Notas de clase Cálculo integral
UFM
SECCIÓN A

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Capítulo 1

5.4 La Integral Indefinida

Una antiderivada de f es una función F(x) cuya derivada es f(x).

$$F'(x) = f(x)$$

$f(x) = 10x^4$ una antiderivada. $F(x) = 2x^5$

$F'(x) = 10x^4$

$F(x) = 2x^5 + \pi$ $\quad F(x) = 2x^5 - 1,215$

Antiderivada General: $F(x) = 2x^5 + C.$

constante de integr.

La Integral Indefinida de f(x) respecto a x, es la antiderivada más general de f.

$$\boxed{\int f(x)\,dx = F(x) + C} \qquad C \in \mathbb{R}.$$

signa elongada $\int$ $\quad dx$ diferencial

integre respecto a x

$\int f(x,y)\,dx$ $\quad \int f(x,y)\,dy.$

$f(x,y) = 2x + y.$

$\int (2x+y)\,dx = x^2 + yx + C.$

$\int (2x+y)\,dy = 2xy + \frac{y^2}{2} + C.$

Integrar significa encontrar la antiderivada general de f $\quad \int dx.$

Integrales indefinidas Básicas.

Reescriba las reglas de Antiderivadas.
Voltear la tabla de derivación.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C. \quad , \quad \int \frac{1}{x} \, dx = \ln|x| + C.$$

$n \neq -1$

$$\int e^x \, dx = e^x + C \qquad \int a^x \, dx = \frac{a^x}{\ln(a)} + C.$$

$$\int \cos x \, dx = \sin x + C. \qquad \int \tan x \, dx \quad ? \quad \int \sec x \, dx \quad ?$$

$$\int \sin x \, dx = -\cos x + C. \qquad \int \cot x \, dx. \quad ? \quad \int \cot x \, dx \quad ?$$

$$\int \sec^2 x \, dx = \tan x + C. \qquad \int \sec x \tan x \, dx = \sec x + C.$$

$$\int \csc^2 x \, dx = -\cot x + C \qquad \int \csc x \cot x \, dx = -\csc x + C.$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1}(x) + C. \qquad \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + C$$

$$\int \frac{-1}{1+x^2} \, dx = \cot^{-1}(x) + C \qquad \int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1}(x) + C.$$

Suma: $\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx$

Constante: $\int K f(x) \, dx = K \int f(x) \, dx$ $\quad K$ es constante

Ejemplos.

a) $\int (5x^{10} + 2x^5) \, dx = \frac{5x^{11}}{11} + \frac{2x^6}{6} + C.$

b) $\int \frac{1}{x^2} + \frac{1}{x^{1/2}} \, dx = \int x^{-2} + x^{-1/2} \, dx = \frac{x^{-1}}{-1} + \frac{x^{1/2}}{1/2} + C.$

b) Otra respuesta. $-\frac{1}{x} + 2\sqrt{x} + C.$

d. $\int x^{\pi} + x^{\sqrt{2}}\, dx. = \frac{x^{\pi+1}}{\pi+1} + \frac{x^{\sqrt{2}+1}}{\sqrt{2}+1} + C.$

c. $\int \frac{1}{\sqrt[5]{x^2}} + \sqrt[3]{x^2}\, dx$

$\int x^{-2/5} + x^{2/3}\, dx = \frac{5}{3}x^{3/5} + \frac{3}{5}x^{5/3} + C.$

Ejercicio 1: Evalúe. $x^{-1} \rightarrow \frac{x^0}{0}$

a) $\int (\underbrace{x^e}_{\text{potencia}} + \underbrace{e^x}_{\text{exponencial.}})\, dx = \frac{x^{e+1}}{e+1} + e^x + C.$

$a^0 = \frac{a^1}{a^1} = 1$

b) $\int (8 \cdot 10^x - \frac{2}{x})\, dx = \frac{8 \cdot 10^x}{\ln(10)} - 2\ln|x| + C$

c) $\int (x-2)(x+2)(x^2+4)\, dx = \int (x^2-4)(x^2+4)\, dx.$

$\int (x^4 - 16)\, dx = \frac{x^5}{5} - 16x + C.$

d) $\int \frac{e^{4x} + e^{5x}}{e^{4x}}\, dx = \int (1 + e^x)\, dx = x + e^x + C.$

Integral Definida.

Es una integral con límites de integración en $x=a$ & $x=b$.

$\int_a^b f(x)\, dx.$ $\int f(x)\, dx = F(x) + C$

Teorema de Evaluación:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ es un número.

utilice la notación de corchete $\Big]_a^b$ para indicar que la antiderivada se está en ambos límites.

$$\int_a^b f(x)\,dx = F(x)\Big]_{x=a}^{x=b} \qquad F(x)+C\Big]_{x=a}^{x=b} = F(b) + \cancel{C'} - F(a) - \cancel{C}$$

Ejercicio 1: Evalúe los sigs. integrales definidas.

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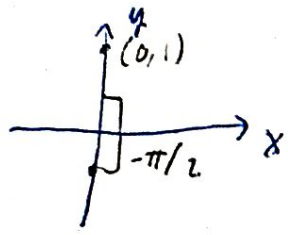
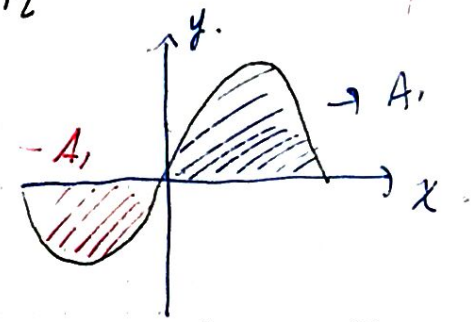
0. $$\int_{-\pi/2}^{\pi/2} \sin x\,dx = -\cos x\Big]_{-\pi/2}^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{-\pi}{2}\right) = 0$$

0 0

✓✓





detalles

a. $$\int_0^3 x^2\,dx = \frac{x^3}{3}\Big]_0^3 = \frac{27}{3} - \frac{0}{3} = 9$$

Antiderive: Luego evalúe. álgebra.

b. $$\int_9^{36} \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big]_9^{36} = \frac{2}{3}\left(6^{2\,3/2} - 3^{2\,3/2}\right)$$

$x^{1/2}$

$$\frac{2}{3}\left(6^3 - 3^3\right) = \frac{2}{3}(216-27)$$

c. $$\int_{\pi/4}^{\pi/2} \frac{1}{1-x^2}\,dx$$ capcioso, no existe.

porque $\frac{1}{1-x^2}$ se indefine en $x=\pm 1$!

$\sin$ en $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

No hay regla para $\int \frac{1}{1-x^2}\,dx = \int \frac{A}{1-x}\,dx + \int \frac{B}{1+x}\,dx$

d. $\int_1^4 \left(\frac{1}{\sqrt{x}} + 3\sqrt{x}\right) dx = 2\cdot x^{1/2} + \frac{6}{3}x^{3/2}\Big]_{x=1}^{x=4}$

invertir en $x=0$

$\int x^{-1/2} + 3x^{1/2}\,dx$

$= 2\sqrt{4} + 2\cdot 4^{3/2} - (2\sqrt{1} + 2\cdot 1^{3/2})$

$= 4 + 16 - (2+2)$

$= 20 - 4 = \underline{\underline{16}}$

Capítulo 2

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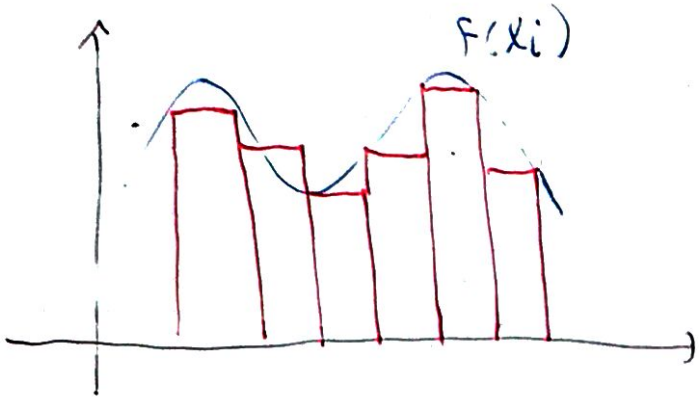
5.4 Áreas y Propiedades de la Integral Definida.



Área

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

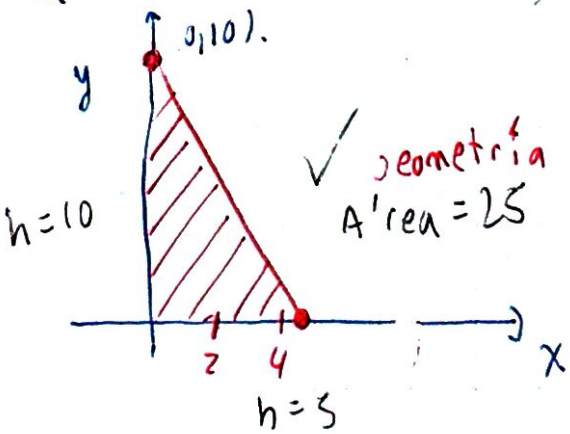
Si f es continua en [a,b], la integral definida es

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

La integral definida es el área de la región bajo la curva $y = f(x)$ en $[a,b]$. $f(x) \geq 0$.

Ejercicio 2: Encuentre el área de las sigs. regiones. Bosqueje cada región.

a. $f(x) = 10 - 2x$, $f(x) \geq 0$ en $0 \leq x \leq 5$.



✓ geometría
Área = 25

$$A = \int_0^5 (10 - 2x)\,dx$$

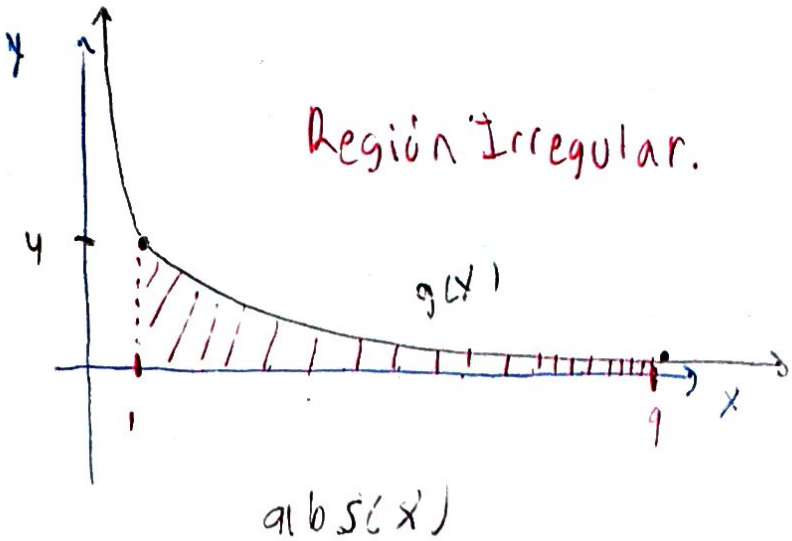
$$A = 10x - x^2 \Big]_0^5 = 50 - 25 - 0$$

$$A = 25.$$

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a).$$

b. $g(x) = \dfrac{4}{\sqrt{x}}$ entre $1 \le x \le 9$

AV en $x = 0$
AH en $y = 0$.



Región Irregular.

$$A = \int_1^9 4x^{-1/2}\,dx$$

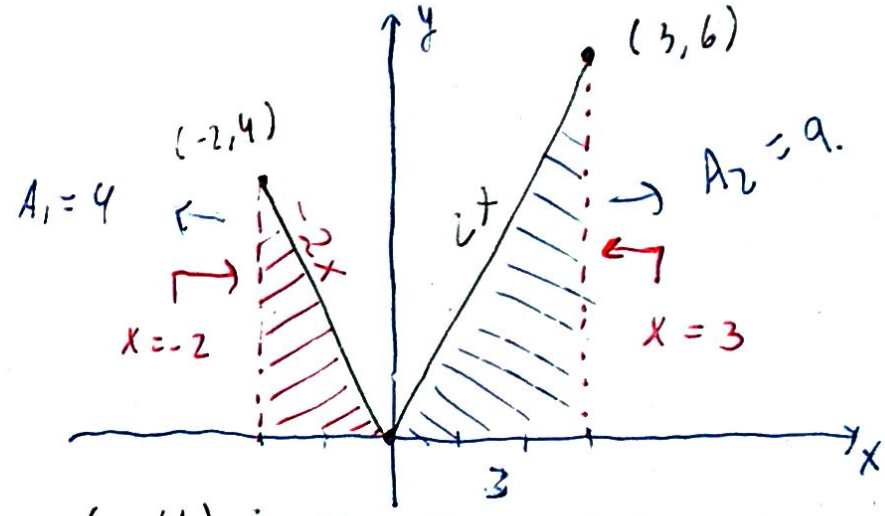
$$A = 8x^{1/2}\Big]_1^9$$

$$A = 8(\sqrt{9} - \sqrt{1}) = 8 \cdot 2 = 16.$$

c. $h(x) = 2|x|$ entre $x = -2$ y $x = 3$.

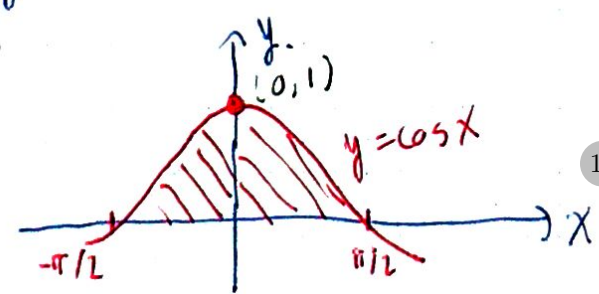
$$A = \int_{-2}^{3} 2|x|\,dx$$

$$A = \int_{-2}^{0} -2x\,dx + \int_{0}^{3} 2x\,dx$$



$$A = -x^2\Big]_{-2}^{0} + x^2\Big]_0^3 = -0 - (-4) + 9 - 0 = 4 + 9 = 13.$$

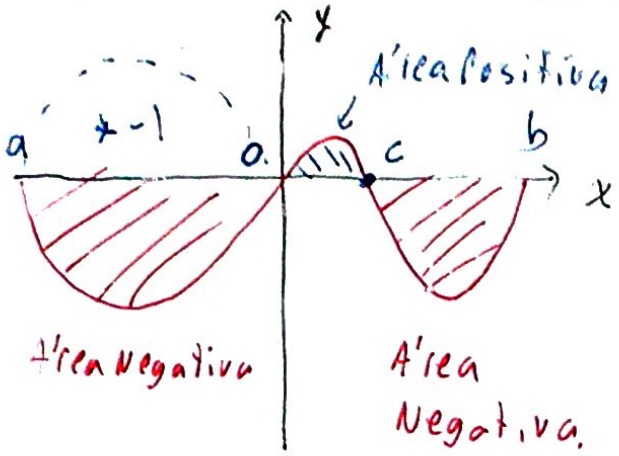
d. $y = \cos x$ entre $x = -\pi/2$ y $x = \pi/2$.



$$A = \int_{-\pi/2}^{\pi/2} \cos x\,dx = \sin x\Big]_{-\pi/2}^{\pi/2}$$

$$A = 1 - (-1) = 2.$$

¿Qué sucede cuando $f(x) < 0$ en partes del intervalo?



Área de la región entre $f(x)$ y el eje-x.

$$A \neq \int_a^b f(x)\,dx$$

$$* \int x f(x)\,dx = x \int f(x)\,dx$$

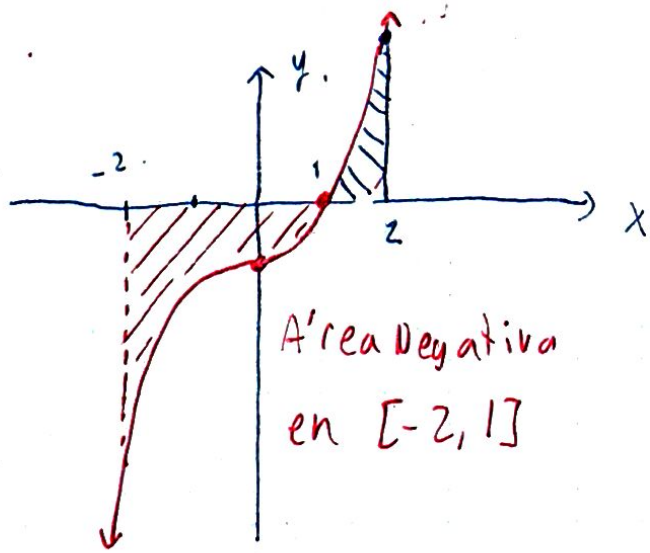
$$A = \int_a^0 -f(x)\,dx + \int_0^c f(x)\,dx - \int_c^b f(x)\,dx$$

Pag 16. Ejercicio 3: Considere $f(x) = 4x^3 - 4$ en $-2 \leq x \leq 2$.

a. $\displaystyle\int_{-2}^{2} (4x^3 - 4)\,dx = x^4 - 4x \Big]_{-2}^{2} = (16 - 8) - (16 + 8)$

$4(x^3 - 1) = 3$ en $x = 2$ $\quad 8 - 24 = -16$

b. Bosqueje la región y explique si la integral definida representa el área de la región.



La integral definida no es el área de la región.

c. Encuentre el área de la región

$$A = \int_{-2}^{1} 4 - 4x^3 \, dx + \int_{1}^{2} (4x^3 - 4) \, dx.$$

$$A = \underbrace{4x - x^4 \Big]_{-2}^{1}}_{A_1} + \underbrace{x^4 - 4x \Big]_{1}^{2}}_{A_2} = \overbrace{(4-1)-(-8-16)}^{A_1} + \underbrace{(16-8)-(1-4)}_{A_2}$$

$$A = (3+24) + (8+3) = 27 + 11 = 38$$

Propiedades Integral Definida.

$$\int_a^b k_1 f(x) \pm k_2 g(x) \, dx = k_1 \int_a^b f(x) \, dx \pm k_2 \int_a^b g(x) \, dx$$

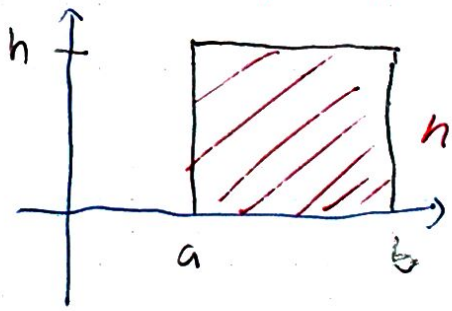
Otras Propiedades. $F(x)\Big]_a^a = F(a) - F(a) = 0$

Ejemplos: $\int_{\pi}^{\pi} \arctan(e^{x^2} + \log|x|) \, dx = 0$

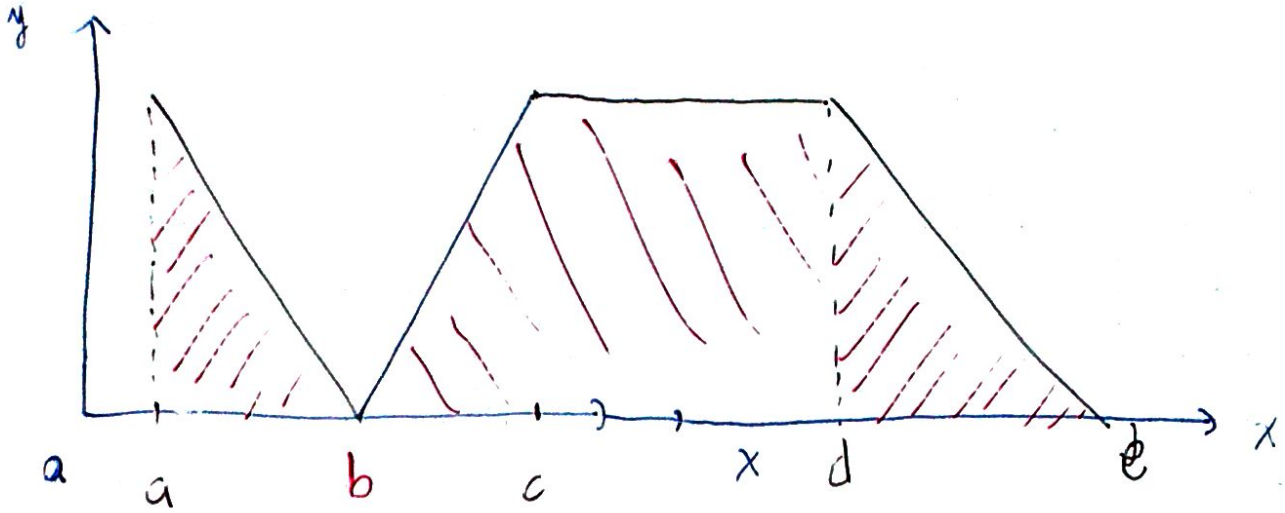
$$\int_a^a f(x) \, dx = 0$$

$h(b-a) = A.$

$$\int_a^b h \, dx = hx\Big]_a^b = h(b-a)$$



p.e. $\int_5^{1005} \sqrt{2} \, dx = \sqrt{2}(1005-5)$



7. $$\int_a^e f(x)\,dx = \int_a^b f\,dx + \int_b^c f\,dx + \int_c^d f\,dx + \int_c^e f\,dx.$$

8. Invertir el orden de integración.

$$\int_a^b f(x)\,dx = \int_b^a -f(x)\,dx.$$

$$F(b) - F(a) = (-F(b) + F(a))(-1) = \int_b^a -f(x)\,dx.$$

p.e. $$\int_0^\pi \sin x\,dx = -\cos x\Big]_0^\pi = -\overbrace{\cos\pi}^{-1} - (-\overbrace{\cos 0}^{1})$$
$$= 1 + 1 = 2.$$

$$\int_\pi^0 -\sin x\,dx = \cos x\Big]_\pi^0 = \cos 0 - \cos\pi = 1 + 1 = 2.$$

Ejercicio 5: Evalúe la sig. integral definida.

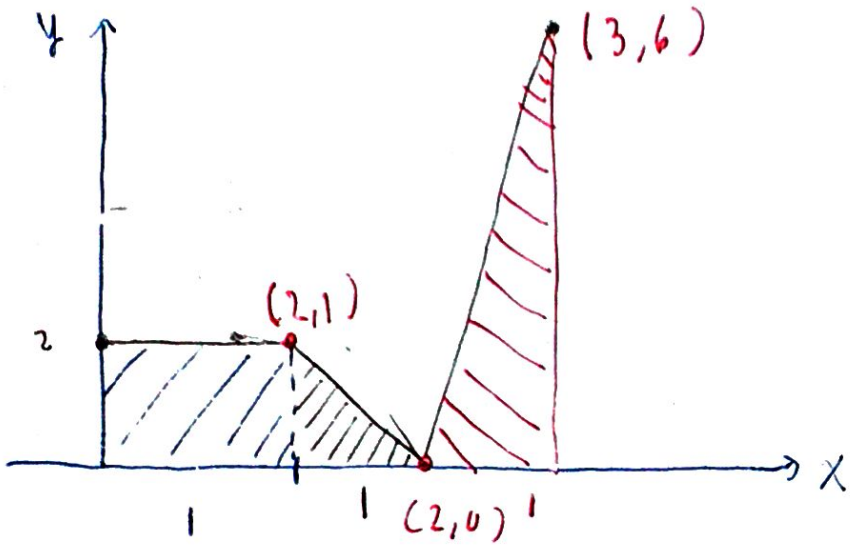
$$\int_0^3 f(x)\,dx \qquad \text{donde} \qquad f(x) = \begin{cases} 2 & \text{si } 0 \le x \le 1 \\ 4-2x & \text{si } 1 < x \le 2 \\ 6x-12 & \text{si } 2 < x \le 3. \end{cases}$$

$$\int_0^3 f\,dx = \int_0^1 2\,dx + \int_1^2 (4-2x)\,dx + \int_2^3 (6x-12)\,dx.$$

$$= 2x\Big]_0^1 + 4x - x^2\Big]_1^2 + 3x^2 - 12x\Big]_2^3$$

$$= 2. \quad -(8-4) - (4-1) + (27-36) - (12-24)$$

$$= 2 \quad + \quad 1 \quad + \quad 3 \quad = \quad 6.$$



$A_1 = 2 \cdot 1$ Rectángulo

$A_2 = \frac{1}{2}\, 2 \cdot 1$ triángulo

$A_3 = \frac{1}{2}\, 6 \cdot 1$ Triángulo

$A = 1 + 2 + 3 = 6$

Capítulo 3

5.4 Desplazamientos y Distancias

Antiderivada de $f'(x)$ es $f(x) + c$.

$$\int_a^b f'(x)\,dx = f(x)\Big]_a^b = \underbrace{f(b) - f(a)}_{\text{cambio neto.}}$$

La integral de la razón de cambio es el cambio neto.

Razón de cambio puede ser la velocidad $s'(t)$

costo marginal $c'(x)$.

cambio poblacional $p'(t)$

Desplazamiento: $s = \int_a^b s'(t)\,dt$ cambio neto en la posición.

Costo Neto: $c = \int_a^b c'(x)\,dx$

Población: $p = \int_a^b p'(t)\,dt$.

Ejemplo: velocidad $v(t) = \dfrac{2}{t^{4/3}}$, $t \geq 1$ m/s.

Encuentre el desplazamiento de la partícula entre 1 y 8 s.

$v(t) = s'(t)$ $\quad s = \int_1^8 2t^{-4/3}\,dt. = -6t^{-1/3}\Big]_1^8 = 6t^{-1/3}\Big]_8^1$

$$s = -6\left(\frac{1}{\sqrt[3]{8}} - \frac{1}{\sqrt[3]{1}}\right) = -6\left(\frac{1}{2} - 1\right) = 6 \cdot \frac{1}{2} = 3 \text{ m.}$$

Ejercicio 1: Velocidad $v(t) = 64 - 32t$ Encuentre el desplazamiento entre 1 y 3 segundos. v - p/s.

$$s = \int_1^3 (64 - 32t)\,dt = 64t - 16t^2\Big]_1^3 = 64\cdot 3 - 16\cdot 9 - (64 - 16)$$

$$= 64\cdot 2 - 16\cdot 4\cdot 2$$

$$128 - 128 = 0 \text{ pies.}$$

$$64t\Big]_1^3 - 16t^2\Big]_1^3$$

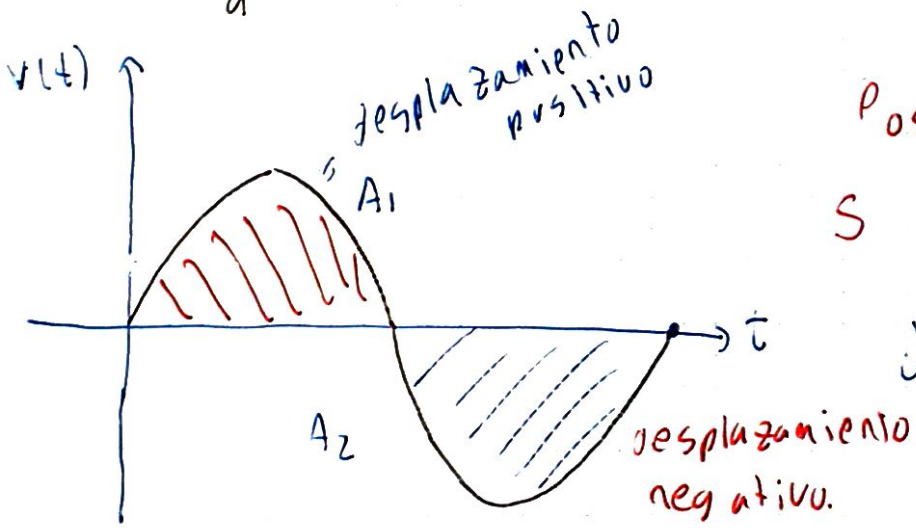
$$64\cdot 2 - 16(8)$$

Distancia vs. Desplazamiento.

$$s = a\hat{i} + b\hat{j}$$
$$d = \sqrt{a^2 + b^2}$$

La distancia es la magnitud del desplazamiento.

$$d = \int_a^b |v(t)|\,dt. \text{ Área} \qquad s = \int_a^b v(t)\,dt. \text{ Cambio Neto.}$$



Posición relativa

$$S = A_1 + A_2 = 0$$

$$d = A_1 - A_2 > 0 \text{ positivo.}$$

Rapidez $|v(t)|$ escalar

Velocidad vector.

Ejercicio 2: Un vehículo da vueltas en un circuito a una velocidad de $v(t) = 27 - 3t^2$ mph.

a. Encuentre la velocidad y rapidez del vehículo a las 5 horas.

velocidad: $v(5) = 27 - 3(25) = 27 - 75 = -48$ mph.

rapidez: $|v(5)| = |-48| = 48$ mph, speed.

b. Encuentre el desplazamiento en $0 \leq t \leq 4$

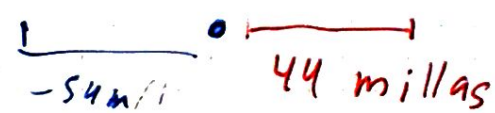
$$s = \int_0^4 (27 - 3t^2)\,dt = 27t\Big]_0^4 - t^3\Big]_0^4 = 27 \cdot 4 - 64$$

$108 - 64 = 44$ millas

c. Encuentre el desplazamiento en $0 \leq t \leq 6$.

$$s = \int_0^6 (27 - 3t^2)\,dt = 27t\Big]_0^6 - t^3\Big]_0^6 = 27 \cdot 6 - 6^3$$

$s = 162 - 216. = -54$ mph.



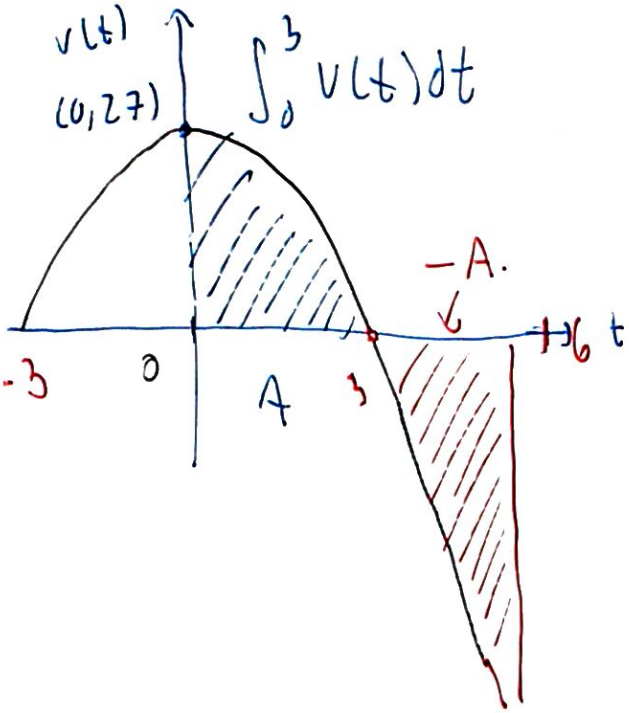
d. Encuentre la distancia recorrida durante las primeras seis horas. $0 \leq t \leq 6$.

$$d = \int_0^6 |v(t)|\,dt \qquad d = \int_0^6 |27 - 3t^2|\,dt.$$

¿Cuándo $v(t) = 0$? $\quad 27 - 3t^2 = 0$

$3t^2 = 27 \Rightarrow t = \pm 3.$

		-3		3	
$27 - 3t^2$	$-$	$\vert$	$+$	$\vert$	$-$



$$d = \int_0^3 (27-3t^2)dt - \int_3^6 (27-3t^2)dt.$$

$$d = \left(27t - t^3\Big]_0^3\right) + \left(t^3 - 27t\Big]_3^6\right)$$

$$d = (81-27) + (216 - 162 - 27 + 81)$$

$$d = 54 + (108) = 162 \text{ millas.}$$

$$\int f(x)dx = F(x) + \underline{C}$$

Integrales Indefinidas. $\int_a^b f(x)dx$ es un número.

aceleración: $a(t)$

velocidad inicial.

Función de velocidad: $v(t) = \int a(t)dt$ use $v(0) = V_0$.

Función de desplazamiento: $s(t) = \int v(t)dt$. use $s(0) = S_0$

posición inicial.

Derivando. $v(t) = s'(t)$ y $a(t) = v'(t) = s''(t)$.

Ejercicio 3: Un cohete despega con una aceleración vertical $a(t) = t^2\left(\frac{72}{t} - 36\right)$, con una posición inicial de 0 pies, y velocidad inicial de 400 pies/s.

a) Encuentre la posición del cohete en cualquier tiempo.

