Refute light speed constancy by math

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Abstract 3D and 1D analytical solutions to Maxwell's equations are given. From these solutions, the propagation speeds of the electromagnetic fields can be analytically derived. The speed formulas show that the speeds are not constant. For a light source with limit strength, the speed rises from 0 to *c* quickly but a transition period is required. The length of the transition period depends on the strength of the source. Numerical calculations using the formulas give a hint that a Dirac delta function could generate a constant propagation speed. Practically we see a light of constant speed, but mathematically the speed is not constant. This mathematical result reveals a fact that the constant light speed *c* is a "good enough" approximation, not an axiom. Once the axiom is removed Einstein's relativity theories collapse due to losing their foundation. This paper only presents results with stationary sources. The results of moving sources are presented in another paper.

Keywords: velocity of light, Maxwell's equations, analytical solution, special relativity, Einstein

Contents

Introduction

On July 14, 1972 the results of the Hafele–Keating experiment was published [1]. The whole world hailed the experiment as a big victory of Einstein. Such an experiment was mentioned as early as 1957 by S.F. Singer [2]. W. Cochran suggested using particles to do similar experiments [3]. Experiments with π mesons and γ-ray proved that Einstein's theory of special relativity was highly accurate [4]. Special Relativity is considered experimentally verified by a vast number of experiments to very high precision, and thus accepted by the physics community [5].

But M. von Laue, N. R. Campbell, and Paul Ehrenfest, all point out that experiment results cannot be used to tell Einstein's theory from Lorentz's theory, [6] : page 510.

Mr. L. Essen points out that those experiment results actually invalidate Special Relativity and Einstein's "thought experiement":

> *If the "experiment" is carried out correctly it should therefore give the same readings for the clocks as they would have when they are in uniform relative motion. The result given by Einstein does not follow from the "experiment" but from an assumption made implicitly that the clock which does the round trip is actually going slower than the one regarded as stationary and does not simply appear to go slower as viewed by the stationary observer [7].*

Professor H. Dingle reminds us that "*the disproof of Einstein's theory … leaves Lorentz's intact*" [8].

Professor W. M. McCrea does not see there is a difference between Einstein's theory and Lorentz's theory [9], [10].
Mathematician Whittaker also does not distinguish the two theories [8].

Today, we, or most of us, consider that there is just one theory because Einstein's Special Relativity are presented in the name of Lorentz: Lorentz transformation, Lorentz Covariance of Maxwell's equations, Lorentz invariant, Lorentz group, etc. [11], [12], [13].
Before 1919, they were considered two distinct theories. Dingle tells us:

> *Poincare, as late as 1912, spoke of "le principle de relativite de Lorentz", even in a paper in which he was discussing Einstein's view of the action of light on molecules [8].*

Kenneth F. Schaffner points out that Lorentz's 1904 paper [14] includes both reciprocal and nonreciprocal transformations:

> *…Lorentz's interpretation not only of the contraction hypothesis and the "local" time hypothesis, but also of any transformation equation (e.g., force, mass, charge density) relating moving and rest systems is a nonreciprocal interpretation and quite antithetical to a theory of relativity. [6]: page 504.*

In Einstein's 1905 paper [15], only reciprocal transformations are included, see the Lorentz transformation formulas at the end of section "§ 3", the Lorentz state transformation formulas at the end of section "§ 6", and the principle of relativity formulas shown by the 1st set and the 3rd set of equations in section "§ 6".

Let's hear what Lorentz said about his theory and Einstein's theory:

> *…the chief difference being that Einstein simply postulates what we have deduced, with some difficulty and not altogether satisfactory, from the fundamental equations of the electromagnetic field. [6]: page 509.*

It seems Einstein took the credit for Lorentz's hard work. Mathematician Whittaker gives the credit back to Lorentz by entitling his chapter on the theory, "The Relativity Theory of Poincare and Lorentz" [8].

But Lorentz recognized Einstein's significant contributions:

> *…By doing so, he may certainly take credit for making us see in the negative result of experiments like those of Michelson, Rayleigh and Brace, not a fortuitous compensation of opposing effects, but the manifestation of a general and fundamental principle. [6]: page 509.*

Einstein's contributions are to use "principles" to explain Lorentz's formulas.
After many years' hard work, Lorentz finally deduced his formulas to match experiment results; but he was facing a tough job of providing causes for his formulas, for example, by molecules deformations, electron deformations, ether properties, etc., "*and not altogether satisfactory*", as Lorentz said.
Einstein uses postulates as principles:

> *We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, …. These two postulates …*
> *The following reflexions are based on the principle of relativity and on the principle of the constancy of the velocity of light. [15]*

A principle in physics is kind of like an axiom in mathematics. That means, no more work is needed to find the causes for Lorentz's formulas.

Einstein made 3 postulates out of what Lorentz deduced:
The 1st postulate: the principle of relativity.
The 2nd postulate: the principle of constancy of the velocity of light.
The 3rd postulate: the principle that nothing moves faster than light.

1. Lorentz uses his coordinate transformation and state transformation to make the electromagnetic field look inertial on the surface of the Earth. Einstein generalizes Lorentz's result to say that everything is inertial, and expresses it as the principle of relativity. By generalizing it to be a "principle" it covers the whole universe for everything, electrodynamics and kinematics alike.
2. Lorentz deduces a result of "constant light speed" from results of experiments. Einstein says that the constancy of the velocity of light is a principle.
3. Lorentz says that his formulas are for any velocity less than c. Einstein generalizes Lorentz's restriction into a principle that nothing can move faster than c.

About the 3rd principle:
Professor H. Dingle points out that it is baseless to claim that nothing can move faster than c. Lorentz never claimed that. Dingle said:

> *Lorentz specifically restricted his theory to "a system moving with any velocity less than that of light", and, from the nature of its effects, it must break down well short of that velocity, just as Boyle's law breaks down well before the volume of a gas shrinks to nothing; it makes the "light barrier" no more necessarily impassable than the "sound barrier" [8].*

About the 1st principle:
The principle of relativity causes a paradox. This paradox was spotted by Paul Langevin in 1911. Papers about this paradox appeared in Nature at least as early as 1939. Lots of prominent physicists joined a debate in Nature and lasted for at least 35 years. Since then numerous authors have claimed that they settled the debate once and for all. But new schemes keep coming up to settle the debate over and over again. In 2022, a new settlement is proposed [16]. I surveyed more than 30 papers from 1904 to 2022 and found that a correct understanding of the paradox must include a juxtaposition of views from Lorentz's nonreciprocal use of his transformation and Einstein's reciprocal use of the same transformation.

About the 2nd principle:
I have not seen an objection to this principle in literature. This principle is taught as a fundamental law [12].

Let's see how Einstein presents the 2nd principle.

In the first paragraph of [15], Einstein tells us that his theory is to fix an asymmetry in "**Maxwell's electrodynamics**" for "**moving bodies**".
In the second paragraph of [15], Einstein says that his theory is "***based on Maxwell's theory for stationary bodies***".
In section 6 of [15], Einstein works on Maxwell's equations for moving frames and deduces Lorentz Covariance of Maxwell's equations, and thus fulfills his promise of removing the asymmetry from Maxwell's electrodynamics, by giving two sets of symmetric Maxwell's equations: the first set and the third set of equations in section 6 of [15].

I need to remind the readers that if you want to introduce other theories, such as quantum mechanics, gravity, etc., into Special Relativity, then you are creating your own theories, not Einstein's theory. Einstein's Special Relativity is self-complete and strictly within the scope of Maxwell's equations.

In the first paragraph of [15], Einstein presents the foundation of his theory:

> *light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. [15]*

In [17], Einstein gives a definite meaning for the above statement, in a footnote,

> ***The principle of the constancy of the velocity of light is of course contained in Maxwell's equations****". [17]*

Prof. D V Redžić gives a remark:

> *The laconic footnote probably reveals Einstein's original train of thought leading him to special relativity: from Maxwell's equations to the constancy of the speed of light, and through the principle of relativity and properties of space and time, to the Lorentz transformations. (arXiv:2404.19566v2 [physics.class-ph] 6 Sep 2024).*

Therefore, if Einstein's above footnote is wrong then the special theory of relativity collapses due to losing its foundation. This is what I am going to present.

