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Magnetic Flux Patterns in the Superconducting Intermediate State

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Abstract

The problem of the flux penetration of an infinite square plate of type I superconductor in a perpendicular magnetic field is analyzed perturbatively via the Ginzburg-Landau equations. It is shown that Landau's classic 1937 textbook model is at best a qualitative guide to the expected flux patterns. If the plate thickness d is near a critical scale d_c, the pattern periodicity p diverges as $p \sim |d - d_c|^{-.5}$. For somewhat larger values of d, essentially all periods occur. It is also shown how the vertical and horizontal coordinates of the plate act like Fourier conjugates, so that the squared superconductor order parameter is similar to a probability distribution in a quantum "phase space."

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1. Prolegomena

The study of the magnetic intermediate state in type I superconductors is a formidable problem which has been little explored. Its solution involves the analysis of highly degenerate ground states and intricate fractal-like patterns on many length scales. Yet for all the complexity of its solution, the problem itself can be formulated with deceptive ease: Consider a large square plate of type I superconducting material in an incident magnetic field. For sufficiently large field strength, the magnetic field must penetrate the superconductor. The problem is to predict this pattern of penetration.

2. Background and synopsis.

It has been almost forty years since Abrikosov [1-3] pointed out that the Ginzburg- Landau model of superconductivity[4] predicts the existence of two different types of superconductors, distinguished by the value of a parameter κ. Type I superconductors (where $\kappa < 1/\sqrt{2}$) have a positive surface energy at a bulk normal-superconducting interface, so domain surface is typically minimized in equilibrium configurations in a magnetic field. Conversely, type II superconductors have $\kappa > 1/\sqrt{2}$ and generally maximize this surface area.

The problem of the pattern of flux penetration in a large plate of type II superconductor in a uniform perpendicular magnetic field was originally treated by Abrikosov[2]. His formalism[3] correctly predicts that the equilibrium configuration is a periodic triangular lattice of flux tubes.

Flux penetration patterns in type I superconductors are a good deal more complex, and are the subject of this paper. The textbook[5] model of this intermediate state problem was originally formulated by Landau[6] in 1937, thirteen years before the Ginzburg-Landau equations appeared. It is au fond an exercise in classical electromagnetism augmented by boundary conditions obtained from the Meissner effect. The Landau model predicts that flux penetrates the plate in a parallel series of stripes whose spacing depends upon the plate thickness.

Experimental studies [7,8] showed that the true situation was far more baroque. Rather than simple stripes, complicated patterns appear which can depend strongly upon sample preparation and history (temperature and magnetic field variation). A detailed theoretical analysis of this problem in the context of the Ginzburg-Landau model is quite intricate and is still incomplete. The approach taken here and previously by Lasher[9], Maki[10] and others[11–14] follows the philosophy of Abrikosov, analyzing the (nonlinear) Ginzburg-Landau equations in a perturbative expansion about the linear regime. This is reasonable when the average flux density B is near its critical value B_c:

$$B \lesssim B_c \tag{1}$$

Above B_c, the bulk superconducting order parameter vanishes; near B_c it should be small. Since the Ginzburg-Landau equations themselves are an expansion of a free energy in powers of this order parameter, analysis based upon such a perturbative scheme should be universal in this regime.

The present study follows and extends these classic works. It will be seen that the flux penetrates a plate of thickness d in a set of periodic patterns. The periodicity is describable in terms of a rectangular unit cell whose area is proportional to p, an integer equal to the number of flux quanta penetrating the cell. For d near a critical scale d_c, $p \sim |d-d_c|^{-.5}$. When d is larger than a second scale d_2, solutions exist for essentially any value of p. It will also be shown that the squared order parameter (superconductor wave function) behaves much like a probability distribution in a quantum "phase space."

3. Landau's problem revisited

The primary assumption of the Ginzburg-Landau approach is that the free energy density $f(\mathbf{x})$ of a superconductor in a magnetic field has an expansion

$$f(\mathbf{x}) \doteq f_n + \frac{1}{2}|\mathbf{D}\psi(\mathbf{x})|^2 + \frac{1}{4}(|\psi(\mathbf{x})|^2 - 1)^2 + \frac{1}{2}\kappa^2 H^2(\mathbf{x}) \tag{2}$$

where $\psi(\mathbf{x})$ is the order parameter (i.e., the superconductor wave function), $\mathbf{H}$ is the magnetic field, $\mathbf{D} = \partial - i\,\mathbf{A}$ is the covariant derivative and f_n is the free energy of the normal state. Units are chosen so as to measure

$$\begin{gathered} \mathbf{x} \text{ in units of } \xi \equiv (\Phi_0/2\pi B_c)^{\frac{1}{2}} \\ \mathbf{A} \text{ in units of } \xi B_c \\ f(\mathbf{x}) \text{ in units of } B_c^2/2\pi\kappa^2 \\ \mathbf{H}(\mathbf{x}) \text{ in units of } B_c \end{gathered} \tag{3}$$

where ξ is the temperature-dependent coherence length and $\Phi_0 = 2\pi\hbar c/e^*$ ($e^* = 2e$) is the elementary flux quantum. Minimization of the free energy leads to the Euler equations of motion:

$$\begin{gathered} \mathbf{D}^2\psi + \psi - |\psi|^2\psi = 0 \\ -\kappa^2[\partial^2 \cdot \mathbf{A} - \partial(\partial \cdot \mathbf{A})] = Im[\psi^* \mathbf{D}\psi] \end{gathered} \tag{4}$$

Note the appearance of the Abrikosov parameter κ. This Ginzburg-Landau formation is actually quite general[15] and can be derived[16] from the BCS theory.

The problem considered here is Landau's original paradigm. A flat square plate of type I superconductor is situated in the xy plane. The boundaries of the superconductor are

$$\begin{gathered} -\infty < x, y < \infty \\ -\frac{d}{2} \leq z \leq \frac{d}{2} \end{gathered} \tag{5}$$

A perpendicular magnetic field which is uniform at large $|z|$ is applied to the plate, i.e.,

$$\lim_{|z| \to \infty} \mathbf{H}(\mathbf{x}) = -B\hat{z} \tag{6}$$

where B is a constant. The problem is to predict the magnetic flux distribution immediately above the plate.

