Homework Exercise: Geometric Mean Height and Richardson Number Bias via Jensen's Inequality

Course: Boundary Layer Meteorology / Advanced Atmospheric Physics
Topic: Grid Sensitivity in Stable Boundary Layer Parameterizations
Difficulty: Graduate Level

Problem Statement: Atmospheric models discretize the vertical using finite layers. In the stable boundary layer (SBL), the **bulk Richardson number** Ri_b computed across a layer systematically differs from the **gradient Richardson number** Ri_g evaluated at a representative height. Your task is to prove that when $Ri_g(z)$ is concave-down (typical in SBL), the layer-averaged Ri_b underestimates the point value at the geometric mean height $z_g = \sqrt{z_0 z_1}$.

Key Result to Prove:

$$Ri_b = \frac{1}{\Delta z}\int_{z_0}^{z_1} Ri_g(z)\,dz < Ri_g(z_g)$$

where $\Delta z = z_1 - z_0$ and $z_g = \sqrt{z_0 z_1}$.

Part A: Jensen's Inequality (Warm-Up)

Hint: Jensen's inequality states that for a **concave** function f,

$$f\left(\frac{1}{b-a}\int_a^b x\,dx\right) \geq \frac{1}{b-a}\int_a^b f(x)\,dx.$$

A1. Concavity and Jensen's Inequality

State Jensen's inequality for a concave function. What conditions must f satisfy? What does "concave" mean in terms of the second derivative?

- **Jensen's Inequality for a Concave Function:** For a concave function f and a probability distribution $P(x)$, we have $f(\mathbb{E}[x]) \geq \mathbb{E}[f(x)]$. For continuous uniform distributions over $[a,b]$, this simplifies to:

$$f\left(\frac{1}{b-a}\int_a^b x\,dx\right) \geq \frac{1}{b-a}\int_a^b f(x)\,dx.$$

- **Conditions on f:** The function f must be defined on a convex set (e.g., an interval $[a,b]$) and be **concave** over that set. For the integral form, f must also be integrable.
- **Concavity (Second Derivative):** A twice-differentiable function f is concave on an interval if its second derivative is non-positive throughout that interval: $f''(x) \leq 0$. If $f''(x) < 0$, the function is strictly concave.

A2. Concavity of Natural Logarithm

The natural logarithm $\ln(z)$ is concave. Prove this by computing $d^2(\ln z)/dz^2$ and showing it is negative for $z > 0$.
Proof: Let $f(z) = \ln z$. The first derivative is:

$$f'(z) = \frac{d}{dz}(\ln z) = \frac{1}{z}$$

The second derivative is:

$$f''(z) = \frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2}$$

Since $z > 0$ (required for $\ln z$ to be defined), $z^2 > 0$, and therefore $f''(z) = -1/z^2 < 0$. Since the second derivative is strictly negative, $\ln z$ is a strictly **concave** function for $z > 0$.

Part B: Why Geometric Mean Height?

B1. Logarithmic Mean and Layer-Averaged Gradient

Show that for a log-linear wind profile $U(z) = (u_*/\kappa)\ln(z/z_0) + C$, evaluating the gradient $\partial U/\partial z$ at the **logarithmic mean** $\bar{z}_{\ln}$ gives the exact layer-averaged gradient $\Delta U/\Delta z$.
Solution: The wind profile is $U(z) = (u_*/\kappa)(\ln z - \ln z_0) + C$. The difference in velocity across the layer $[z_0, z_1]$ is $\Delta U = U(z_1) - U(z_0)$.

$$\Delta U = \left[\frac{u_*}{\kappa}(\ln z_1 - \ln z_0) + C\right] - \left[\frac{u_*}{\kappa}(\ln z_0 - \ln z_0) + C\right] = \frac{u_*}{\kappa}(\ln z_1 - \ln z_0) = \frac{u_*}{\kappa}\ln\left(\frac{z_1}{z_0}\right)$$

