STABILITY FUNCTIONS BASED UPON SHEAR FUNCTIONS

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Abstract. The stability functions for momentum and heat under a Richardson number formulation are derived from the nondimensional shear functions under a Monin–Obukhov formulation. The Prandtl number is also derived as a function of the Richardson number. Previously, this has been done only in a limited sense. Because the Richardson number formulation is expressed in closed form, iterative techniques are no longer needed in numerical models that use Monin–Obukhov similarity theory. This time-saving approach is made possible by deriving expressions for the friction velocity and temperature in terms of the Richardson-number-dependent stability functions. In addition, the Richardson number approximation in the lowest layer is made to depend explicitly upon the surface roughness.

1. Introduction

Under first-order closure, the exchange coefficients for momentum (K_m) and heat (K_h) depend upon the stability of the atmosphere, height above the surface (mixing length) and vertical wind shear. Similarity theory yields shear functions, which depend upon the dimensionless Monin–Obukhov height (ζ) for incorporating stability. The use of shear functions is widespread among boundary-layer analysts, and numerous field programs have been conducted to determine these functions (see e.g., Dyer and Hicks, 1970; Webb, 1970; Businger *et al.*, 1971; Carl *et al.*, 1973; Dyer, 1974; Hicks, 1976; Dyer and Bradley, 1982; Högström, 1985).

The existence of limit-cycle mixing is known to depend upon the turbulent Prandtl number Pr (Mahrt, 1989), which can be expressed in terms of shear functions. Bendixson's criterion can be applied to a two-layer, K-closure model to determine the possible existence of limit-cycles (see England and McNider, 1995). However, the derivatives needed in Bendixson's criterion cannot be resolved analytically, because ζ depends upon the shear functions. Such a circular definition also means that numerical boundary-layer models must use iterative solution procedures. A closed-form method for incorporating stability dependence into the exchange coefficients would therefore be desirable. The use of stability functions depending on the Richardson number (Ri) provides such a method. Relationships between the shear and stability functions, as well as ζ and Ri, are known (see e.g., Dyer, 1974; Paulson, 1970; Blackadar, 1979; Garratt, 1992). To date, little work has been done in expressing Pr as a function of Ri. The main goal of this study is to use Monin–Obukhov similarity theory to derive both the stability functions and Pr in terms of Ri.

Prevalent choices for the shear functions have different forms for the stable boundary layer (SBL) and the unstable boundary layer (UBL). Hence, derivations

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for the different regimes must be carried out separately. The friction velocity and temperature will also be expressed in terms of stability functions and Ri, with explicit dependence upon surface roughness obtained through a logarithmic transformation. In addition, it will be shown that if Pr = 1 in the SBL, the stability functions are the same and have a quadratic form rather than the linear form introduced by Blackadar (1979). This result is important because more modern and accurate experiments indicate that Pr does not significantly deviate from 1.00 in the SBL (see Yaglom, 1977; Högström, 1988; Kaimal and Finnigan, 1994). The quadratic stability function has the advantage of having a continuous derivative in the SBL, which should provide smoother solutions of first-order closure models. Finally, some mesoscale models diminish smoothness in the vertical direction by using a time-consuming, iterative ζ formulation near the surface and an inconsistent Ri formulation for the upper levels. The inconsistency can be eliminated by using the results outlined in this work.

2. Exchange Coefficients

The shear- and stability-function techniques for expressing the exchange coefficients differ only in whether atmospheric stability is parameterized by ζ or Ri. The shear functions are based upon similarity theory, but certain constants must be determined from experiments. Similarity theory is used extensively in numerical models, especially near the ground, because a fine grid resolution is needed to resolve flux quantities. A drawback of this formulation, from which the Ri formulation does not suffer, is that ζ must be found by an iterative procedure. These two formulations are compared and contrasted in the following discussion.

2.1. Monin–Obukhov Formulation

The exchange coefficients for momentum and heat expressed in terms of shear functions are

$$K_m = l_m u_* / \phi_m = l_m^2 s / \phi_m^2, \tag{1a}$$

$$K_h = l_m u_* / \phi_h = l_m^2 s / (\phi_h \phi_m), \tag{1b}$$

where the friction velocity is given by

$$u_* = \frac{l_m(z)}{\phi_m(\zeta)} s, \tag{2}$$

the vertical wind shear by

$$s = |\partial \mathbf{V} / \partial z| = \sqrt{(\partial u/\partial z)^2 + (\partial v/\partial z)^2}, \tag{3}$$

and the mixing length in the surface layer by

$$l_m(z) = \kappa z, \tag{4}$$

where κ is von Kármán's constant, and z is height above the surface. This investigation is restricted to shear functions of the following forms

$$\phi_m(\zeta) = \begin{cases} 1 + \beta_m \zeta, & \zeta > 0 \\ (1 - b_m \zeta)^{a_m} & \zeta < 0 \end{cases} \tag{5a}$$

$$\phi_h(\zeta) = \begin{cases} \alpha_\theta + \beta_h \zeta, & \zeta > 0 \\ \alpha_\theta (1 - b_h \zeta)^{a_h}, & \zeta < 0 \end{cases} \tag{5b}$$

where the constants β_m, b_m, a_m, β_h, b_h, a_h, and α_θ are determined empirically (see Table I). The results obtained by Businger *et al.* (1971) were for a von Kármán constant κ of 0.35. The functions (5) are also referred to as flux-profile relationships. There has been controversy concerning the value of von Kármán's constant and whether $\alpha_\theta \neq 1$ (see Dyer, 1974; Yaglom, 1977; Högström, 1985; Telford and Businger, 1986; Högström, 1986; Högström, 1988; Kaimal and Finnigan, 1994). Results from more precise instrumentation than that used by Businger *et al.* (1971) indicate that $\kappa = 0.40$ and $\alpha_\theta = 1.00$. Nevertheless, the functions in (5) will be taken to be 'known' in this work because of their generality. The Ri-based stability functions will be derived from the functions in (5). A consistent set of values can then be used (such as from Table I) to provide first-order closure. The authors are not advocating that $\alpha_\theta = 0.74$, although this value will be used for comparison of results with those of Businger *et al.* (1971). The Businger *et al.* (1971) results are considered in order to gauge the extent of the deviation of Pr from 1.00 in the SBL. The derivation of the Ri-based stability functions from the flux-profile relationship (5) is kept general with α_θ, β_m, β_h, b_m, and b_h unspecified. However, for the sake of brevity, the stability functions are derived only for the momentum and heat exponents of $a_m = -1/4$ and $a_h = -1/2$, respectively. The momentum exponent $a_m = -1/3$ for the very unstable regime (Carl *et al.*, 1973) is not considered. Changes in the functional form of the shear functions (such as the changes in Equations (5) at $\zeta = 0$) can introduce unwanted noise in numerical solutions in both space and time, because higher-order derivatives are not continuous. It will be shown that the stability functions also change form slightly when Pr = 1.0 in the SBL.

