

## NON-TRIVIALITY OF GAUGE THEORIES WITH ELEMENTARY SCALARS AND UPPER BOUNDS ON HIGGS MASSES

D.J.E. CALLAWAY

*CERN, Geneva, Switzerland*

Received 8 August 1983

Strong evidence supports the idea that pure  $\phi^4$  field theory is trivial (non-interacting) in four dimensions. In the context of perturbation theory when gauge fields or fermions are present the combined theory may become non-trivial for a limited range of values of the renormalized coupling constants. Calculable upper bounds on the masses of elementary Higgs particles are implied by the restrictions on these coupling constants. Other constraints (on, e.g. the number of fermions in the theory) are also implied.

### 1. Prolegomena

The concept of particle mass generation by spontaneous symmetry breaking [1] has become one of the most useful mechanisms in elementary particle physics. One way symmetries may be broken in a quantum field theory is by the introduction of elementary scalar fields (i.e., by coupling the gauge fields of the theory to a  $\phi^4$  field theory). At the classical level these fields interact with themselves in such a fashion as to produce a vacuum state which lacks certain symmetries of the lagrangian. Gauge particles propagating in this asymmetric vacuum may then appear to be massive. It is generally assumed that no significant changes occur in this picture due to quantum effects such as renormalization.

This assumption may not be correct, however. A large body of evidence [2] has been presented which indicates that (at least single-component) pure  $\phi^4$  field theory is trivial in four space-time dimensions. The word “trivial” is taken to mean that the scalar field does not interact with itself; i.e. that the renormalized quartic coupling constant is zero. This triviality seemingly persists for all values of the bare coupling constant, and therefore presumably precludes the existence of spontaneous symmetry breaking in *pure*  $\phi^4$  field theory. In order to salvage the Higgs mechanism with fundamental scalars, a new phenomenon must occur in the theory when gauge fields are present. It is with the structure and scope of this phenomenon that the following analysis is concerned.

It has been known for some time [3] that ultraviolet-stable fixed points exist for zero gauge and quartic coupling in various theories (e.g., SU(3) coupled to a single

fundamental representation scalar]. Theories of this form are asymptotically free in both the gauge and quartic couplings, and triviality is thus avoided. In general, realistic theories which involve spontaneous symmetry breaking do *not* have asymptotically free gauge and quartic couplings; however, as is shown below, it may still be reasonable to study the question of their triviality (or non-triviality) in perturbation theory.

When  $\phi^4$  field theory is coupled to gauge fields (or to fermionic matter fields) it can become non-trivial, but only for certain values of the renormalized coupling constants. If the Higgs mechanism with elementary scalars is to work, there must therefore be restrictions on the renormalized parameters of the theory. These restrictions can be translated into calculable upper bounds on the Higgs mass.

The reason for such a bound can be explained simply. For a pure field theory the Lagrange density is given by

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{1}{2} \lambda_0 \phi^4, \quad (1)$$

where  $m_0$  and  $\lambda_0$  are the bare mass and quartic coupling constant, respectively. Triviality is equivalent to the statement that the renormalized quartic coupling,  $\lambda_R$ , equals zero. In other words, the scalar particles interact in such a fashion as to screen totally any bare charge  $\lambda_0$ .

Consider the effect of coupling a gauge field to the scalar Lagrange density, eq. (1). It might be expected (and it is demonstrated below) that in order for the combined theory to be non-trivial the renormalized quartic coupling must not be too strong. Denote the gauge coupling constant by  $g$ . The breakdown of total screening occurs when the quartic coupling constant is less than the effective quartic coupling generated by the gauge field interaction, i.e. when

$$\lambda_R \leq \zeta_+ g_R^2, \quad (g_R^2 \ll 1) \quad (2)$$

for some calculable constant  $\zeta_+$  which is greater than or equal to zero. Note that eq. (2) gives the correct result ( $\lambda_R = 0$ ) when the gauge field is absent (i.e., when  $g_R$  is zero). In, e.g. the standard model [4] of the weak interaction the squared ratio of Higgs to W-boson mass is given at tree level by

$$\left( \frac{m_H}{m_W} \right)^2 = 16 \frac{\lambda_R}{g_R^2}. \quad (3)$$

Eq. (2) therefore implies the bound

$$\left( \frac{m_H}{m_W} \right)^2 \leq 16 \zeta_+. \quad (4)$$

## 2. Application of the renormalization group

The renormalization group [5, 6] equation for the effective (or “running”) quartic coupling constant  $\bar{\lambda}(t)$  is

$$\frac{d\bar{\lambda}(t)}{dt} = \beta_{\lambda}(\bar{\lambda}). \quad (5)$$

Equations of this form describe the response of coupling constants when the normalization point  $\mu^2$  at which the couplings are defined is changed to  $\mu^2 e^{-t}$ . Such equations apply in the deep euclidean region of the theory where mass terms can be dropped.

Eq. (5) has the boundary conditions

$$\bar{\lambda}(t=0) = \lambda_R, \quad (6a)$$

$$\bar{\lambda}(t \rightarrow \infty) = \lambda_0, \quad (6b)$$

where  $\lambda_R$  is the renormalized quartic coupling constant and  $\lambda_0$  the corresponding bare coupling. Eq. (6b) simply states that at high-momentum transfer an incident scalar particle interacts with the bare charge of the target scalar.

