Summer 2025 – Understanding Analysis Solutions

David J Chen

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1 The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

No exercises in this section.

1.2 Some Preliminaries

1.2.1 **Exercise 1**

(a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?
(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

(a) Assume for sake of contradiction (AFSOC) that $\sqrt{3} \in \mathbb{Q}$. This implies that $\sqrt{3} = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, and $\gcd(p, q) = 1$.

Therefore, $p^2 = 3q^2$, which means that 3 divides p^2.

Since 3 is prime, p must be divisible by 3. Therefore for some $k \in \mathbb{Z}$,

$$p = 3k \Rightarrow 9k^2 = 3q^2 \Rightarrow 3k^2 = q^2.$$

This implies that q^2 and thus q is also divisible by 3, which is a contradiction.

A similar proof does not quite work for $\sqrt{6}$ and needs to be adjusted, since 6 is not prime and thus we cannot directly say that 6 divides p^2 implies 6 divides p.

(b) It is exactly the step where we try to show that 4 divides q^2 implies that 4 divides q. In fact, if we have just that 2 (and not 4) divides q, then clearly 4 still divides q^2.

1.2.2 **Exercise 2**

Show that there is no rational number r satisfying $2^r = 3$.

AFSOC that there does exist $r = \frac{p}{q}$, with coprime $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Then $2^r = 2^{p/q} = 3$, which implies that $2^p = 3^q$. This is false.

1.2.3 **Exercise 3**

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \ldots$ are all sets containing an infinite number of elements, then the interesection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.
(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \ldots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.
(c) $A \cap (B \cup C) = (A \cap B) \cup C$.
(d) $A \cap (B \cap C) = (A \cap B) \cap C$.
(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

TODO: skipped

1.2.4 **Exercise 4**

Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

Assume we have infinite primes. Since they are a subset of $\mathbb{N}$, they are enumerable ($p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$).

Also assume we have unique prime decomposition.

Now let

$$A_i = \{n \in \mathbb{N} \mid p_i \text{ is the smallest prime in the decomposition of } n\},$$

with the additional modification that A_1 includes 1.

They are all disjoint, since there can only be one smallest prime factor of each number.

Their union forms the natural numbers, since every natural number n has a unique finite prime factor decomposition, and by the the fact that every non-empty subset of the natural numbers will have a smallest element, n must be an element of some A_i.

Clearly, every set is also infinite, since we can consider that each A_i contains the powers of p_i.

1.2.5 (De Morgan's Laws) **Exercise 5**

Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cup B)^{\mathsf{C}}$, explain why $x \in A^{\mathsf{C}} \cup B^{\mathsf{C}}$. This shows that $(A \cap B)^{\mathsf{C}} \subseteq A^{\mathsf{C}} \cup B^{\mathsf{C}}$.

(b) Prove the reverse inclusion $(A \cap B)^{\mathsf{C}} \supseteq A^{\mathsf{C}} \cup B^{\mathsf{C}}$, and conclude that $(A \cap B)^{\mathsf{C}} = A^{\mathsf{C}} \cup B^{\mathsf{C}}$.

(c) Show $(A \cup B)^{\mathsf{C}} = A^{\mathsf{C}} \cup B^{\mathsf{C}}$ by demonstrating inclusion both ways.

TODO: skipped

1.2.6 **Exercise 6**

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.

(d) Prove $||a| - |b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

(a) **TODO: part skipped**

(b) **TODO: part skipped**

(c) **TODO: part skipped**

(d) Using the "unremarkable identity", for any a and b,

$$\begin{aligned} |a| &= |a-b+b| \\ &\leq |a-b| + |b|. \end{aligned}$$

So first we have $|a| - |b| \le |a - b|$. Next, we proceed the same exact way using $|b|$, and we get that $|b| - |a| \le |b - a|$.

Since $|a - b| = |b - a|$, we can combine the above two facts and get that

$$||a| - |b|| \le |a - b|.$$

1.2.7 **Exercise 7**

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R} : 0 \le x \le 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

(a) $A \cap B = [1, 2]$. $f(A) = [0, 4]$, and $f(B) = [1, 16]$. So therefore, $f(A) \cap f(B) = [1, 4]$.

$f(A \cap B) = [1, 4]$ as well. So equality holds.

$A \cup B = [0, 4]$, so $f(A \cup B) = [0, 16] = f(A) \cup f(B)$.

Therefore equality holds in both cases.

(b) Let $A = \{1\}$, and $B = \{-1\}$.

Then $A \cap B = \emptyset$, but $f(A) = \{1\} = f(B)$, so $f(A) \cap f(B) = \{1\} \neq \emptyset$.

(c) For arbitrary $y \in g(A \cap B)$, we have that $y = g(x)$, where $x \in A \cap B$.

Therefore, $x \in A$ and $x \in B$, which implies that $g(x) \in g(A)$ and $g(x) \in g(B)$.

This further implies that $y = g(x) \in g(A) \cap g(B)$.

Thus we have that $g(A \cap B) \subseteq g(A) \cap g(B)$.

This doesn't work the other way around, since we could have some $y = g(x) = g(z)$, where $x \neq z$, and $x \in A$ and $z \in B$, and neither exists in the other set.