The dimensionless Monin–Obukhov height is $\zeta = z/L$, where $L = u_*^2/\kappa B\theta_*$ is the Monin–Obukhov length and $B = g/\theta_0$ is the buoyancy parameter, and θ_0 a scaling temperature. The friction temperature,

$$\theta_* = \frac{l_m(z)}{\phi_h(\zeta)} \frac{\partial \theta}{\partial z}, \tag{6}$$

TABLE I
Some estimated values for the unknown constants in the shear functions (5)

Author	Constants								Range of ζ
	κ	α_θ	a_m	b_m	a_h	b_h	β_m	β_h	
Dyer and Hicks (1970)	0.41	1.00	−1/4	16	−1/2	16			−1.0; 0.0
Webb (1970)	0.41	1.00					5.2	5.2	−0.03; 1.0
Businger *et al.* (1971)	0.35	0.74	−1/4	15	−1/2	9			−1.0; 0.0
							4.7	4.7	0.0; 2.0
Carl *et al.* (1973)	0.41	1.00	−1/3	16					−10.0; −2.0
	0.41	1.00	−1/4	16					−2.0; 0.0
Dyer (1974)	0.41	1.00	−1/4	16	−1/2	16	5.0	5.0	
Hicks (1976)							5.0	5.0	0.02; 1.0
Dyer and Bradley (1982)	0.40	1.00	−1/4	28	−1/2	14			

is analogous to the friction velocity in (2), where θ is potential temperature. The definition of the friction parameters (2) and (6) is circular (not closed-form) because they are expressed in terms of the shear functions (5), which depend upon ζ, which in turn depends on the friction parameters. In numerical computations, an iterative scheme is usually required to solve (2) and (6). Furthermore, such a circular formulation does not easily lend itself to the application of Bendixson's criterion, which addresses the existence of limit cycles in a two-dimensional dynamical system, because derivatives cannot be resolved analytically (see England and McNider, 1995).

2.2. Richardson Number Formulation

The Richardson number is defined by

$$\mathrm{Ri} = B\frac{\partial \theta}{\partial z}\Big/ s^2. \tag{7}$$

Using (2), (6) and (7), a well-known relationship can be derived that expresses Ri as a function of ζ, i.e.,

$$\mathrm{Ri}(\zeta) = \zeta\frac{\phi_h(\zeta)}{\phi_m^2(\zeta)}. \tag{8}$$

An analogous expression for ζ as a function of Ri will be derived after the stability functions have been discussed.

Monin–Obukhov theory holds for all values of ζ. However, a Ri formulation breaks down past a critical value. The critical Richardson number (Ri_c), which corresponds to an infinite ζ, is found by taking the following limit

$$\mathrm{Ri}_c = \lim_{\zeta\to\infty} \mathrm{Ri}(\zeta) = \lim_{\zeta\to\infty} \zeta \frac{\phi_h(\zeta)}{\phi_m^2(\zeta)} = \frac{\beta_h}{\beta_m^2}. \tag{9}$$

The critical value is usually taken to be between 1/5 and 1/4.

Substitution of the profiles (5) for the SBL into (8) results in a quadratic equation,

$$(\beta_m^2\mathrm{Ri} - \beta_h)\zeta^2 + (2\beta_m\mathrm{Ri} - \alpha_\theta)\zeta + \mathrm{Ri} = 0, \tag{10}$$

which can be solved to yield

$$\zeta(\mathrm{Ri}) = \frac{\alpha_\theta - 2\beta_m\mathrm{Ri}\sqrt{\alpha_\theta^2 + 4(\beta_h - \alpha_\theta\beta_m)\mathrm{Ri}}}{2(\beta_m^2\mathrm{Ri} - \beta_\mathrm{h})}, \quad 0 < \mathrm{Ri} < \mathrm{Ri}_c = \frac{\beta_h}{\beta_m^2}. \tag{11}$$

The negative root has been taken because $\zeta(0) = 0$. Also, when $\alpha_\theta = 1$ and $\beta = \beta_m = \beta_h$, the relationship (11) is simply

$$\zeta(\mathrm{Ri}) = \frac{\mathrm{Ri}}{1 - \beta\mathrm{Ri}}, \quad 0 < \mathrm{Ri} < \mathrm{Ri}_c = \frac{1}{\beta}, \tag{12}$$

a form used by Paulson (1970) and Blackadar (1979) among others. Repeating the procedure used to obtain (10) for the SBL, the following cubic equation is obtained:

$$b_m\zeta^3 - \zeta^2 - b_h\left(\frac{\mathrm{Ri}}{\alpha_\theta}\right)^2\zeta + \left(\frac{\mathrm{Ri}}{\alpha_\theta}\right)^2 = 0, \qquad \mathrm{Ri} < 0, \tag{13}$$

for the UBL. The algebraic solution of (13) is given in the Appendix.

