Time Space Theorem From Maxwell's Equations

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Abstract Maxwell's equations give a relation between the first order curls and the first order temporal derivatives. This paper shows that such a relationship exists for any orders, not just for the first order. Formulas for such relationships are presented and proved.

Contents

Introduction

Maxwell's equations give a relation between the first order curls and the first order temporal derivatives. In 2018, while developing an arbitrary order FDTD algorithm, I found that such a relationship exists for any order, not just for the first order. I call it a "Time-Space Theorem".

This theorem can be used in the following applications.

1. Develop new FDTD algorithms; the order of the new FDTD algorithms can to be higher than the second. (for FDTD, see [2] and [3])
2. Solve Maxwell's equations analytically.

Time Space Theorem

Consider curl-part of Maxwell's equations in free space [1],

$\frac{\partial H(x,y,z,t)}{\partial t} = -\frac{1}{\mu}\nabla \times E(x,y,z,t)$	(1.1)
$\frac{\partial E(x,y,z,t)}{\partial t} = \frac{1}{\varepsilon}\nabla \times H(x,y,z,t) - \frac{1}{\varepsilon}J(x,y,z,t)$	(1.2)
$E,H,J \in R^3$ $x,y,z,t,\varepsilon,\mu \in R$	

For notational simplicity, I omitted the subscript of 0 for permittivity ε_0, and permeability μ_0, used in [1], because only free space is involved in this paper.

To further simplify notations, and make formula deductions less error-prone, a time scale of *ct* is used. Maxwell's equations become

$c = \frac{1}{\sqrt{\varepsilon\mu}}$	(1.3)
$\eta = \sqrt{\frac{\mu}{\varepsilon}}$	(1.4)
$\theta = ct$	(1.5)
$\frac{\partial H(x,y,z,\theta)}{\partial \theta} = -\frac{1}{\eta}\nabla \times E(x,y,z,\theta)$	(1.6)
$\frac{\partial E(x,y,z,\theta)}{\partial \theta} = \eta\nabla \times H - \eta J(x,y,z,\theta)$	(1.7)

For (1.6) and (1.7) to be compatible with (1.1) and (1.2), I should have used $E\left(x,y,z,\frac{1}{c}\theta\right)$, $H\left(x,y,z,\frac{1}{c}\theta\right)$, and $J\left(x,y,z,\frac{1}{c}\theta\right)$. Since I will not use (1.1) and (1.2) in formula deductions, I'll let (1.6) and (1.7) use the simpler notations.

(1.6) and (1.7) use one constant η, (1.1) and (1.2) use two constants ε and μ. The new variable θ is in meters. For convenience, I still call θ a "time", to distinguish it from the conventional 3D space dimensions (x, y, z). θ does function as a time.

Introduce a concept of "curl order" $\nabla^m \times$ similar to the concept of derivative order $\partial^m/\partial x^m$. An m-th order curl is defined for a 3-D vector F by

$\nabla^0 \times F \equiv F$ $\nabla^m \times F \equiv \underbrace{\nabla \times \nabla \times \ldots \times \nabla \times}_{m} F, m \geq 0$	(1.8)

Time-Space Theorem. For an electromagnetic field, if the sequence of derivative and curl is interchangeable, $\frac{\partial}{\partial t}\nabla \times = \nabla \times \frac{\partial}{\partial t}$, then

1. A derivative of any odd order of the magnetic field with respect to time is formed by a curl of the electric field of the same order, as described by (t.1);
2. A derivative of any odd order of the electric field with respect to time is formed by a curl of the magnetic field of the same order, as described by (t.3);
3. A derivative of any even order of the magnetic field with respect to time is formed by a curl of the magnetic field of the same order, as described by (t.2);
4. A derivative of any even order of the electric field with respect to time is formed by a curl of the electric field of the same order, as described by (t.4);

