Summer 2025 – Understanding Analysis Solutions

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2025-08-10

Contents

1 The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

No exercises in this section.

1.2 Some Preliminaries

1.2.1 **Exercise 1**

(a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?
(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

(a) Assume for sake of contradiction (AFSOC) that $\sqrt{3} \in \mathbb{Q}$. This implies that $\sqrt{3} = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, and $\gcd(p, q) = 1$.

Therefore, $p^2 = 3q^2$, which means that 3 divides p^2.

Since 3 is prime, p must be divisible by 3. Therefore for some $k \in \mathbb{Z}$,

$$p = 3k \Rightarrow 9k^2 = 3q^2 \Rightarrow 3k^2 = q^2.$$

This implies that q^2 and thus q is also divisible by 3, which is a contradiction.

A similar proof does not quite work for $\sqrt{6}$ and needs to be adjusted, since 6 is not prime and thus we cannot directly say that 6 divides p^2 implies 6 divides p.

(b) It is exactly the step where we try to show that 4 divides q^2 implies that 4 divides q. In fact, if we have just that 2 (and not 4) divides q, then clearly 4 still divides q^2.

1.2.2 **Exercise 2**

Show that there is no rational number r satisfying $2^r = 3$.

AFSOC that there does exist $r = \frac{p}{q}$, with coprime $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Then $2^r = 2^{p/q} = 3$, which implies that $2^p = 3^q$. This is false.

1.2.3 **Exercise 3**

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \ldots$ are all sets containing an infinite number of elements, then the interesection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.
(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \ldots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.
(c) $A \cap (B \cup C) = (A \cap B) \cup C$.
(d) $A \cap (B \cap C) = (A \cap B) \cap C$.
(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

TODO: skipped

1.2.4 **Exercise 4**

Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

Assume we have infinite primes. Since they are a subset of $\mathbb{N}$, they are enumerable ($p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$).

Also assume we have unique prime decomposition.

Now let

$$A_i = \{n \in \mathbb{N} \mid p_i \text{ is the smallest prime in the decomposition of } n\},$$

with the additional modification that A_1 includes 1.

They are all disjoint, since there can only be one smallest prime factor of each number.

Their union forms the natural numbers, since every natural number n has a unique finite prime factor decomposition, and by the the fact that every non-empty subset of the natural numbers will have a smallest element, n must be an element of some A_i.

Clearly, every set is also infinite, since we can consider that each A_i contains the powers of p_i.

1.2.5 (De Morgan's Laws) **Exercise 5**

Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^{\mathsf{C}}$, explain why $x \in A^{\mathsf{C}} \cup B^{\mathsf{C}}$. This shows that $(A \cap B)^{\mathsf{C}} \subseteq A^{\mathsf{C}} \cup B^{\mathsf{C}}$.
(b) Prove the reverse inclusion $(A \cap B)^{\mathsf{C}} \supseteq A^{\mathsf{C}} \cup B^{\mathsf{C}}$, and conclude that $(A \cap B)^{\mathsf{C}} = A^{\mathsf{C}} \cup B^{\mathsf{C}}$.
(c) Show $(A \cup B)^{\mathsf{C}} = A^{\mathsf{C}} \cap B^{\mathsf{C}}$ by demonstrating inclusion both ways.

TODO: skipped

1.2.6 **Exercise 6**

(a) Verify the triangle inequality in the special case where a and b have the same sign.
(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.
(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.
(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

(a) **TODO: part skipped**
(b) **TODO: part skipped**
(c) **TODO: part skipped**
(d) Using the "unremarkable identity", for any a and b,

$$\begin{aligned} |a| &= |a-b+b| \\ &\leq |a-b| + |b|. \end{aligned}$$

So first we have $|a| - |b| \leq |a-b|$. Next, we proceed the same exact way using $|b|$, and we get that $|b| - |a| \leq |b-a|$.

Since $|a-b| = |b-a|$, we can combine the above two facts and get that

$$||a|-|b|| \leq |a-b|.$$

1.2.7 **Exercise 7**

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.
(a) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?
(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
(c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.
(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

(a) $A \cap B = [1,2]$. $f(A) = [0,4]$, and $f(B) = [1,16]$. So therefore, $f(A) \cap f(B) = [1,4]$.

$f(A \cap B) = [1,4]$ as well. So equality holds.

$A \cup B = [0,4]$, so $f(A \cup B) = [0,16] = f(A) \cup f(B)$.

Therefore equality holds in both cases.
(b) Let $A = \{1\}$, and $B = \{-1\}$.

Then $A \cap B = \emptyset$, but $f(A) = \{1\} = f(B)$, so $f(A) \cap f(B) = \{1\} \neq \emptyset$.
(c) For arbitrary $y \in g(A \cap B)$, we have that $y = g(x)$, where $x \in A \cap B$.

Therefore, $x \in A$ and $x \in B$, which implies that $g(x) \in g(A)$ and $g(x) \in g(B)$.

This further implies that $y = g(x) \in g(A) \cap g(B)$.

Thus we have that $g(A \cap B) \subseteq g(A) \cap g(B)$.

This doesn't work the other way around, since we could have some $y = g(x) = g(z)$, where $x \neq z$, and $x \in A$ and $z \in B$, and neither exists in the other set.

(d) My conjecture is that

$$g(A \cap B) = g(A) \cup g(B).$$

To show this, I first prove that $g(A \cup B) \subseteq g(A) \cup g(B)$, then the other way around.

$\boldsymbol{g(A \cup B) \subseteq g(A) \cup g(B)}$**:**

For arbitrary $y \in g(A \cup B)$, we have that $y = g(x)$ such that x in A or B. In either case, it must be such that y is in $g(A)$ or $g(B)$ and thus be in $g(A) \cup g(B)$.

$\boldsymbol{g(A) \cup g(B) \subseteq g(A \cup B)}$**:**

If $y \in g(A)$, then we have that $y = g(x)$ where $x \in A$, and therefore $x \in A \cup B \Longrightarrow y = g(x) \in g(A \cup B)$. Same for $y \in g(B)$.

Thus we have proved both directions and shown set equality.

1.2.8 **Exercise 8**

Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1–1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1–1 but not onto.
(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1–1.
(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1–1 and onto.

TODO: skipped

1.2.9 **Exercise 9**

Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?
(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(a) **TODO: part skipped**
(b) Let $x \in g^{-1}(A \cap B)$. This implies that $g(x) \in A \cap B$, which implies that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$.

From this we can conclude that $x \in g^{-1}(A) \cap g^{-1}(B)$.

Going backwards, we see that if $x \in g^{-1}(A) \cap g^{-1}(B)$, then it must be the case that $g(x) \in A$ and $g(x) \in B$, which leads us to conclude that $x \in g^{-1}(A \cap B)$.

For union, we have if $x \in g^{-1}(A \cup B)$, then $g(x) \in A \cup B$. From the two cases, we will have that either $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$, which lets us conclude that $x \in g^{-1}(A) \cup g^{-1}(B)$.

Backwards, we have that either $g(x) \in A$ or $g(x) \in B$ depending on the cases, so therefore $g(x) \in A \cup B$ and thus $x \in g^{-1}(A \cup B)$.

1.2.10 **Exercise 10**

Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.
(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.
(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

1.2.11 **Exercise 11**

Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that…" in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + 1/n < b$.
(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.
(c) Between every two distinct real numbers there is a rational number.

TODO: skipped

1.2.12 **Exercise 12**

Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.
(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

TODO: skipped

1.2.13 **Exercise 13**

For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^C = A_1^C \cap A_2^C \cap \cdots \cap A_n^C$$

for any finite $n \in \mathbb{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^C = \bigcap_{i=1}^{\infty} A_i^C,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

(a) **TODO: part skipped**

(b) Let $B_i = \mathbb{N} \setminus \{i\}$. Any finite intersection will still have infinitely many elements, but the entire infinite intersection cannot have any elements.

(c) Let $x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathsf{C}}$. Then we know that for all i, $x \neq A_i$. (Otherwise, we would have that $x \in \bigcup_{i=1}^{\infty} A_i$.)

Therefore, for all i, $x \in A_i^{\mathsf{C}}$, which lets us conclude that $x \in \bigcap_{i=1}^{\infty} A_i^{\mathsf{C}}$.

For the other direction, we just proceed from each step backwards and see that it works fine.

1.3 The Axiom of Completeness

1.3.1 **Exercise 14**

(a) Write a formal definition in the style of Definition 1.3.2 or the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

TODO: skipped

1.3.2 **Exercise 15**

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

(a) Let $B = \{1\}$, hmm...

(b) This cannot be possible. Since there are finite elements, there is necessarily a maximum and minimum, so the set must contain both of them.

(c) Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. The supremum is 1, which is contained. The infimum is clearly 0, which is not contained.

1.3.3 **Exercise 16**

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

(a) First, we know that the supremum of B must exist, since it is bounded above by any element of A.

So let $b' = \sup B$, and $a' = \inf A$.

AFSOC that there exists some $a \in A$ such that $a < b'$. Let $\epsilon = b' - a > 0$, and then we know that there must be some $b \in B$ such that $b > b' - \epsilon = a$, so we have $b > a$. This is a contradiction, since we assumed that b is a lower bound for all elements in A.

Therefore, we have shown that b' is a lower bound for A, and since it is a supremum of B, it must be the greatest such lower bound. This is exactly the definition of infimum of A.

(b) For any set bounded from below, we can take the set of all lower bounds, and use part (a) to show that the greatest lower bound is the smallest upper bound of the set of lower bounds.

1.3.4 **Exercise 17**

Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

(a) $\sup(A_1 \cup A_2) = \max(\sup A_1, \sup A_2)$.

Extended to $n \in \mathbb{N}$, we have

$$\sup\left(\bigcup_{k=1}^{n} A_k\right) = \max_{k\in[n]}(\sup A_k).$$

(b) This does not extend to infinite max, since it may be possible for the infinite max to exist. For example, if we have each A_k simply consist of the natural number k, then there is no supremum and no max.

1.3.5 **Exercise 18**

As in Example 1.3.7, let $A \in \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

TODO: skipped

1.3.6 **Exercise 19**

Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an pper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

(a) Let $c \in A + B$. Then $c = a + b$, with $a \in A$, and $b \in B$.

Now, we have that $a \leq s$ and $b \leq t$, so therefore, $c \leq s + t$.

(b) For all $b \in B$, we have that $a + b \leq u$. Thus, $u - a \geq b$, so $u - a$ is an upper bound for B.

Since t is the least upper bound for B, we now have that $t \leq u - a$.

(c) Let u be an arbitrary upper bound for $A + B$. By (b), we have that for all $a \in A$, $t \leq u - a$.

Therefore we also have that $a \leq u - t$, showing that $u - t$ is an upper bound on A. Since s is the least upper bound on A, we have $s \leq u - t$, and thus have $s + t \leq u$. This shows that $s + t$ must be the least upper bound and therefore is the supremum of $A + B$.

(d) Choose arbitrary $\epsilon > 0$. For $\frac{\epsilon}{2}$, there must exist $a \in A$ and $b \in B$ such that $a \geq s - \frac{\epsilon}{2}$ and $b \geq t - \frac{\epsilon}{2}$.

Therefore, $s + t - \epsilon \leq a + b$ for some $a + b$ in $A + B$.

But from part (a), we know that $s + t$ itself is an upper bound of $A + B$. Therefore, it must be that $s + t$ is the supremum of $A + B$.

1.3.7 **Exercise 20**

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup(A)$.

TODO: skipped

1.3.8 **Exercise 21**

Compute, without proofs, the suprema and infima (if they exist) of the following sets:
(a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$.
(b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.
(c) $\{n/(3n + 1) : n \in \mathbb{N}\}$.
(d) $\{m/(m + n) : m, n \in \mathbb{N}\}$.

(a) supremum is 1, infimum is 0.
(b) supremum is 1, infimum is -1.
(c) supremum is $\frac{1}{3}$, infimum is $\frac{1}{4}$.
(d) supremum is 1, infimum is 0.

1.3.9 **Exercise 22**

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.
(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

TODO: skipped

1.3.10 (Cut Property) **Exercise 23**

The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.
(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.
(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbb{R}$ is replaced by $\mathbb{Q}$.

(a) A is clearly bounded by above, just pick any element in B.

Using the Axiom of Completeness, there must exist some $c = \sup A$. By definition, $c \geq x$ for all $x \in A$.

By 1.3.3, c is the infimum of B, so for all $x \in B$, we have that $b \geq c$.

(b) Assume the Cut Property.

Let B be the set of upper bounds of E. Now let $A = \mathbb{R} \setminus B$.

Now note that for any $a \in A$ and $b \in B$, we have that $a < b$. This is because if we assume otherwise, then we see that a is an upper bound for E and should have been an element of B in the first place.

Now, from the Cut Property, we have that there exists a c such that $a \leq c \leq b$.

Now, I show that c is an upper bound for e.

AFSOC that there exists some $e \in E$ such that $e > c$. Examine $\epsilon = e - c > 0$.

Since $\frac{\epsilon}{2} + c > c$, it must be a member of B and thus be an upper bound for E.

However, we also have that $\frac{\epsilon}{2} + c < e$, so it cannot be an upper bound for E. Contradiction!

Thus, since c is an upper bound and is less than or equal to all upper bounds of E, we have that c exists and is the supremum of E.

(c) Let A be $\{x \in \mathbb{Q} : x^2 \leq 2\}$, and $B := \{x \in \mathbb{Q} : x^2 > 2\}$.

They clearly are disjoint sets that form the rationals.

But we have proven that there cannot be such a $c \in \mathbb{Q}$ such that it exists in the middle of these two sets.

1.3.11 **Exercise 24**

Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.
(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
(c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

(a) This is true. AFSOC false. Then there must exist some $a \in A$ such that it is greater than $\sup B$ but less than or equal to $\sup A$.

But since $A \subseteq B$, it must be an element of B as well, which leads us to a contradiction since we assumed it would be greater than $\sup B$.

(b) True. **TODO: skipped**

(c) False. **TODO: skipped**

1.4 Consequences of Completeness

1.4.1 **Exercise 25**

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

(a) Let $a = \frac{m}{n}$, $b = \frac{p}{q}$. Then $mp \in \mathbb{Z}$, and $nq \in \mathbb{N}$.

Therefore $ab = \frac{mp}{nq} \in \mathbb{Q}$.

$mq \in \mathbb{Z}$, and $np \in \mathbb{Z}$, so therefore $a + b = \frac{mq+np}{nq} \in \mathbb{Q}$.

(b) **TODO: skipped**

(c) **TODO: skipped**

1.4.2 **Exercise 26**

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

AFSOC $s > \sup A$.

Then there must exist some $n \in \mathbb{N}$ such that $\frac{1}{n} < s - \sup A$.

So then, $s - \frac{1}{n} > \sup A$, which is a contradiction with the condition that $s - \frac{1}{n}$ cannot be an upper bound.

The other direction $s < \sup A$ works the same way.

Therefore it must be that $s = \sup A$.

1.4.3 **Exercise 27**

Prove that $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

AFSOC there exists some $x \in \bigcap_{n=1}^{\infty} (0, 1/n)$.

It must be that $x > 0$, and therefore, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

However, x would then be excluded from the interval $\left(0, \frac{1}{n}\right)$, which is a contradiction.

1.4.4 **Exercise 28**

Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

b is clearly an upper bound.

Let $\epsilon > 0$, and also choose $\epsilon < b - a$.

There must exist $r \in \mathbb{Q}$ such that $b - \varepsilon < r < b$. Therefore $r \in T$, which shows that $\sup T = b$.

1.4.5 **Exercise 29**

Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real $a - \sqrt{2}$ and $b - \sqrt{2}$.

First, choose a rational y such that $a - \sqrt{2} < y < b - \sqrt{2}$. Next, we see clearly that $y + \sqrt{2} \in \mathbb{I}$.

Now, we can see that $a < y + \sqrt{2} < b$.

1.4.6 **Exercise 30**

Recall that a set B is *dense* in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.
(a) The set of all rational numbers p/q with $q \leq 10$.
(b) The set of all rational numbers p/q with q a power of 2.
(c) The set of all rational numbers p/q with $10|p| \geq q$.

TODO: skipped

1.4.7 **Exercise 31**

Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Claim: If we choose $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$, then we can show that $\alpha - \frac{1}{n_0}$ is still an upper bound for $T = \{t \in \mathbb{R} : t^2 < 2\}$.

Proof: Consider the following:

$$\begin{aligned} \left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}. \end{aligned}$$

We want to choose an n such that $\alpha^2 - \frac{2\alpha}{n} > 2$. Note that if we choose n_0 as in the claim, we get that the inequality holds.

Thus, $\alpha - \frac{1}{n_0}$ is actually an upper bound on T that is smaller than α, contradicting the assumption that $\alpha = \sup T$. ∎

1.4.8 **Exercise 32**

Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

(a) Let $A = \{q \in \mathbb{Q} \mid q < 0\}$, and $B = \{r \in \mathbb{R} \setminus \mathbb{Q} \mid r < 0\}$.

(b) Let $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then the only element in the intersection can be 0.

(c) Let $L_n = [n, \infty)$. This cannot have any element.

(d) This is **impossible**, and we can prove it using the nested interval property.

Proof: First, we use the fact that a non-empty intersection of two closed, bounded intervals must itself be a closed bounded interval.

Now, let

$$I'_n = \bigcap_{m=1}^{n} I_m$$

define a new sequence of closed bounded intervals, which are nested by construction.

By the assumption that every finite intersection is non-empty, every I'_n must also be a non-empty closed, bounded, interval.

It is also important to note that the finite and infinite intersection of this sequence is exactly equal to the finite and infinite intersection of the original sequence.

Now, we can apply the NIP to deduce that the infinite intersection must be non-empty, which disproves the original claim. ■

1.5 Cardinality

1.5.1 **Exercise 33**

Finish the following proof for Theorem 1.5.7.

Assume B is a countable set. Thus, there exists $f : \mathbb{N} \to B$, which is 1–1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1–1 function g from $\mathbb{N}$ onto A.

For $i > 1$, let $n_i = \min\{n \in \mathbb{N} : f(n) \in A, n > n_{i-1}\}$. This must exist since A is an infinite set, thus there cannot be an upper bound on n such that $f(n) \in A$.

Now, just let $g(i) = f(n_i)$. This is an injective function, since each n_i is distinct and f is an injective function.

1.5.2 **Exercise 34**

Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbb{Q}$ is uncountable:

Assume, for contradiction, that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbb{Q}$ must therefore be uncountable.

NIP is not true in general over the rationals, since the element could be an irrational.

1.5.3 **Exercise 35**

Use the following outline to supply proofs for the statements in Theorem 1.5.8.

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this s that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Now, explain how the more general statement in (i) follows.

(b) Explain why induction *cannot* be used to prove part (ii) of Theorem 1.5.8 from part (i).

(c) Show how arranging $\mathbb{N}$ into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof of Theorem 1.5.8 (ii).

(a) Select $B_2 = A_2 \setminus A_1$. For enumeration purposes, alternate elements from A_1 and B_2. If B_2 is finite, after all elements are enumerated, continue enumerating from A_1.

Now, continue this enumeration strategy all the way to A_m by doing this same alternating enumeration strategy for $\bigcup_{i=1}^{n} A_i$ and A_{n+1}.

(b) Induction may not work, since induction only makes it hold for finite unions, not infinite unions.

(c) If we arrange the countable sets in rows, then we can visit by zig-zag pattern.

1.5.4 **Exercise 36**

(a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1–1, onto functions, but it is not really necesssary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1–1 onto function between the two sets.