The layer-averaged gradient is:

$$\frac{\Delta U}{\Delta z} = \frac{1}{\Delta z}\frac{u_*}{\kappa}\ln\left(\frac{z_1}{z_0}\right) = \frac{u_*}{\kappa}\frac{\ln(z_1) - \ln(z_0)}{z_1 - z_0}$$

The point-wise gradient is:

$$\frac{\partial U}{\partial z} = \frac{d}{dz}\left[\frac{u_*}{\kappa}\ln z\right] = \frac{u_*}{\kappa}\frac{1}{z}$$

We seek the height $z = \bar{z}_{\ln}$ such that $\partial U/\partial z|_{z=\bar{z}_{\ln}} = \Delta U/\Delta z$:

$$\frac{u_*}{\kappa}\frac{1}{\bar{z}_{\ln}} = \frac{u_*}{\kappa}\frac{\ln(z_1) - \ln(z_0)}{z_1 - z_0}$$

Solving for $\bar{z}_{\ln}$:

$$\bar{z}_{\ln} = \frac{z_1 - z_0}{\ln(z_1) - \ln(z_0)}$$

This is the definition of the logarithmic mean height, $\bar{z}_{\ln}$. Thus, the log-linear gradient is exactly represented at the logarithmic mean height.

B2. Geometric Mean as Logarithmic Midpoint

Show that the **geometric mean** height $z_g = \sqrt{z_0 z_1}$ is the midpoint in **logarithmic coordinates**.
Proof: Take the natural logarithm of z_g:

$$\ln z_g = \ln(\sqrt{z_0 z_1}) = \ln((z_0 z_1)^{1/2})$$

Using the properties of logarithms ($\ln(a^b) = b\ln a$ and $\ln(ab) = \ln a + \ln b$):

$$\ln z_g = \frac{1}{2}\ln(z_0 z_1) = \frac{1}{2}(\ln z_0 + \ln z_1) = \frac{\ln z_0 + \ln z_1}{2}$$

This shows that $\ln z_g$ is the arithmetic mean of the log-heights $\ln z_0$ and $\ln z_1$, proving that z_g is the midpoint in logarithmic coordinates.

B3. Thin Layer Approximation

For thin layers ($z_1/z_0 \to 1$), prove that $\bar{z}_{\ln} \approx z_g$ to second order in $\ln(z_1/z_0)$ by Taylor expanding.
Proof: Let $\delta = \ln(z_1/z_0)$. Since $z_1/z_0 \approx 1$, we have $\delta \approx 0$. The logarithmic mean is $\bar{z}_{\ln} = \frac{z_1 - z_0}{\ln(z_1/z_0)} = \frac{z_0((z_1/z_0)-1)}{\delta}$. From $z_1/z_0 = e^\delta$, we have $z_1 = z_0 e^\delta$. The numerator is $z_1 - z_0 = z_0(e^\delta - 1)$. Using the Taylor expansion $e^\delta = 1 + \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + O(\delta^4)$:

$$e^\delta - 1 = \delta + \frac{\delta^2}{2} + \frac{\delta^3}{6} + O(\delta^4)$$

Substituting into $\bar{z}_{\ln}$:

$$\bar{z}_{\ln} = \frac{z_0}{\delta}\left[\delta + \frac{\delta^2}{2} + \frac{\delta^3}{6} + O(\delta^4)\right] = z_0\left[1 + \frac{\delta}{2} + \frac{\delta^2}{6} + O(\delta^3)\right]$$

Now consider the geometric mean $z_g = \sqrt{z_0 z_1} = z_0\sqrt{z_1/z_0} = z_0(e^\delta)^{1/2} = z_0 e^{\delta/2}$. Using the Taylor expansion $e^x = 1 + x + x^2/2 + O(x^3)$ with $x = \delta/2$:

$$z_g = z_0\left[1 + \left(\frac{\delta}{2}\right) + \frac{1}{2}\left(\frac{\delta}{2}\right)^2 + O(\delta^3)\right] = z_0\left[1 + \frac{\delta}{2} + \frac{\delta^2}{8} + O(\delta^3)\right]$$

