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Random Matrix Formulation of the Fractional Quantum Hall Effect

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Abstract

A random-matrix model is formulated and shown to yield directly the Laughlin theory of the fractional quantum Hall effect. Quasihole excitations which display fractional statistics can be easily included. This reformulation connects two large sets of results and and should lead to simplifications for both analytical and numerical studies.

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A number of peculiar physical phenomena occur in two-dimentional systems to which a perpendicular magnetic field has been applied[1]. One particularly exciting example is the fractional quantum Hall effect (FQHE), discovered by Tsui, Störmer, and Gossard in 1982[2]. The effect occurs in high-quality semiconducting heterostructures to which a strong magnetic field is applied at very low temperatures. It is observed that the Hall resistance develops plateaus at quantized values $h/\nu e^2$ where the filling factor ν is a rational fraction with odd denominator. The FQHE is quite distinct from the integer quantum Hall effect[3,4].

The generally accepted theory of the FQHE [4,5] is based upon a ground state wavefunction proposed by Laughlin[6]. Briefly summarized, the main idea is that the electrons in a semiconducting heterostructure can, at certain "magic" ratios ν of electron density to flux density, form an incompressible quantum fluid.

Each such state has precisely m flux quanta for each electron, yielding $\nu = 1/m$. In addition, there are vortex excitations which behave like a flux tube with a single quantum of flux. These vortices thus each exclude precisely $1/m$ of an electron. The composite has fractional charge and exhibits fractional statistics[7].

Interestingly enough, the original Laughlin model can be reformulated as a random matrix problem. Random matrix models have recently been employed as theories of strings and quantum gravity[9,10]. The basis for this connection is the observation[10] that the problem of diagram enumeration in random matrix models is equivalent to that of constructing surfaces by assembling together flat elementary polygons. This association makes it possible to construct easily quantities like correlation functions and to apply powerful combinatorial methods. It also permits more efficient computer algorithms to be formulated. The connection between random matrix models and those of the fractional quantum Hall effect is not too surprising, since the FQHE involves topological objects (magnetic flux quanta) in a plane. It has been noticed [6] previously that electrons in the FQHE behave much like a Coulomb gas in a harmonic potential, which in turn is a property of the eigenvalues of random matrices[11,12]. Similarities between the $S = \frac{1}{2}$ Heisenberg antiferromagnet and random matrices have also been observed[13]. In what follows the connection between random matrices and the (fractional statistics) Laughlin model of the FQHE will be made explicit by a direct construction.

A central feature of Laughlin's picture of the FQHE is a set of wavefunctions of N electrons in a magnetic field B, each located at plane coordinates $z_j = x_j + iy_j$, with $j = 0, \ldots, n-1$. These wavefunctions describe circular droplets of an incompressible quantum fluid. In the symmetric gauge $A = \frac{1}{2} B \times r$ these wavefunctions are (apart from a normalization constant):

$$|m> = \psi_m(z_0, \ldots, z_{N-1}) = (\Delta_N)^m \; exp \; \{-\frac{1}{4} \sum_{j=0}^{N-1} |z_j|^2\} \qquad (1)$$

where Δ_N is the Vandermonde determinant

$$\Delta_N = \prod_{j<k}^{N-1} (z_j - z_k) \qquad (2)$$

and $m = 2p+1$ is an <u>odd</u> integer [distances are measured in terms of the magnetic length $(\hbar \; c/eB)^{\frac{1}{2}}$]. The wavefunction $|m>$ is antisymmetric in each pair of coordinates. When one z is rotated about another by θ the wave function changes by exp $(im\theta)$, and so there are m flux quanta per electron. If all of the z are rotated by exp $(i\theta)$, the wavefunction changes by exp $[iN(N-1)m \; \theta/2]$.

To connect Laughlin's picture to the theory of random matrices, consider the electrons as objects of indefinite position described by a coordinate matrix Q_{jk} of rank N whose eigenvalues are the complex coordinates z_j. (Since Q has complex eigenvalues it is, in particular, not Hermitian). Choose the matrices Q according to a probability distribution $P(Q)\mu(dQ)$, where $\mu(dQ)$ is the linear measure