Velocidad: $v(t) = \int a(t)\,dt = \int (72t - 36t^2)\,dt.$

Use $v(t) = 36t^2 - 12t^3 + \cancel{C}.\ 400.$

$v(0) = 400$: $v(0) = 0 - 0 + C = 400$

Posición: $s(t) = \int v(t)\,dt = \int (36t^2 - 12t^3 + 400)\,dt.$

$s(t) = 12t^3 - 3t^4 + 400t + C_2.$

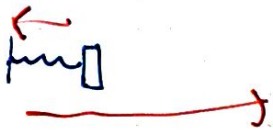
Use $s(0) = 0$: $s(0) = 0 + C_2 = 0$

Posición $\boxed{s(t) = 12t^3 - 3t^4 + 400t}$

b) Velocidad y rapidez del cohete a los 10s.

$v(10) = 36(100) - 12(1000) = 3{,}600 - 12{,}000 = -8{,}400$ p/s.

Rapidez: $|v(10)| = 8{,}400$ pies/s.



Ejercicio 4: Un resorte $a(t) = 4\cos t - 3\sin t.$

Encuentre la velocidad y desplazamiento del resorte si parte del reposo y su posición inicial es cero.

velocidad: $v(t) = \int a(t)\,dt = 4\sin t + 3\cos t + C_1$

$v(0) = 0.$ $v(0) = 0 + 3 + C_1 = 0 \Rightarrow C_1 = -3.$

desplazamiento: $s(t) = \int v(t)\,dt = -4\cos t + 3\sin t - 3t + C_2.$

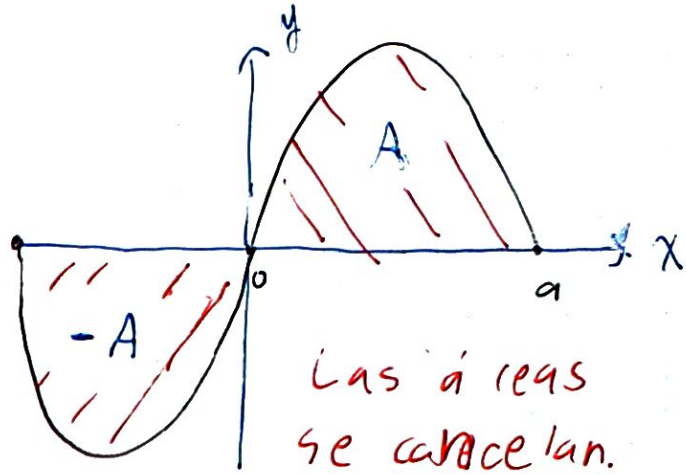
$s(0) = 0$ $s(0) = -4 + 0 + C_2 = 0 \Rightarrow C_2 = 4.$

$v(t) = 4\sin t + 3\cos t - 3$

$s(t) = -4\cos t + 3\sin t - 3t + 4.$

Integrales Definidas para funciones Impares en intervalos simétricos.

$$\int_{-a}^{a} f(x)\,dx = 0$$



$f(x)$ es impar.

$f(-x) = -f(x)$

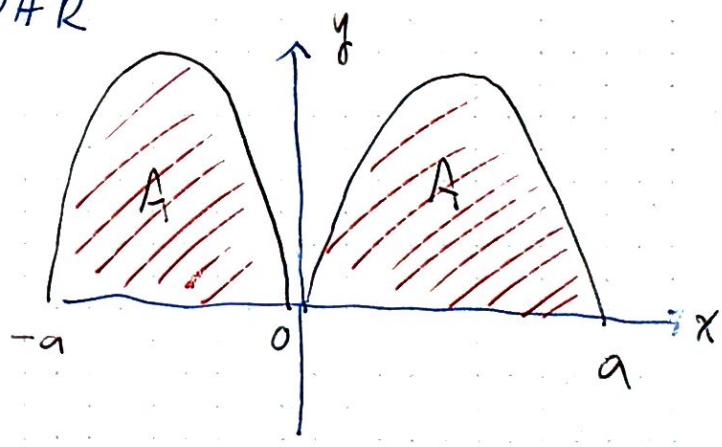
$$\int_{-50}^{50} \underbrace{\left(x^3 + \sinh^5 x + \tan\frac{x}{200}\right)}_{\text{es impar}}{}^{a}\,dx = 0$$

$\tan x = \dfrac{\sin x}{\cos x}$ $\quad \dfrac{\text{IMPAR}}{\text{PAR}} = \text{impar}$ $\quad \tan(-x) = \dfrac{-\sin x}{\cos x} = -\tan x$

Integral Definida Función PAR

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$f(-x) = f(x)$



Capítulo 4

5.3 Teorema fundamental del Cálculo (TFC)

Si $f(x)$ es continua en $[a, x]$ entonces

Parte 1: $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$. ✓ $\quad F^{-1}[F(x)] = x$

La integración y la derivación son procesos inversos entre sí.

La antiderivada de la derivada es la función original

$$\int \frac{df}{dx}\,dx = f(x) + C.$$

La derivada de la integral es la función original.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Parte 2: $\int_a^b f(x)\,dx = F(x)\Big]_{x=a}^{x=b}$. ✓ Ya se había visto

Ejemplo: derive la siguiente función $f(x) = \int_a^x t^2\,dt$

Método 1: Integrar y luego derivar

$$f(x) = \frac{t^3}{3}\Big]_{t=a}^{t=x} = \frac{x^3}{3} - \frac{a^3}{3} \qquad \boxed{f'(x) = x^2} + 0$$

Método 2: Use TFC parte 1: $\quad f'(x) = \frac{d}{dx}\left(\int_a^x t^2\,dt\right) = x^2.$

$t \to x$

t es una variable temporal de integración.

$$\int_a^x t^2\,dt = \frac{t^3}{3}\Big]_a^x \qquad \int_a^b f(t)\,dt = \int_a^b f(y)\,dy = \int_a^b f(x)\,dx$$

$\cancel{\int_a^x x^2\,dx}$ forma incorrecta de escribirla.

Varias funciones se definen por medio de integrales que no se pueden evaluar.

$$\int e^t\,dt = e^t + C$$

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\,dt.$$

5.5 $\int e^{t^2} t\,dt = \frac{1}{2} e^{t^2} + C.$

Distribución normal media cero y varianza 1.

$$P'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Ejercicio 1: Derive las siguientes funciones.

a. $h(x) = \int_a^x 3\sqrt{t}\,dt.$ $\qquad h'(x) = 3\sqrt{x}$

$t \to x$

b. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt.$ $\qquad S'(x) = \sin\left(\frac{\pi x^2}{2}\right)$

c. $H(w) = \int_{-5}^{w} \frac{t+4}{t^4+t^2+2}\,dt$ $\qquad H'(w) = \frac{w+4}{w^4+w^2+2}.$

Derivadas de Funciones Compuestas y definidas por integrales.

$$f(x) = \int_{100}^{x^5} e^t \, dt. = e^t \Big]_{t=100}^{t=x^5} = e^{x^5} - e^{100}.$$

$$f'(x) = e^{x^5} 5x^4 + 0$$ por la regla de la cadena.

$$\frac{d}{dx}\left(\int_{100}^{x^5} e^t \, dt\right) = \frac{d}{dx}\int_{100}^{u} e^t \, dt. = e^u \frac{du}{dx}$$

$u(x) = x^5$

$$= e^{x^5} \cdot 5x^4$$

TFC parte I y la regla de la cadena.

$$\frac{d}{dx}\int_{a}^{b(x)} f(t)\,dt = f(b(x))\, b'(x)$$

$t \to b(x)$

$$\int_1^{\ln x} * \, dt = \frac{t^2}{2}\Big]_1^{\ln x} = \frac{(\ln x)^2}{2} - 1$$

Ejercicio 2: Derive.

a. $g(x) = \int_5^{\ln x} \sqrt{t^2+1}\, dt.$ $\quad y'(x) = \sqrt{(\ln x)^2+1} \cdot \frac{1}{x}$

$u \to \ln x$

b. $h(x) = \int_{\sec x}^{4} \tan^{-1} t \, dt. - \int_4^{\sec x} \tan^{-1} t \, dt.$

$$h'(x) = \tan^{-1}(\sec x) \sec x \tan x$$

Generalizaciones del TFC.

Derive $\int_{x^5}^{x^{10}} \cos t \, dt = \sin t \Big]_{t=x^5}^{t=x^{10}} = \sin(x^{10}) - \sin(x^5)$

$$\frac{d}{dx}\left(\int_{x^5}^{x^{10}} \cos t \, dt\right) = \cos(x^{10})\, 10x^9 - \cos(x^5)\, 5x^4.$$

Ejercicio 3: Derive las sigs. expresiones.

a. $f(x) = \int_{\sin x}^{e^x} \sqrt{10 + 4t^4}\, dt.$

$$f(x) = \underbrace{\sqrt{10 + 4e^{4x}} \cdot e^x}_{t \to e^x} - \underbrace{\sqrt{10 + 4\sin^4 x}}_{t \to \sin x} \cos x$$

b. $g(x) = \int_{1/x}^{\cosh x} \sinh(\theta^3)\, d\theta$

$$g'(x) = \sinh(\cosh^3 x) \cdot \sinh x - \sinh\left(\frac{1}{x^3}\right)\left(\frac{-1}{x^2}\right)$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\, dt. = \underbrace{f(b(x))\, b'(x)}_{t \to b(x)} - \underbrace{f(a(x))\, a'(x)}_{t \to a(x)}.$$

Ejercicio 4: Encuentre la ecuación de la recta tangente a $f(x) = \int_0^x e^{-t^2/2}\, dt$. en $x = 0$.

Ec. Recta Tangente

pendiente $f'(0)$
punto sobre recta $f(0)$

$$y = f(0) + f'(0)(x - 0).$$

$$f(0) = \int_0^0 e^{-t^2/2}\, dt. = 0.$$

$$f'(x) = e^{-x^2/2} \qquad f'(0) = e^{-0^2} = 1.$$

Ec. Recta Tangente. $y = 0 + x = x$

$$\boxed{y = x}$$

Capítulo 5

5.5 Regla de la Sustitución.

a $\int 3(x+2)^2\,dx = \int (3x^2+12x+12)\,dx$

$x^3+6x^2+12x+C.$

$= (x+2)^3 + C$

b. $\int 5(x+2)^4\,dx = (x+2)^5 + C.$

Reglas de Integración para funciones compuestas

$f(g(x)) \longrightarrow F(g(x)) + C.$

Regla de la Potencia para derivadas.

$$\frac{d}{dx}\left([g(x)]^{n+1}\right) = (n+1)[g(x)]^n g'(x).$$

$$\int [g(x)]^n g'(x)\,dx = \frac{1}{n+1}[g(x)]^{n+1} + C.$$

Nueva variable $u = g(x)$ $\quad du = g'(x)\,dx$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\text{ó } \ln|u| + C \qquad \text{si } n = -1.$$

Ejercicio 1: Integre

a. $\int 5(\overbrace{x+2}^{u})^4\,dx = \int 5u^4\,du = u^5 + C$

$u = x+2 \qquad du = d$

$= \boxed{(x+2)^5 + C.}$

20. $\int \overbrace{(x^2+3)}^{u}{}^5 \overbrace{2x\,dx}^{du} = \int u^5\,du = \frac{u^6}{6} + C.$

$u = x^2+3 \qquad du = 2x\,dx \qquad \frac{(x^2+3)^6}{6} + C.$

a. $\int \underbrace{(y^3+3y^2+11)^{2/3}}_{u^{2/3}} \underbrace{(3y^2+6y)\,dy}_{du} = \int u^{2/3}\,du$

$= \frac{3}{5} u^{5/3} + C.$

$u = y^3+3y^2+11 \qquad du = (3y^2+6y)\,dy. \qquad = \frac{3}{5}(y^3+3y^2+11)^{5/3} + C.$

b. $\int (10w^3-8)^{15} \underbrace{w^2\,dw}_{du/30} = \int \frac{u^{15}}{30}\,du$ sólo en términos de u.

$u = 10w^3-8$

$\frac{du}{30} = w^2\,dw \qquad dw = \frac{du}{30w^2}.$

$\int (10w^3-8)^{15} w^2\,dw = \int u^{15} \frac{du}{30} = \frac{u^{16}}{30 \cdot 16} + C.$

$= \frac{(10w^3-8)^{16}}{480} + C.$

c. $\int 8x^3 \sqrt{8+x^4}\,dx = \int 8\cancel{x^3}\, u^{1/2} \frac{du}{4\cancel{x^3}} = \int 2u^{1/2}\,du.$

$u = 8+x^4 \qquad du = 4x^3\,dx \Rightarrow dx = \frac{du}{4x^3}$

$\int 2\sqrt{\underbrace{8+x^4}_{u}}\, \underbrace{4x^3\,dx}_{du.} = \int 2u^{1/2}\,du. = \frac{4}{3} u^{3/2} + C.$

$= \frac{4}{3}(8+x^4)^{3/2} + C.$

d. $\int (\underbrace{4x^3+x}_{u})^2\,dx. = \int (16x^6 + 8x^4 + x^2)\,dx$

$= \frac{16}{7}x^7 + \frac{8}{5}x^5 + \frac{1}{3}x^3 + C.$

Regla de la cadena. $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\, g'(x)$

Regla de la Sustitución: si $g(x)$ es derivable entonces.

$$\int f'(g(x))\, g'(x)\, dx = \int f'(u)\, du = f(u) + C.$$

$$f(g(x)) + C.$$

$u = g(x) \qquad du = g'(x)\, dx$

Ejercicio 2: Integre.

0. $\int \frac{\overbrace{(2+4x+12x^2)\, dx}^{2\cdot du}}{x+x^2+2x^3} = \int \frac{2du}{u} = 2\ln|u| + C.$

$u = x + x^2 + 2x^3 \qquad du = (1+2x+6x^2)\, dx$

$2du = (2+4x+12x^2)\, dx$

La integral es $2\ln|x+x^2+2x^3| + C$

01. $\int e^{x^8} x^7\, dx = \int e^u \frac{du}{8} = \frac{e^u}{8} + C = \frac{e^{x^8}}{8} + C$

$du/8$

$u = x^8 \qquad du = 8x^7\, dx.$

a2 $\int e^{x^8} \underbrace{x^6}_{\neq x^7.}\, dx$ No se puede integrar

b. $\int (x^4+3)^2 \sin(x^4+3)^3 x^3\, dx = \int u^2 \sin u^3 \frac{du}{4}.$

$u = x^4 + 3 \qquad du = 4x^3\, dx$

$$\int u^2 \sin u^3 \frac{du}{4} = \int \sin w \frac{dw}{12} = \frac{-\cos(w)}{12} + C$$

$w = u^3 \quad dw = 3u^2 du$

$u^2 du = \frac{dw}{3}$

$$= -\frac{1}{12}\cos(u^3) + C$$

$$= -\frac{1}{12}\cos(x^4+3)^3 + C.$$

c. $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{u} du = \ln|u| + C.$

$= \ln|\sin x| + C.$

$u = \sin x, \; du = \cos x \, dx$

d. $\int \sec^2(\ln x) \frac{1}{x} dx = \int \sec^2 u \, du = \tan u + C$

$= \tan(\ln x) + C.$

$u = \ln x, \; du = \frac{dx}{x}$

e $\int (1000x + 2{,}000)^{1000} dx = \frac{1}{1000} \frac{(1000x + 2000)^{1001}}{1001} + C.$

$\int e^{x^2} 2x \, dx = e^{x^2} + C.$

Sustitución incompleta.

$x = u - 3.$

f. $\int 28x(x+3)^{1/3} dx. = \int 28x \, u^{1/3} du.$

$u = x+3 \quad du = dx \quad = \int 28(u-3) u^{1/3} du.$

$= \int 28 u^{4/3} - 28 \cdot 3 u^{1/3} du.$

$= \frac{28 \cdot 3}{7} u^{7/3} - 28 \cdot \frac{3 \cdot 3}{4} u^{4/3} + C.$

$= 12 u^{7/3} - 63 u^{4/3} + C.$

$= 12(x+3)^{7/3} - 63(x+3)^{4/3} + C.$

Regla de la Sustitución para Integrales Definidas

por ejemplo.

Integre respecto a u.

$$\int_0^{1/2} \cos(\pi x)\, \underbrace{dx}_{du/\pi} = \int_0^{\pi/2} \cos u \, \frac{du}{\pi} = \frac{1}{\pi} \sin u \Big]_0^{\pi/2}$$

$u = \pi x$ $\quad u(1/2) = \pi/2$

$du = \pi\, dx$ $\quad u(0) = 0$

$$= \frac{1}{\pi}\left(\overset{1}{\cancel{\sin \frac{\pi}{2}}} - \overset{0}{\cancel{\sin 0}}\right) = \frac{1}{\pi}.$$

Regla de la Sustitución

$$\boxed{\int_a^b f(\underbrace{g(x)}_{u})\, \underbrace{g'(x)\,dx}_{du} = \int_{u(a)}^{u(b)} f(u)\, du.}$$

Ejercicio 3: Evalúe

a. $$\int_{-4}^{0} \frac{1}{3x-2}\, dx = \int_{-14}^{-2} \frac{1}{3}\frac{du}{u} = \frac{1}{3}\ln|u|\Big]_{-14}^{-2} = \frac{1}{3}\ln 2 - \frac{1}{3}\ln 14.$$

$u = 3x-2$, $\quad du = 3\,dx$ $\quad u(0) = -2$ $\quad u(-4) = -14$

b. $$\int_0^1 \frac{8\sin^{-1} t}{\sqrt{1-t^2}}\, dt = \int_0^{\pi/2} 8u\, du. = 4u^2\Big]_0^{\pi/2} = 4\frac{\pi^2}{4} = \pi^2.$$

$u = \sin^{-1} t$ $\quad u(1) = \sin^{-1}(1) = \pi/2$

$du = \frac{1}{\sqrt{1-t^2}}\, dt.$ $\quad u(0) = \sin^{-1}(0) = 0$

Capítulo 6

7.1 Integración por partes (Pág 39)

La regla de sustitución no se puede usar para integrar

$\int \ln x \, dx \qquad \int x^n e^x \, dx \qquad \int x^n \sin x \, dx \qquad \int \sin^{-1} x \, dx$

Regla del Producto para la Diferenciación.

$(fg)' = f'g + fg'$

$(fg)' - f'g = fg'$ Integre respecto a x.

$\int fg' \, dx = \int (fg)' \, dx - \int f'g \, dx$

$$\boxed{\int fg' \, dx = fg - \int f'g \, dx}$$ Regla del Producto para Integración

Objetivo: Pasar de una integral compleja $\int fg' \, dx$ a una más sencilla $\int f'g \, dx$

Integración por partes. IPP.

$\int f(x) \overbrace{g'(x) \, dx}^{dv} = uv - \int v \, du \Rightarrow \boxed{\int u \, dv = uv - \int v \, du.}$

sea $u = f(x) \qquad dv = g'(x) \, dx$

$du = f'(x) \, dx \qquad v = g(x)$

ILATE.

derivar $\longrightarrow$

I = inversas $\sin^{-1} x$, $\tan^{-1} x$.

L = logaritmos. $\ln x$, $\log_a x$.

A = algebraicas, potencias x^2, x^n, x^r

T = trigonométricas $\sin x$, $\cos x$. E = exponenciales e^x

Ejercicio 1: Integre $\int x e^x \, dx$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$u = e^x \qquad dv = x^1 dx$

$du = e^x dx \qquad v = \frac{x^2}{2}$

No es el adecuado $\int v\,du$

$$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = \underbrace{xe^x}_{u \cdot v} - \underbrace{\int e^x dx}_{v \cdot du} = \boxed{xe^x - e^x + C.}$$

Compruebe: $(xe^x - e^x)' = e^x + xe^x - \cancel{e^x} = xe^x$

Ejercicio 2: Integre (Pág 40.)

Int. $2x^3$

Der. $12x$.

a $\int \underbrace{6x^2}_{dv} \underbrace{\ln x}_{u} \, dx = \overset{u}{(\ln x)} \overset{v}{2x^3} - \int \overset{v}{2x^3} \overset{dy}{\frac{dx}{x}}$

$u = \ln x \qquad dv = 6x^2 dx$

$du = \frac{dx}{x} \qquad v = 2x^3$

$= 2x^3 \ln x - \int 2x^2 dx$

$= 2x^3 \ln x - \frac{2}{3} x^3 + C.$

En Resumen $\int u\,dv = uv - \int v\,du$ más fácil

b. $\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv.} = x \ln x - \int x \frac{dx}{x} = x \ln x - \int dx$

$= x \ln x - x + C.$

$u = \ln x \qquad dv = dx$

$du = \frac{dx}{x} \qquad v = x$

c. $\int \underbrace{\sin^{-1}x}_{u} \, \underbrace{dx}_{dv.}$

b2. $\int (1+x)\underbrace{(x+\frac{1}{2}x^2)^4}\, dx = \int u^4 du = \frac{1}{5}u^5 + C.$

$u' = 1+x \qquad u = x+\frac{1}{2}x^2$

c. $\int \underbrace{\sin^{-1}x}_{u} \, \underbrace{dx}_{dv.} = x\sin^{-1}x - \int \overbrace{\frac{x}{\sqrt{1-x^2}}\, dx}^{-du/2.}$

$u = 1-x^2$

$du = -2x\,dx$

$\frac{du}{-2} = x\,dx$

$\frac{1}{\sqrt{u}} = u^{-1/2}.$

$du = \frac{1}{\sqrt{1-x^2}}, v = x \qquad = x\sin^{-1}x + \frac{1}{2}\int u^{-1/2}du$

$= x\sin^{-1}x + \frac{1}{2}\cdot 2u^{1/2} + C$

$= x\sin^{-1}x + \sqrt{1-x^2} + C.$

I L A T E

b

d. $\int x^2\cos x\, dx = x^2\sin x - \int 2x\sin x\, dx$

utilice IPP otra vez.

$u = x^2 \qquad dv = \cos x\, dx$

$du = 2x\,dx \qquad v = \sin x$

$\int 2x\sin x\, dx = -2x\cos x + \int 2\cos x\, dx = -2x\cos x + 2\sin x + C_1$

$u = 2x \qquad dv = \sin x\, dx$

$du = 2 \qquad v = -\cos x$

$\int x^2\cos x\, dx = x^2\sin x + 2x\cos x - 2\sin x - C$

IPP

Integrales Definidas

$$\int_a^b u\,dv = vu\Big]_a^b - \int_a^b v\,du$$

Pág 41.

Ejercicio 3: Evalúe.

derive.

ILATE

a. $\int_1^e \sqrt{x}\,\ln x^9\,dx = \int_1^e 9x^{1/2}\cdot \underbrace{\ln x}_{x\ln x - x}\,dx$

$\ln(x^9) = 9\ln x$

$u = \ln x \qquad dv = 9x^{1/2}\,dx$

$du = x^{-1}\,dx \qquad v = 6\cdot x^{3/2}$

$uv - \int v\,du = 6x^{3/2}\ln x\Big]_1^e - \int_1^e 6x^{3/2}x^{-1}\,dx$

$= 6e^{3/2}\cancel{\ln e}^{\,1} - 6\cdot 1^{3/2}\cancel{\ln 1}^{\,0} - \int_1^e 6x^{1/2}\,dx$

$= 6e^{3/2} - 6\cdot\underbrace{\frac{2}{3}}_{4}x^{3/2}\Big]_1^e = 6e^{3/2} - 4e^{3/2} + 4\cdot 1^{3/2}.$

$= \boxed{2e^{3/2} + 4}$

Integre $\int e^x\cos x\,dx$ Pág. 44

$u = \cos x \qquad dv = e^x\,dx$

$du = -\sin x\,dx \qquad v = e^x$

$\int e^x\cos x\,dx = e^x\cos x + \int e^x\sin x\,dx.$

$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx$

$u = \sin x \qquad dv = e^x\,dx$

$du = \cos x\,dx \qquad v = e^x$

$\int e^x\cos x\,dx = e^x\cos x + e^x\sin x - \int e^x\cos x\,dx$

$2\int e^x\cos x\,dx = e^x\cos x + e^x\sin x$

$\int e^x\cos x\,dx = \boxed{\frac{1}{2}(e^x\cos x + e^x\sin x) + C.}$

Capítulo 7

7.2 Integrales Trigonométricas.

Algunas integrales requieren el uso de identidades trigonométricas.

$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x$

$\div \cos^2 x \quad \tan^2 x + 1 = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$\div \sin^2 x \quad 1 + \cot^2 x = \csc^2 x \Rightarrow \cot^2 x = \csc^2 x - 1$

I. Forma $\int \sin^n x \cos^m x \, dx$.

II. Forma $\int \tan^n x \sec^m x \, dx$

III. Forma $\int \cot^n x \sec^m x \, dx$

Ia) Potencias Impares de Seno y Coseno.

$$\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx$$

$u = \cos x$
$du = -\sin x \, dx$

$$\sin^4 x = (\sin^2 x)^2 = (1 - \cos^2 x)^2$$

Sustitución.
$u = \cos x$
$du = -\sin x \, dx$

$$= \int (1 - \cos^2 x)^2 \sin x \, dx$$

$$= \int (1 - u^2)^2 (-du)$$

$\cos^3 x = \cos x^3$

$$= -\int (1 - 2u^2 + u^4) \, du.$$

$$= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C.$$

$$= \boxed{-\cos x + \frac{2}{3}(\cos x)^3 - \frac{1}{5}\cos^5 x + C.}$$

Ejercicio 1: Evalúe Pág. 46.

a. $\int \cos^3 x \sin^6 x \, dx$

$\int \cos^2 x \sin^6 x \cos x \, dx$ ó $\int \cos^3 x \sin^5 x \sin x \, dx$

$\cos^2 x = 1 - \sin^2 x$ $\quad u = \sin x \quad du = \cos x \, dx$

$$\int \cos^2 x \sin^6 x \cos x \, dx = \int (1-\sin^2 x) \sin^6 x \cos x \, dx$$
$$= \int (1-u^2) u^6 \, du.$$
$$= \int (u^6 - u^8) \, du.$$
$$= \frac{1}{7} u^7 - \frac{1}{9} u^9 + C.$$
$$= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C.$$

b. $\int \cos^3 x \sin^3 x \, dx$

$\int \cos^2 x \sin^3 x \cos x \, dx$ ó $\int \cos^3 x \sin^2 x \sin x \, dx$

$\cos^2 x = 1 - \sin^2 x$ $\quad \sin^2 x = 1 - \cos^2 x$

$$\int (1-\sin^2 x) \sin^3 x \cos x \, dx$$

$u = \sin x \quad du = \cos x \, dx$

$$\int (1-u^2) u^3 \, du.$$
$$\int (u^3 - u^5) \, du$$
$$\frac{1}{4} u^4 - \frac{1}{6} u^6 + C.$$
$$\frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C.$$

$$\int \cos^3 x (1-\cos^2 x) \sin x \, dx$$

$u = \cos x \quad du = -\sin x \, dx$

$$-\int u^3 (1-u^2) \, du.$$
$$\int -u^3 + u^5 \, du$$
$$-\frac{1}{4} u^4 + \frac{1}{6} u^6 + C.$$
$$-\frac{1}{4} \cos^4 x + \frac{1}{6} \cos^6 x + C.$$

Siempre y cuando haya un término impar de seno y coseno.

(b) Potencias Pares de seno y Coseno.

$$\int \cos^2 x \, dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \boxed{\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C.}$$

Utilice las identidades de doble ángulo.

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\cos(x+x) = -\sin x \sin x + \cos x \cos x$$

$$\cos 2x = -\sin^2 x + \cos^2 x.$$

$$1 = \sin^2 x + \cos^2 x$$

$$1+\cos(2x) = 2\cos^2 x.$$

Ejercicio 2: Evalúe. $(1-\cos 2x)(1+\cos 2x)$

a. $\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)\,dx$

Vuelva a utilizarla $= \frac{1}{4}\int (1-\cos^2 2x)\,dx$

$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$ $= \frac{1}{4}\int \left(\overbrace{1-\frac{1}{2}}^{1/2} + \frac{1}{2}\cos(4x)\right)dx$

$$= \frac{1}{4}\left(\frac{x}{2} + \frac{1}{2\cdot 4}\sin(4x) + C\right)$$

$$= \frac{x}{8} + \frac{1}{32}\sin(4x) + \frac{C}{4}$$

b. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 x\,dx = 2\int_0^{\pi/4} \sin^2 x\,dx$

PAR.

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$= \int_0^{\pi/4} (1-\cos 2x)\,dx$$

$$= x - \frac{1}{2}\sin 2x \Big]_0^{\pi/4}$$

$$= \frac{\pi}{4} - \frac{1}{2}\cancelto{1}{\sin\frac{\pi}{2}} - 0 + \frac{1}{2}\cancelto{0}{\sin 0}$$

$$= \boxed{\frac{\pi}{4} - \frac{1}{2}.}$$

II. Integrales de la forma $\int \sec^m x \tan^n x\,dx$

$\frac{d}{dx}\tan x = \sec^2 x\,dx$ $\qquad$ $\frac{d}{dx}\sec x = \sec x \tan x$

$\sec^2 x = \tan^2 x + 1$ $\qquad$ $\tan^2 x = \sec^2 x - 1$

Aparte $\sec^2 x$ ó $\sec x \tan x$

Utilice las identidades: $\sec^2 x = \tan^2 x + 1$

ó $\tan^2 x = \sec^2 x - 1$

Ejercicio 3: Evalúe las sigs. integrales.

a. $\int \tan^6 x \cdot \sec^4 x\,dx = \int \tan^6 x \sec^2 x (\sec^2 x\,dx)$

$\int \tan^6 x \sec^2 x \sec^2 x\,dx$ ✓ ; $\int \tan^5 x \sec^5 x \sec x \tan x\,dx$ ✗

Use $\sec^2 x = \tan^2 x + 1$

$$\int \tan^6 x\,(\tan^2 x + 1)\,(\sec^2 x\, dx)$$

$u = \tan x \qquad du = \sec^2 x\, dx$

$$\int u^6 (u^2+1)\, du = \int u^8 + u^6\, du$$
$$= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C.$$
$$= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C.$$

b. $\int \tan^5 x \sec^6 x\, dx$

$\int \tan^5 x\, \underbrace{\sec^4 x}\, (\sec^2 x\, dx)$ ó $\int \tan^4 x \sec^5 x (\sec x \tan x\, dx)$

$(\sec^2 x)^2 = (1 + \tan^2 x)^2 \qquad (\tan^2 x)^2 = (\sec^2 x - 1)^2.$

$$\int \underbrace{\tan^5 x}_{u = \tan x}\, (1 + \tan^2 x)^2\, \underbrace{(\sec^2 x\, dx)}_{du = \sec^2 x\, dx}$$

$$\int u^5 (1+u^2)^2\, du = \int u^5 (1 + 2u^2 + u^4)\, du$$
$$= \int (u^5 + 2u^7 + u^9)\, du.$$
$$= \frac{1}{6}u^6 + \frac{2}{8}u^8 + \frac{1}{10}u^{10} + C.$$
$$= \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^8 x + \frac{1}{10}\tan^{10} x + C.$$

$$\therefore \int \tan^3 x \sec^3 x\, dx = \int \tan^2 x \sec^2 x\, (\sec x \tan x\, dx)$$

$\tan^2 x = \sec^2 x - 1 \qquad = \int (\sec^2 x - 1)\sec^2 x\, (\sec x \tan x\, dx)$

$u = \sec x \qquad = \int (u^2 - 1)\, u^2\, du.$

$du = \sec x \tan x\, dx$

$$= \int (u^4 - u^2)\,du.$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C.$$

$$= \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.}$$

$-\ln a = \ln(a^{-1})$

Casos especiales $\int \sec^n x\,dx$ $\int \tan^n x\,dx$

a. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\ln|u| + C.$

$u = \cos x$ $du = -\sin x\,dx$ $= -\ln|\cos x| + C.$

$= \ln|\sec x| + C.$

b. $\int \sec x\,dx = \int \sec x \frac{\overbrace{(\sec x + \tan x)}^{1}}{\tan x + \sec x}\,dx$

$\int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}\,dx$

$u = \tan x + \sec x$ $du = (\sec^2 x + \sec x \tan x)\,dx$

$= \int \frac{du}{u} = \ln|u| + C$

$= \boxed{\ln|\tan x + \sec x| + C.}$

c. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \boxed{\tan x - x + C.}$

d. $\int \sec^2 x\,dx = \boxed{\tan x + C.}$

e. $\int \tan^3 x\,dx = \int \tan x(\tan^2 x)\,dx$ $\int f - g\,dx$

$= \int \tan x(\sec^2 x - 1)\,dx$ $\int f\,dx - \int g\,dx$

$$= \int \underbrace{\tan x}_{u} \underbrace{\sec^2 x \, dx}_{du} - \int \tan x \, dx \qquad \frac{\sin}{\cos}$$

$w = \cos x$

$$= \frac{1}{2} u^2 + \ln|w| + C$$

$$= \frac{1}{2} \tan^2 x + \ln|\cos x| + C.$$

Integral de $\sec^3 x$

$$\int \sec x \sec^2 x \, dx = \sec x \tan x - \int \underbrace{\tan^2 x}_{v} \sec x \, dx$$

IPP.

