MATH H113: Honors Introduction to Abstract Algebra

2016-03-07

- The Alternating Group
- Group Actions
- Cayley's Theorem

Last Time

Theorem: Let $n \geq 2$. Then there is a unique homomorphism $\epsilon : S_n \to \{\pm 1\}$$ such that $\epsilon(\tau) = -1$ for all transpositions of $\tau \in S_n$. It is surjective.

See pictures

Group Actions

Recall: An action of a group G on a set A is a function $G \times A \to A$, denoted $(g, a) \mapsto g \cdot a$, such that:

1. $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a \; \forall g_1, g_2 \in G; a \in A$, and
2. $1 \cdot a = a \; \forall a \in A$.

Giving an action of G on A is equivalent to giving a homomorphism $\phi : G \to S_A$, $g \mapsto (a \mapsto g \cdot a)$. *New*: Such a map is called a *permutation representation* of G.

The *kernel* of a group action $= \{g \in G : g \cdot a = a \; \forall a \in A\} = \ker \phi$.
The *stabilizer* of an element $a \in A$ is the subgroup
$G_a = \{g \in G : g \cdot a = a\}$
(so the kernel of the action is $\bigcap\limits_{a \in A} G_a$).
(*Note*: G_a need not be normal.)
Definition: An action of a group on a set is:

- *faithful* if its kernel is the trivial subgroup, and
- *transitive* if it has only one orbit ($\forall a, b \in A. \exists g \in G$ s.t. $b = g \cdot a$)

Proposition: Let G act on a set A, let $a \in A$, and let O be the orbit of a under the action of G. Then, for any $b \in O$, the set $\{g \in G : g \cdot a = b\}$ is a left coset of G_a in G, and there is a natural well-defined bijection from the set G/G_a (set of left cosets) to O, given by $gG_a \mapsto g \cdot a$. Therefore, $|O| = |G : G_a|$ (and $|O| = \infty \iff |G : G_a| = \infty$).

Proof: Given $b \in O$ and $h \in G$ such that $b = h \cdot a$, we have $\{g \in G : g \cdot a = b\} = hG_a$ because
$g \in hG_a \iff gG_a = hG_a \iff g^{-1}h \in G_a \iff g^{-1}h \cdot a = a \iff g^{-1}(h(a)) = a \iff g^{-1}(b) = a \iff g(a) = b \iff g \in \{g' \in G : g' \cdot a = b\}$.
Then the above map is well defined $gG_a = hG_a \iff g(a) = h(a)$ (from the above). It's injective by the above. It's surjective by the definition of the orbit: $O \implies h \cdot a = b$ for some $h \in G \implies hG_a \mapsto b$.
The last sentence is immediate.

Cayley's Theorem

Look at ghe multiplication table for V_4 (Klein group):

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

(Generally, in a multiplication table, the entry row x and column y is the product xy (not yx)).
Each row of the multiplication table contains every element of the group exactly once (not counting row labels). Similarly for columns. (by Prop. 2 on p. 20)

By (1) each $g \in G$ gives rise to a permutation of G called σ_g: $\sigma_g(h) = gh$.
So we get a *function* $\phi : G \to S_G$.
By (2), this function is injective. In fact, ϕ is a homomorphism, because it is the permutation representation of G associated to the action of G on itself by left multiplication: $\sigma_g(h) = g \cdot h = gh$
Check that it's a group action:

1. $g_1 \cdot (g_2 \cdot a) = g_1(g_2a) = (g_1g_2)a = (g_1g_2) \cdot a \; \forall g_1, g_2, a \in G$, and
2. $1 \cdot a = a \; \forall a \in G$ because $1a = a$.

This proves: **Theorem (Cayley)**: Every group is isomorphic to a subgroup of some permutation group. (In fact, one such isomorphism is given by the permutation representation of G associated to the action of G on itself by left multiplication.) So, if G is a finite group and $n = |G|$, then one such representation is the isomorphism of G with a subgroup of S_n.
Proof: Let $\phi : G \to S_G$ be as in the parentesized sentence. Since ϕ is injective, it gives an isomorphism to $\mathrm{Im}\phi$, which is a subgroup of S_G.
Example: If $G = S_3$, then the map $G \to S_G$ gives an isomorphism of S_3 with a subgroup of S_6 (on homework: Ex. 4.2.2).

More general theorem: Let G be a group, let H be a subgroup of G, and let $A = G/H$(= set of left coset of H in G). Let G act on A by $g \cdot (aH) = (ga)H$ for all $g, a \in G$ (this is well defined because $(ga)H = g(aH) = \{gx : x \in aH\}$). Let $\pi_H : G \to S_A = S_{G/H}$ be the associated permutation representation. Then:

1. G acts transitively on A;
2. The stabilizer of $1H = H$ is H
3. $\ker \pi_H = \bigcap\limits_{x \in G} xHx^{-1}$. This is the largest normal subgroup of G contained in H.

Proof:

1. Let $gH \in A$. Then $gH = g \cdot 1H$, so gH is in the same orbit as $1H$, so there is only one orbit.
2. $x \cdot 1H = 1H \iff (x1)H = H \iff xH = H \iff x \in H$ so $x \in G_{1H} \iff x \in H$, so $G_{1H} = H$.
3. From your homework, $xHx^{-1} = xG_{1H}x^{-1} = G_{x \cdot 1H} = G_{xH}$:
So $\bigcap\limits_{x \in G} xHx^{-1} = \bigcap\limits_{x \in G} G_{xH} = \bigcap_{a \in A} G_a = \ker \pi_H$.
Second part:
$\bigcap\limits_{x \in G} xHx^{-1}$ isnormal in G (it's in $\ker \pi_H$), and $\subseteq H$.
Let $N \trianglelefteq G$ with $N \subseteq H$.
Then $N = gNg^{-1} \subseteq gHg^{-1}$ $\forall g \in G$, $\therefore N \subseteq \bigcap\limits_{x \in G} xHx^{-1}$.

When $H = 1$ you get back Cayley's theorem because G bijective to G/H by $\psi(g) = gH \in G/H \forall g \in g \in G$. and the actions correspond $x\psi(g) = \psi(xg)$ $\forall g, x \in G$.
So $\ker \pi_H = 1$ (subgroup of $H = 1$)
so π_H is injective.