Set 11: Splines – Part 1

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Foundations of Computational Math 1

Fall 2017

Interpolation and Smoothness

- The piecewise Hermite interpolant is cubic locally but still only $\mathcal{C}^{(1)}$ globally.
- Derivative values may not be available.
- To get $\mathcal{C}^{(2)}$ globally and maintain piecewise cubic polynomial we must give up something.
- Give up interpolating f_i'.
- Interpolate f_i at nodes.
- Require piecewise cubic polynomial.
- Use continuity of first and second derivatives as constraints but do not specify values.
- Family of interpolatory cubic splines.

Polynomial Splines

- Polynomial splines are the subject of a large body of literature.
- In addition to the text the following have useful discussions at the appropriate level for this class. and have been used as source material:
  - P. M. Prenter, Splines and Variational Methods, Wiley.
  - C. W. Ueberhuber, Numerical Computation, Springer
- An excellent more advanced reference is: Carl de Boor, A Practical Guide to Splines, Springer-Verlag, 1978.
- A second standard text is Larry Schumaker, Spline Functions: Basic Theory, Wiley 1981 and Cambridge University Press 2007.

Polynomial Splines

Definition 11.1. Given $[a, b]$ let the distinct points $a = x_0 < x_1 < \cdots < x_n$ define a partition into intervals $[x_{i-1}, x_i)$ denoted π. A polynomial spline, $s(t)$, of degree d is a piecewise polynomial of degree d, $s(t) = p_{i,d}(t)$ on $[x_{i-1}, x_i)$ for $1 \leq i \leq n$.

Further, the polynomials are such that their values and the values of their first to $(d-1)$-st derivatives match at x_i for $1 \leq i \leq n-1$.

Polynomial Splines

Definition 11.2. A subspline of degree d is a piecewise polynomial that satisfies all of the conditions of a spline but is only continuous to the m-th derivatives with $m < d - 1$.

Note. Piecewise Hermite interpolating polynomials are cubic subsplines since they are only $\mathcal{C}^{(1)}$.

Cubic Splines

Lemma. *Given a partition, π, the set of cubic splines, $S_3(\pi)$ is a linear space with dimension $n + 3$.*

Informal Argument:

- n intervals each with a cubic polynomial require $4n$ parameters.
- continuity of $s(t)$ at x_i, $1 \leq i \leq n - 1$, imposes $n - 1$ constraints
- continuity of $s'(t)$ at x_i, $1 \leq i \leq n - 1$, imposes $n - 1$ constraints
- continuity of $s''(t)$ at x_i, $1 \leq i \leq n - 1$, imposes $n - 1$ constraints
- $4n - 3(n - 1) = n + 3$ degrees of freedom

Note. A proof requires exhibiting a basis with $n + 3$ linearly independent functions.

Interpolating Cubic Spline

Definition 11.3. An interpolating cubic spline is a cubic spline that satisfies

$$s(x_i) = f_i \quad 0 \le i \le n$$

where $a = x_0 < x_1 < \cdots < x_n = b$ are distinct points.

- Interpolation imposes $n + 1$ constraints.
- 2 degrees of freedom remain
- Typically two boundary conditions are specified.

Boundary Conditions

- Natural boundary condition

$$s''(a) = s''(b) = 0$$

- Periodic boundary condition – assumes $f(a) = f(b)$

$$s''(a) = s''(b) \text{ and } s'(a) = s'(b)$$

- Hermite boundary conditions

$$s' = f'(a) \text{ and } s'(b) = f'(b) \tag{1}$$

$$s''(a) = f''(a) \text{ and } s''(b) = f''(b) \tag{2}$$

Boundary Conditions

- Hermite boundary conditions (derivative-free form)
  - Define two cubic interpolation polynomials, $c_1(x)$ and $c_2(x)$, based on (x_0, x_1, x_2, x_3) and $(x_{n-3}, x_{n-2}, x_{n-1}, x_n)$.
  - Use the value of the first or second derivatives of $c_1(x)$ and $c_2(x)$

$$s' = c_1'(a) \text{ and } s'(b) = c_2'(b) \tag{3}$$

$$s''(a) = c_1''(a) \text{ and } s''(b) = c_2''(b) \tag{4}$$

- Not-a-knot boundary conditions

$$s_-'''(x_1) = s_+'''(x_1) \quad \text{and} \quad s_-'''(x_{n-1}) = s_+'''(x_{n-1}) \tag{5}$$

Boundary Conditions

- (1) is called the complete spline by De Boor and by others.
- (1) is called “the” interpolatory spline by Prenter.
- (2) is called the natural spline by Prenter.
- (3) is called the Lagrangian spline by Prenter and by Swartz and Varga.

Spline Construction

Let $h_i = x_i - x_{i-1}$. Denote $s'_i = s'(x_i)$ and $s''_i = s''(x_i)$.

