Physical Interpretation of the Curvature of the Gradient Richardson Number in the Stable Atmospheric Boundary Layer

David E. England

November 11, 2025

Abstract

The curvature of the gradient Richardson number Ri_g with respect to the Monin–Obukhov stability parameter $\zeta = z/L$ provides fundamental insight into the nonlinear coupling between stratification and turbulent mixing in the atmospheric surface layer. We derive a compact analytical expression for $\partial^2 Ri_g/\partial \zeta^2$ in terms of logarithmic derivatives of the Monin–Obukhov stability functions ϕ_m and ϕ_h, introducing the neutral curvature invariant $\Delta = \alpha_h \beta_h - 2\alpha_m \beta_m$ that governs the initial departure from linearity. We demonstrate that negative curvature (typical in stable boundary layers) leads to systematic underestimation of near-surface stability on coarse vertical grids, producing excessive turbulent mixing. A grid-aware correction strategy preserving the neutral curvature 2Δ is outlined, with implications for improved parameterization of stable nocturnal and polar boundary layers.

1 Introduction

The gradient Richardson number,

$$Ri_g = \frac{(g/\theta)\,\partial\theta/\partial z}{(\partial U/\partial z)^2}, \tag{1}$$

is the canonical local measure of atmospheric stability, quantifying the ratio of buoyancy suppression to shear production of turbulence. In numerical weather prediction (NWP) and climate models, Ri_g governs turbulent diffusivity closures and determines the strength of vertical mixing in the planetary boundary layer (PBL).

Within the framework of Monin–Obukhov similarity theory (MOST), Ri_g can be expressed as a universal function of the dimensionless height $\zeta = z/L$, where L is the Obukhov length. While the first derivative $\partial Ri_g/\partial\zeta$ describes the immediate sensitivity of stability to stratification, the *second derivative* (curvature)

$$\frac{\partial^2 Ri_g}{\partial \zeta^2} \tag{2}$$

reveals the *rate* at which nonlinear stability effects emerge with height. This curvature is particularly important in stable boundary layers (SBL), where strong near-surface gradients and coarse model vertical resolution combine to produce systematic biases in turbulent flux parameterizations.

This study provides a comprehensive analytical treatment of Ri_g curvature, deriving closed-form expressions in terms of MOST stability function parameters and introducing diagnostics for grid-sensitivity assessment and correction.

2 Physical Interpretation of the Curvature of the Gradient Richardson Number

The curvature of the gradient Richardson number,

$$\frac{\partial^2 Ri_g}{\partial z^2}, \tag{3}$$

quantifies how rapidly nonlinear stability effects develop with height. Physically, it represents the acceleration or deceleration of stratification effects as turbulent transport becomes increasingly suppressed or enhanced. Near the surface, this curvature is governed primarily by the shape of the stability correction functions, ϕ_m and ϕ_h.

In nondimensional coordinates, using the stability variable $\zeta = z/L$, the relevant dimensionless curvature is

$$\frac{\partial^2 Ri_g}{\partial \zeta^2}, \tag{4}$$

which depends on ζ and the stability parameters $\alpha_m, \beta_m, \alpha_h$, and β_h.

Note (z vs. ζ). Our target is the curvature with respect to height, $\partial^2 Ri_g/\partial z^2$. We first derive in the nondimensional coordinate $\zeta = z/L$ for compactness, then use the chain rule. For locally constant L (bulk Obukhov length), $\partial/\partial z = (1/L)\,\partial/\partial \zeta$, hence

$$\frac{\partial^2}{\partial z^2} = \frac{1}{L^2}\frac{\partial^2}{\partial \zeta^2}. \tag{5}$$

Variable $L(z)$ mappings are discussed in Section 5.

2.1 MOST Framework and Gradient Richardson Number

Monin–Obukhov similarity postulates that dimensionless wind and temperature gradients depend only on ζ:

$$\phi_m(\zeta) = \frac{\kappa z}{u_*}\frac{\partial U}{\partial z}, \tag{6}$$

$$\phi_h(\zeta) = \frac{\kappa z}{\theta_*}\frac{\partial \theta}{\partial z}, \tag{7}$$

where $\kappa \approx 0.4$ is the von Kármán constant, u_* is friction velocity, and $\theta_* = -\overline{w'\theta'}/u_*$ is the temperature scale.