$u = \sec x \qquad dv = \sec^2 x \, dx \qquad \tan^2 x = \sec^2 x - 1$

$du = \sec x \tan x \, dx \qquad v = \tan x$

$$\int \sec^3 x \, dx = \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$\underline{\underline{\int \sec^3 x \, dx}} = \sec x \tan x - \underline{\underline{\int \sec^3 x \, dx}} + \int \sec x \, dx$$

Ciclica.

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| + C.$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

$$= \frac{1}{2} \text{derivada}(\sec) + \frac{1}{2} \text{integral}(\sec)$$

Capítulo 8

Lunes 26 de agosto Simulacro Parcial
Lunes 3 de septiembre Parcial 1, capítulos 5 y 7.
Libro Págs. 9-70.

Integrales Trigonométricas.

Forma $\int \operatorname{sen}^n x \cos^m x \, dx$

$\int \sec^n x \tan^m x \, dx$

$\int \csc^n x \cot^m x \, dx$

$D(\csc x) = -\csc x \cot x$ aparte $\csc x \cot x$. Pág 50.

$\cot^2 x = \csc^2 x - 1$ $u = \csc x$ $du = -\csc x \cot x$

$D(\cot x) = -\csc^2 x$

$\csc^2 x = \cot^2 x + 1$ $u = \cot x$, $du = -\csc^2 x$.

$\sec^2 x = \tan^2 x + 1$

Ejercicio 4: Integre. $\cot x \csc x$ ó $\csc^2 x$

a $\int \cot^2 x \csc^4 x \, dx = \int \cot^2 x \csc^2 x \csc^2 x \, dx$.

$\csc^2 x = \cot^2 x + 1$ $= \int \cot^2 x (\cot^2 x + 1) \csc^2 x \, dx$

$u = \cot x$, $du = -\csc^2 x \, dx$ $= -\int u^2 (u^2 + 1) \, du$

$= \int (-u^4 - u^2) \, du$.

$= -\frac{1}{5} u^5 - \frac{1}{3} u^3 + C$.

$= -\frac{1}{5} \cot^5 x - \frac{1}{3} \cot^3 x + C$.

b. $\int \csc^3 x \cot^3 x \, dx = \int \csc^2 x \cot^2 x (\csc x \cot x \, dx)$

$\cot^2 x = \csc^2 x - 1 \quad = \int \csc^2 x (\csc^2 x - 1)(\csc x \cot x \, dx)$

$u = \csc x, \; du = -\csc x \cot x \, dx \quad = -\int u^2 (u^2 - 1) \, du.$

$= -\int (u^4 - u^2) \, du.$

$= -\left(\frac{u^5}{5} - \frac{u^3}{3} + C \right)$

$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$

$\boxed{= -\frac{\csc^5 x}{5} + \frac{\csc^3 x}{3} - C.}$

c. $\int \csc x \, dx = \int \csc x \frac{(\csc x + \cot x)}{\cot x + \csc x} \, dx$

$= \int \frac{\csc^2 x + \csc x \cot x}{\cot x + \csc x} \, dx$

$u = \cot x + \csc x \qquad du = -(\csc^2 x + \csc x \cot x) \, dx$

$= -\int \frac{du}{u} = -\ln|u| + C.$

$= -\ln|\cot x + \csc x| + C.$

$\int \sec^3 x \, dx = \frac{1}{2} d(\sec x) + \frac{1}{2} \int \sec x \, dx$

$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.$

$\int \csc^3 x \, dx = \frac{1}{2} d(\csc x) + \frac{1}{2} \int \csc x \, dx$

$= -\frac{1}{2} \csc x \cot x - \frac{1}{2} \ln|\cot x + \csc x| + C.$

Integrales de la forma $\int \sin(mx)\cos(px)\,dx$

utilice la identidad trigonométrica apropiada.

$$\sin(mx)\cos(px) = \frac{1}{2}[\sin(m-p)x + \sin(m+p)x].$$

$$\sin(mx)\sin(px) = \frac{1}{2}[\cos(m-p)x - \cos(m+p)x]$$

$$\cos(mx)\cos(px) = \frac{1}{2}[\underbrace{\cos(m-p)x + \cos(m+p)x}_{\text{se pueden integrar.}}].$$

Ejercicio 5: Evalúe. pág 51.

a. $$\int_0^{\pi/4} \sin 8x \cos 4x\,dx = \frac{1}{2}\int_0^{\pi/4} (\sin 4x + \sin 12x)\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{4}\cos 4x\Big]_{\pi/4}^{0} + \frac{1}{2}\cdot\frac{1}{12}\cos(12x)\Big]_{\pi/4}^{0}$$

$$= \frac{1}{8}(\overset{1}{\cos 0} - \overset{+1}{\cos \pi}) + \frac{1}{24}(\overset{1}{\cos 0} - \overset{+1}{\cos 3\pi})$$

$$= \frac{2}{8} + \frac{2}{24} = \frac{3}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}.$$

b. $$\int_{-\pi}^{\pi} \overset{\text{PAR}}{\cos mx}\ \overset{\text{PAR.}}{\cos nx}\,dx = 2\int_0^{\pi} \cos mx \cos nx\,dx$$

$$= \int_0^{\pi} \cos(m-n)x + \cos(m+n)x\,dx$$

m, n son enteros diferentes $$= \frac{1}{m-n}\sin(m-n)x + \frac{\sin(m+n)x}{m+n}\Big]_0^{\pi}$$

$$= \frac{1}{m-n}(\overset{0}{\sin(m-n)\pi} - \overset{0}{\sin 0}) + \frac{1}{m+n}(\overset{0}{\sin(m+n)\pi} - \overset{0}{\sin 0})$$

$\sin K\pi = 0$ multiplos de $180°$ son iguales a cero

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0.$$

si $m=n$ $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos mx \, dx = 2\int_{0}^{\pi} \cos^2 mx \, dx$

$$= \int_0^{\pi} 1 + \cos(2mx) \, dx$$

$$x + \frac{\sin(2mx)}{2m}\Big]_0^{\pi} = \pi + \frac{\sin(2m\pi)}{2m} - 0 - 0$$

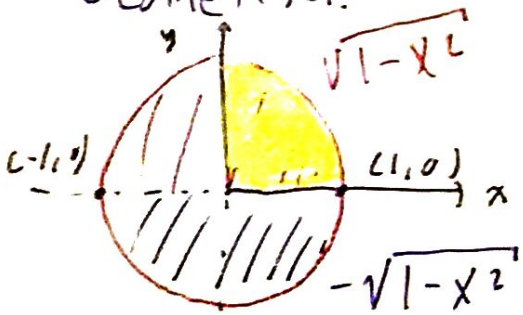
$360°$

$$\int_{-\pi}^{\pi} \cos^2 mx \, dx = \pi.$$

Pág 51. $\displaystyle\int_{-\pi}^{\pi} \underbrace{\sin 8x}_{\text{impar}} \overbrace{\cos 4x}^{\text{impar.}} \, dx = 0$

impar * par

Área de un Circulo de Unitario sin utilizar Geometría.



$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2.$$

$$y = \pm\sqrt{1-x^2} \quad \text{en } [-1, 1]$$

$$A = \int_{-1}^{1} \sqrt{1-x^2} \, dx - \int_{-1}^{1} -\sqrt{1-x^2} \, dx$$

$$A = 2\int_{-1}^{1} \sqrt{1-x^2} \, dx = 4\int_0^1 \underline{\sqrt{1-x^2}} \, dx$$

$u = 1 - x^2 \qquad du = -2x\,dx$ } No Sustitución

$u = \sqrt{1-x^2} \qquad dv = dx$

$du = \dfrac{-x}{\sqrt{1-x^2}} \qquad v = x$ } No IPP

$$A = 4\int_0^1 \sqrt{1-x^2}\,dx \qquad (x = \sin\theta) \qquad dx = \cos\theta\,d\theta.$$

$$1 - x^2 = 1 - \sin^2\theta = \cos^2\theta.$$

$$A = 4\int_0^{\pi/2} \underbrace{\sqrt{1-\sin^2\theta}}_{\cos\theta.}\cos\theta\,d\theta = \boxed{4\int_0^{\pi/2} \cos^2\theta\,d\theta.}$$

$\sin\theta = x \qquad \sin\theta = 1 \Rightarrow \theta = \sin^{-1}(1) = \pi/2.$

$\sin\theta = 0 \qquad \theta = 0$

$$A = 4\int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{4}{2}\int_0^{\pi/2}(1 + \cos 2\theta)\,d\theta.$$

$$A = 2\left(\theta + \frac{1}{2}\sin 2\theta\Big]_0^{\pi/2}\right) = 2\left(\frac{\pi}{2} - \frac{1}{2}\cancel{\sin\pi}^{\,0} - 0 - 0\right)$$

$A = \dfrac{2\pi}{2} = \pi.$ ✓ consistente con la geometría $\qquad A = \pi(1)^2.$

7.3 Sustitución Trigonométrica o Sustitución Inversa.

$$\int f(x)\,dx = \int \overbrace{f(u(\theta))}\,u'(\theta)\,d\theta.$$

$x = u(\theta) \qquad dx = u'(\theta)\,d\theta$

$u = \sin\theta,\ \tan\theta,\ \sec\theta.$

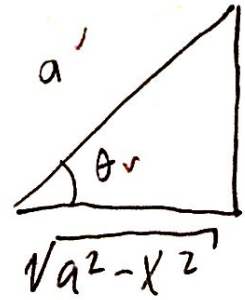
utilice identidades o triángulos para simp. $f(u(\theta))$.

Forma $\sqrt{a^2-x^2}$

Hipotenusa: a

Cateto O: $x = a\sin\theta.$

Cateto A: $\sqrt{a^2-x^2} = a\cdot\cos\theta.$



$\sin\theta = \frac{C.O}{H}$

$\sin\theta = \frac{x}{a.}$

$\cos\theta = \frac{\sqrt{a^2-x^2}}{a.}$

Sustitución $x = a\sin\theta.$

Diferencial $dx = a\cos\theta\, d\theta.$

Sustituya. $\sqrt{a^2-x^2} = a\cdot\cos\theta.$

$\sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a\cdot\cos\theta.$

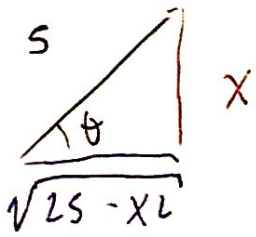
Ejercicio 1: Evalúe.

a. $\int \frac{x}{\sqrt{25-x^2}}\,dx = \int \frac{5\sin\theta}{5\cos\theta}\, 5\cos\theta\, d\theta. = 5\int \sin\theta\, d\theta.$

$= -5\cos\theta + C.$

$= -\sqrt{25-x^2} + C.$

Método 1: Sustitución. Trig.



$x = 5\sin\theta.$ $dx = 5\cos\theta\, d\theta$

$\sqrt{25-x^2} = 5\cos\theta.$

Método 2: Regla de la sustitución.

$u = 25 - x^2$

$du = -2x\, dx$

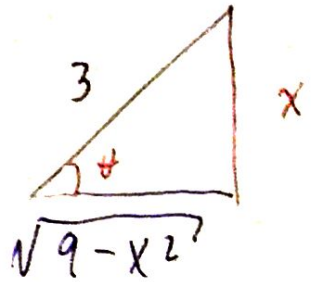
$\int \frac{-du/2}{u^{1/2}} = -\frac{1}{2}\int u^{-1/2}\, du.$

$= -\frac{2}{2}u^{1/2} + C.$

$= -\sqrt{25-x^2} + C.$

7.

a. $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{27\sin^3\theta \cdot 3\cancel{\cos\theta}\,d\theta}{3\cancel{\cos\theta}} = \int 27\sin^3\theta\,d\theta.$



$\sin\theta = \frac{x}{3} \Rightarrow x = 3\sin\theta, \quad dx = 3\cos\theta\,d\theta$

$3\cos\theta = \sqrt{9-x^2}$

$\displaystyle\int 27\sin^2\theta\,\sin\theta\,d\theta. = 27\int (1-\cos^2\theta)\sin\theta\,d\theta.$

$1-\cos^2\theta = \sin^2\theta, \qquad \boxed{u = \cos\theta}, \quad du = -\sin\theta\,d\theta.$

$\displaystyle = -27\int 1-u^2\,du$

$\displaystyle = -27u + \frac{27}{3}u^3 + C$

u

$\theta \quad \cos\theta = \frac{\sqrt{9-x^2}}{3}$

x

$= -27\cos\theta + 9\cos^3\theta + C.$

$= -9\sqrt{9-x^2} + \frac{1}{3}(9-x^2)^{3/2} + C.$

b. $\displaystyle\int \frac{u}{\sqrt{4-u^2}}\,du. = -\frac{1}{2}\int w^{-1/2}\,dw = -\frac{2}{2}\,w^{1/2} + C$

$= -\sqrt{4-u^2} + C.$

sólo sustitución $\quad w = 4-u^2$

$dw = -2u\,du$

PRACTIQUE $\quad u = 2\sin\theta.$

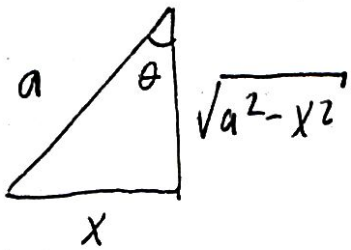
Capítulo 9

Sustitución Trigonométrica.

Forma $\sqrt{a^2-x^2}$

$H = a$

$C.O. = x$



$\sin\theta = \dfrac{x}{a}$

$\cos\theta = \dfrac{\sqrt{a^2-x^2}}{a}$

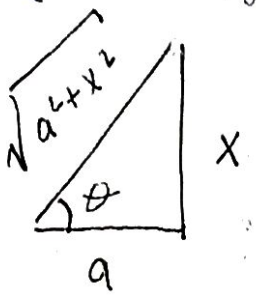
$x = a\cdot\sin\theta \qquad dx = a\cdot\cos\theta\, d\theta \qquad \sqrt{a^2-x^2} = a\cdot\cos\theta.$

Forma $\sqrt{a^2+x^2}$ Pág. 58.

$H: \sqrt{a^2+x^2}$

$C.O. = x$

$C.A. = a.$



$\tan\theta = \dfrac{x}{a} \Rightarrow x = a\cdot\tan\theta.$

$dx = a\cdot\sec^2\theta\, d\theta.$

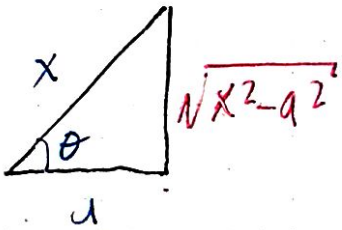
$a\cdot\sec\theta = \sqrt{a^2+x^2} \qquad \sec\theta = \dfrac{H}{C.A.}$

Forma $\sqrt{x^2-a^2}$

$H = x$

$C.O. =$

$C.A. = a$



$\dfrac{x}{a} = \sec\theta.$

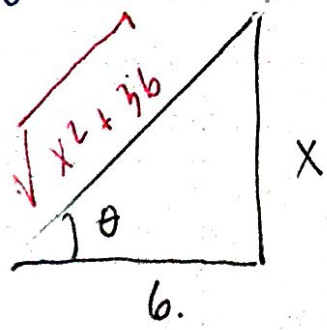
$x = a\cdot\sec\theta.$

$dx = a\cdot\sec\theta\tan\theta\, d\theta.$

$\sqrt{x^2-a^2} = a\cdot\tan\theta.$

Ejercicio 2 y 3.

2. $\displaystyle\int \frac{1}{x^2+36}\,dx. = \int \frac{6\sec^2\theta\, d\theta}{36\tan^2\theta+36} = \frac{6}{36}\int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta.$



$\dfrac{x}{6} = \tan\theta. \qquad x = 6\cdot\tan\theta.$

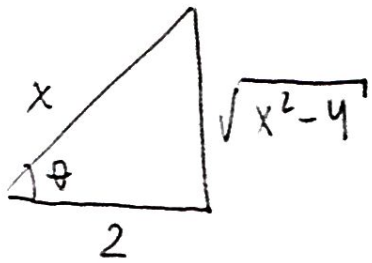
$dx = 6\sec^2\theta\, d\theta.$

$\dfrac{\sqrt{x^2+36}}{6} = \sec\theta \Rightarrow x^2+36 = 36\sec^2\theta.$

$$\frac{1}{6}\int d\theta = \frac{1}{6}\theta + C. = \boxed{\frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right) + C.}$$

$$\tan^{-1}\left(\frac{x}{6}\right) = \theta.$$

3a. $$\int \frac{(x^2-4)^{3/2}}{x^6}\,dx = \int \frac{2^3\tan^3\theta}{2^6\cdot\sec^6\theta}\; 2\cdot\sec\theta\tan\theta\,d\theta.$$



$$\frac{2}{x} = \cos\theta. \rightarrow x = \frac{2}{\cos\theta} = 2\sec\theta.$$

$$dx = 2\sec\theta\tan\theta\,d\theta.$$

$$\sqrt{x^2-4} = 2\tan\theta.$$

$$\left[(x^2-4)^{1/2}\right]^3 = 8\tan^3\theta.$$

$$\frac{1}{2^2}\int \frac{\tan^4\theta}{\sec^5\theta}\,d\theta. = \frac{1}{4}\int \frac{\sin^4\theta}{\cos^4\theta}\cos^5\theta\,d\theta.$$

$u = \sin\theta \quad du = \cos\theta\,d\theta$

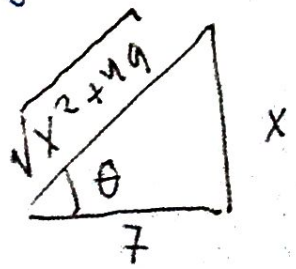
$\frac{1}{4}\int u^4\,du.$

$$= \frac{1}{4}\int \underbrace{\sin^4\theta}_{u^4}\,\underbrace{\cos\theta\,d\theta}_{du}. = \boxed{\frac{1}{4}\cdot\frac{1}{5}\sin^5\theta + C.}$$

Regrese a la variable x $\quad \sin\theta = \frac{\sqrt{x^2-4}}{x}, \quad \sin^5\theta = \frac{(x^2-4)^{5/2}}{x^5}$

$$\int \frac{(x^2-4)^{3/2}}{x^6}\,dx = \boxed{\frac{1}{20}\,\frac{(x^2-4)^{5/2}}{x^5} + C.}$$

2a $$\int \frac{49}{x^2\sqrt{x^2+49}}\,dx = \int \frac{49\;\cancel{7}\sec^2\theta}{49\tan^2\theta\cdot\cancel{7}\sec\theta}\,d\theta. = \int \frac{\sec\theta}{\underbrace{\tan^2\theta}_{\sec^2\theta-1}}\,d\theta.$$



$$\frac{x}{7} = \tan\theta \qquad x = 7\tan\theta.$$

$$dx = 7\sec^2\theta\,d\theta.$$

$$\frac{H}{C.A.} = \sec\theta. \qquad \sqrt{x^2+49} = 7\sec\theta.$$

3. $\int \sec^m\theta \tan\theta^n x \, dx$ No está disponible.

$$\int \frac{\sec\theta}{\tan^2\theta}\, d\theta. = \int \frac{1}{\cos\theta.} \frac{\cos^2\theta}{\sin^2\theta} + \int \frac{\cos\theta}{\sin\theta} \frac{1}{\sin\theta}\, d\theta.$$

$$= \int \cot\theta \csc\theta \, d\theta$$

$$= -\csc\theta + C.$$

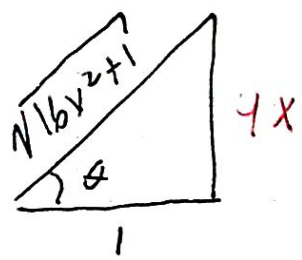
$$= -\frac{\sqrt{x^2+49}}{x} + C.$$

$$\csc\theta = \frac{H}{C.O.} = \frac{\sqrt{x^2+49}}{x}$$

$$\int \frac{\cos\theta}{\sin^2\theta.}\, d\theta = \int \frac{du}{u^2} = \int u^{-2} du = \frac{u^{-1}}{-1} + C = \frac{-1}{u} = \frac{-1}{\sin\theta} + C.$$

$u = \sin\theta$, $du = \cos\theta \, d\theta.$

b. $$\int \frac{1}{x\sqrt{16x^2+1}}\, dx = \int \frac{0.25 \sec^2\theta \, d\theta}{0.25 \tan\theta . \sec\theta} = \int \frac{\sec\theta}{\tan\theta}\, d\theta.$$



$\tan\theta = 4x \Rightarrow x = \frac{1}{4}\tan\theta \, d\theta.$

$dx = \frac{1}{4}\sec^2\theta \, d\theta.$

$$\frac{\sqrt{16x^2+1}}{1} = \sec\theta.$$

Reescriba la integral.

$$\int \frac{\sec\theta}{\tan\theta}\, d\theta = \int \frac{1}{\cos\theta} \frac{\cos\theta}{\sin\theta}\, d\theta = \int \frac{d\theta}{\sin\theta} = \int \csc\theta \, d\theta.$$

$$= -\ln|\csc\theta + \cot\theta| + C.$$

$$\csc\theta = \frac{H}{C.O.} = \frac{\sqrt{16x^2+1}}{4x} \qquad \cot\theta = \frac{1}{4x}.$$

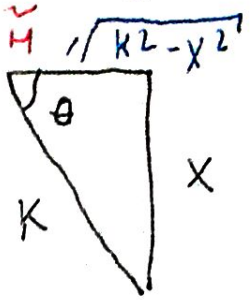
$$= -\ln\left|\frac{\sqrt{16x^2+1}}{4x} + \frac{1}{4x}\right| + C.$$

7.3 Sustitución Trigonométrica

Forma $\sqrt{\underset{H}{K^2} - x^2}$

$H = K$

$C.O = x$

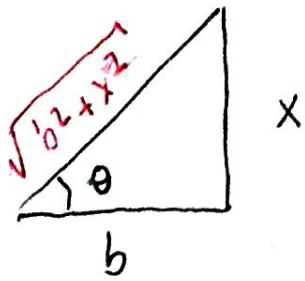


$\frac{C.O}{H} = \sin\theta = \frac{x}{K} \Rightarrow x = K\sin\theta.$

$dx = K\cos\theta\, d\theta.$

$\frac{\sqrt{K^2-x^2}}{K} = \cos\theta \Rightarrow \sqrt{K^2-x^2} = K\cdot\cos\theta.$

Forma $\sqrt{b^2 + x^2}$

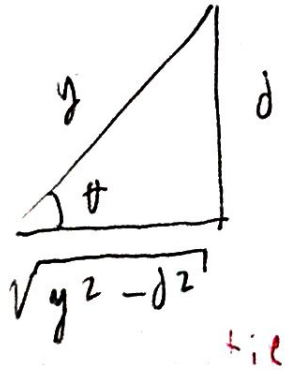


$\frac{x}{b} = \tan\theta. \Rightarrow x = b\cdot\tan\theta.$

$dx = b\cdot\sec^2\theta\, d\theta.$

$\frac{\sqrt{b^2+x^2}}{b} = \sec\theta \Rightarrow \sqrt{b^2+x^2} = b\sec\theta.$

Forma $\sqrt{y^2 - d^2}$



$\operatorname{sen}\theta = \frac{d}{y}$

$y = d\cdot\csc\theta.$

$dy = -d\cdot\csc\theta\cot\theta.$

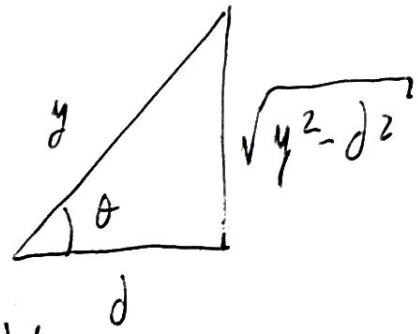
tiene signos negativos

$\frac{y}{d} = \sec\theta.$

$y = d\sec\theta.$

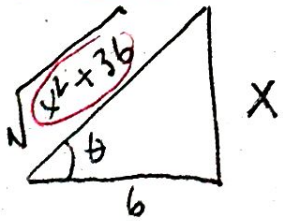
$dy = d\cdot\sec\theta\tan\theta.$

$\sqrt{y^2-d^2} = d\tan\theta.$



Ejercicios 2 y 3 Pág 58 y 59.

2o) $\int \frac{1}{x^2+36}\,dx = \int \frac{6\sec^2\theta\, d\theta}{36\sec^2\theta} = \int \frac{d\theta}{6} = \frac{\theta}{6} + C$



$x = 6\cdot\tan\theta.$

$dx = 6\sec^2\theta\, d\theta.$

$x^2 + 36 = 36\tan^2\theta + 36) = 36\sec^2\theta$

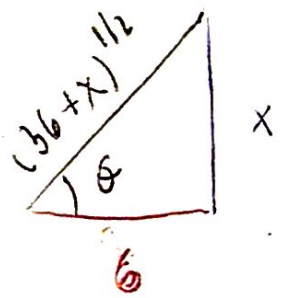
Capítulo 10

Parcial Lunes 2:30 D-504

Corto 4 1 Problema Sustitución Trigonométrica.

Práctica Problemas Integración. Pág 59.

2a) $\displaystyle\int_0^6 \frac{72}{(36+x^2)^{3/2}}\,dx = \int_0^{\pi/4} \frac{72\cdot 6\sec^2\theta\,d\theta}{36^{3/2}\sec^3\theta} = 2\int_0^{\pi/4} \frac{1}{\sec\theta}\,d\theta.$



$x = 6\tan\theta.$ $\qquad \dfrac{x}{6} = \tan\theta$

$dx = 6\sec^2\theta\,d\theta.$ $\qquad \theta = \tan^{-1}\left(\dfrac{x}{6}\right)$

$(36+x)^{1/2} = 6\sec\theta$

$(36+x)^{3/2} = 6^3\sec^3\theta$ $\qquad \theta = \tan^{-1}(1) = \dfrac{\pi}{4}$

$\theta = \tan^{-1}(0) = 0$

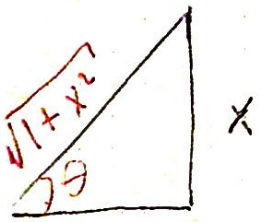
$$2\int_0^{\pi/4} \frac{1}{\sec\theta}\,d\theta = 2\int_0^{\pi/4} \cos\theta\,d\theta. = 2\sin\theta\Big]_0^{\pi/4} = 2\sin\frac{\pi}{4} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

2c) $\displaystyle\int_0^1 \frac{1}{(1+x)^2}\,dx = \int_1^2 u^{-2}\,du = -\frac{1}{u}\Big]_1^2 = -\frac{1}{2} + \frac{1}{1} = \frac{1}{2}.$

$u = 1+x$ $\qquad u(1) = 2.$

$du = dx$ $\qquad u(0) = 1$

2c1) $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \int_0^{\pi/4} \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,d\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int_0^{\pi/4} \frac{1}{\sec^2\theta}\,d\theta.$



$x = \tan\theta.$ $\qquad dx = \sec^2\theta\,d\theta.$

$1 = \tan\theta.$ $\Rightarrow$ $\theta = \pi/4$

$0 = \tan\theta$ $\qquad \theta = 0$

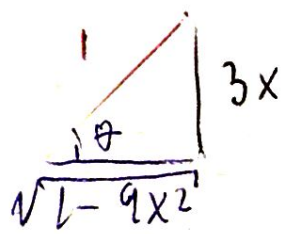
$$\int_0^{\pi/4} \cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/4} (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\Big]_0^{\pi/4}\right)$$

doble ángulo

$$\int_0^{\pi/4} \cos^2\theta \, d\theta = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\overset{1}{\cancel{\sin\frac{\pi}{2}}} - 0\right) = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) = \frac{\pi}{8} + \frac{1}{4}$$

3c) $$\int_0^{1/3} 3^6 x^5 \underbrace{\sqrt{1-9x^2}}_{H} \, dx = \int_0^{\pi/2} \frac{3^6}{3^5} \sin^5\theta \cos\theta \, \frac{1}{3} \cos\theta \, d\theta.$$

Pág 61.



$\sin\theta = 3x \Rightarrow x = \frac{1}{3}\sin\theta.$

$\cos\theta \, d\theta = 3\,dx \qquad dx = \frac{1}{3}\cos\theta \, d\theta.$

$\sqrt{1-9x^2} = \cos\theta. \qquad x^5 = \frac{1}{3^5}\sin^5\theta.$

$\sin\theta = 1$

$\theta = \pi/2$

$\sin\theta = 0$

$\theta = 0$

$$\frac{3^6}{3^6}\int_0^{\pi/2} \sin^5\theta \cdot \cos^2\theta \, d\theta. = \int_0^{\pi/2} \underbrace{\sin^4\theta}_{(\sin^2\theta)^2 = (1-\cos^2\theta)^2} \cos^2\theta \, (\sin\theta \, d\theta)$$

$$\int_0^{\pi/2} (1-\cos^2\theta)^2 \cos^2\theta \, (\underbrace{\sin\theta \, d\theta}_{-du}) = -\int_1^0 (1-u^2)^2 u^2 \, du.$$

$u = \cos\theta \qquad u = \cos\pi/2 = 0$

$du = -\sin\theta \, d\theta \qquad u = \cos 0 = 1$

$$= \int_0^1 (1-2u^2+u^4) u^2 \, du.$$

$$= \int_0^1 u^2 - 2u^4 + u^6 \, du.$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 \Big]_0^1 \leftarrow$$

$$= \frac{1}{3} - \frac{2}{5} + \frac{1}{7}.$$

$$-\int_b^a f \, dx = \int_a^b f \, dx$$

$$\int 3^6 x^5 \sqrt{1-9x^2} \, dx$$

$$\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C.$$

$\cos\theta = \sqrt{1-9x^2}$

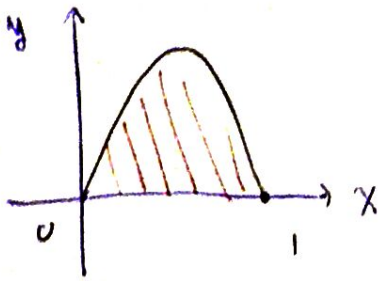
$$\frac{1}{3}\cos^3\theta - \frac{2}{5}\cos^5\theta + \frac{1}{7}\cos^7\theta + C.$$

$$\frac{1}{3}(1-9x^2)^{3/2} - \frac{2}{5}(1-9x^2)^{3/2} + \frac{1}{7}(1-9x^2)^{7/2} + C$$

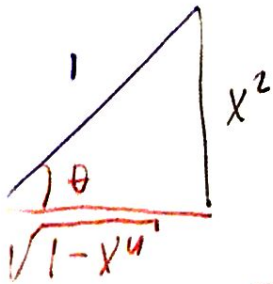
Triángulos "Interesantes"

Ejercicio 4.

