MATH H113: Honors Introduction to Abstract Algebra

2016-03-30

- Zorn's Lemma
- Rings of fractions

Zorn's Lemma

Version 1 (most convenient): Let A be a nonempty partially ordered set in which every nonempty chain has an upper bound. Then A has a maximal element.
Version 2: Let A be a partially ordered set in which every chain has an upper bound. Then A has a maximal element.
Book's Version: Let A be a nonempty partially ordered set in which every chain has an upper bound. Then A has a maximal element.
These three versions are all equivalent: first sentences all describe the same condition on A.
Caution: B (the choice of chains) don't have to be countable.
Proposition: Let R be a ring with 1. Then every *proper* ideal is contained in some maximal ideal $\neq R$.

Proof: Let R be a ring with 1 and let I be a proper ideal of R. Let A be the set of proper ideals of R that contain I: $A \cdot \{\text{ideals} J : I \subseteq J \subsetneq R\}$. Then $A \neq \emptyset$ because it contains I. The set A is partially ordered under inclusion.
Let B be a nonempty chain in A.
Let $K = \bigcup_{J \in B} J$
Then K is an ideal in R:
$B \neq \emptyset$, so $B \ni J$ and $J \subseteq K$, so $K \neq \emptyset$.
Let $x, yinK$. Then there are elements $J_1, J_2 \in B$ such that $x \in J_1, y \in J_2$. Let J be the larger of J_1 and J_2. Then $x \in J_1 \in J$ and $y \in J_2 \in J$, so $x - y \in J \subseteq K$.
$\therefore K$ is an additive subgroup of R.
Now let $x \in K$ and $r \in R$. Then $x \in J$ for some $J \in B$, and $\therefore xr \in J$ (since J is an ideal), so $xr \in K$ (since $J \subseteq K$). $\therefore K$ is an ideal of R.
Also $K \supseteq I$ because $K \subseteq J \supseteq I$ for some $J \in B$. Clearly K is an upper bound for B in A.
K is an ideal $\supseteq I$, is it a *proper* ideal? Yes, because $1 \notin K$ ($1 \notin J \forall J \in A$). $\therefore$ by Zorn's lemma, A contains a maximal element, which is a maximal ideal of R that contains I.
Let M be a max element of A. THen M is an ideal of R that contains I. Also $M \neq R$ (since $M \in A$), and it's maximal, because if it's not then $\exists$ an ideal N such that $M \subsetneq N \subsetneq R$, but then $N \in A$ ($N \supseteq M \supseteq I$ and N is a proper ideal), contradicting the maximality of M in A.

Corollary: Let R be a ring with 1. Then R has a maximal ideal $\iff R \neq 0$.
Proof: $R = 0 \implies$ its only ideal is 0, which is not maximal. $R \neq 0 \implies I = 0$ is a proper ideal, so it's contained in some maximal ideal.

Corollary: Same for prime ideals.
$R = 0 \implies$ no proper ideals, $\therefore$ no prime ideals.
$R \neq 0 \implies$ it has a maximal ideal, which is prime.

Rings of Fractions

Ex. 7.1.5(1): is the set $\{r \in \mathbb{Q} : r$ has odd denominator (when written in lowest terms)$\}$ a subring of $\mathbb{Q}$? Yes.
We'd like to be able to make such subrings, if if we don't have something to play the role of $\mathbb{Q}$.
(And we'd like to generalize to generalize the consturction of $\mathbb{Q}$).
Given a ring R we'd like to construct a ring Q, containing a ring isomorphic to) R as a subring, such that certain nonzero elements of R are invertible in Q.
Which elements *can* we want to invert in Q?
Don't want 0 to become a unit in Q. Likewise, if a is a zero divisor in R, say $ab = 0$ with $b \neq 0$, then $\frac{1}{a} = \frac{b}{ab} = \frac{b}{0}$, which is bad.
We want our set of possible denominators to be closed under multiplication.
And, we want to construct Q without using any known ring that contains Q. $\nvDash$