf{u} + \mathbf{w}$
  - $\mathbf{u} - \mathbf{w} - \mathbf{v}$



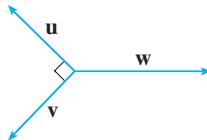
- Copy the vectors in the figure and use them to draw the following vectors.
  - $\mathbf{a} + \mathbf{b}$
  - $\mathbf{a} - \mathbf{b}$
  - $\frac{1}{2}\mathbf{a}$
  - $-3\mathbf{b}$
  - $\mathbf{a} + 2\mathbf{b}$
  - $2\mathbf{b} - \mathbf{a}$



- In the figure, the tip of  $\mathbf{c}$  and the tail of  $\mathbf{d}$  are both the midpoint of  $QR$ . Express  $\mathbf{c}$  and  $\mathbf{d}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .



- If the vectors in the figure satisfy  $|\mathbf{u}| = |\mathbf{v}| = 1$  and  $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$ , what is  $|\mathbf{w}|$ ?



- Find a vector  $\mathbf{a}$  with representation given by the directed line segment  $\vec{AB}$ . Draw  $\vec{AB}$  and the equivalent representation starting at the origin.

- $A(-2, 1), B(1, 2)$
- $A(3, -1), B(2, 3)$
- $A(0, 3, 1), B(2, 3, -1)$
- $A(0, 6, -1), B(3, 4, 4)$

- Find the sum of the given vectors and illustrate geometrically.

- $\langle -1, 4 \rangle, \langle 6, -2 \rangle$
- $\langle 3, -1 \rangle, \langle -1, 5 \rangle$
- $\langle 3, 0, 1 \rangle, \langle 0, 8, 0 \rangle$
- $\langle 1, 3, -2 \rangle, \langle 0, 0, 6 \rangle$

- Find  $\mathbf{a} + \mathbf{b}$ ,  $4\mathbf{a} + 2\mathbf{b}$ ,  $|\mathbf{a}|$ , and  $|\mathbf{a} - \mathbf{b}|$ .

- $\mathbf{a} = \langle -3, 4 \rangle, \mathbf{b} = \langle 9, -1 \rangle$
- $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j}, \mathbf{b} = -\mathbf{i} - 2\mathbf{j}$
- $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} - 4\mathbf{k}$
- $\mathbf{a} = \langle 8, 1, -4 \rangle, \mathbf{b} = \langle 5, -2, 1 \rangle$

- Find a unit vector that has the same direction as the given vector.

- $\langle 6, -2 \rangle$
- $-5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
- $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

- Find the vector that has the same direction as  $\langle 6, 2, -3 \rangle$  but has length 4.

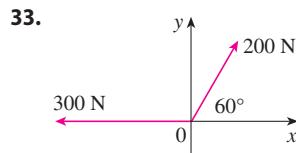
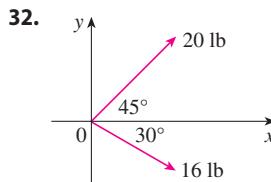
- What is the angle between the given vector and the positive direction of the  $x$ -axis?

- $\mathbf{i} + \sqrt{3}\mathbf{j}$
- $8\mathbf{i} + 6\mathbf{j}$

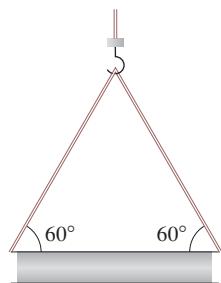
- If  $\mathbf{v}$  lies in the first quadrant and makes an angle  $\pi/3$  with the positive  $x$ -axis and  $|\mathbf{v}| = 4$ , find  $\mathbf{v}$  in component form.

- If a child pulls a sled through the snow on a level path with a force of 50 N exerted at an angle of  $38^\circ$  above the horizontal, find the horizontal and vertical components of the force.
- A quarterback throws a football with angle of elevation  $40^\circ$  and speed 60 ft/s. Find the horizontal and vertical components of the velocity vector.

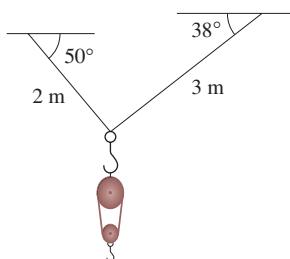
- 32–33** Find the magnitude of the resultant force and the angle it makes with the positive  $x$ -axis.



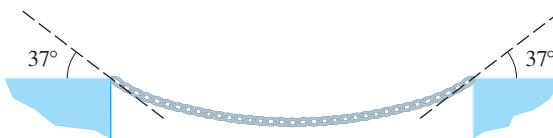
- 34.** The magnitude of a velocity vector is called *speed*. Suppose that a wind is blowing from the direction  $N45^\circ W$  at a speed of 50 km/h. (This means that the direction from which the wind blows is  $45^\circ$  west of the northerly direction.) A pilot is steering a plane in the direction  $N60^\circ E$  at an airspeed (speed in still air) of 250 km/h. The *true course*, or *track*, of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The *ground speed* of the plane is the magnitude of the resultant. Find the true course and the ground speed of the plane.
- 35.** A woman walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 22 mi/h. Find the speed and direction of the woman relative to the surface of the water.
- 36.** A crane suspends a 500-lb steel beam horizontally by support cables (with negligible weight) attached from a hook to each end of the beam. The support cables each make an angle of  $60^\circ$  with the beam. Find the tension vector in each support cable and the magnitude of each tension.



- 37.** A block-and-tackle pulley hoist is suspended in a warehouse by ropes of lengths 2 m and 3 m. The hoist weighs 350 N. The ropes, fastened at different heights, make angles of  $50^\circ$  and  $38^\circ$  with the horizontal. Find the tension in each rope and the magnitude of each tension.



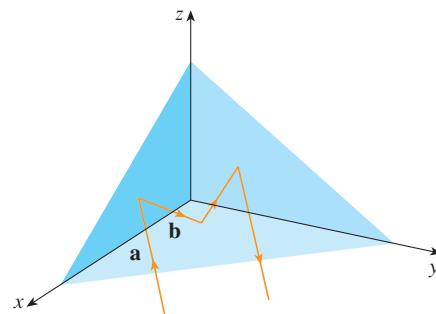
- 38.** The tension  $\mathbf{T}$  at each end of a chain has magnitude 25 N (see the figure). What is the weight of the chain?



- 39.** A boatman wants to cross a canal that is 3 km wide and wants to land at a point 2 km upstream from his starting point. The current in the canal flows at 3.5 km/h and the speed of his boat is 13 km/h.
- In what direction should he steer?
  - How long will the trip take?
- 40.** Three forces act on an object. Two of the forces are at an angle of  $100^\circ$  to each other and have magnitudes 25 N and 12 N. The third is perpendicular to the plane of these two forces and has magnitude 4 N. Calculate the magnitude of the force that would exactly counterbalance these three forces.
- 41.** Find the unit vectors that are parallel to the tangent line to the parabola  $y = x^2$  at the point  $(2, 4)$ .
- 42.**
  - Find the unit vectors that are parallel to the tangent line to the curve  $y = 2 \sin x$  at the point  $(\pi/6, 1)$ .
  - Find the unit vectors that are perpendicular to the tangent line.
  - Sketch the curve  $y = 2 \sin x$  and the vectors in parts (a) and (b), all starting at  $(\pi/6, 1)$ .
- 43.** If  $A$ ,  $B$ , and  $C$  are the vertices of a triangle, find
- $$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$$
- 44.** Let  $C$  be the point on the line segment  $AB$  that is twice as far from  $B$  as it is from  $A$ . If  $\mathbf{a} = \overrightarrow{OA}$ ,  $\mathbf{b} = \overrightarrow{OB}$ , and  $\mathbf{c} = \overrightarrow{OC}$ , show that  $\mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ .
- 45.**
  - Draw the vectors  $\mathbf{a} = \langle 3, 2 \rangle$ ,  $\mathbf{b} = \langle 2, -1 \rangle$ , and  $\mathbf{c} = \langle 7, 1 \rangle$ .
  - Show, by means of a sketch, that there are scalars  $s$  and  $t$  such that  $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ .
  - Use the sketch to estimate the values of  $s$  and  $t$ .
  - Find the exact values of  $s$  and  $t$ .
- 46.** Suppose that  $\mathbf{a}$  and  $\mathbf{b}$  are nonzero vectors that are not parallel and  $\mathbf{c}$  is any vector in the plane determined by  $\mathbf{a}$  and  $\mathbf{b}$ . Give a geometric argument to show that  $\mathbf{c}$  can be written as  $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$  for suitable scalars  $s$  and  $t$ . Then give an argument using components.
- 47.** If  $\mathbf{r} = \langle x, y, z \rangle$  and  $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ , describe the set of all points  $(x, y, z)$  such that  $|\mathbf{r} - \mathbf{r}_0| = 1$ .
- 48.** If  $\mathbf{r} = \langle x, y \rangle$ ,  $\mathbf{r}_1 = \langle x_1, y_1 \rangle$ , and  $\mathbf{r}_2 = \langle x_2, y_2 \rangle$ , describe the set of all points  $(x, y)$  such that  $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2| = k$ , where  $k > |\mathbf{r}_1 - \mathbf{r}_2|$ .
- 49.** Figure 16 gives a geometric demonstration of Property 2 of vectors. Use components to give an algebraic proof of this fact for the case  $n = 2$ .

50. Prove Property 5 of vectors algebraically for the case  $n = 3$ . Then use similar triangles to give a geometric proof.
51. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.
52. Suppose the three coordinate planes are all mirrored and a light ray given by the vector  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  first strikes the  $xz$ -plane, as shown in the figure. Use the fact that the angle of incidence equals the angle of reflection to show that the direction of the reflected ray is given by  $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$ . Deduce that, after being reflected by all three mutually perpendicular mirrors, the resulting ray is parallel to the initial ray. (American space scientists used this principle, together with laser beams

and an array of corner mirrors on the moon, to calculate very precisely the distance from the earth to the moon.)



## 12.3 The Dot Product

So far we have added two vectors and multiplied a vector by a scalar. The question arises: is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows. Another is the cross product, which is discussed in the next section.

**1 Definition** If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , then the **dot product** of  $\mathbf{a}$  and  $\mathbf{b}$  is the number  $\mathbf{a} \cdot \mathbf{b}$  given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Thus, to find the dot product of  $\mathbf{a}$  and  $\mathbf{b}$ , we multiply corresponding components and add. The result is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the **scalar product** (or **inner product**). Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

### EXAMPLE 1

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = (-1)(6) + 7(2) + 4\left(-\frac{1}{2}\right) = 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = 7$$

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

**2 Properties of the Dot Product** If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_3$  and  $c$  is a scalar, then

1.  $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2.  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3.  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4.  $(ca) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5.  $\mathbf{0} \cdot \mathbf{a} = 0$

These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1.  $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$
3. 
$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \end{aligned}$$

The proofs of the remaining properties are left as exercises. ■

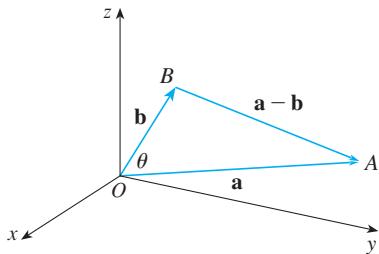


FIGURE 1

The dot product  $\mathbf{a} \cdot \mathbf{b}$  can be given a geometric interpretation in terms of the **angle  $\theta$  between  $\mathbf{a}$  and  $\mathbf{b}$** , which is defined to be the angle between the representations of  $\mathbf{a}$  and  $\mathbf{b}$  that start at the origin, where  $0 \leq \theta \leq \pi$ . In other words,  $\theta$  is the angle between the line segments  $\overrightarrow{OA}$  and  $\overrightarrow{OB}$  in Figure 1. Note that if  $\mathbf{a}$  and  $\mathbf{b}$  are parallel vectors, then  $\theta = 0$  or  $\theta = \pi$ .

The formula in the following theorem is used by physicists as the *definition* of the dot product.

**3 Theorem** If  $\theta$  is the angle between the vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

**PROOF** If we apply the Law of Cosines to triangle  $OAB$  in Figure 1, we get

**4** 
$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta$$

(Observe that the Law of Cosines still applies in the limiting cases when  $\theta = 0$  or  $\pi$ , or  $\mathbf{a} = \mathbf{0}$  or  $\mathbf{b} = \mathbf{0}$ .) But  $|OA| = |\mathbf{a}|$ ,  $|OB| = |\mathbf{b}|$ , and  $|AB| = |\mathbf{a} - \mathbf{b}|$ , so Equation 4 becomes

**5** 
$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$$

Using Properties 1, 2, and 3 of the dot product, we can rewrite the left side of this equation as follows:

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \end{aligned}$$

Therefore Equation 5 gives

$$|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$$

Thus

$$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}| \cos \theta$$

or

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

**EXAMPLE 2** If the vectors  $\mathbf{a}$  and  $\mathbf{b}$  have lengths 4 and 6, and the angle between them is  $\pi/3$ , find  $\mathbf{a} \cdot \mathbf{b}$ .

**SOLUTION** Using Theorem 3, we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\pi/3) = 4 \cdot 6 \cdot \frac{1}{2} = 12$$

The formula in Theorem 3 also enables us to find the angle between two vectors.

**6 Corollary** If  $\theta$  is the angle between the nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

**EXAMPLE 3** Find the angle between the vectors  $\mathbf{a} = \langle 2, 2, -1 \rangle$  and  $\mathbf{b} = \langle 5, -3, 2 \rangle$ .

**SOLUTION** Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \quad \text{and} \quad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

we have, from Corollary 6,

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84^\circ)$$

Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  are called **perpendicular** or **orthogonal** if the angle between them is  $\theta = \pi/2$ . Then Theorem 3 gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\pi/2) = 0$$

and conversely if  $\mathbf{a} \cdot \mathbf{b} = 0$ , then  $\cos \theta = 0$ , so  $\theta = \pi/2$ . The zero vector  $\mathbf{0}$  is considered to be perpendicular to all vectors. Therefore we have the following method for determining whether two vectors are orthogonal.

**7**

Two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal if and only if  $\mathbf{a} \cdot \mathbf{b} = 0$ .

**EXAMPLE 4** Show that  $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  is perpendicular to  $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ .

**SOLUTION** Since

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular by (7). ■

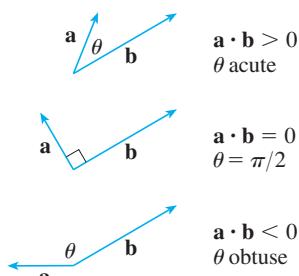


FIGURE 2

**TEC** Visual 12.3A shows an animation of Figure 2.

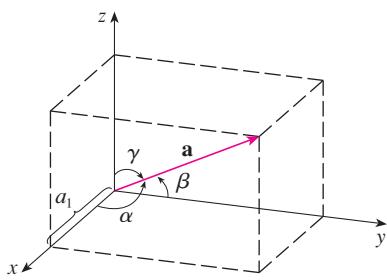


FIGURE 3

Because  $\cos \theta > 0$  if  $0 \leq \theta < \pi/2$  and  $\cos \theta < 0$  if  $\pi/2 < \theta \leq \pi$ , we see that  $\mathbf{a} \cdot \mathbf{b}$  is positive for  $\theta < \pi/2$  and negative for  $\theta > \pi/2$ . We can think of  $\mathbf{a} \cdot \mathbf{b}$  as measuring the extent to which  $\mathbf{a}$  and  $\mathbf{b}$  point in the same direction. The dot product  $\mathbf{a} \cdot \mathbf{b}$  is positive if  $\mathbf{a}$  and  $\mathbf{b}$  point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2). In the extreme case where  $\mathbf{a}$  and  $\mathbf{b}$  point in exactly the same direction, we have  $\theta = 0$ , so  $\cos \theta = 1$  and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}|$$

If  $\mathbf{a}$  and  $\mathbf{b}$  point in exactly opposite directions, then we have  $\theta = \pi$  and so  $\cos \theta = -1$  and  $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}| |\mathbf{b}|$ .

### ■ Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector  $\mathbf{a}$  are the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  (in the interval  $[0, \pi]$ ) that  $\mathbf{a}$  makes with the positive  $x$ -,  $y$ -, and  $z$ -axes, respectively. (See Figure 3.)

The cosines of these direction angles,  $\cos \alpha$ ,  $\cos \beta$ , and  $\cos \gamma$ , are called the **direction cosines** of the vector  $\mathbf{a}$ . Using Corollary 6 with  $\mathbf{b}$  replaced by  $\mathbf{i}$ , we obtain

8

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}| |\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

(This can also be seen directly from Figure 3.)

Similarly, we also have

9

$$\cos \beta = \frac{a_2}{|\mathbf{a}|} \quad \cos \gamma = \frac{a_3}{|\mathbf{a}|}$$

By squaring the expressions in Equations 8 and 9 and adding, we see that

10

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

We can also use Equations 8 and 9 to write

$$\begin{aligned} \mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}| \cos \alpha, |\mathbf{a}| \cos \beta, |\mathbf{a}| \cos \gamma \rangle \\ &= |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle \end{aligned}$$

Therefore

11

$$\frac{1}{|\mathbf{a}|} \mathbf{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

which says that the direction cosines of  $\mathbf{a}$  are the components of the unit vector in the direction of  $\mathbf{a}$ .

