Gradient and Bulk Richardson Numbers: ζ Mapping and Curvature Basis

1. Definitions

Gradient:

$$Ri_g = \frac{(g/\theta)\,\partial\theta/\partial z}{(\partial U/\partial z)^2}.$$

Bulk (layer $z_0 \to z_1$):

$$Ri_b = \frac{g}{\theta}\frac{\Delta\theta\,\Delta z}{(\Delta U)^2}.$$

2. MOST Relation

$$Ri_g(\zeta) = \zeta\frac{\phi_h(\zeta)}{\phi_m(\zeta)^2} = \zeta F(\zeta), \qquad \zeta = z/L.$$

3. Near-Neutral Series

Let

$$\phi_{m,h} = 1 + a_{m,h}\zeta + b_{m,h}\zeta^2 + O(\zeta^3).$$

Then

$$Ri_g = \zeta + \Delta\zeta^2 + \tfrac{1}{2}(\Delta^2 + c_1)\zeta^3 + O(\zeta^4),$$

$$\Delta = a_h - 2a_m, \quad c_1 = b_h - 2b_m.$$

4. Inversion ζ(Ri)

$$\zeta = Ri_g - \Delta Ri_g^2 + \left(\tfrac{3}{2}\Delta^2 - \tfrac{1}{2}c_1\right) Ri_g^3 + O(Ri_g^4).$$

Seed for Newton refinement when evaluating ϕ at given Ri.

5. Curvature

Log derivatives:

$$V_{\log} = \frac{\phi_h'}{\phi_h} - 2\frac{\phi_m'}{\phi_m}, \quad W_{\log} = V_{\log}'.$$

Curvature:

$$\frac{d^2 Ri_g}{d\zeta^2} = F\left[2V_{\log} + \zeta(V_{\log}^2 - W_{\log})\right], \quad F = \frac{\phi_h}{\phi_m^2}.$$

Neutral:

$$\left.\frac{d^2 Ri_g}{d\zeta^2}\right|_0 = 2\Delta.$$

6. Bulk vs Point Bias

Concave-down ($\Delta < 0$) ⇒

$$Ri_b < Ri_g(z_g), \qquad z_g = \sqrt{z_0 z_1}, \quad B = \frac{Ri_g(z_g)}{Ri_b} > 1.$$

7. Correction Principle

Grid damping $G(\zeta, \Delta z)$ with $G(0, \Delta z) = 1$, $\partial_\zeta G|_0 = 0$ preserves 2Δ while reducing B at coarse Δz.

8. Critical Richardson Number

Fixed Ri_c vs dynamic Ri_c^* informed by curvature growth (magnitude of ζ(V_{\log}^2-W_{\log})) and inversion strength.

9. Key Identities

$$Pr_t = \frac{\phi_h}{\phi_m}, \quad f_m(Ri_g) = \frac{1}{\phi_m(\zeta(Ri_g))^2}, \quad f_h(Ri_g) = \frac{1}{\phi_m(\zeta(Ri_g))\phi_h(\zeta(Ri_g))}.$$

10. Generic Scalar Closure

For any scalar q:

$$f_q(Ri_g) = \frac{1}{\phi_m(\zeta(Ri_g))\,\phi_q(\zeta(Ri_g))}, \quad K_q = l_m^2 S f_q.$$

Schmidt number:

$$Sc_t^{(q)} = \frac{\phi_q}{\phi_m}.$$

Series (near-neutral):

$$f_q \approx 1 + a_q Ri_g + (b_q - a_q\Delta + 2a_m a_q)Ri_g^2.$$

Concise algorithm (near-neutral):

1. Compute Ri_g.
2. ζ ≈ Ri_g – Δ Ri_g² (cubic if needed).
3. Evaluate ϕ_m, ϕ_h; obtain K_m,K_h.

11. Numerical Estimation of Ri_g and Ri_b

Given discrete z_k, U_k, θ_k:

Point gradient (centered):

$$\partial U/\partial z\Big|_{z_k} \approx (U_{k+1} - U_{k-1})/(z_{k+1} - z_{k-1}).$$

Bulk Ri_b (layer $[z_0, z_1]$):

- Definition: $Ri_b = \frac{1}{\Delta z}\int_{z0}^{z1} Ri_g(z)\,dz$.
- Trapezoid: $Ri_b \approx \frac{1}{2}[Ri_g(z_0) + Ri_g(z_1)]$.
- Simpson (3-pt): $Ri_b \approx \frac{1}{6}[Ri_g(z_0) + 4Ri_g(z_g) + Ri_g(z_1)]$.

