Combinatorial approach of certain generalized Stirling numbers

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Abstract

A combinatorial methods are used to investigate some properties of certain generalized Stirling numbers, including explicit formula and recurrence relations. Furthermore, an expression of these numbers with symmetric function is deduced.

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1 Introduction

The Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$, the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}$ and the Third kind $\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \rfloor$ which are know as Lah numbers. These numbers are coefficients in the expression of the raising-falling factorial of x, see [9, pp 204],

$$(x)^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \tag{1}$$

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)^{\underline{k}}, \tag{2}$$

$$(x)^{\overline{n}} = \sum_{k=0}^{n} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor (x)^{\underline{k}},$$

where $(x)^{\overline{n}} = x(x+1)\cdots(x+n-1)$ and $(x)^{\underline{n}} = x(x-1)\cdots(x-n+1)$, with $(x)^{\overline{0}} = (x)^{\underline{0}} = 1$.

These three sequences satisfy respectively the following recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}, \tag{3}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}, \tag{4}$$

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rfloor + (n+k-1)\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor. \tag{5}$$

with $\left[\begin{smallmatrix}n\\0\end{smallmatrix}\right]=\left\{\begin{smallmatrix}n\\0\end{smallmatrix}\right\}=\left\lfloor\begin{smallmatrix}n\\0\end{smallmatrix}\right\rfloor=\delta_{n,0}$, where δ is the Kronecker delta, and for $n\neq0$ $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]=\left\{\begin{smallmatrix}n\\k\end{smallmatrix}\right\}=\left\lfloor\begin{smallmatrix}n\\k\end{smallmatrix}\right\rfloor=0$ when $k\notin\{0,\ldots,n\}$.

The Stirling numbers of the both kinds and the Lah numbers have a combinatorial interpretation, see for instance [4, 2, 1], as follows

- $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]$ count the number of permutations of the set $\{1,\ldots,n\}$ with k cycles,

- $\left\{\begin{smallmatrix}n\\k\end{smallmatrix}\right\}$ count the number of partitions of the set $\{1,\ldots,n\}$ into k subsets,

- $\left\lfloor\begin{smallmatrix}n\\k\end{smallmatrix}\right\rfloor$ count the number of partitions of the set $\{1,\ldots,n\}$ into k ordered lists.

Many generalizations of the Stirling numbers were developed using: combinatorial approach, see Howard [10], Broder [6]; the falling factorial, see Belbachir et *al* [3], Benoumhani [5]; or generating functions, see Carlitz [8, 7].

As a unified approach to the generalized Stirling numbers, Hsu and Shiue [11], defined a Stirling-type pair $\left\{S^1\left(n,k\right),S^2\left(n,k\right)\right\}\equiv\left\{S\left(n,k;\alpha,\beta,r\right),S\left(n,k;\beta,\alpha,-r\right)\right\}$ by the inverse relations

$$(x|\alpha)^{\underline{n}}=\sum_{k=0}^{n}S^1\left(n,k\right)\left(x-r|\beta\right)^{\underline{k}}, \tag{6}$$

$$(x|\beta)^{\underline{n}}=\sum_{k=0}^{n}S^2\left(n,k\right)\left(x+r|\alpha\right)^{\underline{k}}, \tag{7}$$

where α,β and r are arbitrary parameters with $(\alpha,\beta,r)\neq(0,0,0)$ and $(x|\theta)^{\underline{n}}$ is the generalized n-th falling factorial of x with increment θ defined by

$$\begin{aligned}(x|\theta)^{\underline{n}}&=&x\left(x-\theta\right)\left(x-2\theta\right)\cdots\left(x-\theta\left(n-1\right)\right),\quad n\geq1,\\(x|\theta)^{\underline{0}}&=&1.\end{aligned}$$

The numbers $S\left(n,k;\alpha,\beta,r\right)$ satisfy the following triangular recurrence relation

$$S\left(n,k;\alpha,\beta,r\right)=S\left(n-1,k-1;\alpha,\beta,r\right)+\left(\beta k-\alpha n+r\right)S\left(n-1,k;\alpha,\beta,r\right),$$

with $S\left(0,0;\alpha,\beta,r\right)=S\left(n,n;\alpha,\beta,r\right)=1$ and $S\left(1,0;\alpha,\beta,r\right)=r$.

