Homework 7

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MAP5345: Partial Differential Equations I

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Problem 1

Consider the beam equation for the vertical deflection u(x,t) of an elastic beam

$$u_{tt} + Ku_{xxxx} = 0, \quad \text{for } 0 < x < L \tag{1}$$

where $K > 0$ is a constant. Suppose the boundary conditions are given by

$$u(0,t) = u_x(0,t) = 0 \tag{2}$$
$$u_{xx}(L,t) = u_{xxx}(L,t) = 0 \tag{3}$$

and the initial conditions are

$$u(x,0) = u_0(x) \tag{4}$$
$$u_t(x,0) = \dot{u}_0(x) \tag{5}$$

Use separation of variables to find the general solution to the PDE.

Assuming our solution $u(x,t)$ can be expressed in product form $X(x))T(t)$ we get the following

$$X(x)\frac{d^2X(x)}{dx^2} + KX(x)\frac{d^4T(t)}{dt^4} = 0$$

where this can only be true if they equal some constant. Setting them equal to λ we arrive at the differential equations

$$-\frac{1}{KT(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{X(x)}\frac{d^4X(x)}{dx^4} = -\lambda$$

Solving for $T(t)$ we get

$$T(t) = Acos(\sqrt{K\lambda}t) + Bsin(\sqrt{K\lambda t})$$
$$u(x,0) = u_0 \Rightarrow A = u_0(x), \quad u_t(x,0) = \dot{u}_0(x) \Rightarrow B = \dot{u}_0(x)$$

and then solving for $X(x)$ using the boundary conditions we get

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -cos(\sqrt[4]{\lambda}L) & -sin(\sqrt[4]{\lambda}L) & cosh(\sqrt[4]{\lambda}L) & sinh(\sqrt[4]{\lambda}L) \\ sin(\sqrt[4]{\lambda}L) & -cos(\sqrt[4]{\lambda}L) & sinh(\sqrt[4]{\lambda}L) & cosh(\sqrt[4]{\lambda}L) \end{pmatrix} \begin{pmatrix} C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where we use the fact that the spatial eigenfucntion is

$$X(x) = Ccos(\sqrt[4]{\lambda}x) + Dsin(\sqrt[4]{\lambda}x) + Ecosh(\sqrt[4]{\lambda}x) + Fsinh(\sqrt[4]{\lambda}x).$$

From the first two rows of the linear system we get that $C = -E$ and $D = -F$. We can then rewrite the linear system as

$$\begin{pmatrix} (cosh(\sqrt[4]{\lambda}L) + cos(\sqrt[4]{\lambda}L)) & (sinh(\sqrt[4]{\lambda}L) + sin(\sqrt[4]{\lambda}L)) \\ (sin(\sqrt[4]{\lambda}L) - sinh(\sqrt[4]{\lambda}L)) & (cosh(\sqrt[4]{\lambda}L) + cos(\sqrt[4]{\lambda}L)) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the above linear system we get that

$$D = -C\frac{cosh(\sqrt[4]{\lambda}L) + cos(\sqrt[4]{\lambda}L)}{sinh(\sqrt[4]{\lambda}L) + sin(\sqrt[4]{\lambda}L)}$$

where we will denote the above as $-CD_n$ so it does not become cumbersome. We also need to determine λ_n and we can do so by setting the determinant of the above matrix to zero. We do this to avoid the trivial solution. Doing so gives us the

$$(cosh(\sqrt[4]{\lambda}L) + cos(\sqrt[4]{\lambda}L))^2 - (sinh(\sqrt[4]{\lambda}L) - sin(\sqrt[4]{\lambda}L))(sinh(\sqrt[4]{\lambda}L) + sin(\sqrt[4]{\lambda}L)) = 0$$
$$\Rightarrow \lambda_n : \; cos(\sqrt[4]{\lambda_n}L)cosh(\sqrt[4]{\lambda_n}L) = -1$$

Now we can derive the general solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n T_n(t) X_n(x)$$
$$T_n(t) = u_0(x)cos(\sqrt{K\lambda_n}t) + \dot{u}_0(x)sin(\sqrt{K\lambda_n}t)$$
$$X_n(x) = \left[\left(cosh(\sqrt[4]{\lambda_n}x) - cos(\sqrt[4]{\lambda_n}x)\right) - D_n\left(sinh(\sqrt[4]{\lambda_n}x) - sin(\sqrt[4]{\lambda_n}x)\right)\right]$$

where the coefficients A_n can be found by projection against our spatial eigenfunction X_m.