(a) We can map (a,b) to $(0,1)$.

Alternatively, for a direct bijection, consider the function

$$f(x) = \tan\left(\left(\frac{x-a}{b-a} - \frac{1}{2}\right)\pi\right).$$

(b) Choose the function $\ln(x-a)$.

(c) Let $f : [0,1) \to (0,1)$ be defined the following way:

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ x & \text{else.} \end{cases}$$

1.5.5 **Exercise 37**

(a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

(a) Use identity bijection.

(b) Bijection has bijection inverse.

(c) Let $f : A \to B$ be a bijection, and $g : B \to C$ be another. Then $g \circ f$ is also a bijection.

Proof:

Injective: If $x \neq y$ for $x, y \in A$ then $f(x) \neq f(y)$. Since g is also injective we also have $g(f(x)) \neq g(f(y))$.

Surjective: For all $c \in C$, there exists $b \in B$ such that $g(b) = c$. Since f is also surjective, there must exist $a \in A$ such that $f(a) = b$. Thus, for all $c \in C$, there exists $a \in A$ such that $g(f(a)) = c$. ■

1.5.6 **Exercise 38**

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

(a) Let $I_n = (n, n+1)$. Clearly disjoint, and open intervals.

(b) This cannot exist. If it did, then consider by density of rational numbers that there would exist a distinct rational in each interval, thus implying there would be uncountable rationals.

1.5.7 **Exercise 39**

Consider the open interval $(0,1)$, and let S be the set of points in the open unit square; that is, $S = \{(x,y) : 0 < x, y < 1\}$.

(a) Find a 1–1 function that maps $(0,1)$ into, but not necessarily onto, S. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to produce a 1–1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999....)

The Schröder–Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

(a) Consider $f(x) = \left(\frac{x}{2}, x\right)$.

(b) For $(x, y) \in S$, consider the (potentially countable) decimal expansion of each. Let's label the expansion of x as $0.d_1 d_2 d_3 d_4 \ldots$, and the second as $0.d'_1 d'_2 d'_3 d'_4 \ldots$. We consider the terminating decimal expansion representations, rather than one with inifinite 9s.

Now we simply map (x, y) to the following real number:

$$f((x, y)) = 0.d_1 d'_1 d_2 d'_2 d_3 d'_3 \ldots.$$

To see that this is injective, note that if two intervals differ from each other, that at least one of the left or right endpoints must differ. Since they differ, they must have a different decimal expansion, and thus the resulting real number will also have a different digit and be a different real number.

This logic only fails if we somehow produce a real number that ends in repeating 9s, which is impossible since it would imply that both of our original expansions were of that form.

However, this function is not surjective.

Consider a real number that, for example, ends in alternating 1s and 9s. This itself is a unique real number with no other decimal representation, but the only way to construct it would be with a decimal with repeating 9s. This representation is not in our domain, so there is no way to output this real number.

1.5.8 **Exercise 40**

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

First, note that $B \subseteq (0, 2]$.

For arbitrary $n \in \mathbb{N}$, consider the subset $\left(\frac{1}{n}, 2\right] \cap B$. This can only have finite elements, since otherwise, we could choose $2n$ elements from the subset to sum to greater than 2.

Note that this holds true for all n.

Now also note that $B = \bigcup_{n=1}^{\infty} \left[\left(\frac{1}{n}, 2\right] \cap B\right]$. This is a countable union of finite sets, which is countable.

1.5.9 **Exercise 41**

A real number $x \in \mathbb{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3}+\sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbb{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

(a) Consider the following:

$$x^2 - 2 = 0, \quad x^3 - 2 = 0, \quad x^4 - 10x^2 + 1 = 0.$$

(b) There are a countable number of integer polynomials of degree n, since it can be defined uniquely with a finite product of countable sets.

Since each has finite solutions, the total number of solutions and thus elements of A_n is a countable union of finite sets and is countable.

(c) We simply take the countable union of all A_n for all n.

Again, a countable union of countable sets is countable.

Since the algebraic numbers are countable, the rest of the reals (transcendental) must be uncountable.

1.5.10 **Exercise 42**

(a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

(a) AFSOC that there does not exist such an a.

Then for all $a \in (0,1)$, $C \cap [a,1]$ is countable.

Examine the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{1}{n+1}$.

Clearly, we have that $C = \left(\bigcup_{n=1}^{\infty} C \cap \left[\frac{1}{n+1}, 1\right]\right) \cup (C \cap \{0\})$.

This is simply a countable union of countable sets, which implies that C is countable.

Contradiction!

Therefore there must exist some $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Not necessarily. Consider if $C = [0,1]$. Then any $a \in (0,1)$ will produce an uncountable set $[a,1]$. The supremum of A is 1. But $C \cap [1,1] = \{1\}$, which is finite.

(c) No. Let $C = \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$. All choices of a lead to a finite intersection.

1.5.11 (Schröder–Bernstein Theorem) **Exercise 43**

> Assume there exists a 1–1 function $f : X \to Y$ and another 1–1 function $g : Y \to X$. Follow the steps to show that there exists a 1–1, onto function $h : X \to Y$ and hence $X \sim Y$.
>
> The strategy is to partition X and Y into components
>
> $$X = A \cup A' \qquad \text{and} \qquad Y = B \cup B'$$
>
> with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.
> (a) Explain how achieving this would lead to a proof that $X \sim Y$.
> (b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is a similar collection in Y.
> (c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.
> (d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

(a) If we restrict the domain of f to A, then it is a bijection between A and B.

Similarly, if we restrict g to B', then g is a bijection between B' and A'.

Now, we can just define $h : X \to Y$ the following way:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{else.} \end{cases}$$

This is clearly a bijection.

(b) First, if $A_1 = \emptyset$, then we are done. This is because $g(Y) = X$, which implies that g is a bijection and we are done.

Assuming that A_1 is non-empty, we can proceed by induction.

Proof: **Base case:** Notice that $f(A_1) \subseteq Y$. Thus $g(f(A_1)) \subseteq g(Y)$, so $(X \setminus g(Y)) \cap (g(f(A_1))) = \emptyset$.

Inductive hypothesis: Assume for some n that $A_1, ..., A_n$ are pairwise disjoint. Thus, $f(A_n)$ is also disjoint from all $f(A_1), ..., f(A_{n-1})$, since f is injective. By the same logic, $g(f(A_1)), ..., g(f(A_n))$ are all also disjoint.

Since $g(f(A_i)) = A_{i+1}$, we have that A_{n+1} is disjoint from all $A_2, ..., A_n$.

It is also disjoint with A_1 by similar logic from the base case.

Note for completeness, if at any point any A_i is empty, then we can just stop with finite A_i that are all pairwise disjoint. ■

Also note that this implies that all $\{f(A_n) : n \in \mathbb{N}\}$ are pairwise disjoint since f is injective.

(c) If $b \in B$, then it must exist in exactly one $f(A_i)$. This means that there must be some $a \in A_i$ such that $f(a) = b$, which shows that f is surjective.

(d) First, let's note that $X \setminus A$ is a subset of $g(Y)$. This is because if $a \in X \setminus A$, then $a \in X \setminus A_1 = X \setminus (X \setminus g(Y)) = g(Y)$.

So we know there must **exist** some $b \in Y$ such that $g(b) = a$.

We should also argue that this b cannot be in B.

AFSOC that $b \in B$. Then $b \in f(A_n)$ for some n, and thus $g(b) \in g(f(A_n)) = A_{n+1}$. However, this is clearly disjoint with $X \setminus A$, so it must be the case that $b \notin B \implies b \in Y \setminus B$.

1.6 Cantor's Theorem

1.6.1 **Exercise 44**

Show that $(0, 1)$ is uncountable if and only if $\mathbb{R}$ is uncountable. This shows that Theorem 1.6.1 is equivalent to Theorem 1.5.6.

($\Rightarrow$) This direction is easy, since if $(0, 1)$ is uncountable, then clearly since $(0, 1) \subseteq \mathbb{R}$, the real numbers must also be uncountable.

($\Leftarrow$) If $(0, 1)$ is countable, then $\mathbb{R}$ must be countable, since we can construct $\mathbb{R}$ from $(0, 1)$ using a countable union of the integers plus $(0, 1)$.

1.6.2 **Exercise 45**

(a) Explain why the real number $x = .b_1 b_2 b_3 b_4 \ldots$ cannot be $f(1)$.
(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbb{N}$.
(c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

(a) It must differ from $f(1)$ at the first digit by construction.
(b) It must differ from the nth digit of $f(n)$ by construction.
(c) This shows that there must be a real number that is not in our enumeration. But we assumed we could enumerate them. This is the contradiction, and thus $(0, 1)$ is uncountable.

1.6.3 **Exercise 46**

Supply rebuttals to the following complaints about the proof of Theorem 1.6.1

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.
(b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, 1/2 can be written as 0.5 or as .4999…. Doesn't this cause some problems?

(a) In general, the number that is produced may not be a rational.
(b) No, this is fine. Let's just only consider non-repeating 9's representation, and note that with our construction, we will never produce a number that runs into this issue.

1.6.4 **Exercise 47**

Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \ldots)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, \ldots)$.

Give a rigorous argument showing that S is uncountable.

Cantor's diagonalization argument.

Produce a new binary sequence that differs from all other sequences at the nth element.

1.6.5 **Exercise 48**

(a) Let $A = \{a, b, c\}$. List the eight elements of $\mathcal{P}(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)
(b) If A is finite with n elements, show that $\mathcal{P}(A)$ has 2^n elements.

(a) $\emptyset, \{a\}, \{a, b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.
(b) An element can either be in or out of a subset, which gives us two choices per element. Thus there are 2^n distinct subsets.

1.6.6 **Exercise 49**

(a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1–1 mappings from A into $\mathcal{P}(A)$.
(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1–1 map $g : C \to \mathcal{P}(C)$.
(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are *onto*.

(a) One mapping:

$$a \to \{a\}, \quad b \to \{b\}, \quad c \to \{c\}.$$

Another mapping:

$$a \to \{a, b\}, \quad b \to \{b\}, \quad c \to \{c\}.$$

(b) $$1 \to \{1\}, \quad 2 \to \{2\}, \quad 3 \to \{3\}, \quad 4 \to \{4\}.$$
(c) There are strictly more elements in the range than the domain.

1.6.7 **Exercise 50**

Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule. In each case, note that B is not in the range of the function used.

TODO: skipped

1.6.8 **Exercise 51**

(a) First, show that the case $a' \in B$ leads to a contradiction.
(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

(a) If $a' \in B$, then it must be that $a' \notin f(a')$ by definition of B.

However, $f(a') = B$ by assumption, so we have shown that $a' \in B$ and $a' \notin B$ which is a contradiction.
(b) If $a' \notin B$, then it must be that $a' \notin f(a')$. This implies that it must be in B which is again a contradiction.

1.6.9 **Exercise 52**

Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

First, we construct an injection from $(0, 1)$ to the set of infinite binary sequences. We do this by considering the decimal expansion.

Next, we construct an injection from the set of infinite binary sequences to $(0, 1)$.

This is a little trickier, as a direct conversion would result in some numbers that are actually the same real number. (For example, 0.0111… and 0.1).

We can first consider all sequences that do not end in repeating 1's. This will map into $[0, 1)$, which we know has a bijection with $(0, 1)$. We can divide the result by 3 to get an injection into $\left(0, \frac{1}{3}\right)$.

Next, we map the sequences that end in infinite 1's to their representative real number, divide by 3, and then add $\frac{1}{3}$ to get an injection into $\left(\frac{1}{3}, \frac{2}{3}\right]$.

This completes the injection into $(0, 1)$, so using Schröder–Bernstein we can conclude that the set of infinite binary sequences has the same cardinality as $(0, 1)$, and we can use transitivity of this equivalence relation to deduce that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

1.6.10 **Exercise 53**

As a final exercise, answer each of the following by establishing a 1–1 correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ countable or uncountable?

(b) Is the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ countable or uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $\mathcal{P}(B)$ is called an *antichain* if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $\mathcal{P}(\mathbb{N})$ contain an uncountable antichain?

(a) This is countable, since there only needs to be two natural numbers to specify the function fully. This essentially reduces to the set with $(n, m) \in \mathbb{N}^2$.

(b) This is uncountable. This is equivalent to the set of infinite sequences of 0's and 1's, which is shown to be uncountable due to a diagonalization argument.

(c) There exists an uncountable antichain. Consider the following bijection between an infinite binary sequence and a subset of the natural numbers:

$$f((b_n)) = \{n : n = 2i + b_i, i \in \mathbb{N}\}.$$

In plain English, for every distinct pair of adjacent natural numbers, we select only one of them based off of the ith value of the binary sequence. If a binary sequence is distinct from another binary sequence, then transformed into subset world, each subset will have an element that is not included in the other.

Considering this bijection, this antichain must be uncountable.

1.7 Epilogue

No exercises in this section.

2 Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

No exercises in this section.

2.2 The Limit of a Sequence

2.2.1 **Exercise 54**

What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

An example is the sequence of alternating 0's and 1's.

This is vercongent to any real number. We can just select large enough ϵ and it will work out.

I believe that this is actually describing bounded sequences.

2.2.2 **Exercise 55**

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

(a) I claim we need to choose $N > \frac{3}{25\epsilon} - \frac{4}{5}$.

Proof: Let $\epsilon > 0$. Choose $N > \frac{3}{25\epsilon} - \frac{4}{5}$. Now for $n \geq N$, we can verify that:

$$\begin{aligned} \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &= \left|\frac{3/5}{5n+4}\right| \\ &= \frac{3/5}{5n+4} \\ &< \frac{3/5}{5\left(\frac{3}{25\epsilon} - \frac{4}{5}\right) + 4} \\ &= \epsilon \end{aligned}$$

as desired. ■

(b) I claim we choose $N > \frac{2}{\epsilon}$.

Proof: Let $\epsilon > 0$. Choose $N > \frac{2}{\epsilon}$. Notice that $\left|\frac{2n^2}{n^3+3}\right|$ is always positive if $n > 0$. For $n \geq N$, we have that:

$$\left|\frac{2n^2}{n^3+3}\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} < \epsilon$$

as desired. ■

(c) I claim we choose $N > \frac{1}{\epsilon^3}$.

Proof: Let $\epsilon > 0$. Choose $N > \frac{1}{\epsilon^3}$. Notice that $|\sin(n^2)| \leq 1$ always.

If $n \geq N$, we can see that $n > \frac{1}{\epsilon^3}$ or alternatively $\sqrt[3]{n} > \frac{1}{\epsilon}$.

Therefore, we have that:

$$\begin{aligned}\left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| &= \frac{|\sin(n^2)|}{\sqrt[3]{n}} \\ &\leq \frac{1}{\sqrt[3]{n}} < \epsilon.\end{aligned}$$

as desired. ■

2.2.3 **Exercise 56**

Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.
(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.
(c) There exists a college in the United States where every student is at least six feet tall.

(a) Find a college in the US where all students are below 7 feet tall.
(b) Find a college in the US where all professors give out grades other than A or B.
(c) Show that all colleges have a student under 6 feet tall.

2.2.4 **Exercise 57**

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.
(b) A sequence with an infinite number of ones that converges to a limit not equal to one.
(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

(a) Alternating 0's and 1's.
(b) Not possible. If we select $\epsilon < |x-1|$, where x is the "limit", then we can see that there can never be a N such that every element in the sequence after that is within that ϵ-neighborhood. This is because there must be infinite ones, which cannot all be in the first N elements.
(c) Yes, just do $1, 0, 1, 1, 0, 1, 1, 1, 0, \ldots$. This can never converge due to a similar argument to part (b). But by construction, we can always find n consecutive ones.

2.2.5 **Exercise 58**

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.
(a) $a_n = [[5/n]]$,
(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

(a) Claim: $\lim a_n = 0$.

Proof: After $N > 5$, all $n \geq N$ will be such that $a_n = 0$. ■

(b) Claim: $\lim a_n = 1$.

Proof: After $N > 6$, for $n \geq N$, the inner part of a_n will be less than 2. In addition, the inner part will always be greater than $4/3$. Therefore after $N > 6$ every element in the sequence will equal 1 exactly. ■

2.2.6 **Exercise 59**

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

We start with the stated assumptions.

AFSOC $a \neq b$, then we could choose $\epsilon < \frac{|a-b|}{2}$.

By the definition of limits, there would exist N and N' such that any $n \geq \max(N, N')$ satisfies $|x_n - a| < \epsilon$ and $|x_n - b| < \epsilon$.

Using the triangle inequality, we know that

$$|a - b| = |a - x_n + x_n - b| \leq |x_n - a| + |x_n - b| < 2\epsilon < |a - b|.$$

In other words, we have shown that $|a - b| < |a - b|$. This is a **contradiction**.

Therefore, it must be the case that $a = b$.

2.2.7 **Exercise 60**

Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

(a) Frequently.
(b) Eventually implies frequently. To see this, notice that for any natural number, if it is less than or equal to N, then we can just use any number after N as our n, and if it is greater than N, then any number greater than our current number should work.
(c) A sequence (a_n) converges to a if for any ϵ-neighborhood of a, the sequence is eventually in it.
(d) (x_n) is not necessarily eventually in it, as we could have also an infinite number of terms that are 2.2 for example.

However, it is definitely the case that (x_n) is frequently within those bounds.

2.2.8 **Exercise 61**

For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 0, 1, \ldots)$ zero-heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counter example.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if....

(a) The given sequence is zero-heavy. Consider $M = 1$. Since there are never two 1's in a row, this is a valid M.

(b) A zero-heavy sequence must contain an infinite amount of 0's. Otherwise, we could consider the first index N after which there are no more 0's, and see that no value of M will produce an interval that contains a 0.

(c) No. Consider the sequence $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \ldots)$. Given any M, once we are far enough in the sequence, we will always be able to find a string of 1's that is longer than M.

(d) A sequence is not zero-heavy if for all $M \in \mathbb{N}$, there exists a $N \in \mathbb{N}$ such that for all n satisfying $N \leq n \leq N + M$, $x_n \neq 0$.

2.3 The Algebraic and Order Limit Theorems

2.3.1 **Exercise 62**

Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

(a) Let arbitrary $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n| < \epsilon^2$. Thus we can reuse the same N for $|\sqrt{x_n}| < \epsilon$.

(b) Assume $x > 0$. (This is valid due to Order Limit Theorem). Let arbitrary $\epsilon > 0$. Now observe the following:

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \frac{|x_n - x|}{|\sqrt{x_n} + \sqrt{x}|} \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{|x_n - x|}{\sqrt{x}} \\ &< \frac{\epsilon'}{\sqrt{x}} \text{ for } n \text{ larger than some } N \in \mathbb{N}. \end{aligned}$$

If we choose $\epsilon' = \epsilon\sqrt{x}$, then we get that for $n \geq$ some $N \in \mathbb{N}$ that

$$|\sqrt{x_n} - \sqrt{x}| < \epsilon.$$

2.3.2 **Exercise 63**

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

(a)
$$\left|\frac{2x_n - 1}{3} - 1\right| = \left|\frac{2x_n - 4}{3}\right| = \frac{2}{3}|x_n - 2| < \frac{2}{3}\epsilon'.$$

Choose $\epsilon' = \frac{3}{2}\epsilon$.

