14.5 Regla de la Cadena

$y = f(g(t))$ $\quad y = f(x) \quad x = g(t)$

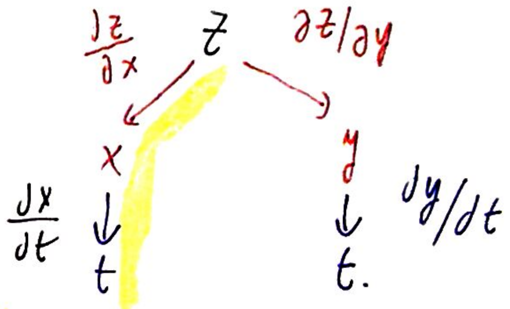
$y \to x \to t$ $\quad \dfrac{dy}{dt} = \underbrace{\dfrac{dy}{dx}}_{\text{externa}} \underbrace{\dfrac{dx}{dt}}_{\text{interna}}$

Caso 1: $\quad z = f(x, y) \quad x = g(t) \quad y = h(t)$

¿Cómo se encuentra $\partial z/dt$?

$z = f(x(t), y(t))$

Diagrama de Árbol.

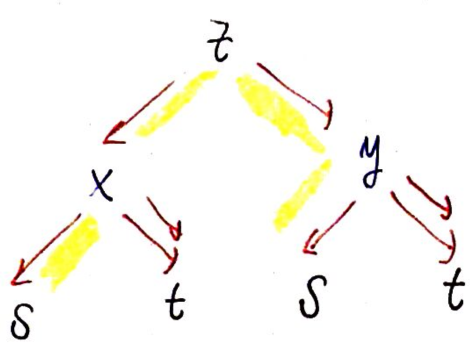


Variable dependiente z.
Variables intermedias x, y.
Variable independiente t

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.}$$

Sume cada trayectoria.

Caso 2: $\quad z = f(x, y) \quad x = g(s, t) \quad y = h(s, t)$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Ejercicio 1: suponga que el costo de producir x uds. de A. y y uds. de B es:

$$C(x,y) = (3x^2 + y^3 + 4)^{1/3}$$ explícita

Las funciones de producción para cada producto son:

$$x = 10KL \qquad y = 5K^2 + 4L.$$ explícita

Encuentre la razón de cambio de C respecto al capital y al trabajo.

$$\begin{tikzcd}[column sep=small]
 & & C \arrow[dl] \arrow[dr] & & \\
 & x \arrow[dl] \arrow[d] & & y \arrow[d] \arrow[dr] & \\
K & L & & K & L
\end{tikzcd}$$

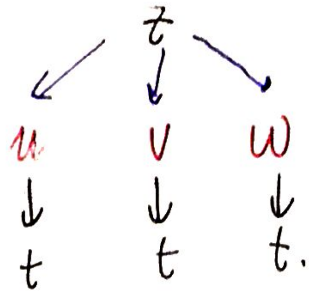
$$\frac{\partial C}{\partial K} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial K} + \frac{\partial C}{\partial y}\cdot\frac{\partial y}{\partial K}.$$

$$\frac{\partial C}{\partial L} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial L} + \frac{\partial C}{\partial y}\frac{\partial y}{\partial L}.$$

$$\frac{\partial C}{\partial K} = \frac{1}{3}6x\,(3x^2+y^3+4)^{-2/3}\,10L + \frac{1}{3}\frac{3y^2}{(3x^2+y^3+4)^{2/3}}\,10K.$$

$$\frac{\partial C}{\partial L} = \frac{2x}{(3x^2+y^3+4)^{2/3}}\,10K + \frac{y^2}{(3x^2+y^3+4)^{2/3}}(4)$$

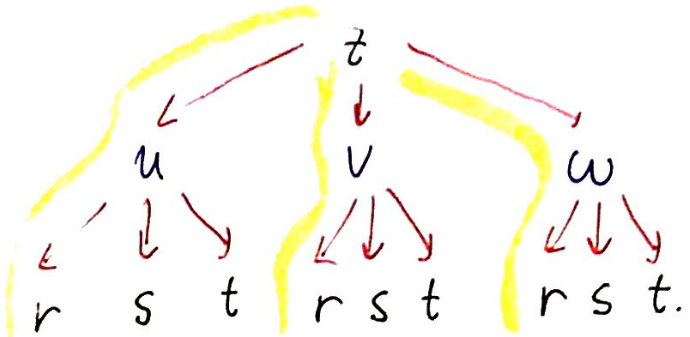
Ejercicio 3: Suponga que $z = f(u, v, w)$ y que u, v, w son funciones de t. Encuentre dz/dt.



$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t}.$$

Ejercicio 4: Suponga ahora que $z = f(u, v, w)$ y que u, v, w son funciones de r, s, t.

Encuentre las derivadas parciales de z resp. a r, s & t.



