Art Of Problem Solving

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Problem
For a certain value of k, the system

$$\begin{aligned} x+y+3z&=10, \\ -4x+2y+5z&=7, \\ kx+z&=3 \end{aligned}$$

has no solutions. What is this value of k?
Solution

Let A= $\begin{bmatrix} 1 & 1 & 3 \\ -4 & 2 & 5 \\ k & 0 & 1 \end{bmatrix}$ be the matrix of the system and

$\overline{A} = \begin{bmatrix} 1 & 1 & 3 & 10 \\ -4 & 2 & 5 & 7 \\ k & 0 & 1 & 3 \end{bmatrix}$ the extended matrix.

$As \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix}$ =6$\neq$ 0 we get that $rankA \in \{2,3\}$

$detA = 6-k$
If $k=6$ then $rankA = 2$

As $\begin{vmatrix} 1 & 1 & 10 \\ -4 & 2 & 7 \\ k & 0 & 3 \end{vmatrix}$ =$18-13k \neq 0$ for $k=6$ so rank $\overline{A}$ =3$\neq$ $rankA$

From Kronecker-Capelli we get that the system is incompatible(it has no solutions), so $k=6$ is a solution.

If $k \neq 6$ then $rankA = rank\overline{A} = 3$ and again from Kronecker-Capelli we have that the system is compatible so there is at least a solution.

Therefore $k=6$ is the unique value that makes the system incompatible.

Kronecker-Capelli Theorem
A linear system is compatible if and only if $rankA = rank\overline{A}$

Property $f : R \to R$ a continuous and periodic function, with period $T > 0$, $a \in R$ and F_0 a primitive for $f|_{[a,a+T]}$. Then, function $F : R \to R$, $F(x) = F_0(x - kT) + k(F_0(a+T) - F_0(a)), \forall x \in (a+kT, a+(k+1)T)$, $k \in Z$ is a primitive for f.

Proof: Let $x_0 \in R$. If $k \in Z$ so that $x_0 \in (a+kT, a+(k+1)T)$, then F is differentiable at x_0 and $F'(x_0) = F_0'(x_0 - kT) = f(x_0 - kT) = f(x_0)$. If $x_0 = a+(k+1)T, k \in Z$, then: $\lim_{x\to x_0, x<x0} \frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to x_0, x<x0} \frac{F_0(x-kT)-F_0(a+T)}{x-kT-(a+T)} = F_0'(a+T) = f(a+T) = f(x_0)$ and $\lim_{x\to x_0, x>x0} \frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to x_0, x>x0} \frac{F_0(x-(k+1)T)-F_0(a)}{x-(k+1)T-a} = F_0'(a) = f(a) = f(x_0)$ Then F is differentiable at x_0 and $F'(x_0) = f(x_0), \forall x_0 \in R$, so F is a primitive for$f$.

Example: Find a primitive on R of function $f : R \to R, f(x) = \sqrt{1 - sinx}$

Solution: Function f is continuous and periodic with period $T = 2\pi$. We have: $f(x) = \sqrt{\left(sin\frac{x}{2} - cos\frac{x}{2}\right)^2} = \left|sin\frac{x}{2} - cos\frac{x}{2}\right|$.

For $x \in \left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$, so $\frac{x}{2} \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ we have

$f(x) = sin\frac{x}{2} - cos\frac{x}{2}$ and $F_0(x) = -2cos\frac{x}{2} - 2sin\frac{x}{2}$ is a primitive of function $f|_{[\frac{\pi}{2}, \frac{5\pi}{2}]}$. Because $F_0\left(\frac{5\pi}{2}\right) - F_0\left(\frac{\pi}{2}\right) = 4\sqrt{2}$ it results that

$F : R \to R, F(x) = F_0(x - 2k\pi) + k \cdot 4\sqrt{2}, \forall x \in \left(\frac{\pi}{2} + 2k\pi, \frac{5\pi}{2} + 2k\pi\right]$ and $k \in Z$, is a primitive on R for f.