(b)
$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \frac{|x_n - 2|}{2|x_n|}$$

Choose N_1 such that we get $|x_n| > \frac{|x|}{2}$. Now choose $\epsilon' = |x|\epsilon$. So for $\max(N_1, N_2)$ we have

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| < \frac{|x_n - 2|}{|x|} < \frac{\epsilon'}{|x|} = \epsilon.$$

2.3.3 (Squeeze Theorem) **Exercise 64**

Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Since $x_n \leq y_n \leq z_n$, we also get that

$$x_n - l \leq y_n - l \leq z_n - l.$$

Choose large enough N such that for $n \geq N$ we get that $z_n - l \leq |z_n| - l < \epsilon$, as well as $x_n - l > -|x_n - l| > \epsilon$. This leaves us with:

$$-\epsilon < y_n - l < \epsilon, \quad \Longrightarrow \quad |y_n - l| < \epsilon.$$

Thus y_n converges and it must converge to l.

2.3.4 **Exercise 65**

Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$

(a) 1.

(b)
$$\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) = \lim\left(\frac{a_n^2+4a_n}{a_n}\right) = \lim(a_n+4) = 4.$$

(c)
$$\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) = \lim\left(\frac{2+3a_n}{1+5a_n}\right) = 2.$$

2.3.5 **Exercise 66**

Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, ..., x_n, y_n, ...)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

($\Rightarrow$) Assume that (z_n) is convergent. Then after some $N \in \mathbb{N}$, for $n \geq N$ we have that all $|z_n - z| < \epsilon$ for some $z \in \mathbb{R}$ and arbitrary ϵ.

Then we also get that for $n > \frac{N}{2}$, both $|x_n - z| < \epsilon$ and $|y_n - z| < \epsilon$, which shows they both converge to z.

($\Leftarrow$) Assume that x_n and y_n both converge to z. For arbitrary $\epsilon > 0$, pick $N = \max\{N_1, N_2\}$ such that for $n \geq N$, we have $|x_n - z| < \epsilon$ and $|y_n - z| < \epsilon$.

Therefore for $n \geq 2N$, we have that $z_n = x_{\lfloor \frac{n}{2} \rfloor}$ or $y_{\lfloor \frac{n}{2} \rfloor}$ is such that $|z_n - z| < \epsilon$, and we have shown that (z_n) converges to z.

2.3.6 **Exercise 67**

Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Multiply top and bottom by the conjugate:

$$\begin{aligned} n-\sqrt{n^2+2n} &= \frac{n^2-n^2-2n}{n+\sqrt{n^2+2n}} \\ &= \frac{-2n}{n+\sqrt{n^2+2n}} \\ &= \frac{-2}{1+\sqrt{1+\frac{2}{n}}}. \end{aligned}$$

The sequence defined by $1+\frac{2}{n}$ is positive and approaches the limit 1, so therefore the square root of the sequence does as well.

Thus, the original sequence for the entire expression approaches -1 by the ALT.

2.3.7 **Exercise 68**

Give and example of each of the following, or state the such a request is impossible by referencing the proper theorems(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum (x_n+y_n) converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and (x_n+y_n) converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with (a_n-b_n) bounded;

(e) two sequences (a_n) and (b_n), where (a_nb_n) and (a_n) converge but (b_n) does not.

(a) Yes, consider $x_n=(-1)^n$, and $y_n=(-1)^{n+1}$. Their sum is simply 0.

(b) No, by ALT we would have that $(x_n+y_n-x_n)$ converges.

(c) Consider $b_n=\frac{1}{n}$. Then $\frac{1}{b_n}=n$ which clearly diverges.

(d) Since every convergent sequence is bounded, we know that $|b_n| \leq M$, where M is the bound on b_n and N is the bound on a_n-b_n.

So therefore

$$|a_n|=|a_n-b_n+b_n| \leq |a_n-b_n|+|b_n| \leq M+N,$$

and (a_n) must be bounded as well.

(e) Let $a_n=0$ for all n, and let $b_n=n$.

Clearly their product is 0 for all n.

2.3.8 **Exercise 69**

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

(a) Follows directly from ALT, since a polynomial is simply a combination of multiplications and additions.

(b) Let $f(x)$ be the following:

$$f(x) = \begin{cases} 5 \text{ if } x = 0 \\ x \text{ else} \end{cases}.$$

Now, $f(0) = 5$, but any sequence that approaches 0 but never reaches it will instead approach the limit value 0.

2.3.9 **Exercise 70**

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?
(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?
(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

(a) Since a_n is bounded, we have that $|a_n| \leq M$ for all n.

Now for arbitrary $\epsilon' > 0$, there is some N such that $n \geq N$ implies

$$|a_n b_n| = |a_n||b_n| \leq M|b_n| < M\epsilon'.$$

Choose $\epsilon' = \frac{\epsilon}{M}$, and we have that $(a_n b_n)$ converges to 0.

We can't use ALT since (a_n) is not necessarily convergent, just bounded.

(b) No. Consider the constant sequence $b_n = 1$, which is clearly convergent. However, given any bounded and not convergent sequence (a_n), we have that $(a_n b_n) = (a_n)$. However I do believe that the product sequence is still bounded… I won't prove this.

(c) Since all convergent sequences are bounded, we can just use our result from (a) directly.

2.3.10 **Exercise 71**

Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.
(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.
(b) If $(b_n) \to b$, then $|b_n| \to |b|$.
(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.
(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \to b$.

(a) False, consider $a_n = n$, $b_n = n$. They have no limit, but their difference is simply 0.

(b) $||b_n| - |b|| \leq |b_n - b|$ by reverse triangle inequality.

(c) Directly follows from ALT.

(d) Yes, because $|b_n - b| \leq a_n \leq |a_n| < \epsilon$ for all $n \geq N \in \mathbb{N}$ for any arbitrary ϵ.

2.3.11 (Cesaro Means) **Exercise 72**

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \dots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

(a) Assume $(x_n) \to x$.

Let $\epsilon > 0$ be arbitrary.

Past some $N_1 \in \mathbb{N}$ we have that $|x_n - x| < \frac{\epsilon}{2}$.

For all x_i for $i < N_1$, let $M = \max\{|x_i - x|\}$.

Let $N_2 \in \mathbb{N}$ be such that for $n \geq N_2$, we have $\frac{1}{n} < \frac{\epsilon}{2N_1M}$.

Now for $N = \max\{N_1, N_2\}$, we have for $n \geq N$

$$\begin{aligned}\left|\frac{x_1 + x_2 + \dots + x_n}{n} - x\right| &= \frac{1}{n}|x_1 + x_2 + \dots + x_n - nx| \\ &\leq \frac{1}{n}(|x_1 - x + x_2 - x + \dots + x_i - x| + |x_{i+1} - x + \dots + x_n - x|) \\ &< \frac{1}{n}\left(N_1M + (n - N_1)\frac{\epsilon}{2}\right) \\ &\leq \frac{1}{n}(N_1M) + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

(b) The alternating sequence of 0's and 1's does not converge. However, the sequence of averages will converge to $1/2$.

2.3.12 **Exercise 73**

A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

(a) True. AFSOC that a is not an upper bound for the set B. Then there is some $b \in B$ such that $b > a$. Let $\epsilon = b - a$. Then there is some $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n - a| < \epsilon$. Clearly all a_n must be larger than a, so we have that $a_n - a < b - a$, so then $a_n < b$, and we have shown our contradiction since we assumed that all a_n would also be upper bounds.

(b) True. AFSOC that $a \in (0,1)$. Then choose $\epsilon = \frac{1}{2}\min\{a, 1-a\} > 0$. There must be an a_n within that ϵ-neighborhood, which is clearly not in the complement of $(0,1)$.

(c) False, consider the sequence defined by the decimal approximation of π.

2.3.13 (Iterated Limits) **Exercise 74**

Given a doubly indexed array a_{mn} where $m, n \in \mathbb{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the *iterated* limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right).$$

Define $\lim_{m,n} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2+n^2)$.

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

(a) The first limit is equal to 1, while the second limit is equal to 0.

(b) Yes, the limit is 0. Yes, both iterated limits exist and are 0.

For $a_{mn} = mn/(m^2+n^2)$, $\lim_{m,n} a_{mn}$ does not exist, since we can make the sequence approach different values. (This is not super rigorous, but it is if we assume the result that a limit can only have one value.)

However, the iterated limits exist and are both 0.

(c) Choose the following:

$$a_{mn} = (-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right)$$

The iterated limits do not exist, as they will oscillate between $\frac{1}{m}$ and $-\frac{1}{m}$ or $\frac{1}{n}$ and $-\frac{1}{n}$.

However, $\lim_{m,n} a_{mn} = 0$, which can be easily proven by triangle inequality.

(d) Let $\epsilon > 0$ be arbitrary.

Let's use the triangle inequality:

$$\begin{aligned} |b_m - a| &= |b_m - a_{mn} + a_{mn} - a| \\ &\leq |a_{mn} - b_m| + |a_{mn} - a| \\ &\leq \epsilon' + \epsilon'' \end{aligned}$$

Find N and M such that we approach $\epsilon' = \epsilon'' = \frac{\epsilon}{2}$, and take $m \geq \max\{N, M\}$ to finish the proof.

(e) This is just part (d).

2.4 The Monotone Convergence Theorem and Infinite Series

2.4.1 **Exercise 75**

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

(a) I claim that the sequence is monotone decreasing.

Proof: BC: $x_2 = 1 < 3 = x_1$.

IH: Assume true for some n (that $x_n \leq x_{n-1}$). Then:

$$4 - x_n \geq 4 - x_{n-1} \implies \frac{1}{4 - x_n} \leq \frac{1}{4 - x_{n-1}} \implies x_{n+1} \leq x_n.$$

■

It is also bounded below by $\frac{1}{4}$, which can also be proved by induction. The base case is obvious, and if we assume $x \geq \frac{1}{4} > 0$, then

$$x_{n+1} = \frac{1}{4 - x_n} \geq \frac{1}{4}.$$

So now MCT finishes the argument.

(b) It's literally the same, just missing the first term.

For $n \geq N \in \mathbb{N}$, we can see that $\max\{0, N - 1\}$ works in the limit argument.

(c)
$$x = \frac{1}{4 - x} \implies x = 2 - \sqrt{3}.$$

$2 + \sqrt{3}$ is not valid since it is greater than 3.

2.4.2 **Exercise 76**

(a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$.

What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

(a) There may not be a limit in the first place. In this case, there is not, since it oscillates between 1 and 2.

(b) Yes, we can use a similar approach to 2.4.1 to show that it is monotone increasing and bounded above by 3.

2.4.3 **Exercise 77**

(a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

(a) The recursive formula for the sequence is $x_{n+1} = \sqrt{2 + x_n}$.

By induction (using the fact that the square root function is increasing), the sequence is increasing.

I also claim it is bounded above by 2.

Proof: BC: $\sqrt{2} < 2$.

IH: assume true for some n.

Now:

$$\sqrt{2 + x_n} < \sqrt{4} = 2.$$

∎

Now we apply MCT.

Taking the limit of both sides of the recursive formula, we get that $x = 2$.

(b) By similar argument to above, we claim the sequence is monotone increasing and the upper bound is 2. The limit is 2.

2.4.4 **Exercise 78**

(a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbb{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.

> These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

(a) Assume $y > 0$, otherwise we can just choose any $n \in \mathbb{N}$ and be done.

Now notice that the sequence $\frac{1}{n}$ is bounded below by 0, and is also monotone decreasing.

This suggests that it converges to a limit by MCT.

To produce the limit, notice that $\lim \frac{1}{n} = \lim \frac{1}{n+1}$.

We can recursively see that:

$$\frac{1}{n+1} = \frac{1}{n} \cdot \left(\frac{n}{n+1}\right) = \frac{1}{n} \cdot \left(1 - \frac{1}{n+1}\right).$$

So by ALT we can see that the limit s must be such that:

$$s = s(1-s) \implies s = 0.$$

Therefore by the definition of a limit we get that for arbitrary $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$:

$$\left|\frac{1}{n}\right| < \epsilon.$$

Take $\epsilon = y$, and note that $\frac{1}{n}$ is always positive to get the Archimedean Property:

$$\frac{1}{n} < y \text{ for large enough } n \in \mathbb{N}.$$

(b) Note that the sequences (a_n) and (b_n) are monotone increasing/decreasing and bounded above/below.

Therefore by MCT they must converge to some a and b respectively. In addition, its clear that $a_n \leq a$ and $b_n \geq b$ for all n.

I claim that $a \leq b$. If it were the case that $a > b$, then we could set $\epsilon = a - b$, and select some b_n such that $b_n - b = |b_n - b| < a - b$.

With some algebra we get:

$$b_n - a_n < a - a_n \leq 0 \quad \implies \quad b_n < a_n.$$

This is impossible for an interval, which is a **contradiction**. Therefore it must be that $a \leq b$ and thus we can choose any x such that $a \leq x \leq b$, and it will be present in every interval. Thus the interval will not be empty.

2.4.5 **Exercise 79**

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

(a) Clearly $x_1^2 = 4$. Let's work out x_{n+1}^2:

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) \\ &= \frac{1}{4}\left(x_n^2 - 4 + \frac{4}{x_n^2}\right) + 2 \\ &= \frac{1}{4}\left(x_n - \frac{2}{x_n}\right)^2 + 2 \\ &\geq 2. \end{aligned}$$

This applies for all n.

Now, let's look at $x_n - x_{n+1}$:

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= \frac{1}{2}x_n - \frac{1}{x_n} \\ &= \frac{x_n^2 - 2}{2x_n} \\ &\geq 0. \end{aligned}$$

The last inequality relies on the fact that $x_n^2 \geq 0$, as well as the fact that $x_n > 0$ for all n (this is easy to see).

Thus, we have that $x_{n+1} \leq x_n$ and the sequence is monotone decreasing, while being bounded below by $\sqrt{2}$.

It therefore has a limit, and we can take the limit of both sides of the recursive formula to work it out:

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right) \implies x = \sqrt{2}.$$

(b) I claim

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

works.

Through similar steps to part (a), we first show that $x_n^2 \geq c$ for all n:

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{c^2}\left(x_n^2 + 2c + \frac{c^2}{x_n}\right) \\ &= \frac{1}{4}\left(x_n^2 - 2c + \frac{c^2}{x_n}\right) + c \\ &\geq c. \end{aligned}$$

Then we show that the sequence is monotone decreasing:

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{c}{x_n}\right) \\ &= \frac{1}{2}\left(x_n - \frac{c}{x_n}\right) \\ &= \frac{1}{2}\left(\frac{x_n^2 - c}{x_n}\right) \\ &\geq 0. \end{aligned}$$

Then by MCT the limit exists, and we can compute it:

$$x = \frac{1}{2}\left(x + \frac{c}{x}\right) \implies x = \sqrt{c}.$$

2.4.6 (Arithmetic–Geometric Mean) **Exercise 80**

(a) Explain why $\sqrt{xy} \leq (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

(a)

$$\begin{gathered} \left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0 \iff \\ x - 2\sqrt{xy} + y \geq 0 \iff \\ \frac{x+y}{2} \geq \sqrt{xy}. \end{gathered}$$

(b) First, note that both y_n and x_n are bounded below by 0 for all n by closure of positive numbers under addition, multiplication, and square root.

First, by AM-GM inequality, $x_n \leq y_n$ for all n.

Next, I claim that (y_n) is monotone decreasing.

Proof:

$$y_{n+1} = \frac{x_n + y_n}{2} \leq y_n.$$

■

I claim that (x_n) is monotone increasing.

Proof:

$$x_{n+1} = \sqrt{x_n y_n} \geq x_n.$$

■

Note also that (x_n) is bounded above by y_1, since every $x_n \leq y_n \leq y_1$.

Therefore by MCT, the limit exists for both sequences. To find the limit, let's solve for the limits in one of the recursive formulas:

$$y = \frac{x+y}{2} \implies x = y.$$

This checks out with the other formula.

2.4.7 (Limit Superior) **Exercise 81**

Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n.$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

(a) (y_n) must be bounded below, otherwise that would imply that (a_n) is not bounded below.

In addition, (y_n) is monotone decreasing. From y_n to y_{n+1}, we are only ignoring one element, which can never increase the supremum, only possibly decrease it (or keep it the same).

Thus, by MCT this sequence converges.

(b) Let (y_n) be defined as $y_n = \inf\{a_k : k \geq n\}$.

Then $\liminf a_n = \lim y_n$.

This exists by similar argument to part (a).

(c) It is clear that for every n, $\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$. Thus by the OLT their limits must follow the same inequality.

One example where equality holds is simply the constant 0 sequence.

(d) ($\Rightarrow$) We can directly apply the squeeze theorem for this direction, since $\inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\}$.

($\Leftarrow$) Since $\lim a_n$ exists, we know that for arbitrary $\epsilon' > 0$ there exists an N after which all a_n exist within the ϵ'-neighborhood of a.

The supremum of all of those points must also exist either within that neighborhood or on its boundary.

Therefore the $\limsup a_n$ must converge to a as well, if we just select $0 < \epsilon' < \epsilon$.

The same argument applies for $\liminf a_n$.

2.4.8 **Exercise 82**

For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

(a) $$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

(c) $$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$$

(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

(a) $s_n = 1 - \frac{1}{2^n}$.

This converges to 1.

(b) $s_n = 1 - \frac{1}{n+1}$.

This converges to 1.

(c) $s_n = \log(n+1)$.

This does not converge, as it grows unbounded. ($\log n$ is unbounded above, it is easily shown that it contains an unbounded subsequence that grows like n).

2.4.9 **Exercise 83**

Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges.

Let's take a closer look at the partial sums of $\sum_{n=1}^{\infty} b_n$.

Particularly, let's look at the sequence of partial sums defined by s_{2^k}:

$$\begin{aligned} s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + \cdots + b_8) + \cdots + \left(b_{2^{k-1}+1} + \cdots + b_{2^k}\right) \\ &\geq b_1 + 1 \cdot b_2 + 2 \cdot b_4 + 4 \cdot b_8 + \cdots + 2^{k-1} \cdot b_{2^k} \\ &\geq \frac{1}{2}\left(b_1 + 2 \cdot b_2 + 4 \cdot b_4 + 8 \cdot b_8 + \cdots + 2^k \cdot b_{2^k}\right) \end{aligned}$$

This is unbounded, otherwise we could show that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges because its partial sums converge.

2.4.10 (Infinite Products) **Exercise 84**

A close relative of infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \ldots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \ldots b_m .$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots, \quad \text{where } a_n \geq 0.$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

(a) From a few calculations, and verified by induction, we can see that $p_m = m + 1$. This clearly does not converge.

For $1/n^2$:

$$\begin{aligned}
p_1 &= (1+1) &&= 2 \\
p_2 &= (1+1)\left(1 + \frac{1}{4}\right) &&= \frac{5}{2} \\
p_3 &= \left(\frac{5}{2}\right)\left(\frac{10}{9}\right) &&= \frac{25}{9} \\
p_4 &= \left(\frac{25}{9}\right)\left(\frac{17}{16}\right) &&= \frac{425}{144}
\end{aligned}$$

My conjecture is that this converges. **Not proved.**

(b) ($\Rightarrow$) I wish to show that if the sequence of partial products converges, then the infinite sum converges.

Reminder, we have that $a_n \geq 0$.

It is easy to see that in p_m, it contains the partial sum $s_m = \sum_{n=1}^{m} a_n$.

To see this, we can simply expand out the product and see that s_m exists as a subset of the terms.

Thus, $p_m \geq s_m \geq 0$. Since p_m is convergent, it must be bounded above. So s_m is also bounded above, and furthermore, is monotone increasing. Thus it is also convergent.