Representative heights:

- $z_g = \sqrt{z_0 z_1}$ (geometric mean, midpoint in $\ln z$).
- $z_L = (z_1 - z_0)/\ln(z_1/z_0)$ (logarithmic mean, exact for ΔU in log wind).
- $z_a = (z_0 + z_1)/2$ (arithmetic mean, biases high for log profiles).

Use z_g for point Ri_g evaluation; use z_L for exact layer-averaged gradient matching.

12. Practical Estimation Techniques & Jensen reminder

- Representative heights:
  - z_g = √(z0 z1) (geometric mean) → use for evaluating Ri_g for log/power-law profiles.
  - z_L = (z1-z0)/ln(z1/z0) (log mean) → use when matching ΔU exactly.
- Finite-difference estimates:
  - Centered (interior): (U_{k+1}-U_{k-1})/(z_{k+1}-z_{k-1})
  - First-layer forward difference: (U1-U0)/(z1-z0) with z_rep = z_g or z_L
- Bulk vs gradient correction workflow:
  1. Compute Ri_g(z_g).
  2. Compute Ri_b (bulk formula or integral).
  3. Compute B = Ri_g(z_g)/Ri_b. If B>1.1, flag for curvature-aware correction.
- Numerical integration: prefer Simpson/trapezoid on Ri_g(z) over bulk formula for curved profiles.

13. Quick decision tree

- If B ≤ 1.05 → no correction.

- If 1.05 < B ≤ 1.3 → mild correction (K multiplier with small γ).
- If B > 1.3 and strong inversion (Γ large) → apply mixing-length reduction + K damping; consider raising Ri_c*.

14. Mixed Concavity and Inflection Handling

If $d^2 Ri_g/d\zeta^2$ changes sign inside a layer (inflection at ζ_{inf}):

- Split layer at $z_{\text{inf}} = \zeta_{\text{inf}} L$.
- Apply bias logic (Ri_b < Ri_g midpoint) only to concave-down segment.
- Concave-up segment may yield $Ri_b > Ri_g$ locally.
  Report:

$$Ri_b = \frac{(z_{\text{inf}} - z_0)Ri_{b1} + (z_1 - z_{\text{inf}})Ri_{b2}}{\Delta z}, \quad B_1 = \frac{Ri_g(z_{g1})}{Ri_{b1}},\ B_2 = \frac{Ri_g(z_{g2})}{Ri_{b2}}.$$

  Use damping only below ζ_{inf}.

X. ϕ-Agnostic Diagnosis & Correction (practical cookbook)

When ϕ(ζ) is unknown inside a model, use these robust, model-agnostic steps:

1. Representative heights

- z_g = sqrt(z0*z1) (use for Ri_g point evaluation)
- z_L = (z1 - z0)/ln(z1/z0) (use if you need exact ΔU reconstruction)

2. Finite-difference estimators (use available levels)

- centered shear at z_k:
  U_z = (U_{k+1} - U_{k-1})/(z_{k+1} - z_{k-1})
  theta_z = (TH_{k+1} - TH_{k-1})/(z_{k+1} - z_{k-1})
- point Ri_g:
  Ri_g = (g/theta_k) * theta_z / (U_z**2)

3. Bulk Ri_b (two-level):

- Ri_b = (g/theta_ref) * (TH1 - TH0) * (z1 - z0) / ((U1 - U0)**2)

4. Bias check and correction (spreadsheet formula friendly)

- B = Ri_g(z_g) / Ri_b
- If B ≤ 1.05 → no change
- Else apply K modifier:
  - G = EXP(-D*(Δz/Δz_ref)^p * (ζ/ζ_ref)^q)
  - K_new = K_old * G

5. Safe default surrogate (if you must produce f_m(Ri) or f_h(Ri))

- Exponential Ri closure (pole-free):
  f_m(Ri) = exp(-γ_m * Ri / Ri_c*)
  f_h(Ri) = exp(-γ_h * Ri / Ri_c*)
  suggested γ_m ≈ 1.8, γ_h ≈ 1.5; Ri_c* dynamic or 0.25 default

