MAP5345: Partial Differential Equations I

Homework 2

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Problem 1

We have the following Taylor Series expansions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad cos(x) = (-1)^n \frac{x^{2n}}{(2n)!}, \quad sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Letting $x = i\theta$ we get

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\
&= \sum_{m=4n}^{\infty} \frac{\theta^m}{m!} - \sum_{m=4n+2}^{\infty} \frac{\theta^m}{m!} + i\left(\sum_{m=4n+1}^{\infty} \frac{\theta^m}{m!} - \sum_{m=4n+3}^{\infty} \frac{\theta^m}{m!} \right) \\
&= \sum_{m=2n}^{\infty} (-1)^{m/2} \frac{\theta^m}{m!} + i\left(\sum_{m=2n+1}^{\infty} (-1)^{(m-1)/2} \frac{\theta^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i\left(\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \right) \\
&= cos(\theta) + isin(\theta)
\end{aligned}$$

$\square$

Problem 2

Letting D be the unit cube $[0,1]^3$ and $F = (x, y, z)$ we get

$$\begin{aligned}
\int_D \nabla \cdot \vec{F} dV &= \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \big(\partial_x(x) + \partial_y(y) + \partial_z(z)\big)\ dxdydz \\
&= \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 3\ dxdydz \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\int_{\partial D} \vec{F} \cdot \hat{n}\ dS &= \int_0^1 \int_0^1 (0, y, z) \cdot (-1, 0, 0)\ dydz + \int_0^1 \int_0^1 (1, y, z) \cdot (1, 0, 0)\ dydz \\
&+ \int_0^1 \int_0^1 (x, y, z) \cdot (0, -1, 0)\ dxdz + \int_0^1 \int_0^1 (x, y, z) \cdot (0, 1, 0)\ dxdz \\
&+ \int_0^1 \int_0^1 (x, y, z) \cdot (0, 0, -1)\ dxdy + \int_0^1 \int_0^1 (x, y, z) \cdot (0, 0, 1)\ dxdy \\
&= 3\bigg(\int_0^1 \int_0^1 0\ dS + \int_0^1 \int_0^1 1\ dS \bigg) \\
&= 3
\end{aligned}$$

Now letting D be the unit cube S^2 and using the coordinate transformation

$$x = rsin(\theta)cos(\phi), \quad y = rsin(\theta)sin(\phi), \quad z = rcos(\theta)$$

we get the following

$$\begin{aligned}
\int_D \nabla \cdot \vec{F} dV &= \int_D \big(\partial_x(x) + \partial_y(y) + \partial_z(z)\big)\ dV \\
&= 3 \int_D dV \\
&= 4\pi
\end{aligned}$$

$$\begin{aligned}
\int_{\partial D} \vec{F} \cdot \hat{n}\ dS &= \int_{\partial D} \big((x, y, z) \cdot (sin(\theta)cos(\phi), sin(\theta)sin(\phi), cos(\theta))\big) \\
&= \int_{\partial D} \big((sin(\theta)cos(\phi), sin(\theta)sin(\phi), cos(\theta)) \cdot (sin(\theta)cos(\phi), sin(\theta)sin(\phi), cos(\theta))\big)\ dS \\
&= \int_{\partial D} |r|\ dS \\
&= 4\pi
\end{aligned}$$

where we have used the fact that the volume of a sphere is $\frac{4}{3}\pi r^3$ and the area is $4\pi r^2$. $\square$

Problem 3

(a) We have the IBVP PDE

$$\partial_t u = k\partial_{xx}u, \quad x \in (0, L), t \geq 0$$
$$u(x, 0) = u_0(x) = 2sin(\pi x/L) - 0.5sin(2\pi x/L) + 0.2sin(3\pi x/L)$$
$$u(0, t) = u(L, t) = 0$$

Assuming it has a solution of the form $u(x, t) = X(x)T(t)$ and plugging this into the PDE we get

$$X(x)T'(t) = kX''(x)T(t)$$

The only way that the above is possible is when both the RHS and LHS equal some constant $-\lambda$. Using this fact and simplifying we are left with

$$\frac{1}{kT(t)}\frac{\partial T}{\partial t} = \frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} = -\lambda$$