I am going to deal with stationary light sources here. For "*independent of the state of motion of the emitting body*", I handle the moving source in another paper which gives much more interesting math results.

Because Einstein uses words "**of course**", hardly anyone would ask why Maxwell's equations give "constancy of the velocity of light". The origin of it was from wave equations derived from Maxwell's equations [18].

For 1D fields, the wave equation becomes

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f(x,t)}{\partial t^2}$$

The above wave equation has an analytical solution which is of constant speed c:

$$f(x,t) = f(x - ct)$$

It does not say that any solution is of constant speed c.

Actually from a control engineer's point of view, $f(x - ct)$ is a stable solution, not a dynamic solution. It is not driven by an input, i.e., a light source.

In control engineering, a dynamic system is often described by constant linear ordinary differential equations. It is possible to get analytical solutions and get a deep understanding of the characteristics of the solutions, see [19].

Thus, every control engineer knows that a dynamic system has a dynamic period. A concept of "constant speed" for dynamic systems is highly questionable due to the existence of dynamic periods.

But in physics, solutions to Maxwell's equations are hard to get without multi-level integrations, Oleg D. Jefimenko (1992) [20], Valery Yakhno (2020) [21]. The generic characteristics of the solutions are also not understood as much as presented in [19] for ordinary differential equations. Powerful analysis tools, such as the root-locus analysis, in control engineering [19], are scarce for Maxwell's equations.

Unlike states of ordinary differential equations, the 6 electromagnetic components to be solved are coupled spatially. The state of the art approach is to evade the problem by taking derivatives on Maxwell's equations to get 6 wave equations. Each wave equation is for one electromagnetic component only. Thus, only single level integrations are needed. M. Lax, et al (1975) [22] get solutions for paraxial waves in this approach. For given spectrums, Roger L. Garay-Avendaño, et al (2014) [23], Michel Zamboni-Rached, et al (2017) [24] and J. Nobre-Pereira (2024) [25] solve the integrations by getting Fourier transformation coefficients, and thus get analytical solutions. These excellent results have important engineering applications:

> *Applications for these structured non-diffracting beams are many, such as in optical tweezers, optical atom guiding, medical imaging, medical treatments, remote sensing, optical communications, military applications and so on. [25]*

Roger L. Garay-Avendaño, et al [23] choose Lorentz gauge, get a vector potential, and analytical solutions to Maxwell's equations are obtained, the electric field and magnetic field are linked by the vector potential.

Note that, unlike wave equations, even for 1D Maxwell's equations, the electric field and the magnetic field are still coupled spatially, just that the curl is reduced to the spatial derivative.

After 7 years of research I have developed new techniques and skills to solve Maxwell's equations analytically and generically, in open space, without using integrations, without wave equations and gauges. I am not going to teach the readers how to use my technology to solve Maxwell's equations in this paper. I am going to present several analytical solutions I thus obtained, and show that the principle of the constancy of the velocity of light is not true, just as in the case of control engineering.

My math work proves that "constant light speed" is a good-enough approximation for Lorentz's work, but it is not an axiom and cannot be used as a principle to explain Lorentz's work by symmetry. We must go back to Lorentz's original idea of asymmetry: absolute movement caused time-dilation.

For math calculations, the difference between a good-enough approximation and an axiom is subtle.
But for Special Relativity, the difference prevents it from being a valid theory.
When "constant light speed" is not an axiom, Einstein's theory collapses leaving the following debris ([15]: "I. KINEMATICAL PART").

- The relativistic clock synchronization defined by $t_B - t_A = t'_A - t_B$ is no longer valid
- The universal constant defined by $c = \frac{2AB}{t'_A - t_A}$ is no longer valid
- The Lorentz invariant $x^2 + y^2 + z^2 - c^2t^2$ is not an invariant anymore
- All formulas involving constant c are no longer valid
- Formulas for time-dilation/length-contraction do not belong to SR anymore:
$$time\ dilation\ and\ length\ contraction \neq special\ relativity$$
- Experiments of time-dilation/length-contraction do not verify SR anymore. They only verify Lorentz's formulas and his original ideas.

Einstein said that Special Relativity is an approximation to General Relativity in low gravity:

> *the special theory of relativity provides only an approximation to reality; it should apply only in the limit case where differences in the gravitational potential in the space-time region under consideration are not too great. [26, V4, D13, p.153]*

The word "approximation" used in this way by no means can deny that Special Relativity is a rigorous theory built on axioms. If any of the axioms does not hold then the theory collapses.

General Relativity also faces this "axiom crisis".

You may not agree that the new knowledge I present leads to an "axiom crisis". But at least for those researchers who's research involves light speed c, this new knowledge should be included in their knowledge database.

Let me show and prove the mathematical fact which refutes light speed constancy to be an axiom.

3D stable solutions

Even for stable solutions, the speeds are not necessarily constant *c*. I'll give two examples.

The two examples given below show true analytical 3D solutions to Maxwell's equations other than the form of $f(x - ct)$. Had such analytical solutions been available and their speeds been noticed then the belief in the concept of "constancy of light speed" would have been questioned already.

I present them for demonstration purposes only. The strict research should be done on dynamic solutions, which is provided in the next section.

Example 1. Sustained waves

$$\theta = ct;\ \eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{e1.1}$$

$$H(x,y,z,\theta) = -\frac{\sqrt{3}}{\eta}\sin\left(\sqrt{3}\theta\right)\begin{bmatrix} -\cos(x)\sin(y)\sin(z) \\ 0 \\ \sin(x)\sin(y)\cos(z) \end{bmatrix} \tag{e1.2}$$

$$E(x,y,z,\theta) = \cos\left(\sqrt{3}\theta\right)\begin{bmatrix} \sin(x)\cos(y)\cos(z) \\ -2\cos(x)\sin(y)\cos(z) \\ \cos(x)\cos(y)\sin(z) \end{bmatrix} \tag{e1.3}$$

Because the phases of the above waves do not change, the phase speeds are 0.

Example 2. Ever growing fields

$$H(x,y,z,\theta) = -\frac{1}{\eta}\theta\begin{bmatrix} x(z^2+2y^2) \\ y(x^2-z^2) \\ -z(x^2+2y^2) \end{bmatrix} + \frac{1}{\eta}\theta^3\begin{bmatrix} -x \\ 0 \\ z \end{bmatrix} \tag{e2.1}$$

$$E(x,y,z,\theta) = \begin{bmatrix} x^2yz \\ -2xy^2z \\ xyz^2 \end{bmatrix} - \theta^2\begin{bmatrix} -yz \\ 2xz \\ -xy \end{bmatrix} \tag{e2.2}$$

The speeds in different directions are different.

The readers can easily verify that the above two 3D fields satisfy Maxwell's equations.

I have formed a stable solution with decaying fields. The formulas are too long, not easy to be verified by the readers. I am not presenting it here to avoid blurring this paper's focus.

I'll show a dynamic solution driven by a Gaussian source. This is the major result I am presenting.

1D Gaussian source solution

Consider 1D fields using the x-axis.
By applying a time scale, Maxwell's equations are given below.

For derivation of this time-scaled version of Maxwell's equations, see **Appendix A**. and **Appendix B**.