$\frac{\partial^{2n+1} H}{\partial \theta^{2n+1}} = \frac{1}{\eta}(-1)^{n+1}\, \nabla^{2n+1} \times E + J_{h,2n+1}$	(t.1)
$\frac{\partial^{2(n+1)} H}{\partial \theta^{2(n+1)}} = (-1)^{n+1} \nabla^{2(n+1)} \times H + J_{h,2(n+1)}$	(t.2)
$\frac{\partial^{2n+1} E}{\partial \theta^{2n+1}} = \eta(-1)^n \nabla^{2n+1} \times H + \eta J_{e,2n+1}$	(t.3)
$\frac{\partial^{2(n+1)} E}{\partial \theta^{2(n+1)}} = (-1)^{n+1} \nabla^{2(n+1)} \times E + \eta J_{e,2(n+1)}$	(t.4)
$n = 0,1,2, \ldots$	

Where the field sources are given by

$J_{h,2n+1} = \begin{cases} \vec{0}, & n = 0 \\ \sum_{m=1}^{n} (-1)^{m+1} \nabla^{2m-1} \times \frac{\partial^{2(n-m)+1} J}{\partial \theta^{2(n-m)+1}}, & n > 0 \end{cases}$	(t.1J)
$J_{h,2(n+1)} = \sum_{m=0}^{n} (-1)^m \nabla^{2m+1} \times \frac{\partial^{2(n-m)} J}{\partial \theta^{2(n-m)}}$	(t.2J)
$J_{e,2n+1} = \sum_{m=0}^{n} (-1)^{m+1} \nabla^{2m} \times \frac{\partial^{2(n-m)} J}{\partial \theta^{2(n-m)}}$	(t.3J)
$J_{e,2(n+1)} = \sum_{m=0}^{n} (-1)^{m+1} \nabla^{2m} \times \frac{\partial^{2(n-m)+1} J}{\partial \theta^{2(n-m)+1}}$	(t.4J)
$n = 0,1,2, \ldots$	

Proof. This theorem can be proved by induction based on Maxwell's equations (1.6) and (1.7).

Let

$$\mathrm{n} = 0$$

(t.1) and (t.1J) lead to (1.6), thus (t.1) holds for $\mathrm{n} = 0$.

(t.3) and (t.3J) lead to (1.7), thus (t.3) holds for $\mathrm{n} = 0$.

Take derivative with respect to θ on both sides of (1.6), we have

$$\frac{\partial^2 H}{\partial \theta^2} = -\frac{1}{\eta} \nabla \times \frac{\partial E}{\partial \theta}$$

Substitute (1.7) into the right side, we have

$\frac{\partial^2 H}{\partial \theta^2} = -\nabla^2 \times H + \nabla \times J$	(1.9)

The above is (t.2) and (t.2J) for $\mathrm{n}=0$. Thus (t.2) holds for $\mathrm{n}=0$.

Take derivative with respect to θ on both sides of (1.7), we have

$$\frac{\partial^2 E}{\partial \theta^2}=\eta\nabla\times\frac{\partial H}{\partial\theta}-\eta\frac{\partial J}{\partial\theta}$$

Substitute (1.6) into the right side, we have

$$\frac{\partial^2 E}{\partial \theta^2}=-\nabla^2\times E-\eta\frac{\partial J}{\partial\theta} \tag{1.10}$$

The above is (t.4) and (t.4J) for $\mathrm{n}=0$. Thus (t.4) holds for $\mathrm{n}=0$.

Thus the theorem holds for $\mathrm{n}=0$.

Now consider the case of $\mathrm{n}=1$.

Take derivative with respect to θ on both sides of (1.9) and substitute (1.6) into it, we have

$$\frac{\partial^3 H}{\partial \theta^3}=-\nabla^2\times\left(-\frac{1}{\eta}\nabla\times E\right)+\nabla\times\frac{\partial J}{\partial\theta}$$

$$\frac{\partial^3 H}{\partial \theta^3}=\frac{1}{\eta}\nabla^3\times E+\nabla\times\frac{\partial J}{\partial\theta} \tag{1.11}$$

The above is (t.1) and (t.1J) with $\mathrm{n}=1$. Thus (t.1) holds for $\mathrm{n}=1$.

