Set 10: Piecewise Polynomial Interpolation

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Summary

- Approximation of $f(x) \in \mathcal{C}^{(0)}$ with polynomials.
- Various metrics possible:
  - $\sum_i |f(x_i) - p(x_i)|$
  - $\sum_i |f(x_i) - p(x_i)| + \cdots + |f^{(k)}(x_i) - p^{(k)}(x_i)|$
  - $\|f - p\|_\infty$
  - $\|f - p\|_{L_2}$

Summary

- Interpolation
  - $f(x_i) = p(x_i)$
  - $f(x_i) = p(x_i), \quad \ldots, \quad f^{(k)}(x_i) = p^{(k)}(x_i)$
  - more general combinations of function values and derivatives
- Various interpolation forms of unique polynomials
  - Lagrange – standard or barycentric
  - Newton
  - Hermite-Birkoff
- $\|f - p\|_\infty \to 0$: convergent sequence of polynomial family representations
  - Bernstein polynomials for $f \in \mathcal{C}^{(0)}$
  - interpolatory strategies for more constrained class of f

Polynomial Interpolation

Problems:

- Pointwise error too large at important points
- $\|f - p\|_\infty$ too large on interval of interest
- erratic variation, i.e., not smooth enough
- excessive computational complexity
- ill-conditioning and instability

Polynomial Interpolation

Solutions – Complications:

- choose better points – may not be possible
- increase n – may or may not improve error, may not converge
- interpolate derivatives – values may not be available

Piecewise Lagrange Interpolation

Use local interpolants of lower order rather than one global polynomial.

- $a = x_0 < x_1 < \cdots < x_n = b$
- $[a, b] = \cup_s I_s$: union of disjoint subintervals (intersect only at subset of grid points)
- $g_k(x)$, on $I_s = [x_{i_s}, x_{i_s+k}]$ is in $\mathbb{P}_k$
- $g_k(x)$ is a piecewise polynomial
- local interpolant $p_{k,i_s}(x_j) = f(x_j), \quad i_s \leq j \leq i_s + k$
- global interpolant $g_k(x_i) = f(x_i), \quad 0 \leq i \leq n$

Choices

- Form of $p_{k,i}(x)$
- In practice, each interval is independent in construction and evaluation.
- For analysis the form matters, e.g., basis choice
- When used to define a set of relationships between unknown $f(x_i), \ldots, f^{(k)}(x_i)$ the form determines the structure of equations to be solved.

Forms and Bases

- monomial

$$p_{k,i_s}(x) = \alpha_0^{(i_s)} + \alpha_1^{(i_s)} x + \cdots + \alpha_{k-1}^{(i_s)} x^{k-1} + \alpha_k^{(i_s)} x^k$$

- Newton

$$p_{k,i_s}(x) = f_{i_s} + f[x_{i_s}, x_{i_s+1}](x - x_{i_s}) + \cdots + f[x_{i_s}, \ldots, x_{i_s+k}]\omega_k^{(i_s)}$$

- Lagrange

$$p_{k,i_s}(x) = \sum_{j=0}^{k} \ell_j^{(i_s)}(x) f_{i_s+j}$$

- basis form for analysis and implicit equations

$$g_k(x) = \sum_{i=0}^{n} f_i \phi_i(x) = \sum_{i=0}^{n} \gamma_i \psi_i(x)$$

where $\phi_i(x)$ and $\psi_i(x)$ are piecewise polynomials.

Error

If $f \in \mathcal{C}^{(k+1)}[a,b]$

$$\forall a \le x \le b, \quad f(x) - g_k(x) = f(x) - p_{k,i_s}(x) = \frac{f^{(k+1)}(\xi)}{(k+1)!}\omega_{k+1}^{(i_s)}(x)$$

$$x \in [x_{i_s}, x_{i_s+k}]$$

The local error expressions can be combined to get a global error

$$\|f - g_k\|_\infty \le Ch^{k+1}\|f^{(k+1)}\|_\infty$$

where h is maximum size of intervals I_i

Error

This is easily shown:

$$|\frac{f^{(k+1)}(\xi)}{(k+1)!}| \leq C\|f^{(k+1)}\|_\infty$$

$$\omega_{k+1}^{(i_s)}(x) = (x - x_{i_s}) \cdots (x - x_{i_s+k})$$

$$|(x - x_j)| \leq (x_{i_s+k} - x_{i_s}) \leq h, \quad i_s \leq j \leq i_s + k$$

$$\therefore \|f - g_k\|_\infty \leq Ch^{k+1}\|f^{(k+1)}\|_\infty$$

Reducing Error

- Increasing k, the order of the local polynomial, may not improve things.
- Shrinking the intervals by increasing the number of points causes the error to go to 0, i.e.,

$$\lim_{h \to 0} \|f - g_k\|_\infty = 0$$

- This avoids problems with increasing the order of an interpolating polynomial.
- Order vs. accuracy vs. number of points can be analyzed in terms of error bounds.