4. The perturbative solution and a glimpse of quantum phase space.

When the incident magnetic field is slightly below its critical value (above which the material goes normal), the order parameter $\psi(\mathbf{x})$ is expected to be small. The Ginzburg-Landau equations can then be solved perturbatively by expanding in a power series in a fictitious parameter ϵ[9–14]:

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \mathbf{A}^{(0)}(\mathbf{x}) + \epsilon\mathbf{A}^{(1)}(\mathbf{x}) + \epsilon^2\mathbf{A}^{(2)}(\mathbf{x}) + \ldots \\ \psi(\mathbf{x}) &= \sqrt{\epsilon}(\phi^{(0)} + \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \ldots) \\ f(\mathbf{x}) &= f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots \end{aligned} \tag{7}$$

At the end of the calculation, ϵ is set to one. The result is a perturbative expansion in powers of $(1-B)$.

The symmetry of the problem allows the choice

$$\mathbf{A}(\mathbf{x}) = A(\mathbf{x})\hat{e}_y \tag{8}$$

The lowest-order equation is then

$$(\mathbf{D}_0^2 + 1)\phi^{(0)}(x,y) = 0$$

where

$$\mathbf{D}_0 = \partial - i\mathbf{A}^{(0)}$$
$$\mathbf{A}^{(0)} = -B\hat{e}_y \tag{9}$$

Near the critical field the order parameter is independent of z and Eq. (9) is essentially the Schrödinger equation for a one-dimensional harmonic oscillator, with an important difference – the "energy" is one! The normalizable solutions are the familiar eigenfunctions, whence

$$(2n+1)B = 1 \tag{10}$$

and thus B must not exceed its critical value of one (in the present units). The inclusion of nonlinear terms will, of course, modify Eq. (10).

The appropriate lower-order solution occurs when $B = 1$, and is given by

$$\phi^{(0)}(x,y) = \int dk\, C(k)\, exp[-iky - \frac{1}{2}(x-k)^2] \tag{11}$$

where the function $C(k)$ is arbitrary. The function $\phi^{(0)}\,(x,y)$ has [14] an unusual property related to quantum phase-space distributions. Construct $\rho_1(x)$, the x density of the order parameter

$$\rho_1(x) \equiv \int_{-\infty}^{\infty} |\phi^{(0)}(x,y)|^2 dy$$
$$= 2\pi^{3/2}\int dk\, |C(k)|^2 \Delta(x-k) \tag{12}$$

where

$$\Delta(s) = exp(-s^2)/\sqrt{\pi} \tag{13}$$

is a sharply localized smearing function with unit volume. Similarly

$$\rho_2(y) \equiv \int_{-\infty}^{\infty} |\phi^{(0)}(x,y)|^2 dx$$

$$= 2\pi^{3/2}\int dq\, |\tilde{C}(q)|^2 \Delta(y-q) \tag{14}$$

where

$$\tilde{C}(q) \equiv \int \frac{dk}{\sqrt{2\pi}} e^{-ikq} C(k) \tag{15}$$

is just the Fourier transform of $C(k)$. Since distances are measured in units of the (microscopic) coherence length, at macroscopic scales $\Delta(s) \to \delta(s)$ and

$$\rho_1(x) \approx 2\pi^{3/2}|C(x)|^2$$
$$\rho_2(y) \approx 2\pi^{3/2}|\tilde{C}(y)|^2 \tag{16}$$

At macroscopic scales the coordinates (x,y) can be thought of as Fourier conjugates, much like position and momentum in quantum mechanics. The squared order parameter therefore behaves like a probability distribution in a quantum phase space[14].

5. The Abrikosov order parameter

The function $C(k)$ in Eq. (11) must be chosen so that the wave function $\phi^{(0)}(x,y)$ fills the xy plane. This requirement leads to the function chosen by Abrikosov[2]:

$$C(k) = \sum_{n=-\infty}^{\infty} C_{n/p}\, \delta(k - nk_0) \tag{17}$$

where n and p are integers ($p \geq 1$). Here k_0 is a real parameter and the complex parameters $C_{n/p}$ are periodic

$$C_{n/p} = C_{(n+p)/p} \tag{18}$$

and are normalized by

$$\sum_{n=0}^{p-1} |C_{n/p}|^2 = 1 \tag{19}$$

The corresponding Abrikosov order parameter

$$\psi_p(x,y) \equiv \sum_{n=-\infty}^{\infty} C_{n/p}\, exp[-ink_0 y - \frac{1}{2}(x - nk_0)^2] \tag{20}$$

exhibits the properties

$$\psi_p(x, y + 2\pi/k_0) = \psi_p(x,y)$$

$$\psi_p(x + pk_0, y) = exp(-ipk_0y)\psi_p(x, y) \tag{21}$$

Thus $|\psi_p(x,y)|^2$ is periodic [and $\psi_p(x,y)$ is quasiperiodic] on a rectangle of size $(\Delta x, \Delta y) = (pk_0, 2\pi/k_0)$. Each rectangle is penetrated by p units of flux

$$Flux = \Delta x \cdot \Delta y \cdot B_c \cdot \xi^2 = p\Phi_0 \tag{22}$$

and each ψ_p is specified by $2p$ independent real parameters $\{C_{n/p}, k_0\}$, as expected of p-vortex solutions [17] because of the Atiyah-Singer theorem [18].

