Homework 9

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MAP5345: Partial Differential Equations I

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Problem 1

For each function f(x) below, defined on the interval $x \in [-\pi, \pi]$, consider the periodic extension $f_{ext}(x) : \mathbb{R} \to \mathbb{R}$ In each case, answer the following questions:
i) Is $f_{ext}(x)$ continuous? Is it piecewise continuous?
ii) Is $f'_{ext}(x)$ continuous? Is it piecewise continuous?
iii) Is $f_{ext}(x)$ a C^1 function?
iv) By the convergence theorems we have covered, are you guaranteed pointwise convergence of the Fourier series? Are you guaranteed uniform convergence? Are you guaranteed convergence in L^2?
v) Does the truncated Fourier series exhibit Gibbs phenomenon? If so, at what x values?

a) $f(x) = |x|$

We have $f_{ext} \in C^0, f'_{ext} \notin C^0$ but is piecewise continuous. From this $f_{ext} \notin C^1$. We are guaranteed pointwise but not uniform convergence. The truncated Fourier series does not exhibit Gibb's phenomenon.

b) $f(x) = x$

We have f_{ext} is piecewise continuous, $f'_{ext} \in C^0$. From this $f_{ext} \notin C^1$. We are guaranteed pointwise convergence but not uniform convergence. The truncated Fourier series does exhibit gibbs phenomenon at the points $n\pi$ for $n \in \mathbb{Z}$.

c) $f(x) = \pi^2 - x^2$

We have $f_{ext} \in C^0, f'_{ext}$ is piecewise continuous. From this we have $f_{ext} \notin C^1$. We are guaranteed pointwise convergence but not uniform convergence. The truncated Fourier series does nt exhibit Gibb's phenomenon.

d) $f(x) = \sqrt{\pi^2 - x^2}$

We have $f_{ext} \in C^0, f'_{ext}$ is not even piecewise continuous since it has singularities at the boundaries. From this we have $f_ext \notin C^1$. We are not guaranteed pointwise or uniform convergence. The truncated Fourier series does not exhibit Gibb's phenomenon.

e) $f(x) = x(\pi^2 - x^2)$

We have $f_{ext} \in C^1, f'_{ext} \in C^0$.From this we have $f_{ext} \in C^1$. We are guaranteed pointwise and uniform convergence. The truncated Fourier series does not exhibit Gibb's phenomenon.

f) $f(x) = (\pi^2 - x^2)^2$

We have $f_{ext} \in C^2, f'_{ext} \in C^1$. From this we have $f_{ext} \in C^2$. We are guaranteed pointwise and uniform convergence. The truncated Fourier series does not exhibit Gibb's phenomenon.

Problem 2

Finish the proof of the 'decay-rate theorem' from class, i.e., if $f_{ext} \in C^n$ then a bound on the decay-rate of the Fourier coefficients is given by

$$|c_k| = \frac{M}{|k|^n} \text{ for all } k \tag{1}$$

To prove this, we shall use induction. Let $f_{ext} \in C^1$ then we have

$$\mathcal{F}(f') = \int_{-L}^{L} f'_{ext}(x)e^{ikx}\,dx = ike^{ikx}f_{ext}(x)\Big|_{-L}^{L} - ik\int_{-L}^{L} f_{ext}(x)e^{ikx}\,dx = -ik\mathcal{F}(f)$$

where $\mathcal{F}(f)$ is the Fourier transform of f. From this we get

$$|c_k| = \frac{|c'_k|}{|k|}$$

where we use the fact that if two Fourier series are the same, their coefficients are the same (uniqueness). Now assume $\mathcal{F}(f^{(n)}) = i^n k^n \mathcal{F}(f)$ and let $f_{ext}(x) \in C^{n+1}$ then

$$\mathcal{F}(f^{(n+1)}) = \int_{-L}^{L} f_{ext}^{(n+1)}(x)e^{ikx}\,dx = -ik\int_{L}^{L} f_{ext}^{(n)}(x)e^{ikx}\,dx = -ik\mathcal{F}(f_k^{(n)}) = -i^{n+1}k^{n+1}\mathcal{F}(f)$$

where we utilized integration by parts and the first term going to zero since $f_{ext}^{(n+1)}(x)$ must agree on the boundaries. We are now left with

$$|c_k| = \frac{|c_k^{(n+1)}|}{|k|^{n+1}}$$

where $c_k^{(n+1)}$ can be expressed as

$$\frac{1}{2L}\int_{-L}^{L} f_{ext}^{(n+1)}(x)e^{inx}\,dx, \quad f_{ext}^{(n+1)}(x) \in C^0.$$

Since $f_{ext}^{(n+1)}$ is closed and bounded we can bound it above by some constant M and therefore bound the above integral by M times the length of the integral yielding

$$|c_k| = \frac{|c_k^{(n+1)}|}{|k|^{n+1}} \le \frac{M}{|k|^{n+1}}$$

Problem 3

Consider the same 6 functions from problem 1. In each case, determine the largest integer n, such that $f_{ext} \in C^n$. Based on this information, what can you say about the decay rate of the Fourier coefficients?