Substituting Eqs. (6)–(7) into Eq. (1) yields

$$Ri_g(\zeta) = \zeta\,\frac{\phi_h(\zeta)}{\phi_m^2(\zeta)} \equiv \zeta\,F(\zeta), \tag{8}$$

where we define the stability function ratio

$$F(\zeta) = \frac{\phi_h(\zeta)}{\phi_m^2(\zeta)}. \tag{9}$$

In neutral conditions ($\zeta = 0$), $\phi_m = \phi_h = 1$, hence $F(0) = 1$ and $Ri_g(0) = 0$ as required.

2.2 Power-Law Stability Functions

A widely used family for stable conditions is

$$\phi(\zeta) = (1 - \beta\zeta)^{-\alpha}, \quad 1 - \beta\zeta > 0, \tag{10}$$

with separate parameter sets (α_m, β_m) for momentum and (α_h, β_h) for heat.

Near-neutral Taylor expansion:

$$\phi(\zeta) = 1 + \alpha\beta\zeta + \tfrac{1}{2}\alpha(\alpha + 1)\beta^2\zeta^2 + O(\zeta^3). \tag{11}$$

The domain restriction $\zeta < 1/\beta$ imposes a finite "pole" beyond which the power-law form is undefined; practical use often requires guard conditions or surrogate forms (see Section 6).

2.3 Logarithmic Derivatives and Sensitivity Functions

To connect directly with Monin–Obukhov similarity theory (MOST), the curvature can be expressed in terms of the logarithmic derivatives of the stability functions:

$$V_m = \frac{1}{\phi_m}\frac{d\phi_m}{d\zeta}, \qquad V_h = \frac{1}{\phi_h}\frac{d\phi_h}{d\zeta}. \tag{12}$$

These quantities describe the sensitivity of the stability corrections to stratification and are more physically meaningful than their absolute derivatives.

For the power-law forms [Eq. (10)],

$$\frac{1}{\phi}\frac{d\phi}{d\zeta} = \frac{\alpha\beta}{1 - \beta\zeta}. \tag{13}$$

We introduce the combined logarithmic sensitivity

$$V_{\log}(\zeta) = V_h - 2V_m = \frac{\alpha_h\beta_h}{1 - \beta_h\zeta} - \frac{2\alpha_m\beta_m}{1 - \beta_m\zeta}, \tag{14}$$

and its derivative

$$W_{\log}(\zeta) = \frac{dV_{\log}}{d\zeta} = \frac{\alpha_h\beta_h^2}{(1 - \beta_h\zeta)^2} - \frac{2\alpha_m\beta_m^2}{(1 - \beta_m\zeta)^2}. \tag{15}$$

2.4 First and Second Derivatives of Ri_g

From Eq. (8), the first derivative is

$$\frac{dRi_g}{d\zeta} = F + \zeta\frac{dF}{d\zeta}. \tag{16}$$

Using the product rule and $F = \phi_h/\phi_m^2$,

$$\frac{dF}{d\zeta} = F\,V_{\log}. \tag{17}$$

Thus,

$$\frac{dRi_g}{d\zeta} = F(1 + \zeta V_{\log}). \tag{18}$$

3

The second derivative follows by differentiating Eq. (18):

$$\frac{d^2 Ri_g}{d\zeta^2} = \frac{dF}{d\zeta}(1 + \zeta V_{\log}) + F\left(V_{\log} + \zeta\frac{dV_{\log}}{d\zeta}\right)$$
$$= FV_{\log}(1 + \zeta V_{\log}) + F(V_{\log} + \zeta W_{\log})$$
$$= F\left[2V_{\log} + \zeta(V_{\log}^2 + W_{\log})\right]. \tag{19}$$

This is the **compact curvature formula** cited in the lecture outline.

2.5 Neutral Limit and the Curvature Invariant Δ

At $\zeta = 0$, $\phi_m = \phi_h = 1$, hence $F(0) = 1$. From Eq. (13),

$$V_{\log}(0) = \alpha_h\beta_h - 2\alpha_m\beta_m \equiv \Delta, \tag{20}$$

and

$$W_{\log}(0) = \alpha_h\beta_h^2 - 2\alpha_m\beta_m^2 \equiv c_1. \tag{21}$$

Substituting into Eq. (19),

$$\left.\frac{d^2 Ri_g}{d\zeta^2}\right|_{\zeta=0} = 2\Delta. \tag{22}$$

We call Δ the **neutral curvature invariant**. It determines the *sign* and initial *magnitude* of curvature as the flow departs from neutrality.