Take derivative with respect to θ on both sides of (1.11) and substitute (1.7) into it, we have

$$\frac{\partial^4 H}{\partial \theta^4}=\frac{1}{\eta}\nabla^3\times(\eta\nabla\times H-\eta J)+\nabla\times\frac{\partial^2 J}{\partial\theta^2}$$

$$\frac{\partial^4 H}{\partial \theta^4}=\nabla^4\times H-\nabla^3\times J+\nabla\times\frac{\partial^2 J}{\partial\theta^2} \tag{1.12}$$

The above is (t.2) and (t.2J) with $\mathrm{n}=1$. Thus (t.2) holds for $\mathrm{n}=1$.

Take derivative with repect to θ on both sides of (1.10) and substitute (1.7) into it, we have

$$\frac{\partial^3 E}{\partial \theta^3}=-\nabla^2\times(\eta\nabla\times H-\eta J)-\eta\frac{\partial^2 J}{\partial\theta^2}$$

$$\frac{\partial^3 E}{\partial \theta^3}=-\eta\nabla^3\times H+\eta\left(\nabla^2\times J-\frac{\partial^2 J}{\partial\theta^2}\right) \tag{1.13}$$

The above is (t.3) and (t.3J) with $\mathrm{n}=1$. Thus (t.3) holds for $\mathrm{n}=1$.

Take derivative with repect to θ on both sides of (1.13) and substitute (1.6) into it, we have

$$\frac{\partial^4 E}{\partial \theta^4}=-\eta\nabla^3\times\left(-\frac{1}{\eta}\nabla\times E\right)+\eta\left(\nabla^2\times\frac{\partial J}{\partial\theta}-\frac{\partial^3 J}{\partial\theta^3}\right)$$

$$\frac{\partial^4 E}{\partial \theta^4}=\nabla^4\times E+\eta\left(\nabla^2\times\frac{\partial J}{\partial\theta}-\frac{\partial^3 J}{\partial\theta^3}\right) \tag{1.14}$$

The above is (t.4) and (t.4J) with $\mathrm{n}=1$. Thus (t.4) holds for $\mathrm{n}=1$.

Thus the theorem holds for $\mathrm{n}=0$ and $\mathrm{n}=1$.

Suppose the theorem holds for an integer $\mathrm{n}>1$. Let's check for a case of $\mathrm{n}+1$.

Take derivative with repect to θ on both sides of (t.2) and substitute (1.6) into it, we have

$$\frac{\partial^{2(n+1)+1}H}{\partial\theta^{2(n+1)+1}}=(-1)^{n+1}\nabla^{2(\mathrm{n}+1)}\times\left(-\frac{1}{\eta}\nabla\times E\right)+\sum_{m=0}^{n}(-1)^m\nabla^{2m+1}\times\frac{\partial^{2(n-m)+1}J}{\partial\theta^{2(n-m)+1}}$$

$$\frac{\partial^{2(n+1)+1}H}{\partial\theta^{2(n+1)+1}}=\frac{1}{\eta}(-1)^{(n+1)+1}\nabla^{2(\mathrm{n}+1)+1}\times E+\sum_{m=1}^{n+1}(-1)^{m+1}\nabla^{2m-1}\times\frac{\partial^{2(n+1-m)+1}J}{\partial\theta^{2(n+1-m)+1}} \tag{1.15}$$

The above is (t.1) and (t.1J) when substituting n with $\mathrm{n}+1$. Thus (t.1) holds for $\mathrm{n}+1$.

Take derivative with repect to θ on both sides of (1.15) and substitute (1.7) into it, we have

$$\frac{\partial^{2(n+1)+2}H}{\partial\theta^{2(n+1)+2}}=\frac{1}{\eta}(-1)^{(n+1)+1}\nabla^{2(n+1)+1}\times(\eta\nabla\times H-\eta J)+\sum_{m=1}^{n+1}(-1)^{m+1}\nabla^{2m-1}\times\frac{\partial^{2(n+1-m)+2}J}{\partial\theta^{2(n+1-m)+2}}$$

$$\frac{\partial^{2(n+2)}H}{\partial\theta^{2(n+2)}}=(-1)^{(n+2)}\nabla^{2(n+2)}\times H+(-1)^{(n+2)+1}\nabla^{2(n+1)+1}\times J+\sum_{m=0}^{n}(-1)^{m+1+1}\nabla^{2m+1}\times\frac{\partial^{2(n+1-m)}J}{\partial\theta^{2(n+1-m)}}$$

$$\frac{\partial^{2(n+2)}H}{\partial\theta^{2(n+2)}}=(-1)^{(n+2)}\nabla^{2(n+2)}\times H+\sum_{m=0}^{n+1}(-1)^{m}\nabla^{2m+1}\times\frac{\partial^{2(n+1-m)}J}{\partial\theta^{2(n+1-m)}}$$