Symbols (A.3), (A.4) and (A.5) in Appendix A are collected into (1.0) below.
Equations (B.1) and (B.2) in Appendix B are listed below as (1.1) and (1.2).

$$\begin{gathered} \theta = ct \\ c = \frac{1}{\sqrt{\varepsilon\mu}} \\ \eta = \sqrt{\frac{\mu}{\varepsilon}} \end{gathered} \tag{1.0}$$

$$\frac{\partial H_y(x,\theta)}{\partial\theta} = \frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} \tag{1.1}$$

$$\frac{\partial E_z(x,\theta)}{\partial\theta} = \eta\frac{\partial H_y(x,\theta)}{\partial x} - \eta J_z(x,\theta) \tag{1.2}$$

A Gaussian source is used to generate the electromagnetic field:

$$J_z(x,\theta) = e^{-ax^2};\ a > 0 \tag{1.3}$$

$$\begin{gathered} H_y(x,0) = 0 \\ E_z(x,0) = 0 \end{gathered} \tag{1.4}$$

The solution to the above Maxwell's equations (1.1) and (1.2), using the source (1.3) and the initial values (1.4), is given below.

$$\xi = \sqrt{a}\theta \tag{1.5}$$

$$\varrho = 2\sqrt{a}x \tag{1.6}$$

$$H_y(x,\theta) = \frac{1}{\sqrt{a}}e^{-ax^2}esinh(\xi,\varrho)_1 \tag{1.7}$$

$$E_z(x,\theta) = -\eta\frac{1}{\sqrt{a}}e^{-ax^2}eicoshi(\xi,\varrho) \tag{1.8}$$

Where $esinh(\xi,\varrho)_1$ and $eicoshi(\xi,\varrho)$ are hyper-exponential functions. The two hyper-exponential functions are given by summations, as shown below.

$$esinh(\xi,\varrho)_1 = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2(n+1)}}{(2(n+1))!} \sum_{k=0}^{n} (-1)^k p_{0,n,k} \varrho^{2k+1} \tag{1.9}$$

$$eicoshi(\xi,\varrho) = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n+1}}{(2n+1)!} \sum_{k=0}^{n} (-1)^k q_{0,n,k} \varrho^{2k} \tag{1.10}$$

Where $p_{0,n,k}$ is named "odd binomial coefficients", and $q_{0,n,k}$ is named "even binomial coefficients", as given below. Collectively I name them "double binomial coefficients".

$$p_{0,n,k} = \frac{(2n+1)!}{(n-k)!\,(2k+1)!} \tag{1.11}$$

$$q_{0,n,k} = \frac{(2n)!}{(n-k)!\,(2k)!} \tag{1.12}$$

I am not going to show how to use my techniques to solve Maxwell's equations and get the above solution. For this paper it is more important to prove that the above is indeed the solution. That is, Eqs. (1.5) to (1.12) satisfy Eqs. (1.0) to (1.4).
The proof of the above solution is given in **Appendix C**.

The above solution certainly cannot be expressed by $f(x-ct)$. Its speeds are not constant c as Einstein wanted.

I'll use both analytical methods and numerical calculations to investigate the speeds of these fields. First, let's get a feeling of how these fields propagate in space over time.

Propagation of the electromagnetic field

Since we have a closed-form analytical solution to Maxwell's equations, we may calculate the fields precisely. To make calculations in long ranges of time and space, high performance computing (HPC) is needed.
For this purpose I assembled a desktop computer from scratch with an Intel 24 core CPU (Core i-9 14900K) and a Nvidia 3036 core GPU (GeForce 4096 TI).

I spent a lot of effort on the Nvidia GPU, coding in CUDA. But I could not get good HPC performance out of the GPU.
On reading some reports from the web, including a report from University of Tsukuba (Daisuke Takahashi, FFT and Parallel Numerical Libraries, Center for Computational Sciences, University of Tsukuba, 2014/2/19), I gave up my efforts to try to use GeForce GPU.
Even though Takahashi's report was from 10 years ago, its results matched what I got with my newer CPU and GPU.
I ended up relying on Intel's CPU to make the calculations.

Boost multiprecision library (https://gmplib.org/) was used to make calculations in decimal precisions of 4000 digits.

Fig.1. and Fig.2. show the propagations of the magnetic field and the electric field, respectively.

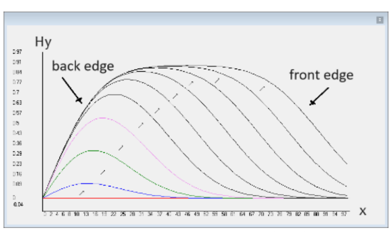

Fig.1. propagation of $H_y(x,\theta)$

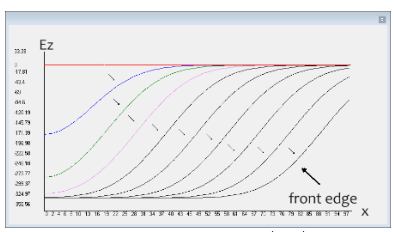

Fig.2. propagation of $E_z(x,\theta)$

Each curve is a snapshot of the field at one time.
The next curve on the right side of each curve shows the propagation of the field during the time interval between the two curves.

The propagations are visually demonstrated by the moving of the front edges; I use small arrows to mark the directions of the moving of the front edges.
The same time interval is used in Fig.1. and Fig.2., so, the closer the two curves the slower the speed of the propagation.

Let's find out math formulas for the speeds.

Speed formulas

From Fig.1 and Fig.2 we can see that the front edges appear at the times when the magnitudes of the fields lose their growths.

Let’s show field derivatives with respect to space.

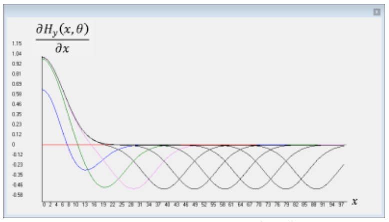

Fig.3. propagation of $\partial H_y(x,\theta)/\partial x$

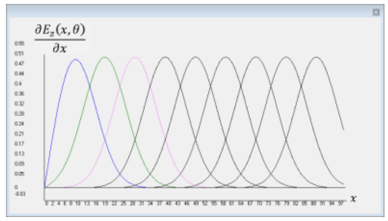

Fig.4. propagation of $\partial E_z(x,\theta)/\partial x$

The crests in Fig.3 and Fig.4 show where/when the fields get their maximum growths.

The crests in Fig.3 is given by

$$\frac{\partial^2 H_y(x,\theta)}{\partial x^2} = 0$$

From the above formula we can find out the speed formula of $\frac{dx}{d\theta}$ for $H_y(x,\theta)$.

The crests in Fig.4 is given by

$$\frac{\partial^2 E_z(x,\theta)}{\partial x^2} = 0$$

From the above formula we can find out the speed formula of $\frac{dx}{d\theta}$ for $E_z(x,\theta)$.

Speed of the electric field

In Appendix C while proving the solution to Maxwell’s equations, it gives

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} = e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k}$$

See section “Verify (1.1)” in Appendix C.

By (1.11), the definition of $p_{0,n,k}$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k} = \sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\xi^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k\varrho^{2k+1}}{(n-k)!\,(2k+1)!}$$

Apply the summation rule (D.1), see Appendix D, we have

$$\sum_{n=0}^{\infty}(-1)^n\xi^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k\varrho^{2k+1}}{(n-k)!\,(2k+1)!} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^n\xi^{2n+1}(-1)^k\varrho^{2k+1}}{(n-k)!\,(2k+1)!}$$

Let

$$m = n - k$$

We have

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^n\xi^{2n+1}(-1)^k\varrho^{2k+1}}{(n-k)!\,(2k+1)!} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+k}\xi^{2(m+k)+1}(-1)^k\varrho^{2k+1}}{m!\,(2k+1)!}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\xi^{2m+2k+1}\varrho^{2k+1}}{m!\,(2k+1)!}$$

$$= \sum_{k=0}^{\infty}\frac{(\xi\varrho)^{2k+1}}{(2k+1)!}\sum_{m=0}^{\infty}\frac{(-1)^m\xi^{2m}}{m!}$$

$$= \sinh(\varrho\xi)\, e^{-\xi^2}$$

We have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} = e^{-(ax^2+\xi^2)}\sinh(\varrho\xi) \tag{3.1}$$

From the above formula, we can get the following result.