In the language of the renormalization group, the triviality of a  $\phi^4$  theory is essentially equivalent to two statements:

(i) *The bare coupling constant  $\lambda_0$  is finite* [or the running coupling is finite at all momentum scales]. It would indeed be difficult to make sense out of a theory whose effective coupling increased truly *without limit* as the momentum scale became larger;

(ii) *The beta function  $\beta_{\lambda}(\bar{\lambda})$  for the running quartic coupling constant is positive and equals zero only when  $\bar{\lambda}$  is zero, i.e.*

$$\begin{aligned} \beta_{\lambda}(\bar{\lambda}) &> 0, & \text{if } \bar{\lambda} > 0, \\ &= 0, & \text{if } \bar{\lambda} = 0. \end{aligned} \quad (7)$$

Specifically, eqs. (5)–(7) imply that the renormalized coupling  $\lambda_R$  is zero for any sensible (i.e. finite and positive) value of the bare coupling  $\lambda_0$ .

Strictly speaking the current evidence for the triviality of  $\phi^4$  theory is generally valid only for a single-component theory. This deficiency is probably due to the fact that more complicated (i.e. multicomponent) scalar field theories have not been analyzed with this question in mind (but see ref. [7]). It is therefore assumed in what follows that *any*  $\phi^4$  field theory is trivial in four dimensions (when gauge fields or fermions are absent).

This presumed fact is so remarkable that it deserves further scrutiny. Recall that in the original classical Lagrange density (eq. (1)) there are two bare parameters,  $\mu_0$  and  $\lambda_0$ . When quantum effects are accounted for (i.e. when the theory is renormalized), all that remains is one parameter, the renormalized mass  $\mu_R$ . Quantum effects

have determined that  $\lambda_R$  is zero; i.e. the process of renormalization has “adjusted” one of the parameters of the theory. In, e.g. the standard model [4] of the weak interaction one parameter, the Higgs mass, is not determined by low-energy phenomenology in the classical (tree-level) approximation. Thus it might be hoped that a similar quantum “parameter reduction” occurs and determines the Higgs mass. In fact, as is shown below, renormalization effects may generally *bound* the Higgs mass from above.

### 3. Examples

#### 3.1. $SU(N)$ WITH A FUNDAMENTAL REPRESENTATION SCALAR

In order to illustrate the above points, it is useful to consider the case of an  $SU(N)$  gauge theory coupled to a fundamental (vector) representation scalar. Define the gauge coupling constant via a minimal coupling generalization of eq. (1). Then the renormalization group equations can be written in the form

$$16\pi^2 \frac{d\bar{\lambda}}{dt} = A\bar{\lambda}^2 + B\bar{\lambda}\bar{g}^2 + C\bar{g}^4 + \dots, \quad (8a)$$

$$16\pi^2 \frac{d\bar{g}}{dt} = -\frac{1}{2}b_0\bar{g}^3 + \dots, \quad (8b)$$

where [3, 8]

$$A = 2(N+4), \quad B = -\frac{6}{N}(N^2-1), \quad C = \frac{6}{4N^2}(N-1)(N^2+2N-2), \quad (9)$$

and

$$b_0 = \frac{11}{3}C_2(G) - \sum_{\text{fermions}} \frac{4}{3} \frac{C_2(F)d(F)}{r} - \sum_{\text{scalars}} \frac{1}{6} \frac{C_2(S)d(S)}{r}. \quad (10)$$

In eq. (10),  $d$  is the dimension of the representation and  $r$  the rank (number of generators) of the group, while  $C_2$  is the value of the Casimir operator. For the case under consideration,

$$SU(N) \text{ adjoint: } C_2(G) = N,$$

$$SU(N) \text{ fundamental: } \frac{C_2(S)d(S)}{r} = \frac{1}{2}. \quad (11)$$

Eqs. (8) are most easily studied in terms of a new variable  $\zeta$  defined by

$$\zeta \equiv \frac{\lambda}{g^2}. \quad (12)$$

In terms of this new variable, eq. (8a) is

$$\begin{aligned} \frac{16\pi^2}{g^2} \frac{d\bar{\zeta}}{dt} &= A\bar{\zeta}^2 + (B + b_0)\bar{\zeta} + C \\ &= A(\bar{\zeta} - \zeta_+)(\bar{\zeta} - \zeta_-). \end{aligned} \quad (13)$$

Eq. (13) possesses two fixed points  $\zeta_+$  and  $\zeta_-$  (with  $\zeta_+$  greater than  $\zeta_-$ ) which are real and positive whenever

$$b_0 < -B - (4AC)^{1/2}. \quad (14)$$

Consider first the case when the condition eq. (14) is satisfied. If the renormalized coupling constants  $\lambda_R$  and  $g_R$  are such that

$$\zeta_R \equiv \frac{\lambda_R}{g_R^2} < \zeta_+, \quad (15)$$

then it follows from eq. (13) that  $\bar{\zeta}$  approaches  $\zeta_-$  as  $t$  increases without bound. (Of course if  $\zeta_R$  equals  $\zeta_+$  then  $\bar{\zeta}$  does not change as  $t$  increases.) The existence of the ultraviolet-stable fixed point  $\zeta_-$  implies that a non-trivial theory can be constructed whenever  $\zeta_R$  is within the domain of attraction of this fixed point (i.e. whenever  $\zeta_R < \zeta_+$ ) provided that the bare gauge coupling  $g_0$  is finite.