Encuentre el área entre $f(x) = 4\pi x\sqrt{1-x^4}$, el eje x, y las rectas $x=0$ & $x=1$.



$$A = \int_0^1 f(x)\,dx = 2\pi \int_0^1 \sqrt{1-x^4}\; 2x\,dx$$



$x^2 = \sin\theta.$

$2x\,dx = \cos\theta\,d\theta.$

$\sqrt{1-x^4} = \cos\theta.$

$\sin\theta = 1$

$\theta = \pi/2$

$\sin\theta = 0$

$\theta = 0$

$$A = 2\pi \int_0^{\pi/2} \cos\theta\cos\theta\,d\theta. = \frac{2\pi}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta.$$

$$A = \pi\left(\theta + \frac{\sin 2\theta}{2}\Big]_0^{\pi/2}\right) = \pi\left(\frac{\pi}{2} + \frac{\overset{0}{\sin\pi}}{2} - 0 - \frac{\overset{0}{\sin 0}}{2}\right)$$

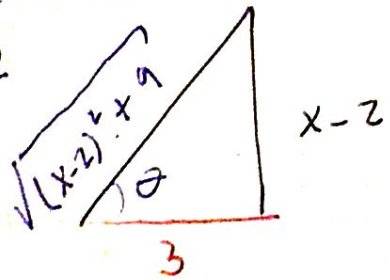
$$A = \pi\cdot\frac{\pi}{2} = \frac{\pi^2}{2}$$

$$\int \frac{(x-2)^3}{(x^2-4x+13)^{1/2}}\,dx. = \int \frac{(x-2)^3}{\sqrt{(x-2)^2+9}}\,dx = \int \frac{3^3\tan^3\theta}{3\sec\theta}\,3\sec^2\theta\,d\theta.$$

Complete al cuadrado $(x^2-4x+4)+13-4 = (x-2)^2+9$

C. $x-2$

C. 3



$3\cdot\tan\theta = x-2$

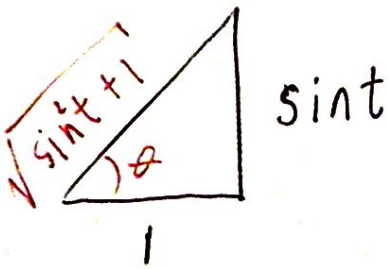
$3\sec^2\theta\,d\theta = dx.$

$\sqrt{(x-2)^2+9} = 3\sec\theta.$

$$3^3\int \tan^3\theta\cdot\sec\theta\,d\theta. = 27\int \tan^2\theta\,(\tan\theta\sec\theta\,d\theta)$$

Problema 2 b) Simulacro.

$$\int_0^{\pi/2} \frac{\cos t}{\sqrt{\sin^2 t + 1}}\, dt = \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int_0^{\pi/4} \sec\theta\, d\theta.$$



$\sin\theta$

$\tan\theta = \sin t.$

$\sec^2\theta\, d\theta = \cos t\, dt.$

$\sqrt{\sin^2 t + 1} = \sec\theta.$

$\tan\theta = \sin\frac{\pi}{2} = 1$

$\theta = \tan^{-1}(1) = \pi/4$

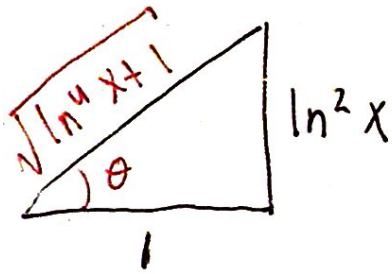
$\tan\theta = \sin 0 = 0$

$\theta = 0$

$$\int_0^{\pi/4} \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} = \ln|\sec\pi/4 + \tan\pi/4) - \ln(\sec 0 + \tan 0)$$

$$= \ln(\sqrt{2}+1) - \cancel{\ln(1)}^{\,0} = \underline{\ln(\sqrt{2}+1)}$$

$$\int \frac{4}{\sqrt{\ln^4 x + 1}} \frac{2(\ln x)}{x}\, dx = 4\int \frac{\sec^2\theta\, d\theta}{\sec\theta} = 4\int \sec\theta\, d\theta.$$



$\tan\theta = [\ln x]^2$

$\sec^2\theta\, d\theta = 2\ln x \frac{1}{x}\, dx.$

$\sqrt{\ln^4 x + 1} = \sec\theta.$

$$4\int \sec\theta\, d\theta = 4\ln|\sec\theta + \tan\theta| + C.$$

$$4\ln|\sqrt{\ln^4 x + 1} + \ln^2 x| + C.$$

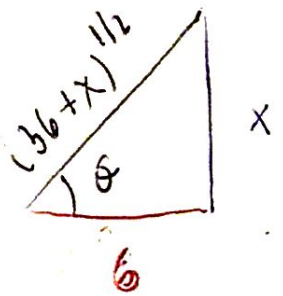
Capítulo 11

Parcial Lunes 2:30 D-504

Corto 4 1 Problema Sustitución Trigonométrica.

Práctica Problemas Integración. Pág 59.

2a) $\displaystyle\int_0^6 \frac{72}{(36+x^2)^{3/2}}\,dx = \int_0^{\pi/4} \frac{72\cdot 6\sec^2\theta\,d\theta}{36\cdot 6\sec^3\theta} = 2\int_0^{\pi/4} \frac{1}{\sec\theta}\,d\theta.$



$x = 6\tan\theta.$ $\qquad \frac{x}{6} = \tan\theta$

$dx = 6\sec^2\theta\,d\theta.$ $\qquad \theta = \tan^{-1}\left(\frac{x}{6}\right)$

$(36+x)^{1/2} = 6\sec\theta$

$(36+x)^{3/2} = 6^3\sec^3\theta$ $\qquad \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$\theta = \tan^{-1}(0) = 0$

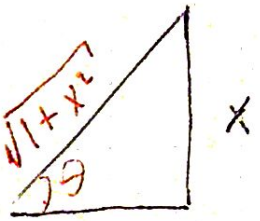
$$2\int_0^{\pi/4} \frac{1}{\sec\theta}\,d\theta = 2\int_0^{\pi/4} \cos\theta\,d\theta. = 2\sin\theta\Big]_0^{\pi/4} = 2\sin\frac{\pi}{4} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

2c) $\displaystyle\int_0^1 \frac{1}{(1+x)^2}\,dx = \int_1^2 u^{-2}\,du = -\frac{1}{u}\Big]_1^2 = -\frac{1}{2} + \frac{1}{1} = \frac{1}{2}.$

$u = 1+x$ $\qquad u(1) = 2.$

$du = dx$ $\qquad u(0) = 1$

2c1) $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \int_0^{\pi/4} \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,d\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int_0^{\pi/4} \frac{1}{\sec^2\theta}\,d\theta.$



$x = \tan\theta.$ $\qquad dx = \sec^2\theta\,d\theta.$

$1 = \tan\theta.$ $\Rightarrow$ $\theta = \pi/4$

$0 = \tan\theta$ $\qquad \theta = 0$

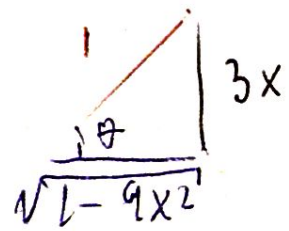
$$\int_0^{\pi/4} \cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/4} (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\Big]_0^{\pi/4}\right)$$

doble ángulo

$$\int_0^{\pi/4} \cos^2\theta\, d\theta = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - 0\right) = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) = \frac{\pi}{8} + \frac{1}{4}$$

3c) $\displaystyle\int_0^{1/3} 3^6 x^5 \sqrt{\underbrace{1-9x^2}_{u}}\, dx = \int_0^{\pi/2} \frac{3^6}{3^5} \sin^5\theta \cos\theta \frac{1}{3}\cos\theta\, d\theta.$

Pág 61.



$\sin\theta = 3x \Rightarrow x = \frac{1}{3}\sin\theta.$

$\cos\theta\, d\theta = 3\, dx \qquad dx = \frac{1}{3}\cos\theta\, d\theta.$

$\sqrt{1-9x^2} = \cos\theta. \qquad x^5 = \frac{1}{3^5}\sin^5\theta.$

$\sin\theta = 1$
$\theta = \pi/2$
$\sin\theta = 0$
$\theta = 0$

$$\frac{3^6}{3^6}\int_0^{\pi/2} \sin^5\theta \cdot \cos^2\theta\, d\theta. = \int_0^{\pi/2} \underbrace{\sin^4\theta}_{(\sin^2\theta)^2 = (1-\cos^2\theta)^2} \cos^2\theta\, (\sin\theta\, d\theta)$$

$$\int_0^{\pi/2} (1-\cos^2\theta)^2 \cos^2\theta\, (\underbrace{\sin\theta\, d\theta}_{-du}) = -\int_1^0 (1-u^2)^2 u^2\, du.$$

$u = \cos\theta \qquad u = \cos\pi/2 = 0$

$du = -\sin\theta\, d\theta \qquad u = \cos 0 = 1$

$$= \int_0^1 (1-2u^2+u^4)\, u^2\, du.$$

$$= \int_0^1 u^2 - 2u^4 + u^6\, du.$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 \Big]_0^1 \leftarrow$$

$$= \frac{1}{3} - \frac{2}{5} + \frac{1}{7}.$$

$$-\int_b^a f\,dx = \int_a^b f\,dx$$

$$\int 3^6 x^5 \sqrt{1-9x^2}\, dx$$

$$\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C.$$

$\cos\theta = \sqrt{1-9x^2}$

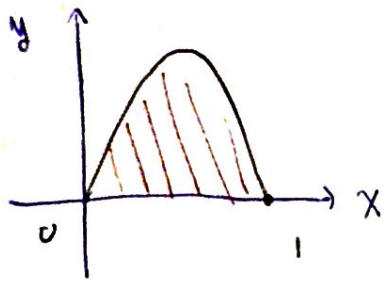
$$\frac{1}{3}\cos^3\theta - \frac{2}{5}\cos^5\theta + \frac{1}{7}\cos^7\theta + C.$$

$$\frac{1}{3}(1-9x^2)^{3/2} - \frac{2}{5}(1-9x^2)^{3/2} + \frac{1}{7}(1-9x^2)^{7/2} + C$$

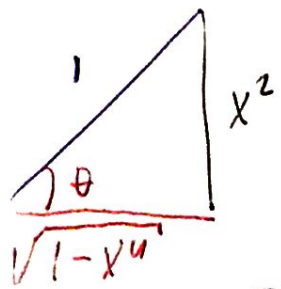
Triángulos "Interesantes"

Ejercicio 4.

Encuentre el área entre $f(x) = 4\pi x\sqrt{1-x^4}$, el eje x, y las rectas $x=0$ & $x=1$.



$$A = \int_0^1 f(x)\,dx = 2\pi \int_0^1 \sqrt{1-x^4}\; 2x\,dx$$



$x^2 = \sin\theta.$ $\quad \sin\theta = 1$

$2x\,dx = \cos\theta\,d\theta.$ $\quad \theta = \pi/2$

$\sqrt{1-x^4} = \cos\theta.$ $\quad \sin\theta = 0$

$\theta = 0$

$$A = 2\pi \int_0^{\pi/2} \cos\theta\cos\theta\,d\theta. = \frac{2\pi}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta.$$

$$A = \pi\left(\theta + \frac{\sin 2\theta}{2}\Big]_0^{\pi/2}\right) = \pi\left(\frac{\pi}{2} + \cancelto{0}{\frac{\sin\pi}{2}} - 0 - \cancelto{0}{\frac{\sin 0}{2}}\right)$$

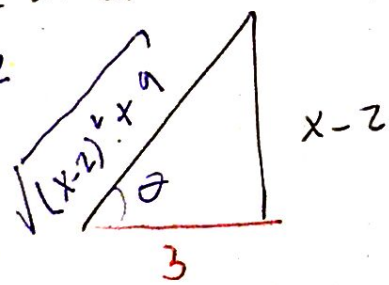
$$A = \pi\cdot\frac{\pi}{2} = \frac{\pi^2}{2}$$

$$\int \frac{(x-2)^3}{(x^2-4x+13)^{1/2}}\,dx. = \int \frac{(x-2)^3}{\sqrt{(x-2)^2+9}}\,dx = \int \frac{3^3\tan^3\theta}{3\sec\theta}\,3\sec^2\theta\,d\theta.$$

Complete al cuadrado $(x^2-4x+4)+13-4 = (x-2)^2+9$

c. $x-2$

c. 3



$3\cdot\tan\theta = x-2$

$3\sec^2\theta\,d\theta = dx.$

$\sqrt{(x-2)^2+9} = 3\sec\theta.$

$$3^3\int \tan^3\theta\cdot\sec\theta\,d\theta. = 27\int \tan^2\theta\,(\tan\theta\sec\theta\,d\theta)$$

$$27\int(\sec^2\theta - 1)(\tan\theta\sec\theta\,d\theta) \leftarrow du$$

$u = \sec\theta \qquad du = \sec\theta\tan\theta\,d\theta.$

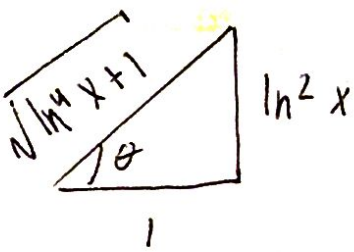
$$\int(27u^2 - 27)\,du = 9u^3 - 27u + C.$$

$$= 9\sec^3\theta - 27\sec\theta + C.$$

$\sec\theta = \dfrac{\sqrt{x^2-4x+13}}{3}$

$$\boxed{= \frac{9}{27}(x^2-4x+13)^{3/2} - 9(x^2-4x+13)^{1/2} + C.}$$

$$\int\sqrt{\ln^4 x + 1}\,\ln x\,\frac{dx}{x} = \int\sec\theta\cdot\frac{\sec^2\theta\,d\theta}{2}$$



$\ln^2 x = \tan\theta. \qquad \sqrt{\ln^4 x+1} = \sec\theta.$

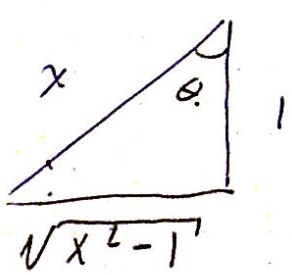
$\ln x\,\dfrac{dx}{x} = \dfrac{\sec^2\theta\,d\theta}{2}.$

$$\frac{1}{2}\int\sec^3\theta\,d\theta. = \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C\right)$$

$$\frac{1}{2}(\text{Der} + \text{Int}) = \frac{1}{4}\left(\ln^2 x\sqrt{\ln^4 x+1} + \ln\left|\sqrt{\ln^4 x+1} + \ln^2 x\right| + C\right).$$

$$\int\sqrt{x^2-1}\,dx = \int\tan^2\theta\cdot\sec\theta\,d\theta. = \int(\sec^2\theta - 1)\sec\theta\,d\theta.$$

$$= \int\sec^3\theta - \sec\theta\,d\theta.$$



$\dfrac{H}{C.A.} = x = \sec\theta.$

$dx = \sec\theta\tan\theta\,d\theta.$

$\sqrt{x^2-1} = \tan\theta.$

Capítulo 12

Simulacro Parcial 1.

1. Evalúe

$uv - \int v\,du.$ IPP.

a $\int x \tan^{-1} x^2 \, dx = \frac{x^2}{2}\tan^{-1}(x^2) - \int \frac{x^2}{2}\frac{2x}{1+x^4}\,dx$

$u = \tan^{-1}(x^2) \qquad dv = x\,dx$

$du = \frac{2x}{1+x^4} \qquad v = \frac{x^2}{2}.$

$\frac{x^2}{2}\tan^{-1}(x^2) - \int \frac{x^3}{1+x^4}\,dx$

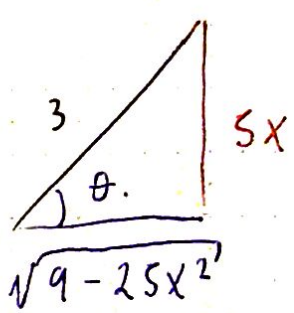
$w = 1+x^4$

$\frac{dw}{4} = x^3\,dx$

$\frac{1}{2}x^2\tan^{-1}(x^2) - \frac{1}{4}\int\frac{dw}{w} = \frac{1}{2}x^2\tan^{-1}(x^2) - \frac{1}{4}\ln|w| + C.$

$= \frac{1}{2}x^2\tan^{-1}(x^2) - \frac{1}{4}\ln|1+x^4| + C.$

b. $\int \frac{x^2}{\sqrt{9-25x^2}}\,dx = \int \frac{(9/25)\sin^2\theta}{\cancel{3}\cos\theta}\,\frac{\cancel{3}}{5}\cos\theta\,d\theta.$



$\sin\theta = \frac{5x}{3} \qquad x = \frac{3}{5}\sin\theta., \qquad dx = \frac{3}{5}\cos\theta\,d\theta.$

$\cos\theta = \frac{\sqrt{9-25x^2}}{3} \qquad \sqrt{9-25x^2} = 3\cdot\cos\theta.$

$\frac{9}{125}\int \sin^2\theta\,d\theta. = \frac{9}{250}\int(1-\cos 2\theta)\,d\theta. = \frac{9}{250}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$

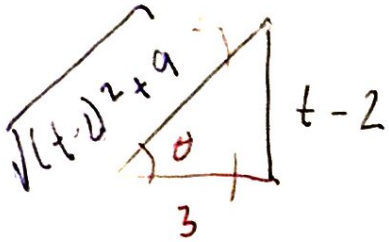
I Doble Ángulo.

$= \frac{9}{250}(\theta - \sin\theta\cos\theta) + C.$

$\theta = \sin^{-1}\left(\frac{5x}{3}\right)$

$$\boxed{= \frac{9}{250}\left(\sin^{-1}\left(\frac{5x}{3}\right) - \frac{5x\sqrt{9-25x^2}}{9}\right) + C.}$$

c. $\displaystyle\int \frac{1}{\sqrt{(t-2)^2+9}}\,dt.$; $\displaystyle\int \frac{\sqrt{(t-2)^2+9}}{t^3}\,dt.$



$$\boxed{\begin{aligned}\tan\theta &= \frac{t-2}{3}\\ \sec\theta &= \frac{\sqrt{(t-2)^2+9}}{3}\end{aligned}}$$

$(t-2) = 3\tan\theta.$

$dt = 3\sec^2\theta\, d\theta.$

$$\int \frac{1}{\sqrt{(t-2)^2+9}}\,dt = \int \frac{3\sec^2\theta\, d\theta}{3\sec\theta} = \int \sec\theta\, d\theta.$$

$$= \ln|\sec\theta + \tan\theta| + C.$$

Regrese a la variable t.

$$= \boxed{\ln\left|\frac{\sqrt{(t-2)^2+9}}{3} + \frac{t-2}{3}\right| + C.}$$

J. $\displaystyle\int \frac{x e^x}{(x+1)^2}\,dx$

2 funciones. $\displaystyle\int \frac{x}{(x+1)^2}\,dx.$ Derivada $\dfrac{x}{(x+1)^2}$ es larga

sigue siendo dificil

Derivar xe^x Integre $(x+1)^{-2}. \rightarrow -(x+1)^{-1}$

$$\int \frac{x e^x\,dx}{(x+1)^2} = -\frac{x}{x+1}e^x + \int \frac{e^x + xe^x}{x+1}\,dx$$

$u = xe^x$ $\qquad dv = (x+1)^{-2}dx$

$du = (e^x + xe^x)dx$ $\qquad v = \dfrac{-1}{(x+1)}$

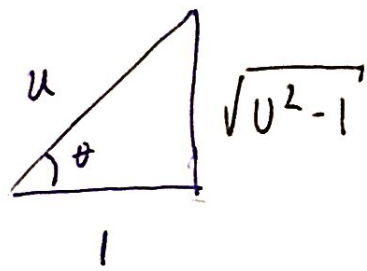
$$= -\frac{x}{x+1}e^x + \int \frac{e^x(1+\cancel{x})}{(\cancel{x+1})}\,dx$$

$$= \boxed{-\frac{x}{x+1}e^x + e^x + C.}$$

Miranda $\displaystyle\int_0^{\ln 2} \frac{e^{4x}}{\sqrt{e^{2x}-1}}\,dx = \int_1^2 \frac{u^3}{\sqrt{u^2-1}}\,du.$

$u = e^x \qquad u(\ln 2) = e^{\ln 2} = 2. \qquad u^3 = e^{3x}$

$du = e^x\,dx \qquad u(0) = e^0 = 1$



$u = \sec\theta.$

$du = \sec\theta\tan\theta\,d\theta.$

$\sqrt{u^2-1} = \tan\theta.$

$$\int \frac{u^3}{\sqrt{u^2-1}}\,du = \int \frac{\sec^3\theta}{\tan\theta}\cdot\sec\theta\tan\theta\,d\theta. = \int \sec^4\theta\,d\theta.$$

$\int \tan^m x \sec^n x\,dx$ aparte $\sec^2 x$ o $\sec x\tan x$.

$u = \tan\theta \qquad du = \sec^2\theta\,d\theta.$

$$\int \sec^2\theta\,\sec^2\theta\,d\theta. = \int (\tan^2\theta + 1)\sec^2\theta\,d\theta.$$

$\sec^2\theta = \tan^2\theta + 1$

$$= \frac{1}{3}\tan^3\theta. + \theta + C.$$

$$= \frac{1}{3}(u^2-1)^{3/2} + \sec^{-1}(u). + C.$$

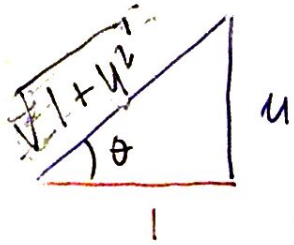
$$\int_1^2 \frac{u^3}{\sqrt{u^2-1}}\,du = \frac{1}{3}\left(3^{3/2} + \sec^{-1}(2) - \cancel{\frac{1}{3}0^{3/2}}^{\,0} + \cancel{\sec^{-1}(1)}^{\,0}\right)$$

$$= 3^{1/2} + \sec^{-1}(2).$$

2 b) $\displaystyle\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt = \int_0^1 \frac{du}{\sqrt{1+u^2}} = \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{\sec\theta}.$

$u = \sin t \qquad u(\pi/2) = \sin(\pi/2) = 1$

$du = \cos t\,dt \qquad u(0) = \sin(0) = 0$



$u = \tan\theta \qquad du = \sec^2\theta\,d\theta.$

$\tan\theta = 1 \Rightarrow \theta = \pi/4. \qquad \sqrt{1+u^2} = \sec\theta.$

$\tan\theta = 0 \Rightarrow \theta = 0$

$$\int_0^{\pi/4} \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4}$$

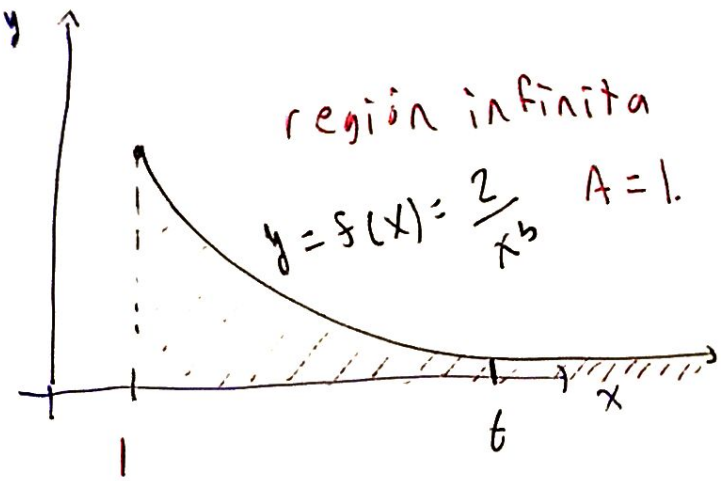
$$= \ln|\sec\pi/4 + \tan\pi/4| - \ln|\overset{1}{\cancel{\sec 0}} + \overset{0}{\cancel{\tan 0}}|$$

$$= \ln\left|\frac{2}{\sqrt{2}} + 1\right| - \ln(1) = \ln(\sqrt{2}+1).$$

Capítulo 13

7.8 Integrales Impropias.

Considere la región bajo la curva $y = \frac{2}{x^3}$, encima del eje-x y a la derecha de la recta $x=1$.



$$A = \int_1^t 2x^{-3}\,dx.$$

$$A = \frac{2}{-2}x^{-2}\Big]_1^t$$

$$A = -\frac{1}{t^2} + \frac{2}{2}\frac{1}{1^2} = 1 - \frac{1}{t^2}.$$

A medida que t aumenta. $\frac{1}{t^2} \to 0$

$$\lim_{t\to\infty} A = \lim_{t\to\infty} 1 - \frac{1}{t^2} = 1$$

Concluyendo $\int_1^\infty \frac{2}{x^3}\,dx = 1.$

Integral Impropia Tipo 1: (Pág 73.)

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

CONVERGENTE si el límite existe.

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$$

DIVERGENTE. si el límite no existe.

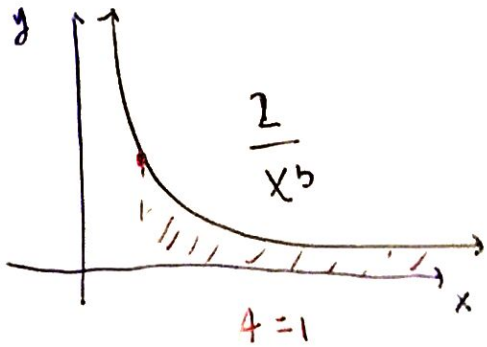
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

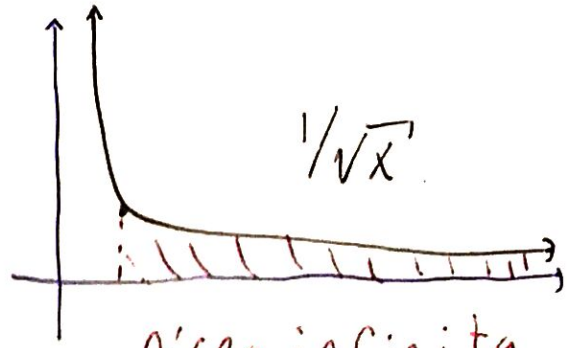
Ejercicio 1: Evalúe $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ y determine si la integral converge.

$$\int_1^\infty x^{-1/2}\,dx = 2x^{1/2}\Big]_1^\infty = \lim_{x\to\infty} 2x^{1/2} - 2 = +\infty.$$

$\sqrt{\infty} \to \infty.$

DIVERGENTE





Área infinita.

$f(x) \to 0$ a medida que $x \to \infty$ $\quad \int_a^\infty f(x)\,dx$ es posible que no exista.

Ej 2: Análisis de la integral $\int_1^\infty \frac{1}{x^p}\,dx$ (Pág. 79).

$p=1$ $\quad \int_1^\infty \frac{1}{x}\,dx = \ln x\Big|_1^\infty = \lim_{x\to\infty} \ln x - \cancel{\ln(1)}^{\,0} = +\infty.$

$+\infty$ DIVERGE.

$p=0.99$ $\quad \int_1^\infty x^{-0.99}\,dx = \frac{x^{0.01}}{0.01}\Big|_1^\infty = \lim_{x\to\infty} x^{0.01} - \frac{1}{0.01} = +\infty.$

DIVERGE.

Reglas límites $\quad \lim_{x\to\infty} x^r = +\infty.$ $\quad r>0.$ $\quad \lim_{x\to\infty} \frac{1}{x^r} = 0.$

$p=1.01$ $\quad \int_1^\infty x^{-1.01}\,dx = \frac{x^{-0.01}}{-0.01}\Big|_1^\infty = \lim_{x\to\infty} \frac{1}{-0.01}\frac{1}{x^{0.01}} + \frac{1}{0.01} = \frac{1}{0.01}$

$0 + \frac{1}{0.01}$ CONVERGE

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{DIVERGE} & \text{si } p \leq 1 \\ \text{CONVERGE} & \text{si } p > 1. \end{cases}$$

Ejercicio 3: Evalúe.

a. $$\int_{-\infty}^{0} e^{-x^2} x\,dx = \int_{-\infty}^{0} e^{u} \frac{du}{-2} = -\frac{1}{2} e^{u}\Big]_{-\infty}^{0}$$

$$e^{-\infty} = \frac{1}{e^{\infty}} \to 0$$

$u = -x^2$ $\quad u(0) = 0^2$

$du = -2x\,dx$ $\quad u(-\infty) = -\infty.$

$$= -\frac{1}{2} e^{0} + \cancel{\lim_{u \to -\infty} \frac{1}{2} e^{u}}^{\,0}$$

$$= -\frac{1}{2} \cdot 1 = -\frac{1}{2}.$$

$$\int_{-\infty}^{0} e^{-x^2} x\,dx = -\frac{1}{2} \quad \text{converge.}$$

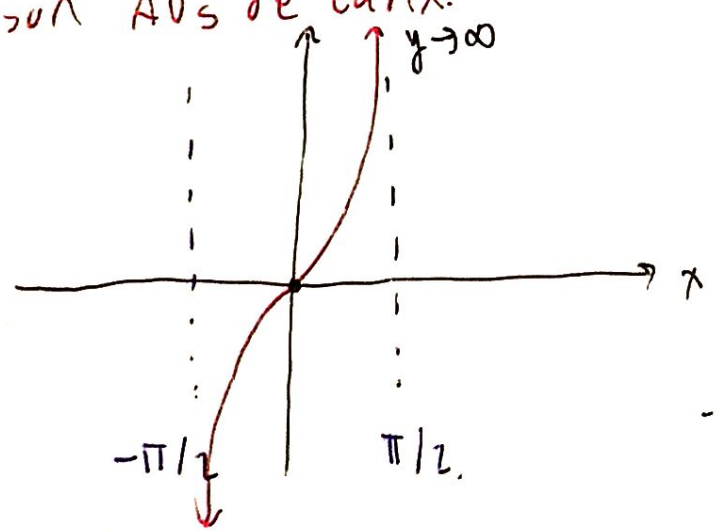
CONVERGE

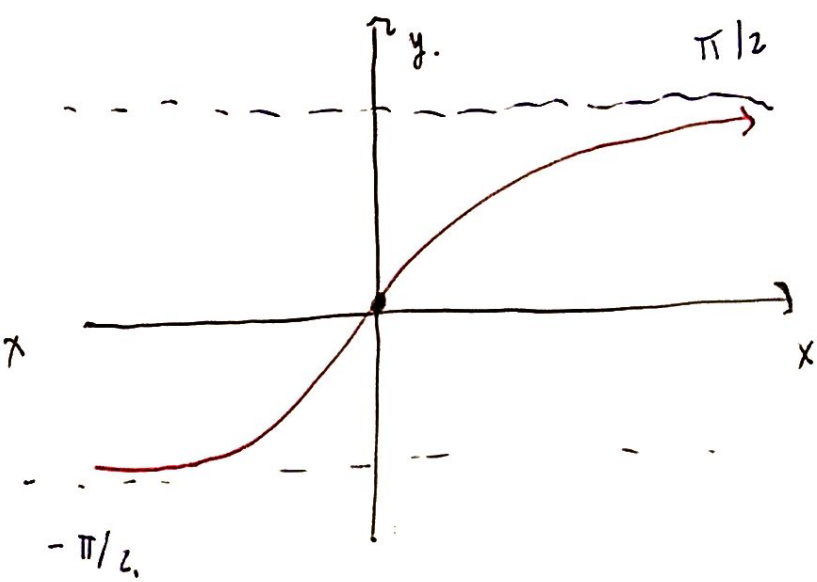
b. $$\int_{-\infty}^{\infty} \frac{2}{1+x^2}\,dx = 2\tan^{-1}(x)\Big]_{-\infty}^{\infty} = \pi + \frac{2\pi}{2} = 2\pi.$$

$$= 2\lim_{x \to \infty} \underbrace{\tan^{-1}(x)}_{\pi/2} - 2\lim_{x \to -\infty} \underbrace{\tan^{-1}(x)}_{-\pi/2}.$$

AHs de $\tan^{-1}(x)$

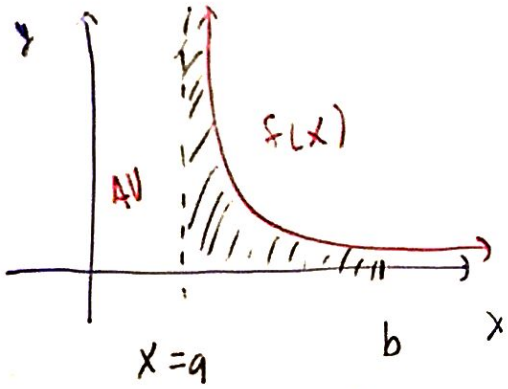
son AVs de $\tan x$.





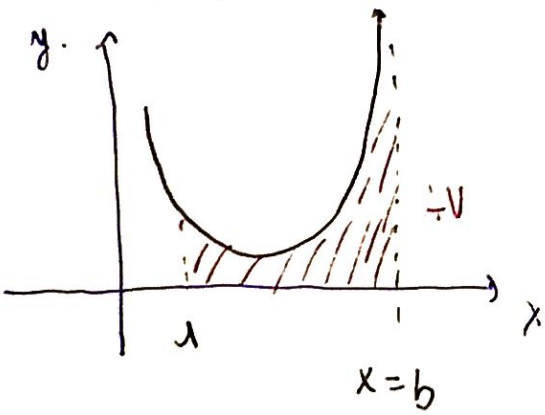
Integrales impropias Tipo 2.

Hay una A.V en $x = a$.