Since $s(t)$ is piecewise cubic each $p''_i(x)$ is linear.

Imposing that $s''(x)$ is continuous at the internal nodes, we have $1 \le i \le n$,

$$p''_i(x) = s''_{i-1}\frac{x_i - x}{h_i} + s''_i\frac{x - x_{i-1}}{h_i}$$
$$x_{i-1} \le x \le x_i$$

Spline Construction

Integrating yields $p_i(x)$ and $p'_i(x)$

$$p'_i(x) = -s''_{i-1}\frac{(x-x_i)^2}{2h_i} + s''_i\frac{(x-x_{i-1})^2}{2h_i} + \gamma_{i-1}$$

$$p_i(x) = s''_{i-1}\frac{(x_i-x)^3}{6h_i} + s''_i\frac{(x-x_{i-1})^3}{6h_i} + \gamma_{i-1}(x-x_{i-1}) + \tilde{\gamma}_{i-1}$$

So each $p_i(x)$ has 4 parameters as required.

Spline Construction

Next, impose interpolation conditions and continuity of $s(t)$:

$$s(x_i) = f_i, \quad 0 \le i \le n \quad n+1 \text{ constraints}$$

$$p_i(x_i) = p_{i+1}(x_i), \quad 1 \le i \le n-1 \quad n-1 \text{ constraints}$$

$$\Updownarrow$$

$$p_1(x_0) = f_0, \quad 1 \text{ constraint}$$

$$p_n(x_n) = f_n, \quad 1 \text{ constraint}$$

$$p_i(x_i) = p_{i+1}(x_i) = f_i, \quad 1 \le i \le n-1 \quad 2n-2 \text{ constraints}$$

Spline Construction

These are equivalent to:

For $1 \leq i \leq n$,

$$p_i(x_{i-1}) = f_{i-1} = \frac{h_i^2}{6} s''_{i-1} + \tilde{\gamma}_{i-1}$$

$$p_i(x_i) = f_i = \frac{h_i^2}{6} s''_i + h_i \gamma_{i-1} + \tilde{\gamma}_{i-1}$$

Spline Construction

To enforce $s'(x_i^-) = s'(x_i^+)$ continuity of $s'(x)$ we have the equations, $1 \leq i \leq n-1$

$$p_i'(x) = -s_{i-1}'' \frac{(x - x_i)^2}{2h_i} + s_i'' \frac{(x - x_{i-1})^2}{2h_i} + \gamma_{i-1}$$

$$p_i'(x_i) = p_{i+1}'(x_i)$$

$$p_i'(x_i) = s_i'' \frac{h_i}{2} + \gamma_{i-1} = p_{i+1}'(x_i) = -s_i'' \frac{h_{i+1}}{2} + \gamma_i$$

Spline Construction

$3n+1$ unknowns $s_0'', \ldots, s_n''$, $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ and $\gamma_1, \ldots, \gamma_n$.

$3n-1$ equations

$$s_i'' \frac{h_i}{2} + \gamma_{i-1} = -s_i'' \frac{h_{i+1}}{2} + \gamma_i, \quad 1 \le i \le n-1$$

$$f_{i-1} = \frac{h_i^2}{6} s_{i-1}'' + \tilde{\gamma}_{i-1}, \quad 1 \le i \le n$$

$$f_i = \frac{h_i^2}{6} s_i'' + h_i \gamma_{i-1} + \tilde{\gamma}_{i-1}, \quad 1 \le i \le n$$

Spline Construction

The $\tilde{\gamma}_i$ and $\tilde{\gamma}_i$ can be found due to the structure of those $2n$ equations:

$$\tilde{\gamma}_{i-1} = f_{i-1} - s''_{i-1}\frac{h_i^2}{6}, \quad 1 \leq i \leq n$$

$$\therefore \gamma_{i-1} = \frac{f_i - f_{i-1}}{h_i} - \frac{h_i}{6}(s''_i - s''_{i-1}), \quad 1 \leq i \leq n$$

thereby eliminating $2n$ variables.

Spline Construction

Subsituting the $\tilde{\gamma}_i$ and $\tilde{\gamma}_i$, $1 \le i \le n$ yields for$1 \le i \le n-1$

$$\frac{h_i}{6} s''_{i-1} + \frac{h_i}{3} s''_i + \frac{f_i - f_{i-1}}{h_i} = -\frac{h_{i+1}}{3} s''_i - \frac{h_{i+1}}{6} s''_{i+1} + \frac{f_{i+1} - f_i}{h_{i+1}}$$

which are the $n-1$ equations that along with the boundary conditions will define the $n+1$ variables $s''_0, \ldots, s''_n$.

