Module 6: Combination of Functions and Inverses

Combination of Functions:

Note: $(f+g)(x)$ is not distribution!!

1. Using the given functions, perform the indicated operations and give the domains of each resulting function.

Functions	$f(x) = x + 7$ and $g(x) = x^2 - 9$	Domain
$(f+g)(x)$		
$(f-g)(x)$		
$(f \cdot g)(x)$		
$\left(\frac{f}{g}\right)(x)$		
$(f+g)(-2)$		

2. Using the given functions, perform the indicated operations and give the domains of each resulting function.

Functions	$f(x) = \sqrt{x+5}$ and $g(x) = 3x - 4$	Domain
$(f-g)(x)$		
$(f-g)(-1)$		

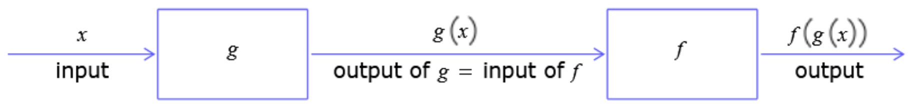
3. A small publishing company is releasing a new book. The production costs will include a one-time fixed cost for editing and an additional cost for each book printed. The total production cost C (in dollars) is given by the function $C = 750 + 16.95N$, where N is the number of books. The total revenue earned (in dollars) from selling the books is given by the function $R = 32.80N$. Let P represent the profit made (in dollars). Write an equation relating P to N. Simplify your answer as much as possible.

Composition of Functions:

The composition of f and g, denoted $f \circ g$ is defined by **Note: This is not multiplication!**

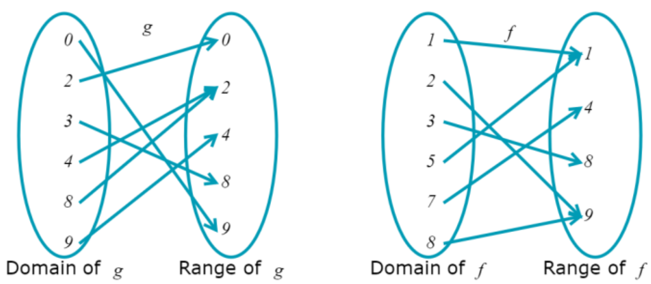
$$(f \circ g)(x) = f(g(x))$$

What value is used as the input of f? This is the output of what function?



To find the value of this function, first evaluate the function g at x to obtain $g(x)$. Then evaluate the function f at $g(x)$ to obtain $f(g(x))$.

Domain and Range of a Composed Function



a) $(f \circ g)(3) =$

b) $(f \circ g)(4) =$

c) $(f \circ g)(9) =$

The domain of $g \circ f$ contains all the inputs of f whose outputs are in the domain of g.

Domain of $(f \circ g)$: Range of $(f \circ g)$:

Evaluating Composed Functions:

1. Let $f(x) = -2x + 1$ and $g(x) = -x^2$. Find $(f \circ g)(3)$.

2. Let $u(x) = x^2 + 6$ and $v(x) = \sqrt{x + 9}$.
   a. Find $(u(v(7))$.

   b. Find $v(u(7))$.

Using the Equations:

Assume that all of the following functions are real values.

1. If $f(x) = x^2 - 2$ and $g(x) = x + 1$, find the composition $f \circ g$ and specify its domain using interval notation.

   $(f \circ g)(x) =$

   Domain of $f \circ g$:

Note: To avoid errors when finding the domain, consider the composition before any simplification!

2. If $f(x) = x^2 + 1$ and $g(x) = \sqrt{x+2}$, find the composition $f \circ g$ and specify its domain using interval notation.

   $(f \circ g)(x) =$

   Domain of $f \circ g$:

https://www.desmos.com/calculator/mliczlvmgu

3. Let $g(x) = \frac{x+6}{x-5}$ and $h(x) = 4x + 7$. Find $(g \circ h)(x)$ and specify its domain using interval notation.

