Non-linearity

EDUC 643: Unit 5 Part II

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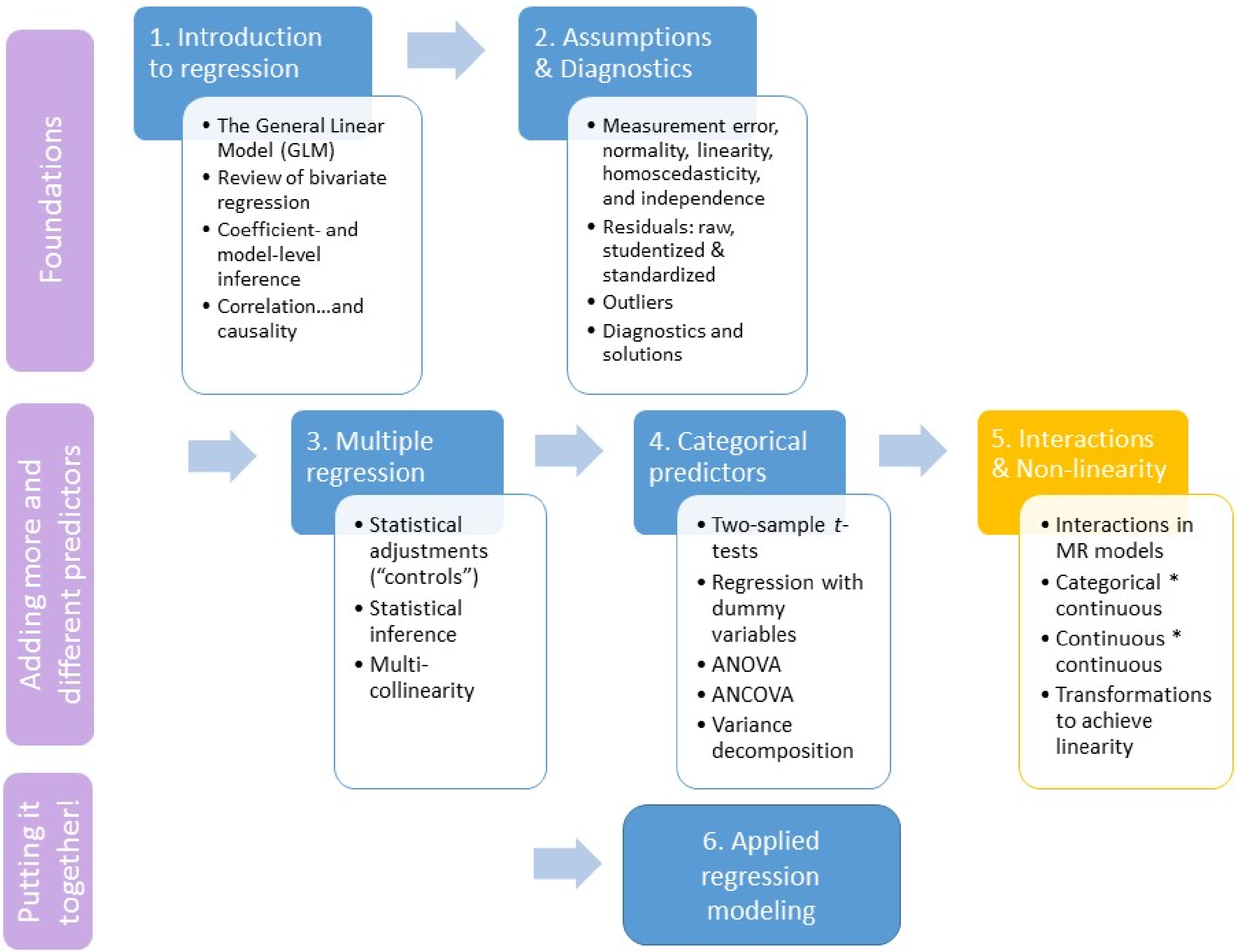
Roadmap



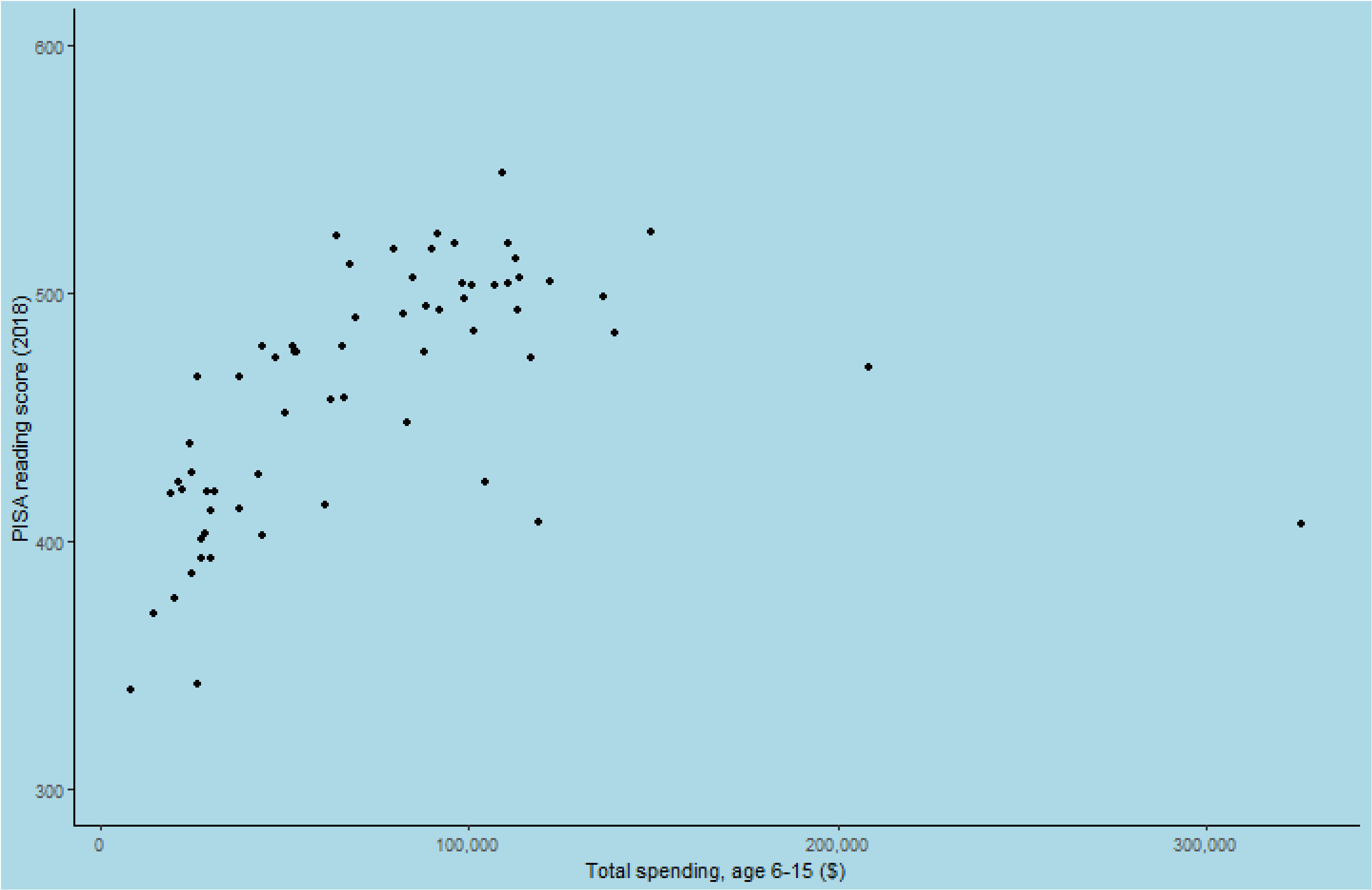
Foundations
Adding more and different predictors
Putting it together!
1. Introduction to regression
• The General Linear Model (GLM)
• Review of bivariate regression
• Coefficient- and model-level inference
• Correlation…and causality
2. Assumptions & Diagnostics
• Measurement error, normality, linearity, homoscedasticity, and independence
• Residuals: raw, studentized & standardized
• Outliers
• Diagnostics and solutions
3. Multiple regression
• Statistical adjustments ("controls")
• Statistical inference
• Multi-collinearity
4. Categorical predictors
• Two-sample t-tests
• Regression with dummy variables
• ANOVA
• ANCOVA
• Variance decomposition
5. Interactions & Non-linearity
• Interactions in MR models
• Categorical * continuous
• Continuous * continuous
• Transformations to achieve linearity
6. Applied regression modeling

Goals of the unit

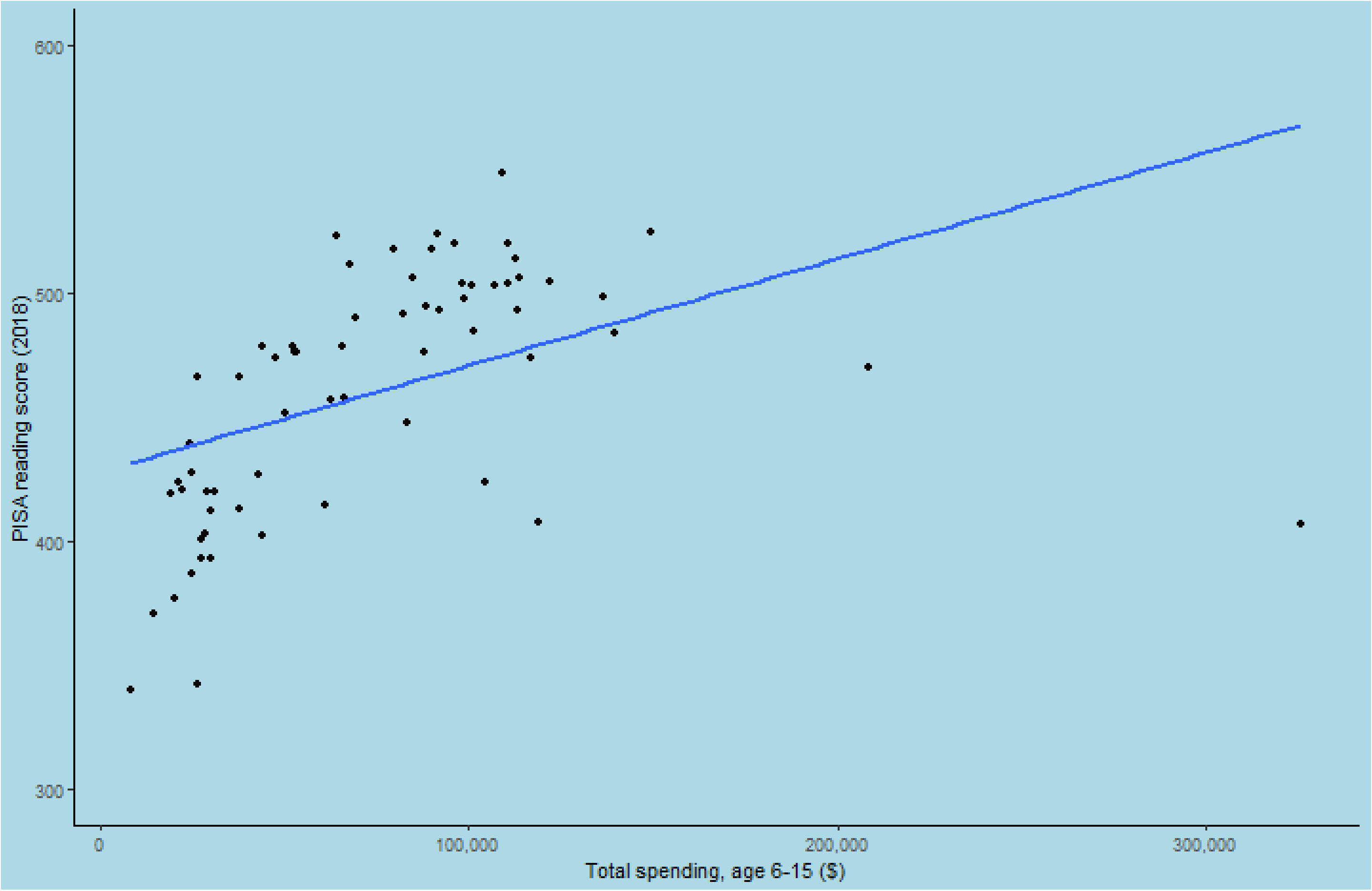
- Describe in writing and verbally the assumptions we violate when we fit a non-linear relationship with a linear model
- Transform non-linear relationships into linear ones by using logarithmic scales
- Estimate regression models using logarithmic scales and interpret the results
- Estimate models with quadratic and higher-order polynomial terms (special kinds of interactions)
- Select between transformation options

Non-linearity

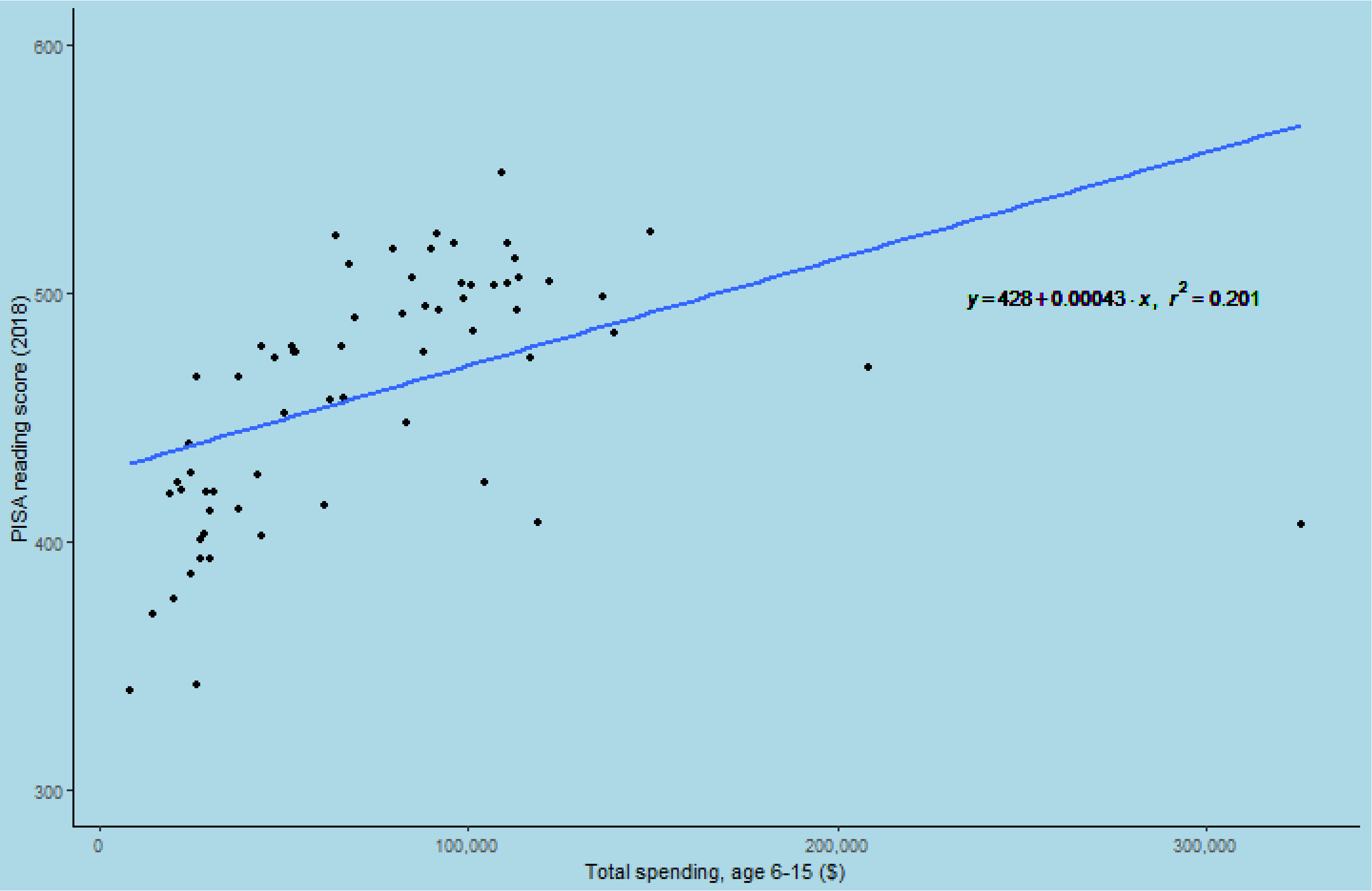
$ and learning



$ and learning



$ and learning



If assumptions hold, each $10,000 diff in total spending associated, on average, with 4.3 scale score point difference in reading scores. **But do they?**