**EXAMPLE 5** Find the direction angles of the vector  $\mathbf{a} = \langle 1, 2, 3 \rangle$ .

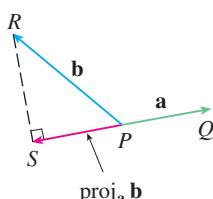
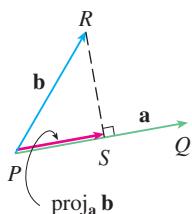
**SOLUTION** Since  $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ , Equations 8 and 9 give

$$\cos \alpha = \frac{1}{\sqrt{14}} \quad \cos \beta = \frac{2}{\sqrt{14}} \quad \cos \gamma = \frac{3}{\sqrt{14}}$$

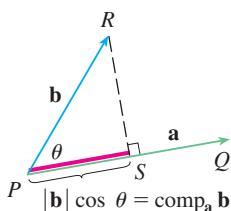
and so

$$\alpha = \cos^{-1} \left( \frac{1}{\sqrt{14}} \right) \approx 74^\circ \quad \beta = \cos^{-1} \left( \frac{2}{\sqrt{14}} \right) \approx 58^\circ \quad \gamma = \cos^{-1} \left( \frac{3}{\sqrt{14}} \right) \approx 37^\circ$$

**TEC** Visual 12.3B shows how Figure 4 changes when we vary  $\mathbf{a}$  and  $\mathbf{b}$ .



**FIGURE 4**  
Vector projections



**FIGURE 5**  
Scalar projection

## ■ Projections

Figure 4 shows representations  $\vec{PQ}$  and  $\vec{PR}$  of two vectors  $\mathbf{a}$  and  $\mathbf{b}$  with the same initial point  $P$ . If  $S$  is the foot of the perpendicular from  $R$  to the line containing  $\vec{PQ}$ , then the vector with representation  $\vec{PS}$  is called the **vector projection** of  $\mathbf{b}$  onto  $\mathbf{a}$  and is denoted by  $\text{proj}_{\mathbf{a}} \mathbf{b}$ . (You can think of it as a shadow of  $\mathbf{b}$ .)

The **scalar projection** of  $\mathbf{b}$  onto  $\mathbf{a}$  (also called the **component of  $\mathbf{b}$  along  $\mathbf{a}$** ) is defined to be the signed magnitude of the vector projection, which is the number  $|\mathbf{b}| \cos \theta$ , where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . (See Figure 5.) This is denoted by  $\text{comp}_{\mathbf{a}} \mathbf{b}$ . Observe that it is negative if  $\pi/2 < \theta \leq \pi$ . The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = |\mathbf{a}| (|\mathbf{b}| \cos \theta)$$

shows that the dot product of  $\mathbf{a}$  and  $\mathbf{b}$  can be interpreted as the length of  $\mathbf{a}$  times the scalar projection of  $\mathbf{b}$  onto  $\mathbf{a}$ . Since

$$|\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of  $\mathbf{b}$  along  $\mathbf{a}$  can be computed by taking the dot product of  $\mathbf{b}$  with the unit vector in the direction of  $\mathbf{a}$ . We summarize these ideas as follows.

Scalar projection of  $\mathbf{b}$  onto  $\mathbf{a}$ :  $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of  $\mathbf{b}$  onto  $\mathbf{a}$ :  $\text{proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of  $\mathbf{a}$ .

**EXAMPLE 6** Find the scalar projection and vector projection of  $\mathbf{b} = \langle 1, 1, 2 \rangle$  onto  $\mathbf{a} = \langle -2, 3, 1 \rangle$ .

**SOLUTION** Since  $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$ , the scalar projection of  $\mathbf{b}$  onto  $\mathbf{a}$  is

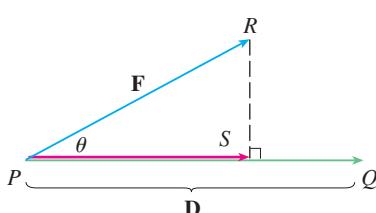
$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of  $\mathbf{a}$ :

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{3}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14} \mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

One use of projections occurs in physics in calculating work. In Section 6.4 we defined the work done by a constant force  $\mathbf{F}$  in moving an object through a distance  $d$  as  $W = \mathbf{F}d$ , but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector  $\mathbf{F} = \vec{PR}$  pointing in some other direction, as in Figure 6. If the force moves the object from  $P$  to  $Q$ , then the **displacement vector** is  $\mathbf{D} = \vec{PQ}$ . The **work** done by this force is defined to be the product of the component of the force along  $\mathbf{D}$  and the distance moved:

$$W = (|\mathbf{F}| \cos \theta) |\mathbf{D}|$$



**FIGURE 6**

But then, from Theorem 3, we have

12

$$W = |\mathbf{F}| |\mathbf{D}| \cos \theta = \mathbf{F} \cdot \mathbf{D}$$

Thus the work done by a constant force  $\mathbf{F}$  is the dot product  $\mathbf{F} \cdot \mathbf{D}$ , where  $\mathbf{D}$  is the displacement vector.

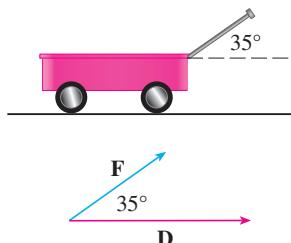


FIGURE 7

**EXAMPLE 7** A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of  $35^\circ$  above the horizontal. Find the work done by the force.

**SOLUTION** If  $\mathbf{F}$  and  $\mathbf{D}$  are the force and displacement vectors, as pictured in Figure 7, then the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}| |\mathbf{D}| \cos 35^\circ \\ &= (70)(100) \cos 35^\circ \approx 5734 \text{ N} \cdot \text{m} = 5734 \text{ J} \end{aligned}$$

**EXAMPLE 8** A force is given by a vector  $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$  and moves a particle from the point  $P(2, 1, 0)$  to the point  $Q(4, 6, 2)$ . Find the work done.

**SOLUTION** The displacement vector is  $\mathbf{D} = \vec{PQ} = \langle 2, 5, 2 \rangle$ , so by Equation 12, the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{D} = \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle \\ &= 6 + 20 + 10 = 36 \end{aligned}$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 J.

## 12.3 EXERCISES

1. Which of the following expressions are meaningful? Which are meaningless? Explain.

- (a)  $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$       (b)  $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$   
 (c)  $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$       (d)  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$   
 (e)  $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$       (f)  $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$

- 2-10 Find  $\mathbf{a} \cdot \mathbf{b}$ .

2.  $\mathbf{a} = \langle 5, -2 \rangle$ ,  $\mathbf{b} = \langle 3, 4 \rangle$

3.  $\mathbf{a} = \langle 1.5, 0.4 \rangle$ ,  $\mathbf{b} = \langle -4, 6 \rangle$

4.  $\mathbf{a} = \langle 6, -2, 3 \rangle$ ,  $\mathbf{b} = \langle 2, 5, -1 \rangle$

5.  $\mathbf{a} = \langle 4, 1, \frac{1}{4} \rangle$ ,  $\mathbf{b} = \langle 6, -3, -8 \rangle$

6.  $\mathbf{a} = \langle p, -p, 2p \rangle$ ,  $\mathbf{b} = \langle 2q, q, -q \rangle$

7.  $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$ ,  $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

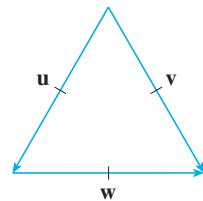
8.  $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ ,  $\mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$

9.  $|\mathbf{a}| = 7$ ,  $|\mathbf{b}| = 4$ , the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $30^\circ$

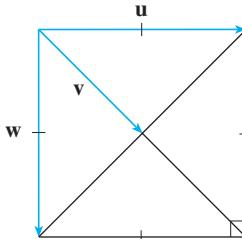
10.  $|\mathbf{a}| = 80$ ,  $|\mathbf{b}| = 50$ , the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $3\pi/4$

- 11-12 If  $\mathbf{u}$  is a unit vector, find  $\mathbf{u} \cdot \mathbf{v}$  and  $\mathbf{u} \cdot \mathbf{w}$ .

11.



12.



13. (a) Show that  $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$ .  
 (b) Show that  $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ .

14. A street vendor sells  $a$  hamburgers,  $b$  hot dogs, and  $c$  soft drinks on a given day. He charges \$4 for a hamburger, \$2.50 for a hot dog, and \$1 for a soft drink. If  $\mathbf{A} = \langle a, b, c \rangle$  and  $\mathbf{P} = \langle 4, 2.5, 1 \rangle$ , what is the meaning of the dot product  $\mathbf{A} \cdot \mathbf{P}$ ?

- 15-20 Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

15.  $\mathbf{a} = \langle 4, 3 \rangle$ ,  $\mathbf{b} = \langle 2, -1 \rangle$

16.  $\mathbf{a} = \langle -2, 5 \rangle$ ,  $\mathbf{b} = \langle 5, 12 \rangle$

17.  $\mathbf{a} = \langle 1, -4, 1 \rangle$ ,  $\mathbf{b} = \langle 0, 2, -2 \rangle$

18.  $\mathbf{a} = \langle -1, 3, 4 \rangle$ ,  $\mathbf{b} = \langle 5, 2, 1 \rangle$

19.  $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$

20.  $\mathbf{a} = 8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ ,  $\mathbf{b} = 4\mathbf{j} + 2\mathbf{k}$

21–22 Find, correct to the nearest degree, the three angles of the triangle with the given vertices.

21.  $P(2, 0)$ ,  $Q(0, 3)$ ,  $R(3, 4)$

22.  $A(1, 0, -1)$ ,  $B(3, -2, 0)$ ,  $C(1, 3, 3)$

23–24 Determine whether the given vectors are orthogonal, parallel, or neither.

23. (a)  $\mathbf{a} = \langle 9, 3 \rangle$ ,  $\mathbf{b} = \langle -2, 6 \rangle$

(b)  $\mathbf{a} = \langle 4, 5, -2 \rangle$ ,  $\mathbf{b} = \langle 3, -1, 5 \rangle$

(c)  $\mathbf{a} = -8\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$ ,  $\mathbf{b} = 6\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$

(d)  $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ ,  $\mathbf{b} = 5\mathbf{i} + 9\mathbf{j} - 2\mathbf{k}$

24. (a)  $\mathbf{u} = \langle -5, 4, -2 \rangle$ ,  $\mathbf{v} = \langle 3, 4, -1 \rangle$

(b)  $\mathbf{u} = 9\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$ ,  $\mathbf{v} = -6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

(c)  $\mathbf{u} = \langle c, c, c \rangle$ ,  $\mathbf{v} = \langle c, 0, -c \rangle$

25. Use vectors to decide whether the triangle with vertices  $P(1, -3, -2)$ ,  $Q(2, 0, -4)$ , and  $R(6, -2, -5)$  is right-angled.

26. Find the values of  $x$  such that the angle between the vectors  $\langle 2, 1, -1 \rangle$ , and  $\langle 1, x, 0 \rangle$  is  $45^\circ$ .

27. Find a unit vector that is orthogonal to both  $\mathbf{i} + \mathbf{j}$  and  $\mathbf{i} + \mathbf{k}$ .

28. Find two unit vectors that make an angle of  $60^\circ$  with  $\mathbf{v} = \langle 3, 4 \rangle$ .

29–30 Find the acute angle between the lines.

29.  $2x - y = 3$ ,  $3x + y = 7$

30.  $x + 2y = 7$ ,  $5x - y = 2$

31–32 Find the acute angles between the curves at their points of intersection. (The angle between two curves is the angle between their tangent lines at the point of intersection.)

31.  $y = x^2$ ,  $y = x^3$

32.  $y = \sin x$ ,  $y = \cos x$ ,  $0 \leq x \leq \pi/2$

33–37 Find the direction cosines and direction angles of the vector. (Give the direction angles correct to the nearest degree.)

33.  $\langle 2, 1, 2 \rangle$

34.  $\langle 6, 3, -2 \rangle$

35.  $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

36.  $\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}$

37.  $\langle c, c, c \rangle$ , where  $c > 0$

38. If a vector has direction angles  $\alpha = \pi/4$  and  $\beta = \pi/3$ , find the third direction angle  $\gamma$ .

39–44 Find the scalar and vector projections of  $\mathbf{b}$  onto  $\mathbf{a}$ .

39.  $\mathbf{a} = \langle -5, 12 \rangle$ ,  $\mathbf{b} = \langle 4, 6 \rangle$

40.  $\mathbf{a} = \langle 1, 4 \rangle$ ,  $\mathbf{b} = \langle 2, 3 \rangle$

41.  $\mathbf{a} = \langle 4, 7, -4 \rangle$ ,  $\mathbf{b} = \langle 3, -1, 1 \rangle$

42.  $\mathbf{a} = \langle -1, 4, 8 \rangle$ ,  $\mathbf{b} = \langle 12, 1, 2 \rangle$

43.  $\mathbf{a} = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

44.  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ ,  $\mathbf{b} = 5\mathbf{i} - \mathbf{k}$

45. Show that the vector  $\text{orth}_{\mathbf{a}} \mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}} \mathbf{b}$  is orthogonal to  $\mathbf{a}$ . (It is called an **orthogonal projection** of  $\mathbf{b}$ .)

46. For the vectors in Exercise 40, find  $\text{orth}_{\mathbf{a}} \mathbf{b}$  and illustrate by drawing the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\text{proj}_{\mathbf{a}} \mathbf{b}$ , and  $\text{orth}_{\mathbf{a}} \mathbf{b}$ .

47. If  $\mathbf{a} = \langle 3, 0, -1 \rangle$ , find a vector  $\mathbf{b}$  such that  $\text{comp}_{\mathbf{a}} \mathbf{b} = 2$ .

48. Suppose that  $\mathbf{a}$  and  $\mathbf{b}$  are nonzero vectors.

(a) Under what circumstances is  $\text{comp}_{\mathbf{a}} \mathbf{b} = \text{comp}_{\mathbf{b}} \mathbf{a}$ ?

(b) Under what circumstances is  $\text{proj}_{\mathbf{a}} \mathbf{b} = \text{proj}_{\mathbf{b}} \mathbf{a}$ ?

49. Find the work done by a force  $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$  that moves an object from the point  $(0, 10, 8)$  to the point  $(6, 12, 20)$  along a straight line. The distance is measured in meters and the force in newtons.

50. A tow truck drags a stalled car along a road. The chain makes an angle of  $30^\circ$  with the road and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

51. A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of  $40^\circ$  above the horizontal moves the sled 80 ft. Find the work done by the force.

52. A boat sails south with the help of a wind blowing in the direction S36°E with magnitude 400 lb. Find the work done by the wind as the boat moves 120 ft.

53. Use a scalar projection to show that the distance from a point  $P_1(x_1, y_1)$  to the line  $ax + by + c = 0$  is

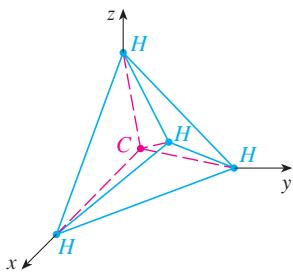
$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Use this formula to find the distance from the point  $(-2, 3)$  to the line  $3x - 4y + 5 = 0$ .

54. If  $\mathbf{r} = \langle x, y, z \rangle$ ,  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ , and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , show that the vector equation  $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$  represents a sphere, and find its center and radius.

55. Find the angle between a diagonal of a cube and one of its edges.

56. Find the angle between a diagonal of a cube and a diagonal of one of its faces.
57. A molecule of methane,  $\text{CH}_4$ , is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the  $\text{H}-\text{C}-\text{H}$  combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about  $109.5^\circ$ . [Hint: Take the vertices of the tetrahedron to be the points  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ , and  $(1, 1, 1)$ , as shown in the figure. Then the centroid is  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ .]



58. If  $\mathbf{c} = |\mathbf{a}| \mathbf{b} + |\mathbf{b}| \mathbf{a}$ , where  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are all nonzero vectors, show that  $\mathbf{c}$  bisects the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .
59. Prove Properties 2, 4, and 5 of the dot product (Theorem 2).

60. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

61. Use Theorem 3 to prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}|$$

62. The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

- (a) Give a geometric interpretation of the Triangle Inequality.  
 (b) Use the Cauchy-Schwarz Inequality from Exercise 61 to prove the Triangle Inequality. [Hint: Use the fact that  $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$  and use Property 3 of the dot product.]

63. The Parallelogram Law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

- (a) Give a geometric interpretation of the Parallelogram Law.  
 (b) Prove the Parallelogram Law. (See the hint in Exercise 62.)  
 64. Show that if  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - \mathbf{v}$  are orthogonal, then the vectors  $\mathbf{u}$  and  $\mathbf{v}$  must have the same length.

65. If  $\theta$  is the angle between vectors  $\mathbf{a}$  and  $\mathbf{b}$ , show that

$$\text{proj}_{\mathbf{a}} \mathbf{b} \cdot \text{proj}_{\mathbf{b}} \mathbf{a} = (\mathbf{a} \cdot \mathbf{b}) \cos^2 \theta$$

## 12.4 The Cross Product

Given two nonzero vectors  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , it is very useful to be able to find a nonzero vector  $\mathbf{c}$  that is perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ , as we will see in the next section and in Chapters 13 and 14. If  $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$  is such a vector, then  $\mathbf{a} \cdot \mathbf{c} = 0$  and  $\mathbf{b} \cdot \mathbf{c} = 0$  and so

1

$$a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$$

2

$$b_1 c_1 + b_2 c_2 + b_3 c_3 = 0$$

To eliminate  $c_3$  we multiply (1) by  $b_3$  and (2) by  $a_3$  and subtract:

3

$$(a_1 b_3 - a_3 b_1) c_1 + (a_2 b_3 - a_3 b_2) c_2 = 0$$

Equation 3 has the form  $p c_1 + q c_2 = 0$ , for which an obvious solution is  $c_1 = q$  and  $c_2 = -p$ . So a solution of (3) is

$$c_1 = a_2 b_3 - a_3 b_2 \quad c_2 = a_3 b_1 - a_1 b_3$$

Substituting these values into (1) and (2), we then get

$$c_3 = a_1 b_2 - a_2 b_1$$

This means that a vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$  is

$$\langle c_1, c_2, c_3 \rangle = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

The resulting vector is called the *cross product* of  $\mathbf{a}$  and  $\mathbf{b}$  and is denoted by  $\mathbf{a} \times \mathbf{b}$ .