These differential equation $T(t)$ has the solution

$$T(t) = ce^{-\lambda kt}$$

while $X(x)$ can take on 3 different solutions

$\underline{\text{Case 1} : \lambda > 0}$
$$X(x) = c_1cos(\sqrt{\lambda}x) + c_2sin(\sqrt{\lambda}x)$$
$$X(0) = 0 \Rightarrow c_1 = 0$$
$$X(L) = 0 \Rightarrow \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3\ldots$$
$$\Rightarrow X(x) = c_2sin(n\pi x/L)$$
$\underline{\text{Case 2} : \lambda = 0}$
$$X(x) = c_1x + c_2$$
$$X(0) = 0 \Rightarrow c_2 = 0$$
$$X(L) = 0 \Rightarrow c_1 = 0$$
$$\Rightarrow X(x) = 0$$
$\underline{\text{Case 3} : \lambda < 0}$
$$X(x) = c_1cosh(\sqrt{-\lambda}x) + c_2sinh(\sqrt{-\lambda}x)$$
$$X(0) = 0 \Rightarrow c_1 = 0$$
$$X(L) = 0 \Rightarrow c_2 = 0$$
$$\Rightarrow X(x) = 0$$

Case 1 is the only one that does not return the trivial solution. Therefore our product solution is

$$u_n(x, t) = B_nsin(\frac{n\pi x}{L})e^{-\frac{n^2\pi^2}{L^2}kt}$$

By the Principle of Superposition we obtain the general solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n sin(\frac{n\pi x}{L})e^{-\frac{n^2\pi^2}{L^2}kt} \tag{1}$$

To satisfy $u_0(x)$ we have that $B_1 = 2$, $B_2 = -0.5$, $B_3 = 0.2$, and $B_n = 0$ otherwise. Putting all this together, we get that the solution to our IBVP is

$$u(x,t) = 2sin(\frac{\pi x}{L})e^{-\frac{\pi^2}{L^2}kt} - 0.5sin(\frac{2\pi x}{L})e^{-\frac{4\pi^2}{L^2}kt} + 0.2sin(\frac{3\pi x}{L})e^{-\frac{9\pi^2}{L^2}kt} \tag{2}$$

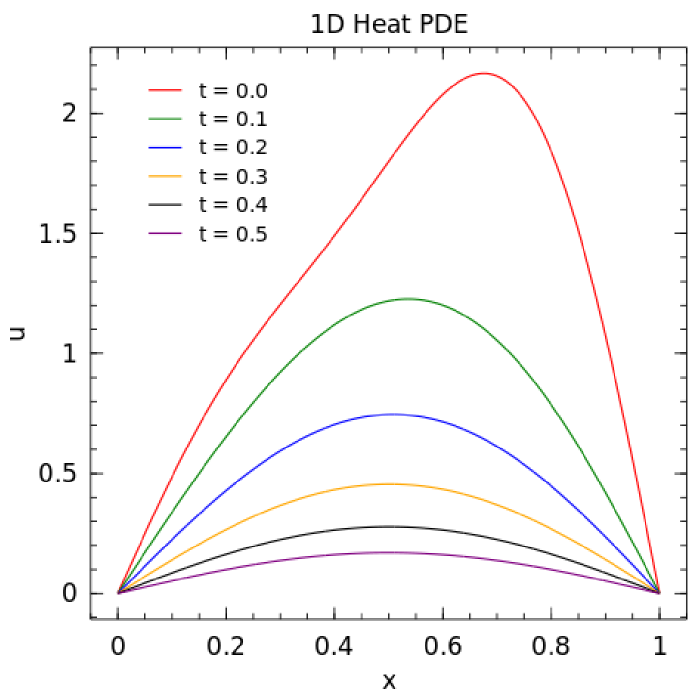


Figure 1: Numerical representation of solution (1)

Figure 1 shows plots the solution (1) at different time values. As we can see $u(x,t) \to 0$ as $t \to \infty$ as we expected. This is because heats dissipates on any object and reaches the equilibrium temperature zero. We can also see that $u(x,t)$ tends to a damped sin graph which is due to heat distribution becoming uniform over our domainx.

(b) Now we are given parabolic initial condition

$$u_0(x) = x(L-x)$$

which we can use to find our coefficient B_n. As we do later in problem 4, we take the inner product of both sides with respect to the eigenfunction $X_n = sin(\frac{n\pi x}{L})$. Doing so yields

