MATH H113: Honors Introduction to Abstract Algebra

2016-03-04

- Isomorphism theorems
- Even & odd permutations

Homework due 3/11

- 3.3: 1, 3, 7, 8
- 3.5: 5, 12
- 4.1: 2, 4
- 4.2: 2, 8, 11

Theorem (Second Isomorphism Theorem):
Let A and B be subgroups of a group G, and assume that $A \leq N_G(B)$ (e.g. $B \trianglelefteq G$) Then $AB \leq G$, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \cong A/(A \cap B)$ (via $aB \mapsfrom a(A \cap B)$) **Proof** (continued):
We already showed $AB \leq G$ and $B \trianglelefteq AB$.
Define a homomorphism $\phi : A \to AB/B$ by $\phi(a) = aB$. This is onto, because if $x \in AB/B$ then $x = (ab)B$ with $a \in A$ and $b \in B$. Then $x = a(bB) = aB = \phi(a)$.
The kernel of ϕ is $\ker \phi = \{a \in A : aB = B\} = \{a \in A : a \in B\} = A \cap B$.
Then, by the First Isomorphism Theorem, $A \cap B \trianglelefteq A$ and $A \cap B \cong AB/B$.

$$
\begin{tikzcd}
 & AB \arrow[dl, dash] \arrow[dr, dash] & \\
A \arrow[dr, dash] & & B \arrow[dl, dash] \\
 & A \cap B &
\end{tikzcd}
$$

Theorem (Third Isomorphism Theorem): Let H and K be normal subgroups of a group G, and assume that $H \leq K$. Then $K/H \trianglelefteq G/H$ and $(G/H)/(K/H) \cong (G/K)$ via $(gh)(K/H) \mapsto gK)$.
Proof: Define $\phi : G/H \to G/K$ by $\phi(gH) = gK$ Need to check that ϕ is well defined. It is well defined and onto for the same reasons as in Ex. 3.2.11 We have $gH \in \ker \phi \iff gK = K \iff g \in K$, so $\ker \phi = K/H$.
Then, by the $1st$ Isomorphism Theorem:
$(G/H)/(K/H) \cong G/K$.

Theorem (Fourth Isomorphism Theorem, "lattice"):
Let $N \trianglelefteq G$. Then, for subgroups of A of G containing N, A/N is a subgroup of G/N. Call it $\bar{A}$. Then $A \mapsto \bar{A}$ is a bijection from $\{$subgroups of G containing $N\}$

to \${\text{subgroup of G/N}\$, and its inverse is $H \mapsto \pi^{-1}(H)$, where $\pi : G \to G/N$ is the natural projection.

Moreover, this bijection satisfies:

1. $A \leq B \iff \bar{A} \leq \bar{B}$ for all subgroups A, B of G containing N (it preserves inclusion).
2. $|B : A| = |\bar{B} : \bar{A}|$ for all A, B as above with $A \leq B$ (it preserves index).
3. $\overline{\langle A, B \rangle} = \langle \bar{A}, \bar{B} \rangle$ (it preserves join)
4. $\overline{A \cap B} = \bar{A} \cap \bar{B}$ for all A, B (it preserves intersection).
5. $A \trianglelefteq B \iff \bar{A} \trianglelefteq \bar{B}$ for all A, B as above (it preserves normality).
6. A is conjugate to B in $C \iff \bar{A}$ is conjugate to $\bar{B}$ in $\bar{C}$ for all subgroups A, B, C of G containing N such that $A \leq C$ and $B \leq C$ (it preserves conjugacy).

(Partial) **Proof**: $A/N \leq G/N$ because $A/N = \pi(A)$ (proved earlier).
If $H \leq G/N$ then $\pi^{-1}(H) \leq G$ (also proved earlier) and $\pi^{-1}(H) \supseteq N$ because if $n \in N$ then $\pi(n) = nN = N$ which is the identity element of G/N, so it's in H.
If we map A ($A \leq G$ and $A \supseteq N$) to $\bar{A}$ and back to a subgroup of G, we get $\pi^{-1}(A/N)$. To show $\pi^{-1}(A/N) = A$: if $a \in A$ then $aNsubseteqA/N$, so $\pi(a) \in A/N$, $\therefore a \in \pi^{-1}(A/N)$. Thus $A \subseteq \pi^{-1}(A/N)$. To show that $\pi^{-1}(A/N) \leq A$ if $b \in \pi^{-1}(A/N)$ then $\pi(b) = bN$ lies in A/N so $bN = aN$ for some $a \in A$. $\therefore b \in aN$, so $b = an$ for some $n \in N$. Then $b \in A$ since $n \in N \subseteq A$, and $a \in A$. $\therefore \pi^{-1}(A/N) \subseteq A$. So they're equal.
Also we need to show: if $H \leq G/N$ then $\pi^{-1}(H) = H$. Let $x \in G/N$. Then $x = gN$ for some $g \in G$. $x \in \pi^{-1}(H)/N \implies gN = hN$ with $h \in \pi^{-1}(H)$, so $hN = \pi(h) \in H \implies x \in H$. $\therefore \pi^{-1}(H)/N \subseteq H$.
$x \in H \implies \pi(g) = gN = x \in H \implies g \in \pi^{-1}(H) \therefore x = gN \in \pi^{-1}(H)/N$.
This shows $H \subseteq \pi^{-1}(H)/N \therefore H = \pi^{-1}(H)/N = \overline{\pi^{-1}(H)}$.
Proofs of (1) - (6) are left as an exercise.

Example: Let $G = D_{16} = D_{2n}$ with $n = 8 = \langle r, s | r^8 = s^2 = 1, rs = sr^{-1} \rangle$.
Let $N = \langle r^4 \rangle$. This is the center of G (Ex 1.2.4 or 2.2.7). We have $N \trianglelefteq G$. This is because the center of a group is always a normal subgroup. $gNg^{-1} = \{gng^{-1} : n \in N\} = \{ngg^{-1} : n \in N\} = \{n : n \in N\} = N$.
Then $G/N = \langle r, s | r^8 = s^2 = 1, rs = sr^{-1}, r^4 = 1 \rangle = \langle r, s | r^4 = s^2 = 1, rs = sr^{-1} \rangle = D_8$.
The lattice of subgroups of D_8 is on p. 69:

$$
\begin{tikzcd}
 & & D_8 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
 & \langle s, r^2 \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \langle r \rangle \arrow[d, dash] & \langle rs, r^2 \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle s \rangle \arrow[drr, dash] & \langle r^2 s \rangle \arrow[dr, dash] & \langle r^2 \rangle \arrow[d, dash] & \langle rs \rangle \arrow[dl, dash] & \langle r^3 s \rangle \arrow[dll, dash] \\
 & & 1 & &
\end{tikzcd}
$$

and the lattice for $D_{16} = G$ is on p. 70
All of the subgroups $\supseteq \langle r^4 \rangle$ form a lattice that's identical to the subgroup lattice of D_8.

Next: we will prove: **Theorem**: Let $n \geq 2$. Then there is a unique homomorphism $S_n \to \{\pm 1\}$ such that $\epsilon((i\ j)) = -1\ \forall 1 \leq i < j \leq n$. *But* we'll do it differently from the book (expect handout on Mon.).