This last statement deserves a word of explanation. If  $\bar{\zeta}$  approaches the fixed value  $\zeta_+$  as  $t$  increases without bound, then the ratio of the bare quartic coupling constant  $\lambda_0$  to the squared bare coupling constant  $g_0^2$  is finite. When, in addition, the gauge theory in question is asymptotically free (i.e. when  $b_0$  is greater than zero) both of the bare couplings are zero. In that case both of the running couplings  $\bar{\lambda}$  and  $\bar{g}^2$  approach zero in a fixed ratio  $\zeta_-$ . More generally  $\bar{g}^2$  approaches some finite value  $g_0^2$  in the limit of large  $t$ .

If either of the bounds eqs. (14) or (15) is *not* satisfied,  $\bar{\zeta}$  should increase without bound as  $t$  increases. For large  $\bar{\zeta}$  the gauge coupling apparently cannot stop  $\bar{\lambda}$  from increasing without bound, and so the bare coupling is infinite (cf. eqs. (6)). Since by assumption no sensible theory can be constructed with an infinite bare coupling constant, it follows that eqs. (14) and (15) must be satisfied in a sensible theory of this form.

For the case of  $SU(N)$  coupled to a fundamental representation scalar, the bound eq. (14) gives a restriction on  $b_0$  (equivalent, via eq. (10), to a lower bound on the number of fermions in the theory),

$$b_0 < b_0^{\max}(N), \quad (16)$$

with  $b_0^{\max}(N)$  given in table 1 for various values of  $N$ . Note that for  $N$  equals 2 (the only case where symmetry breaking is complete)  $b_0$  is negative. In general it seems to

be likely that complete symmetry breaking demands an asymptotically non-free gauge coupling [3, 8] (but see ref. [10]).

In the most interesting case ( $N = 2$ ), where spontaneous symmetry breaking is complete (i.e. no massless vector bosons remain) the above discussion implies the bound

$$\zeta_R \equiv \frac{\lambda_R}{g_R^2} \leq \zeta_+, \quad (17a)$$

with

$$\zeta_+ = \frac{1}{12} \left[ \frac{9}{2} - \frac{1}{2} b_0 + \left( \left( \frac{9}{2} - \frac{1}{2} b_0 \right)^2 - 27 \right)^{1/2} \right]. \quad (17b)$$

Eqs. (17) imply the upper bound eq. (3) on the Higgs mass.

In the above analysis the existence of an ultraviolet-stable fixed point in the renormalization group equation for the gauge coupling is assumed. Of course, this fixed point occurs at zero coupling in an asymptotically free theory. If the gauge theory in question is *not* asymptotically free, then such a fixed point may either appear in the gauge-Higgs system itself, or (at higher momentum scales) in the unification of the gauge theory with other gauge fields or quantum gravity. If such a fixed point did *not* appear, then the gauge theory would be “trivial” in the same sense that pure  $\phi^4$  theory appears to be. The apparent existence of quantum electrodynamics may be taken as evidence that a theory which is *not* asymptotically free in perturbation theory can become non-trivial in one of these ways.

A more relevant question is whether this presumed fixed point occurs within the region of validity of perturbation theory. As is demonstrated below, for typical theories this region of validity extends to momentum scales higher than the Planck mass; thus perturbation theory should give reliable information about stable fixed points up to the momentum scale where quantum gravitation plays an important rôle. In other words, the analysis given here is expected to be valid if a fixed point in

TABLE I  
Upper bounds on  $b_0$  for  $SU(N)$  coupled to a fundamental representation scalar

$N$	$b_0^{\max}(N)$
2	$2\left(\frac{9}{2} - 3\sqrt{3}\right) \sim -1.4$
3	$2\left(8 - \sqrt{\frac{182}{3}}\right) \sim 0.42$
4	$2\left(\frac{45}{4} - 3\sqrt{11}\right) \sim 2.6$
$\infty$ (large- $N$ limit)	$2(3 - \sqrt{3})N \sim 2.5N$

the gauge coupling is approached at or before momentum scales of the order of the Planck mass.

Perturbation theory presumably loses its meaning when the running gauge coupling constant becomes of order unity, i.e. when the momentum scale  $Q$  divided by the renormalization point  $\mu$  exceeds the bound ( $b_0$  is assumed to be negative)

$$\frac{Q_{\max}}{\mu} \sim \exp\left(\frac{-16\pi^2}{g_R^2 b_0}\right). \quad (18a)$$

For typical non-asymptotically-free theories  $-g_R^2 b_0$  is positive and of order unity at momenta of a few GeV. Thus perturbation theory may be appropriate up to momenta of the order of

$$Q_{\max} \sim 10^{70} \text{ GeV}, \quad (18b)$$

which is far greater than the Planck mass  $m_{\text{Planck}} \sim 10^{19} \text{ GeV}$ . Hence perturbation theory in  $g$  should be valid up to at least the point where unification with gravity presumably occurs (actually much further than this scale if eqs. (18) are taken literally).