Problem 2

Consider the interval $0 < x < 5$. In each case, calculate the L^1, L^2, and L^∞ norm of each function on the interval (0,5):

(a) $f(x) = x(x-5)$

The $\|f(x)\|_1, \|f(x)\|_2$, and $\|f(x)\|_\infty$ over $D = (0,5)$ are

$$\|f(x)\|_1 = \int_D |f(x)|\,dx = \int_0^5 |x(x-5)|\,dx = \left(-\frac{1}{3}x^3 + \frac{5}{2}x^2\right)\Big|_0^5 = \frac{125}{6}$$

$$\begin{aligned}\|f(x)\|_2 &= \left(\int_D |x(x-5)|^2\,dx\right)^{1/2} = \left(\int_0^5 (x^2-5x)^2\,dx\right)^{1/2} = \left(\frac{1}{5}x^5 - \frac{5}{2}x^4 + \frac{25}{3}x^3\Big|_0^5\right)^{1/2} \\ &= \left(5^4 - \frac{5^5}{2} + \frac{5^5}{3}\right)^{1/2} = \sqrt{\frac{625}{6}} = \frac{25\sqrt{6}}{6}\end{aligned}$$

To calculate $\|f(x)\|_\infty = \sup\limits_{x\in\bar{D}}|f(x)|$, where $\bar{D}$ is the closure of D, we can graph the function $|x(x-5)|$ and visually determine it.

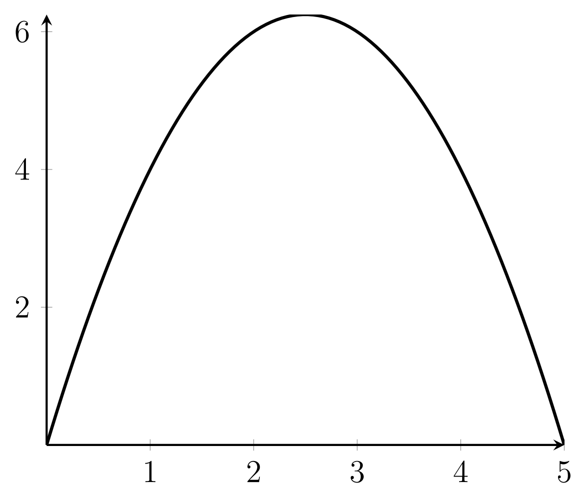

Figure 1: Plot of $|f(x)| = |x^2 - 5x|$

From the graph we can see that $\|f(x)\|_\infty$ occurs at the vertex $x = 2.5$, so $\|f(x)\|_\infty = 6.25$. It is also easy to determine this value since extrema for a parabola happen at its vertex.

(b) $f(x) = x^{-1/2}$

The $\|f(x)\|_1, \|f(x)\|_2$, and $\|f(x)\|_\infty$ over $D = (0, 5)$ are

$$\|f(x)\|_1 = \int_D |f(x)|\, dx = \int_0^5 |x^{-1/2}|\, dx = 2\sqrt{x}\,\Big|_0^5 = 2\sqrt{5}$$

$$\|f(x)\|_2 = \left(\int_D |f(x)|^2\, dx\right)^{1/2} = \left(\int_0^5 |x^{-1/2}|^2\, dx\right)^{1/2} = ln(x)\,\Big|_1^5 - ln(x)\,\Big|_0^1 = ln(4) + \infty = \infty$$

To calculate $\|f(x)\|_\infty = \sup_{x \in \bar{D}} |f(x)|$, where $\bar{D}$ is the closure of D, we can graph the function $|1/\sqrt{x}|$ and visually determine it.

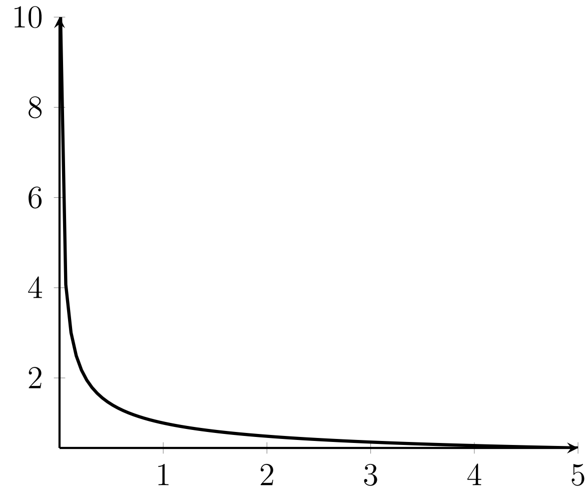

Figure 2: Plot of $|f(x) = |x^{-1/2}||$

We can see from the graph that $x = 0$ so then $\|f(x)\|_\infty = \infty$. We can also determine this analytically by simply realizing that the derivative $f'(x) = -\frac{1}{2}x^{-3/2}$ is decreasing over our interval D so the left bound must be the maximum value.