It is convenient to use the Fourier representation for the superconductor density [9]:

$$|\psi_p(x,y)|^2 = N_p \sum_k g_p(\mathbf{k})\, e^{ik\cdot\rho} \tag{23}$$

with the sum taken over values

$$\mathbf{k} = [(2\pi/pk_0)N_x, k_0N_y] \tag{24}$$

where (N_x, N_y) are integers ranging from negative to positive infinity, and ρ is the two-dimensional vector (x,y). Also

$$N_p \equiv \overline{|\psi_p(x,y)|^2} \tag{25}$$

is the xy spatial average of $|\psi_p(x,y)|^2$. Then [14]

$$g_p(\mathbf{k}) = exp(-k^2/4 + ik_xk_y/2)\, h_p(N_x, N_y)$$

$$h_p(N_x, N_y) = \sum_{n=0}^{p-1} exp(2\pi i n N_x/p)\, C^*_{(n+Ny)/p}\, C_{n/p} \tag{26}$$

Also define the finite Fourier transform matrix

$$F_{mn} \equiv \frac{1}{\sqrt{p}} exp(2\pi i mn/p) \tag{27}$$

and $\tilde{C}_{m/p}$, the finite Fourier transform of $C_{n/p}$:

$$\tilde{C}_{m/p} \equiv \sum_{n=0}^{p-1} F_{mn} C_{n/p} \tag{28}$$

whence

$$h_p(M,N) = \sum_{m,n=0}^{p-1} F_{Mm} \tilde{h}_p(m,n) F_{nN} \tag{29}$$

$$\tilde{h}_p(m,n) = F_{mn} C_{m/p} \tilde{C}^*_{n/p} \tag{30}$$

It follows that $|\psi_p|^2$ is $[x \leftrightarrow y]$ symmetric when [14]

$$(i)\; k_0^2 = 2\pi/p \; and \tag{31}$$

$$(ii)\; \tilde{C}_{n/p} = e^{i\theta} C^*_{n/p} \tag{32}$$

where θ is an arbitrary real parameter.

Equation (32) can be solved in terms of the eigenvectors of F. Note that F is symmetric and satisfies

$$F^4 = F^+F = 1 \tag{33}$$

The eigenvectors $f_{n/p}(M)$ of F can be taken real, and the corresponding eigenvalues are $\lambda_M = exp(iM\pi/2)$, where M is an integer. Thus,

$$\hat{C}_{n/p}(M) = exp[i(2\theta - M\pi)/4]\, f_{n/p}(M) \tag{34}$$

satisfies Eq. (32) [as do linear combinations of the $\hat{C}_{n/p}(M)$ with real coefficients]. The (unnormalized) $f_{n/p}(M)$ for a given p are contained in the set

$$f_{n/p}(M) = \sum_{N=-\infty}^{\infty} exp[-\pi p(N - n/p)^2]\, H_M[\sqrt{2\pi p}\,(N - n/p)] \tag{35}$$

where the H_M are the Hermite polynomials. Equation (35) follows from the Poisson sum rule identity

$$\sum_{N=-\infty}^{\infty} exp[-\frac{s^2}{2}(N - n/p)^2]$$

$$= \frac{\sqrt{2\pi}}{s} \sum_{M=-\infty}^{\infty} exp(-\frac{2\pi^2}{s^2} M^2 + 2\pi i \frac{Mn}{p}) \tag{36}$$

and the generating function for the $H_M(x)$. In what follows, the parameter θ is set to zero.

Finally, note that the order parameter $\chi_p(x,y)$ of a state consisting entirely of vortices of quantum number p can be cast in this formalism. As pointed out by Lasher [9]:

$$\chi_p(x,y) = [\psi_1(x/\sqrt{p}, y/\sqrt{p})]^p \tag{37}$$

The order parameter χ_p is proportional to to the Abrikosov wave function $\psi_p(x,y)$ of Eq.(20) with coefficients $C_{n/p} = V_{n/p}$ satisfying the recursion relation [14]:

$$V_{n/(p+1)} = \sum_{m=0}^{p-1} exp\{\frac{-k_0^2}{2p(p+1)}[(1+p)m - pn]^2\}\, V_{m/p} \qquad (38)$$

When $k_0^2 = 2\pi$, the $V_{n/p}$ are linear combinations of the $\hat{C}_{n/p}(4I)$ with eigenvalue $\chi_{4I} = 1$.

6. Finding the optimum Abrikosov solution by minimizing the free energy.

The classic approach followed by Abrikosov [2] and others [9-14] involves a perturbative calculation of the free energy in the fashion of Eq. (7). Typically, this free energy is evaluated to order $(1-B)^2$. Minimization of the free energy for a given flux density B, plate thickness d, and Abrikosov parameter κ yields the parameter set $\{p, k_0, C_{n/p}\}$. The order $(1-B)^3$ contribution is proportional to $(1-2\kappa^2)$ and is numerically negligible for typical cases, suggesting that the perturbative method is reliable [9] for $B \sim 1$.

The results of the order $(1-B)^2$ calculation in the notation Eqs. (23)-(26) follow from the work of Lasher [9]. The average free energy density in the xy plane is

$$\overline{f(x)} = d(f_n + \Delta f) \qquad (39)$$

where in the present units

$$\Delta f = \frac{1}{4} + \frac{1}{2}\kappa^2 B^2 - \frac{1}{2}\kappa^2(1-B)^2/(2\kappa^2 D)$$

$$2\kappa^2 D \equiv 2\kappa^2\beta - \sum_k{}' |g_k|^2(e^{-kd} - 1 + kd)/kd$$

$$\beta \equiv \sum_k |g_k|^2 \qquad (40)$$

and the primed summation excludes $\mathbf{k} = 0$. The magnetic flux pattern immediately above the plate ($z = d/2 + \delta, \delta \tilde{>} 0$) has

$$-H_z = B + \frac{N_p}{2\kappa^2}\Delta H$$

$$\Delta H(x,y) \equiv \sum_k{}' e^{i\mathbf{k}\cdot\rho}(sinh\frac{1}{2}kd)g_k \qquad (41)$$

and ρ is the two-dimensional vector (x,y). Note that the xy average of ΔH vanishes:

$$\overline{\Delta H} = 0 \qquad (42)$$

and that since $g_k \sim exp(-k^2/4)$, the function ΔH is maximized when $|\mathbf{k}| \simeq d$. There is also a constraint equation

$$N_p = (1-B)/(2\kappa^2 D) \qquad (43)$$

which, in conjunction with Eq. (40), implies that

$$2\kappa^2 D \geq 0 \qquad (44)$$

in the superconducting state.

