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$\sqrt{2}$ RENORMALIZATION GROUP TRANSFORMATION FOR LATTICE GAUGE THEORIES IN FOUR DIMENSIONS

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ABSTRACT

A real space renormalization group transformation with scale factor $\sqrt{2}$ for four-dimensional lattice gauge theories is presented. No such transformation on a hypercube whose block lattice points are also site points can have a smaller scale factor. The transformation is utilized here to study the critical behaviour of SU(2) lattice gauge theory. Because of the small scale factor, lattices as small as 2^4 sites can be considered. Results are in excellent agreement with standard conclusions.

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1. PROLEGOMENA

The real space renormalization group[1] is an excellent tool for the analysis of lattice gauge theories[2]. By dealing directly with the scaling properties of the system, this renormalization group permits the approach to the continuum limit to be handled in a direct and economical fashion. Because of the complexity of typical lattice gauge theories, at present the renormalization must be performed by numerical means such as Monte Carlo simulation[3]. This approach in fact originally led to the Monte Carlo renormalization group, or MCRG[4)-8)].

In the real space renormalization group the original or "site" system is divided into "blocks" or sets of site variables. A smaller number of block variables are then defined by averaging in some fashion over the site variables. The thermodynamic description of the system can be made either in terms of the site variables (governed by a site action S) or in terms of the block variables (governed by a block action S'). The block system is then itself blocked, and a third action S" is generated. By studying the sequence of actions (S,S',S"...) as the site degrees of freedom are systematically integrated out the critical properties of the system can be determined. For a lattice gauge theory, this critical behaviour dictates the nature of the continuum limit.

In practical calculations on finite lattices it is important that the blocking procedure integrate out as few degrees of freedom as is feasible. Equivalently the scale factor $b > 1$ of the transformation -- the lattice spacing of the block lattice in units of the site lattice spacing -- should be as small as possible. In this fashion finite volume effects are reduced and the finite original system can be blocked a large number of times.

Here a blocking transformation with scale factor $\sqrt{2}$ for four-dimensional lattice gauge theories is described. This blocking procedure is sufficiently economical in its integration over the degrees of freedom in the system that it can be applied to a lattice with as few as 2^4 sites and still produce reasonable results. To date no other blocking procedure has been suggested which can even be defined upon such a small lattice. It is in fact shown below that, under the assumption that the points of the block lattice are also points in the original lattice, no blocking transformation can be constructed on a four-dimensional hypercube whose scale factor b is smaller than $\sqrt{2}$. Blocking procedures with the larger scale factors of 2[9] and $\sqrt{3}$[10)-12)] have, however, been constructed.

The general philosophy behind the construction of such transformations is discussed in detail in the sequel. Following this orientation, the new procedure is applied to SU(2) gauge theory and the critical exponent corresponding to its asymptotically free continuum behaviour is calculated.

2. FORMALISM

The method used to construct the block lattice from the original lattice is next described. Denote the four orthogonal unit vectors in the four directions of the orginal hypercubic lattice by e_i, with i running from 0 to 3. Thus for example e_0 can also be written (1,0,0,0) in component notation. Each point r in the site lattice can be represented as a sum of integers n_i times these unit vectors:

$$r = \sum_{i=0}^{3} n_i e_i \tag{1a}$$

while the orthogonality of the e_i implies that

$$e_i \cdot e_j = \delta_{ij} \tag{1b}$$

Conceivably the system could be blocked to a smaller one whose lattice points are not also points in the original lattice. However, such a transformation must confront the problem of gauge invariance, and it is not immediately obvious how to proceed. Thus in the sequel it is tacitly assumed that points in the block lattice are also points in the original lattice.

Not all of the original lattice points r given by Eq. (1a) are necessarily also points in the block lattice. Thus it makes sense to define orthogonal vectors e'_i for the block lattice as well. The norm of these vectors is the scale factor b of the blocking transformation:

$$e'_i \cdot e'_j = b^2 \delta_{ij} \tag{2}$$

and the fact that each point in the block lattice is also a point in the original lattice is reflected by the equation

$$e'_i \cdot e_j = \text{integer} \equiv R_{ij} \tag{3}$$

All points r' in the block lattice can then be specified in terms of four integers n'_i:

$$r' = \sum_{i=0}^{3} n'_i e'_i \tag{4}$$

The constraint Eq.(3) implies that each block vector can also be decomposed as

$$e'_i = \sum_j \mathbb{R}_{ij} e_j \tag{5a}$$

and so Eq. (2) gives

$$b^2 = \frac{1}{4} \sum_{j,i} (\mathbb{R}_{ij})^2 \tag{5b}$$

Recall that the R_{ij} are all integers. Then if the trivial solution with b equal to one (corresponding to no blocking) is excluded, it is easy to see that the minimum possible value for b is $\sqrt{2}$. Thus the blocking transformation outlined below is the most compact one possible on a hypercube if the block lattice points are also site lattice points.