$$\frac{1}{\eta}\frac{\partial^2 E_z(x,\theta)}{\partial x^2} = -2axe^{-(ax^2+\xi^2)}\sinh(\varrho\xi) + e^{-(ax^2+\xi^2)}\cosh(\varrho\xi)\, 2\sqrt{a}\xi$$

$$= \sqrt{a}e^{-(ax^2+\xi^2)}\left(-2\sqrt{a}x\sinh(\varrho\xi) + \cosh(\varrho\xi)\, 2\xi\right)$$

Because $\varrho = 2\sqrt{a}x$ we have

$$\frac{1}{\eta}\frac{\partial^2 E_z(x,\theta)}{\partial x^2} = \sqrt{a}e^{-(ax^2+\xi^2)}(2\xi\ \cosh(\varrho\xi) - \varrho\sinh(\varrho\xi)) \tag{3.2}$$

The moving of the front edge is given by

$$\frac{\partial^2 E_z(x,\theta)}{\partial x^2} = 0 \tag{3.3}$$

From (3.2) and (3.3) we have

$$\varrho\sinh(\varrho\xi) = 2\xi\ \cosh(\varrho\xi) \tag{3.4}$$

Take derivative on both sides of (3.4) with respect to ξ, we have

$$\frac{d\varrho}{d\xi}\sinh(\varrho\xi) + \varrho\cosh(\varrho\xi)\left(\varrho + \xi\frac{d\varrho}{d\xi}\right) = 2\cosh(\varrho\xi) + 2\xi\ \sinh(\varrho\xi)\left(\varrho + \xi\frac{d\varrho}{d\xi}\right)$$

$$\frac{d\varrho}{d\xi}\sinh(\varrho\xi) + \varrho^2\cosh(\varrho\xi) + \varrho\xi\frac{d\varrho}{d\xi}\cosh(\varrho\xi) = 2\cosh(\varrho\xi) + 2\varrho\xi\ \sinh(\varrho\xi) + 2\xi^2\frac{d\varrho}{d\xi}\sinh(\varrho\xi)$$

$$\frac{d\varrho}{d\xi}\sinh(\varrho\xi) - 2\xi^2\frac{d\varrho}{d\xi}\sinh(\varrho\xi) + \varrho\xi\frac{d\varrho}{d\xi}\cosh(\varrho\xi) = 2\cosh(\varrho\xi) + 2\varrho\xi\ \sinh(\varrho\xi) - \varrho^2\cosh(\varrho\xi)$$

$$\frac{d\varrho}{d\xi}\left((1-2\xi^2)\sinh(\varrho\xi) + \varrho\xi\cosh(\varrho\xi)\right) = (2-\varrho^2)\cosh(\varrho\xi) + 2\varrho\xi\ \sinh(\varrho\xi)$$

$$\frac{d\varrho}{d\xi} = \frac{(2-\varrho^2)\cosh(\varrho\xi) + 2\varrho\xi\ \sinh(\varrho\xi)}{(1-2\xi^2)\sinh(\varrho\xi) + \varrho\xi\cosh(\varrho\xi)}$$

Substitute (3.4) into above, we have

$$\frac{d\varrho}{d\xi} = \frac{(2-\varrho^2)\cosh(\varrho\xi) + 4\xi^2\ \cosh(\varrho\xi)}{\frac{(1-2\xi^2)2\xi}{\varrho}\ \cosh(\varrho\xi) + \varrho\xi\cosh(\varrho\xi)}$$

$$= \frac{(2-\varrho^2) + 4\xi^2}{\frac{(1-2\xi^2)2\xi}{\varrho} + \varrho\xi} = \frac{\varrho}{\xi}\frac{(2-\varrho^2) + 4\xi^2}{(1-2\xi^2)2 + \varrho^2}$$

We have

$$\frac{d\varrho}{d\xi} = -\frac{\varrho}{\xi}\cdot\frac{\varrho^2 - 4\xi^2 - 2}{\varrho^2 - 4\xi^2 + 2} \tag{3.5}$$

Because

$$\frac{dx}{d\theta} = \frac{1}{2}\frac{d\varrho}{d\xi}$$

We have the speed formula for the electric field:

$$\frac{dx}{d\theta} = -\frac{\varrho}{2\xi}\cdot\frac{\varrho^2 - 4\xi^2 - 2}{\varrho^2 - 4\xi^2 + 2} \tag{3.6}$$

Let’s see the speeds calculated by formula (3.6).

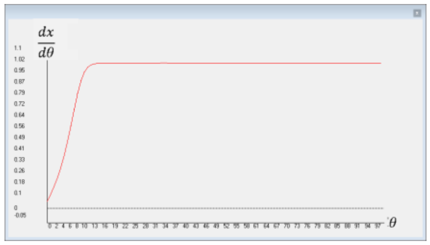

Fig.5. propagation speed of the electric field by formula (3.6)

The time range in Fig.5 is

$$t \in \left(0, \frac{100}{c}\right)$$

$$\left.\frac{dx}{dt}\right|_{t=\frac{10}{c}} = (0.99999999999995559\ldots)c$$

In a very short time the speed rises very close to *c*. When we physically measure the speed of the electric field, without extremely high precision measurement instruments, we will see that the speed looks a constant *c*.

Formula (3.6) tells us that the speed actually is not a constant *c*.

But it is a good enough approximation for Lorentz to assume $\frac{dx}{dt} \equiv c$ for deriving the Lorentz Transformation.

Now we see that the "constant light speed" is a good-enough approximation, not an axiom.

Speed of the magnetic field

In Appendix C while proving the solution to Maxwell's equations, it gives

$$\frac{\partial H_y(x,\theta)}{\partial x} = -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} + e^{-ax^2}$$

See section "Verify (1.2)" for the above formula.

The summations in the above formula can be further simplified as shown below.

$$\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} =$$

$$= \sum_{n=0}^{\infty}(-1)^n\xi^{2n}\sum_{k=0}^{n}\frac{(-1)^k\varrho^{2k}}{(n-k)!\,(2k)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n\xi^{2n}(-1)^k\varrho^{2k}}{(n-k)!\,(2k)!}$$

Apply the summation exchange rule (D.1), see Appendix D, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n\xi^{2n}(-1)^k\varrho^{2k}}{(n-k)!\,(2k)!} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^n\xi^{2n}(-1)^k\varrho^{2k}}{(n-k)!\,(2k)!}$$

Let

$$m = n - k$$

We have

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^n\xi^{2n}(-1)^k\varrho^{2k}}{(n-k)!\,(2k)!} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+k}\xi^{2m+2k}(-1)^k\varrho^{2k}}{m!\,(2k)!}$$

$$= \sum_{k=0}^{\infty}\frac{(\varrho\xi)^{2k}}{(2k)!}\sum_{m=0}^{\infty}\frac{(-1)^m\xi^{2m}}{m!} = \cosh(\varrho\xi)\,e^{-\xi^2}$$

We have

$$\frac{\partial H_y(x,\theta)}{\partial x} = e^{-ax^2}\left(1 - e^{-\xi^2}\cosh(\varrho\xi)\right) \tag{4.1}$$

From the above formula we can get the following result.

$$\frac{\partial^2 H_y(x,\theta)}{\partial x^2} = \sqrt{a}e^{-ax^2}(e^{-\xi^2}(\varrho\cosh(\varrho\xi) - 2\xi\sinh(\varrho\xi)) - \varrho) \tag{4.2}$$

The moving of the front edge is given by

$$\frac{\partial^2 H_y(x,\theta)}{\partial x^2} = 0 \tag{4.3}$$

From (4.2) and (4.3) we have

$$\varrho\cosh(\varrho\xi) - 2\xi\sinh(\varrho\xi) = \varrho\, e^{\xi^2} \tag{4.4}$$

Take derivative on both sides of (4.4) with respect to ξ, we have

$$\frac{d\varrho}{d\xi} = 2\frac{4\varrho\xi e^{\xi^2} + (4\xi^2 - \varrho^2 + 2)\sinh(\varrho\xi)}{(2 - 4\xi^2 + \varrho^2)\cosh(\varrho\xi) - (\varrho^2 + 2)e^{\xi^2}} \tag{4.5}$$

Because

$$\frac{dx}{d\theta} = \frac{1}{2}\frac{d\varrho}{d\xi}$$

We have the formula for the speed of the magnetic field:

$$\frac{dx}{d\theta} = \frac{4\varrho\xi e^{\xi^2} + (4\xi^2 - \varrho^2 + 2)\sinh(\varrho\xi)}{(2 - 4\xi^2 + \varrho^2)\cosh(\varrho\xi) - (\varrho^2 + 2)e^{\xi^2}} \tag{4.6}$$

Let's see the speed calculated by formula (4.6).

