Transient speeds of propagations of electromagnetic fields

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ABSTRACT

Maxwell's equations can form a dynamic system of input-driving-output. The outputs of a dynamic system, when driven by the inputs, inevitably go into a transient period and then approach a stable period. In the transient period, the speeds of the propagations of the outputs cannot be constant. In this report, analytical solutions to Maxwell's equations driven by sources are present to demonstrate the transient behavior of the propagations of the electromagnetic fields. From the analytical solutions, the propagation speeds of the electromagnetic fields can be analytically derived. For a field source with limit strength, the speed rises from 0 to *c* quickly but a transient period is required. The length of the transient period depends on the strength of the source. Numerical calculations using the speed formulas give a hint that a Dirac delta function could generate a constant propagation speed. Practically we see a wave of constant speed *c*, but mathematically the speed is not constant, if the strength of the source is not infinite.

Keywords: Maxwell's equations, electromagnetic field, electromagnetic wave, Electrodynamics, velocity of light

Contents

Introduction

It is well known that the velocity of light is a constant given by $c = 1/\sqrt{\varepsilon_0 \mu_0}$, see university textbooks:

...James Clerk Maxwell, in 1861, produced his electromagnetic theory of light. It now became possible to predict the numerical value of the speed of light for any given medium, in terms of measurable electric and magnetic properties of the medium. [1]: page 40

...Maxwell's electromagnetic theory-the same theory that extracted the correct value of the speed of light from the physics of basic electric and magnetic phenomena. [1]: page 13

...we see that $V_p = (\mu_0 \varepsilon_0)^{-1/2}$, which is equal to the velocity of light in free space c. [2]:page 31

The speed of light, c, has long been recognized as one of the fundamental constants of nature....It may be seen that the accuracies of the results for different photon energies differ widely. The most accurate determinations are for visible light and for microwaves of about 1 cm wavelength ...[1]: page 12

Einstein also teaches us:

light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. [3]

The principle of the constancy of the velocity of light is of course contained in Maxwell's equations [4]

In university classrooms, the above understanding is taught to be from source-free wave equations derived from Maxwell's equations:

we shall investigate solutions to the Maxwell equations in regions devoid of source… Sources must exist outside the regions of interest in order to produce fields in these regions. [2]: page 24

On page 24 of [2], the wave equation is derived from Maxwell's equations. On page 25 of [2], the wave equation is solved:

The wave equation it satisfies follows from (1.2.6) which becomes

$$\frac{\partial^2 E_x}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0$$

The simplest solution to (1.2.7) takes the form

$$\bar{E} = \hat{x} E_x(z,t) = \hat{x} E_0 \cos(kz - \omega t)$$

…the dispersion relation, must be satisfied:

$$k^2 = \mu_0 \varepsilon_0 \omega^2$$

From the above relation, we know that the wave speed is $c = 1/\sqrt{\varepsilon_0 \mu_0}$. Actually, any function of constant speed *c* is a solution to the wave equation:

$$E_x(z,t) = f(z - ct)$$

But there are issues in the above understanding.

- We do not have a conclusion that any solution to the wave equation is of constant speed *c*.
- A source outside a source-free region still has an impact on the region. It is incorrect to use source-free equations for the region.
- Solutions to wave equations do not necessarily satisfy Maxwell's equations

Because Einstein uses words "**of course**", the above understanding is taken for granted and hardly anyone would ask why Maxwell's equations give "*constancy of the velocity of light*" and pay attention to the above issues.

I gained some knowledge of control engineering while I was doing research on optimal control for anti-ship missiles. From the point of view of control engineering, $f(z - ct)$ is a stable solution, not a dynamic solution. It is not driven by an input. The concept of "constant speed" for dynamic systems is highly questionable due to the existence of transient periods.

From the point of view of control engineering, Maxwell's equations ([5], [2]:page 3) can be arranged to represent a typical situation of "input driving output", as shown below.

$$\begin{bmatrix} \nabla \times E + \frac{\partial B}{\partial t} \\ \nabla \times B - \frac{1}{c^2}\frac{\partial E}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\varepsilon_0 c^2} J(x,y,z,t) \end{bmatrix}; t \geq 0$$

$$\begin{bmatrix} E \\ B \end{bmatrix}(x,y,z,t)_{t=0} = \vec{0}$$

Where $J(x,y,z,t)$ is the input, $[E,B]^T(x,y,z,t)$ is the output. The minus sign in the equation serves as a negative feedback needed to make the dynamic system stable.

Field source $J(x,y,z,t)$ can be seen as the source of light; usually it is a point source, for example, a single frequency light source could be $J(x,y,z,t) = e^{-a(x^2+y^2+z^2)}\sin(\omega t)$.

From the perspective of control engineering, no matter what $J(x,y,z,t)$ is, the propagation of states $[E,B]^T(x,y,z,t)$ will go into a transient period first and then approach a stable state. In the transient period, the propagation speed cannot be constant.

Therefore, if $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is understood as that the velocity of electromagnetic wave is constant then it could be a misunderstanding of Maxwell's equations. This misunderstanding can be easily demonstrated if the solution $[E,B]^T(x,y,z,t)$ in response to $J(x,y,z,t)$ is available.

Due to the difficulty of solving Maxwell's equations analytically, this misunderstanding has not been warned and noticed.

In control engineering, a dynamic system is often described by ordinary differential equations. It is possible to get analytical solutions and get a deep understanding of the characteristics of the solutions. When I was doing research for fault detection in dynamic systems, the states of the systems I worked on were accurately available, and could be compared with sensor signals, even the signals had to go through FFT to filter out noises, see [6], [7].

However, solutions to Maxwell's equations are hard to get without multi-level integrations, see Oleg D. Jefimenko (1992) [8], Valery Yakhno (2020) [9]. The generic characteristics of the solutions are also not understood as much as presented in [10] for ordinary differential equations. Powerful analysis tools, such as the root-locus analysis, in control engineering [10], are scarce for Maxwell's equations.

The favorable approaches adopted are to use potentials and wave equations, which are quite different from the approach of "input driving output" in control engineering. Let me briefly review them.

From Gauss' law $\nabla \cdot B = 0$ there exists a vector A such that

$$B = \nabla \times A$$

By Faraday's law $\frac{\partial B}{\partial t} + \nabla \times E = 0$, together with $B = \nabla \times A$ we have $\nabla \times \left(\frac{\partial A}{\partial t} + E\right) = 0$. Thus, there exists a scalar φ such that $\frac{\partial A}{\partial t} + E = -\nabla\varphi$. Thus we have

$$E = -\nabla\varphi - \frac{\partial A}{\partial t}$$

Thus the 6 field elements are represented by 4 potential elements. The problem is simplified but arbitrariness is introduced. For an arbitrary scalar function ψ, the following potentials give the same electromagnetic fields as given by A and φ:

$$\tilde{A} = A + \nabla\psi$$

$$\tilde{\varphi} = \varphi - \frac{\partial \psi}{\partial t}$$

From the perspective of control engineering, the potential equations can be written as "input driving output":

$$c^2\nabla(\nabla \cdot A) - c^2\nabla^2 A + \nabla\left(\frac{\partial \varphi}{\partial t}\right) + \frac{\partial^2 A}{\partial t^2} = \frac{1}{\varepsilon_0}J$$

