

MATH H113: Honors Introduction to Abstract Algebra

2016-03-02

- Finish normal subgroups
- Isomorphism theorems

Note: We'll *skip* Section 3.4

Proof of Ex. 3.2.11 (p. 96):

Let $H \leq K \leq G$. Show that $|G : H| = |G : K||K : H|$.

Note: that we can't assume that G is finite. We also can't assume that H or K is a normal subgroup of G . We'll assume that $|G : H|$ is finite.

Map G/H to G/K by $uH \mapsto uK$.

This is well defined because $uH = vH \implies u^{-1}v \in H \implies u^{-1}v \in K \implies uK = vK$.

It's onto (surjective) because $\forall u \in G. uK = f(uH)$ (where f is the function we're defining).

Since $K/H \leq G/H$, $|K : H| < \infty$.

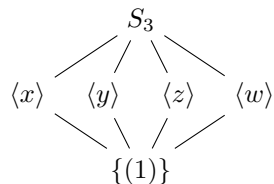
Let a_1H, \dots, a_nH be the elements of K/H (without repetition). Then $K = a_1H \cup a_2H \cup \dots \cup a_nH$, and these cosets are disjoint (a_1, \dots, a_n are called a set of *coset representatives* of H in K). Then $\forall u \in G. uK = (ua_1)H \cup \dots \cup (ua_n)H$, and again these cosets are disjoint.

So the fibers of $f : G/H \rightarrow G/K$ have n elements each ($vH \in f^{-1}(uK) \iff f(vH) = vK = uK \iff v \in uK \iff v \in ua_iH$ for some $i \iff vH = ua_iH$ for some $i = 1, \dots, n$).

$\therefore |G : H| = n|G : K| = |G : K||K : H|$.

(f sorts the elements of G/H into piles of n elements each, and the number of piles is $|G/K| = |G : K|$)

Back to Subgroups of S_3



We spent a lot of time showing that there were no subgroups between $\langle x \rangle, \langle y \rangle, \langle z \rangle, \langle w \rangle$ and S_3 . This is now immediate, because $|S_3 : \langle x \rangle| = |S_3 : \langle y \rangle| = |S_3 : \langle z \rangle| = 3$ and $|S_3 : \langle w \rangle| = 2$ are all prime. Which subgroups of S_3 are normal in S_3 ?

From earlier examples:

$\{1\}$ and S_3 are normal (in S_3).

$\langle x \rangle$ is not normal (in S_3).

$\langle y \rangle, \langle z \rangle$ are also not normal (same reason as for $\langle x \rangle$).

$\langle w \rangle$ is normal. This is because $|S_3 : \langle w \rangle| = 2$, and any subgroup of index 2 is normal.

If $|G : H| = 2$ then the left cosets of H are H and the other one must be $G - H$.

Likewise the right cosets are H and $G - H$.

$\therefore gH = Hg$ because if $g \in H$ then $gH = Hg = H$ otherwise $gH = Hg$ because both $= G - H$.

Lagrange's theorem: if $H \leq G$ and $|G| = n$ is finite, then $|H|$ divides n . You don't necessarily have a subgroup of every order dividing n .

However you do have:

Theorem (Cauchy): If G is a finite group and p is a prime divisor of $|G|$, then G contains an element of order p (\therefore it has a subgroup of order p : $\langle x \rangle$ where $|x| = p$).

Proof: On your homework.

Proposition: If H and K are finite subgroups of a group G , then $|HK| = \frac{|HK|}{|H \cap K|}$.

Proof: See book, but it's the same idea as Ex. 3.2.11: map H to HK/K . The fibers have $|H \cap K|$ elements each.

Note: $|HK|$ might not be a subgroup of G ($|HK|$ is defined as $\{hk : h \in H, k \in K\}$, $HK/K = \{uK : u \in HK\}$).

Prop: Let H and K be subgroups of a group G . Then $|HK| \leq G \iff HK = KH$.

Proof (sketch): If $HK \leq G$, then $\forall x \in HK$, $x^{-1} \in HK$, so $x^{-1} = hk$ with $h \in H, k \in K$.

$\therefore x = k^{-1}h^{-1}$ lies in KH .

$\therefore HK \subseteq KH$.

$KH \subseteq HK$, let $x \in KH$; $x = kh$ $k \in K, h \in H$.

Then $x^{-1} = h^{-1}k^{-1}$ lies in HK , so $x \in HK$ since $HK \leq G$.

$\therefore KH \subseteq HK$. This proves " \implies ".

" \impliedby ": Clearly $HK \neq \emptyset$,

Let $a, b \in HK$. Then $a = h_1k_1$ and $b = h_2k_2$ $h_1, h_2 \in H; k_1, k_2 \in K$.

Then $ab^{-1} = h_1k_1k_2^{-1}h_2^{-1}$

$k_1k_2^{-1}h_2^{-1} \in KH$, \therefore it's in HK , so it equals h_3k_3 ($h_3 \in H, k_3 \in K$)

$\therefore ab^{-1} = h_1h_3k_3$ lies in HK .

$\therefore HK$ is a subgroup.

Corollary: If $H \leq N_G(K)$ then HK is a subgroup. In particular, if $K \trianglelefteq G$ then $HK \leq G$.

Proof: $HK = \bigcup_{h \in H} hK = \bigcup_{h \in H} Kh = KH$.

Isomorphism Theorems

Theorem (1st isomorphism theorem): Let $\phi : G \rightarrow H$ be a homomorphism. Then $\ker \phi$ is a normal subgroup of G , and $\text{Im } \phi \cong G / \ker \phi$ (via $\phi(u) \mapsto u(\ker \phi)$)

Proof: This was done in Sect. 3.1

Corollary 1: $\phi : G \rightarrow H$ is injective $\iff \ker \phi = 1$.

Proof: If $\ker \phi = 1$ then fibers of ϕ all are cosets of $\ker \phi$ (or \emptyset), \therefore they have $|\ker \phi| = 1$ element each (or 0), so ϕ is injective.

If $\ker \phi \neq 1$ then $\ker \phi$ has two distinct elements, and they map to $1 \in H$.

Corollary: $|G : \ker \phi| = |\text{Im } \phi|$

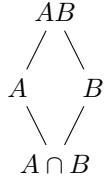
Proof: $|G : \ker \phi| = |G/(\ker \phi)| = |\text{Im } \phi|$ (use isomorphism).

Anatomy of a homomorphism: let $\phi : G \rightarrow H$ be a homomorphism.

$$G \xrightarrow[\text{surjective}]{\pi} G/(\ker \phi) \xrightarrow[\text{injective}]{\sim} \text{Im } \phi \xrightarrow{\quad} H$$

$\text{Im } \phi \rightarrow H$ is the inclusion map $x \mapsto x$. It is a homomorphism.

Theorem (2nd isomorphism theorem, “diamond”): Let A and B be subgroups of G , and assume that $A \subseteq N_G(B)$ (for example, $B \trianglelefteq G$). Then AB is a subgroup of G , $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$, and $AB/B \cong A/(A \cap B)$ via $aB \mapsto a(A \cap B)$.



Proof: $AB \leq G$ is proved already.

$B \trianglelefteq AB$ because $N_{AB}(B) \supseteq B$ ($B \leq AB$)

$N_{AB}(B) \supseteq A$ by assumption ($N_{AB}(B) = N_G(B) \cap AB$)

$\therefore N_{AB}(B) = AB$ because “ \supseteq ” is proved above, “ \leq ” is from the def.

To be continued