

Summer 2025 – Understanding Analysis Solutions

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1 The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

No exercises in this section.

1.2 Some Preliminaries

1.2.1

Exercise 1

- (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?
- (b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

- (a) Assume for sake of contradiction (AFSOC) that $\sqrt{3} \in \mathbb{Q}$. This implies that $\sqrt{3} = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, and $\gcd(p, q) = 1$.

Therefore, $p^2 = 3q^2$, which means that 3 divides p^2 .

Since 3 is prime, p must be divisible by 3. Therefore for some $k \in \mathbb{Z}$,

$$p = 3k \Rightarrow 9k^2 = 3q^2 \Rightarrow 3k^2 = q^2.$$

This implies that q^2 and thus q is also divisible by 3, which is a contradiction.

A similar proof does not quite work for $\sqrt{6}$ and needs to be adjusted, since 6 is not prime and thus we cannot directly say that 6 divides p^2 implies 6 divides p .

- (b) It is exactly the step where we try to show that 4 divides q^2 implies that 4 divides q . In fact, if we have just that 2 (and not 4) divides q , then clearly 4 still divides q^2 .

1.2.2

Exercise 2

Show that there is no rational number r satisfying $2^r = 3$.

AFSOC that there does exist $r = \frac{p}{q}$, with coprime $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Then $2^r = 2^{p/q} = 3$, which implies that $2^p = 3^q$. This is false.

1.2.3

Exercise 3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

- (a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.
- (b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.
- (c) $A \cap (B \cup C) = (A \cap B) \cup C$.
- (d) $A \cap (B \cap C) = (A \cap B) \cap C$.
- (e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

TODO: skipped

1.2.4

Exercise 4

Produce an infinite collection of sets A_1, A_2, A_3, \dots with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

Assume we have infinite primes. Since they are a subset of \mathbb{N} , they are enumerable ($p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \dots$).

Also assume we have unique prime decomposition.

Now let

$$A_i = \{n \in \mathbb{N} \mid p_i \text{ is the smallest prime in the decomposition of } n\},$$

with the additional modification that A_1 includes 1.

They are all disjoint, since there can only be one smallest prime factor of each number.

Their union forms the natural numbers, since every natural number n has a unique finite prime factor decomposition, and by the fact that every non-empty subset of the natural numbers will have a smallest element, n must be an element of some A_i .

Clearly, every set is also infinite, since we can consider that each A_i contains the powers of p_i .

1.2.5 (De Morgan's Laws)

Exercise 5

Let A and B be subsets of \mathbb{R} .

- If $x \in (A \cup B)^c$, explain why $x \in A^c \cap B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.
- Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.
- Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

TODO: skipped

1.2.6

Exercise 6

- Verify the triangle inequality in the special case where a and b have the same sign.
- Find an efficient proof for all the cases at once by first demonstrating $(a + b)^2 \leq (|a| + |b|)^2$.
- Prove $|a - b| \leq |a - c| + |c - d| + |d - b|$ for all a, b, c , and d .
- Prove $||a| - |b|| \leq |a - b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

- TODO: part skipped**
- TODO: part skipped**
- TODO: part skipped**
- Using the “unremarkable identity”, for any a and b ,

$$\begin{aligned} |a| &= |a - b + b| \\ &\leq |a - b| + |b|. \end{aligned}$$

So first we have $|a| - |b| \leq |a - b|$. Next, we proceed the same exact way using $|b|$, and we get that $|b| - |a| \leq |b - a|$.

Since $|a - b| = |b - a|$, we can combine the above two facts and get that

$$||a| - |b|| \leq |a - b|.$$

1.2.7

Exercise 7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A ; that is, $f(A) = \{f(x) : x \in A\}$.

- Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?
- Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
- Show that, for an arbitrary function $g : \mathbb{R} \rightarrow \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.
- Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g .

- $A \cap B = [1, 2]$. $f(A) = [0, 4]$, and $f(B) = [1, 16]$. So therefore, $f(A) \cap f(B) = [1, 4]$.

$f(A \cap B) = [1, 4]$ as well. So equality holds.

