Speeds of Electromagnetic Fields of a Gaussian Source

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Abstract Based on the closed form analytical solution to Maxwell's equations in response to a Gaussian source, formulas for the propagation speeds of the electric field and the magnetic field are derived. The formulas tell us that for this source the propagation speeds rise quickly from 0 to the standard light speed c, but never reach c. Therefore, when we measure the field speeds, without extremely high precision instruments we will get a constant speed of c. High precision calculations are made based on those formulas to study the characteristics of the propagation speeds.

Contents

Introduction

Given a point source of a Gaussian function, we can get the closed-form analytical solutions to Maxwell's equations, thus, we can analytically study the speed of propagation of electromagnetic fields. Let's start by presenting the analytical electromagnetic field. For proof of this solution to Maxwell's equations, see [1].

Consider 1D fields using the x-axis. Maxwell's equations are

$\theta = ct$ $c = \frac{1}{\sqrt{\varepsilon\mu}}$ $\eta = \sqrt{\frac{\mu}{\varepsilon}}$	(1.0)
$\frac{\partial H_y(x,\theta)}{\partial\theta} = \frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x}$	(1.1)
$\frac{\partial E_z(x,\theta)}{\partial\theta} = \eta\frac{\partial H_y(x,\theta)}{\partial x} - \eta J_z(x,\theta)$	(1.2)

A Gaussian source is used to generate the electromagnetic field:

$J_z(x,\theta) = e^{-ax^2}; a > 0$	(1.3)
$H_y(x,0) = 0$ $E_z(x,0) = 0$	(1.4)

The solution to the above Maxwell's equations is

$\xi = \sqrt{a}\theta$	(1.5)
$\varrho = 2\sqrt{a}x$	(1.6)

$H_y(x,\theta) = \frac{1}{\sqrt{a}} e^{-ax^2} esinh(\xi,\varrho)_1$	(1.7)
$E_z(x,\theta) = -\eta \frac{1}{\sqrt{a}} e^{-ax^2} eicoshi(\xi,\varrho)$	(1.8)

Where $esinh(\xi,\varrho)_1$ and $eicoshi(\xi,\varrho)$ are hyper-exponential functions, see [1]. The two hyper-exponential functions are given by summations, as shown below.

$esinh(\xi,\varrho)_1 = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2(n+1)}}{(2(n+1))!} \sum_{k=0}^{n} (-1)^k p_{0,n,k} \varrho^{2k+1}$	(1.9)
$eicoshi(\xi,\varrho) = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n+1}}{(2n+1)!} \sum_{k=0}^{n} (-1)^k q_{0,n,k} \varrho^{2k}$	(1.10)

Where $p_{0,n,k}$ is "odd binomial coefficients", and $q_{0,n,k}$ is "even binomial coefficients", as given below. Collectively we call them "double binomial coefficients".

$p_{0,n,k} = \frac{(2n+1)!}{(n-k)!\,(2k+1)!}$	(1.11)
$q_{0,n,k} = \frac{(2n)!}{(n-k)!\,(2k)!}$	(1.12)

I'll use both analytical methods and numerical calculations to investigate the speeds of these fields. First, let's get a feeling of how these fields propagate in space over time.

Propagation of the electromagnetic field

Since we have a closed-form analytical solution to Maxwell's equations, we may calculate the fields precisely. From such precise calculations, not estimations, we can get a visual and exact view of the electromagnetic field propagating in space over time. We can also see the dynamics of the propagation speeds visually.

To make calculations in long ranges of time and space, high performance computing (HPC) is needed. I assembled a desktop computer with an Intel 24 core CPU (Core i-9 14900K) and a Nvidia 3036 core GPU (GeForce 4096 TI). I spent a lot of effort on the GPU. But I could not get good HPC performance from the GPU. On reading some reports from other people, including a report from University of Tsukuba ([2]), I gave up my efforts to try to use GeForce GPU. Even though [2] is a report from 10 years ago, its results match what I got with my newer CPU and GPU. I ended up relying on Intel's CPU to make the calculations. Boost multiprecision library [3] was used to make calculations in decimal precisions of 2000 to 8000 digits. When different numbers of digits are used and the calculation results do not change, we know the calculation results are reliable.

Fig.1. and Fig.2. show the propagations of the magnetic field and the electric field, respectively.

