Homework 6

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MAP5345: Partial Differential Equations I

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Problem 1

Analyze the following dynamical systems. In each case sketch the vector field on the real line, find all the fixed points and classify their stability.

(a) $\dot{x} = x - x^3$

Fixed Points: $x = 0, \pm 1$ with $d\dot{x}/dx = 1 - 3x^2$.

$$\longrightarrow \quad \bullet \quad \longleftarrow \quad \bullet \quad \longrightarrow \quad \bullet \quad \longleftarrow$$

Therefore we have that $x = \pm 1$ is stale fixed point and $x = 0$ is unstable fixed point.

(b) $\dot{x} = 1 + cos(x)/2$

There are no fixed points since $\max(cos\, x) = 1$.

(c) $\dot{x} = 1 - x^{14}$

Fixed Points: $x = \pm 1$ with $d\dot{x}/dx = -14x^{13}$

$$\longleftarrow \quad \bullet \quad \longrightarrow \quad \bullet \quad \longleftarrow$$

Therefore we have that $x = 1$ is a stable fixed point and $x = -1$ is an unstable fixed point.

Problem 2

Discuss the bifurcation properties of the following dynamical systems

The main idea behind bifurcation theory is to determine how some parameter a can affect the dynamical system

$$\dot{x} = f(x, a)$$

given that we can mess around with the value of a. So what we do is set $\dot{x} = 0$ and derive the equations

$$a = f(x), \quad \frac{d\dot{x}}{dx} = g(x)$$

and plot them. Doing so yields bifurcation points and tells us which fixed points are stable and which are unstable. It is important to note that we care when $d\dot{x}/dx < 0$ because that tells us our dynamical system will go back to its fixed point.

(a) $\dot{x} = a - x - e^{-x}$

From the graphs we can see that $(x, a) = (1, 0)$ is a bifurcation point where the stable fixed points occur when $x > 0$ and the unstable fixed points occur when $x < 0$. Note how $d\dot{x}/dx$ is independent of a.

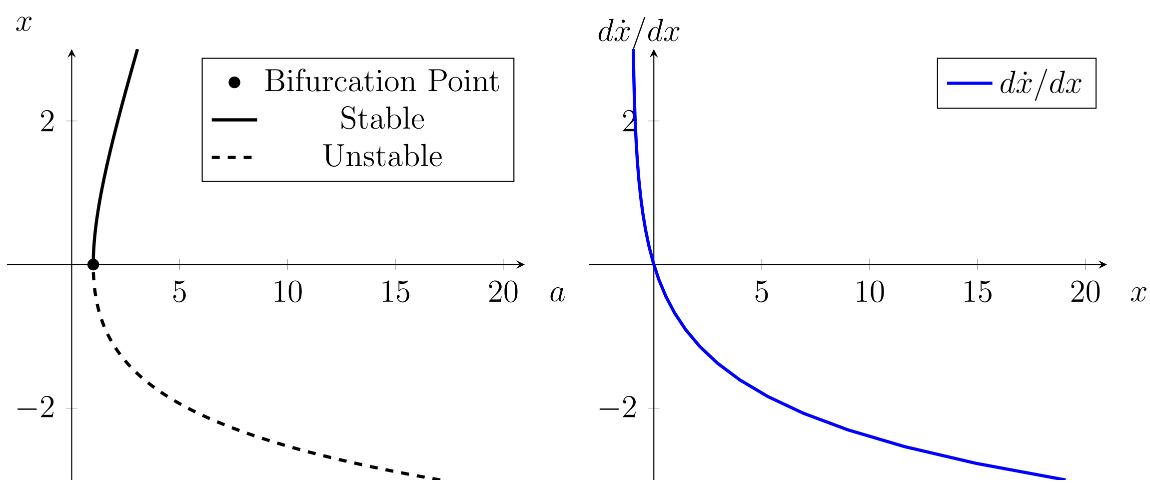


Figure 1: Plot of $a = x + e^{-x}$ for $x \in [-3, 3]$ (left) and $f'(x) = -1 + e^{-x}$

(b) $\dot{x} = rlnx + x - 1$

We can see that Also $\dot{x} = 0$ when $x = 1$ regardless of the value r. Also, the function $d\dot{x}/dx = r/x + 1$ is negative in the area of interest when $x < r$. Therefore we know that fixed points above the line $x = r$ will be unstable fixed points.