Problem
$\int_{-1}^{3} \frac{x}{x^3+1} dx$

Solution

The integral does not converge

Indefinite integral :

$\frac{x}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

We want the following relation to be true in order to establish the equality from above

$A(x^2-x+1) + (Bx+C)(x+1) = x$ $(A+B)x^2 + (-A+B+C)x + (A+C) = x$

So we have the system $\begin{cases} A+B=0 \\ -A+B+C=1 \\ A+C=0 \end{cases}$ with solutions: $A = -\frac{1}{3}$

$B = \frac{1}{3}$

$C = \frac{1}{3}$ So the fraction is rewritten as: $\frac{x+1}{3(x^2-x+1)} - \frac{1}{3(x+1)}$ $I = \int \frac{x+1}{3(x^2-x+1)} - \frac{1}{3(x+1)} = \frac{1}{3}\left(\int \frac{x}{x^2-x+1} + \int \frac{1}{x^2-x+1} - \int \frac{1}{x+1}\right)$

$I = \frac{1}{3}\left(\frac{1}{2} \cdot \int \frac{2x-1+1}{x^2-x+1} + \int \frac{1}{x^2-x+1} - \int \frac{1}{x+1}\right)$

$I = \frac{1}{3}\left(\frac{1}{2} \cdot \int \frac{\frac{d}{dx}(x^2-x+1)}{x^2-x+1} + \frac{1}{2} \cdot \int \frac{1}{x^2-x+1} + \int \frac{1}{x^2-x+1} - \int \frac{1}{x+1}\right)$

$I = \frac{1}{3}\left(\frac{1}{2} ln|x^2-x+1| + \frac{3}{2}\int \frac{1}{x^2-x+1} - ln|x+1|\right)$

$I = \frac{1}{6}\left(ln|x^2-x+1| - 2ln|x+1| + 3\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\right)$

$I = \frac{1}{6}\left(ln|x^2-x+1| - 2ln|x+1| + 2\sqrt{3} \cdot arctan\frac{2x-1}{\sqrt{3}}\right) + C$

When we try to compute $F(-1)$ we obtain $ln\ 0$ which does not exist.

Problem

Evaluate: $\int_0^\infty \frac{x^3}{1+x^6}\ dx$

Solution

Using the formula for $x^{2n}+y^{2n}$ we obtain $1+x^6=(x^2+\sqrt{3}x+1)(x^2+1)(x^2-\sqrt{3}x+1)$ so the fraction can be written as: $\frac{x}{6(x^2-\sqrt{3}x+1)}-\frac{x}{3(x^2+1)}+\frac{x}{6(x^2+\sqrt{3}x+1)}$

General way: $\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\int \frac{2ax}{a^2x+bx+c}dx = \frac{1}{2a}\int \frac{2ax+b-b}{ax^2+bx+c}dx = \frac{1}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx - \frac{b}{2a}\int \frac{1}{ax^2+bx+c}dx$

$\frac{1}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx = \frac{1}{2a}\int \frac{\frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c}dx = \frac{1}{2a}ln|ax^2+bx+c|+C$

$\int \frac{1}{ax^2+bx+c}dx = \int \frac{1}{a\left(x+\frac{b}{2a}\right)^2+\frac{-\Delta}{4a}}dx = \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2+\frac{-\Delta}{4a^2}}dx$

Substituting $x+\frac{b}{2a}=t$ we get

$\frac{1}{a}\int \frac{1}{t^2-\frac{\Delta}{4a^2}}dt$

If $\Delta<0$ we use the following formula

$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}tan^{-1}\frac{x}{a}+C$

If $\Delta>0$ we use the following formula

$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}ln\left|\frac{x-a}{x+a}\right|+C$ or $-\frac{1}{a}coth^{-1}\frac{x}{a}$

Finally, using the above formulas, we compute the definite integrals to get: $\lim_{x->\infty} I - F(0) = 0-(\frac{-1}{\sqrt{3}}\cdot tan^{-1}(\sqrt{3})) = \frac{\pi}{3\sqrt{3}}$

Problem

Evaluate: $\int \frac{1}{\sin x+\tan x}\ dx$

Solution

$sinx = \frac{2tan(\frac{x}{2})}{1+tan^2\frac{x}{2}}$ $cosx = \frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}$ So $tanx = \frac{2tan(\frac{x}{2})}{1-tan^2\frac{x}{2}}$ $I = \int \frac{1}{sinx+tanx}dx = \int \frac{1}{\frac{2tan\frac{x}{2}}{1+tan^2\frac{x}{2}}+\frac{2tan\frac{x}{2}}{1-tan^2\frac{x}{2}}}dx = \int \frac{(1+tan^2\frac{x}{2})(1-tan^2\frac{x}{2})}{4tan(\frac{x}{2})}dx$ Knowing that $\frac{d}{dx}tan\frac{x}{2} = \frac{1+tan^2\frac{x}{2}}{2}$ we obtain $\int \frac{(\frac{d}{dx}tan\frac{x}{2})(1-tan^2\frac{x}{2})}{2tan(\frac{x}{2})}dx$
$u(x) = tan\frac{x}{2}$ $I = \frac{1}{2}\int \frac{1-u^2}{u}du$ $I = \frac{1}{2}(lnu - \frac{u^2}{2}) + C = \frac{1}{2}(ln(tan\frac{x}{2}) - \frac{tan^2\frac{x}{2}}{2}) + C$