(c) $f(x) = e^{-kx}$

The $\|f(x)\|_1, \|f(x)\|_2$, and $\|f(x)\|_\infty$ over $D = (0, 5)$ are

$$\|f(x)\|_1 = \int_D |f(x)|\,dx = \int_0^5 |e^{-kx}|\,dx = -\frac{1}{k}e^{-kx}\Big|_0^5 = -\frac{1}{k}(e^{-5k} - 1)$$

$$\|f(x)\|_2 = \left(\int_D |f(x)|^2\,dx\right)^{1/2} = \left(\int_0^5 |e^{-kx}|^2\,dx\right)^{1/2} = \sqrt{-\frac{1}{2k}e^{-2k}\Big|_0^5} = \sqrt{-\frac{1}{2k}(e^{-10k} - 1)}$$

To calculate $\|f(x)\|_\infty = \sup_{x \in \bar{D}} |f(x)|$, where $\bar{D}$ is the closure of D, we can graph the function $|e^{-kx}|$ and visually determine it.

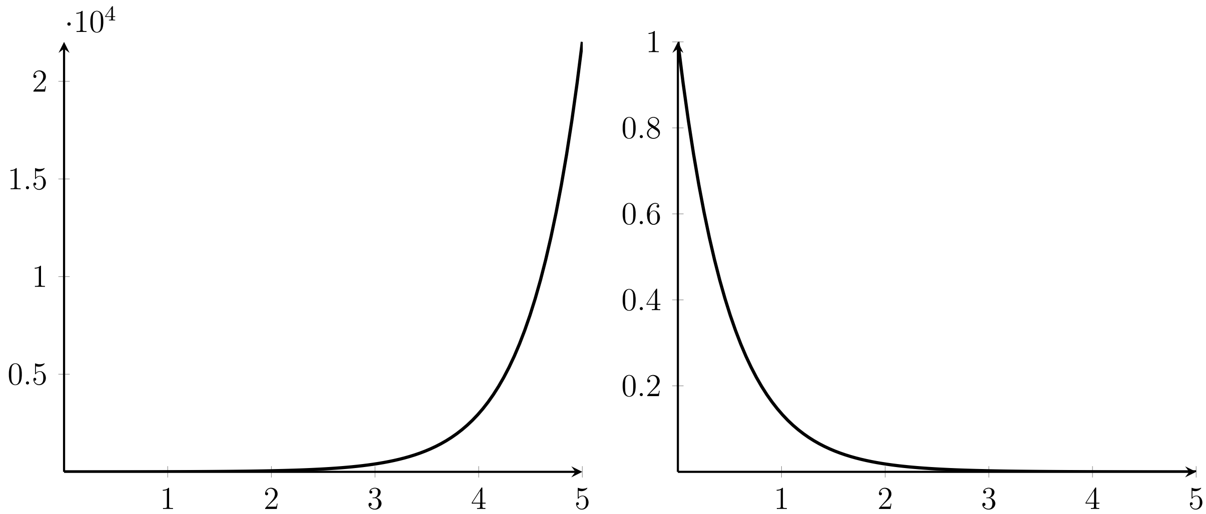

Figure 3: Plot of $|f(x)| = |e^{kx}|$ (left) and $|f(x)| = |e^{-kx}|$ (right) for $k = 2$

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For the graphs in figure 3, we have set the value $k = 2$, but the graphs describe the general behavior for $k > 0$ (right) and $k < 0$ (left). So we have that

$$\|f(x)\|_\infty = \begin{cases} 1, & k < 0 \\ e^{5k}, & k > 0 \\ 1, & k = 0 \end{cases}$$

This can also be check analytically by showing $f'(x) = -ke^{-kx}$ and is therefore strictly increasing for $k < 0$, strictly decreasing if $k > 0$ and constant if $k = 0$. Therefore we pick the left and right bounds for $\|f(x)\|_\infty$.

Problem 3

Suppose $f(x) : (a, b) \to \mathbb{R}$, where (a, b) is a finite interval. Suppose that $f_n(x)$ converges uniformly to $f(x)$ on the interval (a, b) as $n \to \infty$.

(a) Show that $f_n(x)$ must also converge pointwise to $f(x)$ on the interval (a, b).

Pointwise convergence follows trivially from uniform convergence. To show pointwise convergence, it must be shown that given any $\epsilon > 0$ we can find some N so that for all $n > N$ we have $|f_n(x) - f(x)| < \epsilon$ for all x. However, if we are given uniform convergence we are already given such an N no matter the choice of x. Thus uniform convergence implies pointwise convergence.

(b) Show that $f_n(x)$ must also converge to $f(x)$ in the L^2 norm.

Since we have uniform convergence we know that $\sup_{x\in(a,b)} |f_n(x) - f(x)| \to 0$, then

$$\|f_n(x) - f(x)\|_2 = \left(\int_a^b |f_n x - f(x)|^2 \, dx \right)^{1/2} \leq C \sup_{x\in(a,b)} |f_n(x) - f(x)| \to 0$$

where $C = \sqrt{b-a}$. Therefore we have uniform convergence implies L^2 convergence.

(c) Now consider an unbounded interval. Does uniform convergence still imply pointwise convergence? Does it imply L^2 convergence? If not, give a counterexample.

By definition of uniform convergence, we take an arbitrary set D for all x to converge and therefore has no dependence on whether it is bounded or not. Therefore uniform convergence still implies pointwise if D is unbounded..