3. Stability Functions

In the Ri formulation, the exchange coefficients are

$$K_m = f_m(\mathrm{Ri})l_m^2 s, \tag{14a}$$

$$K_h = f_h(\mathrm{Ri})l_m^2 s, \tag{14b}$$

where f_m and f_h are stability functions for momentum and heat. Comparison of (1) and (14) yields

$$f_m(\mathrm{Ri}) = \phi_m^{-2}(\zeta), \tag{15a}$$

$$f_h(\mathrm{Ri}) = \phi_h^{-1}(\zeta)\phi_m^{-1}(\zeta). \tag{15b}$$

The stability functions are now known implicitly, because for a given Ri, (11), (12) or (13) can be used to solve for ζ, which in turn can be used in (5) and then (15) to give the corresponding values of the stability functions. However, one of the goals of this work is to obtain an explicit functional form of the stability functions so that they can be physically analyzed. This goal is only partially achieved because of the cubic equation (13) for the UBL. A systematic approach is outlined that may seem to be mathematically excessive but is followed because of its generality, i.e., the same approach can be used to derive expressions for the stability functions for different exponent values such as $a_m = -1/3$. In fact, the approach may also work for shear functions that do not have the form of (5). Also, in a numerical setting the Ri approximation in the lower level(s) has not been made to depend explicitly upon the surface roughness as is customary for ζ.

The relationships (2), (6) and (15) give a friction velocity and temperature

$$u_* = \sqrt{f_m(\mathrm{Ri})} l_m s = \sqrt{f_m(\mathrm{Ri})}\kappa z\left|\frac{\partial \mathbf{V}}{\partial z}\right| = \sqrt{f_m(\mathrm{Ri})}\kappa\left|\frac{\partial \mathbf{V}}{\partial \ln z}\right|, \tag{16a}$$

$$\theta_* = \frac{f_h(\mathrm{Ri})}{\sqrt{f_m(\mathrm{Ri})}} l_m \frac{\partial \theta}{\partial z} = \frac{f_h(\mathrm{Ri})}{\sqrt{f_m(\mathrm{Ri})}}\kappa\frac{\partial \theta}{\partial \ln z}, \tag{16b}$$

which are expressed in terms of stability functions and Ri. Equations (16), along with $\zeta = z/L = B\kappa z\theta_*/u_*^2$, can be used to relate ζ to Ri, i.e.,

$$\zeta(\mathrm{Ri}) = \mathrm{Ri}\frac{f_h(\mathrm{Ri})}{f_m^{3/2}(\mathrm{Ri})}. \tag{17}$$

Equation (17) can be considered to be the dual of (8) and will be instrumental in deriving the stability functions from (5) and (15). To the authors' knowledge, this relationship has not appeared previously in the literature, although Paulson (1970) did present a relationship similar to (17) based upon the KEYPS equation

$$\phi_m^4(\zeta) - \gamma_m\zeta\phi_m^3(\zeta) = 1 \;\; (\gamma_m \approx b_m). \tag{18}$$

3.1. Stable Boundary Layer

Before deriving the stability functions in the SBL, it is necessary to discuss the (turbulent) Prandtl number, because the functional form of the stability functions depend upon it. The Prandtl number, defined by

$$\mathrm{Pr} = \frac{K_m}{K_h}, \tag{19}$$

(see e.g., Garratt, 1992) is a ratio that expresses the relative efficiency of the mixing of heat and momentum. In general, it depends upon atmospheric stability, i.e.,

$$\mathrm{Pr} = \frac{K_m}{K_h} = \frac{f_m(\mathrm{Ri})}{f_h(\mathrm{Ri})} = \frac{\phi_h(\zeta)}{\phi_m(\zeta)}. \tag{20}$$

In the SBL, a ζ formulation for (20) is

$$\mathrm{Pr}(\zeta) = \frac{\phi_h(\zeta)}{\phi_m(\zeta)} = \frac{\alpha_\theta + \beta_h\zeta}{1 + \beta_m\zeta}. \tag{21}$$

3.1.1. $\mathrm{Pr} \neq 1$

If $\alpha_\theta\beta_m < \beta_h$, then d Pr/$d\zeta > 0$. Hence, Pr increases asymptotically from Pr(0) = α_θ for a neutral atmosphere to $\beta_h/\beta_m \cong 1$ as stability increases, i.e., Pr $\uparrow \beta_h/\beta_m$ as $\zeta \uparrow \infty$. The small variation of Pr is justification for assuming it to be constant in the SBL. The assumption may not be valid for the UBL.

The stability functions for the SBL based upon the profiles (5) will be derived before considering the more difficult UBL regime. From Table I, it can be seen that $\beta_m = \beta_h$ in the SBL. In the subsequent analysis, therefore, we set $\beta = \beta_m = \beta_h$. The analysis can be carried out without this assumption, but the authors feel that it is unwarranted. From (5), (15) and (17), we have

$$\frac{1}{\sqrt{f_m}} = \phi_m = 1 + \beta\zeta = 1 + \beta\mathrm{Ri}\frac{f_h}{(\sqrt{f_m})^3}, \tag{22a}$$

$$\frac{\sqrt{f_m}}{f_h} = \phi_h = \alpha_\theta + \beta\zeta = \alpha_\theta + \beta\mathrm{Ri}\frac{f_h}{(\sqrt{f_m})^3}. \tag{22b}$$

Equation (22a) can be manipulated to yield

$$f_h = \frac{1 - \sqrt{f_m}}{\beta\mathrm{Ri}} f_m. \tag{23}$$

Substitution of (23) into (22b) gives the following quadratic equation in $\sqrt{f_m}$, i.e.,

$$(1 - \alpha_\theta)(\sqrt{f_m})^2 + (\alpha_\theta - 2)\sqrt{f_m} + (1 - \beta\mathrm{Ri}) = 0, \tag{24}$$

which can be solved to yield

$$\sqrt{f_m(\mathrm{Ri})} = \frac{(2 - \alpha_\theta) - \sqrt{\alpha_\theta^2 + 4(1 - \alpha_\theta)\beta\mathrm{Ri}}}{2(1 - \alpha_\theta)}. \tag{25}$$

The negative root has been taken because (15a) implies that $f_m(0) = 1$. The stability function for momentum is then simply

$$f_m(\mathrm{Ri}) = \left[\frac{(2-\alpha_\theta) - \sqrt{\alpha_\theta^2 + 4(1-\alpha_\theta)\beta \mathrm{Ri}}}{2(1-\alpha_\theta)} \right]^2 . \tag{26}$$