(d) My conjecture is that

$$g(A \cap B) = g(A) \cup g(B).$$

To show this, I first prove that $g(A \cup B) \subseteq g(A) \cup g(B)$, then the other way around.

$\boldsymbol{g(A \cup B) \subseteq g(A) \cup g(B)}$**:**

For arbitrary $y \in g(A \cup B)$, we have that $y = g(x)$ such that x in A or B. In either case, it must be such that y is in $g(A)$ or $g(B)$ and thus be in $g(A) \cup g(B)$.

$\boldsymbol{g(A) \cup g(B) \subseteq g(A \cup B)}$**:**

If $y \in g(A)$, then we have that $y = g(x)$ where $x \in A$, and therefore $x \in A \cup B \implies y = g(x) \in g(A \cup B)$. Same for $y \in g(B)$.

Thus we have proved both directions and shown set equality.

1.2.8 **Exercise 8**

Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1–1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:
(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1–1 but not onto.
(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1–1.
(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1–1 and onto.

TODO: skipped

1.2.9 **Exercise 9**

Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.
(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?
(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(a) **TODO: part skipped**
(b) Let $x \in g^{-1}(A \cap B)$. This implies that $g(x) \in A \cap B$, which implies that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$.

From this we can conclude that $x \in g^{-1}(A) \cap g^{-1}(B)$.

Going backwards, we see that if $x \in g^{-1}(A) \cap g^{-1}(B)$, then it must be the case that $g(x) \in A$ and $g(x) \in B$, which leads us to conclude that $x \in g^{-1}(A \cap B)$.

For union, we have if $x \in g^{-1}(A \cup B)$, then $g(x) \in A \cup B$. From the two cases, we will have that either $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$, which lets us conclude that $x \in g^{-1}(A) \cup g^{-1}(B)$.

Backwards, we have that either $g(x) \in A$ or $g(x) \in B$ depending on the cases, so therefore $g(x) \in A \cup B$ and thus $x \in g^{-1}(A \cup B)$.

1.2.10 **Exercise 10**

Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:
(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.
(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.
(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

1.2.11 **Exercise 11**

Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that…" in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + 1/n < b$.
(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.
(c) Between every two distinct real numbers there is a rational number.

TODO: skipped

1.2.12 **Exercise 12**

Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.
(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

TODO: skipped

1.2.13 **Exercise 13**

For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^{\mathsf{C}} = A_1^{\mathsf{C}} \cap A_2^{\mathsf{C}} \cap \cdots \cap A_n^{\mathsf{C}}$$

for any finite $n \in \mathbb{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathsf{C}} = \bigcap_{i=1}^{\infty} A_i^{\mathsf{C}},$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

(a) **TODO: part skipped**

(b) Let $B_i = \mathbb{N} \setminus \{i\}$. Any finite intersection will still have infinitely many elements, but the entire infinite intersection cannot have any elements.

(c) Let $x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathsf{C}}$. Then we know that for all i, $x \neq A_i$. (Otherwise, we would have that $x \in \bigcup_{i=1}^{\infty} A_i$.)

Therefore, for all i, $x \in A_i^{\mathsf{C}}$, which lets us conclude that $x \in \bigcap_{i=1}^{\infty} A_i^{\mathsf{C}}$.

For the other direction, we just proceed from each step backwards and see that it works fine.

1.3 The Axiom of Completeness

1.3.1 **Exercise 14**

(a) Write a formal definition in the style of Definition 1.3.2 or the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

TODO: skipped

1.3.2 **Exercise 15**

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

(a) Let $B = \{1\}$, hmm...

(b) This cannot be possible. Since there are finite elements, there is necessarily a maximum and minimum, so the set must contain both of them.

(c) Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. The supremum is 1, which is contained. The infimum is clearly 0, which is not contained.

1.3.3 **Exercise 16**

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

(a) First, we know that the supremum of B must exist, since it is bounded above by any element of A.

So let $b' = \sup B$, and $a' = \inf A$.

AFSOC that there exists some $a \in A$ such that $a < b'$. Let $\epsilon = b' - a > 0$, and then we know that there must be some $b \in B$ such that $b > b' - \epsilon = a$, so we have $b > a$. This is a contradiction, since we assumed that b is a lower bound for all elements in A.

Therefore, we have shown that b' is a lower bound for A, and since it is a supremum of B, it must be the greatest such lower bound. This is exactly the definition of infimum of A.

(b) For any set bounded from below, we can take the set of all lower bounds, and use part (a) to show that the greatest lower bound is the smallest upper bound of the set of lower bounds.

1.3.4 **Exercise 17**

Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

(a) $\sup(A_1 \cup A_2) = \max(\sup A_1, \sup A_2)$.

Extended to $n \in \mathbb{N}$, we have

$$\sup\left(\bigcup_{k=1}^{n} A_k\right) = \max_{k\in[n]}(\sup A_k).$$

(b) This does not extend to infinite max, since it may be possible for the infinite max to exist. For example, if we have each A_k simply consist of the natural number k, then there is no supremum and no max.