6a. Normal Region

Some simple limits follow directly from Eqs. (39-44). First, note that as $d \to \infty$

$$\lim_{d\to\infty} 2\kappa^2 D = (2\kappa^2 - 1)\beta + 1 \qquad (45)$$

and thus from Eq. (44)

$$d \to \infty: \quad 2\kappa^2 \geq 1 - 1/\beta \cong 1 - 1./1.160 \qquad (46)$$

in order for a superconducting state to exist. The minimum value that β may take ($\beta_{min} \sim 1.160$) occurs [3] when $p = 2$ and, e.g.,

$$k_0 = (\pi/\sqrt{3})^{\frac{1}{2}}$$

$$C_{0/2} = 1/\sqrt{2}$$

$$C_{1/2} = i/\sqrt{2} \qquad (47)$$

and yields a triangular lattice of flux tubes. Thus an infinitely thick superconducting plate must have $\kappa > \kappa_{min}$, where $\kappa_{min} \sim .263$. For finite d the plate will go normal when $\kappa < \kappa_{min}$ if

$$Normal: \quad d > d_{max}(\kappa) \sim .2\kappa(\kappa_{min} - \kappa)^{-1.0} \qquad (48)$$

as shown in Fig. 1. The specific form of Eq. (48) follows from numerical evaluation.

6b. Region S_I

Another simple limit occurs when $d \to 0$

$$d \to 0 \ : \ 2\kappa^2 D \to 2\kappa^2 \beta \tag{49}$$

and implies that for small enough d the free energy is minimized for the triangular lattice solution Eq. (47), as originally predicted by Tinkham [12] and confirmed by others [9–11]. More generally, the triangular flux lattice Eq. (47) is the optimum solution when

$$S_I(triangular) : d < d_1(\kappa) \tag{50}$$

where, as $\kappa \to \kappa_c \equiv 1/\sqrt{2}$

$$d_1(\kappa) \sim .2(\kappa_c - \kappa)^{-1.0} \tag{51}$$

as follows from numerical evaluation. The region S_I is displayed in Fig. 1.

6c. Region S_{III}

In region S_{III} (displayed in Fig. 1) the optimum solution occurs when D vanishes

$$S_{III} : D \tilde{>} 0 \tag{52}$$

Equation (52) can be made to hold when

$$S_{III} : d_2 < d < d_{max}(\kappa) \tag{53}$$

where as $\kappa \to \kappa_c$

$$d_2(\kappa) \sim .3(\kappa_c - \kappa)^{-1.5} \tag{54}$$

as follows from numerical evaluation. As $d \to \infty$, this solution is characterized by

$$\beta = 1/(1 - 2\kappa^2) \tag{55}$$

for $\kappa > \kappa_{min}$ (note $\beta_{min} \leq \beta \leq \infty$). Properties of the superconducting solution set S_{III} have been extensively explored elsewhere [14], and are therefore not considered in detail here. A few remarks are in order, however.

Recall that Landau's original model [5,6] predicted that the optimal flux distribution was highly elongated. Specifically, the Landau model yields a flux pattern which is a (horizontal or vertical) series of stripes, whose periodicity is a calculable function of d. By contrast, the Ginzburg-Landau equations in region S_{III} typically do not have a unique solution [at least in a perturbative calculation to order $(1 - B)^2$]. There are a large number of flux distributions which, to this order in $(1 - B)$, have the same free energy. Most of these distributions have some degree of elongation, but some are in fact $[x \leftrightarrow y]$ symmetric. The observed flux distribution in S_{III} thus depends in a highly sensitive way upon the way the state is created (reminiscent of problems related to ergodicity in classical statistical mechanics).

These statements can be made more explicit. Take as an example the case $\kappa = 0.5$, and consider first the result of a free energy calculation with p fixed at ($p = 1$). Then there is only one $C_{n/1}$, which can be set to unity. In Fig. 2, the value of k_0 which minimizes the free energy for $p = 1$ is plotted as a function of d. From Eq. (21) the unit cell periodicity is $(\Delta x, \Delta y) = (k_0, 2\pi/k_0)$ and so $[x \leftrightarrow y]$ symmetry is manifest only when $k_0 = \sqrt{2\pi}$. For $d > d_2 \sim 2.7$ (region S_{III}) the minimum occurs when $D = 0$, so no solution with larger p can have a smaller free energy. For $d_2 < d < \infty$, k_0 never equals $\sqrt{2\pi}$, so all these solutions are $[x \leftrightarrow y]$ asymmetric (note that a solution with $k_0 \to 2\pi/k_0$ also always exists). Indeed, at $d = d_2$, the solution seemingly has $k_0 = 0$ and thus infinite asymmetry.

Now take the case $p = 8$, and evaluate D separately for the two cases $\hat{C}_{n/8}(0)$ and $\hat{C}_{n/8}(1)$ of Eqs. (34-35) with $k_0^2 = 2\pi/8$ and $\kappa = 0.5$. For $d > d_2 \sim 2.7$ the $\hat{C}(1)$ solution gives $D > 0$, while the $\hat{C}(0)$ solution gives $D < 0$. Both yield distributions which are $[x \leftrightarrow y]$ symmetric, as will any normalized linear combination $r_0\hat{C}(0) + r_1\hat{C}(1)$ with real coefficients r_0 and r_1. By continuity there is always some such combination which yields $D = 0$. This linear combination is thus an absolute minimum of the free energy, degenerate with the $p = 1$ case. By construction, however, it is $[x \leftrightarrow y]$ symmetric for all d. These two examples show that there is at least one asymmetric flux pattern and at least one $[x \leftrightarrow y]$ symmetric pattern for all d in S_{III}. The pattern chosen depends on the preparation of the system. Landau's model is thus an inexact guide to the flux patterns in this region.

6d. Region S_{II}

The most dramatic behavior of the flux patterns occurs in region S_{II}. In this region the quartic function D is always positive. For each p, the parameter set $\{C_{n/p}, k_0\}$ was determined numerically by a second-order gradient minimization scheme using all possible sums of the independent $\hat{C}_{n/p}(M)$ of Eq. (34) as starting points. Previously Lasher [9] evaluated this function using the $V_{n/p}$ of Eq. (38). The results presented here extend and refine his classic analysis.

