Homework Exercise:
Geometric Mean Height and Richardson Number Bias via Jensen's Inequality

Boundary Layer Meteorology / Advanced Atmospheric Physics

Graduate Level — Estimated Time: 2–3 hours

Abstract

This exercise explores systematic bias in bulk Richardson number computation on coarse vertical grids in stable boundary layers. Students will apply Jensen's inequality to prove that layer-averaged Ri_b underestimates the point value at the geometric mean height $z_g = \sqrt{z_0 z_1}$ when $Ri_g(z)$ is concave-down, and verify this bias numerically for realistic stable layer profiles.

1 Problem Statement

Atmospheric models discretize the vertical using finite layers. In the stable boundary layer (SBL), the **bulk Richardson number** Ri_b computed across a layer systematically differs from the **gradient Richardson number** Ri_g evaluated at a representative height. Your task is to prove that when $Ri_g(z)$ is concave-down (typical in SBL), the layer-averaged Ri_b underestimates the point value at the geometric mean height $z_g = \sqrt{z_0 z_1}$.

$$\boxed{Ri_b = \frac{1}{\Delta z} \int_{z_0}^{z_1} Ri_g(z)\, dz < Ri_g(z_g)} \tag{1}$$

where $\Delta z = z_1 - z_0$ and $z_g = \sqrt{z_0 z_1}$.

2 Part A: Jensen's Inequality (Warm-Up)

Hint: Jensen's inequality states that for a **concave** function f,

$$f\left(\frac{1}{b-a}\int_a^b x\, dx\right) \geq \frac{1}{b-a}\int_a^b f(x)\, dx. \tag{2}$$

A1. State Jensen's Inequality

State Jensen's inequality for a concave function. What conditions must f satisfy? What does "concave" mean in terms of the second derivative?

Answer space:

A2. Logarithm Concavity

The natural logarithm $\ln(z)$ is concave. Prove this by computing $d^2(\ln z)/dz^2$ and showing it is negative for $z > 0$.

Proof:

3 Part B: Why Geometric Mean Height?

B1. Logarithmic Mean for Exact Gradient Matching

The **logarithmic mean** of z_0 and z_1 is defined as

$$\bar{z}_{\ln} = \frac{z_1 - z_0}{\ln(z_1) - \ln(z_0)} = \frac{\Delta z}{\ln(z_1/z_0)}. \tag{3}$$

Show that for a log-linear wind profile $U(z) = (u_*/\kappa)\ln(z/z_0) + C$, evaluating the gradient $\partial U/\partial z$ at $\bar{z}_{\ln}$ gives the **exact** layer-averaged gradient:

$$\frac{\Delta U}{\Delta z} = \left.\frac{\partial U}{\partial z}\right|_{z=\bar{z}_{\ln}}. \tag{4}$$

Derivation:

B2. Geometric Mean as Logarithmic Midpoint

The **geometric mean** height is

$$z_g = \sqrt{z_0 z_1}. \tag{5}$$

Show that z_g is the midpoint in **logarithmic coordinates**:

$$\ln z_g = \frac{\ln z_0 + \ln z_1}{2}. \tag{6}$$

Proof:

B3. Thin-Layer Approximation

For thin layers ($z_1/z_0 \to 1$), prove that $\bar{z}_{\ln} \approx z_g$ to second order in $\ln(z_1/z_0)$ by Taylor expanding:

$$\frac{\Delta z}{\ln(z_1/z_0)} \approx z_g \left[1 + O\big((\ln(z_1/z_0))^2\big)\right]. \tag{7}$$

Hint: Use $\ln(z_1/z_0) = \ln(1 + (z_1 - z_0)/z_0) \approx (z_1 - z_0)/z_0 - \dots$

Derivation:

4 Part C: Concave-Down Ri_g and Bias

Assume $Ri_g(z)$ is **concave-down** (i.e., $d^2 Ri_g/dz^2 < 0$) over the interval $[z_0, z_1]$.

C1. Jensen Applied to Ri_g in z Coordinates

Apply Jensen's inequality to $Ri_g(z)$ treated as a concave function. Specifically, show that

$$\frac{1}{\Delta z}\int_{z_0}^{z_1} Ri_g(z)\,dz < Ri_g\!\left(\frac{1}{\Delta z}\int_{z_0}^{z_1} z\,dz\right). \tag{8}$$

What is the arithmetic mean height $\bar{z}_a = (z_0 + z_1)/2$ in this context?