System of Equations

Separating knowns from unknowns yields:

$$\frac{h_i}{6}s''_{i-1} + \frac{(h_i + h_{i+1})}{3}s''_i + \frac{h_{i+1}}{6}s''_{i+1} = \frac{(f_{i+1} - f_i)}{h_{i+1}} - \frac{(f_i - f_{i-1})}{h_i}$$

multiply by $\dfrac{6}{h_i + h_{i+1}}$

$$\frac{h_i}{h_i + h_{i+1}}s''_{i-1} + 2s''_i + \frac{h_{i+1}}{h_i + h_{i+1}}s''_{i+1}$$

$$= \frac{6}{h_i + h_{i+1}}\left[\frac{(f_{i+1} - f_i)}{h_{i+1}} - \frac{(f_i - f_{i-1})}{h_i}\right]$$

System of Equations

$n-1$ equations for $n+1$ unknowns

$$\mu_i s''_{i-1} + 2s''_i + \lambda_i s''_{i+1} = d_i, \quad 1 \leq i \leq n-1$$

$$\mu_i = \frac{h_i}{h_i + h_{i+1}} < 1 \text{ and } \lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}} < 1$$

$$d_i = \frac{6}{h_i + h_{i+1}} \Big[\frac{(f_{i+1} - f_i)}{h_{i+1}} - \frac{(f_i - f_{i-1})}{h_i} \Big]$$

$$= 6f[x_{i-1}, x_i, x_{i+1}]$$

Note. The left-hand side is a combination of second derivatives and the right hand side is a second divided difference – scales are consistent.

System of Equations

$$\begin{pmatrix} \mu_1 & 2 & \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & \mu_2 & 2 & \lambda_2 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & \mu_{n-2} & 2 & \lambda_{n-2} & 0 \\ 0 & \dots & \dots 0 & 0 & \mu_{n-1} & 2 & \lambda_{n-1} \end{pmatrix} \begin{pmatrix} s''_0 \\ s''_1 \\ \vdots \\ s''_{n-1} \\ s''_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_{n-1} \end{pmatrix}$$

Need 2 more equations or 2 more constraints.

Boundary Conditions

To enforce $s''(x_0) = s''(x_n) = 0$ is trivial and yields $n-1$ equations in the $n-1$ unknowns s''_i $1 \le i \le n-1$.

$$\begin{pmatrix} 2 & \lambda_1 & 0 & \dots & 0 \\ \mu_2 & 2 & \lambda_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \mu_{n-2} & 2 & \lambda_{n-2} \\ 0 & \dots & 0 & \mu_{n-1} & 2 \end{pmatrix} \begin{pmatrix} s''_1 \\ \vdots \\ s''_{n-1} \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_{n-1} \end{pmatrix}$$

Boundary Conditions

Hermite boundary conditions on $s''(x)$ are handled similarly in that only the right-hand side vector need be modified.

$$\begin{pmatrix} 2 & \lambda_1 & 0 & \dots & 0 \\ \mu_2 & 2 & \lambda_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \mu_{n-2} & 2 & \lambda_{n-2} \\ 0 & \dots & 0 & \mu_{n-1} & 2 \end{pmatrix} \begin{pmatrix} s_1'' \\ \vdots \\ s_{n-1}'' \end{pmatrix} = \begin{pmatrix} d_1 - \mu_1 s_0'' \\ \vdots \\ d_{n-1} - \lambda_{n-1} s_n'' \end{pmatrix}$$

Boundary Conditions

To enforce $s'(x_0) = f'_0$ and $s'(x_n) = f'_n$ use the expression defined by $p'_1(x_0) = f'_0$ as the equation $i = 0$ and similarly for a derivative boundary condition at x_n.

To enforce more general boundary conditions add equations

$$2s''_0 + \lambda_0 s''_1 = d_0 \text{ and } \mu_n s''_{n-1} + 2s''_n = d_n$$
$$0 \leq \lambda_0 \leq 1, \;\; 0 \leq \mu_n \leq 1$$

System of Equations

$$\begin{pmatrix} 2 & \lambda_0 & 0 & \cdots & 0 \\ \mu_1 & 2 & \lambda_1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \mu_{n-1} & 2 & \lambda_{n-1} \\ 0 & \cdots & 0 & \mu_n & 2 \end{pmatrix} \begin{pmatrix} s_0'' \\ s_1'' \\ \vdots \\ s_{n-1}'' \\ s_n'' \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix}$$

The matrix is diagonally dominant and therefore nonsingular. A unique solution exists.

Another Form

Recall Hermite cubic for two points on $[x_{i-1}, x_i]$. It can be written

$$p_i(x) = \psi_{L,i}(x) f_{i-1} + \psi_{R,i}(x) f_i + \Psi_{L,i}(x) s'_{i-1} + \Psi_{R,i}(x) s'_i$$

$$\psi_{L,i}(x) = \frac{(x - x_i)^2}{h_i^2}\left[1 + \frac{2}{h_i}(x - x_{i-1})\right]$$

$$\psi_{R,i}(x) = \frac{(x - x_{i-1})^2}{h_i^2}\left[1 - \frac{2}{h_i}(x - x_i)\right]$$

$$\Psi_{L,i}(x) = \frac{(x - x_i)^2}{h_i^2}(x - x_{i-1})$$

$$\Psi_{R,i}(x) = \frac{(x - x_{i-1})^2}{h_i^2}(x - x_i)$$

This form enforces, interpolation and $\mathcal{C}^1$.