Piecewise Linear Lagrange

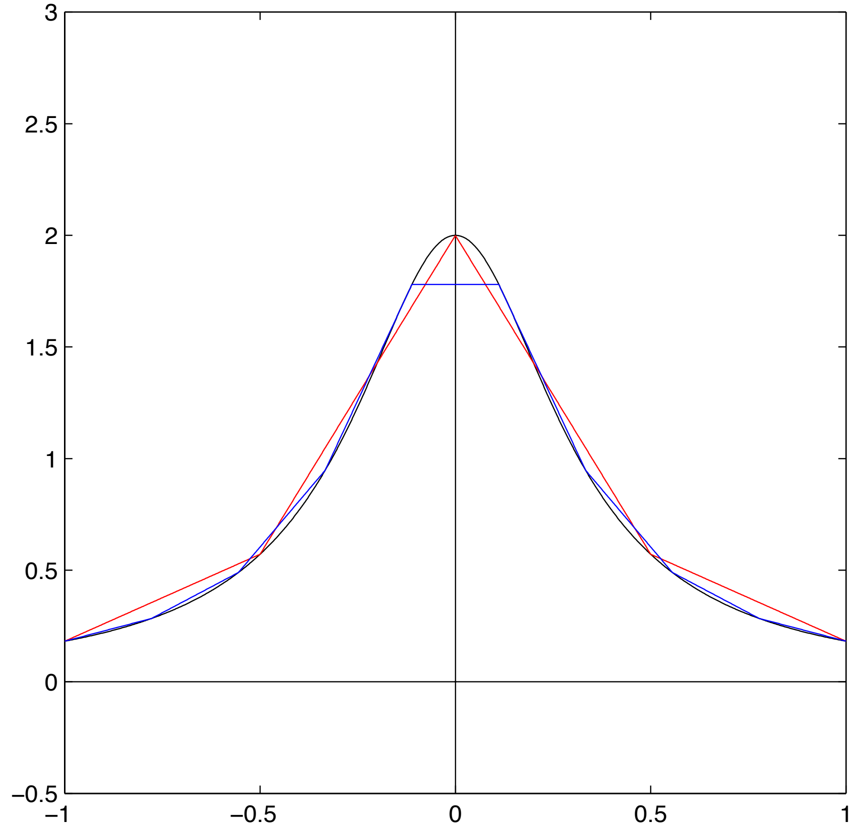
- Lagrange is used here to refer interpolation of function values only.
- interval $I_i = [x_i, x_{i+1}]$
- interpolate (x_i, f_i) and (x_{i+1}, f_{i+1})
- Newton form on I_i

$$p_{1,i}(x) = f_i + f[x_i, x_{i+1}](x - x_i)$$

- Standard Lagrange form

$$p_{1,i}(x) = f_i \frac{(x - x_{i+1})}{(x_i - x_{i+1})} + f_{i+1} \frac{(x - x_i)}{(x_{i+1} - x_i)}$$

Piecewise Linear Lagrange



Piecewise linear, intervals: 4 (red) and 9 (blue), $f(x) = \frac{2}{1+10x^2}$ (black)

Piecewise Linear Lagrange

- Runge phenomenon caused significant problems before with equidistant points.
- Equidistant points, i.e., uniform h_i, are used here
- Very quickly the approximation is good (at least from a visual p.o.v.)
- The piecewise linear polynomial $g_1(x) \in \mathcal{C}^{(0)}$ but clearly $g_1(x) \notin \mathcal{C}^{(1)}$
- Note local variation in quality, 9-interval g_1 chops off peak while 5-interval g_1 OK there.
- 9-interval g_1 is, in general, better everywhere else.