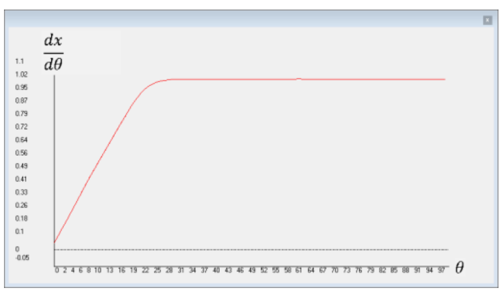

Fig.6. propagation speed of the magnetic field by formula (4.6)

Fig.6 shows that when we physically measure the speed of the magnetic field, we will see that the speed looks a constant *c*.

Formula (4.6) tells us that the speed actually is not a constant *c*.

But it is a good enough approximation for Lorentz to assume $\frac{dx}{dt} \equiv c$ for deriving the Lorentz Transformation.

Again, we see that the "constant light speed" is a good-enough approximation, not an axiom.

Length of dynamic period

Fig. 5 shows that the speed of the electric field has a dynamic period followed by a stable period.
Fig. 6 shows the same behavior of the speeds of the magnetic field.

Let's investigate the length of the dynamic periods. The only parameter we can change is the factor for the Gaussian function. In all the above figures, I am using

$$a = 1 \tag{5.1}$$

Let's examine $a\ in\ (0,\infty)$ to see how it affects the length of the dynamic periods.

For $a \to \infty$, $J_z(x,\theta) = e^{-ax^2} \to 0$. To keep the magnitudes of the output fields, let's change the source to

$$J_z(x,\theta) = \sqrt{a}e^{-ax^2} \tag{5.2}$$

The solution (1.7) and (1.8) become

$$H_y(x,\theta) = e^{-\frac{1}{4}\varrho^2} esinh(\xi,\varrho)_1 \tag{5.3}$$

$$E_z(x,\theta) = -\eta e^{-\frac{1}{4}\varrho^2} eicoshi(\xi,\varrho) \tag{5.4}$$

By (5.2), we have

$$\int_0^\infty J_z(x,\theta)dx = \int_0^\infty e^{-(\sqrt{a}x)^2} d\sqrt{a}x = \frac{\sqrt{\pi}}{2}$$

Thus, we have

$$\lim_{a\to\infty} J_z(x,\theta) = \sqrt{\pi}\delta(x) \cong \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \tag{5.5}$$

where $\delta(x)$ is the Dirac delta function.

The following figures show the effects of value a on the lengths of the dynamic periods.

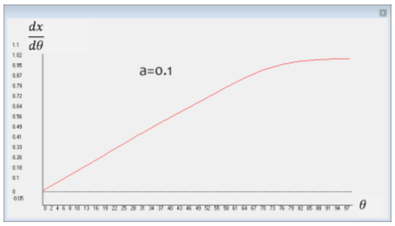

Fig.7. speed of Hy, a=0.1

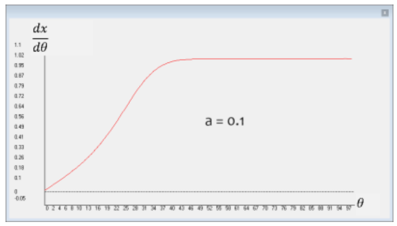

Fig.8. speed of Ez, a=0.1

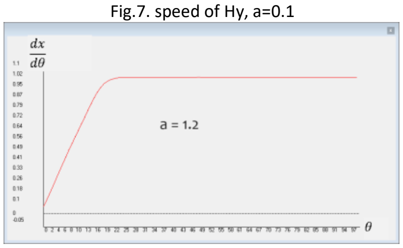

Fig.9. speed of Hy, a=1.2

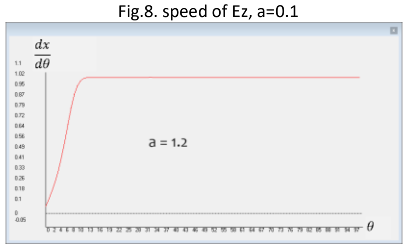

Fig.10. speed of Ez, a=1.2

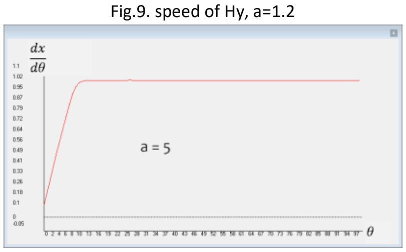

Fig.11. speed of Hy, a=5

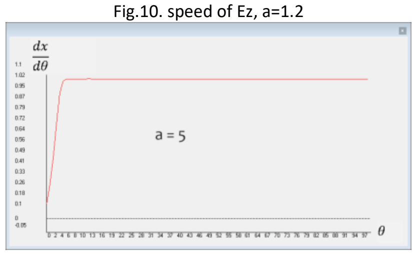

Fig.12. speed of Ez, a=5

From these drawings we can see that the larger value a the shorter the dynamic periods. Thus, it is reasonable to think that when $a \to \infty$ the lengths of the dynamic periods are 0. That is, the Dirac delta function generates a constant speed solution to Maxwell's equations.

Summary

The electric field and the magnetic field generated by a Gaussian source have the following characteristics.

1. their propagation speeds are not constant.
2. their propagation speeds approach the standard light speed *c* when the time approaches infinity.
3. At lower ranges of time and space, their propagation speeds change greatly. We may call this range the dynamic period. Their speeds are instantaneous speeds.
4. At higher ranges of time and space, their propagation speeds do not change much, the speeds are very close to the standard light speed *c*. We may call this range the stable period. The speed *c* is their stable speeds.
5. The transition from the dynamic period to the stable period can be smooth or sharp, depending on the Gaussian factor.
6. The lengths of the dynamic periods depend on the Gaussian factor. The larger the factor the shorter the dynamic period.
7. When the Dirac delta function is used as the source, the length of the dynamic period is 0. This is my observation from the numerical data, not theoretically proven.

Einstein based his relativity theories on his claim:

The principle of the constancy of the velocity of light is of course contained in Maxwell's equations. [17]

On solving Maxwell's equations, we know Einstein's claim is totally denied.

It is not that we found one specifically arranged counter-example.
It is that hardly a dynamic solution to Maxwell's equations, driven by a light source, can be of constant speed.

Maxwell's equations form a dynamic system. A dynamic system when driven by an input will always have a dynamic period. In the dynamic period, the speeds of its states can hardly be constant.

Einstein's relativity theories collapse on removing their foundation.

This paper only deals with the rest source. Einstein also claims:

> *light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. [15]*

I handle moving source solutions in another paper, discovering a new phenomenon of electromagnetic fields I call "pseudo inertial". The solutions to Maxwell's equations do exhibit an independence of source moving, but it is limited to the vincenity of the source.

Can we modify Special Relativity to make it consistent with the math result I provided above? Considering the logical error embedded in Special Relativity shown by the twin's paradox, my vote is to follow Lorentz's path:

> *Yet, I think, something may also be claimed in favor of the form in which I have presented the theory. I cannot but regard the ether, which can be the seat of an electromagnetic field with its energy and its vibrations, as endowed with a certain degrees of substantiality, however different it may be from all ordinary matter. [6] : page 510*

We need to resume Lorentz's efforts of finding the causes for time-dilation and length-contraction.