Linear?

```
# Fit the model
fit <- lm(read_score ~ total_spending, data=pisa)
# Generate residual vs fitted plot
pisa$resid <- resid(fit)
pisa$fitted <- fitted(fit)
ggplot(pisa, aes(fitted, resid)) + geom_point() +
  geom_hline(yintercept = 0, color = "red", linetype="dashed")
```

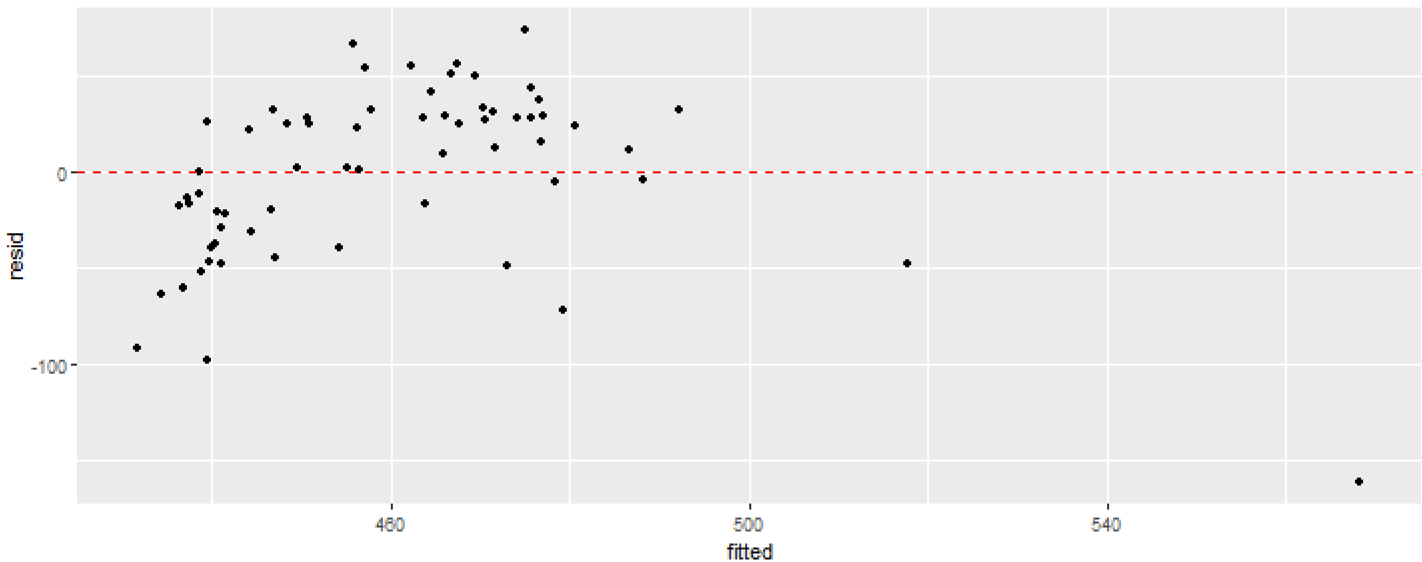
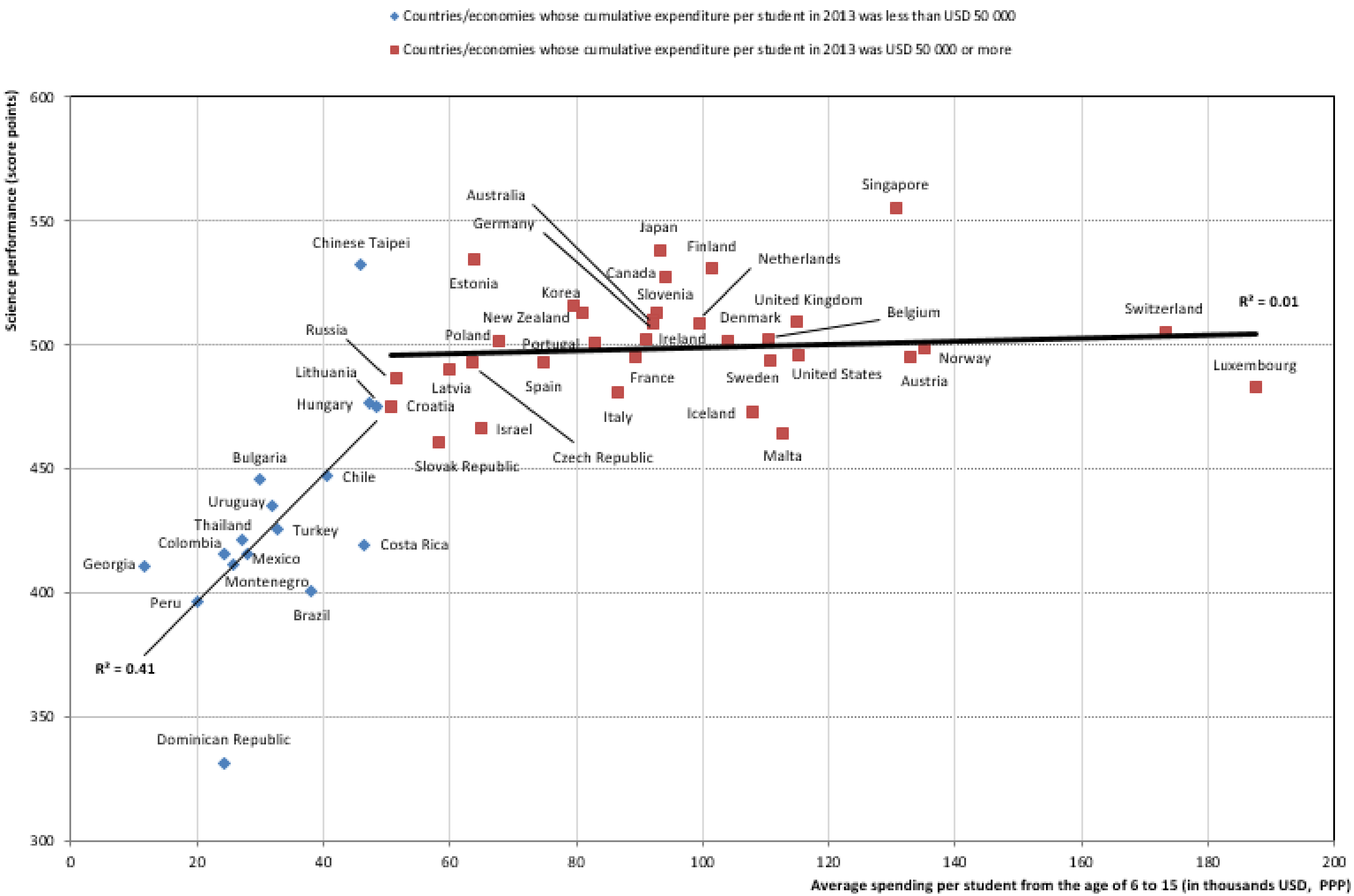


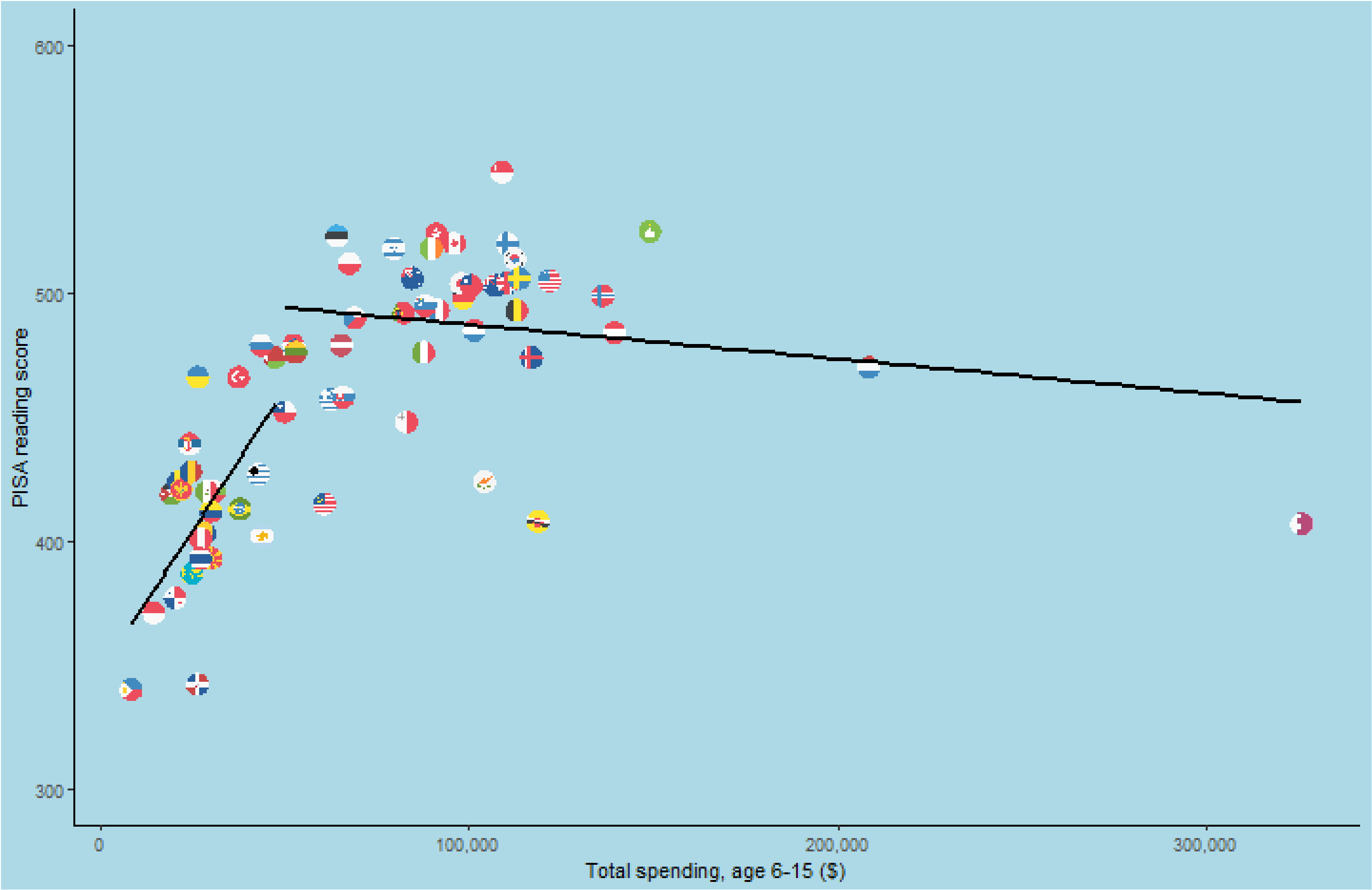
Figure II.6.2
Spending per student from the age of 6 to 15 and science performance



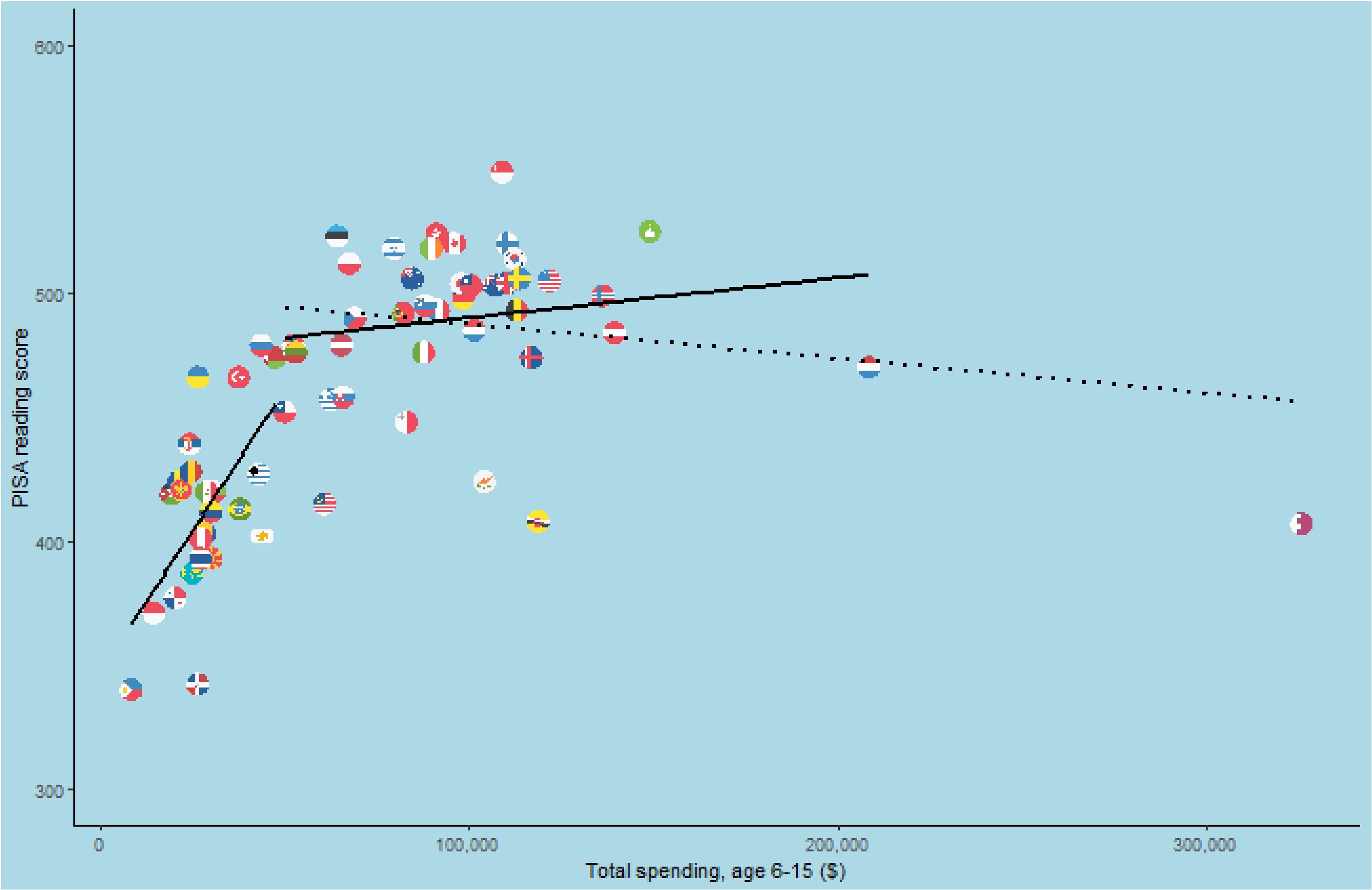
Make it nice

At low levels of spending the relationship between ***total_spending*** and ***read_score*** has a big magnitude. At higher levels of spending, it seems much more modest (negative?).