The above is a set of 3 equations for 4 variables. So, we need one more equation. It is called to fix the gauge. The Coulomb gauge is to only use transversal potential, that is, let the longitudinal potential be 0:

$$\nabla \cdot A = 0$$

The Lorenz gauge is to use the following condition

$$\nabla \cdot A + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$$

Apply Gauss' law $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ the Lorenz gauge gives the following wave equations for the potentials:

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2\varphi = \frac{\rho}{\varepsilon_0}$$

$$\frac{\partial^2 A}{\partial t^2} - c^2\nabla^2 A = \frac{1}{\varepsilon_0}J$$

The wave equations for the electromagnetic fields, as shown below, can be obtained by taking temporal derivatives on the curl part of Maxwell's equations.

$$\frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = \nabla^2 H + \nabla \times J$$

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \nabla^2 E + \frac{1}{\varepsilon_0}\nabla\int \nabla \cdot J dt - \mu_0\frac{\partial J}{\partial t}$$

Because the Laplacian operator ∇^2 does not mix the vector elements,

$$\nabla^2 A = \begin{bmatrix} \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \\ \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} \\ \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \end{bmatrix} = \begin{bmatrix} \nabla^2 A_x \\ \nabla^2 A_y \\ \nabla^2 A_z \end{bmatrix}$$

the wave equations become independent single variable equations, and thus are much easier to solve than Maxwell's equations. M. Lax, et al (1975) [11] get solutions for paraxial waves in this approach. For given spectrums, Roger L. Garay-Avendaño, et al (2014) [12], Michel Zamboni-Rached, et al (2017) [13] and J. Nobre-Pereira (2024) [14] solve the integrations by getting Fourier transformation coefficients, and thus get analytical solutions. Roger L. Garay-Avendaño, et al [12] choose Lorenz gauge, get a vector potential, and analytical solutions to Maxwell's equations are obtained.

Not all solutions to wave equations satisfy Maxwell's equations. For example, the solution for a differential equation

$$\frac{dy(t)}{dt} = \sin(t)\,; y(0) = 0$$

is

$$y(t) = -\cos(t) + 1$$

The solution for its second order equation

$$\frac{d^2y(t)}{dt^2} = \cos(t)\,; y(0) = 0$$

is

$$y(t) = -\cos(t) + At + 1$$

Where *A* is an arbitrary constant; it is not necessarily the original solution.
However, to solve the problem of "input-driving-output", arbitrariness is not allowed. From the perspective of control engineering, introducing arbitrariness is unimaginable; for controlling a flying object, such arbitrariness is disastrous.

I am not saying that arbitrariness is always a bad thing merely for evading math difficulties. Innovations involve arbitrariness. The approaches popularly taken produced important results.
Arbitrariness led to important engineering applications:

> *Applications for these structured non-diffracting beams are many, such as in optical tweezers, optical atom guiding, medical imaging, medical treatments, remote sensing, optical communications, military applications and so on. [14].*

Arbitrariness led to fundamental theory of quantum mechanics, the Yang-Mills theory:

> *We define 'isotopic gauge' as an arbitrary way of choosing the orientation of the isotopic spin axes at all spacetime points, in analogy with the electromagnetic gauge which represents an arbitrary way of choosing the complex phase factor of a charged field at all space-time points. [15].*

I am just trying to show that the approaches popularly taken do not help in solving a math problem of getting analytical solutions to Maxwell's equations for given initial values and field sources.

Let me summarize the above survey:
A century-old difficulty in mathematics hides a century-old misunderstanding of Maxwell's equations.

My research on higher order FDTD led to an arbitrary order algorithm, which gives infinite order of precision equivalent to an analytical solution to Maxwell's equations, in open space, without using integrations, without involving potentials and gauges, without wave equations, and without arbitrariness. I am not going to show the readers how to use my technology to solve Maxwell's equations in this report. I am going to present examples of analytical solutions I thus obtained, and show that the speeds of field propagations are not constant when the fields are driven by sources.
The mathematical proofs and calculation configurations are presented in **Methods** for the readers to verify and reproduce the results presented in this report.

Results

1D wave for single frequency moving source

For simplicity, let's consider 1D fields using the x-axis. By applying a time scale, Maxwell's equations are given below.

$$c = \frac{1}{\sqrt{\varepsilon\mu}}; \eta = \sqrt{\frac{\mu}{\varepsilon}};\ \theta = ct \tag{1.0}$$

$$\frac{\partial H_y(x,\theta)}{\partial\theta} = \frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} \tag{1.1}$$

$$\frac{\partial E_z(x,\theta)}{\partial\theta} = \eta\frac{\partial H_y(x,\theta)}{\partial x} - \eta J_z(x,\theta) \tag{1.2}$$

Using my technology, I solved Maxwell's equations for a single-frequency moving source:

$$J_z(x,\theta) = e^{-a(x-v\theta)^2 + i\omega\theta}; a > 0$$

Where v/c is the speed of source-moving, and ω/c is the angular frequency of the wave. Here I only provide the solution for the cosine source shown below.

$$J_z(x,\theta) = e^{-a(x-v\theta)^2}\cos(\omega\theta)\,;\ a > 0, \omega > 0 \tag{1.3}$$

$$\varrho = 2\sqrt{a}x \tag{1.4}$$

$$\xi = \omega\theta \tag{1.5}$$

$$\varphi = \frac{\sqrt{a}}{\omega} \tag{1.6}$$

$$\begin{aligned} H_y(x,\theta,v) = {} & \frac{e^{-ax^2}}{\sqrt{a}} \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n+2}}{(2n+2)!} \sum_{m=0}^{n} \varphi^{2m+2} \sum_{h=0}^{n-m} \binom{2n-2m}{2h} (v\varphi)^{2h} \sum_{k=0}^{m+h} (-1)^k \varrho^{2k+1} p_{0,m+h,k} \\ & - \frac{e^{-ax^2}}{\omega} \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n+3}}{(2n+3)!} \sum_{m=0}^{n} \varphi^{2m+1} \sum_{h=0}^{n-m} \binom{2n-2m+1}{2h+1} (v\varphi)^{2h+1} \sum_{k=0}^{m+h+1} (-1)^k \varrho^{2k} q_{0,m+h+1,k} \end{aligned} \tag{1.7}$$

$$\begin{aligned} E_z(x,\theta,v) = {} & -\eta \frac{e^{-ax^2}}{\sqrt{a}} \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n+2}}{(2n+2)!} \sum_{m=0}^{n} \varphi^{2m+1} \sum_{h=0}^{n-m} \binom{2n-2m+1}{2h+1} (v\varphi)^{2h+1} \sum_{k=0}^{m+h} (-1)^k \varrho^{2k+1} p_{0,m+h,k} \\ & - \eta \frac{e^{-ax^2}}{\omega} \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n+1}}{(2n+1)!} \sum_{m=0}^{n} \varphi^{2m} \sum_{h=0}^{n-m} \binom{2n-2m}{2h} (v\varphi)^{2h} \sum_{k=0}^{m+h} (-1)^k \varrho^{2k} q_{0,m+h,k} \end{aligned} \tag{1.8}$$

Speeds of wave propagations

Let's show the field propagations calculated by formulas (1.7) and (1.8), using field snapshots at two times.