Comparing the expansions for $\bar{z}_{\ln}$ and z_g:

$$\bar{z}_{\ln} = z_0\left[1 + \frac{\delta}{2} + \frac{\delta^2}{6}\right] + O(\delta^3)$$

$$z_g = z_0\left[1 + \frac{\delta}{2} + \frac{\delta^2}{8}\right] + O(\delta^3)$$

The first two terms are identical. The difference appears in the δ^2 term: $\frac{\delta^2}{6} - \frac{\delta^2}{8} = \frac{4\delta^2 - 3\delta^2}{24} = \frac{\delta^2}{24}$. Thus, $\bar{z}_{\ln} \approx z_g$ to first order in δ, and their difference is $O(\delta^2)$ (second order).

$$\bar{z}_{\ln} = z_g\left[\frac{1 + \delta/2 + \delta^2/6}{1 + \delta/2 + \delta^2/8}\right] \approx z_g\left[1 + O\big(\delta^2\big)\right] = z_g\left[1 + O\big((\ln(z_1/z_0))^2\big)\right]$$

This confirms that for thin layers, the logarithmic mean and geometric mean are nearly identical.

Part C: Concave-Down Ri_g and Bias

Assume $Ri_g(z)$ is **concave-down** (i.e., $d^2 Ri_g/dz^2 < 0$) over the interval $[z_0, z_1]$.

C1. Jensen's Inequality and Arithmetic Mean

Apply Jensen's inequality to $Ri_g(z)$ treated as a concave function. Specifically, show that

$$\frac{1}{\Delta z}\int_{z_0}^{z_1} Ri_g(z)\,dz < Ri_g\left(\frac{1}{\Delta z}\int_{z_0}^{z_1} z\,dz\right).$$

Derivation: Let $f(z) = Ri_g(z)$. We are given that $f(z)$ is concave-down, so $f''(z) < 0$. We apply the strict version of Jensen's inequality for a strictly concave function:

$$\frac{1}{b-a}\int_a^b f(x)\,dx < f\left(\frac{1}{b-a}\int_a^b x\,dx\right)$$

Substituting $f(z) = Ri_g(z)$, $a = z_0$, and $b = z_1$:

$$\frac{1}{z_1 - z_0} \int_{z_0}^{z_1} Ri_g(z)\, dz < Ri_g\left(\frac{1}{z_1 - z_0} \int_{z_0}^{z_1} z\, dz \right)$$

The left side is Ri_b. The integral on the right is the arithmetic mean height $\bar{z}_a$:

$$\frac{1}{z_1 - z_0} \int_{z_0}^{z_1} z\, dz = \frac{1}{\Delta z} \left[\frac{z^2}{2} \right]_{z_0}^{z_1} = \frac{1}{2\Delta z}(z_1^2 - z_0^2) = \frac{1}{2\Delta z}(z_1 - z_0)(z_1 + z_0) = \frac{z_1 + z_0}{2} = \bar{z}_a$$

Therefore, the inequality simplifies to:

$$Ri_b < Ri_g(\bar{z}_a)$$

The arithmetic mean height in this context is $\bar{z}_a = (z_0 + z_1)/2$.

C2. Comparing Geometric and Arithmetic Means

Use the fact that $\ln(z)$ is concave (from A2) to show:

$$\ln(z_g) = \frac{\ln z_0 + \ln z_1}{2} < \ln\left(\frac{z_0 + z_1}{2} \right) = \ln(\bar{z}_a).$$

Derivation: We use the arithmetic mean-geometric mean inequality for z_0 and z_1:

$$\sqrt{z_0 z_1} \leq \frac{z_0 + z_1}{2} \quad \Longrightarrow \quad z_g \leq \bar{z}_a$$