Another Form

We have

$$\therefore p_i''(x) = \psi_{L,i}''(x) f_{i-1} + \psi_{R,i}''(x) f_i + \Psi_{L,i}''(x) s_{i-1}' + \Psi_{R,i}''(x) s_i'$$

$$\Psi_{L,i}''(x) = \frac{4(x-x_i)}{h_i^2} + \frac{2(x-x_{i-1})}{h_i^2}$$

$$\Psi_{R,i}''(x) = \frac{2(x-x_i)}{h_i^2} + \frac{4(x-x_{i-1})}{h_i^2}$$

$$\psi_{L,i}''(x) = \frac{8(x-x_i)}{h_i^3} + \frac{4(x-x_{i-1})}{h_i^3} + \frac{2}{h_i^2}$$

$$\psi_{R,i}''(x) = -\frac{8(x-x_{i-1})}{h_i^3} - \frac{4(x-x_i)}{h_i^3} + \frac{2}{h_i^2}$$

Another Form

Equations come from enforcing continuity of $s''(t)$.

Setting $p_i''(x_i) = p_{i+1}''(x_i)$ yields

$$\Psi_{L,i}'' s_{i-1}' + (\Psi_{R,i}'' - \Psi_{L,i+1}'') s_i' - \Psi_{R,i+1}'' s_{i+1}' =$$
$$-\psi_{L,i}'' f_{i-1} + (\psi_{L,i+1}'' - \psi_{R,i}'') f_i + \psi_{R,i+1}'' f_{i+1}$$

We have $n-1$ equations defining s_i' for $1 \le i \le n-1$ via a tridiagonal system of equations. s_0' and s_n' are still free. (Note the argument x_i has been suppressed on the second derivatives of the basis functions.)

Coefficients

$$\Psi''_{L,i}(x_i) = \frac{2}{h_i} \qquad \Psi''_{R,i}(x_i) = \frac{4}{h_i}$$

$$\Psi''_{L,i+1}(x_i) = -\frac{4}{h_{i+1}} \qquad \Psi''_{R,i+1}(x_i) = -\frac{2}{h_{i+1}}$$

$$\psi''_{L,i}(x_i) = \frac{6}{h_i^2} \qquad \psi''_{R,i}(x_i) = -\frac{6}{h_i^2}$$

$$\psi''_{L,i+1}(x_i) = -\frac{6}{h_{i+1}^2} \qquad \psi''_{R,i+1}(x_i) = \frac{6}{h_{i+1}^2}$$

System of Equations

The basis values and continuity of $s''(x)$ yields for $1 \le i \le n-1$,

$$\frac{2}{h_i}s'_{i-1} + (\frac{4}{h_i} + \frac{4}{h_{i+1}})s'_i + \frac{2}{h_{i+1}}s'_{i+1} = -\frac{6}{h_i^2}f_{i-1} + (\frac{6}{h_i^2} - \frac{6}{h_{i+1}^2})f_i + \frac{6}{h_{i+1}^2}f_{i+1}$$

$$h_{i+1}s'_{i-1} + 2(h_i + h_{i+1})s'_i + h_i s'_{i+1} = 3\Big[-\frac{h_{i+1}}{h_i}f_{i-1} + (\frac{h_{i+1}}{h_i} - \frac{h_i}{h_{i+1}})f_i + \frac{h_i}{h_{i+1}}f_{i+1}\Big]$$

$$h_{i+1}s'_{i-1} + 2(h_i + h_{i+1})s'_i + h_i s'_{i+1} = 3\Big[h_{i+1}\frac{(f_i - f_{i-1})}{h_i} + h_i\frac{(f_{i+1} - f_i)}{h_{i+1}}\Big]$$

$$\lambda_i s'_{i-1} + 2s'_i + \mu_i s'_{i+1} = 3\Big[\lambda_i f[x_{i-1}, x_i] + \mu_i f[x_i, x_{i+1}]\Big] = g_i$$

$$\lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}} < 1 \text{ and } \mu_i = \frac{h_i}{h_i + h_{i+1}} < 1$$

Note. The two sides have consistent scaling.

System of Equations

$n-1$ equations and $n+1$ unknowns:

$$\begin{pmatrix} \lambda_1 & 2 & \mu_1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 2 & \mu_2 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & \lambda_{n-2} & 2 & \mu_{n-2} & 0 \\ 0 & \dots & \dots 0 & 0 & \lambda_{n-1} & 2 & \mu_{n-1} \end{pmatrix} \begin{pmatrix} s'_0 \\ s'_1 \\ \vdots \\ s'_{n-1} \\ s'_n \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_{n-1} \end{pmatrix}$$

Boundary Conditions

Boundary conditions where s'_0 and s'_n are given specific values are easily imposed and yields $n-1$ equations in the $n-1$ unknowns s'_i $1 \leq i \leq n-1$.