$A \cup B = [0, 4]$, so $f(A \cup B) = [0, 16] = f(A) \cup f(B)$.

Therefore equality holds in both cases.

- Let $A = \{1\}$, and $B = \{-1\}$.

Then $A \cap B = \emptyset$, but $f(A) = \{1\} = f(B)$, so $f(A) \cap f(B) = \{1\} \neq \emptyset$.

- For arbitrary $y \in g(A \cap B)$, we have that $y = g(x)$, where $x \in A \cap B$.

Therefore, $x \in A$ and $x \in B$, which implies that $g(x) \in g(A)$ and $g(x) \in g(B)$.

This further implies that $y = g(x) \in g(A) \cap g(B)$.

Thus we have that $g(A \cap B) \subseteq g(A) \cap g(B)$.

This doesn't work the other way around, since we could have some $y = g(x) = g(z)$, where $x \neq z$, and $x \in A$ and $z \in B$, and neither exists in the other set.

- My conjecture is that

$$g(A \cap B) = g(A) \cap g(B).$$

To show this, I first prove that $g(A \cup B) \subseteq g(A) \cup g(B)$, then the other way around.

$g(A \cup B) \subseteq g(A) \cup g(B)$:

For arbitrary $y \in g(A \cup B)$, we have that $y = g(x)$ such that x in A or B . In either case, it must be such that y is in $g(A)$ or $g(B)$ and thus be in $g(A) \cup g(B)$.

$g(A) \cup g(B) \subseteq g(A \cup B)$:

If $y \in g(A)$, then we have that $y = g(x)$ where $x \in A$, and therefore $x \in A \cup B \implies y = g(x) \in g(A \cup B)$. Same for $y \in g(B)$.

Thus we have proved both directions and shown set equality.

1.2.8

Exercise 8

Here are two important definitions related to a function $f : A \rightarrow B$. The function f is *one-to-one* (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B . The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:

- (a) $f : \mathbb{N} \rightarrow \mathbb{N}$ that is 1-1 but not onto.
- (b) $f : \mathbb{N} \rightarrow \mathbb{N}$ that is onto but not 1-1.
- (c) $f : \mathbb{N} \rightarrow \mathbb{Z}$ that is 1-1 and onto.

TODO: skipped

1.2.9

Exercise 9

Given a function $f : D \rightarrow \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B ; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B .

- (a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?
- (b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbb{R} \rightarrow \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(a) **TODO: part skipped**

- (b) Let $x \in g^{-1}(A \cap B)$. This implies that $g(x) \in A \cap B$, which implies that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$.

From this we can conclude that $x \in g^{-1}(A) \cap g^{-1}(B)$.

Going backwards, we see that if $x \in g^{-1}(A) \cap g^{-1}(B)$, then it must be the case that $g(x) \in A$ and $g(x) \in B$, which leads us to conclude that $x \in g^{-1}(A \cap B)$.

For union, we have if $x \in g^{-1}(A \cup B)$, then $g(x) \in A \cup B$. From the two cases, we will have that either $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$, which lets us conclude that $x \in g^{-1}(A) \cup g^{-1}(B)$.

Backwards, we have that either $g(x) \in A$ or $g(x) \in B$ depending on the cases, so therefore $g(x) \in A \cup B$ and thus $x \in g^{-1}(A \cup B)$.

1.2.10

Exercise 10

Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

- (a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.
- (b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.
- (c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

1.2.11

Exercise 11

Form the logical negation of each claim. One trivial way to do this is to simply add “It is not the case that...” in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word “not” altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

- (a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + 1/n < b$.
- (b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.
- (c) Between every two distinct real numbers there is a rational number.

TODO: skipped

1.2.12

Exercise 12

Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

- (a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.
- (b) Use another induction argument to show the sequence (y_1, y_2, y_3, \dots) is decreasing.

TODO: skipped

1.2.13

Exercise 13

For this exercise, assume Exercise 1.2.5 has been successfully completed.