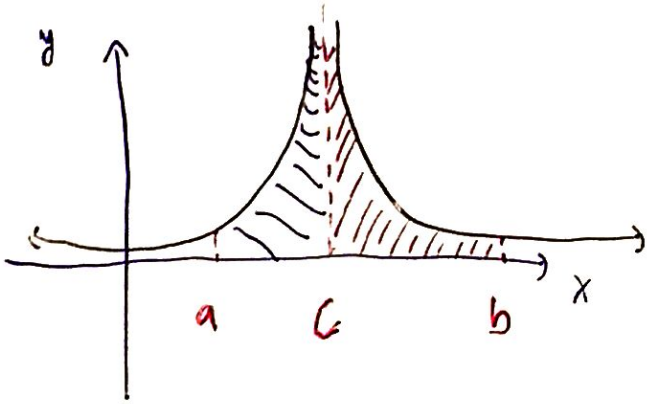


$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx.$$

existe, es CONVERGENTE.



$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$



AV en $x = c$ y está en medio del intervalo.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

2 integrales impropias.

Ejercicio 4: Evalúe las sigs. integrales. Indique donde son discontinuas.

a. $\displaystyle\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^8 u^{-1/3}\,du. = \frac{3}{2} u^{2/3}\Big]_{0^+}^{8} = \frac{3}{2}(8)^{2/3} - \lim_{u \to 0^+} \frac{3}{2} u^{2/3}$

$u = x - 1 \qquad u(9) = 8$

$du = dx \qquad u(1) = 0$

indefinida en $x = 0$

$\frac{1}{0}$ indefinido.

$$\frac{3}{2}(64)^{1/3} - \frac{3}{2}0^{2/3} = \frac{3}{2}\cdot 4 - 0 = 6. \quad \text{converge.}$$

b. $$\int_{-2}^{3} \frac{3}{x^4}\,dx = \int_{-2}^{0} 3x^{-4}\,dx + \int_{0}^{3} 3x^{-4}\,dx$$

$\frac{1}{0}$ indefinida en $x=0$.

$$\int_{-2}^{0} 3x^{-4}\,dx = -x^{-3}\Big]_{-2}^{0^-} = \underbrace{\lim_{x\to 0^-} -\frac{1}{x^3}}_{+\infty.} + \frac{1}{(-2)^3}$$

NO EXISTE.

DIVERGE.

DIVERGE $\to \pm\infty$.

CONVERGE $\to$ número, límite existe.

c. $$\int_{0}^{1} \ln x\,dx$$

$\ln 1 = 0$

indefinida en $x=0$.

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C.$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$\int_{0}^{1} \ln x\,dx = x\ln x - x\Big]_{0^+}^{1} = 1\cdot\cancel{\ln(1)}^{0} - 1 - \lim_{x\to 0^+} x\ln x - \cancel{x}^{0}$$

$$= -1 - \lim_{x\to 0^+} \cancel{x\ln x}^{0}. \qquad 0\cdot\infty$$

Regla de L'Hospital $$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \underset{LH}{=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}}$$

$$\lim_{x\to 0^+} -x = -0$$

$$\boxed{\int_{0}^{1} \ln x\,dx = -1 \quad \text{CONVERGE}}$$

$0 \cdot \infty \neq 0$ $\quad \lim_{x\to\infty} \frac{1}{x} e^x = \lim_{x\to\infty} \frac{e^x}{x} \underset{LH}{=} \lim_{x\to\infty} \frac{e^x}{1} = +\infty.$

$0 \cdot \infty.$ $\quad \infty/\infty$

Problema 2: $\quad u(t) = 10t\cos\left(\frac{t}{2}\right)$

$v(t) = \int 10t\cos\left(\frac{t}{2}\right)dt. = 20t\sin\frac{t}{2} - \int 20\sin\frac{t}{2}\,dt.$

$u = 10\cdot t \qquad dv = \cos\left(\frac{t}{2}\right)dt. \qquad 20t\sin\frac{t}{2} + 40\cos\frac{t}{2} + C.$

$du = 10\,dt. \qquad v = 2\sin\left(\frac{t}{2}\right)$

a) Reposo $\quad v(0) = 0. \qquad v(0) = 0 + 40 + C = 0$

$C = -40.$

b) $s(t) = \int\left(20t\sin\frac{t}{2} + 40\cos\frac{t}{2} - 40\right)dt.$

$s(0) = 5.$

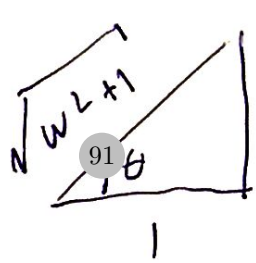
Problema 6: $\quad \int_1^e \frac{24\ln^2 x}{(\ln^6 x + 1)^2}\frac{dx}{x} = \int_0^1 \frac{24u^2}{(u^6+1)^2}\,du.$

$u = \ln x \qquad u = \ln^3 x \qquad \ln^3 x = \tan\theta.$

$du = \frac{dx}{x} \qquad u(e) = 1,\ u(1) = 0 \qquad u^3 = \tan\theta.$

$w = u^3 \qquad w(1) = 1 \qquad \frac{24}{3}\int_0^1 \frac{1}{(w^2+1)^{4/2}}\frac{dw}{3}.$

$dw = 3u^2\,du \qquad w(0) = 0$



$w = \tan\theta.$

$dw = \sec^2\theta.\,d\theta.$

$\sqrt{w^2+1} = \sec\theta.$

$$\frac{24}{3}\int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = 8\int_0^{\pi/4} \cos^2\theta\, d\theta$$

$$= 4\int_0^{\pi/4} (1+\cos 2\theta)\, d\theta$$

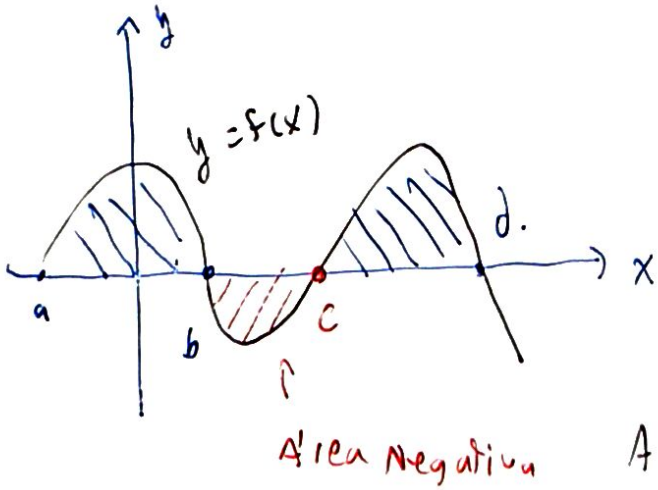
$$= 4\left(\theta + \frac{1}{2}\sin 2\theta \Big]_0^{\pi/4}\right)$$

$$= \frac{4\pi}{4} + 2\sin\frac{\pi}{2} - 0 - 0$$

$$= \pi + 2.$$

Capítulo 14

6.1 Áreas entre Curvas (P 79).



Área entre $y = f(x)$ y el eje-x

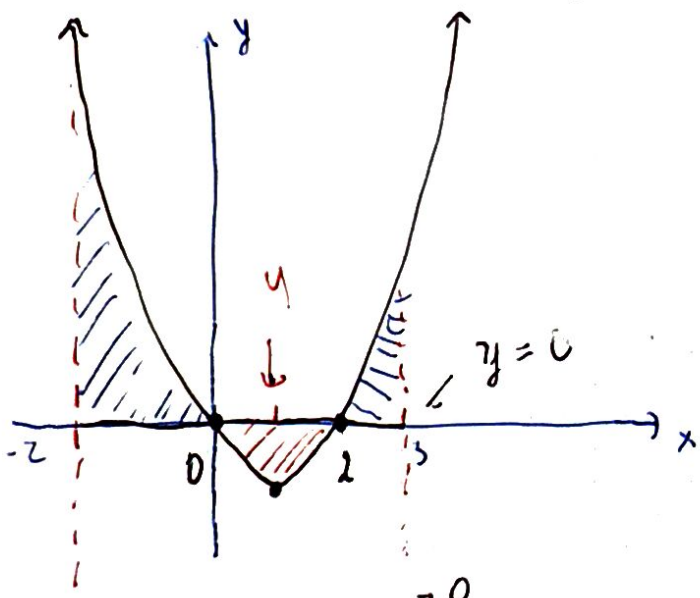
$$A = \int_a^d |f(x)|\, dx$$

$$A = \int_a^b f\, dx - \int_b^c f\, dx + \int_c^d f\, dx.$$

Ejercicio 1: Bosqueje y encuentre el área de la región delimitada por $y = 3x^2 - 6x$, $x = -2$, $x = 3$ y el eje-x.

Bosqueje la curva $y = 3x^2 - 6x$

interceptos-x: $y = 3x^2 - 6x = 0$ $\quad 3x(x-2) = 0$ $\quad x = 0, 2.$



$$A = \int_{-2}^{3} |3x^2 - 6x|\, dx$$

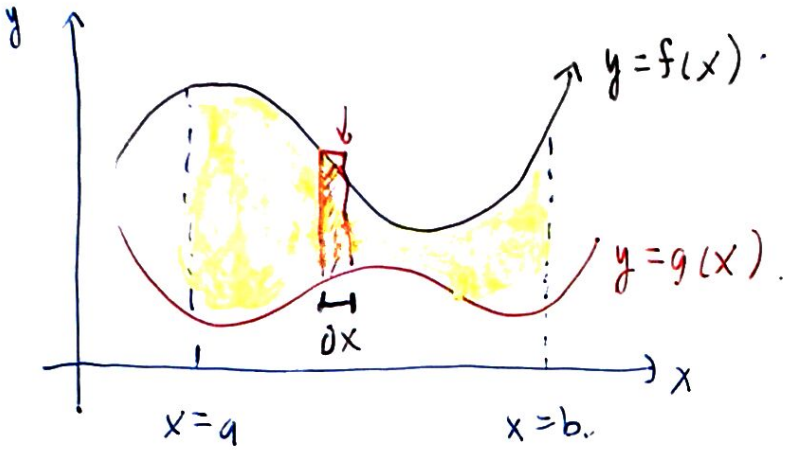
$$A = \int_{-2}^{0} 3x^2 - 6x\, dx - \int_0^2 3x^2 - 6x\, dx + \int_2^3 3x^2 - 6x\, dx.$$

$$A = x^3 - 3x^2 \Big]_{-2}^{0} + 3x^2 - x^3 \Big]_0^2 + x^3 - 3x^2 \Big]_2^3$$

$$A = 0 - (-8 - 3(4)) + (12 - 8 - 0) + (27 - 27 - (8 - 12))$$

$$A = 20 \quad + 4 \quad + 4. \quad = 28$$

Área entre dos curvas (p. 80)



Región está entre las dos curvas $g(x) \leq y \leq f(x)$ y las rectas verticales $a \leq x \leq b$.

Considere un rectángulo infinitesimal, ancho dx, altura $f(x) - g(x)$.

$$dA = [f(x) - g(x)]\, dx$$

Integrando en x desde $x = a$ hasta $x = b$.

$$\int dA = A = \int_a^b [f(x) - g(x)]\, dx = \underbrace{\int_a^b f\, dx}_{\text{amarillo + blanco}} - \underbrace{\int_a^b g\, dx}_{\text{blanco.}}$$

Área entre dos curvas:

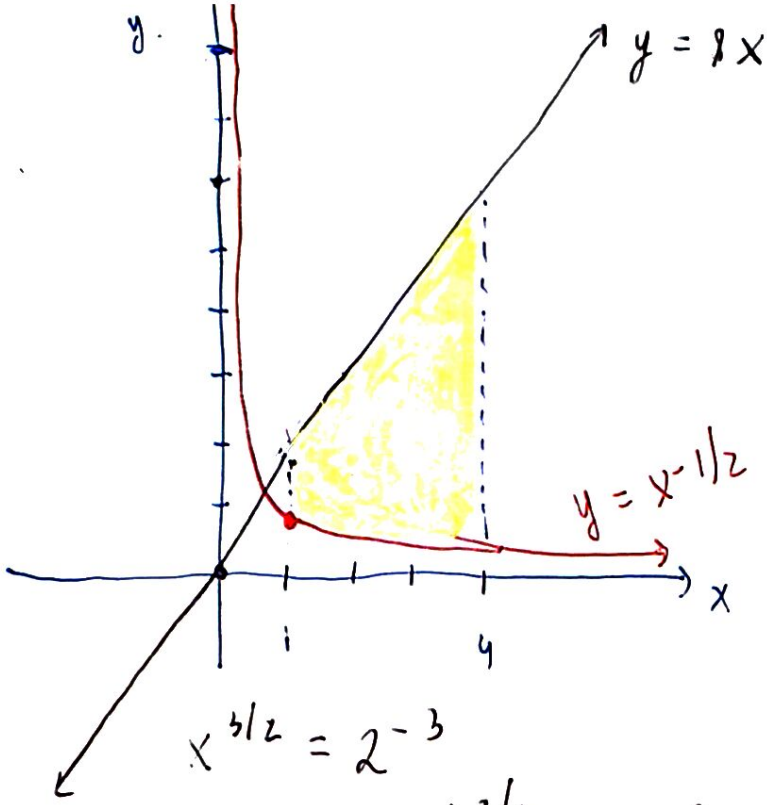
$$A = \int_a^b y_{sup} - y_{inf.}\, dx$$

$$(-2) - (-8) = -2 + 8 = 6.$$

Ejemplo: Bosqueje y encuentre el área de la región.

$$y_1 = \frac{1}{\sqrt{x}}, \quad y_2 = 8x \quad \text{en } [1, 4].$$

Realice los bosquejos de las curvas.



en $x = 1$

$y_1(1) = 1 \qquad y_2(1) = 8$

intersección ocurre fuera del intervalo $[1, 4]$.

$y_1 = y_2.$

$x^{-1/2} = 8x$

$1 = 8x^{3/2}.$

$x^{3/2} = \frac{1}{8} = 2^{-3}$

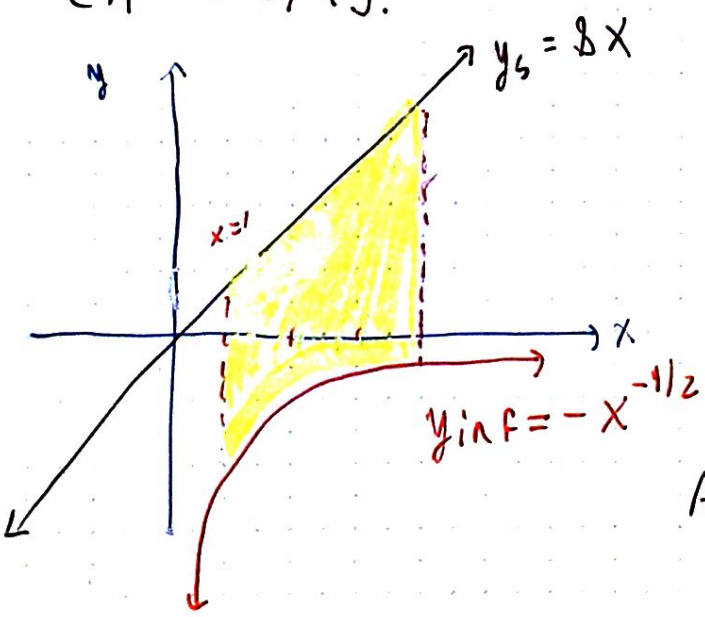
$x^{3/2} = 2^{-3}$

$x = (2^{-3})^{2/3} = 2^{-2} = \frac{1}{4}.$ intersecto. (no afecta el problema)

$$A = \int_1^4 (8x - x^{-1/2})\,dx = 4x^2 - 2x^{1/2}\Big]_1^4$$

$$A = 4^3 - 2\sqrt{4} - (4-2) = 64 - 4 - 2 = 58$$

Emilio: Área de la región entre $y_2 = 8x$ & $y_1 = -\frac{1}{\sqrt{x}}$ en $[1, 4]$.



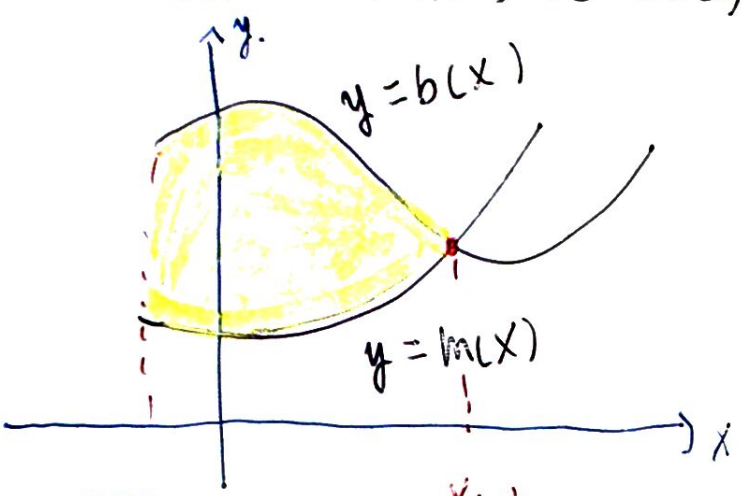
$$A = \int_1^4 y_s - y_{inf}\,dx$$

$$A = \int_1^4 8x + x^{-1/2}\,dx$$

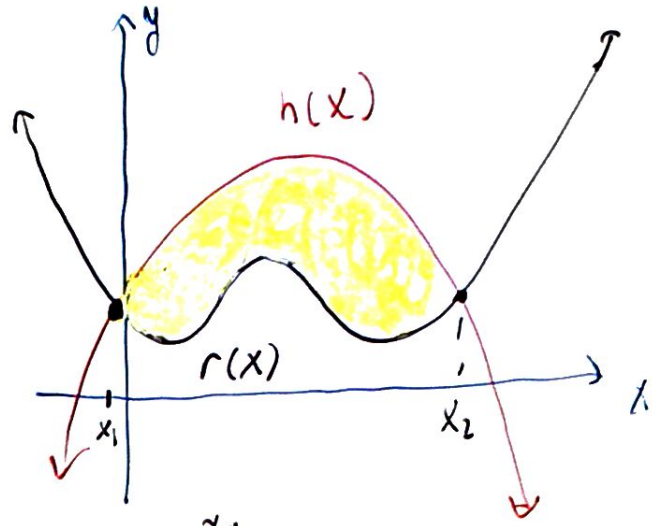
$$A = 4x^2 + 2x^{1/2}\Big]_1^4$$

$A = 64 + 4 - 4 - 2 = 62.$ más grande.

Regiones con puntos de intersección entre las curvas. límites de integración no están explícitos.



$$A = \int_a^{x_{int}} b - m \, dx$$

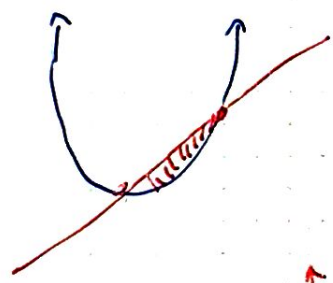


$$A = \int_{x_1}^{x_2} h - r \, dx$$

Es necesario encontrar las intersecciones entre las 2 curvas

Ejemplo: Encuentre el área de la región entre

$y_1 = 3x$ & $y_2 = 3x^2$.

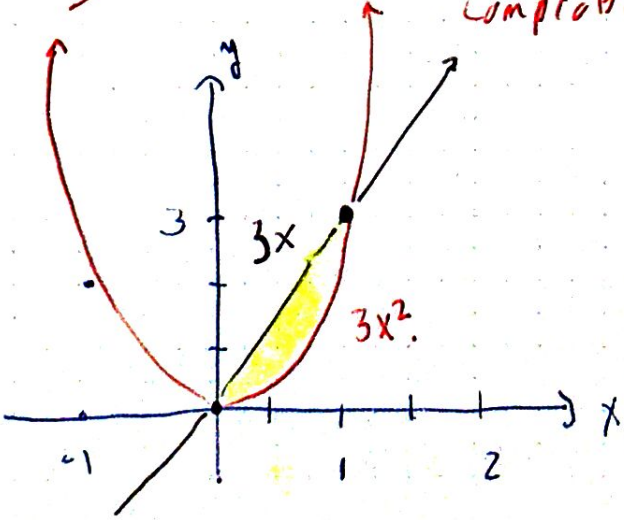


Intersecciones $y_2 = y_1$

$3x^2 = 3x \Rightarrow 3x^2 - 3x = 3x(x-1) = 0$

$x = 0, 1.$ $\quad y_1(0) = 0$ ✓ $\quad y_2(0) = 0$ ✓

Comprobando. $\quad y_1(1) = 3$ ✓ $\quad y_2(1) = 3$ ✓



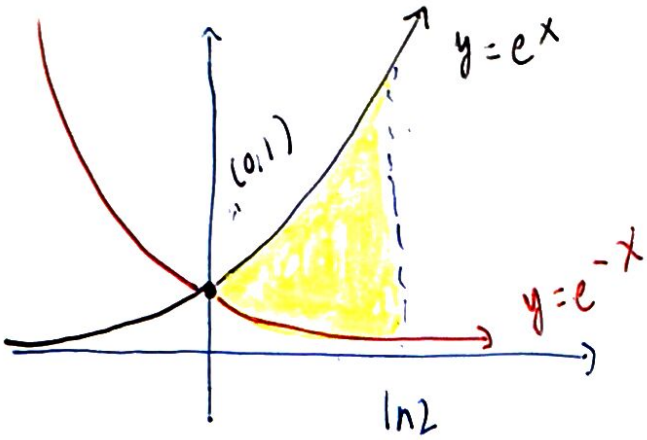
$$A = \int_0^1 3x - 3x^2 \, dx$$

$$A = \frac{3}{2}x^2 - x^3 \Big]_0^1$$

$$A = \frac{3}{2} - 1 = 0 = \frac{1}{2}.$$

Ejercicio 2: Bosqueje y encuentre de la región (P. 81)

a) $y_1 = e^x$, $y_2 = e^{-x}$, $x = 0$ & $x = \ln 2$.



$$\frac{1}{e^{\ln 2}} = \frac{1}{2}$$

$$A = \int_0^{\ln 2} e^x - e^{-x}\, dx$$

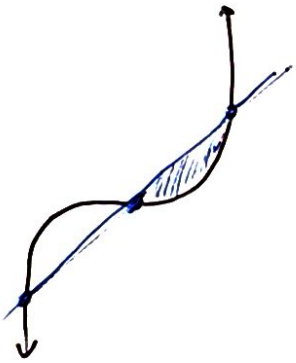
$$A = e^x + e^{-x} \Big|_0^{\ln 2}$$

$$A = e^{\ln 2} + e^{-\ln 2} - (e^0 + e^{-0})$$

$$A = 2 + \frac{1}{2} - (1 + 1)$$

$$A = 2 + \frac{1}{2} - 2 = \frac{1}{2}.$$

b.) $y_1 = x^3$ & $y_2 = 4x$.

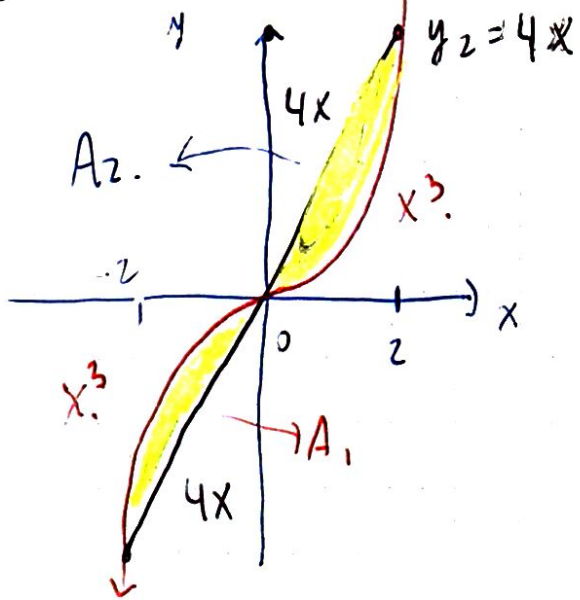


Intersecciones entre y_1 & y_2.

$x^3 = 4x$

$x^3 - 4x = x(x^2 - 4) = 0 \quad x = 0, \pm 2.$

$y_1(0) = y_2(0) = 0 \qquad y_1(2) = 8 \quad y_2(2) = 8.$



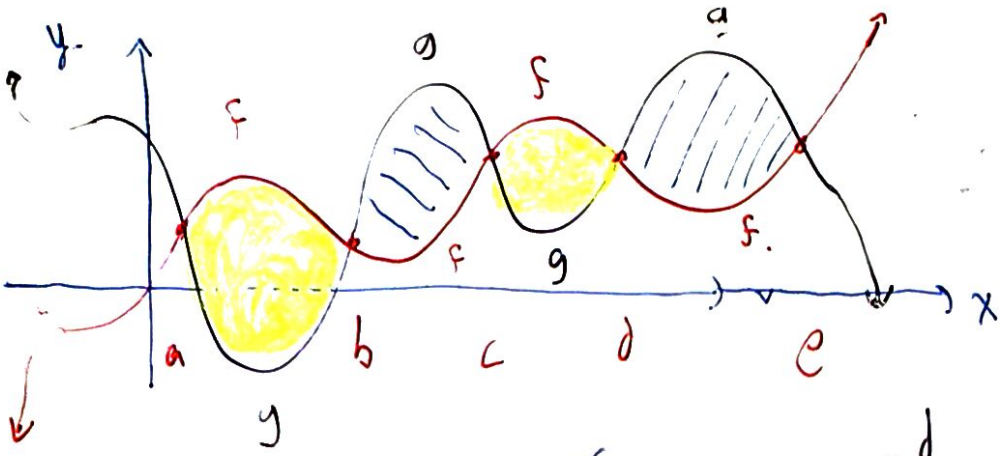
$$A = \underbrace{\int_{-2}^{0} x^3 - 4x\, dx}_{} + \underbrace{\int_0^2 4x - x^3\, dx}_{}$$

son iguales.

$$A = 2\int_0^2 4x - x^3\, dx$$

$$A = 2\left(2x^2 - \frac{x^4}{4}\Big]_0^2\right) = 2\left(8 - \frac{16}{4} - 0\right) = 2(4) = 8.$$

Regiones con dos curvas superiores diferentes.



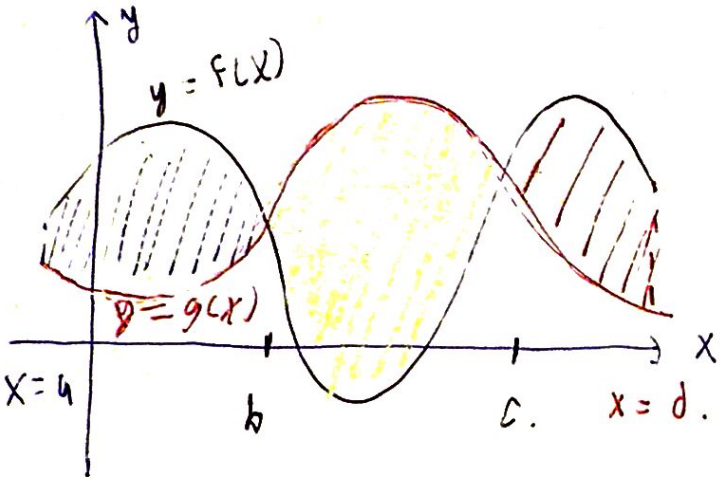
Área entre estas dos curvas

$$A = \int_a^b f - g\,dx + \int_b^c g - f\,dx + \int_c^d f - g\,dx + \int_d^e g - f\,dx.$$

$$A = \int_a^e |f - g|\,dx$$

Capítulo 15

Áreas entre curvas.

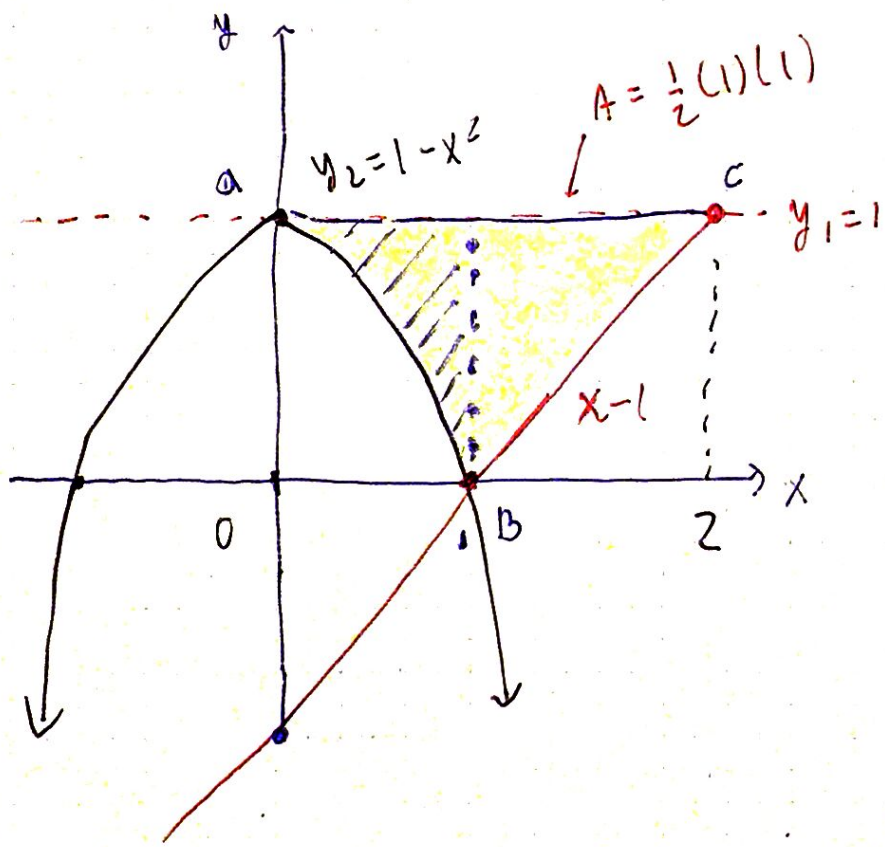


$$A = \int_a^b f-g\,dx + \int_b^c g-f\,dx + \int_c^d f-g\,dx$$

$$A = \int_a^b |f-g|\,dx \checkmark$$

Ejercicio 3: Encuentre el área de la región entre las curvas dadas.

b. $y_1 = 1$, $y_2 = 1-x^2$, $y_3 = x-1$. P (83)



a. $y_1 = y_2$

$1 = 1 - x^2$

$x^2 = 0 \Rightarrow x = 0$

c. $y_1 = y_3$.

$1 = x - 1 \Rightarrow x = 2$.

o. $y_2 = y_3 \Rightarrow x = 1$

$1 - x^2 = x - 1$

$-x^2 - x + 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, \quad x = 1$

$$A = \left\{ \int_0^1 1-(1-x^2)\,dy + \int_1^2 1-(x-1)\,dx \right\} 1/2.$$

$$A = \int_0^1 x^2\,dx + \int_1^2 2-x\,dx$$

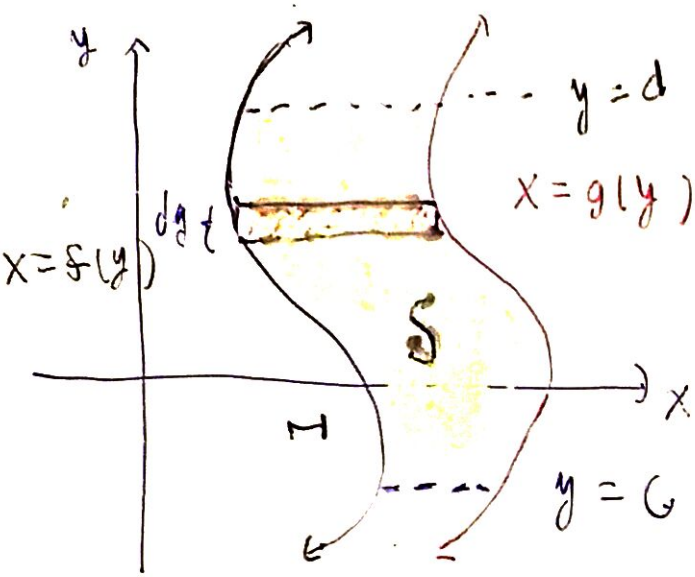
$$A = \frac{1}{3}x^3\Big]_0^1 + 2x - \frac{x^2}{2}\Big]_1^2$$

$$A = \frac{1}{3} + 4 - 2 - \left(2 - \frac{1}{2}\right) \Big\} \; 1/2.$$

$$A = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

P (83)

Integración en el eje-y, franjas Horizontales.



Región S: $c \leq y \leq d$.