The solution (25) is not valid for (24) if $\alpha_\theta = 1$. This case will later be shown to coincide with the assumption that Pr = 1 in the SBL. Now that the stability function for momentum is known, the one for heat can be found from (23). Comparison of equations (20) and (23) gives Pr as a function of Ri in the SBL, i.e.,

$$\mathrm{Pr}(\mathrm{Ri}) = \frac{\beta \mathrm{Ri}}{1 - \sqrt{f_m(\mathrm{Ri})}} = \frac{2(1-\alpha_\theta)\beta \mathrm{Ri}}{\sqrt{\alpha_\theta^2 + 4(1-\alpha_\theta)\beta \mathrm{Ri}} - \alpha_\theta} . \tag{27}$$

Using the fact that $f_h(\mathrm{Ri}) = f_m(\mathrm{Ri})/\mathrm{Pr}(\mathrm{Ri})$, (26) can be divided by (27) to give

$$f_h(\mathrm{Ri}) = \frac{(\alpha_\theta - 4)(1-\alpha_\theta)\beta \mathrm{Ri} - \alpha_\theta + [1 - (1-\alpha_\theta)\beta \mathrm{Ri}]\sqrt{\alpha_\theta^2 + 4(1-\alpha_\theta)\beta \mathrm{Ri}}}{2(1-\alpha_\theta)^3 \beta \mathrm{Ri}} . \tag{28}$$

Graphs of the stability functions and Pr are given in Figure 1, based upon the profiles (5).

3.1.2. Pr = 1

In this section, we consider the stability functions for a Prandtl number identically equal to one in the SBL, i.e.,

$$\mathrm{Pr}(\zeta) = \frac{\phi_h(\zeta)}{\phi_m(\zeta)} = \frac{\alpha_\theta + \beta_h \zeta}{1 + \beta_m \zeta} = 1, \quad \forall \zeta > 0. \tag{29}$$

The above assumption, which is supported by the data, implies that we must have $\alpha_\theta = 1$ and $\beta = \beta_m = \beta_h$. In this case, substituting (12) into (15a) or solving (24) gives

$$f_m(\mathrm{Ri}) = (1 - \beta \mathrm{Ri})^2, \quad 0 < \mathrm{Ri} < 1/\beta, \tag{30}$$

or

$$f_m(\mathrm{Ri}) = f_h(\mathrm{Ri}) = \begin{cases} \left(\dfrac{\mathrm{Ri}_c - \mathrm{Ri}}{\mathrm{Ri}_c} \right)^2, & 0 < \mathrm{Ri} < \mathrm{Ri}_c \\ 0, & \mathrm{Ri} > \mathrm{Ri}_c \end{cases} \tag{31}$$

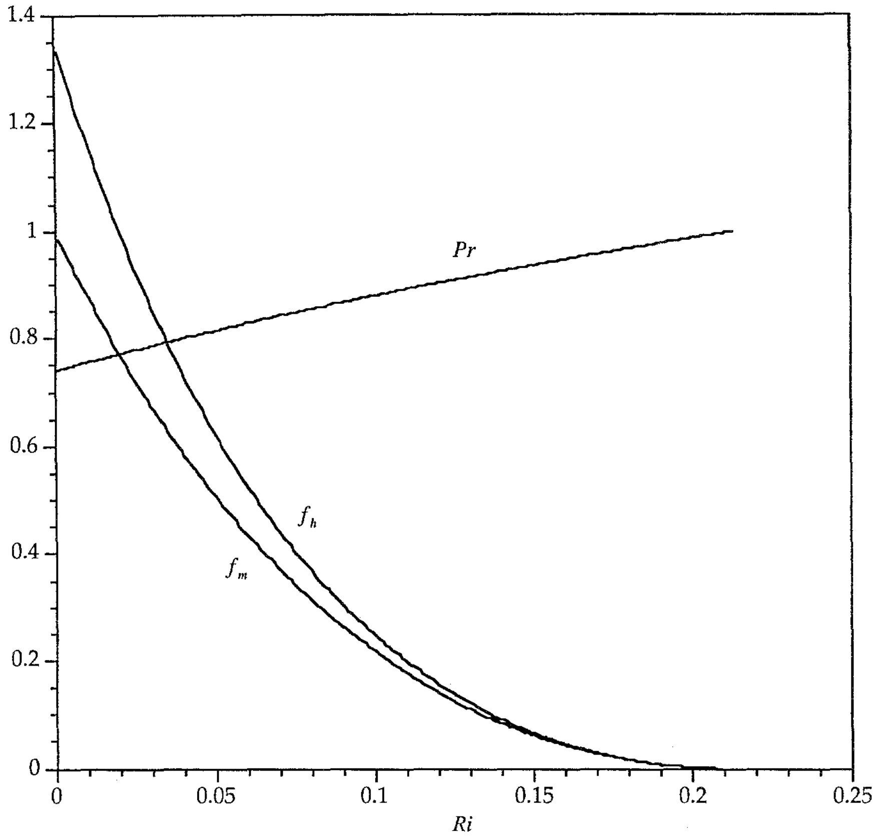

Fig. 1. Stability functions for the stable boundary based upon the profiles (5).

since $\mathrm{Ri}_c = 1/\beta$. For comparison with Figure 1, the functions f_m and 1.33 f_m are shown in Figure 4. Notice that the quadratic form (31) for f_m is smoother than the linear form proposed by Blackadar (1979), i.e.,

$$f_B(\mathrm{Ri}) = \begin{cases} \dfrac{\mathrm{Ri}_c - \mathrm{Ri}}{\mathrm{Ri}_c}, & 0 < \mathrm{Ri} < \mathrm{Ri}_c \\ 0. & \mathrm{Ri} > \mathrm{Ri}_c \end{cases} \tag{32}$$

The stability function (31) is preferrable to (32) for two reasons. First, it is derived from the flux-profile relationships, which are determined from data, under the assumption of Pr = 1 in the SBL. Second, (31) is smoother than (32), i.e., (31) has a continuous derivative in the SBL, while the derivative of (32) has a discontinuity at the critical Richardson number. The advantages of this additional degree of smoothness are discussed later. Also, numerical models that employ (32) can be modified to (31) by the addition of a single line of code where the stability function is merely squared in the SBL.

