MATH H113: Honors Introduction to Abstract Algebra

2016-02-08

- Group Actions
- Subgroups

Sample First Midterm

On the first question, if you answered $D_{2n} = \langle \{r, s\} : r^n = 1, s^2 = 1, rs = sr^{-1} \rangle$ you would only get partial credit (this isn't the definition).

Group Actions Continued

Remember: A group action is defined by:

- $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$
- $1 \cdot a = a$

More examples of group actions:

2. Let $n \in \mathbb{Z}_{>0}$, $G = S_n$, and $A = \{1, 2, \ldots, n\}$. Then $\sigma \cdot a = \sigma(a)$ is a group action of G on A.
   $\sigma_1 \cdot (\sigma_2 \cdot a) = \sigma_1(\sigma_2(a))$
   $(\sigma_1 \sigma_2)(a) = (\sigma_1 \circ \sigma_2)(a) = \sigma_1(\sigma_2(a))$
   $1 \cdot a = 1(a) = a$ (where 1 is the identity map).
   This also works for $G = S_\Omega$ and $A = \Omega$, for any set Ω.
3. Let G be any group and A be any set. Then $g \cdot a = a \ \forall a \in A, g \in G$ is a group action, called the trivial action.

Lemma: Let a group G act on a set A. For all $g \in G$, define a function $\sigma_g : A \to A$ by $\sigma_g(a) = g \cdot a$. Then σ_g is a bijection $\forall g \in G$.
Proof: $\sigma_{g^{-1}} \circ \sigma_g$ maps a to $g^{-1} \cdot (g \cdot a) = (g^{-1} g) \cdot a = 1 \cdot a = a$. So $\sigma_{g^{-1}} \circ \sigma_g$ is the identity map on A. Therefore $\sigma_{g^{-1}}$ is a left inverse for σ_g. Similarly, $\sigma_g \circ \sigma_{g^{-1}}$ maps a to $\sigma_g(\sigma_{g^{-1}}(a)) = g \cdot (g^{-1} \cdot a) = (gg^{-1}) \cdot a = 1 \cdot a = a$, so $\sigma_{g^{-1}}$ is also a right inverse of σ_g, $\therefore \sigma_g$ is bijective. ($\therefore \sigma_g$ is a permuation of A.)

Proposition: Giving an action of a group G on a set A is equivalent to giving a homomorphism $G \to S_A$.
Proof:

1. Let G act on A. Then, for each $g \in G$, $\sigma_g \in S_A$, so we get a function $\phi : G \to S_A$. This is a homomorphism, because $\phi(g_1 g_2) = \sigma_{g_1 g_2}$, which maps a to $(g_1 g_2) \cdot a$ and $\phi(g_1)\phi(g_2) = \sigma_{g_1} \circ \sigma_{g_2}$, which maps a to $g_1 \cdot (g_2 \cdot a)$ this is true $\forall a \in A$, so $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.
2. Let $\phi : G \to S_A$ be a homomorphism. Define a group action of G on A by letting $\sigma_g = \phi(g)$, and then let $g \cdot a = \sigma_g(a) \in A$. Check that this is a group action. Since ϕ is a homomorphism, $\phi(g_1 g_2) = \sigma_{g_1} g_2 = \phi(g_1)\phi(g_2) = \sigma_{g_1} \sigma g_2$.

Then (1) is true because because $g_1 \cdot (g_2 \cdot a) = \sigma_{g_1}(\sigma_{g_2}(a)) = (\sigma_{g_1}\sigma_{g_2})(a) = \sigma_{g_1 g_2}(a) = (g_1 g_2) \cdot a$. Also (2) is true because $\phi(1_g)$ is the identity map on A, so $1 \cdot a = \sigma_1 \cdot a =$ (identity map on A)$(a) = a$. So, we have functions: $\{$actions of G on $A\} \leftrightarrows \{$homorphisms $G \to S_A\}$. Need to check that these are mutually inverse. (Composing the two maps reverses the steps in each case).

Exercise 14: If G is a *non-abelian* group and $A = G$, then the operation $g \cdot a = ag$ does *not* satisfy the conditions for a group action. This is because $g_1 \cdot (g_2 \cdot a) = g_1 \cdot (ag_2) = (ag_2)g_2 = ag_2 g_1$ but $(g_1 g_2) \cdot a = ag_1 g_2$. These are equal $\iff g_2 g_1 = g_1 g_2$ (cancel a: recall $a \in G$).

So condition (1) for a group operation will not always be true, since G is non-abelian.

Exercise 15: Let G be any group, and let $A = G$. Show that $g \cdot a = ag^{-1}$ is a group action of G on A.

1. $g_1 \cdot (g_2 \cdot a) = g_1 \cdot (ag_2^{-1}) = (ag_2^{-1})g_1^{-1} = ag_2^{-1}g_1^{-1}$ and $(g_1 g_2) \cdot a = a(g_1 g_2)^{-1} = a(g_2^{-1}g_1^{-1})$ which are always equal, so (1) is true.
2. $1 \cdot a = a1^{-1} = a1 = a$.

Definition: Let G act on a set A. Then the *kernel* of the group action is the set $\{g \in G : g \cdot a = a \ \forall a \in A\}$. Let H be such a kernel. Then $1 \in H$ (by (2)). If $x, y \in H$ then $xy \in H$ because $(xy) \cdot a = x \cdot (y \cdot a) = x \cdot a = a \ \forall a \in A$ and $x^{-1} \in H$ because σ_x is the identity map on $A \therefore \sigma_x^{-1}$ is the identity map. But we showed $\sigma_x^{-1} = \sigma_{x^{-1}}$, so $\sigma_{x^{-1}}$ is the identity map, $\therefore x^{-1} \in H$. Incidentally, the kernel is also equal to the set $\{g \in G : \sigma_g$ is the identity map on A$\} = \{g \in G : \phi(g) = 1\}$, where $\phi : G \to S_A$ is the homomorphism that corresponds to the group action.

In example (2). $G = S_A$ acts on A by $\sigma \cdot a = \sigma(a)$ the kernel is the trivial subgroup ($\sigma \cdot a = a \ \forall a \in A \implies \phi =$ identity). The map $\phi : G \to S_A = \phi : S_A \to S_A$ is also the identity map.
In example (3) (the trivial action), $g \cdot a = a \ \forall a, \forall g$. The kernel is all of G. The map $\phi : G \to S_A$ is the trivial homorphism (the constant function $\phi(g) = 1 \ \forall g$).

Definition: A *subgroup* of a group G is a subset of H of G which is:

i. nonempty

ii. closed under the group operation ($x, y \in H \implies xy \in H$)
iii. closed under inversion ($x \in H \implies x^{-1} \in H$)

The notation $H \leq G$ means that H is a subgroup of G, and $H < G$ means that $H \leq G$ and $H \neq G$ (proper subgroup).

Examples:

1. $\mathbb{Z} < \mathbb{Q} < \mathbb{R} < \mathbb{C}$ (under addition).
   $\mathbb{Q}^* < \mathbb{R}^* < \mathbb{C}^*$ (under multiplication)
2. The kernel of a group action is a subgroup (of the group that is doing the acting).
3. If $\phi : G \to G'$ is a homomorphism, then $\ker \phi = \{g \in G : \phi(g) = 1\}$ is a subgroup of G. This is called the *kernel* of ϕ.