1.3.5 **Exercise 18**

As in Example 1.3.7, let $A \in \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

TODO: skipped

1.3.6 **Exercise 19**

Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an pper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

(a) Let $c \in A + B$. Then $c = a + b$, with $a \in A$, and $b \in B$.

Now, we have that $a \leq s$ and $b \leq t$, so therefore, $c \leq s + t$.

(b) For all $b \in B$, we have that $a + b \leq u$. Thus, $u - a \geq b$, so $u - a$ is an upper bound for B.

Since t is the least upper bound for B, we now have that $t \leq u - a$.

(c) Let u be an arbitrary upper bound for $A + B$. By (b), we have that for all $a \in A$, $t \leq u - a$.

Therefore we also have that $a \leq u - t$, showing that $u - t$ is an upper bound on A. Since s is the least upper bound on A, we have $s \leq u - t$, and thus have $s + t \leq u$. This shows that $s + t$ must be the least upper bound and therefore is the supremum of $A + B$.

(d) Choose arbitrary $\epsilon > 0$. For $\frac{\epsilon}{2}$, there must exist $a \in A$ and $b \in B$ such that $a \geq s - \frac{\epsilon}{2}$ and $b \geq t - \frac{\epsilon}{2}$.

Therefore, $s + t - \epsilon \leq a + b$ for some $a + b$ in $A + B$.

But from part (a), we know that $s + t$ itself is an upper bound of $A + B$. Therefore, it must be that $s + t$ is the supremum of $A + B$.

1.3.7 **Exercise 20**

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup(A)$.

TODO: skipped

1.3.8 **Exercise 21**

Compute, without proofs, the suprema and infima (if they exist) of the following sets:
(a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$.
(b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.
(c) $\{n/(3n+1) : n \in \mathbb{N}\}$.
(d) $\{m/(m+n) : m, n \in \mathbb{N}\}$.

(a) supremum is 1, infimum is 0.
(b) supremum is 1, infimum is -1.
(c) supremum is $\frac{1}{3}$, infimum is $\frac{1}{4}$.
(d) supremum is 1, infimum is 0.

1.3.9 **Exercise 22**

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.
(b) Give an example to show that this is not always the case if we only assume $\sup A \le \sup B$.

TODO: skipped

1.3.10 (Cut Property) **Exercise 23**

The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \le c$ whenever $x \in A$ and $x \ge c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.
(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.
(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbb{R}$ is replaced by $\mathbb{Q}$.

(a) A is clearly bounded by above, just pick any element in B.

Using the Axiom of Completeness, there must exist some $c = \sup A$. By definition, $c \ge x$ for all $x \in A$.

By 1.3.3, c is the infimum of B, so for all $x \in B$, we have that $b \ge c$.

(b) Assume the Cut Property.

Let B be the set of upper bounds of E. Now let $A = \mathbb{R} \setminus B$.

Now note that for any $a \in A$ and $b \in B$, we have that $a < b$. This is because if we assume otherwise, then we see that a is an upper bound for E and should have been an element of B in the first place.

Now, from the Cut Property, we have that there exists a c such that $a \leq c \leq b$.

Now, I show that c is an upper bound for e.

AFSOC that there exists some $e \in E$ such that $e > c$. Examine $\epsilon = e - c > 0$.

Since $\frac{\epsilon}{2} + c > c$, it must be a member of B and thus be an upper bound for E.

However, we also have that $\frac{\epsilon}{2} + c < e$, so it cannot be an upper bound for E. Contradiction!

Thus, since c is an upper bound and is less than or equal to all upper bounds of E, we have that c exists and is the supremum of E.

(c) Let A be $\{x \in \mathbb{Q} : x^2 \leq 2\}$, and $B := \{x \in \mathbb{Q} : x^2 > 2\}$.

They clearly are disjoint sets that form the rationals.

But we have proven that there cannot be such a $c \in \mathbb{Q}$ such that it exists in the middle of these two sets.

1.3.11 **Exercise 24**

Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

(a) This is true. AFSOC false. Then there must exist some $a \in A$ such that it is greater than $\sup B$ but less than or equal to $\sup A$.

But since $A \subseteq B$, it must be an element of B as well, which leads us to a contradiction since we assumed it would be greater than $\sup B$.

(b) True. **TODO: skipped**

(c) False. **TODO: skipped**

1.4 Consequences of Completeness

1.4.1 **Exercise 25**

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

(a) Let $a = \frac{m}{n}$, $b = \frac{p}{q}$. Then $mp \in \mathbb{Z}$, and $nq \in \mathbb{N}$.

Therefore $ab = \frac{mp}{nq} \in \mathbb{Q}$.

$mq \in \mathbb{Z}$, and $np \in \mathbb{Z}$, so therefore $a + b = \frac{mq+np}{nq} \in \mathbb{Q}$.

(b) **TODO: skipped**

(c) **TODO: skipped**

1.4.2 **Exercise 26**

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

AFSOC $s > \sup A$.

Then there must exist some $n \in \mathbb{N}$ such that $\frac{1}{n} < s - \sup A$.

