

Concrete Mathematics

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Contents

1	Sums	2
1.1	Notation	2
1.2	Manipulation Of Sums	8
1.3	Multiple Sums	10
1.4	General Methods	12

1 Sums

1.1 Notation

2.1 NOTATION

In Chapter 1 we encountered the sum of the first n integers, which we wrote out as $1 + 2 + 3 + \cdots + (n - 1) + n$. The ‘ \cdots ’ in such formulas tells us to complete the pattern established by the surrounding terms. Of course we have to watch out for sums like $1 + 7 + \cdots + 41.7$, which are meaningless without a mitigating context. On the other hand, the inclusion of terms like 3 and $(n - 1)$ was a bit of overkill; the pattern would presumably have been clear if we had written simply $1 + 2 + \cdots + n$. Sometimes we might even be so bold as to write just $1 + \cdots + n$.

We’ll be working with sums of the general form

$$a_1 + a_2 + \cdots + a_n, \tag{2.1}$$

where each a_k is a number that has been defined somehow. This notation has the advantage that we can “see” the whole sum, almost as if it were written out in full, if we have a good enough imagination.

Each element a_k of a sum is called a *term*. The terms are often specified implicitly as formulas that follow a readily perceived pattern, and in such cases we must sometimes write them in an expanded form so that the meaning is clear. For example, if

$$1 + 2 + \cdots + 2^{n-1}$$

is supposed to denote a sum of n terms, not of 2^{n-1} , we should write it more explicitly as

$$2^0 + 2^1 + \cdots + 2^{n-1}.$$

The three-dots notation has many uses, but it can be ambiguous and a bit long-winded. Other alternatives are available, notably the delimited form

$$\sum_{k=1}^n a_k, \tag{2.2}$$

which is called Sigma-notation because it uses the Greek letter \sum (upper-case sigma). This notation tells us to include in the sum precisely those terms a_k whose index k is an integer that lies between the lower and upper limits 1 and n , inclusive. In words, we “sum over k , from 1 to n .” Joseph Fourier introduced this delimited \sum -notation in 1820, and it soon took the mathematical world by storm.

Incidentally, the quantity after \sum (here a_k) is called the *summand*.

The index variable k is said to be *bound* to the \sum sign in (2.2), because the k in a_k is unrelated to appearances of k outside the Sigma-notation. Any other letter could be substituted for k here without changing the meaning of (2.2). The letter i is often used (perhaps because it stands for “index”), but we’ll generally sum on k since it’s wise to keep i for $\sqrt{-1}$.

It turns out that a generalized Sigma-notation is even more useful than the delimited form: We simply write one or more conditions under the \sum , to specify the set of indices over which summation should take place. For example, the sums in (2.1) and (2.2) can also be written as

$$\sum_{1 \leq k \leq n} a_k. \tag{2.3}$$

In this particular example there isn’t much difference between the new form and (2.2), but the general form allows us to take sums over index sets that aren’t restricted to consecutive integers. For example, we can express the sum of the squares of all odd positive integers below 100 as follows:

$$\sum_{\substack{1 \leq k < 100 \\ k \text{ odd}}} k^2.$$

The delimited equivalent of this sum,

$$\sum_{k=0}^{49} (2k+1)^2,$$

The biggest advantage of general Sigma-notation is that we can manipulate it more easily than the delimited form. For example, suppose we want to change the index variable k to $k + 1$. With the general form, we have

$$\sum_{1 \leq k \leq n} a_k = \sum_{1 \leq k+1 \leq n} a_{k+1};$$

it's easy to see what's going on, and we can do the substitution almost without thinking. But with the delimited form, we have

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} a_{k+1};$$

it's harder to see what's happened, and we're more likely to make a mistake.

On the other hand, the delimited form isn't completely useless. It's nice and tidy, and we can write it quickly because (2.2) has seven symbols compared with (2.3)'s eight. Therefore we'll often use \sum with upper and lower delimiters when we state a problem or present a result, but we'll prefer to work with relations-under- \sum when we're manipulating a sum whose index variables need to be transformed.

