Summer 2025 – Understanding Analysis Solutions

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Contents

1 The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

No exercises in this section.

1.2 Some Preliminaries

1.2.1 **Exercise 1**

(a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?
(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

(a) Assume for sake of contradiction (AFSOC) that $\sqrt{3} \in \mathbb{Q}$. This implies that $\sqrt{3} = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, and $\gcd(p, q) = 1$.

Therefore, $p^2 = 3q^2$, which means that 3 divides p^2.

Since 3 is prime, p must be divisible by 3. Therefore for some $k \in \mathbb{Z}$,

$$p = 3k \Rightarrow 9k^2 = 3q^2 \Rightarrow 3k^2 = q^2.$$

This implies that q^2 and thus q is also divisible by 3, which is a contradiction.

A similar proof does not quite work for $\sqrt{6}$ and needs to be adjusted, since 6 is not prime and thus we cannot directly say that 6 divides p^2 implies 6 divides p.

(b) It is exactly the step where we try to show that 4 divides q^2 implies that 4 divides q. In fact, if we have just that 2 (and not 4) divides q, then clearly 4 still divides q^2.

1.2.2 **Exercise 2**

Show that there is no rational number r satisfying $2^r = 3$.

AFSOC that there does exist $r = \frac{p}{q}$, with coprime $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Then $2^r = 2^{p/q} = 3$, which implies that $2^p = 3^q$. This is false.

1.2.3 **Exercise 3**

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \ldots$ are all sets containing an infinite number of elements, then the interesection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.
(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \ldots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.
(c) $A \cap (B \cup C) = (A \cap B) \cup C$.
(d) $A \cap (B \cap C) = (A \cap B) \cap C$.
(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

TODO: skipped

1.2.4 **Exercise 4**

Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

Assume we have infinite primes. Since they are a subset of $\mathbb{N}$, they are enumerable ($p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$).

Also assume we have unique prime decomposition.

Now let

$$A_i = \{n \in \mathbb{N} \mid p_i \text{ is the smallest prime in the decomposition of } n\},$$

with the additional modification that A_1 includes 1.

They are all disjoint, since there can only be one smallest prime factor of each number.

Their union forms the natural numbers, since every natural number n has a unique finite prime factor decomposition, and by the the fact that every non-empty subset of the natural numbers will have a smallest element, n must be an element of some A_i.

Clearly, every set is also infinite, since we can consider that each A_i contains the powers of p_i.

1.2.5 (De Morgan's Laws) **Exercise 5**

Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cup B)^{\mathsf{C}}$, explain why $x \in A^{\mathsf{C}} \cup B^{\mathsf{C}}$. This shows that $(A \cap B)^{\mathsf{C}} \subseteq A^{\mathsf{C}} \cup B^{\mathsf{C}}$.
(b) Prove the reverse inclusion $(A \cap B)^{\mathsf{C}} \supseteq A^{\mathsf{C}} \cup B^{\mathsf{C}}$, and conclude that $(A \cap B)^{\mathsf{C}} = A^{\mathsf{C}} \cup B^{\mathsf{C}}$.
(c) Show $(A \cup B)^{\mathsf{C}} = A^{\mathsf{C}} \cup B^{\mathsf{C}}$ by demonstrating inclusion both ways.

TODO: skipped

1.2.6 **Exercise 6**

(a) Verify the triangle inequality in the special case where a and b have the same sign.
(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.
(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.
(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

(a) **TODO: part skipped**
(b) **TODO: part skipped**
(c) **TODO: part skipped**
(d) Using the "unremarkable identity", for any a and b,

$$\begin{aligned} |a| &= |a-b+b| \\ &\leq |a-b| + |b|. \end{aligned}$$

So first we have $|a| - |b| \leq |a-b|$. Next, we proceed the same exact way using $|b|$, and we get that $|b| - |a| \leq |b-a|$.

Since $|a-b| = |b-a|$, we can combine the above two facts and get that

$$||a| - |b|| \leq |a-b|.$$

1.2.7 **Exercise 7**

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.
(a) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?
(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
(c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.
(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

(a) $A \cap B = [1,2]$. $f(A) = [0,4]$, and $f(B) = [1,16]$. So therefore, $f(A) \cap f(B) = [1,4]$.

$f(A \cap B) = [1,4]$ as well. So equality holds.

$A \cup B = [0,4]$, so $f(A \cup B) = [0,16] = f(A) \cup f(B)$.

Therefore equality holds in both cases.
(b) Let $A = \{1\}$, and $B = \{-1\}$.

Then $A \cap B = \emptyset$, but $f(A) = \{1\} = f(B)$, so $f(A) \cap f(B) = \{1\} \neq \emptyset$.
(c) For arbitrary $y \in g(A \cap B)$, we have that $y = g(x)$, where $x \in A \cap B$.

Therefore, $x \in A$ and $x \in B$, which implies that $g(x) \in g(A)$ and $g(x) \in g(B)$.

This further implies that $y = g(x) \in g(A) \cap g(B)$.

Thus we have that $g(A \cap B) \subseteq g(A) \cap g(B)$.

This doesn't work the other way around, since we could have some $y = g(x) = g(z)$, where $x \neq z$, and $x \in A$ and $z \in B$, and neither exists in the other set.

(d) My conjecture is that

$$g(A \cap B) = g(A) \cup g(B).$$

To show this, I first prove that $g(A \cup B) \subseteq g(A) \cup g(B)$, then the other way around.

$\boldsymbol{g(A \cup B) \subseteq g(A) \cup g(B)}$:

For arbitrary $y \in g(A \cup B)$, we have that $y = g(x)$ such that x in A or B. In either case, it must be such that y is in $g(A)$ or $g(B)$ and thus be in $g(A) \cup g(B)$.

$\boldsymbol{g(A) \cup g(B) \subseteq g(A \cup B)}$:

If $y \in g(A)$, then we have that $y = g(x)$ where $x \in A$, and therefore $x \in A \cup B \implies y = g(x) \in g(A \cup B)$. Same for $y \in g(B)$.

Thus we have proved both directions and shown set equality.

1.2.8 **Exercise 8**

Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1–1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1–1 but not onto.

(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1–1.

(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1–1 and onto.

TODO: skipped

1.2.9 **Exercise 9**

Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(a) **TODO: part skipped**

(b) Let $x \in g^{-1}(A \cap B)$. This implies that $g(x) \in A \cap B$, which implies that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$.

From this we can conclude that $x \in g^{-1}(A) \cap g^{-1}(B)$.

Going backwards, we see that if $x \in g^{-1}(A) \cap g^{-1}(B)$, then it must be the case that $g(x) \in A$ and $g(x) \in B$, which leads us to conclude that $x \in g^{-1}(A \cap B)$.

For union, we have if $x \in g^{-1}(A \cup B)$, then $g(x) \in A \cup B$. From the two cases, we will have that either $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$, which lets us conclude that $x \in g^{-1}(A) \cup g^{-1}(B)$.

Backwards, we have that either $g(x) \in A$ or $g(x) \in B$ depending on the cases, so therefore $g(x) \in A \cup B$ and thus $x \in g^{-1}(A \cup B)$.

1.2.10 **Exercise 10**

Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.
(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.
(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

1.2.11 **Exercise 11**

Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that…" in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + 1/n < b$.
(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.
(c) Between every two distinct real numbers there is a rational number.

TODO: skipped

1.2.12 **Exercise 12**

Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.
(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

TODO: skipped

1.2.13 **Exercise 13**

For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^{\mathsf{C}} = A_1^{\mathsf{C}} \cap A_2^{\mathsf{C}} \cap \cdots \cap A_n^{\mathsf{C}}$$

for any finite $n \in \mathbb{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathsf{C}} = \bigcap_{i=1}^{\infty} A_i^{\mathsf{C}},$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

(a) **TODO: part skipped**

(b) Let $B_i = \mathbb{N} \setminus \{i\}$. Any finite intersection will still have infinitely many elements, but the entire infinite intersection cannot have any elements.

(c) Let $x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^{C}$. Then we know that for all i, $x \neq A_i$. (Otherwise, we would have that $x \in \bigcup_{i=1}^{\infty} A_i$.)

Therefore, for all i, $x \in A_i^{C}$, which lets us conclude that $x \in \bigcap_{i=1}^{\infty} A_i^{C}$.

For the other direction, we just proceed from each step backwards and see that it works fine.

1.3 The Axiom of Completeness

1.3.1 **Exercise 14**

(a) Write a formal definition in the style of Definition 1.3.2 or the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

TODO: skipped

1.3.2 **Exercise 15**

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

(a) Let $B = \{1\}$, hmm...

(b) This cannot be possible. Since there are finite elements, there is necessarily a maximum and minimum, so the set must contain both of them.

(c) Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. The supremum is 1, which is contained. The infimum is clearly 0, which is not contained.

1.3.3 **Exercise 16**

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

(a) First, we know that the supremum of B must exist, since it is bounded above by any element of A.

So let $b' = \sup B$, and $a' = \inf A$.

AFSOC that there exists some $a \in A$ such that $a < b'$. Let $\epsilon = b' - a > 0$, and then we know that there must be some $b \in B$ such that $b > b' - \epsilon = a$, so we have $b > a$. This is a contradiction, since we assumed that b is a lower bound for all elements in A.

Therefore, we have shown that b' is a lower bound for A, and since it is a supremum of B, it must be the greatest such lower bound. This is exactly the definition of infimum of A.

(b) For any set bounded from below, we can take the set of all lower bounds, and use part (a) to show that the greatest lower bound is the smallest upper bound of the set of lower bounds.

1.3.4 **Exercise 17**

Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

(a) $\sup(A_1 \cup A_2) = \max(\sup A_1, \sup A_2)$.

Extended to $n \in \mathbb{N}$, we have

$$\sup\left(\bigcup_{k=1}^{n} A_k\right) = \max_{k \in [n]}(\sup A_k).$$

(b) This does not extend to infinite max, since it may be possible for the infinite max to exist. For example, if we have each A_k simply consist of the natural number k, then there is no supremum and no max.

1.3.5 **Exercise 18**

As in Example 1.3.7, let $A \in \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

TODO: skipped

1.3.6 **Exercise 19**

Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an pper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

(a) Let $c \in A + B$. Then $c = a + b$, with $a \in A$, and $b \in B$.

