Solving Maxwell’s Equations in Open Space

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Date: August 15, 2023

Abstract A generic analytical solution to Maxwell’s equations in open space is deduced, which is expressed in initial field values and field sources. The generic solution is in summations of series. From the summations of series, closed form analytical solutions are obtained for given initial values and field sources. Several typical examples of 3D closed form analytical solutions are obtained in this approach.

Contents

Introduction

I am going to present a generic analytical solution to Maxwell’s equations in open space. The solution is expressed in initial field values and field sources, in summations of series. From the summations of series, closed form analytical solutions are obtained for given initial values and field sources.

Getting a generic analytical solution to Maxwell’s equations has not been reported to my knowledge. The closest solution probably is the Kirchoff’s formula, the solution at one space point can be calculated by a sphere surface integration on functions of initial values, the radius of the sphere is proportional to time. Integration operations are still in the solutions. I saw a wide range of research including using Cauchy’s formula ([1],[2]) and Kirchhoff’s formula ([3]), reaching various forms of solutions, but far from the solution I am going to present.

Maxwell’s equations give a relation between the first order curls and the first order temporal derivatives. I have proved that such a relation exists for any orders, not just for the first order. I call it a “Time-Space Theorem” (see [4]). This theorem does not give new information other than what Maxwell’s equations already give. However, from this theorem, a generic solution to Maxwell’s equations can be deduced, which is expressed purely in initial field values and field source inputs.

Several typical examples of closed form analytical solutions are obtained from the generic solution. One example presents an ever growing 3D field; one example presents oscillating 3D fields of sustained strength; one example presents 3D fields with ever decreasing strength.

Solution to Maxwell's equations

Problem formulation

Consider curl-part of Maxwell's equations in free space [5],

$\frac{\partial H(x,y,z,t)}{\partial t} = -\frac{1}{\mu}\nabla \times E(x,y,z,t)$	(1.1)
$\frac{\partial E(x,y,z,t)}{\partial t} = \frac{1}{\varepsilon}\nabla \times H(x,y,z,t) - \frac{1}{\varepsilon}J(x,y,z,t)$	(1.2)
$E, H, J \in R^3$ $x, y, z, t, \varepsilon, \mu \in R$	

For notational simplicity, I omitted the subscript of 0 for permittivity ε_0, and permeability μ_0, used in [5], because only free space is involved in this paper.

To further simplify notations, and make formula deductions less error-prone, a time scale of ct is used. Maxwell's equations become

$c = \frac{1}{\sqrt{\varepsilon\mu}}$	(1.3)
$\eta = \sqrt{\frac{\mu}{\varepsilon}}$	(1.4)
$\theta = ct$	(1.5)
$\frac{\partial H(x,y,z,\theta)}{\partial \theta} = -\frac{1}{\eta}\nabla \times E(x,y,z,\theta)$	(1.6)
$\frac{\partial E(x,y,z,\theta)}{\partial \theta} = \eta\nabla \times H - \eta J(x,y,z,\theta)$	(1.7)

For (1.6) and (1.7) to be compatible with (1.1) and (1.2), I should have used $E\left(x,y,z,\frac{1}{c}\theta\right)$, $H\left(x,y,z,\frac{1}{c}\theta\right)$, and $J\left(x,y,z,\frac{1}{c}\theta\right)$. Since I will not use (1.1) and (1.2) in formula deductions, I'll let (1.6) and (1.7) use the simpler notations.

The new variable θ is in meters. For convenience, I still call θ a "time", to distinguish it from the conventional 3D space dimensions (x,y,z). θ does function as a time.

The solution to Maxwell's equations is determined by following given values

$J(x,y,z,\theta)$ *is given*	(2.1)
$E(x,y,z,0)$ *is given*	(2.2)
$H(x,y,z,0)$ *is given*	(2.3)
$J, E,$ *and* H *are* differentiable with respect to time and space in any order	

(2.1) is the field source; (2.2) and (2.3) are initial values.

Also, it is required that

$$\nabla \cdot E(x,y,z,0) = 0$$

$$\nabla \cdot H(x,y,z,0) = 0$$

for satisfying the divergence parts of Maxwell's equations.

To solve Maxwell's equations is to find 3D functions

$$E(x,y,z,\theta)$$

$$H(x,y,z,\theta)$$

which satisfy equations (1.6) and (1.7) for the given values (2.1), (2.2) and (2.3).

Note that there is not a boundary condition presented in the above problem formulation. In case some readers might raise a question of "well-posed-ness" of the problem, I'll explain a little bit about this issue of boundary condition and "well-posed-ness".

First and foremost, mathematically, a boundary condition is not needed for the above problem to have a solution.

