Set 13: Parametric Curves

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Foundations of Computational Math 1
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Parametric Curves

- Consider a curve in $\mathbb{R}^2$.
- Parametric form $(x(t), y(t))$ for $t_0 \leq t \leq t_n$ and $t \in \mathbb{R}$.
- Implies an ordering to the points.
- The interval can be adjusted to yield any length or rate of motion.
- Used in phase plane representation of behavior of a dynamical system.
- May have sample points of an underlying system.
- May have points in a plane from a graphics application.
- Order can be chosen and imposed on the parameter.
- Want a smooth curve to indicate the shape of the point collection.

Parametric Curves

A circle can be parameterized as

$$(\sin \omega t, \quad \cos \omega t)$$

Frequency ω can be set to dictate the velocity around the circle as a function of t.

Suppose you had points on the circle

$$(x_i, y_i), \quad 0 \leq i \leq n$$

Could draw the curve with

- simple linear connections
- piecewise Lagrange of any degree
- splines of any degree

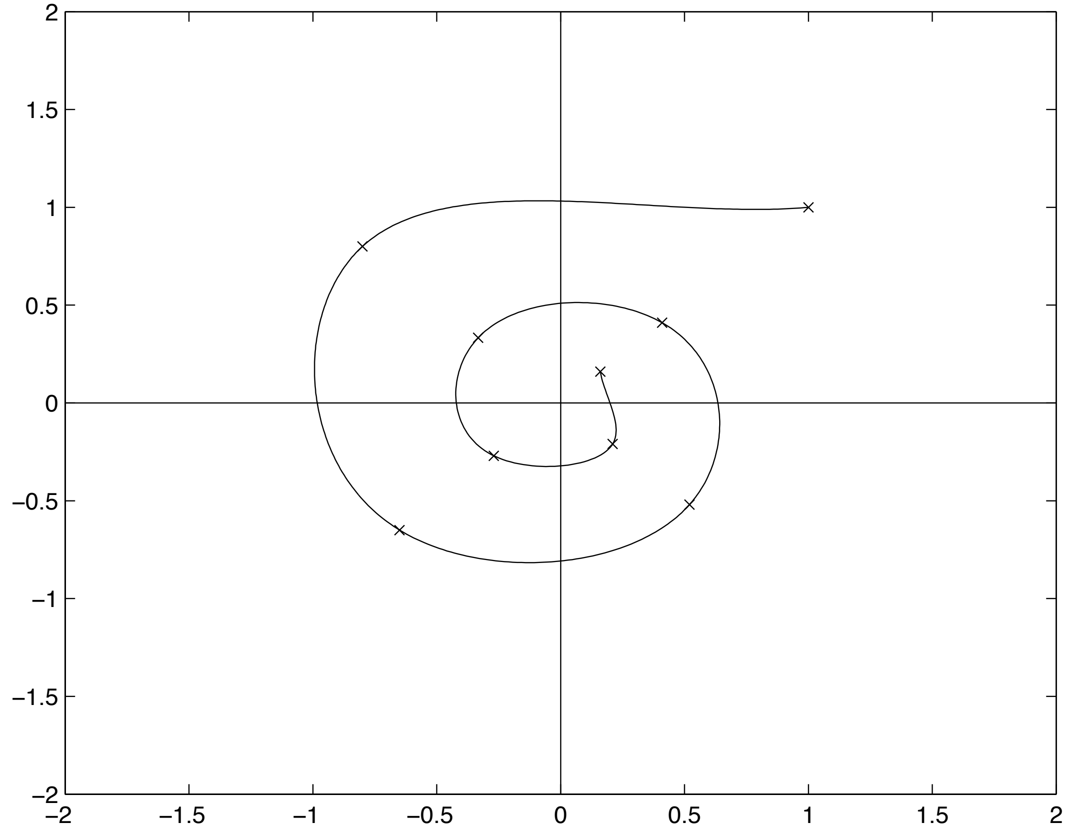
Parameterization

Given $(x_i, y_i), \quad 0 \le i \le n$

Want parametric form $(x(t), y(t))$ for $t_0 \le t \le t_n$ but t is not implicit in the data.