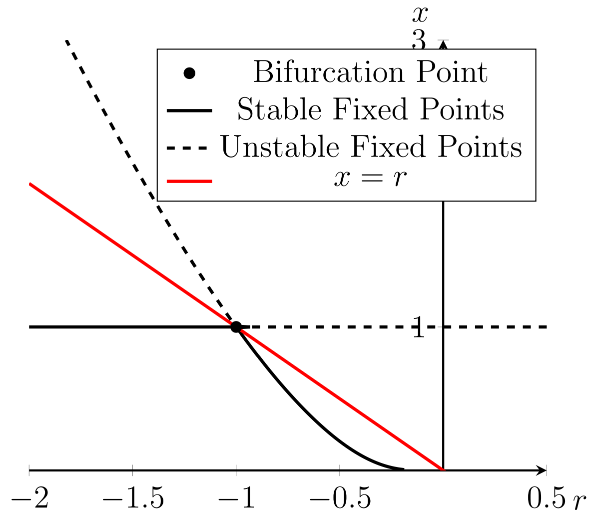


Figure 2: Plot of $r = \frac{1-x}{ln(x)}$ for $x \in [0, 3]$

(c) $\dot{x} = -x + rtanhx$

We can see that $\dot{x} = 0$ when $x = 0$ regardless of the value of r. Also, the function $d\dot{x}/dx = -1 + r(1 - tanh^2(x))$ is negative in the area of interest when $x = 0$ and $r > 1$.

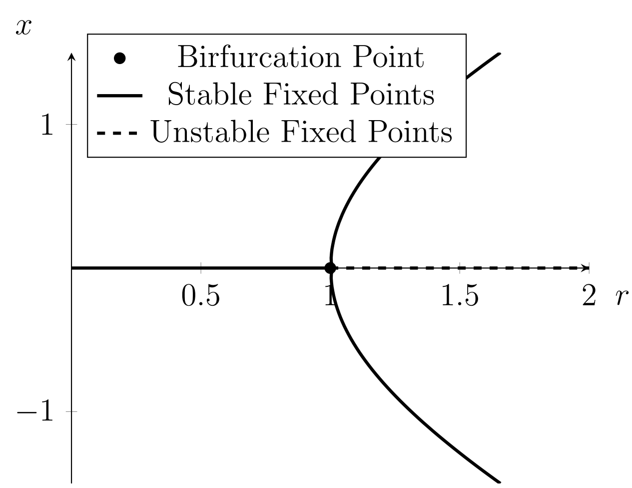


Figure 3: Plot of $r = \frac{x}{tanh(x)}$ for $x \in [-1.5, 1.5]$

(d) $\dot{x} = rx + x^3 - x^5$

We can easily see that the fixed points are $(r, x) = (0, 0)$ and $(0, \pm 1)$. Some number crunching tells us that for $x \in (-1, 1)$ the fixed points are unstable since $d\dot{x}/dx > 0$.