($\Leftarrow$)

Assume the infinite sum is convergent, to some limit a.

$$p_m = \prod_{n=1}^{m} (1 + a_n) \leq \prod_{n=1}^{m} 3^{a_n} = 3^{\sum_{n=1}^{m} a_n} = 3^{s_m}.$$

I won't finish the proof rigorously, but this clearly also converges, which we use to show that the infinite product also converges.

2.5 Subsequences and the Bolzano–Weierstrass Theorem

2.5.1 **Exercise 85**

Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, ...\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, ...\}$, and no subsequences converging to points outside of this set.

(a) Impossible, since that bounded subsequence itself would have a subsequence that converges.

(b) Yes, let $a_n = \frac{1}{n+1}$ and $b_n = 1 + \frac{1}{n}$. Now alternate these.

(c) Yes, we can just choose the enumeration of the rationals between 0 and 1. We can always choose a subsequence that gets arbitrarily close to any of the numbers in the set.

(d) False, since any such sequence must also converge to 0, which I don't think is in the set.

If 0 is allowed to be in the set, then consider the following sequence:

$$(1, 1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, 1, ...)$$

Thus, every number in our sequence will appear an infinite number of times.

Any subsequence cannot converge to any number outside of this set (if it includes 0).

Assume we have a subsequence with limit $0 < x < 1$. Find the number in our infinite set that is closest to it, say $1/n$, and AFSOC $1/n \neq x$. (This is only possible if $x \neq 0$).

Now choose positive $\epsilon < \left|\frac{1}{n} - x\right|$. Since we assumed that $1/n$ is the closest possible number in our sequence to x, there are no numbers in our sequence within this neighborhood.

Thus it must be the case that $x = 1/n$ for some n.

2.5.2 **Exercise 86**

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

(a) True, since we could create a proper subsequence by discarding finite elements from the beginning. Since that converges, then clearly the original sequence also converges.

(b) True. This shows that for arbitrary $\epsilon > 0$, for all $N \in \mathbb{N}$ there will always exist some $n \geq N$ such that x_n is outside of the ϵ-neigborhood of any proposed limit, and we can choose that n from the divergent subsequence.

(c) True. Since the sequence is bounded and diverges, $\limsup x_n$ and $\liminf x_n$ must exist and differ. Thus we can also find subsequences that converge to those different values.

(d) True, since the convergent subsequence is bounded. The original sequence must obey the same bounds, and since it is monotone it must also be convergent.

2.5.3 **Exercise 87**

(a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \ldots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$\left(a_1 + a_2 + \ldots + a_{n_1}\right) + \left(a_{n_1+1} + \ldots + a_{n_2}\right) + \left(a_{n_2+1} + \ldots + a_{n_3}\right) + \ldots$$

leads to a series that also converges to L.

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

(a) The regrouping gives us a sequence of partial sums $\left(s_{n_k}\right)$. This is a subsequence of the original sequence of partial sums (s_n). Since we know that is convergent, then the subsequence must also be convergent to the same limit.

(b) The proof only works in one direction, from convergence to associativity. If we only have subsequence convergence, then we cannot say anything about the convergence of the original series.

2.5.4 **Exercise 88**

The Bolzano–Weirstrass Theorem is extremely important, and so is the strategy employed in the proof. To gain some more experience with this technique, assume the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

Assume we have a set that is bounded above by M. Choose any element x of our given set X.

If $x = M$ we are done, so assume $x < M$. Let $l = M - x$.

Now form the closed interval $I_1 = [x, M]$. Bisect it into two halves, and select I_2 based on the following criteria: If the right-most half has elements in X, choose it. Otherwise choose the left half. Either way, I_k should always include an element in X.

By the NIP, there exists a real number s in every I_k, and every I_k should contain an element from X.

I claim that s is the supremum of X.

Proof: Assume there was some x' such that $x' > s$. Then there would be some interval I_k which contained both s and x', and some I_{k+1} such that it only contained s. This would imply that x' existed in the right half of I_k while s exists in the left half of I_k. However, by construction, we would have picked the right half of I_k, which is a **contradiction**. Therefore it must be that any $x' \in X$ is such that $x' \leq s$.

To show it is the least upper bound, suppose we have some upper bound $b < s$. Then choose $\epsilon < s - b$. Because the length of I_k (which is $\frac{l}{2^k}$) converges to 0, we can choose some I_k such that the length is less than ϵ. By construction, it must contain some element x' of X. It must be that $x' < s$, otherwise we would immediately run into a contradiction.

Thus, we have:

$$|s - x'| < s - b \Longleftrightarrow s - x' < s - b \Longleftrightarrow x' > b.$$

This is a **contradiction**, so therefore it must be that for any upper bound b, that $b \geq s$. ■

2.5.5 **Exercise 89**

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

If (a_n) diverged, then due to the fact that it is bounded, there would be two subsequences converging to the lim sup and lim inf respectively, which would be different values.

Thus (a_n) must converge. Therefore it itself is a convergent subsequence, and must converge to a.

2.5.6 **Exercise 90**

Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

If $b > 1$, then $b^{1/n}$ is decreasing and bounded below by 1.

If $b < 1$, then $b^{\frac{1}{n}}$ is increasing and bounded above by 1.

If $b = 1$, then we just have the constant sequence of 1.

By MCT there must be some limit l

We can choose the subsequence $b^{\frac{1}{2n}} = \sqrt{b^{1/n}}$, which should have the same limit l.

By Exercise 2.3.1, we know that $\left(\sqrt{b^{1/n}}\right) \to \sqrt{l}$, and the only value where $\sqrt{l} = l$ is either 0 or 1. It is clearly not zero, so the limit must be 1.

2.5.7 **Exercise 91**

Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

($\Leftarrow$)

Assume $-1 < b < 1$. Let $\epsilon > 0$ be arbitrary.

From Example 2.5.3 we know that $|b|^n$ converges to 0. Using that result:

$$|b^n| = ||b|^n| < \epsilon$$

for all n greater than some $N \in \mathbb{N}$. This proves that b^n converges to 0.

($\Rightarrow$)

First, if $b = 1$ then we clearly converge to 1. If $b = -1$ we diverge since we alternate between -1 and 1.

Assume $b > 1$ or $b < -1$. Then it must be true that $|b| > 1$. Choose $0 < \epsilon < 1$.

Clearly it cannot converge to 0 then, since with our given value of ϵ it can never be the case that $|b^n| < \epsilon$.

2.5.8 **Exercise 92**

Another way to prove the Bolzano–Weierstrass Theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of a *peak term*. Given a sequence (x_n), a particular term x_m is a peak term if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.

(a) Find examples of sequences with zero, one, and two peak terms. Find an example of a sequnce with infinitely many peak terms that is not monotone.

(b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano–Weierstrass Theorem.

(a) Zero peak terms:

$$\left(-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right)$$

One peak term:

$$\left(0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right)$$

Two peak terms:

$$\left(1, 0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right)$$

Infinitely many peak terms, not monotone:

$$\left(1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \ldots\right)$$

(b) First, in the case that we have finite peak terms, choose the term after the last peak term, call it x_{n_1}.

We know that we can find another term x_{n_2} after that term ($n_2 > n_1$) such that $x_k > x_n$. Otherwise, x_n would be a peak term. The same logic applies for all n_k and n_{k-1}. Thus, we have found a monotone increasing subsequence.

In the case that we have infinite peak terms, we simply choose our subset as the peak terms, since each one must be less than or equal to the previous peak term. Thus this gives us a monotone decreasing subsequence.

Therefore, we can conclude that since every bounded sequence has a monotone subsequence, that subsequence is itself bounded and by MCT convergent.

2.5.9 **Exercise 93**

Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence $\left(a_{n_k}\right)$ converging to $s = \sup S$. (This is a direct proof of the Bolzano–Weierstrass Theorem using the Axiom of Completeness).

Notice s is clearly bounded above by the upper bound of (a_n), so by AoC $s = \sup S$ exists.

Since s is the supremum, we know that given arbitrary $n \in \mathbb{N}$, that there must exist an element $x_n \in S$ such that $x_n > s - \frac{1}{n}$. Rearranging, we get:

$$s - x_n < \frac{1}{n}.$$

Note also that there must be infinite elements in (a_n) that are less than $s + \frac{1}{n}$. Otherwise, we could find an element $x \in S$ such that $s < x < s + \frac{1}{n}$.

Let's choose the following subsequence. Let a_{n_k} be the first term after the first n_{k-1} such that $x_k < a_{n_k} < s + \frac{1}{k}$.

Thus, we have that $s - a_{n_k} < s - x_k < \frac{1}{k}$

If $s < a_{n_k}$, then $a_{n_k} - s < \frac{1}{k}$, so essentially we have that

$$\left|a_{n_k} - s\right| < \frac{1}{k}.$$

Now for arbitrary ϵ, we can just choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, and now we see that for all $k \geq N$ that our subsequence converges to s:

$$\left|a_{n_k} - s\right| < \epsilon.$$

2.6 The Cauchy Criterion

2.6.1 **Exercise 94**

Supply a proof for Theorem 2.6.2.

Assume $(x_n) \to x$.

Let ϵ be arbitrary.

Find $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - x| < \frac{\epsilon}{2}$.

Then for $n, m \geq N$, we have that:

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus our sequence is also Cauchy.

2.6.2 **Exercise 95**

Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

(c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

(a) $x_n = \frac{(-1)^n}{n}$.

(b) Impossible, since all Cauchy sequences are bounded.

(c) Impossible, since the subsequence would have an upper bound, and all terms in the original sequence would also have to obey that upper bound and thus the original sequence would be convergent by MCT.

(d) $(0, 1, 0, 2, 0, 3, 0, 4, \ldots)$

2.6.3 **Exercise 96**

If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

(a) Let $\epsilon > 0$ be arbitrary. Choose $n, m \geq \max\{N_1, N_2\}$ where $N_1, N_2 \in \mathbb{N}$ are such that $|x_n - x_m| < \frac{\epsilon}{2}$ and $|y_n - y_m| < \frac{\epsilon}{2}$ respectively.

Then:

$$|(x_n + y_n) - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| < \epsilon.$$

(b) Since they are both Cauchy, we know they are bounded by some M_1 and M_2 respectively. Let $\epsilon > 0$ be arbitrary. Choose $n, m \geq \max\{N_1, N_2\}$ where $N_1, N_2 \in \mathbb{N}$ are such that $|x_n - x_m| < \frac{\epsilon}{2M_1}$ and $|y_n - y_m| < \frac{\epsilon}{2M_2}$ respectively.

$$
\begin{aligned}
|x_n y_n - x_m y_m| &= |x_n y_n - x_m y_m + x_n y_m - x_n y_m| \\
&\le |x_n y_n - x_n y_m| + |x_n y_m - x_m y_m| \\
&= |x_n||y_n - y_m| + |y_m||x_n - x_m| \\
&\le M_2|y_n - y_m| + M_1|x_n - x_m| \\
&< \epsilon.
\end{aligned}
$$

2.6.4 **Exercise 97**

Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

(a) $c_n = |a_n - b_n|$

(b) $c_n = (-1)^n a_n$

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

(a) Yup, by using reverse triangle inequality followed by triangle inequality.

$$||a_n - b_n| - |a_m - b_m|| \le |a_n - b_n - a_m + b_m| \le |a_n - a_m| + |b_n - b_m|.$$

I don't prove this rigorously.

(b) No, consider counter-example where $a_n = 1$. Then c_n will alternate between -1 and 1.

(c) No, consider the counter-example where $a_n = \frac{(-1)^n}{n}$. Then c_n will alternate between -1 and 0.

2.6.5 **Exercise 98**

Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\epsilon > 0$, there exists an N such that if $n \ge N$, then $|s_{n+1} - s_n| < \epsilon$.

Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

(ii) is true.

Proof: Let $\epsilon > 0$ be arbitrary. Let $N \in \mathbb{N}$ be such that for $n \ge N$, both $|x_{n+1} - x_n|$ and $|y_{n+1} - y_n|$ are less than $\frac{\epsilon}{2}$. Then we can check that:

$$|x_{n+1} + y_{n+1} - x_n - y_n| \le |x_{n+1} - x_n| + |y_{n+1} - y_n| < \epsilon.$$

■

For the counterexample, consider the sequence of partial sums for the Harmonic series. We know it is unbounded, but the difference between subsequence terms is simply $\frac{1}{n}$, so we can clearly make the difference as small as we'd like.

2.6.6 **Exercise 99**

Let's call a sequence (a_n) *quasi-increasing* if for all $\epsilon > 0$ there exists an N such that whenever $n > m \geq N$ it follows that $a_n > a_m - \epsilon$.

(a) Give an example of a sequence that is quasi-increasing but not monotone or eventually monotone.

(b) Give an example of a quasi-increasing sequence that is divergent and not monotone or eventually monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasi-increasing sequences? give an example of a bounded, quasi-increasing sequence that doesn't converge, or prove that no such sequence exists.

(a)
$$\left(1, \frac{1}{2}, 1, \frac{3}{4}, 1, \frac{7}{8}, 1, \frac{15}{16}, \dots\right)$$

(b)
$$\left(1, \frac{1}{2}, 2, \frac{7}{4}, 3, \frac{23}{8}, 4, \frac{63}{16}, \dots\right)$$

(c) I claim that if our sequence is bounded from above and is quasi-increasing, then it converges.

Proof: Let $\epsilon > 0$ be arbitrary.

Since the sequence is quasi-increasing, there exists N_1 such that for any $n > m \geq N_1$ we have $a_n > a_m - \frac{\epsilon}{2}$.

First, note that we have that

$$a_n - a_m > -\frac{\epsilon}{2} > -\epsilon.$$

Now, consider $s_{N_1} = \sup\{a_k : k \geq N_1\}$. This must exist since the sequence is bounded. By some lemma, there must be some $N_2 \geq N_1$ such that $s_{N_1} - \frac{\epsilon}{2} < a_{N_2}$.

Note that for $m \geq N_2$ we have that $a_m - a_{N_2} > -\frac{\epsilon}{2}$, so with some rearranging we get:

$$-a_m < -a_{N_2} + \frac{\epsilon}{2}.$$

Now, for any $n, m \geq N_2$, we have the following:

$$\begin{aligned} a_n - a_m &\leq s_{N_1} - a_m \\ &< s_{N_1} - a_{N_2} + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned}$$

So putting everything together, we have:

$$-\epsilon < a_n - a_m < \epsilon \quad \Longrightarrow \quad |a_n - a_m| < \epsilon$$

for $n > m \geq N_2$.

For $n < m$, we can simply switch the symbols and it works out fine.

■

2.6.7 **Exercise 100**

Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano–Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC, and MCT are all equivalent.

(b) Use the Cauchy Criterion to prove the Bolzano–Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of the completeness discussed at the end of Section 2.6.

(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

(a) Assume that we have a bounded sequence (a_n) such that it is monotonically increasing.

Then we know by BW that there is a convergent subsequence $\left(a_{n_k}\right) \to a$.

I claim that (a_n) also converges to a.

Proof: Let $\epsilon > 0$ be arbitrary. Now there must exist $K \in \mathbb{N}$ such that for every $k \geq K$,

$$\left|a_{n_k} - a\right| < \epsilon.$$

Notice also that $a_{n_k} \leq a$, since otherwise, we could find an element of the subsequence such that it contradicts the increasing assumption.

Consider $N = n_k$. It must also be that every $a_n \leq a$, otherwise again, we could find a counterexample to the increasing assumption.

Thus:

$$|a_n - a| = a - a_n = a - a_{n_k} + a_{n_k} - a_n < a - a_{n_k} \leq \epsilon.$$

This shows that $(a_n) \to a$. ■

(b) Assume that we have a bounded sequence. We wish to show that there is a subsequence that is Cauchy, thus implying that it converges.

Proof: Assume our interval is bounded by M. First, we select any element, with index n_1. We proceed by bisecting the interval, and selecting the half that has infinite terms in the sequence that occur *after* our selected element a_{n_1}. We repeat the above steps.

For arbitrary $\epsilon > 0$, we repeat this process K times until $\frac{M}{2^K} < \epsilon$. We now know that for any $k \geq K$, $\left(a_{n_k}\right)$ as chosen by our iterative process above will be contained within the same ϵ-neighborhood. This is the step where the Archimedean Property is implicitly required.

Now, it should be clear that for any $k_1, k_2 \geq K$, since they are contained within the same ϵ-neighborhood, that it must be such that

$$\left|a_{k_1} - a_{k_2}\right| < \epsilon.$$

This implies the subsequence is Cauchy and therefore converges by the CC. ■

(c) The rationals are not complete, but the Archimedean Property is valid over $\mathbb{Q}$.

2.7 Properties of Infinite Series

2.7.1 **Exercise 101**

Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

(a) Let $\epsilon > 0$ be arbitrary. Since $(a_n) \to 0$, we can find N s.t. $n \geq N$ implies

$$|a_n| = a_n < \epsilon.$$

Now for any $n > m \geq N$, note that $|s_n - s_m|$ is equivalent to:

$$\left|a_{m+1} - a_{m+2} + \cdots \pm a_n\right|.$$

Using the triangle inequality, we can split the above into the following:

$$\leq \left|a_{m+1} - a_{m+2}\right| + \cdots + \left|a_{n-1} - a_n\right|$$

The above assumes $n - m$ is even, but if it is odd, the last two terms are instead $|a_{n-2} - a_{n-1}| + |a_n|$.

Inside all of the absolute values, all the expressions are actually non-negative, so we can remove the absolute values and regroup the terms:

$$= a_{m+1} + (-a_{m+2} + a_{m+3}) + \cdots + (-a_n).$$

Now notice that all the expressions with in the parentheses are ≤ 0. This allows us to finish the inequality:

$$|s_n - s_m| \leq a_{m+1} < \epsilon.$$

Therefore the sequence of partial sums is Cauchy and thus convergent.

(b) I construct a sequence of closed intervals such that it contains an infinite number of the partial sums, as well as having the property that their lengths converge to 0 so that the sequence converges to a single limit. Let I_1 be $[0, a_1]$. Clearly since $a_2 \leq a_1$, $a_1 - a_2 \geq 0$, and $a_1 - a_2 \leq a_1$. The next interval $[a_1 - a_2, a_1]$ clearly contains s_2 and also has a length of a_2.

We continue in this manner, with the intervals being clearly nested. Since the length of the n th interval is exactly a_n, all of the intervals are valid and we can always find one smaller than any $\epsilon > 0$.

By the NIP, this shows that there must exist a *single* real number s such that it is contained within all the intervals.

Furthermore, by construction for arbitrary $\epsilon > 0$ we can always find N such that for $n \geq N$, we have $|s_n - s| < \epsilon$.

This shows that the sequence of partial sums converges to s, which is a result of NIP.

(c) I don't prove this rigorously.

But clearly (s_{2n}) is lower bounded by 0 and monotonically increasing, while (s_{2n+1}) is upper bounded by a_1 and monotonically decreasing.

Thus by MCT they must converge respectively to some s_1 and s_2.

It must also be the case that $s_1 = s_2$, otherwise we could find two partial sums s_n and s_{n+1} such that they are too far from each other, which contradicts the fact that the original sequence converges to 0.

2.7.2 **Exercise 102**

Decide whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

(a)
$$0 \leq a_k \leq \frac{1}{2^k}.$$

The RHS gives us a geometric series with $\left|\frac{1}{2}\right| < 1$, so by comparison test this converges.

(b)
$$0 \leq |a_k| \leq \frac{1}{k^2}$$

We know the RHS gives us a series which converges, so therefore, using the comparison and absolute convergence test we can conclude that the original series converges.

(c) This does not converge, since we can always find two adjacent partial sums and notice that they will always be at least $\frac{1}{2}$ distance apart.