The above is (t.2) and (t.2J) when substituting n with $n+1$. Thus (t.2) holds for $n+1$.

Take derivative with repect to θ on both sides of (t.4) and substitute (1.7) into it, we have

$$\frac{\partial^{2(n+1)+1}E}{\partial\theta^{2(n+1)+1}}=(-1)^{n+1}\nabla^{2(n+1)}\times(\eta\nabla\times H-\eta J)+\eta\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n-m)+2}J}{\partial\theta^{2(n-m)+2}}$$

$$\frac{\partial^{2(n+1)+1}E}{\partial\theta^{2(n+1)+1}}=\eta(-1)^{n+1}\nabla^{2(n+1)+1}\times H+\eta(-1)^{(n+1)+1}\nabla^{2(n+1)}\times J+\eta\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n+1-m)}J}{\partial\theta^{2(n+1-m)}}$$

$\frac{\partial^{2(n+1)+1}E}{\partial\theta^{2(n+1)+1}}=\eta(-1)^{n+1}\nabla^{2(n+1)+1}\times H+\eta\sum_{m=0}^{n+1}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n+1-m)}J}{\partial\theta^{2(n+1-m)}}$	(1.16)

The above is (t.3) and (t.3J) when substituting n with $n+1$. Thus (t.3) holds for $n+1$.

Take derivative with repect to θ on both sides of (1.16) and substitute (1.6) into it, we have

$$\frac{\partial^{2(n+1)+2}E}{\partial\theta^{2(n+1)+2}}=\eta(-1)^{n+1}\nabla^{2(n+1)+1}\times\left(-\frac{1}{\eta}\nabla\times E\right)+\eta\sum_{m=0}^{n+1}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n+1-m)+1}J}{\partial\theta^{2(n+1-m)+1}}$$

$$\frac{\partial^{2(n+1+1)}E}{\partial\theta^{2(n+1+1)}}=(-1)^{n+1+1}\nabla^{2(n+1+1)}\times E+\eta\sum_{m=0}^{n+1}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n+1-m)+1}J}{\partial\theta^{2(n+1-m)+1}}$$

The above is (t.4) and (t.4J) when substituting n with $n+1$. Thus (t.4) holds for $n+1$.

Thus, the theorem holds for $n+1$.

Thus, the theorem holds for $n\geq 0$.

QED

Without scaling time *t* with *ct*, the Time-Space Theorem for Maxwell's equations (1.1) and (1.2) is given below.

$\frac{\partial^{2n+1}H}{\partial t^{2n+1}}=\frac{1}{\mu}\frac{(-1)^{n+1}}{(\varepsilon\mu)^{n}}\nabla^{2n+1}\times E+\begin{cases}\vec{0}, n=0\\ \sum_{m=1}^{n}\frac{(-1)^{m+1}}{(\varepsilon\mu)^{m}}\nabla^{2m-1}\times\frac{\partial^{2(n-m)+1}J}{\partial t^{2(n-m)+1}}, n>0\end{cases}$	(0.1)
$\frac{\partial^{2(n+1)}H}{\partial t^{2(n+1)}}=\frac{(-1)^{n+1}}{(\varepsilon\mu)^{n+1}}\nabla^{2(n+1)}\times H+\sum_{m=0}^{n}\frac{(-1)^{m}}{(\varepsilon\mu)^{m+1}}\nabla^{2m+1}\times\frac{\partial^{2(n-m)}J}{\partial t^{2(n-m)}}$	(0.2)
$\frac{\partial^{2n+1}E}{\partial t^{2n+1}}=\frac{1}{\varepsilon}\frac{(-1)^{n}}{(\varepsilon\mu)^{n}}\nabla^{2n+1}\times H+\frac{1}{\varepsilon}\sum_{m=0}^{n}\frac{(-1)^{m+1}}{(\varepsilon\mu)^{m}}\nabla^{2m}\times\frac{\partial^{2(n-m)}J}{\partial t^{2(n-m)}}$	(0.3)
$\frac{\partial^{2(n+1)}E}{\partial t^{2(n+1)}}=\frac{(-1)^{n+1}}{(\varepsilon\mu)^{n+1}}\nabla^{2(n+1)}\times E+\frac{1}{\varepsilon}\sum_{m=0}^{n}\frac{(-1)^{m+1}}{(\varepsilon\mu)^{m}}\nabla^{2m}\times\frac{\partial^{2(n-m)+1}J}{\partial t^{2(n-m)+1}}$	(0.4)
$n=0,1,2,\ldots$	