Now, we have that $a \leq s$ and $b \leq t$, so therefore, $c \leq s + t$.

(b) For all $b \in B$, we have that $a + b \leq u$. Thus, $u - a \geq b$, so $u - a$ is an upper bound for B.

Since t is the least upper bound for B, we now have that $t \leq u - a$.

(c) Let u be an arbitrary upper bound for $A + B$. By (b), we have that for all $a \in A$, $t \leq u - a$.

Therefore we also have that $a \leq u - t$, showing that $u - t$ is an upper bound on A. Since s is the least upper bound on A, we have $s \leq u - t$, and thus have $s + t \leq u$. This shows that $s + t$ must be the least upper bound and therefore is the supremum of $A + B$.

(d) Choose arbitrary $\epsilon > 0$. For $\frac{\epsilon}{2}$, there must exist $a \in A$ and $b \in B$ such that $a \geq s - \frac{\epsilon}{2}$ and $b \geq t - \frac{\epsilon}{2}$.

Therefore, $s + t - \epsilon \leq a + b$ for some $a + b$ in $A + B$.

But from part (a), we know that $s + t$ itself is an upper bound of $A + B$. Therefore, it must be that $s + t$ is the supremum of $A + B$.

1.3.7 **Exercise 20**

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup(A)$.

TODO: skipped

1.3.8 **Exercise 21**

Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.

(c) $\{n/(3n+1) : n \in \mathbb{N}\}$.

(d) $\{m/(m+n) : m, n \in \mathbb{N}\}$.

(a) supremum is 1, infimum is 0.

(b) supremum is 1, infimum is -1.

(c) supremum is $\frac{1}{3}$, infimum is $\frac{1}{4}$.

(d) supremum is 1, infimum is 0.

1.3.9 **Exercise 22**

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

TODO: skipped

1.3.10 (Cut Property) **Exercise 23**

The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.
(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.
(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbb{R}$ is replaced by $\mathbb{Q}$.

(a) A is clearly bounded by above, just pick any element in B.

Using the Axiom of Completeness, there must exist some $c = \sup A$. By definition, $c \geq x$ for all $x \in A$.

By 1.3.3, c is the infimum of B, so for all $x \in B$, we have that $b \geq c$.

(b) Assume the Cut Property.

Let B be the set of upper bounds of E. Now let $A = \mathbb{R} \setminus B$.

Now note that for any $a \in A$ and $b \in B$, we have that $a < b$. This is because if we assume otherwise, then we see that a is an upper bound for E and should have been an element of B in the first place.

Now, from the Cut Property, we have that there exists a c such that $a \leq c \leq b$.

Now, I show that c is an upper bound for e.

AFSOC that there exists some $e \in E$ such that $e > c$. Examine $\epsilon = e - c > 0$.

Since $\frac{\epsilon}{2} + c > c$, it must be a member of B and thus be an upper bound for E.

However, we also have that $\frac{\epsilon}{2} + c < e$, so it cannot be an upper bound for E. Contradiction!

Thus, since c is an upper bound and is less than or equal to all upper bounds of E, we have that c exists and is the supremum of E.

(c) Let A be $\{x \in \mathbb{Q} : x^2 \leq 2\}$, and $B := \{x \in \mathbb{Q} : x^2 > 2\}$.

They clearly are disjoint sets that form the rationals.

But we have proven that there cannot be such a $c \in \mathbb{Q}$ such that it exists in the middle of these two sets.

1.3.11 **Exercise 24**

Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.
(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.
(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
(c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

(a) This is true. AFSOC false. Then there must exist some $a \in A$ such that it is greater than $\sup B$ but less than or equal to $\sup A$.

But since $A \subseteq B$, it must be an element of B as well, which leads us to a contradiction since we assumed it would be greater than $\sup B$.

(b) True. **TODO: skipped**

(c) False. **TODO: skipped**

1.4 Consequences of Completeness

1.4.1 **Exercise 25**

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

(a) Let $a = \frac{m}{n}$, $b = \frac{p}{q}$. Then $mp \in \mathbb{Z}$, and $nq \in \mathbb{N}$.

Therefore $ab = \frac{mp}{nq} \in \mathbb{Q}$.

$mq \in \mathbb{Z}$, and $np \in \mathbb{Z}$, so therefore $a + b = \frac{mq+np}{nq} \in \mathbb{Q}$.

(b) **TODO: skipped**

(c) **TODO: skipped**

1.4.2 **Exercise 26**

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

AFSOC $s > \sup A$.

Then there must exist some $n \in \mathbb{N}$ such that $\frac{1}{n} < s - \sup A$.

So then, $s - \frac{1}{n} > \sup A$, which is a contradiction with the condition that $s - \frac{1}{n}$ cannot be an upper bound.

The other direction $s < \sup A$ works the same way.

Therefore it must be that $s = \sup A$.

1.4.3 **Exercise 27**

Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

AFSOC there exists some $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$.

It must be that $x > 0$, and therefore, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

However, x would then be excluded from the interval $\left(0, \frac{1}{n}\right)$, which is a contradiction.

1.4.4 **Exercise 28**

Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

b is clearly an upper bound.

Let $\epsilon > 0$, and also choose $\epsilon < b - a$.

There must exist $r \in \mathbb{Q}$ such that $b - \varepsilon < r < b$. Therefore $r \in T$, which shows that $\sup T = b$.

1.4.5 **Exercise 29**

Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real $a - \sqrt{2}$ and $b - \sqrt{2}$.

First, choose a rational y such that $a - \sqrt{2} < y < b - \sqrt{2}$. Next, we see clearly that $y + \sqrt{2} \in \mathbb{I}$.

Now, we can see that $a < y + \sqrt{2} < b$.

1.4.6 **Exercise 30**

Recall that a set B is *dense* in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.
(a) The set of all rational numbers p/q with $q \leq 10$.
(b) The set of all rational numbers p/q with q a power of 2.
(c) The set of all rational numbers p/q with $10|p| \geq q$.

TODO: skipped

1.4.7 **Exercise 31**

Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Claim: If we choose $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$, then we can show that $\alpha - \frac{1}{n_0}$ is still an upper bound for $T = \{t \in \mathbb{R} : t^2 < 2\}$.

Proof: Consider the following:

$$\begin{aligned} \left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}. \end{aligned}$$

We want to choose an n such that $\alpha^2 - \frac{2\alpha}{n} > 2$. Note that if we choose n_0 as in the claim, we get that the inequality holds.

Thus, $\alpha - \frac{1}{n_0}$ is actually an upper bound on T that is smaller than α, contradicting the assumption that $\alpha = \sup T$. ■

1.4.8 **Exercise 32**

Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

(a) Let $A = \{q \in \mathbb{Q} \mid q < 0\}$, and $B = \{r \in \mathbb{R} \backslash \mathbb{Q} \mid r < 0\}$.

(b) Let $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then the only element in the intersection can be 0.

(c) Let $L_n = [n, \infty)$. This cannot have any element.

(d) This is **impossible**, and we can prove it using the nested interval property.

Proof: First, we use the fact that a non-empty intersection of two closed, bounded intervals must itself be a closed bounded interval.

Now, let

$$I_n' = \bigcap_{m=1}^{n} I_m$$

define a new sequence of closed bounded intervals, which are nested by construction.

By the assumption that every finite intersection is non-empty, every I_n' must also be a non-empty closed, bounded, interval.

It is also important to note that the finite and infinite intersection of this sequence is exactly equal to the finite and infinite intersection of the original sequence.

Now, we can apply the NIP to deduce that the infinite intersection must be non-empty, which disproves the original claim. ■

1.5 Cardinality

1.5.1 **Exercise 33**

Finish the following proof for Theorem 1.5.7.

Assume B is a countable set. Thus, there exists $f : \mathbb{N} \to B$, which is 1–1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1–1 function g from $\mathbb{N}$ onto A.

For $i > 1$, let $n_i = \min\{n \in \mathbb{N} : f(n) \in A, n > n_{i-1}\}$. This must exist since A is an infinite set, thus there cannot be an upper bound on n such that $f(n) \in A$.

Now, just let $g(i) = f(n_i)$. This is an injective function, since each n_i is distinct and f is an injective function.

1.5.2 **Exercise 34**

Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbb{Q}$ is uncountable:

Assume, for contradiction, that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbb{Q}$ must therefore be uncountable.

NIP is not true in general over the rationals, since the element could be an irrational.

1.5.3 **Exercise 35**

Use the following outline to supply proofs for the statements in Theorem 1.5.8.

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this s that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Now, explain how the more general statement in (i) follows.

(b) Explain why induction *cannot* be used to prove part (ii) of Theorem 1.5.8 from part (i).

(c) Show how arranging $\mathbb{N}$ into the two-dimensional array

$$\begin{array}{cccccc}
1 & 3 & 6 & 10 & 15 & \cdots \\
2 & 5 & 9 & 14 & \cdots & \\
4 & 8 & 13 & \cdots & & \\
7 & 12 & \cdots & & & \\
11 & \cdots & & & & \\
\vdots & & & & &
\end{array}$$

leads to a proof of Theorem 1.5.8 (ii).

(a) Select $B_2 = A_2 \setminus A_1$. For enumeration purposes, alternate elements from A_1 and B_2. If B_2 is finite, after all elements are enumerated, continue enumerating from A_1.

Now, continue this enumeration strategy all the way to A_m by doing this same alternating enumeration strategy for $\bigcup_{i=1}^{n} A_i$ and A_{n+1}.

(b) Induction may not work, since induction only makes it hold for finite unions, not infinite unions.

(c) If we arrange the countable sets in rows, then we can visit by zig-zag pattern.

1.5.4 **Exercise 36**

(a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1–1, onto functions, but it is not really necesssary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1–1 onto function between the two sets.

(a) We can map (a, b) to $(0, 1)$.

Alternatively, for a direct bijection, consider the function

$$f(x) = \tan\left(\left(\frac{x-a}{b-a} - \frac{1}{2}\right)\pi\right).$$

(b) Choose the function $\ln(x - a)$.

(c) Let $f : [0, 1) \to (0, 1)$ be defined the following way:

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ x & \text{else.} \end{cases}$$

1.5.5 **Exercise 37**

(a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation.*

(a) Use identity bijection.

(b) Bijection has bijection inverse.