6. Minimal pseudocode

```
# φ-agnostic correction pseudocode
z_g = sqrt(z0*z1)
Ri_g_zg = compute_point_Ri(z_g)  # centered diffs or interpolation
Ri_b = compute_bulk_Ri(z0,z1)
B = Ri_g_zg / Ri_b
if B > 1.1:
    G = exp(-D*(dz/10.0)**p * (zeta/zeta_ref)**q)
    K_star = K * G
else:
    K_star = K

# fallback Ri-based closure
f_m = exp(-gamma_m * Ri / Ric_star)
```

Notes

- Always verify neutral-curvature preservation by testing near ζ→0 that your G does not change first derivative (G′(0)=0).
- For Excel: use LOG(), SQRT(), EXP() and user-defined constants in header cells to allow easy tuning.

13. Mixed Concavity Handling

If $\frac{d^2 Ri_g}{d\zeta^2}$ changes sign:

- Split layer at inflection ζ_{inf}.
- Apply bias logic separately to concave-down and concave-up segments.
- Recombine weighted averages.

14. ϕ-Agnostic Surrogate

When ϕ-functions are unknown:

- Use exponential Ri closures:

$$f_m(Ri) = e^{-\gamma_m Ri/Ri_c^*}, \quad f_h(Ri) = e^{-\gamma_h Ri/Ri_c^*}$$

- Suggested: $\gamma_m \approx 1.8, \gamma_h \approx 1.5$.
- Ri_c^* dynamic or default 0.25.

15. Minimal Pseudocode

```
z_g = sqrt(z0*z1)
Ri_g_zg = compute_point_Ri(z_g)
Ri_b = compute_bulk_Ri(z0,z1)
B = Ri_g_zg / Ri_b

if B > 1.1:
    G = exp(-D*(dz/dz_ref)**p * (zeta/zeta_ref)**q)
    K_star = K * G
else:
    K_star = K

# φ-agnostic fallback
f_m = exp(-gamma_m * Ri / Ric_star)
```

Curvature in ζ versus z

Curvature in ζ is natural in MOST because the similarity functions are defined in terms of the non-dimensional height $\zeta = z/L$. If L is treated as locally constant across a thin layer, then derivatives transform simply: $\frac{d}{dz} = \frac{1}{L}\frac{d}{d\zeta}$ and $\frac{d^2}{dz^2} = \frac{1}{L^2}\frac{d^2}{d\zeta^2}$. In that case, the sign and relative magnitude of curvature are preserved between z and ζ. When L varies with height, curvature in z picks up extra terms involving dL/dz, so ζ-based diagnostics cleanly separate profile shape from coordinate effects.

Near-neutral coefficients from Businger–Dyer

For the classic Businger–Dyer (BD) stable formulations near $\zeta \to 0^+$:

$$\phi_m = 1 + a_m\zeta, \quad \phi_h = 1 + a_h\zeta,$$

with commonly used values $a_m \approx 4.7$ and $a_h \approx 7.8$. If you retain only the linear terms (i.e., $b_m = b_h = 0$), then

$$\Delta = a_h - 2a_m = 7.8 - 9.4 = -1.6, \qquad c_1 = b_h - 2b_m = 0.$$

Implications:

- **Neutral curvature:** $\left.\frac{d^2 Ri_g}{d\zeta^2}\right|_0 = 2\Delta = -3.2$ → concave-down.
- **Bias direction:** concave-down implies $Ri_b < Ri_g(z_g)$ and a bulk-versus-point bias factor $B > 1$.
- **ζ inversion (cubic):**

$$\zeta \approx Ri_g - \Delta Ri_g^2 + \left(\tfrac{3}{2}\Delta^2 - \tfrac{1}{2}c_1\right)Ri_g^3 = Ri_g + 1.6\,Ri_g^2 + 3.84\,Ri_g^3.$$

For unstable BD, ϕ functions are non-linear (e.g., $\phi_m = (1-16\zeta)^{-1/4}$, $\phi_h = (1-16\zeta)^{-1/2}$). A Taylor expansion about $\zeta = 0^-$ yields finite linear coefficients as well:

$$\phi_m \approx 1 + 4\zeta + 10\zeta^2 + \dots, \quad \phi_h \approx 1 + 8\zeta + 48\zeta^2 + \dots,$$

so near-neutral on the unstable side you'd have $a_m \approx 4$, $a_h \approx 8$, giving $\Delta \approx 0$ (specifically $\Delta = 8 - 2 \cdot 4 = 0$), i.e., weak curvature in the immediate neutral limit and rapidly increasing nonlinearity at larger $|\zeta|$. If you prefer other empirical constants (e.g., 5 and 5 for modified BD), update a_m, a_h and recompute Δ, c_1 accordingly.