The matrix R_{ij} can also simply be multiplied into the column vector e_j:

$$(\mathbb{R} e)_i = e'_i \tag{6a}$$

$$(\tilde{e} \tilde{\mathbb{R}})_i = \tilde{e}'_i \tag{6b}$$

where the tilde (~) denotes transpose. The block vectors e'_i can be ordered in such a fashion that R is symmetric, i.e., so that R equals $\tilde{R}$. Equation (2) then implies that

$$(\mathbb{R}^2)_{ij} = b^2 \delta_{ij} \tag{7}$$

It therefore follows that if the blocking transformation is applied a second time,

$$(\mathbb{R} e')_i = e''_i \tag{8}$$

the resulting lattice vectors are just

$$e''_i = b^2 e_i \tag{9}$$

In other words, two blocking transformations bring the lattice back to its original axes. The matrix R can be thought of as a combination of a rotation and dilatation which defines the series of block lattices necessary for the real space renormalization group. It is also an obvious consequence of the formalism that the vectors e'_i are just the rows (and columns) of the matrix R.

Thus in order to specify the block lattice for the $\sqrt{2}$ transformation it is only necessary to write down the matrix R:

$$\mathbb{R}_{\sqrt{2}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \tag{10}$$

From Eq. (10) it can be seen that the four vectors e'_i are mutually orthogonal and each has norm $\sqrt{2}$, as claimed. The subscript "$\sqrt{2}$" of course refers to the scale factor.

Other block transformations reported previously in the literature can also be expressed in the form Eq. (10). For example, block axes of the transformation given in Ref.(9) correspond to the matrix

$$\mathbb{R}_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{11}$$

(the scale factor b equals two), while the block axes for the $\sqrt{3}$ transformation given in Ref.(10) can be represented as

$$\mathbb{R}_{\sqrt{3}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \tag{12}$$

3. THE CALCULATION

After the delineation of the block lattice, the next step in the calculation is the construction of the block links. This task is considerably simplified in the case of SU(2) for, as is well known [see, e.g. Ref.(13)], the sum of elements of the group is proportional to another element of the group.

A specific example may make matters more clear. Suppose the SU(2) block link from the origin to the point $e'_0 = (1,0,1,0)$ is to be constructed. The constraint of gauge invariance and the serendipity mentioned above suggest the procedure of defining the block link as a sum of paths connecting the two points but divided by an appropriate normalization factor so that the determinant of the block link is unity. The simplest possibility is to take this weighted sum of the two shortest paths — one going through the point (1,0,0,0) and the other through the point (0,0,1,0). In such a fashion a complete set of block links for the lattice can be generated.

The efficacy of the transformation is demonstrated by calculating the critical exponent associated with asymptotic freedom. Specifically, the block action S' and site action S near a critical point are expanded in terms of a set of basis functions,

$$S'\{U'\} = \sum_n \beta'_n S_n \{U'\} \tag{13a}$$

$$S\{U\} = \sum_n \beta_n S_n \{U\} \tag{13b}$$

The first term is taken as the simple plaquette term[13a)],

$$S_0\{U\} = \sum_{\text{plaquettes}} \left[1 - \frac{1}{2} \mathrm{Tr}\left(U U U^+ U^+\right)\right] \tag{14}$$

$$\beta_0 \equiv \beta$$

Other terms include larger loops and higher (e.g, adjoint) representation contributions.