Proof. The above formulas can also be proved by induction. Since (t.1) – (t.4) have been proved, I'll take a shortcut. Use E_0, H_0 *and* J_0 in (1.1) and (1.2) to make (1.1) and (1.2) compatible with (1.6) and (1.7),

$\frac{\partial H_0(x,y,z,t)}{\partial t}=-\frac{1}{\mu}\nabla\times E_0(x,y,z,t)$	(2.1)

$\frac{\partial E_0(x,y,z,t)}{\partial t} = \frac{1}{\varepsilon}\nabla \times H_0(x,y,z,t) - \frac{1}{\varepsilon}J_0(x,y,z,t)$	(2.2)

Where

$H_0(x,y,z,t) = H(x,y,z,\theta) = H(x,y,z,ct)$	(2.3)
$E_0(x,y,z,t) = E(x,y,z,\theta) = E(x,y,z,ct)$	(2.4)
$J_0(x,y,z,t) = J(x,y,z,\theta) = J(x,y,z,ct)$	(2.5)

From (2.3), (2.4) and (2.5) we have

$\frac{\partial^k H_0(x,y,z,t)}{\partial t^k} = \frac{\partial^k H(x,y,z,\theta)}{\partial \theta^k}c^k$	(2.6)
$\frac{\partial^k E_0(x,y,z,t)}{\partial t^k} = \frac{\partial^k E(x,y,z,\theta)}{\partial \theta^k}c^k$	(2.7)
$\frac{\partial^k J_0(x,y,z,t)}{\partial t^k} = \frac{\partial^k J(x,y,z,\theta)}{\partial \theta^k}c^k$	(2.8)
$\nabla^k \times H_0(x,y,z,t) = \nabla^k \times H(x,y,z,\theta)$	(2.9)
$\nabla^k \times E_0(x,y,z,t) = \nabla^k \times E(x,y,z,\theta)$	(2.10)
$\nabla^k \times J_0(x,y,z,t) = \nabla^k \times J(x,y,z,\theta)$	(2.11)
$k \geq 0$	

Substitute (2.6) to (2.11) into (t.1) and (t.1J), we have

$$\frac{\partial^{2n+1}H_0}{\partial t^{2n+1}} = \left(\frac{1}{\eta}(-1)^{n+1}\,\nabla^{2n+1} \times E + \begin{cases} \vec{0}, & n=0 \\ \sum_{m=1}^{n}(-1)^{m+1}\nabla^{2m-1} \times \frac{\partial^{2(n-m)+1}J_0}{\partial \theta^{2(n-m)+1}}c^{-(n-m)-1}, & n>0 \end{cases}\right)c^{2n+1}$$

We have

$$\frac{\partial^{2n+1}H_0}{\partial t^{2n+1}} = \frac{1}{\mu}\frac{(-1)^{n+1}}{(\varepsilon\mu)^n}\,\nabla^{2n+1} \times E_0 + \begin{cases} \vec{0}, & n=0 \\ \sum_{m=1}^{n}\frac{(-1)^{m+1}}{(\varepsilon\mu)^m}\nabla^{2m-1} \times \frac{\partial^{2(n-m)+1}J_0}{\partial \theta^{2(n-m)+1}}, & n>0 \end{cases}$$

The above is (0.1) when substitute H with H_0, E with E_0, and J with J_0. Thus (0.1) holds for (2.1) and (2.2).