We know that with estimation algorithms, such as FDTD algorithms, a boundary condition must be present so that the algorithms may work, because there is a computing or programming problem. A computer does not have unlimited memory, and a limited computing domain must be defined. An FDTD algorithm uses near-by spaces of a space point to do estimation. At the computing domain boundary, some "near-by" space points are outside the boundary due to symmetric selection of "near-by" space points. For those unavailable space points, a boundary condition must be used to provide values at those space points. For many engineering problems, the boundary conditions already exist physically. For open-space problems, we get into big trouble. Some imaginary boundary conditions can be used, i.e., perfect conductors. Adding an imaginary boundary condition alters the original problem of open-space. So, we are really not solving the original problem.

Must a boundary condition be present for FDTD? Not really. There is a smart algorithm which uses a moving window to solve this programming problem, see [6], [7]. Suppose we are only interested in one direction of radio transmission, then during the time of forward simulations, the memories for the opposite direction can be discarded and re-used for the interested direction. Thus, there is unlimited memory for the direction we are interested in. A boundary condition for that direction is thus no longer needed.

A full discussion of "well-posed-ness" is out of scope of this paper. I'll prove that the solution I deduced is unique. From the solution I am going to present, we can see that the solution's behavior changes continuously with the initial values and field source.

Generic solution

Introduce a concept of "curl order" $\nabla^m \times$ similar to the concept of derivative order $\partial^m/\partial x^m$. An m-th order curl is defined for a 3-D vector F by

$$\begin{gathered}\nabla^0 \times F \equiv F \\ \nabla^m \times F \equiv \underbrace{\nabla \times \nabla \times \ldots \times \nabla \times}_{m} F, m \geq 0\end{gathered} \tag{1.8}$$

My first step to deduce a generic solution to Maxwell's equations is to express a solution in Taylor's series.

For simplicity, denote

$$H(\theta) = H(x, y, z, \theta)$$

$$E(\theta) = E(x, y, z, \theta)$$

$$J(\theta) = J(x, y, z, \theta)$$

By Taylor's series, we have

$$H(\theta) = \sum_{h=0}^{\infty} \frac{\partial^h H(0)}{\partial \theta^h} \frac{\theta^h}{h!}$$

$$E(\theta) = \sum_{h=0}^{\infty} \frac{\partial^h E(0)}{\partial \theta^h} \frac{\theta^h}{h!}$$

Re-group the summations, we have

$$H(\theta) = H(0) + \sum_{n=0}^{\infty} \frac{\partial^{2(n+1)} H(0)}{\partial \theta^{2(n+1)}} \frac{\theta^{2(n+1)}}{(2(n+1))!} + \sum_{n=0}^{\infty} \frac{\partial^{2n+1} H(0)}{\partial \theta^{2n+1}} \frac{\theta^{2n+1}}{(2n+1)!} \tag{1.9}$$

$$E(\theta) = E(0) + \sum_{n=0}^{\infty} \frac{\partial^{2(n+1)} E(0)}{\partial \theta^{2(n+1)}} \frac{\theta^{2(n+1)}}{(2(n+1))!} + \sum_{n=0}^{\infty} \frac{\partial^{2n+1} E(0)}{\partial \theta^{2n+1}} \frac{\theta^{2n+1}}{(2n+1)!} \tag{1.10}$$

The **Time-Space Theorem** says that the following formulas hold, see [4].

$$\frac{\partial^{2n+1} H(\theta)}{\partial \theta^{2n+1}} = \frac{1}{\eta}(-1)^{n+1}\, \nabla^{2n+1} \times E(\theta) + J_{h,2n+1}(\theta) \tag{t.1}$$

$$\frac{\partial^{2(n+1)} H(\theta)}{\partial \theta^{2(n+1)}} = (-1)^{n+1} \nabla^{2(n+1)} \times H(\theta) + J_{h,2(n+1)}(\theta) \tag{t.2}$$

$$\frac{\partial^{2n+1} E(\theta)}{\partial \theta^{2n+1}} = \eta(-1)^n \nabla^{2n+1} \times H(\theta) + \eta J_{e,2n+1}(\theta) \tag{t.3}$$

$$\frac{\partial^{2(n+1)} E(\theta)}{\partial \theta^{2(n+1)}} = (-1)^{n+1} \nabla^{2(n+1)} \times E(\theta) + \eta J_{e,2(n+1)}(\theta) \tag{t.4}$$

$$n = 0,1,2, \ldots$$

Where the field sources are given by

$$J_{h,2n+1}(\theta) = \begin{cases} \vec{0}, & n = 0 \\ \sum_{m=1}^{n} (-1)^{m+1} \nabla^{2m-1} \times \frac{\partial^{2(n-m)+1} J(\theta)}{\partial \theta^{2(n-m)+1}}, & n > 0 \end{cases} \tag{t.1J}$$