(c) Let $f : A \to B$ be a bijection, and $g : B \to C$ be another. Then $g \circ f$ is also a bijection.

Proof:

Injective: If $x \neq y$ for $x, y \in A$ then $f(x) \neq f(y)$. Since g is also injective we also have $g(f(x)) \neq g(f(y))$.

Surjective: For all $c \in C$, there exists $b \in B$ such that $g(b) = c$. Since f is also surjective, there must exist $a \in A$ such that $f(a) = b$. Thus, for all $c \in C$, there exists $a \in A$ such that $g(f(a)) = c$. ■

1.5.6 **Exercise 38**

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

(a) Let $I_n = (n, n + 1)$. Clearly disjoint, and open intervals.

(b) This cannot exist. If it did, then consider by density of rational numbers that there would exist a distinct rational in each interval, thus implying there would be uncountable rationals.

1.5.7 **Exercise 39**

Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1–1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to produce a 1–1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999….)

The Schröder–Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

(a) Consider $f(x) = \left(\frac{x}{2}, x\right)$.

(b) For $(x, y) \in S$, consider the (potentially countable) decimal expansion of each. Let's label the expansion of x as $0.d_1 d_2 d_3 d_4 \ldots$, and the second as $0.d'_1 d'_2 d'_3 d'_4 \ldots$. We consider the terminating decimal expansion representations, rather than one with inifinite 9s.

Now we simply map (x, y) to the following real number:

$$f((x, y)) = 0.d_1 d'_1 d_2 d'_2 d_3 d'_3 \ldots.$$

To see that this is injective, note that if two intervals differ from each other, that at least one of the left or right endpoints must differ. Since they differ, they must have a different decimal expansion, and thus the resulting real number will also have a different digit and be a different real number.

This logic only fails if we somehow produce a real number that ends in repeating 9s, which is impossible since it would imply that both of our original expansions were of that form.

However, this function is not surjective.

Consider a real number that, for example, ends in alternating 1s and 9s. This itself is a unique real number with no other decimal representation, but the only way to construct it would be with a decimal with repeating 9s. This representation is not in our domain, so there is no way to output this real number.

1.5.8 **Exercise 40**

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

First, note that $B \subseteq (0, 2]$.

For arbitrary $n \in \mathbb{N}$, consider the subset $\left(\frac{1}{n}, 2\right] \cap B$. This can only have finite elements, since otherwise, we could choose $2n$ elements from the subset to sum to greater than 2.

Note that this holds true for all n.

Now also note that $B = \bigcup_{n=1}^{\infty} \left[\left(\frac{1}{n}, 2\right] \cap B\right]$. This is a countable union of finite sets, which is countable.

1.5.9 **Exercise 41**

A real number $x \in \mathbb{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3}+\sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbb{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

(a) Consider the following:

$$x^2 - 2 = 0, \quad x^3 - 2 = 0, \quad x^4 - 10x^2 + 1 = 0.$$

(b) There are a countable number of integer polynomials of degree n, since it can be defined uniquely with a finite product of countable sets.

Since each has finite solutions, the total number of solutions and thus elements of A_n is a countable union of finite sets and is countable.

(c) We simply take the countable union of all A_n for all n.

Again, a countable union of countable sets is countable.

Since the algebraic numbers are countable, the rest of the reals (transcendental) must be uncountable.

1.5.10 **Exercise 42**

(a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha,1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

(a) AFSOC that there does not exist such an a.

Then for all $a \in (0,1)$, $C \cap [a,1]$ is countable.

Examine the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{1}{n+1}$.

Clearly, we have that $C = \left(\bigcup_{n=1}^{\infty} C \cap \left[\frac{1}{n+1}, 1\right]\right) \cup (C \cap \{0\})$.

This is simply a countable union of countable sets, which implies that C is countable.

Contradiction!

Therefore there must exist some $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Not necessarily. Consider if $C = [0,1]$. Then any $a \in (0,1)$ will produce an uncountable set $[a,1]$. The supremum of A is 1. But $C \cap [1,1] = \{1\}$, which is finite.

(c) No. Let $C = \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$. All choices of a lead to a finite intersection.

1.5.11 (Schröder–Bernstein Theorem) **Exercise 43**

> Assume there exists a 1–1 function $f : X \to Y$ and another 1–1 function $g : Y \to X$. Follow the steps to show that there exists a 1–1, onto function $h : X \to Y$ and hence $X \sim Y$.
>
> The strategy is to partition X and Y into components
>
> $$X = A \cup A' \qquad \text{and} \qquad Y = B \cup B'$$
>
> with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.
> (a) Explain how achieving this would lead to a proof that $X \sim Y$.
> (b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is a similar collection in Y.
> (c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.
> (d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

(a) If we restrict the domain of f to A, then it is a bijection between A and B.

Similarly, if we restrict g to B', then g is a bijection between B' and A'.

Now, we can just define $h : X \to Y$ the following way:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{else.} \end{cases}$$

This is clearly a bijection.

(b) First, if $A_1 = \emptyset$, then we are done. This is because $g(Y) = X$, which implies that g is a bijection and we are done.

Assuming that A_1 is non-empty, we can proceed by induction.

Proof: **Base case:** Notice that $f(A_1) \subseteq Y$. Thus $g(f(A_1)) \subseteq g(Y)$, so $(X \setminus g(Y)) \cap (g(f(A_1))) = \emptyset$.

Inductive hypothesis: Assume for some n that $A_1, \ldots, A_n$ are pairwise disjoint. Thus, $f(A_n)$ is also disjoint from all $f(A_1), \ldots, f(A_{n-1})$, since f is injective. By the same logic, $g(f(A_1)), \ldots, g(f(A_n))$ are all also disjoint.

Since $g(f(A_i)) = A_{i+1}$, we have that A_{n+1} is disjoint from all $A_2, \ldots, A_n$.

It is also disjoint with A_1 by similar logic from the base case.

Note for completeness, if at any point any A_i is empty, then we can just stop with finite A_i that are all pairwise disjoint. ■

Also note that this implies that all $\{f(A_n) : n \in \mathbb{N}\}$ are pairwise disjoint since f is injective.

(c) If $b \in B$, then it must exist in exactly one $f(A_i)$. This means that there must be some $a \in A_i$ such that $f(a) = b$, which shows that f is surjective.

(d) First, let's note that $X \setminus A$ is a subset of $g(Y)$. This is because if $a \in X \setminus A$, then $a \in X \setminus A_1 = X \setminus (X \setminus g(Y)) = g(Y)$.

So we know there must **exist** some $b \in Y$ such that $g(b) = a$.

We should also argue that this b cannot be in B.

AFSOC that $b \in B$. Then $b \in f(A_n)$ for some n, and thus $g(b) \in g(f(A_n)) = A_{n+1}$. However, this is clearly disjoint with $X \setminus A$, so it must be the case that $b \notin B \implies b \in Y \setminus B$.

1.6 Cantor's Theorem

1.6.1 **Exercise 44**

Show that $(0, 1)$ is uncountable if and only if $\mathbb{R}$ is uncountable. This shows that Theorem 1.6.1 is equivalent to Theorem 1.5.6.

($\Rightarrow$) This direction is easy, since if $(0, 1)$ is uncountable, then clearly since $(0, 1) \subseteq \mathbb{R}$, the real numbers must also be uncountable.

($\Leftarrow$) If $(0, 1)$ is countable, then $\mathbb{R}$ must be countable, since we can construct $\mathbb{R}$ from $(0, 1)$ using a countable union of the integers plus $(0, 1)$.

1.6.2 **Exercise 45**

(a) Explain why the real number $x = .b_1 b_2 b_3 b_4 \ldots$ cannot be $f(1)$.
(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbb{N}$.
(c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

(a) It must differ from $f(1)$ at the first digit by construction.
(b) It must differ from the nth digit of $f(n)$ by construction.
(c) This shows that there must be a real number that is not in our enumeration. But we assumed we could enumerate them. This is the contradiction, and thus $(0, 1)$ is uncountable.

1.6.3 **Exercise 46**

Supply rebuttals to the following complaints about the proof of Theorem 1.6.1
(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.
(b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9′s. For instance, 1/2 can be written as 0.5 or as .4999…. Doesn't this cause some problems?

(a) In general, the number that is produced may not be a rational.
(b) No, this is fine. Let's just only consider non-repeating 9′s representation, and note that with our construction, we will never produce a number that runs into this issue.

1.6.4 **Exercise 47**

Let S be the set consisting of all sequences of 0′s and 1′s. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \ldots)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, \ldots)$.

Give a rigorous argument showing that S is uncountable.

Cantor's diagonalization argument.

Produce a new binary sequence that differs from all other sequences at the nth element.

1.6.5 **Exercise 48**

(a) Let $A = \{a, b, c\}$. List the eight elements of $\mathcal{P}(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)
(b) If A is finite with n elements, show that $\mathcal{P}(A)$ has 2^n elements.

(a) $\emptyset, \{a\}, \{a, b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.
(b) An element can either be in or out of a subset, which gives us two choices per element. Thus there are 2^n distinct subsets.

1.6.6 **Exercise 49**

(a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1–1 mappings from A into $\mathcal{P}(A)$.
(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1–1 map $g : C \to \mathcal{P}(C)$.
(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are *onto*.

(a) One mapping:

$$a \to \{a\}, \quad b \to \{b\}, \quad c \to \{c\}.$$

Another mapping:

$$a \to \{a, b\}, \quad b \to \{b\}, \quad c \to \{c\}.$$

(b) $$1 \to \{1\}, \quad 2 \to \{2\}, \quad 3 \to \{3\}, \quad 4 \to \{4\}.$$

(c) There are strictly more elements in the range than the domain.

1.6.7 **Exercise 50**

Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule. In each case, note that B is not in the range of the function used.

TODO: skipped

1.6.8 **Exercise 51**

(a) First, show that the case $a' \in B$ leads to a contradiction.
(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

(a) If $a' \in B$, then it must be that $a' \notin f(a')$ by definition of B.

However, $f(a') = B$ by assumption, so we have shown that $a' \in B$ and $a' \notin B$ which is a contradiction.
(b) If $a' \notin B$, then it must be that $a' \notin f(a')$. This implies that it must be in B which is again a contradiction.