$f(y) \leq x \leq g(y)$

rectángulo. altura dy.

ancho $g(y) - f(y)$

$$dA = [g(y) - f(y)]\,dy.$$

$$\boxed{A = \int_c^d [g(y) - f(y)]\,dy.}$$

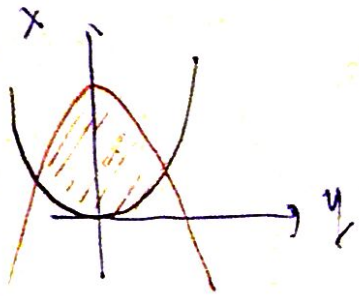
$$A = \int_a^b y_{arr} - y_{aba}\,dx$$

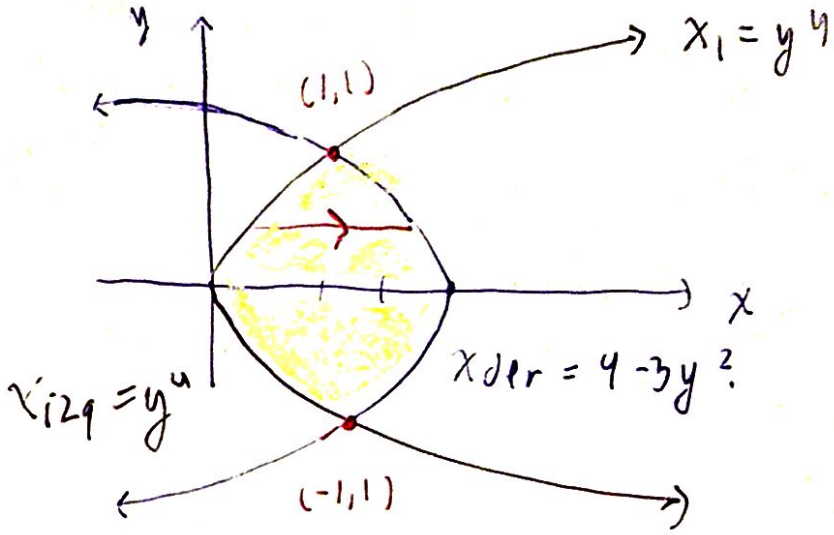
$$A = \int_c^d x_{der} - x_{izq}\,dy.$$

$x = f(y) \Rightarrow y = f^{-1}(x)$

Cuite las inversas.

Ejemplo: Encuentre el área entre $x_1 = y^4$ & $x_2 = 4-3y^2$.





$$A = \int_{-1}^{1} x_{der} - x_{izq}\, dy = \int_{-1}^{1} 4-3y^2-y^4\, dy.$$

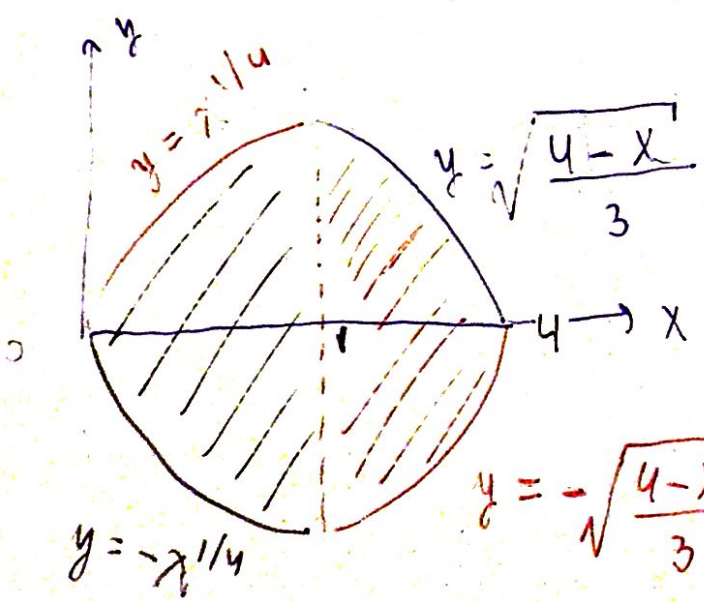
Intersecciones $\quad y^4 = 4-3y^2.$

$$y^4 + 3y^2 - 4 = 0$$
$$y^2 + 4 = 0$$
$$y^2 - 1 = 0$$

$$\int_{-\infty}^{\infty} \frac{1}{9+x^2}\, dx$$

$$y^2 \neq -4$$
$$y^2 = 1$$
$$y = \pm 1$$

$$A = 2\int_0^1 4-3y^2-y^4\, dy. = 2\left(4y - y^3 - \frac{1}{5}y^5\Big]_0^1\right)$$
$$= 2\left(4-1-\frac{1}{5}\right) = 2\left(\frac{15}{5}-\frac{1}{5}\right) = \frac{28}{5}.$$



$x = y^4$ Resuelva para x:

$$y = \pm \sqrt[4]{x}$$
$$x = 4-3y^2.$$

$$3y^2 = 4-x$$
$$y^2 = \frac{4-x}{3} \qquad y = \pm\sqrt{\frac{4-x}{3}}$$

Área de la región

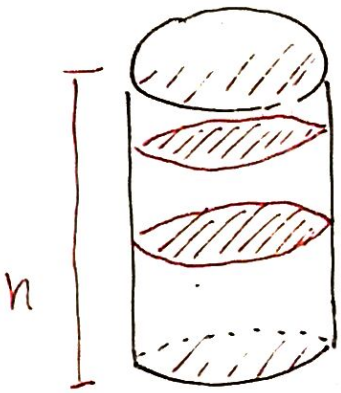
$$A = \int_0^1 x^{1/4} - (-x^{1/4})\,dx + \int_1^4 \sqrt{\frac{4-x}{3}} - \left(-\sqrt{\frac{4-x}{3}}\right) dx$$

$$A = 2\int_0^1 x^{1/4}\,dx + 2\int_1^4 \left(\frac{4-x}{3}\right)^{1/2} dx \Big\} = 68/5$$

< 40 2pts. P1 $\qquad$ > 60 1 pt. neto.

Volúmenes de sólido. Encuentre el área, utilizando integrales.

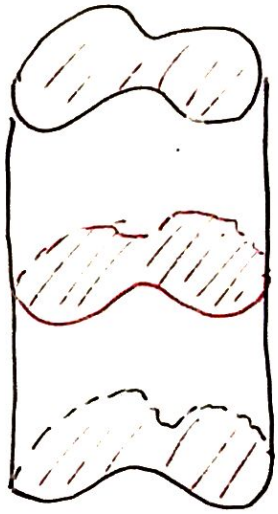
Volumen de un cilindro.



Área transversal πr^2.

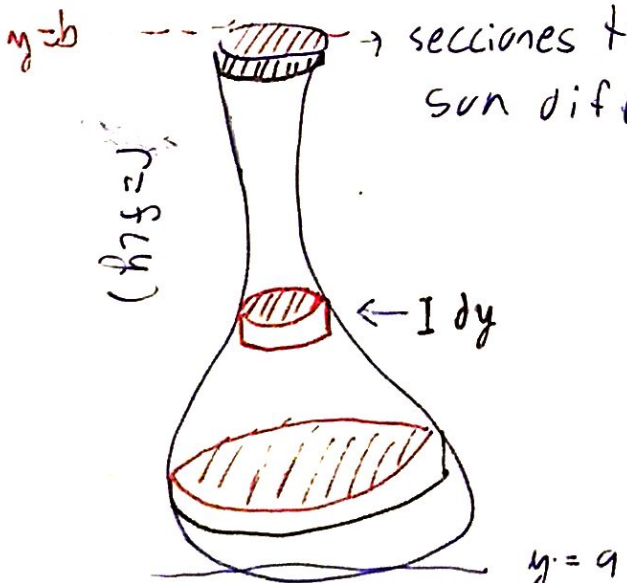
$V = Ah = \pi r^2 h.$

cilindro circular



sección transversal

$V = Ah.$



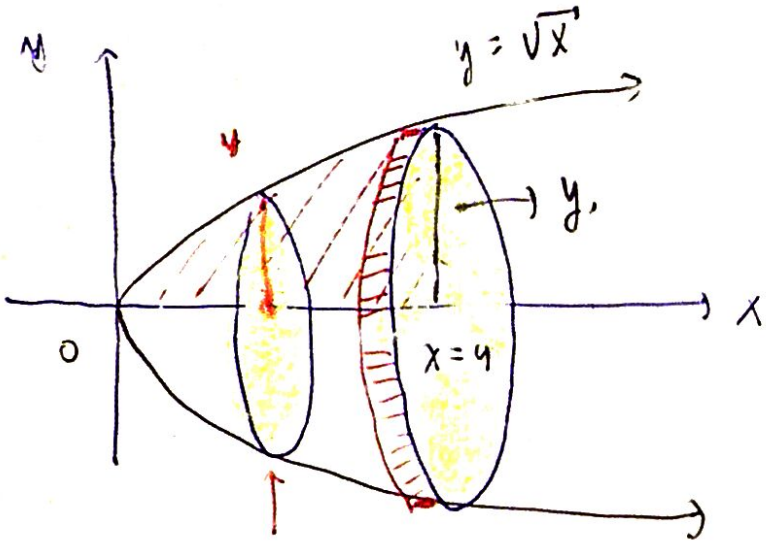
volumen de una sección infinitesimal, el cual es un cilindro con altura dy. y área $A(y)$.

$$dV = A(y)\,dy$$

Integrando en $a \leq y \leq b$.

Volumen total del sólido $\boxed{V = \int_a^b A(y)\, dy.}$

Ejemplo: Considere la región $0 \leq y \leq \sqrt{x}$
$0 \leq x \leq 4$.



rote la región respecto al eje-x obtiene un sólido de revolución.

sección transversal círculo de radio y.

$A(y) = \pi y^2$. grosor/altura dx.

$$V = \int_0^4 \pi y^2\, dx = \int_0^4 \pi (\sqrt{x})^2\, dx = \pi \int_0^4 x\, dx$$

$$V = \frac{\pi}{2} x^2 \Big]_0^4 = \frac{\pi}{2} 16 = 8\pi.$$

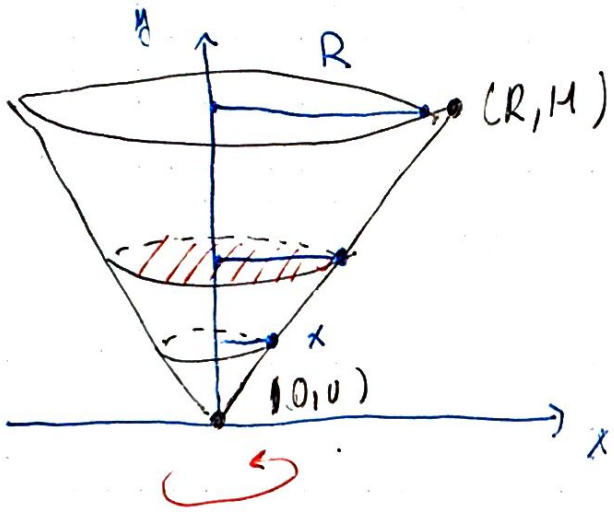
Capítulo 16

Volúmenes:

$$V = \int_a^b A(x)\,dx$$

$A(x)$ es el área de la sección transversal del sólido.

Ejemplo: Encuentre el volumen de un cono de altura H y base circular de radio R.



Rebanando el cono círculos de radio x.

Área $\pi r^2(y)$

$$V = \int_0^H \pi x^2\,dy.$$

Ec. Recta $y = mx + b$

$$m = \frac{H-0}{R-0} = \frac{H}{R}$$

$$0 = 0 + b \Rightarrow b = 0$$

$$\boxed{y = \frac{H}{R}x}$$

$$x = \frac{Ry}{H}$$

$$V = \int_0^H \pi x^2\,dy.$$

$$V = \int_0^H \pi \frac{R^2 y^2}{H^2}\,dy. = \pi \frac{R^2}{H^2}\int_0^H y^2\,dy.$$

$$\pi \frac{R^2}{H^2}\left.\frac{y^3}{3}\right]_0^H = \frac{\pi R^2}{H^2}\frac{H^3}{3}.$$

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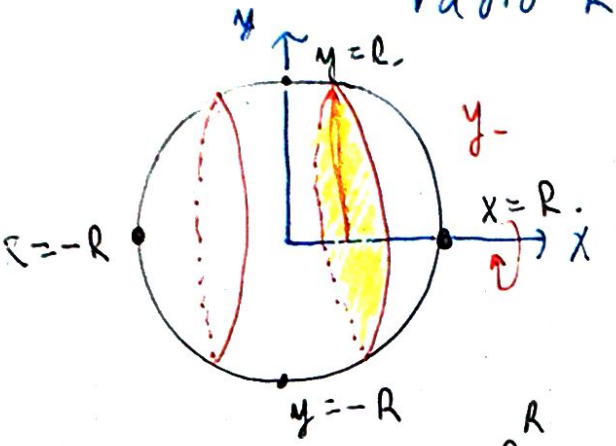
$$\boxed{V = \frac{1}{3}\pi R^2 H.}$$

Ejercicio 1: Encuentre el volumen de una esfera de radio R.



Ec. esfera $x^2+y^2+z^2=R^2$.

Rebanando la esfera, se obtiene un círculo de radio de radio y.

Ec. círculo $x^2+y^2=R^2$.

Volumen: $V=\int_{-R}^{R} A(x)\,dx = \int_{-R}^{R} \pi y^2\,dx$

$-R \leq x \leq R$.

$y=0 \quad x^2=R^2$.

$y^2 = R^2 - x^2$.

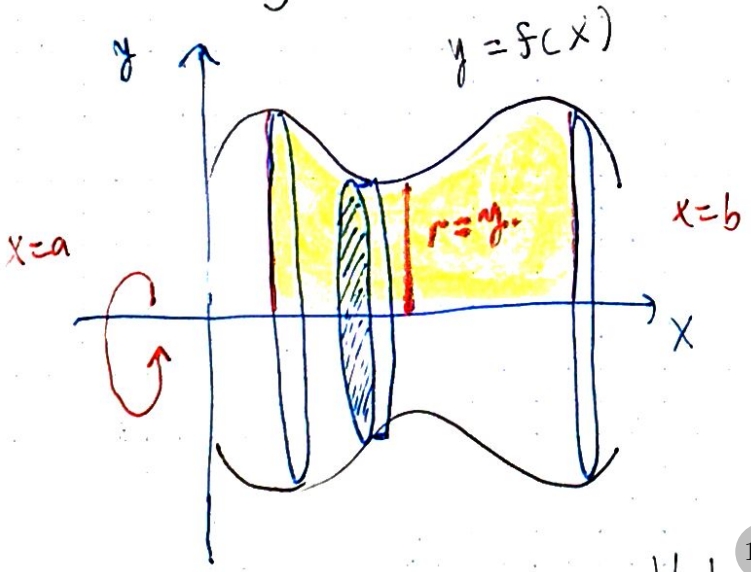
$$V=\int_{-R}^{R} \pi (R^2-x^2)\,dx. = 2\pi\int_0^R (R^2-x^2)\,dx$$

R es constante

$1-\frac{1}{3}=\frac{2}{3}$

$$V = 2\pi\left(R^2x-\frac{1}{3}x^3\Big]_0^R\right) = 2\pi\left(R^3-\frac{1}{3}R^3\right) = 2\pi\,\frac{2}{3}R^3$$

$$V=\frac{4\pi}{3}R^3$$



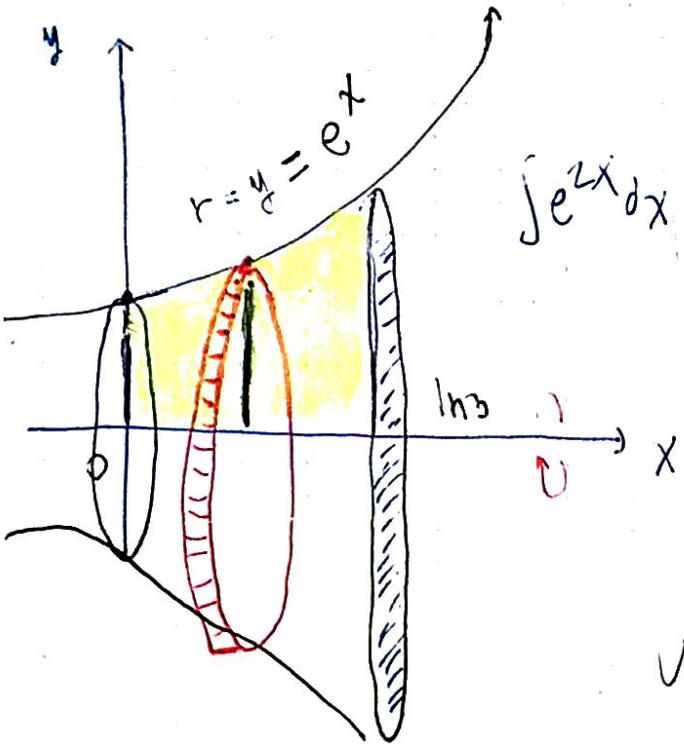
Sólidos rellenos discos (secciones transversales)

$R: a \leq x \leq b.$
$0 \leq y \leq f(x)$

Área $= \pi y^2 = \pi f^2(x)$.

Volumen $\boxed{V=\int_a^b \pi y^2\,dx}$

Ejercicio 2: (P 90) Encuentre el volumen del sólido obtenido al girar la región $R: 0 \le x \le \ln 3,\ 0 \le y \le e^x$ respecto al eje-x.



$$V = \pi \int_a^b r^2\, dx$$

$$\int e^{2x}\, dx = \frac{e^{2x}}{2} \qquad a = 0,\ b = \ln 3,\ r = e^x$$

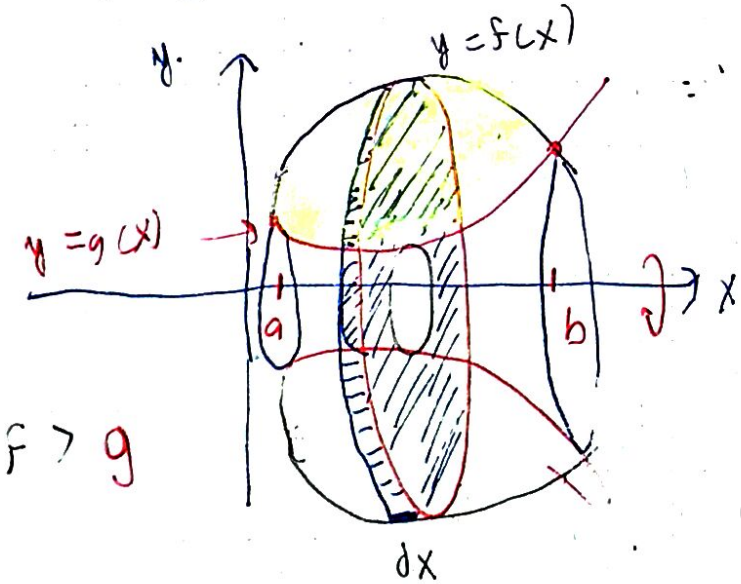
$$V = \pi \int_0^{\ln 3} e^x e^x\, dx$$

$$V = \pi \int_0^{\ln 3} e^{2x}\, dx$$

$$V = \frac{\pi}{2} e^{2x} \Big]_0^{\ln 3} = \frac{\pi}{2}\left(e^{2\ln 3} - e^0\right)$$

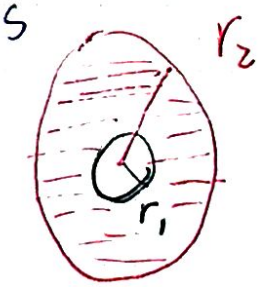
$$= \frac{\pi}{2}(9-1) = 4\pi.$$

Sólidos que se generan girando una región que está entre dos curvas. "Sólidos huecos"



$f > g$

La sección de este sólido es un anillo con dos radios



reste el área del círculo hueco

$$A = \pi r_2^2 - \pi r_1^2.$$

El volumen de la sección transversal es

$r_{ext} = f(x) > r_{int} = g(x)$.

$$V = \int_a^b A(x)\,dx = \pi \int_a^b \left[f^2(x) - g^2(x)\right] dx$$

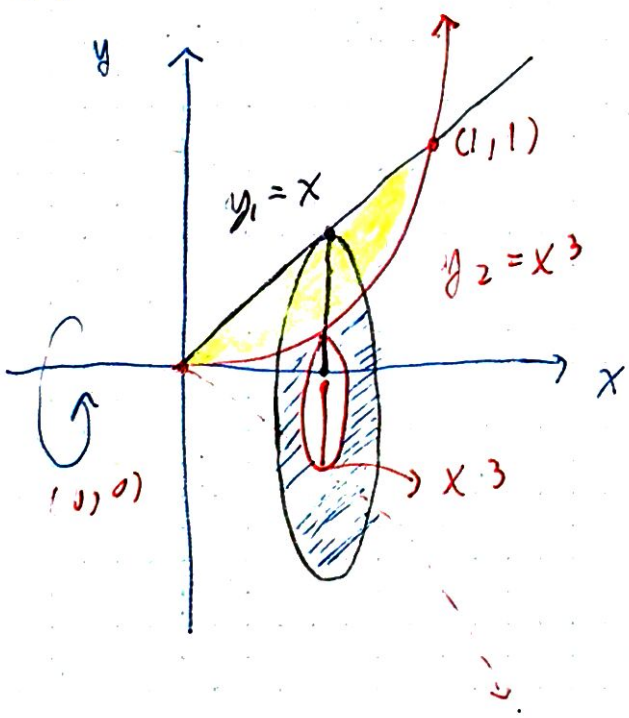
exterior^2 − interior^2

Método de Arandelas $R: a \leq x \leq b,\ g(x) \leq y \leq f(x)$.

el volumen del sólido obtenido al girar R respecto al eje-x es:

$$V = \pi \int_a^b \left(f^2 - g^2\right) dx$$

Ejercicio 4: Calcule el volumen del sólido que se obtiene al girar la región encerrada por las curvas $y_1 = x$ & $y_2 = x^3$ respecto al eje-x. en el 1er cuadrante.



$V = V_{externa} - V_{interna}$.

Área Anillo.

$r_{ext} = x \qquad r_{int} = x^3$.

$A = \pi r_{ext.}^2 - \pi r_{int}^2$

$A = \pi x^2 - \pi x^6$

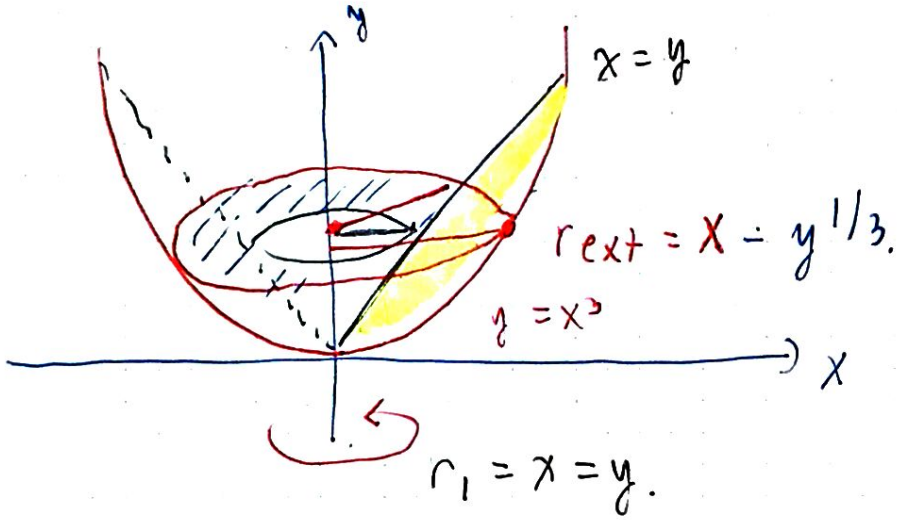
P. I.s. $x^3 = x \Rightarrow x(x^2-1) = 0$

$x = 0, \pm 1$

$$V = \int_0^1 A(x)\,dx = \pi \int_0^1 (x^2 - x^6)\,dx$$

$$V = \pi \left(\frac{x^3}{3} - \frac{x^7}{7} \Big]_0^1 \right) = \pi \left(\frac{1}{3} - \frac{1}{7} \right)$$

Ejercicio 5: <u>Plantee</u> la integral para encontrar el volumen del sólido que se obtiene al girar la región del ejercicio 4 respecto al eje- y.



$y = x^3$.

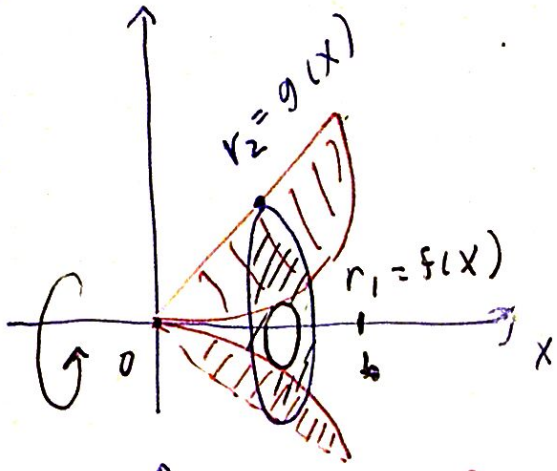
$x = y^{1/3}$

Área Anillo $\pi r_{ext}^2 - \pi r_{int}^2 = \pi y^{2/3} - \pi y^2$.

$$V = \int_0^1 A(y)\,dy. = \int_0^1 (\pi y^{2/3} - \pi y^2)\,dy.$$

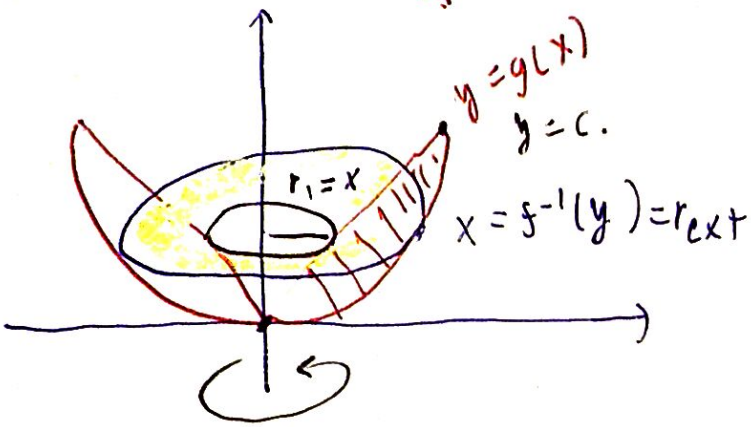
Capítulo 17

Volúmenes



$$A = \pi r_2^2 - \pi r_1^2$$

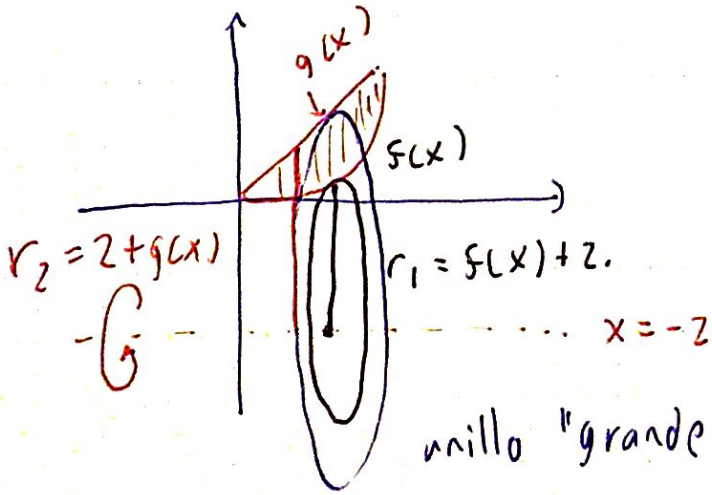
$$V = \pi \int_0^b r_2^2 - r_1^2 \, dx.$$



$$r_1 = g^{-1}(y)$$
$$r_2 = f^{-1}(y).$$

$$V = \pi \int_0^c r_2^2 - r_1^2 \, dy.$$

Rotación respecto a una recta horizontal o vertical (P 96).

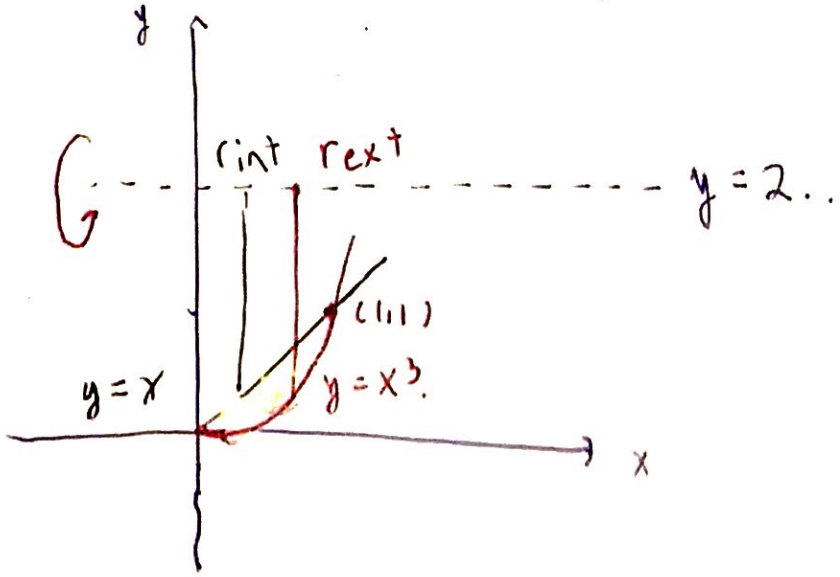


$$A = \pi r_2^2 - \pi r_1^2$$

$$V = \pi \int_0^b (2+g)^2 - (2+f)^2 \, dx$$

Ejercicio 6: Considere la región entre $f(x)=x$ y $g(x)=x^3$.

a. Plantee la integral para encontrar el volumen del sólido al girar la región respecto a $y=2$. en el 1er cuadrante.



$r_{ext} = 2-g(x)$
$= 2-x^3$.

$r_{int.} = 2-f(x)$
$= 2-x$.

$f(x)=g(x)$

$x^3=x \qquad x^3-x=0$

$x(x^2-1)=0$

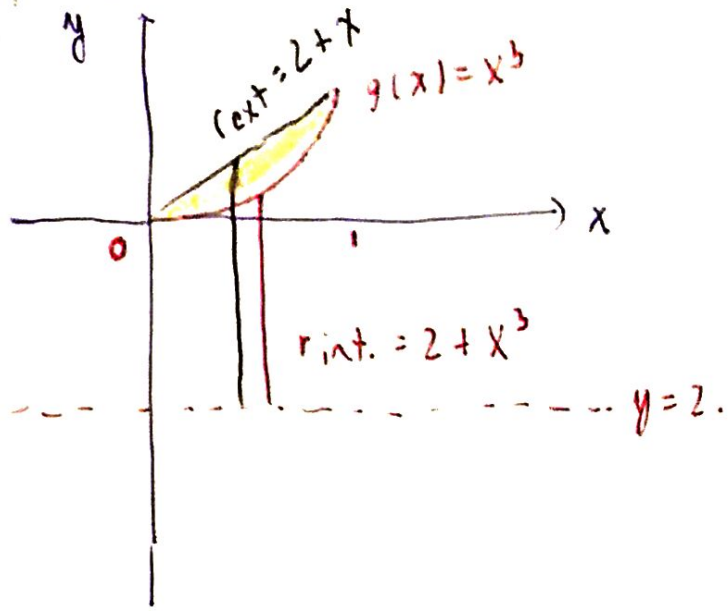
Límites $0 \leq x \leq 1 \qquad x=0,1$

Área Anillo: $\pi r_{ext}^2 - \pi r_{int}^2$

Volumen: $V=\pi\int_0^1 (2-x^3)^2-(2-x)^2\,dx$.

$$V=\pi\int_0^1 (x^6-4x^3-x^2+4x)\,dx = \frac{17\pi}{21}$$

b. Encuentre el volumen obtenido al girar la región respecto a $y=-2$.



$r_{ext} = 2 + x$ ✓

$r_{int} = 2 + x^3$ ✓

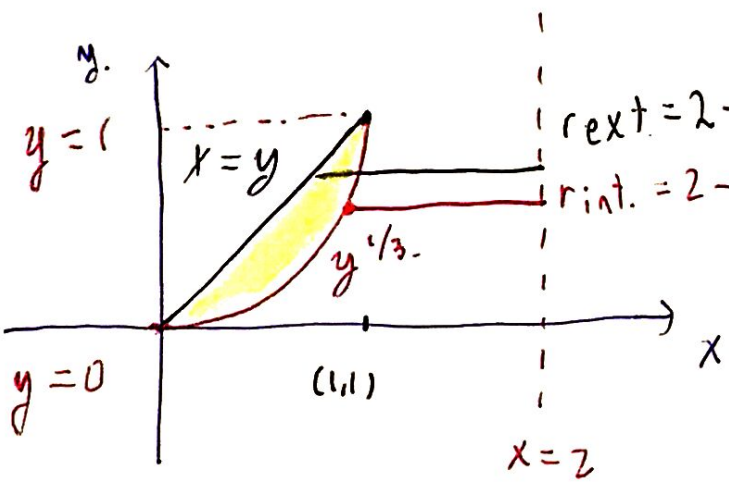
$0 \leq x \leq 1.$

$$V = \pi \int_0^1 r_{ext}^2 - r_{int}^2 \, dx.$$

$$V = \pi \int_0^1 (2+x)^2 - (2+x^3)^2 \, dx$$

$$V = \pi \int_0^1 (4x + x^2 - 4x^3 - x^6) \, dx = \frac{32\pi}{21}.$$

c. Plantee la integral del volumen del sólido girando la región alrededor de $x = 2$.