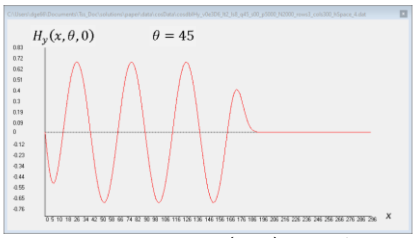

Fig.1. magnetic wave $H_y(x,\theta,v)$ at $\theta = 45$

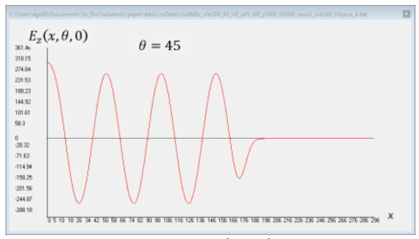

Fig.2. electric wave $E_z(x,\theta,v)$ at $\theta = 45$

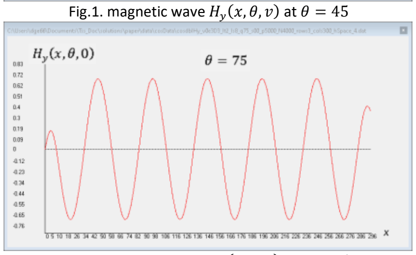

Fig.3. magnetic wave $H_y(x,\theta,v)$ at $\theta = 75$

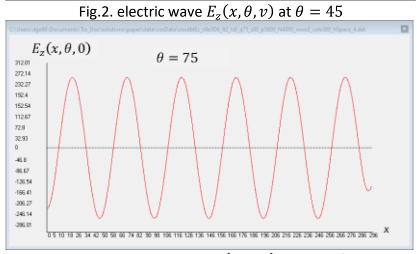

Fig. 4. electric wave $E_z(x,\theta,v)$ at $\theta = 75$

Fig.1 shows a snapshot of the magnetic wave at time $\theta = 45$.
Fig.3 shows a snapshot of the magnetic wave at time $\theta = 75$. It shows the moving forward of the magnetic wave.

Fig.2 shows a snapshot of the electric wave at time $\theta = 45$.
Fig.4 shows a snapshot of the electric wave at time $\theta = 75$. It shows the moving forward of the electric wave.

These figures show the waves of fields propagating in space over time. We may use the moving of the crests to represent the propagation of the wave.
For a wave of single frequency with phase velocity v_p,

$$f(x,t) = \sin\left(\omega\left(x - v_p t\right)\right) \tag{1.9}$$

its crests can be calculated by

$$\frac{\partial f(x,t)}{\partial x} = 0 \tag{1.10}$$

We have

$$\omega \cos\left(\omega\left(x - v_p t\right)\right) = 0$$

$$\omega\left(x - v_p t\right) = \frac{\pi}{2}$$

We get the propagation speed:

$$\frac{dx}{dt} = v_p$$

The above speed calculation process is valid for any perfectly periodic waves $f\left(x - v_p t\right)$.

The waves shown in Fig.1 to Fig.4 are not perfectly periodic, and their phase speeds are not constant. Using (1.10) to calculate the propagation speeds, the waves do not need to be perfectly periodic, and we can calculate the instantaneous phase speeds. Also, we can see that the following formula gives the same result.

$$\frac{\partial^n f(x,t)}{\partial x^n} = 0; n > 0 \tag{1.11}$$

In some situations, (1.11) has to be used.

Using the above process, we can get the speed for the propagation of electric field for a stationary single frequency source:

$$\frac{d\varrho}{d\xi} = -\varrho \frac{\xi^2 + \sum_{n=1}^{\infty} \frac{(-1)^n \xi^{2n+2}}{(2n+2)!} \left(2n + 2 + \sum_{m=1}^{n} \sum_{k=1}^{m} (-1)^k \varrho^{2k} \frac{(2m+2)!}{(m-k)!\,(2k)!}\right)}{\sum_{n=1}^{\infty} \frac{(-1)^n \xi^{2n+3}}{(2n+3)!} \left(\sum_{m=1}^{n} \sum_{k=1}^{m} (-1)^k \varrho^{2k} \frac{(2m+2)!}{(m-k)!\,(2k-1)!}\right)}$$

$$\frac{\xi^3}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \xi^{2n+3}}{(2n+3)!} \left(2n + 2 + \sum_{m=1}^{n} \sum_{k=1}^{m} (-1)^k \varrho^{2k} \frac{(2m+2)!}{(m-k)!\,(2k)!}\right) = 0$$

For the above formulas $\varphi = 1$ is used. The speed does not look constant.

Next, I'll present a simpler case.

1D wave for Gaussian source

Let's consider the most simple case: $v = 0$ and $\omega = 0$. That is enough to show the points of this report. The source becomes a Gaussian function:

$$J_z(x, \theta) = e^{-ax^2}; a > 0 \tag{2.1}$$

$$H_y(x, 0) = 0;\ E_z(x, 0) = 0 \tag{2.2}$$

The solution to Maxwell's equations (1.1) and (1.2), using the source (2.1) and the initial values (2.2), is given below.

$$\xi = \sqrt{a}\theta \tag{2.3}$$

$$\varrho = 2\sqrt{a}x \tag{2.4}$$

$$H_y(x, \theta) = \frac{1}{\sqrt{a}} e^{-ax^2} esinh(\xi, \varrho)_1 \tag{2.5}$$

$$E_z(x, \theta) = -\eta \frac{1}{\sqrt{a}} e^{-ax^2} eicoshi(\xi, \varrho) \tag{2.6}$$

Where $esinh(\xi, \varrho)_1$ and $eicoshi(\xi, \varrho)$ are hyper-exponential functions. The two hyper-exponential functions are given by summations, as shown below.

$$esinh(\xi, \varrho)_1 = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2(n+1)}}{(2(n+1))!} \sum_{k=0}^{n} (-1)^k p_{0,n,k} \varrho^{2k+1} \tag{2.7}$$

$$eicoshi(\xi, \varrho) = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n+1}}{(2n+1)!} \sum_{k=0}^{n} (-1)^k q_{0,n,k} \varrho^{2k} \tag{2.8}$$

Where $p_{0,n,k}$ is named "odd binomial coefficients", and $q_{0,n,k}$ is named "even binomial coefficients", as given below. Collectively I name them "double binomial coefficients".

$$p_{0,n,k} = \frac{(2n+1)!}{(n-k)!\,(2k+1)!} \tag{2.9}$$

$$q_{0,n,k} = \frac{(2n)!}{(n-k)!\,(2k)!} \tag{2.10}$$

The proof of the above solution is given in **Methods**.

I'll use both analytical methods and numerical calculations to investigate the speeds of these fields. First, let's get a feeling of how these fields propagate in space over time.

Propagation of the electromagnetic field

Fig.5. and Fig.6. show the propagations of the magnetic field and the electric field, respectively.