Piecewise Quadratic Lagrange

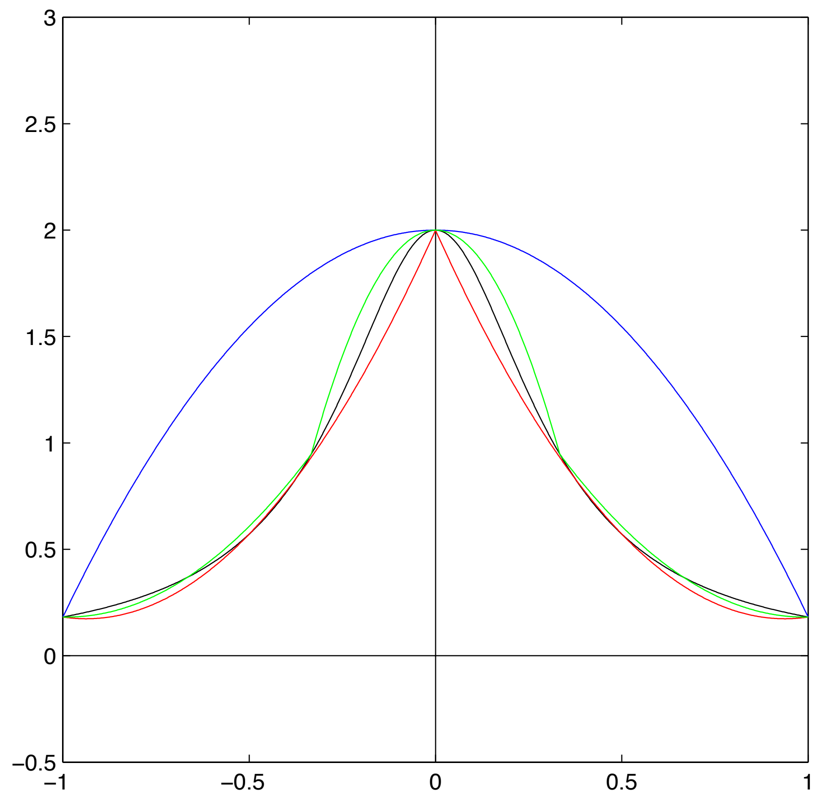
- interval $I_i = I_{2j} = [x_{2j}, x_{2j+2}]$
- n must be even
- interpolate (x_i, f_i),(x_{i+1}, f_{i+1}), (x_{i+2}, f_{i+2})
- Newton form on I_i

$$p_{2,i}(x) = f_i + f[x_i, x_{i+1}](x - x_i) + f[x_i, x_{i+1}, x_{i+2}](x - x_i)(x - x_{i+1})$$

- Standard Lagrange form

$$\begin{aligned} p_{2,i}(x) = f_i \frac{(x - x_{i+1})(x - x_{i+2})}{(x_i - x_{i+1})(x_i - x_{i+2})} + f_{i+1} \frac{(x - x_i)(x - x_{i+2})}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} \\ + f_{i+2} \frac{(x - x_i)(x - x_{i+1})}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} \end{aligned}$$

Piecewise Quadratic Lagrange



Piecewise quadratic, intervals: 1 (blue), 2 (red), 3 (green), $f(x) = \frac{2}{1+10x^2}$, (black)

Piecewise Quadratic Lagrange

- Equidistant points, i.e., uniform h_i, are used here.
- Very quickly the approximation is good (at least from a visual p.o.v.)
- Locally $p_{2,i}(x) \in \mathcal{C}^{(2)}$
- $g_2(x) \in \mathcal{C}^{(0)}$ but $g_2(x) \notin \mathcal{C}^{(1)}$
- Note local variation in quality due to change in curvature, e.g., 2-interval vs 3-interval near peak.
- Quality good but not as simple a function of number of intervals as expected.

Piecewise Linear Lagrange Error Bound

Assume equidistant points, $h = (2\pi)/n$,

$$f(x) = \sin x, \quad -\pi \leq x \leq \pi$$

$$x_i \leq x \leq x_{i+1} \leftrightarrow 0 \leq s \leq 1, \quad x = x_i + sh$$

$$|f(x) - g_1(x)| = |\frac{1}{2} f^{(2)}(\xi)(x - x_i)(x - x_{i+1})|$$

$$\leq \frac{1}{2} \|f^{(2)}(\xi)\|_\infty \|(x - x_i)(x - x_{i+1})\|_\infty = \frac{1}{2} h^2 \|f^{(2)}\|_\infty \|(s^2 - s)\|_\infty$$

$$0 \leq s \leq \rightarrow \|(s^2 - s)\|_\infty \leq \frac{1}{4}$$

Piecewise Linear Lagrange Error Bound

$$\therefore \|f(x) - g_1(x)\|_\infty \le \frac{1}{8}h^2\|f^{(2)}\|_\infty$$

$$f(x) = \sin x, \quad -\pi \le x \le \pi$$

$$f'(x) = \cos x, \quad f^{(2)}(x) = -\sin x$$

$$\therefore \|f^{(2)}\|_\infty \le 1$$

$$\frac{1}{8}h^2 \le 10^{-d} \to h \le \sqrt{8} \times 10^{-d/2} \to \|f(x) - g_1(x)\|_\infty \le 10^{-d}$$