$$J_{h,2(n+1)}(\theta) = \sum_{m=0}^{n} (-1)^m \nabla^{2m+1} \times \frac{\partial^{2(n-m)} J(\theta)}{\partial \theta^{2(n-m)}} \tag{t.2J}$$

$$J_{e,2n+1}(\theta) = \sum_{m=0}^{n} (-1)^{m+1} \nabla^{2m} \times \frac{\partial^{2(n-m)} J(\theta)}{\partial \theta^{2(n-m)}} \tag{t.3J}$$

$$J_{e,2(n+1)}(\theta)=\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n-m)+1}J(\theta)}{\partial\theta^{2(n-m)+1}}$$	(t.4J)
$n=0,1,2,\ldots$	

Substitute (t.1) to (t.4) and (t.1J) to (t.4J) into (1.9) and (1.10), we reach a generic solution to Maxwell's equations.

$$H(\theta)=H(0)+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2(n+1)}}{\big(2(n+1)\big)!}\nabla^{2(n+1)}\times H(0)+\frac{1}{\eta}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!}\nabla^{2n+1}\times E(0)$$ $$+\sum_{n=0}^{\infty}\frac{\theta^{2(n+1)}}{\big(2(n+1)\big)!}\sum_{m=0}^{n}(-1)^{m}\nabla^{2m+1}\times\frac{\partial^{2(n-m)}J(0)}{\partial\theta^{2(n-m)}}+\sum_{n=1}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\sum_{m=1}^{n}(-1)^{m+1}\nabla^{2m-1}\times\frac{\partial^{2(n-m)+1}J(0)}{\partial\theta^{2(n-m)+1}}$$	(1.11)
$$E(\theta)=E(0)+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2(n+1)}}{\big(2(n+1)\big)!}\nabla^{2(n+1)}\times E(0)+\eta\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n+1}}{(2n+1)!}\nabla^{2n+1}\times H(0)$$ $$+\eta\sum_{n=0}^{\infty}\frac{\theta^{2(n+1)}}{\big(2(n+1)\big)!}\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n-m)+1}J(0)}{\partial\theta^{2(n-m)+1}}+\eta\sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n-m)}J(0)}{\partial\theta^{2(n-m)}}$$	(1.12)

The generic solution presented in (1.11) and (1.12) is expressed purely in given values (2.1), (2.2) and (2.3). From the generic solution, closed form analytical solutions can be obtained for given initial values and field sources.

Verification of the solution

Before going for a close-form analytical solution, I noticed that the generic solution (1.11) and (1.12) contradicts the "Principle of locality". I'll discuss it at the end of this paper. This contradiction made me doubt the correctness of the solution. So, let's verify the solution first.

Solution to Maxwell's equations. The solution to Maxwell's equations (1.6) and (1.7) for given values (2.1), (2.2) and (2.3) is given uniquely by (1.11) and (1.12).

Proof. Due to the linearity of Maxwell's equations, let's separate the solution (1.11) and (1.12) into two solutions, one for the initial values and one for the field source, and verify them separately.

Initial values solution:

$$H(\theta)=H(0)+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2(n+1)}}{\big(2(n+1)\big)!}\nabla^{2(n+1)}\times H(0)+\frac{1}{\eta}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!}\nabla^{2n+1}\times E(0)$$	(1.11i)
$$E(\theta)=E(0)+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2(n+1)}}{\big(2(n+1)\big)!}\nabla^{2(n+1)}\times E(0)+\eta\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n+1}}{(2n+1)!}\nabla^{2n+1}\times H(0)$$	(1.12i)

Field source solution:

$$H(\theta)=\sum_{n=0}^{\infty}\frac{\theta^{2(n+1)}}{\big(2(n+1)\big)!}\sum_{m=0}^{n}(-1)^{m}\nabla^{2m+1}\times\frac{\partial^{2(n-m)}J(0)}{\partial\theta^{2(n-m)}}+\sum_{n=1}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\sum_{m=1}^{n}(-1)^{m+1}\nabla^{2m-1}\times\frac{\partial^{2(n-m)+1}J(0)}{\partial\theta^{2(n-m)+1}}$$	(1.11f)
$$E(\theta)=\eta\sum_{n=0}^{\infty}\frac{\theta^{2(n+1)}}{\big(2(n+1)\big)!}\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n-m)+1}J(0)}{\partial\theta^{2(n-m)+1}}+\eta\sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n-m)}J(0)}{\partial\theta^{2(n-m)}}$$	(1.12f)