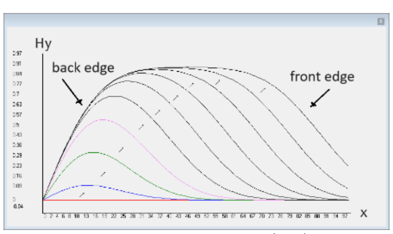

Fig.5. propagation of $H_y(x, \theta)$

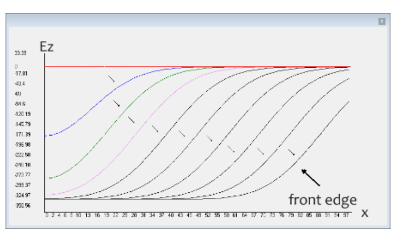

Fig.6. propagation of $E_z(x, \theta)$

Each curve is a snapshot of the field at one time.
The next curve on the right side of each curve shows the propagation of the field during the time interval between the two curves.

The propagations are visually demonstrated by the moving of the front edges; I use small arrows to mark the directions of the moving of the front edges.

The same time interval is used in Fig.5. and Fig.6., so, the closer the two curves the slower the speed of the propagation. Let's find out math formulas for the speeds.

Speeds of field propagations

Let's show field derivatives with respect to space.

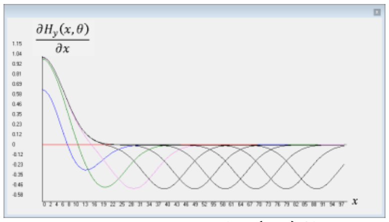

Fig.7. propagation of $\partial H_y(x,\theta)/\partial x$

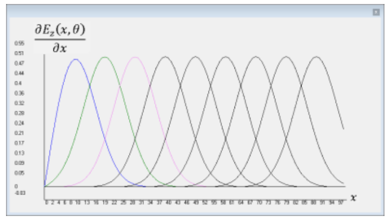

Fig.8. propagation of $\partial E_z(x,\theta)/\partial x$

The crests in Fig.7 and Fig.8 show where/when the fields get their maximum growths.

The crests in Fig.7 is given by

$$\frac{\partial^2 H_y(x,\theta)}{\partial x^2} = 0$$

From the above formula we can find out the speed formula of $\frac{dx}{d\theta}$ for $H_y(x,\theta)$.

The crests in Fig.8 is given by

$$\frac{\partial^2 E_z(x,\theta)}{\partial x^2} = 0$$

From the above formula we can find out the speed formula of $\frac{dx}{d\theta}$ for $E_z(x,\theta)$.

Speed of the electric field

See section "Derive speed of electric field" under **Methods** for the following speed formula:

$$\frac{dx}{d\theta} = -\frac{\varrho}{2\xi} \cdot \frac{\varrho^2 - 4\xi^2 - 2}{\varrho^2 - 4\xi^2 + 2} \tag{3.6}$$

The above speed formula looks quite strange. I had no idea what it meant until I made calculations with it. Let's see the speeds calculated by formula (3.6).

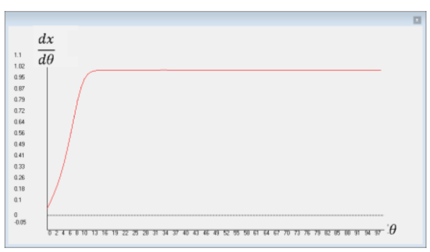

Fig.9. propagation speed of the electric field by formula (3.6)

The time range in Fig.9 is

$$t \in \left(0, \frac{100}{c}\right)$$

$$\left.\frac{dx}{dt}\right|_{t=\frac{10}{c}} = (0.99999999999995559 \ldots)c$$

In a very short time the speed rises very close to *c*. When we physically measure the speed of the electric field, without extremely high precision measurement instruments, we will see that the speed looks a constant *c*.

Formula (3.6) tells us that the speed actually is not a constant *c*. A transient period exists. It is the same behavior known by control engineers.

Such result gives a good enough approximation for assuming $\frac{dx}{dt} \equiv c$ in most cases.

Speed of the magnetic field

See section “Derive speed of magnetic field” under **Methods** for the following formula:

$$\frac{dx}{d\theta} = \frac{4\varrho\xi e^{\xi^2} + (4\xi^2 - \varrho^2 + 2)\sinh(\varrho\xi)}{(2 - 4\xi^2 + \varrho^2)\cosh(\varrho\xi) - (\varrho^2 + 2)e^{\xi^2}} \tag{4.6}$$

Let’s see the speed calculated by formula (4.6).

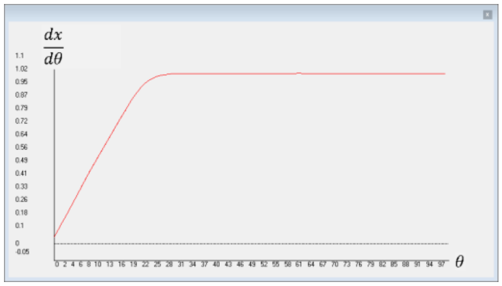

Fig.10. propagation speed of the magnetic field by formula (4.6)

Fig.10 shows that when we physically measure the speed of the magnetic field, we will see that the speed looks a constant *c*.

Formula (4.6) tells us that the speed actually is not a constant *c*. But it is a good enough approximation for assuming $\frac{dx}{dt} \equiv c$ in most cases.

Length of dynamic period

Fig. 9 shows that the speed of the electric field has a dynamic period followed by a stable period.
Fig. 10 shows the same behavior of the speeds of the magnetic field.

Let’s investigate the length of the dynamic periods. The only parameter we can change is the factor for the Gaussian function. In all the above figures, I am using

$$a = 1 \tag{5.1}$$

Let’s examine $a \ in \ (0, \infty)$ to see how it affects the length of the dynamic periods.

For $a \to \infty$, $J_z(x,\theta) = e^{-ax^2} \to 0$. To keep the magnitudes of the output fields, let’s change the source to

$$J_z(x,\theta) = \sqrt{a}e^{-ax^2} \tag{5.2}$$

The solution (2.5) and (2.6) become

$$H_y(x,\theta) = e^{-\frac{1}{4}\varrho^2} esinh(\xi,\varrho)_1 \tag{5.3}$$

$$E_z(x,\theta) = -\eta e^{-\frac{1}{4}\varrho^2} eicoshi(\xi,\varrho) \tag{5.4}$$

By (5.2), we have

$$\int_0^\infty J_z(x,\theta)dx = \int_0^\infty e^{-(\sqrt{a}x)^2} d\sqrt{a}x = \frac{\sqrt{\pi}}{2}$$

Thus, we have

$$\lim_{a\to\infty} J_z(x,\theta) = \sqrt{\pi}\delta(x) \cong \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \tag{5.5}$$

where $\delta(x)$ is the Dirac delta function.

The following figures show the effects of value a on the lengths of the dynamic periods.