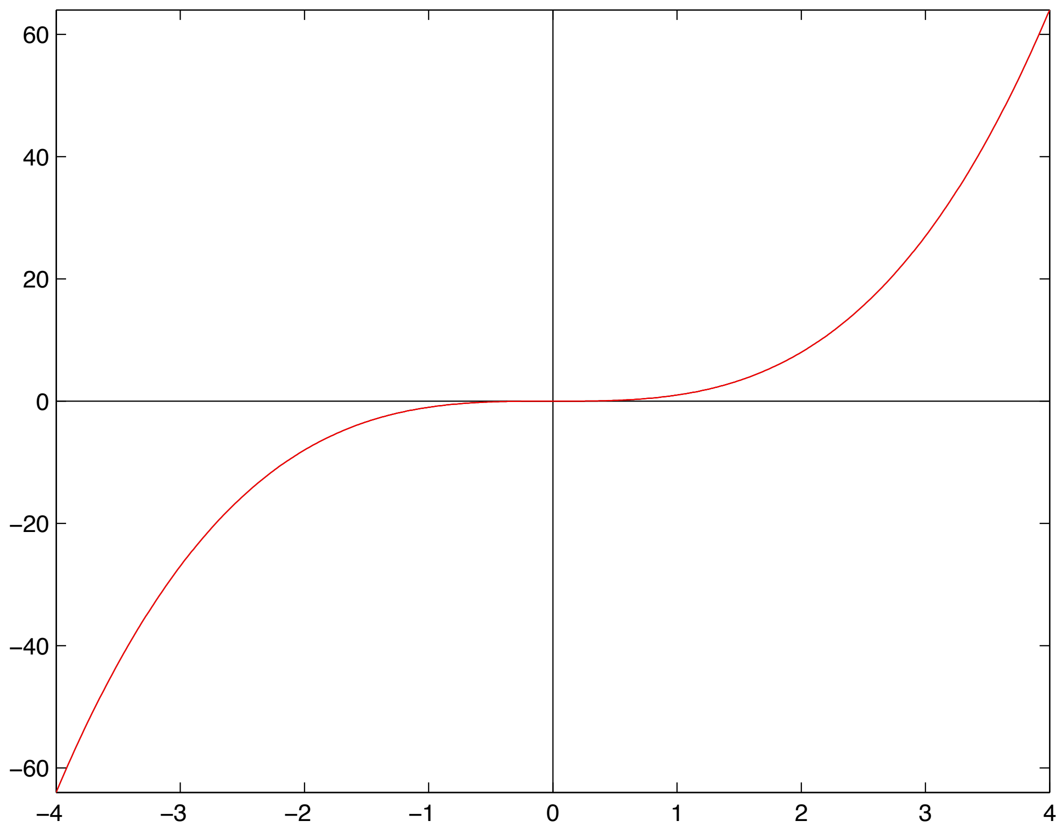
To enforce $s''(x_0) = f''_0$ and $s''(x_n) = f''_n$ use the expression defined by $p''_1(x_0) = f''_0$ as the equation $i = 0$ and similarly for a derivative boundary condition at x_n.

Boundary Conditions

Imposing specific values on s'_0 and s'_n yields:

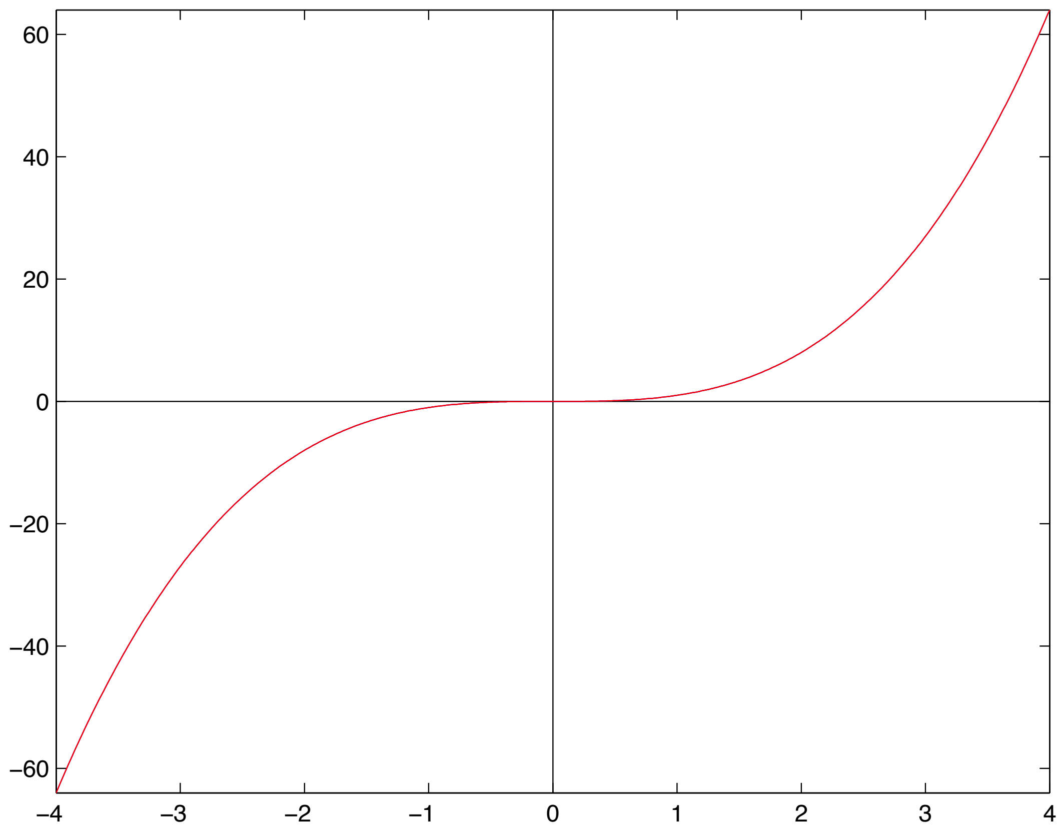
$$\begin{pmatrix} 2 & \mu_1 & 0 & \dots & 0 \\ \lambda_2 & 2 & \mu_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \lambda_{n-2} & 2 & \mu_{n-2} \\ 0 & \dots & 0 & \lambda_{n-1} & 2 \end{pmatrix} \begin{pmatrix} s'_1 \\ \vdots \\ s'_{n-1} \end{pmatrix} = \begin{pmatrix} g_1 - \lambda_1 s'_0 \\ \vdots \\ g_{n-1} - \mu_{n-1} s'_n \end{pmatrix}$$

Cubic Spline



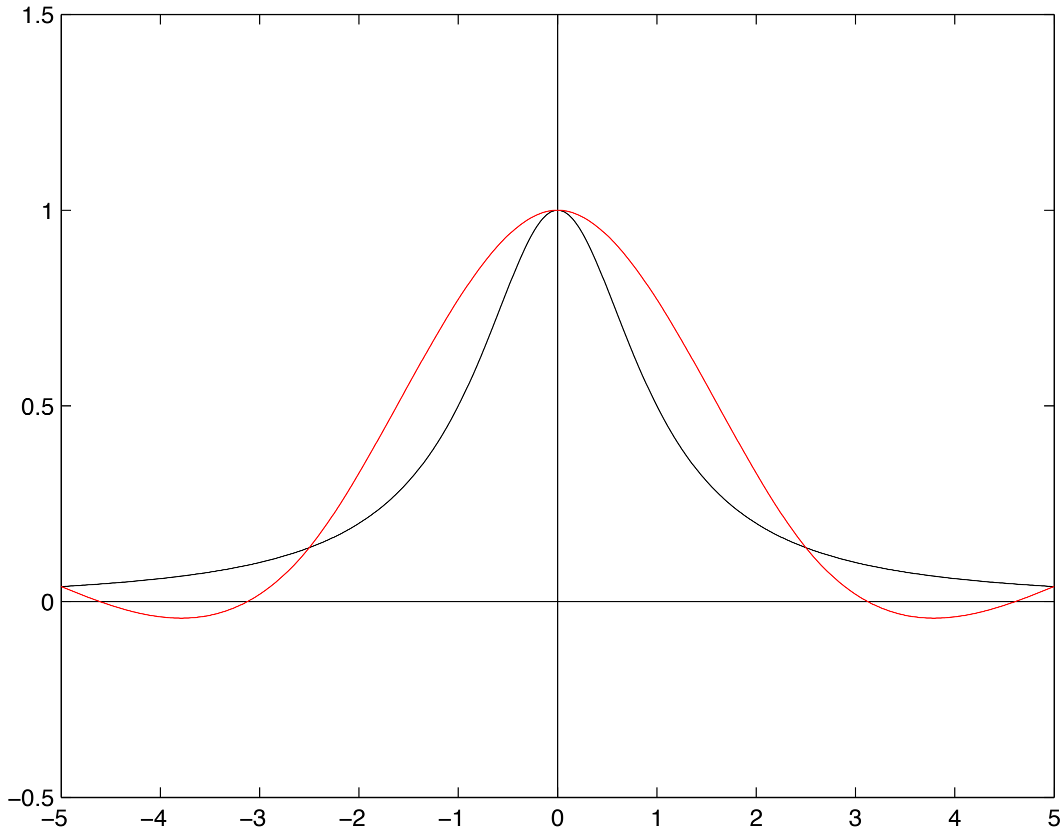
$Ts'' = d$ form, 5 points, $s(x)$ (red) $f(x) = x^3$ (black), $\|e\| = 10^{-14}$.