**Hamilton**

The cross product was invented by the Irish mathematician Sir William Rowan Hamilton (1805–1865), who had created a precursor of vectors, called quaternions. When he was five years old Hamilton could read Latin, Greek, and Hebrew. At age eight he added French and Italian and when ten he could read Arabic and Sanskrit. At the age of 21, while still an undergraduate at Trinity College in Dublin, Hamilton was appointed Professor of Astronomy at the university and Royal Astronomer of Ireland!

**4 Definition** If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , then the **cross product** of  $\mathbf{a}$  and  $\mathbf{b}$  is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Notice that the **cross product**  $\mathbf{a} \times \mathbf{b}$  of two vectors  $\mathbf{a}$  and  $\mathbf{b}$ , unlike the dot product, is a vector. For this reason it is also called the **vector product**. Note that  $\mathbf{a} \times \mathbf{b}$  is defined only when  $\mathbf{a}$  and  $\mathbf{b}$  are *three-dimensional* vectors.

In order to make Definition 4 easier to remember, we use the notation of determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(Multiply across the diagonals and subtract.) For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\begin{array}{c} \boxed{5} \\ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \end{array}$$

Observe that each term on the right side of Equation 5 involves a number  $a_i$  in the first row of the determinant, and  $a_i$  is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which  $a_i$  appears. Notice also the minus sign in the second term. For example,

$$\begin{array}{c} \begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix} \\ = 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) = -38 \end{array}$$

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ , we see that the cross product of the vectors  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  and  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  is

$$\boxed{6} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

In view of the similarity between Equations 5 and 6, we often write

$$\boxed{7} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain

Equation 6. The symbolic formula in Equation 7 is probably the easiest way of remembering and computing cross products.

**EXAMPLE 1** If  $\mathbf{a} = \langle 1, 3, 4 \rangle$  and  $\mathbf{b} = \langle 2, 7, -5 \rangle$ , then

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k} \\ &= (-15 - 28) \mathbf{i} - (-5 - 8) \mathbf{j} + (7 - 6) \mathbf{k} = -43 \mathbf{i} + 13 \mathbf{j} + \mathbf{k} \quad \blacksquare\end{aligned}$$

**EXAMPLE 2** Show that  $\mathbf{a} \times \mathbf{a} = \mathbf{0}$  for any vector  $\mathbf{a}$  in  $V_3$ .

**SOLUTION** If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ , then

$$\begin{aligned}\mathbf{a} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= (a_2 a_3 - a_3 a_2) \mathbf{i} - (a_1 a_3 - a_3 a_1) \mathbf{j} + (a_1 a_2 - a_2 a_1) \mathbf{k} \\ &= 0 \mathbf{i} - 0 \mathbf{j} + 0 \mathbf{k} = \mathbf{0} \quad \blacksquare\end{aligned}$$

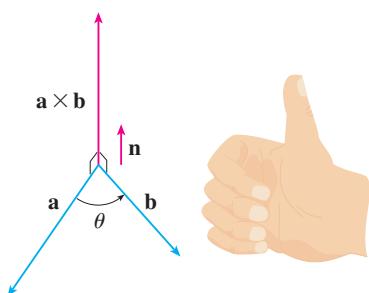
We constructed the cross product  $\mathbf{a} \times \mathbf{b}$  so that it would be perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ . This is one of the most important properties of a cross product, so let's emphasize and verify it in the following theorem and give a formal proof.

**8 Theorem** The vector  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ .

**PROOF** In order to show that  $\mathbf{a} \times \mathbf{b}$  is orthogonal to  $\mathbf{a}$ , we compute their dot product as follows:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2 b_3 - a_3 b_2) - a_2(a_1 b_3 - a_3 b_1) + a_3(a_1 b_2 - a_2 b_1) \\ &= a_1 a_2 b_3 - a_1 b_2 a_3 - a_1 a_2 b_3 + b_1 a_2 a_3 + a_1 b_2 a_3 - b_1 a_2 a_3 \\ &= 0\end{aligned}$$

A similar computation shows that  $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$ . Therefore  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ . ■



**FIGURE 1**

The right-hand rule gives the direction of  $\mathbf{a} \times \mathbf{b}$ .

If  $\mathbf{a}$  and  $\mathbf{b}$  are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 8 says that the cross product  $\mathbf{a} \times \mathbf{b}$  points in a direction perpendicular to the plane through  $\mathbf{a}$  and  $\mathbf{b}$ . It turns out that the direction of  $\mathbf{a} \times \mathbf{b}$  is given by the *right-hand rule*: if the fingers of your right hand curl in the direction of

a rotation (through an angle less than  $180^\circ$ ) from  $\mathbf{a}$  to  $\mathbf{b}$ , then your thumb points in the direction of  $\mathbf{a} \times \mathbf{b}$ .

Now that we know the direction of the vector  $\mathbf{a} \times \mathbf{b}$ , the remaining thing we need to complete its geometric description is its length  $|\mathbf{a} \times \mathbf{b}|$ . This is given by the following theorem.

**9 Theorem** If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$  (so  $0 \leq \theta \leq \pi$ ), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

**TEC** Visual 12.4 shows how  $\mathbf{a} \times \mathbf{b}$  changes as  $\mathbf{b}$  changes.

**PROOF** From the definitions of the cross product and length of a vector, we have

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\ &= a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + a_3^2 b_1^2 - 2a_1 a_3 b_1 b_3 + a_1^2 b_3^2 \\ &\quad + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2 \theta \quad (\text{by Theorem 12.3.3}) \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2 \theta) \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2 \theta \end{aligned}$$

Taking square roots and observing that  $\sqrt{\sin^2 \theta} = \sin \theta$  because  $\sin \theta \geq 0$  when  $0 \leq \theta \leq \pi$ , we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

#### Geometric characterization of $\mathbf{a} \times \mathbf{b}$

Since a vector is completely determined by its magnitude and direction, we can now say that  $\mathbf{a} \times \mathbf{b}$  is the vector that is perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ , whose orientation is determined by the right-hand rule, and whose length is  $|\mathbf{a}| |\mathbf{b}| \sin \theta$ . In fact, that is exactly how physicists *define*  $\mathbf{a} \times \mathbf{b}$ .

**10 Corollary** Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

**PROOF** Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  are parallel if and only if  $\theta = 0$  or  $\pi$ . In either case  $\sin \theta = 0$ , so  $|\mathbf{a} \times \mathbf{b}| = 0$  and therefore  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ .

The geometric interpretation of Theorem 9 can be seen by looking at Figure 2. If  $\mathbf{a}$  and  $\mathbf{b}$  are represented by directed line segments with the same initial point, then they determine a parallelogram with base  $|\mathbf{a}|$ , altitude  $|\mathbf{b}| \sin \theta$ , and area

$$A = |\mathbf{a}| (|\mathbf{b}| \sin \theta) = |\mathbf{a} \times \mathbf{b}|$$

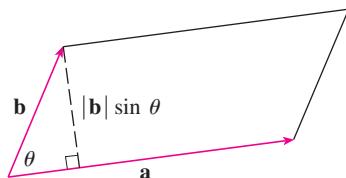


FIGURE 2

Thus we have the following way of interpreting the magnitude of a cross product.

The length of the cross product  $\mathbf{a} \times \mathbf{b}$  is equal to the area of the parallelogram determined by  $\mathbf{a}$  and  $\mathbf{b}$ .

**EXAMPLE 3** Find a vector perpendicular to the plane that passes through the points  $P(1, 4, 6)$ ,  $Q(-2, 5, -1)$ , and  $R(1, -1, 1)$ .

**SOLUTION** The vector  $\vec{PQ} \times \vec{PR}$  is perpendicular to both  $\vec{PQ}$  and  $\vec{PR}$  and is therefore perpendicular to the plane through  $P$ ,  $Q$ , and  $R$ . We know from (12.2.1) that

$$\vec{PQ} = (-2 - 1)\mathbf{i} + (5 - 4)\mathbf{j} + (-1 - 6)\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\vec{PR} = (1 - 1)\mathbf{i} + (-1 - 4)\mathbf{j} + (1 - 6)\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\mathbf{i} - (15 - 0)\mathbf{j} + (15 - 0)\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

So the vector  $\langle -40, -15, 15 \rangle$  is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as  $\langle -8, -3, 3 \rangle$ , is also perpendicular to the plane. ■

**EXAMPLE 4** Find the area of the triangle with vertices  $P(1, 4, 6)$ ,  $Q(-2, 5, -1)$ , and  $R(1, -1, 1)$ .

**SOLUTION** In Example 3 we computed that  $\vec{PQ} \times \vec{PR} = \langle -40, -15, 15 \rangle$ . The area of the parallelogram with adjacent sides  $PQ$  and  $PR$  is the length of this cross product:

$$|\vec{PQ} \times \vec{PR}| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area  $A$  of the triangle  $PQR$  is half the area of this parallelogram, that is,  $\frac{5}{2}\sqrt{82}$ . ■

If we apply Theorems 8 and 9 to the standard basis vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  using  $\theta = \pi/2$ , we obtain

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

∅ Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products. The following theorem summarizes the properties of vector products.

**11 Properties of the Cross Product** If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors and  $c$  is a scalar, then

1.  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2.  $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3.  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4.  $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5.  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6.  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product. We give the proof of Property 5 and leave the remaining proofs as exercises.

**PROOF OF PROPERTY 5** If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ ,  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , and  $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ , then

$$\begin{aligned} \mathbf{12} \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$

### ■ Triple Products

The product  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$  that occurs in Property 5 is called the **scalar triple product** of the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ . Notice from Equation 12 that we can write the scalar triple product as a determinant:

$$\mathbf{13} \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ . (See Figure 3.) The area of the base parallelogram is  $A = |\mathbf{b} \times \mathbf{c}|$ . If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b} \times \mathbf{c}$ , then the height  $h$  of the parallelepiped is  $h = |\mathbf{a}| |\cos \theta|$ . (We must use  $|\cos \theta|$  instead of  $\cos \theta$  in case  $\theta > \pi/2$ .) Therefore the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}| |\mathbf{a}| |\cos \theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus we have proved the following formula.

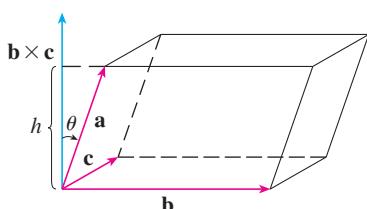


FIGURE 3

**14** The volume of the parallelepiped determined by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

If we use the formula in (14) and discover that the volume of the parallelepiped determined by  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

**EXAMPLE 5** Use the scalar triple product to show that the vectors  $\mathbf{a} = \langle 1, 4, -7 \rangle$ ,  $\mathbf{b} = \langle 2, -1, 4 \rangle$ , and  $\mathbf{c} = \langle 0, -9, 18 \rangle$  are coplanar.

**SOLUTION** We use Equation 13 to compute their scalar triple product:

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} \\ &= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} \\ &= 1(18) - 4(36) - 7(-18) = 0 \end{aligned}$$

Therefore, by (14), the volume of the parallelepiped determined by  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is 0. This means that  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are coplanar. ■

The product  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$  that occurs in Property 6 is called the **vector triple product** of  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ . Property 6 will be used to derive Kepler's First Law of planetary motion in Chapter 13. Its proof is left as Exercise 50.

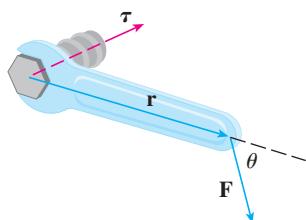


FIGURE 4

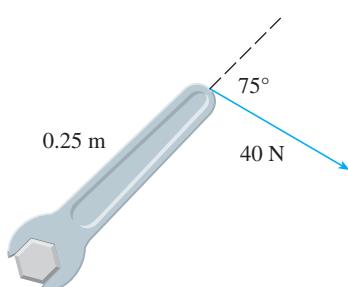


FIGURE 5

### ■ Torque

The idea of a cross product occurs often in physics. In particular, we consider a force  $\mathbf{F}$  acting on a rigid body at a point given by a position vector  $\mathbf{r}$ . (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.) The **torque**  $\tau$  (relative to the origin) is defined to be the cross product of the position and force vectors

$$\tau = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. According to Theorem 9, the magnitude of the torque vector is

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin \theta$$

where  $\theta$  is the angle between the position and force vectors. Observe that the only component of  $\mathbf{F}$  that can cause a rotation is the one perpendicular to  $\mathbf{r}$ , that is,  $|\mathbf{F}| \sin \theta$ . The magnitude of the torque is equal to the area of the parallelogram determined by  $\mathbf{r}$  and  $\mathbf{F}$ .

**EXAMPLE 6** A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5. Find the magnitude of the torque about the center of the bolt.

**SOLUTION** The magnitude of the torque vector is

$$\begin{aligned} |\tau| &= |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin 75^\circ = (0.25)(40) \sin 75^\circ \\ &= 10 \sin 75^\circ \approx 9.66 \text{ N}\cdot\text{m} \end{aligned}$$

If the bolt is right-threaded, then the torque vector itself is

$$\tau = |\tau| \mathbf{n} \approx 9.66 \mathbf{n}$$

where  $\mathbf{n}$  is a unit vector directed down into the page (by the right-hand rule). ■

## 12.4 EXERCISES

**1–7** Find the cross product  $\mathbf{a} \times \mathbf{b}$  and verify that it is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ .

1.  $\mathbf{a} = \langle 2, 3, 0 \rangle, \mathbf{b} = \langle 1, 0, 5 \rangle$
2.  $\mathbf{a} = \langle 4, 3, -2 \rangle, \mathbf{b} = \langle 2, -1, 1 \rangle$
3.  $\mathbf{a} = 2\mathbf{j} - 4\mathbf{k}, \mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
4.  $\mathbf{a} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}, \mathbf{b} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$
5.  $\mathbf{a} = \frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{4}\mathbf{k}, \mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$
6.  $\mathbf{a} = t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}, \mathbf{b} = \mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}$
7.  $\mathbf{a} = \langle t, 1, 1/t \rangle, \mathbf{b} = \langle t^2, t^2, 1 \rangle$

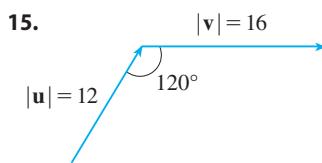
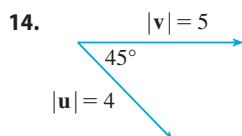
8. If  $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$  and  $\mathbf{b} = \mathbf{j} + \mathbf{k}$ , find  $\mathbf{a} \times \mathbf{b}$ . Sketch  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{a} \times \mathbf{b}$  as vectors starting at the origin.

**9–12** Find the vector, not with determinants, but by using properties of cross products.

9.  $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$
10.  $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$
11.  $(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})$
12.  $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$

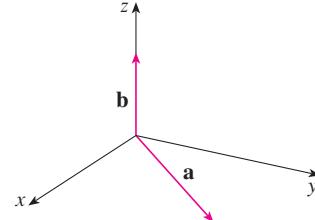
13. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.
- (a)  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
  - (b)  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
  - (c)  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
  - (d)  $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$
  - (e)  $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$
  - (f)  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

**14–15** Find  $|\mathbf{u} \times \mathbf{v}|$  and determine whether  $\mathbf{u} \times \mathbf{v}$  is directed into the page or out of the page.



16. The figure shows a vector  $\mathbf{a}$  in the  $xy$ -plane and a vector  $\mathbf{b}$  in the direction of  $\mathbf{k}$ . Their lengths are  $|\mathbf{a}| = 3$  and  $|\mathbf{b}| = 2$ .
- (a) Find  $|\mathbf{a} \times \mathbf{b}|$ .

- (b) Use the right-hand rule to decide whether the components of  $\mathbf{a} \times \mathbf{b}$  are positive, negative, or 0.



17. If  $\mathbf{a} = \langle 2, -1, 3 \rangle$  and  $\mathbf{b} = \langle 4, 2, 1 \rangle$ , find  $\mathbf{a} \times \mathbf{b}$  and  $\mathbf{b} \times \mathbf{a}$ .
18. If  $\mathbf{a} = \langle 1, 0, 1 \rangle$ ,  $\mathbf{b} = \langle 2, 1, -1 \rangle$ , and  $\mathbf{c} = \langle 0, 1, 3 \rangle$ , show that  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ .
19. Find two unit vectors orthogonal to both  $\langle 3, 2, 1 \rangle$  and  $\langle -1, 1, 0 \rangle$ .
20. Find two unit vectors orthogonal to both  $\mathbf{j} - \mathbf{k}$  and  $\mathbf{i} + \mathbf{j}$ .
21. Show that  $\mathbf{0} \times \mathbf{a} = \mathbf{0} = \mathbf{a} \times \mathbf{0}$  for any vector  $\mathbf{a}$  in  $V_3$ .
22. Show that  $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$  for all vectors  $\mathbf{a}$  and  $\mathbf{b}$  in  $V_3$ .
- 23–26 Prove the property of cross products (Theorem 11).
23. Property 1:  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
24. Property 2:  $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
25. Property 3:  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
26. Property 4:  $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
27. Find the area of the parallelogram with vertices  $A(-3, 0)$ ,  $B(-1, 3)$ ,  $C(5, 2)$ , and  $D(3, -1)$ .
28. Find the area of the parallelogram with vertices  $P(1, 0, 2)$ ,  $Q(3, 3, 3)$ ,  $R(7, 5, 8)$ , and  $S(5, 2, 7)$ .
- 29–32 (a) Find a nonzero vector orthogonal to the plane through the points  $P$ ,  $Q$ , and  $R$ , and (b) find the area of triangle  $PQR$ .
29.  $P(1, 0, 1)$ ,  $Q(-2, 1, 3)$ ,  $R(4, 2, 5)$
30.  $P(0, 0, -3)$ ,  $Q(4, 2, 0)$ ,  $R(3, 3, 1)$
31.  $P(0, -2, 0)$ ,  $Q(4, 1, -2)$ ,  $R(5, 3, 1)$

32.  $P(2, -3, 4)$ ,  $Q(-1, -2, 2)$ ,  $R(3, 1, -3)$

33–34 Find the volume of the parallelepiped determined by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ .

33.  $\mathbf{a} = \langle 1, 2, 3 \rangle$ ,  $\mathbf{b} = \langle -1, 1, 2 \rangle$ ,  $\mathbf{c} = \langle 2, 1, 4 \rangle$

34.  $\mathbf{a} = \mathbf{i} + \mathbf{j}$ ,  $\mathbf{b} = \mathbf{j} + \mathbf{k}$ ,  $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

35–36 Find the volume of the parallelepiped with adjacent edges  $PQ$ ,  $PR$ , and  $PS$ .

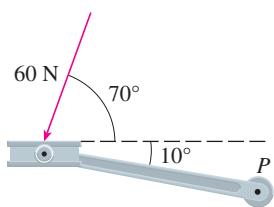
35.  $P(-2, 1, 0)$ ,  $Q(2, 3, 2)$ ,  $R(1, 4, -1)$ ,  $S(3, 6, 1)$

36.  $P(3, 0, 1)$ ,  $Q(-1, 2, 5)$ ,  $R(5, 1, -1)$ ,  $S(0, 4, 2)$

37. Use the scalar triple product to verify that the vectors  $\mathbf{u} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ ,  $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$ , and  $\mathbf{w} = 5\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$  are coplanar.

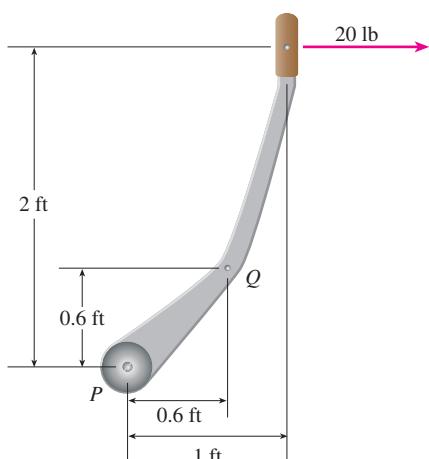
38. Use the scalar triple product to determine whether the points  $A(1, 3, 2)$ ,  $B(3, -1, 6)$ ,  $C(5, 2, 0)$ , and  $D(3, 6, -4)$  lie in the same plane.

39. A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about  $P$ .



40. (a) A horizontal force of 20 lb is applied to the handle of a gearshift lever as shown. Find the magnitude of the torque about the pivot point  $P$ .

(b) Find the magnitude of the torque about  $P$  if the same force is applied at the elbow  $Q$  of the lever.



41. A wrench 30 cm long lies along the positive  $y$ -axis and grips a bolt at the origin. A force is applied in the direction  $\langle 0, 3, -4 \rangle$  at the end of the wrench. Find the magnitude of the force needed to supply 100 N·m of torque to the bolt.

42. Let  $\mathbf{v} = 5\mathbf{j}$  and let  $\mathbf{u}$  be a vector with length 3 that starts at the origin and rotates in the  $xy$ -plane. Find the maximum and minimum values of the length of the vector  $\mathbf{u} \times \mathbf{v}$ . In what direction does  $\mathbf{u} \times \mathbf{v}$  point?

43. If  $\mathbf{a} \cdot \mathbf{b} = \sqrt{3}$  and  $\mathbf{a} \times \mathbf{b} = \langle 1, 2, 2 \rangle$ , find the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

44. (a) Find all vectors  $\mathbf{v}$  such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, -5 \rangle$$

(b) Explain why there is no vector  $\mathbf{v}$  such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, 5 \rangle$$

45. (a) Let  $P$  be a point not on the line  $L$  that passes through the points  $Q$  and  $R$ . Show that the distance  $d$  from the point  $P$  to the line  $L$  is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

where  $\mathbf{a} = \overrightarrow{QR}$  and  $\mathbf{b} = \overrightarrow{QP}$ .

(b) Use the formula in part (a) to find the distance from the point  $P(1, 1, 1)$  to the line through  $Q(0, 6, 8)$  and  $R(-1, 4, 7)$ .

46. (a) Let  $P$  be a point not on the plane that passes through the points  $Q$ ,  $R$ , and  $S$ . Show that the distance  $d$  from  $P$  to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

where  $\mathbf{a} = \overrightarrow{QR}$ ,  $\mathbf{b} = \overrightarrow{QS}$ , and  $\mathbf{c} = \overrightarrow{QP}$ .

(b) Use the formula in part (a) to find the distance from the point  $P(2, 1, 4)$  to the plane through the points  $Q(1, 0, 0)$ ,  $R(0, 2, 0)$ , and  $S(0, 0, 3)$ .

47. Show that  $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$ .

48. If  $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ , show that

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

49. Prove that  $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$ .

50. Prove Property 6 of cross products, that is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

51. Use Exercise 50 to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

52. Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

53. Suppose that  $\mathbf{a} \neq \mathbf{0}$ .

- If  $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ , does it follow that  $\mathbf{b} = \mathbf{c}$ ?
- If  $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ , does it follow that  $\mathbf{b} = \mathbf{c}$ ?
- If  $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$  and  $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ , does it follow that  $\mathbf{b} = \mathbf{c}$ ?

54. If  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  are noncoplanar vectors, let

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \quad \mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$\mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

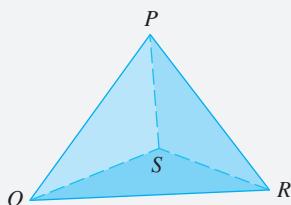
(These vectors occur in the study of crystallography. Vectors of the form  $n_1\mathbf{v}_1 + n_2\mathbf{v}_2 + n_3\mathbf{v}_3$ , where each  $n_i$  is an integer, form a *lattice* for a crystal. Vectors written similarly in terms of  $\mathbf{k}_1$ ,  $\mathbf{k}_2$ , and  $\mathbf{k}_3$  form the *reciprocal lattice*.)

- Show that  $\mathbf{k}_i$  is perpendicular to  $\mathbf{v}_j$  if  $i \neq j$ .
- Show that  $\mathbf{k}_i \cdot \mathbf{v}_i = 1$  for  $i = 1, 2, 3$ .

$$(c) \text{ Show that } \mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \frac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}.$$

## DISCOVERY PROJECT

## THE GEOMETRY OF A TETRAHEDRON



A tetrahedron is a solid with four vertices,  $P$ ,  $Q$ ,  $R$ , and  $S$ , and four triangular faces, as shown in the figure.

- Let  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\mathbf{v}_3$ , and  $\mathbf{v}_4$  be vectors with lengths equal to the areas of the faces opposite the vertices  $P$ ,  $Q$ ,  $R$ , and  $S$ , respectively, and directions perpendicular to the respective faces and pointing outward. Show that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

- The volume  $V$  of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.

- Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices  $P$ ,  $Q$ ,  $R$ , and  $S$ .
- Find the volume of the tetrahedron whose vertices are  $P(1, 1, 1)$ ,  $Q(1, 2, 3)$ ,  $R(1, 1, 2)$ , and  $S(3, -1, 2)$ .

- Suppose the tetrahedron in the figure has a trirectangular vertex  $S$ . (This means that the three angles at  $S$  are all right angles.) Let  $A$ ,  $B$ , and  $C$  be the areas of the three faces that meet at  $S$ , and let  $D$  be the area of the opposite face  $PQR$ . Using the result of Problem 1, or otherwise, show that

$$D^2 = A^2 + B^2 + C^2$$

(This is a three-dimensional version of the Pythagorean Theorem.)

## 12.5 Equations of Lines and Planes

### ■ Lines

A line in the  $xy$ -plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.

Likewise, a line  $L$  in three-dimensional space is determined when we know a point  $P_0(x_0, y_0, z_0)$  on  $L$  and the direction of  $L$ . In three dimensions the direction of a line is conveniently described by a vector, so we let  $\mathbf{v}$  be a vector parallel to  $L$ . Let  $P(x, y, z)$  be an arbitrary point on  $L$  and let  $\mathbf{r}_0$  and  $\mathbf{r}$  be the position vectors of  $P_0$  and  $P$  (that is, they have representations  $\overrightarrow{OP_0}$  and  $\overrightarrow{OP}$ ). If  $\mathbf{a}$  is the vector with representation  $\overrightarrow{P_0P}$ , as in Figure 1, then the Triangle Law for vector addition gives  $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$ . But, since  $\mathbf{a}$  and  $\mathbf{v}$  are parallel vectors, there is a scalar  $t$  such that  $\mathbf{a} = t\mathbf{v}$ . Thus

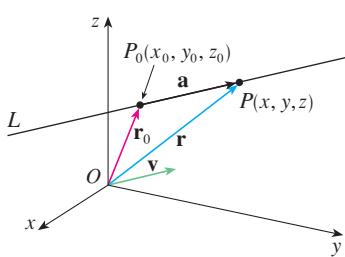


FIGURE 1

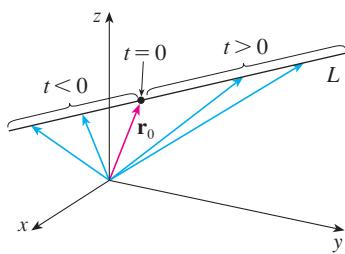


FIGURE 2

1

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

which is a **vector equation** of  $L$ . Each value of the **parameter**  $t$  gives the position vector  $\mathbf{r}$  of a point on  $L$ . In other words, as  $t$  varies, the line is traced out by the tip of the vector  $\mathbf{r}$ . As Figure 2 indicates, positive values of  $t$  correspond to points on  $L$  that lie on one side of  $P_0$ , whereas negative values of  $t$  correspond to points that lie on the other side of  $P_0$ .

If the vector  $\mathbf{v}$  that gives the direction of the line  $L$  is written in component form as  $\mathbf{v} = \langle a, b, c \rangle$ , then we have  $t\mathbf{v} = \langle ta, tb, tc \rangle$ . We can also write  $\mathbf{r} = \langle x, y, z \rangle$  and  $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ , so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

where  $t \in \mathbb{R}$ . These equations are called **parametric equations** of the line  $L$  through the point  $P_0(x_0, y_0, z_0)$  and parallel to the vector  $\mathbf{v} = \langle a, b, c \rangle$ . Each value of the parameter  $t$  gives a point  $(x, y, z)$  on  $L$ .

2 Parametric equations for a line through the point  $(x_0, y_0, z_0)$  and parallel to the direction vector  $\langle a, b, c \rangle$  are

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

Figure 3 shows the line  $L$  in Example 1 and its relation to the given point and to the vector that gives its direction.

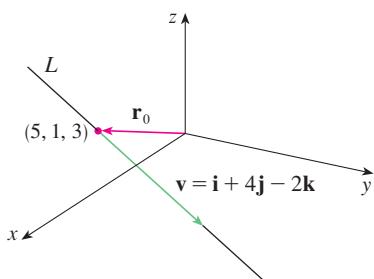


FIGURE 3

### EXAMPLE 1

- (a) Find a vector equation and parametric equations for the line that passes through the point  $(5, 1, 3)$  and is parallel to the vector  $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ .  
 (b) Find two other points on the line.

### SOLUTION

- (a) Here  $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$  and  $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ , so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or  $\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$

Parametric equations are

$$x = 5 + t \quad y = 1 + 4t \quad z = 3 - 2t$$

- (b) Choosing the parameter value  $t = 1$  gives  $x = 6$ ,  $y = 5$ , and  $z = 1$ , so  $(6, 5, 1)$  is a point on the line. Similarly,  $t = -1$  gives the point  $(4, -3, 5)$ . ■

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change. For instance, if, instead of  $(5, 1, 3)$ , we choose the point  $(6, 5, 1)$  in Example 1, then the parametric equations of the line become

$$x = 6 + t \quad y = 5 + 4t \quad z = 1 - 2t$$

Or, if we stay with the point  $(5, 1, 3)$  but choose the parallel vector  $2\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$ , we arrive at the equations

$$x = 5 + 2t \quad y = 1 + 8t \quad z = 3 - 4t$$

In general, if a vector  $\mathbf{v} = \langle a, b, c \rangle$  is used to describe the direction of a line  $L$ , then the numbers  $a$ ,  $b$ , and  $c$  are called **direction numbers** of  $L$ . Since any vector parallel to  $\mathbf{v}$  could also be used, we see that any three numbers proportional to  $a$ ,  $b$ , and  $c$  could also be used as a set of direction numbers for  $L$ .

Another way of describing a line  $L$  is to eliminate the parameter  $t$  from Equations 2. If none of  $a$ ,  $b$ , or  $c$  is 0, we can solve each of these equations for  $t$ :

$$t = \frac{x - x_0}{a} \quad t = \frac{y - y_0}{b} \quad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

3

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called **symmetric equations** of  $L$ . Notice that the numbers  $a$ ,  $b$ , and  $c$  that appear in the denominators of Equations 3 are direction numbers of  $L$ , that is, components of a vector parallel to  $L$ . If one of  $a$ ,  $b$ , or  $c$  is 0, we can still eliminate  $t$ . For instance, if  $a = 0$ , we could write the equations of  $L$  as

$$x = x_0 \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that  $L$  lies in the vertical plane  $x = x_0$ .

Figure 4 shows the line  $L$  in Example 2 and the point  $P$  where it intersects the  $xy$ -plane.

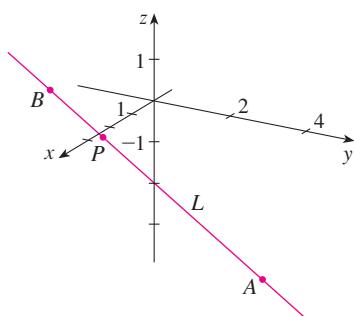


FIGURE 4

### EXAMPLE 2

- Find parametric equations and symmetric equations of the line that passes through the points  $A(2, 4, -3)$  and  $B(3, -1, 1)$ .
- At what point does this line intersect the  $xy$ -plane?

#### SOLUTION

- We are not explicitly given a vector parallel to the line, but observe that the vector  $\mathbf{v}$  with representation  $\overrightarrow{AB}$  is parallel to the line and

$$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3) \rangle = \langle 1, -5, 4 \rangle$$

Thus direction numbers are  $a = 1$ ,  $b = -5$ , and  $c = 4$ . Taking the point  $(2, 4, -3)$  as  $P_0$ , we see that parametric equations (2) are

$$x = 2 + t \quad y = 4 - 5t \quad z = -3 + 4t$$

and symmetric equations (3) are

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$

- The line intersects the  $xy$ -plane when  $z = 0$ , so we put  $z = 0$  in the symmetric equations and obtain

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{3}{4}$$

This gives  $x = \frac{11}{4}$  and  $y = \frac{1}{4}$ , so the line intersects the  $xy$ -plane at the point  $(\frac{11}{4}, \frac{1}{4}, 0)$ . ■

In general, the procedure of Example 2 shows that direction numbers of the line  $L$  through the points  $P_0(x_0, y_0, z_0)$  and  $P_1(x_1, y_1, z_1)$  are  $x_1 - x_0$ ,  $y_1 - y_0$ , and  $z_1 - z_0$  and so symmetric equations of  $L$  are

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Often, we need a description, not of an entire line, but of just a line segment. How, for instance, could we describe the line segment  $AB$  in Example 2? If we put  $t = 0$  in the parametric equations in Example 2(a), we get the point  $(2, 4, -3)$  and if we put  $t = 1$  we get  $(3, -1, 1)$ . So the line segment  $AB$  is described by the parametric equations

$$x = 2 + t \quad y = 4 - 5t \quad z = -3 + 4t \quad 0 \leq t \leq 1$$

or by the corresponding vector equation

$$\mathbf{r}(t) = \langle 2 + t, 4 - 5t, -3 + 4t \rangle \quad 0 \leq t \leq 1$$

In general, we know from Equation 1 that the vector equation of a line through the (tip of the) vector  $\mathbf{r}_0$  in the direction of a vector  $\mathbf{v}$  is  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ . If the line also passes through (the tip of)  $\mathbf{r}_1$ , then we can take  $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$  and so its vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

The line segment from  $\mathbf{r}_0$  to  $\mathbf{r}_1$  is given by the parameter interval  $0 \leq t \leq 1$ .

**4** The line segment from  $\mathbf{r}_0$  to  $\mathbf{r}_1$  is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

The lines  $L_1$  and  $L_2$  in Example 3, shown in Figure 5, are skew lines.