From problem 1 we have

$$
\begin{aligned}
f_{ext}(x) &= |x| \in C^0 \\
f_{ext}(x) &= x \text{ is piecewise continuous} \\
f_{ext}(x) &= \pi^2 - x^2 \in C^0 \\
f_{ext}(x) &= \sqrt{\pi^2 - x^2} \in C^0 \\
f_{ext}(x) &= x(\pi^2 - x^2) \in C^1 \\
f_{ext}(x) &= (\pi^2 - x^2)^2 \in C^2
\end{aligned}
$$

where these are the largest n. Using these values for n we can bound the Fourier coefficients using

$$f_{ext} \in C^n \Rightarrow |c_k| \leq \frac{M}{k^n} \text{ for some } M \in \mathbb{R}$$

where we have determined the largest n for our functions. Therefore

$$
\begin{aligned}
c_k &\leq M \text{ for } f(x) = x \\
c_k &\leq M \text{ for } f(x) = \pi^2 - x^2 \\
c_k &\leq M \text{ for } f(x) = \sqrt{\pi^2 - x^2} \\
c_k &\leq \frac{M}{k} \text{ for } x(\pi^2 - x^2) \\
c_k &\leq \frac{M}{k^2} \text{ for } (\pi^2 - x^2)^2
\end{aligned}
$$

Problem 4

Suppose we have a function $f(x)$ on the interval $0 < x < L$ that is not even continuous, but it has a finite L^2 norm. Note that we are guaranteed that the Fourier series of $f(x)$ converges in L^2. Someone comes into the room and says they have calculated the Fourier coefficients c_n and found that

$$|c_n| = 1/\sqrt{n}\,\text{for } n > 1 \tag{2}$$

Without checking their tedious calculation, can you say why this result must be wrong?

From previous homeworks we know that the coefficients will be $a_n = \frac{2}{\sqrt{n}}$ and $b_n = 0$. Applying Perseval's equality yields

$$\|f\|_2^2 = \sum_{n=1}^{\infty} A_n^2 \|X_n\|_2^2 = \sum_{n=1}^{\infty} \frac{4}{n} \|cos(\frac{n\pi x}{L})\|_2^2$$

where $\|cos(\frac{n\pi x}{L})\|_2^2 = \langle cos(\frac{n\pi x}{L}), cos(\frac{n\pi x}{L})\rangle = L$. Plugging this back in we get

$$\|f\|_2^2 = \sum_{n=1}^{\infty} \frac{4L}{n}.$$

We know the left side of Perseval's equality converges but the right side diverges by the $p-$series test. Therefore $c_n = 1/\sqrt{n}$ can not be the correct coefficients.

Problem 5

Consider the heat conduction in a rod on the interval $x \in [0, L]$ with vanishing Dirichlet boundary conditions. As done in class, consider an initial condition $u_0(x)$ that is 'sufficiently smooth', and consider its odd reflection $f(x)$ defined on the interval $[-L, L]$.

a) Prove that f_{ext} is C^1 and that f''_{ext} is piecewise continuous.

Applying the odd reflection we get

$$f_{ext}(x) = \begin{cases} u_0(x) & x \in (0, L) \\ -u_0(-x) & x \in (-L, 0) \\ 0 & x = n\pi \end{cases}$$

Taking the derivative we get

$$f'_{ext}(x) = \begin{cases} u'_0(x) & x \in (0 + 2Ln, L + 2Ln) \\ u'_0(-x) & x \in (-L + 2Ln, 0 + 2Ln) \end{cases}$$

We just need to verify that the derivative on the boundaries match. To verify this we take the limit

$$\begin{aligned} f'_{ext}(0) &= \lim_{x \to 0^+} f'_{ext}(x) = u'_0(0) = \lim_{x \to 0^-} f'_{ext}(x) = f'_{ext}(0) \\ f'_{ext}(L) &= \lim_{x \to L^-} f'_{ext}(x) = u'(L) = \lim_{x \to -L^+} f'_{ext}(-L) = f'_{ext}(-L) \end{aligned}$$

Since the derivative match at the boundaries and $f'_{ext}(x)$ is continuous we have that $f_{ext}(x) \in C^1$. Taking the second derivative we get

$$f''_{ext}(x) = \begin{cases} u''(x) & x \in (0, L) \\ -u''_0(x) & x \in (-L, 0) \end{cases}$$