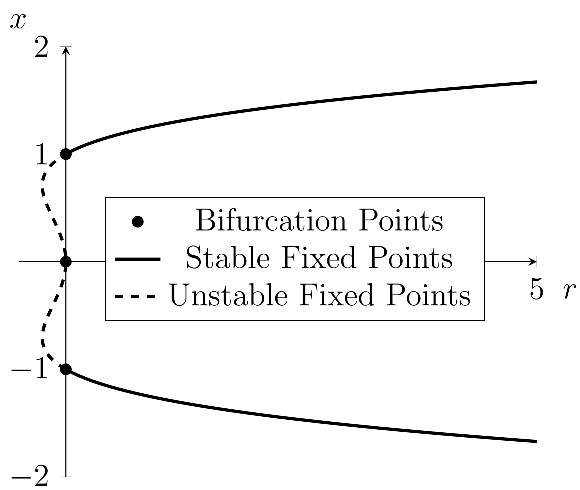


Figure 4: Plot of $r = \frac{x^5 - x^3}{x}$ for $x \in [-3, 3]$

(e) $\dot{x} = 1 + rx + x^2$

From the function $d\dot{x}/dx = 2x + r$, we can see that it is negative when only one of them is negative and $|r| > |x|/2$. From the graphs we can then see that $(r, x) = (-2, 1)$ is a bifurcation point with stable fixed points such that $x < 1$ and $(r, x) = (2, -1)$ is a bifurcation point with stable fixed points such that $x < -1$.

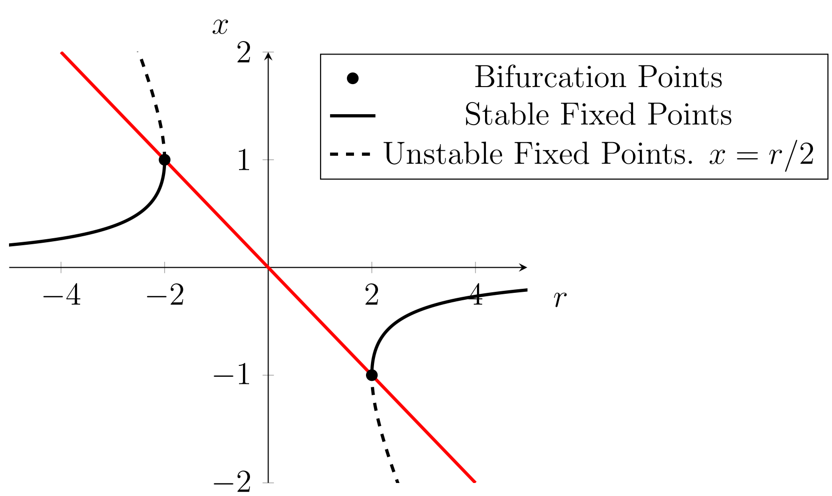


Figure 5: Plot of $r = \frac{-x^2 - 1}{x}$ for $x \in [-3, 3]$

(f) $\dot{x} = r - cosh x$

It is easy to check that $(r, x) = (1, 1)$ is a fixed point for this system. Graphing $d\dot{x}/dx$ we see that for all values $r > 1$ the system has stable fixed points regardless of the value of x.