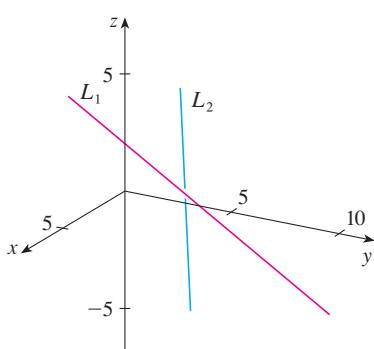


FIGURE 5

**EXAMPLE 3** Show that the lines  $L_1$  and  $L_2$  with parametric equations

$$L_1: \quad x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$$

$$L_2: \quad x = 2s \quad y = 3 + s \quad z = -3 + 4s$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

**SOLUTION** The lines are not parallel because the corresponding direction vectors  $\langle 1, 3, -1 \rangle$  and  $\langle 2, 1, 4 \rangle$  are not parallel. (Their components are not proportional.) If  $L_1$  and  $L_2$  had a point of intersection, there would be values of  $t$  and  $s$  such that

$$1 + t = 2s$$

$$-2 + 3t = 3 + s$$

$$4 - t = -3 + 4s$$

But if we solve the first two equations, we get  $t = \frac{11}{5}$  and  $s = \frac{8}{5}$ , and these values don't satisfy the third equation. Therefore there are no values of  $t$  and  $s$  that satisfy the three equations, so  $L_1$  and  $L_2$  do not intersect. Thus  $L_1$  and  $L_2$  are skew lines. ■

## ■ Planes

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe. A single vector parallel to a plane is not enough to convey the

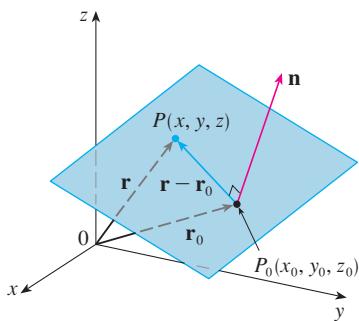


FIGURE 6

“direction” of the plane, but a vector perpendicular to the plane does completely specify its direction. Thus a plane in space is determined by a point  $P_0(x_0, y_0, z_0)$  in the plane and a vector  $\mathbf{n}$  that is orthogonal to the plane. This orthogonal vector  $\mathbf{n}$  is called a **normal vector**. Let  $P(x, y, z)$  be an arbitrary point in the plane, and let  $\mathbf{r}_0$  and  $\mathbf{r}$  be the position vectors of  $P_0$  and  $P$ . Then the vector  $\mathbf{r} - \mathbf{r}_0$  is represented by  $\overrightarrow{P_0P}$ . (See Figure 6.) The normal vector  $\mathbf{n}$  is orthogonal to every vector in the given plane. In particular,  $\mathbf{n}$  is orthogonal to  $\mathbf{r} - \mathbf{r}_0$  and so we have

5

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can be rewritten as

6

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

To obtain a scalar equation for the plane, we write  $\mathbf{n} = \langle a, b, c \rangle$ ,  $\mathbf{r} = \langle x, y, z \rangle$ , and  $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ . Then the vector equation (5) becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

7 A **scalar equation of the plane** through point  $P_0(x_0, y_0, z_0)$  with normal vector  $\mathbf{n} = \langle a, b, c \rangle$  is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

**EXAMPLE 4** Find an equation of the plane through the point  $(2, 4, -1)$  with normal vector  $\mathbf{n} = \langle 2, 3, 4 \rangle$ . Find the intercepts and sketch the plane.

**SOLUTION** Putting  $a = 2$ ,  $b = 3$ ,  $c = 4$ ,  $x_0 = 2$ ,  $y_0 = 4$ , and  $z_0 = -1$  in Equation 7, we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$

To find the  $x$ -intercept we set  $y = z = 0$  in this equation and obtain  $x = 6$ . Similarly, the  $y$ -intercept is 4 and the  $z$ -intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7). ■

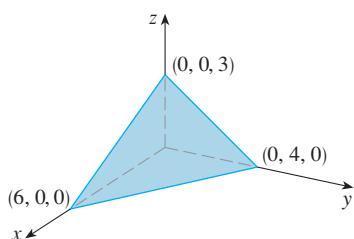


FIGURE 7

By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

8

$$ax + by + cz + d = 0$$

where  $d = -(ax_0 + by_0 + cz_0)$ . Equation 8 is called a **linear equation** in  $x$ ,  $y$ , and  $z$ . Conversely, it can be shown that if  $a$ ,  $b$ , and  $c$  are not all 0, then the linear equation (8) represents a plane with normal vector  $\langle a, b, c \rangle$ . (See Exercise 83.)

Figure 8 shows the portion of the plane in Example 5 that is enclosed by triangle  $PQR$ .

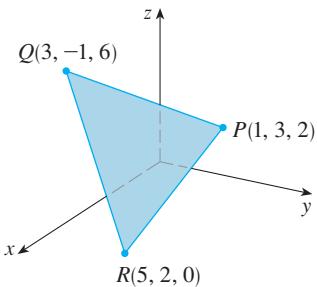


FIGURE 8

**EXAMPLE 5** Find an equation of the plane that passes through the points  $P(1, 3, 2)$ ,  $Q(3, -1, 6)$ , and  $R(5, 2, 0)$ .

**SOLUTION** The vectors  $\mathbf{a}$  and  $\mathbf{b}$  corresponding to  $\vec{PQ}$  and  $\vec{PR}$  are

$$\mathbf{a} = \langle 2, -4, 4 \rangle \quad \mathbf{b} = \langle 4, -1, -2 \rangle$$

Since both  $\mathbf{a}$  and  $\mathbf{b}$  lie in the plane, their cross product  $\mathbf{a} \times \mathbf{b}$  is orthogonal to the plane and can be taken as the normal vector. Thus

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}$$

With the point  $P(1, 3, 2)$  and the normal vector  $\mathbf{n}$ , an equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

**EXAMPLE 6** Find the point at which the line with parametric equations  $x = 2 + 3t$ ,  $y = -4t$ ,  $z = 5 + t$  intersects the plane  $4x + 5y - 2z = 18$ .

**SOLUTION** We substitute the expressions for  $x$ ,  $y$ , and  $z$  from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

This simplifies to  $-10t = 20$ , so  $t = -2$ . Therefore the point of intersection occurs when the parameter value is  $t = -2$ . Then  $x = 2 + 3(-2) = -4$ ,  $y = -4(-2) = 8$ ,  $z = 5 - 2 = 3$  and so the point of intersection is  $(-4, 8, 3)$ .

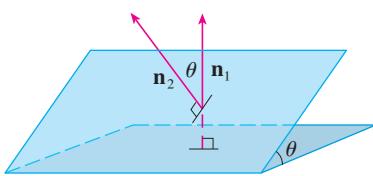


FIGURE 9

Two planes are **parallel** if their normal vectors are parallel. For instance, the planes  $x + 2y - 3z = 4$  and  $2x + 4y - 6z = 3$  are parallel because their normal vectors are  $\mathbf{n}_1 = \langle 1, 2, -3 \rangle$  and  $\mathbf{n}_2 = \langle 2, 4, -6 \rangle$  and  $\mathbf{n}_2 = 2\mathbf{n}_1$ . If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see angle  $\theta$  in Figure 9).

**EXAMPLE 7**

- Find the angle between the planes  $x + y + z = 1$  and  $x - 2y + 3z = 1$ .
- Find symmetric equations for the line of intersection  $L$  of these two planes.

**SOLUTION**

- The normal vectors of these planes are

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle \quad \mathbf{n}_2 = \langle 1, -2, 3 \rangle$$

and so, if  $\theta$  is the angle between the planes, Corollary 12.3.6 gives

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1} \sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

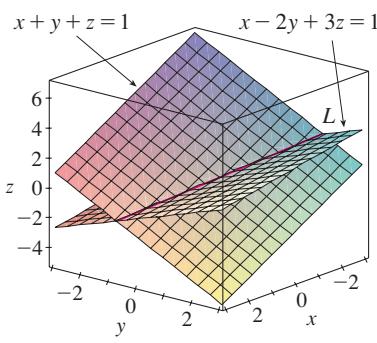


FIGURE 10

- We first need to find a point on  $L$ . For instance, we can find the point where the line intersects the  $xy$ -plane by setting  $z = 0$  in the equations of both planes. This gives the

equations  $x + y = 1$  and  $x - 2y = 1$ , whose solution is  $x = 1, y = 0$ . So the point  $(1, 0, 0)$  lies on  $L$ .

Now we observe that, since  $L$  lies in both planes, it is perpendicular to both of the normal vectors. Thus a vector  $\mathbf{v}$  parallel to  $L$  is given by the cross product

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

and so the symmetric equations of  $L$  can be written as

$$\frac{x - 1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

■

**NOTE** Since a linear equation in  $x, y$ , and  $z$  represents a plane and two nonparallel planes intersect in a line, it follows that two linear equations can represent a line. The points  $(x, y, z)$  that satisfy both  $a_1x + b_1y + c_1z + d_1 = 0$  and  $a_2x + b_2y + c_2z + d_2 = 0$  lie on both of these planes, and so the pair of linear equations represents the line of intersection of the planes (if they are not parallel). For instance, in Example 7 the line  $L$  was given as the line of intersection of the planes  $x + y + z = 1$  and  $x - 2y + 3z = 1$ . The symmetric equations that we found for  $L$  could be written as

$$\frac{x - 1}{5} = \frac{y}{-2} \quad \text{and} \quad \frac{y}{-2} = \frac{z}{-3}$$

which is again a pair of linear equations. They exhibit  $L$  as the line of intersection of the planes  $(x - 1)/5 = y/(-2)$  and  $y/(-2) = z/(-3)$ . (See Figure 11.)

In general, when we write the equations of a line in the symmetric form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

we can regard the line as the line of intersection of the two planes

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \text{and} \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

### ■ Distances

**EXAMPLE 8** Find a formula for the distance  $D$  from a point  $P_1(x_1, y_1, z_1)$  to the plane  $ax + by + cz + d = 0$ .

**SOLUTION** Let  $P_0(x_0, y_0, z_0)$  be any point in the given plane and let  $\mathbf{b}$  be the vector corresponding to  $\overrightarrow{P_0P_1}$ . Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

From Figure 12 you can see that the distance  $D$  from  $P_1$  to the plane is equal to the absolute value of the scalar projection of  $\mathbf{b}$  onto the normal vector  $\mathbf{n} = \langle a, b, c \rangle$ . (See Section 12.3.) Thus

$$\begin{aligned} D &= |\text{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

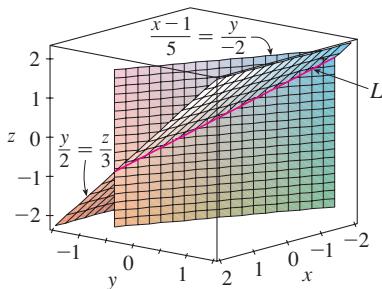


FIGURE 11

Figure 11 shows how the line  $L$  in Example 7 can also be regarded as the line of intersection of planes derived from its symmetric equations.

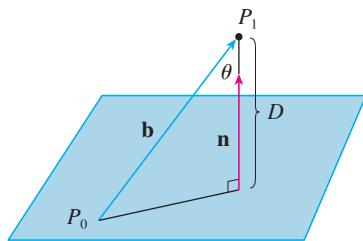


FIGURE 12

Since  $P_0$  lies in the plane, its coordinates satisfy the equation of the plane and so we have  $ax_0 + by_0 + cz_0 + d = 0$ . Thus the formula for  $D$  can be written as

9

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

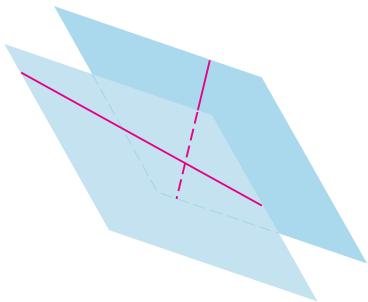
■

**EXAMPLE 9** Find the distance between the parallel planes  $10x + 2y - 2z = 5$  and  $5x + y - z = 1$ .

**SOLUTION** First we note that the planes are parallel because their normal vectors  $\langle 10, 2, -2 \rangle$  and  $\langle 5, 1, -1 \rangle$  are parallel. To find the distance  $D$  between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular, if we put  $y = z = 0$  in the equation of the first plane, we get  $10x = 5$  and so  $(\frac{1}{2}, 0, 0)$  is a point in this plane. By Formula 9, the distance between  $(\frac{1}{2}, 0, 0)$  and the plane  $5x + y - z - 1 = 0$  is

$$D = \frac{|5(\frac{1}{2}) + 1(0) - 1(0) - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

So the distance between the planes is  $\sqrt{3}/6$ . ■



**FIGURE 13**

Skew lines, like those in Example 10, always lie on (nonidentical) parallel planes.

**EXAMPLE 10** In Example 3 we showed that the lines

$$L_1: \quad x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$$

$$L_2: \quad x = 2s \quad y = 3 + s \quad z = -3 + 4s$$

are skew. Find the distance between them.

**SOLUTION** Since the two lines  $L_1$  and  $L_2$  are skew, they can be viewed as lying on two parallel planes  $P_1$  and  $P_2$ . The distance between  $L_1$  and  $L_2$  is the same as the distance between  $P_1$  and  $P_2$ , which can be computed as in Example 9. The common normal vector to both planes must be orthogonal to both  $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$  (the direction of  $L_1$ ) and  $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$  (the direction of  $L_2$ ). So a normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

If we put  $s = 0$  in the equations of  $L_2$ , we get the point  $(0, 3, -3)$  on  $L_2$  and so an equation for  $P_2$  is

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0 \quad \text{or} \quad 13x - 6y - 5z + 3 = 0$$

If we now set  $t = 0$  in the equations for  $L_1$ , we get the point  $(1, -2, 4)$  on  $P_1$ . So the distance between  $L_1$  and  $L_2$  is the same as the distance from  $(1, -2, 4)$  to  $13x - 6y - 5z + 3 = 0$ . By Formula 9, this distance is

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$

■

## 12.5 EXERCISES

1. Determine whether each statement is true or false in  $\mathbb{R}^3$ .
- Two lines parallel to a third line are parallel.
  - Two lines perpendicular to a third line are parallel.
  - Two planes parallel to a third plane are parallel.
  - Two planes perpendicular to a third plane are parallel.
  - Two lines parallel to a plane are parallel.
  - Two lines perpendicular to a plane are parallel.
  - Two planes parallel to a line are parallel.
  - Two planes perpendicular to a line are parallel.
  - Two planes either intersect or are parallel.
  - Two lines either intersect or are parallel.
  - A plane and a line either intersect or are parallel.

2-5 Find a vector equation and parametric equations for the line.

- The line through the point  $(6, -5, 2)$  and parallel to the vector  $\langle 1, 3, -\frac{2}{3} \rangle$
- The line through the point  $(2, 2.4, 3.5)$  and parallel to the vector  $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- The line through the point  $(0, 14, -10)$  and parallel to the line  $x = -1 + 2t, y = 6 - 3t, z = 3 + 9t$
- The line through the point  $(1, 0, 6)$  and perpendicular to the plane  $x + 3y + z = 5$

6-12 Find parametric equations and symmetric equations for the line.

- The line through the origin and the point  $(4, 3, -1)$
- The line through the points  $(0, \frac{1}{2}, 1)$  and  $(2, 1, -3)$
- The line through the points  $(1, 2.4, 4.6)$  and  $(2.6, 1.2, 0.3)$
- The line through the points  $(-8, 1, 4)$  and  $(3, -2, 4)$
- The line through  $(2, 1, 0)$  and perpendicular to both  $\mathbf{i} + \mathbf{j}$  and  $\mathbf{j} + \mathbf{k}$
- The line through  $(-6, 2, 3)$  and parallel to the line  $\frac{1}{2}x = \frac{1}{3}y = z + 1$
- The line of intersection of the planes  $x + 2y + 3z = 1$  and  $x - y + z = 1$

- Is the line through  $(-4, -6, 1)$  and  $(-2, 0, -3)$  parallel to the line through  $(10, 18, 4)$  and  $(5, 3, 14)$ ?
- Is the line through  $(-2, 4, 0)$  and  $(1, 1, 1)$  perpendicular to the line through  $(2, 3, 4)$  and  $(3, -1, -8)$ ?
- (a) Find symmetric equations for the line that passes through the point  $(1, -5, 6)$  and is parallel to the vector  $\langle -1, 2, -3 \rangle$ .  
(b) Find the points in which the required line in part (a) intersects the coordinate planes.
- (a) Find parametric equations for the line through  $(2, 4, 6)$  that is perpendicular to the plane  $x - y + 3z = 7$ .

- In what points does this line intersect the coordinate planes?

- Find a vector equation for the line segment from  $(6, -1, 9)$  to  $(7, 6, 0)$ .
- Find parametric equations for the line segment from  $(-2, 18, 31)$  to  $(11, -4, 48)$ .

19-22 Determine whether the lines  $L_1$  and  $L_2$  are parallel, skew, or intersecting. If they intersect, find the point of intersection.

19.  $L_1: x = 3 + 2t, y = 4 - t, z = 1 + 3t$   
 $L_2: x = 1 + 4s, y = 3 - 2s, z = 4 + 5s$

20.  $L_1: x = 5 - 12t, y = 3 + 9t, z = 1 - 3t$   
 $L_2: x = 3 + 8s, y = -6s, z = 7 + 2s$

21.  $L_1: \frac{x - 2}{1} = \frac{y - 3}{-2} = \frac{z - 1}{-3}$   
 $L_2: \frac{x - 3}{1} = \frac{y + 4}{3} = \frac{z - 2}{-7}$

22.  $L_1: \frac{x}{1} = \frac{y - 1}{-1} = \frac{z - 2}{3}$   
 $L_2: \frac{x - 2}{2} = \frac{y - 3}{-2} = \frac{z}{7}$

23-40 Find an equation of the plane.