The striking property of the solutions in region S_{II} is that the minimum free energy occurs for a series of patterns whose period p diverges at a length scale d_c like

$$p \sim |d - d_c|^{-0.5} \tag{56}$$

with

$$d_c(\kappa) \sim .7\, d_2(\kappa) \sim .2(\kappa_c - \kappa)^{-1.5} \tag{57}$$

where Eqs. (56-57) are determined numerically. The parameter set for $d \sim d_c$ is essentially $C_{n/p} = \hat{C}_{n/p}(0)$ [cf. Eq. (35)] and $k_0^2 = 2\pi/p$. Note in particular that the optimal parameter set is distinct from that assumed in Ref. [9].

For clarity, the case $\kappa = 0.5$ is considered in detail. Here

$$d_2(0.5) = 2.7$$

$$d_c(0.5) = 2.1$$

$$d_1(0.5) = 1.0 \tag{58}$$

The results of the numerical minimization are displayed in Fig. 4, where the minimum value of $2\kappa^2 D$ is plotted as a function of d labelled with various values of p. When $d < d_1$, the appropriate solution is the $p = 2$ triangular lattice described in section 6b. For $d_1 < d \tilde{<} 1.4$, the optimal solution has $p = 1$ with values of k_0 as given in Fig. 1. (Note in particular that since $k_0 \neq \sqrt{2\pi}$, the solutions are elongated).

For $1.4 < d < d_2$, the solutions are essentially those with $[x \leftrightarrow y]$ symmetric parameters $C_{n/p} = \hat{C}_{n/p}(0)$ [cf. Eq. (31)] with $k_0^2 = 2\pi/p$. The period diverges as per Eq. (56). (The optimal value of p is plotted in Fig. 5). For $d_c < d < d_2$ the period p decreases, and for $d \geq d_2$ the function $D \sim 0$, reverting to the analysis of Sec. (6c). This complicated series of $[x \leftrightarrow y]$ symmetric patterns is much more intricate than the behavior expected from Landau's model.

The actual flux patterns are evident from the plots displayed in Fig. 6. These plots are generated as follows. The function $\Delta H(x, y)$ of Eq. (41) is rescaled to lie between zero and one. Everywhere that this rescaled function exceeds 0.3 the plot is shaded. Note that the flux pattern is $[x \leftrightarrow y]$ symmetric for $d > 1.4$.

7. Conclusions

It is clear that the problem of magnetic flux penetration in type I superconductors is far more intricate than one might expect from Landau's textbook model. This is not too surprising, since he presented the model thirteen years before the Ginzburg-Landau equations, which allow a more sophisticated analysis to be performed.

These Ginzburg-Landau equations were here studied perturbatively, extending and refining the classic works on the subject. A number of interesting phenomena appear in this perturbative scheme. For instance, the flux penetration pattern through a large square plate of thickness d is periodic. The unit cell area is proportional to an integer p which equals the number of flux quanta per cell. The integer p apparently diverges like $p \sim |d - d_c|^{-1/2}$ near a critical length scale d_c. The patterns themselves are essentially $[x \leftrightarrow y]$ symmetric in this region, but do possess a degree of elongation.

For plates thicker than a somewhat larger scale $d_2 > d_c$, a large (perhaps infinite) number of flux patterns are degenerate in free energy. Thus, the pattern observed depends most strongly on sample history and preparation. Some of the degenerate patterns are $[x \leftrightarrow y]$ symmetric, but most are not. Thus, experimentally (or in a numerical simulation), the expected result is a complicated ensemble of domains of elongated structures. These domains can be aligned [8] by applying a magnetic field at an oblique angle, as per the Meissner effect. However, elongated patterns need not form in a perpendicular magnetic field, in contrast to Landau's prediction. (The situation is somewhat reminiscent of problems related to ergodicity in classical dynamics).

The Landau model is simplistic from another perspective as well. Recall that Landau predicted a flux penetration pattern which was periodic in one plane coordinate and independent of the other. Yet (as shown in Sec. 4), the squared superconducting wave function behaves much as if the x- and y- directions in the plate were Fourier conjugates, like position and momentum in a one-dimensional quantum-mechanics problem. This, if the squared wavefunction is independent of one plane coordinate, it will be sharply localized in the other, not periodic.

The present analysis is, of course, based upon a perturbative scheme. Although corrections are expected to be small [9], it is imperative to perform direct numerical simulations. Some progress in this direction has already been made [19,20].

Interesting questions as to the dynamics of this system (and its behavior when voltages or currents are present) were left unanswered. It may be useful to consider this system as a giant array of Josephson junctions with dynamic boundaries. The study of these systems might lead to interesting applications, such as a novel type of dark-matter detector for particle physics [14].

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Figure Captions

Figure 1
Division of the (κ, d) parameter plane into regions labelled Normal, S_I, S_{II}, and S_{III} and separated by solid lines. The dashed line is a plot of $d_c(\kappa)$ versus κ.

Figure 2
Value of k_0 which minimizes the free energy Eq. (40) for $p = 1$, plotted as a function of d for $\kappa = 0.5$.

Figure 3
Plot of $2\kappa^2 D$ versus d for $\kappa = 0.5$ for the two parameter sets $\hat{C}_{n/p}(0)$ and $\hat{C}_{n/p}(1)$.

Figure 4
Plot of minimum value of $2\kappa^2 D$ versus d for $\kappa = 0.5$ and various values of p.

Figure 5
Value of p for the parameter set which minimizes the free energy, plotted as a function of d for $\kappa = 0.5$.

Figure 6
Flux patterns for various values of d for $\kappa = 0.5$. Width of picture is 17 coherence lengths.

