MATH H113: Honors Introduction to Abstract Algebra

2016-03-02

- Finish normal subgroups
- Isomorphism theorems

Note: We'll *skip* Section 3.4

Proof of Ex. 3.2.11 (p. 96):
Let $H \leq K \leq G$. Show that $|G : H| = |G : K||K : H|$.
Note: that we can't assume that G is finite. We also can't assume that H or K is a normal subgroup of G. We'll assume that $|G : H|$ is finite.
Map G/H to G/K by $uH \mapsto uK$.
This is well defined because $uH = vH \implies u^{-1}v \in H \implies u^{-1}v \in K \implies uK = vK$.
It's onto (surjective) because $\forall u \in G.\, uK = f(uH)$ (where f is the function we're definining).
Since $K/H \leq G/H$, $|K : H| < \infty$.
Let $a_1H, \ldots, a_nH$ be the elements of K/H (without repetition). Then $K = a_1H \cup a_2H \cup \ldots \cup a_nH$, and these cosets are disjoint ($a_1, \ldots, a_n$) are called a set of *coset representatives* of H in K). Then $\forall u \in G.\, uK = (ua_1)H \cup \ldots \cup (ua_n)H$, and again these cosets are disjoint.
So the fibers of $f : G/H \to G/K$ have n elements each ($vH \in f^{-1}(uK) \iff f(vH) = vK = uK \iff v \in uK \iff v \in ua_1H)$ for some $i \iff vH = ua_iH$ for some $i = 1, \ldots, n$.
$\therefore |G : H| = n|G : K| = |G : K||K : H|$.
(f sorts the elements of G/H into piles of n elements each, and the number of piles is $|G/K| = |G : K|$

Back to Subgroups of S_3

$$
\begin{tikzcd}[column sep=small]
 & & & S_3 \arrow[dlll, dash] \arrow[dl, dash] \arrow[dr, dash] \arrow[drrr, dash] & & & \\
\langle x\rangle & & \langle y\rangle & & \langle z\rangle & & \langle w\rangle \\
 & & & \{(1)\} \arrow[ulll, dash] \arrow[ul, dash] \arrow[ur, dash] \arrow[urrr, dash] & & &
\end{tikzcd}
$$

We spent a lot of time showing that there were no subgroups between $\langle x\rangle, \langle y\rangle, \langle z\rangle, \langle w\rangle$ and S_3. This is now immediate, because $|S_3 : \langle x\rangle| = |S_3 : \langle y\rangle| = |S_3 : \langle z\rangle| = 3$ and $|S_3 : \langle w\rangle| = 2$ are all prime. Which subgroups of S_3 are normal in S_3?

From earlier examples:
$\{1\}$ and S_3 are normal (in S_3).
$\langle x\rangle$ is not normal (in S_3).
$\langle y\rangle, \langle z\rangle$ are also not normal (same reason as for $\langle x\rangle$).
$\langle w\rangle$ is normal. This is because $|S_3 : \langle w\rangle| = 2$, and any subgroup of index 2 is normal.
If $|G : H| = 2$ then the left cosets of H are H and the other one must be $G - H$. Likewise the right cosets are H and $G - H$.
$\therefore$ $gH = Hg$ because if $g \in H$ then $gH = Hg = H$ otherwise $gH = Hg$ because both $= G - H$.

Lagrange's theorem: if $H \leq G$ and $|G| = n$ is finite, then $|H|$ divides n. You don't necessarily have a subgroup of every order dividing n.
However you do have:
Theorem (Cauchy): If G is a finite group and p is a prime divisor of $|G|$, then G contains an element of order p ($\therefore$ it has a subgroup of order p: $\langle x\rangle$ where $|x| = p$).
Proof: On you homework.

Proposition: If H and K are finite subgroups of a group G, then $|HK| = \frac{|HK|}{|H\cap K|}$.
Proof: *See book*, but it's the same idea as Ex. 3.2.11: map H to HK/K. The fibers have $|H \cap K|$ elements each.
Note: $|HK|$ might not be a subgroup of G ($|HK|$ is defined as $\{hk : h \in H, k \in K\}$, $HK/K = \{uK : u \in HK\}$).

