4.6 Variación de Parámetros. (VP)

Se utiliza para encontrar y_p para cualquier ED lineal inhomogénea.

ED lineal 2do orden: $y'' + P(x)y' + Q(x)y = g(x)$

Asume que se conoce la solución complementaria.

$$y_c = c_1 y_1 + c_2 y_2.$$

Asume la sig. forma de soln. particular

$$y_p = \underline{u_1(x)}\, y_1 + \underline{u_2(x)}\, y_2.$$

Sustituya y_p en la ED, agrupe términos semejantes.
u_1' & u_2' se encuentran al resolver sig. sistema de ecs:

$$\underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}}_{\vec{u}} = \underbrace{\begin{bmatrix} 0 \\ g(x) \end{bmatrix}}_{b}$$

Es invertible.

Como y_1 & y_2 son L.I. $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$

$$\boxed{u_1' = -\frac{y_2\, g(x)}{W} \qquad u_2' = \frac{y_1\, g(x)}{W}.}$$

Soln gral. $y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2.$

Ejercicio 1: Resuelva.

a. $y'' + y = \underbrace{\sin x}_{g(x)}$

soln. complementaria: $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$y_c = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}.$$

soln. particular. $y_p = u_1(x)\, y_1 + u_2(x)\, y_2$

Wronskiano: $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$$W = \cos^2 x + \sin^2 x = 1$$

Construya u_1' & u_2' $\quad$ $g(x)$ término inhomogéneo.

$$u_1' = -\frac{y_2\, g}{W} = \frac{-\sin x \sin x}{1} = -\sin^2 x$$

$$u_2' = \frac{y_1\, g}{W} = \frac{\cos x \sin x}{1} = \cos x \sin x$$

Integre

$$u_1 = \int -\sin^2 x\, dx = -\frac{1}{2}\int (1 - \cos 2x)\, dx = -\frac{x}{2} + \frac{1}{4}\sin 2x$$

$$u_2 = \int \underbrace{\sin x}_{w} \underbrace{\cos x\, dx}_{dw} = \frac{1}{2}\sin^2 x$$

Soln. Particular. $y_p = u_1 y_1 + u_2 y_2$.

$$y_p = \boxed{\frac{-x}{2}\cos x} + \frac{1}{4}\sin 2x \cos x + \frac{1}{2}\sin^2 x \sin x$$

con coeficientes indeterminados

E.D. $y'' + y = \sin x$ $\quad y_p = Ax\sin x + Bx\cos x$

use $\sin 2x = 2\sin x\cos x$ $\quad \sin^2 x + \cos^2 x = 1$

$$y_p = -\frac{x}{2}\cos x + \frac{1}{2}\sin x\cos^2 x + \frac{1}{2}\sin^2 x \sin x$$

$$y_p = \frac{-x}{2}\cos x + \frac{1}{2}\sin x\,(\underbrace{\cos^2 x + \sin^2 x}_{1})$$

Soln
Gral.

$$\boxed{y = c_1\cos x + \underbrace{c_2 \sin x + \frac{1}{2}\sin x}_{c_2 \sin x} - \frac{x}{2}\cos x}$$

b. $y'' + y = \sec x \tan x$ } no se puede utilizar. coeficientes indeterminados.

$y_c = c_1 \cos x + c_2 \sin x$

Wronskiano: $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$$u_1' = -\frac{g y_2}{W} = \frac{-1}{\cos x}\tan x \sin x = -\tan^2 x$$

$$u_2' = \frac{g y_1}{W} = \frac{\sec x \tan x \cos x}{1} = \tan x$$

Integre: $\sec^2 x = \tan^2 x + 1$

$\tan^2 x = \sec^2 x - 1$

no es necesario agregar c.

$$u_1 = \int -\tan^2 x\, dx = \int 1 - \sec^2 x\, dx = x - \tan x$$

$$u_2 = \int \tan x\, dx = \ln|\sec x| \quad ó \quad -\ln|\cos x|$$

$$\int \frac{\sin x}{\cos x}\, dx$$

Soln particular $\quad y_p = u_1 \cos x + u_2 \sin x$

$$y_p = x\cos x - \underbrace{\tan x \cos x}_{\sin x} + \ln|\sec x| \sin x$$

Soln. general:

$$y = c_1 \cos x + c_2 \sin x - \sin x + x\cos x + \ln|\sec x| \sin x$$

$$y = A_1 \cos x + A_2 \sin x + x\cos x + \ln|\sec x| \sin x$$

c. $\quad y'' = \dfrac{1}{x}$

no se puede usar coefs. indeterminados.