Piecewise


PISA reading score
600
500
400
300
0
100,000
200,000
300,000
Total spending, age 6-15 ($)

Piecewise



PISA reading score
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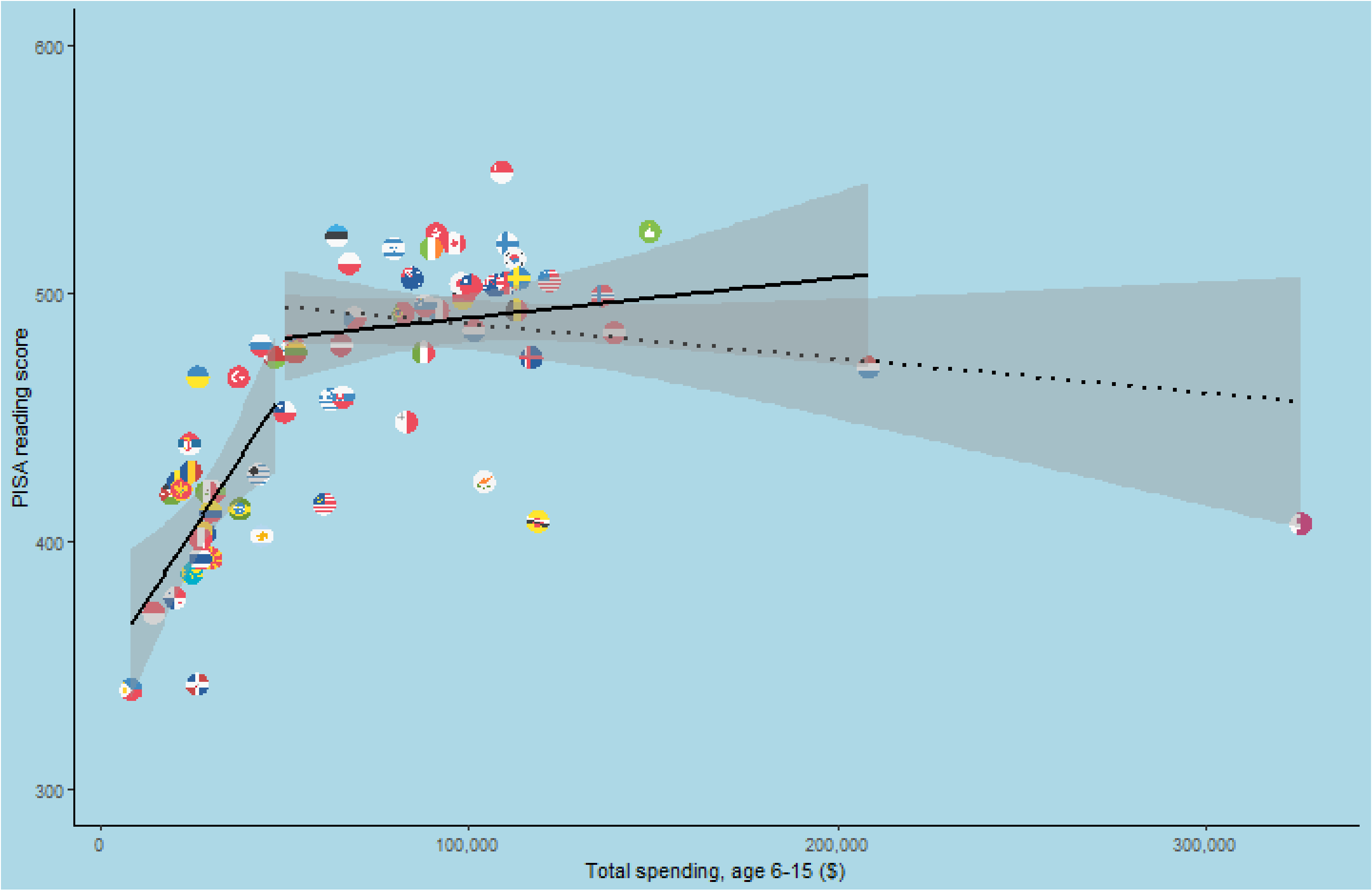
200,000
300,000
Total spending, age 6-15 ($)

Piecewise

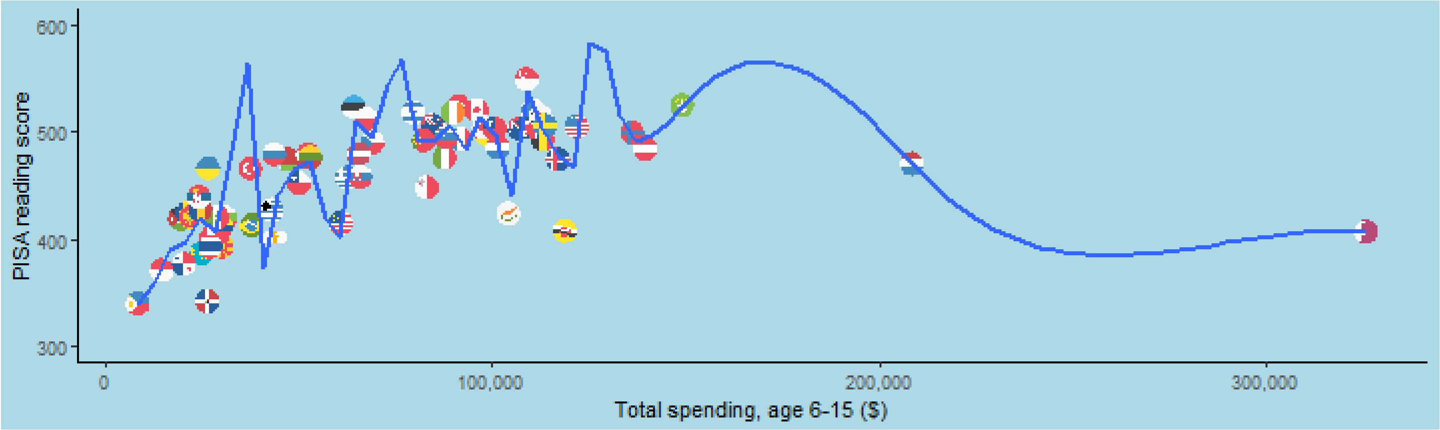


600
500
400
300
PISA reading score
0
100,000
200,000
300,000
Total spending, age 6-15 ($)

Piecewise

While it is true, as we've said before that *locally all relationships are linear*, we've identified some emerging issues:

- Cut points arbitrary and these choices may substantially alter nature of observed relationship
- With large data "eyeballing" linear sub-segments impossible
- Increasing loss of power (larger standard errors and confidence intervals, greater influence of outliers)
- **Overfitting** risks increase
  - Analysis conforms to particularly to your specific data, but generalizes poorly to population of inference



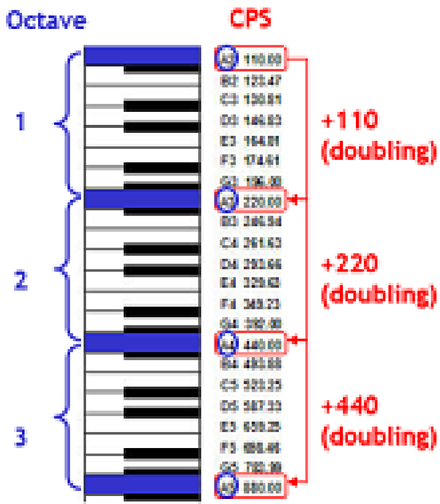
Solutions: transformations and polynomials

Logarithmic transformations in X

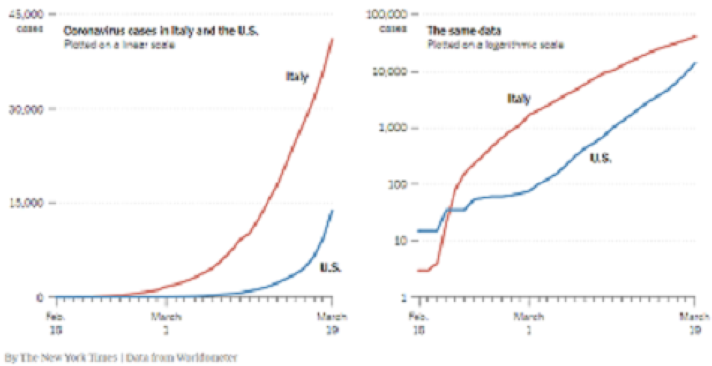
Log transformations

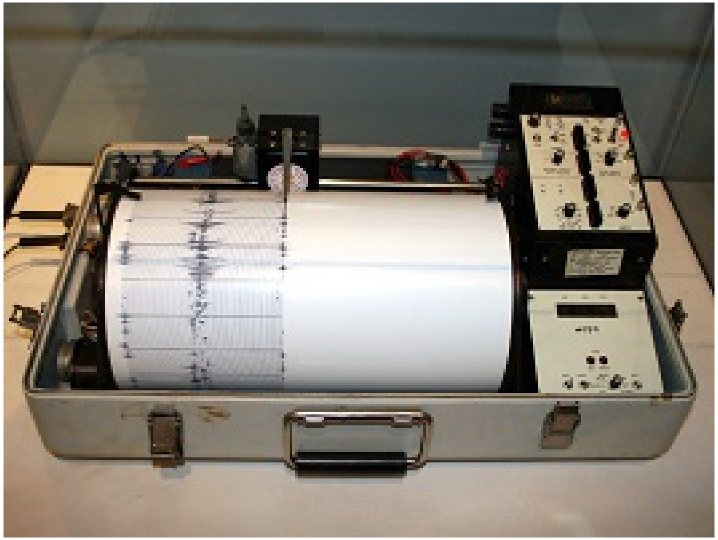
- We can posit a **non-linear relationship** between X and Y *in the population*
- Any non-linear relationship implies that the relationship between X and Y is relative to a particular value of X and/or Y, not absolute (the slope is non-constant)
- **Transformations** (i.e., spreading out in some cases and compressing in others the values of our X and Y variables) allow us to fit non-linear relationships within the existing machinery of the general linear model

Log transformations in life



↑ 1 octave = doubling of cycles-per-second





Seismic-wave amplitude	Location	Richter Scale
1,000,000	Christchurch, 2010	6.0
10,000,000	Port-au-Prince, 2010	7.0
100,000,000	Sichuan, 2008	8.0
1,000,000,000	Sumatra, 2004	9.0

↑ 1 Richter = 10x ↑ SWA

A log 🌲 you say??

Logs are the function we can perform to "undo" raising a number to a power. If a number is equal to a base raised to a power ($x = base^{power}$), then a logarithim of a given base is the number you would have to raise to that power to get x:

Exponents

$10 = 10^1$

$100 = 10^2$

$1,000 = 10^3$

$10,000 = 10^4$

$100,000 = 10^5$

Logarithms

$\log_{10}(10) = 1$

$\log_{10}(100) = 2$

$\log_{10}(1,000) = 3$

$\log_{10}(100,000) = 4$

$\log_{10}(100,000) = 5$

Each 1 unit increase in a base-10 logarithm represents a 10-fold increase in x. Can have logarithms of different base.

A log 🌲 you say??

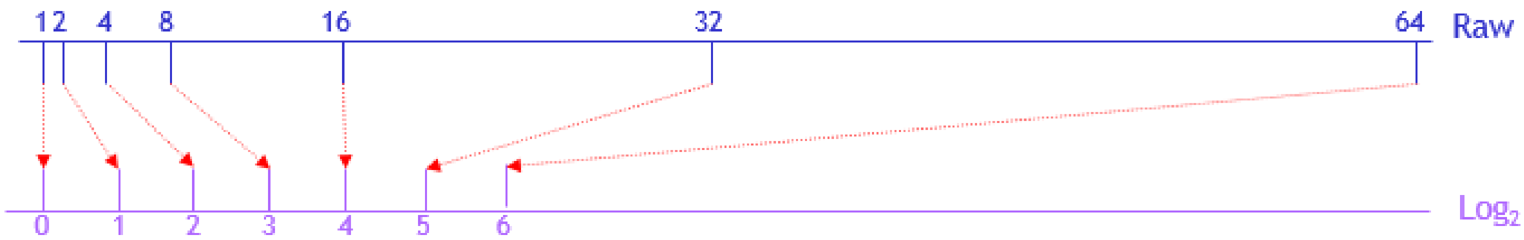
Logs are the function we can perform to "undo" raising a number to a power. If a number is equal to a base raised to a power ($x = base^{power}$), then a logarithim of a given base is the number you would have to raise to that power to get x:

Exponents	**Logarithms**
$2 = 2^1$	$\log_2(2) = 1$
$4 = 2^2$	$\log_2(4) = 2$
$8 = 2^3$	$\log_2(8) = 3$
$16 = 2^4$	$\log_2(16) = 4$
$32 = 2^5$	$\log_2(32) = 5$

Each 1 unit increase in a base-2 logarithm represents a doubling of x.

Can say this as: “Log base 2 of 32 is 5” or “Log base 10 of 1,000 is 3”

Understanding logs



Some key concepts:

- Taking logs spreads out the distance between small (closer to 0) values and compresses the distance between large (further from zero) values.
- Log base anything(1) is = 0
- Log base anything(0) is undefined (can't raise anything to a power and get 0)
- Log base anything>0(negative number) is undefined (technically a complex number)
- Taking logs is a **monotonic** transformation; doesn't change the order of any of the underlying raw values

$ and scores?

Let's try transforming our X variable (*total_spending*) on a logarithmic scale; can do this directly in our plot:

```
log_flag <- flag +
              xlab("Total spending, age 6-15 (Log10 $)") +
              scale_x_log10(breaks=c(10000, 50000, 100000, 300000),
                            label=scales::comma)
```

$ and scores?

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$ and scores?