$$\begin{aligned}
B_n &= \frac{2}{L} < u_0(x), X_n > \\
&= \frac{2}{L}\int_0^L x(L-x)sin(\frac{n\pi x}{L})dx \\
&= \frac{2}{L}\int_0^L xLsin(\frac{n\pi x}{L})dx - \frac{2}{L}\int_0^L x^2 sin(\frac{n\pi x}{L})dx \\
&= 2\Big(-\frac{xL}{n\pi}cos(\frac{n\pi x}{L})\Big|_0^L + \frac{L}{n\pi}\int_0^L cos(\frac{n\pi x}{L})dx\Big) - \frac{2}{L}\Big(-x^2\frac{L}{n\pi}cos(\frac{n\pi x}{L})\Big|_0^L - \frac{2L}{n\pi}\int_0^L xcos(\frac{n\pi x}{L})dx\Big) \\
&= 2\Big(\frac{2L^2}{n\pi} + \frac{L^2}{n\pi}sin(\frac{n\pi x}{L})\Big|_0^L\Big) - \frac{2}{L}\left(\frac{2L^3}{n\pi} - \frac{2L}{n\pi}\left(\frac{xL}{n\pi}sin(\frac{n\pi x}{L})\Big|_0^L - \frac{L}{n\pi}\int_0^L sin(\frac{n\pi x}{L})dx\right)\right) \\
&= \frac{4L^2}{n\pi} - \frac{2}{L}\left(\frac{2L^3}{n\pi} - \frac{2L}{n\pi}\left(-\frac{L^2}{n^2\pi^2}cos(\frac{n\pi x}{L})\Big|_0^L\right)\right) \\
&= \frac{4L^2}{n\pi} - \frac{2}{L}\left(\frac{2L^3}{n\pi} - \frac{2L}{n\pi}\left(\frac{2L^2}{n^2\pi^2}\right)\right) \\
&= \frac{4L^2}{n\pi} - \frac{2}{L}\left(\frac{2L^3}{n\pi} - \frac{4L^3}{n^3\pi^3}\right) \\
&= \frac{8L^2}{n^3\pi^3}
\end{aligned}$$

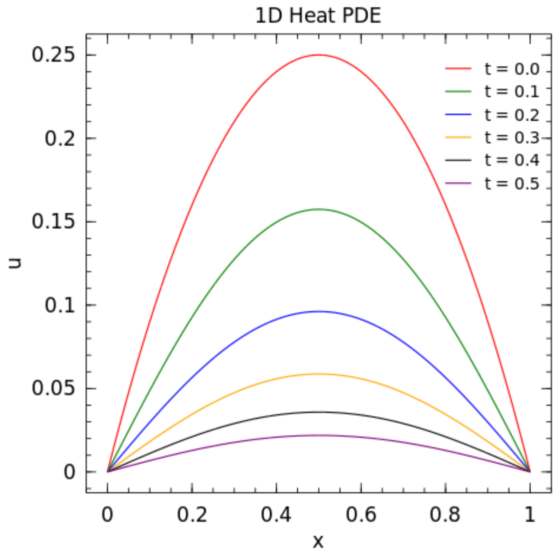
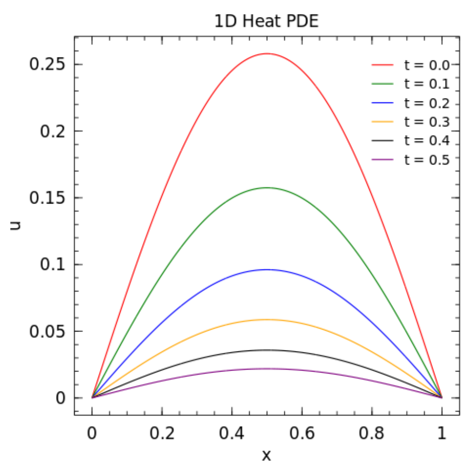
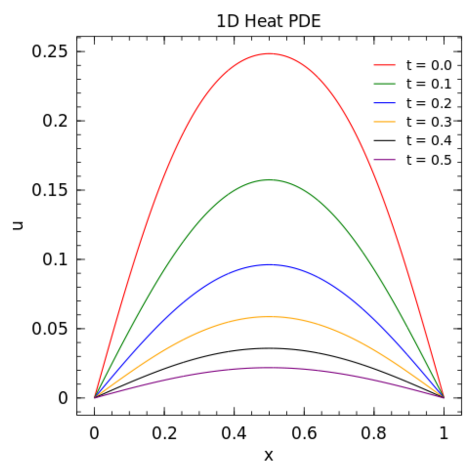


Figure 2: Numerical representation of 1D Heat Equation with parabolic initial condition

The above graph had a truncation at $N = 1000$ However, truncation values $N \geq 3$ have little visual difference from figure 2.