Piecewise Linear Lagrange Cardinal Basis

- Each interval has local form of $p_{k,i}(x)$
- $g_k(x)$ is an element of a linear space
- search for basis to express $g_k(x)$ in terms of linear combination of other interpolants
- can be done starting from various forms depending on desired coefficients
- cardinal basis is general form of what we have called Lagrange form
- coefficients are function values (and derivatives when extended to piecewise Hermite)
- consider the derivation of these bases for $k = 1$ and $k = 2$

Piecewise Linear Lagrange Cardinal Basis

- Use Lagrange to find coefficient of f_i in $g_1(x)$
- intervals $[x_i, x_{i+1}]$ and $[x_{i-1}, x_i]$

$$p_{1,i}(x) = f_i \frac{(x - x_{i+1})}{(x_i - x_{i+1})} + f_{i+1} \frac{(x - x_i)}{(x_{i+1} - x_i)} \text{ On interval } [x_i, x_{i+1}]$$

$$p_{1,i-1}(x) = f_{i-1} \frac{(x - x_i)}{(x_{i-1} - x_i)} + f_i \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \text{ On interval } [x_{i-1}, x_i]$$

No other interval involves f_i

Piecewise Linear Lagrange Cardinal Basis

Weight of f_i

$$\phi_{1,i}(x) = \frac{(x - x_{i+1})}{(x_i - x_{i+1})} \text{ On interval } [x_i, x_{i+1}]$$

$$\phi_{1,i}(x) = \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \text{ On interval } [x_{i-1}, x_i]$$

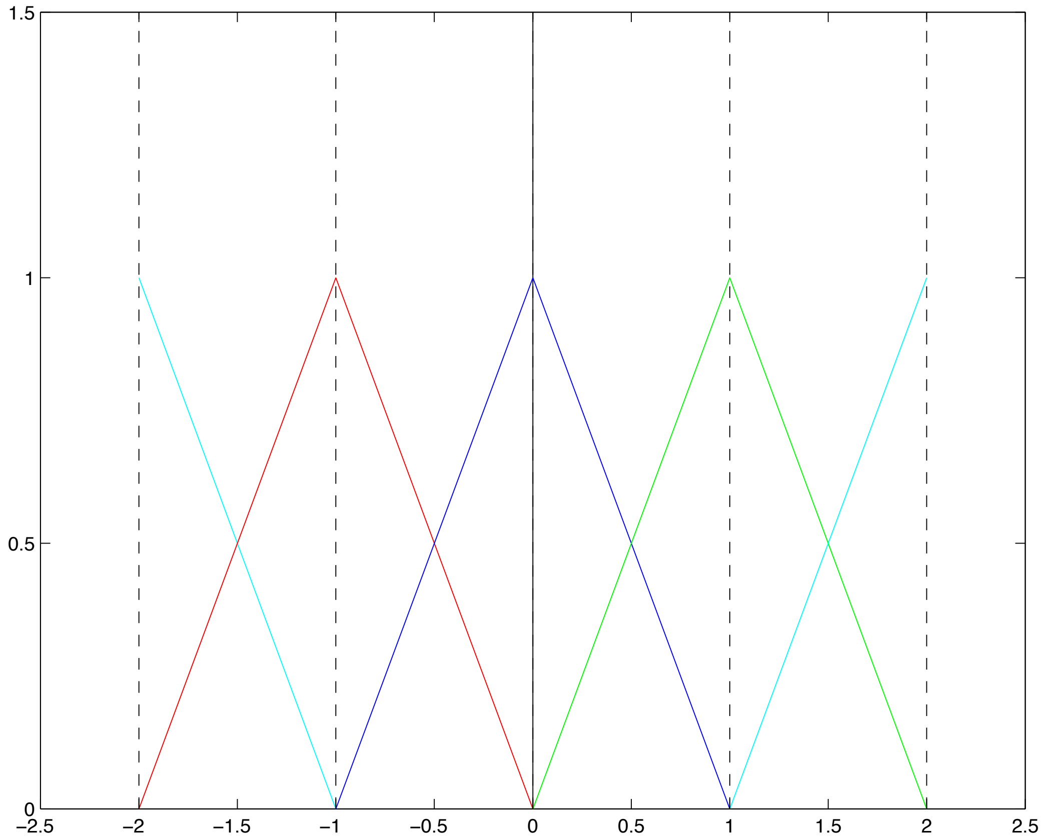
$$\phi_{1,i}(x) = 0 \text{ for } x < x_{i-1} \text{ and } x > x_{i+1}$$