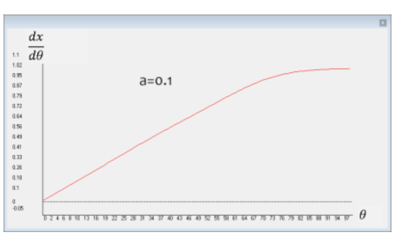

Fig.11. speed of Hy, a=0.1

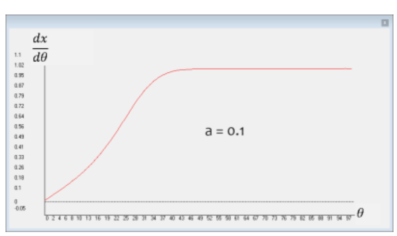

Fig.12. speed of Ez, a=0.1

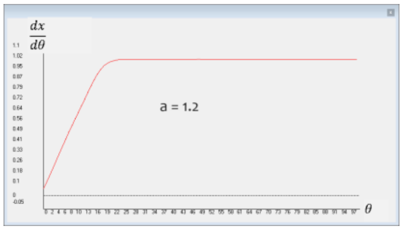

Fig.13. speed of Hy, a=1.2

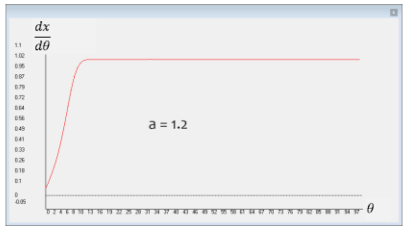

Fig.14. speed of Ez, a=1.2

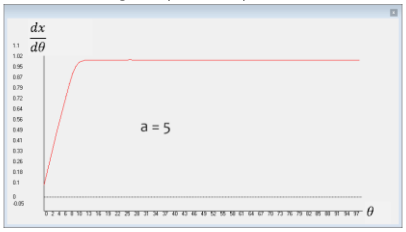

Fig.15. speed of Hy, a=5

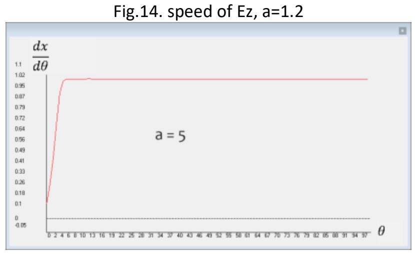

Fig.16. speed of Ez, a=5

From these drawings we can see that the larger value a the shorter the dynamic periods. Thus, it is reasonable to think that when $a \to \infty$ the lengths of the dynamic periods are 0. That is, the Dirac delta function generates a constant speed solution to Maxwell's equations.

Discussion

The analytical solutions to Maxwell's equations presented in this report reveal the following facts.
The electromagnetic wave generated by a single frequency source has a varying instantaneous phase speed.
The electric field and the magnetic field generated by a Gaussian source have the following characteristics.

1. their propagation speeds are not constant.
2. their propagation speeds approach the standard light speed *c* when the time approaches infinity.
3. At lower ranges of time and space, their propagation speeds change greatly. We may call this range the dynamic period. Their speeds are instantaneous speeds.
4. At higher ranges of time and space, their propagation speeds do not change much, the speeds are very close to the standard light speed *c*. We may call this range the stable period. The speed *c* is their stable speeds.
5. The transition from the dynamic period to the stable period can be smooth or sharp, depending on the Gaussian factor.
6. The lengths of the dynamic periods depend on the Gaussian factor. The larger the factor the shorter the dynamic period.
7. When the Dirac delta function is used as the source, the length of the dynamic period is 0. This is my observation from the numerical data, not theoretically proven.

Maxwell's equations form a dynamic system. A dynamic system when driven by an input will always have a dynamic period. In the dynamic period, the speeds of its states can hardly be constant. This math fact is not aware of in the current textbooks for university students [1], [2], and by Einstein [4].

For electromagnetic waves this report shows that "speed constancy" is a good-enough approximation to be used in most cases including Lorentz's work on electrodynamics published in 1904 [16].

The calculation results presented in this report were produced by my desktop computer. Research institutes may use their powerful computing resources to produce much more interesting results. For example, propagations of 3D electromagnetic waves driven by moving sources, accurate numerical simulation of Michelson-Morley experiment, etc.

Methods

Numerical calculations

Since we have closed-form analytical solutions to Maxwell's equations, we may calculate the fields precisely. To make calculations in long ranges of time and space, high performance computing (HPC) is needed which is not available to me. I assembled a desktop computer from scratch with an Intel 24 core CPU (Core i-9 14900K) and a Nvidia 3036 core GPU (GeForce 4096 TI).
I spent a lot of effort on the Nvidia GPU, coding in CUDA. But I could not get good HPC performance out of the GPU.
On reading some reports from the web, including a report from University of Tsukuba (Daisuke Takahashi, FFT and Parallel Numerical Libraries, Center for Computational Sciences, University of Tsukuba, 2014/2/19), I gave up my efforts to try to use GeForce GPU.
Even though Takahashi's report was from 10 years ago, its results matched what I got with my newer CPU and GPU.
I ended up relying on Intel's CPU to make the calculations, crunching out hundreds GB of data from the analytical solutions. Many interesting results are revealed by the data. Among them the propagation speeds presented in this report are just the simpliest ones.

Boost multiprecision library (https://gmplib.org/) was used to make calculations in decimal precision of 4000 digits.

1D solution of Gaussian source

Proof of Lemma 1

Lemma 1:

$$\sum_{k=0}^{n}(-1)^k \varrho^{2k+1} q_{0,n,k} - \sum_{k=1}^{n}(-1)^k 4k\varrho^{2k-1} q_{0,n,k} = \sum_{k=0}^{n}(-1)^k \varrho^{2k+1} p_{0,n,k} \tag{C.1}$$

Proof.

By (2.10), we have

$$\sum_{k=0}^{n}(-1)^k \varrho^{2k+1} q_{0,n,k} - \sum_{k=1}^{n}(-1)^k 4k\varrho^{2k-1} q_{0,n,k} =$$

$$= \sum_{k=0}^{n}(-1)^k \varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!} - \sum_{k=1}^{n}(-1)^k 4k\varrho^{2k-1}\frac{(2n)!}{(n-k)!\,(2k)!}$$

$$= \sum_{k=n}^{n}(-1)^k \varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!} - \sum_{k=0}^{n-1}(-1)^{k+1}4(k+1)\varrho^{2k+1}\frac{(2n)!}{(n-k-1)!\,(2k+2)!}$$

$$= (-1)^n\varrho^{2n+1} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!} - \sum_{k=0}^{n-1}(-1)^{k+1}4(k+1)\varrho^{2k+1}\frac{(2n)!}{(n-k-1)!\,(2k+2)!}$$

$$= (-1)^n\varrho^{2n+1} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!} + \sum_{k=0}^{n-1}(-1)^{k}4(k+1)\varrho^{2k+1}\frac{(2n)!}{(n-k-1)!\,(2k+2)!}$$

$$= (-1)^n\varrho^{2n+1} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}(2n)!\left(\frac{1}{(n-k)!\,(2k)!} + \frac{4(k+1)}{(n-k-1)!\,(2k+2)!}\right)$$

$$= (-1)^n\varrho^{2n+1} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}(2n)!\left(\frac{1}{(n-k)!\,(2k)!} + \frac{4(k+1)(n-k)}{(n-k)!\,(2k)!\,(2k+2)(2k+1)}\right)$$