Tsylova [12] gave a partial statistical interpretation of the numbers $A_{\beta,\alpha}\left(k,n\right)$ which coincide with the special case $S\left(n,k;\alpha,\beta,0\right)$, also Belbachir and Bousbaa [3] define the translated Whitney numbers of the three kinds using a combinatorial approach: the translated Whitney numbers of the first kind, denoted $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]^{(\alpha)}$, count the number of permutations of n elements with k cycles such that the elements of each cycle can be colored in α ways except the smallest one; the translated Whitney numbers of second kind, denoted $\left\{\begin{smallmatrix}n\\k\end{smallmatrix}\right\}^{(\alpha)}$, count the partitions of the set $\{1,2,...,n\}$ into k subsets such the elements of each subset can mute in α ways except the smallest one; and the translated Whitney-Lah numbers, denoted $\left\lfloor\begin{smallmatrix}n\\k\end{smallmatrix}\right\rfloor^{(\alpha)}$, count the number of ways to distribute the set $\{1,2,...,n\}$ into k ordered lists such that the elements of each list can mute with α ways, except the dominant one. These three kinds of numbers correspond, respectively, to $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]^{(\alpha)}=S(n,k;-\alpha,0,0)$, $\left\{\begin{smallmatrix}n\\k\end{smallmatrix}\right\}^{(\alpha)}=S(n,k;0,\alpha,0)$ and $\left\lfloor\begin{smallmatrix}n\\k\end{smallmatrix}\right\rfloor^{(\alpha)}=S(n,k;-\alpha,\alpha,0)$. Motivated by the previews works, our aim is to find combinatorial meaning for the numbers $S\left(n,k,\alpha,\beta,0\right)$ which are denoted by $\left\lfloor\begin{smallmatrix}n\\k\end{smallmatrix}\right\rfloor^{\alpha,\beta}$. We start by giving the combinatorial interpretation in the first section, also we prove combinatorially the explicit formula using the inclusion and exclusion principle. In sections 2 and 3, we derive some recurrence relations and an expression using the symmetric functions. We give, in the last section, a convolution identity.

2 Combinatorial Interpretation and explicit formula

Let $\Omega_{n,k}$ to be the set of all possible ways to distribute n elements, denoted $1, 2, \ldots, n$, into k ordered *no empty* lists, one element at a time, such that:

(i) we assign a weight of β to the head list,

(ii) the remaining elements in the list have weight α,

(iii) the first element putted in the list have a weight 1.

Given a distribution $\varepsilon \in \Omega_{n,k}$, we define the weight of ε, denoted by $w(\varepsilon)$, to be the product of the weights of its elements. Since the total weight of $\Omega_{n,k}$ is given by the sum of weights of all distributions.

To clarify the interpretation given above, we illustrate the case when $n = 3$ and $k = 1$. First, the weight of the first element is $w(\{(1)\}) = 1$. Next, there are two ways to add the second element either after the first element $\varepsilon_1 = \{(1,2)\}$ with $w(\varepsilon_1) = \alpha$ or before the first element $\varepsilon_2 = \{(2,1)\}$ with $w(\varepsilon_2) = \beta$. Finally, to add the third element

$$\varepsilon_{11} = \{(1,2,3)\} \dashrightarrow w(\varepsilon_{11}) = \alpha^2, \quad \varepsilon_{12} = \{(1,3,2)\} \dashrightarrow w(\varepsilon_{12}) = \alpha^2,$$
$$\varepsilon_{13} = \{(3,1,2)\} \dashrightarrow w(\varepsilon_{13}) = \alpha\beta, \quad \varepsilon_{21} = \{(2,1,3)\} \dashrightarrow w(\varepsilon_{21}) = \alpha\beta,$$
$$\varepsilon_{22} = \{(2,3,1)\} \dashrightarrow w(\varepsilon_{22}) = \alpha\beta, \quad \varepsilon_{23} = \{(3,2,1)\} \dashrightarrow w(\varepsilon_{23}) = \beta^2.$$

Thus, the total weight of $\Omega_{3,1}$ is $2\alpha^2 + 3\alpha\beta + \beta^2 = (\alpha + \beta)(2\alpha + \beta)$.