So then, $s - \frac{1}{n} > \sup A$, which is a contradiction with the condition that $s - \frac{1}{n}$ cannot be an upper bound.

The other direction $s < \sup A$ works the same way.

Therefore it must be that $s = \sup A$.

1.4.3 **Exercise 27**

Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

AFSOC there exists some $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$.

It must be that $x > 0$, and therefore, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

However, x would then be excluded from the interval $\left(0, \frac{1}{n}\right)$, which is a contradiction.

1.4.4 **Exercise 28**

Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

b is clearly an upper bound.

Let $\epsilon > 0$, and also choose $\epsilon < b - a$.

There must exist $r \in \mathbb{Q}$ such that $b - \varepsilon < r < b$. Therefore $r \in T$, which shows that $\sup T = b$.

1.4.5 **Exercise 29**

Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real $a - \sqrt{2}$ and $b - \sqrt{2}$.

First, choose a rational y such that $a - \sqrt{2} < y < b - \sqrt{2}$. Next, we see clearly that $y + \sqrt{2} \in \mathbb{I}$.

Now, we can see that $a < y + \sqrt{2} < b$.

1.4.6 **Exercise 30**

Recall that a set B is *dense* in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.
(b) The set of all rational numbers p/q with q a power of 2.
(c) The set of all rational numbers p/q with $10|p| \geq q$.

TODO: skipped

1.4.7 **Exercise 31**

Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Claim: If we choose $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$, then we can show that $\alpha - \frac{1}{n_0}$ is still an upper bound for $T = \{t \in \mathbb{R} : t^2 < 2\}$.

Proof: Consider the following:

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

We want to choose an n such that $\alpha^2 - \frac{2\alpha}{n} > 2$. Note that if we choose n_0 as in the claim, we get that the inequality holds.

Thus, $\alpha - \frac{1}{n_0}$ is actually an upper bound on T that is smaller than α, contradicting the assumption that $\alpha = \sup T$. ∎

1.4.8 **Exercise 32**

Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.
(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.
(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.)
(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

(a) Let $A = \{q \in \mathbb{Q} \mid q < 0\}$, and $B = \{r \in \mathbb{R} \setminus \mathbb{Q} \mid r < 0\}$.
(b) Let $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then the only element in the intersection can be 0.
(c) Let $L_n = [n, \infty)$. This cannot have any element.
(d) This is **impossible**, and we can prove it using the nested interval property.

Proof: First, we use the fact that a non-empty intersection of two closed, bounded intervals must itself be a closed bounded interval.

Now, let

$$I_n' = \bigcap_{m=1}^{n} I_m$$

define a new sequence of closed bounded intervals, which are nested by construction.

By the assumption that every finite intersection is non-empty, every I_n' must also be a non-empty closed, bounded, interval.

It is also important to note that the finite and infinite intersection of this sequence is exactly equal to the finite and infinite intersection of the original sequence.

Now, we can apply the NIP to deduce that the infinite intersection must be non-empty, which disproves the original claim. ■

1.5 Cardinality

1.5.1 **Exercise 33**

Finish the following proof for Theorem 1.5.7.

Assume B is a countable set. Thus, there exists $f : \mathbb{N} \to B$, which is 1–1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1–1 function g from $\mathbb{N}$ onto A.

For $i > 1$, let $n_i = \min\{n \in \mathbb{N} : f(n) \in A, n > n_{i-1}\}$. This must exist since A is an infinite set, thus there cannot be an upper bound on n such that $f(n) \in A$.

Now, just let $g(i) = f(n_i)$. This is an injective function, since each n_i is distinct and f is an injective function.

1.5.2 **Exercise 34**

Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbb{Q}$ is uncountable:

Assume, for contradiction, that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbb{Q}$ must therefore be uncountable.

NIP is not true in general over the rationals, since the element could be an irrational.

1.5.3 **Exercise 35**

Use the following outline to supply proofs for the statements in Theorem 1.5.8.

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this s that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Now, explain how the more general statement in (i) follows.

(b) Explain why induction *cannot* be used to prove part (ii) of Theorem 1.5.8 from part (i).

(c) Show how arranging $\mathbb{N}$ into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof of Theorem 1.5.8 (ii).

(a) Select $B_2 = A_2 \setminus A_1$. For enumeration purposes, alternate elements from A_1 and B_2. If B_2 is finite, after all elements are enumerated, continue enumerating from A_1.

Now, continue this enumeration strategy all the way to A_m by doing this same alternating enumeration strategy for $\bigcup_{i=1}^{n} A_i$ and A_{n+1}.

(b) Induction may not work, since induction only makes it hold for finite unions, not infinite unions.

(c) If we arrange the countable sets in rows, then we can visit by zig-zag pattern.

1.5.4 **Exercise 36**

(a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1–1, onto functions, but it is not really necesssary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1–1 onto function between the two sets.

(a) We can map (a, b) to $(0, 1)$.

Alternatively, for a direct bijection, consider the function

$$f(x) = \tan\left(\left(\frac{x-a}{b-a} - \frac{1}{2}\right)\pi\right).$$

(b) Choose the function $\ln(x - a)$.