Substitute (2.6) to (2.11) into (t.2) and (t.2J), we have

$$\frac{\partial^{2(n+1)}H_0}{\partial t^{2(n+1)}} = \left((-1)^{n+1}\nabla^{2(n+1)} \times H + \sum_{m=0}^{n}(-1)^m\nabla^{2m+1} \times \frac{\partial^{2(n-m)}J_0}{\partial t^{2(n-m)}}c^{-2(n-m)}\right)c^{2(n+1)}$$

We have

$$\frac{\partial^{2(n+1)}H_0}{\partial t^{2(n+1)}} = \left(\frac{(-1)^{n+1}}{(\varepsilon\mu)^{n+1}}\nabla^{2(n+1)} \times H_0 + \sum_{m=0}^{n}\frac{(-1)^m}{(\varepsilon\mu)^{m+1}}\nabla^{2m+1} \times \frac{\partial^{2(n-m)}J_0}{\partial t^{2(n-m)}}\right)$$

The above is (0.2) when substitute H with H_0, E with E_0, and J with J_0. Thus (0.2) holds for (2.1) and (2.2).

Substitute (2.6) to (2.11) into (t.3) and (t.3J), we have

$$\frac{\partial^{2n+1}E_0}{\partial \theta^{2n+1}} = \left(\eta(-1)^n\nabla^{2n+1} \times H_0 + \eta\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m} \times \frac{\partial^{2(n-m)}J_0}{\partial \theta^{2(n-m)}}c^{-2(n-m)}\right)c^{2n+1}$$

We have

$$\frac{\partial^{2n+1}E_0}{\partial \theta^{2n+1}} = \frac{1}{\varepsilon}\frac{(-1)^n}{(\varepsilon\mu)^n}\nabla^{2n+1} \times H_0 + \frac{1}{\varepsilon}\sum_{m=0}^{n}\frac{(-1)^{m+1}}{(\varepsilon\mu)^m}\nabla^{2m} \times \frac{\partial^{2(n-m)}J_0}{\partial \theta^{2(n-m)}}$$

The above is (0.3) when substitute H with H_0, E with E_0, and J with J_0. Thus (0.3) holds for (2.1) and (2.2).

Substitute (2.6) to (2.11) into (t.4) and (t.4J), we have

$$\frac{\partial^{2(n+1)}E_0}{\partial t^{2(n+1)}} = \left((-1)^{n+1}\nabla^{2(n+1)} \times E_0 + \eta\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m} \times \frac{\partial^{2(n-m)+1}J_0}{\partial \theta^{2(n-m)+1}}c^{-2(n-m)-1}\right)c^{2(n+1)}$$

$$\frac{\partial^{2(n+1)}E_0}{\partial t^{2(n+1)}} = \frac{(-1)^{n+1}}{(\varepsilon\mu)^{n+1}}\nabla^{2(\mathrm{n}+1)} \times E_0 + \frac{1}{\varepsilon}\sum_{m=0}^{n}\frac{(-1)^{m+1}}{(\varepsilon\mu)^{m}}\nabla^{2m} \times \frac{\partial^{2(n-m)+1}J_0}{\partial\theta^{2(n-m)+1}}$$

The above is (0.4) when substitute H with H_0, E with E_0, and J with J_0. Thus (0.4) holds for (2.1) and (2.2).

QED

References

[1] Maxwell's Equations, http://hyperphysics.phy-astr.gsu.edu/hbase/electric/maxeq.html

[2] Hannes Kohlmann, 2020, FDTD Simulation of Lightning Electromagnetic Fields An approach with the software package MEEP, MASTER THESIS, TECHNISCHE UNIVERSITAT WIEN Vienna Austria

[3] Alessandro Fedeli, Claudio Montecucco and Gian Luigi Gragnani 1, Open-Source Software for Electromagnetic Scattering Simulation: The Case of Antenna Design, Electronics 2019, 8, 1506; doi:10.3390/electronics8121506