MATH H113: Honors Introduction to Abstract Algebra

2016-03-07

- More Rubik
- Free groups

For Friday: Read Sect. 7.1
Two Handouts

Rubik's Cubes (cont)

Number of Positions = $\frac{8! \times 3^8 \times 12! \times 12^{12}}{12}$
Parity of permutations of corner pieces = parityf of permutations of edge pieces ($= (-1)^{\text{number of permutations}}$).
Twists of corner pieces
Similar thing for twists (flips) of edge pieces.

Free groups, generators, and relations.

We'd like to define what group is $\langle(\text{generators})|(\text{relations})\rangle$.
Start with no relations (just generators).
Proposition: Let $\phi : G \to H$ be a homomorphism. If a set $S \subseteq G$ generates G, then $\phi(S)$ generates $\phi(G) = \text{im}\phi$.
Proof: $\text{im}\phi$ is a subgroup and it contains $\phi(S)$, so $\langle\phi(s)\rangle \leq \text{im}\phi$.
On the other hand, let $h \in \text{im}\phi$. Then $h = \phi(g)$ for some $g \in G$, and we can write $g = s_1^{n_1} \cdots s_m^{n_m}$ with $m \in \mathbb{N}$, and $s_i \in S$, $n_i \in \mathbb{Z}\ \forall i$.
Then $h = \phi(g) = \phi(s_1)^{n_1} \cdots \phi(s_m)^{n_m}$ lies in $\langle\phi(S)\rangle$. $\therefore$ $\text{im}phi \subseteq \langle\phi(S)\rangle$.
Corollary: Let G be a grup, let $N \trianglelefteq G$, and let $\pi : G \to G/N$ be the natural projection. If S generates G, then $\pi(S)$ generates $G/N = \text{im}\pi$.
Corollary (not needed today, but useful): Let $\phi_1, \phi_2 : G \to H$ be homomorphisms. If S generates G and $\phi_1(S) = \phi_2(S)\ \forall s \in S$, then $\phi_1 = \phi_2$.
Proof: Exercise.
Back to the first corollary:
Going from G to G/N, you have the same number of generators, but more relations. $\cdots \to G \to G/N$ (in order of fewer to more relations).
This motivates:
Definition: Let S be a set. The *free group on S* (if it exists) is a group G, together with a function $i : S \to G$, that satisfies the following *universal property*: For all groups G' and all functions $\phi : S \to G'$, there is a *unique* homomorphism $\Phi : G \to G'$ such that $\phi = \Phi \circ i$.

(*Note*: I'm *not* requiring i to be injective, I'm not requiring that $i(S)$ generate G or that $\phi(S)$ generate G'. But I am requiring Φ to be unique.)