$$\mu(dQ) = \prod_{j,k=0}^{N-1} dQ_{jk} \; dQ^*_{jk} \qquad (3a)$$

and

$$P(Q) \equiv exp \; [-S_o(Q)] \qquad (3b)$$

$$S_o(Q) \equiv \frac{1}{2} \; Tr \; (Q^\dagger Q) \qquad (3c)$$

Thus Q varies over all complex matrices C. Except in regions of lower dimensionality which are irrelevant to the probability distribution, the eigenvalues are distinct. Define the matrix $E = \text{diag}\,(z_0, \ldots, z_{N-1})$ and let X be the $N \times N$ matrix whose columns are the eigenvectors of Q (i.e., essentially the electron wavefunctions). Then X is nonsingular and $Q = XEX^{-1}$. The distribution of P_c of eigenvalues of Q is then obtained by integrating over all complex matrices X, and is given by [14,12]:

$$P_c(z_0, \ldots, z_{N-1}) = K_c \; |\Delta_N|^2 \; exp \; \{-\frac{1}{2} \sum_{\ell=0}^{N-1} |z_\ell|^2\}$$

$$K_c^{-1} = (2\pi)^N \prod_{j=0}^{N-1} (j+1)! \qquad (4)$$

Of course $P_c = <1|1>$ is just the square of the $m = 1$ Laughlin wavefunction. Note in particular that the Fermi statistics of electrons is automatically accounted for by the well-known "repulsion" of eigenvalues of a random matrix[12].

More is needed, however, to complete the full Laughlin picture for general m. The major obstacle is the above-mentioned eigenvalue repulsion, which has the practical consequence in the random matrix model Eq. (3) that powers of z_j greater than $(N-1)$ are not allowed. In other words, each coordinate taken separately has at most angular momentum $(N-1)$. It is necessary to add up to an additional $2p(N-1)$ units of angular momentum to each of the fermionic degrees of freedom in the system $[2p+1 = m$, as in Eq. (1)].

Define a set of matrices F_{IJ}; $I, J = 0, \ldots, N-1$

$$F_{I,J} = Tr \; (Q^{I+J}) \qquad (5a)$$

and their Fourier transform

$$\tilde{F}_{K,L} = \frac{1}{N} \sum_{I,J=0}^{N-1} exp[2\pi(IK + JL)/N] \; F_{IJ} \qquad (5b)$$

Both F and $\tilde{F}$ depend only on the eigenvalues of Q and are totally symmetric in them. Thus under a rotation $z \rightarrow z \; exp \; (i\theta) \; by \; \theta = 2\pi/N$

$$F_{I,J} \rightarrow exp \; [2\pi i \; (I+J)/N] \; F_{I,J} \qquad (6a)$$

$$\tilde{F}_{K,L} \rightarrow \tilde{F}_{K+1,L+1} \qquad (6b)$$

Thus if a term

$$S_F(p) = -\sum_{I,J,K=0}^{N-1} \bar{\psi}_I (\tilde{F}^p)^{\dagger}_{IJ} (\tilde{F}^p)_{JK} \psi_K \tag{7}$$

is added to the action S_0, a rotation by $2\pi/Np$ reduce to a cyclic permutation of the $\{\bar{\psi}\}$ and $\{\psi\}$, thus accounting for the Pauli principle. These Grassmann variables are taken as spinless as it is assumed that the magnetic field freezes out the spin degrees of freedom. The interaction term Eq. (7) thus couples up to max $|(I+J)p| = 2p(N-1)$ units of angular momentum to fermionic degrees of freedom, as required.

When the integration over the 2N Grassmann coordinates $\{\bar{\psi}, \psi\}$ is performed, the result is

$$\int D\bar{\psi}\, D\psi\, exp\,[-S_F(p)] = |Det\tilde{F}|^p \tag{8}$$

The right-hand side of Eq. (8) can be expressed simply as follows:

$$\begin{aligned} |Det\tilde{F}| &= |DetF| \\ &= |Det_{IJ} Tr(Q^{I+J})| \\ &= |Det_{IJ} Tr(E^{I+J})| \\ &= |Det_{IJ} \textstyle\sum_{j=0}^{N-1} z_j^{I+J}| \\ &= |(Det_{iI} z_i^I)(Det_{jJ} z_j^J)| \\ &= |(\Delta_N)^2| \end{aligned} \tag{9}$$

Thus the eigenvalues of complex matrices Q chosen with weight $\exp\,[-S_o - S_F\,(p)]\ \mu\ (dQ) D\bar{\psi}\ D\psi$ are distributed according to

$$|\Delta_N|^{4p+2}\, exp\,\{-\frac{1}{2}\sum_j |z_j|^2\} = <2p+1|2p+1> \tag{10}$$

which is perforce the absolute square of the Laughlin wavefunction for $m = 2p+1$. Note that the construction Eq. (6) with spinless fermions ensures that $m = 2p+1$ is an odd integer. Had fermions of spin s been used, the result would have been $m = 2p(2s+1) = 1$. The allowed values of m thus depend upon the number of the fermionic degrees of freedom to which the magnetic field couples. This restriction may have implications for the allowed values of m in the FQHE.