- (a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \dots \cup A_n)^c = A_1^c \cap A_2^c \cap \dots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

- (b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets B_1, B_2, B_3, \dots where $\bigcap_{i=1}^n B_i \neq \emptyset$ is true for every $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

- (c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

(a) **TODO: part skipped**

- (b) Let $B_i = \mathbb{N} \setminus \{i\}$. Any finite intersection will still have infinitely many elements, but the entire infinite intersection cannot have any elements.

- (c) Let $x \in \left(\bigcup_{i=1}^{\infty} A_i \right)^c$. Then we know that for all i , $x \notin A_i$. (Otherwise, we would have that $x \in \bigcup_{i=1}^{\infty} A_i$.)

Therefore, for all i , $x \in A_i^c$, which lets us conclude that $x \in \bigcap_{i=1}^{\infty} A_i^c$.

For the other direction, we just proceed from each step backwards and see that it works fine.

1.3 The Axiom of Completeness

1.3.1

Exercise 14

- (a) Write a formal definition in the style of Definition 1.3.2 or the *infimum* or *greatest lower bound* of a set.
- (b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

TODO: skipped

1.3.2

Exercise 15

Give an example of each of the following, or state that the request is impossible.

- (a) A set B with $\inf B \geq \sup B$.
- (b) A finite set that contains its infimum but not its supremum.
- (c) A bounded subset of \mathbb{Q} that contains its supremum but not its infimum.

- (a) Let $B = \{1\}$, hmm...
- (b) This cannot be possible. Since there are finite elements, there is necessarily a maximum and minimum, so the set must contain both of them.
- (c) Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. The supremum is 1, which is contained. The infimum is clearly 0, which is not contained.

1.3.3

Exercise 16

- (a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.
- (b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

- (a) First, we know that the supremum of B must exist, since it is bounded above by any element of A .

So let $b' = \sup B$, and $a' = \inf A$.

AFSOC that there exists some $a \in A$ such that $a < b'$. Let $\epsilon = b' - a > 0$, and then we know that there must be some $b \in B$ such that $b > b' - \epsilon = a$, so we have $b > a$. This is a contradiction, since we assumed that b is a lower bound for all elements in A .

Therefore, we have shown that b' is a lower bound for A , and since it is a supremum of B , it must be the greatest such lower bound. This is exactly the definition of infimum of A .
- (b) For any set bounded from below, we can take the set of all lower bounds, and use part (a) to show that the greatest lower bound is the smallest upper bound of the set of lower bounds.

1.3.4

Exercise 17

Let A_1, A_2, A_3, \dots be a collection of nonempty sets, each of which is bounded above.

- (a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^n A_k)$.
- (b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

(a) $\sup(A_1 \cup A_2) = \max(\sup A_1, \sup A_2).$

Extended to $n \in \mathbb{N}$, we have

$$\sup\left(\bigcup_{k=1}^n A_k\right) = \max_{k \in [n]}(\sup A_k).$$

- (b) This does not extend to infinite max, since it may be possible for the infinite max to exist. For example, if we have each A_k simply consist of the natural number k , then there is no supremum and no max.

1.3.5

Exercise 18

As in Example 1.3.7, let $A \in \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

- (a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.
 (b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

TODO: skipped

1.3.6

Exercise 19

Given sets A and B , define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

- (a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.
 (b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.
 (c) Finally, show $\sup(A + B) = s + t$.
 (d) Construct another proof of this same fact using Lemma 1.3.8.

- (a) Let $c \in A + B$. Then $c = a + b$, with $a \in A$, and $b \in B$.

Now, we have that $a \leq s$ and $b \leq t$, so therefore, $c \leq s + t$.

- (b) For all $b \in B$, we have that $a + b \leq u$. Thus, $u - a \geq b$, so $u - a$ is an upper bound for B .

Since t is the least upper bound for B , we now have that $t \leq u - a$.

- (c) Let u be an arbitrary upper bound for $A + B$. By (b), we have that for all $a \in A$, $t \leq u - a$.

Therefore we also have that $a \leq u - t$, showing that $u - t$ is an upper bound on A . Since s is the least upper bound on A , we have $s \leq u - t$, and thus have $s + t \leq u$. This shows that $s + t$ must be the least upper bound and therefore is the supremum of $A + B$.