1.6.9 **Exercise 52**

Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

First, we construct an injection from $(0, 1)$ to the set of infinite binary sequences. We do this by considering the decimal expansion.

Next, we construct an injection from the set of infinite binary sequences to $(0, 1)$.

This is a little trickier, as a direct conversion would result in some numbers that are actually the same real number. (For example, 0.0111… and 0.1).

We can first consider all sequences that do not end in repeating 1's. This will map into $[0, 1)$, which we know has a bijection with $(0, 1)$. We can divide the result by 3 to get an injection into $\left(0, \frac{1}{3}\right)$.

Next, we map the sequences that end in infinite 1's to their representative real number, divide by 3, and then add $\frac{1}{3}$ to get an injection into $\left(\frac{1}{3}, \frac{2}{3}\right]$.

This completes the injection into $(0, 1)$, so using Schröder–Bernstein we can conclude that the set of infinite binary sequences has the same cardinality as $(0, 1)$, and we can use transitivity of this equivalence relation to deduce that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

1.6.10 **Exercise 53**

As a final exercise, answer each of the following by establishing a 1–1 correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ countable or uncountable?
(b) Is the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ countable or uncountable?
(c) Given a set B, a subset $\mathcal{A}$ of $\mathcal{P}(B)$ is called an *antichain* if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $\mathcal{P}(\mathbb{N})$ contain an uncountable antichain?

(a) This is countable, since there only needs to be two natural numbers to specify the function fully. This essentially reduces to the set with $(n, m) \in \mathbb{N}^2$.
(b) This is uncountable. This is equivalent to the set of infinite sequences of 0's and 1's, which is shown to be uncountable due to a diagonalization argument.
(c) There exists an uncountable antichain. Consider the following bijection between an infinite binary sequence and a subset of the natural numbers:

$$f((b_n)) = \{n : n = 2i + b_i, i \in \mathbb{N}\}.$$

In plain English, for every distinct pair of adjacent natural numbers, we select only one of them based off of the ith value of the binary sequence. If a binary sequence is distinct from another binary sequence, then transformed into subset world, each subset will have an element that is not included in the other.

Considering this bijection, this antichain must be uncountable.

1.7 Epilogue

No exercises in this section.

2 Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

No exercises in this section.

2.2 The Limit of a Sequence

2.2.1 **Exercise 54**

What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

An example is the sequence of alternating 0's and 1's.

This is vercongent to any real number. We can just select large enough ϵ and it will work out.

I believe that this is actually describing bounded sequences.

2.2.2 **Exercise 55**

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

(a) I claim we need to choose $N > \frac{3}{25\epsilon} - \frac{4}{5}$.

Proof: Let $\epsilon > 0$. Choose $N > \frac{3}{25\epsilon} - \frac{4}{5}$. Now for $n \geq N$, we can verify that:

$$\begin{aligned} \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &= \left|\frac{3/5}{5n+4}\right| \\ &= \frac{3/5}{5n+4} \\ &< \frac{3/5}{5\left(\frac{3}{25\epsilon} - \frac{4}{5}\right) + 4} \\ &= \epsilon \end{aligned}$$

as desired. ■

(b) I claim we choose $N > \frac{2}{\epsilon}$.

Proof: Let $\epsilon > 0$. Choose $N > \frac{2}{\epsilon}$. Notice that $\left|\frac{2n^2}{n^3+3}\right|$ is always positive if $n > 0$. For $n \geq N$, we have that:

$$\left|\frac{2n^2}{n^3+3}\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} < \epsilon$$

as desired. ■

(c) I claim we choose $N > \frac{1}{\epsilon^3}$.

Proof: Let $\epsilon > 0$. Choose $N > \frac{1}{\epsilon^3}$. Notice that $\left|\sin(n^2)\right| \le 1$ always.

If $n \ge N$, we can see that $n > \frac{1}{\epsilon^3}$ or alternatively $\sqrt[3]{n} > \frac{1}{\epsilon}$.

Therefore, we have that:

$$\begin{aligned}\left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| &= \frac{\left|\sin(n^2)\right|}{\sqrt[3]{n}} \\ &\le \frac{1}{\sqrt[3]{n}} < \epsilon.\end{aligned}$$

as desired. ■

2.2.3 **Exercise 56**

Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.
(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.
(c) There exists a college in the United States where every student is at least six feet tall.

(a) Find a college in the US where all students are below 7 feet tall.
(b) Find a college in the US where all professors give out grades other than A or B.
(c) Show that all colleges have a student under 6 feet tall.

2.2.4 **Exercise 57**

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.
(b) A sequence with an infinite number of ones that converges to a limit not equal to one.
(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

(a) Alternating 0's and 1's.
(b) Not possible. If we select $\epsilon < |x - 1|$, where x is the "limit", then we can see that there can never be a N such that every element in the sequence after that is within that ϵ-neighborhood. This is because there must be infinite ones, which cannot all be in the first N elements.
(c) Yes, just do $1, 0, 1, 1, 0, 1, 1, 1, 0, \ldots$. This can never converge due to a similar argument to part (b). But by construction, we can always find n consecutive ones.

2.2.5 **Exercise 58**

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.
(a) $a_n = [[5/n]]$,
(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

(a) Claim: $\lim a_n = 0$.

Proof: After $N > 5$, all $n \geq N$ will be such that $a_n = 0$. ■

(b) Claim: $\lim a_n = 1$.

Proof: After $N > 6$, for $n \geq N$, the inner part of a_n will be less than 2. In addition, the inner part will always be greater than $4/3$. Therefore after $N > 6$ every element in the sequence will equal 1 exactly. ■

2.2.6 **Exercise 59**

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

We start with the stated assumptions.

AFSOC $a \neq b$, then we could choose $\epsilon < \frac{|a-b|}{2}$.

By the definition of limits, there would exist N and N' such that any $n \geq \max(N, N')$ satisfies $|x_n - a| < \epsilon$ and $|x_n - b| < \epsilon$.

Using the triangle inequality, we know that

$$|a - b| = |a - x_n + x_n - b| \leq |x_n - a| + |x_n - b| < 2\epsilon < |a - b|.$$

In other words, we have shown that $|a - b| < |a - b|$. This is a **contradiction**.

Therefore, it must be the case that $a = b$.

2.2.7 **Exercise 60**

Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

(a) Frequently.
(b) Eventually implies frequently. To see this, notice that for any natural number, if it is less than or equal to N, then we can just use any number after N as our n, and if it is greater than N, then any number greater than our current number should work.
(c) A sequence (a_n) converges to a if for any ϵ-neighborhood of a, the sequence is eventually in it.
(d) (x_n) is not necessarily eventually in it, as we could have also an infinite number of terms that are 2.2 for example.

However, it is definitely the case that (x_n) is frequently within those bounds.

2.2.8 **Exercise 61**

For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 0, 1, \ldots)$ zero-heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counter example.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if….

(a) The given sequence is zero-heavy. Consider $M = 1$. Since there are never two 1's in a row, this is a valid M.

(b) A zero-heavy sequence must contain an infinite amount of 0's. Otherwise, we could consider the first index N after which there are no more 0's, and see that no value of M will produce an interval that contains a 0.

(c) No. Consider the sequence $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \ldots)$. Given any M, once we are far enough in the sequence, we will always be able to find a string of 1's that is longer than M.

(d) A sequence is not zero-heavy if for all $M \in \mathbb{N}$, there exists a $N \in \mathbb{N}$ such that for all n satisfying $N \leq n \leq N + M$, $x_n \neq 0$.

2.3 The Algebraic and Order Limit Theorems

2.3.1 **Exercise 62**

Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $\left(\sqrt{x_n}\right) \to 0$.

(b) If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

(a) Let arbitrary $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n| < \epsilon^2$. Thus we can reuse the same N for $|\sqrt{x_n}| < \epsilon$.

(b) Assume $x > 0$. (This is valid due to Order Limit Theorem). Let arbitrary $\epsilon > 0$. Now observe the following:

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \frac{|x_n - x|}{|\sqrt{x_n} + \sqrt{x}|} \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{|x_n - x|}{\sqrt{x}} \\ &< \frac{\epsilon'}{\sqrt{x}} \text{ for } n \text{ larger than some } N \in \mathbb{N}. \end{aligned}$$

If we choose $\epsilon' = \epsilon\sqrt{x}$, then we get that for $n \geq$ some $N \in \mathbb{N}$ that

$$|\sqrt{x_n} - \sqrt{x}| < \epsilon.$$

2.3.2 **Exercise 63**

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

(a)

$$\left|\frac{2x_n - 1}{3} - 1\right| = \left|\frac{2x_n - 4}{3}\right| = \frac{2}{3}|x_n - 2| < \frac{2}{3}\epsilon'.$$

Choose $\epsilon' = \frac{3}{2}\epsilon$.

(b)

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \frac{|x_n - 2|}{2|x_n|}$$

Choose N_1 such that we get $|x_n| > \frac{|x|}{2}$. Now choose $\epsilon' = |x|\epsilon$. So for $\max(N_1, N_2)$ we have

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| < \frac{|x_n - 2|}{|x|} < \frac{\epsilon'}{|x|} = \epsilon.$$

2.3.3 (Squeeze Theorem) **Exercise 64**

Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Since $x_n \leq y_n \leq z_n$, we also get that

$$x_n - l \leq y_n - l \leq z_n - l.$$

Choose large enough N such that for $n \geq N$ we get that $z_n - l \leq |z_n| - l < \epsilon$, as well as $x_n - l > -|x_n - l| > \epsilon$. This leaves us with:

$$-\epsilon < y_n - l < \epsilon, \quad \Longrightarrow \quad |y_n - l| < \epsilon.$$

Thus y_n converges and it must converge to l.

2.3.4 **Exercise 65**

Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$

(a) 1.

(b)
$$\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) = \lim\left(\frac{a_n^2+4a_n}{a_n}\right) = \lim(a_n+4) = 4.$$

(c)
$$\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) = \lim\left(\frac{2+3a_n}{1+5a_n}\right) = 2.$$

2.3.5 **Exercise 66**

Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, ..., x_n, y_n, ...)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

($\Rightarrow$) Assume that (z_n) is convergent. Then after some $N \in \mathbb{N}$, for $n \geq N$ we have that all $|z_n - z| < \epsilon$ for some $z \in \mathbb{R}$ and arbitrary ϵ.