Appendix A. 3D Maxwell's equations with time scale

The curl-part of Maxwell's equations in free space are given below, [12]: page 3.

$$\frac{\partial H(x,y,z,t)}{\partial t} = -\frac{1}{\mu_0}\nabla \times E(x,y,z,t) \tag{A.1}$$

$$\frac{\partial E(x,y,z,t)}{\partial t} = \frac{1}{\varepsilon_0}\nabla \times H(x,y,z,t) - \frac{1}{\varepsilon_0}J(x,y,z,t) \tag{A.2}$$

$$E, H, J \in R^3$$

$$x, y, z, t, \varepsilon, \mu \in R$$

To further simplify notations, and make formula deductions less error-prone, a time scale of *ct* is use:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \tag{A.3}$$

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{A.4}$$

$$\theta = ct \tag{A.5}$$

Below I'll derive Maxwell's equations under the above time scaling.

Time scale on Eq.(A.1)

By (A.5) we have

$$\frac{\partial H(x,y,z,t)}{\partial \theta} = \frac{\partial H(x,y,z,t)}{\partial t}\frac{\partial t}{\partial \theta} = \frac{\partial H(x,y,z,t)}{\partial t}\frac{1}{c}$$

Eq. (A.1) becomes

$$\frac{\partial H(x,y,z,t)}{\partial \theta} = -\frac{1}{\mu_0}\nabla \times E(x,y,z,t)\frac{1}{c}$$

By (A.3) we have

$$\frac{\partial H(x,y,z,t)}{\partial \theta} = -\frac{1}{\mu_0}\nabla \times E(x,y,z,t)\sqrt{\varepsilon_0\mu_0} = -\nabla \times E(x,y,z,t)\sqrt{\frac{\varepsilon_0}{\mu_0}}$$

By (A.4) we have

$$\frac{\partial H(x,y,z,t)}{\partial \theta} = -\nabla \times E(x,y,z,t)\frac{1}{\eta}$$

The above can be written as

$$\frac{\partial H\left(x,y,z,\frac{1}{c}\theta\right)}{\partial \theta} = -\frac{1}{\eta}\nabla \times E\left(x,y,z,\frac{1}{c}\theta\right) \tag{A.1a}$$

Time scale on Eq.(A.2)

By (A.5) we have

$$\frac{\partial E(x,y,z,t)}{\partial \theta} = \frac{\partial E(x,y,z,t)}{\partial t}\frac{\partial t}{\partial \theta} = \frac{\partial E(x,y,z,t)}{\partial t}\frac{1}{c}$$

Eq. (A.2) becomes

$$\frac{\partial E(x,y,z,t)}{\partial \theta} = \left(\frac{1}{\varepsilon_0}\nabla \times H(x,y,z,t) - \frac{1}{\varepsilon_0}J(x,y,z,t)\right)\frac{1}{c}$$

By (A.3) we have

$$\frac{\partial E(x,y,z,t)}{\partial \theta} = \left(\frac{1}{\varepsilon_0}\nabla \times H(x,y,z,t) - \frac{1}{\varepsilon_0}J(x,y,z,t)\right)\sqrt{\varepsilon_0\mu_0}$$

$$= \sqrt{\frac{\mu_0}{\varepsilon_0}}\nabla \times H(x,y,z,t) - \sqrt{\frac{\mu_0}{\varepsilon_0}}J(x,y,z,t)$$

By (A.4) we have

$$\frac{\partial E(x,y,z,t)}{\partial \theta} = \eta\nabla \times H(x,y,z,t) - \eta J(x,y,z,t)$$

The above can be written as

$$\frac{\partial E\left(x,y,z,\frac{1}{c}\theta\right)}{\partial \theta} = \eta\nabla \times H\left(x,y,z,\frac{1}{c}\theta\right) - \eta J\left(x,y,z,\frac{1}{c}\theta\right) \quad \text{(A.2a)}$$

3D Maxwell's equations with time scale

Use new function names:

$$H_0(x,y,z,\theta) = H\left(x,y,z,\frac{1}{c}\theta\right)$$
$$E_0(x,y,z,\theta) = E\left(x,y,z,\frac{1}{c}\theta\right)$$
$$J_0(x,y,z,\theta) = J\left(x,y,z,\frac{1}{c}\theta\right)$$

Equations (A.1a) and (A.2a) can be written, using new function names H_0, E_0 and J_0, as

$$\frac{\partial H_0(x,y,z,\theta)}{\partial \theta} = -\frac{1}{\eta}\nabla \times E_0(x,y,z,\theta) \quad \text{(A.1b)}$$

$$\frac{\partial E_0(x,y,z,\theta)}{\partial \theta} = \eta\nabla \times H_0(x,y,z,\theta) - \eta J_0(x,y,z,\theta) \quad \text{(A.2b)}$$

Since I will not use (A.1) and (A.2) in formula deductions, I'll let time-scaled version use the simple format by writing (A.1b) and (A.2b) as (A.3) and (A.4):

$$\frac{\partial H(x,y,z,\theta)}{\partial \theta} = -\frac{1}{\eta}\nabla \times E(x,y,z,\theta) \quad \text{(A.3)}$$

$$\frac{\partial E(x,y,z,\theta)}{\partial \theta} = \eta\nabla \times H(x,y,z,\theta) - \eta J(x,y,z,\theta) \quad \text{(A.4)}$$

Note that (A.3) and (A.4) are not compatible with (A.1) and (A.2) in math notations. In the cases where time *t* has to be used, it is very important to pay attention to use the proper scale conversion given by (A.5).

It is a common practice to use time scaling. MIT does it in its FDTD software MEEP (https://meep.readthedocs.io/en/master/). Mr. Hannes Kohlmann makes proper time scale conversions accordingly. See section "2.2.3 Dimensions and units" of his Master Thesis "FDTD Simulation of Lightning Electromagnetic Fields An approach with the software package MEEP".

Appendix B. 1D Maxwell's equations with time scale

Let

$$H = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}, E = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, J = \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}$$

where

$$H_x, H_y, H_z, E_x, E_y, E_z, J_x, J_y, J_z \in R$$

Suppose the electromagnetic fields are only propagating along the x-axis, the restrictions are listed below:

$$J_x = 0, J_y = 0$$

$$H_x = 0, H_z = 0, \frac{\partial H_y}{\partial y} = 0, \frac{\partial H_y}{\partial z} = 0$$

$$E_x = 0, E_y = 0, \frac{\partial E_z}{\partial y} = 0, \frac{\partial E_z}{\partial z} = 0$$

We have

$$\frac{\partial H(x, y, z, \theta)}{\partial \theta} = \begin{bmatrix} \frac{\partial H_x}{\partial \theta} \\ \frac{\partial H_y}{\partial \theta} \\ \frac{\partial H_z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\partial H_y}{\partial \theta} \\ 0 \end{bmatrix}$$

$$\frac{\partial E(x, y, z, \theta)}{\partial \theta} = \begin{bmatrix} \frac{\partial E_x}{\partial \theta} \\ \frac{\partial E_y}{\partial \theta} \\ \frac{\partial E_z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial E_z}{\partial \theta} \end{bmatrix}$$

$$\nabla \times E(x, y, z, \theta) = \begin{bmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 - 0 \\ 0 - \frac{\partial E_z}{\partial x} \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\partial E_z}{\partial x} \\ 0 \end{bmatrix}$$

$$\nabla \times H(x, y, z, \theta) = \begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 - 0 \\ 0 - 0 \\ \frac{\partial H_y}{\partial x} - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial H_y}{\partial x} \end{bmatrix}$$

Eq. (A.3) becomes

$$\begin{bmatrix} 0 \\ \frac{\partial H_y}{\partial \theta} \\ 0 \end{bmatrix} = -\frac{1}{\eta} \begin{bmatrix} 0 \\ -\frac{\partial E_z}{\partial x} \\ 0 \end{bmatrix}$$

Eq. (A.4) becomes

$$\begin{bmatrix} 0 \\ 0 \\ \frac{\partial E_z}{\partial \theta} \end{bmatrix} = \eta \begin{bmatrix} 0 \\ 0 \\ \frac{\partial H_y}{\partial x} \end{bmatrix} - \eta \begin{bmatrix} 0 \\ 0 \\ J_z \end{bmatrix}$$

The variables y and z can be dropped:

$$H_y(x, y, z, \theta) \to H_y(x, \theta)$$

$$E_z(x, y, z, \theta) \to E_z(x, \theta)$$

We get 1D Maxwell's equations:

$$\frac{\partial H_y(x, \theta)}{\partial \theta} = \frac{1}{\eta} \frac{\partial E_z(x, \theta)}{\partial x} \tag{B.1}$$

$$\frac{\partial E_z(x, \theta)}{\partial \theta} = \eta \frac{\partial H_y(x, \theta)}{\partial x} - \eta J_z(x, \theta) \tag{B.2}$$

Appendix C. Proof of 1D solution of Gaussian source

Proof of Lemma 1

Lemma 1:

$$\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}q_{0,n,k}-\sum_{k=1}^{n}(-1)^k 4k\varrho^{2k-1}q_{0,n,k}=\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k} \tag{C.1}$$

Proof.

