MATH H113: Honors Introduction to Abstract Algebra

2016-02-19

- Subgroups of S_3
- Normalizers, centralizers, etc.
- Subgroups of cyclic groups

Homework due 2016-02-26:

- 2.1: 8, 14, 15
- 2.2: 6, 10
- 2.3: 2, 9, 11, 16
- 2.4: 3, 7, 15

Subgroups of S_3

$S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (3\ 2\ 1)\}$
We've found the following subgroups so far:
$\{(1)\}, \langle x\rangle, \langle y\rangle, \langle z\rangle, \langle w\rangle, S_3$
Claim: There are no other subgroups.
Let $H < S_3$
- If $|H \cap \{x, y, z\}| = 3$ then $H = S_3$ (last time)
- If $|H \cap \{x, y, z\}| = 2$ then if $x, y \in H$ then $z = xyz \in H$, contradiction.
Similarly if $x, y \in H$ use $y = zxz$
" " if $y, z \in H$ use $x = yzy$ - If $|H \cap \{x, y, z\}| = 1$ If $x \in H$ ($\implies H \supseteq \langle x\rangle = \{1, 2\}$) and $H \neq \langle x\rangle$ then H must contain w or w^{-1} $\therefore H \ni w$, so $y = wx \in H$, contradiction.
Similarly if $y \in H$, get a contradiction using $z = wy$
" " $z \in H$, " " $x = wz$ - If $|H \cap \{x, y, z\}| = 0$ then $H \subseteq \{(1), w, w^{-1}\}$ so either $H = \{(1)\}$ or H contains w or w^{-1} ($H \ni w, \therefore H \supseteq \langle w\rangle$)
$\therefore H = \langle w\rangle$

Diagram of subgroups of S_3:

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      S_3
  /  /   \  \
<x> <y> <z> <w>
 \   \   /   /
     {(1)}
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where the vertical or slanted lines are all of the inclusions.

Note: $\langle x \rangle \cup \langle y \rangle$ is not a subgroup of S_3 because $xy = (1\ 2)(1\ 3) = (1\ 3\ 2) = w^{-1}$ is not in $\langle x \rangle \cup \langle y \rangle$. So the union of some subgroups is usually *not* a subgroup. However, the intersection of subgroups is a subgroup.
Let $\{H_i : i \in I\}$ be a nonempty collection of subgroups of a group G. Then $H = \bigcap\limits_{i \in I} H_i$ is a subgroup of G.
Proof: $i \in H_i\ \forall i$, so $1 \in \bigcap\limits_{i \in I} H_i$; in particular, $H \neq \emptyset$
Also, if $x, y \in H$ then $x, y \in H_i \forall i$, $\therefore\ xy^{-1} \in H_i\ \forall i$, $\therefore\ xy^{-1} \in \bigcap\limits_{i \in I} H_i = H$. Therefore $H \leq G$.

Centralizers and Normalizers

Definition: Let G be a group and let A be a *subset* of G. Then:

a. The *centralizer* of A in G is the set $C_G(A) = \{g \in G : gag^{-1} = a\ \forall a \in A\}$.
b. The *normalizer* of A in G is the set $N_G(A) = \{g \in G : gAg^{-1} = A\}$. Here $gAg^{-1} = \{gag^{-1} : a \in A\}$.

Variations: If $a \in G$, then we write $C_G(a) = C_G(\{a\}) = \{g \in G : gag^{-1} \in a\} = \{g \in G : g$ commutes with $a\}$. $\therefore\ C_G(A) = \bigcap\limits_{a \in A} C_G(a)$ (if $A \neq \emptyset$).
Also, if $A = G$ then $C_G(G)$ is called the center of G, and is written $Z(G)$.
All of these are subgroups of G.
For any set $A \in G$, $C_G(A) \subseteq N_G(A)$.
For example, $C_G(G) = Z(G)$ but $N_G(G) = G$.
$Z(G) = G \iff G$ is abelian ($ghg^{-1} = a \iff h = g^{-1}ag$)

Definition: Let a group G act on a set S. Then for $s \in S$, the stabilizer of s in G is $G_s = \{g \in G : g \cdot s = s\}$. This is a subgroup of G.
Examples:

1. If we let G act on itself by conjugation:
   $g \cdot s = gsg^{-1}\ \forall s \in G = S, g \in G$ then $G_s = C_G(s)$.
2. If $G = S_3$ and $S = \{1, 2, 3\}$ with the usual action $\sigma \cdot a = \sigma(a)$ ($\sigma \in G, a \in S$), then $G_3 = \langle (1\ 2) \rangle$
3. Similarly if $G = S_4$ and $S = \{1, 2, 3, 4\}$, with G acting on S via σ, then $G_4 \cong S_3$ (via $\sigma \in G \mapsto \sigma|_{\{1,2,3\}} \in S_3$)

Note also: Let H_1 and H_2 be subgroups of G. Then $H_1 \leq H_2 \iff H_1 \subseteq H_2$.
Proof:

- $\implies$ is part of the definition of a subgroup
- $\impliedby$ group operation on H_1 and H_2 are compatible since they both come from the operation on G: $H_1 \leq G \implies H_1 \neq \emptyset$ and $xy^{-1} \in H_1\ \forall x, y \in H_1 \implies H_1 \leq H_2$ (since $H_1 \subseteq H_2$).

Subgroups of Cyclic Groups

Definition: A group (or subgroup) H is *cyclic* if there is an element $x \in H$ such that $H = \langle x \rangle = \{x^n : n \in \mathbb{Z}\}$ = image of $\mathbb{Z} \to H$ given by $n \mapsto x^n$.
Such an element x is called a *cyclic generator* for H.
($\langle x \rangle = \{nx : x \in \mathbb{Z}\}$ if H is written additively)
If x is a cyclic generator for H_1 then so is x^{-1} ($\{x^n : n \in \mathbb{Z}\} = \{(x^{-1})^m : m \in \mathbb{Z}\}$$
(take $m = -n$)).
We saw a week ago that:
if $|x| = \infty$ then $\langle x \rangle \cong \mathbb{Z}$
if $|x| = m < \infty$ then $\langle x \rangle \cong \mathbb{Z}/m\mathbb{Z}$
in either case, $|x| = |\langle x \rangle|$.

Corollary: Any two cyclic groups of the same order are isomorphic.
Proposition: Any subgroup of $\mathbb{Z}$ is cyclic. If it's non-trivial, then it is generated by its smallest positive element.
Proof: If $H = \{0\}$, then $H = \langle 0 \rangle$ is cyclic.
to be continued…