It is interesting to interpret the above discussion in terms of random matrix surface models. The fermionic action $S_F(p)$ generates diagrams with vertices which have up to $2p(N-1)$ legs. Each leg corresponds to a unit of angular momentum. Thus in some sense the random matrix approach is equivalent to counting the number of ways a surface can be covered by various polygons, with each covering weighted by the Pauli principle manifest in the Grassmann algebra. This might be a reflection of the underlying "string theoretic" nature of the interaction of magnetic vortices and electrons.

In order to complete the analogy with the Laughlin model, the effects of quasihole excitations are now considered. In the original droplet wavefunctions $|m>$, there are m flux quanta per electron. Quasiholes are two-dimensional bubbles in the fluid which act like flux tubes with a single unit of flux. Each quasihole removes $1/m$ of an electron charge, and thus displays "fractional statistics"[7].

The effect of a quasihole excitation at complex coordinate η can be included in $S_F(p)$ by replacing the ψ in Eq. (7) with

$$\psi_I \to \sum_{J=0}^{N-1} (\eta\, \delta_{IJ} - Q_{IJ})\, \psi_J \tag{11}$$

and similarly for the $\{\bar{\psi}\}$. This leads to an additional factor of

$$|Det\ (\eta - Q)|^2 = |\prod_{j=0}^{N-1} (\eta - z_j)|^2 \tag{12}$$

in the overall probability distribution, just as in the Laughlin model. Of course, it is possible to continue the hierarchy[5] by taking the η to be dynamic variables equal to the eigenvalues of a new matrix.

In conclusion, therefore, a random matrix version of the Laughlin model of the FQHE has been formulated, thus demonstrating that fractional statistics can occur in random matrix models. Such a formulation may lead to new insights and connections with existing computations. For instance, the random-matrix formulation is very similar to a lattice field theory, for which numerical techniques are readily available [15].

References

1. D.J.E. Callaway, *Nucl.Phys.* **B344** (1990) 627; *Nucl. Phys. B (Proc. Suppl.)* **17** (1990) 720.

2. D.C. Tsui, H.L. Störmer, and A.C. Gossard, *Phys. Rev. Lett.* **48** (1982) 1559.

3. K. von Klitzing, G. Dorda, and M. Pepper, *Phys. Rev. Lett.* **45** (1980) 494.

4. *"The Quantum Hall Effect,"* R.E. Prange and S.M. Girvin, editors (Springer-Verlag, Berlin, 1988).

5. F.D.M. Haldane, *Phys. Rev. Lett.* **51** (1983) 605; B.I. Halperin, *Phys. Rev. Lett.* **52** (1983) 1583; **52** (1984) 2390 (E); R.B. Laughlin, *Surf. Sci.* **41** (1984) 11.

6. R.B. Laughlin, *Phys. Rev. Lett.* **50** (1983) 1395; and in Ref. 4.

7. D. Arovas, J.R. Schrieffer, and F. Wilczek, *Phys. Rev. Lett.* **53** (1984) 722.

8. D.J.E. Callaway, *Rockefeller preprint $RU90/B_1/2$*, to appear in *Physical Review B*.

9. D.J. Gross and A. Migdal, *Phys. Rev. Lett.* **64** (1990) 127.

10. V. Kazakov, *Phys. Rev. Lett.* **150B** (1985) 282; F. David, *Nucl. Phys.* **257B** (1985) 45; V. Kazakov, I. Kostov, and A. Migdal, *Phys. Lett.* **157B** (1985) 295; D. Weingarten, *Phys. Lett.* **90B** (1980) 280.

11. F. Dyson, *J. Math. Phys.* **3** (1962) 1191.

12. M. Mehta, *"Random Matrices and the Statistical Theory of Energy Levels,"* (Academic, New York, 1967).

13. B.S. Shastry, *Phys. Rev. Lett.* **60** (1988) 639; F.D.M. Haldane, *Phys. Rev. Lett.* **60** (1988) 635.

14. J. Ginibre, *J. Math. Phys.* **6** (1965) 440.

15. D.J.E. Callaway and A. Rahman, *Phys. Rev. Lett.* **49** (1982) 613; *Phys. Rev.* **D28** (1983) 1506; D.J.E. Callaway, *Contemp. Phys.* **26** (1985) 23; ibid. 95.