Ampliació d'Algorísmia

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Problem 4. Weighted partial vertex cover with penalties. Consider a connected undirected graph $G = (V, E)$, having an edge cost $c_e \geq 0$, for each $e \in E$, and a weight $p_i \geq 0$, for each $i \in V$. The values c_e, for $e = (u, v)$, represent the cost of not covering edge e. The value p_u is an estimation of the cost of including vertex u in a set. We want to solve a variant of the vertex cover problem, the partial vertex cover problem in which we look for a set S of vertices that minimizes the total weight of its vertices, plus the total weight of the edges that S does not cover. Provide a 3-approximation for this problem.

Solution. To give an integer formulation, we define the variables x_v for all $v \in V$ and y_e for all $e \in E$.

Integer Programming Formulation

minimize

$$cost = v\_cost + e\_penalty$$

where

$$v\_cost = \sum_{v \in V} p_v \times x_v$$

$$e\_penalty = \sum_{e \in E} c_e \times (1 - y_e)$$

s.t.

$x_v \in \{0, 1\}$, $\forall v \in V$, whether vertex v is in S

$y_e \in \{0, 1\}$, $\forall e \in E$, whether edge e is covered

$x_i + x_j \geq y_{(i,j)}$, $\forall (i, j) \in E$, an edge is covered if either end is on S

We obtain an LP formulation by relaxing the constraints on x_v and y_e.

Linear Programming Formulation

minimize

$$cost = v\_cost + e\_penalty$$

where

$$v\_cost = \sum_{v \in V} p_v \times x_v$$

$$e\_penalty = \sum_{e \in E} c_e \times (1 - y_e)$$

s.t.

$$x_v \in [0,1], \quad \forall v \in V$$

$$y_e \in [0,1], \quad \forall e \in E$$

$$x_i + x_j \geq y_{(i,j)}, \quad \forall (i,j) \in E$$

Given a solution for the LP problem, $\{cost^*, x^*, y^*\}$, approximate the solution of the IP problem as $\{cost^A, x^A, y^A\}$, with $y^A_{(i,j)} = 1$ iff $y^*_e \geq \alpha$.

As $x^*_i + x^*_j \geq y^*_{(i,j)}$, it follows that, if $y^*_e \geq \alpha$, either $x^*_i \geq y^*_{(i,j)}/2$ and/or $x^*_j \geq y^*_{(i,j)}/2$. Equivalently, $x^*_i \geq \alpha/2$ and/or $x^*_j \geq \alpha/2$. Hence, we set $x^A_v = 1$ iff $x^*_v \geq \alpha/2$.

Approximate v_cost

$$
\begin{aligned}
cost^A_v &= p_v \times x^A_v \\
x^A_v &= 1 && \text{iff } x^*_v \geq \frac{\alpha}{2} \\
cost^A_v &= p_v \times 1 && \text{iff } x^*_v \geq \frac{\alpha}{2} \\
1 &\leq x^*_v \times \frac{2}{\alpha} && \text{iff } x^*_v \geq \frac{\alpha}{2} \\
cost^A_v &\leq p_v \times x^*_v \times \frac{2}{\alpha} \\
v\_cost^A &\leq \frac{2}{\alpha} \sum_{v \in V} p_v \times x^*_v
\end{aligned}
$$

Approximate *e_penalty*

$$
\begin{aligned}
penalty_e^A &= c_e \times (1 - y_e^A) && \\
y_e^A &= 1 && \text{iff } y_e^* \geq \alpha \\
penalty_e^A &= c_e \times (1 - 1) && \text{iff } y_e^* \geq \alpha \\
1 - 1 &\leq \frac{1 - y_e^*}{1 - \alpha} && \text{iff } y_e^* \geq \alpha \\
e\_penalty &\leq \frac{1}{1 - \alpha} \sum_{e \in E} c_e \times (1 - y_e^*) &&
\end{aligned}
$$

Approximate *cost*

$$
\begin{aligned}
&cost = v\_cost + e\_penalty \\
&cost \leq \frac{2}{\alpha} \sum_{v \in V} p_v \times x_v^* + \frac{1}{1 - \alpha} \sum_{e \in E} c_e \times (1 - y_e^*)
\end{aligned}
$$

This is optimized when:

$$
\frac{2}{\alpha} = \frac{1}{1 - \alpha} \implies \frac{\alpha}{2} = \frac{1 - \alpha}{1} \implies \alpha = 2 - 2\alpha \implies 3\alpha = 2 \implies \alpha = \frac{2}{3}
$$

And so, the cost of the approximation is, w.r.t. the LP cost:

$$
\begin{aligned}
&cost \leq \frac{2}{\alpha} \sum_{v \in V} p_v \times x_v^* + \frac{1}{1 - \alpha} \sum_{e \in E} c_e \times (1 - y_e^*) \\
&cost \leq 3 \sum_{v \in V} p_v \times x_v^* + 3 \sum_{e \in E} c_e \times (1 - y_e^*) \\
&cost \leq 3 \left(\sum_{v \in V} p_v \times x_v^* + \sum_{e \in E} c_e \times (1 - y_e^*) \right) \\
&cost \leq 3 \times cost^*
\end{aligned}
$$

And, as we're minimizing, the LP cost is lesser or equal than the integer OPT.

$$
cost \leq 3 \times cost^* \leq 3 \times \text{OPT}
$$

$\square$