(c) Let $f : [0, 1) \to (0, 1)$ be defined the following way:

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ x & \text{else.} \end{cases}$$

1.5.5 **Exercise 37**

(a) Why is $A \sim A$ for every set A?
(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.
(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

(a) Use identity bijection.
(b) Bijection has bijection inverse.
(c) Let $f : A \to B$ be a bijection, and $g : B \to C$ be another. Then $g \circ f$ is also a bijection.

Proof:

Injective: If $x \neq y$ for $x, y \in A$ then $f(x) \neq f(y)$. Since g is also injective we also have $g(f(x)) \neq g(f(y))$.

Surjective: For all $c \in C$, there exists $b \in B$ such that $g(b) = c$. Since f is also surjective, there must exist $a \in A$ such that $f(a) = b$. Thus, for all $c \in C$, there exists $a \in A$ such that $g(f(a)) = c$. ■

1.5.6 **Exercise 38**

(a) Give an example of a countable collection of disjoint open intervals.
(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

(a) Let $I_n = (n, n+1)$. Clearly disjoint, and open intervals.
(b) This cannot exist. If it did, then consider by density of rational numbers that there would exist a distinct rational in each interval, thus implying there would be uncountable rationals.

1.5.7 **Exercise 39**

Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.
(a) Find a 1–1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy.)
(b) Use the fact that every real number has a decimal expansion to produce a 1–1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999….)

The Schröder–Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

(a) Consider $f(x) = (\frac{x}{2}, x)$.
(b) For $(x, y) \in S$, consider the (potentially countable) decimal expansion of each. Let's label the expansion of x as $0.d_1 d_2 d_3 d_4 \ldots$, and the second as $0.d'_1 d'_2 d'_3 d'_4 \ldots$. We consider the terminating decimal expansion representations, rather than one with inifinite 9s.

Now we simply map (x, y) to the following real number:

$$f((x, y)) = 0.d_1 d'_1 d_2 d'_2 d_3 d'_3 \ldots$$

To see that this is injective, note that if two intervals differ from each other, that at least one of the left or right endpoints must differ. Since they differ, they must have a different decimal expansion, and thus the resulting real number will also have a different digit and be a different real number.

This logic only fails if we somehow produce a real number that ends in repeating 9s, which is impossible since it would imply that both of our original expansions were of that form.

However, this function is not surjective.

Consider a real number that, for example, ends in alternating 1s and 9s. This itself is a unique real number with no other decimal representation, but the only way to construct it would be with a decimal with repeating 9s. This representation is not in our domain, so there is no way to output this real number.

1.5.8 **Exercise 40**

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

First, note that $B \subseteq (0, 2]$.

For arbitrary $n \in \mathbb{N}$, consider the subset $\left(\frac{1}{n}, 2\right] \cap B$. This can only have finite elements, since otherwise, we could choose $2n$ elements from the subset to sum to greater than 2.

Note that this holds true for all n.

Now also note that $B = \bigcup_{n=1}^{\infty} \left[\left(\frac{1}{n}, 2\right] \cap B\right]$. This is a countable union of finite sets, which is countable.

1.5.9 **Exercise 41**

A real number $x \in \mathbb{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, ..., a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbb{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

(a) Consider the following:

$$x^2 - 2 = 0, \quad x^3 - 2 = 0, \quad x^4 - 10x^2 + 1 = 0.$$

(b) There are a countable number of integer polynomials of degree n, since it can be defined uniquely with a finite product of countable sets.

Since each has finite solutions, the total number of solutions and thus elements of A_n is a countable union of finite sets and is countable.

(c) We simply take the countable union of all A_n for all n.

Again, a countable union of countable sets is countable.

Since the algebraic numbers are countable, the rest of the reals (transcendental) must be uncountable.

1.5.10 — Exercise 42

(a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.
(b) Now let A be the set of all $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?
(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

(a) AFSOC that there does not exist such an a.

Then for all $a \in (0, 1)$, $C \cap [a, 1]$ is countable.

Examine the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{1}{n+1}$.

Clearly, we have that $C = \left(\bigcup_{n=1}^{\infty} C \cap \left[\frac{1}{n+1}, 1\right]\right) \cup (C \cap \{0\})$.

This is simply a countable union of countable sets, which implies that C is countable.

Contradiction!

Therefore there must exist some $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(b) Not necessarily. Consider if $C = [0, 1]$. Then any $a \in (0, 1)$ will produce an uncountable set $[a, 1]$. The supremum of A is 1. But $C \cap [1, 1] = \{1\}$, which is finite.

(c) No. Let $C = \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$. All choices of a lead to a finite intersection.

1.5.11 (Schröder–Bernstein Theorem) — Exercise 43

Assume there exists a 1–1 function $f : X \to Y$ and another 1–1 function $g : Y \to X$. Follow the steps to show that there exists a 1–1, onto function $h : X \to Y$ and hence $X \sim Y$.

The strategy is to partition X and Y into components

$$X = A \cup A' \qquad \text{and} \qquad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.
(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is a similar collection in Y.
(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.
(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

(a) If we restrict the domain of f to A, then it is a bijection between A and B.

Similarly, if we restrict g to B', then g is a bijection between B' and A'.