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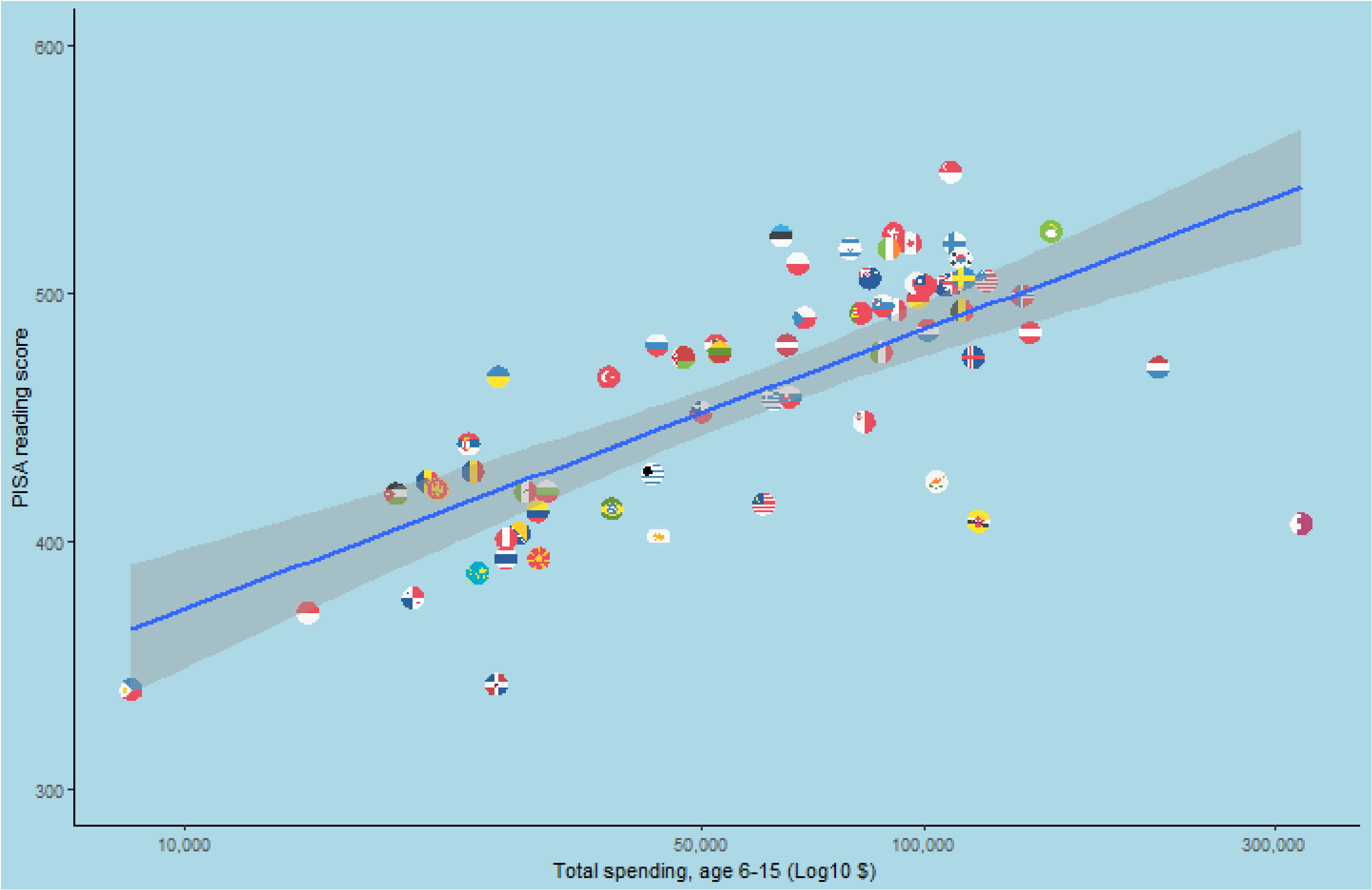
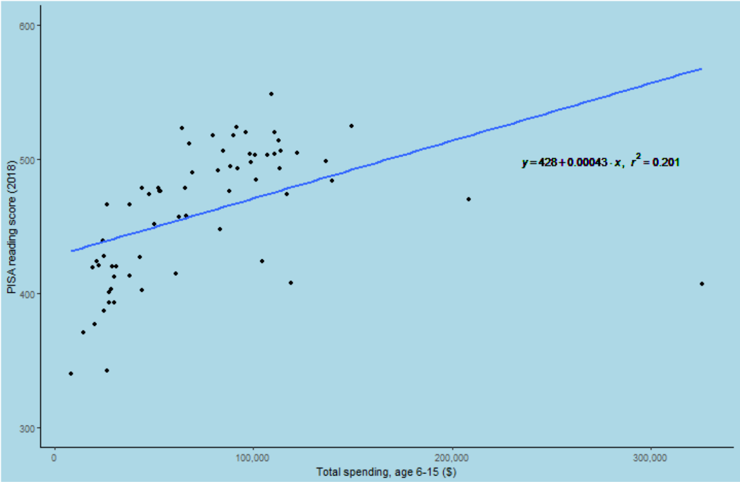


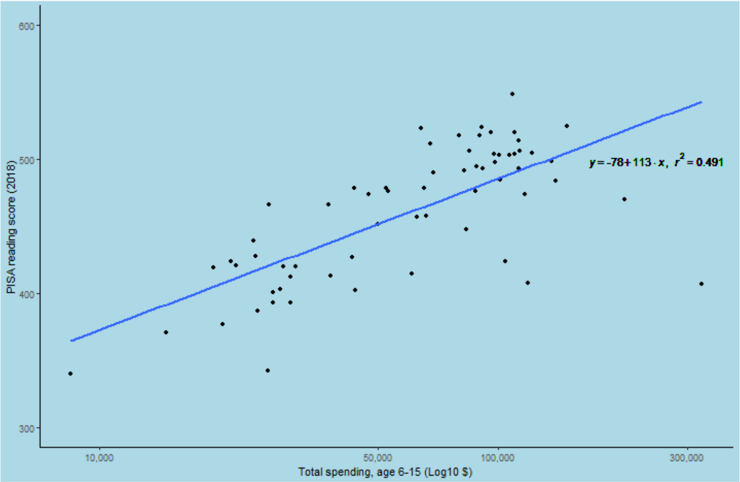
Regress read on $\log_{10}(spend)$

```
summary(lm(read_score ~ log10(total_spending), data=pisa))
```

```
##
## Call:
## lm(formula = read_score ~ log10(total_spending), data = pisa)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -136.50  -20.83   11.00   22.42   59.11
##
## Coefficients:
##                       Estimate Std. Error t value Pr(>|t|)
## (Intercept)             -78.03      69.14  -1.129    0.263
## log10(total_spending)   112.74      14.46   7.798 8.06e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 35.59 on 63 degrees of freedom
## Multiple R-squared:  0.4911,    Adjusted R-squared:  0.4831
## F-statistic:  60.8 on 1 and 63 DF,  p-value: 8.062e-11
```

Conceptually



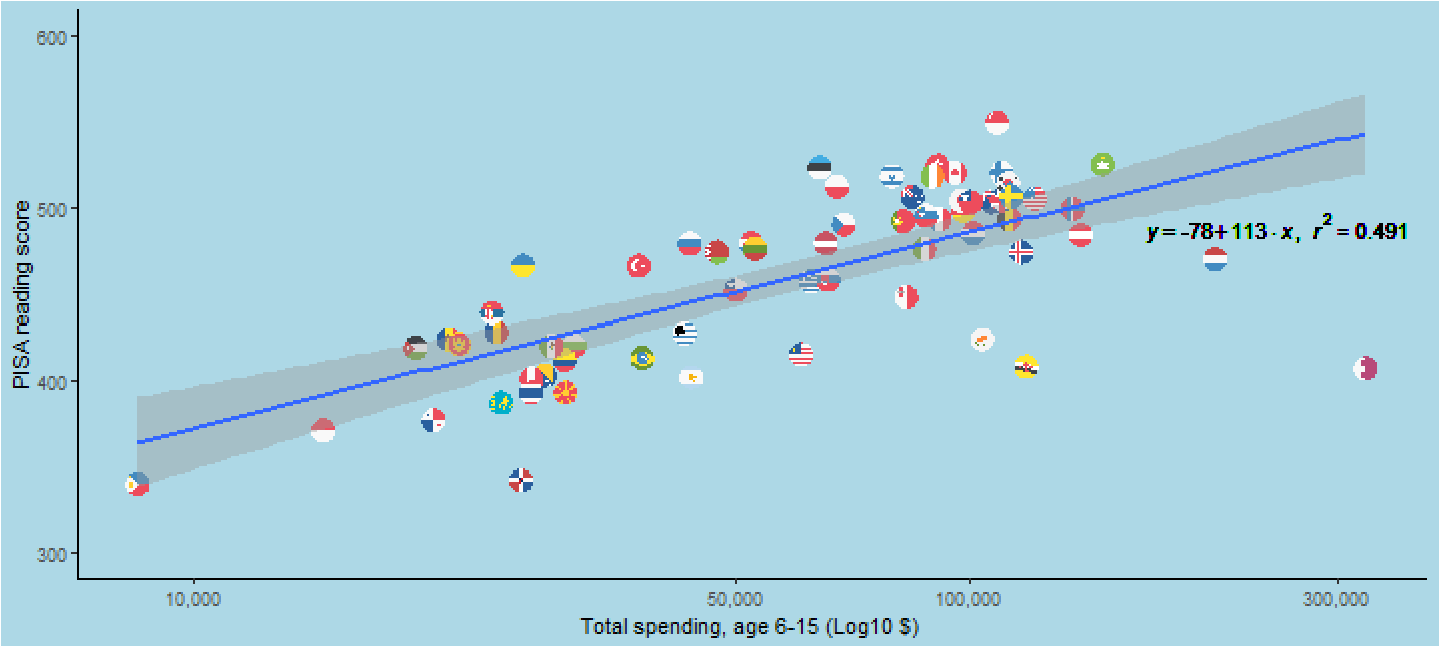


$$\widehat{READ}_j = 428 + 0.00043 \times SPEND_j$$

$$\widehat{READ}_j = -78.03 + 112.74 \times \log_{10}(SPEND_j)$$

- In ed/dev psych this kind of curve is called “learning curve”; represents standard rate of learning
- More broadly, increasing exponential decay or diminishing marginal returns

Interpret



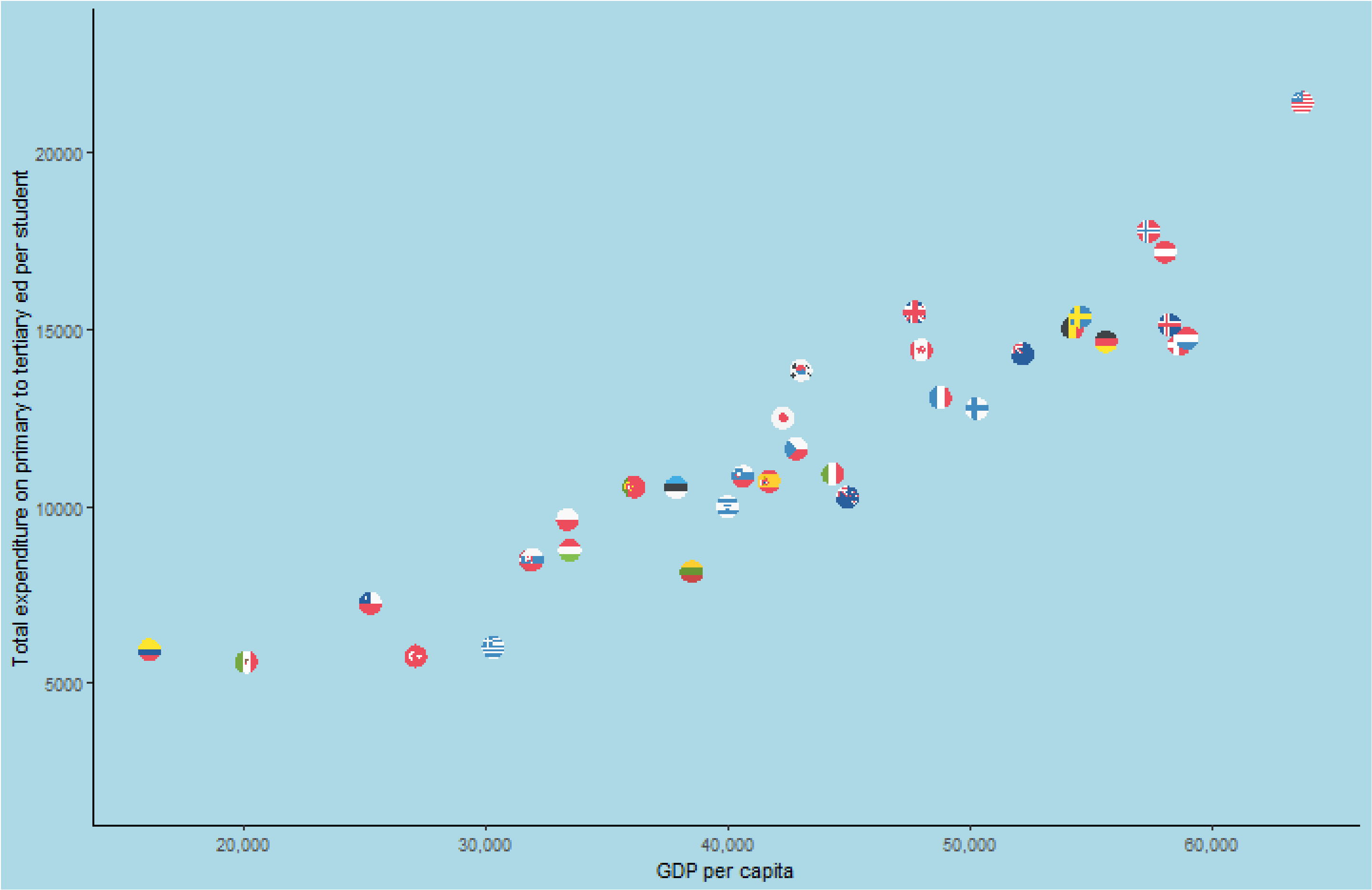
Some alternative ways to describe this relationship:

- Average reading scores in the population of countries sitting for the 2018 PISA reading test were 112.7 points higher for every ten-fold increase in cumulative educational spending on children aged 6-15.
- As cumulative education spending on children aged 6-15 is ten times higher, reading scores in the population of countries sitting for the 2018 PISA reading test were 112.7 points higher, on average.
- We predict that two countries that spend an order of magnitude (e.g., $10,000 vs. $100,00) apart on cumulative educational expenditures on children aged 6-15 will have PISA reading scores 112.7 points apart.