Piecewise Linear Lagrange Cardinal Basis

- By construction, $\phi_{1,i}(x_j) = \delta_{ij}$
- $\phi_{1,i}(x) \in \mathcal{C}^{(0)}$ and $\phi_{1,i}(x) \notin \mathcal{C}^{(1)}$
- $\phi_{1,i}(x)$ are piecewise linear interpolants defined by points $(x_0, 0), (x_1, 0) \ldots, (x_{i-1}, 0), (x_i, 1), (x_{i+1}, 0), \ldots, (x_n, 0)$.
- The dimension of the space containing $g_1(x)$ is $n + 1$

$$g_1(x) = \sum_{i=0}^{n} f_i \phi_{1,i}(x)$$

Piecewise Linear Lagrange Cardinal Basis



Piecewise linear basis functions

Piecewise Quadratic Lagrange Cardinal Basis

- Use Lagrange to find coefficient of f_i in $g_2(x)$ for $i = 2j$.
- Use Lagrange to find coefficient of f_{i+1} in $g_2(x)$.
- Only intervals $[x_i, x_{i+2}]$ and $[x_{i-2}, x_i]$ must be considered.
- Derive a basis of piecewise quadratic interpolants $\phi_{2,i}(x)$ and $\phi_{2,i+1}(x)$.
- The dimension of the space is $n + 1$, independent of k.

$$g_2(x) = \sum_{i=0}^{n} f_i \phi_{2,i}(x)$$

Piecewise Quadratic Lagrange Cardinal Basis

Weights of f_i:

$$\frac{(x - x_{i+1})(x - x_{i+2})}{(x_i - x_{i+1})(x_i - x_{i+2})} \text{ On interval } [x_i, x_{i+2}]$$

$$\frac{(x - x_{i-2})(x - x_{i-1})}{(x_i - x_{i-2})(x_i - x_{i-1})} \text{ On interval } [x_{i-2}, x_i]$$

f_{i+1} only appears in $[x_i, x_{i+2}]$ with weight:

$$\frac{(x - x_i)(x - x_{i+2})}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})}$$

Piecewise Quadratic Lagrange Cardinal Basis

We have, $i = 2j$:

$$\phi_{2,i}(x) = \frac{(x - x_{i+1})(x - x_{i+2})}{(x_i - x_{i+1})(x_i - x_{i+2})} \text{ On interval } [x_i, x_{i+2}]$$