Prop: Let H and K be subgroups of a group G. Then $|HK| \leq G \iff HK = KH$.
Proof (sketch): If $HK \leq G$, then $\forall x \in HK$, $x^{-1} \in HK$, so $x^{-1} = hk$ with $h \in H, k \in K$.
$\therefore$ $x = k^{-1}h^{-1}$ lies in KH.
$\therefore$ $HK \subseteq KH$.
$KH \subseteq HK$, let $x \in KH$; $x = kh$ $k \in K, h \in H$.
Then $x^{-1} = h^{-1}k^{-1}$ lies in HK, so $x \in HK$ since $HK \leq G$.
$\therefore$ $KH \subseteq HK$. This proves "$\implies$".
"$\impliedby$": Clearly $HK \neq \emptyset$,
Let $a, b \in HK$. Then $a = h_1k_2$ and $b = h_2k_2$ $h_1, h_2 \in H; k_1, k_2 \in K$.
Then $ab^{-1} = h_1k_1k_2^{-1}h_2^{-1}$
$k_1k_2^{-1}h_2^{-1} \in KH$, $\therefore$ it's in HK, so it equals h_3k_3 ($h_3 \in H, k_3 \in K$)
$\therefore$ $ab^{-1} = h_1h_3k_3$ lies in HK.
$\therefore$ HK is a subgroup.

Corollary: If $H \leq N_G(K)$ then HK is a subgroup. In particular, if $K \trianglelefteq G$ then $HK \leq G$.
Proof: $HK = \bigcup_{h\in H} hK = \bigcup_{h\in H} Kh = KH$.

Isomorphism Theorems

Theorem (1st isomorphism theorem): Let $\phi : G \to H$ be a homomorphism. Then $\ker\phi$ is a normal subgroup of G, and $\operatorname{Im}\phi \cong G/\ker\phi$ (via $\phi(u) \leftarrow\!\shortmid u(\ker\phi)$)
Proof: This was done in Sect. 3.1

Corollary 1: $\phi : G \to H$ is injective $\iff$ $\ker\phi = 1$.
Proof: If $\ker\phi = 1$ then fibers of ϕ all are cosets of $\ker\phi$ (or $\emptyset$), $\therefore$ they have $|\ker\phi| = 1$ element each (or 0), so ϕ is injective.
If $\ker\phi \neq 1$ then $\ker\phi$ has two distinct elements, and they map to $1 \in H$.

Corollary: $|G : \ker\phi| = |\operatorname{Im}\phi|$
Proof: $|G : \ker\phi| = |G/(\ker\phi)| = |\operatorname{Im}\phi|$ (use isomorphism).

Anatomy of a homomorphism: let $\phi : G \to H$ be a homomorphism.
$G \xrightarrow[\text{surjective}]{\pi} G/(\ker\phi) \xrightarrow{\sim} \operatorname{Im}\phi \xrightarrow[\text{injective}]{} H$
$\operatorname{Im}\phi \to H$ is the inclusion map $x \mapsto x$. It is a homomorphism.

Theorem (2nd isomorphism theorem, "diamond"): Let A and B be subgroups of G, and assume that $A \subseteq N_G(B)$ (for example, $B \trianglelefteq G$). Then AB is a subgroup of G, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$, and $AB/B \cong A/(A\cap B)$ via $aB \leftarrow\!\shortmid a(A \cap B)$.

$$\begin{array}{ccc} & AB & \\ / & & \backslash \\ A & & B \\ \backslash & & / \\ & A \cap B & \end{array}$$

Proof: $AB \leq G$ is proved already.
$B \trianglelefteq AB$ because $N_{AB}(B) \supseteq B$ ($B \leq AB$)
$N_{AB}(B) \supseteq A$ by assumption ($N_{AB}(B) = N_G(B) \cap AB$)
$\therefore N_{AB}(B) = AB$ because "$\supseteq$" is proved above, "$\leq$" is from the def.
To be continued