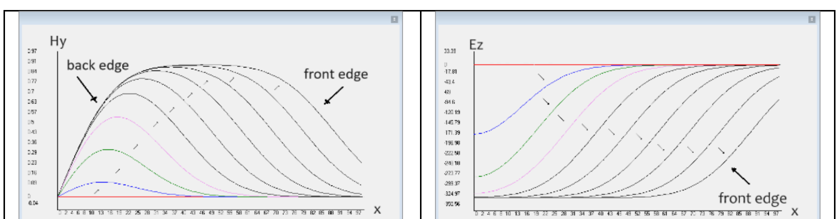

Fig.1. propagation of $H_y(x,\theta)$ | Fig.2. propagation of $E_z(x,\theta)$

Each curve is a snapshot of the field at one time. The next curve on the right side of each curve shows the propagation of the field during the time interval between the two curves. The propagations are visually demonstrated by the moving of the front edges; I use small arrows to mark the directions of the moving of the front edges.

The same time interval is used in Fig.1. and Fig.2., so, the closer the two curves the slower the speed of the propagation. Let's find out math formulas for the speeds.

Front edge speed formulas

From Fig.1 and Fig.2 we can see that the front edges occur at the times when the magnitudes of the fields start to drop. That is, the front edges are at the time when the fields lost their accelerations in space. Let's show field derivatives with respect to space.

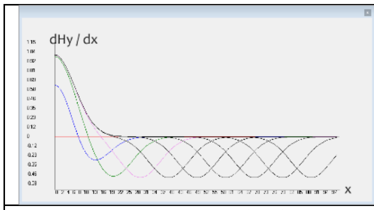

Fig.3. propagation of $\partial H_y(x,\theta)/\partial x$

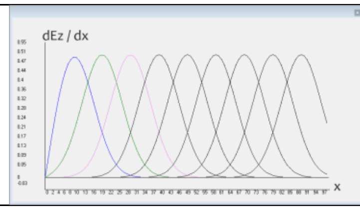

Fig.4. propagation of $\partial E_z(x,\theta)/\partial x$

The crests in Fig.3 and Fig.4 show the space and the time where/when the front edges begin.

So, the moving speed of the front edges are the moving speeds of crests for field derivatives with respect to space:

$$\frac{\partial^2 F(x,t)}{\partial x^2} = 0$$

Front edge speed of the electric field

In [1] while proving the solution to Maxwell's equations, it gives

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} = e^{-ax^2} eisinh(\xi,\varrho)$$

By (i.9) in [1]

$$eisinh(\xi,\varrho) = e^{-\xi^2}\sinh(\varrho\xi)$$

We have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} = e^{-(ax^2+\xi^2)}\sinh(\varrho\xi) \tag{3.1}$$

Using function identities in [1], we can get following result.

$$\frac{1}{\eta}\frac{\partial^2 E_z(x,\theta)}{\partial x^2} = \sqrt{a}e^{-(ax^2+\xi^2)}(2\xi\ \cosh(\varrho\xi) - \varrho\sinh(\varrho\xi)) \tag{3.2}$$

The moving of the front edge is given by

$$\frac{\partial^2 E_z(x,\theta)}{\partial x^2} = 0 \tag{3.3}$$

From (3.2) and (3.3) we have

$$\varrho\sinh(\varrho\xi) = 2\xi\ \cosh(\varrho\xi) \tag{3.4}$$

Take derivative on both sides of (3.4) with respect to ξ, we have

$$\frac{d\varrho}{d\xi} = -\frac{\varrho}{\xi}\cdot\frac{\varrho^2 - 4\xi^2 - 2}{\varrho^2 - 4\xi^2 + 2} \tag{3.5}$$

Because

$$\frac{dx}{d\theta} = \frac{1}{2}\frac{d\varrho}{d\xi}$$

We have the speed formula for the electric field:

$$\frac{dx}{d\theta} = -\frac{\varrho}{2\xi}\cdot\frac{\varrho^2 - 4\xi^2 - 2}{\varrho^2 - 4\xi^2 + 2} \tag{3.6}$$

Let's make a drawing of the crest curve by (3.4).