Special case — identical linear ϕ (ϕ_h = ϕ_m = 1 + β ζ)

Assume

$$\phi_h(\zeta) = \phi_m(\zeta) = 1 + \beta\zeta, \qquad \beta = 4.7.$$

Then

- F = ϕ_h/ϕ_m^2 = 1/(1+\beta\zeta).
- Closed form:

$$Ri_g(\zeta) = \frac{\zeta}{1 + \beta\zeta}.$$

- Near-neutral Taylor series:

$$Ri_g(\zeta) = \zeta - \beta\zeta^2 + \beta^2\zeta^3 + O(\zeta^4).$$

  Hence the near-neutral coefficients are a_m=a_h=β and

$$\Delta = a_h - 2a_m = -\beta = -4.7, \qquad 2\Delta = -9.4.$$

- ζ(Ri) inversion (exact rational form and series):

$$\zeta = \frac{Ri}{1 - \beta Ri} = Ri + \beta Ri^2 + \beta^2 Ri^3 + O(Ri^4).$$

- Turbulent Prandtl: Pr_t = ϕ_h/ϕ_m = 1 (unit Prandtl in this special case).

Implications (one line)

- This symmetric linear choice yields strong negative neutral curvature (2Δ = −9.4), so Jensen bias is significant for coarse Δz; apply curvature-aware damping G(ζ,Δz) or the Q-SBL surrogate as described in the main text.

Curvature mapping: ζ to z

- **If L is uniform in the layer:**
  - **Label:** Derivative scaling
  - $$\frac{d^2 Ri_g}{dz^2} = \frac{1}{L^2}\frac{d^2 Ri_g}{d\zeta^2}$$
  - **Result:** Same concavity and bias logic; only magnitude rescales by $1/L^2$.
- **If L varies with z:**
  - **Label:** Extra terms
  - [
    \frac{d^2 Ri_g}{dz^2} = \frac{1}{L^2}\frac{d^2 Ri_g}{d\zeta^2}
    - 2\frac{1}{L^3}\frac{dL}{dz}\frac{d Ri_g}{d\zeta}
    - \text{terms with } \frac{d^2 L}{dz^2}
      ]
  - **Result:** Curvature in z combines profile shape and stability variation; using ζ isolates the MOST shape. Practically, use ζ-curvature for diagnostics and treat $L(z)$ variability via layer splitting or effective L.

Practical guidance

- **Near-neutral diagnostics:** Use BD-derived a_m, a_h to compute Δ and neutral curvature 2Δ. This sets the expected sign of the bulk vs point bias.
- **Bias correction:** Apply damping $G(\zeta, \Delta z)$ designed so $G(0) = 1$ and $G'(0) = 0$, preserving neutral curvature while reducing coarse-grid bias.
- **Representative height:** Evaluate point Ri_g at $z_g = \sqrt{z_0 z_1}$ and use Simpson/trapezoid integration for Ri_b when curvature is non-negligible.

- **ζ inversion for closures:** Use $\zeta(Ri_g)$ from the series as a seed for Newton to evaluate ϕ_m, ϕ_h robustly when Ri is the control variable.

Quick check with BD-stable

- **Coefficients:** $a_m = 4.7,\ a_h = 7.8 \Rightarrow \Delta = -1.6$.
- **Neutral curvature:** $2\Delta = -3.2 \rightarrow$ concave-down; expect $B > 1$.
- **ζ seed:** $\zeta \approx Ri_g + 1.6\, Ri_g^2 + 3.84\, Ri_g^3$.
- **Action:** Prefer Simpson for Ri_b; if $B > 1.05$, apply mild damping; if $B > 1.3$ with strong inversion, reduce mixing length and consider raising Ri_c^*.

If you're using a different BD constant set, share them and I'll plug them in to give you the updated Δ, curvature, and ζ-inversion coefficients.

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