- (d) Choose arbitrary $\epsilon > 0$. For $\frac{\epsilon}{2}$, there must exist $a \in A$ and $b \in B$ such that $a \geq s - \frac{\epsilon}{2}$ and $b \geq t - \frac{\epsilon}{2}$.

Therefore, $s + t - \epsilon \leq a + b$ for some $a + b$ in $A + B$.

But from part (a), we know that $s + t$ itself is an upper bound of $A + B$. Therefore, it must be that $s + t$ is the supremum of $A + B$.

1.3.7

Exercise 20

Prove that if a is an upper bound for A , and if a is also an element of A , then it must be that $a = \sup(A)$.

TODO: skipped

1.3.8

Exercise 21

Compute, without proofs, the suprema and infima (if they exist) of the following sets:

- (a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$.
- (b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.
- (c) $\{n/(3n+1) : n \in \mathbb{N}\}$.
- (d) $\{m/(m+n) : m, n \in \mathbb{N}\}$.

- (a) supremum is 1, infimum is 0.
- (b) supremum is 1, infimum is -1 .
- (c) supremum is $\frac{1}{3}$, infimum is $\frac{1}{4}$.
- (d) supremum is 1, infimum is 0.

1.3.9

Exercise 22

- (a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A .
- (b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

TODO: skipped

1.3.10 (Cut Property)

Exercise 23

The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

- (a) Use the Axiom of Completeness to prove the Cut Property.
- (b) Show that the implication goes the other way; that is, assume \mathbb{R} possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.
- (c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when \mathbb{R} is replaced by \mathbb{Q} .

- (a) A is clearly bounded by above, just pick any element in B .

Using the Axiom of Completeness, there must exist some $c = \sup A$. By definition, $c \geq x$ for all $x \in A$.

By 1.3.3, c is the infimum of B , so for all $x \in B$, we have that $b \geq c$.

- (b) Assume the Cut Property.

Let B be the set of upper bounds of E . Now let $A = \mathbb{R} \setminus B$.

Now note that for any $a \in A$ and $b \in B$, we have that $a < b$. This is because if we assume otherwise, then we see that a is an upper bound for E and should have been an element of B in the first place.

Now, from the Cut Property, we have that there exists a c such that $a \leq c \leq b$.

Now, I show that c is an upper bound for e .

AFSOC that there exists some $e \in E$ such that $e > c$. Examine $\epsilon = e - c > 0$.

Since $\frac{\epsilon}{2} + c > c$, it must be a member of B and thus be an upper bound for E .

However, we also have that $\frac{\epsilon}{2} + c < e$, so it cannot be an upper bound for E . Contradiction!

Thus, since c is an upper bound and is less than or equal to all upper bounds of E , we have that c exists and is the supremum of E .

(c) Let A be $\{x \in \mathbb{Q} : x^2 \leq 2\}$, and $B := \{x \in \mathbb{Q} : x^2 > 2\}$.

They clearly are disjoint sets that form the rationals.

But we have proven that there cannot be such a $c \in \mathbb{Q}$ such that it exists in the middle of these two sets.

1.3.11

Exercise 24

Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

- (a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.
- (b) If $\sup A < \inf B$ for sets A and B , then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
- (c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

(a) This is true. AFSOC false. Then there must exist some $a \in A$ such that it is greater than $\sup B$ but less than or equal to $\sup A$.

But since $A \subseteq B$, it must be an element of B as well, which leads us to a contradiction since we assumed it would be greater than $\sup B$.

(b) True. **TODO: skipped**

(c) False. **TODO: skipped**

1.4 Consequences of Completeness

1.4.1

Exercise 25

Recall that \mathbb{I} stands for the set of irrational numbers.

- (a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of \mathbb{Q} as well.
- (b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.
- (c) Part (a) can be summarized by saying that \mathbb{Q} is closed under addition and multiplication. Is \mathbb{I} closed under addition and multiplication? Given two irrational numbers s and t , what can we say about $s + t$ and st ?

(a) Let $a = \frac{m}{n}$, $b = \frac{p}{q}$. Then $mp \in \mathbb{Z}$, and $nq \in \mathbb{N}$.