By (1.12), we have

$$\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}q_{0,n,k}-\sum_{k=1}^{n}(-1)^k 4k\varrho^{2k-1}q_{0,n,k}=$$

$$=\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}-\sum_{k=1}^{n}(-1)^k 4k\varrho^{2k-1}\frac{(2n)!}{(n-k)!\,(2k)!}$$

$$=\sum_{k=n}^{n}(-1)^k\varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}-\sum_{k=0}^{n-1}(-1)^{k+1}4(k+1)\varrho^{2k+1}\frac{(2n)!}{(n-k-1)!\,(2k+2)!}$$

$$=(-1)^n\varrho^{2n+1}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}-\sum_{k=0}^{n-1}(-1)^{k+1}4(k+1)\varrho^{2k+1}\frac{(2n)!}{(n-k-1)!\,(2k+2)!}$$

$$=(-1)^n\varrho^{2n+1}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}+\sum_{k=0}^{n-1}(-1)^k 4(k+1)\varrho^{2k+1}\frac{(2n)!}{(n-k-1)!\,(2k+2)!}$$

$$=(-1)^n\varrho^{2n+1}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}(2n)!\left(\frac{1}{(n-k)!\,(2k)!}+\frac{4(k+1)}{(n-k-1)!\,(2k+2)!}\right)$$

$$=(-1)^n\varrho^{2n+1}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}(2n)!\left(\frac{1}{(n-k)!\,(2k)!}+\frac{4(k+1)(n-k)}{(n-k)!\,(2k)!\,(2k+2)(2k+1)}\right)$$

$$=(-1)^n\varrho^{2n+1}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}(2n)!\frac{1}{(n-k)!\,(2k)!}\left(1+\frac{4(k+1)(n-k)}{(2k+2)(2k+1)}\right)$$

$$=(-1)^n\varrho^{2n+1}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}\left(1+\frac{2(2k+2)(n-k)}{(2k+2)(2k+1)}\right)$$

$$=(-1)^n\varrho^{2n+1}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}\left(1+\frac{2n-2k}{(2k+1)}\right)$$

$$=(-1)^n\varrho^{2n+1}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}\left(\frac{2k+1}{(2k+1)}+\frac{2n-2k}{(2k+1)}\right)$$

$$=(-1)^n\varrho^{2n+1}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}\frac{(2n)!}{(n-k)!\,(2k)!}\left(\frac{2n+1}{(2k+1)}\right)$$

$$=(-1)^n\varrho^{2n+1}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}$$

$$=\sum_{k=n}^{n}(-1)^k\varrho^{2k+1}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}+\sum_{k=0}^{n-1}(-1)^k\varrho^{2k+1}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}$$

$$=\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}\frac{(2n+1)!}{(n-k)!\,(2k+1)!}$$

$$= \sum_{k=0}^{n} (-1)^k \varrho^{2k+1} p_{0,n,k}$$

The last step is by (1.11)

QED

Proof of Lemma 2

Lemma 2:

$$\sum_{k=0}^{n} (-1)^k \varrho^{2k+2} p_{0,n,k} - \sum_{k=0}^{n} (-1)^k 2(2k+1) \varrho^{2k} p_{0,n,k} = -\sum_{k=0}^{n+1} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} \tag{C.2}$$

Proof

From the odd-binomial-coefficient (1.11)

$$p_{0,n,k} = \frac{(2n+1)!}{(n-k)!\,(2k+1)!}$$

We have

$$p_{0,n,k+1} = \frac{(2n+1)!}{(n-k-1)!\,(2k+3)!}$$

$$p_{0,0,0} = 1$$

$$p_{0,n,n} = 1$$

$$p_{0,n,0} = \frac{(2n+1)!}{n!}$$

We will use the above formulas in the proof of this lemma.

$$\sum_{k=0}^{n} (-1)^k \varrho^{2k+2} p_{0,n,k} - \sum_{k=0}^{n} (-1)^k 2(2k+1) \varrho^{2k} p_{0,n,k} =$$

$$= \sum_{k=0}^{n} (-1)^k \varrho^{2k+2} p_{0,n,k} - \sum_{k=0}^{0} (-1)^k 2(2k+1) \varrho^{2k} p_{0,n,k} - \sum_{k=1}^{n} (-1)^k 2(2k+1) \varrho^{2k} p_{0,n,k}$$

$$= \sum_{k=0}^{n} (-1)^k \varrho^{2k+2} p_{0,n,k} - 2p_{0,n,0} - \sum_{k=0}^{n-1} (-1)^{k+1} 2(2(k+1)+1) \varrho^{2k+2} p_{0,n,k+1}$$

$$= \sum_{k=n}^{n} (-1)^k \varrho^{2k+2} p_{0,n,k} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} p_{0,n,k} - 2\frac{(2n+1)!}{n!} - \sum_{k=0}^{n-1} (-1)^{k+1} 2(2k+3) \varrho^{2k+2} p_{0,n,k+1}$$

$$= (-1)^n \varrho^{2n+2} p_{0,n,n} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} p_{0,n,k} - 2\frac{(2n+1)!}{n!} + \sum_{k=0}^{n-1} (-1)^k 2(2k+3) \varrho^{2k+2} p_{0,n,k+1}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=0}^{n-1} \left((-1)^k \varrho^{2k+2} p_{0,n,k} + (-1)^k 2(2k+3) \varrho^{2k+2} p_{0,n,k+1} \right) - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} \left(p_{0,n,k} + 2(2k+3) p_{0,n,k+1} \right) - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} \left(\frac{(2n+1)!}{(n-k)!\,(2k+1)!} + 2(2k+3) \frac{(2n+1)!}{(n-k-1)!\,(2k+3)!} \right) - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} \left(\frac{(2n+1)!}{(n-k)!\,(2k+1)!} + 2 \frac{(2n+1)!}{(n-k-1)!\,(2k+2)!} \right) - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} \left(\frac{(2n+1)!}{(n-k)!\,(2k+1)!} + 2\frac{(2n+1)!\,(n-k)}{(n-k)!\,(2k+2)!} \right) - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} \left(\frac{(2n+1)!}{(n-k)!\,(2k+1)!} + 2\frac{(2n+1)!\,(n-k)}{(n-k)!\,(2k+1)!\,(2k+2)} \right) - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} \frac{(2n+1)!}{(n-k)!\,(2k+1)!} \left(1 + 2\frac{(n-k)}{(2k+2)} \right) - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} \frac{(2n+1)!}{(n-k)!\,(2k+1)!} \left(\frac{(2k+2)}{(2k+2)} + \frac{(2n-2k)}{(2k+2)} \right) - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} \frac{(2n+1)!}{(n-k)!\,(2k+1)!} \left(\frac{(2n+2)}{(2k+2)} \right) - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=0}^{n-1} (-1)^k \varrho^{2k+2} \frac{(2n+2)!}{(n-k)!\,(2k+2)!} - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} + \sum_{k=1}^{n} (-1)^{k-1} \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} - \sum_{k=1}^{n} (-1)^{k} \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} - 2\frac{(2n+1)!}{n!}$$