It must be pointed out, however, that phenomena which are intrinsically non-perturbative may conceivably alter the above discussion. More infrared-stable fixed points in  $\zeta$  (cf. eq. (13)), typically associated with new phases of the theory, may appear for larger  $\zeta$  for example. However, it has been argued [9] that the requirement of unitarity places an upper bound (larger than  $\zeta_+$ ) on  $\zeta$ . Additionally the above picture displays the simplest mechanism by which triviality can be avoided. It is therefore worthwhile to examine a realistic model in this fashion.

### 3.2. THE STANDARD MODEL OF THE WEAK INTERACTION

The analysis of subsect. 3.1 is easily generalized to more realistic models. The renormalization group equations for the standard model [4] of the weak interaction follow from the work of ref. [8]. They are

$$16\pi^2 \frac{d\bar{\lambda}}{dt} = 12\bar{\lambda}^2 + (-9)\bar{g}^2\bar{\lambda} + (-3)\bar{g}'^2\bar{\lambda} + \frac{9}{4}\bar{g}^4 + \frac{3}{8}\bar{g}^2\bar{g}'^2 + \frac{3}{4}\bar{g}'^4, \quad (19a)$$

$$16\pi^2 \frac{d\bar{g}'}{dt} = \frac{1}{2}b'_0\bar{g}'^3, \quad (19b)$$

$$16\pi^2 \frac{d\bar{g}}{dt} = -\frac{1}{2}b_0\bar{g}^3, \quad (19c)$$

where  $g$  and  $g'$  are the coupling constants for the SU(2) and U(1) gauge fields respectively, while  $b_0$  and  $b'_0$  are positive.

These equations are simpler to analyze when written in terms of the variables

$$\rho \equiv \frac{\lambda}{g'^2}, \quad (20a)$$

$$\sigma \equiv \frac{g^2}{g'^2}, \quad (20b)$$

whence,

$$\begin{aligned} 16\pi^2 \frac{d\bar{\rho}}{dt} &= \bar{g}'^2 \left[ 12\bar{\rho}^2 - (3 + 9\bar{\sigma} + b_0)\bar{\rho} + \frac{9}{4}\bar{\sigma}^2 + \frac{3}{2}\bar{\sigma} + \frac{3}{4} \right] \\ &\equiv 12\bar{g}'^2 [\bar{\rho} - \rho_+(\bar{\sigma})][\bar{\rho} - \rho_-(\bar{\sigma})], \end{aligned} \quad (21a)$$

$$16\pi^2 \frac{d\bar{\sigma}}{dt} = -\bar{g}'^2 \bar{\sigma} (b_0 + b'_0 \bar{\sigma}), \quad (21b)$$

where the functions  $\rho_{\pm}(\sigma)$  are given by

$$\rho_{\pm}(\sigma) = \frac{1}{24} \left\{ 9\sigma + 3 + b_0 \pm \left[ 9(\sigma_+ - \sigma)(\sigma - \sigma_-) \right]^{1/2} \right\}, \quad (21c)$$

while

$$\sigma_{\pm} = \frac{1}{3} (b'_0 - 1) \pm \frac{2}{3} \left[ \frac{1}{3} (b_0'^2 - 6) \right]^{1/2}. \quad (21d)$$

Note that eqs. (21) imply the restrictions

$$\sigma_- < \sigma < \sigma_+, \quad (22a)$$

$$b_0'^2 > 6, \quad (22b)$$

if the  $\rho_{\pm}(\sigma)$  are real, and that

$$\rho_{\pm} > 0, \quad \text{if} \quad \sigma > \frac{1}{9} (3 + b'_0). \quad (22c)$$

To lowest order  $\sigma_R [\equiv \sigma(t=0)]$  is related to the weak mixing angle by

$$\sigma_R = \frac{1}{\sin^2 \theta_w} - 1. \quad (23)$$

Eqs. (21) imply that

$$-\frac{d\bar{\rho}}{d\bar{\sigma}} = 12 \frac{[\bar{\rho} - \rho_+(\bar{\sigma})][\bar{\rho} - \rho_-(\bar{\sigma})]}{\bar{\sigma}(b'_0 + b_0 \bar{\sigma})}. \quad (24)$$