   $(g \circ h)(x) =$

   Domain of $g \circ h$:

4. The braking distance $D(v)$ (in meters) for a certain car moving at velocity v (in meters/second) is given by $D(v) = \frac{v^2}{34}$. The car's velocity $B(t)$ (in meters/second) t seconds after starting is given by $B(t) = 3t$. Write a formula for the braking distance $S(t)$ (in meters) after t seconds. It is not necessary to simplify.

Function	**Input**	**Output**

Decomposing Functions:

Find two functions f and g such that $H(x) = (f \circ g)(x)$.
Neither can be the identity function (i.e. $f(x) \neq x$ and $g(x) \neq x$).

1. $H(x) = (5x - 3)^4$

   $f(x) =$

   $g(x) =$

2. $H(x) = \sqrt{9 - 4x^2}$

   $f(x) =$

   $g(x) =$

3. $H(x) = 6x^2 + 6$

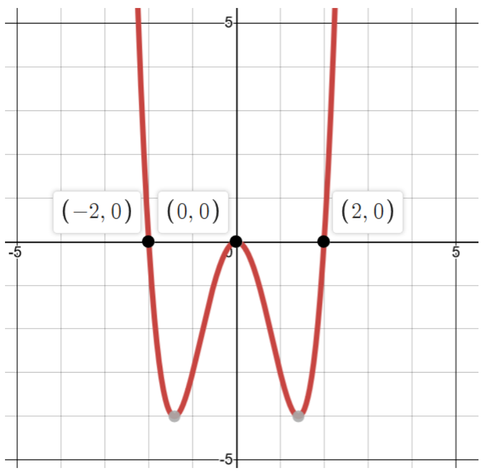
   $f(x) =$

   $g(x) =$

Identifying One-to-One Functions:

Review: What makes a relation a function?

Definition: A function f is ***one-to-one*** if, for x_1 and x_2 in the domain of f, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. In other words, a function is one-to-one if ____ correspond to ____.



For example, the graph to the left contains the points $(-2,0)$, $(0,0)$, and $(2,0)$.

Is this the graph of a function? Why or why not?

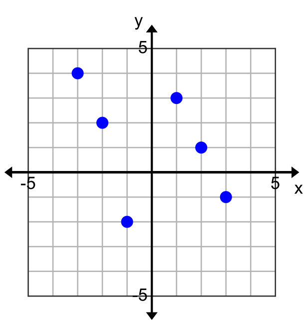
However, this graph __________________________ be one-to-one, since the y −coordinate 0 is paired with the x −coordinates $-2, 0$, and 2.

So just because a graph is a function, this does not guarantee that the function is __________________________.

Horizontal Line Test: A function $y = f(x)$ is a one-to-one function if **<u>NO</u>** horizontal line intersects the ____.

Directions: For each function graphed below, state whether it is one-to-one.

1.



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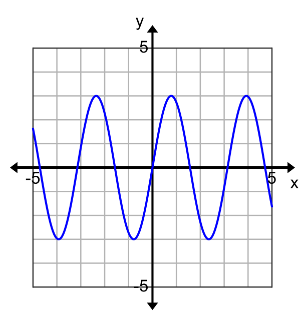

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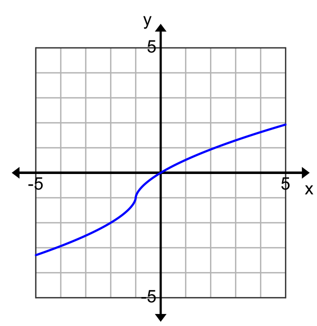
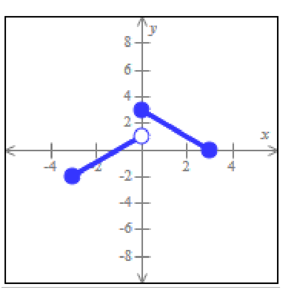
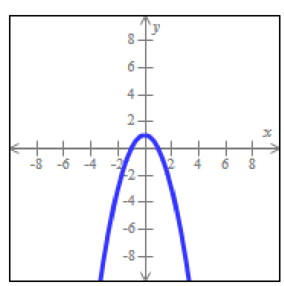


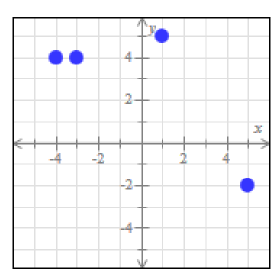
4.