Cubic Spline



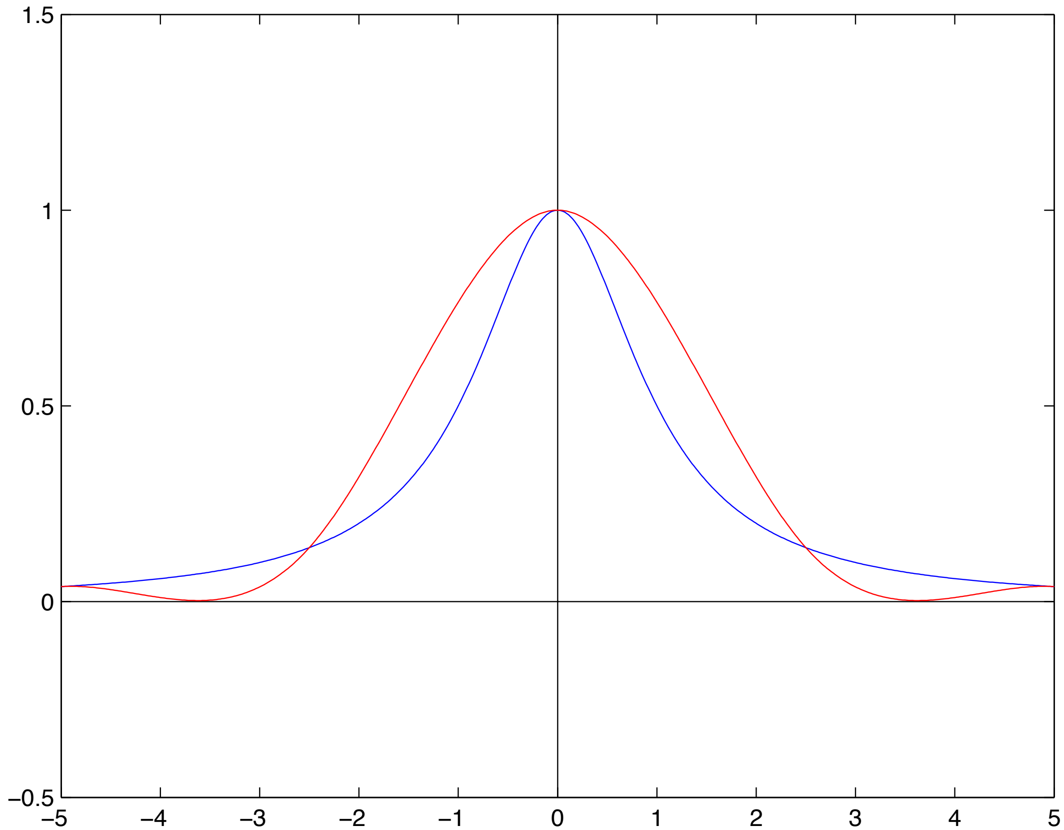
$Ts' = d$ form, 5 points, $s(x)$ (red) $f(x) = x^3$ (blue), $\|e\| = 10^{-14}$.

Cubic Spline



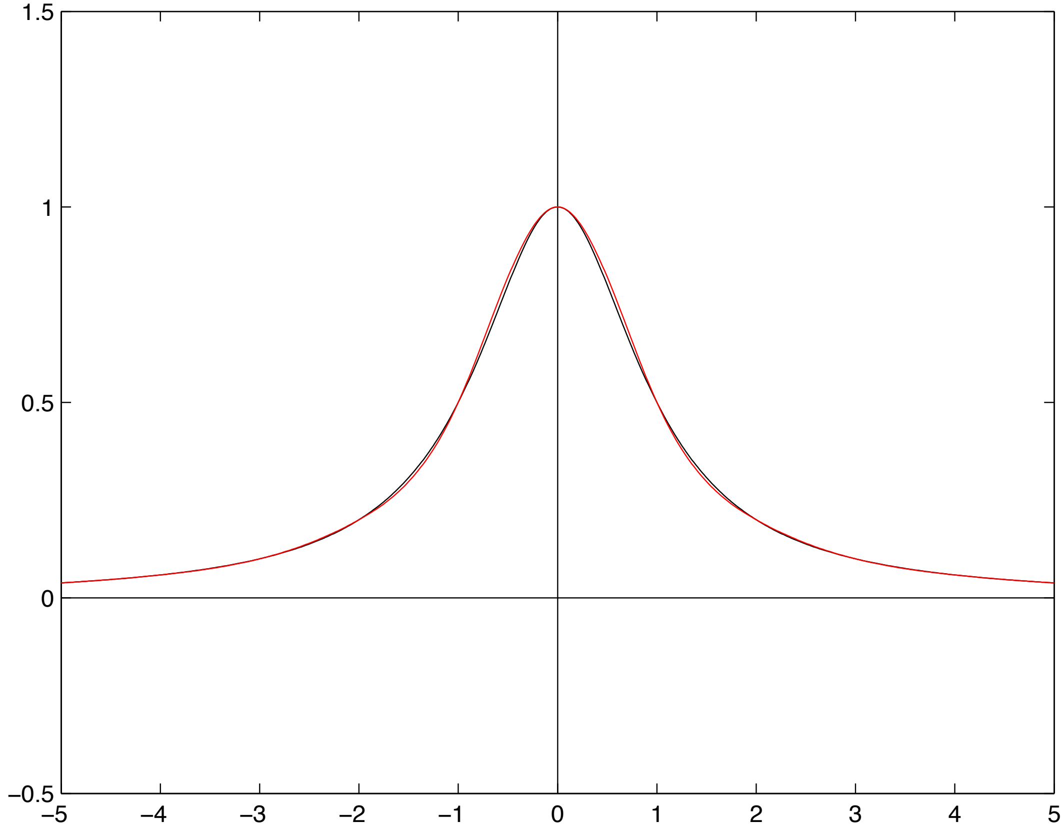
$Ts'' = d$ form, 5 points, $s(x)$ (red) $f(x) = 1/(1+y^2)$ (black), $\|e\| = 0.279$.