The \sum sign occurs more than 1000 times in this book, so we should be sure that we know exactly what it means. Formally, we write

$$\sum_{P(k)} a_k \tag{2.4}$$

as an abbreviation for the sum of all terms a_k such that k is an integer satisfying a given property $P(k)$. (A "property $P(k)$ " is any statement about k that can be either true or false.) For the time being, we'll assume that only finitely many integers k satisfying $P(k)$ have $a_k \neq 0$; otherwise infinitely many nonzero numbers are being added together, and things can get a bit

tricky. At the other extreme, if $P(k)$ is false for all integers k , we have an “empty” sum; the value of an empty sum is defined to be zero.

A slightly modified form of (2.4) is used when a sum appears within the text of a paragraph rather than in a displayed equation: We write ‘ $\sum_{P(k)} a_k$ ’, attaching property $P(k)$ as a subscript of \sum , so that the formula won’t stick out too much. Similarly, ‘ $\sum_{k=1}^n a_k$ ’ is a convenient alternative to (2.2) when we want to confine the notation to a single line.

People are often tempted to write

$$\sum_{k=2}^{n-1} k(k-1)(n-k) \quad \text{instead of} \quad \sum_{k=0}^n k(k-1)(n-k)$$

because the terms for $k = 0, 1$, and n in this sum are zero. Somehow it seems more efficient to add up $n - 2$ terms instead of $n + 1$ terms. But such temptations should be resisted; efficiency of computation is not the same as efficiency of understanding! We will find it advantageous to keep upper and lower bounds on an index of summation as simple as possible, because sums can be manipulated much more easily when the bounds are simple. Indeed, the form $\sum_{k=2}^{n-1}$ can even be dangerously ambiguous, because its meaning is not at all clear when $n = 0$ or $n = 1$ (see exercise 1). Zero-valued terms cause no harm, and they often save a lot of trouble.

So far the notations we’ve been discussing are quite standard, but now we are about to make a radical departure from tradition. Kenneth E. Iverson introduced a wonderful idea in his programming language APL [191, page 11; see also 220], and we’ll see that it greatly simplifies many of the things we want to do in this book. The idea is simply to enclose a true-or-false statement in brackets, and to say that the result is **1** if the statement is true, **0** if the statement is false. For example,

$$[p \text{ prime}] = \begin{cases} 1, & \text{if } p \text{ is a prime number;} \\ 0, & \text{if } p \text{ is not a prime number.} \end{cases}$$

Iverson’s convention allows us to express sums with no constraints whatever on the index of summation, because we can rewrite (2.4) in the form

$$\sum_k a_k [P(k)]. \tag{2.5}$$

If $P(k)$ is false, the term $a_k [P(k)]$ is zero, so we can safely include it among the terms being summed. This makes it easy to manipulate the index of summation, because we don’t have to fuss with boundary conditions.

A slight technicality needs to be mentioned: Sometimes a_k isn’t defined for all integers k . We get around this difficulty by assuming that $[P(k)]$ is “very strongly zero” when $P(k)$ is false; it’s so much zero, it makes $a_k [P(k)]$

equal to zero even when a_k is undefined. For example, if we use Iverson's convention to write the sum of reciprocal primes $\leq N$ as

$$\sum_p [p \text{ prime}] [p \leq N] / p,$$

there's no problem of division by zero when $p = 0$, because our convention tells us that $[0 \text{ prime}] [0 \leq N] / 0 = 0$.

Let's sum up what we've discussed so far about sums. There are two good ways to express a sum of terms: One way uses ' \dots ', the other uses ' \sum '. The three-dots form often suggests useful manipulations, particularly the combination of adjacent terms, since we might be able to spot a simplifying pattern if we let the whole sum hang out before our eyes. But too much detail can also be overwhelming. Sigma-notation is compact, impressive to family and friends, and often suggestive of manipulations that are not obvious in three-dots form. When we work with Sigma-notation, zero terms are not generally harmful; in fact, zeros often make \sum -manipulation easier.

1.2 Manipulation Of Sums

2.3 MANIPULATION OF SUMS

The key to success with sums is an ability to change one \sum into another that is simpler or closer to some goal. And it's easy to do this by learning a few basic rules of transformation and by practicing their use.