Then we also get that for $n > \frac{N}{2}$, both $|x_n - z| < \epsilon$ and $|y_n - z| < \epsilon$, which shows they both converge to z.

($\Leftarrow$) Assume that x_n and y_n both converge to z. For arbitrary $\epsilon > 0$, pick $N = \max\{N_1, N_2\}$ such that for $n \geq N$, we have $|x_n - z| < \epsilon$ and $|y_n - z| < \epsilon$.

Therefore for $n \geq 2N$, we have that $z_n = x_{\lfloor \frac{n}{2} \rfloor}$ or $y_{\lfloor \frac{n}{2} \rfloor}$ is such that $|z_n - z| < \epsilon$, and we have shown that (z_n) converges to z.

2.3.6 **Exercise 67**

Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Multiply top and bottom by the conjugate:

$$\begin{aligned} n-\sqrt{n^2+2n} &= \frac{n^2-n^2-2n}{n+\sqrt{n^2+2n}} \\ &= \frac{-2n}{n+\sqrt{n^2+2n}} \\ &= \frac{-2}{1+\sqrt{1+\frac{2}{n}}}. \end{aligned}$$

The sequence defined by $1+\frac{2}{n}$ is positive and approaches the limit 1, so therefore the square root of the sequence does as well.

Thus, the original sequence for the entire expression approaches -1 by the ALT.

2.3.7 **Exercise 68**

Give and example of each of the following, or state the such a request is impossible by referencing the proper theorems(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

(a) Yes, consider $x_n = (-1)^n$, and $y_n = (-1)^{n+1}$. Their sum is simply 0.

(b) No, by ALT we would have that $(x_n + y_n - x_n)$ converges.

(c) Consider $b_n = \frac{1}{n}$. Then $\frac{1}{b_n} = n$ which clearly diverges.

(d) Since every convergent sequence is bounded, we know that $|b_n| \leq M$, where M is the bound on b_n and N is the bound on $a_n - b_n$.

So therefore

$$|a_n| = |a_n - b_n + b_n| \leq |a_n - b_n| + |b_n| \leq M + N,$$

and (a_n) must be bounded as well.

(e) Let $a_n = 0$ for all n, and let $b_n = n$.

Clearly their product is 0 for all n.

2.3.8 **Exercise 69**

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

(a) Follows directly from ALT, since a polynomial is simply a combination of multiplications and additions.

(b) Let $f(x)$ be the following:

$$f(x) = \begin{cases} 5 \text{ if } x = 0 \\ x \text{ else} \end{cases}.$$

Now, $f(0) = 5$, but any sequence that approaches 0 but never reaches it will instead approach the limit value 0.

2.3.9 **Exercise 70**

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?
(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?
(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

(a) Since a_n is bounded, we have that $|a_n| \leq M$ for all n.

Now for arbitrary $\epsilon' > 0$, there is some N such that $n \geq N$ implies

$$|a_n b_n| = |a_n||b_n| \leq M|b_n| < M\epsilon'.$$

Choose $\epsilon' = \frac{\epsilon}{M}$, and we have that $(a_n b_n)$ converges to 0.

We can't use ALT since (a_n) is not necessarily convergent, just bounded.

(b) No. Consider the constant sequence $b_n = 1$, which is clearly convergent. However, given any bounded and not convergent sequence (a_n), we have that $(a_n b_n) = (a_n)$. However I do believe that the product sequence is still bounded… I won't prove this.

(c) Since all convergent sequences are bounded, we can just use our result from (a) directly.

2.3.10 **Exercise 71**

Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.
(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.
(b) If $(b_n) \to b$, then $|b_n| \to |b|$.
(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.
(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \to b$.

(a) False, consider $a_n = n$, $b_n = n$. They have no limit, but their difference is simply 0.

(b) $||b_n| - |b|| \leq |b_n - b|$ by reverse triangle inequality.

(c) Directly follows from ALT.

(d) Yes, because $|b_n - b| \leq a_n \leq |a_n| < \epsilon$ for all $n \geq N \in \mathbb{N}$ for any arbitrary ϵ.

2.3.11 (Cesaro Means) **Exercise 72**

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

(a) Assume $(x_n) \to x$.

Let $\epsilon > 0$ be arbitrary.

Past some $N_1 \in \mathbb{N}$ we have that $|x_n - x| < \frac{\epsilon}{2}$.

For all x_i for $i < N_1$, let $M = \max\{|x_i - x|\}$.

Let $N_2 \in \mathbb{N}$ be such that for $n \geq N_2$, we have $\frac{1}{n} < \frac{\epsilon}{2N_1 M}$.

Now for $N = \max\{N_1, N_2\}$, we have for $n \geq N$

$$\begin{aligned} \left|\frac{x_1 + x_2 + \ldots + x_n}{n} - x\right| &= \frac{1}{n}|x_1 + x_2 + \ldots + x_n - nx| \\ &\leq \frac{1}{n}(|x_1 - x + x_2 - x + \ldots + x_i - x| + |x_{i+1} - x + \ldots + x_n - x|) \\ &< \frac{1}{n}\left(N_1 M + (n - N_1)\frac{\epsilon}{2}\right) \\ &\leq \frac{1}{n}(N_1 M) + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

(b) The alternating sequence of 0's and 1's does not converge. However, the sequence of averages will converge to $1/2$.

2.3.12 **Exercise 73**

A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

(a) True. AFSOC that a is not an upper bound for the set B. Then there is some $b \in B$ such that $b > a$. Let $\epsilon = b - a$. Then there is some $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n - a| < \epsilon$. Clearly all a_n must be larger than a, so we have that $a_n - a < b - a$, so then $a_n < b$, and we have shown our contradiction since we assumed that all a_n would also be upper bounds.

(b) True. AFSOC that $a \in (0,1)$. Then choose $\epsilon = \frac{1}{2}\min\{a, 1-a\} > 0$. There must be an a_n within that ϵ-neighborhood, which is clearly not in the complement of $(0,1)$.

(c) False, consider the sequence defined by the decimal approximation of π.

2.3.13 (Iterated Limits) **Exercise 74**

Given a doubly indexed array a_{mn} where $m, n \in \mathbb{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the *iterated* limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right).$$

Define $\lim_{m,n} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2+n^2)$.

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

(a) The first limit is equal to 1, while the second limit is equal to 0.

(b) Yes, the limit is 0. Yes, both iterated limits exist and are 0.

For $a_{mn} = mn/(m^2+n^2)$, $\lim_{m,n} a_{mn}$ does not exist, since we can make the sequence approach different values. (This is not super rigorous, but it is if we assume the result that a limit can only have one value.)

However, the iterated limits exist and are both 0.

(c) Choose the following:

$$a_{mn} = (-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right)$$

The iterated limits do not exist, as they will oscillate between $\frac{1}{m}$ and $-\frac{1}{m}$ or $\frac{1}{n}$ and $-\frac{1}{n}$.

However, $\lim_{m,n} a_{mn} = 0$, which can be easily proven by triangle inequality.

(d) Let $\epsilon > 0$ be arbitrary.

Let's use the triangle inequality:

$$\begin{aligned} |b_m - a| &= |b_m - a_{mn} + a_{mn} - a| \\ &\leq |a_{mn} - b_m| + |a_{mn} - a| \\ &\leq \epsilon' + \epsilon'' \end{aligned}$$

Find N and M such that we approach $\epsilon' = \epsilon'' = \frac{\epsilon}{2}$, and take $m \geq \max\{N, M\}$ to finish the proof.

(e) This is just part (d).

2.4 The Monotone Convergence Theorem and Infinite Series

2.4.1 **Exercise 75**

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

(a) I claim that the sequence is monotone decreasing.

Proof: BC: $x_2 = 1 < 3 = x_1$.

IH: Assume true for some n (that $x_n \leq x_{n-1}$). Then:

$$4 - x_n \geq 4 - x_{n-1} \implies \frac{1}{4 - x_n} \leq \frac{1}{4 - x_{n-1}} \implies x_{n+1} \leq x_n.$$

■

It is also bounded below by $\frac{1}{4}$, which can also be proved by induction. The base case is obvious, and if we assume $x \geq \frac{1}{4} > 0$, then

$$x_{n+1} = \frac{1}{4 - x_n} \geq \frac{1}{4}.$$

So now MCT finishes the argument.

(b) It's literally the same, just missing the first term.

For $n \geq N \in \mathbb{N}$, we can see that $\max\{0, N - 1\}$ works in the limit argument.

(c)

$$x = \frac{1}{4 - x} \implies x = 2 - \sqrt{3}.$$

$2 + \sqrt{3}$ is not valid since it is greater than 3.

2.4.2 **Exercise 76**

(a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$.

What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

(a) There may not be a limit in the first place. In this case, there is not, since it oscillates between 1 and 2.

(b) Yes, we can use a similar approach to 2.4.1 to show that it is monotone increasing and bounded above by 3.

2.4.3 **Exercise 77**

(a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

(a) The recursive formula for the sequence is $x_{n+1} = \sqrt{2 + x_n}$.

By induction (using the fact that the square root function is increasing), the sequence is increasing.

I also claim it is bounded above by 2.

Proof: BC: $\sqrt{2} < 2$.

IH: assume true for some n.

Now:

$$\sqrt{2 + x_n} < \sqrt{4} = 2.$$

■

Now we apply MCT.

Taking the limit of both sides of the recursive formula, we get that $x = 2$.

(b) By similar argument to above, we claim the sequence is monotone increasing and the upper bound is 2. The limit is 2.

2.4.4 **Exercise 78**

(a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbb{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.

> These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

(a) Assume $y > 0$, otherwise we can just choose any $n \in \mathbb{N}$ and be done.

Now notice that the sequence $\frac{1}{n}$ is bounded below by 0, and is also monotone decreasing.

This suggests that it converges to a limit by MCT.

To produce the limit, notice that $\lim \frac{1}{n} = \lim \frac{1}{n+1}$.

We can recursively see that:

$$\frac{1}{n+1} = \frac{1}{n} \cdot \left(\frac{n}{n+1}\right) = \frac{1}{n} \cdot \left(1 - \frac{1}{n+1}\right).$$

So by ALT we can see that the limit s must be such that:

$$s = s(1-s) \implies s = 0.$$

Therefore by the definition of a limit we get that for arbitrary $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$:

$$\left|\frac{1}{n}\right| < \epsilon.$$

Take $\epsilon = y$, and note that $\frac{1}{n}$ is always positive to get the Archimedean Property:

$$\frac{1}{n} < y \text{ for large enough } n \in \mathbb{N}.$$

(b) Note that the sequences (a_n) and (b_n) are monotone increasing/decreasing and bounded above/below.