$$= (-1)^n\varrho^{2n+1} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}(2n)!\frac{1}{(n-k)!\,(2k)!}\left(1 + \frac{4(k+1)(n-k)}{(2k+2)(2k+1)}\right)$$

$$= (-1)^n\varrho^{2n+1} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}\left(1 + \frac{2(2k+2)(n-k)}{(2k+2)(2k+1)}\right)$$

$$= (-1)^n\varrho^{2n+1} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}\left(1 + \frac{2n-2k}{(2k+1)}\right)$$

$$= (-1)^n\varrho^{2n+1} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}\left(\frac{2k+1}{(2k+1)} + \frac{2n-2k}{(2k+1)}\right)$$

$$= (-1)^n\varrho^{2n+1} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}\left(\frac{2n+1}{(2k+1)}\right)$$

$$= (-1)^n\varrho^{2n+1} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}$$

$$= \sum_{k=n}^{n}(-1)^k \varrho^{2k+1}\frac{(2n+1)!}{(n-k)!\,(2k+1)!} + \sum_{k=0}^{n-1}(-1)^k \varrho^{2k+1}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}$$

$$= \sum_{k=0}^{n}(-1)^k \varrho^{2k+1}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}$$

$$= \sum_{k=0}^{n}(-1)^k \varrho^{2k+1} p_{0,n,k}$$

The last step is by (2.9)

QED

Proof of Lemma 2

Lemma 2:

$$\sum_{k=0}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k}-\sum_{k=0}^{n}(-1)^k 2(2k+1)\varrho^{2k}p_{0,n,k}=-\sum_{k=0}^{n+1}(-1)^k\varrho^{2k}\frac{(2n+2)!}{(n-k+1)!\,(2k)!} \tag{C.2}$$

Proof

From the odd-binomial-coefficient (2.9) we have

$$p_{0,n,k+1}=\frac{(2n+1)!}{(n-k-1)!\,(2k+3)!}$$

$$p_{0,0,0}=1$$

$$p_{0,n,n}=1$$

$$p_{0,n,0}=\frac{(2n+1)!}{n!}$$

We will use the above formulas in the proof of this lemma. Apply (2.9), we have

$$\sum_{k=0}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k}-\sum_{k=0}^{n}(-1)^k 2(2k+1)\varrho^{2k}p_{0,n,k}=$$

$$=\sum_{k=0}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k}-\sum_{k=0}^{0}(-1)^k 2(2k+1)\varrho^{2k}p_{0,n,k}-\sum_{k=1}^{n}(-1)^k 2(2k+1)\varrho^{2k}p_{0,n,k}$$

$$=\sum_{k=0}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k}-2p_{0,n,0}-\sum_{k=0}^{n-1}(-1)^{k+1}2(2(k+1)+1)\varrho^{2k+2}p_{0,n,k+1}$$

$$=\sum_{k=n}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}p_{0,n,k}-2\frac{(2n+1)!}{n!}-\sum_{k=0}^{n-1}(-1)^{k+1}2(2k+3)\varrho^{2k+2}p_{0,n,k+1}$$

$$=(-1)^n\varrho^{2n+2}p_{0,n,n}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}p_{0,n,k}-2\frac{(2n+1)!}{n!}+\sum_{k=0}^{n-1}(-1)^k 2(2k+3)\varrho^{2k+2}p_{0,n,k+1}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=0}^{n-1}\left((-1)^k\varrho^{2k+2}p_{0,n,k}+(-1)^k 2(2k+3)\varrho^{2k+2}p_{0,n,k+1}\right)-2\frac{(2n+1)!}{n!}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}\left(p_{0,n,k}+2(2k+3)p_{0,n,k+1}\right)-2\frac{(2n+1)!}{n!}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}\left(\frac{(2n+1)!}{(n-k)!\,(2k+1)!}+2(2k+3)\frac{(2n+1)!}{(n-k-1)!\,(2k+3)!}\right)-2\frac{(2n+1)!}{n!}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}\left(\frac{(2n+1)!}{(n-k)!\,(2k+1)!}+2\frac{(2n+1)!}{(n-k-1)!\,(2k+2)!}\right)-2\frac{(2n+1)!}{n!}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}\left(\frac{(2n+1)!}{(n-k)!\,(2k+1)!}+2\frac{(2n+1)!\,(n-k)}{(n-k)!\,(2k+2)!}\right)-2\frac{(2n+1)!}{n!}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}\left(\frac{(2n+1)!}{(n-k)!\,(2k+1)!}+2\frac{(2n+1)!\,(n-k)}{(n-k)!\,(2k+1)!\,(2k+2)}\right)-2\frac{(2n+1)!}{n!}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}\left(1+2\frac{(n-k)}{(2k+2)}\right)-2\frac{(2n+1)!}{n!}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}\left(\frac{(2k+2)}{(2k+2)}+\frac{(2n-2k)}{(2k+2)}\right)-2\frac{(2n+1)!}{n!}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}\left(\frac{(2n+2)}{(2k+2)}\right)-2\frac{(2n+1)!}{n!}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+2}\frac{(2n+2)!}{(n-k)!\,(2k+2)!}-2\frac{(2n+1)!}{n!}$$

$$=(-1)^n\varrho^{2n+2}+\sum_{k=1}^{n}(-1)^{k-1}\varrho^{2k}\frac{(2n+2)!}{(n-k+1)!\,(2k)!}-2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} - \sum_{k=1}^{n} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} - 2\frac{(2n+1)!}{n!}$$

Because

$$\sum_{k=0}^{0} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} = (-1)^0 \varrho^0 \frac{(2n+2)!}{(n-0+1)!\,(0)!} = \frac{(2n+2)!}{(n+1)!}$$

$$= \frac{(2n+2)!}{(n+1)\cdot n!} = 2\frac{(2n+2)!}{(2n+2)n!} = 2\frac{(2n+1)!}{n!}$$

We have

$$(-1)^n \varrho^{2n+2} - \sum_{k=1}^{n} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} - \sum_{k=1}^{n} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} - \sum_{k=0}^{0} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!}$$

$$= -(-1)^{n+1} \varrho^{2n+2} - \sum_{k=0}^{n} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!}$$

$$= -\sum_{k=n+1}^{n+1} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} - \sum_{k=0}^{n} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!}$$

$$= -\sum_{k=0}^{n+1} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!}$$

QED

Verify initlal values

Proof

From (2.7) and (2.8), we have

$$esinh(0,\varrho)_1 = 0$$

$$eicoshi(0,\varrho) = 0$$

Thus, (2.5) and (2.6) satisfy initial values (2.2):

$$H_y(x,0) = 0$$

$$E_z(x,0) = 0$$

QED

Verify Faraday's law

Lemma 3: Solution (2.5) and (2.6) satisfy Maxwell's equation (1.1)