This approach suggests us the following result.

Theorem 1 *For any non-negative integers n and k, we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{\alpha,\beta} = \sum_{\varepsilon \in \Omega_{n,k}} w(\varepsilon). \tag{8}$$

It is clear that

$$\left\lfloor \begin{matrix} n \\ 1 \end{matrix} \right\rfloor^{\alpha,\beta} = \prod_{j=1}^{n-1} (j\alpha + \beta). \tag{9}$$

The following result gives an explicit formulation of $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor^{\alpha,\beta}$. The proof is based on inclusion-exclusion principle. Without loose the generality, we can suppose $\alpha, \beta \in$ (or $\in \Re$).

Theorem 2 *For any non-negative integers n, k, we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{\alpha,\beta} = \frac{1}{\beta^k k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\beta(k-j)|\alpha)^{\overline{n}}. \tag{10}$$

where $(\beta(k-j)|\alpha)^{\overline{n}} = \beta(k-j)(\beta(k-j) + \alpha) \cdots (\beta(k-j) + (n-1)\alpha)$.

Proof. Let ϕ denote the set of all possible ways to distribute n elements, denoted $1, 2, \ldots, n$, into k ordered lists (labeled and not necessary non-empty), one element at a time, such that:

- the first element putted in the list have a weight β,

- we assign a weight of β to the element inserted as head list,

- the remaining elements in the list have a weight α.

The total weight of set ϕ is the sum of all weights of all the distributions. There are three differences with the initial interpretation: the first one concerns the no empty lists and the second one concerns the weight β assigned to the first element putted in each list and the third one is the order between the lists. the two last differences will be considered at the end of the proof.

Now, let Δ be the subset of ϕ which have non-empty lists. We want to count the total weight of subset Δ.

For j $(1 \leq j \leq k)$, let A_j be the subset of k labeled lists of ϕ such that the j-th list is empty. Then

$$\Delta = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k},$$

where

$$\overline{A_j} = \phi \backslash A_j \quad \text{and} \quad \left| \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \right| = |\phi| - |A_1 \cup A_2 \cup \cdots \cup A_k|.$$

Applying inclusion-exclusion principle, we get

$$|\Delta| = |\phi| - \sum_{j=1}^{k} (-1)^j \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} \left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j} \right|. \tag{11}$$

We compute the general term $\sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} \left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j} \right|$, for a fixed j, there are $\binom{k}{j}$ ways to select j empty lists from k. And to distribute n elements in the remaining $k-j$ lists, so the first element have $k-j$ choices with the weight β which gives a total weight of $\beta(k-j)$, the second one have a weight of $\beta(k-j)+\alpha$ coming from the $(k-j)$ choices as head list with weight β or after the inserted element with weight α and so on until the last element which has a weight of $\beta(k-j)+(n-1)\alpha$. So, the total weight of this distribution is $\beta(k-j)(\beta(k-j)+\alpha)\cdots(\beta(k-j)+(n-1)\alpha) = (\beta(k-j)|\alpha)^{\overline{n}}$. Thus

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} \left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j} \right| = \binom{k}{j} (\beta(k-j)|\alpha)^{\overline{n}},$$

and we get

$$|\Delta| = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\beta(k-j)|\alpha)^{\overline{n}}.$$

We divide by $k!$ to avoid the repeated permutations and by β^k to give the k first elements inserted in the lists the weight 1. $\qquad \square$

3 Recurrence relations

In this section, we give combinatorial proofs of the three types of recurrence relations : the triangular, the horizontal and the vertical recurrence relation.