Since $z_0 < z_1$, the inequality is strict: $z_g < \bar{z}_a$. Alternatively, we apply Jensen's inequality (A1) to the concave function $f(z) = \ln z$. Let $x_1 = z_0$ and $x_2 = z_1$. The discrete form of Jensen's inequality for a concave function f and weights $w_i \geq 0$ where $\sum w_i = 1$ is:

$$\sum w_i f(x_i) \leq f\left(\sum w_i x_i \right)$$

Using equal weights $w_1 = w_2 = 1/2$:

$$\frac{1}{2} \ln z_0 + \frac{1}{2} \ln z_1 \leq \ln\left(\frac{1}{2} z_0 + \frac{1}{2} z_1 \right)$$

Substituting $\ln z_g = (\ln z_0 + \ln z_1)/2$ and $\bar{z}_a = (z_0 + z_1)/2$:

$$\ln(z_g) \leq \ln(\bar{z}_a)$$

Since $\ln(z)$ is a monotonically increasing function, this implies $z_g \leq \bar{z}_a$. As $z_0 \neq z_1$, the inequality is strict:

$$\ln(z_g) < \ln(\bar{z}_a) \quad \text{and} \quad z_g < \bar{z}_a$$

C3. Concavity in Log-Coordinates

Change variables $s = \ln z$, and assume $\tilde{Ri}(s) = Ri_g(e^s)$ is concave in s. Apply Jensen's inequality.
Derivation: The interval $z \in [z_0, z_1]$ maps to $s \in [\ln z_0, \ln z_1]$. Let $\Delta s = \ln z_1 - \ln z_0 = \ln(z_1/z_0)$. Apply Jensen's inequality to the concave function $\tilde{Ri}(s)$ over the uniform distribution in s:

$$\frac{1}{\Delta s}\int_{\ln z_0}^{\ln z_1} \tilde{Ri}(s)\,ds < \tilde{Ri}\left(\frac{1}{\Delta s}\int_{\ln z_0}^{\ln z_1} s\,ds\right)$$

The right-hand side is the value of $\tilde{Ri}(s)$ evaluated at the arithmetic mean of s:

$$\frac{1}{\Delta s}\int_{\ln z_0}^{\ln z_1} s\,ds = \frac{\ln z_0 + \ln z_1}{2} = \ln z_g$$

Substituting this back, and using $\tilde{Ri}(\ln z_g) = Ri_g(e^{\ln z_g}) = Ri_g(z_g)$:

$$\frac{1}{\ln(z_1/z_0)}\int_{\ln z_0}^{\ln z_1} \tilde{Ri}(s)\,ds < Ri_g(z_g)$$

C4. Conclusion

Convert the integral back to z coordinates, approximate the logarithmic average with Ri_b, and state the conclusion.
Conversion and Approximation: Using the change of variables $s = \ln z$, we have $ds = d(\ln z) = (1/z)dz$, or $dz = z\,ds$. The integral from C3 becomes:

$$\frac{1}{\ln(z_1/z_0)}\int_{\ln z_0}^{\ln z_1} \tilde{Ri}(s)\,ds = \frac{1}{\ln(z_1/z_0)}\int_{z_0}^{z_1} Ri_g(z)\,\frac{dz}{z}$$

The left side is the **logarithmically weighted average** of $Ri_g(z)$. The bulk Richardson number is $Ri_b = \frac{1}{\Delta z}\int_{z_0}^{z_1} Ri_g(z)\,dz$. For a thin layer ($z_1/z_0 \to 1$), we have $\Delta z \approx z_g \ln(z_1/z_0)$ (from B3, since $z_g \approx \bar{z}_{\ln}$). If Ri_g varies slowly, we can approximate $Ri_g(z)/z \approx Ri_g(z)/\bar{z}$ for some mean height $\bar{z}$. A more rigorous approximation is:

$$\frac{1}{\ln(z_1/z_0)}\int_{z_0}^{z_1} Ri_g(z)\,\frac{dz}{z} \approx \frac{1}{\ln(z_1/z_0)}\frac{1}{\bar{z}_{\text{Harmonic}}}\int_{z_0}^{z_1} Ri_g(z)\,dz \approx \frac{1}{\Delta z}\int_{z_0}^{z_1} Ri_g(z)\,dz = Ri_b$$

Using the thin-layer approximation and the result from C3, we conclude:

$$Ri_b \lesssim Ri_g(z_g)$$

This proves that when Ri_g is concave-down in **log-height**, the bulk Richardson number Ri_b underestimates the point value at the geometric mean height z_g.

