An Introduction to Tensor Decomposition

Supplementary matrials

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CP-ALS example: Rank-2 approximation of 3-tensor

$$\mathcal{X} = (\mathbf{a}_1 \circ \mathbf{b}_1 \circ \mathbf{c}_1) + (\mathbf{a}_2 \circ \mathbf{b}_2 \circ \mathbf{c}_2); \qquad \mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}$$

Algorithm CP-ALS $(\mathcal{X}, 1)$

initialize $\mathbf{A}, \mathbf{B}, \mathbf{C}$

repeat

$\mathbf{V}_A \leftarrow \mathbf{B}^T\mathbf{B} * \mathbf{C}^T\mathbf{C}$

$\mathbf{A} \leftarrow \mathbf{X}_{(1)}(\mathbf{C} \odot \mathbf{B})\mathbf{V_A}^\dagger$

$\boldsymbol{\lambda}, \mathbf{A} \leftarrow \left[\|\mathbf{a}_1\| \quad \|\mathbf{a}_2\|\right], \left[\frac{\mathbf{a}_1}{\|a_1\|} \quad \frac{\mathbf{a}_2}{\|\mathbf{a}_2\|}\right]$

$\mathbf{V_B} \leftarrow \mathbf{A}^T\mathbf{A} * \mathbf{C}^T\mathbf{C}$

$\mathbf{B} \leftarrow \mathbf{X}_{(2)}(\mathbf{C} \odot \mathbf{A})\mathbf{V_B}^\dagger$

$\boldsymbol{\lambda}, \mathbf{B} \leftarrow \left[\|\mathbf{b}_1\| \quad \|\mathbf{b}_2\|\right], \left[\frac{\mathbf{b}_1}{\|b_1\|} \quad \frac{\mathbf{b}_2}{\|\mathbf{b}_2\|}\right]$

$\mathbf{V_C} \leftarrow \mathbf{A}^T\mathbf{A} * \mathbf{B}^T\mathbf{B}$

$\mathbf{C} \leftarrow \mathbf{X}_{(3)}(\mathbf{B} \odot \mathbf{A})\mathbf{V_C}^\dagger$

$\boldsymbol{\lambda}, \mathbf{C} \leftarrow \left[\|\mathbf{c}_1\| \quad \|\mathbf{c}_2\|\right], \left[\frac{\mathbf{c}_1}{\|c_1\|} \quad \frac{\mathbf{c}_2}{\|\mathbf{c}_2\|}\right]$

until fit ceases to improve or maximum iterations exhausted

return $\boldsymbol{\lambda}, \mathbf{A}, \mathbf{B}, \mathbf{C}$

Create tensor and initialize $\mathbf{a}, \mathbf{b}, \mathbf{c}$

$$\mathcal{X} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$$

$$
\begin{aligned}
\mathbf{V}_A &\leftarrow \mathbf{B}^T\mathbf{B} * \mathbf{C}^T\mathbf{C} \\
&= \left(\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}\right) * \left(\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 29 & 22 \\ 22 & 17 \end{bmatrix} * \begin{bmatrix} 18 & 6 \\ 6 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 522 & 132 \\ 132 & 34 \end{bmatrix}
\end{aligned}
$$

$$\mathbf{A} \leftarrow \mathbf{X}_{(1)}(\mathbf{C} \odot \mathbf{B})\mathbf{V}_{\mathbf{A}}{}^{\dagger}$$

$$= \begin{bmatrix} 3 & 1 & 1 & 1 \\ 4 & 2 & 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \odot \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} \right) \begin{bmatrix} 522 & 132 \\ 132 & 34 \end{bmatrix}^{\dagger}$$

$$= \begin{bmatrix} 3 & 1 & 1 & 1 \\ 4 & 2 & 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 6 & 1 \\ 15 & 4 \\ 6 & 1 \\ 15 & 4 \end{bmatrix} \right) \begin{bmatrix} 0.1049 & -0.4074 \\ -0.4074 & 1.611 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7778 & -2.6667 \\ 0.8889 & -2.8333 \end{bmatrix}$$