Log transformations in Y

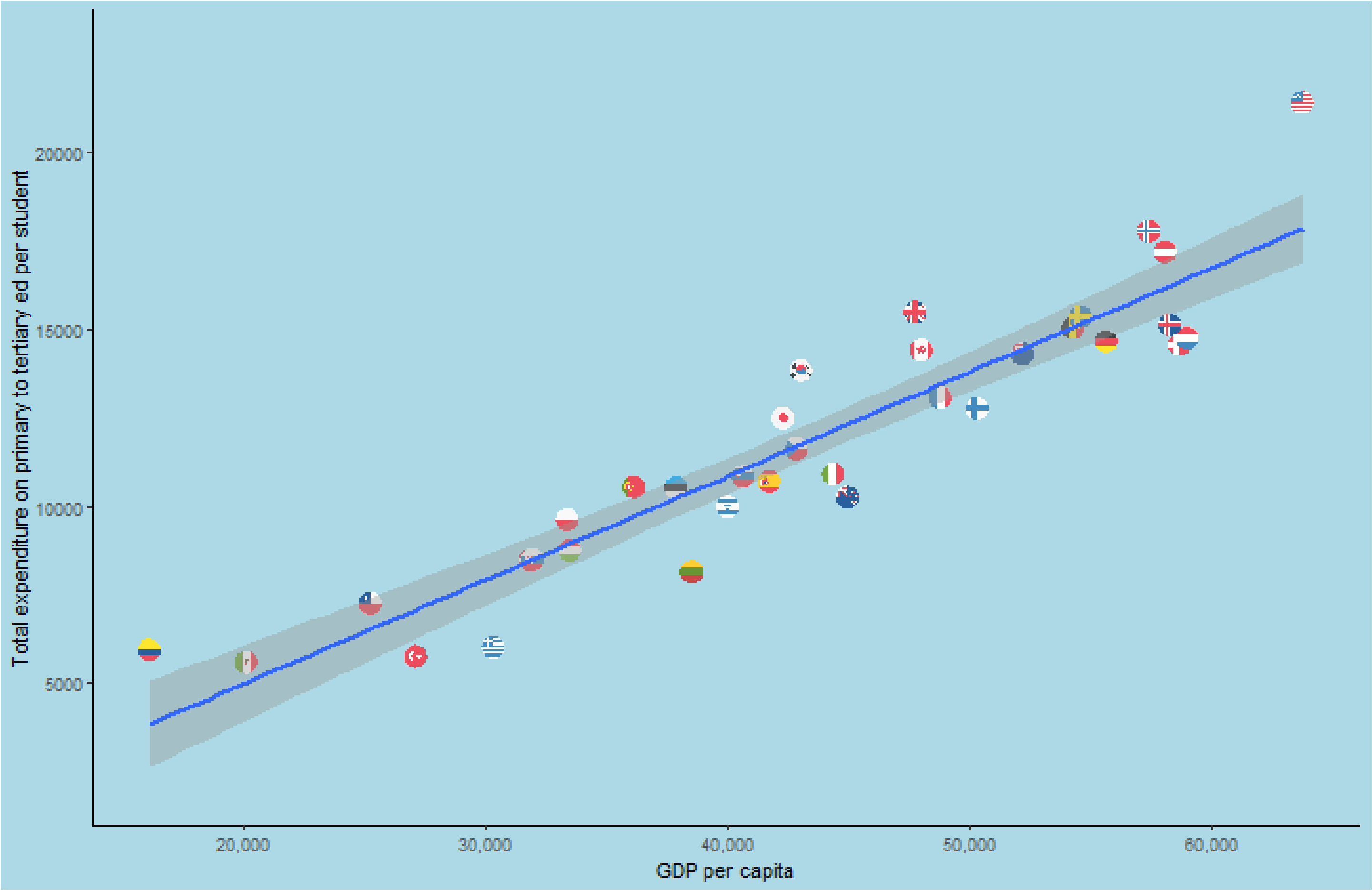
aka Exponential growth curve

GDP and PPE



Total expenditure on primary to tertiary ed per student
20000
15000
10000
5000
20,000
30,000
40,000
50,000
60,000
GDP per capita

GDP and PPE



Total expenditure on primary to tertiary ed per student
20000
15000
10000
5000
20,000
30,000
40,000
50,000
60,000
GDP per capita

An alternative model

The relationship of GDP and PPE are relative to their respective values. The relationship has a smaller magnitude when GDP per capita is smaller and a larger magnitude when GDP per capita is larger. Can use a log transformation to capture the non-absolute (non-constant) nature of the slope:

$$PPE_j = \beta_0 * 2^{(\beta_1 GDP_j + \varepsilon)}$$

$$\log_2(PPE_j) = \log_2 \beta_0 + \beta_1 GDP_j + \varepsilon$$

Interpreting this

Can interpret log outcomes as percent changes because:

$$Y_1 = \beta_0 2^{\beta_1 X_1}$$

$$Y_2 = \beta_0 2^{\beta_1 (X+1)} = \beta_0 2^{\beta_1 X} 2^{\beta_1}$$

$$\frac{Y_2}{Y_1} = \frac{\beta_0 2^{\beta_1 X} 2^{\beta_1}}{\beta_0 2^{\beta_1 X}} = 2^{\beta_1}$$

So, Y_2 is 2^{β_1} times larger than Y_1! Depends on key properties of logs:

- log(xy) = log(x) + log(y)
- $\log(x^p)$ = p*log(x)

Percent growth rate = $100(2^{\beta_1} - 1)$

Regress log(Y) on X and substitute the estimated slope into the equation for the percent growth rate to obtain the estimated percent growth rate per unit change in X.

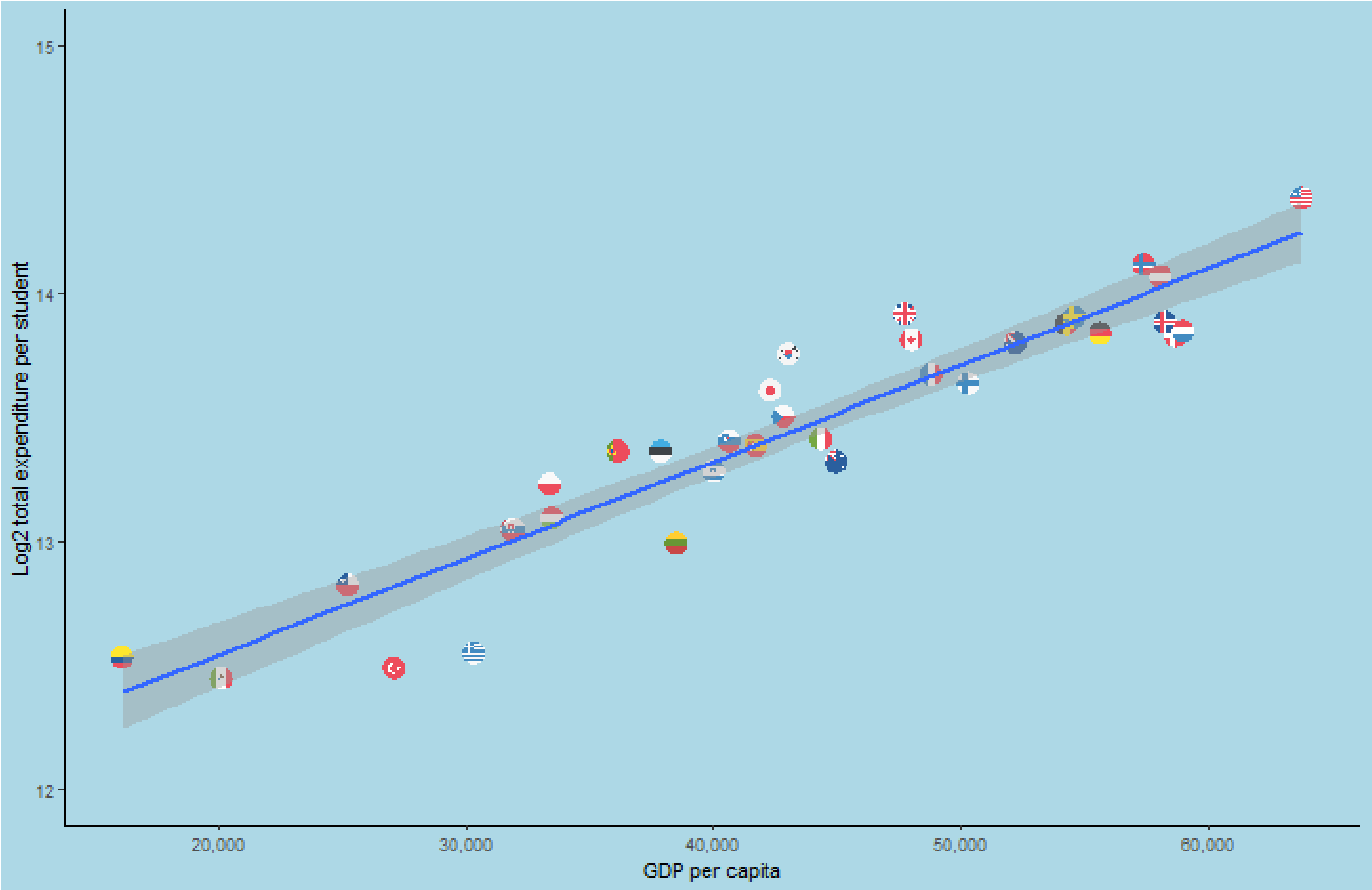
$Y_2 = 2^{\beta_1} Y_1$ is the same thing as saying the percent growth rate is $100(2^{\beta_1} - 1)$

Visualized Y transformation

```
oecd$log2ppe <- log2(oecd$ppe)

log_ppe <- ggplot(oecd, aes(x=gdp, y=log2ppe))
```

Visualized Y transformation



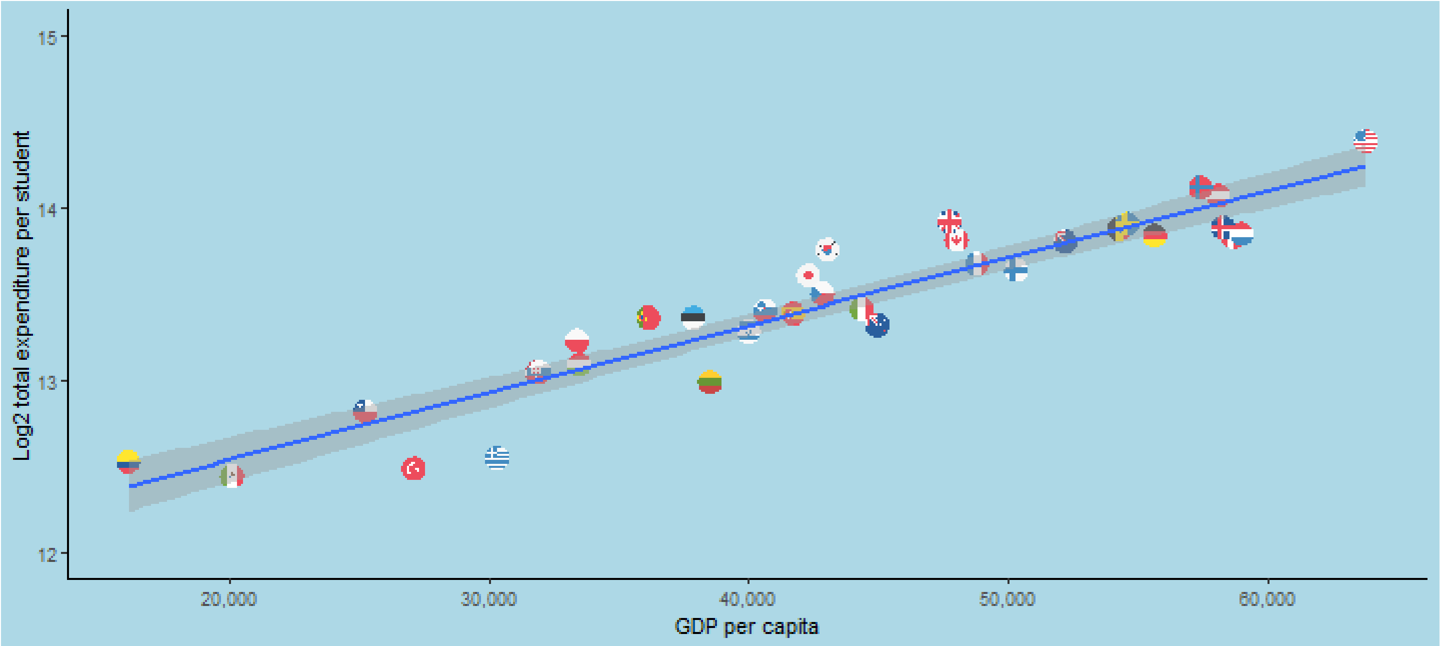
Regress $\log_2(ppe)$ on gdp

```
summary(lm(log2(ppe) ~ gdp, oecd))
```

```
...
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.39728 -0.09378  0.01867  0.11920  0.31357
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 1.176e+01  1.113e-01   105.7   <2e-16 ***
## gdp         3.899e-05  2.484e-06    15.7   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.1712 on 32 degrees of freedom
## Multiple R-squared:  0.8851,    Adjusted R-squared:  0.8815
## F-statistic: 246.5 on 1 and 32 DF,  p-value: < 2.2e-16
...
```

Percent growth rate: $100(2^{0.000039} - 1) = 0.0027\%$; for each $1 more of GDP per person, PPE is 0.0027% higher; or for each $1,000 more of GDP per person, PPE is 2.7% higher

Interpreting log Y results



$$\log_2(\hat{PPE_j}) = 11.8 + 0.000039 * GDP_j$$

> Per capita gross domestic product (GDP) is a strong predictor of yearly per-student expenditure from primary through tertiary education. In particular, if we compare two countries whose GDPs differ by $1,000, we would predict that the wealthier country would have per pupil expenditure that is 2.7 ***percent*** higher than the country with the smaller economy.

Log-log transformations

aka proportional growth

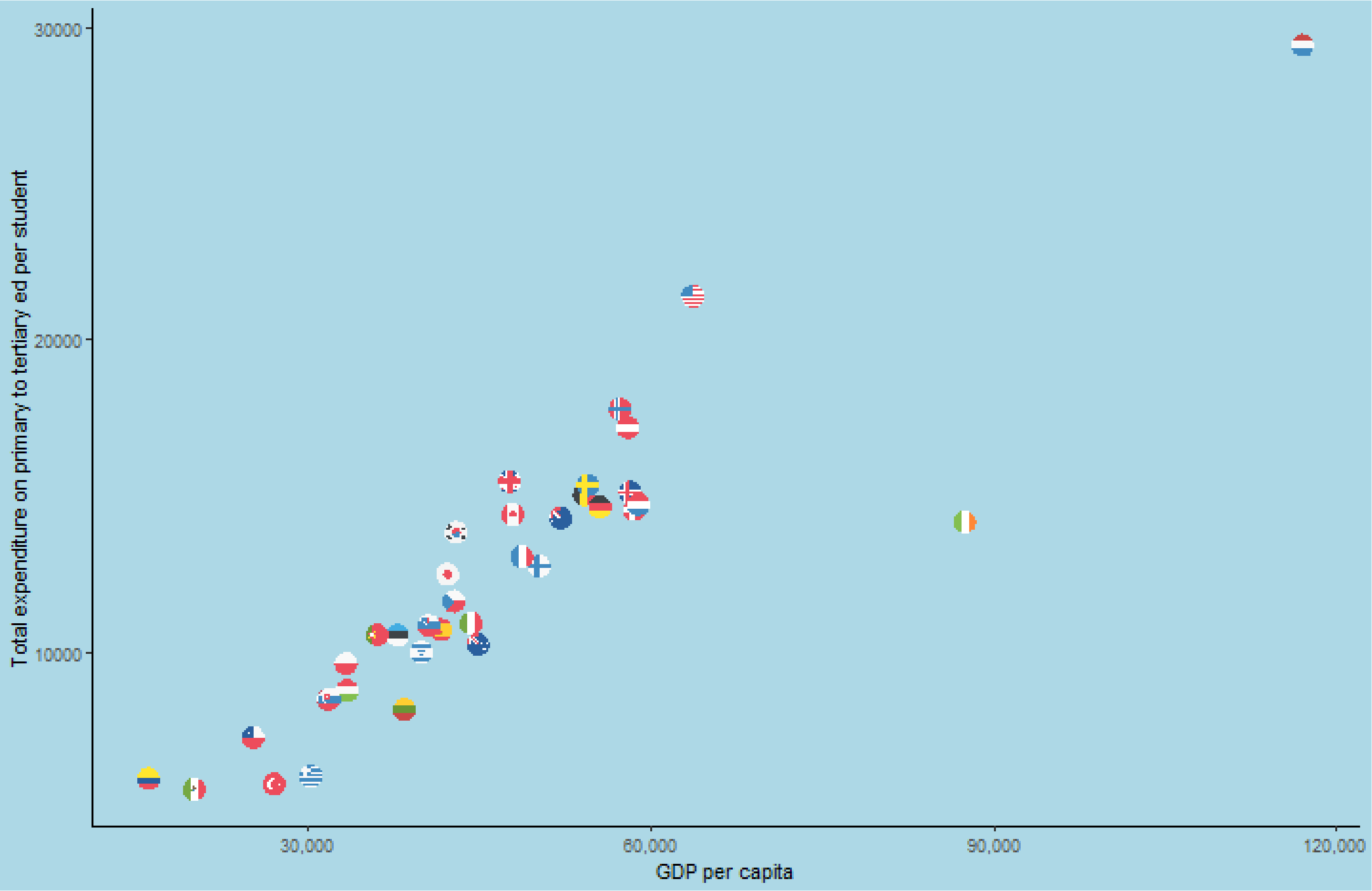
Which 🌳 to harvest?