Therefore $ab = \frac{mp}{nq} \in \mathbb{Q}$.

$mq \in \mathbb{Z}$, and $np \in \mathbb{Z}$, so therefore $a + b = \frac{mq+np}{nq} \in \mathbb{Q}$.

(b) **TODO: skipped**

(c) **TODO: skipped**

1.4.2

Exercise 26

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A . Show $s = \sup A$.

AFSOC $s > \sup A$.

Then there must exist some $n \in \mathbb{N}$ such that $\frac{1}{n} < s - \sup A$.

So then, $s - \frac{1}{n} > \sup A$, which is a contradiction with the condition that $s - \frac{1}{n}$ cannot be an upper bound.

The other direction $s < \sup A$ works the same way.

Therefore it must be that $s = \sup A$.

1.4.3

Exercise 27

Prove that $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

AFSOC there exists some $x \in \bigcap_{n=1}^{\infty} (0, 1/n)$.

It must be that $x > 0$, and therefore, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

However, x would then be excluded from the interval $(0, \frac{1}{n})$, which is a contradiction.

1.4.4

Exercise 28

Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

b is clearly an upper bound.

Let $\epsilon > 0$, and also choose $\epsilon < b - a$.

There must exist $r \in \mathbb{Q}$ such that $b - \epsilon < r < b$. Therefore $r \in T$, which shows that $\sup T = b$.

1.4.5

Exercise 29

Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real $a - \sqrt{2}$ and $b - \sqrt{2}$.

First, choose a rational y such that $a - \sqrt{2} < y < b - \sqrt{2}$. Next, we see clearly that $y + \sqrt{2} \in \mathbb{I}$.

Now, we can see that $a < y + \sqrt{2} < b$.

1.4.6

Exercise 30

Recall that a set B is *dense* in \mathbb{R} if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in \mathbb{R} ? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

- (a) The set of all rational numbers p/q with $q \leq 10$.
- (b) The set of all rational numbers p/q with q a power of 2.
- (c) The set of all rational numbers p/q with $10|p| \geq q$.

TODO: skipped

1.4.7

Exercise 31

Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Claim: If we choose $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$, then we can show that $\alpha - \frac{1}{n_0}$ is still an upper bound for $T = \{t \in \mathbb{R} : t^2 < 2\}$.

Proof: Consider the following:

$$\begin{aligned} \left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}. \end{aligned}$$

We want to choose an n such that $\alpha^2 - \frac{2\alpha}{n} > 2$. Note that if we choose n_0 as in the claim, we get that the inequality holds.

Thus, $\alpha - \frac{1}{n_0}$ is actually an upper bound on T that is smaller than α , contradicting the assumption that $\alpha = \sup T$. ■

1.4.8

Exercise 32

Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

- (a) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.
- (b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \dots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.
- (c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \dots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.)
- (d) A sequence of closed bounded (not necessarily nested) intervals I_1, I_2, I_3, \dots with the property that $\bigcap_{n=1}^N I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

- (a) Let $A = \{q \in \mathbb{Q} \mid q < 0\}$, and $B = \{r \in \mathbb{R} \setminus \mathbb{Q} \mid r < 0\}$.
- (b) Let $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then the only element in the intersection can be 0.
- (c) Let $L_n = [n, \infty)$. This cannot have any element.
- (d) This is **impossible**, and we can prove it using the nested interval property.

Proof: First, we use the fact that a non-empty intersection of two closed, bounded intervals must itself be a closed bounded interval.

Now, let

$$I'_n = \bigcap_{m=1}^n I_m$$

define a new sequence of closed bounded intervals, which are nested by construction.

By the assumption that every finite intersection is non-empty, every I'_n must also be a non-empty closed, bounded, interval.

It is also important to note that the finite and infinite intersection of this sequence is exactly equal to the finite and infinite intersection of the original sequence.

Now, we can apply the NIP to deduce that the infinite intersection must be non-empty, which disproves the original claim. ■

1.5 Cardinality

1.5.1

Exercise 33

Finish the following proof for Theorem 1.5.7.