Truncation N = 1



Truncation N = 2

We see that we can essentially truncate at N = 3 because $|B_n e^{(\frac{-kn^2\pi^2}{L^2})}| \approx 10^{-20}$ for $n = 3$ (only odd n contribute). This means that there are very small contributions to $u(x, t)$ for any $N \geq 2$.

Problem 4

(a) Assume that $u(x,t)$ and $v(x,t)$ solve the wave equation. Now let $w(x,t) = \alpha u(x,t) + \beta v(x,t)$ and take partial derivatives

$$w_t = \alpha u_t + \beta v_t, \qquad w_{tt} = \alpha u_{tt} + \beta v_{tt}$$
$$w_x = \alpha u_x + \beta v_x, \qquad w_{xx} = \alpha u_{xx} + \beta v_{xx}$$

Plugging into the wave equation we get

$$\begin{aligned}
&w_{tt} - c^2 w_{xx} = 0 \Rightarrow \\
&(\alpha u_{tt} + \beta v_{tt}) - c^2(\alpha u_{xx} + \beta v_{xx}) \Rightarrow \\
&(\alpha u_{tt} + c^2 \alpha u_{xx}) + (\beta v_{tt} - c^2 \beta v_{xx}) \Rightarrow \\
&\alpha(u_{tt} + c^2 u_{xx}) + \beta(v_{tt} - c^2 v_{xx}) = \alpha(0) + \beta(0) = 0
\end{aligned}$$

$\square$

(b) The wave equation is given by

$$\begin{aligned}
&\partial_{tt} u = c^2 \partial_{xx} u, \quad x \in (0, L), t \geq 0 \\
&u(0,t) = u(L,t) = 0 \\
&u(x,0) = u_0(x) \\
&u_t(x,0) = \dot{u}_0(x)
\end{aligned}$$

Now we assume that the solution has the form $u(x,t) = X(x)T(t)$ and plug back into the PDE and get

$$\frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T(t)} \frac{\partial^2 T}{\partial t^2} = \lambda$$

for some constant λ. The solution for $X(x)$ is the same eigenvalue problem solved in problem 3. Therefore we have the same countable set of solutions

$$X_n(x) = csin(\frac{n\pi x}{L}), \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3 \ldots$$

The time function $T(t)$ is solved similarly in problem 3 where we consider the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. The only difference is that we do not have explicit initial conditions so we are left with

$$T_n(t) = c_1 cos(\frac{cn\pi t}{L}) + c_2 sin(\frac{cn\pi t}{L}), \quad \lambda_n = \frac{c^2 n^2 \pi^2}{L^2}, \quad n = 1, 2, 3 \ldots$$

Putting all this together we get

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n cos(\frac{cn\pi t}{L}) + B_n sin(\frac{cn\pi t}{L}) \right) sin(\frac{n\pi x}{L})$$

Using the two initial conditions we get

$$\begin{aligned}
&u_0(x) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{L}) \\
&\dot{u}_0(x) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} B_n sin(\frac{n\pi x}{L})
\end{aligned}$$

To determine A_n and B_n we take inner products with respect to the eigenfunction $X_m = sin(m\pi x/L)$.

$$\begin{aligned}
\int_0^L u_0(x) X_m dx &= \int_0^L \sum_{n=1}^{\infty} A_n X_n X_m \\
&= \sum_{n=1}^{\infty} A_n \int_0^L X_n X_m dx
\end{aligned}$$

$$\begin{aligned}
\int_0^L \dot{u}_0(x) X_m dx &= \int_0^L \sum_{n=1}^{\infty} \frac{cn\pi}{L} B_n X_n X_m \\
&= \sum_{n=1}^{\infty} \frac{cn\pi}{L} B_n \int_0^L X_n X_m dx
\end{aligned}$$

To simplify the expressions we determine the inner product $< X_n, X_m >$ over the L^2 norm

