Homework 2

David Miller
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Problem 1

Find the fixed points and classify them.

$$\dot{x} = y - x^3 + x, \quad \dot{y} = -x - y^3 + y \tag{1}$$

The point (0,0) is an obvious fixed point. Now lets check for other fixed points:

$$\begin{aligned} x = 0 = y - x^3 + x &\Rightarrow y = x^3 - x \\ y = 0 = -x - y^3 + y &\Rightarrow 0 - x - (x^3 - x)^3 + x^3 - x \\ &\Rightarrow 0 = -1 - x^2(x+1)^3(x-1)^3 + (x+1)(x-1) \end{aligned}$$

From this we can see (from plugging into Wolfram|Alpha) that there are no other fixed points. Therefore letting $f = y - x^3 + x$ and $g = -x - y^3 + y$ we get

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -3x^2 + 1 & 1 \\ -1 & -3y^2 + 1 \end{pmatrix} \xrightarrow{\text{Plugging in (0,0)}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \xrightarrow{\text{Compute eigenvalue } \lambda} \begin{pmatrix} 1 - \lambda & 1 \\ -1 & 1 - \lambda \end{pmatrix}$$

From this we have that $(1 - \lambda)^2 = -1$ and therefore $\lambda = 1 \pm i$. Therefore the fixed point is an unstable spiral.

Problem 2

A particle is confined on the half line $x \geq 0$ and moves according to the following equation of motion: $\dot{x} = -x^\alpha$ where α is a real constant.

(a) Find all the values of α for which the origin $(x = 0)$ is a stable fixed point.

If we want the origin to be a stable fixed point then we need two conditions

- If $x > 0$ then $\dot{x}$ must be less than zero to force it back to the origin,
- If $x < 0$ then $\dot{x}$ must be greater than zero to force it back to the origin.

When $x > 0$ we get that α can be any positive real number. If it is negative than the problem does ot admit a fixed point. When $x < 0$ we get that α must be some rational $\frac{p}{q}$ where p is an odd integer. Putting this together we get that α can take on any positive number on the positive half-line.

(b) Consider the values of α found in part (a). Can the particle ever reach the origin in finite time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$ as a function of α.

$$\begin{aligned}
\frac{dx}{dt} &= -x^\alpha \\
-\int \frac{dx}{x^\alpha} &= \int dt \\
-\frac{x^{-\alpha+1}}{-\alpha+1} &= t + c
\end{aligned}$$

for some $c \in \mathbb{R}$. The time it takes to get from $x = 1$ to $x = 0$ is

$$t_0 - t_1 = \frac{1}{1-\alpha}$$

Therefore we do have that the particle can reach the origin in finite time if we allow $\alpha \in (0, 1)$.

Problem 3

Consider the equation

$$\dot{x} = cx + x^3 \tag{2}$$

where $x(t) \in \mathbb{R}$ and $c > 0$ is real and fixed. Prove rigorously that $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x_0 \neq 0$.

$$\begin{aligned}
\frac{dx}{dt} &= cx + x^3 \\
\int \frac{dx}{x(c+x^2)} &= \int dt \\
\int \frac{dx}{cx} - \int \frac{xdx}{c(c+x^2)} &= t + a \\
\frac{ln(x)}{c} - \frac{ln(c+x^2)}{2c} &= t + a \\
\frac{1}{c} ln\left(\frac{x}{c+x^2}\right) &= t + a
\end{aligned}$$

for some $a \in \mathbb{R}$. We can not have that $x_0 = 0$ since we can not evaluate $ln(0)$. As $x \to \pm\infty$ we have that $ln(\frac{x}{c+x^2}) \to 0$. Therefore there is some finite t^* such that $x(t^*) \to \pm\infty$.

Problem 4

Find and classify all equilibrium points and sketch the local phase diagrams (find all phase paths whenever possible):

(a) $\dot{x} = sin(y), \quad \dot{y} = -sin(x)$

From the problem we have that

$$sin(y) = 0 \text{ and} - sin(x) = 0$$

and therefore the fixed points are $(x, y) = (n\pi, m\pi)$ for $n, m \in \mathbb{Z}$. Setting $f = sin(y)$ and $g = -sin(x)$ we get

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 0 & cos(y) \\ cos(x) & 0 \end{pmatrix} \xrightarrow{\text{Plugging in } (n\pi, m\pi)} \begin{pmatrix} 0 & cos(m\pi) \\ cos(n\pi) & 0 \end{pmatrix} \xrightarrow{\text{Compute } \lambda} \begin{pmatrix} -\lambda & cos(m\pi) \\ cos(n\pi) & \lambda \end{pmatrix}$$

Solving this we get $\lambda^2 - cos(n\pi)cos(m\pi) = 0$ where $cos(n\pi)cos(m\pi)$ is always 1 or -1. If $cos(n\pi)cos(m\pi) = 1$ we have that $\lambda = \pm i$ and therefore we have center fixed point. If $cos(n\pi)cos(m\pi) =$ -1 then we have that $\lambda = \pm 1$. This is a saddle point and therefore can not be a stable fixed point.