Assume B is a countable set. Thus, there exists $f : \mathbb{N} \rightarrow B$, which is 1-1 and onto. Let $A \subseteq B$ be an infinite subset of B . We must show that A is countable.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \rightarrow A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from \mathbb{N} onto A .

For $i > 1$, let $n_i = \min\{n \in \mathbb{N} : f(n) \in A, n > n_{i-1}\}$. This must exist since A is an infinite set, thus there cannot be an upper bound on n such that $f(n) \in A$.

Now, just let $g(i) = f(n_i)$. This is an injective function, since each n_i is distinct and f is an injective function.

1.5.2

Exercise 34

Review the proof of Theorem 1.5.6, part (ii) showing that \mathbb{R} is uncountable, and then find the flaw in the following erroneous proof that \mathbb{Q} is uncountable:

Assume, for contradiction, that \mathbb{Q} is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3, \dots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies \mathbb{Q} must therefore be uncountable.

NIP is not true in general over the rationals, since the element could be an irrational.

1.5.3

Exercise 35

Use the following outline to supply proofs for the statements in Theorem 1.5.8.

- (a) First, prove statement (i) for two countable sets, A_1 and A_2 . Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Now, explain how the more general statement in (i) follows.

- (b) Explain why induction *cannot* be used to prove part (ii) of Theorem 1.5.8 from part (i).
 (c) Show how arranging \mathbb{N} into the two-dimensional array

1	3	6	10	15	...
2	5	9	14	...	
4	8	13	...		
7	12	...			
11	...				
⋮					

leads to a proof of Theorem 1.5.8 (ii).

TODO: July 2

1.5.4

Exercise 36

- (a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b) .
 (b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as \mathbb{R} as well.
 (c) Using open intervals makes it more convenient to produce the required 1–1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1–1 onto function between the two sets.

1.5.5

Exercise 37

- (a) Why is $A \sim A$ for every set A ?
 (b) Given sets A and B , explain why $A \sim B$ is equivalent to asserting $B \sim A$.
 (c) For three sets A , B , and C , show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that \sim is an *equivalence relation*.

1.5.6

Exercise 38

- (a) Give an example of a countable collection of disjoint open intervals.
 (b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

1.5.7

Exercise 39

Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

- (a) Find a 1–1 function that maps $(0, 1)$ into, but not necessarily onto, S . (This is easy.)
 (b) Use the fact that every real number has a decimal expansion to produce a 1–1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999....)

The Schröder–Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

1.5.8

Exercise 40

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

1.5.9

Exercise 41

A real number $x \in \mathbb{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \dots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

- Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.
- Fix $n \in \mathbb{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n . Using the fact that every polynomial has a finite number of roots, show that A_n is countable.
- Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

1.5.10

Exercise 42

- Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.
- Now let A be the set of all $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?
- Does the statement in (a) remain true if “uncountable” is replaced by “infinite”?

1.5.11 (Schröder–Bernstein Theorem)

Exercise 43

Assume there exists a 1–1 function $f : X \rightarrow Y$ and another 1–1 function $g : Y \rightarrow X$. Follow the steps to show that there exists a 1–1, onto function $h : X \rightarrow Y$ and hence $X \sim Y$.

The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B , and g maps B' onto A' .

- Explain how achieving this would lead to a proof that $X \sim Y$.
- Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X , while $\{f(A_n) : n \in \mathbb{N}\}$ is a similar collection in Y .
- Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B .
- Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A' .

1.6 Cantor's Theorem

1.6.1

Exercise 44

Show that $(0, 1)$ is uncountable if and only if \mathbb{R} is uncountable. This shows that Theorem 1.6.1 is equivalent to Theorem 1.5.6.

1.6.2

Exercise 45

- Explain why the real number $x = .b_1b_2b_3b_4\dots$ cannot be $f(1)$.
- Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbb{N}$.
- Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

1.6.3

Exercise 46

Supply rebuttals to the following complaints about the proof of Theorem 1.6.1

- Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of \mathbb{Q} must be countable, the proof of Theorem 1.6.1 must be flawed.
- Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as 0.5 or as .4999.... Doesn't this cause some problems?