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial r}.$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial s}.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t}$$

Ejercicio 5: Encuentre las derivadas parciales indicadas.

a. $w = \sqrt{x^2+y^2}$ $\qquad \left.\dfrac{\partial w}{\partial p}\right|_{(p=1,\, q=0,\, r=3)}$

$x = p^2 - q^3 + r - 1$

$y = \ln(p) + e^{q} + e^{\ln r}$

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}$$

$$\frac{\partial w}{\partial p} = \frac{x}{(x^2+y^2)^{1/2}}\,2p + \frac{y}{(x^2+y^2)^{1/2}}\,\frac{1}{p}$$

$x(1,0,3) = 1^2 - 0^3 + 3 - 1 = 3.$

$y(1,0,3) = \ln(1) + e^0 + e^{\ln 3} = 0 + 1 + 3 = 4.$

$$\left.\frac{\partial w}{\partial p}\right|_{(1,0,3)} = \frac{3}{\sqrt{9+16}}\,2 + \frac{4}{5}\,\frac{1}{1} = \frac{6}{5} + \frac{4}{5} = 2$$

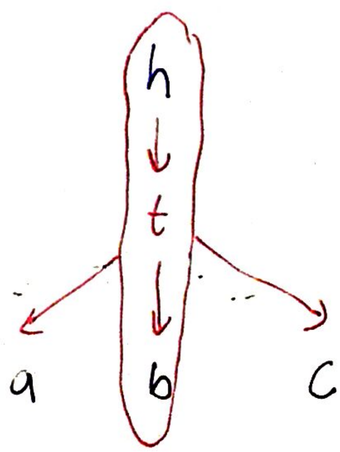
b. $h = 4 - t^2$, $\quad t = 2a + 3b + 4c$, $\quad \left.\dfrac{\partial h}{\partial b}\right|_{(4,2,3)}$

$h(a,b,c) = 4 - (2a+3b+4c)^2$

$$\frac{\partial h}{\partial b} = -2(2a+3b+4c)\,3$$

$$\left.\frac{\partial h}{\partial b}\right|_{(4,2,3)} = -2(8+6+12)\cdot 3$$

$h_b(4,2,3) = -2(26) = -52\cdot 3$



c. $w = \ln(xyz)$ $\quad x = r^2 - s^2,\ y = rs,\ z = r^2 + s^2$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$w_x = \frac{yz}{xyz} = \frac{1}{x} \qquad w = \ln(xyz) = \ln x + \ln y + \ln z.$$

$$\boxed{\frac{\partial w}{\partial r} = \frac{2r}{x} + \frac{s}{y} + \frac{2r}{z}}$$

Ejercicios Derivación Implícita, Planos y Rectas Tangentes.

Ejercicio 6: Encuentre las ecs. paramétricas de las rectas tangentes a $z = \sin x \tan y$ en la dirección de x & y en el punto $(\pi/6, \pi/4)$

En la dirección de x: $m_x = z_x(\pi/6, \pi/4)$

de y: $m_y = z_y(\pi/6, \pi/4)$

$z_x = \cos x \tan y.$ $\quad z_x(\pi/6, \pi/4) = \frac{\sqrt{3}}{2}\ 1$

$z(\pi/6, \pi/4) = \frac{1}{2} \cdot 1$

No es una recta

Intento: $z = z(\pi/6, \pi/4) + z_x(\pi/6, \pi/4)(x - \pi/6)$

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6)$$

En la dirección de x no hay cambio en y.

$x = t.$

$y = \pi/4$

$z = \sin t \tan(\pi/4)$

$\vec{r}(t)$

$\vec{r}\,'(t) = \langle 1, 0, \cos t \rangle.$

$\vec{r}\,'(\pi/6) = \langle 1, 0, \sqrt{3}/2 \rangle.$

$L = \vec{r}(\pi/6) + \vec{r}\,'(\pi/6)\, t.$

$= \langle \pi/6, \pi/4, 1/2 \rangle + t \langle 1, 0, \sqrt{3}/2 \rangle.$

$$\boxed{\begin{aligned} x &= \pi/6 + t. \\ y &= \pi/4. \\ z &= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\, t. \end{aligned}}$$

Recta Tangente a z en la dirección de x.

En la dirección de y. $x = \pi/6$ use $t = \pi/4$.

$x = \pi/6$

$y = t$

$z = \sin\frac{\pi}{6} \tan t.$

$\vec{r}\,'(t) = \langle 0, 1, \frac{1}{2}\sec^2 t \rangle.$

$r'(\pi/4) = \langle 0, 1, \underbrace{\tfrac{1}{2}(\sqrt{2})^2}_{1} \rangle.$

$L = \vec{r}(\pi/4) + \vec{r}\,'(\pi/4)\, t$

$\vec{r}(\pi/4) = \langle \pi/6, \pi/4, 1/2 \rangle.$

$$\boxed{\begin{aligned} x &= \pi/6 \\ y &= \pi/4 + t. \\ z &= \tfrac{1}{2} + t \end{aligned}}$$

Ec. del Plano Tangente ó Aproximación Lineal

$$L(x, y) = z(\pi/6, \pi/4) + z_x(x - \pi/6) + z_y(y - \pi/4)$$

$z(x, y) = \sin x \tan y.$

$z(\pi/6, \pi/4) = \frac{1}{2} \cdot 1$

$z_x = \cos x \tan y$ $\quad z_x(\pi/6, \pi/4) = \frac{\sqrt{3}}{2} \cdot 1$

$z_y = \sin x \sec^2 y$ $\quad z_y(\pi/6, \pi/4) = \frac{1}{2}\, 2 = 1$

Plano Tangente: $\boxed{L(x,y) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) + 1 \cdot (y - \pi/4)}$

$t = 3$

$x = 2$
$y = t$
$z = -1 - 2t^2.$

$r(3) = \langle 2, 3, -19 \rangle$
$r'(t) = \langle 0, 1, -4t \rangle.$
$r'(3) = \langle 0, 1, -12 \rangle.$

$x = 2$
$y = 3 + t.$
$z = -19 - 12t.$