Now, we can just define $h : X \to Y$ the following way:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{else.} \end{cases}$$

This is clearly a bijection.

(b) First, if $A_1 = \emptyset$, then we are done. This is because $g(Y) = X$, which implies that g is a bijection and we are done.

Assuming that A_1 is non-empty, we can proceed by induction.

Proof: **Base case:** Notice that $f(A_1) \subseteq Y$. Thus $g(f(A_1)) \subseteq g(Y)$, so $(X \setminus g(Y)) \cap (g(f(A_1))) = \emptyset$.

Inductive hypothesis: Assume for some n that $A_1, ..., A_n$ are pairwise disjoint. Thus, $f(A_n)$ is also disjoint from all $f(A_1), ..., f(A_{n-1})$, since f is injective. By the same logic, $g(f(A_1)), ..., g(f(A_n))$ are all also disjoint.

Since $g(f(A_i)) = A_{i+1}$, we have that A_{n+1} is disjoint from all $A_2, ..., A_n$.

It is also disjoint with A_1 by similar logic from the base case.

Note for completeness, if at any point any A_i is empty, then we can just stop with finite A_i that are all pairwise disjoint. ■

Also note that this implies that all $\{f(A_n) : n \in \mathbb{N}\}$ are pairwise disjoint since f is injective.

(c) If $b \in B$, then it must exist in exactly one $f(A_i)$. This means that there must be some $a \in A_i$ such that $f(a) = b$, which shows that f is surjective.

(d) First, let's note that $X \setminus A$ is a subset of $g(Y)$. This is because if $a \in X \setminus A$, then $a \in X \setminus A_1 = X \setminus (X \setminus g(Y)) = g(Y)$.

So we know there must **exist** some $b \in Y$ such that $g(b) = a$.

We should also argue that this b cannot be in B.

AFSOC that $b \in B$. Then $b \in f(A_n)$ for some n, and thus $g(b) \in g(f(A_n)) = A_{n+1}$. However, this is clearly disjoint with $X \setminus A$, so it must be the case that $b \notin B \Longrightarrow b \in Y \setminus B$.

1.6 Cantor's Theorem

1.6.1 **Exercise 44**

Show that $(0, 1)$ is uncountable if and only if $\mathbb{R}$ is uncountable. This shows that Theorem 1.6.1 is equivalent to Theorem 1.5.6.

($\Rightarrow$) This direction is easy, since if $(0, 1)$ is uncountable, then clearly since $(0, 1) \subseteq \mathbb{R}$, the real numbers must also be uncountable.

($\Leftarrow$) If $(0, 1)$ is countable, then $\mathbb{R}$ must be countable, since we can construct $\mathbb{R}$ from $(0, 1)$ using a countable union of the integers plus $(0, 1)$.

1.6.2 **Exercise 45**

(a) Explain why the real number $x = .b_1 b_2 b_3 b_4 ...$ cannot be $f(1)$.
(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbb{N}$.
(c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

(a) It must differ from $f(1)$ at the first digit by construction.
(b) It must differ from the nth digit of $f(n)$ by construction.
(c) This shows that there must be a real number that is not in our enumeration. But we assumed we could enumerate them. This is the contradiction, and thus $(0, 1)$ is uncountable.

1.6.3 **Exercise 46**

Supply rebuttals to the following complaints about the proof of Theorem 1.6.1
(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.
(b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating $9'$s. For instance, $1/2$ can be written as 0.5 or as $.4999...$. Doesn't this cause some problems?

(a) In general, the number that is produced may not be a rational.
(b) No, this is fine. Let's just only consider non-repeating $9'$s representation, and note that with our construction, we will never produce a number that runs into this issue.

1.6.4 **Exercise 47**

Let S be the set consisting of all sequences of $0'$s and $1'$s. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, ...) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, ...)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, ...)$.

Give a rigorous argument showing that S is uncountable.

Cantor's diagonalization argument.

Produce a new binary sequence that differs from all other sequences at the nth element.

1.6.5 **Exercise 48**

(a) Let $A = \{a, b, c\}$. List the eight elements of $\mathcal{P}(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)
(b) If A is finite with n elements, show that $\mathcal{P}(A)$ has 2^n elements.

(a) $\emptyset, \{a\}, \{a, b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.

(b) An element can either be in or out of a subset, which gives us two choices per element. Thus there are 2^n distinct subsets.

1.6.6 **Exercise 49**

(a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1–1 mappings from A into $\mathcal{P}(A)$.
(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1–1 map $g : C \to \mathcal{P}(C)$.
(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are *onto*.

(a) One mapping:

$$a \to \{a\}, \quad b \to \{b\}, \quad c \to \{c\}.$$

Another mapping:

$$a \to \{a, b\}, \quad b \to \{b\}, \quad c \to \{c\}.$$

(b)
$$1 \to \{1\}, \quad 2 \to \{2\}, \quad 3 \to \{3\}, \quad 4 \to \{4\}.$$

(c) There are strictly more elements in the range than the domain.

1.6.7 **Exercise 50**

Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule. In each case, note that B is not in the range of the function used.