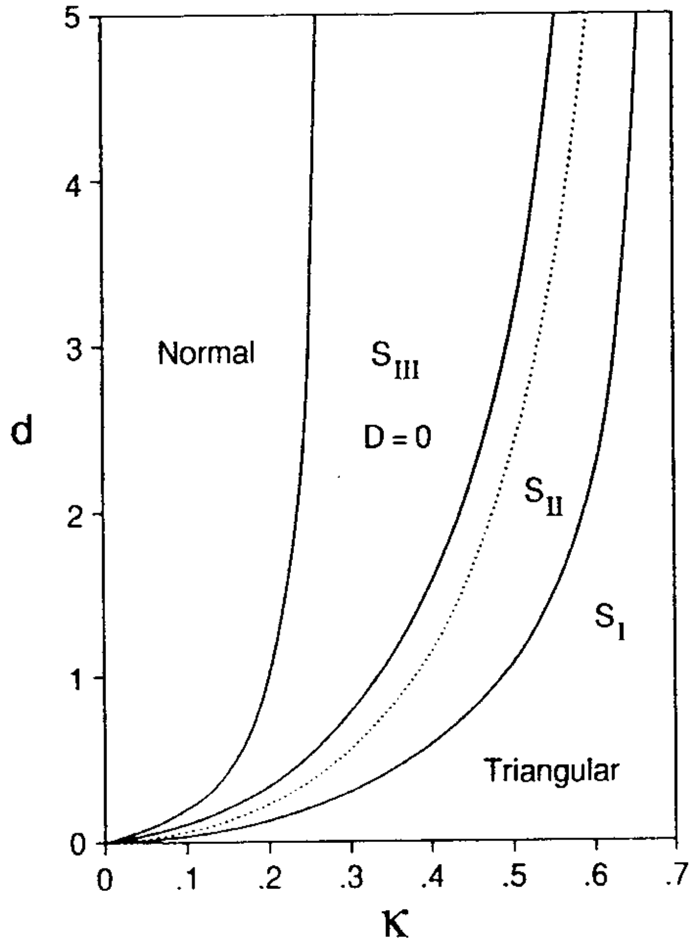

Figure 1

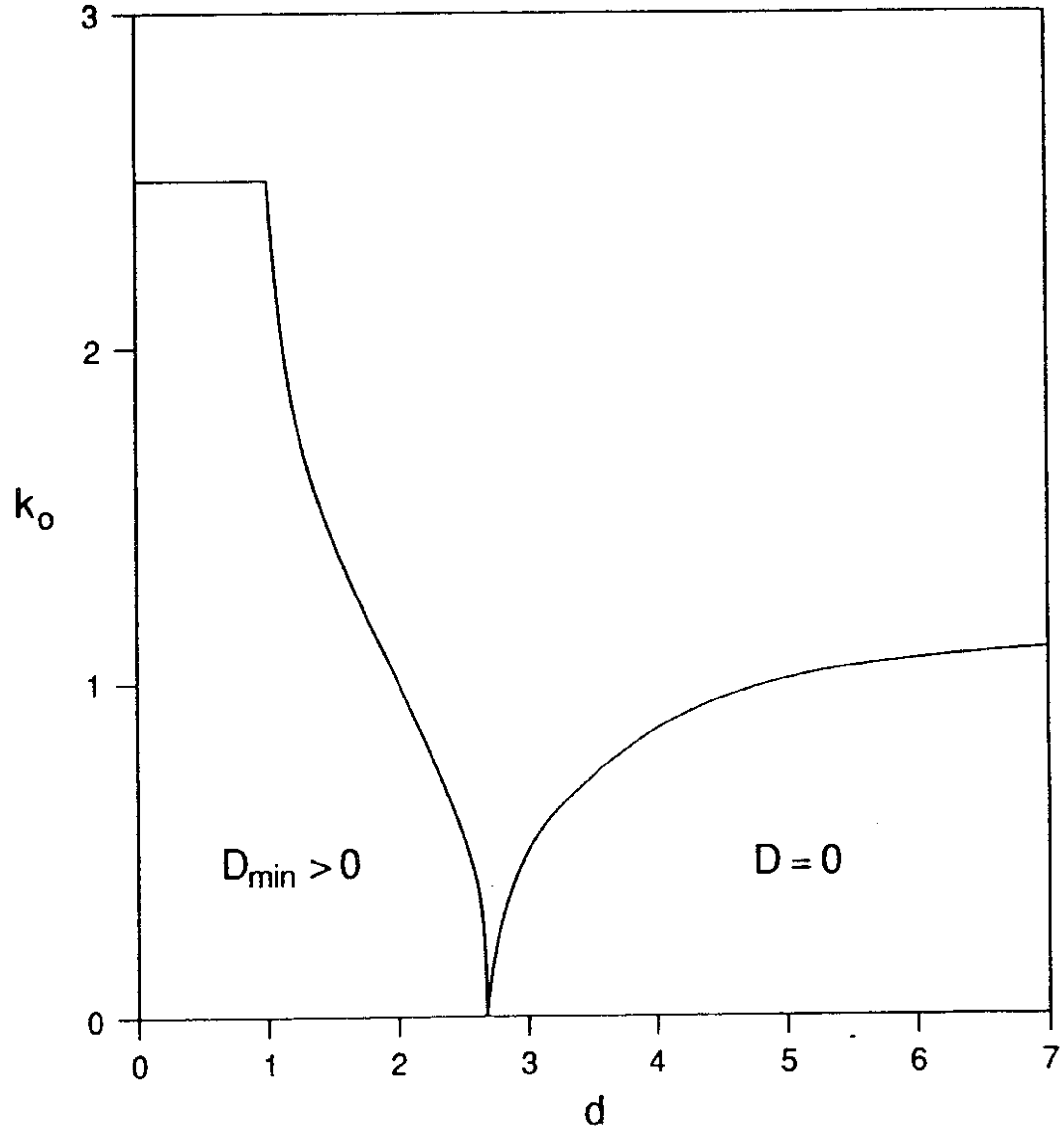

Figure 2