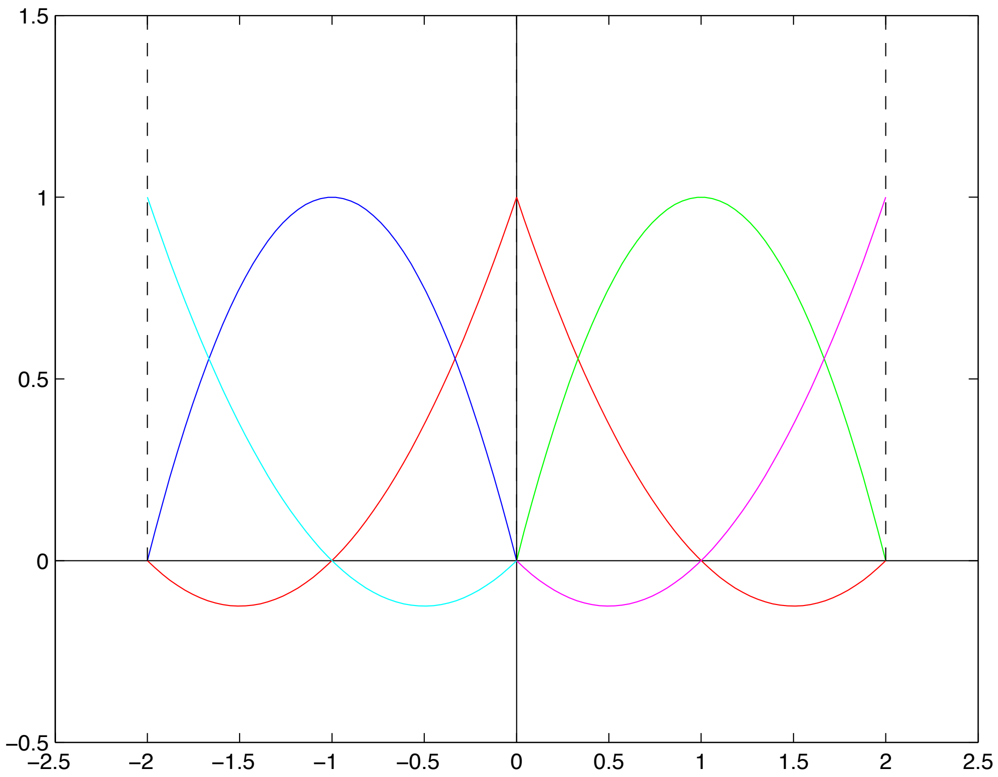
$$\phi_{2,i}(x) = \frac{(x - x_{i-2})(x - x_{i-1})}{(x_i - x_{i-2})(x_i - x_{i-1})} \text{ On interval } [x_{i-2}, x_i]$$

$$\phi_{2,i}(x) = 0 \text{ for } x < x_{i-2} \text{ and } x > x_{i+2}$$

$$\phi_{2,i+1}(x) = \frac{(x - x_i)(x - x_{i+2})}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} \text{ On interval } [x_i, x_{i+2}]$$

$$\phi_{2,i+1}(x) = 0 \text{ for } x < x_i \text{ and } x > x_{i+2}$$

Piecewise Quadratic Lagrange Cardinal Basis



Piecewise quadratic basis functions

Piecewise Lagrange Interpolation

- $g_k(x) \in \mathcal{C}^{(k)}$ on each interval.
- $g_k(x) \in \mathcal{C}^{(0)}[a, b]$
- at the nodes $g_k(x) \notin \mathcal{C}^{(1)}$ generally
- Some applications require $g_k(x) \in \mathcal{C}^{(2)}[a, b]$, e.g., mechanics
- piecewise Lagrange not appropriate there
- piecewise Lagrange is good usually where only global continuity required and nodes can be chosen
- if nodes fixed or higher order continuity required then must consider how to get smoothness
  - piecewise Hermite
  - splines

Piecewise Hermite Interpolation

- Suppose derivative values are available at nodes, $f'(x_i) = f'_i$
- $g_k(x) \in \mathcal{C}^{(1)}[a,b]$ can be achieved via Hermite interpolation on each interval $[x_i, x_{i+1}]$
- Create a piecewise cubic polynomial interpolant.
- $H_3(x_i) = f_i$, $H'_3(x_i) = f'_i$ and $H_3(x_{i+1}) = f_{i+1}$, $H'_3(x_{i+1}) = f'_{i+1}$
- As before, this is expected to smooth the approximation.