Take derivative with respect to θ on both sides of (1.11i), and perform curl on both sides of (1.12i), we have

$$\frac{\partial}{\partial\theta}H(\theta)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!}\nabla^{2(n+1)}\times H(0)+\frac{1}{\eta}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n}}{(2n)!}\nabla^{2n+1}\times E(0)$$

$$\nabla\times E(\theta)=\nabla\times E(0)+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2(n+1)}}{\big(2(n+1)\big)!}\nabla^{2n+3}\times E(0)+\eta\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n+1}}{(2n+1)!}\nabla^{2n+2}\times H(0)$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n}}{(2n)!}\nabla^{2n+1}\times E(0)-\eta\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!}\nabla^{2n+2}\times H(0)$$

$$=-\eta\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!}\nabla^{2n+2}\times H(0)+\frac{1}{\eta}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n}}{(2n)!}\nabla^{2n+1}\times E(0)\right)=-\eta\frac{\partial}{\partial\theta}H(\theta)$$

The above leads to

$$\frac{\partial}{\partial\theta}H(\theta)=-\frac{1}{\eta}\nabla\times E(\theta)$$

Thus, (1.11i) and (1.12i) satisfy (1.6).

Take derivative with respect to θ on both sides of (1.12i), and perform curl on both sides of (1.11i), we have

$$\frac{\partial}{\partial\theta}E(\theta)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!}\nabla^{2(n+1)}\times E(0)+\eta\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n}}{(2n)!}\nabla^{2n+1}\times H(0)$$

$$\nabla\times H(\theta)=\nabla\times H(0)+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2(n+1)}}{\big(2(n+1)\big)!}\nabla^{2n+3}\times H(0)+\frac{1}{\eta}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!}\nabla^{2n+2}\times E(0)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n}}{(2n)!}\nabla^{2n+1}\times H(0)+\frac{1}{\eta}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!}\nabla^{2n+2}\times E(0)$$

$$=\frac{1}{\eta}\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!}\nabla^{2n+2}\times E(0)+\eta\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n}}{(2n)!}\nabla^{2n+1}\times H(0)\right)=\frac{1}{\eta}\frac{\partial}{\partial\theta}E(\theta)$$

The above leads to

$$\nabla\times H(\theta)=\frac{1}{\eta}\frac{\partial}{\partial\theta}E(\theta)$$

Thus, (1.11i) and (1.12i) satisfy (1.7).

Thus, the generic solution for initial values satisfies Maxwell's equations (1.6) and (1.7).

Take derivative with respect to θ on both sides of (1.11f), and perform curl on both sides of (1.12f), we have

$$\frac{\partial}{\partial\theta}H(\theta)=\sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\sum_{m=0}^{n}(-1)^{m}\nabla^{2m+1}\times\frac{\partial^{2(n-m)}J(0)}{\partial\theta^{2(n-m)}}+\sum_{n=1}^{\infty}\frac{\theta^{2n}}{(2n)!}\sum_{m=1}^{n}(-1)^{m+1}\nabla^{2m-1}\times\frac{\partial^{2(n-m)+1}J(0)}{\partial\theta^{2(n-m)+1}}$$

$$\nabla\times E(\theta)=\eta\sum_{n=0}^{\infty}\frac{\theta^{2(n+1)}}{\big(2(n+1)\big)!}\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m+1}\times\frac{\partial^{2(n-m)+1}J(0)}{\partial\theta^{2(n-m)+1}}+\eta\sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m+1}\times\frac{\partial^{2(n-m)}J(0)}{\partial\theta^{2(n-m)}}$$

$$=\eta\sum_{n=1}^{\infty}\frac{\theta^{2n}}{(2n)!}\sum_{m=1}^{n}(-1)^{m}\nabla^{2m-1}\times\frac{\partial^{2(n-m)+1}J(0)}{\partial\theta^{2(n-m)+1}}+\eta\sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m+1}\times\frac{\partial^{2(n-m)}J(0)}{\partial\theta^{2(n-m)}}$$

$$=-\eta\frac{\partial}{\partial\theta}H(\theta)$$

The above leads to

$$\nabla\times E(\theta)=-\eta\frac{\partial}{\partial\theta}H(\theta)$$

Thus, (1.11f) and (1.12f) satisfy (1.6).