Since (by eq. (21b))  $\bar{\sigma}(t)$  is a monotonically decreasing function of  $t$ , the existence of an ultraviolet-stable fixed point (and a non-trivial theory) is related to the behaviour of  $\rho$  as  $\sigma$  tends to zero. In order for  $\rho$  to approach a fixed point, it is necessary that

$$\bar{\rho} \leq \rho_+(\bar{\sigma}), \quad (25a)$$

and in particular that

$$\rho_R \leq \rho_+(\sigma_R). \quad (25b)$$

Note also that  $\rho_+(\sigma_R)$  must be real, which implies that

$$\sigma_R \geq \sigma_-. \quad (25c)$$

When four fermion flavours are present the gauge theory parameters are

$$b'_0 = \frac{163}{18}, \quad (26a)$$

$$b_0 = \frac{55}{6}. \quad (26b)$$

Since  $\sigma_-$  is less than zero, eq. (25c) essentially gives the trivial bound  $\sin^2\theta_w \leq 1$ . However, eq. (25b) gives a useful bound; for  $\sin^2\theta_w = \frac{1}{4}$  this bound is

$$\rho_R = \frac{\lambda_R}{g_R^2} \leq \rho_+(\sigma_R) \sim 2.4. \quad (27)$$

Eq. (27) in turn gives the bound

$$\left(\frac{m_H}{m_W}\right)^2 = 16 \frac{\lambda_R}{g_R^2} = 16 \frac{\rho_R}{\sigma_R} \leq 12.8,$$

that is

$$m_H \leq 3.6 m_W \sim 270 \text{ GeV}, \quad (28)$$

using  $m_W \sim 37 \text{ GeV}/\sin\theta_w \sim 74 \text{ GeV}$ .

### 3.3. GRAND UNIFICATION

As the gauge groups and particle content of a theory grow, so does its complexity. The renormalization group equations for most grand unified theories provide no exception to this rule. There are certain features of the preceding analysis which seems to persist in this more general case, however, and it is with these features that this section is concerned.

The major difficulty with studying more complicated theories involves the proliferation of quartic coupling constants. For example, in an  $SU(N)$  gauge theory coupled to an adjoint representation scalar, there are two quartic couplings

$$\mathcal{L}(\text{adj}) \sim -\frac{1}{2}\lambda_1 [\text{Tr}[\Phi^2]]^2 - \lambda_2 \text{Tr}[\Phi^4]. \quad (29)$$

If a fundamental representation scalar is present as well, the number of quartic couplings goes up to five (for simplicity, couplings of the form  $\Phi^3$  are omitted):

$$\begin{aligned} \mathcal{L}(\text{fund} + \text{adj}) \sim & -\lambda_3 (H^\dagger H)^2 - \lambda_4 H^\dagger \Phi^2 H \\ & - \lambda_5 H^\dagger H \text{Tr}[\Phi^2] + \mathcal{L}(\text{adj}). \end{aligned} \quad (30)$$

Obviously finding the ultraviolet-stable fixed points and their respective domains of attraction in grand unified theories is a technically more difficult (and probably numerical) problem.

Many of the techniques discussed in the last section can be applied to such theories, however. For instance, in the case of an  $SU(N)$  gauge theory with an adjoint representation scalar, the renormalization group equations for the theory can be written [3, 8]

$$\frac{16\pi^2}{\bar{g}^2} \frac{d\bar{g}}{dt} = A_1 \bar{g}_1^2 + (B_1 + b_0) \bar{g}_1 + C_1 \bar{g}_2^2 + D_1 \bar{g}_1 \bar{g}_2 + E_1, \quad (31a)$$

$$\frac{16\pi^2}{\bar{g}^2} \frac{d\bar{g}_2}{dt} = A_2 \bar{g}_2^2 + (B_2 + b_0) \bar{g}_2 + C_2 \bar{g}_1^2 + D_2 \bar{g}_1 \bar{g}_2 + E_2, \quad (31b)$$

with

$$\xi_i \equiv \frac{\lambda_i}{g^2}, \quad (31c)$$

$$16\pi^2 \frac{d\bar{g}}{dt} = -\frac{1}{2} b_0 \bar{g}^3, \quad (31d)$$

and the constants  $A_i, B_i, C_i, \dots$  depend only [3, 8] on  $N$  (and not on  $b_0$ ). Eqs. (31a) and (31b) do *not* possess an ultraviolet-stable fixed point for positive  $b_0$  if  $N < 6$ . However, as is shown below, for  $b_0$  sufficiently large and *negative* such a fixed point always exists in this case. Perturbation theory is a valid method for identifying such fixed points when  $g_R^2 b_0$  is “small” (cf. eq. (18)); thus the limit of large and negative  $b_0$  must be approached with caution.

It is easiest to obtain the limit of large negative  $b_0$  if eqs. (31) are rescaled as follows,

$$-\frac{1}{b_0} \xi_i \equiv \chi_i, \quad \frac{-\bar{g}^2 dt}{16\pi^2 b_0} \equiv ds. \quad (32)$$

Eqs. (31) become

$$\begin{aligned}\frac{d\chi_1}{ds} &= \chi_1(A_1\chi_1 - 1 + D_1\chi_2) + C_1\chi_2^2, \\ \frac{d\chi_2}{ds} &= \chi_2(A_2\chi_2 - 1 + D_2\chi_1) + C_2\chi_1^2,\end{aligned}\quad (33)$$

plus terms which are negligible in this limit. Eqs. (33) have a fixed point when both  $\chi_1$  and  $\chi_2$  vanish. If the equations are expanded for small  $\chi_i$  ( $\simeq \delta\chi_i$ ) the result is

$$\frac{d}{ds}\delta\chi_i = -\delta\chi_i, \quad (i = 1, 2). \quad (34)$$

This fixed point is therefore *attractive* and represents an ultraviolet-stable fixed point. Any point in the domain of attraction of this fixed point represents a non-trivial theory given the usual assumption of a perturbative fixed point in the gauge coupling. Thus for large and negative  $b_0$  a non-trivial theory is possible. Of course, it is always possible to achieve such a value of  $b_0$  by adding enough fermions to the theory. In order to make  $b_0$  negative enough for an ultraviolet-stable fixed point to appear, a lower bound on the number of fermions in the theory may therefore need to be satisfied. The boundary on the domain of attraction of the fixed point also furnishes constraints on the quartic couplings of the theory.