Let K be any finite set of integers. Sums over the elements of K can be transformed by using three simple rules:

$$\sum_{k \in K} ca_k = c \sum_{k \in K} a_k; \quad (\text{distributive law}) \quad (2.15)$$

$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k; \quad (\text{associative law}) \quad (2.16)$$

$$\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}. \quad (\text{commutative law}) \quad (2.17)$$

The distributive law allows us to move constants in and out of a \sum . The associative law allows us to break a \sum into two parts, or to combine two \sum 's into one. The commutative law says that we can reorder the terms in any way we please; here $p(k)$ is any permutation of the set of all integers. For example, if $K = \{-1, 0, +1\}$ and if $p(k) = -k$, these three laws tell us respectively that

$$ca_{-1} + ca_0 + ca_1 = c(a_{-1} + a_0 + a_1); \quad (\text{distributive law})$$

$$(a_{-1} + b_{-1}) + (a_0 + b_0) + (a_1 + b_1) \\ = (a_{-1} + a_0 + a_1) + (b_{-1} + b_0 + b_1); \quad (\text{associative law})$$

$$a_{-1} + a_0 + a_1 = a_1 + a_0 + a_{-1}. \quad (\text{commutative law})$$

Gauss's trick in Chapter 1 can be viewed as an application of these three basic laws. Suppose we want to compute the general sum of an *arithmetic progression*,

$$S = \sum_{0 \leq k \leq n} (a + bk).$$

By the commutative law we can replace k by $n - k$, obtaining

$$S = \sum_{0 \leq n-k \leq n} (a + b(n-k)) = \sum_{0 \leq k \leq n} (a + bn - bk).$$

These two equations can be added by using the associative law:

$$2S = \sum_{0 \leq k \leq n} ((a + bk) + (a + bn - bk)) = \sum_{0 \leq k \leq n} (2a + bn).$$

And we can now apply the distributive law and evaluate a trivial sum:

$$2S = (2a + bn) \sum_{0 \leq k \leq n} 1 = (2a + bn)(n + 1).$$

Dividing by 2, we have proved that

$$\sum_{k=0}^n (a + bk) = (a + \frac{1}{2}bn)(n + 1). \quad (2.18)$$

The right-hand side can be remembered as the average of the first and last terms, namely $\frac{1}{2}(a + (a + bn))$, times the number of terms, namely $(n + 1)$.

It's important to bear in mind that the function $p(k)$ in the general commutative law (2.17) is supposed to be a permutation of all the integers. In other words, for every integer n there should be exactly one integer k such that $p(k) = n$. Otherwise the commutative law might fail; exercise 3 illustrates

1.3 Multiple Sums

2.4 MULTIPLE SUMS

The terms of a sum might be specified by two or more indices, not just by one. For example, here's a double sum of nine terms, governed by two indices j and k :

$$\begin{aligned} \sum_{1 \leq j, k \leq 3} a_j b_k = & a_1 b_1 + a_1 b_2 + a_1 b_3 \\ & + a_2 b_1 + a_2 b_2 + a_2 b_3 \\ & + a_3 b_1 + a_3 b_2 + a_3 b_3. \end{aligned}$$

We use the same notations and methods for such sums as we do for sums with a single index. Thus, if $P(j, k)$ is a property of j and k , the sum of all terms $a_{j,k}$ such that $P(j, k)$ is true can be written in two ways, one of which uses Iverson's convention and sums over *all* pairs of integers j and k :

$$\sum_{P(j,k)} a_{j,k} = \sum_{j,k} a_{j,k} [P(j,k)].$$

Only one \sum sign is needed, although there is more than one index of summation; \sum denotes a sum over all combinations of indices that apply.

We also have occasion to use two \sum 's, when we're talking about a sum of sums. For example,

$$\sum_j \sum_k a_{j,k} [P(j,k)]$$

is an abbreviation for

$$\sum_j \left(\sum_k a_{j,k} [P(j,k)] \right),$$

which is the sum, over all integers j , of $\sum_k a_{j,k} [P(j,k)]$, the latter being the sum over all integers k of all terms $a_{j,k}$ for which $P(j,k)$ is true. In such cases we say that the double sum is "summed first on k ." A sum that depends on more than one index can be summed first on any one of its indices.

