An Introduction to Mathematical Proofs

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Contents

1 Logic

1.1 Propositions, Logical Connectives, and Truth Tables

1.1.1 Propositions

***Propostional logic* studies how the truth of a complex statement is determined by the truth or falsehoods of its parts.**

A *propositon* is a statement that is either true or false, but not both.

1.1.2 Propositional Forms

An expression which is built up by combining *propositional variables* (capital letters) using logical symbols, is called a *propositional form*

The following tables shows the symbols used in propositional logic and their meaning:

Logical Symbol	English Translation
$\neg P$	P is not true
$P \wedge Q$	P and Q.
$P \vee Q$	P or Q.
$P \oplus Q$	P or Q, but not both
$P \implies Q$	if P, then Q
$P \iff Q$	P if and only if Q

We can also combine this symbols:
$(P \wedge (\neg Q)) \vee ((\neg P) \wedge Q)$

Any expression which is built up by combining *propositional variables* (capital letters) using logical symbols, is called a *propositional form*

1.1.3 Definitions of NOT and AND

In order to properly *define* the exact meaning of logical connective words like NOT, AND, OR, IF, etc. we use so-called *truth tables.* that show how to combine the truth values of propositions to obtain the truth value of a new proposition built from these using a logical connective. The letters T and F stand for *true* and *false* respectively.

Definition of NOT. For any proposition P, the truth value of $\neg P$ (”not P”) is determined by the following table

P	$\neg P$
T	F
F	T

Definition of AND. For any propositions, P, Q, the truth value of $P \wedge Q$ (”P and Q”) is determined by the following table.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

1.1.4 Definitions of OR and XOR

$\vee$ stands for *inclusive*-OR, and $\oplus$ stands for *exclusive*-OR.

Definition of OR. For any propositions, P, Q, the truth value of PQ (”P or Q”) is determined by the following table.

P	Q	$P \wedge Q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition of XOR. For any propositions, P, Q, the truth value of $P \oplus Q$ ("P xor Q") is determined by the following table.

P	Q	$P \wedge Q$
T	T	F
T	F	T
F	T	T
F	F	F

1.1.5 Formal Definition of Propositional Forms

In the main text, we informally defined a propositional form to be any expression built up by combining propositional variables using logical symbols. To give a precise, rigorous definition of propositional forms, we need a *recursive definition.*

Specifically, propositional forms are defined recursively via the following rules:

(a) A single capital italic letter is a propositional form.

(b) If A is a propositional form, then $(\neg A)$ is a propositional form.

(c) If A and B are any propositional forms, then $(A \wedge B)$ is a propositional form.

(d) If A and B are any propositional forms, then $(A \vee B)$ is a propositional form.

(e) If A and B are any propositional forms, then $(A \oplus B)$ is a propositional form.

(f) If A and B are any propositional forms, then $(A \implies B)$ is a propositional form.

(g) If A and B are any propositional forms, then $(A \iff B)$ is a propositional form.

(h) An expression is a propositional form only if it can be formed by applying rules (a) through (g) finetly many times.

1.1.6 Remark: Terminology for Propositional Forms

Let A and B be any propositional forms. In some logic texts, $(\neg A)$ is called the *negation* of A; $(A \wedge B)$ is called the *conjunction* of A and B; $(A \vee B)$ is called the *disjunction* of A and B; $(A \implies B)$ is called an *implication* or *conditional* with *hypothesis* of A and B.

1.2 Logical Equivalences and IF-Statements

1.2.1 Logical Equivalence

Definition: Logically Equivalent Propositional Forms

Two propositional forms A and B are *logically equivalent* when the truth tables for A and B have outputs that agree in every row. We write $\boxed{\text{A} \equiv B}$ when A and B are logically equivalent; we write $\boxed{\text{A} \not\equiv B}$ when A and B are not logically equivalent.

Theorem on Logical Equivalence. For all propositional forms P, Q, and R, the following lo

(a) *Commutative Laws:* $P \wedge Q \equiv Q \wedge P$, $P \vee Q \equiv Q \vee P$, and $P \oplus Q \equiv Q \oplus P$.

(b) *Associative Laws:* $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$, $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$, and $P \oplus (Q \oplus R) \equiv (P \oplus Q) \oplus R$.

(c) *Distributive Laws:* $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$, $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$, and $P \wedge (Q \oplus R) \equiv (P \wedge Q) \oplus (P \wedge R)$.

(d) *Idempotent Laws:* $P \wedge P \equiv P$ and $P \vee P \equiv P$

(e) *Absorption Laws:* $P \wedge (P \vee Q) \equiv P$ and $P \vee (P \wedge Q) \equiv P$.

(f) *Negation Laws:* $\neg(\neg P) \equiv P$, $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$, and $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$.

1.2.2 Converse and Contrapositive

Definition: Converse and Contrapositive

For any propositional forms P and Q: the $\boxed{\text{converse of } P \implies Q}$ is $\boxed{Q \implies P}$; the $\boxed{\text{contrapositive of } P \implies Q}$ is $\boxed{(\neg Q) \implies (\neg P)}$.

Theorem on IF. Let P and Q be distinct propositional variables.

(a) *Non-equivalence of Converse:* $P \Rightarrow Q \not\equiv Q \Rightarrow P$

(b) *Equivalence of Contrapositive:* $P \Rightarrow Q \equiv (\neg Q) \Rightarrow (\neg P)$

(c) *Elimination of IF:* $P \Rightarrow Q \equiv (\neg P) \vee Q$

(d) *Denial of IF:* $\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q)$

1.2.3 Five Remarks on Logical Equivalence

2 Proofs

3 Sets

4 Integers

5 Relations and Functions

6 Equivalence Relations and Partial Orders

7 Cardinality

8 Real Numbers (Optional)