Homework 3

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MAP 5345: Partial Differential Equations I

October 2, 2017

Problem 1

Consider the wave equation $u_{tt} = c^2 u_{xx}$. D'Alembert's solution is $u(x,t) = f(x+ct)+g(x-ct)$ for any functions f and g that are twice differentiable.

(a) Verify D'Alemberts solution directly by simply inserting it into the PDE.

Using D'Alembert's solution we get the following partials

$$\begin{aligned}
\partial_t(f(x+ct)+g(x-ct)) &= \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} \\
&= \frac{\partial f}{\partial(x+ct)}\frac{\partial(x+ct)}{\partial t} + \frac{\partial g}{\partial(x-ct)}\frac{\partial(x-ct)}{\partial t} \\
&= cf'(x+ct) - cg'(x+ct) \\
\partial_{tt}(f(x+ct)+g(x+ct)) &= c(\frac{\partial f'}{\partial t} - \frac{\partial g'}{\partial t}) \\
&= c\left(\frac{\partial f'}{\partial(x+ct)}\frac{\partial(x+ct)}{\partial t} - \frac{\partial g'}{\partial(x-ct)}\frac{\partial(x-ct)}{\partial t}\right) \\
&= c^2 f''(x+ct) + c^2 g''(x-ct) \\
\partial_x(f(x+ct)+g(x-ct)) &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} \\
&= \frac{\partial f}{\partial(x+ct)}\frac{\partial(x+ct)}{\partial x} + \frac{\partial g}{\partial(x-ct)}\frac{\partial(x-ct)}{\partial x} \\
&= f'(x+ct) + g'(x+ct) \\
\partial_{xx}(f(x+ct)+g(x-ct)) &= \frac{\partial f'}{\partial x} + \frac{\partial g'}{\partial x} \\
&= \frac{\partial f'}{\partial(x+ct)}\frac{\partial(x+ct)}{\partial x} + \frac{\partial g'}{\partial(x-ct)}\frac{\partial(x-ct)}{\partial x} \\
&= f''(x+ct) + g''(x-ct)
\end{aligned}$$

Plugging this into the PDE we get

$$c^2 f''(x+ct) + c^2 g''(x-ct) = c^2(f''(x+ct) + g''(x-ct)) \quad \checkmark$$

(b) Now consider the 'free-space' initial value problem

$$\begin{aligned} &u_{tt} = c^2 u_{xx} \quad \text{for } -\infty < x < \infty, t > 0 \\ &u(x,0) = \phi(x) \\ &u_t(x,0) = \psi(x) \end{aligned}$$

Use D'Alembert's solution to solve the IVP and determine f and g.

Using the initial conditions we get

$$\begin{aligned} &f(x) + g(x) = \phi(x) \\ &cf'(x) - cg'(x) = \psi(x) \end{aligned}$$

Integrating the second

$$\int_{x_0}^{x_f} \Big(f'(x) - g'(x)\Big) dx = \frac{1}{c} \int_{x_0}^{x_f} \psi(x) dx$$

$$f(x) = g(x) + \frac{1}{c} \int_{x_0}^{x_f} \psi(x) dx \quad (*)$$

Plugging this back into the initial conditions

$$\underbrace{g(x) + \frac{1}{c} \int_{x_0}^{x_f} \psi(x) dx}_{f(x)} + g(x) = \phi(x)$$

$$\Rightarrow \boxed{g(x) = \frac{1}{2}\phi(x) - \frac{1}{2c} \int_{x_0}^{x_f} \psi(x) dx}$$

Plugging this back into $(*)$

$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2c} \int_{x_0}^{x_f} \psi(x) dx + \frac{1}{c} \int_{x_0}^{x_f} \psi(x) dx$$

$$\Rightarrow \boxed{f(x) = \frac{1}{2}\phi(x) + \frac{1}{2c} \int_{x_0}^{x_f} \psi(x) dx}$$

(c) Set $c = 1$ and consider the initial conditions

$$u(x,0) = e^{-x^2/2}$$
$$u_t(x,0) = 0$$

Determine the solution to this IVP. First, plot the solution by hand for a few time values, without the aid of a computer. Next, plot the solution with Julia (choose a reasonable x-interval) and verify your had plot.