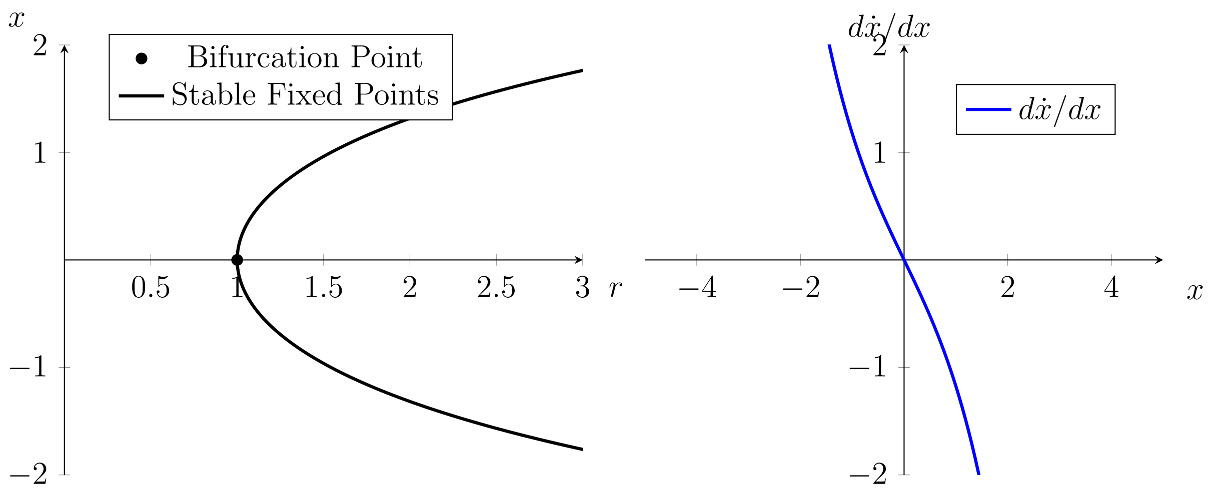


Figure 6: Plot of $r = cosh(x)$ for $x \in [-3, 3]$

(g) $\dot{x} = x - rx(1 - x)$

We can see that $(r, x) = (r, 0)$ are fixed points, but $d\dot{x}/dx = 1 + r(2x - 1)$ sets $(1, 0)$ to the bifurcation point. In fact, $d\dot{x}/dx$ is only negative when $x > 1$ and $r > 1$. From this we get our bifurcation map.

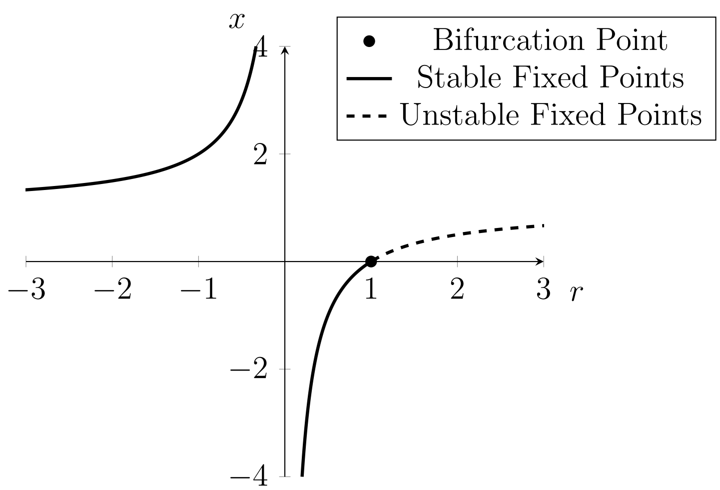


Figure 7: Plot of $r = \frac{x}{x(1-x)}$ for $x \in [-4, 4]$

(h) $\dot{x} = x(r - e^x)$

It is easy to see that $(r, x) = (1, 0)$ and $(r, 0)$ are the fixed points. The function e^x grows faster than r for $x > 1$, therefore fixed points will become stable for $x > 1$ for the first fixed point. For the second fixed point, if we perturb x just a little we see that $d\dot{x}/dx > 0$ and thus becomes unstable on the axis.

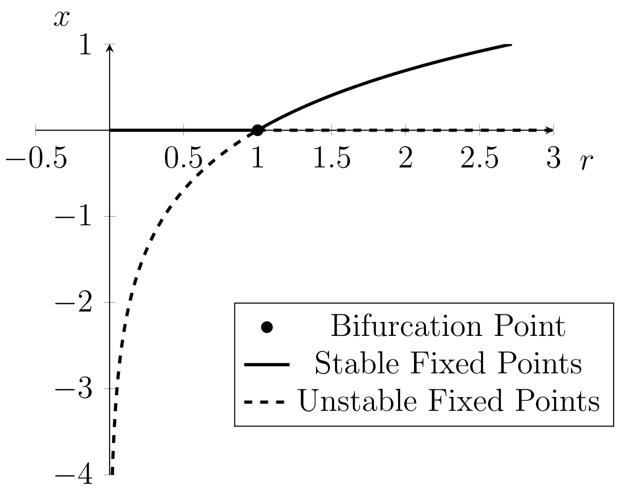


Figure 8: Plot of $r = e^x$ for $x \in [-4, 1]$

(i) $\dot{x} = rx - 4ax^3, \quad a = \pm 1$

The only fixed point for $a = 1$ is $(r, x) = (0, 0)$. The only time when $d\dot{x}/dx > 0$ is when $x = 0$ and $r > 0$, otherwise it is always negative. Since $\dot{x}$ is an odd function, flip it across the y-axis and we have the solution for when $a = -1$. Both are pitchfork bifurcations at $(0, 0)$ where the only unstable fixed points is $(r, x) = (r, 0)$ ($r > 0$ when $a = 1$ and $r < 0$ when $a = -1$).