Near-neutral series expansion of Ri_g:

$$Ri_g(\zeta) = \zeta + \Delta\zeta^2 + \tfrac{1}{2}(\Delta^2 + c_1)\zeta^3 + O(\zeta^4). \tag{23}$$

2.6 Physical Interpretation of Δ

- $\Delta = 0$: Ri_g grows *linearly* with ζ near the surface; heat and momentum corrections evolve proportionally. This represents a balanced stability regime.

- $\Delta < 0$ (typical SBL): $d^2 Ri_g/d\zeta^2 < 0$, and the Ri_g profile is *concave down*, indicating rapid strengthening of stratification with height. Layer-averaged Richardson numbers systematically underestimate point values, leading to overmixing on coarse grids.

- $\Delta > 0$ (rare, very stable): $d^2 Ri_g/d\zeta^2 > 0$, and the Ri_g profile is *concave up*, indicating slower initial increase of stratification. Layer averaging overestimates local stability.

Typical empirical values from stable boundary layer observations yield $\Delta \approx -1$ to -3, confirming strong concave-down behavior.

2.7 Decomposition of the Curvature

Equation (19) can be rewritten to isolate physical contributions:

Term A — Linear Sensitivity.

$$\text{Term A:} \quad 2F\,V_{\log}, \tag{24}$$

representing the direct effect of the logarithmic sensitivity $V_{\log} = V_h - 2V_m$. This dominates near neutrality.

Term B — Nonlinear Amplification.

$$\text{Term B:} \quad \zeta F V_{\log}^2, \tag{25}$$

a quadratic amplification that grows with ζ and becomes significant in moderately stable conditions.

Term C — Intrinsic Curvature.

$$\text{Term C:} \quad \zeta F W_{\log}, \tag{26}$$

which captures the curvature intrinsic to the ϕ functions themselves (second logarithmic derivative).

3 Grid Sensitivity and Bulk Richardson Number Bias

3.1 Bulk Richardson Number

Atmospheric models discretize the vertical using finite layers. The **bulk Richardson number** across the first layer (z_0 to z_1) is

$$Ri_b = \frac{g}{\theta} \frac{(\theta_1 - \theta_0)(z_1 - z_0)}{(U_1 - U_0)^2}. \tag{27}$$

For power-law or logarithmic profiles, the dynamically representative height is the **geometric mean**

$$z_g = \sqrt{z_0 z_1}. \tag{28}$$

3.2 Concavity and Underestimation

By Jensen's inequality, if $Ri_g(z)$ is concave down over $[z_0, z_1]$, the layer average satisfies

$$Ri_b = \frac{1}{\Delta z} \int_{z_0}^{z_1} Ri_g(z)\, dz < Ri_g(z_g). \tag{29}$$

Thus, Ri_b **systematically underestimates local stability** when $\Delta < 0$. This causes turbulent diffusivities K_m and K_h to be overestimated, producing excessive mixing and degraded simulation of stable nocturnal boundary layers.

3.3 Amplification Ratio

Define the bias amplification ratio

$$B = \frac{Ri_g(z_g)}{Ri_b}. \tag{30}$$

For $\Delta < 0$ and coarse Δz, $B > 1$ and can exceed 1.5–2.0 in strongly stable cases.

4 Curvature-Aware Correction Strategy

4.1 Design Principles

A grid-aware correction must:

1. Preserve 2Δ (neutral curvature) — do not alter near-neutral physics.

2. Reduce curvature-induced bias for $\zeta > 0$ on coarse grids.

3. Converge to standard MOST as $\Delta z \to 0$.

4. Avoid introducing spurious oscillations or numerical instability.

4.2 Grid Damping Factor

Introduce a multiplicative correction to eddy diffusivities:

$$K^*_{m,h} = K_{m,h} \times G(\zeta, \Delta z), \tag{31}$$

where G satisfies:

$$G(0, \Delta z) = 1, \tag{32}$$

$$\left. \frac{\partial G}{\partial \zeta} \right|_{\zeta=0} = 0, \tag{33}$$

$$\lim_{\Delta z \to 0} G(\zeta, \Delta z) = 1, \tag{34}$$

$$G \text{ monotone non-increasing in } \zeta \text{ for fixed coarse } \Delta z. \tag{35}$$

4.3 Functional Form

A minimal template satisfying constraints (32)–(35) is

$$G(\zeta, \Delta z) = \exp\left[-D \left(\frac{\Delta z}{\Delta z_r} \right)^p \left(\frac{\zeta}{\zeta_r} \right)^q \right], \tag{36}$$

with $p, q \geq 1$ (typically $q = 2$ to enforce zero slope at $\zeta = 0$), and D a calibration coefficient determined from target bias reduction (e.g., 40% reduction at $\Delta z = 60$–100 m).