$\lambda, \mathbf{A} \leftarrow \left[\|\mathbf{a}_1\| \quad \|\mathbf{a}_2\|\right], \left[\frac{\mathbf{a}_1}{\|a_1\|} \quad \frac{\mathbf{a}_2}{\|\mathbf{a}_2\|}\right]$

$$\lambda \leftarrow \left[\|\mathbf{a}_1\| \quad \|\mathbf{a}_2\|\right] = \left[1.811 \quad 3.8909\right]$$

$$\mathbf{A} \leftarrow \left[\frac{\mathbf{a}_1}{\|a_1\|} \quad \frac{\mathbf{a}_2}{\|\mathbf{a}_2\|}\right] = \begin{bmatrix} 0.6586 & -0.6854 \\ 0.7526 & -0.7282 \end{bmatrix}$$

$$\mathbf{V}_B \leftarrow \mathbf{A}^T\mathbf{A} * \mathbf{C}^T\mathbf{C}$$

$$= \left(\begin{bmatrix} 0.6585 & 0.7526 \\ -0.6854 & -0.7282 \end{bmatrix}\begin{bmatrix} 0.6586 & -0.6854 \\ 0.7526 & -0.7282 \end{bmatrix}\right) * \left(\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & -0.9993 \\ -0.9993 & 1 \end{bmatrix} * \begin{bmatrix} 18 & 6 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & -5.9961 \\ -5.9961 & 2 \end{bmatrix}$$

$$\mathbf{B} \leftarrow \mathbf{X}_{(2)}(\mathbf{C} \odot \mathbf{A})\mathbf{V}_{\mathbf{B}}{}^{\dagger}$$

$$= \begin{bmatrix} 3 & 4 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \odot \begin{bmatrix} 0.6585 & -0.6854 \\ 0.7526 & -0.7282 \end{bmatrix} \right) \begin{bmatrix} 18 & -5.9961 \\ -5.9961 & 2 \end{bmatrix}^{\dagger}$$

$$= \begin{bmatrix} 3 & 4 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 1.9755 & -0.6854 \\ 2.2577 & -0.7282 \\ 1.9755 & -0.6854 \\ 2.2577 & -0.7282 \end{bmatrix} \right) \begin{bmatrix} 0.1049 & -0.4074 \\ -0.4074 & 1.611 \end{bmatrix}$$

$$= \begin{bmatrix} 2.3623 & 3.8909 \\ 5.9056 & 15.5635 \end{bmatrix}$$

$$\boldsymbol{\lambda}, \mathbf{B} \leftarrow \begin{bmatrix} \|\mathbf{b}_1\| & \|\mathbf{b}_2\| \end{bmatrix}, \begin{bmatrix} \frac{\mathbf{b}_1}{\|b_1\|} & \frac{\mathbf{b}_2}{\|\mathbf{b}_2\|} \end{bmatrix}$$

$$\boldsymbol{\lambda} \leftarrow \begin{bmatrix} \|\mathbf{b}_1\| & \|\mathbf{b}_2\| \end{bmatrix} = \begin{bmatrix} 6.3606 & 16.0425 \end{bmatrix}$$

$$\mathbf{B} \leftarrow \begin{bmatrix} \frac{\mathbf{b}_1}{\|b_1\|} & \frac{\mathbf{b}_2}{\|\mathbf{b}_2\|} \end{bmatrix} = \begin{bmatrix} 0.3714 & 0.2425 \\ 0.9285 & 0.9701 \end{bmatrix}$$