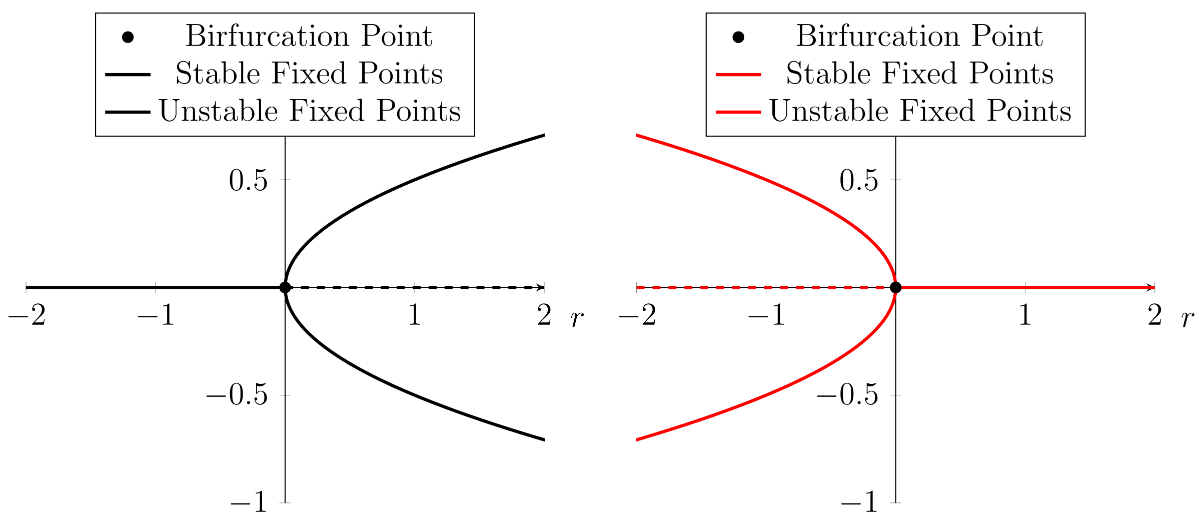


Figure 9: Plot of $r = 4x^2$ (black) and $r = -4x^2$ (red) for $x \in [-1, 1]$

(j) $\dot{x} = x + \frac{rx}{1+x^2}$

The fixed points of the dynamical system is when $x = 0$. The function $d\dot{x}/dx = 1 + r(1 - x^2)/(1+x^2)^2$ is zero when $r = -1$; this is the bifurcation point. We set $d\dot{x}/dx < 0$ and get $r < (x^2-1)/(1+x^2)^2$. This is plotted in red so any fixed points to the left of the graph will be stable.