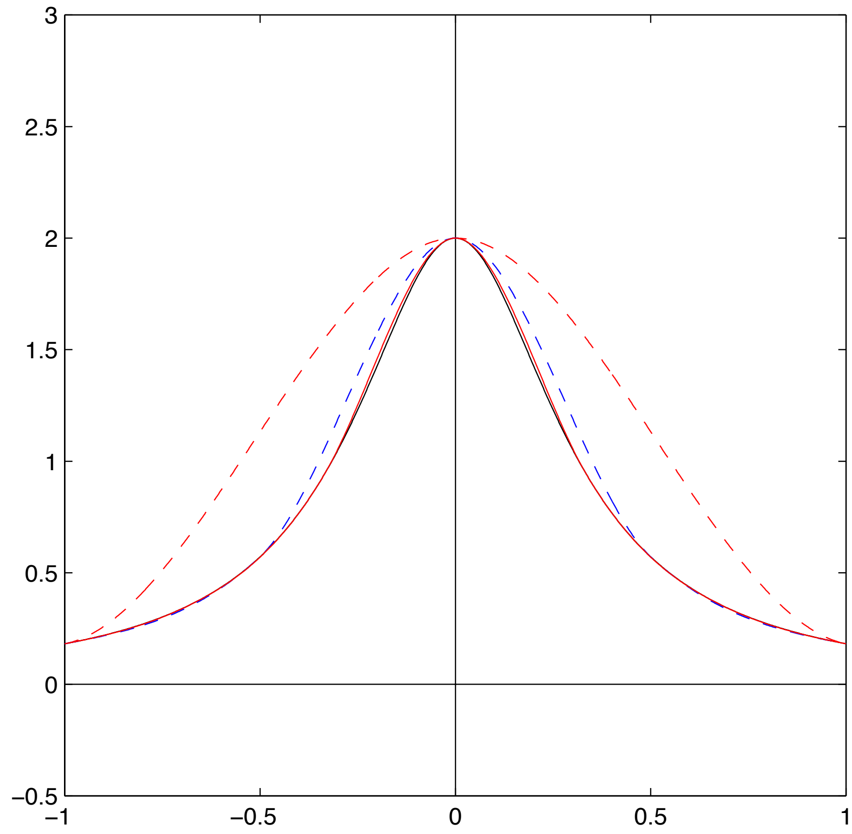
Piecewise Hermite Interpolation

$H_3(x)$ restricted to the interval $[x_i, x_{i+1}]$ is the previously discussed Hermite interpolant taken as a cubic to satisfy the 4 constraints. On the interval the Newton form is:

$$
\begin{aligned}
H_{3,i}(x) = f_i + f_i'(x - x_i) + f[x_i, x_i, x_{i+1}](x - x_i)^2 \\
+ f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^2(x - x_{i+1})
\end{aligned}
$$

The cardinal basis must revisit the associated form of the Hermite interpolating polynomial.

Piecewise Hermite



Piecewise Hermite, intervals: 2 (–red), 4 (–blue), 6 (red), $f(x) = \frac{2}{1+10x^2}$, (black)

Piecewise Hermite Interpolation

We have on $[x_i, x_{i+1}]$

$$H_{3,i}(x) = f_i \psi_{L,i}(x) + f_i' \Psi_{L,i}(x) + f_{i+1} \psi_{R,i}(x) + f_{i+1}' \Psi_{R,i}(x)$$

$$\psi_{L,i}(x) = \ell_{L,i}^2(x)\Big[1 - 2\ell_{L,i}'(x_i)(x - x_i)\Big]$$

$$\psi_{R,i}(x) = \ell_{R,i}^2(x)\Big[1 - 2\ell_{R,i}'(x_{i+1})(x - x_{i+1})\Big]$$

$$\Psi_{L,i}(x) = \ell_{L,i}^2(x)(x - x_i) \text{ and } \Psi_{R,i}(x) = \ell_{R,i}^2(x)(x - x_{i+1})$$

$$\ell_{L,i}(x) = \frac{(x - x_{i+1})}{(x_i - x_{i+1})} \text{ and } \ell_{R,i}(x) = \frac{(x - x_i)}{(x_{i+1} - x_i)}$$

$$\ell_{L,i}'(x) = 1/(x_i - x_{i+1}) \text{ and } \ell_{R,i}'(x) = 1/(x_{i+1} - x_i)$$

Piecewise Hermite Interpolation

On $[x_i, x_{i+1}]$

$$\psi_{L,i}(x) = \frac{(x - x_{i+1})^2}{(x_i - x_{i+1})^2}\left[1 - 2\frac{(x - x_i)}{(x_i - x_{i+1})}\right]$$

$$\psi_{R,i}(x) = \frac{(x - x_i)^2}{(x_{i+1} - x_i)^2}\left[1 - 2\frac{(x - x_{i+1})}{(x_{i+1} - x_i)})\right]$$

$$\Psi_{L,i}(x) = \frac{(x - x_{i+1})^2}{(x_i - x_{i+1})^2}(x - x_i)$$

$$\Psi_{R,i}(x) = \frac{(x - x_i)^2}{(x_{i+1} - x_i)^2}(x - x_{i+1})$$

Piecewise Hermite Cardinal Basis

- for $1 \le i \le n-1$
  - f_i is weighted by $\psi_{L,i}(x)$ on $[x_i, x_{i+1}]$
  - f_i is weighted by $\psi_{R,i-1}(x)$ on $[x_{i-1}, x_i]$
  - f_i' is weighted by $\Psi_{L,i}(x)$ on $[x_i, x_{i+1}]$
  - f_i' is weighted by $\Psi_{R,i-1}(x)$ on $[x_{i-1}, x_i]$
- for $i = 0$ or $i = n$ you have only the terms from intervals that exist, i.e., you lose one term for each.