- Could theoretically select a log of any base to transform outcome or predictor or both to a linear relationship
- Much more sensible to restrict yourself to base_10, base_2 or the natural log; comes from Euler's number (e)

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n \approx 2.718281828459...$$

- Natural log: $\log_{2.718...}(x) = \log_e(x) = \ln(x)$

All the countries

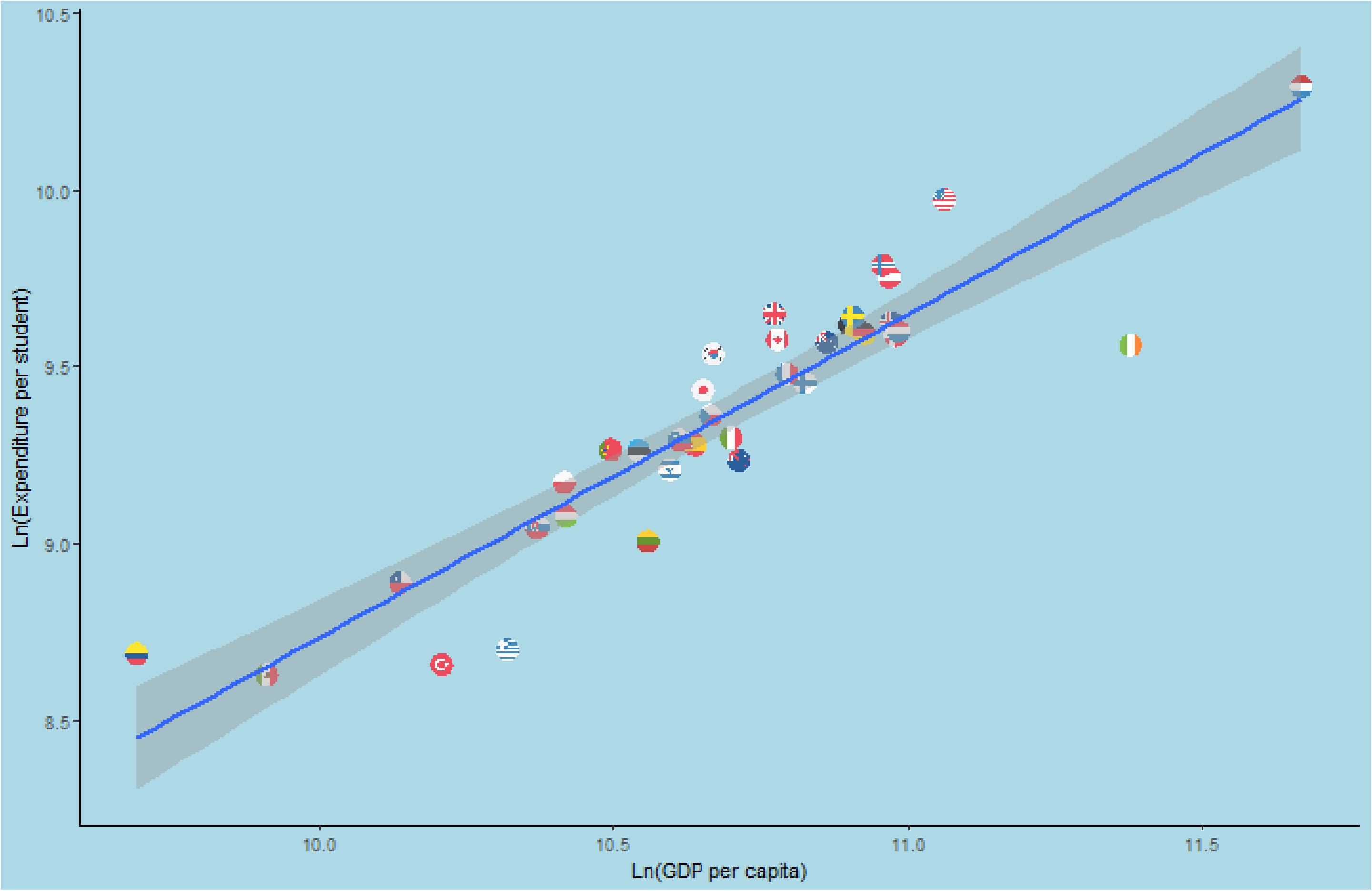


Log-log transformations

```
oecd2$lngdp <- log(oecd2$gdp)
oecd2$lnppe <- log(oecd2$ppe)

ln_ppe <- ggplot(oecd2, aes(x=lngdp, y=lnppe))
```

Log-log transformations



Regress $\ln(ppe)$ on $\ln(gdp)$

```r
summary(lm(log(ppe) ~ log(gdp), oecd2))
```

```
...
## Residuals:
##      Min       1Q   Median       3Q      Max 
## -0.43570 -0.04076  0.01302  0.07489  0.26542 
## 
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) -0.39273    0.72674   -0.54    0.592    
## log(gdp)     0.91274    0.06801   13.42 3.83e-15 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 0.1509 on 34 degrees of freedom
## Multiple R-squared:  0.8412,    Adjusted R-squared:  0.8365 
## F-statistic: 180.1 on 1 and 34 DF,  p-value: 3.826e-15
...
```

$$Ln\hat{P}PE_j = -0.39 + 0.91 * LnGDP_j$$

Interpreting this

Can interpret log-log relationships in percent terms. β_1 **represents the % change in Y per 1% change in X.**

Postulated model:

- $Y = \beta_0 X^{\beta_1} e^{\varepsilon}$
- $\ln(Y) = \ln(\beta_0 X^{\beta_1} e^{\varepsilon})$
- $\ln(Y) = \ln(\beta_0) + \ln(X^{\beta_1}) + \ln(e^{\varepsilon})$
- $\ln(Y) = \ln(\beta_0) + \beta_1 \ln(X) + \varepsilon$

Imagine `(Y_1)` and `(Y_2)` are 1% (or 0.01) apart:

- $Y_1 = \beta_0 X^{\beta_1}$
- $Y_2 = \beta_0 (1.01X)^{\beta_1} = \beta_0 X^{\beta_1} (1.01)^{\beta_1}$
- $\frac{Y_2}{Y_1} = \frac{\beta_0 X^{\beta_1}}{\beta_0 X^{\beta_1}} = (1.01)^{\beta_1}$

So Y_2 is $(1.01)^{\beta_1}$ times larger than Y_1

Regress ln(Y) on ln(X) and the slope estimate is the estimated percent difference in Y per 1 percent difference in X

Interpret log-log relationship

```
summary(lm(log(ppe) ~ log(gdp), oecd2))
```

```
...
## Residuals:
##      Min       1Q   Median       3Q      Max 
## -0.43570 -0.04076  0.01302  0.07489  0.26542 
## 
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) -0.39273    0.72674   -0.54    0.592    
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## F-statistic: 180.1 on 1 and 34 DF,  p-value: 3.826e-15
...
```
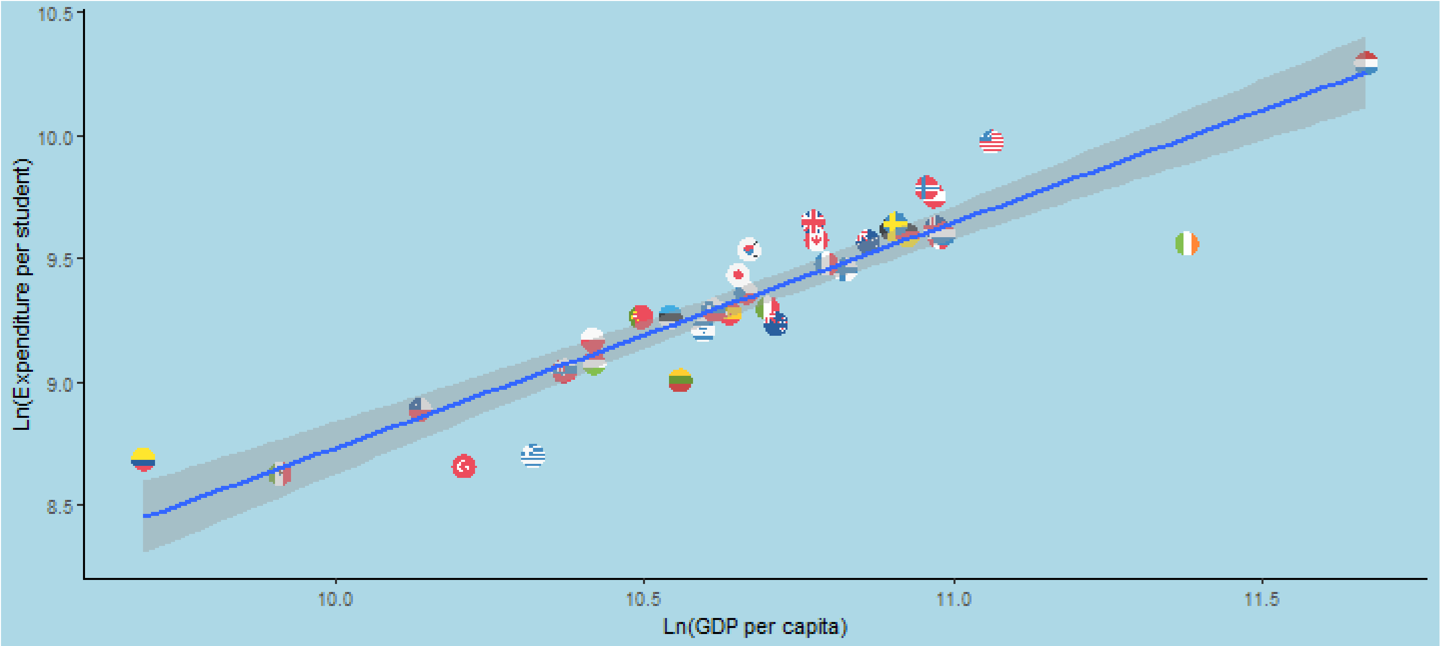

"1 percent change in GDP predicts 0.91 percent change in PPE"

Interpret log-log relationship

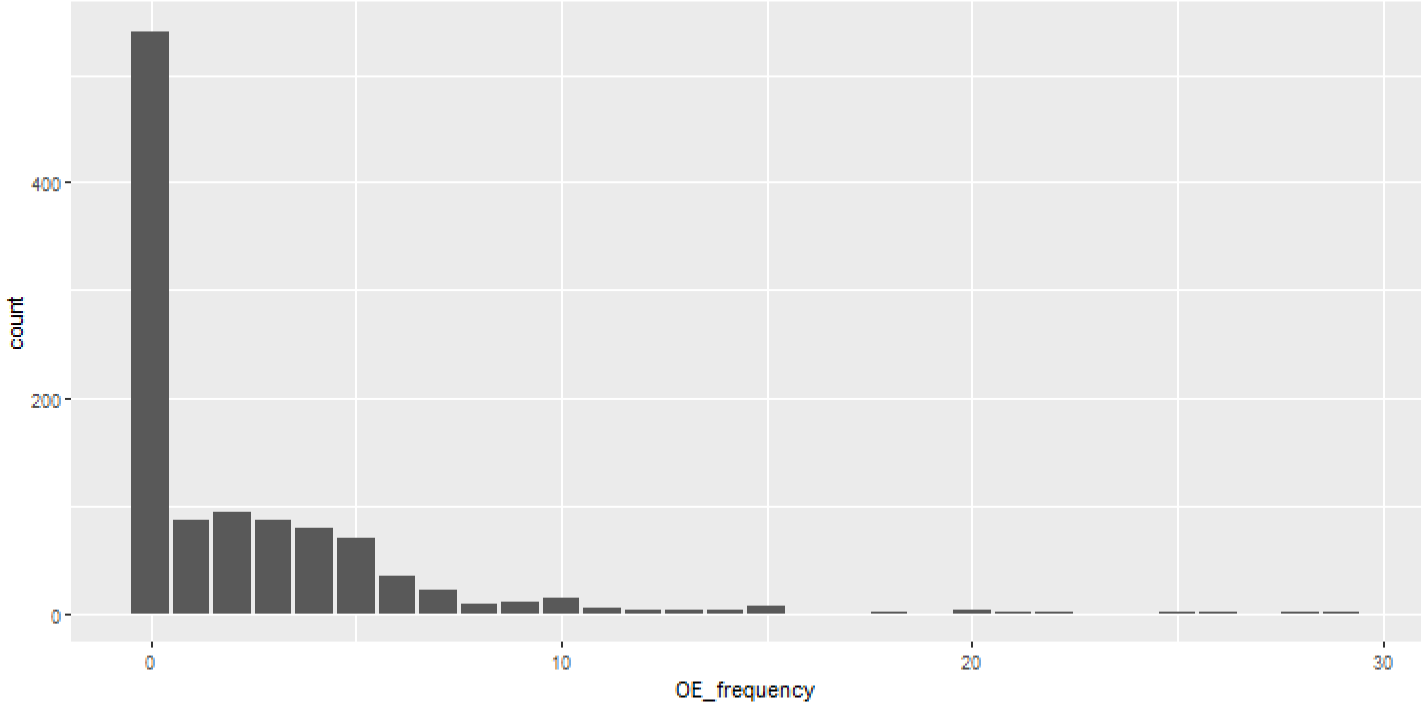


$$\ln(P\hat{P}E_j) = \ln(\beta_0) + \beta_1 \ln(GDP_j) + \varepsilon$$

> We predict that, on average, comparing two countries with GDP per capita separated by 1 percent the wealthier country will spend 0.91 percent more on its pupils across primary through tertiary education.

"Forbidden" log transformations

So far, we've been dealing with situations in which all the variables we needed to transform were non-zero. In fact this is often not the case:



Many other instances: counts of behaviors, individual income, absences, scale scores, etc.

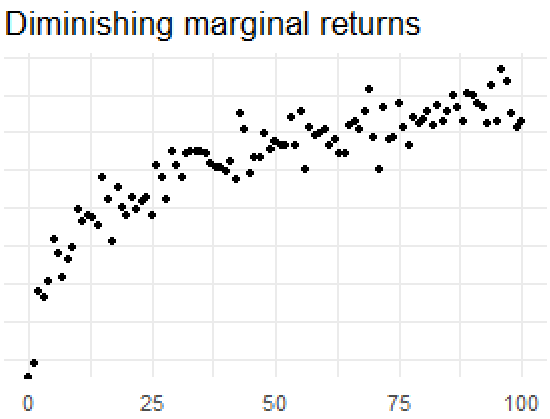
"Forbidden" log transformations

Traditional approach:

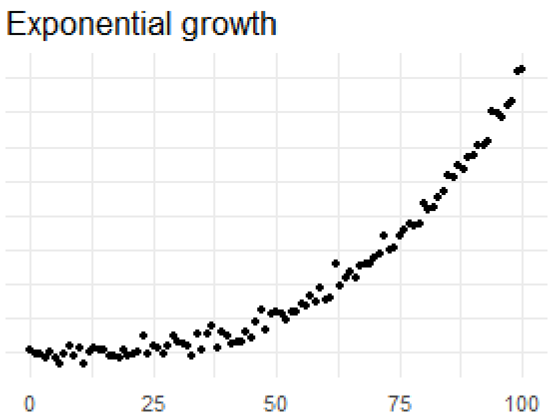
- Add a small "starter" value to all raw values (+1, +0.1, +0.01, +0.001, etc.)
- Take log of this "zero-inflated" variable

DO NOT DO THIS!!!

