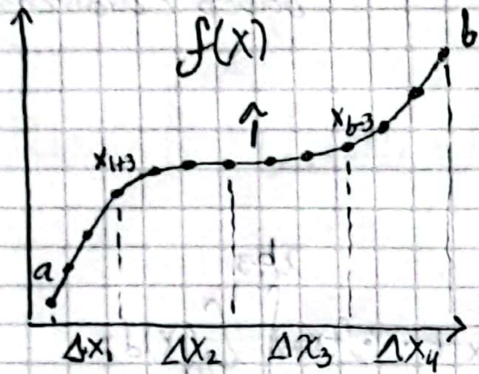
Deducción Regla de Simpson 3/8 para Subintervalos

Sea $f(x)$ La función a integrar:

$$\int_a^b f(x) \approx \int_a^{x_{i+3}} f(x)\,dx + \dots + \int_{x_{b-3}}^{b} f(x)\,dx$$



La integral es la suma de las integrales en intervalos de 4 puntos, donde podemos aproximar la función en cada intervalo por la interpolación de Lagrange

$$\int_{x_i}^{x_{i+3}} f(x) \approx f(x_i)\int_{x_i}^{x_{i+3}} L_i(x)\,dx + \dots + f(x_n)\int_{x_i}^{x_{i+3}} L_n(x)\,dx$$

Vamos a deducir la fórmula para cada subintervalo:

$$L_i(x) = \frac{(x-x_{i+1})(x-x_{i+2})(x-x_{i+3})}{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i+3})}, \quad L_{i+1}(x) = \frac{(x-x_i)(x-x_{i+2})(x-x_{i+3})}{(x_{i+1}-x_i)(x_{i+1}-x_{i+2})(x_{i+1}-x_{i+3})}$$

$$L_{i+2}(x) = \frac{(x-x_i)(x-x_{i+1})(x-x_{i+3})}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})(x_{i+2}-x_{i+3})}, \quad L_{i+3}(x) = \frac{(x-x_i)(x-x_{i+1})(x-x_{i+2})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x-x_{i+2})}$$

Si definimos $h = x_{j+1} - x_j$ y representamos cada x dentro del intervalo $[x_i, x_{i+3}]$ como $x = x_i + th$

Podemos establecer cada diferencia $x - x_j = x - x_i - th$

$x - x_i = th, \quad x - x_{i+1} = x - x_i - h = th - h$

$x - x_{i+2} = x - x_i - 2h = th - 2h, \quad x - x_{i+3} = x - x_i - 3h = th - 3h$

Podemos escribir cada polinomio como:

$$L_i(x) = \frac{h^3(t-1)(t-2)(t-3)}{-6h^3} = \frac{(t-1)(t-2)(t-3)}{-6}$$

$$\mathcal{L}_{i+1} = \frac{h^3 t(t-2)(t-3)}{2h^3} = \frac{t(t-2)(t-3)}{2}$$

$$\mathcal{L}_{i+2} = \frac{h^3 t(t-1)(t-3)}{-2h^3} = \frac{t(t-1)(t-3)}{-2}$$

$$\mathcal{L}_{i+3} = \frac{h^3 t(t-1)(t-2)}{6h^3} = \frac{t(t-1)(t-2)}{6}$$

Ya tenemos las bases Cardinales

Para Cada Subintervalo tenemos:

$$\int_{x_i}^{x_{i+3}} f(x)\,dx = \int_{x_i}^{x_{i+3}} P(x)\,dx \rightarrow$$ Un polinomio aproximador con la base de Lagrange.

$$\int_{x_i}^{x_{i+3}} f(x)\,dx = f(x_i)\int_{x_i}^{x_{i+3}} L_i(x)\,dx + f(x_{i+1})\int_{x_i}^{x_{i+3}} L_{i+1}(x)\,dx + \int_{x_i}^{x_{i+3}} L_{i+2}(x)\,dx \cdot f(x_{i+2})$$

$$+ f(x_{i+3})\int_{x_i}^{x_{i+3}} L_{i+3}(x)\,dx$$

Vamos a realizar un Cambio de Variable:

Si $x = x_i$, Según $x = x_i + th \rightarrow t = 0$

Si $x = x_{i+3} \rightarrow x_{i+3} - x_i = th \rightarrow 3h = th \rightarrow t = 3$

y Según la misma ecuación $dx = h\,dt$

Realizamos estos Cambios para Cada Integral:

$$f(x_i)\int_{x_i}^{x_{i+3}} L_i(x)\,dx = -\frac{f(x_i)h}{6}\int_0^3 (t-1)(t-2)(t-3)\,dt \rightarrow \text{Si expandimos...}$$

$$= \int_0^3 t^3 - 6t^2 + 11t - 6\,dt = \left(\frac{t^4}{4} - \frac{2t^3}{3} + \frac{11t^2}{2} - 6t\right)\Bigg|_0^3 = -\frac{9}{4}$$

$$f(x_i)\int_{x_i}^{x_{i+3}} L_i(x)\,dx = -\frac{f(x_i)h}{6}\cdot\left(-\frac{9}{4}\right) = \frac{f(x_i)h}{6}\left(\frac{9}{4}\right)$$

$$f(x_{i+1})\int_{x_i}^{x_{i+3}} L_{i+1}(x)\,dx = \frac{f(x_{i+1})h}{2}\int_0^3 t(t-2)(t-3)\,dt \rightarrow \text{Si expandimos...}$$

$$\int_0^3 t^3 - 5t^2 + 6t\,dt = \left(\frac{t^4}{4} - \frac{5t^3}{3} + 3t^2\right)\Bigg|_0^3 = \frac{9}{4}$$

$$f(x_{i+1})\int_{x_i}^{x_{i+3}} L_{i+1}(x) = \frac{f(x_{i+1})h}{2}\left(\frac{9}{4}\right)$$

$$f(x_{i+2})\int_{x_i}^{x_{i+3}} L_{i+2}(X) = \frac{f(x_{i+2})h}{-2}\int_0^3 t(t-1)(t-3)\,dt \Rightarrow \text{Si expandimos...}$$

$$\int_0^3 t^3-4t^2+3t\,dt = \left(\frac{t^4}{4}-\frac{4t^3}{3}+\frac{3t^2}{2}\right)\Bigg|_0^3 = -\frac{9}{4}$$

$$f(x_{i+2})\int_{x_i}^{x_{i+3}} L_{i+2}(X) = \frac{f(x_{i+2})h}{2}\left(\frac{9}{4}\right)$$

$$f(x_{i+3})\int_{x_i}^{x_{i+3}} L_{i+3}(X) = \frac{f(x_{i+3})h}{6}\int_0^3 t(t-1)(t-2)\,dt \Rightarrow \text{Si expandimos...}$$

$$\int_0^3 t^3-3t^2+2t\,dt = \left(\frac{t^4}{4}-t^3+t^2\right)\Bigg|_0^3 = \frac{9}{4}$$

$$f(x_{i+3})\int_{x_i}^{x_{i+3}} L_{i+3}(X)\,dx = \frac{f(x_{i+3})h}{6}\left(\frac{9}{4}\right)$$

$$\int_{x_i}^{x_{i+3}} f(x)\,dx \approx \frac{f(x_i)h}{6}\left(\frac{9}{4}\right)+\frac{f(x_{i+1})h}{2}\left(\frac{9}{4}\right)+\frac{f(x_{i+2})h}{2}\left(\frac{9}{4}\right)+\frac{f(x_{i+3})h}{6}\left(\frac{9}{4}\right)$$

$$\boxed{\approx \frac{3h}{8}\left(f(x_i)+3f(x_{i+1})+3f(x_{i+2})+f(x_{i+3})\right)}$$