Therefore by MCT they must converge to some a and b respectively. In addition, its clear that $a_n \leq a$ and $b_n \geq b$ for all n.

I claim that $a \leq b$. If it were the case that $a > b$, then we could set $\epsilon = a - b$, and select some b_n such that $b_n - b = |b_n - b| < a - b$.

With some algebra we get:

$$b_n - a_n < a - a_n \leq 0 \quad \implies \quad b_n < a_n.$$

This is impossible for an interval, which is a **contradiction**. Therefore it must be that $a \leq b$ and thus we can choose any x such that $a \leq x \leq b$, and it will be present in every interval. Thus the interval will not be empty.

> **2.4.5** **Exercise 79**
>
> Let $x_1 = 2$, and define
>
> $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$
>
> (a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

(a) Clearly $x_1^2 = 4$. Let's work out x_{n+1}^2:

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) \\ &= \frac{1}{4}\left(x_n^2 - 4 + \frac{4}{x_n^2}\right) + 2 \\ &= \frac{1}{4}\left(x_n - \frac{2}{x_n}\right)^2 + 2 \\ &\geq 2. \end{aligned}$$

This applies for all n.

Now, let's look at $x_n - x_{n+1}$:

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= \frac{1}{2}x_n - \frac{1}{x_n} \\ &= \frac{x_n^2 - 2}{2x_n} \\ &\geq 0. \end{aligned}$$

The last inequality relies on the fact that $x_n^2 \geq 0$, as well as the fact that $x_n > 0$ for all n (this is easy to see).

Thus, we have that $x_{n+1} \leq x_n$ and the sequence is monotone decreasing, while being bounded below by $\sqrt{2}$.

It therefore has a limit, and we can take the limit of both sides of the recursive formula to work it out:

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right) \implies x = \sqrt{2}.$$

(b) I claim

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

works.

Through similar steps to part (a), we first show that $x_n^2 \geq c$ for all n:

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{c^2}\left(x_n^2 + 2c + \frac{c^2}{x_n}\right) \\ &= \frac{1}{4}\left(x_n^2 - 2c + \frac{c^2}{x_n}\right) + c \\ &\geq c. \end{aligned}$$

Then we show that the sequence is monotone decreasing:

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{c}{x_n}\right) \\ &= \frac{1}{2}\left(x_n - \frac{c}{x_n}\right) \\ &= \frac{1}{2}\left(\frac{x_n^2 - c}{x_n}\right) \\ &\geq 0. \end{aligned}$$

Then by MCT the limit exists, and we can compute it:

$$x = \frac{1}{2}\left(x + \frac{c}{x}\right) \implies x = \sqrt{c}.$$

2.4.6 (Arithmetic–Geometric Mean) **Exercise 80**

(a) Explain why $\sqrt{xy} \leq (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

(a)

$$\begin{gathered} \left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0 \iff \\ x - 2\sqrt{xy} + y \geq 0 \iff \\ \frac{x+y}{2} \geq \sqrt{xy}. \end{gathered}$$

(b) First, note that both y_n and x_n are bounded below by 0 for all n by closure of positive numbers under addition, multiplication, and square root.

First, by AM-GM inequality, $x_n \leq y_n$ for all n.

Next, I claim that (y_n) is monotone decreasing.

Proof:

$$y_{n+1} = \frac{x_n + y_n}{2} \leq y_n.$$

■

I claim that (x_n) is monotone increasing.

Proof:

$$x_{n+1} = \sqrt{x_n y_n} \geq x_n.$$

■

Note also that (x_n) is bounded above by y_1, since every $x_n \leq y_n \leq y_1$.

Therefore by MCT, the limit exists for both sequences. To find the limit, let's solve for the limits in one of the recursive formulas:

$$y = \frac{x+y}{2} \implies x = y.$$

This checks out with the other formula.

2.4.7 (Limit Superior) **Exercise 81**

Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n.$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

(a) (y_n) must be bounded below, otherwise that would imply that (a_n) is not bounded below.

In addition, (y_n) is monotone decreasing. From y_n to y_{n+1}, we are only ignoring one element, which can never increase the supremum, only possibly decrease it (or keep it the same).

Thus, by MCT this sequence converges.

(b) Let (y_n) be defined as $y_n = \inf\{a_k : k \geq n\}$.

Then $\liminf a_n = \lim y_n$.

This exists by similar argument to part (a).

(c) It is clear that for every n, $\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$. Thus by the OLT their limits must follow the same inequality.

One example where equality holds is simply the constant 0 sequence.

(d) $(\Rightarrow)$ We can directly apply the squeeze theorem for this direction, since $\inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\}$.

$(\Leftarrow)$ Since $\lim a_n$ exists, we know that for arbitrary $\epsilon' > 0$ there exists an N after which all a_n exist within the ϵ'-neighborhood of a.

The supremum of all of those points must also exist either within that neighborhood or on its boundary.

Therefore the $\limsup a_n$ must converge to a as well, if we just select $0 < \epsilon' < \epsilon$.

The same argument applies for $\liminf a_n$.

2.4.8 **Exercise 82**

For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

(a) $$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

(c) $$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$$

(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

(a) $s_n = 1 - \frac{1}{2^n}$.

This converges to 1.

(b) $s_n = 1 - \frac{1}{n+1}$.

This converges to 1.

(c) $s_n = \log(n+1)$.

This does not converge, as it grows unbounded. ($\log n$ is unbounded above, it is easily shown that it contains an unbounded subsequence that grows like n).

2.4.9 **Exercise 83**

Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges.

Let's take a closer look at the partial sums of $\sum_{n=1}^{\infty} b_n$.

Particularly, let's look at the sequence of partial sums defined by s_{2^k}:

$$\begin{aligned} s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + \cdots + b_8) + \cdots + \left(b_{2^{k-1}+1} + \cdots + b_{2^k}\right) \\ &\geq b_1 + 1 \cdot b_2 + 2 \cdot b_4 + 4 \cdot b_8 + \cdots + 2^{k-1} \cdot b_{2^k} \\ &\geq \frac{1}{2}\left(b_1 + 2 \cdot b_2 + 4 \cdot b_4 + 8 \cdot b_8 + \cdots + 2^k \cdot b_{2^k}\right) \end{aligned}$$

This is unbounded, otherwise we could show that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges because its partial sums converge.

2.4.10 (Infinite Products) **Exercise 84**

A close relative of infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \ldots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \ldots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)\cdots, \quad \text{where } a_n \geq 0.$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

(a) From a few calculations, and verified by induction, we can see that $p_m = m + 1$. This clearly does not converge.

For $1/n^2$:

$$\begin{aligned}
p_1 &= (1+1) &&= 2 \\
p_2 &= (1+1)\left(1+\frac{1}{4}\right) &&= \frac{5}{2} \\
p_3 &= \left(\frac{5}{2}\right)\left(\frac{10}{9}\right) &&= \frac{25}{9} \\
p_4 &= \left(\frac{25}{9}\right)\left(\frac{17}{16}\right) &&= \frac{425}{144}
\end{aligned}$$

My conjecture is that this converges. **Not proved.**

(b) (⇒) I wish to show that if the sequence of partial products converges, then the infinite sum converges.

Reminder, we have that $a_n \geq 0$.

It is easy to see that in p_m, it contains the partial sum $s_m = \sum_{n=1}^{m} a_n$.

To see this, we can simply expand out the product and see that s_m exists as a subset of the terms.

Thus, $p_m \geq s_m \geq 0$. Since p_m is convergent, it must be bounded above. So s_m is also bounded above, and furthermore, is monotone increasing. Thus it is also convergent.

(⇐)

Assume the infinite sum is convergent, to some limit a.

$$p_m = \prod_{n=1}^{m}(1+a_n) \leq \prod_{n=1}^{m} 3^{a_n} = 3^{\sum_{n=1}^{m} a_n} = 3^{s_m}.$$

I won't finish the proof rigorously, but this clearly also converges, which we use to show that the infinite product also converges.

2.5 Subsequences and the Bolzano–Weierstrass Theorem

2.5.1 **Exercise 85**

Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$, and no subsequences converging to points outside of this set.

(a) Impossible, since that bounded subsequence itself would have a subsequence that converges.

(b) Yes, let $a_n = \frac{1}{n+1}$ and $b_n = 1 + \frac{1}{n}$. Now alternate these.

(c) Yes, we can just choose the enumeration of the rationals between 0 and 1. We can always choose a subsequence that gets arbitrarily close to any of the numbers in the set.

(d) False, since any such sequence must also converge to 0, which I don't think is in the set.

If 0 is allowed to be in the set, then consider the following sequence:

$$(1, 1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, 1, \ldots)$$

Thus, every number in our sequence will appear an infinite number of times.

Any subsequence cannot converge to any number outside of this set (if it includes 0).

Assume we have a subsequence with limit $0 < x < 1$. Find the number in our infinite set that is closest to it, say $1/n$, and AFSOC $1/n \neq x$. (This is only possible if $x \neq 0$).

Now choose positive $\epsilon < \left|\frac{1}{n} - x\right|$. Since we assumed that $1/n$ is the closest possible number in our sequence to x, there are no numbers in our sequence within this neighborhood.

Thus it must be the case that $x = 1/n$ for some n.

2.5.2 **Exercise 86**

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

(a) True, since we could create a proper subsequence by discarding finite elements from the beginning. Since that converges, then clearly the original sequence also converges.

(b) True. This shows that for arbitrary $\epsilon > 0$, for all $N \in \mathbb{N}$ there will always exist some $n \geq N$ such that x_n is outside of the ϵ-neigborhood of any proposed limit, and we can choose that n from the divergent subsequence.

(c) True. Since the sequence is bounded and diverges, $\limsup x_n$ and $\liminf x_n$ must exist and differ. Thus we can also find subsequences that converge to those different values.

(d) True, since the convergent subsequence is bounded. The original sequence must obey the same bounds, and since it is monotone it must also be convergent.