Take derivative with respect to θ on both sides of (1.12f), and perform curl on both sides of (1.11f), we have

$$\frac{\partial}{\partial\theta}E(\theta)=\eta\sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n-m)+1}J(0)}{\partial\theta^{2(n-m)+1}}+\eta\sum_{n=0}^{\infty}\frac{\theta^{2n}}{(2n)!}\sum_{m=0}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n-m)}J(0)}{\partial\theta^{2(n-m)}}$$

$$=\eta\Bigg(-\sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\frac{\partial^{2n+1}J(0)}{\partial\theta^{2n+1}}-\sum_{n=0}^{\infty}\frac{\theta^{2n}}{(2n)!}\frac{\partial^{2n}J(0)}{\partial\theta^{2n}}$$
$$+\sum_{n=1}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\sum_{m=1}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n-m)+1}J(0)}{\partial\theta^{2(n-m)+1}}+\sum_{n=0}^{\infty}\frac{\theta^{2(n+1)}}{(2(n+1))!}\sum_{m=0}^{n}(-1)^{m}\nabla^{2m+2}\times\frac{\partial^{2(n-m)}J(0)}{\partial\theta^{2(n-m)}}\Bigg)$$

$$=\eta\left(-J(\theta)+\sum_{n=1}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\sum_{m=1}^{n}(-1)^{m+1}\nabla^{2m}\times\frac{\partial^{2(n-m)+1}J(0)}{\partial\theta^{2(n-m)+1}}+\sum_{n=0}^{\infty}\frac{\theta^{2(n+1)}}{(2(n+1))!}\sum_{m=0}^{n}(-1)^{m}\nabla^{2m+2}\times\frac{\partial^{2(n-m)}J(0)}{\partial\theta^{2(n-m)}}\right)$$

$$\nabla \times H(\theta) = \sum_{n=0}^{\infty} \frac{\theta^{2(n+1)}}{(2(n+1))!} \sum_{m=0}^{n} (-1)^m \nabla^{2m+2} \times \frac{\partial^{2(n-m)} J(0)}{\partial \theta^{2(n-m)}} + \sum_{n=1}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!} \sum_{m=1}^{n} (-1)^{m+1} \nabla^{2m} \times \frac{\partial^{2(n-m)+1} J(0)}{\partial \theta^{2(n-m)+1}}$$

The above leads to

$$\frac{\partial}{\partial \theta} E(\theta) = \eta\big(-J(\theta) + \nabla \times H(\theta)\big)$$

Thus, (1.11f) and (1.12f) satisfy (1.7).

Thus, the generic solution for field source satisfies Maxwell's equations (1.6) and (1.7).

Thus, the generic solution (1.11) and (1.12) satisfies Maxwell's equations (1.6) and (1.7).

For the given values (2.1), (2.2) and(2.3), suppose there is another solution, $H_2(x,y,z,\theta)$ and $E_2(x,y,z,\theta)$, to Maxwell's equations (1.6) and (1.7), and $H_2(x,y,z,\theta)$ and $E_2(x,y,z,\theta)$ are not formed by (1.11) and (1.12).

If $H_2(x,y,z,\theta)$ and $E_2(x,y,z,\theta)$ satisfy the Time-Space theorem formulas (t.1) to (t.4) and (t.1J) to (t.4J) then $H_2(x,y,z,\theta)$ and $E_2(x,y,z,\theta)$ can be formed by (1.11) and (1.12). Thus, $H_2(x,y,z,\theta)$ and $E_2(x,y,z,\theta)$ do not satisfy (t.1) to (t.4) and (t.1J) to (t.4J). Thus, $H_2(x,y,z,\theta)$ and $E_2(x,y,z,\theta)$ do not satisfy Maxwell's equations (1.6) and (1.7). The uniqueness of the solution (1.11) and (1.12) is thus proved.

QED

We have completed a cycle of deductions. From Maxwell's equations, we deduced the Time-Space-Theorem, from the Time-Space Theorem formulas, we deduced the generic solution to Maxwell's equations, from the generic solution formulas, we proved that the generic solution formulas uniquely satisfy Maxwell's equations. Now we have confidence that all the formulas we deduced so far are correct.

To my knowledge, the only analytical solution in the history of Maxwell's equations was given by Maxwell himself. It was any 1D field with a constant speed of c. I'll show that this only analytical solution in the history is also contained in the above generic solution.

Suppose the x-axis is used. 1D source-free Maxwell's equations become

$\frac{\partial H_y}{\partial \theta} = \frac{1}{\eta}\frac{\partial E_z}{\partial x}$	(1d.1)
$\frac{\partial E_z}{\partial \theta} = \eta\frac{\partial H_y}{\partial x}$	(1d.2)

The solution Maxwell found is given in (1d.3) and (1d.4).

$H_y(x,\theta) = f(x+\theta) + A$	(1d.3)
$E_z(x,\theta) = \eta f(x+\theta)$	(1d.4)

Where A is a constant.

From (1d.3) and (1d.4), we know that the following initial values are used to produce this solution.

$H_y(x,0) = f(x) + A$	(1d.5)
$E_z(x,0) = \eta f(x)$	(1d.6)

For the 1D case, the generic solution (1.11i) and (1.12i) become (1d.7) and (1d.8).