#### 4. Yukawa couplings and triviality

The examples given above illustrate the method by which gauge fields can turn a trivial scalar theory into a non-trivial one. Yet the addition of a gauge field is not the only way to make a  $\phi^4$  theory non-trivial. Another way is to couple fermions to the scalars; in that case the Yukawa couplings can stabilize the quartic coupling so that the theory becomes non-trivial (see also refs. [10]).

The standard model of the weak interaction provides an example of this phenomenon. Consider a version of this model with one generation of leptons, and a Yukawa coupling of strength  $h$  between these and a scalar field of the form

$$\sum_i h e_R \varphi_i e_{L,i}. \quad (35)$$

The renormalization group equation for  $h$  follows from standard results [8]

$$16\pi^2 \frac{d\bar{h}}{dt} = \frac{5}{2}\bar{h}^3 - \left(\frac{15}{4}\bar{g}^{\prime 2} + \frac{9}{4}\bar{g}^2\right)\bar{h}, \quad (36)$$

where the evolution of the gauge couplings  $g$  and  $g'$  is described by eqs. (19).

The coupling  $g'$  increases with  $t$ , while  $g$  decreases to zero for large  $t$ . Thus for a suitable range of parameters it is a reasonable approximation to neglect  $g$  relative to  $g'$ . Define a new variable  $\gamma$  by

$$\gamma = \frac{h}{g'}. \quad (37a)$$

Eqs. (36) and (19) then give the result

$$\frac{1}{g'^2} 16\pi^2 \frac{d\bar{\gamma}}{dt} = \bar{\gamma} \left[ \frac{5}{2} \bar{\gamma}^2 - \left( \frac{15}{4} + b'_0 \right) \right], \quad (38)$$

in this approximation. Note that for  $\gamma$  equal to the fixed point value,

$$(\gamma^*)^2 \equiv \frac{3}{2} + \frac{2}{5} b'_0, \quad (39)$$

the couplings  $h$  and  $g'$  evolve in this fixed ratio. The quartic coupling in turn evolves according to

$$16\pi^2 \frac{d\bar{\lambda}}{dt} = 12\bar{\lambda}^2 + 12\bar{\lambda}\bar{h}^2 - 3\bar{\lambda}\bar{g}'^2 - 12\bar{h}^4 + \frac{3}{4}\bar{g}'^4, \quad (40)$$

where  $\bar{g}$  is set to zero as above. If  $\bar{h}$  is set equal to  $\gamma^* \bar{g}'$  then eq. (40) can be written in terms of  $\bar{\xi}$ , where

$$\bar{\xi} = \frac{\bar{\lambda}}{\bar{g}'^2}, \quad (41a)$$

$$\begin{aligned} \frac{16\pi^2}{g'^2} \frac{d\bar{\xi}}{dt} &= 12\bar{\xi}^2 + (12\gamma^{*2} - 3 - 2b'_0)\bar{\xi} + \left( \frac{3}{4} - 2\gamma^{*4} \right), \\ &= 12(\bar{\xi} - \xi_+)(\bar{\xi} - \xi_-). \end{aligned} \quad (41b)$$

If  $b'_0$  is nearly zero, the fixed points  $\xi_{\pm}$  are given by

$$\xi_{\pm} = \frac{1}{8} \left[ -5 \pm (11 \times 15)^{1/2} \right] \sim 0.97, -2.2. \quad (42)$$

Note that the roots are of opposite sign, and thus  $\bar{\xi}$  approaches zero as  $t$  increases provided that  $\xi_R [\equiv \bar{\xi}(t=0)]$  is less than  $\xi_+$ .

Since  $\bar{\xi}$  approaches zero in the limit of large  $t$  when  $\xi_R$  is less than  $\xi_+$ , and because  $\bar{g}^2$  increases in this limit,  $\bar{\lambda}$  approaches zero. Thus the bare coupling  $\lambda_0$  is zero, and the scalar theory is not only non-trivial but is asymptotically free as well. It must be pointed out, however, that this result occurs only when  $\gamma_R$  is *exactly* equal to  $\gamma^*$ . In the standard model, a Yukawa coupling  $h_R = \gamma^* g'_R$  implies the existence of a lepton of mass

$$\begin{aligned} m_L &= 2^{-3/4} G_F^{-1/2} h_R \\ &\sim 176 \text{ GeV} \times h_R \\ &\sim 315 \text{ GeV}. \end{aligned} \quad (43)$$

Note that this mass is precisely determined by an eigenvalue condition on  $\gamma^*$ .