Theorem 3 *The numbers $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor^{\alpha,\beta}$ satisfy the following triangular recurrence relation*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{\alpha,\beta} = \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rfloor^{\alpha,\beta} + (\alpha(n-1)+\beta k) \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor^{\alpha,\beta}, \tag{12}$$

where $\left\lfloor \begin{smallmatrix} n \\ n \end{smallmatrix} \right\rfloor^{\alpha,\beta} = 1$ and $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor^{\alpha,\beta} = 0$ for $k \in \{0, \ldots, n\}$.

Proof. We count the total distribution weight of the set $\{1, 2, \ldots, n\}$ into k ordered, non-empty, lists according to the situation of the last element $\{n\}$.

- If $\{n\}$ is a singleton (with weight 1), the remaining $n-1$ elements have to be distribute into $k-1$ ordered lists with weight $\left\lfloor{n-1\atop k-1}\right\rfloor^{\alpha,\beta}$.

- If $\{n\}$ is not a singleton, the element n is in one of the k lists with some other elements. Total weight of distributing set $\{1,2,\ldots,n-1\}$ into k ordered no empty lists is $\left\lfloor{n-1\atop k}\right\rfloor^{\alpha,\beta}$, and there are $n-1$ choices to insert the element n after any of the elements $1,2,\ldots,n-1$ with weight α and k choices to insert the element n as head list with weight β. Hence, the weight is $(\alpha(n-1)+\beta k)\left\lfloor{n-1\atop k}\right\rfloor^{\alpha,\beta}$. $\square$

In particular, for $(-\alpha,1)$ and $(-1,\beta)$, the Theorem 3 will reduce to triangular recurrence relation for degenerate Stirling numbers [7]. Furthermore, for $(\alpha,0),(0,\beta)$ and (α,α), we obtain the triangular recurrence relation of the translated Whitney numbers of the both kinds and of the Whitney-Lah numbers [3] respectively.

Next, we give a horizontal recurrence relation using combinatorial proof.

Theorem 4 *For non negative integers n,k, we have*

$$\left\lfloor{n\atop k}\right\rfloor^{\alpha,\beta} = \sum_{j=0}^{n-k} (-1)^j \left((k+1)\beta + n\alpha|\alpha\right)^{\overline{j}} \left\lfloor{n+1\atop k+j+1}\right\rfloor^{\alpha,\beta}. \tag{13}$$

Proof. We proceed by construction, the total weight of distributing a set $\{1,2,\ldots,n\}$ into k ordered lists can be obtained from a total weight of distributing the set $\{1,2,\ldots,n+1\}$ into $k+1$ ordered lists excluding the weight of distributions which does not contain element $n+1$ as singleton. Then, we obtain $\left\lfloor{n\atop k}\right\rfloor^{\alpha,\beta} = \left\lfloor{n+1\atop k+1}\right\rfloor^{\alpha,\beta} - ((k+1)\beta - n\alpha)\left\lfloor{n\atop k+1}\right\rfloor^{\alpha,\beta}$.

Now, $\left\lfloor{n\atop k+1}\right\rfloor^{\alpha,\beta}$ corresponds to the total weight of distribution a set $\{1,2,\ldots,n\}$ into $k+1$ ordered lists, it can be obtained from a total weight of distribution set $\{1,2,\ldots,n+1\}$ into $k+2$ ordered lists excluding the weight of distributions which does not contain element $n+1$ as singleton, which gives:

$$
\begin{aligned}
\left\lfloor{n\atop k}\right\rfloor^{\alpha,\beta} &= \left\lfloor{n+1\atop k+1}\right\rfloor^{\alpha,\beta} - ((k+1)\beta - n\alpha)\left(\left\lfloor{n+1\atop k+2}\right\rfloor^{\alpha,\beta} - ((k+2)\beta - n\alpha)\left\lfloor{n\atop k+2}\right\rfloor^{\alpha,\beta}\right) \\
&= \left\lfloor{n+1\atop k+1}\right\rfloor^{\alpha,\beta} - ((k+1)\beta - n\alpha)\left\lfloor{n+1\atop k+2}\right\rfloor^{\alpha,\beta} + ((k+1)\beta - n\alpha)((k+2)\beta - n\alpha)\left\lfloor{n\atop k+2}\right\rfloor^{\alpha,\beta} \\
&\vdots \\
&= \sum_{j=0}^{n-k}(-1)^j((k+1)\beta + n\alpha|\alpha)^{\overline{j}}\left\lfloor{n+1\atop k+j+1}\right\rfloor^{\alpha,\beta}.
\end{aligned}
$$

$\square$

The last theorem in this Section is a vertical recurrence relation with combinatorial proof.

Theorem 5 *Let n and k be non negative integers, we have*

$$\left\lfloor{n+1\atop k+1}\right\rfloor^{\alpha,\beta} = \sum_{i=k}^{n}(\alpha+\beta|\alpha)^{\overline{n-i}}\binom{n}{i}\left\lfloor{i\atop k}\right\rfloor^{\alpha,\beta}. \tag{14}$$

Proof. Let us consider the i $(k \leqslant i \leqslant n)$ elements not in the same list of the element $n+1$. We have $\binom{n}{i}$ ways to choose the i elements and the total weight to constitute the k lists is $\left\lfloor{i\atop k}\right\rfloor^{\alpha,\beta}$, the remaining $n-i+1$ elements belong to the same list and we have $\left\lfloor{n+1-i\atop 1}\right\rfloor^{\alpha,\beta} = \prod_{j=1}^{n-i}(j\alpha+\beta)$ (see 9). We conclude by summing. $\square$

For $(\alpha,\beta) = (1,0),(0,1)$ and $(1,1)$ we get the identities [6, eq. 30], [6, eq. 35] and [4, eq. 11] respectively.

4   Relation with symmetric functions

The generalized Stirling numbers $\left\lfloor {n+k \atop n} \right\rfloor^{\alpha,\beta}$, for fixed n, are the elementary symmetric functions of the numbers $1, \ldots, n$.

Theorem 6 *For non negative integers n, k, α, β, we have*

$$
\left\lfloor {n+k \atop n} \right\rfloor^{\alpha,\beta} = \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n} \prod_{j=1}^{k} \left((\alpha+\beta)\, i_j + \alpha\, (j-1) \right),
$$

$$
= \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n} (\alpha+\beta)\, i_1 \left((\alpha+\beta)\, i_2 + \alpha \right) \cdots \left((\alpha+\beta)\, i_k + \alpha\, (k-1) \right).
$$

Proof. The left hand side $\left\lfloor {n+k \atop n} \right\rfloor^{\alpha,\beta}$ counts total weight of distributing a set $1, 2, \ldots, n+k$ into n ordered non empty lists.

In the right hand side, we constitute n lists from the elements $1, \ldots, n$ (one by list having each one a weight 1). Now, we discuss the weight of the remaining elements $n+1, \ldots, n+k$.

To insert the element $n+1$ to a list i_1 ($1 \leq i_1 \leq n$), we must consider all the possible situations of the element already in the list i_1 and we distingue two situations:

(A1) first, the initial element holds in the list i_1 and we put the element $n+1$ before the initial one with a weight β or after it with a weight α.

(A2) second, move the initial element to one of the first $i_1 - 1$ lists with a weight $(\alpha+\beta)(i_1 - 1)$ and put the element $n+1$ in the list i_1 with a weight 1. Note that, we move the elements only from right to left to avoid the double counting of situations.

Thus from (A1) and (A2) the weight of the element $n+1$ is $(\alpha+\beta)\, i_1$. We sum over all the possible insertions of the element $n+1$, we get the total weight of the $1, \ldots, n+1$ elements as $\sum_{1 \leq i_1 \leq n} (\alpha+\beta)\, i_1$.