- The plane through the origin and perpendicular to the vector  $\langle 1, -2, 5 \rangle$
- The plane through the point  $(5, 3, 5)$  and with normal vector  $2\mathbf{i} + \mathbf{j} - \mathbf{k}$
- The plane through the point  $(-1, \frac{1}{2}, 3)$  and with normal vector  $\mathbf{i} + 4\mathbf{j} + \mathbf{k}$
- The plane through the point  $(2, 0, 1)$  and perpendicular to the line  $x = 3t, y = 2 - t, z = 3 + 4t$
- The plane through the point  $(1, -1, -1)$  and parallel to the plane  $5x - y - z = 6$
- The plane through the point  $(3, -2, 8)$  and parallel to the plane  $z = x + y$
- The plane through the point  $(1, \frac{1}{2}, \frac{1}{3})$  and parallel to the plane  $x + y + z = 0$
- The plane that contains the line  $x = 1 + t, y = 2 - t, z = 4 - 3t$  and is parallel to the plane  $5x + 2y + z = 1$
- The plane through the points  $(0, 1, 1), (1, 0, 1)$ , and  $(1, 1, 0)$
- The plane through the origin and the points  $(3, -2, 1)$  and  $(1, 1, 1)$
- The plane through the points  $(2, 1, 2), (3, -8, 6)$ , and  $(-2, -3, 1)$

- 34.** The plane through the points  $(3, 0, -1)$ ,  $(-2, -2, 3)$ , and  $(7, 1, -4)$
- 35.** The plane that passes through the point  $(3, 5, -1)$  and contains the line  $x = 4 - t$ ,  $y = 2t - 1$ ,  $z = -3t$
- 36.** The plane that passes through the point  $(6, -1, 3)$  and contains the line with symmetric equations  
 $x/3 = y + 4 = z/2$
- 37.** The plane that passes through the point  $(3, 1, 4)$  and contains the line of intersection of the planes  $x + 2y + 3z = 1$  and  $2x - y + z = -3$
- 38.** The plane that passes through the points  $(0, -2, 5)$  and  $(-1, 3, 1)$  and is perpendicular to the plane  $2z = 5x + 4y$
- 39.** The plane that passes through the point  $(1, 5, 1)$  and is perpendicular to the planes  $2x + y - 2z = 2$  and  $x + 3z = 4$
- 40.** The plane that passes through the line of intersection of the planes  $x - z = 1$  and  $y + 2z = 3$  and is perpendicular to the plane  $x + y - 2z = 1$

**41–44** Use intercepts to help sketch the plane.

**41.**  $2x + 5y + z = 10$       **42.**  $3x + y + 2z = 6$   
**43.**  $6x - 3y + 4z = 6$       **44.**  $6x + 5y - 3z = 15$

**45–47** Find the point at which the line intersects the given plane.

**45.**  $x = 2 - 2t$ ,  $y = 3t$ ,  $z = 1 + t$ ;  $x + 2y - z = 7$   
**46.**  $x = t - 1$ ,  $y = 1 + 2t$ ,  $z = 3 - t$ ;  $3x - y + 2z = 5$   
**47.**  $5x = y/2 = z + 2$ ;  $10x - 7y + 3z + 24 = 0$

**48.** Where does the line through  $(-3, 1, 0)$  and  $(-1, 5, 6)$  intersect the plane  $2x + y - z = -2$ ?

- 49.** Find direction numbers for the line of intersection of the planes  $x + y + z = 1$  and  $x + z = 0$ .
- 50.** Find the cosine of the angle between the planes  $x + y + z = 0$  and  $x + 2y + 3z = 1$ .

**51–56** Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them. (Round to one decimal place.)

**51.**  $x + 4y - 3z = 1$ ,  $-3x + 6y + 7z = 0$   
**52.**  $9x - 3y + 6z = 2$ ,  $2y = 6x + 4z$   
**53.**  $x + 2y - z = 2$ ,  $2x - 2y + z = 1$   
**54.**  $x - y + 3z = 1$ ,  $3x + y - z = 2$   
**55.**  $2x - 3y = z$ ,  $4x = 3 + 6y + 2z$   
**56.**  $5x + 2y + 3z = 2$ ,  $y = 4x - 6z$

**57–58** (a) Find parametric equations for the line of intersection of the planes and (b) find the angle between the planes.

**57.**  $x + y + z = 1$ ,  $x + 2y + 2z = 1$   
**58.**  $3x - 2y + z = 1$ ,  $2x + y - 3z = 3$

**59–60** Find symmetric equations for the line of intersection of the planes.

**59.**  $5x - 2y - 2z = 1$ ,  $4x + y + z = 6$   
**60.**  $z = 2x - y - 5$ ,  $z = 4x + 3y - 5$

**61.** Find an equation for the plane consisting of all points that are equidistant from the points  $(1, 0, -2)$  and  $(3, 4, 0)$ .

**62.** Find an equation for the plane consisting of all points that are equidistant from the points  $(2, 5, 5)$  and  $(-6, 3, 1)$ .

**63.** Find an equation of the plane with  $x$ -intercept  $a$ ,  $y$ -intercept  $b$ , and  $z$ -intercept  $c$ .

**64.** (a) Find the point at which the given lines intersect:

$$\mathbf{r} = \langle 1, 1, 0 \rangle + t\langle 1, -1, 2 \rangle$$

$$\mathbf{r} = \langle 2, 0, 2 \rangle + s\langle -1, 1, 0 \rangle$$

(b) Find an equation of the plane that contains these lines.

**65.** Find parametric equations for the line through the point  $(0, 1, 2)$  that is parallel to the plane  $x + y + z = 2$  and perpendicular to the line  $x = 1 + t$ ,  $y = 1 - t$ ,  $z = 2t$ .

**66.** Find parametric equations for the line through the point  $(0, 1, 2)$  that is perpendicular to the line  $x = 1 + t$ ,  $y = 1 - t$ ,  $z = 2t$  and intersects this line.

**67.** Which of the following four planes are parallel? Are any of them identical?

$$P_1: 3x + 6y - 3z = 6 \quad P_2: 4x - 12y + 8z = 5$$

$$P_3: 9y = 1 + 3x + 6z \quad P_4: z = x + 2y - 2$$

**68.** Which of the following four lines are parallel? Are any of them identical?

$$L_1: x = 1 + 6t, \quad y = 1 - 3t, \quad z = 12t + 5$$

$$L_2: x = 1 + 2t, \quad y = t, \quad z = 1 + 4t$$

$$L_3: 2x - 2 = 4 - 4y = z + 1$$

$$L_4: \mathbf{r} = \langle 3, 1, 5 \rangle + t\langle 4, 2, 8 \rangle$$

**69–70** Use the formula in Exercise 12.4.45 to find the distance from the point to the given line.

**69.**  $(4, 1, -2)$ ;  $x = 1 + t$ ,  $y = 3 - 2t$ ,  $z = 4 - 3t$   
**70.**  $(0, 1, 3)$ ;  $x = 2t$ ,  $y = 6 - 2t$ ,  $z = 3 + t$

**71–72** Find the distance from the point to the given plane.

**71.**  $(1, -2, 4)$ ,  $3x + 2y + 6z = 5$

**72.**  $(-6, 3, 5)$ ,  $x - 2y - 4z = 8$

**73–74** Find the distance between the given parallel planes.

**73.**  $2x - 3y + z = 4$ ,  $4x - 6y + 2z = 3$

**74.**  $6z = 4y - 2x$ ,  $9z = 1 - 3x + 6y$

**75.** Show that the distance between the parallel planes

$ax + by + cz + d_1 = 0$  and  $ax + by + cz + d_2 = 0$  is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

**76.** Find equations of the planes that are parallel to the plane  $x + 2y - 2z = 1$  and two units away from it.

**77.** Show that the lines with symmetric equations  $x = y = z$  and  $x + 1 = y/2 = z/3$  are skew, and find the distance between these lines.

**78.** Find the distance between the skew lines with parametric equations  $x = 1 + t$ ,  $y = 1 + 6t$ ,  $z = 2t$ , and  $x = 1 + 2s$ ,  $y = 5 + 15s$ ,  $z = -2 + 6s$ .

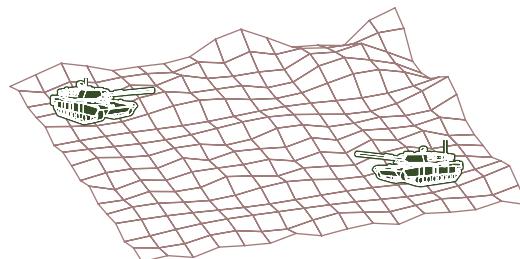
**79.** Let  $L_1$  be the line through the origin and the point  $(2, 0, -1)$ . Let  $L_2$  be the line through the points  $(1, -1, 1)$  and  $(4, 1, 3)$ . Find the distance between  $L_1$  and  $L_2$ .

**80.** Let  $L_1$  be the line through the points  $(1, 2, 6)$  and  $(2, 4, 8)$ . Let  $L_2$  be the line of intersection of the planes  $P_1$  and  $P_2$ , where  $P_1$  is the plane  $x - y + 2z + 1 = 0$  and  $P_2$  is the plane

through the points  $(3, 2, -1)$ ,  $(0, 0, 1)$ , and  $(1, 2, 1)$ . Calculate the distance between  $L_1$  and  $L_2$ .

**81.** Two tanks are participating in a battle simulation. Tank A is at point  $(325, 810, 561)$  and tank B is positioned at point  $(765, 675, 599)$ .

- Find parametric equations for the line of sight between the tanks.
- If we divide the line of sight into 5 equal segments, the elevations of the terrain at the four intermediate points from tank A to tank B are 549, 566, 586, and 589. Can the tanks see each other?



**82.** Give a geometric description of each family of planes.

- $x + y + z = c$
- $x + y + cz = 1$
- $y \cos \theta + z \sin \theta = 1$

**83.** If  $a$ ,  $b$ , and  $c$  are not all 0, show that the equation  $ax + by + cz + d = 0$  represents a plane and  $\langle a, b, c \rangle$  is a normal vector to the plane.

*Hint:* Suppose  $a \neq 0$  and rewrite the equation in the form

$$a\left(x + \frac{d}{a}\right) + b(y - 0) + c(z - 0) = 0$$

## LABORATORY PROJECT PUTTING 3D IN PERSPECTIVE



Computer graphics programmers face the same challenge as the great painters of the past: how to represent a three-dimensional scene as a flat image on a two-dimensional plane (a screen or a canvas). To create the illusion of perspective, in which closer objects appear larger than those farther away, three-dimensional objects in the computer's memory are projected onto a rectangular screen window from a viewpoint where the eye, or camera, is located. The viewing volume—the portion of space that will be visible—is the region contained by the four planes that pass through the viewpoint and an edge of the screen window. If objects in the scene extend beyond these four planes, they must be truncated before pixel data are sent to the screen. These planes are therefore called *clipping planes*.

- Suppose the screen is represented by a rectangle in the  $yz$ -plane with vertices  $(0, \pm 400, 0)$  and  $(0, \pm 400, 600)$ , and the camera is placed at  $(1000, 0, 0)$ . A line  $L$  in the scene passes through the points  $(230, -285, 102)$  and  $(860, 105, 264)$ . At what points should  $L$  be clipped by the clipping planes?
- If the clipped line segment is projected onto the screen window, identify the resulting line segment.

3. Use parametric equations to plot the edges of the screen window, the clipped line segment, and its projection onto the screen window. Then add sight lines connecting the viewpoint to each end of the clipped segments to verify that the projection is correct.
4. A rectangle with vertices  $(621, -147, 206)$ ,  $(563, 31, 242)$ ,  $(657, -111, 86)$ , and  $(599, 67, 122)$  is added to the scene. The line  $L$  intersects this rectangle. To make the rectangle appear opaque, a programmer can use *hidden line rendering*, which removes portions of objects that are behind other objects. Identify the portion of  $L$  that should be removed.

## 12.6 Cylinders and Quadric Surfaces

We have already looked at two special types of surfaces: planes (in Section 12.5) and spheres (in Section 12.1). Here we investigate two other types of surfaces: cylinders and quadric surfaces.

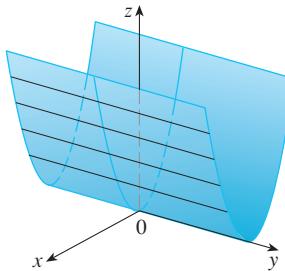
In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.

### ■ Cylinders

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

**EXAMPLE 1** Sketch the graph of the surface  $z = x^2$ .

**SOLUTION** Notice that the equation of the graph,  $z = x^2$ , doesn't involve  $y$ . This means that any vertical plane with equation  $y = k$  (parallel to the  $xz$ -plane) intersects the graph in a curve with equation  $z = x^2$ . So these vertical traces are parabolas. Figure 1 shows how the graph is formed by taking the parabola  $z = x^2$  in the  $xz$ -plane and moving it in the direction of the  $y$ -axis. The graph is a surface, called a **parabolic cylinder**, made up of infinitely many shifted copies of the same parabola. Here the rulings of the cylinder are parallel to the  $y$ -axis.



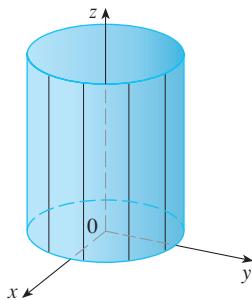
**FIGURE 1**  
The surface  $z = x^2$  is a parabolic cylinder.

We noticed that the variable  $y$  is missing from the equation of the cylinder in Example 1. This is typical of a surface whose rulings are parallel to one of the coordinate axes. If one of the variables  $x$ ,  $y$ , or  $z$  is missing from the equation of a surface, then the surface is a cylinder.

**EXAMPLE 2** Identify and sketch the surfaces.

(a)  $x^2 + y^2 = 1$

(b)  $y^2 + z^2 = 1$



**FIGURE 2**  
 $x^2 + y^2 = 1$

### SOLUTION

(a) Since  $z$  is missing and the equations  $x^2 + y^2 = 1, z = k$  represent a circle with radius 1 in the plane  $z = k$ , the surface  $x^2 + y^2 = 1$  is a circular cylinder whose axis is the  $z$ -axis. (See Figure 2.) Here the rulings are vertical lines.

(b) In this case  $x$  is missing and the surface is a circular cylinder whose axis is the  $x$ -axis. (See Figure 3.) It is obtained by taking the circle  $y^2 + z^2 = 1, x = 0$  in the  $yz$ -plane and moving it parallel to the  $x$ -axis. ■

**NOTE** When you are dealing with surfaces, it is important to recognize that an equation like  $x^2 + y^2 = 1$  represents a cylinder and not a circle. The trace of the cylinder  $x^2 + y^2 = 1$  in the  $xy$ -plane is the circle with equations  $x^2 + y^2 = 1, z = 0$ .

### ■ Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in three variables  $x, y$ , and  $z$ . The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where  $A, B, C, \dots, J$  are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane. (See Section 10.5 for a review of conic sections.)

**EXAMPLE 3** Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

**SOLUTION** By substituting  $z = 0$ , we find that the trace in the  $xy$ -plane is  $x^2 + y^2/9 = 1$ , which we recognize as an equation of an ellipse. In general, the horizontal trace in the plane  $z = k$  is

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \quad z = k$$

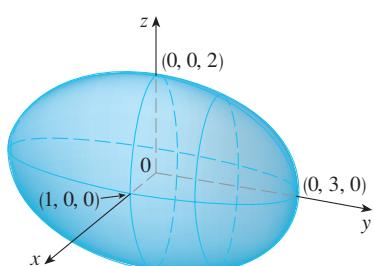
which is an ellipse, provided that  $k^2 < 4$ , that is,  $-2 < k < 2$ .

Similarly, vertical traces parallel to the  $yz$ - and  $xz$ -planes are also ellipses:

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \quad x = k \quad (\text{if } -1 < k < 1)$$

$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \quad y = k \quad (\text{if } -3 < k < 3)$$

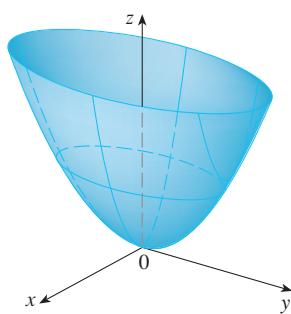
Figure 4 shows how drawing some traces indicates the shape of the surface. It's called an **ellipsoid** because all of its traces are ellipses. Notice that it is symmetric with respect to each coordinate plane; this is a reflection of the fact that its equation involves only even powers of  $x, y$ , and  $z$ . ■



**FIGURE 4**  
The ellipsoid  $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

**EXAMPLE 4** Use traces to sketch the surface  $z = 4x^2 + y^2$ .

**SOLUTION** If we put  $x = 0$ , we get  $z = y^2$ , so the  $yz$ -plane intersects the surface in a parabola. If we put  $x = k$  (a constant), we get  $z = y^2 + 4k^2$ . This means that if we

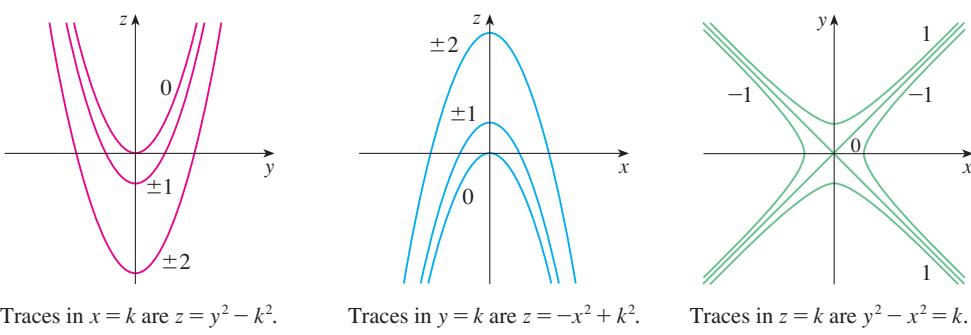
**FIGURE 5**

The surface  $z = 4x^2 + y^2$  is an elliptic paraboloid. Horizontal traces are ellipses; vertical traces are parabolas.

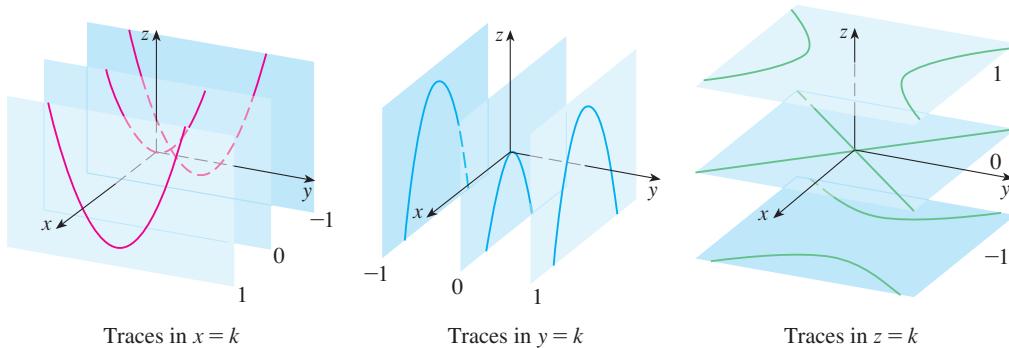
slice the graph with any plane parallel to the  $yz$ -plane, we obtain a parabola that opens upward. Similarly, if  $y = k$ , the trace is  $z = 4x^2 + k^2$ , which is again a parabola that opens upward. If we put  $z = k$ , we get the horizontal traces  $4x^2 + y^2 = k$ , which we recognize as a family of ellipses. Knowing the shapes of the traces, we can sketch the graph in Figure 5. Because of the elliptical and parabolic traces, the quadric surface  $z = 4x^2 + y^2$  is called an **elliptic paraboloid**. ■

**EXAMPLE 5** Sketch the surface  $z = y^2 - x^2$ .

**SOLUTION** The traces in the vertical planes  $x = k$  are the parabolas  $z = y^2 - k^2$ , which open upward. The traces in  $y = k$  are the parabolas  $z = -x^2 + k^2$ , which open downward. The horizontal traces are  $y^2 - x^2 = k$ , a family of hyperbolas. We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.