Proof

Note the symbols used:

$$\varrho = 2\sqrt{a}x$$

$$\xi = \sqrt{a}\theta$$

From (2.6) and (2.8), we have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} = \frac{1}{\eta}\frac{\partial}{\partial x}\left(-\eta\frac{1}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k q_{0,n,k}\varrho^{2k}\right)$$

$$= -\frac{-2ax}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k}$$

$$= 2\sqrt{a}xe^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\sum_{n=0}^{0}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{0}(-1)^k\varrho^{2k}q_{0,n,k} + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\xi + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\xi + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\left(\sum_{k=0}^{0}(-1)^k\varrho^{2k}q_{0,n,k} + \sum_{k=1}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\xi + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\left(1 + \sum_{k=1}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\xi + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!} + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=1}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=1}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}2\sqrt{a}\sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=1}^{n}(-1)^k 2k\varrho^{2k-1}q_{0,n,k}$$

$$= e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}q_{0,n,k} - e^{-ax^2}2\sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=1}^{n}(-1)^k 2k\varrho^{2k-1}q_{0,n,k}$$

$$= e^{-ax^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}q_{0,n,k} - \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=1}^{n}(-1)^k 4k\varrho^{2k-1}q_{0,n,k}\right)$$

$$= e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\left(\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}q_{0,n,k} - \sum_{k=1}^{n}(-1)^k 4k\varrho^{2k-1}q_{0,n,k}\right)$$

Substitute (C.1) into above, we have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} = e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k} \tag{C.3}$$

From (2.5) and (2.7) we have

$$\frac{\partial H_y(x,\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{1}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k p_{0,n,k}\varrho^{2k+1}\right)$$

$$= \frac{1}{\sqrt{a}}e^{-ax^2}\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k}$$

We have

$$\frac{\partial H_y(x,\theta)}{\partial\theta} = e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k} \tag{C.4}$$

(C.3) and (C.4) lead to

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} = \frac{\partial H_y(x,\theta)}{\partial\theta}$$

The above is (1.1). Thus, from (2.5) and (2.6) we get Maxwell's equation (1.1)

QED

Verify Ampere's law

Lemma 4: Solution (2.5) and (2.6) satisfy Maxwell's equation (1.2) and source (2.1)

Proof

From (2.5) and (2.7), we have

$$\frac{\partial H_y(x,\theta)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k}\right)$$

$$= \frac{-2ax}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k} + \frac{1}{\sqrt{a}}e^{-ax^2}2\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k(2k+1)\varrho^{2k}p_{0,n,k}$$

$$= -2\sqrt{a}xe^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k} + e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k 2(2k+1)\varrho^{2k}p_{0,n,k}$$

$$= -\varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k} + e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k 2(2k+1)\varrho^{2k}p_{0,n,k}$$

$$= -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k} + e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k 2(2k+1)\varrho^{2k}p_{0,n,k}$$

$$= -e^{-ax^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k} - \sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k 2(2k+1)\varrho^{2k}p_{0,n,k}\right)$$

$$= -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\left(\sum_{k=0}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k} - \sum_{k=0}^{n}(-1)^k 2(2k+1)\varrho^{2k}p_{0,n,k}\right)$$

Substitute (C.2) into above, we have

$$\frac{\partial H_y(x,\theta)}{\partial x} = -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\left(-\sum_{k=0}^{n+1}(-1)^k\varrho^{2k}\frac{(2n+2)!}{(n-k+1)!\,(2k)!}\right)$$

$$= e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n+1}(-1)^k\varrho^{2k}\frac{(2n+2)!}{(n-k+1)!\,(2k)!}$$

$$= e^{-ax^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!}$$

$$= -e^{-ax^2}\sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!}$$

$$= -e^{-ax^2}\sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} - e^{-ax^2}\sum_{n=0}^{0}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} + e^{-ax^2}\sum_{n=0}^{0}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!}$$

$$= -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} + e^{-ax^2}\sum_{n=0}^{0}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!}$$

$$= -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} + e^{-ax^2}$$

$$= -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} + e^{-ax^2}$$

We have

$$\frac{\partial H_y(x,\theta)}{\partial x} - e^{-ax^2} = -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} \tag{C.5}$$

From (2.6) and (2.8), we have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial\theta} = \frac{1}{\eta}\frac{\partial}{\partial\theta}\left(-\eta\frac{1}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)$$

$$= -\frac{1}{\sqrt{a}}e^{-ax^2}\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k}$$

We have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial\theta} = -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} \tag{C.6}$$

(C.5) and (C.6) lead to

$$\frac{\partial H_y(x,\theta)}{\partial x} - e^{-ax^2} = \frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial\theta}$$

The above is (1.2) and (2.1). Thus, from (2.5) and (2.6) we get Maxwell's equation (1.2) and source (2.1).

QED

Summation exchange rule

Lemma 5. Summation exchange rule:

$$\sum_{i=0}^{I_{max}}\sum_{j=0}^{i}V(i,j)=\sum_{j=0}^{I_{max}}\sum_{i=j}^{I_{max}}V(i,j) \tag{D.1}$$
$$0\le I_{max}\le\infty$$

Proof.

I'll just use a table to prove this simple rule.

$i\downarrow j\rightarrow$	0	1	$\cdots$	I_{max}
0	$V(0,0)$			
1	$V(1,0)$	$V(1,1)$		
$\vdots$	$\vdots$	$\vdots$		
I_{max}	$V(I_{max},0)$	$V(I_{max},1)$	$\cdots$	$V(I_{max},I_{max})$

The summation on the left side of (D.1) is to go row by row; for each row, go column by column.
The summation on the right side of (D.1) is to go column by column; for each column, go row by row.

Thus the two sides go through the same elements in the above table.

QED

Derive speed of the electric field

Section "Verify Faraday's law" gives

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x}=e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k}$$

By the definition of $p_{0,n,k}$, (2.9), we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k}=\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}$$

$$=\sum_{n=0}^{\infty}(-1)^n\xi^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k\varrho^{2k+1}}{(n-k)!\,(2k+1)!}$$

Apply (D.1) given in "Summation exchange rule", we have

$$\sum_{n=0}^{\infty}(-1)^n\xi^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k\varrho^{2k+1}}{(n-k)!\,(2k+1)!}=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^n\xi^{2n+1}(-1)^k\varrho^{2k+1}}{(n-k)!\,(2k+1)!}$$

Let

$$m=n-k$$

We have

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^n\xi^{2n+1}(-1)^k\varrho^{2k+1}}{(n-k)!\,(2k+1)!}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+k}\xi^{2(m+k)+1}(-1)^k\varrho^{2k+1}}{m!\,(2k+1)!}$$

$$=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\xi^{2m+2k+1}\varrho^{2k+1}}{m!\,(2k+1)!}=\sum_{k=0}^{\infty}\frac{(\xi\varrho)^{2k+1}}{(2k+1)!}\sum_{m=0}^{\infty}\frac{(-1)^m\xi^{2m}}{m!}=\sinh(\varrho\xi)\,e^{-\xi^2}$$

We have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x}=e^{-(ax^2+\xi^2)}\sinh(\varrho\xi) \tag{3.1}$$

From the above formula, we can get the following result.

$$\frac{1}{\eta}\frac{\partial^2 E_z(x,\theta)}{\partial x^2}=-2axe^{-(ax^2+\xi^2)}\sinh(\varrho\xi)+e^{-(ax^2+\xi^2)}\cosh(\varrho\xi)\,2\sqrt{a}\xi$$

$$=\sqrt{a}e^{-(ax^2+\xi^2)}\left(-2\sqrt{a}x\sinh(\varrho\xi)+\cosh(\varrho\xi)\,2\xi\right)$$