(d) This is equal to $2 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. We know by Alternating Series Test that this converges.

(e) The series diverges.

I don't prove this ultra-rigorously, but here are the high-level steps.

First, I show that $1 + \frac{1}{3} + \frac{1}{5} + \cdots$ diverges, by using the ALT and comparison tests against the harmonic series.

Next, I show that $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$ converges using comparison test vs $\sum \frac{1}{n^2}$, which we know to converge using the p-test with $p > 1$.

Thus, the negative of that series should also converge by ALT.

Finally, we view $1 + \frac{1}{3} + \frac{1}{5} + \cdots$ as the sum of our original unknown series and our known convergent series.

Therefore, it cannot converge, otherwise by ALT we would show that the harmonic series converges.

2.7.3 **Exercise 103**

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

(a) Assume $\sum b_k$ converges. Let $\epsilon > 0$ be arbitrary.

Then we know by Cauchy Criterion that there exists N such that any $n > m \geq N$ gives us

$$|b_{m+1} + \cdots + b_n| < \epsilon.$$

Clearly, we can chain the following inequalities due to the non-negative assumption of (a_n):

$$|a_{m+1} + \cdots + a_n| = a_{m+1} + \cdots + a_n \leq b_{m+1} + \cdots + b_n \leq |b_{m+1} + \cdots + b_n| < \epsilon.$$

The opposite direction proceeds by contradiction proof. Assume that $\sum a_k$ diverges, and AFSOC that $\sum b_k$ converges. Then by the first part we get that $\sum a_k$ actually converges.

(b) Since $\sum b_k$ converges, let's call the limit of the partial sums B.

This is an upper bound on all the partial sums since it is monotonically increasing.

Thus it is also an upper bound on all partial sums in the series $\sum a_k$.

This sequence is also monotonically increasing, so by the MCT the partial sums must converge.

The opposite direction follows the same argument as before.

2.7.4 **Exercise 104**

Give an example of each or explain why the request is impossible referencing the proper theorems(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but $\sum y_n$ diverges.

(d) A sequences (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum(-1)^n x_n$ diverges.

(a)
$$(x_n) = (0, 1, 0, 1, \ldots),$$
$$(y_n) = (1, 0, 1, 0, \ldots).$$

(b)
$$\sum x_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$
$$y_n = (-1)^n.$$

(c) This is false, since if $\sum x_k = x$, then $\sum -x_k = -x$ by ALT, and this implies that if $\sum(x_k + y_k)$ converges then $\sum(x_k + y_k - x_k) = \sum y_k$ also converges.

(d)
$$(x_n) = \left(1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \ldots\right)$$

This diverges in the same manner as the harmonic series.

2.7.5 **Exercise 105**

Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7.

($\Rightarrow$)

Proof: Assume $\sum_{n=1}^{\infty} 1/n^p$ converges. AFSOC that $p \leq 1$.

Then, we have that $1/n \leq 1/n^p$.

By comparison test, this implies that the harmonic series converges. **Contradiction!** Therefore it must be that $p > 1$. ■

($\Leftarrow$)

Proof: Assume $p > 1$.

By grouping terms, $\sum_{n=1}^{\infty} 1/n^p$ is equivalent to the following series:

$$\sum_{n=1}^{\infty} 1/n^p = \sum_{k=0}^{\infty} \left(\frac{1}{(2^k)^p} + \frac{1}{(2^k + 1)^p} + \cdots + \frac{1}{(2^{k+1} - 1)^p} \right).$$

Working with each term in the new sequence, we see that:

$$\frac{1}{(2^k)^p} + \frac{1}{(2^k + 1)^p} + \cdots + \frac{1}{(2^{k+1} - 1)^p} \leq 2^k \cdot \frac{1}{(2^k)^p} = \left(\frac{1}{2^{p-1}} \right)^k.$$

We can recognize this as a term from a geometric series, which must converge since $p > 1$ implies that $0 < 1/2^{p-1} < 1$.

By comparison test, our original series must converge. ■

2.7.6 **Exercise 106**

Let's say that a series *subverges* if the sequence of partial sums contains a subsequence that converges. Consider this (invented) defintion for a moment, and then decide which of the following statements are valid propositions about subvergent series:

(a) If (a_n) is bounded, then $\sum a_n$ subverges.

(b) All convergent series are subvergent.

(c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.

(d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

(a) False, just consider $a_n = 1$. Then all subsequences of partial sums increase without bound.

(b) True, since the sequence of partial sums converges, and it is a subsequence itself.

(c) True. If the sequence of partial absolute sums contains a convergent subsequence, then the partial absolute sums also converge.

Thus, the series converges absolutely, implying that the original series converges.

By part (b), we know that the series must also subverge.

(d) Consider $(a_n) = (1, -2, 2, -3, 3, -4, 4, ...)$. The partial sums look like:

$$(1, -1, 1, -2, 1, -3, 1, ...).$$

So the partial sums have a convergent subsequence, but the sequence (a_n) does not.

2.7.7 **Exercise 107**

(a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

(a) Fix $0 < \epsilon < l$.

Now, AFSOC that $\sum a_n$ converges.

By ALT we know that $(na_n - l) \to 0$.

Thus we know that after some N, all $n \geq N$ is such that

$$|na_n - l| < l - \epsilon.$$

Alternatively,

$$\begin{aligned} & -l + \epsilon < na_n - l < l - \epsilon \\ \implies & \epsilon < na_n \\ \implies & \frac{1}{n} < \frac{1}{\epsilon} a_n. \end{aligned}$$

By ALT, we know starting from $n = N$, that $\sum a_n / \epsilon$ converges.

So we deduce that the harmonic series after N must also converge, by comparison test. This is a **contradiction!** Thus, $\sum a_n$ must actually diverge.

(b) Assume $n^2 a_n = l \geq 0$. This is valid from the Order Limit Theorem. Now choose $\epsilon > l \geq 0$.

For some $N \in \mathbb{N}$ we know that $n^2 a_n - l < \epsilon - l$, so therefore $a_n < \frac{\epsilon}{n^2}$.

The series $\sum_{n=N}^{\infty} \frac{\epsilon}{n^2}$ must converge, so thus $\sum_{n=N}^{\infty} a_n$ must also converge by comparison test. Because the tail converges, the full series must converge as well.

2.7.8 **Exercise 108**

Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

(a) True. Note first that $\left|a_n^2\right| = a_n^2$ in the reals. After some point, it must be that $|a_n| < 1$. This allows us to say that $a_n^2 < |a_n|$. By comparison test on the tails, $\sum a_n^2$ converges.

(b) False. Consider the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ By AST this converges. In addition, the sequence $b_n = (-1)^{n+1} \frac{1}{\sqrt{n}}$ clearly converges to 0. However, the series $\sum a_n b_n = \sum \frac{1}{n}$ which diverges.

(c) AFSOC that $\sum n^2 a_n$ converges. Then we must have that $(n^2 a_n) \to 0$. So for some $\epsilon = 1$, we have $N \in \mathbb{N}$ after which

$$\left|n^2 a_n\right| < 1 \implies |a_n| < \frac{1}{n^2}$$

By comparison test this means that the tail of $\sum |a_n|$ converges, which implies that $\sum a_n$ converges absolutely. This is a **contradiction**, so therefore $\sum n^2 a_n$ must diverge.

2.7.9 (Ratio Test) **Exercise 109**

Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = r < 1,$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $\left|a_{n+1}\right| \leq |a_n| r'$.

(b) Why does $|a_N| \sum (r')^n$ converge?

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

(a) Let $\epsilon = r' - r > 0$. Then choose N such that $\left|\left|\frac{a_{n+1}}{a_n}\right| - r\right| < \epsilon$. With some manipulation, we get: $\left|a_{n+1}\right| < |a_n|(\epsilon + r) = |a_n| r'$.

(b) This is a geometric series, and we know that $|r'| = r' < 1$.

(c) We first see that for $n \geq N$, that $|a_n| \leq |a_N| (r')^{n-N}$.

The series formed from the RHS converges (if we take out the factor of $(r')^{-N}$), so by comparison test, $\sum |a_n|$ must converge.

2.7.10 (Infinite Products) **Exercise 110**

Review Exercise 2.4.10 about infinite products and then answer the following questions:

(a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$ converge?

(b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots$ certainly converges. (Why?) Does it converge to zero?

(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}.$$

Show that the left side of this identity at least converges to something. (A complete proof of this result is taken up in Section 8.3.)

(a) By the result of 2.4.10, we know that this infinite product converges if and only if $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ converges.

This is simply the geometric series with $r = \frac{1}{2}$.

(b) The partial products are monotonically decreasing and bounded below by 0, so by MCT it must converge.

It does converge to 0.

Proof: The product can be rewritten as:

$$\prod_{n=1}^{\infty} \frac{2n-1}{2n} = 1/\prod_{n=1}^{\infty} \frac{2n}{2n-1} = 1/\prod_{n=1}^{\infty} \left[1 + \frac{1}{2n-1}\right]$$

Let $\epsilon > 0$. The sum $\sum \frac{1}{2n-1}$ diverges by comparison test, so the sequence of partial products (in the demoninator) must diverge too.

Let $N \in \mathbb{N}$ such that $m \geq N$ means $\prod_{n=1}^{m} \left[1 + \frac{1}{2n-1}\right] \geq \frac{2}{\epsilon} > \frac{1}{\epsilon}$.

Then this implies that $0 < p_m < \epsilon$ as desired. ∎

(c) Each term in the product is defined as:

$$x_n = \frac{(2n)^2}{(2n-1)(2n+1)} = \frac{4n^2}{4n^2-1} = 1 + \frac{1}{4n^2-1}.$$

Using 2.4.10, we can see that $a_n = \frac{1}{4n^2-1} \leq \frac{1}{3n^2}$, (true for $n \geq 1$). Thus by comparison test the series $\sum a_n$ converges, so the original product converges as well.

2.7.11 **Exercise 111**

Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

Consider this series:

$$1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots + \frac{1}{42^2} + \cdots$$

The other series is:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+(36 \text{ duplicates of}) \ \frac{1}{6^2}+\frac{1}{42^2}+\frac{1}{43^2}+\cdots$$

Clearly the minimum of each individual term gives us $\frac{1}{n^2}$, of which, the series we know to converge.

However, each individual series has an infinite number of finite length portions that add to 1.

2.7.12 (Summation-by-parts) **Exercise 112**

Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).$$

$$\begin{aligned}
\sum_{j=m}^{n} x_j y_j &= \sum_{j=m}^{n} [s_j y_j - s_{j-1} y_j] \\
&= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_{j-1} y_j \\
&= \sum_{j=m}^{n} s_j y_j - \sum_{j=m-1}^{n-1} s_j y_{j+1} \\
&= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n-1} s_j y_{j+1} - s_{m-1} y_m \\
&= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_j y_{j+1} - s_{m-1} y_m + s_n y_{n+1} \\
&= s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).
\end{aligned}$$

2.7.13 (Abel's Test) **Exercise 113**

Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0,$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}),$$

where $s_n = x_1 + x_2 + \cdots + x_n$.

(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_k (y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

(a) This follows directly, if $s_0 = 0$. Just let $m = k = 1$.

(b) Recall that since (s_k) converges, it means that there exists M such that for all k, $|s_k| \leq M$.

$$|s_k(y_k - y_{k+1})| = |s_k||y_k - y_{k+1}| \leq M(y_k - y_{k+1})$$

The partial sums of the series of the RHS are:

$$M \sum_{k=1}^{n} (y_k - y_{k+1}) = M(y_1 - y_{n+1}) \leq M y_1.$$

This is true for every n, so this is an upper bound. It is also true that we are monotonically increasing, so by MCT this must converge. By comparison test this implies that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ converges absolutely, and thus itself converges.

Finally, we can apply ALT of sequences to deduce that the sequence of partial sums converges as well.

2.7.14 (Dirichlet's Test) **Exercise 114**

Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if (y_k) is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$ with $\lim y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test in Exercise 2.7.13, but show that essentially the same strategy can be used to provide a proof.

(b) Show how the Alternating Series Test (Theorem 2.7.7) can be derived as a special case of Dirichlet's Test.

(a) This differs in two ways. First, the partial sums are bounded but don't converge. Second, the limit of (y_n) is 0 and not some other non-negative value.

This has no effect on the proof, since all we needed was that $|s_n|$ was bounded, and in the end, the ALT is still valid as the first term will converge to 0.

(b) The partial sums of the sequence $x_n = (-1)^{n+1}$ gives us a bounded sequence. ($(1, 0, 1, 0, 1, \ldots)$)

The sequence (y_k) is the sequence of interest (a_n) in the AST.

By Dirichlet's Test, we can conclude that $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

2.8 Double Summations and Products of Infinite Series

2.8.1 **Exercise 115**

Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{nn}$. How does this value compare to the two iterated values for the sum already computed?

The result seems to be equivalent to the following series:

$$-1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \cdots$$

This converges to -2. This gives the same result as if we summed all the columns first and then then added them all up.

2.8.2 **Exercise 116**

Show that f the iterate series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbb{N}$ the series $\sum_{j=1}^{\infty}|a_{ij}|$ converges to some real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$$

converges.

Fix arbitrary i. We know that $\sum_{j=1}^{\infty}|a_{ij}|$ converges to some b_i. (Also note that this $b_i > 0$).

By ACT, we then know that $\sum_{j=1}^{\infty} a_{ij}$ also converges, let's call this value b_i'.

Note that if the sequence of partial sums $\left(\sum_{j=1}^{n} a_{ij}\right)_{n\in\mathbb{N}}$ converges to b_i', then the sequence of absolute partial sums $\left(\left|\sum_{j=1}^{n} a_{ij}\right|\right)_{n\in\mathbb{N}}$ converges to $|b_i'|$. This directly follows from the ϵ definition of convergence and the reverse triangle inequality.

Now, I argue that $|b_i'| \leq b_i$.

When comparing partial sums, we see the following due to triangle inequality:

$$\left|\sum_{j=1}^{n} a_{ij}\right| \leq \sum_{j=1}^{n}|a_{ij}|$$

So by order limit theorem, it must be that $|b_i'| \leq b_i$.

Since both are non-negative and this holds for any i, we can use the comparison test to justify that $\sum_{i=1}^{\infty}|b_i'|$ converges.

Therefore by ACT, we know that $\sum_{i=1}^{\infty} b_i'$ also converges.

2.8.3 **Exercise 117**

(a) Prove that (t_{nn}) converges.

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges.

(a) The sequence $(t_{nn})_{n\in\mathbb{N}}$ is defined as the partial sums

$$t_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|$$

First, note that we get for free that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$ converges. Let's assume this is the same definition of convergence as in exercise 2.8.2.

The first thing we can note is that the sequence (t_{nn}) is monotone increasing. Second, we can upper bound it.

Note that for any fixed i, that $\sum_{j=1}^{n}|a_{ij}| \leq \sum_{j=1}^{\infty}|a_{ij}|$. Then similar logic upper bounds the series indexed by i.

So therefore we have:

$$t_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|.$$

Thus, we can conclude that (t_{nn}) converges by MCT.

(b) Let $\epsilon > 0$ be arbitrary. Since (t_{nn}) is Cauchy, there exists some $N \in \mathbb{N}$ such that for $n > m \geq N$, we get that:

$$|t_{nn} - t_{mm}| < \epsilon.$$

Let's expand out the terms and see what we can deduce:

$$\begin{aligned}
&|t_{nn} - t_{mm}| \\
&= \left|\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}| - \sum_{i=1}^{m}\sum_{j=1}^{m}|a_{ij}|\right| \\
&= \left|\sum_{i=1}^{n}\left[\sum_{j=1}^{m}|a_{ij}| + \sum_{j=m+1}^{n}|a_{ij}|\right] - \sum_{i=1}^{m}\sum_{j=1}^{m}|a_{ij}|\right| \\
&= \left|\sum_{i=1}^{m}\sum_{j=1}^{m}|a_{ij}| + \sum_{i=1}^{m}\sum_{j=m+1}^{n}|a_{ij}| + \sum_{i=m+1}^{n}\sum_{j=1}^{m}|a_{ij}| + \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}|a_{ij}| - \sum_{i=1}^{m}\sum_{j=1}^{m}|a_{ij}|\right| \\
&= \left|\sum_{i=1}^{m}\sum_{j=m+1}^{n}|a_{ij}| + \sum_{i=m+1}^{n}\sum_{j=1}^{m}|a_{ij}| + \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}|a_{ij}|\right| \\
&= \sum_{i=1}^{m}\sum_{j=m+1}^{n}|a_{ij}| + \sum_{i=m+1}^{n}\sum_{j=1}^{m}|a_{ij}| + \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}|a_{ij}| \\
&\geq \left|\sum_{i=1}^{m}\sum_{j=m+1}^{n}a_{ij} + \sum_{i=m+1}^{n}\sum_{j=1}^{m}a_{ij} + \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}a_{ij}\right| && \text{by triangle inequality} \\
&= \left|\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{m}a_{ij}\right| && \text{by similar logic to previous steps.} \\
&= |s_{nn} - s_{mm}|.
\end{aligned}$$

Therefore, we can conclude that:

$$|s_{nn} - s_{mm}| < \epsilon,$$

such that $(s_{nn})_{n\in\mathbb{N}}$ is Cauchy and converges.

2.8.4 **Exercise 118**

(a) Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

(b) Now, show that there exists an N such that

$$|s_{mn}| < \epsilon$$

for all $m, n \geq N$.

(a) Since B is a supremum, clearly we have that for any $m, n \in \mathbb{N}$ that $t_{mn} \leq B$.

Now, recall that for $B - \frac{\epsilon}{2}$, there must be some element such that it is strictly greater. Let's call that element $t_{MM'}$. Since in both series, t is monotonically increasing, we have that $t_{mn} \geq t_{N_1 N_1}$ for $m, n \geq N_1 = \max\{M, M'\}$.

Thus we have N_1 such that for $m, n \geq N_1$,

$$B - \frac{\epsilon}{2} < t_{mn} \leq B.$$

(b) Let $\epsilon > 0$ be arbitrary. Choose N_2 such that for $n \geq N_2$ we get that $|s_{nn} - S| < \frac{\epsilon}{2}$. Choose N_1 from part (a), and let $N = \max\{N_1, N_2\}$. Now using the triangle inequality, we get that:

$$\begin{aligned} |s_{mn} - S| &= |s_{mn} - s_{nn} + s_{nn} - S| \\ &\leq |s_{mn} - s_{nn}| + |s_{nn} - S| \\ &< |s_{mn} - s_{nn}| + \frac{\epsilon}{2}. \end{aligned}$$

I don't write out the entire chain of inequalities since it is tedious, but we can show that

$$|s_{mn} - s_{nn}| < |t_{mn} - t_{nn}| < \frac{\epsilon}{2}.$$

The last inequality is because both terms must live in the same $\frac{\epsilon}{2}$-length interval from $(B - \frac{\epsilon}{2}, B]$. Thus, we can finish the inequality:

$$|s_{mn} - S| < |s_{mn} - s_{nn}| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

2.8.5 **Exercise 119**

(a) Show that for all $m \geq N$

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \epsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges to S.

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$, converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

(a) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ from the result of Exercise 2.8.4 such that for $m, n \geq N$, we get that

$$|s_{mn} - S| < \frac{\epsilon}{2}$$

Fix arbitrary $m \geq N$.

Now for every r_i with $i \in [m]$, it must be the case that the partial sums $\left(\sum_{j=1}^{n} a_{ij}\right) \to r_i$.