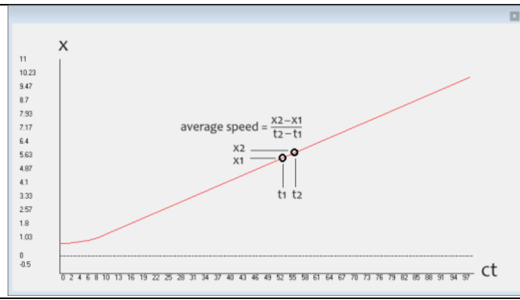

Fig.5.Crest curve by (3.4)

The average moving speed of the front edge can be calculated from the crest curve, as it is shown in Fig.5. Let's see the calculations, and also the speeds calculated by formula (3.6).

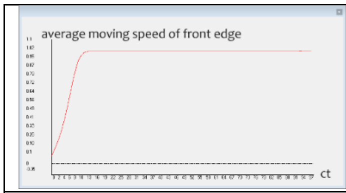

Fig.6. average moving speed of the front edge

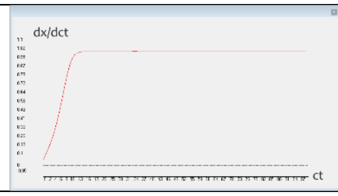

Fig.7. moving speed of the front edge by formula (3.6)

Compare Fig.6 and Fig.7 we can see they are quite similar, showing the correctness of the speed formula (3.6).

The time range in Fig.6 and Fig.7 is

$$t \in \left(0, \frac{10}{c}\right)$$

$$\left.\frac{dx}{dt}\right|_{t=\frac{10}{c}} = (0.99999999999995559\ldots)c$$

The speed calculations were carried out in 4000 digits. In a very short time the speed rises very close to c. When we physically measure the speed of the electric field, without extremely high precision measurement instruments, we will see that the speed is a constant c. But formula (3.6) tells us that the speed actually is never c.

Front edge speed of the magnetic field

In [1] while proving the solution to Maxwell's equations, it gives

$$\frac{\partial H_y(x,\theta)}{\partial x} = e^{-ax^2}\left(-esinhi(\xi,\varrho)_{1,1,1} + 2esinhi(\xi,\varrho)_1\right)$$

By (i.18) in [1]

$$e^{-\xi^2}\cosh(\varrho\xi) + 2\,esinhi(\xi,\varrho)_1 - esinhi(\xi,\varrho)_{1,1,1} = 1$$

We have

$$\frac{\partial H_y(x,\theta)}{\partial x} = e^{-ax^2}\left(1 - e^{-\xi^2}\cosh(\varrho\xi)\right) \tag{4.1}$$

Using function identities in [1], we can get following result.

$$\frac{\partial^2 H_y(x,\theta)}{\partial x^2} == \sqrt{a}e^{-ax^2}\left(e^{-\xi^2}(\varrho\cosh(\varrho\xi) - 2\xi\sinh(\varrho\xi)) - \varrho\right) \tag{4.2}$$

The moving of the front edge is given by

$$\frac{\partial^2 H_y(x,\theta)}{\partial x^2} = 0 \tag{4.3}$$

From (4.2) and (4.3) we have

$$\varrho\cosh(\varrho\xi) - 2\xi\sinh(\varrho\xi) = \varrho\, e^{\xi^2} \tag{4.4}$$

Take derivative on both sides of (4.4) with respect to ξ, we have

$$\frac{d\varrho}{d\xi} = 2\frac{4\varrho\xi e^{\xi^2} + (4\xi^2 - \varrho^2 + 2)\sinh(\varrho\xi)}{(2 - 4\xi^2 + \varrho^2)\cosh(\varrho\xi) - (\varrho^2 + 2)e^{\xi^2}} \tag{4.5}$$

Because

$$\frac{dx}{d\theta} = \frac{1}{2}\frac{d\varrho}{d\xi}$$

We have the formula for the speed of the magnetic field:

$$\frac{dx}{d\theta} = \frac{4\varrho\xi e^{\xi^2} + (4\xi^2 - \varrho^2 + 2)\sinh(\varrho\xi)}{(2 - 4\xi^2 + \varrho^2)\cosh(\varrho\xi) - (\varrho^2 + 2)e^{\xi^2}} \tag{4.6}$$

Let's make a drawing of the crest curve by (4.4).