Now, to insert the element $n+2$, we consider the elements of the lists $1, \ldots, i_1$ as fixed ones due to the insertion of the previous element $n+1$ where we consider all the situations. We have two possibilities :

(B1) If we add the element $n+2$ to one of the lists $1, \ldots, i_1$ with weight $((\alpha+\beta)\, i_1 + \alpha)$.

(B2) Else, it belongs to a list i_2 ($i_1 + 1 \leq i_2 \leq n$), with weight $((\alpha+\beta)\, i_2 + \alpha)$ (indeed, the weight of element $n+2$ is $(\beta+\alpha)$ if it's inserted before or after the initial element of the list i_2 or $(\alpha+\beta)(i_2 - 1) + \alpha$ if it's inserted in the list i_2 and the initial elements of the list i_2 is moved to the previous lists).

Than from (B1) and (B2) the weight of the element $n+2$ is

$$
((\alpha+\beta)\, i_1 + \alpha) + \sum_{i_2 = i_1 + 1}^{n} (\alpha+\beta)\, i_2 + \alpha = \sum_{i_2 = i_1}^{n} (\alpha+\beta)\, i_2 + \alpha.
$$

Altogether, the weight of the elements $n+1$ and $n+2$ is

$$
\sum_{i_1 = 1}^{n} (\alpha+\beta)\, i_1 \sum_{i_2 = i_1}^{n} ((\alpha+\beta)\, i_2 + \alpha) = \sum_{1 \leq i_1 \leq i_2 \leq n} ((\alpha+\beta)\, i_1)\, ((\alpha+\beta)\, i_2 + \alpha).
$$

We carry on by the same process for the remaining $k-2$ elements. So, for the last element $n+k$ we consider the elements of the lists $1, \ldots, i_{k-1}$ as fixed ones, then the weight of the element $n+k$

is $(\alpha + \beta)\,(i_{k-1} - 1) + \alpha(k-1)$ if it's inserted in these lists. Or $(\alpha + \beta)\,i_k + \alpha(k-1)$ if it's inserted in a list i_k $(i_{k-1} + 1 \le i_k \le n)$. This gives the total weight of distributing a set $1, 2, \ldots, n+k$ into n ordered non empty lists.

$$\sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} ((\alpha + \beta)\,i_1)\,((\alpha + \beta)\,i_2 + \alpha) \cdots ((\alpha + \beta)\,i_k + \alpha\,(k-1)).$$

$\square$

Note that for $(\alpha, \beta) = (1, 0)\,,\ (0, 1)$ and $(1, 1)$ we get the identities [6, eq. 22], [6, eq. 23] and [4, eq. 5] respectively.

5 Convolution identity

In this section we proved some combinatorial convolution. The first one is a multinomial convolutional type identity.

Theorem 7 *The Generalized Stirling numbers satisfy*

$$\binom{k}{k_1, \ldots, k_p} \left\lfloor \frac{n}{k} \right\rfloor^{\alpha, \beta} = \sum_{l_1 + \cdots + l_p = n} \binom{n}{l_1, \ldots, l_p} \left\lfloor \frac{l_1}{k_1} \right\rfloor^{\alpha, \beta} \cdots \left\lfloor \frac{l_p}{k_p} \right\rfloor^{\alpha, \beta}.$$

Proof. We consider the weight of the partitions of the set $\{1, \ldots, n\}$ into k lists which is $\left\lfloor \frac{n}{k} \right\rfloor^{\alpha, \beta}$. We color the elements of the lists with p colors such that the elements of each k_i $(1 \le i \le p)$ lists have the same color, there are $\binom{k}{k_1, \ldots, k_p}$ possibilities to do. This is equivalent to choose the elements of same color then count the weight of there distribution into lists. So we choose each l_i elements that have the same color and we have $\binom{n}{l_1, \ldots, l_p}$ possibilities, then consider the weight of all the distributions of the l_i elements into k_i lists and we have $\left\lfloor \frac{l_i}{k_i} \right\rfloor^{\alpha, \beta}$. Summing over all possible values of l_i gives the result. $\square$