Answer:

C2. Ordering of Heights

We want to compare Ri_b to Ri_g at the **geometric** mean z_g, not the arithmetic mean. Use the fact that $\ln(z)$ is concave to show:

$$\ln(z_g) = \frac{\ln z_0 + \ln z_1}{2} > \ln\left(\frac{z_0 + z_1}{2}\right) = \ln(\bar{z}_a). \tag{9}$$

Thus $z_g < \bar{z}_a$ for $z_0 < z_1$.

Proof:

C3. Jensen in Logarithmic Coordinates

For a power-law or logarithmic profile structure, $Ri_g(z)$ is better approximated as concave in $\ln z$ rather than z. Change variables $s = \ln z$, so $z \in [z_0, z_1]$ maps to $s \in [\ln z_0, \ln z_1]$. Define

$$\tilde{Ri}(s) = Ri_g(e^s). \tag{10}$$

If $\tilde{Ri}(s)$ is concave in s, apply Jensen to show:

$$\frac{1}{\ln(z_1/z_0)} \int_{\ln z_0}^{\ln z_1} \tilde{Ri}(s)\, ds < \tilde{Ri}\left(\frac{\ln z_0 + \ln z_1}{2}\right) = Ri_g(z_g). \tag{11}$$

Derivation:

C4. Logarithmically Weighted Average

Convert the integral back to z coordinates using $dz = e^s ds = z\, ds$:

$$\int_{\ln z_0}^{\ln z_1} \tilde{Ri}(s)\, ds = \int_{z_0}^{z_1} Ri_g(z)\, \frac{dz}{z}. \tag{12}$$

This is the **logarithmically weighted average**. For a thin layer or when Ri_g varies slowly, approximate

$$\frac{1}{\ln(z_1/z_0)}\int_{z_0}^{z_1} Ri_g(z)\,\frac{dz}{z} \approx \frac{1}{\Delta z}\int_{z_0}^{z_1} Ri_g(z)\,dz = Ri_b. \tag{13}$$

Conclude:

$$\boxed{Ri_b \lesssim Ri_g(z_g)} \tag{14}$$

when Ri_g is concave-down in $\ln z$.

Final argument:

5 Part D: Numerical Verification

D1. Quadratic Profile

Assume a quadratic near-neutral form:

$$Ri_g(z) = c_1 z + c_2 z^2, \tag{15}$$

with $c_1 > 0$, $c_2 < 0$ (concave-down).

Choose $z_0 = 10$ m, $z_1 = 100$ m, $c_1 = 0.01$, $c_2 = -0.0001$.

Compute:

- $z_g = \sqrt{z_0 z_1}$
- $Ri_g(z_g)$
- $Ri_b = \frac{1}{\Delta z}\int_{z_0}^{z_1} Ri_g(z)\,dz$ (evaluate the integral analytically)
- Bias ratio $B = Ri_g(z_g)/Ri_b$

Solution:

D2. Logarithmic Profile

Repeat for a logarithmic structure:

$$Ri_g(z) = A \ln(z/z_{\text{ref}}) + C, \tag{16}$$

with $A > 0$, $C > 0$, $z_{\text{ref}} = 1$ m. Note that $d^2/dz^2[\ln(z)] = -1/z^2 < 0$ (concave).

Compute B again. Is $B > 1$?

Solution:

6 Part E: Physical Interpretation

E1. Overmixing Explanation

In your own words, explain why $Ri_b < Ri_g(z_g)$ leads to **overmixing** in a numerical model. Specifically, how does underestimating stability affect the computed eddy diffusivities K_m and K_h?

Answer:

E2. Arithmetic vs Geometric Mean

If a model uses the **arithmetic mean** height $\bar{z}_a = (z_0 + z_1)/2$ instead of z_g, does the bias worsen or improve? Justify using Part C.

Answer:

E3. Practical Correction Strategy

Suggest one practical correction strategy to mitigate this bias without changing the vertical grid resolution. (Hint: Think about modifying $K_{m,h}$ or using a representative height correction.)