Piecewise Hermite Cardinal Basis

For $1 \le i \le n-1$

$$\phi_i(x) = \frac{(x - x_{i+1})^2}{(x_i - x_{i+1})^2}\left[1 - 2\frac{(x - x_i)}{(x_i - x_{i+1})}\right], \quad x_i \le x \le x_{i+1}$$

$$\phi_i(x) = \frac{(x - x_{i-1})^2}{(x_i - x_{i-1})^2}\left[1 - 2\frac{(x - x_i)}{(x_i - x_{i-1})})\right], \quad x_{i-1} \le x \le x_i$$

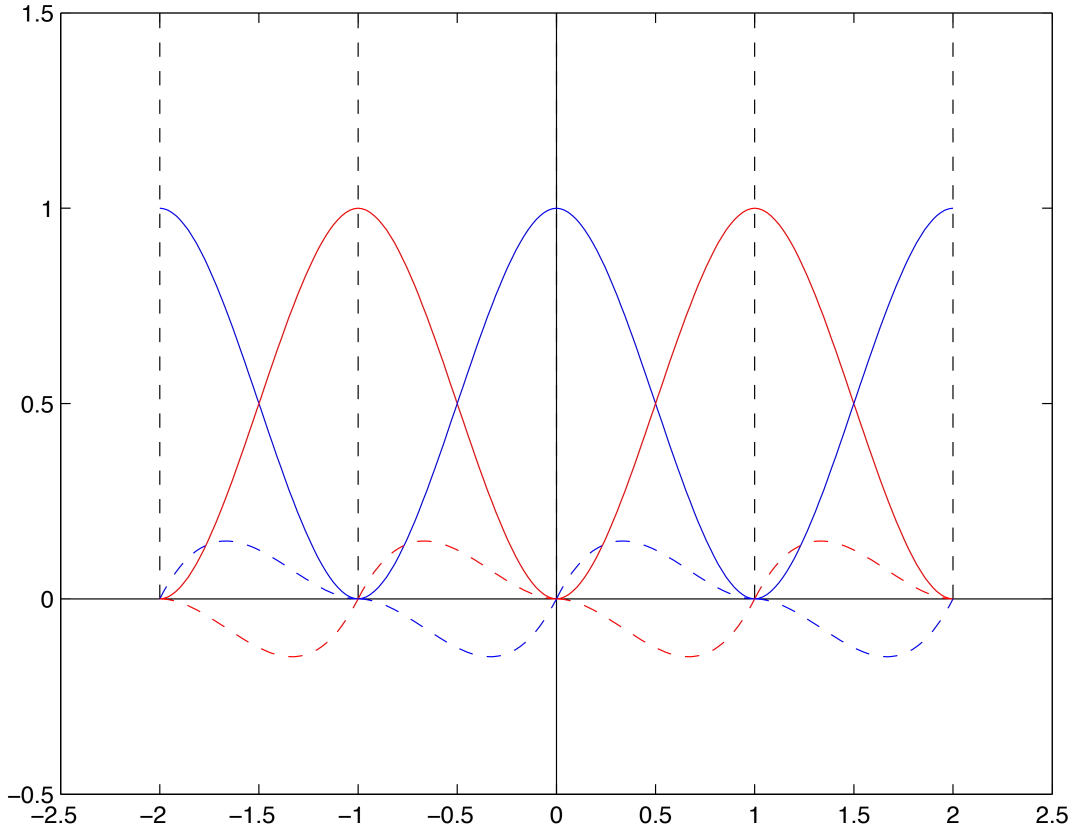
$$\Phi_i(x) = \frac{(x - x_{i+1})^2}{(x_i - x_{i+1})^2}(x - x_i), \quad x_i \le x \le x_{i+1}$$

$$\Phi_i(x) = \frac{(x - x_{i-1})^2}{(x_i - x_{i-1})^2}(x - x_i), \quad x_{i-1} \le x \le x_i$$

$\phi_i(x) = 0$ and $\Phi_i(x) = 0$ elsewhere

$$H_3(x) = \sum_{i=0}^{n}\left[f_i\phi_i(x) + f_i'\Phi_i(x)\right]$$

Piecewise Hermite Cardinal Basis



Piecewise Hermite basis functions, $\phi_i(x)$ (solid) and $\Phi_i(x)$ (dotted)