

An Introduction to Mathematical Proofs

Nicholas A. Loehr

April 2022

Contents

1	Logic	2
1.1	Propositions, Logical Connectives, and Truth Tables	2
1.1.1	Propositions	2
1.1.2	Propositional Forms	3
1.1.3	Definitions of NOT and AND	3
1.1.4	Definitions of OR and XOR	4
1.1.5	Formal Definition of Propositional Forms	5
1.1.6	Remark: Terminology for Propositional Forms	6
2	Proofs	7
3	Sets	7
4	Integers	7
5	Relations and Functions	7
6	Equivalence Relations and Partial Orders	7
7	Cardinality	7
8	Real Numbers (Optional)	7

1 Logic

1.1 Propositions, Logical Connectives, and Truth Tables

1.1.1 Propositions

Propositional logic studies how the truth of a complex statement is determined by the truth or falsehoods of its parts.

A *proposition* is a statement that is either true or false, but not both.

1.1.2 Propositional Forms

An expression which is built up by combining *propositional variables* (capital letters) using logical symbols, is called a *propositional form*

The following tables shows the symbols used in propositional logic and their meaning:

Logical Symbol	English Translation
$\neg P$	P is not true
$P \wedge Q$	P and Q.
$P \vee Q$	P or Q.
$P \oplus Q$	P or Q, but not both
$P \implies Q$	if P, then Q
$P \iff Q$	P if and only if Q

We can also combine this symbols:

$$(P \wedge (\neg Q)) \vee ((\neg P) \wedge Q)$$

Any expression which is built up by combining *propositional variables* (capital letters) using logical symbols, is called a *propositional form*

1.1.3 Definitions of NOT and AND

In order to properly *define* the exact meaning of logical connective words like NOT, AND, OR, IF, etc. we use so-called *truth tables*. that show how to combine the truth values of propositions to obtain the truth value of a new proposition built from these using a logical connective. The letters T and F stand for *true* and *false* respectively.

Definition of NOT. For any proposition P , the truth value of $\neg P$ ("not P ") is determined by the following table

P	$\neg P$
T	F
F	T

Definition of AND. For any propositions, P , Q , the truth value of $P \wedge Q$ (" P and Q ") is determined by the following table.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

1.1.4 Definitions of OR and XOR

\vee stands for *inclusive*-OR, and \oplus stands for *exclusive*-OR.

Definition of OR. For any propositions, P , Q , the truth value of $P \vee Q$ (" P or Q ") is determined by the following table.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition of XOR. For any propositions, P , Q , the truth value of $P \oplus Q$ (" P xor Q ") is determined by the following table.

P	Q	$P \wedge Q$
T	T	F
T	F	T
F	T	T
F	F	F

1.1.5 Formal Definition of Propositional Forms

In the main text, we informally defined a propositional form to be any expression built up by combining propositional variables using logical symbols. To give a precise, rigorous definition of propositional forms, we need a *recursive definition*.

Specifically, propositional forms are defined recursively via the following rules:

- (a) A single capital italic letter is a propositional form.
- (b) If A is a propositional form, then $(\neg A)$ is a propositional form.
- (c) If A and B are any propositional forms, then $(A \wedge B)$ is a propositional form.
- (d) If A and B are any propositional forms, then $(A \vee B)$ is a propositional form.
- (e) If A and B are any propositional forms, then $(A \oplus B)$ is a propositional form.
- (f) If A and B are any propositional forms, then $(A \implies B)$ is a propositional form.
- (g) If A and B are any propositional forms, then $(A \iff B)$ is a propositional form.
- (h) An expression is a propositional form only if it can be formed by applying rules (a) through (g) finitely many times.

1.1.6 Remark: Terminology for Propositional Forms

Let A and B be any propositional forms. In some logic texts, $(\neg A)$ is called the *negation* of A ; $(A \wedge B)$ is called the *conjunction* of A and B ; $(A \vee B)$ is called the *disjunction* of A and B ; $(A \implies B)$ is called an *implication* or *conditional* with *hypothesis* of A and B .

- 2 Proofs
- 3 Sets
- 4 Integers
- 5 Relations and Functions
- 6 Equivalence Relations and Partial Orders
- 7 Cardinality
- 8 Real Numbers (Optional)