The correlations

$$C'_{ab} \equiv \langle S_a\{U'\} S_b\{U'\}\rangle - \tag{15a}$$
$$- \langle S_a\{U'\}\rangle \langle S_b\{U'\}\rangle$$

and

$$C_{ab} \equiv \langle S_a\{U'\} S_b\{U\}\rangle - \langle S_a\{U'\}\rangle\langle S_b\{U\}\rangle \tag{15b}$$

are then calculated, and the derivatives

$$D_{ab} = \frac{\partial \beta'_a}{\partial \beta_b} \tag{16}$$

are determined[5),9)] from the matrix equation,

$$C = C' D \tag{17}$$

If λ is the largest eigenvalue of the matrix D, then the critical exponent y is given by

$$y = \ln\lambda / \ln b \tag{18}$$

Since SU(2) is asymptotically free, λ should equal one and y is thus zero at the critical point. One point on the critical surface (but not necessarily a fixed point of the renormalization group) occurs in the limit of large β.

The systematic errors in this procedure arise from three sources: 1) including a finite number of operators in the expansion Eqs. (13); 2) not being on the exact critical point in the multiparameter space of couplings [this problem can be remedied by following the renormalization group trajectories as in Ref.7) and Ref.8) or by repeated blocking transformations as in Ref.9)]; 3) finite size effects.[*)]

Since the purpose of this calculation is to show the validity of the blocking scheme rather than to demonstrate the asymptotic freedom of SU(2), only the one-plaquette term Eq. (14) was included in the analysis. The matrix D in Eq. (16) was thus one-by-one.

[*)] Note that the shape of the overall volume for the blocked system is not hypercubic for the $\sqrt{2}$ transformation.

4. RESULTS

The SU(2) configurations used as input to the blocking procedures were generated by a heat-bath program written by Creutz and described in Ref.13). Correlation functions [defined in Eqs. (15)] were measured over different lattice sizes. For comparison, the blocking procedure R_2 used in Ref.9) was also applied to SU(2). As with our $R_{\sqrt{2}}$ scheme, the block links were taken as the normalized sum of the chosen paths connecting the two points. In the case of the R_2 transformation, seven paths were averaged. The two procedures required similar amounts of computer time.

The resulting values of $\partial\beta'/\partial\beta$ and y are displayed in the Table. All measurements were taken at β equal to 10. For the case when the linear dimension L equals 8 [L equals 2], the measurement is an average of 1,000 [10,000] configurations following 100 [1,000] equilibration iterations. (The result for L equal to 2 is included simply as a curiosity; no other published block transformation can even be defined upon such a small lattice).

The accuracy of both the $R_{\sqrt{2}}$ and R_2 transformations improves with system size, but, as may be expected, the $R_{\sqrt{2}}$ results are superior. Of course it is conceivable that this may change if the system is blocked a large number of times, if more couplings are included, or if the true fixed point is analyzed in the fashion of Ref.7) or Ref.8).

In conclusion, a blocking transformation with a $\sqrt{2}$ scale factor for four-dimensional lattice gauge theories was presented. It was shown that this is the smallest scale factor possible on a hypercube, provided that the block lattice points are also the original lattice points. The transformation is sufficiently compact that it can be applied to a lattice with as few as 2^4 sites (suggesting the possibility of an analytic calculation!). Results obtained in a preliminary study of SU(2) lattice gauge theory are in good agreement with its known asymptotic freedom. An extension of the above analysis to a coupled gauge-Higgs system [see, e.g, Ref. 14)] is currently underway.

Table of results for the two blocking transformations described in the text. Errors are estimated by comparing the results obtained using different initial configurations, and are purely statistical. Value for R_2 is not shown for L equal to 2 as it is impossible to define this transformation on such a small lattice; the value for $R_{\sqrt{2}}$ for L equal to 2 is displayed largely as a curiosity.

TABLE

BLOCKING TRANSFORMATION

LATTICE SIZE	$R_{\sqrt{2}}$	R_2
L = 2	$\frac{\partial\beta'}{\partial\beta} = 0.80 \pm 0.05$ $y = -0.6 \pm 0.2$	———
L = 8	$\frac{\partial\beta'}{\partial\beta} = 1.1 \pm 0.1$ $y = 0.1 \pm 0.1$	$\frac{\partial\beta'}{\partial\beta} = 1.3 \pm 0.1$ $y = 0.4 \pm 0.1$
EXACT	$\frac{\partial\beta'}{\partial\beta} = 1$ $y = 0$	$\frac{\partial\beta'}{\partial\beta} = 1$ $y = 0$

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