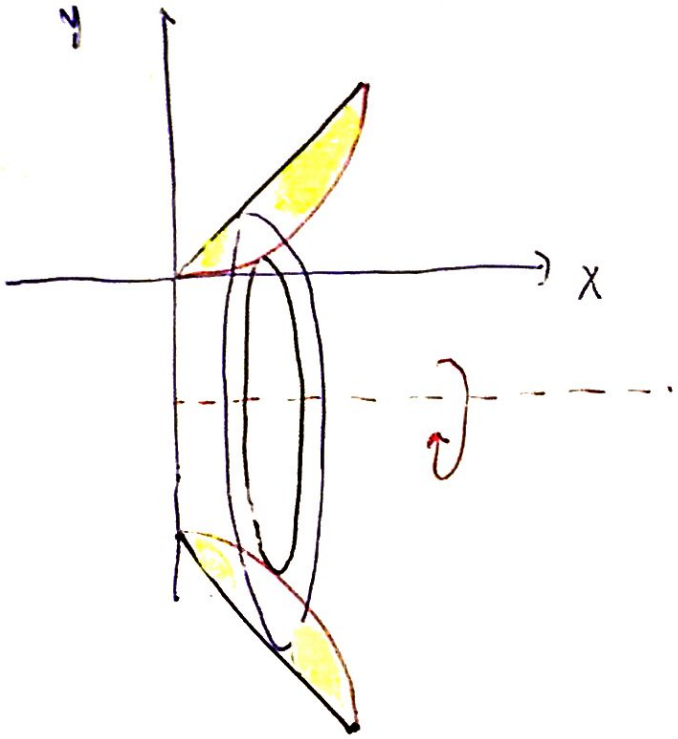


$y = x^3 \Rightarrow x = y^{1/3}.$

Limites $0 \leq y \leq 1.$

$$V = \pi \int_0^1 r_{ext}^2 - r_{int}^2 \, dy.$$

$$V = \pi \int_0^1 (2-y)^2 - (2-y^{1/3})^2 \, dy.$$

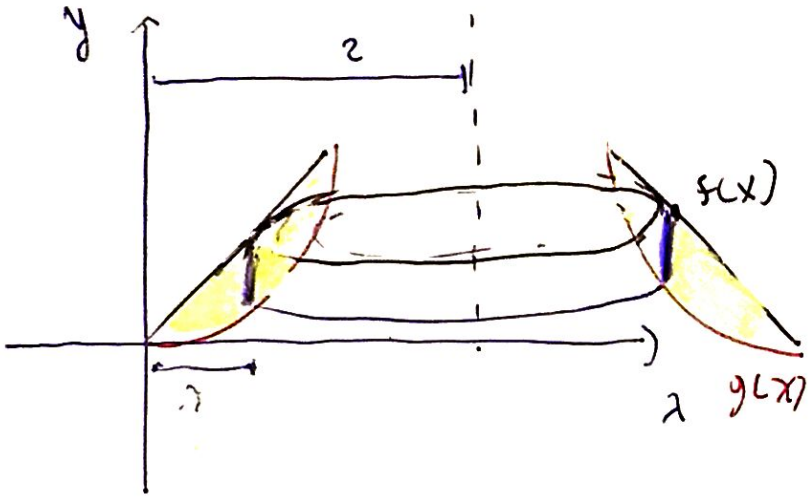


Girar respecto al eje-x o respecto a una recta horizontal.

Es preferible utilizar anillos o discos.

$r_2 = a + f(x)$

$r_1 = a + g(x)$.



anillos si gira respecto a una recta pero hay que encontrar las inversas de f & g.

Encuentre las dimensiones del cilindro.

Altura $h = f(x) - g(x) = x - x^3$. $0 \leq x \leq 1$.

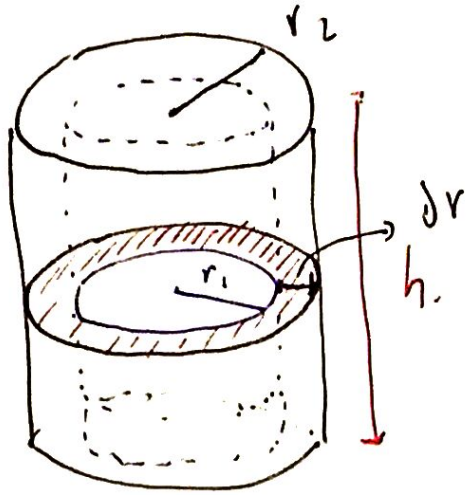
Radio $r = 2 - x$

Volumen: $V = 2\pi \int_0^1 hr \, dx$

Cascarones Cilíndricos.

$$V = 2\pi \int_0^1 (x - x^3)(2 - x) \, dx.$$

6.3 Volúmenes por medio de cascarones cilíndricos. (p 99) (lata).



Volumen Cascarón.

Sección transversal Anillo.

$r_{ext} = r_2 \qquad r_{int} = r_1.$

$$V = A\,h = \pi(r_2^2 - r_1^2)\,h.$$

espesor de la lata.

$$V = \pi h (r_2 + r_1)(r_2 - r_1)$$

$dr = r_2 - r_1$

$r = \frac{1}{2}(r_2 + r_1)$

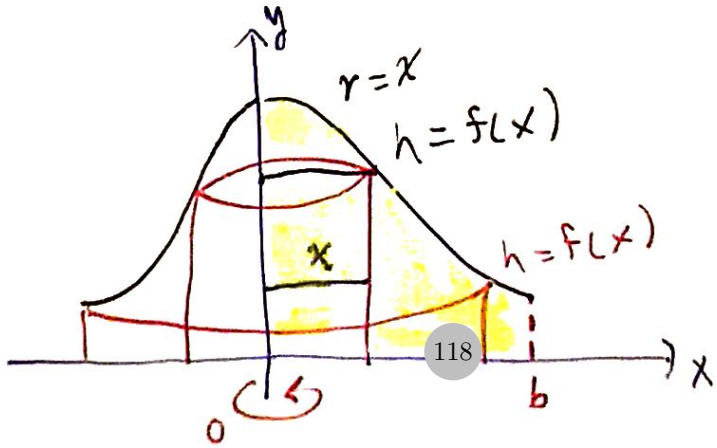
radio promedio r.

$$V = \pi h \; 2r \; dr$$

$$\boxed{V = 2\pi \, h r \, dr}$$ Volumen cascarón cilíndrico/Lata.

si la región $R: a \le x \le b,\ 0 \le y \le f(x)$ se gira respecto al eje-y. el volumen del sólido es:

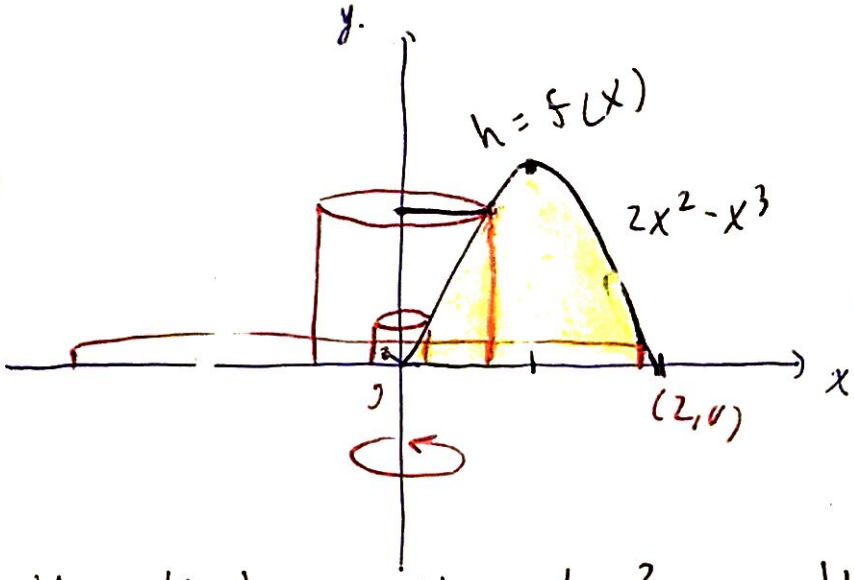
$$V = 2\pi \int_a^b r h \, dr = 2\pi \int_a^b x \, f(x) \, dx.$$



Evitar calcular la inversa de $f(x)$ y de integrar en el eje-y.

Ejemplo: Encuentre el volumen del sólido al girar la región entre el eje x y la curva $f(x)=2x^2-x^3$ respecto al eje-y. (p 100).

Interceptos-x: $2x^2-x^3=0$ $\quad x^2(2-x)=0$ $\quad x=0,2.$



Dimensiones del cilindro.

Altura $h=2x^2-x^3$

radio $r=x$

límites $0\le x\le 2.$

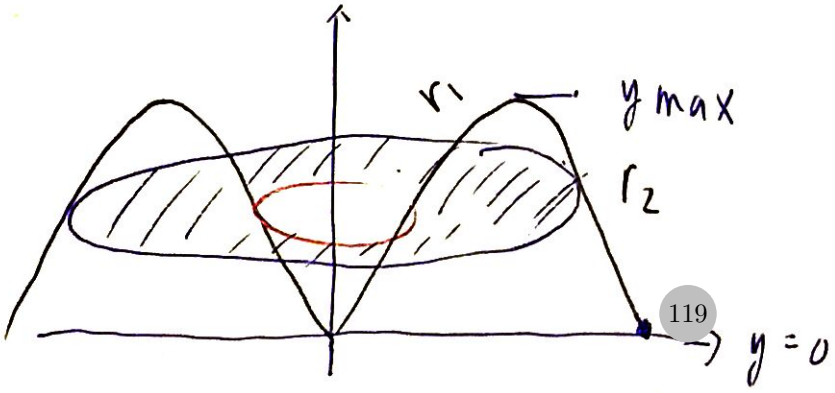
$$V=\int_0^2 2\pi h r\,dr.$$

V cilindro $\quad V=\pi h r^2.$ $\quad dV=\pi h\,2r\,dr.$

$$V=\int_0^2 2\pi(2x^2-x^3)x\,dx = 2\pi\int_0^2 2x^3-x^4\,dx$$

$$V=2\pi\left(\frac{x^4}{2}-\frac{x^5}{5}\Big]_0^2\right)=2\pi\left(\frac{16}{2}-\frac{32}{5}\right)$$

Es difícil de plantear con arandelas.



$0\le y\le y_{max}.$

$r_1=f^{-1}_{izq}(y)$

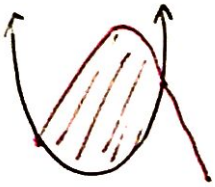
$r_2=f^{-1}_{der}(y)$

$$V=\pi\int_0^{y_{max}} r_2^2-r_1^2\,dy.$$

No se puede encontrar la inversa de $2x^2 - x^3 = y$.

Ejercicio 2: Encuentre el volumen del sólido obtenido al girar la región entre $y_1 = x^2$ & $y_2 = 6x - 2x^2$ alrededor del eje-y.

$y_2 = 0$ cuando $x = 0, 3$

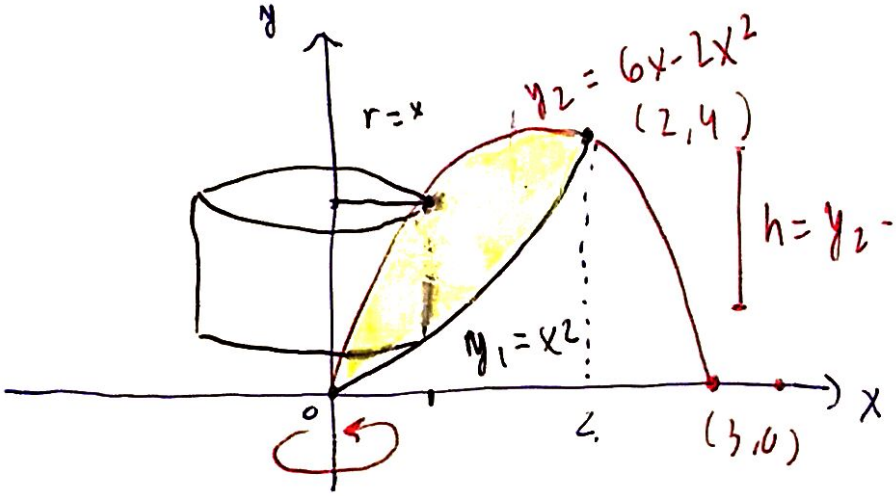


$y_1 = y_2$. $\quad x^2 = 6x - 2x^2$.

$3x^2 - 6x = 3x(x-2) = 0$

$x = 0, 2$.

$y_1(0) = 0 \qquad y_2(2) = 4$.



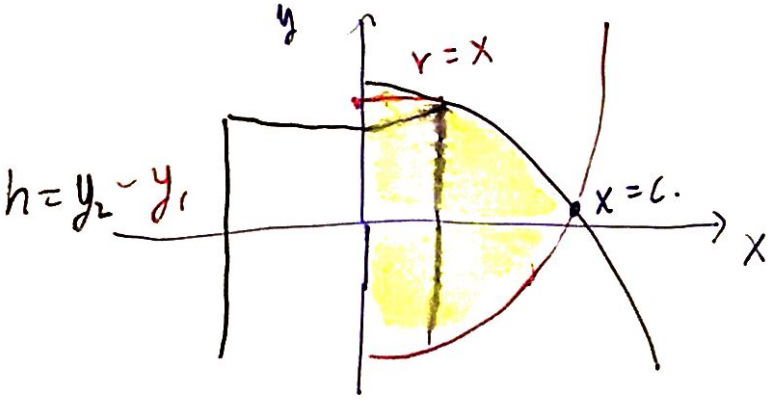
Altura

$h = y_2 - y_1$

$h = 6x - 2x^2 - x^2$.

radio $r = x$.

$0 \leq x \leq 2$.



$$V = \int_0^2 2\pi r h \, dx$$

$$V = 2\pi \int_0^2 (6x - 3x^2)\, x \, dx$$

$$V = 2\pi \int_0^2 (6x^2 - 3x^3)\, dx.$$

$$V = 2\pi \left(2x^3 - \frac{3}{4} x^4 \Big]_0^2 \right) = 2\pi (16 - 3 \cdot 4) = 8\pi$$

Capítulo 18

0.5 Valor Promedio de una función (P 105 - P108)

El valor promedio de una función continua en un intervalo [a,b] es igual a.

$$\boxed{f_{prom} = \frac{1}{b-a} \int_a^b f(x)\,dx}$$

$$f_{prom} = \frac{f_1 + f_2 + \dots f_n}{n} = \sum_{i=1}^{n} \frac{1}{n} f(x_i)$$

Número infinitos y continuo de puntos, dividir el intervalo [a,b] en n subintervalos.

$$\Delta x = \frac{b-a}{n} \Rightarrow \frac{\Delta x}{b-a} = \frac{1}{n}$$

$$f_{prom} \approx \sum_{i=1}^{n} \frac{1}{n} f(x_i) = \sum_{i=1}^{n} \frac{1}{b-a} f(x_i)\,\Delta x$$

Para tener una respuesta $n \to \infty$, $\Delta x \to 0$.

$$f_{prom} = \lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i)\,\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Ejemplo: Encuentre el valor promedio de $f(x) = \csc^2 x$ en $[\pi/4, \pi/2]$.

$b = \frac{\pi}{2}$ $\quad a = \frac{\pi}{4}$ $\quad b - a = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ } Ancho del intervalo.

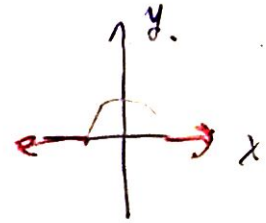
$$f_{prom} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{4}{\pi}\int_{\pi/4}^{\pi/2} \csc^2 x\,dx.$$

$$f_{prom} = \frac{4}{\pi}\int_{\pi/2}^{\pi/4} -\csc^2 x\,dx = \frac{4}{\pi}\cot x\Big]_{\pi/2}^{\pi/4} = \frac{4}{\pi}(1-0) = \frac{4}{\pi}$$

$\cot x = \frac{\cos x}{\sin x}$ $\quad \cot\frac{\pi}{4} = 1$ $\quad \cot\frac{\pi}{2} = \frac{0}{1} = 0$

Ejercicio 6: Encuentre el valor promedio de promedio de las sigs. funciones en el intervalo dado.

a. $f(t) = \cos^4 t \sin t$ en $[0, \pi]$.



$$f_{prom} = \frac{1}{\pi - 0}\int_0^{\pi} \cos^4 t \underbrace{\sin t\,dt}_{du}.$$

$u = \cos t$ $\quad u(\pi) = \cos\pi = -1$

$du = -\sin t\,dt.$ $\quad u(0) = \cos 0 = 1$

$$f_{prom} = \frac{-1}{\pi}\int_1^{-1} u^4\,du = \frac{1}{\pi}\int_{-1}^{1} u^4\,du = \frac{2}{\pi}\int_0^1 u^4\,du.$$

$$f_{prom} = \frac{2}{\pi}\,\frac{u^5}{5}\Big]_0^1 = \boxed{\frac{2}{5}\,\frac{1}{\pi}}$$

b. $g(x) = \dfrac{1}{x}$ en $[e^4, e^{10}]$

$$y_{prom} = \frac{1}{e^{10}-e^4}\int_{e^4}^{e^{10}} \frac{1}{x}\,dx = \frac{1}{e^{10}-e^4}\ln|x|\Big]_{e^4}^{e^{10}}$$

$\ln(e^{10}) = 10 \quad \ln(e^4) = 4 \qquad = \dfrac{1}{e^{10}-e^4}(10-4) = \dfrac{6}{e^{10}-e^4}$

c. $h(x) = \dfrac{3}{(4+x)^{1/2}}$ en $[-4, 5]$ no es continua.

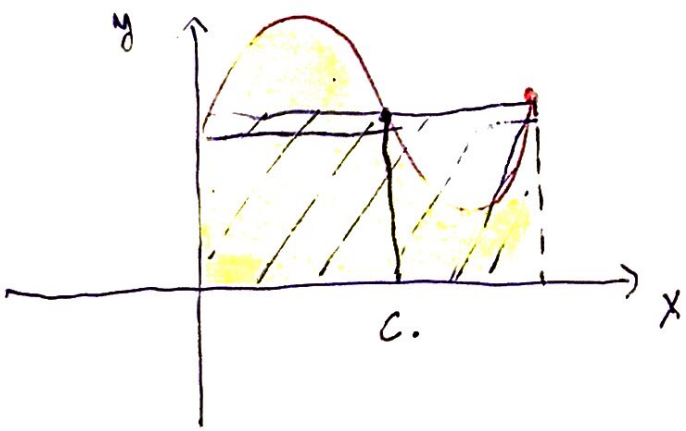
$h(-4) = \dfrac{3}{0}$ indefinido, integral impropia.

$$h_{prom} = \frac{3}{5-(-4)}\int_{-4}^{5}(4+x)^{-1/2}\,dx \qquad -\frac{1}{2}+1 = \frac{1}{2}.$$

$$h_{prom} = \frac{3}{9}\,2(4+x)^{1/2}\Big]_{-4}^{5} \qquad (4-4)^{1/2} = 0^{1/2} = 0$$

$$h_{prom} = \frac{6}{9}\left(4^{1/2} - \lim_{x\to -4^+}(4+x)^{1/2}\right) = \frac{6}{9}\cdot 3 = \boxed{2.}$$

Relación entre el valor promedio y el área de una región



$$f_{prom} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\underbrace{f_{prom}}_{h}\,\underbrace{(b-a)}_{b.} = \underbrace{\int_a^b f(x)\,dx}_{\text{Área Región}}$$

El área de la región es igual al área de un rectángulo con altura f_{prom} y ancho $b-a$.

Teorema del Valor para Integrales.

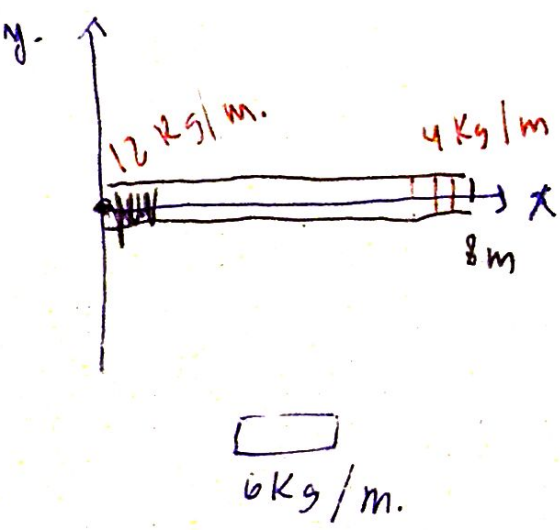
Si f es continua en $[a,b]$ entonces existe por lo menos un número c tal que

$$\boxed{f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

c está entre a y b.

Ejercicio 2: La densidad lineal de una varilla de 8 m de longitud es $\rho = \frac{12}{\sqrt{x+1}}$ kg/m.

a. Encuentre la densidad promedio de la varilla.



$$\rho_{prom} = \frac{1}{8-0}\int_0^8 12(x+1)^{-1/2}\,dx$$

$$\rho_{prom} = \frac{12}{8}\cdot 2\,(x+1)^{1/2}\Big]_0^8$$

$$\rho_{prom} = 3\left(9^{1/2} - 1^{1/2}\right)$$

$$\rho_{prom} \quad 3(3-2) = 6.$$

b. Encuentre la posición x de la varilla donde la densidad lineal es igual a la densidad promedio.

$$\rho(x) = \frac{12}{\sqrt{x+1}} \qquad \rho_{prom} = 6.$$

$$6 = \frac{12}{\sqrt{x+1}} \Rightarrow \sqrt{x+1} = \frac{12}{6} = 2.$$

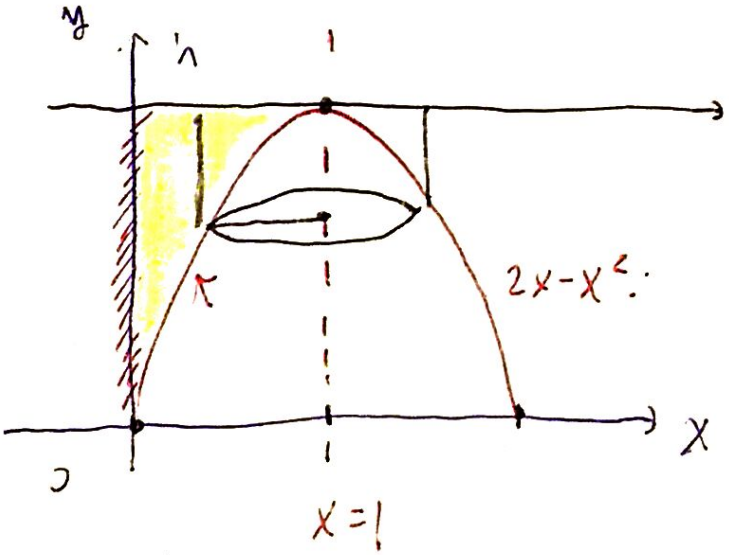
$$x+1 = 4 \Rightarrow x = 3.$$

pág 103 Considere la región entre $y_1 = 2x - x^2$, $y_2 = 1$ & $x = 0$.

Encuentre el volumen al rotar la región respecto a la recta $x = 1$.

$y_1 = y_2$ $\quad 1 = 2x - x^2 \Rightarrow x^2 - 2x + 1 = 0.$

$$(x-1)(x-1) = 0 \Rightarrow x = 1.$$



Altura $h = 1 - (2x - x^2)$

Radio $r = 1 - x$

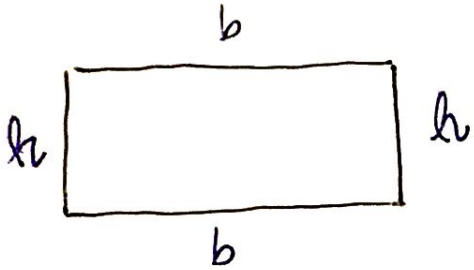
$$V = 2\pi \int_0^1 h\, r\, dx$$

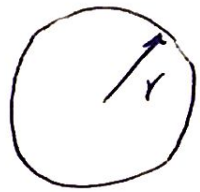
$$V = 2\pi \int_0^1 (1-x)(1-2x+x^2)\, dx$$

Capítulo 19

8.1 Longitud de una curva.

Geometría: longitud o perímetro



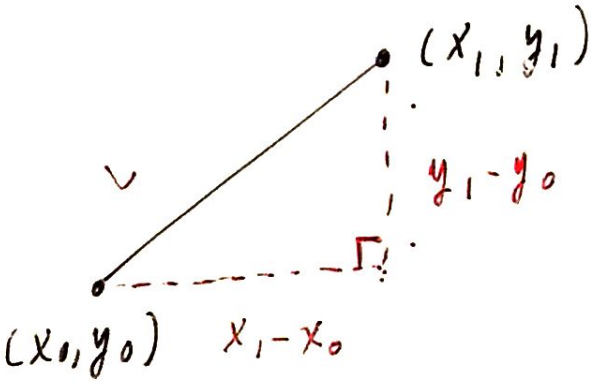


Rectángulo $L = 2b + 2h$

Circunferencia

$L = 2\pi r.$

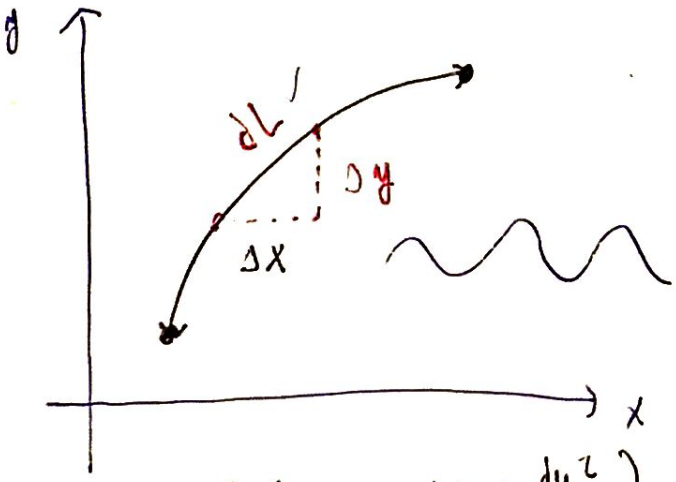
segmento de recta



$$L = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Longitud de una curva: longitud de arco.



curva $\mathcal{C}$: $y = f(x)$
$a \leq x \leq b.$

región $\mathcal{R}$ sólido S.

parte infinitesimal del arco.

$$dL = \sqrt{(dx)^2 + (dy)^2}$$

$$dx^2 + dy^2 = dx^2\left(1 + \frac{dy^2}{dx^2}\right)$$

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Longitud L: integre dL en $a \leq x \leq b$.

Longitud de Arco:

$$\boxed{L = \int_a^b \sqrt{1+[y'(x)]^2}\, \underbrace{dx}_{\text{dado.}}}$$

$$\frac{dy}{dx} = y'(x)$$

Utilice sólo esta fórmula. (P 109.)

Ejemplo: Halle la longitud de la curva $\mathcal{C}: 0 \leq x \leq \frac{8}{9}$

$$y = 1 + 2x^{3/2}.$$

$$y'(x) = \frac{6}{2} x^{1/2} = 3x^{1/2}$$

$$1 + [y'(x)]^2 = 1 + (2x^{1/2})^2 = 1 + 9x.$$

Longitud de Arco:

$$L = \int_0^{8/9} \sqrt{1+(y')^2}\, dx$$

$$L = \int_0^{8/9} \sqrt{1+9x}\, \underbrace{dx}_{du/9.}$$

$u = 1 + 9x$

$du = 9\,dx$

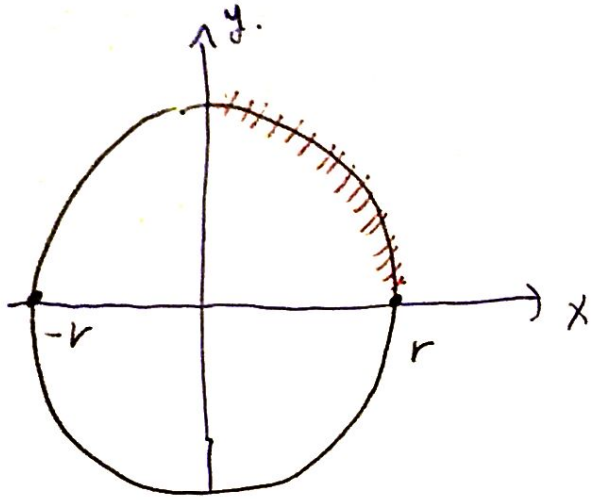
$u(8/9) = 1 + 8 = 9$

$u(0) = 1 + 0 = 1$

$$L = \int_1^9 u^{1/2} \frac{du}{9} = \frac{2}{3} \cdot \frac{1}{9} u^{3/2} \Big]_1^9$$

$$L = \frac{2}{27}\left((3^2)^{3/2} - 1^{3/2}\right) = \frac{2}{27}(3^3 - 1) = \frac{2 \cdot 26}{27}.$$

Ejercicio 1: Encuentre la longitud (o perímetro) de una circunferencia de radio r.



$$x^2 + y^2 = r^2.$$ Ec. Circunferencia.

$$y^2 = r^2 - x^2$$

$$y = +\left(r^2 - x^2\right)^{1/2}$$ ✓ Semi Circunferencia Superior.

$$-r \leq x \leq r.$$

$$L = 4\int_0^r \sqrt{1 + [y'(x)]^2}\, dx.$$

$$y'(x) = \frac{1}{2}\left(r^2 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$[y'(x)]^2 = \frac{x^2}{r^2 - x^2}.$$

$$1 + [y'(x)]^2 = \frac{1}{1} + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$

$$L = 4\int_0^r \sqrt{\frac{r^2}{r^2 - x^2}}\, dx = 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} \qquad \sin^{-1}\left(\frac{x}{r}\right)$$

$$L = 4r \sin^{-1}\left(\frac{x}{r}\right)\Big]_0^r = 4r\left(\overset{\pi/2}{\sin^{-1}(1)} - \overset{0}{\sin^{-1}(0)}\right)$$

$$= 4r\,\frac{\pi}{2} = 2\pi r.$$

Indefine en $x = r$, integral impropia convergente.

Ejercicio 2: p 101. Un cable telefónico cuelga entre dos postes con posiciones horizontales en $x = \pm 25$.