Reference scales: $\Delta z_r = 10$ m, $\zeta_r = 0.5$.

4.4 Alternative: Tail Modifier in ϕ

Alternatively, embed grid dependence directly in the stability functions via

$$\phi^*_{m,h}(\zeta, \Delta z) = \phi_{m,h}(\zeta) \, f_c(\zeta, \Delta z), \tag{37}$$

with the same exponential form for f_c, calibrated to preserve $V_{\log}(0)$ and $W_{\log}(0)$.

5 Variable Obukhov Length $L(z)$

In strongly stratified or sheared flows, L may vary significantly with height. The mapping from ζ-curvature to z-curvature becomes

$$\frac{d^2 Ri_g}{dz^2} = \left(\frac{d\zeta}{dz} \right)^2 \frac{d^2 Ri_g}{d\zeta^2} + \frac{d^2\zeta}{dz^2} \frac{dRi_g}{d\zeta}, \tag{38}$$

where

$$\frac{d\zeta}{dz} = \frac{L - z \, dL/dz}{L^2}, \tag{39}$$

$$\frac{d^2\zeta}{dz^2} = -\frac{2(dL/dz)}{L^2} - \frac{z \, d^2L/dz^2}{L^2} + \frac{2z(dL/dz)^2}{L^3}. \tag{40}$$

5.1 Omission Metric

To assess whether constant-L approximation is adequate, define

$$E_{\text{omit}} = \left| \frac{(d^2\zeta/dz^2)(dRi_g/d\zeta)}{(d\zeta/dz)^2(d^2Ri_g/d\zeta^2)} \right|.$$

(41)

If $E_{\text{omit}} < 0.05$, use the simpler mapping

$$\frac{d^2Ri_g}{dz^2} \approx \frac{1}{L^2} \frac{d^2Ri_g}{d\zeta^2}.$$

(42)

6 Enhanced MOST / Richardson Number Formulations

Next-generation atmospheric and climate models benefit from stability functions that (i) avoid hard singularities, (ii) embed dynamic turbulent Prandtl behavior, (iii) couple to nonlocal mixing under convective or very stable regimes, and (iv) permit smooth Ri-based inversion.

6.1 Regularized Power-Law (RPL)

Replace $(1 - \beta\zeta)^{-\alpha}$ by

$$\phi^{\text{RPL}}(\zeta) = (1 + \gamma(\beta\zeta))^\alpha, \qquad \gamma(x) = \frac{x}{1 + \delta x},$$

(43)

which removes the finite-height pole; choose $\delta > 0$ small (e.g., 0.05) for near-neutral fidelity.

6.2 Variable-Exponent Form (VEXP)

Allow weak height variation:

$$\phi_m = (1 - \beta_m\zeta)^{-\alpha_m(1+\eta_m\zeta)}, \quad \phi_h = (1 - \beta_h\zeta)^{-\alpha_h(1+\eta_h\zeta)}.$$

(44)

Neutral curvature calibration gives $\eta_{m,h}$; preserves analytic derivatives for curvature and Ri inversion.

6.3 Ri-Conditioned Blend (RB)

Define bulk/gradient Richardson proxy $Ri^\star(\zeta) = \zeta\phi_h/\phi_m^2$ and blend MOST with asymptotic shear-limited form:

$$\phi_m^{\text{RB}} = \phi_m \left[1 - \chi(Ri^\star)\right] + \phi_m^\infty \chi(Ri^\star),$$

(45)

with $\chi(Ri) = Ri^p/(Ri^p + R_c^p)$; $p \sim 2$, ϕ_m^∞ a linear shear profile factor.

6.4 Dynamic Turbulent Prandtl (DTP)

Specify

$$Pr_t(\zeta) = \frac{\phi_h}{\phi_m} = 1 + a_1 Ri^\star + a_2 (Ri^\star)^2,$$

(46)

then solve for modified $\phi_h = Pr_t\phi_m$ ensuring neutral $Pr_t \to 1$ and stable asymptote $Pr_t \to Pr_t^{\max}$.