**FIGURE 6**

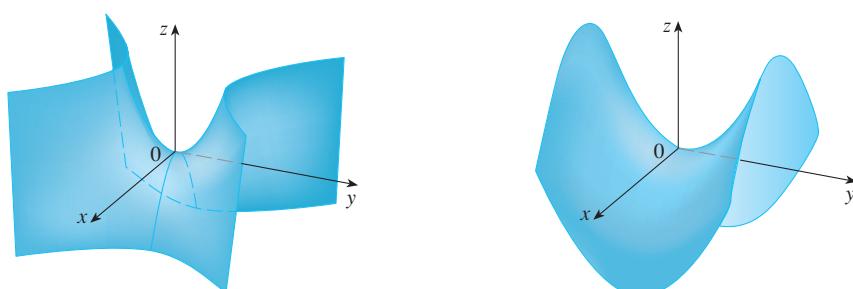
Vertical traces are parabolas; horizontal traces are hyperbolas. All traces are labeled with the value of  $k$ .

**FIGURE 7**

Traces moved to their correct planes

**TEC** In Module 12.6A you can investigate how traces determine the shape of a surface.

In Figure 8 we fit together the traces from Figure 7 to form the surface  $z = y^2 - x^2$ , a **hyperbolic paraboloid**. Notice that the shape of the surface near the origin resembles that of a saddle. This surface will be investigated further in Section 14.7 when we discuss saddle points.

**FIGURE 8**

Two views of the surface  $z = y^2 - x^2$ , a hyperbolic paraboloid

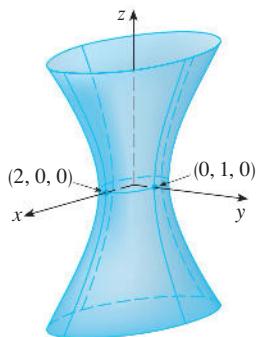


FIGURE 9

**EXAMPLE 6** Sketch the surface  $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$ .

**SOLUTION** The trace in any horizontal plane  $z = k$  is the ellipse

$$\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4} \quad z = k$$

but the traces in the  $xz$ - and  $yz$ -planes are the hyperbolas

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \quad y = 0 \quad \text{and} \quad y^2 - \frac{z^2}{4} = 1 \quad x = 0$$

This surface is called a **hyperboloid of one sheet** and is sketched in Figure 9. ■

The idea of using traces to draw a surface is employed in three-dimensional graphing software. In most such software, traces in the vertical planes  $x = k$  and  $y = k$  are drawn for equally spaced values of  $k$ , and parts of the graph are eliminated using hidden line removal. Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form. All surfaces are symmetric with respect to the  $z$ -axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
<b>Ellipsoid</b> 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$ , the ellipsoid is a sphere.	<b>Cone</b> 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$ .
<b>Elliptic Paraboloid</b> 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	<b>Hyperboloid of One Sheet</b> 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
<b>Hyperbolic Paraboloid</b> 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	<b>Hyperboloid of Two Sheets</b> 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$ . Vertical traces are hyperbolas. The two minus signs indicate two sheets.

**TEC** In Module 12.6B you can see how changing  $a$ ,  $b$ , and  $c$  in Table 1 affects the shape of the quadric surface.

**EXAMPLE 7** Identify and sketch the surface  $4x^2 - y^2 + 2z^2 + 4 = 0$ .

**SOLUTION** Dividing by  $-4$ , we first put the equation in standard form:

$$-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$$

Comparing this equation with Table 1, we see that it represents a hyperboloid of two sheets, the only difference being that in this case the axis of the hyperboloid is the  $y$ -axis. The traces in the  $xy$ - and  $yz$ -planes are the hyperbolas

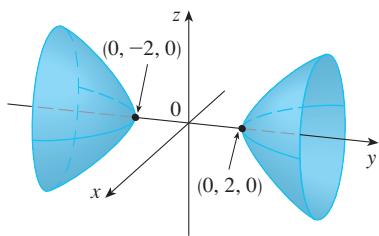
$$-x^2 + \frac{y^2}{4} = 1 \quad z = 0 \quad \text{and} \quad \frac{y^2}{4} - \frac{z^2}{2} = 1 \quad x = 0$$

The surface has no trace in the  $xz$ -plane, but traces in the vertical planes  $y = k$  for  $|k| > 2$  are the ellipses

$$x^2 + \frac{z^2}{2} = \frac{k^2}{4} - 1 \quad y = k$$

which can be written as

$$\frac{x^2}{k^2 - 1} + \frac{z^2}{2\left(\frac{k^2}{4} - 1\right)} = 1 \quad y = k$$



**FIGURE 10**

$$4x^2 - y^2 + 2z^2 + 4 = 0$$

These traces are used to make the sketch in Figure 10. ■

**EXAMPLE 8** Classify the quadric surface  $x^2 + 2z^2 - 6x - y + 10 = 0$ .

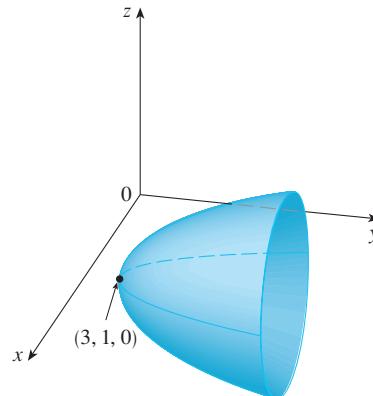
**SOLUTION** By completing the square we rewrite the equation as

$$y - 1 = (x - 3)^2 + 2z^2$$

Comparing this equation with Table 1, we see that it represents an elliptic paraboloid. Here, however, the axis of the paraboloid is parallel to the  $y$ -axis, and it has been shifted so that its vertex is the point  $(3, 1, 0)$ . The traces in the plane  $y = k$  ( $k > 1$ ) are the ellipses

$$(x - 3)^2 + 2z^2 = k - 1 \quad y = k$$

The trace in the  $xy$ -plane is the parabola with equation  $y = 1 + (x - 3)^2$ ,  $z = 0$ . The paraboloid is sketched in Figure 11.



**FIGURE 11**

$$x^2 + 2z^2 - 6x - y + 10 = 0$$

### ■ Applications of Quadric Surfaces

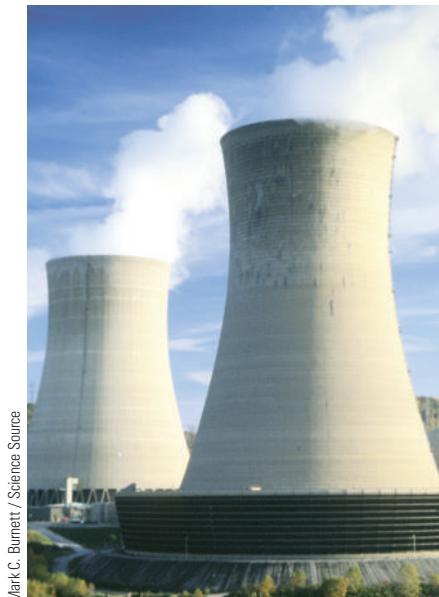
Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example. Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles. (See Exercise 49.)

Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals. In a radio telescope, for instance, signals from distant stars that strike the bowl are all reflected to the receiver at the focus and are therefore amplified. (The idea is explained in Problem 22 on page 273.) The same principle applies to microphones and satellite dishes in the shape of paraboloids.

Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet for reasons of structural stability. Pairs of hyperboloids are used to transmit rotational motion between skew axes. (The cogs of the gears are the generating lines of the hyperboloids. See Exercise 51.)



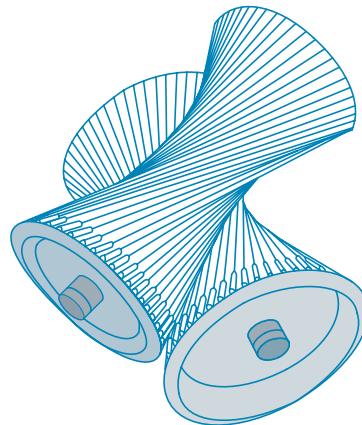
David Frazier / Spirit / Corbis



Mark C. Burnett / Science Source

A satellite dish reflects signals to the focus of a paraboloid.

Nuclear reactors have cooling towers in the shape of hyperboloids.



Hyperboloids produce gear transmission.

## 12.6 EXERCISES

- (a) What does the equation  $y = x^2$  represent as a curve in  $\mathbb{R}^2$ ?  
 (b) What does it represent as a surface in  $\mathbb{R}^3$ ?  
 (c) What does the equation  $z = y^2$  represent?
- (a) Sketch the graph of  $y = e^x$  as a curve in  $\mathbb{R}^2$ .  
 (b) Sketch the graph of  $y = e^x$  as a surface in  $\mathbb{R}^3$ .  
 (c) Describe and sketch the surface  $z = e^y$ .

**3–8** Describe and sketch the surface.

3.  $x^2 + z^2 = 1$

4.  $4x^2 + y^2 = 4$

- $z = 1 - y^2$
- $xy = 1$
- $y = z^2$
- $z = \sin y$

- (a) Find and identify the traces of the quadric surface  $x^2 + y^2 - z^2 = 1$  and explain why the graph looks like the graph of the hyperboloid of one sheet in Table 1.  
 (b) If we change the equation in part (a) to  $x^2 - y^2 + z^2 = 1$ , how is the graph affected?  
 (c) What if we change the equation in part (a) to  $x^2 + y^2 + 2y - z^2 = 0$ ?

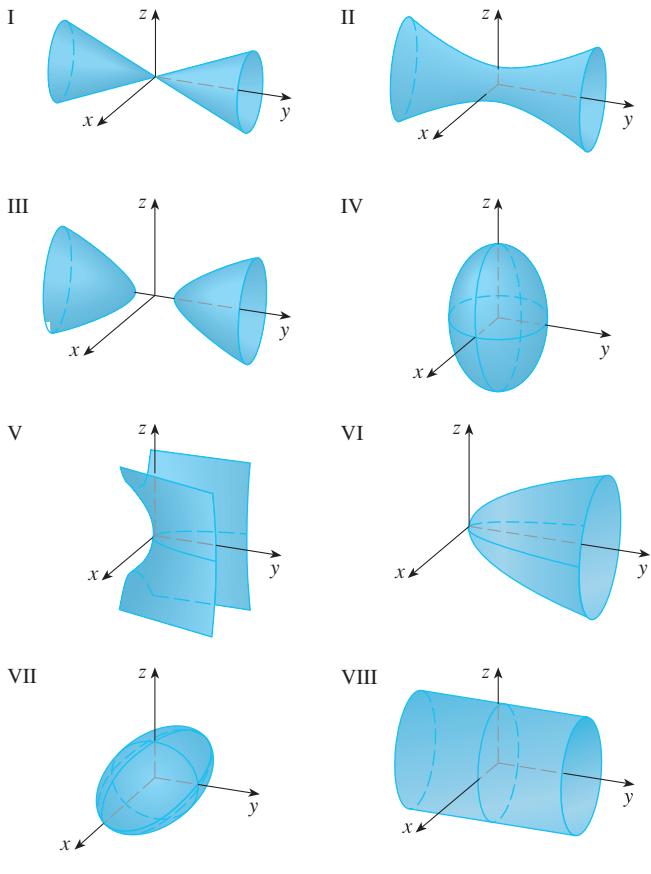
10. (a) Find and identify the traces of the quadric surface  $-x^2 - y^2 + z^2 = 1$  and explain why the graph looks like the graph of the hyperboloid of two sheets in Table 1.  
 (b) If the equation in part (a) is changed to  $x^2 - y^2 - z^2 = 1$ , what happens to the graph? Sketch the new graph.

11–20 Use traces to sketch and identify the surface.

11.  $x = y^2 + 4z^2$       12.  $4x^2 + 9y^2 + 9z^2 = 36$   
 13.  $x^2 = 4y^2 + z^2$       14.  $z^2 - 4x^2 - y^2 = 4$   
 15.  $9y^2 + 4z^2 = x^2 + 36$       16.  $3x^2 + y + 3z^2 = 0$   
 17.  $\frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{4} = 1$       18.  $3x^2 - y^2 + 3z^2 = 0$   
 19.  $y = z^2 - x^2$       20.  $x = y^2 - z^2$

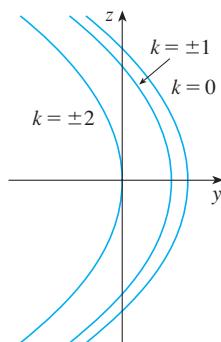
21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choice.

21.  $x^2 + 4y^2 + 9z^2 = 1$       22.  $9x^2 + 4y^2 + z^2 = 1$   
 23.  $x^2 - y^2 + z^2 = 1$       24.  $-x^2 + y^2 - z^2 = 1$   
 25.  $y = 2x^2 + z^2$       26.  $y^2 = x^2 + 2z^2$   
 27.  $x^2 + 2z^2 = 1$       28.  $y = x^2 - z^2$

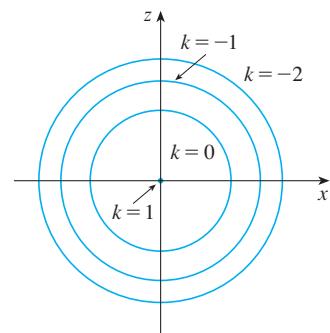


29–30 Sketch and identify a quadric surface that could have the traces shown.

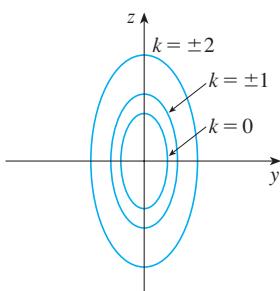
29. Traces in  $x = k$



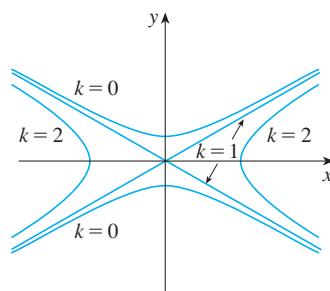
Traces in  $y = k$



30. Traces in  $x = k$



Traces in  $z = k$



31–38 Reduce the equation to one of the standard forms, classify the surface, and sketch it.

31.  $y^2 = x^2 + \frac{1}{9}z^2$       32.  $4x^2 - y + 2z^2 = 0$   
 33.  $x^2 + 2y - 2z^2 = 0$       34.  $y^2 = x^2 + 4z^2 + 4$   
 35.  $x^2 + y^2 - 2x - 6y - z + 10 = 0$   
 36.  $x^2 - y^2 - z^2 - 4x - 2z + 3 = 0$   
 37.  $x^2 - y^2 + z^2 - 4x - 2z = 0$   
 38.  $4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$

39–42 Use a computer with three-dimensional graphing software to graph the surface. Experiment with viewpoints and with domains for the variables until you get a good view of the surface.

39.  $-4x^2 - y^2 + z^2 = 1$       40.  $x^2 - y^2 - z = 0$   
 41.  $-4x^2 - y^2 + z^2 = 0$       42.  $x^2 - 6x + 4y^2 - z = 0$

43. Sketch the region bounded by the surfaces  $z = \sqrt{x^2 + y^2}$  and  $x^2 + y^2 = 1$  for  $1 \leq z \leq 2$ .

44. Sketch the region bounded by the paraboloids  $z = x^2 + y^2$  and  $z = 2 - x^2 - y^2$ .

45. Find an equation for the surface obtained by rotating the curve  $y = \sqrt{x}$  about the  $x$ -axis.
46. Find an equation for the surface obtained by rotating the line  $z = 2y$  about the  $z$ -axis.
47. Find an equation for the surface consisting of all points that are equidistant from the point  $(-1, 0, 0)$  and the plane  $x = 1$ . Identify the surface.
48. Find an equation for the surface consisting of all points  $P$  for which the distance from  $P$  to the  $x$ -axis is twice the distance from  $P$  to the  $yz$ -plane. Identify the surface.
49. Traditionally, the earth's surface has been modeled as a sphere, but the World Geodetic System of 1984 (WGS-84) uses an ellipsoid as a more accurate model. It places the center of the earth at the origin and the north pole on the positive  $z$ -axis. The distance from the center to the poles is 6356.523 km and the distance to a point on the equator is 6378.137 km.
- (a) Find an equation of the earth's surface as used by WGS-84.
- (b) Curves of equal latitude are traces in the planes  $z = k$ . What is the shape of these curves?
- (c) Meridians (curves of equal longitude) are traces in planes of the form  $y = mx$ . What is the shape of these meridians?
50. A cooling tower for a nuclear reactor is to be constructed in the shape of a hyperboloid of one sheet (see the photo on page 839). The diameter at the base is 280 m and the minimum diameter, 500 m above the base, is 200 m. Find an equation for the tower.
51. Show that if the point  $(a, b, c)$  lies on the hyperbolic paraboloid  $z = y^2 - x^2$ , then the lines with parametric equations  $x = a + t$ ,  $y = b + t$ ,  $z = c + 2(b - a)t$  and  $x = a + t$ ,  $y = b - t$ ,  $z = c - 2(b + a)t$  both lie entirely on this paraboloid. (This shows that the hyperbolic paraboloid is what is called a **ruled surface**; that is, it can be generated by the motion of a straight line. In fact, this exercise shows that through each point on the hyperbolic paraboloid there are two generating lines. The only other quadric surfaces that are ruled surfaces are cylinders, cones, and hyperboloids of one sheet.)
52. Show that the curve of intersection of the surfaces  $x^2 + 2y^2 - z^2 + 3x = 1$  and  $2x^2 + 4y^2 - 2z^2 - 5y = 0$  lies in a plane.
53. Graph the surfaces  $z = x^2 + y^2$  and  $z = 1 - y^2$  on a common screen using the domain  $|x| \leq 1.2$ ,  $|y| \leq 1.2$  and observe the curve of intersection of these surfaces. Show that the projection of this curve onto the  $xy$ -plane is an ellipse.