1.6.4

Exercise 47

Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \dots) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \dots)$ is an element of S , as is the sequence $(1, 1, 1, 1, 1, 1, \dots)$.

Give a rigorous argument showing that S is uncountable.

1.6.5

Exercise 48

- Let $A = \{a, b, c\}$. List the eight elements of $\mathcal{P}(A)$. (Do not forget that \emptyset is considered to be a subset of every set.)
- If A is finite with n elements, show that $\mathcal{P}(A)$ has 2^n elements.

1.6.6

Exercise 49

- Using the particular set $A = \{a, b, c\}$, exhibit two different 1-1 mappings from A into $\mathcal{P}(A)$.
- Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1-1 map $g : C \rightarrow \mathcal{P}(C)$.
- Explain why, in parts (a) and (b), it is impossible to construct mappings that are *onto*.

1.6.7

Exercise 50

Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule. In each case, note that B is not in the range of the function used.

1.6.8**Exercise 51**

- (a) First, show that the case $a' \in B$ leads to a contradiction.
 (b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

1.6.9**Exercise 52**

Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

1.6.10**Exercise 53**

As a final exercise, answer each of the following by establishing a 1–1 correspondence with a set of known cardinality.

- (a) Is the set of all functions from $\{0, 1\}$ to \mathbb{N} countable or uncountable?
 (b) Is the set of all functions from \mathbb{N} to $\{0, 1\}$ countable or uncountable?
 (c) Given a set B , a subset \mathcal{A} of $\mathcal{P}(B)$ is called an *antichain* if no element of \mathcal{A} is a subset of any other element of \mathcal{A} . Does $\mathcal{P}(\mathbb{N})$ contain an uncountable antichain?

- (a) This is countable, since there only needs to be two natural numbers to specify the function fully. This essentially reduces to the set with $(n, m) \in \mathbb{N}^2$.
 (b) This is uncountable. This is equivalent to the set of infinite sequences of 0's and 1's, which is shown to be uncountable due to a diagonalization argument.
 (c) There exists an uncountable antichain. Consider the following bijection between an infinite binary sequence and a subset of the natural numbers:

$$f((b_n)) = \{n : n = 2i + b_i\}.$$

If a binary sequence is distinct from another binary sequence, then transformed into subset world, each subset will have an element that is not included in the other.

Considering this bijection, this antichain must be uncountable.

2 Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

2.2 The Limit of a Sequence

2.2.1**Exercise 54**

What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

2.2.2**Exercise 55**

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

- (a) $\lim_{n \rightarrow \infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.
- (b) $\lim_{n \rightarrow \infty} \frac{2n^2}{n^3+3} = 0$.
- (c) $\lim_{n \rightarrow \infty} \frac{\sin(n^2)}{\sqrt[3]{n}}$.

2.2.3

Exercise 56

Describe what we would have to demonstrate in order to disprove each of the following statements.

- (a) At every college in the United States, there is a student who is at least seven feet tall.
- (b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.
- (c) There exists a college in the United States where every student is at least six feet tall.

2.2.4

Exercise 57

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

- (a) A sequence with an infinite number of ones that does not converge to one.
- (b) A sequence with an infinite number of ones that converges to a limit not equal to one.
- (c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

2.2.5

Exercise 58

Let $\llbracket x \rrbracket$ be the greatest integer less than or equal to x . For example, $\llbracket \pi \rrbracket = 3$ and $\llbracket 3 \rrbracket = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

- (a) $a_n = \llbracket 5/n \rrbracket$,
- (b) $a_n = \llbracket (12 + 4n)/3n \rrbracket$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that “the smaller the ϵ -neighborhood, the larger N may have to be.”

2.2.6

Exercise 59

Prove Theorem 2.2.7. To get started, assume $(a_n) \rightarrow a$ and also that $(a_n) \rightarrow b$. Now argue $a = b$.

2.2.7

Exercise 60

Here are two useful definitions:

- (i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.
- (ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.
 - (a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?
 - (b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

- (c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?
- (d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

- (a) Is the sequence $(0, 1, 0, 0, 1, \dots)$ zero heavy?
- (b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.
- (c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counter example.
- (d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if...

Bibliography