From the initial conditions we have

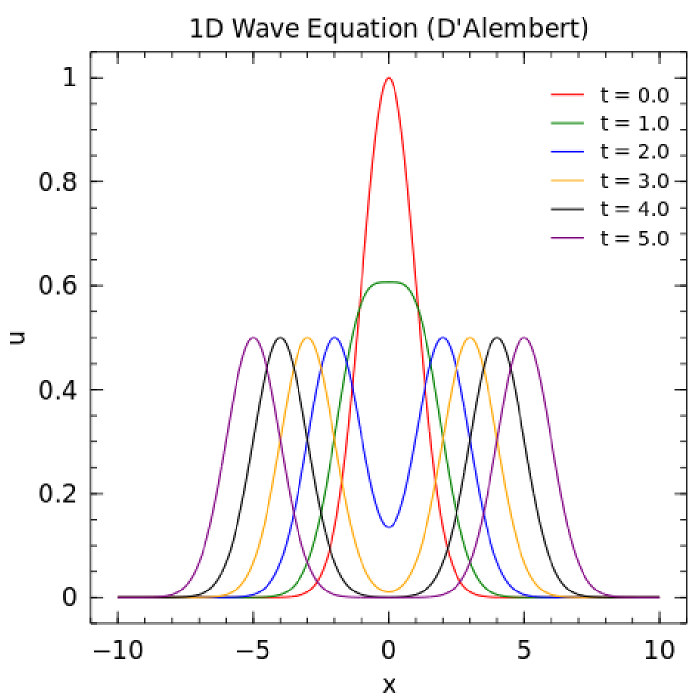
$$f(x) + g(x) = e^{-x^2/2}$$
$$cf(x) - cg(x) = 0 \Rightarrow f(x) = g(x)$$

Plugging the second into the first

$$2f(x) = 2g(x) = e^{-x^2/2}$$
$$\Rightarrow f(x) = g(x) = \frac{1}{2}e^{-x^2/2}$$

Plugging this into D'Alembert's formula we get

$$u(x,t) = f(x+ct) + g(x-ct)$$
$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left(e^{-(x+ct)^2/2} + e^{-(x-ct)^2/2}\right)}$$