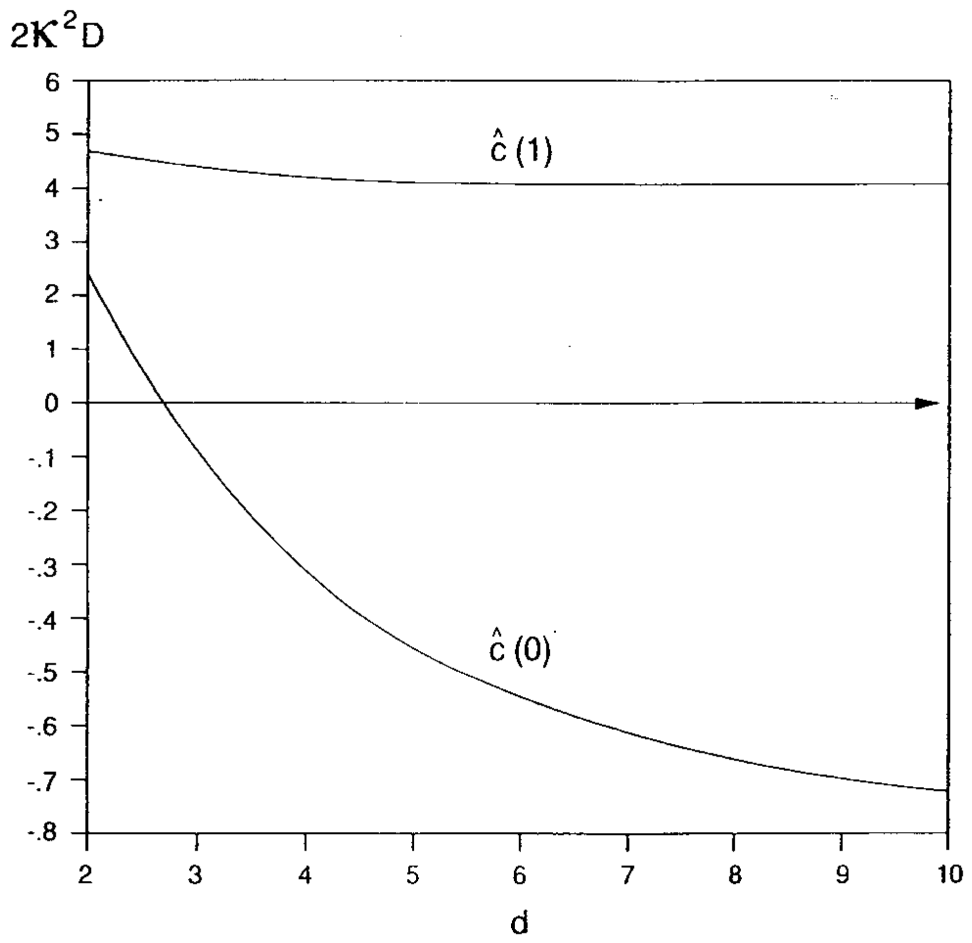

Figure 3

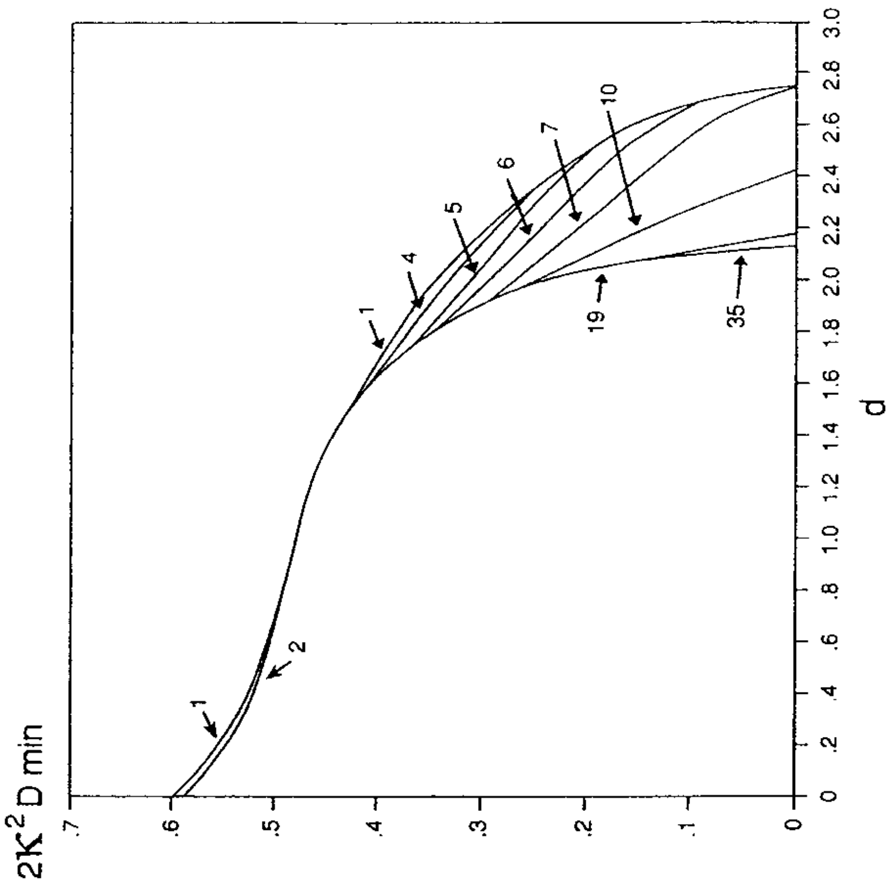

Figure 4