$$\begin{aligned}
< X_n, X_m > &= \int_0^L X_n X_m dx \\
&= \int_0^L \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} \frac{e^{im\pi x/L} - e^{-im\pi x/L}}{2i} dx \\
&= -\frac{1}{4} \int_0^L e^{i(n+m)\pi x/L} + e^{-i(n+m)\pi x/L} - e^{i(m-n)\pi x/L} - e^{i(n-m)\pi x/L} dx \\
&= -\frac{L}{2(n+m)\pi} \int_0^L \frac{e^{i(n+m)\pi x/L} + e^{-i(n+m)\pi x/L}}{2} dx + \frac{L}{2(m-n)\pi} \int_0^L \frac{e^{i(m-n)\pi x/L} + e^{i(n-m)\pi x/L}}{2} dx \\
&= \frac{L}{2\pi} \left(\frac{sin(\frac{(m-n)\pi x}{L})}{m-n} - \frac{sin(\frac{(m+n)\pi x}{L})}{m+n} \right) = 0, \quad m \neq n \\
< X_n, X_n > &= \int_0^L X_n^2 dx \\
&= \int_0^L \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} dx \\
&= -\frac{1}{4} \int_0^L -2 + e^{2in\pi x/L} + e^{-2in\pi x/L} dx \\
&= \frac{x}{2}\Big|_0^L - \frac{L}{2n\pi} \int_0^L \frac{e^{2in\pi x/L} + e^{-2in\pi x/L}}{2} dx \\
&= \frac{L}{2} - \frac{L}{2n\pi} sin(\frac{n\pi x}{L})\Big|_0^L = \frac{L}{2}, \quad m = n
\end{aligned}$$

From this we can conclude that

$$A_n = \frac{< u_0(x), X_n >}{< X_n, X_n >} = \frac{2}{L} < u_0(x), X_n > \qquad B_n = \frac{L}{cn\pi}\frac{< \dot{u}_0(x), X_n >}{< X_n, X_n >} = \frac{2}{cn\pi} < \dot{u}_0(x), X_n >$$

Giving us the desired general solution

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n cos(\frac{cn\pi t}{L}) + B_n sin(\frac{cn\pi t}{L})\right) sin(\frac{n\pi x}{L}), \tag{3}$$

$$A_n = \frac{2}{L} < u_0(x), X_n >, \quad B_n = \frac{2}{cn\pi} < \dot{u}_0(x), X_n > \tag{4}$$

(c) We are given initial conditions

$$u(x,0) = x(L-x)$$
$$u_t(x,0) = 0$$

and from this we can find the solution to (3) by determining A_n and B_n by plugging our initial conditions into (4). Doing so gets us

$$\begin{aligned} A_n &= \frac{2}{L} < u_0(x), X_n > \\ &= \int_0^L x(L-x)sin(\frac{n\pi x}{L})dx \\ &= \frac{8L^2}{n^3\pi^3}, \quad \text{(see problem 3)} \\ B_n &= \frac{2}{cn\pi} < \dot{u}_0(x), X_n > \\ &= \int_0^L 0dx = 0 \end{aligned}$$

Plugging back into the general solution (3) we get the desired result

$$u(x,t) = \sum_{n=0}^{\infty} A_n cos(\frac{cn\pi t}{L}) sin(\frac{n\pi x}{L}), \quad A_n = \frac{8L^2}{n^3\pi^3}$$

(d) To plot the solution to the wave equation for $L = 5$ and $c = 100$, I used the code Dr. Moore has on his personal website. I did this because my personal code was not plotting the solution correctly, but the main idea behind both coding solutions are the same.

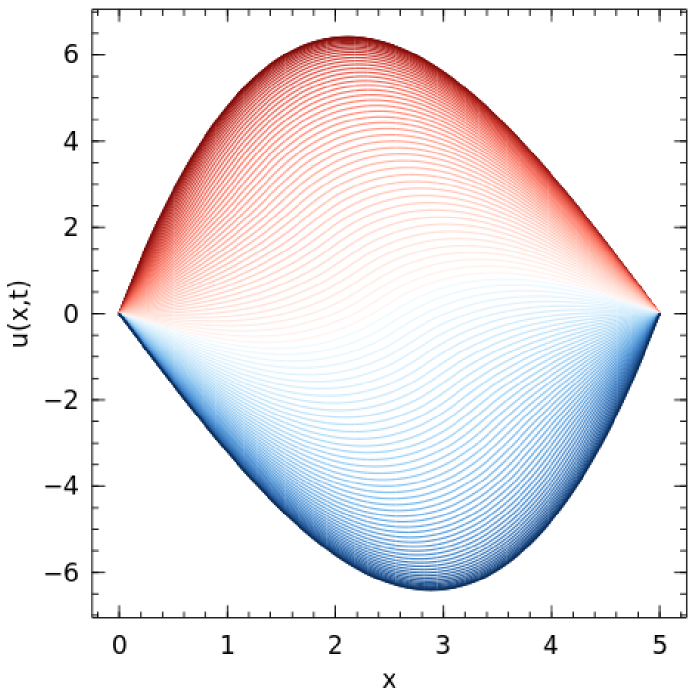


Figure 3: Numerical representation of 1D Wave Equation