Proposal:

7 Submission Guidelines

- **Format:** Typed solutions (LaTeX preferred; Markdown with math acceptable).
- **Code:** Include Python/Julia/MATLAB scripts for Part D with plots of $Ri_g(z)$, z_g, and bias ratio.
- **Figures:** Plot $Ri_g(z)$ vs z with horizontal line at $Ri_g(z_g)$ and shaded region showing layer $[z_0, z_1]$.
- **Length:** ∼5–8 pages including derivations and figures.

8 Additional Challenge (Optional, +10%)

Derive an **exact correction factor** $G(\Delta z, \Delta)$ such that

$$Ri_b^{\text{corrected}} = Ri_b \times G \approx Ri_g(z_g), \tag{17}$$

where $\Delta = d^2 Ri_g/dz^2$ is the local curvature. Express G in terms of z_0, z_1, and Δ to leading order in Δz.

Hint: Use Taylor expansion of Ri_g around z_g.

Derivation:

9 Learning Objectives

By completing this exercise, you will:

1. Understand Jensen's inequality and its application to atmospheric profiles.
2. Justify the use of geometric mean height in log-structured boundary layers.
3. Quantify systematic bias in bulk Richardson number due to curvature.
4. Develop intuition for grid-sensitivity issues in stable layer parameterizations.

Instructor Notes

This problem bridges mathematical analysis (Jensen, concavity) with physical intuition (mixing, stability) and numerical verification. It prepares students for advanced topics in turbulence closures and grid-aware parameterization design.

Solution key available upon request.

A Python Template for Part D

```
import numpy as np
import matplotlib.pyplot as plt

# Part D1: Quadratic profile
z0, z1 = 10.0, 100.0
c1, c2 = 0.01, -0.0001
dz = z1 - z0

def rig_quad(z):
    return c1 * z + c2 * z**2

# Geometric mean
z_g = np.sqrt(z0 * z1)
rig_zg = rig_quad(z_g)

# Bulk Ri (analytic integral)
# int(c1*z + c2*z^2) dz = c1*z^2/2 + c2*z^3/3
def integral_quad(z):
    return c1 * z**2 / 2 + c2 * z**3 / 3

ri_b = (integral_quad(z1) - integral_quad(z0)) / dz
B = rig_zg / ri_b

print(f"D1 Quadratic:")
print(f"  z_g = {z_g:.2f} m")
print(f"  Ri_g(z_g) = {rig_zg:.6f}")
print(f"  Ri_b = {ri_b:.6f}")
print(f"  Bias ratio B = {B:.4f}")
```

```python

# Plot
z_arr = np.linspace(z0, z1, 100)
plt.figure(figsize=(8, 5))
plt.plot(z_arr, rig_quad(z_arr), 'b-', label='$Ri_g(z)$')
plt.axhline(rig_zg, color='r', linestyle='--',
            label=f'$Ri_g(z_g)$ = {rig_zg:.4f}')
plt.axhline(ri_b, color='g', linestyle=':',
            label=f'$Ri_b$ = {ri_b:.4f}')
plt.axvline(z_g, color='orange', linestyle='-.',
            label=f'$z_g$ = {z_g:.1f} m')
plt.axvspan(z0, z1, alpha=0.1, color='gray')
plt.xlabel('Height $z$ (m)')
plt.ylabel('Richardson Number')
plt.title('Part D1: Quadratic $Ri_g$ Profile')
plt.legend()
plt.grid(True, alpha=0.3)
plt.tight_layout()
plt.savefig('hw_partD1_quadratic.pdf')
plt.show()

# Part D2: Logarithmic profile
A, C, z_ref = 0.5, 0.2, 1.0

def rig_log(z):
    return A * np.log(z / z_ref) + C

rig_zg_log = rig_log(z_g)

# Bulk Ri (analytic integral)
# int(A*ln(z/z_ref) + C) dz = A*(z*ln(z/z_ref) - z) + C*z + const
def integral_log(z):
    return A * (z * np.log(z / z_ref) - z) + C * z

ri_b_log = (integral_log(z1) - integral_log(z0)) / dz
B_log = rig_zg_log / ri_b_log

print(f"\nD2 Logarithmic:")
print(f"  z_g = {z_g:.2f} m")
print(f"  Ri_g(z_g) = {rig_zg_log:.6f}")
print(f"  Ri_b = {ri_b_log:.6f}")
print(f"  Bias ratio B = {B_log:.4f}")
```