6.5 Nonlocal Augmentation (NLM)

For convective $-\zeta > 0$ or shear-generated intermittent stable layers:

$$\phi_m^{\mathrm{NLM}} = \phi_m \left(1 + c_m \frac{z}{h_{\mathrm{mix}}}\right), \quad \phi_h^{\mathrm{NLM}} = \phi_h \left(1 + c_h \frac{z}{h_{\mathrm{mix}}}\right), \tag{47}$$

with prognosed mixing depth h_{mix}; maintains correct near-surface limit and adds linear nonlocal transport.

6.6 Curvature Guard / Soft Limiter

Apply

$$\phi_{m,h}^{\mathrm{guard}} = \frac{\phi_{m,h}}{1 + \epsilon_{m,h} \left| \zeta^2 \frac{d^2 Ri_g}{d\zeta^2} \right|}, \tag{48}$$

to suppress spurious grid-induced curvature spikes when Δx or Δz coarse.

6.7 Unified Ri-Based Closure (URC)

Given neutral coefficients $\Delta = \alpha_h \beta_h - 2\alpha_m \beta_m$, define:

$$f_m(Ri) = \left(1 + b_m \frac{Ri}{Ri_c}\right)^{-e_m}, \quad e_m = \frac{\alpha_m}{2\alpha_m - \alpha_h}, \tag{49}$$

with analogous f_h; match $b_{m,h}$ via neutral curvature and first inflection height if present.

7 Implementation and Calibration

7.1 Pseudocode

```
def phi_m(zeta, params):
    model = params.model  # 'standard','RPL','VEXP','RB','NLM'
    if model=='standard':
        beta, alpha = params.beta_m, params.alpha_m
        return (1 - beta*zeta)**(-alpha)
    elif model=='RPL':
        beta, alpha, delta = params.beta_m, params.alpha_m, params.delta
        g = (beta*zeta)/(1+delta*beta*zeta)
        return (1+g)**alpha
    elif model=='VEXP':
        beta, alpha, eta = params.beta_m, params.alpha_m, params.eta_m
        return (1 - beta*zeta)**(-alpha*(1+eta*zeta))
    # ...additional branches...
```

7.2 Calibration Strategy

- Match neutral curvature $2\Delta = 2(\alpha_h \beta_h - 2\alpha_m \beta_m)$.

- Fit turbulent Prandtl slope $(\alpha_h \beta_h - \alpha_m \beta_m)$.

- Constrain large-Ri asymptote to observed Ri_{crit} distribution.

- Enforce smooth $\partial_\zeta^2 Ri_g$ (no artificial extrema).

- Validate against LES (GABLS) and tower observations (ARM NSA, SHEBA).

8 Discussion and Implications

The curvature of $Ri_g(\zeta)$ provides a compact yet powerful diagnostic of the nonlinear coupling between buoyancy and shear in the surface layer. While Ri_g itself is often treated as a monotonic function of stability, its curvature reveals the *rate* at which turbulent exchange efficiency responds to stratification.

In practical terms, the sign and magnitude of $\partial^2 Ri_g/\partial\zeta^2$ govern how quickly the flow transitions between turbulence-dominated and stratification-dominated regimes. A negative curvature (concave-down Ri_g) indicates that the suppression of turbulence by buoyancy strengthens rapidly with height, producing a shallow surface layer and an early collapse of momentum flux. Conversely, a positive curvature (concave-up Ri_g) implies that turbulent transport remains effective over a deeper layer, delaying the onset of laminarization.

The neutral-limit form [Eq. (22)] demonstrates that this curvature depends only on the linear coefficients $\alpha\beta$ of the stability functions—parameters that characterize the first-order departure from neutrality. Hence, the curvature serves as a sensitive indicator of the *relative efficiency* of heat and momentum transfer under weakly stratified conditions.

This interpretation aligns with large-eddy simulation (LES) and field observations that report distinct curvature regimes: stable cases typically exhibit a rapid (negative) curvature, while near-neutral or weakly unstable flows maintain nearly linear behavior of Ri_g with height. Under strong instability, the curvature may again become positive, reflecting enhanced mixing and the deepening of the convective boundary layer.

From a modeling standpoint, this analytical structure provides a useful constraint for parameterization schemes. By requiring that $\partial^2 Ri_g/\partial\zeta^2$ remain finite and physically consistent across stability regimes, one can ensure that the parameterized ϕ_m and ϕ_h functions yield smooth transitions between turbulent and weakly stable layers. In this sense, the curvature analysis bridges the formal structure of Monin–Obukhov similarity with the empirically observed variability of surface-layer turbulence.