3.2. Unstable Boundary Layer

The analysis in this section will only be carried out for the exponents given by Businger *et al.* (1971). Just as for the SBL, (5) and (15) can be used to solve for the stability functions in the UBL, i.e.,

$$\frac{1}{\sqrt{f_m}} = \phi_m = (1 - b_m\zeta)^{-1/4} = \left(1 - b_m \mathrm{Ri}\frac{f_h}{(\sqrt{f_m})^3}\right)^{-1/4}, \tag{33a}$$

$$\frac{\sqrt{f_m}}{f_h} = \phi_h = \alpha_\theta(1 - b_h\zeta)^{-1/2} = \alpha_\theta\left(1 - b_h \mathrm{Ri}\frac{f_h}{(\sqrt{f_m})^3}\right)^{-1/2} \tag{33b}$$

Solving equation (33a) for the heat stability function yields

$$f_h = \frac{f_m^{3/2} - f_m^{7/2}}{b_m \mathrm{Ri}} = \frac{f_m^{1/2} - f_m^{5/2}}{b_m \mathrm{Ri}} f_m, \tag{34}$$

which gives

$$\Pr(\mathrm{Ri}) = \frac{b_m \mathrm{Ri}}{f_m^{1/2}(\mathrm{Ri}) - f_m^{5/2}(\mathrm{Ri})}. \tag{35}$$

Substitution of (34) in (33b) gives a cubic equation in f_m^2, i.e.,

$$(f_m^2)^3 - 2(f_m^2)^2 + \left[1 - b_m b_h\left(\frac{\mathrm{Ri}}{\alpha_\theta}\right)^2\right] f_m^2 + b_m(b_h - b_m)\left(\frac{\mathrm{Ri}}{\alpha_\theta}\right)^2 = 0. \tag{36}$$

Equation (36) could be solved by algebraic methods, but the algebraic functional form is cumbersome and is therefore not presented explicity. We take the equivalent approach of using the solution of (13) for a given (negative) Ri and then using (15) to calculate the values of the stability functions. The stability functions and Pr obtained by this procedure are shown in Figure 2.

3.3. Profile Approximations

The desired solution of (36) is almost a straight line as can be seen by reconsidering (8) for the UBL, i.e.,

$$\mathrm{Ri} = \alpha_\theta\zeta\sqrt{\frac{1 - b_m\zeta}{1 - b_h\zeta}} \cong \alpha_\theta\sqrt{\frac{b_m}{b_h}}\zeta, \tag{37}$$

for $\zeta \ll -1/b_h$, or $\zeta \cong \lambda\mathrm{Ri}$, where $\lambda = \sqrt{b_h/b_m}/\alpha_\theta$. For $b_h = 9$, $b_m = 15$, and $\alpha_\theta = 0.74$, $\lambda \cong 1.05$, i.e., ζ is 5% larger in magnitude than Ri in the UBL. This result agrees quite well with the factor of 4% based upon the Kansas wheat-field

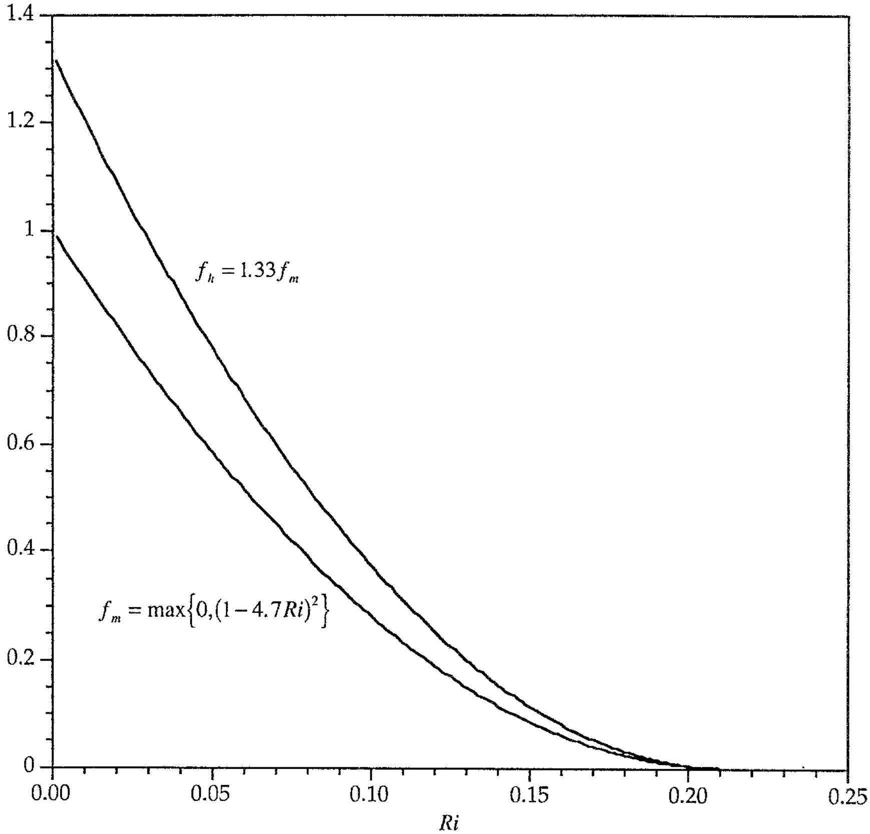

Fig. 2. Approximate stability functions for the stable boundary layer.