Alternative solution

$sec(x) = \frac{1}{cos(x)}$
$I = \int \frac{1}{sin(x)+tan(x)}dx = \int \frac{sec(x)tan(x)}{tan^2(x)+sec(x)tan^2(x)}dx$
Preparing to substitute with $u(x) = sec(x)$, we rewrite the integral using $tan^2(x) = sec^2(x) - 1$
$I = \int \frac{sec(x)tan(x)}{sec^2-1+sec^3(x)-sec(x)}dx = \int \frac{sec(x)tan(x)}{(sec(x)-1)(1+sec(x))^2}dx$
$du = tan(x)sec(x)$
$I = \int \frac{1}{(u-1)(u+1)^2}du$
Using partial fractions we have:
$I = \int \left(-\frac{1}{4(u+1)} - \frac{1}{2(u+1)^2} + \frac{1}{4(u-1)}\right) du$
$I = -\frac{1}{4}\int \frac{1}{u+1}du - \frac{1}{2}\int \frac{1}{(u+1)^2}du + \frac{1}{4}\int \frac{1}{u-1}du$
$I = -\frac{1}{4}ln(u+1) + \frac{1}{2}\cdot\frac{1}{u+1} + \frac{1}{4}ln(u-1) + C$
$I = -\frac{1}{4}ln(sec(x)+1) + \frac{1}{2}\cdot\frac{1}{sec(x)+1} + \frac{1}{4}ln(sec(x)-1) + C$
To obtain an alternative form:
$I = \frac{(sec(x)+1)ln(sec(x)-1)-(sec(x)+1)ln(sec(x)+1)+2}{4sec(x)+4} + C$
$I = \frac{sec(x)ln(sec(x)-1)+ln(sec(x)-1)-ln(sec(x)+1)-sec(x)ln(sec(x)+1)+2}{4sec(x)+4} + C$
$ln(sec(x)-1) - ln(sec(x)+1) = ln\left(\frac{sec(x)-1}{sec(x)+1}\right) = ln\left(\frac{1-cos(x)}{1+cos(x)}\right) = ln\left(tan^2\left(\frac{x}{2}\right)\right)$
$I = \frac{ln\left(tan^2\left(\frac{x}{2}\right)\right)+sec(x)ln(sec(x)-1)-sec(x)ln(sec(x)+1)+2}{4sec(x)+4} + C$
$I = \frac{ln(tan^2\left(\frac{x}{2}\right))}{4} + \frac{1}{2(sec(x)+1)} + C$
$cos(x) - 1 \leq 0, cos(x) + 1 \geq 0$ so $\frac{cos(x)-1}{cos(x)+1} \leq 0.$
To make the natural logarithm exist we should use the absolute value for the fraction and since it's always negative we should consider it $\frac{1-cos(x)}{cos(x)+1}$

Problem

Evaluate:

$$\int x * arcsin(x)dx$$

Solution

By parts: $\int(x \cdot arcsin(x))dx = \frac{x^2}{2} \cdot arcsin(x) - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}}dx$ Then we make the notation: $I = -\int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$ We take into consideration that: $\frac{d}{dx}\sqrt{1-x^2} = -\frac{x}{\sqrt{1-x^2}}$ Therefore: $I = \frac{1}{2}\int x(\frac{d}{dx}\sqrt{1-x^2})dx$ By parts we have: $I = \frac{1}{2} \cdot x\sqrt{1-x^2} - \frac{1}{2}\int \sqrt{1-x^2}dx$ Which is $\frac{1}{2} \cdot x\sqrt{1-x^2} - \frac{1}{2}\int \frac{1-x^2}{\sqrt{1-x^2}}dx$ So, $I = \frac{1}{2} \cdot x\sqrt{1-x^2} - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}dx + \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$ $I = \frac{1}{2}(x\sqrt{1-x^2} - arcsin(x)) - I$ $2I = \frac{1}{2}(x\sqrt{1-x^2} - arcsin(x)) + C$ $I = \frac{1}{4}(x\sqrt{1-x^2} - arcsin(x)) + C$ So the result is: $\int(x \cdot arcsin(x))dx = \frac{x^2}{2} \cdot arcsin(x) + \frac{1}{4}(x\sqrt{1-x^2} - arcsin(x)) + C$ Or: $\frac{2x^2-1}{4} \cdot arcsin(x) + \frac{1}{4}x\sqrt{1-x^2} + C$