$$y' = \int \frac{1}{x}\, dx = \ln x + c_1$$

$$y = \int (\ln x + c_1)\, dx = x\ln x - \underbrace{\int x \frac{dx}{x}}_{\int dx} + c_1 x$$

$u = \ln x$

$dv = dx$

$$\boxed{y = x\ln x - x + c_1 x + c_2.}$$

Método 1

Integre 2 veces.

Método 2: Encuentre y_c & y_p usando VP.

Soln. complementaria $m^2 = 0 \Rightarrow m = 0, 0.$ Raíz Repetida.

$$y_c = c_1 + c_2 x$$

Soln. particular $y_p = u_1 + u_2 x$

$g(x) = \frac{1}{x}$

$y_1 = 1$

$y_2 = x$

Wronskiano. $W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1$

$$u_1' = -\frac{g y_2}{W} = -\frac{1}{x} x = -1 \Rightarrow u_1 = -x$$

$$u_2' = \frac{g y_1}{W} = \frac{1}{x} 1 \Rightarrow u_2 = \ln x$$

Soln. particular: $y_p = -x + x \ln x.$

Soln. general $y = y_c + y_p = c_1 + c_2 x - x + x \ln x$

$$\boxed{y = c_1 + c_2 x + x \ln x}$$

CIs: $y'' + by' + cy = e^x \sin x + x^n + x^n e^x \cos x$

func. trig, exponenciales, polinomios.

VPs: $y'' + by' + cy = \ln x + \sqrt{1 - x^2} + \tan x + x^n$

Integración Directa $y'' = f(x)$, $y^{(n)} = f(x)$

ED Lineal sin coeficientes constantes.

$$a\underline{\underline{x^2}}\, y'' + b\,\underline{\underline{x}}\, y' + c y = f(x). \left.\right\} \begin{array}{c}\text{ED} \\ \text{Cauchy-Euler.}\end{array}$$

La soln no es $y = e^{rx}$

ahora es $y = x^r \qquad y' = r x^{r-1}$

$$y'' = r(r-1)\,x^{r-2}$$

Ejercicio 2: Encuentre la soln de.

$$x^2 y'' - 2x y' - 4y = 30x^3.$$

Soln. complementaria $x^2 y'' - 2xy' - 4y = 0.$

$$y = x^r \;,\quad y' = r x^{r-1} \;,\quad y'' = r(r-1)\,x^{r-2}.$$

Sustituya en la ED homogénea.

$$r(r-1)\,x^r - 2r\,x^r - 4x^r = 0.$$

$$x^r\left[r^2 - r - 2r - 4\right] = 0 \qquad x^r \neq 0$$

$$r^2 - 3r - 4 = (r-4)(r+1) = 0 \Rightarrow r = 4,\ r = -1$$

Soln. complementaria $y_c = c_1 x^{-1} + c_2 x^4$

$y_p \neq Ax^3 + Bx^2 + C.$ Use VP para encontrar y_p

$y_p = u_1 y_1 + u_2 y_2.$ $\qquad g(x) = 30x^3$

$$W = \begin{vmatrix} x^{-1} & x^4 \\ -x^{-2} & 4x^3 \end{vmatrix} = 4x^2 + x^2 = 5x^2$$

Integre.

$$u_1' = -\frac{y_2 g}{W} = \frac{-x^4\, 30x^3}{5x^2} = -6x^5 \Rightarrow u_1 = -x^6.$$

$$u_2' = \frac{y_1 g}{W} = \frac{1}{x}\,\frac{30x^3}{5x^2} = 6 \Rightarrow u_2 = 6x$$

$$y_p = \frac{u_1}{x} + u_2 x^4 = -x^5 + 6x^5 = 5x^5$$

Soln General: $\boxed{y = \frac{c_1}{x} + c_2 x^4 + 5x^5}$