data analyzed by Businger *et al.* (1971). Using (37) in (15a) and squaring produces $f_m^2 = \phi_m^{-4} = 1 - b_m\zeta \cong 1 - \lambda b_m \mathrm{Ri}$, which is approximately linear. Thus, the approximate momentum stability function is

$$f_m(\mathrm{Ri}) \cong \sqrt{1 - \lambda b_m \mathrm{Ri}}, \tag{38}$$

which agrees well with previously obtained results (Blackadar, 1979). The corresponding heat stability function is

$$f_h(\mathrm{Ri}) \cong \frac{1}{\alpha_\theta}(1 - \lambda b_h \mathrm{Ri})^{1/2}(1 - \lambda b_m \mathrm{Ri})^{1/4}, \tag{39}$$

giving

$$\mathrm{Pr}(\mathrm{Ri}) \cong \frac{\alpha_\theta (1 - \lambda b_m \mathrm{Ri})^{1/4}}{(1 - \lambda b_h \mathrm{Ri})^{1/2}}. \tag{40}$$

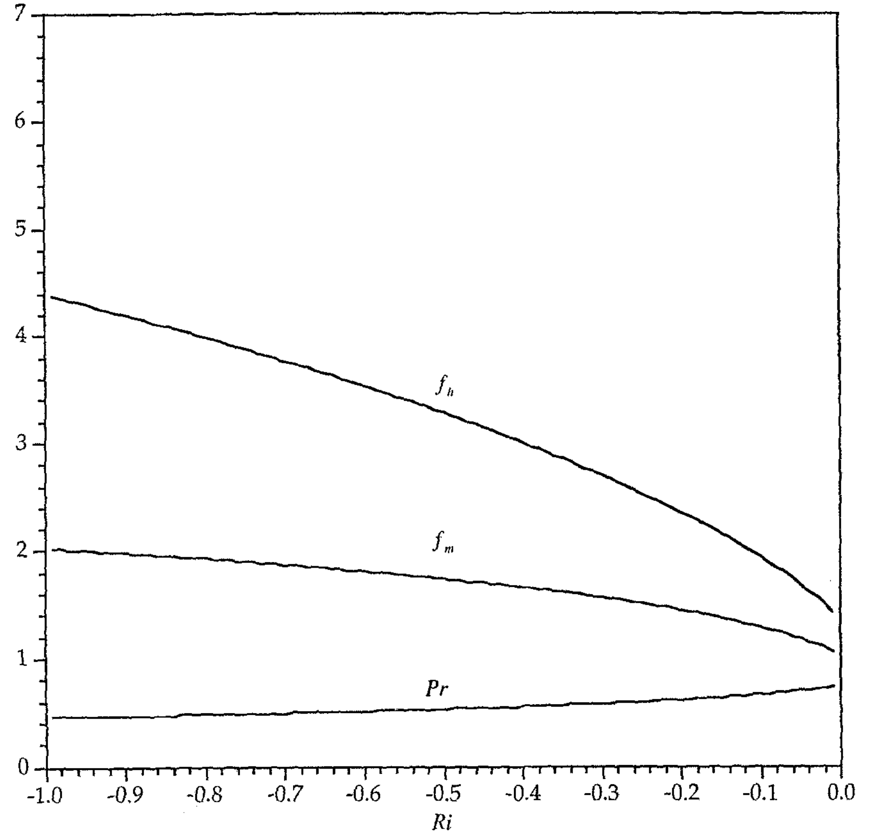

Fig. 3. Stability functions for the unstable boundary based upon the profiles (5).

These approximate functions are shown in Figure 3. For smaller values ($-1/b_m \ll \zeta \leq 0$), Equation (37) gives the approximation $\zeta \cong \mathrm{Ri}/\alpha_\theta$, i.e., in the near-neutral regime, $\lambda = 1/\alpha_\theta \cong 1.33$.

To test the validity of the approximate stability function (38), a relationship will be derived, which the induced shear functions would have to satisfy. Substitution of (8) into (38) and using (15a), implies that

$$\phi_m^4 - \gamma_m \zeta \phi_m^2 \phi_h = 1, \tag{41}$$

where $\gamma_m = \lambda b_m$. [The process could be repeated for (39) to obtain a higher-order multinomial, which is not presented.] Note that if $\phi_h = \phi_m$ in (41), then the KEYPS Equation (18) is obtained. For the profiles (5), Equation (41) would be exactly satisfied in the UBL if $b_m = b_h$ and $\gamma_m = b_m/\alpha_\theta$. This indicates that the KEYPS equation may be valid only in the near-neutral regime where $\mathrm{Pr} \cong 1$.

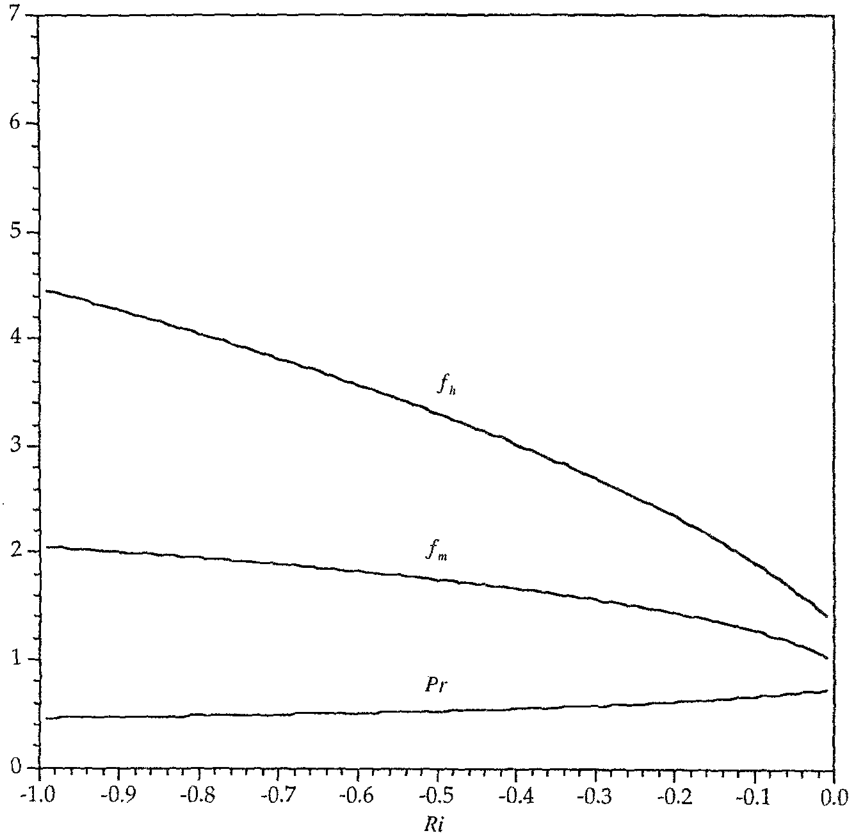

Fig. 4. Approximate stability functions for the unstable boundary.