$H_y(x,\theta) = H_y(x,0) + \sum_{n=0}^{\infty} \frac{\theta^{2(n+1)}}{(2(n+1))!}\frac{\partial^{2(n+1)} H_y(x,0)}{\partial x^{2(n+1)}} + \frac{1}{\eta}\sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!}\frac{\partial^{2n+1} E_z(x,0)}{\partial x^{2n+1}}$	(1d.7)
$E_z(x,\theta) = E_z(x,0) + \sum_{n=0}^{\infty} \frac{\theta^{2(n+1)}}{(2(n+1))!}\frac{\partial^{2(n+1)} E_z(x,0)}{\partial x^{2(n+1)}} + \eta\sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!}\frac{\partial^{2n+1} H_y(x,0)}{\partial x^{2n+1}}$	(1d.8)

Substitute initial values (1d.5) and (1d.6) into (1d.7) and (1d.8), we have

$$H_y(x,\theta) = A + \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!}\frac{\partial^{2n} f(x)}{\partial x^{2n}} + \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!}\frac{\partial^{2n+1} f(x)}{\partial x^{2n+1}}$$

$$E_z(x,\theta) = \eta\sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!}\frac{\partial^{2n} \eta f(x)}{\partial x^{2n}} + \eta\sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!}\frac{\partial^{2n+1} f(x)}{\partial x^{2n+1}}$$

The above is (1d.3) and (1d.4).

Examples of closed form analytical solutions

To my knowledge, except for the fields of 1D constant speed, there is not a closed form analytical electromagnetic field presented in literature. Below, I am providing 3 fields in closed analytical forms: an ever growing field, a sustained field and a decaying field.

Example 1: ever growing fields

The given values are

$J(x,y,z,\theta) = 0$	(e1.1)
$E(x,y,z,0) = \begin{bmatrix} x^2yz \\ -2xy^2z \\ xyz^2 \end{bmatrix}$	(e1.2)
$H(x,y,z,0) = 0$	(e1.3)

We may deduce curls from (e1.2) and insert the curls into the generic solution (1.11) and (1.12).

(1.11) and (1.12) become

$H(x,y,z,\theta) = -\frac{1}{\eta}\theta \begin{bmatrix} x(z^2+2y^2) \\ y(x^2-z^2) \\ -z(x^2+2y^2) \end{bmatrix} + \frac{1}{\eta}\theta^3 \begin{bmatrix} -x \\ 0 \\ z \end{bmatrix}$	(e1.4)
$E(x,y,z,\theta) = \begin{bmatrix} x^2yz \\ -2xy^2z \\ xyz^2 \end{bmatrix} - \theta^2 \begin{bmatrix} -yz \\ 2xz \\ -xy \end{bmatrix}$	(e1.5)

We can see that the fields are getting stronger and stronger over time.

Example 2: sustained fields

The given values are

$J(x,y,z,\theta) = 0$	(e2.1)
$E(x,y,z,0) = \begin{bmatrix} \cos(y)\cos(z)\sin(x) \\ -2\cos(z)\cos(x)\sin(y) \\ \cos(x)\cos(y)\sin(z) \end{bmatrix}$	(e2.2)
$H(x,y,z,0) = 0$	(e2.3)

From (e2.2), we can deduce the following formulas:

$\nabla^{2n} \times E(0) = 3^n \begin{bmatrix} \sin(x)\cos(y)\cos(z) \\ -2\cos(x)\sin(y)\cos(z) \\ \cos(x)\cos(y)\sin(z) \end{bmatrix}; n = 0,1,2,\ldots$	(e2.4)
$\nabla^{2n+1} \times E(0) = 3^{n+1} \begin{bmatrix} -\cos(x)\sin(y)\sin(z) \\ 0 \\ \sin(x)\sin(y)\cos(z) \end{bmatrix}; n = 0,1,2,\ldots$	(e2.5)

Insert the curls in (e2.4) and (e2.5) into the generic solution (1.11) and (1.12). (1.11) and (1.12) become

$H(\theta) = \frac{1}{\eta}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!} 3^{n+1} \begin{bmatrix} -\cos(x)\sin(y)\sin(z) \\ 0 \\ \sin(x)\sin(y)\cos(z) \end{bmatrix}$	(e2.6)
$E(\theta) = \begin{bmatrix} \cos(y)\cos(z)\sin(x) \\ -2\cos(z)\cos(x)\sin(y) \\ \cos(x)\cos(y)\sin(z) \end{bmatrix} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\theta^{2(n+1)}}{(2(n+1))!} 3^{n+1} \begin{bmatrix} \sin(x)\cos(y)\cos(z) \\ -2\cos(x)\sin(y)\cos(z) \\ \cos(x)\cos(y)\sin(z) \end{bmatrix}$	(e2.7)