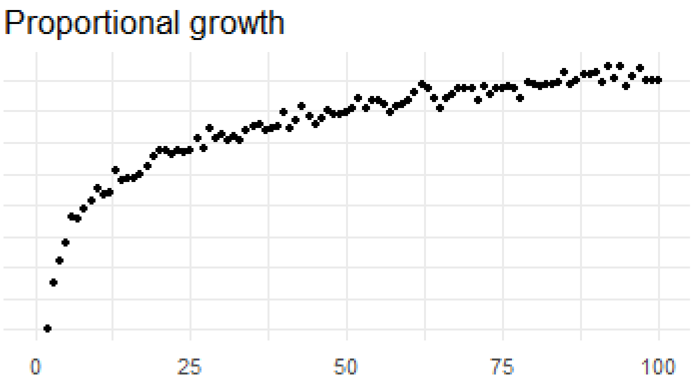
- Value selected for starter and proportion of 0s in your data can results in wildly inconsistent coefficient estimates
- You'll address this issue in EDUC 645 with Poisson regression



- Regress Y on log(X)
- $Y = \hat{\beta}_0 + \hat{\beta}_1 \log(X)$
- "every doubling (or whatever base) of X associated with $\hat{\beta}_1$ diff in Y"



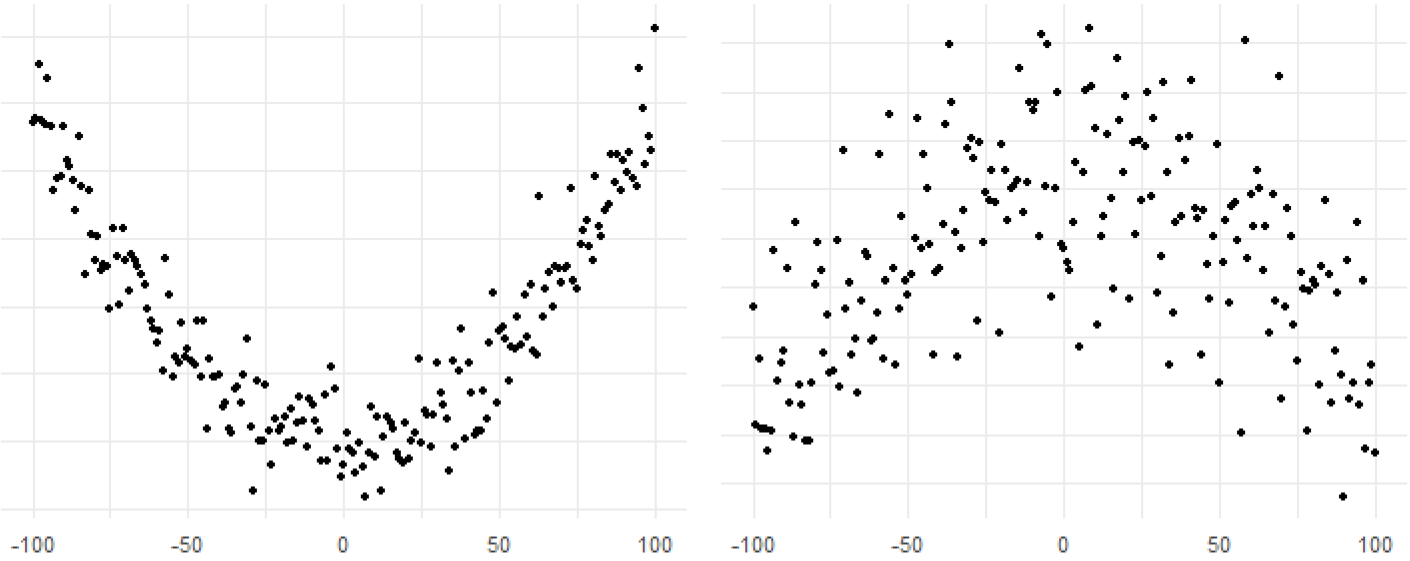
- Regress log(Y) on X
- $\log(Y) = \hat{\beta}_0 + \hat{\beta}_1 X$
- Every 1 unit diff in X associated with $100(e^{\hat{\beta}_1} - 1)$ % diff in Y



- Regress log(Y) on log(X)
- $\log(Y) = \hat{\beta}_0 + \hat{\beta}_1 \log(X)$
- Every 1% diff in X associated with $\hat{\beta}_1$ percent diff in Y

Quadratic terms: a special kind of interaction

Quadratic model



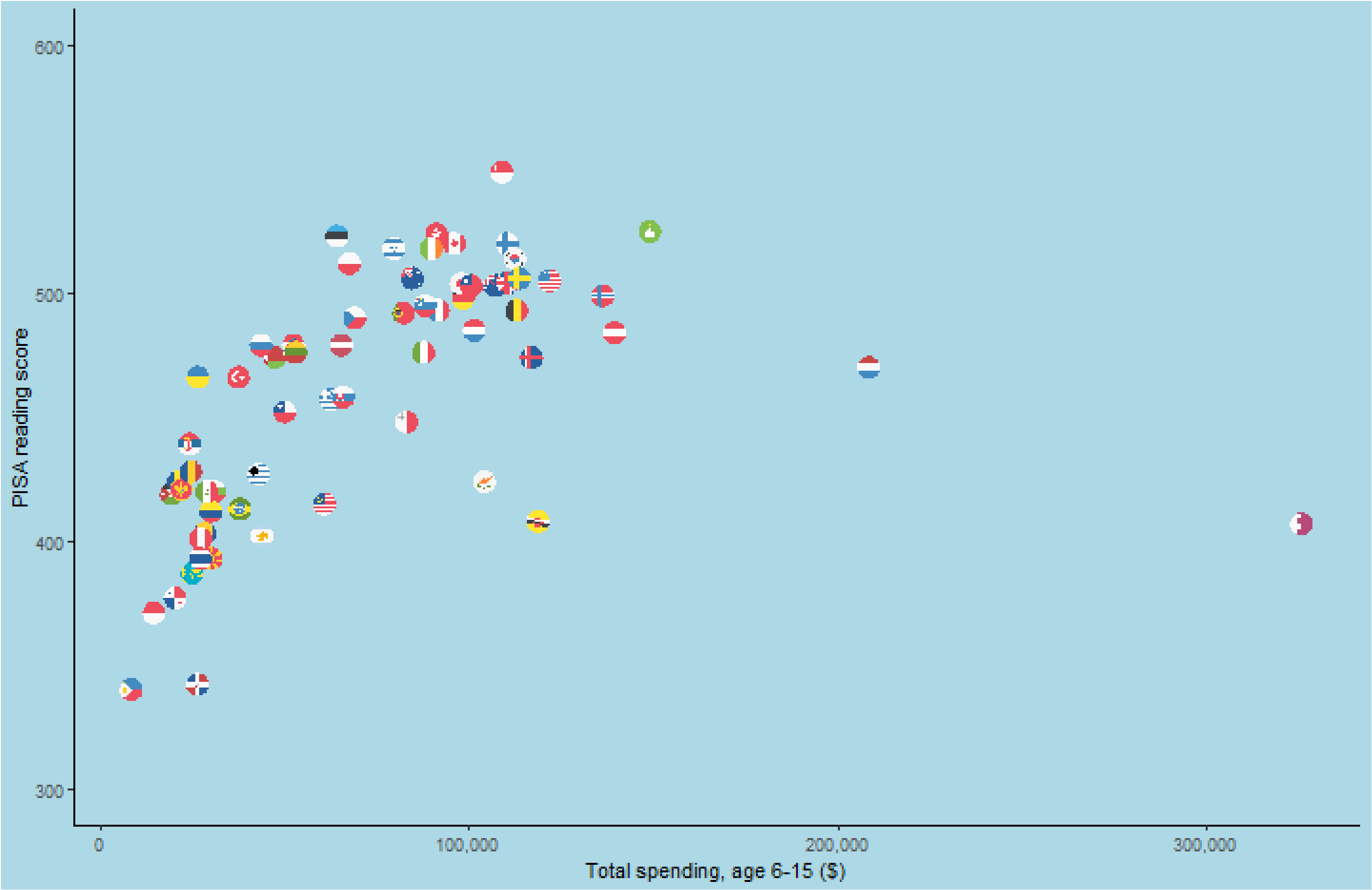
Effects of a predictor can differ by that predictor:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 (X_1 * X_1) + \varepsilon$$

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2) + \varepsilon$$

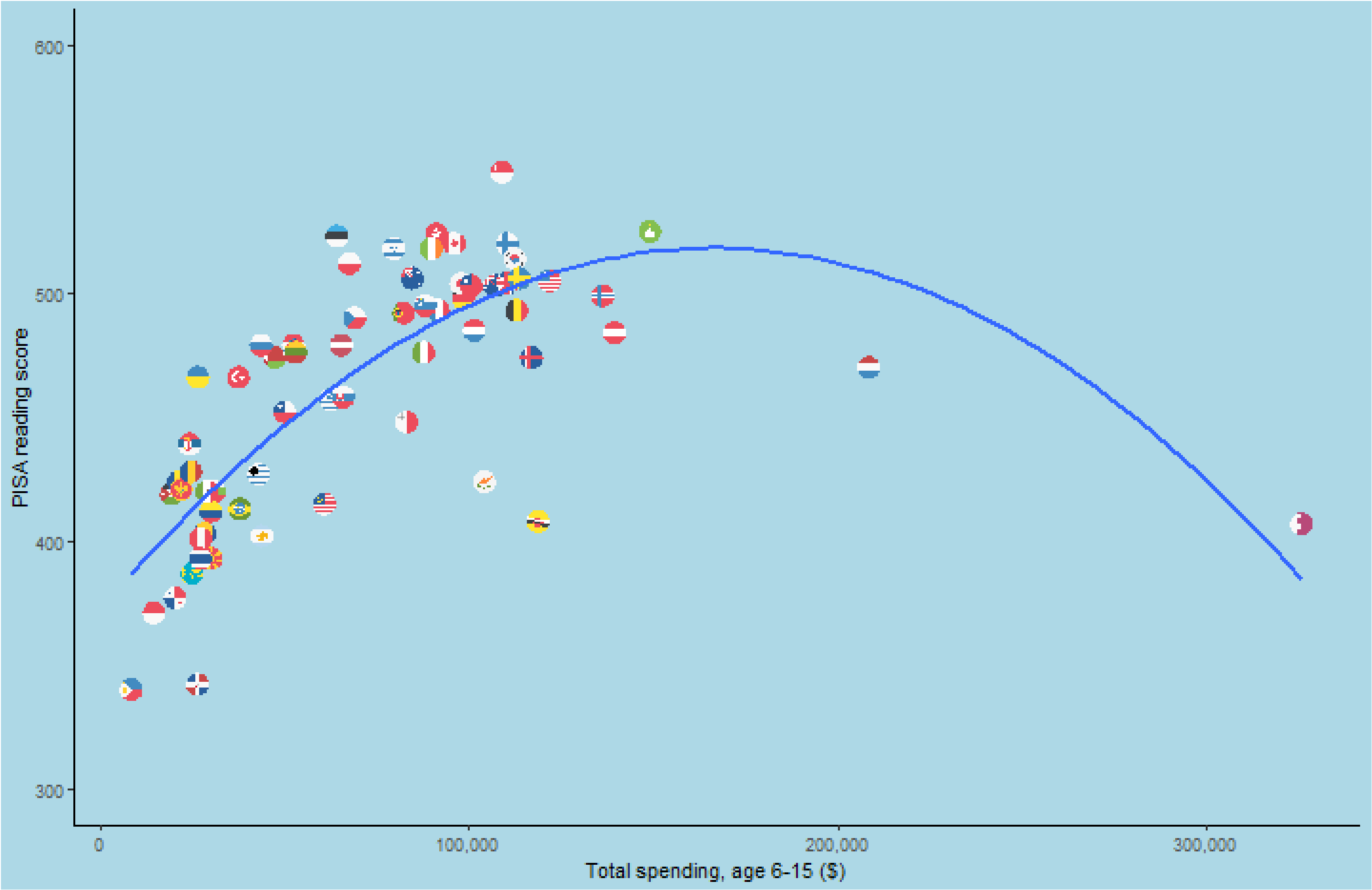
Can point upwards or downwards, but **all quadratic relationships are non-monotic; the relationship both rises and falls (or falls and rises)**

A quadratic relationship

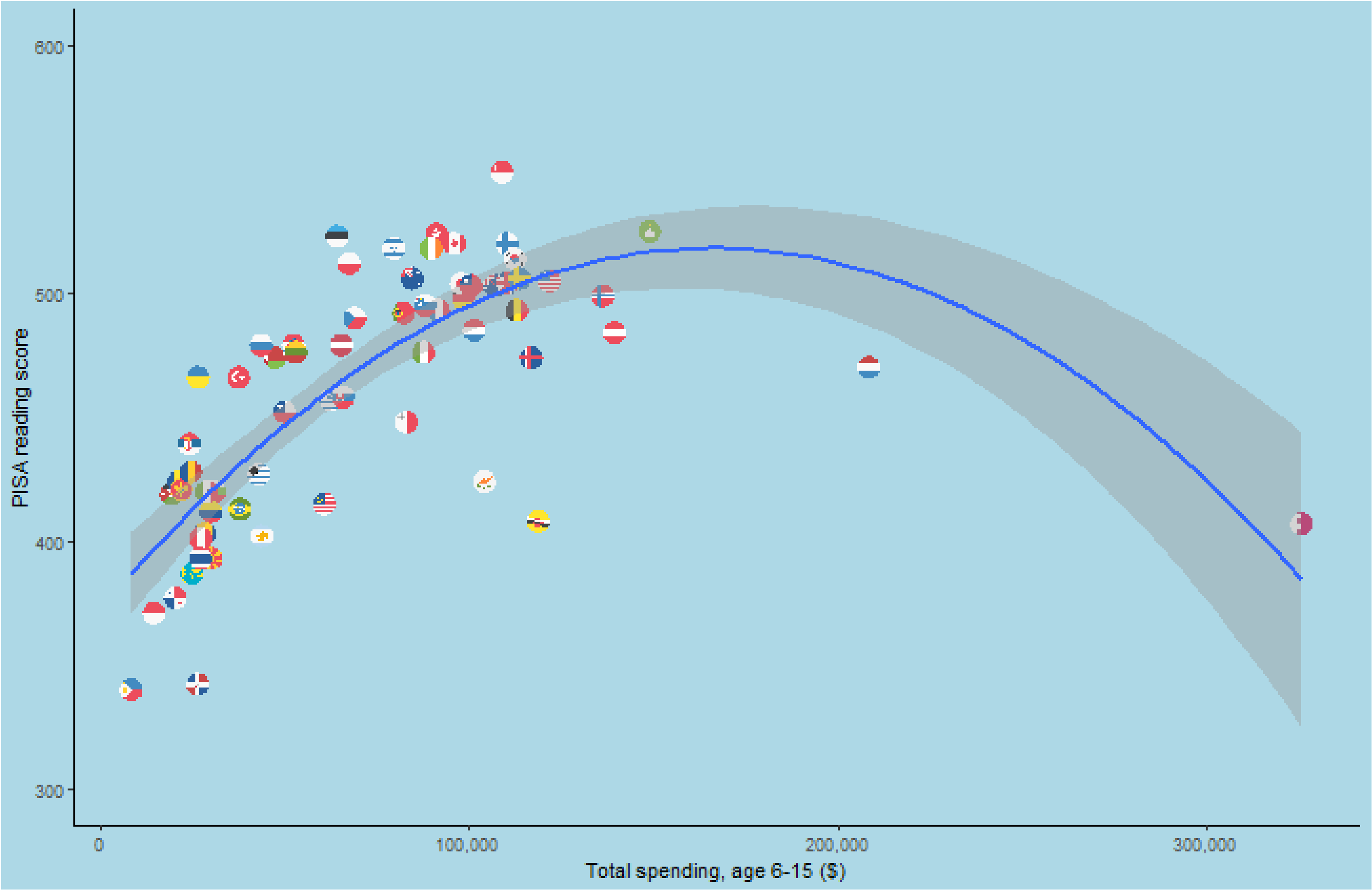


Which direction will the quadratic line of best fit point?