Theorem 8 *The Generalized Stirling numbers satisfy*

$$\left\lfloor \begin{matrix} k + m \\ k \end{matrix} \right\rfloor^{\alpha, \beta} = \sum_{j=0}^{s} \left(\sum_{k-j \le i_1 \le \cdots \le i_{s-j} \le k} \prod_{l=0}^{s-j-1} (\alpha + \beta)\,i_{l+1} + \alpha\,(m - (s - j - l)) \right) \left\lfloor \begin{matrix} k + m - s \\ k - j \end{matrix} \right\rfloor^{\alpha, \beta}.$$

Proof. Let us consider the s $(0 \le s \le k)$ last elements of the set $\{1, 2, \ldots, k, \ldots, k+m\}$, we constitute the last j lists $(0 \le j \le s)$ from these elements. Note that, the weight of distributed the first $k+m-s$ elements into $k - j$ lists is $\left\lfloor \begin{matrix} k+m-s \\ k-j \end{matrix} \right\rfloor^{\alpha, \beta}$. Now, to constitute the j remaining lists, we pick j elements from the s ones all with weight 1 and discuss the insertion of the subset of the remaining $s-j$ elements to the k lists. To insert the first element there are two cases:

1) if we add it in the one of the first $k-j$ lists, then it has the weight $(\alpha + \beta)\,(k - j) + \alpha\,(m - (s - j))$ (In fact, $\beta\,(k - j)$ as head list or $\alpha\,(k - j) + \alpha m - (s - j)$ after each elements).

2) Else, we add it to a list i_1 $(k - j + 1 \le i_1 \le k)$, we have to discuss two other cases:

a) the initial element of the list i_1 stay in the list i_1 so the weight to insert the element $(\alpha + \beta)$.

b) the initial element moves to one of the previous $i_1 - 1$ lists with weight $(\alpha + \beta)\,(i_1 - 1) + \alpha\,(m - (s - j))$.

Thus, we sum over all the possible insertions in the list i_1, we get the weight $\sum_{k-j+1 \le i_1 \le k} (\alpha + \beta)\,i_1 + \alpha\,(m - (s - j))$.

From 1) and 2) we get the weight of the first element

$$\sum_{k-j \le i_1 \le k} (\alpha + \beta)\, i_1 + \alpha\,(m-(s-j)).$$

To insert the second element of the subset, we consider the elements of the lists $1, \ldots, i_1$ as fixed ones due to the insertion of the previous element where we consider all the situations. We have two situations :

a') if we add it to the lists $1, \ldots, i_1$ with weight $(\alpha + \beta)\, i_1 + \alpha\,(m-(s-j)+1)$.

b') else, it belongs to a list i_2 $(i_1 + 1 \le i_2 \le k)$, with weight $(\alpha + \beta)\, i_2 + \alpha\,(m-(s-j)+1)$, that gives $\sum_{i_1+1 \le i_2 \le n} (\alpha + \beta)\, i_2 + \alpha\,(m-(s-j)+1)$.

Thus, from a') and b') we get

$$\sum_{k-j \le i_1 \le i_2 \le n} ((\alpha + \beta)\, i_1 + \alpha\,(m-(s-j)))\,((\alpha + \beta)\, i_2 + \alpha\,(m-(s-j)+1)),$$

by the same way we carry on for the remaining $s-j-2$ elements of the subset, which gives the total weight

$$\sum_{k-j \le i_1 \le i_2 \le \cdots \le i_{s-j} \le k} ((\alpha + \beta)\, i_1 + \alpha\,(m-(s-j))) \cdots ((\alpha + \beta)\, i_{s-j} + \alpha\,(m-1)),$$

then by summing over all possible values of j we get the result. $\qquad\square$

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