4. Smoothness

Smoother stability functions (continuous functions having continuous derivatives) can be obtained from smoother shear functions (5). Smoother exchange coefficients are more desirable because they should produce smoother solutions. For example, in the SBL for Pr = 1, the quadratic form (31) for f_m is smoother than the linear form (31), because the derivative of (31), i.e.,

$$f'_m(\mathrm{Ri}) = \begin{cases} \dfrac{-2}{\mathrm{Ri}_c}\left(\dfrac{\mathrm{Ri}_c - \mathrm{Ri}}{\mathrm{Ri}_c}\right), & 0 < \mathrm{Ri} < \mathrm{Ri}_c \\ 0, & \mathrm{Ri} > \mathrm{Ri}_c \end{cases} \tag{42}$$

is continuous at Ri_c. This smoothness is important in carrying out dynamical analyses of boundary-layer systems (England and McNider, 1995; McNider *et al.*, 1994). It is also expected that numerical solutions using these forms will have less noise in time and space.

Similarly, an added degree of smoothness can be obtained in the near-neutral regime if the following conditions are met:

$$a_m b_m = \beta_m, \tag{43a}$$

$$\alpha_\theta a_h b_h = \beta_h. \tag{43b}$$

From Table I, it can be seen that these conditions are almost met.

The conditions of (43) also provide a means of extrapolating parameter values for the SBL. Specifically, the further assumption of $\beta_m = \beta_h$ produces $a_m b_m = \alpha_\theta a_h b_h$. For the exponent values of $a_m = -1/4$, $a_h = -1/2$, (43b) reduces to $b_h = b_m/2\alpha_\theta \cong 0.65 b_m$. For $b_m = 16.0$, the other parameters are $b_h = 10.4$, $\beta_m = \beta_h = 4.0$, values which are close to those usually cited. These results indicate that the derivatives of the shear (stability) functions are continuous at atmospheric neutrality. This assertion is further supported by the data (see e.g. Businger *et al.*, 1971), which indicate that the shear functions are very smooth. To address the question of smoothness, the optimal values for the constants in (5) would have to be re-estimated with the imposed constraints (43), perhaps by more robust nonlinear estimation techniques.

Ideally, a single, smooth shear (stability) function, one each for momentum and heat, would be best because the change in functional form of the functions from the UBL to the SBL necessitates the use of conditional statements in the source codes of numerical models and hinders optimization of the code.

Some numerical modelers introduce discontinuities in vertical derivatives by using a ζ formulation for the lower level(s) and a Ri formulation for the upper levels that are inconsistent. This problem can be overcome by using the approximations in the next section.

5. Contrast of Formulations

From the preceeding discussion, it is seen that the Ri formulation is equivalent to a ζ formulation. However, in finite-difference models, the lowest level is usually treated differently. Equations (1), (2), (4) and (6) can be combined to produce the following relations

$$K_m \frac{\partial u}{\partial z} = u_*^2 \cos\alpha, \tag{44a}$$

$$K_m \frac{\partial v}{\partial z} = u_*^2 \sin\alpha, \tag{44b}$$

$$K_h \frac{\partial \theta}{\partial z} = u_* \theta_*. \tag{44c}$$

where $\tan\alpha = \partial v/\partial z/\partial u/\partial z$. The friction parameters in the lowest level are approximately

$$u_* = \frac{\kappa|\mathbf{V}_1|}{\ln\dfrac{z_1}{z_0} - \psi_m(\zeta_1)}, \quad \psi_m(\zeta) = \int_0^\zeta \frac{1-\phi_m(\xi)}{\xi}\,\mathrm{d}\xi, \tag{45a}$$

$$\theta_* = \frac{\kappa(\theta_1 - \theta_0)}{\ln\dfrac{z_1}{z_0} - \psi_h(\zeta_1)}, \quad \psi_h(\zeta) = \int_0^\zeta \frac{1-\phi_h(\xi)}{\xi}\,\mathrm{d}\xi, \tag{45b}$$

where z_1, is the height of the lowest level (see Panofsky, 1963). Note that the lower limit of integration of (45a) should be $\zeta_0 = z_0/L$, but the approximation $\zeta_0 \approx 0$ is usually made. The mathematical evaluation of the integrals in (45) is straightforward in the SBL because the shear functions are linear. The process is more difficult in the UBL (see Paulson, 1970; Nickerson and Smiley, 1975; or Benoit, 1977). It should be pointed out that there is a minor difference between the ground temperature, $\theta_g = \theta(0)$, and the temperature at the roughness height, $\theta_0 = \theta(z_0)$. In fact, Blyth *et al.* (1993) indicate that the assumption of equal roughness heights for momentum and heat is a poor one. However, for heat, there is not a 'nice' definition for the roughness height such as $|\mathbf{V}(z_0)| = 0$ for momentum. The roughness height for heat would be the lowest height for which similarity theory is valid for the thermal boundary layer. Also, the ground temperature, $\theta_g = \theta(0)$, is not easily determined because the actual ground surface is a 'fuzzy' boundary. The difficulties encountered by these ambiguities differ depending upon whether one is working with observations or performing numerical simulations. In the case of observations, the heat roughness height is usually taken to be the lowest height that the temperature can be accurately measured. In the numerical case, temperature is parameterized by equations such as Blackadar's (1979) slab model.