The above summations of series can be reformed as following summations.

$$H(\theta) = -\frac{\sqrt{3}}{\eta}\sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{3}\theta)^{2n+1}}{(2n+1)!} \begin{bmatrix} -\cos(x)\sin(y)\sin(z) \\ 0 \\ \sin(x)\sin(y)\cos(z) \end{bmatrix}$$

$$E(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{3}\theta)^{2n}}{(2n)!} \begin{bmatrix} \sin(x)\cos(y)\cos(z) \\ -2\cos(x)\sin(y)\cos(z) \\ \cos(x)\cos(y)\sin(z) \end{bmatrix}$$

The summations of series have closed-form of $\sin(\sqrt{3}\theta)$ and $\cos(\sqrt{3}\theta)$. We have the closed-form solution

$H(x,y,z,\theta) = -\frac{\sqrt{3}}{\eta}\sin(\sqrt{3}\theta) \begin{bmatrix} -\cos(x)\sin(y)\sin(z) \\ 0 \\ \sin(x)\sin(y)\cos(z) \end{bmatrix}$	(e2.8)

$E(x,y,z,\theta)=\cos(\sqrt{3}\theta)\begin{bmatrix}\sin(x)\cos(y)\cos(z)\\-2\cos(x)\sin(y)\cos(z)\\\cos(x)\cos(y)\sin(z)\end{bmatrix}$	(e2.9)

We can see that the fields keep oscillating over time.

Example 3: Decaying fields

The given values are

$J(x,y,z,\theta)=0$	(e3.1)
$E(x,y,z,0)=\begin{bmatrix}yz\\-2zx\\xy\end{bmatrix}e^{-ar^2}$	(e3.2)
$H(x,y,z,0)=0$	(e3.3)
$r^2=x^2+y^2+z^2$	(e3.4)

For this example, curls are given below.

$\nabla^{2n}\times E(0)=\left((2a)^n\sum_{m=0}^{n}(-1)^m\frac{(2n+5)!}{2^{n-m}(n+2)(n+1)}\frac{(m+2)(m+1)}{(2m+5)!}\frac{1}{(n-m)!}(2ar^2)^m\right)\begin{bmatrix}yz\\-2zx\\xy\end{bmatrix}e^{-ar^2}$	(e3.5)
$\nabla^{2n+1}\times E(0)=e^{-ar^2}(2a)^n\begin{bmatrix}x(3P_7(n)-2a(y^2+2z^2)P_9(n))\\y2a(x^2-z^2)P_9(n)\\-z(3P_7(n)-2a(y^2+2x^2)P_9(n))\end{bmatrix}$	(e3.6)
$p_{7nm}=\frac{(2n+5)!\,(m+2)(m+1)}{2^{n-m}(n+2)(n+1)(2m+5)!\,(n-m)!}$	(e3.7)
$p_{9nm}=\frac{(2n+7)!\,(m+3)(m+2)(m+1)}{2^{n-m}(n+3)(n+2)(n+1)(2m+7)!\,(n-m)!}$	(e3.8)
$P_7(n)=\sum_{m=0}^{n}(-1)^m p_{7nm}(2ar^2)^m$	(e3.9)
$P_9(n)=\sum_{m=0}^{n}(-1)^m p_{9nm}(2ar^2)^m$	(e3.10)

Substitute the above curls into the generic solution (1.11) and (1.12), we have the closed-form solution

$H(x,y,z,\theta)=-\frac{e^{-a(r^2+\theta^2)}}{2ar\eta}\left(\sum_{m=0}^{2}2^m p_m(\phi)u_{3m}(v)\begin{bmatrix}3x\\0\\-3z\end{bmatrix}+2a\sum_{m=0}^{3}2^m p_m(\phi)u_{4m}(v)\begin{bmatrix}-x(y^2+2z^2)\\y(x^2-z^2)\\z(y^2+2x^2)\end{bmatrix}\right)$	(e3.11)
$E(x,y,z,\theta)=e^{-a(r^2+\theta^2)}\sum_{m=0}^{3}2^m p_m(\phi)\,\mathrm{s_m}(v)\begin{bmatrix}yz\\-2xz\\xy\end{bmatrix}$	(e3.12)