So we can choose $N' \in \mathbb{N}$ such that for $n \geq \max\{N, N'\}$:

$$\left| r_i - \sum_{j=1}^{n} a_{ij} \right| < \frac{\epsilon}{2m}.$$

Now, we can chain some inequalities:

$$\begin{aligned}
|r_1 + \cdots + r_m - S| &= \left| \sum_{i=1}^{m} r_i - s_{mn} + s_{mn} - S \right| \\
&\leq \left| \sum_{i=1}^{m} r_i - s_{mn} \right| + |s_{mn} - S| \\
&= \left| \sum_{i=1}^{m} \left(r_i - \sum_{j=1}^{n} a_{ij} \right) \right| + |s_{mn} - S| \\
&\leq \sum_{i=1}^{m} \left| \left(r_i - \sum_{j=1}^{n} a_{ij} \right) \right| + |s_{mn} - S| \\
&< m \cdot \frac{\epsilon}{2m} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus, it must be the case that the iterated sum converges to S.

(b) For fixed j, we clearly have that:

$$|a_{ij}| \leq \sum_{k=1}^{\infty} |a_{ik}|,$$

which we know to converge by our hypothesis. Thus by convergence test, $\sum_{i=1}^{\infty} |a_{ij}|$ converges for fixed j.

By ACT, we then can say that $\sum_{i=1}^{\infty} a_{ij}$ converges to c_j.

So now, when we expand out s_{mn}, we have:

$$s_{mn} = \sum_{i=1}^{m} a_{i1} + \cdots + \sum_{i=1}^{m} a_{in}.$$

We can then follow the same steps as in part (a) to show that the iterated sum converges to S.

2.8.6 **Exercise 120**

(a) Assume the hypothesis—and hence the conclusion—of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

(b) Imitate the strategy in the proof of Theorem 2.8.1 to show that $\sum_{k=2}^{\infty} d_k$ converges to $S = \lim_{n\to\infty} s_{nn}$.

(a)
$$\sum_{k=2}^{n}|d_k| = \sum_{k=2}^{n}\left|\sum_{j=1}^{k-1} a_{k-j,j}\right| \le \sum_{k=2}^{n}\sum_{j=1}^{k-1}|a_{k-j,j}| \le \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|.$$

Therefore the sequence of partial sums is monotonically increasing and upper bounded, so it converges.

(b) This is super tedious, so I'll just write out the high level steps.

We want to show that for arbitrary ϵ, there is a N such that for $n \ge N$ we have

$$\left|\sum_{k=2}^{n} d_k - S\right| < \epsilon.$$

First, choose N' such that any $n' \ge N'$ is such that $|s_{n'n'} - S| < \frac{\epsilon}{2}$. Next, choose N'' such that $m, n \ge N''$ is such that $|t_{mm} - t_{nn}| < \frac{\epsilon}{2}$.

Choose $N = \max\{2N', N''\}$. (We need the $2N'$ to make sure that our n', which follows from our choice of n, is big enough.)

We find n' such that $s_{n'n'}$ is contained completely within $\sum_{k=2}^{n} d_k$, while $n' + 1$ is not.

Then we split the expression into:

$$\left|\sum_{k=2}^{n} d_k - S\right| \le \left|\sum_{k=2}^{n} d_k - s_{n'n'}\right| + |s_{n'n'} - S| < \left|\sum_{k=2}^{n} d_k - s_{n'n'}\right| + \frac{\epsilon}{2}.$$

Using similar techniques as seen before, we see that:

$$\left|\sum_{k=2}^{n} d_k - s_{n'n'}\right| \le \left|\sum_{k=2}^{n}|d_k| - t_{n'n'}\right| = \sum_{k=2}^{n}|d_k| - t_{n'n'} \le t_{nn} - t_{n'n'} \le |t_{nn} - t_{n'n'}| < \frac{\epsilon}{2}.$$

Putting it together, we get that for $n \ge N$:

$$\left|\sum_{k=2}^{n} d_k - S\right| < \epsilon.$$

2.8.7 **Exercise 121**

Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_i b_j|$ converges so that we may apply Theorem 2.8.1.

(b) Let $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1$.

(a) For fixed i:

$$|a_i b_j| = |a_i||b_j|$$

which converges by ALT for series, since $\sum_{j=1}^{\infty}|b_j|$ converges.

If $\sum_{j=1}^{\infty}|b_j| = B'$, then $\sum_{j=1}^{\infty}|a_i b_j| = |a_i|B'$.

Now, we look at $\sum_{i=1}^{\infty}|a_i|B'$. By similar argument, we argue that it converges by ALT.

(b) Let $\epsilon > 0$ be arbitrary. Now, choose N_1 such that for $n \geq N_1$ we have

$$\left|\sum_{j=1}^{n} b_j - B\right| < \frac{\epsilon}{2A'},$$

where A' is the limit of $\sum_{i=1}^{n}|a_i|$, and N_2 such that for $n \geq N_2$ we have

$$\left|\sum_{i=1}^{n} a_i - A\right| < \frac{\epsilon}{2|B|}.$$

Then, for $n \geq \max\{N_1, N_2\}$, we have

$$\begin{aligned}
|s_{nn} - AB| &= \left|\sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j - AB\right| \\
&= \left|\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right) - AB\right| \\
&= \left|\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right) - \left(\sum_{i=1}^{n} a_i\right)B + \left(\sum_{i=1}^{n} a_i\right)B - AB\right| \\
&\leq \left|\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right) - \left(\sum_{i=1}^{n} a_i\right)B\right| + \left|\left(\sum_{i=1}^{n} a_i\right)B - AB\right| \\
&= \left|\left(\sum_{i=1}^{n} a_i\right)\right|\left|\left(\sum_{j=1}^{n} b_j\right) - B\right| + |B|\left|\left(\sum_{i=1}^{n} a_i\right) - A\right| \\
&< \left(\sum_{i=1}^{n}|a_i|\right)\cdot\frac{\epsilon}{2A'} - |B|\frac{\epsilon}{|(2|B|)|} \\
&= \epsilon.
\end{aligned}$$

2.9 Epilogue

No exercises in this section.

3 Basic Topology of $\mathbb{R}$

3.1 Discussion: The Cantor Set

No exercises in this section.

3.2 Open and Closed Sets

3.2.1 **Exercise 122**

(a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be *finite* get used?

(b) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, \ldots\}$ whose intersection $\bigcap_{n=1}^{\infty} O_n$ is closed, not empty, and not all of $\mathbb{R}$.

(a) It is used when assuming we can find a non-zero minimum of the ϵ-neighborhoods.

(b) The collection $\left\{\left(-\frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ are all open intervals, yet their intersection is the finite set with one element 0. This is closed, since there are no limit points.

3.2.2 **Exercise 123**

Let

$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots\right\} \quad \text{and} \quad B = \{x \in \mathbb{Q} : 0 < x < 1\}.$$

Answer the following questions for each set:

(a) What are the limit points?

(b) Is the set open? Closed?

(c) Does the set contain any isolated points?

(d) Find the closure of the set.

(a) For A, the limit points are 1 and -1. For B, the limit points are $[0, 1] \subseteq \mathbb{R}$.

(b) A is not open in $\mathbb{R}$, since it contains isolated points. A is not closed either, since it does not contain the limit point -1.

B is not open, since any ϵ-neighborhood of any point will necessarily contain reals, which do not exist in B. B is not closed either, since it contains irrational limit points. Or, we can see that it does not contain 0 or 1.

(c) A contains isolated points, for example, $-\frac{1}{3}$, since we can take $\epsilon = \frac{2}{15}$, and notice that this neighborhood around $-\frac{1}{3}$ contains no other elements of A other than $-\frac{1}{3}$.

B contains no isolated points, due to the density of rationals.

(d) The closure $\overline{A}$ is $A \cup \{-1\}$.

The closure $\overline{B}$ is $[0, 1] \subseteq \mathbb{R}$.

3.2.3 **Exercise 124**

Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbb{Q}$.

(b) $\mathbb{N}$.

(c) $\{x \in \mathbb{R} : x \neq 0\}$.

(d) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\}$.

(e) $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbb{N}\}$.

(a) Not open, just pick any point. Not closed, pick any irrational limit point.

(b) Not open, just pick any point. Closed, there are no limit points.

(c) Open, just pick a small enough ϵ such that it won't contain 0. Not closed, 0 is a limit point.

(d) Not open, since it only contains rationals. Not closed, it does not contain its limit. (If it did, then we could show that it has a limit strictly greater than the proposed limit.)

(e) Not open, since it only contains rationals. Closed, since it does not have any limit points.

3.2.4 **Exercise 125**

Let A be nonempty and bounded above so that $s = \sup A$ exists.

(a) Show that $s \in \overline{A}$.

(b) Can an open set contain its supremum?

(a) If $s \in A$, then clearly $s \in \overline{A}$. Assume $s \notin A$. Then we know that for any $\epsilon > 0$, that there will exist some $a \in A$ such that $a > s - \epsilon$. This means that any ϵ-neighborhood of s will intersect with A, thus showing that s is a limit point and will be included in the closure.

(b) An open set cannot contain its supremum, since any ϵ-neighborhood centered at the supremum will necessarily contain points greater than the supremum, which cannot be in the original set.

3.2.5 **Exercise 126**

Prove Theorem 3.2.8.

We want to show that a set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

($\Rightarrow$)

Proof: Assume F is closed. Every Cauchy sequence in $\mathbb{R}$ must converge to some limit. If the Cauchy sequence converges to a limit, where the limit is already an element of the sequence, then we automatically know it is in F.

If the limit converges to an element not in the sequence, then we know that this must be a limit point, and since F contains all its limit points then the limit must be in F. ■

(⇐)

Proof: Assume that every Cauchy sequence contained in F has a limit that is also an element of F.

Let x be an arbitrary limit point of F. Then we can find some sequence in F that converges to x, where x itself is not present in said sequence.

Every convergent sequence must be Cauchy, so therefore, the limit x must also be in F.

Thus F contains all its limit points. ■

3.2.6 **Exercise 127**

Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.

(b) The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set."

(c) Every nonempty open set contains a rational number.

(d) Every bounded infinite closed set contains a rational number.

(e) The Cantor set is closed.

(a) False. Consider $\mathbb{R} \setminus \{\pi\}$. This is open, since for any point x we can just choose $\epsilon = \frac{|\pi - x|}{2}$, which is clearly non-zero and will be a strict subset of our original set.

(b) False. We can see that the collection of sets $\{k \geq n : k \in \mathbb{N}\}_{n \in \mathbb{N}}$ are all clearly closed, yet there cannot be any element in the infinite intersection.

(c) True. Consider any point x, and then select the ϵ-neighborhood surrounding x such that it is a subset of the original set. Due to the density of rationals, there must exist a rational within that neighborhood.

(d) False. Consider $\left\{\pi + \frac{1}{n} : n \in \mathbb{N}\right\} \cup \{\pi\}$. This is bounded below by π, and above by $\pi + 1$. It is infinite, yet no element can be rational.

(e) True. Each C_n is a finite union of closed intervals, and is thus a closed set. Then, C is an infinite intersection of each C_n, so it is a closed set as well.

3.2.7 **Exercise 128**

Given $A \subseteq \mathbb{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof for Theorem 3.2.12.

(a) Let $x \in \mathbb{R}$ be a limit point of L. Then for any $\epsilon > 0$, it must be such that there is an element $x' \in L$ such that $x' \neq x$ and $x' \in V_\epsilon(x)$.

Let $\epsilon' = \frac{1}{2}\min\{|x - x'|, |x + \epsilon - x'|, |x' - x + \epsilon|\}$.

x' itself is a limit point of A, so choose $a \in A$ such that $a \neq x'$ and $a \in V_{\epsilon'}(x')$.

By construction, we have chosen $V_{\epsilon'}(x') \subseteq V_\epsilon(x)$, as well as ensuring that $a \neq x$. Therefore, x is also a limit point of A, and must be in L as well.

This shows that L contains all its limit points and is therefore closed.

(b) If x is a limit point of $A \cup L$, then we know that for any ϵ-neighborhood of x, it must contain some $x' \in A \cup L$ such that $x' \neq x$.

Either $x' \in A$, or $x' \in L$. In the case that $x' \in L$, we can use part (a) to see that there must be some additional element $a \neq x$ in a ϵ'-neighborhood of x' contained within the original ϵ-neighborhood, and thus we can choose $a \in A$ instead.

Therefore, every limit point x of $A \cup L$ is also a limit point of A itself.

This shows that $A \cup L$ contains all its limit points and is therefore closed.

3.2.8 **Exercise 129**

Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.

(a) $\overline{A \cup B}$

(b) $A \setminus B = \{x \in A : x \notin B\}$

(c) $(A^C \cup B)^C$

(d) $(A \cap B) \cup (A^C \cap B)$

(e) $\overline{A}^C \cap \overline{A^C}$

(a) Definitely closed, since the closure of a set is always closed.

(b) Definitely open, since B^C is open, and $A \setminus B = A \cap B^C$.

(c) Definitely open, since $A^C \cup B$ is closed.

(d) Definitely closed, since this just equals $(A \cup A^C) \cap B = \mathbb{R} \cap B = B$.

(e) Definitely closed.

3.2.9 (De Morgan's Laws) **Exercise 130**

A proof for De Morgan's Laws in the case of two sets is outlined in Exercise 1.2.5. The general argument is similar.

(a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^C = \bigcap_{\lambda\in\Lambda} E_\lambda^C \quad \text{and} \quad \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^C = \bigcup_{\lambda\in\Lambda} E_\lambda^C.$$

(b) Now, provide the details for the proof of Theorem 3.2.14.

(a) Consider any element $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{C}}$. Then we know it must be that $x \notin \bigcup_{\lambda\in\Lambda} E_\lambda$. Therefore, it must be that x cannot be in any given E_λ, else it would appear in the union. Thus, for any λ, we see that $x \notin E_\lambda$, or alternatively, that $x \in E_\lambda^{\mathsf{C}}$. Thus, since $x \in E_\lambda^{\mathsf{C}}$ for every λ, it must be in the intersection of all E_λ^{C}. So therefore, $\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{C}} \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^{\mathsf{C}}$.

Now, if $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^{\mathsf{C}}$, then for all $\lambda \in \Lambda$, $x \in E_\lambda^{\mathsf{C}}$. Alternatively, we see that $x \notin E_\lambda$ for all λ. Thus, if we take the union of all E_λ, x cannot be in that union.

Thus we have that $\bigcap_{\lambda\in\Lambda} E_\lambda^{\mathsf{C}} \subseteq \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{C}}$, so together we have that $\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{C}} = \bigcap_{\lambda\in\Lambda} E_\lambda^{\mathsf{C}}$.

The other law is very similar, so I skip it.

(b) (i) *The union of a finite collection of closed sets is closed.*

Proof: Let $C_1, \dots, C_n$ be the finite collection of closed sets.

We know that C_i^{C} is open. Furthermore, from De Morgan's Laws, we know that $\left(\bigcup_{i=1}^n C_i\right)^{\mathsf{C}} = \bigcap_{i=1}^n C_i^{\mathsf{C}}$, which we know from Theorem 3.2.3 to be open.

Since the complement is open, we know that $\bigcup_{i=1}^n C_i$ must be closed by Theorem 3.2.13. ■

(ii) *The intersection of an arbitrary collection of closed sets is closed.*

Proof: Again, by De Morgan, we see that $\left(\bigcap_{\lambda\in\Lambda} C_\lambda\right)^{\mathsf{C}} = \bigcup_{\lambda\in\Lambda} C_\lambda^{\mathsf{C}}$, which is open by Theorem 3.2.3.

This shows that $\bigcap_{\lambda\in\Lambda} C_\lambda$ must be closed by Theorem 3.2.13. ■

3.2.10 **Exercise 131**

Only one of the following three descriptions can be realized. Provide an example that illustrates the viable description, and explain why the other two cannot exist.

(i) A countable set contained in $[0, 1]$ with no limit points.

(ii) A countable set contained in $[0, 1]$ with no isolated points.

(iii) A set with an uncountable number of isolated points.

We take countable to mean countably infinite.

(i) cannot be realized, since we have a bounded, infinite set, so we know by Bolzano–Weierstrass that any countable sequence (a.k.a, an ordering) of the elements in our set must have a convergent subsequence. This must then converge to a limit point.

(ii) can be realized. Consider the set $\mathbb{Q} \cap [0, 1]$. Any element in this set must be a limit point, since we can always find a sequence of rationals that approach it.

(iii) cannot be realized, since otherwise, we could select a distinct rational number from within the isolated ϵ-neighborhood of each isolated point, and this would indicate that there are uncountably many rationals.

3.2.11 **Exercise 132**

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Does this result about closures extend to infinite unions of sets?

(a) Since $A, B \subseteq A \cup B$, any limit points of A and B must also be limit points of $A \cup B$. So therefore $\overline{A} \subseteq \overline{A \cup B}$, and $\overline{B} \subseteq \overline{A \cup B}$, so $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Now, we know that $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$.

This is because, if any other closed set C contains $A \cup B$, it must contain the limit points of $A \cup B$ as well, so therefore $\overline{A \cup B} \subseteq C$.

Clearly, $A \cup B \subseteq \overline{A} \cup \overline{B}$, and finite union of closed sets is closed, so it must be the case that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Putting the two parts together, we have that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) This does not necessarily extend to infinite unions of sets, since we can no longer have the property that the infinite union is closed.

Consider $A_n = \{1/n\}$.

Clearly, the closure of each is simply $\{1/n\}$, since there are no limit points.

However, the infinite union has a single limit point of 0, which is not present in the infinite union of the closures.

3.2.12* **Exercise 133**

Let A be an uncountable set and let B be the set of real numbers that divides A into two uncountable sets; that is, $s \in B$ if both $\{x : x \in A \text{ and } x < s\}$ and $\{x : x \in A \text{ and } x > s\}$ are uncountable. Show B is nonempty and open.

I produce two sets, B_l and B_r.

$B_l = \{s \in \mathbb{R} : (s, \infty) \cap A \text{ is uncountable}\}$ and $B_r = \{s \in \mathbb{R} : (-\infty, s) \cap A \text{ is uncountable}\}$

Clearly, $B_l \cap B_r$ gives us B.

Neither of them can be empty, since otherwise, we could produce the contradiction that A is countable by constructing it from the countably infinite union of countable sets:

$$A = \bigcup_{n=1}^{\infty} ((-\infty, n) \cap A) \implies A \text{ is countable } ✗,$$

or

$$A = \bigcup_{n=1}^{\infty} ((n, \infty) \cap A) \implies A \text{ is countable } ✗.$$

Moreover, it must be that $B_l \cup B_r = \mathbb{R}$. Otherwise, there would exist some $x \in \mathbb{R}$ such that $(x, \infty) \cap A$ and $(\infty, x) \cap A$ are both countable, giving us the contradiction that

$$A = ((x, \infty) \cap A) \cup ((-\infty, x) \cap A) \cup (\{x\} \cap A) \implies A \text{ is countable } ✗.$$

Note that if $l \in B_l$, then any $l' < l$ is also in B_l. Similarly, any $r' > r \in B_r \implies r' \in B_r$.

I claim both B_l and B_r are both open.

For each, there are two cases. Either the set is unbounded and is therefore the entire set $\mathbb{R}$, or they are upper and lower bounded respectively.

Let's focus on B_l. If it is upper bounded, then select $b_l = \sup B_l$. I claim $b_l \notin B_l$. Otherwise, we could have the countably infinite union as follows:

$$\bigcup_{n=1}^{\infty} \left(\left(b_l + \frac{1}{n}, \infty \right) \cap A \right) = (b_l, \infty) \text{ is countable } ✗.$$

This is a contradiction since $b_l \in B_l$ implies that (b_l, ∞) is uncountable.