5.



6.



Inverse Functions:

Given the function $f(x) = 3x + 6$, do the following:

a. Describe in words what this function tells you to do (in correct order) to the input value of x.

b. Inverse functions "undo" the operations of the original function in reverse order. Describe in words the inverse operations for this function (in correct order).

c. Write the equation for the inverse of f. Call this new function f^{-1}. Then $f^{-1}(x)$ = ___________________.
Note: f^{-1} is read aloud as, "f inverse". $f^{-1}(x)$ is read, "f inverse of x".

<u>Note:</u> the -1 is **NOT** an exponent!

We can also find an inverse function symbolically. Below is the general method to find the inverse of a function that is defined by an equation $f(x) = y$

1. Replace $f(x)$ with y.
2. Switch the names y and x. We are now calling the input y and the output x. This will allow f^{-1} to have an input of x.
3. Solve for y.
4. Replace y with $f^{-1}(x)$.

Find the inverse of $f(x) = 3x + 6$ symbolically. Show work!

So now that you have shown your work symbolically, let's look at some important information about inverses. Find the values of f(x) for the x- values listed below. Do the same for the second table, using $f^{-1}(x)$. What do you notice about the two tables?

x	$f(x) =$
-1	
0	
1	
2	

x	$f^{-1}(x) =$
3	
6	
9	
12	

a. The domain of $f(x)$ is the same as the ____________________ of $f^{-1}(x)$ and

b. The domain of $f^{-1}(x)$ is the same as the ____________________ of $f(x)$.

Testing Potential Inverses:

Now let's look at some compositions.

$f\left(f^{-1}(9)\right) =$

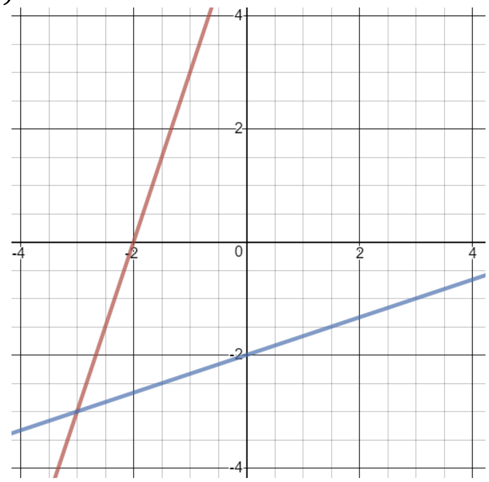
$f^{-1}\left(f(0)\right) =$

$f\left(f^{-1}(x)\right) =$

$f^{-1}\left(f(x)\right) =$

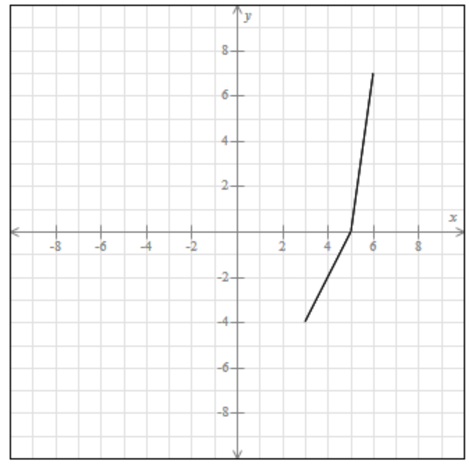
What should happen if two functions are inverses of each other?

Next, let's look at the graphs of both $f(x)$ and $f^{-1}(x)$.



What do you notice about the 2 functions?