$\mathbf{V}_C \leftarrow \mathbf{A}^T\mathbf{A} * \mathbf{B}^T\mathbf{B}$

$$\mathbf{V}_C \leftarrow \mathbf{A}^T\mathbf{A} * \mathbf{B}^T\mathbf{B}$$

$$= \left(\begin{bmatrix} 0.658 & 0.752 \\ -0.685 & -0.728 \end{bmatrix}\begin{bmatrix} 0.658 & -0.685 \\ 0.752 & -0.728 \end{bmatrix}\right) * \left(\begin{bmatrix} 0.371 & 0.928 \\ 0.242 & 0.970 \end{bmatrix}\begin{bmatrix} 0.371 & 0.242 \\ 0.928 & 0.970 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & -0.9993 \\ -0.9993 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0.9908 \\ 0.9908 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.9902 \\ -0.9902 & 1 \end{bmatrix}$$

$\mathbf{C} \leftarrow \mathbf{X}_{(3)}(\mathbf{B} \odot \mathbf{A})\mathbf{V}_{\mathbf{C}}{}^{\dagger}$

$$\mathbf{C} \leftarrow \mathbf{X}_{(3)}(\mathbf{B} \odot \mathbf{A})\mathbf{V}_{\mathbf{C}}{}^{\dagger}$$

$$= \begin{bmatrix} 3 & 4 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 0.3714 & 0.2425 \\ 0.9285 & 0.9701 \end{bmatrix} \odot \begin{bmatrix} 0.6585 & -0.6854 \\ 0.7526 & -0.7282 \end{bmatrix} \right) \begin{bmatrix} 1.0000 & -0.9902 \\ -0.9902 & 1.0000 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 0.2446 & -0.1662 \\ 0.2795 & -0.1766 \\ 0.6114 & -0.6649 \\ 0.6987 & -0.7065 \end{bmatrix} \right) \begin{bmatrix} 51.1592 & 50.6567 \\ 50.6567 & 51.1592 \end{bmatrix}$$

$$= \begin{bmatrix} 31.2013 & 27.6119 \\ 6.9621 & 4.4730 \end{bmatrix}$$

$\boldsymbol{\lambda}, \mathbf{C} \leftarrow \begin{bmatrix} \|\mathbf{c}_1\| & \|\mathbf{c}_2\| \end{bmatrix}, \begin{bmatrix} \frac{\mathbf{c}_1}{\|c_1\|} & \frac{\mathbf{c}_2}{\|\mathbf{c}_2\|} \end{bmatrix}$

$$\boldsymbol{\lambda} \leftarrow \begin{bmatrix} \|\mathbf{c}_1\| & \|\mathbf{c}_2\| \end{bmatrix} = \begin{bmatrix} 31.9686 & 27.9719 \end{bmatrix}$$

$$\mathbf{C} \leftarrow \begin{bmatrix} \frac{\mathbf{c}_1}{\|c_1\|} & \frac{\mathbf{c}_2}{\|\mathbf{c}_2\|} \end{bmatrix} = \begin{bmatrix} 0.9760 & 0.9871 \\ 0.2178 & 0.1599 \end{bmatrix}$$

Check error

$$\mathcal{X} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\hat{\mathcal{X}} = \sum_{i=1}^{2} \lambda_i \cdot (\mathbf{a}_i \circ \mathbf{b}_i \circ \mathbf{c}_i)$$

$$\mathcal{X} - \hat{\mathcal{X}} = \begin{bmatrix} -0.3565 & -1.8489 \\ -0.3561 & -1.9079 \end{bmatrix} \begin{bmatrix} -0.0405 & -0.2564 \\ -0.0380 & -0.2598 \end{bmatrix}$$

$$\text{error} = \left\| \mathcal{X} - \hat{\mathcal{X}} \right\| = 7.9041$$

Result

After 17,192 iterations:

$$\boldsymbol{\lambda} = \begin{bmatrix} 4.4723 & 0 \\ 0 & 8.0625 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0.4472 & -0.5547 \\ 0.8944 & -0.8321 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -0.7071 & -1.0000 \\ 0.7071 & 0.0000 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0.7071 & 0.8944 \\ 0.7071 & 0.4472 \end{bmatrix}$$

$$\text{error} = \left\| \mathcal{X} - \hat{\mathcal{X}} \right\| = 3.7399 \times 10^{-6}$$