Similar logic applies for B_r.

This shows that B_l is an open interval of the form $(-\infty, b_l)$, where b_l can be ∞, and B_r is of the form (b_r, ∞).

Since $B_l \cup B_r = \mathbb{R}$, we must have that $b_l > b_r$, else we would have a gap.

Thus, $B = (b_r, b_l)$ and must be nonempty, and open as well, since it is a finite intersection of two open sets.

3.2.13* **Exercise 134**

Prove that the only sets that are both open and closed are $\mathbb{R}$ and the empty set $\emptyset$.

It is easy to show that $\emptyset$ and $\mathbb{R}$ are closed and open.

Proof: Clearly, $\emptyset$ has no limit points so it is closed. There are also no points that need to have ϵ-neighborhoods around, so it is open. Since $\mathbb{R}$ is $\emptyset^{\complement}$, we have that $\mathbb{R}$ is also open and closed. ■

For the other direction, I show that if A is a nonempty set such that it is closed and open, then it must contain every real number.

First, I show that A can be neither upper nor lower bounded.

Proof: AFSOC A is upper bounded. Then we can find $s = \sup A$ (since A is nonempty). If $s \in A$, then no ϵ-neighborhood of A will be contained in A. However, for any $\epsilon > 0$, we also know that there must be an $a \in A$ such that $a > s - \epsilon$. Additionally, since $s \notin A$, we have that $a \neq s$, and therefore s qualifies as a limit point of A. This is a **contradiction**, since we have now found a limit point such that it is not in A, so A cannot be closed.

Similar logic applies to show that A cannot be lower bounded.

Thus, A is unbounded. ■

Now, I show that if $x \in \mathbb{R}$, then x is a limit point of A.

Proof: AFSOC that x is not a limit point of A. Then there must exist some $\epsilon > 0$ where $V_\epsilon(x)$ does not intersect with A other than at x.

Clearly, $x \notin A$, since otherwise we could choose the same ϵ as a counterexample to A being open.

Now, observe that $V_\epsilon(x) \subseteq (l, r)$, where if $x' \in (l, r)$ then $x' \notin A$.

The largest such interval must be bounded, otherwise, this would imply that A itself is bounded. Assume that r is the infimum of all such upper bounds for the RHS of the interval.

If $r \in A$, then we can see that no ϵ-neighorhood around r is contained in A.

If $r \notin A$, then we can see that r is a limit point of A, which is a contradiction.

Therefore, we have our overall contradiction, and it must be that x is a limit point of A ∎

Thus, for arbitrary $x \in \mathbb{R}$, x is a limit point of A, and since A is closed, $x \in A$.

This shows that any nonempty $A = \mathbb{R}$.

3.2.14 **Exercise 135**

A dual notion to the closure of a set is the interior of a set. The *interior* of E is denoted E° and is defined as

$$E^\circ = \{x \in E : \text{there exists } V_\epsilon(x) \subseteq E\}.$$

Results about closures and interiors possess a useful symmetry.

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$.

(b) Show that $\overline{E}^{\mathrm{C}} = \left(E^{\mathrm{C}}\right)^\circ$, and similarly that $\left(E^\circ\right)^{\mathrm{C}} = \overline{E^{\mathrm{C}}}$.

(a) Clearly, if $\overline{E} = E$, then E is closed since the closure is always closed.

If E is closed, then it contains all its limit points L, such that $L \subseteq E$. Thus, $\overline{E} = L \cup E = E$.

Now, clearly, if $E^\circ = E$, then E is open, since every point in E has an ϵ-neighorhood contained in E.

Now, if E is open, then E° contains every point in E, so $E \subseteq E^\circ$. On the other hand, since E° can only take points from E, then $E \supseteq E^\circ$, so we have that $E^\circ = E$.

(b) Claim: $\overline{E}^{\mathrm{C}} \subseteq \left(E^{\mathrm{C}}\right)^\circ$

Proof: First, assume that $x \in \overline{E}^{\mathrm{C}}$. This implies that $x \notin \overline{E}$, or that $x \notin E \cup L$, where L is the set of limit points of E. From this, we see that $x \in E^{\mathrm{C}} \cap L^{\mathrm{C}}$. If $x \in L^{\mathrm{C}}$, then x is not a limit point of E. This means that either is an interior point of E, or that there is an ϵ-neighborhood of x such that it is fully not in E. Alternatively, we can say that:

$$x \in E \text{ or there exists } \epsilon \text{ such that } V_\epsilon(x) \subseteq E^{\mathrm{C}}.$$

From this, we can see that it must be that $x \in E^{\mathrm{C}}$, and there exists $V_\epsilon(x) \subseteq E^{\mathrm{C}}$. This is the definition of $\left(E^{\mathrm{C}}\right)^\circ$.

Therefore, $\overline{E}^{\mathrm{C}} \subseteq \left(E^{\mathrm{C}}\right)^\circ$. ∎

Claim: $\overline{E}^{\mathrm{C}} \supseteq \left(E^{\mathrm{C}}\right)^\circ$

Proof: Assume $x \in \left(E^{\mathrm{C}}\right)^\circ$. Then, it must be in E^{C}, and there exists $V_\epsilon(x) \subseteq E^c$.

This implies that x is not a limit point of E, so now we have that $x \notin L \Longrightarrow x \in L^{\mathrm{C}}$.

Thus, we have $x \in E^c \cap L^c = (E \cup L)^{\mathrm{C}} = \overline{E}^{\mathrm{C}}$. ∎

Thus, $\overline{E}^{\mathrm{C}} = \left(E^{\mathrm{C}}\right)^\circ$.

Now, I claim that $(E^\circ)^C = \overline{E^C}$.

Claim: $(E^\circ)^C \subseteq \overline{E^C}$

Proof: Assume $x \in (E^\circ)^C$. Then, $x \notin E^\circ$. This implies that either $x \notin E$, or $x \in E$ and for all ϵ, $V_\epsilon(x) \cap E^C \neq \emptyset$. This is the definition of a limit point of E^C, so now we have that either $x \notin E \implies x \in E^c$, or $x \in L'$, where L' is the set of limit points of E^C.

Thus, $x \in E^C \cup L' = \overline{E^C}$. ∎

Claim: $(E^\circ)^C \supseteq \overline{E^C}$

Proof: Assume $x \in \overline{E^C}$. Then $x \in E^C \cup L'$. If $x \in L'$, then it cannot be in the interior of E, since there is no ϵ contained completely in E.

Thus, we have that $x \notin E^\circ$, so $\overline{E^C} \subseteq (E^\circ)^C$. ∎

Thus $(E^\circ)^C = \overline{E^C}$.

3.2.15 **Exercise 136**

A set A is called an F_σ set if it can be written as the countable union of closed sets. A set B is called a G_δ set if it can be written as the countable intersection of open sets.

(a) Show that a closed interval $[a, b]$ is a G_δ set.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.

(c) Show that $\mathbb{Q}$ is an F_σ set, and the set of irrationals $\mathbb{I}$ forms a G_δ set. (We will see in Section 3.5 that $\mathbb{Q}$ is *not* a G_δ set, nor is $\mathbb{I}$ an F_σ set.)

(a) Consider the collection of sets $A_1, A_2, A_3, \ldots$, where $A_i = \left(a - \frac{1}{i}, b + \frac{1}{i}\right)$.

Then, $[a, b]$ is clearly a subset of every set, so it will be in their intersection. In addition, no number outside of $[a, b]$ will be present in the intersection.

(b) For the union of closed sets, consider the collection with $A_i = \left[a + \frac{1}{i}, b\right]$.

For the intersection of open sets, consider the collection with $A_i = \left(a, b + \frac{1}{i}\right)$.

(c) Consider the enumeration of rationals. Now, let each rational be enclosed in its own set. Each is a closed set, and the union of all of them gives us exactly the rationals and nothing else.

Now, consider a set containing just a rational, and now take the complement. This is an open set. The intersection of all such complements gives us the irrationals, as it will give us all real numbers except the rationals.

3.3 Compact Sets

3.3.1 **Exercise 137**

Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

If K is compact and nonempty, then it must be bounded. Since it is bounded, it must be upper and lower bounded. This means that $\sup K$ and $\inf K$ both exist.

We also know that K is closed. Thus, we know from properties of supremum and infimum that they must be limit points of K, and therefore must be contained within K.

3.3.2 **Exercise 138**

Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

(a) $\mathbb{N}$.

(b) $\mathbb{Q} \cap [0,1]$.

(c) The Cantor set.

(d) $\{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in \mathbb{N}\}$.

(e) $\{1, 1/2, 2/3, 3/4, 4/5, \ldots\}$.

(a) Not compact, consider the natural enumeration of natural numbers.

(b) Not compact, consider any sequence converging to $\frac{\sqrt{2}}{2}$.

(c) Compact. I can't think of a proof that relies on the sequence definition, but it is clearly bounded, and also closed, since it is the infinite intersection of closed sets.

(d) Not compact, consider the natural enumeration as a sequence, and we see that it is strictly monotonically increasing and bounded by our results on series. Therefore, the limit cannot exist in the set, as otherwise we would find an element strictly greater than the limit.

(e) By Bolzano–Weierstrass any sequence with elements from this set should have a convergent subsequence.

I claim that any subsequence that converges must converge to an element in this set.

Proof: AFSOC that we have a subsequence that converges to an x not in our set.

Clearly, our x must be be in the interval $[1/2, 1]$.

We find the closest two elements in our set, let's call it $\frac{n}{n+1}$ and $\frac{n+1}{n+2}$. (If x were to be 1 or greater, then this would not be possible. This is a result using the Archimedean property, and if $x > 1$, then we would end up either trying to find a natural number greater than infinity, or a natural number smaller than a negative number.)

Set ϵ to be $\frac{1}{2}\min\left\{x - \frac{n}{n+1}, \frac{n+1}{n+2} - x\right\}$, and now notice that there can be no element in our set within that ϵ-neighborhood of x.

This contradicts the definition of convergence.

Therefore, every subsequence must converge to a value within the set. ∎

3.3.3 **Exercise 139**

Prove the converse of Theorem 3.3.4 by showing that if a set $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

Take any sequence (x_n) with elements from the set K.

Since our set is bounded, we can apply Bolzano–Weierstrass to find a convergent $\left(x_{n_k}\right)$ with elements from K.

Now, there are two cases.

Case 1: The subsequence converges to a limit x already present in our sequence, and we get that $x \in K$ for free.

Case 2: The subsequence converges to a limit x not in our original sequence. Now, notice, that x satisfies the definition for a limit point. Since our set is closed, we then get the result that $x \in K$.

So, in either case, we get that any (x_n) has a subsequence that converges to a point $x \in K$, which is exactly the definition for a compact set.

3.3.4 **Exercise 140**

Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$

(b) $\overline{F^{\mathrm{C}} \cup K^{\mathrm{C}}}$

(c) $K \setminus F = \{x \in K : x \notin F\}$

(d) $\overline{K \cap F^{\mathrm{C}}}$

(a) **Definitely compact** and closed, since the intersection will stay bounded, and the intersection of closed sets is still closed.

(b) $F^{\mathrm{C}} \cup K^{\mathrm{C}}$ will be unbounded (since it is the complement of a bounded set). Thus, it is not compact, but the closure is still **definitely closed**.

(c) We have **no guarantees**, other than boundedness. As an example, consider $K = [-1, 2]$, and $F = (0, 1)^{\mathrm{C}}$. Then $K \setminus F = (0, 1)$, which is open, not closed, and thus not compact.

(d) This is definitely closed, and also bounded, so it must be **compact**.

3.3.5 **Exercise 141**

Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) The arbitrary intersection of compact sets is compact.

(b) The arbitrary union of compact sets is compact.

(c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

(a) True. The intersection of bounded sets cannot be unbounded, and will be bounded (or empty). We also know that the arbitrary intersection of closed sets is closed. So the resulting set is either empty (and thus compact), or nonempty and compact.

(b) False, consider the collection of compact sets of the form:

$$\{[1/n, 1-1/n] : n \in \mathbb{N}, n \geq 2\}.$$

The countable union gives us $(0,1)$, which is clearly not closed and therefore not compact.

(c) False, consider $A = (0,1)$, $K = [0,1]$. Then the intersection is $(0,1)$.

(d) False, since the closed set could be unbounded, for example, $\{[n, \infty] : n \in \mathbb{N}\}$. The countable intersection here is the empty set.

3.3.6 **Exercise 142**

This exercise is meant to illustrate the point made in the opening paragraph to Section 3.3. Verify that the following three statements are true if every blank is filled in with the word "finite". Which are true if every blank is filled in with the word "compact"? Which are true if every blank is filled in with the work "closed"?

(a) Every ______ set has a maximum.

(b) If A and B are ______, then $A + B = \{a + b : a \in A, b \in B\}$ is also ______.

(c) If $\{A_n : n \in \mathbb{N}\}$ is a collection of ______ sets with the property that every finite subcollection has a nonempty intersection, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty as well.

(a) True, since the compact set is bounded, then it must have a supremum. It must also contain the supremum, since we can produce a sequence that converges to the supremum. Thus, the supremum must be contained in the set and is therefore a maximum.

This is not true if it is just closed, since we can have two closed sets that are not upper bounded.

(b) True, $A + B$ will stay bounded, and for every sequence in $A + B$, by Bolzano–Weierstrass it will have a convergent subsequence. For the subsequence, we can split it into the addition of two sequences from A and B respectively. Each will converge to a and b, which exist in A and B respectively, so by ALT the limit of our original subsequence is $a + b$, which we know must exist in $A + B$.

This is not true if we have closed, since we can then have two unbounded but closed sets:

$$A = \mathbb{N}, \quad B = \{-n + 1/n : n \in \mathbb{N}\}.$$

We can produce a sequence in $A + B$ where $(x_n) = 1/n$, which we know to converge to 0. However, 0 is clearly not an element of $A + B$, as it would require $m - n + 1/n = 0$ which cannot be true for $m, n \in \mathbb{N}$.

(c) For finite sets, this must be true, as we can show by induction that every A_n has a shared common element. Therefore, the infinite intersection should also have that shared common element.

For compact sets, this must also be true. We can invoke the Nested Compact Set property, by iteratively constructing $K_n = \bigcap_{i=1}^{n} A_i$, and noting that we now have a nested sequence of nonempty compact sets.

The infinite intersection of this nested sequence is thus nonempty.

For closed sets, this is not necessarily true, just take $A_n = [n, \infty]$.

3.3.7 **Exercise 143**

As some more evidence of the surprising nature of the Cantor set, follow these steps to show that the sum $C + C = \{x + y : x, y \in C\}$ is equal to the closed interval $[0, 2]$. (Keep in mind that C has zero length and contains no intervals.)

Because $C \subseteq [0, 1]$, $C + C \subseteq [0, 2]$, so we only need to prove that the reverse inclusion $[0, 2] \subseteq \{x + y : x, y \in C\}$. Thus, given $s \in [0, 2]$, we must find two elements $x, y \in C$ satisfying $x + y = s$.

(a) Show that there exist $x_1, y_1 \in C_1$ for which $x_1 + y_1 = s$. Show in general that, for an arbitrary $n \in \mathbb{N}$, we can always find $x_n, y_n \in C_n$ for which $x_n + y_n = s$.

(b) Keeping in mind that the sequences (x_n) and (y_n) do not necessarily converge, show how they can nevertheless be used to produce the desired x and y in C satisfying $x + y = s$.

(a) If $s \in C_1$, then clearly we can just take $x_1 = 0$, and $y_1 = s$. If $s \in [1, 4/3] \cup [5/3, 2]$, then we can take $x_1 = 1$ and $y_1 = s - 1$.

Now, notice that $[1/3, 2/3]$ can be covered by $x_1 = 1/3$ and y_1 from $[0, 1/3]$. Likewise, $[4/3, 5/3]$, can be covered by $x_1 = 2/3$ and $y_1 \in [2/3, 1]$.

To show that arbitrary x_n, y_n from C_n can cover s, I use an inductive argument.

Assume that we can find x_n, y_n such that $x_n + y_n = s \in [0, 2]$.

Now, we note the property that if $x \in C_n$, then $x/3 \in C_{n+1}$, as well as $\frac{x+2}{3} \in C_{n+1}$.

Assume we have arbitrary $s \in [0, 2]$. Now, we have three cases:

$$s = \begin{cases} s'/3 & \text{for } s \in [0, 2/3] \\ (s' + 2)/3 & \text{for } s \in (2/3, 4/3], \\ (s' + 4)/3 & \text{for } s \in (4/3, 2]. \end{cases} \quad \text{where } s' \in [0, 2].$$

Now, we can see that there must exist $x_n, y_n \in C_n$ such that $x_n + y_n = s'$.

From our three cases, we can now see that we will have:

$$s = \begin{cases} \frac{x_n}{3} + \frac{y_n}{3} & \text{for } s \in [0, 2/3] \\ \frac{x_n+2}{3} + \frac{y_n}{3} & \text{for } s \in (2/3, 4/3]. \\ \frac{x_n+2}{3} + \frac{y_n+2}{3} & \text{for } s \in (4/3, 2]. \end{cases}$$

In all cases, both terms are members of C_{n+1}.

Thus by induction, we can always find x_n, y_n such that $x_n + y_n = s \in [0, 2]$.

(b) (x_n) is bounded, so it possesses a convergent subsequence.

Looking at the convergent subsequence, let's say that $\left(x_{n_k}\right) \to x$.

Since C_1 is closed, it must be that $x \in C_1$. I claim that $x \in C_n$ for every $n \in \mathbb{N}$.

To see this, just consider the tail of the same subsequence such that $n_k \geq n$, and note that it must share the same x as its limit.

Since every C_n is compact, it must be that $x \in C_n$ for all $n \in \mathbb{N}$.

Thus, since $C = \bigcap_{n=1}^{\infty} C_n$, it must be that $x \in C$.

If we define $y_n = s - x_n$, then for the subsequence $\left(x_{n_k}\right)$, the analogous subsequence $\left(y_{n_k}\right) \to y = s - x$ by ALT.

By similar reasoning to above, we see that $y \in C$ as well.

Therefore, we have produced $x, y \in C$ such that $x + y = s$.

3.3.8 **Exercise 144**

Let K and L be nonempty compact sets, and define

$$d = \inf\{|x - y| : x \in K \text{ and } y \in L\}.$$

This turns out to be a reasonable definition for the *distance* between K and L.

(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

(b) Show that it's possible to have $d = 0$ if we assume only that the disjoint sets K and L are closed.

(a) I claim that absolute value of a closed set gives a closed set.

Proof: Let X be closed. Let $X' = \{|x| : x \in X\}$.

Observe arbitrary limit point $|a|$ of X'. Now, there must be an $(|a_n|) \to |a|$.

The original sequence in $|a_n|$ in X must either have infinite negative terms, infinite non-negative terms, or both.

In any case, we can pick out a subsequence of that sequence to see that it will approach a or $-a$, which indicates that a, $-a$, or both are in X.

This implies that $|x| \in X'$, so all limit points of X' are contained within it, and X' is closed. ■

Clearly, the negation of a compact set is compact, and by Exercise 3.3.6, the addition of two compact sets is compact.

Finally, the absolute value (as just proved) of a compact set must be closed and bounded and therefore still compact.

This immediately gives us the result that it must contain its infimum.

So, if K and L are disjoint, then $d \neq 0$, since that would imply there are two elements $x = y$, which contradicts the disjointness of the two sets.