Because $\varrho=2\sqrt{a}x$ we have

$$\frac{1}{\eta}\frac{\partial^2 E_z(x,\theta)}{\partial x^2}=\sqrt{a}e^{-(ax^2+\xi^2)}(2\xi\ \cosh(\varrho\xi)-\varrho\sinh(\varrho\xi)) \tag{3.2}$$

The moving of the front edge is given by

$$\frac{\partial^2 E_z(x,\theta)}{\partial x^2}=0 \tag{3.3}$$

From (3.2) and (3.3) we have

$$\varrho\sinh(\varrho\xi) = 2\xi\ \cosh(\varrho\xi) \tag{3.4}$$

Take derivative on both sides of (3.4) with respect to ξ, we have

$$\frac{d\varrho}{d\xi}\sinh(\varrho\xi) + \varrho\cosh(\varrho\xi)\left(\varrho + \xi\frac{d\varrho}{d\xi}\right) = 2\cosh(\varrho\xi) + 2\xi\ \sinh(\varrho\xi)\left(\varrho + \xi\frac{d\varrho}{d\xi}\right)$$

$$\frac{d\varrho}{d\xi}\sinh(\varrho\xi) + \varrho^2\cosh(\varrho\xi) + \varrho\xi\frac{d\varrho}{d\xi}\cosh(\varrho\xi) = 2\cosh(\varrho\xi) + 2\varrho\xi\ \sinh(\varrho\xi) + 2\xi^2\frac{d\varrho}{d\xi}\sinh(\varrho\xi)$$

$$\frac{d\varrho}{d\xi}\sinh(\varrho\xi) - 2\xi^2\frac{d\varrho}{d\xi}\sinh(\varrho\xi) + \varrho\xi\frac{d\varrho}{d\xi}\cosh(\varrho\xi) = 2\cosh(\varrho\xi) + 2\varrho\xi\ \sinh(\varrho\xi) - \varrho^2\cosh(\varrho\xi)$$

$$\frac{d\varrho}{d\xi}\big((1-2\xi^2)\sinh(\varrho\xi) + \varrho\xi\cosh(\varrho\xi)\big) = (2-\varrho^2)\cosh(\varrho\xi) + 2\varrho\xi\ \sinh(\varrho\xi)$$

$$\frac{d\varrho}{d\xi} = \frac{(2-\varrho^2)\cosh(\varrho\xi) + 2\varrho\xi\ \sinh(\varrho\xi)}{(1-2\xi^2)\sinh(\varrho\xi) + \varrho\xi\cosh(\varrho\xi)}$$

Substitute (3.4) into above, we have

$$\frac{d\varrho}{d\xi} = \frac{(2-\varrho^2)\cosh(\varrho\xi) + 4\xi^2\ \cosh(\varrho\xi)}{\frac{(1-2\xi^2)2\xi}{\varrho}\cosh(\varrho\xi) + \varrho\xi\cosh(\varrho\xi)}$$

$$= \frac{(2-\varrho^2) + 4\xi^2}{\frac{(1-2\xi^2)2\xi}{\varrho} + \varrho\xi} = \frac{\varrho}{\xi}\frac{(2-\varrho^2) + 4\xi^2}{(1-2\xi^2)2 + \varrho^2}$$

We have

$$\frac{d\varrho}{d\xi} = -\frac{\varrho}{\xi}\cdot\frac{\varrho^2 - 4\xi^2 - 2}{\varrho^2 - 4\xi^2 + 2} \tag{3.5}$$

Because

$$\frac{dx}{d\theta} = \frac{1}{2}\frac{d\varrho}{d\xi}$$

we get the following speed formula for the electric field:

$$\frac{dx}{d\theta} = -\frac{\varrho}{2\xi}\cdot\frac{\varrho^2 - 4\xi^2 - 2}{\varrho^2 - 4\xi^2 + 2} \tag{3.6}$$

Derive speed of the magnetic field

Section “Verify Ampere’s law” gives the following formula

$$\frac{\partial H_y(x,\theta)}{\partial x} = -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} + e^{-ax^2}$$

The summations in the above formula can be further simplified as shown below.

$$\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} =$$

$$= \sum_{n=0}^{\infty}(-1)^n\xi^{2n}\sum_{k=0}^{n}\frac{(-1)^k\varrho^{2k}}{(n-k)!\,(2k)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n\xi^{2n}(-1)^k\varrho^{2k}}{(n-k)!\,(2k)!}$$

Apply (D.1) given in “Summation exchange rule”**,** we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n\xi^{2n}(-1)^k\varrho^{2k}}{(n-k)!\,(2k)!} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^n\xi^{2n}(-1)^k\varrho^{2k}}{(n-k)!\,(2k)!}$$

Let

$$m = n - k$$

We have

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^n\xi^{2n}(-1)^k\varrho^{2k}}{(n-k)!\,(2k)!}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+k}\xi^{2m+2k}(-1)^k\varrho^{2k}}{m!\,(2k)!}$$

$$=\sum_{k=0}^{\infty}\frac{(\varrho\xi)^{2k}}{(2k)!}\sum_{m=0}^{\infty}\frac{(-1)^m\xi^{2m}}{m!}=\cosh(\varrho\xi)\,e^{-\xi^2}$$

We have

$$\frac{\partial H_y(x,\theta)}{\partial x}=e^{-ax^2}\left(1-e^{-\xi^2}\cosh(\varrho\xi)\right) \tag{4.1}$$

From the above formula we can get the following result.

$$\frac{\partial^2 H_y(x,\theta)}{\partial x^2}=\sqrt{a}e^{-ax^2}(e^{-\xi^2}(\varrho\cosh(\varrho\xi)-2\xi\sinh(\varrho\xi))-\varrho) \tag{4.2}$$

The moving of the front edge is given by

$$\frac{\partial^2 H_y(x,\theta)}{\partial x^2}=0 \tag{4.3}$$

From (4.2) and (4.3) we have

$$\varrho\cosh(\varrho\xi)-2\xi\sinh(\varrho\xi)=\varrho\,e^{\xi^2} \tag{4.4}$$

Take derivative on both sides of (4.4) with respect to ξ, we have

$$\frac{d\varrho}{d\xi}=2\frac{4\varrho\xi e^{\xi^2}+(4\xi^2-\varrho^2+2)\sinh(\varrho\xi)}{(2-4\xi^2+\varrho^2)\cosh(\varrho\xi)-(\varrho^2+2)e^{\xi^2}} \tag{4.5}$$

Because

$$\frac{dx}{d\theta}=\frac{1}{2}\frac{d\varrho}{d\xi}$$

We have the formula for the speed of the magnetic field:

$$\frac{dx}{d\theta}=\frac{4\varrho\xi e^{\xi^2}+(4\xi^2-\varrho^2+2)\sinh(\varrho\xi)}{(2-4\xi^2+\varrho^2)\cosh(\varrho\xi)-(\varrho^2+2)e^{\xi^2}} \tag{4.6}$$

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