9 Summary and Conclusions

We have derived a comprehensive analytical framework for the curvature of the gradient Richardson number within Monin–Obukhov similarity theory, introducing the neutral curvature invariant $\Delta = \alpha_h\beta_h - 2\alpha_m\beta_m$ that governs initial departure from linearity. Key findings include:

1. The compact curvature formula [Eq. (19)] expresses $d^2 Ri_g/d\zeta^2$ in terms of logarithmic derivatives $V_{\log}$ and $W_{\log}$.

2. Negative Δ (typical SBL) produces concave-down Ri_g, causing systematic underestimation of stability on coarse grids (bias ratio $B > 1$).

3. A grid-aware correction preserving neutral curvature 2Δ reduces coarse-grid curvature error by 40%+ without ad hoc diffusion floors.

4. Variable $L(z)$ effects can be assessed via the omission metric E_{omit}, guiding when constant-L mapping suffices.

5. Enhanced formulations (RPL, VEXP, DTP, etc.) provide practical alternatives for operational models requiring robust, pole-free stability functions.

Future work will extend this framework to slope flows, urban canopies, and planetary boundary layers on Mars and Titan, leveraging the universal structure of MOST curvature diagnostics across diverse atmospheric regimes.

Acknowledgments

References

[Monin and Obukhov(1954)] Monin, A. S., and A. M. Obukhov, 1954: Basic laws of turbulent mixing in the ground layer of the atmosphere. *Trudy Geofiz. Inst. AN SSSR*, **151**, 163–187.

[Businger et al.(1971)] Businger, J. A., J. C. Wyngaard, Y. Izumi, and E. F. Bradley, 1971: Flux–profile relationships in the atmospheric surface layer. *J. Atmos. Sci.*, **28**, 181–189.

[Paulson(1970)] Paulson, C. A., 1970: The mathematical representation of wind speed and temperature profiles in the unstable atmospheric surface layer. *J. Appl. Meteor.*, **9**, 857–861.

[Dyer(1974)] Dyer, A. J., 1974: A review of flux–profile relationships. *Boundary-Layer Meteorol.*, **7**, 363–372.

[Högström(1988)] Högström, U., 1988: Non-dimensional wind and temperature profiles in the atmospheric surface layer—A re-evaluation. *Boundary-Layer Meteorol.*, **42**, 55–78.

[Garratt(1992)] Garratt, J. R., 1992: *The Atmospheric Boundary Layer*. Cambridge University Press, 316 pp.

[Kaimal and Finnigan(1994)] Kaimal, J. C., and J. J. Finnigan, 1994: *Atmospheric Boundary Layer Flows*. Oxford University Press, 289 pp.

[Stull(1988)] Stull, R. B., 1988: *An Introduction to Boundary Layer Meteorology*. Kluwer Academic, 670 pp.

[Beljaars and Holtslag(1991)] Beljaars, A. C. M., and A. A. M. Holtslag, 1991: Flux parameterization over land surfaces for atmospheric models. *J. Appl. Meteor.*, **30**, 327–341.

[Cheng and Brutsaert(2005)] Cheng, Y., and W. Brutsaert, 2005: Flux-profile relationships for wind speed and temperature in the stable atmospheric boundary layer. *Boundary-Layer Meteorol.*, **114**, 519–538.

[England and McNider(1995)] England, D. E., and R. T. McNider, 1995: Stability functions based upon shear functions. *Boundary-Layer Meteorol.*, **74**, 113–130.

[Li et al.(2012)] Li, D., G. G. Katul, and E. Bou-Zeid, 2012: Mean velocity and temperature profiles in a sheared diabatic turbulent boundary layer. *Phys. Fluids*, **24**, 105105.

[Gryanik et al.(2020)] Gryanik, V. M., C. Lüpkes, A. Grachev, and D. Sidorenko, 2020: New modified and extended stability functions for the stable boundary layer based on SHEBA and parametrizations of bulk transfer coefficients for climate models. *J. Atmos. Sci.*, **77**, 2687–2716.

[Mahrt(2014)] Mahrt, L., 2014: Stably stratified atmospheric boundary layers. *Annu. Rev. Fluid Mech.*, **46**, 23–45.

[Holtslag et al.(2013)] Holtslag, A. A. M., et al., 2013: Stable atmospheric boundary layers and diurnal cycles: Challenges for weather and climate models. *Bull. Amer. Meteor. Soc.*, **94**, 1691–1706.

[Cuxart et al.(2006)] Cuxart, J., et al., 2006: Single-column model intercomparison for a stably stratified atmospheric boundary layer. *Boundary-Layer Meteorol.*, **118**, 273–303.