## 12 REVIEW

### CONCEPT CHECK

- What is the difference between a vector and a scalar?
- How do you add two vectors geometrically? How do you add them algebraically?
- If  $\mathbf{a}$  is a vector and  $c$  is a scalar, how is  $c\mathbf{a}$  related to  $\mathbf{a}$  geometrically? How do you find  $c\mathbf{a}$  algebraically?
- How do you find the vector from one point to another?
- How do you find the dot product  $\mathbf{a} \cdot \mathbf{b}$  of two vectors if you know their lengths and the angle between them? What if you know their components?
- How are dot products useful?
- Write expressions for the scalar and vector projections of  $\mathbf{b}$  onto  $\mathbf{a}$ . Illustrate with diagrams.
- How do you find the cross product  $\mathbf{a} \times \mathbf{b}$  of two vectors if you know their lengths and the angle between them? What if you know their components?
- How are cross products useful?
- (a) How do you find the area of the parallelogram determined by  $\mathbf{a}$  and  $\mathbf{b}$ ?
- (b) How do you find the volume of the parallelepiped determined by  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ ?

Answers to the Concept Check can be found on the back endpapers.

- How do you find a vector perpendicular to a plane?
- How do you find the angle between two intersecting planes?
- Write a vector equation, parametric equations, and symmetric equations for a line.
- Write a vector equation and a scalar equation for a plane.
- (a) How do you tell if two vectors are parallel?  
(b) How do you tell if two vectors are perpendicular?  
(c) How do you tell if two planes are parallel?
- (a) Describe a method for determining whether three points  $P$ ,  $Q$ , and  $R$  lie on the same line.  
(b) Describe a method for determining whether four points  $P$ ,  $Q$ ,  $R$ , and  $S$  lie in the same plane.
- (a) How do you find the distance from a point to a line?  
(b) How do you find the distance from a point to a plane?  
(c) How do you find the distance between two lines?
- What are the traces of a surface? How do you find them?
- Write equations in standard form of the six types of quadric surfaces.

## TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If  $\mathbf{u} = \langle u_1, u_2 \rangle$  and  $\mathbf{v} = \langle v_1, v_2 \rangle$ , then  $\mathbf{u} \cdot \mathbf{v} = \langle u_1 v_1, u_2 v_2 \rangle$ .

2. For any vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V_3$ ,  $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$ .

3. For any vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V_3$ ,  $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}| |\mathbf{v}|$ .

4. For any vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V_3$ ,  $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}|$ .

5. For any vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V_3$ ,  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ .

6. For any vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V_3$ ,  $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$ .

7. For any vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V_3$ ,  $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$ .

8. For any vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V_3$  and any scalar  $k$ ,

$$k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$$

9. For any vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V_3$  and any scalar  $k$ ,

$$k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v}$$

10. For any vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  in  $V_3$ ,

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$$

11. For any vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  in  $V_3$ ,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

12. For any vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  in  $V_3$ ,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

13. For any vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V_3$ ,  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ .

14. For any vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V_3$ ,  $(\mathbf{u} + \mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v}$ .

15. The vector  $\langle 3, -1, 2 \rangle$  is parallel to the plane

$$6x - 2y + 4z = 1$$

16. A linear equation  $Ax + By + Cz + D = 0$  represents a line in space.

17. The set of points  $\{(x, y, z) \mid x^2 + y^2 = 1\}$  is a circle.

18. In  $\mathbb{R}^3$  the graph of  $y = x^2$  is a paraboloid.

19. If  $\mathbf{u} \cdot \mathbf{v} = 0$ , then  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ .

20. If  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ , then  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ .

21. If  $\mathbf{u} \cdot \mathbf{v} = 0$  and  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ , then  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ .

22. If  $\mathbf{u}$  and  $\mathbf{v}$  are in  $V_3$ , then  $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}| |\mathbf{v}|$ .

## EXERCISES

1. (a) Find an equation of the sphere that passes through the point  $(6, -2, 3)$  and has center  $(-1, 2, 1)$ .

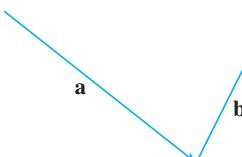
- (b) Find the curve in which this sphere intersects the  $yz$ -plane.

- (c) Find the center and radius of the sphere

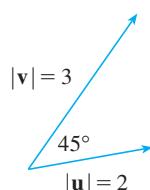
$$x^2 + y^2 + z^2 - 8x + 2y + 6z + 1 = 0$$

2. Copy the vectors in the figure and use them to draw each of the following vectors.

- (a)  $\mathbf{a} + \mathbf{b}$  (b)  $\mathbf{a} - \mathbf{b}$  (c)  $-\frac{1}{2}\mathbf{a}$  (d)  $2\mathbf{a} + \mathbf{b}$



3. If  $\mathbf{u}$  and  $\mathbf{v}$  are the vectors shown in the figure, find  $\mathbf{u} \cdot \mathbf{v}$  and  $|\mathbf{u} \times \mathbf{v}|$ . Is  $\mathbf{u} \times \mathbf{v}$  directed into the page or out of it?



4. Calculate the given quantity if

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

$$\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{c} = \mathbf{j} - 5\mathbf{k}$$

- (a)  $2\mathbf{a} + 3\mathbf{b}$  (b)  $|\mathbf{b}|$   
 (c)  $\mathbf{a} \cdot \mathbf{b}$  (d)  $\mathbf{a} \times \mathbf{b}$   
 (e)  $|\mathbf{b} \times \mathbf{c}|$  (f)  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$   
 (g)  $\mathbf{c} \times \mathbf{c}$  (h)  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$   
 (i)  $\text{comp}_{\mathbf{a}} \mathbf{b}$  (j)  $\text{proj}_{\mathbf{a}} \mathbf{b}$   
 (k) The angle between  $\mathbf{a}$  and  $\mathbf{b}$  (correct to the nearest degree)

5. Find the values of  $x$  such that the vectors  $\langle 3, 2, x \rangle$  and  $\langle 2x, 4, x \rangle$  are orthogonal.

6. Find two unit vectors that are orthogonal to both  $\mathbf{j} + 2\mathbf{k}$  and  $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ .

7. Suppose that  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2$ . Find

- (a)  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$  (b)  $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$   
 (c)  $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$  (d)  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$

8. Show that if  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are in  $V_3$ , then

$$(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2$$

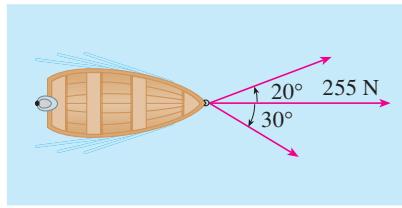
9. Find the acute angle between two diagonals of a cube.

10. Given the points  $A(1, 0, 1)$ ,  $B(2, 3, 0)$ ,  $C(-1, 1, 4)$ , and  $D(0, 3, 2)$ , find the volume of the parallelepiped with adjacent edges  $AB$ ,  $AC$ , and  $AD$ .

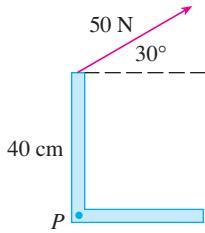
11. (a) Find a vector perpendicular to the plane through the points  $A(1, 0, 0)$ ,  $B(2, 0, -1)$ , and  $C(1, 4, 3)$ .  
 (b) Find the area of triangle  $ABC$ .

12. A constant force  $\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$  moves an object along the line segment from  $(1, 0, 2)$  to  $(5, 3, 8)$ . Find the work done if the distance is measured in meters and the force in newtons.

13. A boat is pulled onto shore using two ropes, as shown in the diagram. If a force of 255 N is needed, find the magnitude of the force in each rope.



14. Find the magnitude of the torque about  $P$  if a 50-N force is applied as shown.



15–17 Find parametric equations for the line.

15. The line through  $(4, -1, 2)$  and  $(1, 1, 5)$   
 16. The line through  $(1, 0, -1)$  and parallel to the line  $\frac{1}{3}(x - 4) = \frac{1}{2}y = z + 2$   
 17. The line through  $(-2, 2, 4)$  and perpendicular to the plane  $2x - y + 5z = 12$

18–20 Find an equation of the plane.

18. The plane through  $(2, 1, 0)$  and parallel to  $x + 4y - 3z = 1$   
 19. The plane through  $(3, -1, 1)$ ,  $(4, 0, 2)$ , and  $(6, 3, 1)$   
 20. The plane through  $(1, 2, -2)$  that contains the line  $x = 2t$ ,  $y = 3 - t$ ,  $z = 1 + 3t$

21. Find the point in which the line with parametric equations  $x = 2 - t$ ,  $y = 1 + 3t$ ,  $z = 4t$  intersects the plane  $2x - y + z = 2$ .

22. Find the distance from the origin to the line  $x = 1 + t$ ,  $y = 2 - t$ ,  $z = -1 + 2t$ .

23. Determine whether the lines given by the symmetric equations

$$\frac{x - 1}{2} = \frac{y - 2}{3} = \frac{z - 3}{4}$$

$$\text{and } \frac{x + 1}{6} = \frac{y - 3}{-1} = \frac{z + 5}{2}$$

are parallel, skew, or intersecting.

24. (a) Show that the planes  $x + y - z = 1$  and  $2x - 3y + 4z = 5$  are neither parallel nor perpendicular.  
 (b) Find, correct to the nearest degree, the angle between these planes.

25. Find an equation of the plane through the line of intersection of the planes  $x - z = 1$  and  $y + 2z = 3$  and perpendicular to the plane  $x + y - 2z = 1$ .

26. (a) Find an equation of the plane that passes through the points  $A(2, 1, 1)$ ,  $B(-1, -1, 10)$ , and  $C(1, 3, -4)$ .  
 (b) Find symmetric equations for the line through  $B$  that is perpendicular to the plane in part (a).  
 (c) A second plane passes through  $(2, 0, 4)$  and has normal vector  $\langle 2, -4, -3 \rangle$ . Show that the acute angle between the planes is approximately  $43^\circ$ .  
 (d) Find parametric equations for the line of intersection of the two planes.

27. Find the distance between the planes  $3x + y - 4z = 2$  and  $3x + y - 4z = 24$ .

28–36 Identify and sketch the graph of each surface.

28.  $x = 3$       29.  $x = z$   
 30.  $y = z^2$       31.  $x^2 = y^2 + 4z^2$   
 32.  $4x - y + 2z = 4$       33.  $-4x^2 + y^2 - 4z^2 = 4$   
 34.  $y^2 + z^2 = 1 + x^2$   
 35.  $4x^2 + 4y^2 - 8y + z^2 = 0$   
 36.  $x = y^2 + z^2 - 2y - 4z + 5$

37. An ellipsoid is created by rotating the ellipse  $4x^2 + y^2 = 16$  about the  $x$ -axis. Find an equation of the ellipsoid.

38. A surface consists of all points  $P$  such that the distance from  $P$  to the plane  $y = 1$  is twice the distance from  $P$  to the point  $(0, -1, 0)$ . Find an equation for this surface and identify it.

## Problems Plus

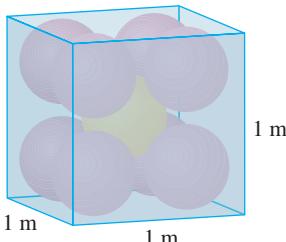


FIGURE FOR PROBLEM 1

- Each edge of a cubical box has length 1 m. The box contains nine spherical balls with the same radius  $r$ . The center of one ball is at the center of the cube and it touches the other eight balls. Each of the other eight balls touches three sides of the box. Thus the balls are tightly packed in the box (see the figure). Find  $r$ . (If you have trouble with this problem, read about the problem-solving strategy entitled *Use Analogy* on page 71.)
- Let  $B$  be a solid box with length  $L$ , width  $W$ , and height  $H$ . Let  $S$  be the set of all points that are a distance at most 1 from some point of  $B$ . Express the volume of  $S$  in terms of  $L$ ,  $W$ , and  $H$ .
- Let  $L$  be the line of intersection of the planes  $cx + y + z = c$  and  $x - cy + cz = -1$ , where  $c$  is a real number.
  - Find symmetric equations for  $L$ .
  - As the number  $c$  varies, the line  $L$  sweeps out a surface  $S$ . Find an equation for the curve of intersection of  $S$  with the horizontal plane  $z = t$  (the trace of  $S$  in the plane  $z = t$ ).
  - Find the volume of the solid bounded by  $S$  and the planes  $z = 0$  and  $z = 1$ .
- A plane is capable of flying at a speed of 180 km/h in still air. The pilot takes off from an airfield and heads due north according to the plane's compass. After 30 minutes of flight time, the pilot notices that, due to the wind, the plane has actually traveled 80 km at an angle  $5^\circ$  east of north.
  - What is the wind velocity?
  - In what direction should the pilot have headed to reach the intended destination?
- Suppose  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are vectors with  $|\mathbf{v}_1| = 2$ ,  $|\mathbf{v}_2| = 3$ , and  $\mathbf{v}_1 \cdot \mathbf{v}_2 = 5$ . Let  $\mathbf{v}_3 = \text{proj}_{\mathbf{v}_1}\mathbf{v}_2$ ,  $\mathbf{v}_4 = \text{proj}_{\mathbf{v}_2}\mathbf{v}_3$ ,  $\mathbf{v}_5 = \text{proj}_{\mathbf{v}_3}\mathbf{v}_4$ , and so on. Compute  $\sum_{n=1}^{\infty} |\mathbf{v}_n|$ .
- Find an equation of the largest sphere that passes through the point  $(-1, 1, 4)$  and is such that each of the points  $(x, y, z)$  inside the sphere satisfies the condition

$$x^2 + y^2 + z^2 < 136 + 2(x + 2y + 3z)$$

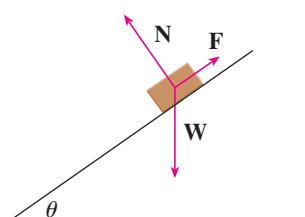


FIGURE FOR PROBLEM 7

- Suppose a block of mass  $m$  is placed on an inclined plane, as shown in the figure. The block's descent down the plane is slowed by friction; if  $\theta$  is not too large, friction will prevent the block from moving at all. The forces acting on the block are the weight  $\mathbf{W}$ , where  $|\mathbf{W}| = mg$  ( $g$  is the acceleration due to gravity); the normal force  $\mathbf{N}$  (the normal component of the reactionary force of the plane on the block), where  $|\mathbf{N}| = n$ ; and the force  $\mathbf{F}$  due to friction, which acts parallel to the inclined plane, opposing the direction of motion. If the block is at rest and  $\theta$  is increased,  $|\mathbf{F}|$  must also increase until ultimately  $|\mathbf{F}|$  reaches its maximum, beyond which the block begins to slide. At this angle  $\theta_s$ , it has been observed that  $|\mathbf{F}|$  is proportional to  $n$ . Thus, when  $|\mathbf{F}|$  is maximal, we can say that  $|\mathbf{F}| = \mu_s n$ , where  $\mu_s$  is called the *coefficient of static friction* and depends on the materials that are in contact.
  - Observe that  $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$  and deduce that  $\mu_s = \tan(\theta_s)$ .
  - Suppose that, for  $\theta > \theta_s$ , an additional outside force  $\mathbf{H}$  is applied to the block, horizontally from the left, and let  $|\mathbf{H}| = h$ . If  $h$  is small, the block may still slide down the plane; if  $h$  is large enough, the block will move up the plane. Let  $h_{\min}$  be the smallest value of  $h$  that allows the block to remain motionless (so that  $|\mathbf{F}|$  is maximal).

By choosing the coordinate axes so that  $\mathbf{F}$  lies along the  $x$ -axis, resolve each force into components parallel and perpendicular to the inclined plane and show that

$$h_{\min} \sin \theta + mg \cos \theta = n \quad \text{and} \quad h_{\min} \cos \theta + \mu_s n = mg \sin \theta$$

- Show that

$$h_{\min} = mg \tan(\theta - \theta_s)$$

Does this equation seem reasonable? Does it make sense for  $\theta = \theta_s$ ? Does it make sense as  $\theta \rightarrow 90^\circ$ ? Explain.

- (d) Let  $h_{\max}$  be the largest value of  $h$  that allows the block to remain motionless. (In which direction is  $\mathbf{F}$  heading?) Show that

$$h_{\max} = mg \tan(\theta + \theta_s)$$

Does this equation seem reasonable? Explain.

- 8.** A solid has the following properties. When illuminated by rays parallel to the  $z$ -axis, its shadow is a circular disk. If the rays are parallel to the  $y$ -axis, its shadow is a square. If the rays are parallel to the  $x$ -axis, its shadow is an isosceles triangle. (In Exercise 12.1.48 you were asked to describe and sketch an example of such a solid, but there are many such solids.) Assume that the projection onto the  $xz$ -plane is a square whose sides have length 1.
- (a) What is the volume of the largest such solid?
  - (b) Is there a smallest volume?