A quadratic relationship



A quadratic relationship



We can represent quadratic fits mathematically in generic form: $y = \beta_0 + \beta_1 x + \beta_3 x^2$.

Challenge: what signs will each of the three coefficients take for the above relationship?

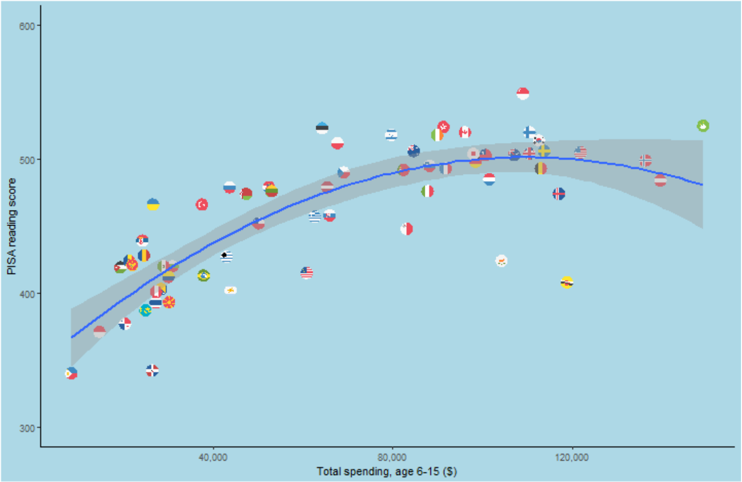
Fitting the quadratic

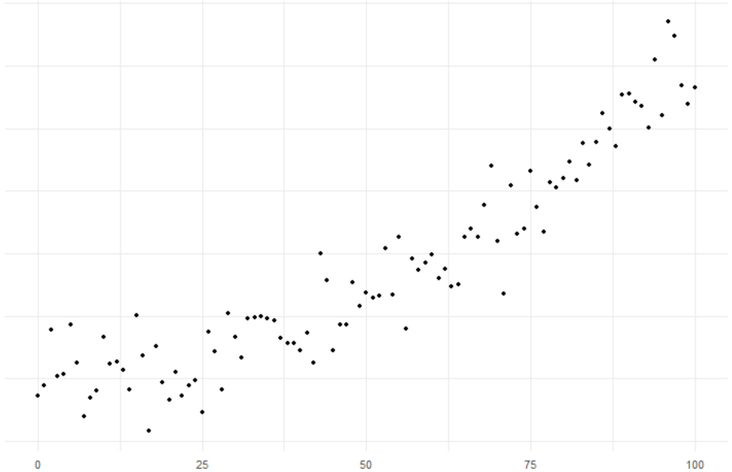
```
summary(lm(read_score ~ poly(total_spending, 2), pisa))
```

```
...
## Residuals:
##     Min      1Q  Median      3Q     Max 
## -98.511 -15.722   3.806  22.651  59.394 
## 
## Coefficients:
##                          Estimate Std. Error t value Pr(>|t|)    
## (Intercept)               459.985      3.887 118.327  < 2e-16 ***
## poly(total_spending, 2)1  177.680     31.341   5.669  4.0e-07 ***
## poly(total_spending, 2)2 -253.728     31.341  -8.096  2.7e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 31.34 on 62 degrees of freedom
## Multiple R-squared:  0.6117,	Adjusted R-squared:  0.5992 
## F-statistic: 48.84 on 2 and 62 DF,  p-value: 1.834e-13
...
```

Fitted equation: $\hat{read} = 460.0 + 177.7 * spend - 253.8 * spend^2$. How do our model fit statistics compare to the linear version?

The "right" fit to data





- A declining relationship between spending and performance doesn't make much substantive sense, so we would probably not use a quadratic fit for our full data
- However, without Qatar and Luxembourg, a quadratic describes the relationship quite nicely

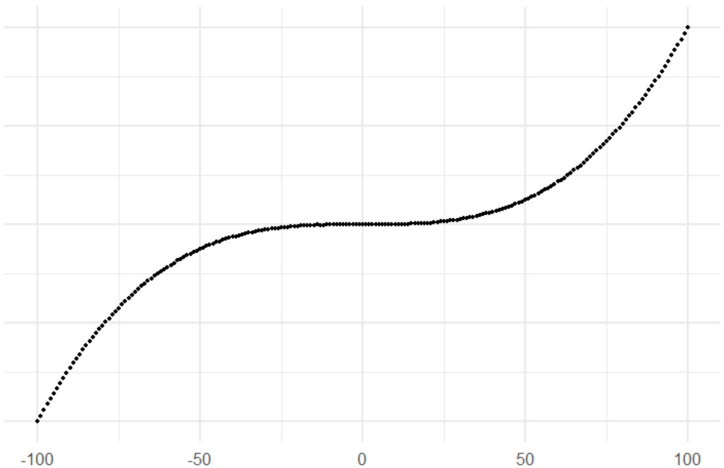
- Don't extrapolate the shape of the parabola to the left of the y-axis
- Shouldn't assume the y values will be higher to the left of the y-axis

Higher-order polynomials

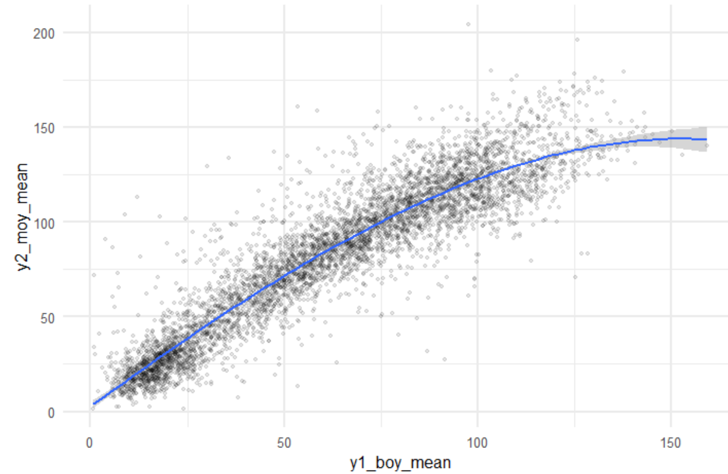
Cubics

We needn't restrict ourselves to transformations to normality to only quadratic relationships. Many relationships, for example are cubic (third-power) in nature. Particularly true when there are measurement issues in the tails and/or floor/ceiling effects.

Strong cubic



Our DIBELS data



$$\hat{W20\_ORF} = 62.3 + 2230 * F19\_ORF + 4.2 * F19\_ORF^2 - 182.0 * F19\_ORF^3$$

Other approaches

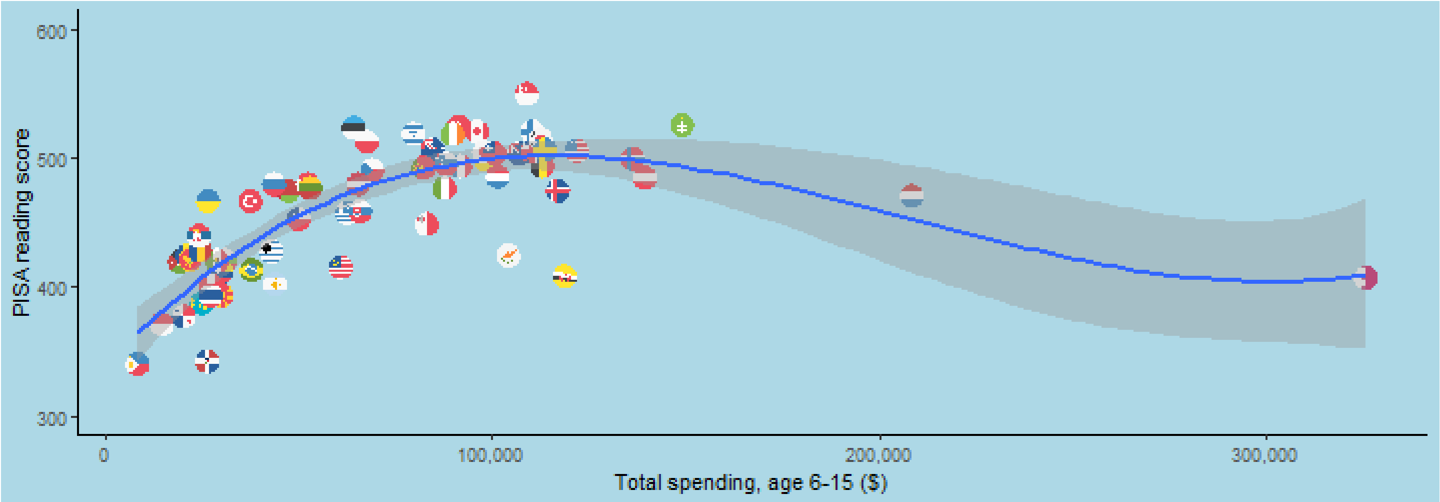
There are an infinite number of potentially effective transformations:

- Squares, cubes, quartic, quintics, …
- Square roots, cube roots, fourth roots, …
- Logarithms (of any base), antilogarithms
- Inverses
- Trigonometric functions
- Hyperbolic functions
- Combinations of above…

Approaches to achieve local linearity:

- Splines
- Local estimated scatterplot smoothing (LOESS)

Some emerging issues:



Synthesis and wrap-up

Different approaches

Empirical approach

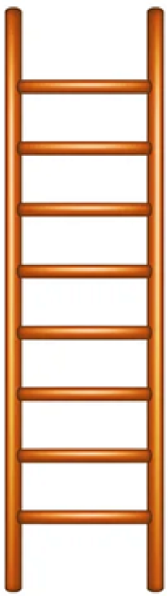
- Notice presence of non-linearity in relationship
- Find an *ad-hoc* transformation of either the predictor, the outcome, or both that renders the relationship linear
- Use OLS in the transformed world, and conduct inference there
- De-transform fitted model to produce sensible plots

Theory-driven approach

- Use theory or knowledge from prior research to postulate a non-linear model
- Use non-linear regression (`nls` or other estimation packages) (part of the Generalized Linear Model family) to fit the postulated trend in the real world and conduct inference there
- Interpret parameter estimates directly
- **We are not learning how to do this, but worth exploring yourself**

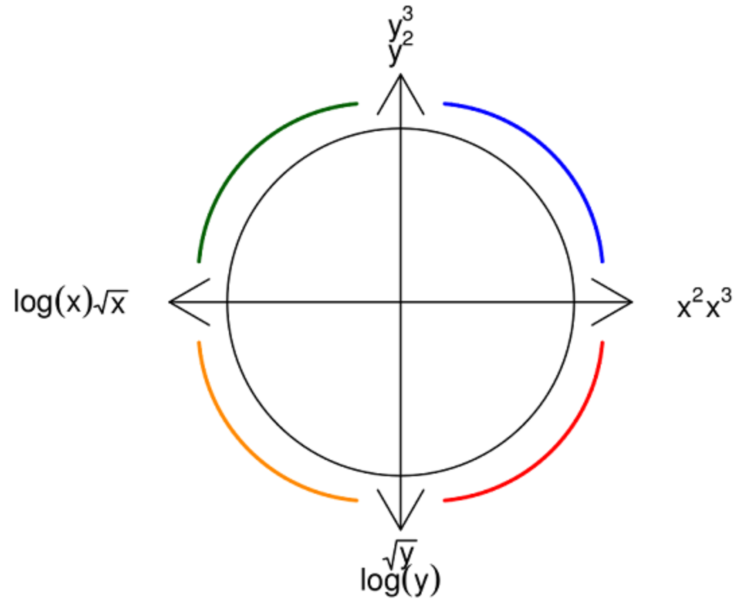
The Ladder and the Bulge

Tukey's Ladder



Power	Transformation	Name
3	x^3	Cubic
2	x^2	Quadratic
1	x	Untransformed
½	$x^{1/2} = \sqrt{x}$	Square root
"0"	$\log(x)$	Logarithm
-1/2	$x^{-1/2} = 1/\sqrt{x}$	Reciprocal root
-1	$x^{-1} = 1/x$	Reciprocal
-2	$x^{-2} = 1/x^2$	Reciprocal square
-3	$x^{-3} = 1/x^3$	Reciprocal cubic

Tukey's Bulge



Putting non-linearity together

- **Remember to check your linearity assumption**
    - Use bivariate scatter plots
    - Use residual and Q-Q plots to diagnose
- **Make sensible transformations**
    - Logarithmic, inverse, root and other functions can allow a return to a world of linearity and permit you to use the GLM tools of OLS to estimate non-linear relationships
    - Best to use transformations that are the most straightforward to interpret
    - Use Tukey's Bulge to guide what kind of transformation you will attempt
    - There is no one "right" transformation for a given data shape
    - Start with transforming x before y
    - Generally, do **not** use a "start" to log transform data that includes 0s
    - Inspect scatter plots post-transformation to check for success in linearizing
        - With large data, can be hard to see; consider binscatter options (by hand or `binsreg`)
- **Predictors can interact with themselves**
    - Quadratic and cubic models provide a flexible strategy for fitting non-linear models, especially those that cannot be linearized by logarithms
    - Be careful about overfitting and model instability with polynomials of order >3!
    - Quadratics and logs will often produce similar fitted lines; quadratic allows direct statistical test for non-linearity, logarithm may fit with theory better and/or can be more readily interpretable

Goals of the unit

- Describe in writing and verbally the assumptions we violate when we fit a non-linear relationship with a linear model
- Transform non-linear relationships into linear ones by using logarithmic scales
- Estimate regression models using logarithmic scales and interpret the results
- Estimate and interpret models with quadratic and higher-order polynomial terms (special kinds of interactions)
- Select between transformation options

To-Dos

Assignment 5:

- Due March 8, 11:59p

Final

- Due March 23, 12:01p

Re- (late) submissions

- Everything due March 17, 9:00a (no exceptions)
- Assignments with scores <10.8 only
- Earn up to 10.8

Log vs. quadratic