(b) Consider the closed but unbounded sets $K = \mathbb{N}$ and $L = \{n + 1/n : n \in \mathbb{N}, n \geq 2\}$.

The distance between any two elements approaches 0, but there are no elements in common.

3.3.9 **Exercise 145**

Follow these steps to prove the final implication in Theorem 3.3.8.

Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.

(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

(c) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

(a) Let I_0 be the proposed closed interval, such that $K \subseteq I_0$. Note that due to our assumption, $I_0 \cap K$ covered by $\{O_\lambda : \lambda \in \Lambda\}$ does not have a finite subcover.

Now, select I_{i+1} in the following way:

Bisect I_i into two closed intervals, and call the left half L_i, and the right half R_i. Both have $|L_i| = |R_i| = |I_i|/2$, and at least one of the two must be such that $L_i \cap K$ and $R_i \cap K$ cannot be covered by a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$. (Otherwise, we could find a finite subcover for I_i.)

Select any one of the half intervals that does not have a finite subcover as described above.

Continue in this manner. Notice that no matter the initial length of I_0, we will eventually find an interval smaller than any positive value. Therefore $\lim_{n\to\infty} |I_n| = 0$.

(b) This follows directly from the Nested Interval Property. We should also note that at any point, $I_n \cap K$ cannot be empty, otherwise we would have the trivial finite subcover.

So the countable intersection is both nonempty and shares an element with K.

(c) Since $x \in K$ and also $x \in I_n$ for all n, it must be the case that some O_{λ_0} from our open cover contains x.

This implies that for some $\epsilon > 0$, we have that $V_\epsilon(x) \subseteq O_{\lambda_0}$.

However, we can also find an interval I_n such that $|I_n| < \epsilon$, such that $x \in I_n$ and also $x \in I_n \cap K$. This implies that $I_n \subseteq V_\epsilon(x) \subset O_{\lambda_0}$.

This is a **contradiction**, since O_{λ_0} itself functions as a finite subcover for $I_n \cap K$, but we chose I_n so that this wouldn't happen.

This means our original assumption that there could be no finite subcover is incorrect.

3.3.10 **Exercise 146**

Here is an alternate proof to the one given in Exercise 3.3.9 for the final implication in the Heine–Borel Theorem.

> Consider the special case where K is a closed interval. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define S to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$.
>
> (a) Argue that S is nonempty and bounded, and thus $s = \sup S$ exists.
>
> (b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.
>
> (c) Finally, prove the theorem for an arbitrary closed and bounded set K.

(a) Clearly, we can see that $a \in S$, since the single point a would need to be covered by some open set in the cover.

S is also clearly bounded, since it is a subset of a bounded interval.

Therefore, we can select $s = \sup S$.

(b) Note that the largest s can be is b.

Now, we check if s is contained in S.

Clearly, s must be covered by some open set in the cover. It is also the case that this open set must cover some points in a small ϵ-neighborhood around s.

We already know that any point less than s must be able to be covered with a finite subcover. So, we can just add our open set covering s to our finite subcover and see that $s \in S$.

Within that small ϵ-neighborhood, it must be that points greater than s should also be covered by that same open set.

However, if there were points greater than s covered by the open set, then s would no longer be the supremum.

Therefore, there cannot be any points greater than s within $[a, b]$, implying that $s = b$.

(c) Since K is closed and bounded, we can find its upper and lower bounds a and b.

Note further, that these points must be contained within K.

Form the interval $[a, b]$, and define S as the set of all $x \in [a, b] \cap K$ such that $[a, x] \cap K$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$.

By similar argument from part (a), $a \in S$, so $s = \sup S$ must exist.

AFSOC $s < b$.

If there exists a small ϵ-neighborhood around s such that it is fully contained in K, then we can use the argument from part (b).

If it is instead the case that there is some (s, r) such that $(s, r) \cap K$ is empty, then we can proceed in the following manner:

We observe $[r, b] \cap K$. This is a compact set, thus it contains its infimum.

Call the infimum y, and note that $s < y \leq b$.

We then work with the following interval:

$$\begin{aligned} [a, y] \cap K &= ([a, s] \cup (s, y) \cup \{y\}) \cap K \\ &= [a, s] \cap K \cup \{y\}. \end{aligned}$$

We know that y must be covered by some O_λ.

Let's just add that O_λ to the finite subcover for $[a, s] \cap K$, and we arrive at our **contradiction**.

Therefore, it must be the case that $s = b$, so therefore, we have a finite subcover for our entire compact set.

3.3.11 **Exercise 147**

Consider each of the sets listed in Exercise 3.3.2. For each one that is not compact, find an open cover for which there is no finite subcover.

(a) $\mathbb{N}$: Consider the open cover where there is simply an interval $\left(n - \frac{1}{2}, n + \frac{1}{2}\right)$ around each natural number

(b) $\mathbb{Q} \cap [0, 1]$:

Consider $\left(\frac{\sqrt{2}}{2}, 2\right)$, and the set of open intervals $\left\{\left(-1, \frac{\sqrt{2}}{2} - \frac{1}{n}\right) : n \in \mathbb{N}\right\}$

(c) The Cantor set: compact.

(d) $\{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in \mathbb{N}\}$:

Consider the set of open sets $\{(0, s_n) : s_n \in S\}$, where S is the set of partial sums as seen above and s_n is the natural enumeration.

(e) $\{1, 1/2, 2/3, 3/4, 4/5, \ldots\}$: compact.

3.3.12 **Exercise 148**

Using the concept of open covers (and explicitly avoiding the Bolzano–Weierstrass Theorem), prove that every bounded infinite set has a limit point.

AFSOC that there are no limit points in our set S.

Then, since we know that a closed set is one that contains all its limit points, we see that our set is closed.

Therefore, since it is closed and bounded, our set is compact.

Now, since there are no limit points, we know that for every point in our set x, there exists an ϵ-neighborhood for which the intersection with S gives us $\{x\}$.

Choose all the ϵ-neighborhoods centered at individual points as our open cover.

Clearly, this covers all points, but any finite subcover will always be missing infinitely many points.

This is a **contradiction**, since we determined that any open cover of a compact set will have a finite subcover.

Thus, it must be that S has at least one limit point.

3.3.13 **Exercise 149**

> Let's call a set *clompact* if it has the property that every *closed* cover (i.e., a cover consisting of closed sets) admits a finite subcover. Describe all of the clompact subsets of $\mathbb{R}$.

A set is clompact if and only if it is finite.

($\Rightarrow$)

Pick the closed cover such that every point x in our set is covered by the closed set $\{x\}$. Now, we choose our finite subcover, and we immediately get that our set can only contain finitely many points.

($\Leftarrow$)

For every point x, simply select a closed set covering it, and in the end we will have finitely many closed sets.

3.4 Perfect Sets and Connected Sets

> **3.4.1** **Exercise 150**
>
> If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

The intersection is always compact, since a perfect set is closed. Thus the intersection will still be closed and bounded and therefore compact.

The set will not always be perfect. As an example, consider $P = [0, 1]$, and $K = \{1\}$. The intersection is an isolated point and thus not perfect.

> **3.4.2** **Exercise 151**
>
> Does there exist a perfect set consisting of only rational numbers?

No, since a perfect set must be uncountable.

> **3.4.3** **Exercise 152**
>
> Review the portion of the proof given in Example 3.4.2 and follow these steps to complete the argument.
>
> (a) Because $x \in C_1$, argue that there exists an $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| \leq 1/3$.
>
> (b) Finish the proof by showing that for each $n \in \mathbb{N}$, there exists $x_n \in C \cap C_n$, different from x, satisfying $|x - x_n| \leq 1/3^n$.

(a) Just choose the closest end-point of the intervals that make up C_1 (that isn't x itself). The furthest away it can be is thus $\frac{1}{6} < \frac{1}{3}$, or exactly $1/3$ if x is an endpoint. We also know that C contains all such endpoints.

(b) The same logic using endpoints can be applied for all n, and this gives us a sequence for any x such that the distance $|x_n - x| \leq 1/3^n$. This proves that x is a limit point, and thus not isolated.

3.4.4 **Exercise 153**

Repeat the Cantor construction from Section 3.1 starting with the interval $[0,1]$. This time, however, remove the open middle *fourth* from each component.

(a) Is the resulting set compact? Perfect?

(b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

TODO: skipped

3.4.5 **Exercise 154**

Let A and B be nonempty subsets of $\mathbb{R}$. Show that if there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.

If U and V are disjoint open sets, then the limit point of one set cannot exist in the other set. WLOG, AFSOC assume U had a limit point x in V.

Now there exists a ball around x fully contained in V, but also from the definition of limit point, there will always be a point $x' \neq x$ in U within any ball around x, contradicting the initial condition that U and V are disjoint.

Thus, the closure of any one set and the original second set will still be disjoint.

The closure of any subset of the original set will be a subset of the original set, so A and B must also be separated.

3.4.6 **Exercise 155**

Prove Theorem 3.4.6.

($\Rightarrow$) I prove the contrapositive. Assume that there exists a non-trivial partition $A \cup B = E$, with the property that all convergent sequences with all elements in A have a limit that cannot be in B, and vice-versa.

Thus, all limit points of A also cannot be in B, and $\overline{A} \cap B = \emptyset$. Similar logic applies for all convergent sequences fully contained in B.

Thus, A and B must be separated, which implies that E is disconnected.

($\Leftarrow$) Assume that for all non-trivial partitions $A \cup B = E$, there always exists a convergent sequence fully contained in one set with a limit in the other.

The limit must be a limit point, and thus, the closure of the set containing the convergent sequence contains said limit point. The intersection must be non-empty, which implies that all partitions cannot be separated, and thus, E must be connected.

3.4.7 **Exercise 156**

A set E is *totally disconnected* if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.

(a) Show that $\mathbb{Q}$ is totally disconnected.

(b) Is the set of irrational numbers totally disconnected?

(a) There will always exist an irrational number x between any two distinct rationals, just partition $\mathbb{Q}$ based on rationals greater and lesser than x.

(b) There will always exist a rational number x between any two distinct irrationals. Partition the irrationals based on those greater and lesser than x.

3.4.8 **Exercise 157**

Follow these steps to show that the Cantor set is totally disconnected in the sense described in Exercise 3.4.7.

Let $C = \bigcap_{n=0}^{\infty} C_n$, as defined in Section 3.1.

(a) Given $x, y \in C$, with $x < y$, set $\epsilon = y - x$. For each $n = 0, 1, 2, \ldots$, the set C_n consists of a finite number of losed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval of C_N.

(b) Show that C is totally disconnected.

(a) The length of the intervals in C_n is exactly $1/3^n$. So just find N large enough such that $1/3^N < \epsilon$.

(b) Since it must be the case that there is a C_N after which $y - x$ cannot exist in the same interval, then it must be the case that there is a $z \notin C_N$ such that $x < z < y$. Thus in $z \notin C$ as well. We can then partition C into all those that are greater or less than z.

3.4.9 **Exercise 158**

Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rational numbers, and for each $n \in \mathbb{N}$ set $\epsilon_n = 1/2^n$. Define $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$, and let $F = O^{\mathsf{C}}$.

(a) Argue that F is a closed, nonempty set consisting only of irrational numbers.

(b) Does F contain any nonempty open intervals? Is F totally disconnected? (See Exercise 3.4.7 for the definition.)

(c) Is it possible to know whether F is perfect? If not, can we modify the construction to produce a nonempty perfect set of irrational numbers?

(a) Clearly F must be closed, since O is the countable union of open sets and thus must be open itself.

F must also not contain any rational numbers by construction, so if there are any elements, it must be irrational.

To prove F is non-empty, we should use the Nested Compact Set Property.

First, using De Morgan's Law, we see that $F = \bigcap_{n=1}^{\infty} V_{\epsilon_n}(r_n)^{\mathsf{C}}$.

Let $C_n = \bigcap_{m=1}^{n} V_{\epsilon_m}(r_m)^{\mathsf{C}} \cap [0, 2]$. This is clearly compact, since $V_{\epsilon_n}(r_n)^{\mathsf{C}}$ is closed, and nested.

To argue that it is nonempty, consider the compact set of points:

$$A_n = \{i/2^n : i \in \mathbb{Z}_+, 0 \leq i \leq 2^{n+1}\}.$$

This is a subset of $[0, 2]$, and has $2^{n+1} + 1$ elements.

Since every $V_{\epsilon_m}(r_m)$ has a length of $1/2^{m-1}$ max, it can contain at most 2^{n-m+1} elements of A_n.

Thus, the number of elements in the union is as follows:

$$\left| A_n \cap \bigcup_{m=1}^{n} V_{\epsilon_m}(r_m) \right| \leq \sum_{m=1}^{n} 2^{n-m+1} = \sum_{i=1}^{n} 2^i = 2^{n+1} - 2.$$

Thus, we can see the following relationship:

$$\begin{aligned}
C_n &= \bigcap_{m=1}^{n} V_{\epsilon_m}(r_m)^{\mathsf{C}} \cap [0, 2] \supseteq \bigcap_{m=1}^{n} V_{\epsilon_m}(r_m)^{\mathsf{C}} \cap A_n \\
&\supseteq A_n \cap \left(\bigcup_{m=1}^{n} V_{\epsilon_m}(r_m) \right)^{\mathsf{C}} = (A_n \cap A_n^{\mathsf{C}}) \cup \left(A_n \cap \left(\bigcup_{m=1}^{n} V_{\epsilon_m}(r_m) \right)^{\mathsf{C}} \right) \\
&= A_n \cap \left(A_n^{\mathsf{C}} \cup \left(\bigcup_{m=1}^{n} V_{\epsilon_m}(r_m) \right)^{\mathsf{C}} \right) = A_n \cap \left(A_n \cap \left(\bigcup_{m=1}^{n} V_{\epsilon_m}(r_m) \right) \right)^{\mathsf{C}} \\
&= A_n \setminus \left(A_n \cap \left(\bigcup_{m=1}^{n} V_{\epsilon_m}(r_m) \right) \right).
\end{aligned}$$

From our previous result, we know that this has at least 3 elements in it, so therefore, for any C_n, it has at least 3 elements in it and is therefore non-empty.

Thus, we have our sequence of nested, compact, and non-empty sets, so their intersection must be non-empty. Their intersection is a subset of F, so F must also be non-empty.

(b) F does not contain any nonempty open intervals, since between any endpoints of any interval there is a rational number, which cannot be in the interval.

F must therefore be totally disconnected as well.

(c) It is possible that F is not perfect. To see this, we can order the rationals in such a way that for any rational close to $\sqrt{2}$, it is far enough down the list such that the ϵ-neighborhood is small enough to not include $\sqrt{2}$. I don't rigorously justify this enumeration, but it should be possible.

Thus, $\sqrt{2} \in F$, but there is no other irrational close to it and it is an isolated point.

To modify the construction such that F is perfect, we must do the following

TODO: not finished

3.5 Baire's Theorem

3.5.1 **Exercise 159**

Argue that a set A is a G_δ set if and only if its complement is an F_δ set.

3.5.2 **Exercise 160**

Replace each ______ with the word *finite* or *countable*, depending on which is more appropriate.

(a) The ______ union of F_σ sets is an F_σ set.

(b) The ______ intersection of F_σ sets is an F_σ set.

(c) The ______ union of G_δ sets is a G_δ set.

(d) The ______ intersection of G_δ sets is a G_δ set.

3.5.3 **Exercise 161**

(This exercise has already appeared as Exercise 3.2.15.)

(a) Show that a clsoed interval $[a, b]$ is a G_δ set.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.

(c) Show that $\mathbb{Q}$ is an F_σ set, and the set of irrationals $\mathbb{I}$ forms a G_δ set.

3.5.4 **Exercise 162**

Starting with $n = 1$, inductively construct a nested sequence of *closed* intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ satisfying $I_n \subseteq G_n$. Give special attention to the issue of endpoints of each I_n. Show how this leads to a proof of the theorem.

3.5.5 **Exercise 163**

Show that it is impossible to write

$$\mathbb{R} = \bigcup_{n=1}^{\infty} F_n,$$

where for each $n \in \mathbb{N}$, F_n is a closed set containing no nonempty open intervals.

3.5.6 **Exercise 164**

Show how the previous exercise implies that the set $\mathbb{I}$ of irrationals cannot be an F_σ set, and $\mathbb{Q}$ cannot be a G_δ set.

3.5.7 **Exercise 165**

Using Exercise 3.5.6 and versions of the statements in Exercise 3.5.2, construct a set that is neither in F_σ nor in G_δ.

3.5.8 **Exercise 166**

Show that a set E is nowhere-dense in $\mathbb{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbb{R}$.

3.5.9 **Exercise 167**

Decide whether the following sets are dense in $\mathbb{R}$, nowher-dense in $\mathbb{R}$, or somewhere in between.

(a) $A = \mathbb{Q} \cap [0, 5]$.

(b) $B = \{1/n : n \in \mathbb{N}\}$.

(c) the set of irrationals.

(d) the Cantor set.

3.5.10 **Exercise 168**

Finish the proof by finding a contradiction to the results in this section.

3.6 Epilogue

No exercises in this section.

4 Functional Limits and Continuity

4.1 Discussion: Examples of Dirichlet and Thomae

No exercises in this section.

4.2 Functional Limits

4.2.1 **Exercise 169**

4.2.2 **Exercise 170**

4.2.3 **Exercise 171**

4.2.4 **Exercise 172**

4.2.5 **Exercise 173**

4.2.6 **Exercise 174**

4.2.7 **Exercise 175**

4.2.8 **Exercise 176**

4.2.9 **Exercise 177**

4.2.10 **Exercise 178**

4.3 Continous Functions

4.3.1 **Exercise 179**

4.3.2 **Exercise 180**

4.3.3 **Exercise 181**

4.3.4 **Exercise 182**

4.3.5 **Exercise 183**

4.3.6 **Exercise 184**

4.3.7 **Exercise 185**

4.3.8 **Exercise 186**

4.3.9 **Exercise 187**

4.3.10 **Exercise 188**

4.3.11 **Exercise 189**

4.3.12 **Exercise 190**

4.3.13 **Exercise 191**

4.3.14 **Exercise 192**

4.4 Continuous Functions on Compact Sets

4.4.1 **Exercise 193**

4.4.2 **Exercise 194**

4.4.3 **Exercise 195**

4.4.4 **Exercise 196**

4.4.5 **Exercise 197**

4.4.6 **Exercise 198**

4.4.7 **Exercise 199**

4.4.8 **Exercise 200**

4.4.9 **Exercise 201**

4.4.10 **Exercise 202**

4.4.11 **Exercise 203**

4.4.12 **Exercise 204**

4.4.13 **Exercise 205**

4.4.14 **Exercise 206**

4.5 The Intermediate Value Theorem

4.5.1 **Exercise 207**

4.5.2 **Exercise 208**

4.5.3 **Exercise 209**

4.5.4 **Exercise 210**

4.5.5 **Exercise 211**

4.5.6 **Exercise 212**

4.5.7 **Exercise 213**

4.5.8 **Exercise 214**

4.6 Sets of Discontinuity

4.6.1 **Exercise 215**

4.6.2 **Exercise 216**

4.6.3 **Exercise 217**

4.6.4 **Exercise 218**

4.6.5 **Exercise 219**

4.6.6 **Exercise 220**

4.6.7 **Exercise 221**

4.6.8 **Exercise 222**

4.6.9 **Exercise 223**

4.6.10 **Exercise 224**

4.6.11 **Exercise 225**

4.7 Epilogue

No exercises in this section.

5 The Derivative

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