(b) $\dot{x} = 4 - 4x^2 - y^2, \quad \dot{y} = 3xy$

From the problem we have that

$$3xy = 0 \Rightarrow x = y = 0 \Rightarrow 4 - 4x^2 = 0 \text{ and } 4 - y^2 = 0 \Rightarrow y = \pm 2,\, x = \pm 1$$

So we have that the four fixed points are $(x, y) = (0, \pm 2)$ and $(x, y) = (\pm 1, 0)$. Setting $f = 4 - 4x^2 - y^2$ and $g = 3xy$ we get

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -8x & -2y \\ 3y & 3x \end{pmatrix} \xrightarrow{\text{Plugging in } (0,\pm 2)} \begin{pmatrix} 0 & \mp 4 \\ \pm 6 & 0 \end{pmatrix} \xrightarrow{\text{Compute eigenvalue } \lambda} \begin{pmatrix} -\lambda & \mp 4 \\ \pm 6 & -\lambda \end{pmatrix}$$

Solving this we get $\lambda^2 + 24 = 0$. Therefore we have that $\lambda = \pm i\sqrt{24}$ which is a center for the fixed point $(0, \pm 2)$. Plugging in $(\pm 1, 0)$ yields $(8 \pm \lambda)(3 \pm \lambda)$ which evaluates to $\lambda = \sqrt{24}$. This implies that $(\pm 1, 0)$ is a saddle point and therefore not a stable fixed point.

Problem 5

Prove that the ODE

$$\dot{x} = 1 + x^{12}, \quad x(0) = 2 \tag{3}$$

blows up in finite time.

We have that

$$\dot{x}_1 = x^{12} \leq \dot{x} = 1 + x^{12}, \quad \forall x$$

Therefore it is sufficient to show $\dot{x}_1$ blows up in finite time.

$$\begin{aligned} \frac{dx_1}{dt} &= x_1^{12} \\ \int \frac{dx_1}{x_1^{12}} &= \int dt \\ -\frac{1}{11x_1^{11}} &= t + c \\ \Rightarrow c &= \frac{-2^{-11}}{11} \\ x_1 &= \frac{-1}{11}\frac{1}{t+c} \end{aligned}$$

Therefore we have that $\dot{x}_1$ blows up in finite time ($t = \frac{1}{11 \star 2^{11}}$) and therefore $\dot{x}$ blows up in finite time.

Problem 6

Prove the following theorem:

Theorem. *Suppose that Γ lies in a simply connected domain in $\mathbb{R}^2$ and there are no fixed points in the area enclosed by Γ, then the index I_Γ must be zero.*

Proof. Since Γ is a closed non-intersecting curve in a simply connected region we can apply Green's Theorem which states

$$\oint_\Gamma (Pdx + Qdy) = \iint_{D\Gamma} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

where D_Γ is the interior region bounded by Γ. In class we were given the equation for calculating the index

$$I_\Gamma = \frac{1}{2\pi} \int\limits_{s_0}^{s} \frac{XY' - YX'}{X^2 + Y^2}\, ds$$

Since we want to evaluate this in $\mathbb{R}^2$ to use Green's Theorem we use the chain rule

$$X' = \frac{dX}{ds} = \frac{dx}{ds} + X_y \frac{dy}{ds}, \quad Y' = \frac{dY}{ds} = Y_x \frac{dx}{ds} + Y_y \frac{dy}{ds}$$

which then turns the index equation into

$$I_\Gamma = \frac{1}{2\pi} \oint_\Gamma \left(\underbrace{\frac{XY_x - YX_x}{X^2 + Y^2}}_{P}\, dx + \underbrace{\frac{XY_y - YX_y}{X^2 + Y^2}}_{Q}\, dy \right).$$

The functions P and Q satisfy Green's Theorem since the denominator of each does not equal zero. Therefore we have

$$I_\Gamma = \frac{1}{2\pi} \iint_\Gamma \left[\left(\frac{\partial}{\partial x} \frac{XY_y - YX_y}{X^2 + Y^2} \right) - \frac{\partial}{\partial y} \left(\frac{XY_x - YX_x}{X^2 + Y^2} \right) \right]$$

Evaluating the inside of the double integral we get

$$\begin{gathered}(X_xY_y - XY_{yx} - Y_xX_y - YX_{yx})(X^2 + Y^2) - (2XX_x + 2YY_x)(XY_y - YX_y) \\ -(X_yY_x + XY_{xy} - Y_yX_x - YX_{xy})(X^2 + Y^2) + (2XX_y + 2YY_y)(XY_x - YX_x)\end{gathered}$$

After some very tedious algebra this expression reduces to zero proving that the index of the curve is zero. $\square$