In this regard we have a basic law called *interchanging the order of summation*, which generalizes the associative law (2.16) we saw earlier:

$$\sum_j \sum_k a_{j,k} [P(j,k)] = \sum_{P(j,k)}^{10} a_{j,k} = \sum_k \sum_j a_{j,k} [P(j,k)]. \quad (2.27)$$

The middle term of this law is a sum over two indices. On the left, $\sum_j \sum_k$ stands for summing first on k , then on j . On the right, $\sum_k \sum_j$ stands for summing first on j , then on k . In practice when we want to evaluate a double sum in closed form, it's usually easier to sum it first on one index rather than on the other; we get to choose whichever is more convenient.

1.4 General Methods

2.5 GENERAL METHODS

Now let's consolidate what we've learned, by looking at a single example from several different angles. On the next few pages we're going to try to find a closed form for the sum of the first n squares, which we'll call \square_n :

$$\square_n = \sum_{0 \leq k \leq n} k^2, \quad \text{for } n \geq 0. \quad (2.37)$$

We'll see that there are at least seven different ways to solve this problem, and in the process we'll learn useful strategies for attacking sums in general.

First, as usual, we look at some small cases.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
n^2	0	1	4	9	16	25	36	49	64	81	100	121	144
\square_n	0	1	5	14	30	55	91	140	204	285	385	506	650

No closed form for \square_n is immediately evident; but when we do find one, we can use these values as a check.

Method 0: You could look it up.

A problem like the sum of the first n squares has probably been solved before, so we can most likely find the solution in a handy reference book. Sure enough, page 36 of the *CRC Standard Mathematical Tables* [28] has the answer:

$$\square_n = \frac{n(n+1)(2n+1)}{6}, \quad \text{for } n \geq 0. \quad (2.38)$$

Just to make sure we haven't misread it, we check that this formula correctly gives $\square_5 = 5 \cdot 6 \cdot 11 / 6 = 55$. Incidentally, page 36 of the *CRC Tables* has further information about the sums of cubes, ..., tenth powers.

The definitive reference for mathematical formulas is the *Handbook of Mathematical Functions*, edited by Abramowitz and Stegun [2]. Pages 813–814 of that book list the values of \square_n for $n \leq 100$; and pages 804 and 809 exhibit formulas equivalent to (2.38), together with the analogous formulas for sums of cubes, ..., fifteenth powers, with or without alternating signs.

But the best source for answers to questions about sequences is an amazing little book called the *Handbook of Integer Sequences*, by Sloane [330], which lists thousands of sequences by their numerical values. If you come up with a recurrence that you suspect has already been studied, all you have to do is compute enough terms to distinguish your recurrence from other famous ones; then chances are you'll find a pointer to the relevant literature in Sloane's *Handbook*. For example, 1, 5, 14, 30, ... turns out to be Sloane's sequence number 1574, and it's called the sequence of "square pyramidal numbers" (because there are \square_n balls in a pyramid that has a square base of n^2 balls). Sloane gives three references, one of which is to the handbook of Abramowitz and Stegun that we've already mentioned.

Still another way to probe the world's store of accumulated mathematical wisdom is to use a computer program (such as Axiom, MACSYMA, Maple, or Mathematica) that provides tools for symbolic manipulation. Such programs are indispensable, especially for people who need to deal with large formulas.

It's good to be familiar with standard sources of information, because they can be extremely helpful. But Method 0 isn't really consistent with the spirit of this book, because we want to know how to figure out the answers

by ourselves. The look-up method is limited to problems that other people have decided are worth considering; a new problem won't be there.

Method 1: Guess the answer, prove it by induction.

Perhaps a little bird has told us the answer to a problem, or we have arrived at a closed form by some other less-than-rigorous means. Then we merely have to prove that it is correct.

We might, for example, have noticed that the values of \square_n have rather small prime factors, so we may have come up with formula (2.38) as something that works for all small values of n . We might also have conjectured the equivalent formula

$$\square_n = \frac{n(n + \frac{1}{2})(n + 1)}{3}, \quad \text{for } n \geq 0, \quad (2.39)$$

which is nicer because it's easier to remember. The preponderance of the evidence supports (2.39), but we must prove our conjectures beyond all reasonable doubt. Mathematical induction was invented for this purpose.