TODO: skipped

1.6.8 **Exercise 51**

(a) First, show that the case $a' \in B$ leads to a contradiction.
(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

(a) If $a' \in B$, then it must be that $a' \notin f(a')$ by definition of B.

However, $f(a') = B$ by assumption, so we have shown that $a' \in B$ and $a' \notin B$ which is a contradiction.

(b) If $a' \notin B$, then it must be that $a' \notin f(a')$. This implies that it must be in B which is again a contradiction.

1.6.9 **Exercise 52**

Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

First, we construct an injection from $(0, 1)$ to the set of infinite binary sequences. We do this by considering the decimal expansion.

Next, we construct an injection from the set of infinite binary sequences to $(0, 1)$.

This is a little trickier, as a direct conversion would result in some numbers that are actually the same real number. (For example, $0.0111...$ and 0.1).

We can first consider all sequences that do not end in repeating 1's. This will map into $[0, 1)$, which we know has a bijection with $(0, 1)$. We can divide the result by 3 to get an injection into $\left(0, \frac{1}{3}\right)$.

Next, we map the sequences that end in infinite 1's to their representative real number, divide by 3, and then add $\frac{1}{3}$ to get an injection into $\left(\frac{1}{3}, \frac{2}{3}\right]$.

This completes the injection into $(0, 1)$, so using Schröder–Bernstein we can conclude that the set of infinite binary sequences has the same cardinality as $(0, 1)$, and we can use transitivity of this equivalence relation to deduce that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

1.6.10 **Exercise 53**

As a final exercise, answer each of the following by establishing a 1–1 correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ countable or uncountable?
(b) Is the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ countable or uncountable?
(c) Given a set B, a subset $\mathcal{A}$ of $\mathcal{P}(B)$ is called an *antichain* if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $\mathcal{P}(\mathbb{N})$ contain an uncountable antichain?

(a) This is countable, since there only needs to be two natural numbers to specify the function fully. This essentially reduces to the set with $(n, m) \in \mathbb{N}^2$.
(b) This is uncountable. This is equivalent to the set of infinite sequences of 0's and 1's, which is shown to be uncountable due to a diagonalization argument.
(c) There exists an uncountable antichain. Consider the following bijection between an infinite binary sequence and a subset of the natural numbers:

$$f((b_n)) = \{n : n = 2i + b_i, i \in \mathbb{N}\}.$$

In plain English, for every distinct pair of adjacent natural numbers, we select only one of them based off of the ith value of the binary sequence. If a binary sequence is distinct from another binary sequence, then transformed into subset world, each subset will have an element that is not included in the other.

Considering this bijection, this antichain must be uncountable.

2 Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

No exercises in this section.

2.2 The Limit of a Sequence

2.2.1 **Exercise 54**

What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

An example is the sequence of alternating 0's and 1's.

This is vercongent to any real number. We can just select large enough ϵ and it will work out.

I believe that this is actually describing bounded sequences.

2.2.2 **Exercise 55**

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.
(b) $\lim \frac{2n^2}{n^3+3} = 0$.
(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}}$.

TODO: July 3

2.2.3 **Exercise 56**

Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.
(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.
(c) There exists a college in the United States where every student is at least six feet tall.

(a) Find a college in the US where all students are below 7 feet tall.
(b) Find a college in the US where all professors give out grades other than A or B.
(c) Show that all colleges have a student under 6 feet tall.

2.2.4 **Exercise 57**

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.
(b) A sequence with an infinite number of ones that converges to a limit not equal to one.
(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

(a) Alternating 0's and 1's.
(b) Not possible. If we select $\epsilon < |x - 1|$, where x is the "limit", then we can see that there can never be a N such that every element in the sequence after that is within that ϵ-neighborhood. This is because there must be infinite ones, which cannot all be in the first N elements.
(c) Yes, just do $1, 0, 1, 1, 0, 1, 1, 1, 0, \ldots$. This can never converge due to a similar argument to part (b). But by construction, we can always find n consecutive ones.

2.2.5 **Exercise 58**

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.
(a) $a_n = [[5/n]]$,
(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

TODO: July 3

2.2.6 **Exercise 59**

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

We start with the stated assumptions.

AFSOC $a \neq b$, then we could choose $\epsilon < \frac{|a-b|}{2}$.

By the definition of limits, there would exist N and N' such that any $n \geq \max(N, N')$ satisfies $|x_n - a| < \epsilon$ and $|x_n - b| < \epsilon$.

Using the triangle inequality, we know that

$$|a - b| = |a - x_n + x_n - b| \leq |x_n - a| + |x_n - b| < 2\epsilon < |a - b|.$$

In other words, we have shown that $|a - b| < |a - b|$. This is a **contradiction**.

Therefore, it must be the case that $a = b$.

2.2.7 **Exercise 60**

Here are two useful definitions:
(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.
(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?
(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?
(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?
(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

(a) Frequently.
(b) Eventually implies frequently. To see this, notice that for any natural number, if it is less than or equal to N, then we can just use any number after N as our n, and if it is greater than N, then any number greater than our current number should work.
(c) A sequence (a_n) converges to a if for any ϵ-neighborhood of a, the sequence is eventually in it.
(d) (x_n) is not necessarily eventually in it, as we could have also an infinite number of terms that are 2.2 for example.