2.5.3 **Exercise 87**

(a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \ldots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$\left(a_1 + a_2 + \ldots + a_{n_1}\right) + \left(a_{n_1+1} + \ldots + a_{n_2}\right) + \left(a_{n_2+1} + \ldots + a_{n_3}\right) + \ldots$$

leads to a series that also converges to L.

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

(a) The regrouping gives us a sequence of partial sums $\left(s_{n_k}\right)$. This is a subsequence of the original sequence of partial sums (s_n). Since we know that is convergent, then the subsequence must also be convergent to the same limit.

(b) The proof only works in one direction, from convergence to associativity. If we only have subsequence convergence, then we cannot say anything about the convergence of the original series.

2.5.4 **Exercise 88**

The Bolzano–Weirstrass Theorem is extremely important, and so is the strategy employed in the proof. To gain some more experience with this technique, assume the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

Assume we have a set that is bounded above by M. Choose any element x of our given set X.

If $x = M$ we are done, so assume $x < M$. Let $l = M - x$.

Now form the closed interval $I_1 = [x, M]$. Bisect it into two halves, and select I_2 based on the following criteria: If the right-most half has elements in X, choose it. Otherwise choose the left half. Either way, I_k should always include an element in X.

By the NIP, there exists a real number s in every I_k, and every I_k should contain an element from X.

I claim that s is the supremum of X.

Proof: Assume there was some x' such that $x' > s$. Then there would be some interval I_k which contained both s and x', and some I_{k+1} such that it only contained s. This would imply that x' existed in the right half of I_k while s exists in the left half of I_k. However, by construction, we would have picked the right half of I_k, which is a **contradiction**. Therefore it must be that any $x' \in X$ is such that $x' \leq s$.

To show it is the least upper bound, suppose we have some upper bound $b < s$. Then choose $\epsilon < s - b$. Because the length of I_k (which is $\frac{l}{2^k}$) converges to 0, we can choose some I_k such that the length is less than ϵ. By construction, it must contain some element x' of X. It must be that $x' < s$, otherwise we would immediately run into a contradiction.

Thus, we have:

$$|s - x'| < s - b \Longleftrightarrow s - x' < s - b \Longleftrightarrow x' > b.$$

This is a **contradiction**, so therefore it must be that for any upper bound b, that $b \geq s$. ∎

2.5.5 **Exercise 89**

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

If (a_n) diverged, then due to the fact that it is bounded, there would be two subsequences converging to the lim sup and lim inf respectively, which would be different values.

Thus (a_n) must converge. Therefore it itself is a convergent subsequence, and must converge to a.

2.5.6 **Exercise 90**

Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

If $b > 1$, then $b^{1/n}$ is decreasing and bounded below by 1.

If $b < 1$, then $b^{\frac{1}{n}}$ is increasing and bounded above by 1.

If $b = 1$, then we just have the constant sequence of 1.

By MCT there must be some limit l

We can choose the subsequence $b^{\frac{1}{2n}} = \sqrt{b^{1/n}}$, which should have the same limit l.

By Exercise 2.3.1, we know that $\left(\sqrt{b^{1/n}}\right) \to \sqrt{l}$, and the only value where $\sqrt{l} = l$ is either 0 or 1. It is clearly not zero, so the limit must be 1.

2.5.7 **Exercise 91**

> Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

($\Leftarrow$)

Assume $-1 < b < 1$. Let $\epsilon > 0$ be arbitrary.

From Example 2.5.3 we know that $|b|^n$ converges to 0. Using that result:

$$|b^n| = ||b|^n| < \epsilon$$

for all n greater than some $N \in \mathbb{N}$. This proves that b^n converges to 0.

($\Rightarrow$)

First, if $b = 1$ then we clearly converge to 1. If $b = -1$ we diverge since we alternate between -1 and 1.

Assume $b > 1$ or $b < -1$. Then it must be true that $|b| > 1$. Choose $0 < \epsilon < 1$.

Clearly it cannot converge to 0 then, since with our given value of ϵ it can never be the case that $|b^n| < \epsilon$.

2.5.8 **Exercise 92**

> Another way to prove the Bolzano–Weierstrass Theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of a *peak term*. Given a sequence (x_n), a particular term x_m is a peak term if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.
>
> (a) Find examples of sequences with zero, one, and two peak terms. Find an example of a sequnce with infinitely many peak terms that is not monotone.
>
> (b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano–Weierstrass Theorem.

(a) Zero peak terms:

$$\left(-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right)$$

One peak term:

$$\left(0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right)$$

Two peak terms:

$$\left(1, 0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right)$$

Infinitely many peak terms, not monotone:

$$\left(1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \ldots\right)$$

(b) First, in the case that we have finite peak terms, choose the term after the last peak term, call it x_{n_1}.

We know that we can find another term x_{n_2} after that term ($n_2 > n_1$) such that $x_k > x_n$. Otherwise, x_n would be a peak term. The same logic applies for all n_k and n_{k-1}. Thus, we have found a monotone increasing subsequence.

In the case that we have infinite peak terms, we simply choose our subset as the peak terms, since each one must be less than or equal to the previous peak term. Thus this gives us a monotone decreasing subsequence.

Therefore, we can conclude that since every bounded sequence has a monotone subsequence, that subsequence is itself bounded and by MCT convergent.

2.5.9 **Exercise 93**

Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence $\left(a_{n_k}\right)$ converging to $s = \sup S$. (This is a direct proof of the Bolzano–Weierstrass Theorem using the Axiom of Completeness).

Notice s is clearly bounded above by the upper bound of (a_n), so by AoC $s = \sup S$ exists.

Since s is the supremum, we know that given arbitrary $n \in \mathbb{N}$, that there must exist an element $x_n \in S$ such that $x_n > s - \frac{1}{n}$. Rearranging, we get:

$$s - x_n < \frac{1}{n}.$$

Note also that there must be infinite elements in (a_n) that are less than $s + \frac{1}{n}$. Otherwise, we could find an element $x \in S$ such that $s < x < s + \frac{1}{n}$.

Let's choose the following subsequence. Let a_{n_k} be the first term after the first n_{k-1} such that $x_k < a_{n_k} < s + \frac{1}{k}$.

Thus, we have that $s - a_{n_k} < s - x_k < \frac{1}{k}$

If $s < a_{n_k}$, then $a_{n_k} - s < \frac{1}{k}$, so essentially we have that

$$\left|a_{n_k} - s\right| < \frac{1}{k}.$$

Now for arbitrary ϵ, we can just choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, and now we see that for all $k \geq N$ that our subsequence converges to s:

$$\left|a_{n_k} - s\right| < \epsilon.$$

2.6 The Cauchy Criterion

2.6.1 **Exercise 94**

Supply a proof for Theorem 2.6.2.

Assume $(x_n) \to x$.

Let ϵ be arbitrary.

Find $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - x| < \frac{\epsilon}{2}$.

Then for $n, m \geq N$, we have that:

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus our sequence is also Cauchy.

2.6.2 **Exercise 95**

Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

(c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

(a) $x_n = \frac{(-1)^n}{n}$.

(b) Impossible, since all Cauchy sequences are bounded.

(c) Impossible, since the subsequence would have an upper bound, and all terms in the original sequence would also have to obey that upper bound and thus the original sequence would be convergent by MCT.

(d) $(0, 1, 0, 2, 0, 3, 0, 4, \ldots)$

2.6.3 **Exercise 96**

If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

(a) Let $\epsilon > 0$ be arbitrary. Choose $n, m \geq \max\{N_1, N_2\}$ where $N_1, N_2 \in \mathbb{N}$ are such that $|x_n - x_m| < \frac{\epsilon}{2}$ and $|y_n - y_m| < \frac{\epsilon}{2}$ respectively.

Then:

$$|(x_n + y_n) - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| < \epsilon.$$

(b) Since they are both Cauchy, we know they are bounded by some M_1 and M_2 respectively. Let $\epsilon > 0$ be arbitrary. Choose $n, m \geq \max\{N_1, N_2\}$ where $N_1, N_2 \in \mathbb{N}$ are such that $|x_n - x_m| < \frac{\epsilon}{2M_1}$ and $|y_n - y_m| < \frac{\epsilon}{2M_2}$ respectively.

$$\begin{aligned}
|x_n y_n - x_m y_m| &= |x_n y_n - x_m y_m + x_n y_m - x_n y_m| \\
&\leq |x_n y_n - x_n y_m| + |x_n y_m - x_m y_m| \\
&= |x_n||y_n - y_m| + |y_m||x_n - x_m| \\
&\leq M_2 |y_n - y_m| + M_1 |x_n - x_m| \\
&< \epsilon.
\end{aligned}$$

2.6.4 **Exercise 97**

Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

(a) $c_n = |a_n - b_n|$

(b) $c_n = (-1)^n a_n$

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

(a)

(b)

(c) No, consider the counter example where $a_n = \frac{(-1)^n}{n}$. Then c_n will alternate between -1 and 0.

2.6.5 **Exercise 98**

Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$.

Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

2.6.6 **Exercise 99**

Let's call a sequence (a_n) *quasi-increasing* if for all $\epsilon > 0$ there exists an N such that whenever $n > m \geq N$ it follows that $a_n > a_m - \epsilon$.

(a) Give an example of a sequence that is quasi-increasing but not monotone or eventually monotone.

(b) Give an example of a quasi-increasing sequence that is dvergent and not monotone oreventually monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasi-increasing sequences? give an example of a bounded, quasi-increasing sequence that doesn't converge, or prove that no such sequence exists.

2.6.7 **Exercise 100**

Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano–Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC, and MCT are all equivalent.

(b) Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of the completeness discussed at the end of Section 2.6.

(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

2.7 Properties of Infinite Series

2.7.1 **Exercise 101**

Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

2.7.2 **Exercise 102**

Decide whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9}$

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2}$

2.7.3 **Exercise 103**

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

2.7.4 **Exercise 104**

Give an example of each or explain why the request is impossible referencing the proper theorems(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but $\sum y_n$ diverges.

(d) A sequences (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

2.7.5 **Exercise 105**

Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7.

2.7.6 **Exercise 106**

Let's say that a series *subverges* if the sequence of partial sums contains a subsequence that converges. Consider this (invented) defintion for a moment, and then decide which of the following statements are valid propositions about subvergent series:

(a) If (a_n) is bounded, then $\sum a_n$ subverges.