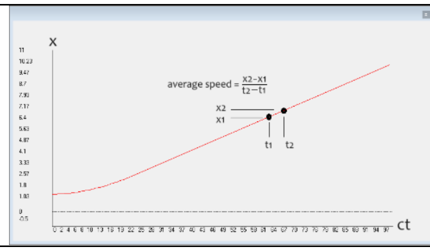

Fig.8. Crest curve by (4.4)

The average moving speed of the front edge can be calculated from the crest curve, as it is shown in Fig.8. Let's see this calculation and the speed calculated by formula (4.6).

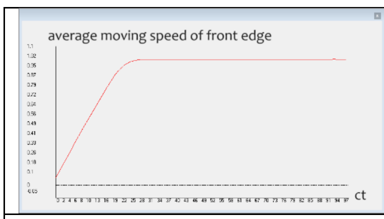

Fig.9. average moving speed of the front edge

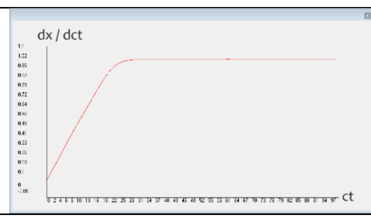

Fig.10. moving speed of the front edge by formula (4.6)

Compare Fig.9 and Fig.10 we can see they are quite similar, showing the correctness of the speed formula (4.6).

Fig.9 and Fig.10 show that when we physically measure the speed of the magnetic field, we will see that the speed is a constant c. But formula (4.6) tells us that the speed actually is never c.

Length of dynamic period

Fig. 7 shows that the speed of the electric field has a dynamic period followed by a stable period.

Fig.10 shows the same behavior of the speeds of the magnetic field.

Let's investigate the length of the dynamic periods. The only parameter we can change is the factor for the Gaussian function. In all the above figures, I am using

$$a = 1 \tag{5.1}$$

Let's examine $a\ in\ (0, \infty)$ to see how it affects the length of the dynamic periods.

For $a \to \infty, J_z(x,\theta) = e^{-ax^2} \to 0$. To keep the magnitudes of the output fields, let's change the source to

$$J_z(x,\theta) = \sqrt{a}e^{-ax^2} \tag{5.2}$$

Now the fields become

$$H_y(x,\theta) = e^{-\frac{1}{4}\varrho^2} esinh(\xi,\varrho)_1 \tag{5.3}$$

$$E_z(x,\theta) = -\eta e^{-\frac{1}{4}\varrho^2} eicoshi(\xi,\varrho) \tag{5.4}$$

By (5.2), we have

$$\int_0^{\infty} J_z(x,\theta)dx = \int_0^{\infty} e^{-(\sqrt{a}x)^2} d\sqrt{a}x = \frac{\sqrt{\pi}}{2}$$

Thus, we have

$$\lim_{a\to\infty} J_z(x,\theta) = \sqrt{\pi}\delta(x) \cong \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \quad (5.5)$$

where $\delta(x)$ is the Dirac delta function.

Following figures show the effects of value a on the lengths of the dynamic periods.