The symbols are given below

$v=2ar\theta$	(e3.13)
$\phi=-a\theta^2$	
$p_0(\phi)=1$	
$p_1(\phi)=\phi$	
$p_2(\phi)=\phi+\phi^2$	
$p_3(\phi)=\phi+3\phi^2+\phi^3$	
$s_0(v)=\cosh v$	
$s_1(v)=\frac{1}{v^5}((3v^4+14v^2+24)\sinh v-(6v^3+24v)\cosh v)$	
$s_2(v)=\frac{1}{v^5}((3v^3+18v)\cosh v-(9v^2+18)\sinh v)$	
$s_3(v)=\frac{1}{v^5}((v^2+3)\sinh v-3v\cosh v)$	
$u_{30}(v)=\sinh v$	
$u_{31}(v)=\frac{1}{v^4}((2v^3+6v)\cosh v-(4v^2+6)\sinh v)$	
$u_{32}(v)=\frac{1}{v^4}((v^2+3)\sinh v-3v\cosh v)$	
$u_{40}(v)=\sinh v$	
$u_{41}(v)=\frac{1}{v^6}\left((3v^5+26v^3+120v)\cosh v-(9v^4+66v^2+120)\sinh v\right)$	
$u_{42}(v)=\frac{1}{v^6}\left((3v^4+45v^2+90)\sinh v-(15v^3+90v)\cosh v\right)$	
$u_{43}(v)=\frac{1}{v^6}\left((v^3+15v)\cosh v-(6v^2+15)\sinh v\right)$	

From the solution, we can see that the strength of this electromagnetic field goes to 0 as time and space approach infinity.

Discussions

I hope this generic solution to Maxwell's equations may find its engineering/academic applications.

Getting this generic solution is just a beginning. There are lots of issues. I'll talk a little bit below.

Principle of locality

The generic solution to Maxwell's equations contradicts the "Principle of locality". For example, consider a pulse source having non-0 value at one space point and 0 for all other space locations, as shown below.

$$J(x,y,z,t)=\begin{cases} J_0(t), & x^2+y^2+z^2=0 \\ 0, & x^2+y^2+z^2>0 \end{cases}$$

It also implies that

$$\frac{\partial^n J(x,y,z,t)}{\partial x^n}=\frac{\partial^n J(x,y,z,t)}{\partial y^n}=\frac{\partial^n J(x,y,z,t)}{\partial z^n}=0; for\ n\geq 0\ and\ \ x^2+y^2+z^2>0$$

Using an FDTD algorithm to estimate the solution to Maxwell's equations for such a pulse source, the non-0 value at one space point generates a radio transmission propagating to the whole universe, demonstrating the effects of the "Principle of locality".

To apply the generic solution to Maxwell's equations for a pulse source, we need to make it differentiable. We may use a Gaussian factor to do it.

$$\tilde{J}(x,y,z,t)=e^{-ar^2}J(t); r^2=x^2+y^2+z^2$$

We have

$$\lim_{a\to+\infty}\tilde{J}(x,y,z,t)=J(x,y,z,t)$$

And

$$\lim_{a\to+\infty}\frac{\partial^n \tilde{J}(x,y,z,t)}{\partial x^n}=\lim_{a\to+\infty}\frac{\partial^n \tilde{J}(x,y,z,t)}{\partial y^n}=\lim_{a\to+\infty}\frac{\partial^n \tilde{J}(x,y,z,t)}{\partial z^n}=0; for\ n\geq 0\ and\ \ x^2+y^2+z^2>0$$

Substituting the above values into the generic solution, we can see that when $a\to+\infty$ there is not a radio transmission occurs, the non-0 electromagnetic field remains at the space point $(x=0,y=0,z=0)$. The electromagnetic field remains 0 for all other space locations. The radio transmission occurs only for $a<\infty$. For $a<\infty$, $\tilde{J}(x,y,z,t)$ is defined for the whole universe. Now we see that a local cause cannot affect other places unless it is pre-defined for the other places before the time starts ticking. An instant connection in the whole universe is required.

Space-restricted problem

In many situations, space boundaries are involved. For example, in the area of lightning research, see [8], the ground is a boundary. One possible way to handle it might be to treat boundary conditions as special field sources.

Presenting sources in differentiable formats

To get closed form solutions to Maxwell's equations, we need to express field sources in differentiable math functions. For example, in the area of antenna design, see [9], we need to use differentiable math expressions to represent an antenna design.

The above 3 issues all have something to do with the math continuity requirement of the solution.

Getting closed-form analytical solutions

For given initial values and field sources, it is possible to get closed form analytical solutions, as the 3 examples I presented. But the process needs patience. I currently do not see a generic approach. It can be an interesting research topic. Probably involving some sort of AI.

Numeric calculations

Precise numerical values of dynamic electromagnetic fields can be helpful in academic and engineering applications. For some solutions, probably many of them, making calculations for large time and space values can be a big challenge. Analytical and numerical research on the unique characteristics of the generic solution given by (1.11) and (1.12) will help the calculations. This is a wide research area.

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