El cable toma la forma de una catenaria con ecuación

$$y = -5 + 25\cosh\left(\frac{x}{25}\right). \qquad \sqrt{1+(y')^2}$$

Halle la longitud del cable.

$$y'(x) = 25\sinh\left(\frac{x}{25}\right)\frac{1}{25} = \sinh\left(\frac{x}{25}\right)$$

$$1+[y'(x)]^2 = 1+\sinh^2\left(\frac{x}{25}\right) = \cosh^2\left(\frac{x}{25}\right),$$

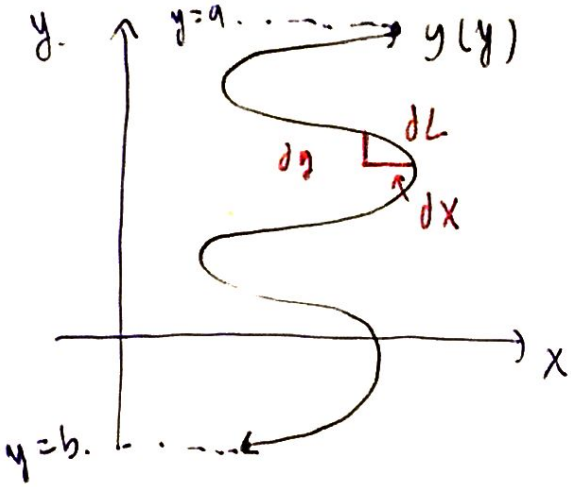
$$L = \int_{-25}^{25}\sqrt{1+(y')^2}\,dx = \int_{-25}^{25}\sqrt{\cosh^2\left(\frac{x}{25}\right)}\,dx$$

$$L = \int_{-25}^{25}\cosh\left(\frac{x}{25}\right)dx = 2\int_{0}^{25}\cosh\left(\frac{x}{25}\right)dx$$

$$L = 2\cdot 25\sinh\left(\frac{x}{25}\right)\Big]_0^{25} = 50\left(\sinh 1 - \overset{0}{\sinh 0}\right)$$

$$= 50\sinh 1 \approx 58.76.$$

Integración en el eje-y



curva $\mathcal{C}$: $a \leq y \leq b$, $x = g(y)$

$$dL = \sqrt{(dx)^2 + (dy)^2}$$

$$dL = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; dy.$$

$$L = \int_a^b \sqrt{[g'(y)]^2 + 1}\; dy.$$

Ejercicio 3: Encuentre la longitud de arco. para las curvas dadas. (P. 112).

a. $\mathcal{C}_1$: $x = \frac{y^3}{6} + \frac{1}{2y}$ $\quad 1 \leq y \leq 2.$

$y^2 y^{-2} = 1$

$$x'(y) = \frac{3y^2}{6} - \frac{1}{2y^2} = \frac{1}{2}y^2 - \frac{1}{2}y^{-2} = \frac{1}{2}(y^2 - y^{-2})$$

$$[x'(y)]^2 = \frac{1}{4}(y^2 - y^{-2})^2 = \frac{1}{4}(y^4 - 2 + y^{-4})$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$1 + [x'(y)]^2 = 1 + \frac{1}{4}(y^4 - 2 + y^{-4})$$

$$= \frac{1}{4}(4 + y^4 - 2 + y^{-4})$$

$$= \frac{1}{4}(y^4 + 2 + y^{-4}) = \frac{1}{4}(y^2 + y^{-2})^2.$$

$$a^2 + 2a + 1 = (a+1)(a+1).$$

$$L = \int_1^2 \sqrt{\frac{1}{4}\left(y^2 + y^{-2}\right)^2}\, dy. = \sqrt{\frac{1}{4}} \int_1^2 \left(y^2 + y^{-2}\right) dy.$$

$$L = \frac{1}{2}\left(\frac{y^3}{3} - \frac{1}{y}\Big]_1^2 \right) = \frac{1}{2}\left(\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + \frac{1}{1} \right) = \frac{17}{6 \cdot 2}.$$

5. $\mathcal{C}_2 \quad y = \ln(\csc\theta), \quad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}.$

$$L = \int_{\pi/6}^{\pi/2} \sqrt{1 + (y')^2}\, d\theta.$$

$$y' = \frac{-\csc\theta}{\csc\theta} \cot\theta = -\cot\theta.$$

$$1 + (y')^2 = 1 + \cot^2\theta = \csc^2\theta.$$

$$L = \int_{\pi/6}^{\pi/2} \sqrt{\csc^2\theta}\, d\theta. = \int_{\pi/6}^{\pi/2} \csc\theta\, d\theta.$$

$$L = -\ln|\csc\theta + \cot\theta|\Big]_{\pi/6}^{\pi/2}.$$

$$L = -\ln|\overset{1}{\csc\pi/2} + \overset{0}{\cot\pi/2}| + \ln|\overset{2}{\csc\pi/6} + \overset{\sqrt{3}}{\cot\pi/6}|.$$

$$\frac{1}{\sin\pi/2} = 1 \qquad \frac{\cos\pi/2}{\sin\pi/2} = 0 \qquad \frac{1}{\sin\pi/6} = 2. \qquad \frac{\cos\pi/6}{\sin\pi/6.} = \frac{\sqrt{3}/2}{1/2.}$$

$$L = -\ln(1) + \ln(2 + \sqrt{3})$$

$$L = \ln(2 + \sqrt{3}).$$

Función longitud de arco: $a \leq x \leq t$.

$$s(t) = \int_a^t \sqrt{1+[y']^2}\, dx$$

$y = \ln(\sin t)$, $\frac{\pi}{2} \leq t \leq x$.

$$L = \int_{\pi/2}^{x} \sqrt{1+(y')^2}\, dt.$$

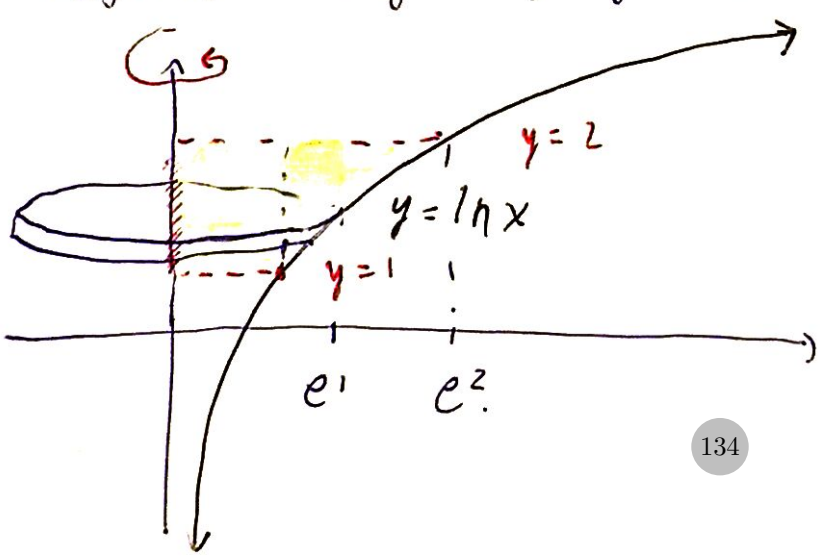
$y' = \frac{-\cos t}{\sin t} = -\cot(t)$ $\quad 1+(y')^2 = 1+\cot^2(t)$
$\csc^2(t)$.

$$L = \int_{\pi/2}^{x} \sqrt{\csc^2(t)}\, dt = \int_{\pi/2}^{x} \csc t\, dt.$$

$$L = -\ln|\csc t + \cot t|\Big]_{\pi/2}^{x} = -\ln|\csc x + \cot x| + \ln(1).$$

Laboratorio 8. Problema 2:

Región $1 \leq y \leq 2$, $y = \ln x$ & $x = 0$.



$y = \ln x$ DISCOS.

$$V = \pi \int_1^2 x^2\, dy.$$

$x = e^y$

$$V = \pi \int_1^2 e^{2y}\, dy.$$

Capítulo 20

1.

Simulacro lunes 7 oct. 2:30 PM CES.
1-2:30 PM Resol Simulacro.

Parcial lunes 14 oct 2:30 PM. CES.
1-2:30 PM Resol. Dudas.

7.6 Integrales Impropias - 8.5 Probabilidad.
7.3 Integración Fracciones Parciales.

8.5 Probabilidad (p. 123).

Una variable aleatoria puede ser:

Discreta: el número de eventos es contable como el lanzamiento de una moneda, un dado, los números de la lotería.

Continua: mediciones como el peso, estatura, volumen. eventos no es contable o es un número real.

Probabilidad de que ocurra a. $P(X=a) = \dfrac{\#\text{ veces que ocurre } a}{\#\text{ total de eventos.}}$

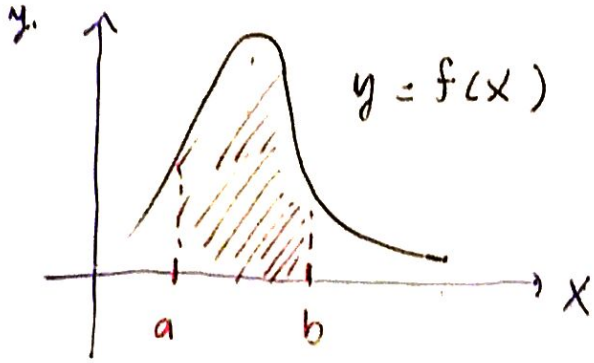
1 dado, 6 eventos posibles. $P(X=\text{par}) = \dfrac{3}{6} = 50\%$

Probabilidad Var. Discreta $P(X=a_1, a_2, \ldots a_n) = \sum_{i=1}^{n} P(X_i)$.

Variable continua: ¿Cuál es la probabilidad de que x ocurra entre a y b?

$$\boxed{P(a \leq x \leq b) = \int_a^b f(x)\,dx}$$

encuentre el área bajo la curva de $y = f(x)$.



$f(x)$ es una función de densidad de probabilidad.

Más Alta del 100% → 1

más Baja del 0% → 0.

Condiciones para que $f(x)$ sea función de densidad de probabilidad.

$f(x) \geq 0$ en todo su dominio $-\infty < x < \infty$.

No negatividad.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$ Probabilidad 1.

Distribuciones de Probabilidad comunes.

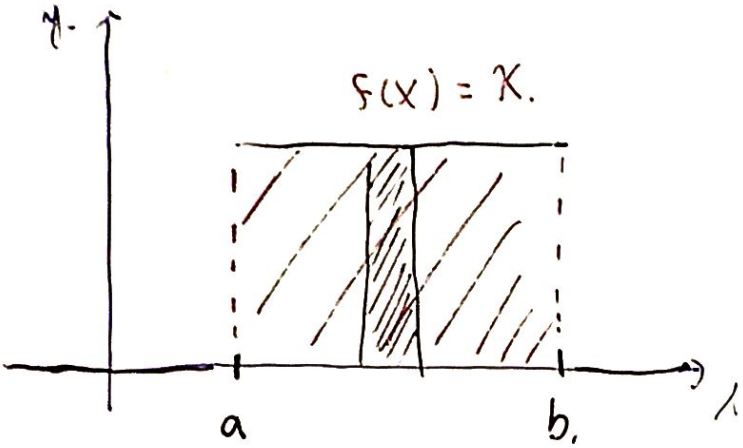
a. Distribución Uniforme.

b. Distribución Exponencial.

c. Distribución Normal

$$\int_a^b f(x)\,dx$$

1. Distribución Uniforme.



Área Rectángulo

base = b - a.

altura = K.

Área = K(b-a) = 1

$K = \frac{1}{b-a}$

100%.

$$U(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b. \\ 0 & x < a \text{ o } x > b. \end{cases}$$

$U(-\infty) = 0$

$U(\infty) = 0.$

si es función de densidad de probabilidad. porque:

$U(x) \geq 0$ en $-\infty \leq x \leq \infty$. ✓

$$\int_{-\infty}^{\infty} U(x)\,dx = 1 \qquad \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}\,x\Big]_a^b$$

$$= \frac{1}{b-a}(b-a) = 1$$

Ejercicio 1: Un contenedor tiene mercancía cuyo peso tiene una distribución uniforme entre 2 y 4 tons.

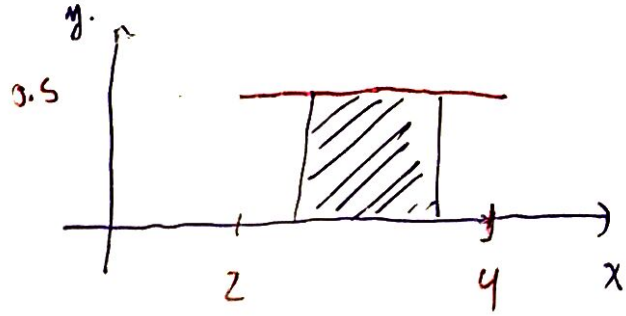
$$U(x) = \frac{1}{b-a} \qquad a \leq x \leq b.$$

a = número mínimo

b = número máximo.

a. Calcule la probabilidad de que un contenedor pese entre 2.5 y 3.5 toneladas.

$$U(x) = \frac{1}{2} \qquad 2 \leq x \leq 4$$



$$P(2.5 \leq x \leq 3.5) = \int_{2.5}^{3.5} U(x)\,dx$$

$$= \frac{1}{2}\int_{2.5}^{3.5} dx = \frac{1}{2}\,x\Big]_{2.5}^{3.5} = \frac{1}{2}(1) = 50\%$$

b. Calcule la probabilidad de que un contenedor pese más de 1 tonelada.

$$P(x \geq 1) = \int_{1}^{\infty} U(x)\,dx = \cancel{\int_{1}^{2} U(x)\,dx}^{\,0} + \int_{2}^{4} \frac{1}{2}\,dx + \cancel{\int_{4}^{\infty} U(x)\,dx}^{\,0}$$

$$P(x \geq 1) = \int_{2}^{4} \frac{1}{2}\,dx = \frac{1}{2}(4-2) = \frac{2}{2} = 100\%.$$

todos los contenedores pesan más de 2 toneladas.

c) Distribución Exponencial.

$$f(t) = \frac{1}{\mu} e^{-t/\mu} \qquad 0 \leq t < \infty.$$

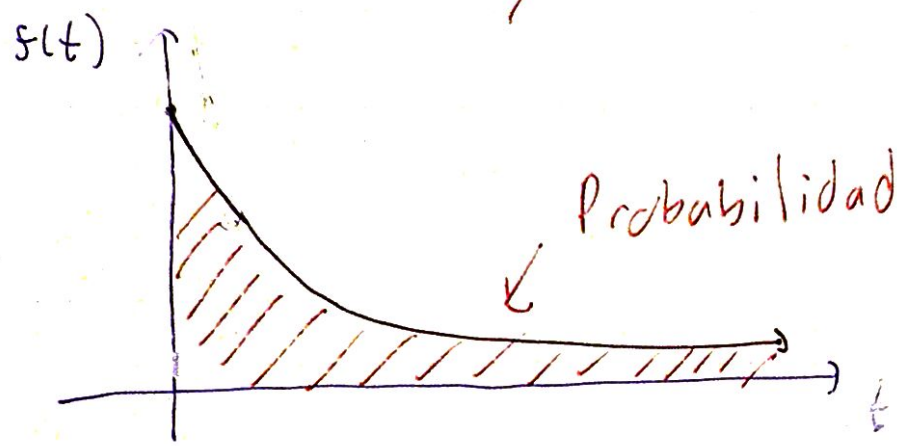
$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{0}^{\infty} \frac{1}{\mu} e^{-t/\mu}\,dt. \qquad u = -t/\mu \qquad du = -dt/\mu.$$

$\mu = mu.$

$$= \int_{0}^{-\infty} -e^{u}\,du. = -e^{u}\Big]_{0}^{-\infty}$$

$e^{-\infty} \to 0$

$$= \lim_{t \to -\infty} -e^{t} + e^{0} = 1$$



Probabilidad 100%
Área igual a 1

Capítulo 21

8.5 Probabilidad

Variable continua, Función de densidad de probabilidad $f(x)$

i. $f(x) \geq 0$ en $-\infty \leq x \leq \infty$.

ii $\int_{-\infty}^{\infty} f(x)\,dx = 1$ 100% probabilidad.

Probabilidad de que x ocurra entre a y b.

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx.$$

probabilidad.

Distribución Exponencial. $f(x) = \frac{1}{\mu} e^{-x/\mu}$. $x \geq 0$.

μ = media.

Ejercicio 2: El tiempo de espera promedio para recibir una orden a domicilio de comida es de 0.5 h y tiene una distribución exponencial.

a. Encuentre la probabilidad de que la comida se reciba después de 0.5 hora.

$\mu = 0.5$ $f(x) = \frac{1}{0.5} e^{-x/0.5} = 2e^{-2x}$ $\frac{1}{0.5} = 2$.

$$P(x \geq 0.5) = \int_{0.5}^{\infty} e^{-2x} \underbrace{2\,dx}_{du} = -e^{-2x}\Big]_{0.5}^{\infty}$$

$u = -2x \qquad du = -2dx$

$$P(X \geq 0.5) = -\lim_{x \to \infty} e^{-2x} + e^{-2(0.5)} = e^{-1} = \frac{1}{e}.$$

$e^{-\infty} \to 0$

36.79 %.

Probabilidad Alta.

b. Encuentre la probabilidad de que la comida se reciba en menos de 15 min, ó 0.25 horas.

$$P(X \leq 0.25) = \int_{-\infty}^{0.25} f(x)dx = \int_{0}^{0.25} e^{-2x}(2dx)$$

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0. \\ 0 & x < 0. \end{cases}$$

$$-e^{-2x}\Big]_0^{0.25} = -e^{-2(0.25)} + e^{-0}$$

$$= 1 - e^{-0.5}.$$

$$\approx 39.34\%.$$

c. El servicio a domicilio se vuelve más eficiente y ahora la media para entregar el producto es de 20 min. ¿Cuál es la probabilidad de que el producto se entregue después de 30 minutos. ó 0.5 h

$\mu = 20 \text{ min} = \frac{20}{60} \text{ hr} = \frac{1}{3} \text{ h} \qquad f(x) = 3e^{-3x}$

$$P(X \geq 0.5) = \int_{0.5}^{\infty} e^{-3x}(3dx) = -e^{-3x}\Big]_{0.5}^{\infty}$$

$$= -\lim_{x \to \infty} \cancel{e^{-3x}}^{\,0} + e^{-3(0.5)}$$

$e^{-\infty} \to 0$

$$= e^{-1.5} \approx 0.2231 \text{ ó } 22.31\%$$

Media o Valor Esperado. (μ)

Variable discreta: media es un promedio ponderado.

$$\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ p(x_1) & p(x_2) & & p(x_n) \end{array} \rightarrow \text{pesos, frecuencias}$$

$$\mu = x_1 p(x_1) + x_2 p(x_2) + \cdots x_n p(x_n)$$

$$\mu = \sum_{i=1}^{n} x_i p(x_i) \qquad p(a \leq x \leq b) = \sum_{i=1}^{n} p(x_i)$$

Si la variable es continua, μ se obtiene al integrar $x f(x)$.

$$\boxed{\mu = \int_{-\infty}^{\infty} x f(x)\, dx}$$

$f(x)$ uniforme
exponencial
normal.

Varianza: Suma de las diferencias al cuadrado de cada x_i respecto a la media μ.

Discreta. $$\sum_{i=1}^{n} (x_i - \mu)^2 p(x_i) = \sigma^2.$$

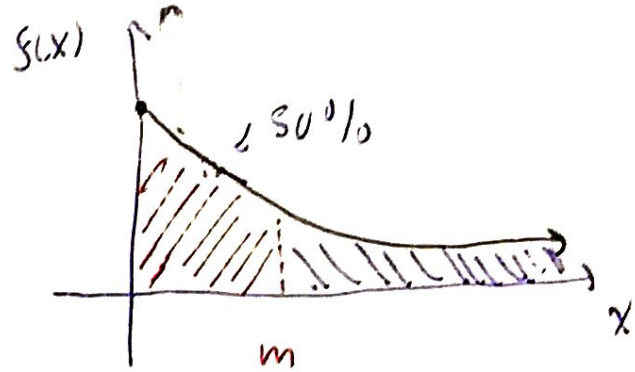
σ sigma $\frac{\sigma}{x}$
μ mu.

continua: $$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx$$

Desviación Estándar: $\sigma = \sqrt{\sigma^2} = \sqrt{\text{Varianza}}$

Mediana: el número m para el cual la probabilidad acumulada es igual al 50% ó a 0.5.

$$\int_{-\infty}^{m} f(x)\,dx = 0.5$$



Ejercicio 3: Considere la función exponencial

$$f(x) = \frac{1}{\mu} e^{-x/\mu}, \quad x \geq 0.$$

parámetro μ.

a. Encuentre la media de f(x). media vs. mediana.

$$\int_0^{\infty} x f(x)\,dx = \int_0^{\infty} \frac{x}{\mu} e^{-x/\mu}\,dx = uv - \int v\,du.$$

$u = x$ $\quad dv = \frac{e^{-x/\mu}}{\mu}\,dx$

$du = dx$ $\quad v = -e^{-x/\mu}$

$$\int \frac{x}{\mu} e^{-x/\mu}\,dx = -x e^{-x/\mu} + \int e^{-x/\mu}\,dx.$$

$$= -x e^{-x/\mu} - \mu e^{-x/\mu} + C.$$

$$\int_0^{\infty} \frac{x}{\mu} e^{-x/\mu}\,dx = -\lim_{x\to\infty} x e^{-x/\mu} - \lim_{x\to\infty} \mu e^{-x/\mu}$$

$e^{-\infty} \to 0$

$$+ 0 \cdot e^{-0} + \mu e^{-0}$$

$$= \mu - \lim_{x\to\infty} x e^{-x/\mu}.$$

$\infty \cdot 0$

Regla LH $\frac{0}{0}$ ó $\frac{\infty}{\infty}$ $\lim_{x\to\infty} \frac{x}{e^{x/\mu}} \underset{LH}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{\mu}e^{x/\mu}} = 0$

Media distribución exponencial: $\int_0^\infty \frac{x}{\mu} e^{-x/\mu}\, dx = \mu.$

b. Encuentre la mediana de $f(x) = \frac{1}{\mu} e^{-x/\mu}.$

Encuentre m tal que $\int_0^m f(x)\, dx = 0.5.$

$$\int_0^m \frac{1}{\mu} e^{-x/\mu}\, dx = -e^{-x/\mu}\Big]_0^m = \boxed{-e^{-m/\mu} + 1}$$

$-e^{-m/\mu} + 1 = 0.5$ Incógnita m.

$+e^{-m/\mu} = +0.5$

$-m/\mu = \ln(0.5)$

$-m = \mu \ln(0.5)$

$m = -\ln(0.5)\,\mu. \approx 0.70\,\mu.$

Integrar $\int (x-\mu)^2 e^{-x/\mu}\, d\mu. = \sigma^2$ $\mu = 1.$

Ejercicio 4: Encuentre la media, mediana, varianza y desviación estándar de la distribución uniforme.

$$f(x) = \frac{1}{b-a} \qquad a \leq x \leq b. \qquad \text{para } b = 6 \text{ y } a = 0.$$

Específico $f(x) = \frac{1}{6} \qquad 0 \leq x \leq 6.$

Probabilidades $P(x_0 \leq x \leq x_1) = \int_{x_0}^{x_1} \frac{1}{6}\, dx$

Media: $\mu = \int_0^6 x\, f(x) = \int_0^6 \frac{x}{6}\, dx = \frac{1}{6} \frac{x^2}{2}\Big]_0^6 = \frac{1}{12} 36 = 3.$

Mediana $\int_0^m f(x)\, dx = 0.5.$

$$\int_0^m \frac{1}{6}\, dx = \frac{x}{6}\Big]_0^m = \frac{m}{6} = 0.5.$$

$$m = 6(0.5) = 3.$$

Media = Mediana $= \frac{1}{2}(b+a).$

Varianza y Desviación Estándar. $\mu = 3$, $f(x) = \frac{1}{6}.$

$$\sigma^2 = \int_0^6 (x-\mu)^2 f(x)\, dx = \frac{1}{6}\int_0^6 \underbrace{(x-3)}_{u}{}^2 \underbrace{dx}_{du.}$$

$$= \frac{1}{6} \cdot \frac{1}{3} (x-3)^3 \Big]_0^6.$$

$$= \frac{1}{18}\left(3^3 - (-3)^3\right)$$

$$\sigma^2 = \frac{1}{18}(27+27) = \frac{54}{18} = 3.$$

Desviación Estándar. $\sigma = \sqrt{3}$.

Laboratorio 9:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx. \qquad f(x) = \frac{9}{x^2}.$$

$$1 \le x \le 3.$$

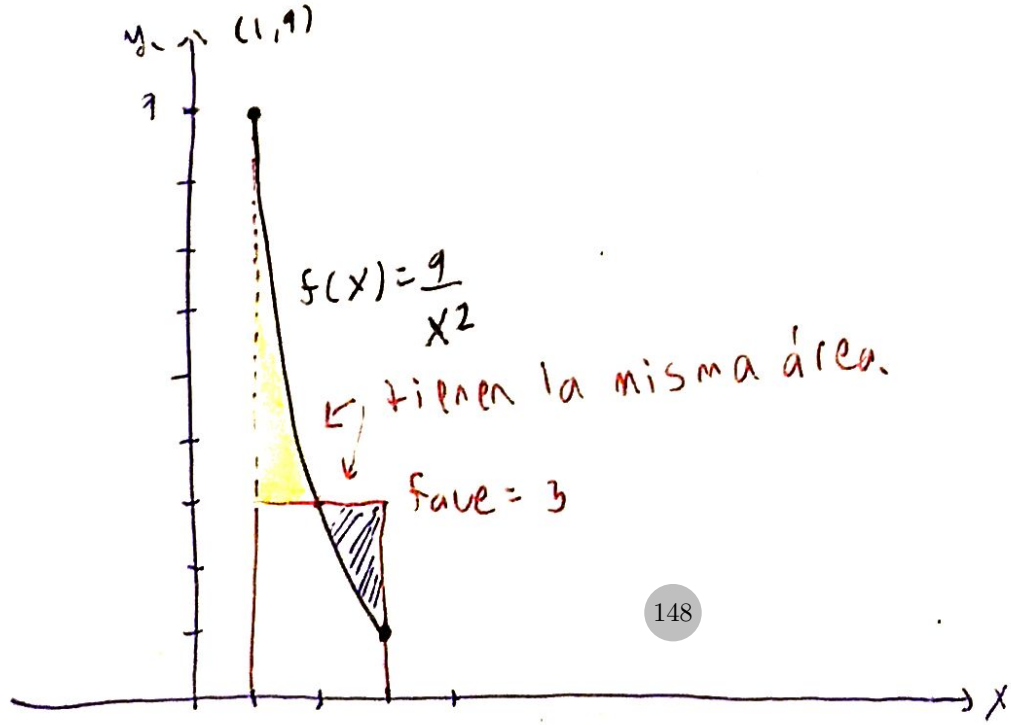
a) $$f_{ave} = \frac{1}{2}\int_1^3 9x^{-2}\,dx = -\frac{1}{2}\frac{9}{x}\Big]_1^3$$

$$= -\frac{1}{2}3 + \frac{1}{2}9 = \frac{6}{2} = 3.$$

b) $f(c) = f_{ave}$ resuelva para c.

$$\frac{9}{c^2} = 3. \Rightarrow 3 = c^2 \Rightarrow c = \sqrt{3} \approx 1.73$$

c).



Capítulo 22

1.4 Fracciones Parciales (P 64).

Función Racional $f(x) = \dfrac{P(x)}{Q(x)}$.

P y Q son polinomios de grado m y n.

Factorice $Q(x) = (x+a)(x+b)(x+c)$

$$\underbrace{\frac{P(x)}{(x+a)(x+b)(x+c)}}_{\text{No se puede integrar}} = \underbrace{\frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}}_{\text{si se pueden}}.$$

A, B y C son constantes.

$$\int \frac{1}{x+x_0}\,dx = \ln|x+x_0| + C.$$

simplifique la función racional en fracciones más simples (fracciones parciales) y luego utilizar reglas de integración conocidas.

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + C.$$

1. Objetivo: Encuentre los coeficientes A, B, ... C.
   Adicional
2. Condición: grado $P(x)$ < grado $Q(x)$

$$\frac{6}{x^2-9} \qquad \frac{9z}{2z^2+7z-4}, \qquad \frac{x^2+1}{x^4+x^2}$$

Utilice la división larga si $P(x)$ tiene un grado igual o mayor a $Q(x)$.

$$\frac{x^2+1}{x^2-9}, \ldots \frac{x^3+x^2+1}{x^2+4}, \ldots;$$

Polinomio: $(x-3)(x-2) = x^2-5x+6.$ Factores Lineal

x^2+4 Factores Cuadráticos.

$(x-3)^2(x^2+4)^2$ Factores Repetidos.

Caso 1: Factores Lineales Distintos.

Caso 2: Factores Lineales Repetidos.

Caso 3: Factores Cuadráticos Distintos

Caso 4: Factores Cuadráticos Repetidos.

$x^3-x = x(x^2-1) = x(x-1)(x+1).$

Ejercicio 1: P. 65

$$\int \frac{9z}{2z^2+7z-4}\,dz. \qquad u \neq 2z^2+7z-4.$$

Factorice el denominador.

$2z^2+7z-4. = (2z-1)(z+4)$

$2z \quad -1$

$z \quad 4$

Los ceros del denominador.

$2z-1=0$

$z+4=0$

$$\boxed{\begin{array}{l} z = 1/2. \\ z = -4 \end{array}}$$

$$\frac{9z}{(2z-1)(z+4)} = \frac{A}{2z-1} + \frac{B}{z+4} = \frac{A(z+4)+B(2z-1)}{(2z-1)(z+4)}.$$

Encuentre A y B. multiplique por $(2z-1)(z+4)$

$$A(z+4) + B(2z-1) = 9z + 0.$$

$$Az + 4A + 2Bz - B = 9z + 0.$$

Igualando términos del mismo grado.

z: $(A+2B) = 9 \Rightarrow A + 2\cdot 4A = 9A = 9 \Rightarrow A = 1$

0: $4A - B = 0 \Rightarrow B = 4A \Rightarrow B = 4.$

Atajo: evalúe la ecuación en los valores de z donde el denominador se hace cero. -4 y $1/2$

$$A(z+4) + B(2z-1) = 9z.$$

$z=-4$: $0 \quad -9B = -36. \Rightarrow B = 4.$

$z = 1/2$: $4.5A + 0 = 4.5 \Rightarrow A = 1$

finalmente, integre.

$$\int \frac{9z}{(2z-1)(z+4)}\,dz = \int \frac{dz}{2z-1} + \int \frac{4\,dz}{z+4}$$

$$= \boxed{\frac{1}{2}\ln|2z-1| + 4\ln|z+4| + C.}$$

Ejercicio 2: Evalúe las sigs. integrales (P 6 5)

a) $\displaystyle\int \frac{5x+13}{x^2+5x+6}\,dx$ $\qquad x^2+5x+6 = (x+3)(x+2)$

$x = -3, -2$ } números clave.

$$\frac{5x+13}{\underbrace{x^2+5x+6}_{(x+3)(x+2)}} = \frac{A}{(x+3)} + \frac{B}{(x+2)}$$

Factores lineales distintos.

Multiplique por $(x+3)(x+2)$

$$5x+13 = A(x+2) + B(x+3)$$

$x=-3$: $\quad -2 = -A + 0 \Rightarrow A = 2.$

$x=-2$: $\quad 3 = 0 + B \Rightarrow B = 3.$

$$\int \frac{5x+13}{(x+2)(x+3)}\,dx = \int \frac{2}{x+3}\,dx + 3\int \frac{dx}{x+2}.$$

$$= \boxed{2\ln|x+3| + 3\ln|x+2| + C.}$$

b) $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx$ $\qquad x^3-x = x(x^2-1)$

$= x(x+1)(x-1)$

3 factores lineales distintos, 3 fracciones parciales.

$$\frac{x^2+2x-1}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}.$$

Multiplique por $x^3 - x = x(x^2-1)$

$$x^2+2x-1 = A(x^2-1) + Bx(x-1) + Cx(x+1)$$

El denominador tiene ceros en $x = 0, 1, -1$.

$x=0$: $\quad -1 = -A + 0 + 0 \Rightarrow A = 1$

$x=1$: $\quad 2 = 0 + 0 + 2C \Rightarrow C = 1$

$x=-1$: $\quad -2 = 0 - B(-2) + 0 \Rightarrow B = \frac{-2}{2} = -1$

$$\int \frac{x^2+2x-1}{x(x-1)(x+1)} = \int \frac{dx}{x} - \int \frac{dx}{x+1} + \int \frac{dx}{x-1}$$

$$= \ln|x| - \ln|x+1| + \ln|x-1| + C.$$

Caso 2: Factores Lineales Repetidos

$$\frac{P(x)}{(x+b)^n} = \frac{A_1}{(x+b)} + \frac{A_2}{(x+b)^2} + \dots + \frac{A_n}{(x+b)^n}$$

n factores cada uno con su respectiva potencia

el valor de los n coeficientes $A_1, \dots, A_n$.

Ejercicio 5: Evalúe.

Repetido 2 veces.

a. $\displaystyle\int \frac{x+2}{(x+3)^2}\,dx \qquad \frac{x+2}{(x+3)^2} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2}$

Multiplique por $(x+3)^2$.

$$(x+2) = A(x+3) + B.$$

Agrupe términos.

$$x + 2 = Ax + 3A + B$$

$$\int u^{-2}\,du = \frac{u^{-1}}{-1}$$

$$x = Ax \Rightarrow A = x/x = 1$$

$$2 = 3A + B. \Rightarrow 2 = 3 + B \Rightarrow B = -1.$$

$$\int \frac{x+2}{(x+3)^2}\,dx = \int \frac{1}{(x+3)}\,dx - 1\int (x+3)^{-2}\,dx$$

$$= \ln|x+3| - \frac{1}{-1}(x+3)^{-1} + C.$$

$$= \ln|x+3| + \frac{1}{(x+3)} + C.$$