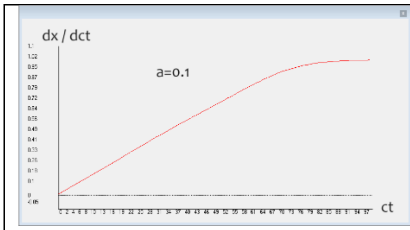

Fig.11. speed of Hy, a=0.1

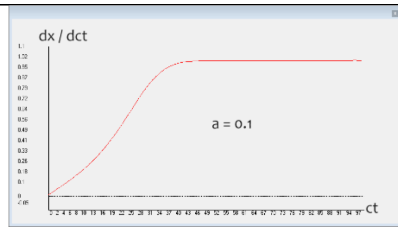

Fig.12. speed of Ez, a=0.1

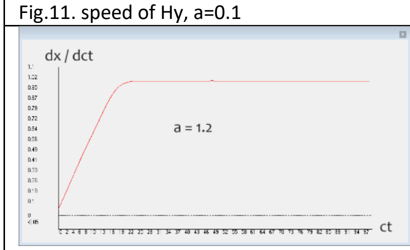

Fig.13. speed of Hy, a=1.2

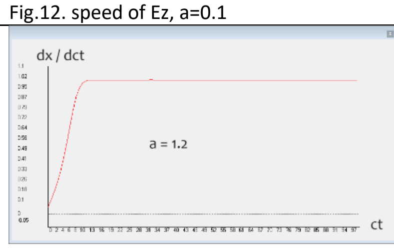

Fig.14. speed of Ez, a=1.2

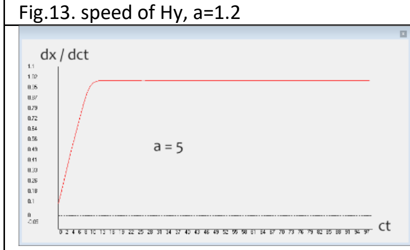

Fig.15. speed of Hy, a=5

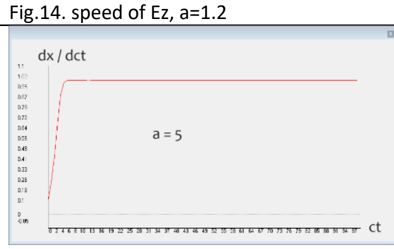

Fig.16. speed of Ez, a=5

From these drawings we can see that the larger value a the shorter the dynamic periods. Thus, it is reasonable to think that when $a \to \infty$ the lengths of the dynamic periods are 0. That is, the Dirac delta function generates a constant speed solution to Maxwell's equations.

Summary

The electric field and the magnetic field generated by a Gaussian source have following characteristics.

1. their propagation speeds are not constant.
2. their propagation speeds approach the standard light speed c when the time approaches infinity.
3. At lower ranges of time and space, their propagation speeds change greatly. We may call this range the dynamic period. Their speeds are instantaneous speeds.
4. At higher ranges of time and space, their propagation speeds do not change much, the speeds are very close to the standard light speed c. We may call this range the stable period. The speed c is their stable speeds.
5. The transition from the dynamic period to the stable period can be smooth or sharp, depending on the Gaussian factor.
6. The lengths of the dynamic periods depend on the Gaussian factor. The larger the factor the shorter the dynamic period.
7. When the Dirac delta function is used as the source, the length of the dynamic period is 0. This is my guess, not proved.

Research topics

1. Stable solution vs instantaneous solution to Maxwell's equations

Function limit. function $g(x,\theta)$ is a limit of function $f(x,\theta)$ if for any $\varepsilon > 0$ there are ε_x and ε_θ such that $|f(x,\theta) - g(x,\theta)| < \varepsilon$ for any $x > \varepsilon_x$ and $\theta > \varepsilon_\theta$.

The function limit can be written as

$$\lim_{\substack{x\to\infty \\ \theta\to\infty}} f(x,\theta) = g(x,\theta)$$

If we can find following function limits

$$S_{hy}(\xi,\varrho) = \lim_{\substack{\xi\to\infty \\ \varrho\to\infty}} e^{-\frac{1}{4}\varrho^2} esinh(\xi,\varrho)_1$$

$$S_{ez}(\xi,\varrho) = \lim_{\substack{\xi\to\infty \\ \varrho\to\infty}} e^{-\frac{1}{4}\varrho^2} eicoshi(\xi,\varrho)$$

then the speeds of $S_{hy}(\xi,\varrho)$ and $S_{ez}(\xi,\varrho)$ will be a constant c. These two functions are the stable solution to Maxwell's equations.

We may expect that the stable solutions are simpler than the instantaneous solutions. Probably, we might have

$$S_{hy}(\xi,\varrho) = F_{hy}(x-ct)$$

$$S_{ez}(\xi,\varrho) = F_{ez}(x-ct)$$

2. Dirac delta function vs length of dynamic period

From F.11 to F.16, it is reasonable to think that when $a\to\infty$ the lengths of the dynamic periods are 0. When $a\to\infty$ the Gaussian source becomes the Dirac delta function. We may try to mathematically prove that using the Dirac delta function as the source, the length of the dynamic period is 0, and we will get a constant speed solution to Maxwell's equations.

References

[1] David Wei Ge, A Closed Form Analytical Solution to Maxwell's Equations in Response to a Time Invariant Gaussian Source, August 2023, DOI: 10.13140/RG.2.2.15985.70245, http://dx.doi.org/10.13140/RG.2.2.15985.70245

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[3] The GNU Multiple Precision Arithmetic Library, 2023, https://gmplib.org/