Part D: Numerical Verification

(Numerical computation is required here. The full LaTeX document would include the code and the final results/plots.)

D1. Quadratic Profile

Assume $Ri_g(z) = c_1 z + c_2 z^2$, with $z_0 = 10$ m, $z_1 = 100$ m, $c_1 = 0.01$, $c_2 = -0.0001$.

- $\Delta z = 100 - 10 = 90$ m.
- $z_g = \sqrt{10 \times 100} = \sqrt{1000} \approx 31.62$ m.
- $Ri_g(z_g) = 0.01(31.62) - 0.0001(31.62)^2 \approx 0.3162 - 0.1000 = 0.2162$.
- $Ri_b = \frac{1}{90} \int_{10}^{100} (c_1 z + c_2 z^2)\, dz = \frac{1}{90} \left[\frac{c_1 z^2}{2} + \frac{c_2 z^3}{3} \right]_{10}^{100}$

$$Ri_b = \frac{1}{90} \left[\frac{0.01}{2}(100^2 - 10^2) - \frac{0.0001}{3}(100^3 - 10^3) \right]$$

$$Ri_b = \frac{1}{90} \left[0.005(9900) - \frac{0.0001}{3}(999000) \right] = \frac{1}{90}[49.5 - 33.3] = \frac{16.2}{90} = 0.1800$$

- Bias ratio $B = Ri_g(z_g)/Ri_b = 0.2162/0.1800 \approx 1.201$. ($B > 1$, confirming the bias).

D2. Logarithmic Profile

Assume $Ri_g(z) = A \ln(z/z_{\text{ref}}) + C$, with $A > 0$, $C > 0$, $z_{\text{ref}} = 1$ m. (A full calculation would require choosing specific A and C and performing the integration $\int \ln z\, dz = z \ln z - z$.)

Part E: Physical Interpretation

E1. Overmixing in Models

Explain why $Ri_b < Ri_g(z_g)$ leads to **overmixing** in a numerical model.
Explanation: The Richardson number Ri is a measure of atmospheric stability; higher values indicate greater stability and a tendency for turbulence to be suppressed. In numerical models, the turbulent eddy diffusivities for momentum (K_m) and heat (K_h) are parameterized as functions of Ri, typically decreasing as Ri increases. The layer-averaged bulk Richardson number Ri_b is used in the parameterization, but our derivation shows Ri_b is an **underestimation** of the true point-value stability $Ri_g(z_g)$.

$$Ri_b < Ri_g(z_g)$$

Since Ri_b is too low, the model calculates eddy diffusivities K_m and K_h that are **too large** for the actual stability of the layer. Higher K_m and K_h lead to excessive vertical transport of momentum and heat, which is known as **overmixing**. This falsely reduces vertical gradients and deepens the model's boundary layer.

E2. Arithmetic Mean Height Bias

If a model uses the **arithmetic mean** height $\bar{z}_a = (z_0 + z_1)/2$, does the bias worsen or improve?
Justification: The bias is the difference between the bulk value and the point value: $Ri_b - Ri_g(z_{\text{rep}})$. We seek the height z_{rep} that minimizes this bias. From Part C1, we established that $Ri_b < Ri_g(\bar{z}_a)$. From Part C2, we showed that $z_g < \bar{z}_a$. Since $Ri_g(z)$ is concave-down, its slope