<u>More practice</u>: The graph of a function h is given to the left. Graph the inverse function.



What is $h(5)$?

What is $h^{-1}(0)$?

Finding More Inverse Functions:

1. Let $g(x) = \frac{x-9}{11}$ Find $g^{-1}(x)$ and $(g^{-1} \circ g)(5)$.

2. Let $f(x) = x^2 + 1$ for the domain $[0, \infty)$. Find $f^{-1}(x)$ and its domain.

Function name	Domain	Range
f	$[0, \infty)$	
f^{-1}		$[0, \infty)$

3. Let $f(x) = \sqrt{x-4} + 8$ for the domain $[4, \infty)$. Find $f^{-1}(x)$ and its domain.

Function name	Domain	Range
f	$[4, \infty)$	
f^{-1}		$[4, \infty)$

4. Let $f(x) = \sqrt[3]{5-x} + 4$. Find $f^{-1}(x)$.

Word Problems with Inverses:

5. Sara is walking. Her distance D in kilometers from Glen City after t hours of walking is given by $D(t) = 13.5 - 5t$. Let D^{-1} be the inverse function of D. Take x to be an output of the function D. That is, $x = D(t)$ and $t = D^{-1}(x)$.

Which statement best describes $D^{-1}(x)$?

a. The reciprocal of her distance from Glen City (in kilometers) after walking x hours.
b. The amount of time she has walked (in hours) when she is x kilometers from Glen City.
c. Her distance from Glen City (in kilometers) after she has walked x hours.
d. The ratio of the amount of time she has walked (in hours) to her distance from Glen City (in kilometers), x.

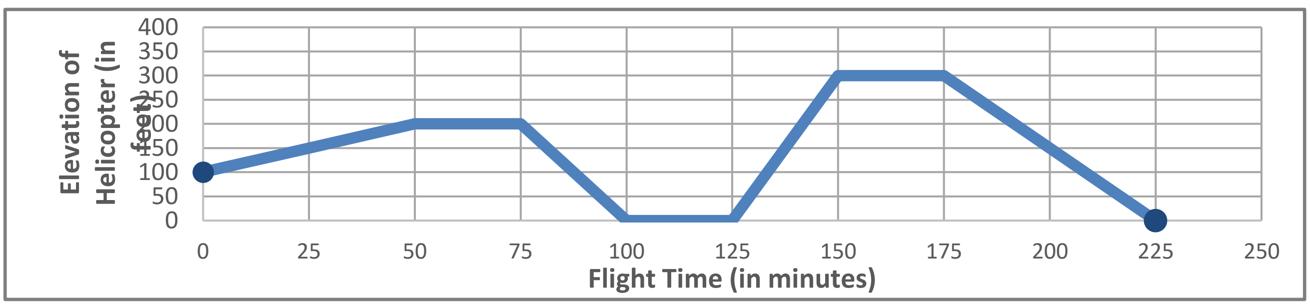
What is $D^{-1}(x)$?

What is $D^{-1}(7.5)$?

Piecewise Defined Functions:

A piecewise-defined function is a function that is defined according to different rules or equations depending on a specified set of input values.

The data in the chart below is from a flight of a helicopter. Let x represent the flight time in minutes and $y = f(x)$ represent the elevation of the helicopter in feet.



Below is a possible equation for this piecewise function:

$$f(x) = \begin{cases} 2x + 100 & 0 \le x < 50 \\ 200 & 50 \le x < 75 \\ -8x + 800 & 75 \le x < 100 \\ 0 & 100 \le x < 125 \\ 12x - 1500 & 125 \le x < 150 \\ 300 & 150 \le x < 175 \\ -6x + 1350 & 175 \le x \le 225 \end{cases}$$

What is happening to the elevation of the helicopter

a. … from 125 to 150 minutes?
b. … from 100 to 125 minutes?
c. …. from 175 to 225 minutes?