Of course, if  $h_R$  is less than  $\gamma^* g'_R$ ,  $\bar{h}$  decreases to zero too fast to affect the quartic coupling. If  $h_R$  is greater than  $\gamma^* g'_R$ , the effective coupling  $\bar{h}$  presumably grows uncontrollably and no sensible theory can result (but see the discussion at the end of subsect. 3.1).

## 5. Conclusions

The assumption that a scalar field theory *without* gauge fields is trivial (i.e. that the renormalized quartic coupling is zero) implies strong constraints on a theory *with* gauge fields. The addition of gauge fields can in fact make a trivial pure scalar theory non-trivial. Indeed, such a phenomenon may occur in realistic theories such as the standard model of the weak interaction and in grand unified theories.

The mechanism by which triviality is eliminated typically works for a small range of the renormalized coupling constants of the theory. Basically a bare scalar particle screens itself *totally*, so that the renormalized scalar charge is zero regardless of its bare value. The addition of a gauge field generates an effective quartic coupling constant; if this effective coupling is at least as large as the original coupling it can destroy the total screening of bare charges.

However, the screening persists if the quartic coupling is much larger than the effective (gauge + quartic) coupling. The necessity of the destruction of the screening phenomenon forces restrictions on the bare couplings; a bound on the *renormalized* couplings follows:

$$\lambda_R \leq \zeta_+ g_R^2, \quad (g_R^2 \ll 1) \quad (44)$$

for some *calculable* constant  $\zeta_R$ . This restriction in turn implies a *calculable* upper bound on the ratio of Higgs to gauge boson mass. For, e.g. the standard model of the weak interaction, this bound is

$$\left( \frac{m_H}{m_W} \right)^2 = 16 \frac{\lambda_R}{g_R^2} \leq 12.8, \quad (45)$$

for reasonable parameter choices, and similarly for other theories.

It was also shown that Yukawa couplings to fermions could rescue a theory from triviality. This mechanism generally leads to an eigenvalue condition [10] for the fermion mass. Thus if such a mechanism is present in a theory it should be possible to calculate various fermion masses (all roughly the order of the gauge boson masses). Such a precise prediction may be aesthetically undesirable, however.

It must be emphasized that the elimination of triviality and the existence of bounds like eq. (45) has nothing to do with the appearance (or disappearance) of asymptotic freedom [3,8,10,11]. Indeed, in order for a theory to be rendered non-trivial by gauge couplings, at least one of the couplings is in general not

asymptotically free. Restrictions on the number of fermions in the theory can follow from such a requirement.

Finally it should be noted that technically speaking the work presented here assumes the presence of an ultraviolet-stable fixed point in the (non-asymptotically free) gauge coupling of the theory. Such a fixed point is necessary in order that the associated gauge field is non-trivial, and may occur either in the gauge theory by itself or when the theory under study is unified with other gauge fields or with gravity.

Simple arguments based on perturbation theory have been shown to yield interesting results. Although perturbation theory appears to be a valid technique up to momentum scales of the size of the Planck mass it would be useful to extend the present discussion beyond a perturbative analysis. In particular, it would be interesting to study the triviality question for a non-asymptotically free gauge theory coupled to a scalar (e.g. the abelian Higgs model) by non-perturbative methods such as a Monte Carlo simulation [12]. Another problem of interest is to study the effects of triviality on supersymmetric theories, where constraints demanded of a non-trivial theory must be even more restrictive.

At the conclusion of this work a preprint [13] was received which gives upper bounds on the mass of Higgs particles. In distinct contrast to this work, it is there assumed that  $\phi^4$  theory remains trivial *even when coupled to a gauge field*. In view of the existence [10,11] of eigenvalue conditions for asymptotic freedom in spontaneously broken theories, as well as theories (e.g. SU(3) coupled to a fundamental representation scalar) which are asymptotically free [3] in both the gauge and quartic couplings, and finally the results of the present work, such an assumption may require some qualification.

It is a pleasure to thank S. Coleman, S.D. Ellis, I. Halliday, B. Halperin, M. Nauenberg, L. O’Raifeartaigh, R. Petronzio, R. Pisarski and F. Wilczek for several useful discussions. Special thanks also go to E.L. Berger, Argonne National Laboratory, and the Institute for Theoretical Physics for their support during the initial stages of this work.

## References

- [1] P.W. Higgs, Phys. Lett. 12 (1964) 132; Phys. Rev. Lett. 13 (1964) 508; Phys. Rev. 145 (1966) 1156;  
P. W. Anderson, Phys. Rev. 130 (1963) 439;  
F. Englert and R. Brout, Phys. Rev. Lett. 13 (1964) 321;  
G.S. Guralnik, C.R. Hagen and T.W.B. Kibble, Phys. Rev. Lett. 13 (1964) 585
- [2] K.G. Wilson, Phys. Rev. B4 (1971) 3184;  
K.G. Wilson and J. Kogut, Phys. Reports 12C (1974) 75;  
R. Schrader, Phys. Rev. B14 (1976) 172;  
G.A. Baker and J. Kincaid, Phys. Rev. Lett. 42 (1979) 1431;  
G.A. Baker and S. Krinsky, J. Math. Phys. 18 (1977) 590;