Because

$$\sum_{k=0}^{0} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} = (-1)^0 \varrho^0 \frac{(2n+2)!}{(n-0+1)!\,(0)!} = \frac{(2n+2)!}{(n+1)!}$$

$$= \frac{(2n+2)!}{(n+1)\cdot n!} = 2\frac{(2n+2)!}{(2n+2)n!} = 2\frac{(2n+1)!}{n!}$$

We have

$$(-1)^n \varrho^{2n+2} - \sum_{k=1}^{n} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} - 2\frac{(2n+1)!}{n!}$$

$$= (-1)^n \varrho^{2n+2} - \sum_{k=1}^{n} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} - \sum_{k=0}^{0} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!}$$

$$= -(-1)^{n+1} \varrho^{2n+2} - \sum_{k=0}^{n} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!}$$

$$= -\sum_{k=n+1}^{n+1} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!} - \sum_{k=0}^{n} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!}$$

$$= -\sum_{k=0}^{n+1} (-1)^k \varrho^{2k} \frac{(2n+2)!}{(n-k+1)!\,(2k)!}$$

QED

Proof of 1D Gaussian source solution

Verify initIal values

Proof

From (1.9) and (1.10), it is easy to see (1.7) and (1.8) satisfy (1.4).

QED

Verify Eq.(1.1)

Lemma: Solution (1.7) and (1.8) satisfy Maxwell's equation (1.1)

Proof

Note the symbols used:

$$\varrho = 2\sqrt{a}x$$

$$\xi = \sqrt{a}\theta$$

From (1.8) and (1.10), we have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} = \frac{1}{\eta}\frac{\partial}{\partial x}\left(-\eta\frac{1}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k q_{0,n,k}\varrho^{2k}\right)$$

$$= -\frac{-2ax}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k}$$

$$= 2\sqrt{a}xe^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\sum_{n=0}^{0}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{0}(-1)^k\varrho^{2k}q_{0,n,k} + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\xi + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\xi + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\left(\sum_{k=0}^{0}(-1)^k\varrho^{2k}q_{0,n,k} + \sum_{k=1}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\xi + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\left(1 + \sum_{k=1}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\xi + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!} + \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=1}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}\frac{\partial}{\partial x}\left(\sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=1}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)$$

$$= \varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} - \frac{1}{\sqrt{a}}e^{-ax^2}2\sqrt{a}\sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=1}^{n}(-1)^k 2k\varrho^{2k-1}q_{0,n,k}$$

$$= e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}q_{0,n,k} - e^{-ax^2}2\sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=1}^{n}(-1)^k 2k\varrho^{2k-1}q_{0,n,k}$$

$$= e^{-ax^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}q_{0,n,k} - \sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=1}^{n}(-1)^k 4k\varrho^{2k-1}q_{0,n,k}\right)$$

$$= e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\left(\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}q_{0,n,k} - \sum_{k=1}^{n}(-1)^k 4k\varrho^{2k-1}q_{0,n,k}\right)$$

Substitute (C.1) into above, we have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x} = e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k} \tag{C.3}$$

From (1.7) and (1.9) we have

$$\frac{\partial H_y(x,\theta)}{\partial\theta}=\frac{\partial}{\partial\theta}\left(\frac{1}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k p_{0,n,k}\varrho^{2k+1}\right)$$

$$=\frac{1}{\sqrt{a}}e^{-ax^2}\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k}$$

We have

$$\frac{\partial H_y(x,\theta)}{\partial\theta}=e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k} \tag{C.4}$$

(C.3) and (C.4) lead to

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial x}=\frac{\partial H_y(x,\theta)}{\partial\theta}$$

The above is (1.1). Thus, from (1.7) and (1.8) we get Maxwell's equation (1.1)

QED

Verify Eq.(1.2)

Lemma: Solution (1.7) and (1.8) satisfy Maxwell's equation (1.2) and source (1.3)

Proof

From (1.7) and (1.9), we have

$$\frac{\partial H_y(x,\theta)}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k}\right)$$

$$=\frac{-2ax}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k}+\frac{1}{\sqrt{a}}e^{-ax^2}2\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k(2k+1)\varrho^{2k}p_{0,n,k}$$

$$=-2\sqrt{a}xe^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k}+e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k2(2k+1)\varrho^{2k}p_{0,n,k}$$

$$=-\varrho e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+1}p_{0,n,k}+e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k2(2k+1)\varrho^{2k}p_{0,n,k}$$

$$=-e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k}+e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k2(2k+1)\varrho^{2k}p_{0,n,k}$$

$$=-e^{-ax^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k}-\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n}(-1)^k2(2k+1)\varrho^{2k}p_{0,n,k}\right)$$

$$=-e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\left(\sum_{k=0}^{n}(-1)^k\varrho^{2k+2}p_{0,n,k}-\sum_{k=0}^{n}(-1)^k2(2k+1)\varrho^{2k}p_{0,n,k}\right)$$

Substitute (C.2) into above, we have

$$\frac{\partial H_y(x,\theta)}{\partial x}=-e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\left(-\sum_{k=0}^{n+1}(-1)^k\varrho^{2k}\frac{(2n+2)!}{(n-k+1)!\,(2k)!}\right)$$

$$=e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2(n+1)}}{(2(n+1))!}\sum_{k=0}^{n+1}(-1)^k\varrho^{2k}\frac{(2n+2)!}{(n-k+1)!\,(2k)!}$$

$$=e^{-ax^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!}$$

$$= -e^{-ax^2}\sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!}$$

$$= -e^{-ax^2}\sum_{n=1}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} - e^{-ax^2}\sum_{n=0}^{0}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!}$$
$$+ e^{-ax^2}\sum_{n=0}^{0}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!}$$

$$= -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} + e^{-ax^2}\sum_{n=0}^{0}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!}$$

$$= -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}\frac{(2n)!}{(n-k)!\,(2k)!} + e^{-ax^2}$$

$$= -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} + e^{-ax^2}$$

We have

$$\frac{\partial H_y(x,\theta)}{\partial x} - e^{-ax^2} = -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} \tag{C.5}$$

From (1.8) and (1.10), we have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial\theta} = \frac{1}{\eta}\frac{\partial}{\partial\theta}\left(-\eta\frac{1}{\sqrt{a}}e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{(2n+1)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k}\right)$$

$$= -\frac{1}{\sqrt{a}}e^{-ax^2}\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k}$$

We have

$$\frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial\theta} = -e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{k=0}^{n}(-1)^k\varrho^{2k}q_{0,n,k} \tag{C.6}$$

(C.5) and (C.6) lead to

$$\frac{\partial H_y(x,\theta)}{\partial x} - e^{-ax^2} = \frac{1}{\eta}\frac{\partial E_z(x,\theta)}{\partial\theta}$$

The above is (1.2) and (1.3). Thus, from (1.7) and (1.8) we get Maxwell's equation (1.2) and source (1.3).

QED

Appendix D. Summation exchange rule

Lemma 3. Summation exchange rule:

$$\sum_{i=0}^{I_{max}}\sum_{j=0}^{i}V(i,j) = \sum_{j=0}^{I_{max}}\sum_{i=j}^{I_{max}}V(i,j) \tag{D.1}$$
$$0 \le I_{max} \le \infty$$

Proof.

I'll just use a table to prove this simple rule.

$i\downarrow j\rightarrow$	0	1	$\cdots$	I_{max}
0	$V(0,0)$			
1	$V(1,0)$	$V(1,1)$		
$\vdots$	$\vdots$	$\vdots$		
I_{max}	$V(I_{max},0)$	$V(I_{max},1)$	$\cdots$	$V(I_{max},I_{max})$

The summation on the left side of (D.1) is to go row by row; for each row, go column by column.
The summation on the right side of (D.1) is to go column by column; for each column, go row by row.

Thus the two sides go through the same elements in the above table.

QED

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