"Well, Your Honor, we know that $\square_0 = 0 = 0(0 + \frac{1}{2})(0 + 1)/3$, so the basis is easy. For the induction, suppose that $n > 0$, and assume that (2.39) holds when n is replaced by $n - 1$. Since

$$\square_n = \square_{n-1} + n^2,$$

we have

$$\begin{aligned} 3\square_n &= (n-1)(n-\frac{1}{2})(n) + 3n^2 \\ &= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\ &= (n^3 + \frac{3}{2}n^2 + \frac{1}{2}n) \\ &= n(n + \frac{1}{2})(n + 1). \end{aligned}$$

Therefore (2.39) indeed holds, beyond a reasonable doubt, for all $n \geq 0$." Judge Wapner, in his infinite wisdom, agrees.

Induction has its place, and it is somewhat more defensible than trying to look up the answer. But it's still not really what we're seeking. All of the other sums we have evaluated so far in this chapter have been conquered without induction; we should likewise be able to determine a sum like \square_n from scratch. Flashes of inspiration should not be necessary. We should be able to do sums even on our less creative days.

Method 2: Perturb the sum.

So let's go back to the perturbation method that worked so well for the geometric progression (2.25). We extract the first and last terms of \square_{n+1} in

order to get an equation for \square_n :

$$\begin{aligned}\square_n + (n+1)^2 &= \sum_{0 \leq k \leq n} (k+1)^2 = \sum_{0 \leq k \leq n} (k^2 + 2k + 1) \\ &= \sum_{0 \leq k \leq n} k^2 + 2 \sum_{0 \leq k \leq n} k + \sum_{0 \leq k \leq n} 1 \\ &= \square_n + 2 \sum_{0 \leq k \leq n} k + (n+1).\end{aligned}$$

Oops—the \square_n 's cancel each other. Occasionally, despite our best efforts, the perturbation method produces something like $\square_n = \square_n$, so we lose.

On the other hand, this derivation is not a total loss; it does reveal a way to sum the first n integers in closed form,

$$2 \sum_{0 \leq k \leq n} k = (n+1)^2 - (n+1),$$

even though we'd hoped to discover the sum of first integers squared. Could it be that if we start with the sum of the integers cubed, which we might call \mathcal{D}_n , we will get an expression for the integers squared? Let's try it.

$$\begin{aligned}\mathcal{D}_n + (n+1)^3 &= \sum_{0 \leq k \leq n} (k+1)^3 = \sum_{0 \leq k \leq n} (k^3 + 3k^2 + 3k + 1) \\ &= \mathcal{D}_n + 3\square_n + 3\frac{(n+1)n}{2} + (n+1).\end{aligned}$$

Sure enough, the \mathcal{D}_n 's cancel, and we have enough information to determine \square_n without relying on induction:

$$\begin{aligned}3\square_n &= (n+1)^3 - 3(n+1)n/2 - (n+1) \\ &= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1) = (n+1)(n + \frac{1}{2})n.\end{aligned}$$

Method 3: Build a repertoire.

A slight generalization of the recurrence (2.7) will also suffice for summands involving n^2 . The solution to

$$\begin{aligned}R_0 &= \alpha; \\ R_n &= R_{n-1} + \beta + \gamma n + \delta n^2, \quad \text{for } n > 0,\end{aligned}\tag{2.40}$$

will be of the general form

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta;\tag{2.41}$$

and we have already determined $A(n)$, $B(n)$, and $C(n)$, because (2.40) is the same as (2.7) when $\delta = 0$. If we now plug in $R_n = n^3$, we find that n^3 is the

solution when $\alpha = 0$, $\beta = 1$, $\gamma = -3$, $\delta = 3$. Hence

$$3D(n) - 3C(n) + B(n) = n^3;$$

this determines $D(n)$.

We're interested in the sum \square_n , which equals $\square_{n-1} + n^2$; thus we get $\square_n = R_n$ if we set $\alpha = \beta = \gamma = 0$ and $\delta = 1$ in (2.41). Consequently $\square_n = D(n)$. We needn't do the algebra to compute $D(n)$ from $B(n)$ and $C(n)$, since we already know what the answer will be; but doubters among us should be reassured to find that

$$3D(n) = n^3 + 3C(n) - B(n) = n^3 + 3\frac{(n+1)n}{2} - n = n(n+\frac{1}{2})(n+1).$$