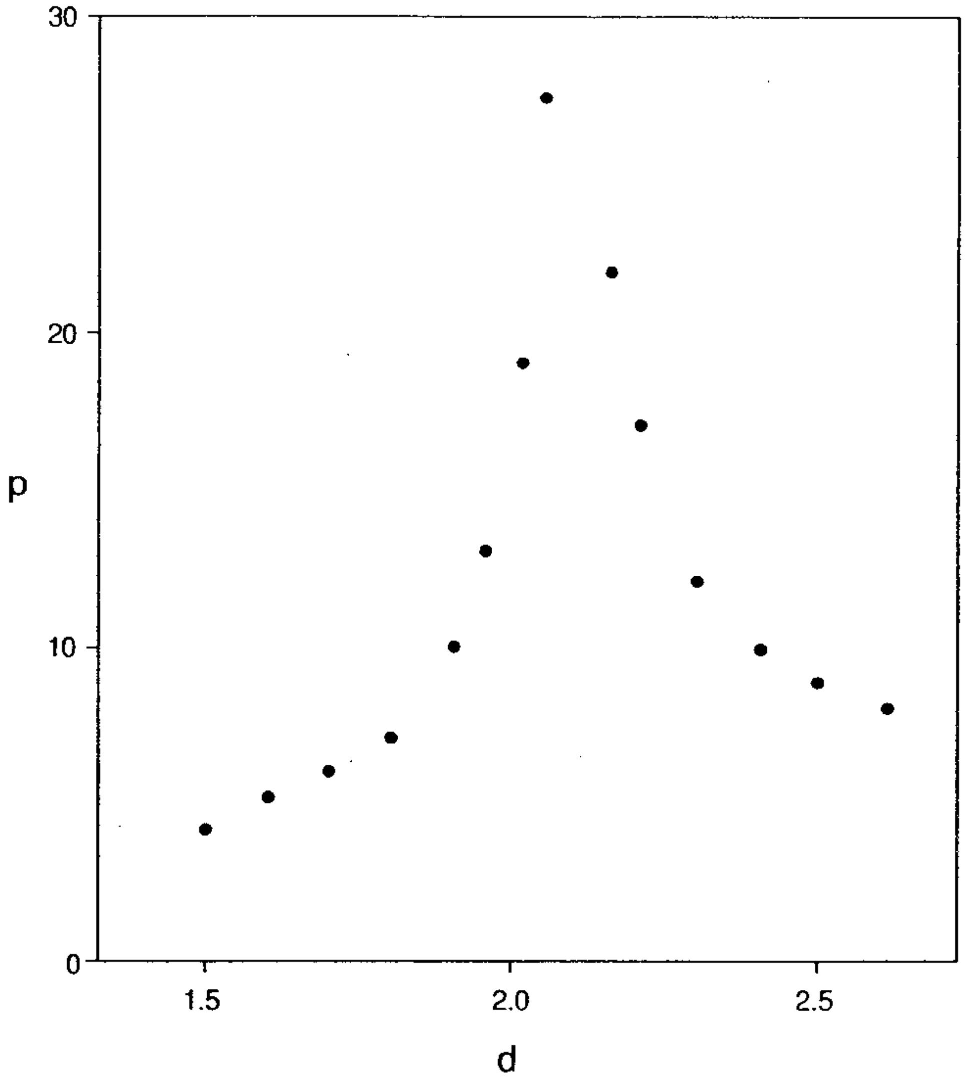

Figure 5

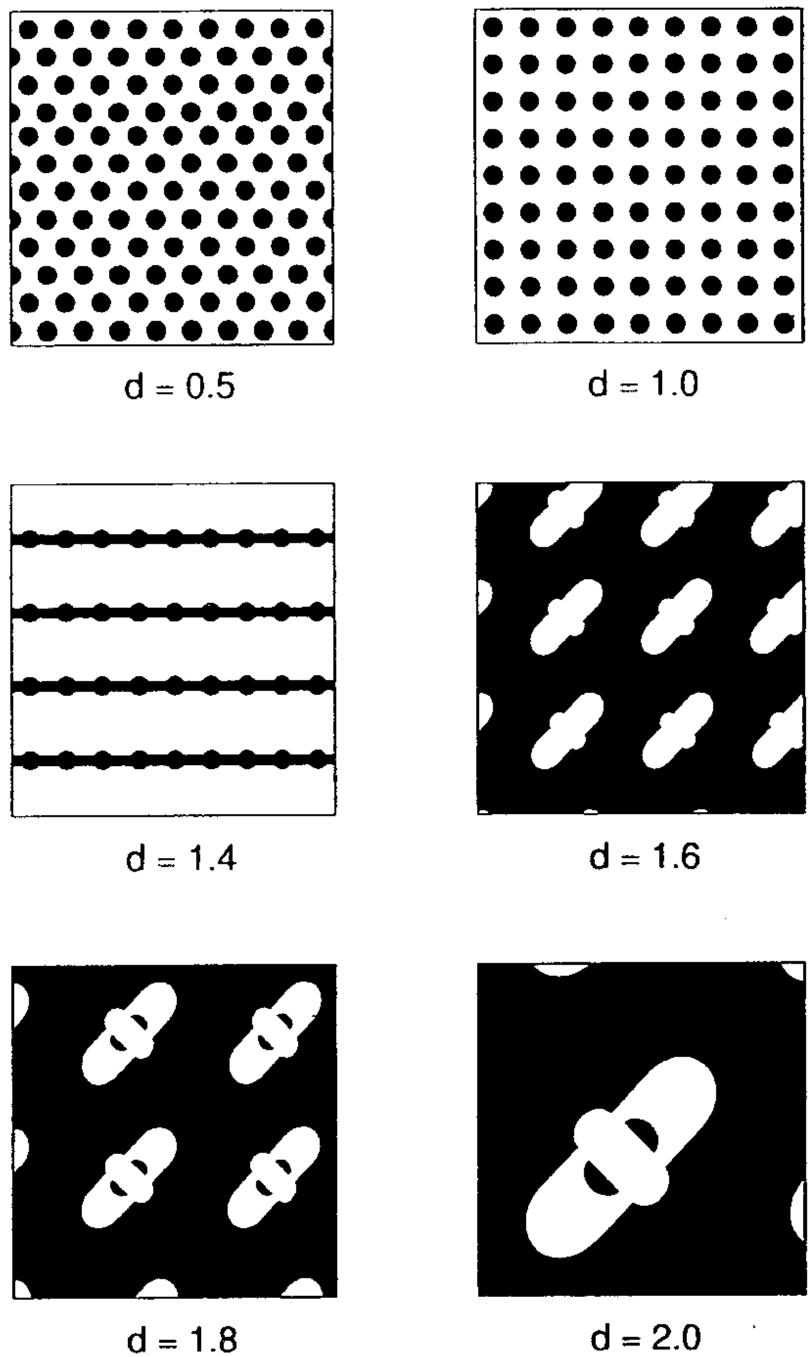

Figure 6