Cubic Spline



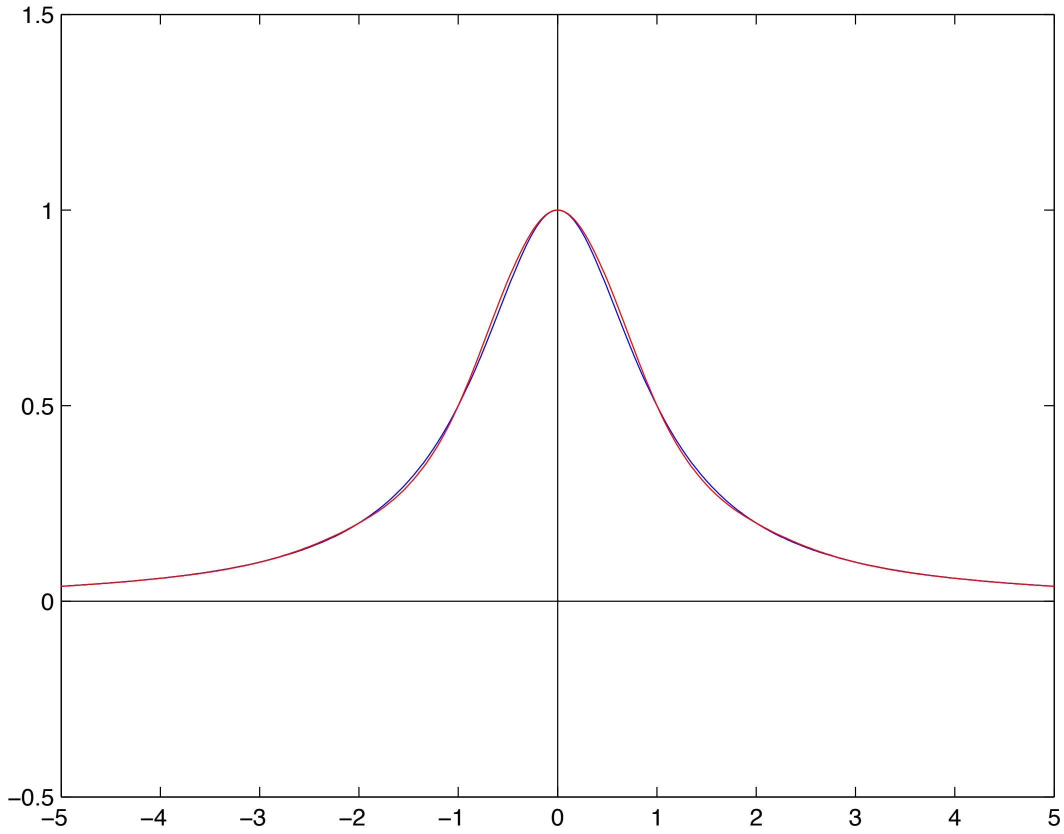
$Ts' = d$ form, 5 points, $s(x)$ (red) $f(x) = 1/(1+y^2)$ (blue), $\|e\| = 0.271$.

Cubic Spline



$Ts'' = d$ form, 11 points, $s(x)$ (red) $f(x) = 1/(1+y^2)$ (black), $\|e\| = 0.022$.

Cubic Spline



$Ts' = d$ form, 11 points, $s(x)$ (red) $f(x) = 1/(1+y^2)$ (blue), $\|e\| = 0.022$.