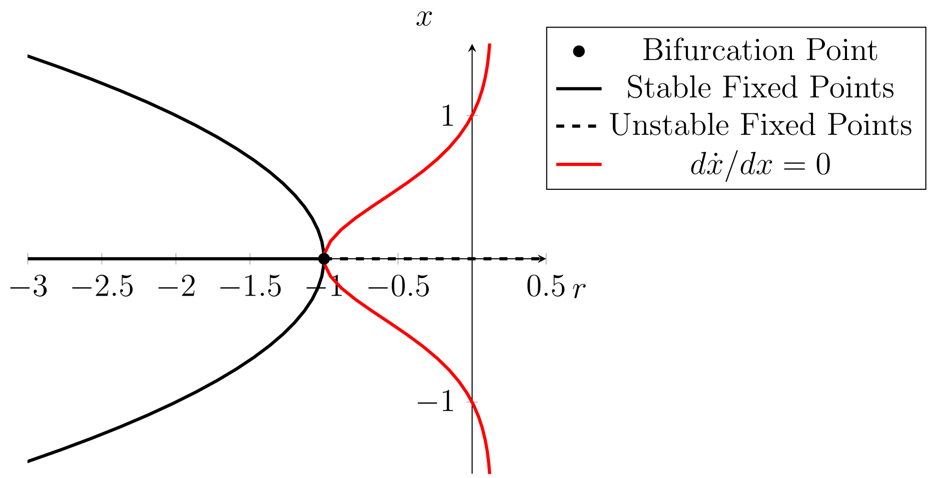


Figure 10: Plot of $r = -(1+x^2)$ for $x \in [-1.5, 1.5]$

Problem 3

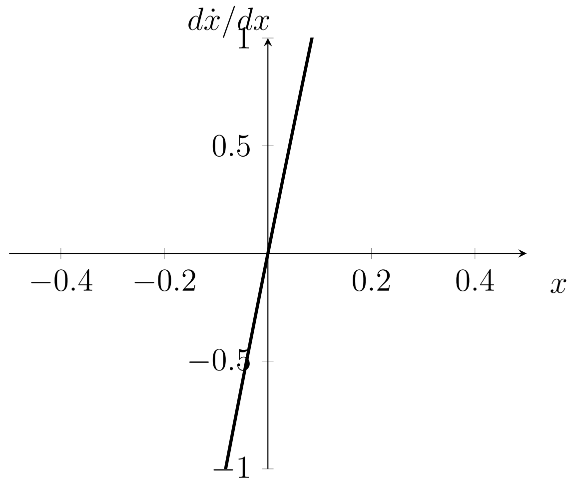
Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x) = 0$, use a graphical argument to determine the stability.

(a) $\dot{x} = x(1-x)$

Fixed Points: $x = 0, 1$. We have that $d\dot{x}/dx = 1 - 2x$ and from this we can conclude that $x = 0$ is an unstable fixed point while $x = 1$ is a stable fixed point.

(b) $\dot{x} = x^2(6-x)$

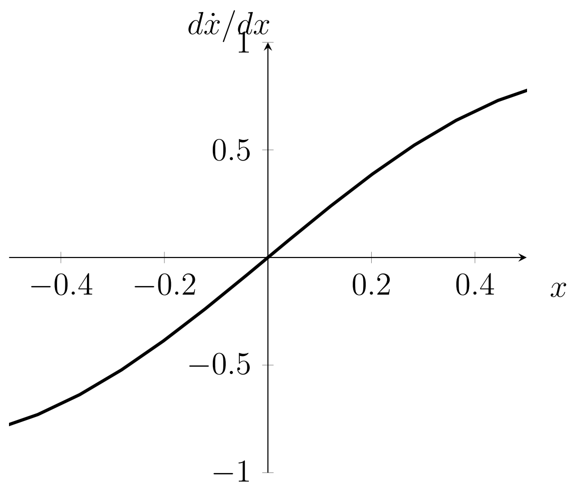
Fixed Points: $x = 0, 6$. We have that $d\dot{x}/dx = 12x - 3x^2$ and from this we can conclude that $x = 6$ is a stable fixed point; w need to further investigate $x = 0$.



From the graph we can see that a small perturbation around $x = 0$ in the negative direction is stable while a small perturbation in the positive direction is unstable.

(c) $\dot{x} = 1 - e^{-x^2}$

Fixed Points: $x = 0$. We have that $d\dot{x}/dx = -2xe^{x^2}$ however we can not conclude anything from this, so we must investigate further.



From the graph we can see that a small perturbation around $x = 0$ in the negative direction is stable while a small perturbation in the positive direction is unstable. It is the same case as before.

(d) $\dot{x} = ax - x^3$

Fixed Points: $x = 0, \pm\sqrt{a}$. We have that $d\dot{x}/dx = a - 3x^2$ and from this we have that $x = \pm\sqrt{a}$ are fixed points. A bit more work must be done. When $a > 0$ we have that $d\dot{x}/dx$ at $x = 0$ is positive, so $x = 0$ is linearly unstable. We also have that $d\dot{x}/dx$ is negative for

$\pm\sqrt{a}$, so $x = \pm\sqrt{a}$ are linearly stable. If $a < 0$, there is no solution to the dynamical system, so we are done.