(b) All convergent series are subvergent.

(c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.

(d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

2.7.7 **Exercise 107**

(a) Show that if $a_n > 0$ and $\lim(n a_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.
(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

2.7.8 **Exercise 108**

Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

2.7.9 (Ratio Test) **Exercise 109**

Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

(b) Why does $|a_N| \sum (r')^n$ converge?

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

2.7.10 (Infinite Products) **Exercise 110**

Review Exercise 2.4.10 about infinite products and then answer the following questions:

(a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$ converge?

(b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots$ certainly converges. (Why?) Does it converge to zero?

(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}.$$

Show that the left side of this identity at least converges to something. (A complete proof of this result is taken up in Section 8.3.)

2.7.11 **Exercise 111**

Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

2.7.12 (Summation-by-parts) **Exercise 112**

Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).$$

2.7.13 (Abel's Test) **Exercise 113**

Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0,$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}),$$

where $s_n = x_1 + x_2 + \cdots + x_n$.

(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_k (y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

2.7.14 (Dirichlet's Test) **Exercise 114**

Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if (y_k) is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$ with $\lim y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test in Exercise 2.7.13, but show that essentially the same strategy can be used to provide a proof.

(b) Show how the Alternating Series Test (Theorem 2.7.7) can be derived as a special case of Dirichlet's Test.

2.8 Double Summations and Products of Infinite Series

2.8.1 **Exercise 115**

Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{nn}$. How does this value compare to the two iterated values for the sum already computed?

2.8.2 **Exercise 116**

Show that f the iterate series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbb{N}$ the series $\sum_{j=1}^{\infty}|a_{ij}|$ converges to some real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

converges.

2.8.3 **Exercise 117**

(a) Prove that (t_{nn}) converges.

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges.

2.8.4 **Exercise 118**

(a) Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

(b) Now, show that there exists an N such that

$$|s_{mn}| < \epsilon$$

for all $m, n \geq N$.

2.8.5 **Exercise 119**

(a) Show that for all $m \geq N$

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \epsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges to S.

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$, converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

2.8.6 **Exercise 120**

(a) Assume the hypothesis—and hence the conclusion—of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

(b) Imitate the strategy in the proof of Theorem 2.8.1 to show that $\sum_{k=2}^{\infty} d_k$ converges to $S = \lim_{n\to\infty} s_{nn}$.

2.8.7 **Exercise 121**

Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges so that we may apply Theorem 2.8.1.

(b) Let $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1$.

2.9 Epilogue

No exercises in this section.

3 Basic Topology of $\mathbb{R}$

3.1 Discussion: The Cantor Set

No exercises in this section.

3.2 Open and Closed Sets

3.2.1 **Exercise 122**

3.2.2 **Exercise 123**

3.2.3 **Exercise 124**

3.2.4 **Exercise 125**

3.2.5 **Exercise 126**

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3.2.11 **Exercise 132**

3.2.12 **Exercise 133**

3.2.13 **Exercise 134**

3.2.14 **Exercise 135**

3.2.15 **Exercise 136**

3.3 Compact Sets

3.3.1 **Exercise 137**

3.3.2 **Exercise 138**

3.3.3 **Exercise 139**

3.3.4 **Exercise 140**

3.3.5 **Exercise 141**

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3.3.10 **Exercise 146**

3.3.11 **Exercise 147**

3.3.12 **Exercise 148**

3.3.13 **Exercise 149**

3.4 Perfect Sets and Connected Sets

3.4.1 **Exercise 150**

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3.4.9 **Exercise 158**

3.5 Baire's Theorem

3.5.1 **Exercise 159**

3.5.2 **Exercise 160**

3.5.3 **Exercise 161**

3.5.4 **Exercise 162**

3.5.5 **Exercise 163**

3.5.6 **Exercise 164**

3.5.7 **Exercise 165**

3.5.8 **Exercise 166**

3.5.9 **Exercise 167**

3.5.10 **Exercise 168**

3.6 Epilogue

No exercises in this section.

4 Functional Limits and Continuity

4.1 Discussion: Examples of Dirichlet and Thomae

No exercises in this section.

4.2 Functional Limits

4.2.1 **Exercise 169**

4.2.2 **Exercise 170**

4.2.3 **Exercise 171**

4.2.4 **Exercise 172**

4.2.5 **Exercise 173**

4.2.6 **Exercise 174**

4.2.7 **Exercise 175**

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4.2.9 **Exercise 177**

4.2.10 **Exercise 178**

4.3 Continous Functions

4.3.1 **Exercise 179**

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4.3.3 **Exercise 181**

4.3.4 **Exercise 182**

4.3.5 **Exercise 183**

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4.3.12 **Exercise 190**

4.3.13 **Exercise 191**

4.3.14 **Exercise 192**

4.4 Continuous Functions on Compact Sets

4.4.1 **Exercise 193**

4.4.2 **Exercise 194**

4.4.3 **Exercise 195**

4.4.4 **Exercise 196**

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4.4.10 **Exercise 202**

4.4.11 **Exercise 203**

4.4.12 **Exercise 204**

4.4.13 **Exercise 205**

4.4.14 **Exercise 206**

4.5 The Intermediate Value Theorem

4.5.1 **Exercise 207**

4.5.2 **Exercise 208**

4.5.3 **Exercise 209**

4.5.4 **Exercise 210**

4.5.5 **Exercise 211**

4.5.6 **Exercise 212**

4.5.7 **Exercise 213**

4.5.8 **Exercise 214**

4.6 Sets of Discontinuity

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4.6.9 **Exercise 223**

4.6.10 **Exercise 224**

4.6.11 **Exercise 225**

4.7 Epilogue

No exercises in this section.

5 The Derivative

5.1 Discussion: Are Derivatives Continuous?

No exercises in this section.

5.2 Derivatives and the Intermediate Value Property

5.2.1 **Exercise 226**

5.2.2 **Exercise 227**

5.2.3 **Exercise 228**

5.2.4 **Exercise 229**

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5.2.10 **Exercise 235**

5.2.11 **Exercise 236**

5.2.12 **Exercise 237**

5.3 The Mean Value Theorems

5.3.1 **Exercise 238**

5.3.2 **Exercise 239**

5.3.3 **Exercise 240**

5.3.4 **Exercise 241**

5.3.5 **Exercise 242**

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5.3.7 **Exercise 244**

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5.3.12 **Exercise 249**

5.4 A Continuous Nowhere-Differentiable Function

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5.4.2 **Exercise 251**

5.4.3 **Exercise 252**

5.4.4 **Exercise 253**

5.4.5 **Exercise 254**

5.4.6 **Exercise 255**

5.4.7 **Exercise 256**

5.4.8 **Exercise 257**

5.5 Epilogue

No exercises in this section.

6 Sequences and Series of Functions

6.1 Discussion: The Power of Power Series

No exercises in this section.

6.2 Uniform Convergence of a Sequence of Functions

6.2.1 **Exercise 258**

6.2.2 **Exercise 259**

6.2.3 **Exercise 260**

6.2.4 **Exercise 261**

6.2.5 **Exercise 262**

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6.2.9 **Exercise 266**

6.2.10 **Exercise 267**

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6.2.12 **Exercise 269**

6.2.13 **Exercise 270**

6.2.14 **Exercise 271**

6.2.15 **Exercise 272**

6.3 Uniform Convergence and Differentiation

6.3.1 **Exercise 273**

6.3.2 **Exercise 274**

6.3.3 **Exercise 275**

6.3.4 **Exercise 276**

6.3.5 **Exercise 277**

6.3.6 **Exercise 278**

6.3.7 **Exercise 279**

6.4 Series of Functions

6.4.1 **Exercise 280**

6.4.2 **Exercise 281**

6.4.3 **Exercise 282**

6.4.4 **Exercise 283**

6.4.5 **Exercise 284**

6.4.6 **Exercise 285**

6.4.7 **Exercise 286**

6.4.8 **Exercise 287**

6.4.9 **Exercise 288**

6.4.10 **Exercise 289**

6.5 Power Series

6.5.1 **Exercise 290**

6.5.2 **Exercise 291**

6.5.3 **Exercise 292**

6.5.4 **Exercise 293**

6.5.5 **Exercise 294**

6.5.6 **Exercise 295**

6.5.7 **Exercise 296**

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6.5.9 **Exercise 298**

6.5.10 **Exercise 299**

6.5.11 **Exercise 300**

6.6 Taylor Series

6.6.1 **Exercise 301**

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6.6.4 **Exercise 304**

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6.6.9 **Exercise 309**

6.6.10 **Exercise 310**

6.7 The Weierstrass Approximation Theorem

6.7.1 **Exercise 311**

6.7.2 **Exercise 312**

6.7.3 **Exercise 313**

6.7.4 **Exercise 314**

6.7.5 **Exercise 315**

6.7.6 **Exercise 316**

6.7.7 **Exercise 317**

6.7.8 **Exercise 318**

6.7.9 **Exercise 319**

6.7.10 **Exercise 320**

6.7.11 **Exercise 321**

6.8 Epilogue

No exercises in this section.

7 The Riemann Integral

7.1 Discussion: How Should Integration be Defined?

No exercises in this section.

7.2 The Definition of the Riemann Integral

7.2.1 **Exercise 322**

7.2.2 **Exercise 323**

7.2.3 **Exercise 324**

7.2.4 **Exercise 325**

7.2.5 **Exercise 326**

7.2.6 **Exercise 327**

7.2.7 **Exercise 328**

7.3 Integrating Functions with Discontinuities

7.3.1 **Exercise 329**

7.3.2 **Exercise 330**

7.3.3 **Exercise 331**

7.3.4 **Exercise 332**

7.3.5 **Exercise 333**

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7.3.7 **Exercise 335**

7.3.8 **Exercise 336**

7.3.9 **Exercise 337**

7.4 Properties of the Integral

7.4.1 **Exercise 338**

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7.4.4 **Exercise 341**

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7.4.9 **Exercise 346**

7.4.10 **Exercise 347**

7.4.11 **Exercise 348**

7.5 The Fundamental Theorem of Calculus

7.5.1 **Exercise 349**

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7.5.3 **Exercise 351**

7.5.4 **Exercise 352**

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7.5.6 **Exercise 354**

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