Taking the limit again to determine derivative values at the boundaries we get

$$\begin{aligned} f''_{ext}(0) &= \lim_{x \to 0^+} f''_{ext}(x) = u''_0(0) \neq -u''_0(0) = \lim_{x \to 0^-} f''_{ext}(x) = f''_{ext}(0) \\ f''_{ext}(L) &= \lim_{x \to L^-} f''_{ext}(x) = u''_0(L) \neq -u''_0(-L) = \lim_{x \to -L^+} f''_{ext}(x) = f''_{ext}(L) \end{aligned}$$

From the above we can see that $f''_{ext}(x)$ is piecewise continuous.

b) How smooth precisely does $u_0(x)$ have to be for this all to work out?

The function $u_0(x)$ must be at least C^2 since $u''_0(x)$ is continuous on $(0, L)$ (f''_{ext} piecewise continuous).

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Problem 6

Consider the wave equation with vanishing Neumann boundary conditions

$$u_{tt} - c^2 u_{xx} = u_0(x) \quad \text{for } 0 < x < L,\, t > 0 \tag{3}$$
$$u_x(0,t) = u_x(L,t) = 0 \quad \text{for } t > 0 \tag{4}$$

and initial conditions

$$u(x,0) = u_0(x) \quad \text{for } 0 < x < L \tag{5}$$
$$u_t(x,t) = \dot{u}_0(x) \quad \text{for } 0 < x < L \tag{6}$$

a) By using reflections and the Fourier convergence theorems, show that the eigenfunctions from separation of variables are complete in the space of all 'sufficiently smooth' initial conditions.

For this problem we essentially need to prove that the eigenfunctions span the space of all 'sufficiently smooth' initial conditions. We can do this by proving $f_{ext}(x)$ (eigenfunction extension) is at least pointwise convergent. The spatial eigenfunction we get for this problem results from the eigenvalue problem

$$X'' + \lambda X = 0$$
$$u_x(0,t) = u_x(L,t) = 0$$

resulting in $X_n = cos(\frac{n\pi x}{L})$. Applying even extension to our eigenfunction we get

$$f_{ext}(x) = \begin{cases} cos(\frac{n\pi x}{L}) & x \in (0,L) \\ cos(-\frac{n\pi x}{L}) & x \in (-L,0) \end{cases}$$

We can quickly check that $f_{ext}(x)$ is continuous

$$f_{ext}(0) = \lim_{x\to 0^+} f_{ext}(x) = cos(0) = \lim_{x\to 0^-} f_{ext}(x) = f_{ext}(0)$$
$$f_{ext}(-L) = \lim_{x\to -L^+} f_{ext}(x) = cos(n\pi) = \lim_{x\to L^-} f(x) = f_{ext}(L)$$

which confirms f_{ext} is continuous. We also have that $f'_{ext}(x)$ is continuous by the Neumann boundary conditions. Checking $f''_{ext}(x)$ we get

$$f''_{ext}(x) = \begin{cases} -\frac{L^2}{n^2\pi^2} cos(\frac{n\pi x}{L}) & x \in (0,L) \\ -\frac{L^2}{n^2\pi^2} cos(-\frac{n\pi x}{L}) & x \in (-L,0) \end{cases}$$

from which we can verify boundary values match

$$f''_{ext}(0) = \lim_{x\to 0^+} f''_{ext}(x) = -\frac{L^2}{n^2\pi^2} cos(0) = \lim_{x\to 0^-} f''_{ext}(x) = f''_{ext}(0)$$
$$f''_{ext}(-L) = \lim_{x\to -L^+} f''(x) = -\frac{L^2}{n^2\pi^2} cos(n\pi) = \lim_{x\to xL^-} f''(x) = f''_{ext}(L)$$

Therefore we have that $f_{ext}(x), f'_{ext}(x)$, and $f''_{ext}(x)$ and thus

$$u_0(x) = \sum_{n=0}^{\infty} A_n X_n \tag{7}$$

is uniformly convergent. This implies our eigenfunctions span the space of our 'sufficiently smooth' initial conditions.

b) If $u_0(x)$ and $\dot{u}_0(x)$ are both C^∞, can you say how smooth the solution $u(x,t)$ will be at later times? What if $u_0(x)$ and $\dot{u}_0(x)$ are only C^n for some $n > 0$?

Since out temporal eigenfunctions T_n are C^∞ the smoothness of our solution $u(x,t)$ is the smoothness of $u_0(x)$ and $\dot{u}_0(x)$. This is because the general solution $u(x,t)$ has the form

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x)$$

and its smoothness is therefore limited by the smoothness of $u_0(x)$ and $\dot{u}_0(x)$.