Use both the graph and the equation in finding the following:

a. $f(25)$
b. $f(105)$
c. $f(200)$

What is the vertical intercept? Explain the meaning of V-intercepts in context of the problem.

What is/are the horizontal intercept(s)? Explain the meaning of H-intercepts in context of the problem.

Evaluating Piecewise-Defined Functions:

Suppose that the function h is defined on the interval $(-2,2]$ as follows.

$$h(x)=\begin{cases}-1 & \text{if } -2<x\le -1\\ 0 & \text{if } -1<x\le 0\\ 1 & \text{if } 0<x\le 1\\ 2 & \text{if } 1<x\le 2\end{cases}$$

Find $h(-1)$, $h(-0.75)$, and $h(2)$.

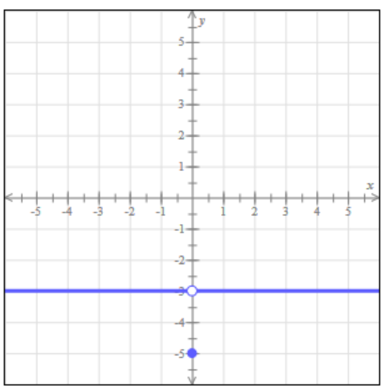
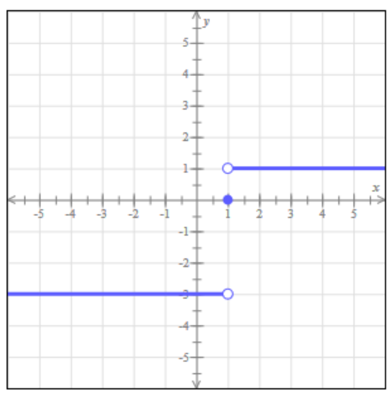
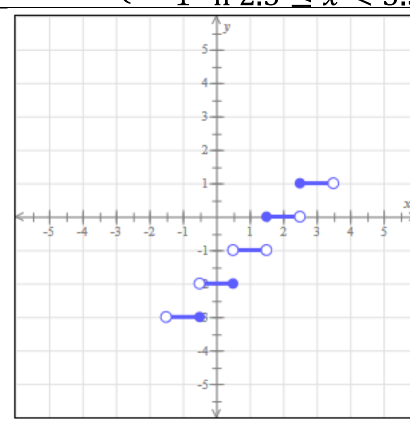
Suppose that the function f is defined, for all real numbers, as follows.

$$f(x)=\begin{cases}\frac{1}{2}x+1 & \text{if } x<-1\\ -(x+1)^2+2 & \text{if } -1\le x\le 2\\ -\frac{1}{2}x-2 & \text{if } x>2\end{cases}$$

Find $f(-1)$, $f(0)$, and $f(5)$.

Graphing Piecewise-Defined Functions:

Some examples of piecewise-defined functions are drawn below. Graph the remaining functions.

1. $f(x)=\begin{cases}-3 & \text{if } x\ne 0\\ -5 & \text{if } x=0\end{cases}$	2. $f(x)=\begin{cases}-3 & if\ x<-1\\ 0 & if\ x=-1\\ 1 & if\ x>-1\end{cases}$	3. $f(x)=\begin{cases}-3 & \text{if } -1.5<x\le -0.5\\ -2 & \text{if } -0.5<x\le 0.5\\ -1 & \text{if } 0.5<x<1.5\\ 0 & \text{if } 1.5\le x<2.5\\ 1 & \text{if } 2.5\le x<3.5\end{cases}$
		
4. $f(x)=\begin{cases}2 & if\ x<-1\\ 3 & if\ x=-1\\ -3 & if\ x>-1\end{cases}$	5. $f(x)=\begin{cases}3 & \text{if } x\ne 0\\ 1 & \text{if } x=0\end{cases}$	6. $f(x)=\begin{cases}0 & \text{if } -3<x\le -2\\ 1 & \text{if } -2<x\le -1\\ 2 & \text{if } -1<x\le 0\\ 3 & \text{if } 0<x\le 1\end{cases}$