

Problem (clean)

Let

$$\mathbb{X} = \{u \in C^1([0, 1]) : u(0) = u(1) = 0\}, \quad F_n(u) = \int_0^1 (|u'(x)|^n + (u(x) - x)^{76} + 2^{-nu(x)}) dx.$$

(1) For each fixed  $n \geq 2$  show  $F_n$  attains a minimiser  $u_n$  on  $\mathbb{X}$ .

(2) Compute  $\lim_{n \rightarrow \infty} m_n$ , where  $m_n = \min_{u \in \mathbb{X}} F_n(u)$ .

Main (clean) answer —  $\Gamma$ -limit and limit of minima

- Admissible set

$$\mathcal{K} = \{u \in W^{1,\infty}(0, 1) : u \geq 0 \text{ a.e.}, |u'| \leq 1 \text{ a.e.}, u(0) = u(1) = 0\}.$$

- $\Gamma$ -limit functional (candidate)

$$F(u) = \begin{cases} \int_0^1 (u - x)^{76} dx + \mathcal{L}(\{|u'| = 1\}) + \mathcal{L}(\{u = 0\}), & u \in \mathcal{K}, \\ +\infty, & \text{otherwise.} \end{cases}$$

- Minimiser of  $F$ : the "tent" function

$$u_T(x) = \min(x, 1 - x).$$

Compute

$$\int_0^1 (u_T - x)^{76} dx = \frac{1}{154}, \quad \mathcal{L}(\{|u'_T| = 1\}) = 1, \quad \mathcal{L}(\{u_T = 0\}) = 0.$$

Hence

$$F(u_T) = 1 + \frac{1}{154} = \frac{155}{154}.$$

- Limit of minima (conclusion)

$$\lim_{n \rightarrow \infty} m_n = \frac{155}{154}.$$

Evaluation of the Reference Answer (concise)

- Correct points

- It identifies the right constraints ( $u \geq 0$ ,  $|u'| \leq 1$  a.e.) and the correct form of the  $\Gamma$ -limit (measure contributions from  $|u'| = 1$  and  $\{u = 0\}$ ).
- Compactness idea and recovery (zig-zag) construction are the right strategies.
- The tent function as minimiser and the numeric value  $155/154$  are correctly identified.
- Gaps / items to tighten
  - Fix the ambient topology/function space (use  $W^{1,\infty}$  or explain the compactness extraction precisely).
  - Make the lower bound estimates rigorous: control of sets  $\{|u'_n| > 1 + \delta\}$  and passage to measure limits needs explicit lemmas (sketched but not detailed).
  - Give an explicit recovery sequence for a generic admissible  $u$  (the sketch indicates how but omits details).
- Verdict: sketch is essentially correct; it requires standard technical lemmas to be fully rigorous.