(d) Do the same for the initial conditions

$$u(x,0) = 0$$
$$u_t(x,0) = xe^{-x^2/2}$$

From the initial conditions we have

$$f(x) + g(x) = 0 \Rightarrow f(x) = -g(x)$$
$$cf'(x) - cg'(x) = xe^{-x^2/2}$$

Plugging the first into the second

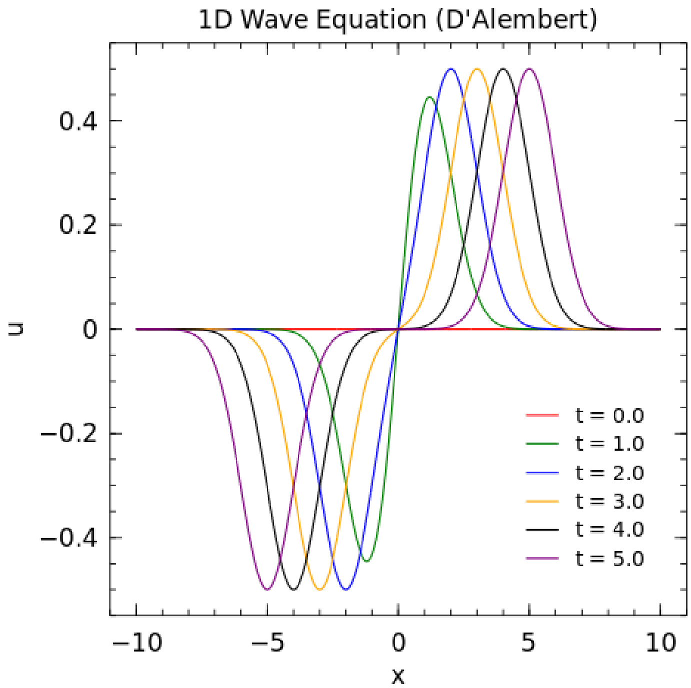
$$2cf'(x) = xe^{-x^2/2}$$

Integrating both sides we get

$$\int_{x_0}^{x_f} f'(x)dx = \frac{1}{2c}\int_{x_0}^{x_f} xe^{-x^2/2}dx$$
$$\Rightarrow f(x) = -\frac{1}{2c}\left(e^{x_f^2/2} - e^{x_0^2/2}\right), \quad g(x) = \frac{1}{2c}\left(e^{x_f^2/2} - e^{x_0^2/2}\right)$$

Plugging this back into D'Alembert's formula we get

$$u(x,t) = f(x+ct) + g(x-ct)$$
$$u(x,t) = -\frac{1}{2c}\left(e^{(x+ct)^2/2} - e^{x_0^2/2}\right) + \frac{1}{2c}\left(e^{(x-ct)^2/2} - e^{x_0^2/2}\right)$$
$$\Rightarrow \boxed{u(x,t) = \frac{1}{2c}\left(e^{(x-ct)^2/2} - e^{(x+ct)^2/2}\right)}$$


1D Wave Equation (D'Alembert)
0.4
0.2
0
−0.2
−0.4
u
−10
−5
0
5
10
x
t = 0.0
t = 1.0
t = 2.0
t = 3.0
t = 4.0
t = 5.0

Problem 2 INCOMPLETE :(

Now consider the vibrating string with clamped ends, given by th IBVP

$$\begin{aligned} &u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < L, t > 0 \\ &u(x,0) = \phi(x) \\ &u_t(x,0) = \psi(x) \\ &u(0,t) = u(L,t) = 0 \end{aligned}$$

Consider a relatively simple case in which $\phi(x) = x(L-x)$ and $\psi(x) = 0$.

(a) Can you use D'Alembert's formula to construct the exact solution to this problem?

Yes, we can use D'Alembert's formula to construct the exact solution to this problem.

Using the boundary conditions we get

$$\begin{aligned} &f(ct) + g(-ct) = 0 \Rightarrow f(ct) = -g(-ct) \\ &f(L+ct) + g(L-ct) = 0 \Rightarrow f(L+ct) = -g(L-ct) \end{aligned}$$

If we let $a = L + ct$ we get

$$f(a) = -g(2L-a) \text{ and } f(ct) = -g(-ct)$$

from this we can conclude some useful functions. For arbitrary values s we get that

$$f(s) = -g(-s) \quad \text{and} \quad f(s) = -g(2L-s)$$

From these two equations we can determine f on the entire real line! This is done by the following

$$\begin{aligned} &f(s) = f(s) && \text{for } 0 \le s \le L \\ &f(s) = -g(2L-s) && \text{for } L \le s \le 2L \\ &f(s) = -g(2L-s) = f(s-2L) && \text{for } 2L \le s \le 3L \\ &f(s) = -g(2L-s) = f(s-2L) = -g(4L-s) && \text{for } 3L \le s \le 4L \\ &\quad\vdots && \vdots \end{aligned}$$

The same can be done for g but on the domain $(-\infty, 0]$. Doing so we have constructed a $u(x,t)$ that satisifies both the boundary conditions in the entire real line.

(b) Using Julia, plot the solution for several time values. You solved the exact same IBVP in HW 2 using separation of variables. Compare your two solutions and make sure they are nearly the same. Is one more accurate than the other?

D'Alembert's formula is more accurate than the seperation of variables solution because their is no truncation. For the seperation of variables gneneral soulution we must pick some value N that we truncate the infinite sum, effectively losing precision, where as D'Alembert's formula is an exact solution.