In practice, calculating the friction parameters from (45) requires an iterative procedure because the upper limit of integration $\zeta_1 = z_1/L = z_1(\kappa B\theta_*/u_*^2)$ also depends upon the friction parameters. This time-comsuming procedure can be circumvented in a Richardson number formulation. However, a forward difference approximation to the lowest-layer Ri produces the bulk Richardson number,

$$\mathrm{Ri} = Bz_1\frac{\theta_1 - \theta_g}{|\mathbf{V}_1|^2}, \tag{46}$$

which does not depend directly upon z_0 (although it should), but does depend upon the ambiguous ground temperature. Recall that the ζ formulation incorporated the z_0 dependence by means of a logarithmic transformation and the chain rule in the form of

$$\frac{\partial}{\partial z} = \frac{\mathrm{d}\ln z}{\mathrm{d}z}\frac{\partial}{\partial \ln z} = \frac{1}{z}\frac{\partial}{\partial \ln z}. \tag{47}$$

Thus, Equations (16a) and (16b) in centered finite-difference approximation form are

$$u_* = \sqrt{f_m(\mathrm{Ri}_{1/2}} \frac{\kappa |\mathbf{V}_1|}{\ln \dfrac{z_1}{z_0}}, \tag{48a}$$

$$\theta_* = \frac{f_h(\mathrm{Ri}_{1/2})}{\sqrt{f_m(\mathrm{Ri}_{1/2})}} \frac{\kappa(\theta_1 - \theta_0)}{\ln \dfrac{z_1}{z_0}}. \tag{48b}$$

Logically, Ri must also go through the same transformation. Under a logarithmic transformation

$$\mathrm{Ri} = B \frac{\partial \theta}{\partial z} \Big/ \left| \frac{\partial \mathbf{V}}{\partial z} \right|^2 = B \frac{1}{z} \frac{\partial \theta}{\partial \ln z} \Big/ \left| \frac{1}{n} \frac{\partial \mathbf{V}}{\partial \ln z} \right|^2, \tag{49}$$

so that the approximation to Ri is

$$\mathrm{Ri}_{1/2} = B \bar{z} \ln \frac{z_1}{z_0} \frac{\theta_1 - \theta_0}{|\mathbf{V}_1|^2} = B h_1 \frac{\theta_1 - \theta_0}{|\mathbf{V}_1|^2}, \tag{50}$$

where $\ln \bar{z} = (\ln z_0 + \ln z_1)/2$, or $\bar{z} = \sqrt{z_0 z_1}$. Thus the new, effective, average depth for Ri is $h_1 = \sqrt{z_0 z_1} \ln z_1 / z_0$. The transformation can be performed without assuming equal roughness heights for heat and momentum, yielding a more involved result than (50). Finally, the closed-form Ri formulation should now be fully equivalent to the ζ formulation. The relations (48) can be manipulated to yield

$$u_*^2 = \kappa^2 f_m(\mathrm{Ri}_{1/2}) |\mathbf{V}_1|^2 \Big/ \left(\ln \frac{z_1}{z_0} \right)^2, \tag{51a}$$

$$u_* \theta_* = \kappa^2 f_h(\mathrm{Ri}_{1/2}) |\mathbf{V}_1| (\theta_1 - \theta_g) \Big/ \left(\ln \frac{z_1}{z_0} \right)^2, \tag{51b}$$

which can then be used in conjunction with (44) to approximate the momentum and heat fluxes.

6. Conclusion

This paper outlines a consistent, closed-form, Richardson-number approach for a smooth, first-order, K-closure, boundary-layer numerical model that does not require iterative procedures for a constant surface roughness. On the whole, the mathematical results agree well with the observations and theoretical results of other researchers. However, the stability function (31) for the SBL is preferrable over (32) for two reasons. First, it was derived from the flux-profile relationships (5) for Pr = 1.0 in the SBL. Second, both the function and its derivative are continuous, which should produce smoother results. Shear or stability functions that do not change form from the SBL to the UBL would add an additional degree of smoothness, and also would allow numerical models based upon first-order K-closure to be greatly optimized. Such functions must be based upon observations. Additional work is required to generalize the Ri-parameterized fluxes to include evaporation as well as sensible heat transfer.

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Appendix

To solve (13), define the following quantities:

$$Q(\mathrm{Ri}) = \frac{-1}{3b_m}\left[\frac{1}{3b_m} + b_h\left(\frac{\mathrm{Ri}}{\alpha_\theta}\right)^2\right], \tag{A1}$$

$$R(\mathrm{Ri}) = \frac{1}{b_m^3}\left[\frac{2}{9} + b_m(b_h - 3b_m)\left(\frac{\mathrm{Ri}}{\alpha_\theta}\right)^2\right]. \tag{A2}$$

There are two negative Richardson numbers a and b such that $R^2(a) + Q^3(a) = R^2(b) + Q^3(b) = 0$. These are given by

$$a = -\alpha_\theta\sqrt{\frac{27b_m^2 - 18b_m b_h + b_h^2 - (9b_m - b_h)\sqrt{(9b_m - b_h)(b_m - b_h)}}{8b_m b_h^3}}, \tag{A3}$$

$$b = -\alpha_\theta\sqrt{\frac{27b_m^2 - 18b_m b_h + b_h^2 - (9b_m - b_h)\sqrt{(9b_m - b_h)(b_m - b_h)}}{8b_m b_h^3}}. \tag{A4}$$

For the profiles (5), the values are approximately $a = -0.025$, $b = -0.21$. The nondimensional height for a given Richardson number in the unstable boundary layer based upon (5) is given by

$$\zeta = \begin{cases} \sqrt[3]{R + \sqrt{R^2 + Q^3}} + \sqrt[3]{R - \sqrt{R^2 + Q^3}} + \dfrac{1}{3b_m}, & b \leq \mathrm{Ri} \leq a \\ 2\sqrt{-Q} \cos\left(\dfrac{\omega + 2\pi}{3}\right) + \dfrac{1}{3b_m}, & a \leq \mathrm{Ri} < 0 \text{ or } \mathrm{Ri} \leq b, \end{cases} \tag{A5}$$

where $\cos \omega = R/\sqrt{-Q^3}$ (ω in radians).

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