- C.M. Bender, F. Cooper, G.S. Guralnik, R. Roskies and D.H. Sharp, *Phys. Rev.* D23 (1981) 2976; D23 (1981) 2999;  
 B. Freedman, P. Smolensky and D. Weingarten, *Phys. Lett.* 113B (1982) 481;  
 M. Aizenmann, *Phys. Rev. Lett.* 47 (1981) 1; *Commun. Math. Phys.* 86 (1982) 1;  
 J. Fröhlich, *Nucl. Phys.* B200[FS4] (1982) 281;  
 A.D. Sokal, *Ann. Inst. Henri Poincaré* 37 (1982) 317;  
 C. Aragão de Carvalho, C.S. Carracciolo and J. Fröhlich, *Nucl. Phys.* B215[FS7] (1983) 209, and references therein;  
 N. Khuri, *Phys. Lett.* 82B (1979) 83;  
 B.S. Shaverdyan and A.G. Ushveridze, *Phys. Lett.* 123B (1983) 316
- [3] D.J. Gross and F. Wilczek, *Phys. Rev.* D8 (1973) 3633
- [4] S.L. Glashow, *Nucl. Phys.* 22 (1961) 579;  
 S. Weinberg, *Phys. Rev. Lett.* 19 (1967) 1264;  
 A. Salam, in *Elementary particle theory*, ed. N. Svartholm (Almqvist and Wiksell, Stockholm, 1968) 367
- [5] M. Gell-Mann and F. Low, *Phys. Rev.* 95 (1954) 1300;  
 E.C.G. Stueckelberg and A. Peterman, *Helv. Phys. Acta* 26 (1953) 499;  
 N.N. Bogolubov and D.V. Shirkov, *Introduction to the theory of quantized fields* (Interscience, New York, 1978) chap. 8
- [6] C.G. Callan, *Phys. Rev.* D2 (1970) 1541;  
 K. Symanzik, *Comm. Math. Phys.* 18 (1970) 227; in *Springer tracts in modern physics*, ed. G. Höhler (Springer, New York, 1971) vol. 57, p. 222;  
 S. Coleman, in *Proc. 1971 Int. Summer School "Ettore Majorana"* (Academic, New York)
- [7] W.A. Bardeen and M. Moshe, *Phys. Rev.* D28 (1983) 1372;  
 S. Coleman, R. Jackiw and H.D. Politzer, *Phys. Rev.* D10 (1974) 2491
- [8] T.P. Cheng, E. Eichten and L.-F. Li, *Phys. Rev.* D9 (1974) 2259
- [9] M. Veltman, *Acta Phys. Pol.* B8 (1977) 475;  
 B.W. Lee, C. Quigg and H.B. Thacker, *Phys. Rev. Lett.* 38 (1977) 883; *Phys. Rev.* D16 (1977) 1519;  
 M.S. Chanowitz, M.A. Furman and I. Hinchliffe, *Phys. Lett.* 78B (1978) 285;  
 D. Dicus and V. Mathur, *Phys. Rev.* D7 (1973) 3111
- [10] N.-P. Chang, *Phys. Rev.* D10 (1974) 2706;  
 E. Ma, *Phys. Rev.* D11 (1974) 322
- [11] N.P. Cheng, A. Das and J. Perez-Mercader, *Phys. Rev.* D22 (1980) 1414, 1429, 1829; D23 (1981) 132;  
 E.S. Fradkin and O.K. Kalashnikov, *Phys. Lett.* 64B (1976) 177;  
 E.S. Fradkin, O.K. Kalashnikov and S.E. Konstein, *Lett. Nuovo Cim.* 21 (1978) 5;  
 O.K. Kalashnikov and S.E. Konstein, *Nucl. Phys.* B166 (1980) 507;  
 A. Zee, *Phys. Rev.* D7 (1973) 3630;  
 N. Cabibbo, L. Maiani, G. Parisi and R. Petronzio, *Nucl. Phys.* B158 (1979) 295;  
 L. Maiani, G. Parisi and R. Petronzio, *Nucl. Phys.* B136 (1978) 115;  
 R. Petronzio, 1979 Erice Lectures;  
 A. Salam and J. Strathdee, *Phys. Rev.* D18 (1978) 4713;  
 J. Lemmon and K.T. Mahanthappa, *Phys. Rev.* D13 (1976) 2907;  
 D.A. Ross, *Nucl. Phys.* B140 (1978) 1;  
 F.A. Bais and H.A. Weldon, *Phys. Rev.* D18 (1978) 1199, and references therein
- [12] D.J.E. Callaway and L.J. Carson, *Phys. Rev.* D25 (1982) 531;  
 K.C. Bowler, G.S. Pawley, B.J. Pendleton, D.J. Wallace and G.W. Thomas, *Phys. Lett* 104B (1982) 481;  
 D.J.E. Callaway and D.J. Maloof, *Phys. Rev.* D27 (1983) 406;  
 D.J.E. Callaway, *Phys. Rev.* D27 (1983) 2974
- [13] R. Dashen and H. Neuberger, *Phys. Rev. Lett.* 50 (1983) 1897