However, it is definitely the case that (x_n) is frequently within those bounds.

2.2.8 **Exercise 61**

For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists n satisfying $N \leq b \leq N + M$ where $x_n = 0$.
(a) Is the sequence $(0, 1, 0, 0, 1, \ldots)$ zero heavy?
(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.
(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counter example.
(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if….

2.3 The Algebraic and Order Limit Theorems

2.3.1 **Exercise 62**

Let $x_n \geq 0$ for all $n \in \mathbb{N}$.
(a) If $(x_n) \to 0$, show that $\left(\sqrt{x_n}\right) \to 0$.
(b) If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

2.3.2 **Exercise 63**

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then
(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;
(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

2.3.3 (Squeeze Theorem) **Exercise 64**

Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

2.3.4 **Exercise 65**

Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$

2.3.5 **Exercise 66**

Let (x_n) and (y_n) be given, and define (x_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, ..., x_n, y_n, ...)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

2.3.6 **Exercise 67**

Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

2.3.7 **Exercise 68**

Give and example of each of the following, or state the such a request is impossible by referencing the proper theorems(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

2.3.8 **Exercise 69**

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

2.3.9 **Exercise 70**

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n, b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?
(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

2.3.10 **Exercise 71**

Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.
(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.
(b) If $(b_n) \to b$, then $|b_n| \to |b|$.
(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.
(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \to b$.

2.3.11 (Cesaro Means) **Exercise 72**

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

also converges to the same limit.
(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

2.3.12 **Exercise 73**

A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.
(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.
(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.
(c) If every a_n is rational, then a is rational.

2.3.13 (Iterated Limits) **Exercise 74**

Given a doubly indexed array a_{mn} where $m, n \in \mathbb{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?
(a) Let $a_{mn} = m/(m + n)$ and compute the *iterated* limits

$$\lim_{n\to\infty} \left(\lim_{m\to\infty} a_{mn} \right) \quad \text{and} \quad \lim_{m\to\infty} \left(\lim_{n\to\infty} a_{mn} \right).$$

Define $\lim_{m,n} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.
(b) Let $a_{mn} = 1/(m + n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2 + n^2)$.
(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.
(d) Assume $\lim_{m,n\to\infty} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

2.4 The Monotone Convergence Theorem and Infinite Series

2.4.1 **Exercise 75**

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{y - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

2.4.2 **Exercise 76**

(a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$.

What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

2.4.3 **Exercise 77**

(a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

2.4.4 **Exercise 78**

(a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbb{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.

These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

2.4.5 **Exercise 79**

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.
(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

2.4.6 (Arithmetic–Geometric Mean) **Exercise 80**

(a) Explain why $\sqrt{xy} \leq (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)
(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

2.4.7 (Limit Superior) **Exercise 81**

Let (a_n) be a bounded sequence.
(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.
(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n.$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.
(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of sequence for which the inequality is strict.
(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

2.4.8 **Exercise 82**

For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.
(a) $$\sum_{n=1}^{\infty} \frac{1}{2^n}$$
(b) $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$
(c) $$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$$

(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

2.4.9 **Exercise 83**

Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

2.4.10 (Infinite Products) **Exercise 84**

A close relative of infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \ldots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \ldots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots, \quad \text{where } a_n \geq 0.$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.
(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

2.5 Subsequences and the Bolzano–Weierstrass Theorem

2.5.x **Exercise 85**

2.5.x **Exercise 86**

2.5.x **Exercise 87**

2.5.x **Exercise 88**

2.5.x **Exercise 89**

2.5.x **Exercise 90**

2.5.x **Exercise 91**

2.5.x **Exercise 92**

2.5.x **Exercise 93**

2.6 The Cauchy Criterion

2.6.x **Exercise 94**

2.6.x **Exercise 95**

2.6.x **Exercise 96**

2.6.x **Exercise 97**

2.6.x **Exercise 98**

2.6.x **Exercise 99**

2.6.x **Exercise 100**

2.7 Properties of Infinite Series

2.7.x **Exercise 101**

2.7.x **Exercise 102**

2.7.x **Exercise 103**

2.7.x **Exercise 104**

2.7.x **Exercise 105**

2.7.x **Exercise 106**

2.7.x **Exercise 107**

2.7.x **Exercise 108**

2.7.x **Exercise 109**

2.7.x **Exercise 110**

2.7.x **Exercise 111**

2.7.x **Exercise 112**

2.7.x **Exercise 113**

2.7.x **Exercise 114**

2.8 Double Summations and Products of Infinite Series

2.8.x **Exercise 115**

2.8.x **Exercise 116**

2.8.x **Exercise 117**

2.8.x **Exercise 118**

2.8.x **Exercise 119**

2.8.x **Exercise 120**

3 Basic Topology of $\mathbb{R}$

3.1 Discussion

4 Functional Limits and Continuity

5 The Derivative

6 Sequences and Series of Functions

7 The Riemann Integral

8 Additional Topics

Bibliography