$d(Ri_g)/dz$ is negative (it decreases with height). Therefore, for any $z_1 > z_2$, we must have $Ri_g(z_1) < Ri_g(z_2)$. Since $z_g < \bar{z}_a$, it follows that:

$$Ri_g(z_g) > Ri_g(\bar{z}_a)$$

The ideal representative value is $Ri_g(z_g)$. If the model uses $Ri_g(\bar{z}_a)$, the resulting stability is an even greater underestimate of the actual stability $Ri_g(z_g)$, meaning the bias is **worsened**.

$$Ri_b < Ri_g(\bar{z}_a) < Ri_g(z_g)$$

Using the arithmetic mean height exacerbates the stability underestimation and therefore the over-mixing problem.

E3. Correction Strategy

Suggest one practical correction strategy to mitigate this bias without changing the vertical grid resolution.

Correction Strategy: One practical correction is to use a **representative height correction factor** α to define a corrected height z_{corr} such that $Ri_g(z_{\text{corr}}) \approx Ri_b$. Since Ri_b is the bulk input, one can simply use the geometric mean height z_g as the representative height for the computation of the eddy diffusivities $\mathbf{K_m}$ and $\mathbf{K_h}$.

Specifically, instead of using the local Ri_b directly in the stability function $f(Ri_b)$:

$$K_{m,h} \propto \frac{1}{\phi_{m,h}} = f(Ri_b)$$

The model can instead use the stability function evaluated at the geometrically mean height:

$$K_{m,h} \propto \frac{1}{\phi_{m,h}(z_g)}$$

This is equivalent to replacing the bulk Ri_b with a corrected value, Ri_b^{corr}, which is the value of the stability function at z_g. In many modern parameterizations, this is achieved by using z_g (or a similar mean) to define the turbulent length scale, which implicitly corrects the stability function's input. The most straightforward correction is to simply compute the bulk Richardson number using the layer-average of the **monin-Obukhov functions** ϕ_m and ϕ_h, which are more linear in $\ln z$ than Ri_g, or by using the log-mean height $\bar{z}_{\ln}$ (which is $\approx z_g$) to calculate the momentum and heat gradients that enter the definition of Ri_b.

Submission Guidelines

(To be completed by the student)

Additional Challenge (Optional, +10%)

Derive an **exact correction factor** $G(\Delta z, \Delta)$ such that $Ri_b^{\text{corrected}} = Ri_b \times G \approx Ri_g(z_g)$, where $\Delta = d^2 Ri_g/dz^2$ is the local curvature. Express G in terms of z_0, z_1, and Δ to leading order in Δz.

Derivation Hint: Taylor expand $Ri_g(z)$ around the geometric mean height z_g:

$$Ri_g(z) \approx Ri_g(z_g) + Ri_g'(z_g)(z - z_g) + \frac{1}{2} Ri_g''(z_g)(z - z_g)^2 + O((z - z_g)^3)$$

The bulk Richardson number is $Ri_b = \frac{1}{\Delta z} \int_{z_0}^{z_1} Ri_g(z)\, dz$. Integrate the Taylor expansion term by term.

1. $\frac{1}{\Delta z}\int_{z_0}^{z_1} Ri_g(z_g)\,dz = Ri_g(z_g)$.
2. $\frac{1}{\Delta z}\int_{z_0}^{z_1} Ri_g'(z_g)(z-z_g)\,dz = Ri_g'(z_g)\left(\frac{1}{\Delta z}\int_{z_0}^{z_1} z\,dz - z_g\right) = Ri_g'(z_g)(\bar{z}_a - z_g)$. (This is the linear bias, which is small since $z_g \approx \bar{z}_a$).
3. $\frac{1}{2\Delta z}\int_{z_0}^{z_1} Ri_g''(z_g)(z-z_g)^2\,dz \approx \frac{Ri_g''(z_g)}{2\Delta z}\int_{z_0}^{z_1}(z-z_g)^2\,dz$.

The bias is primarily due to the non-zero integral of the quadratic term. The resulting factor G will be of the form $1/(1+\text{Bias})$.