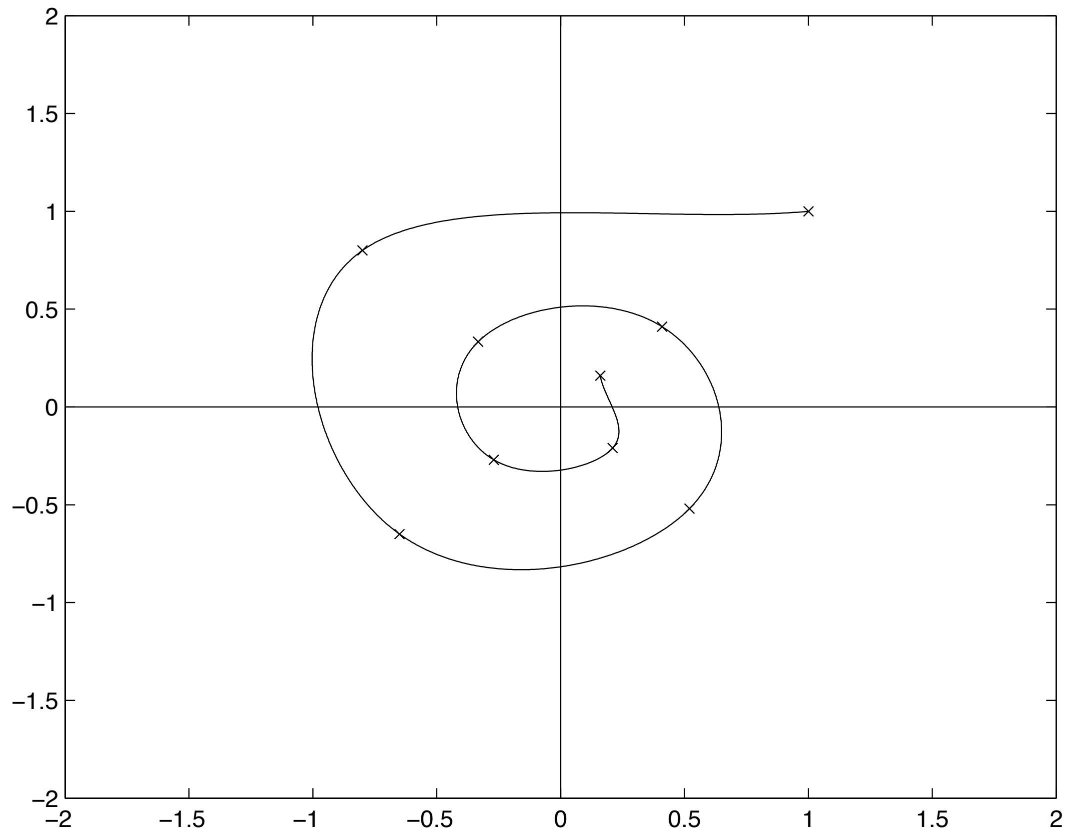
- Simple uniform scale
  - $t_0 \le t \le t_n$ with $t_0 = 0$ and $t_n = 1$
  - (x_i, y_i, t_i) with $t_i = i/n$
- Cumulative length scale
  - $t_0 \le t \le t_n$ with $t_0 = 0$ and $t_n = L$
  - $t_i = \sum_{k=1}^{i} \ell_k \; 1 \le i \le n$ and $\ell_k = \|(x_k, y_k) - (x_{k-1}, y_{k-1})\|_2$
- Fit sets (t_i, x_i) and (t_i, y_i) separately with a spline or other piecewise interpolating curve.
- For a closed curve add (t_{n+1}, x_0) (t_{n+1}, y_0) to each.

Spiral Data



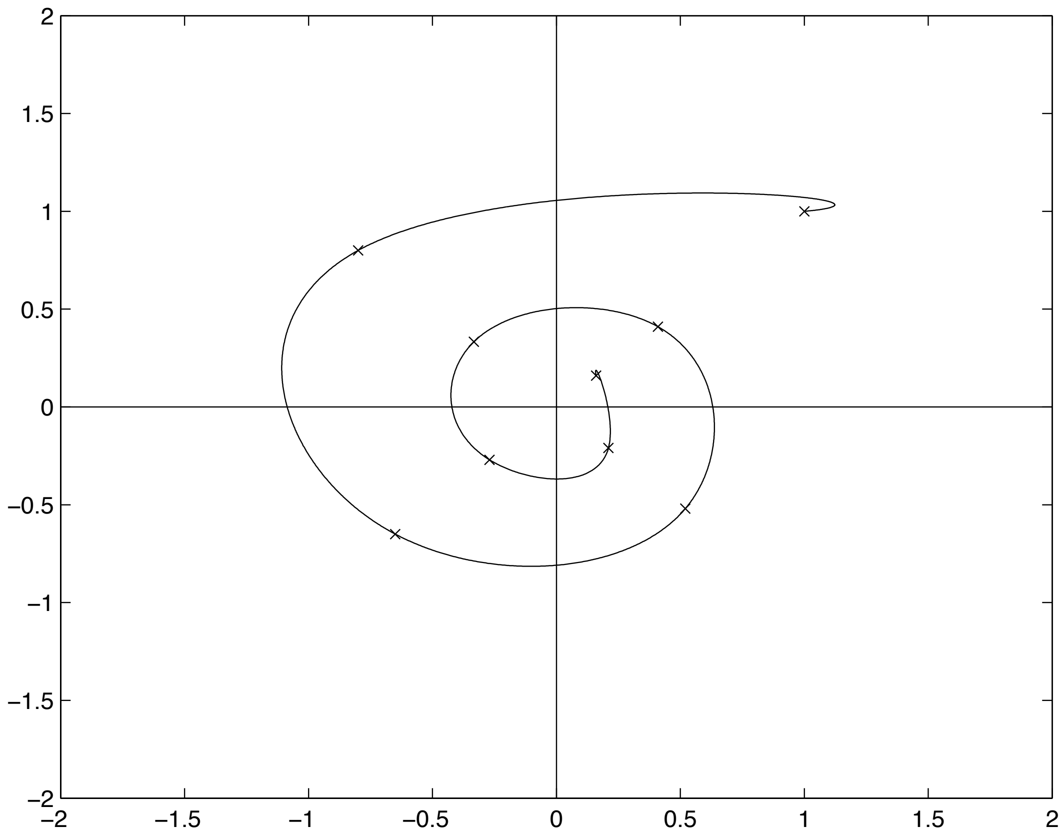
$Ts' = d$ form, first divided difference boundary, arclength

Spiral Data



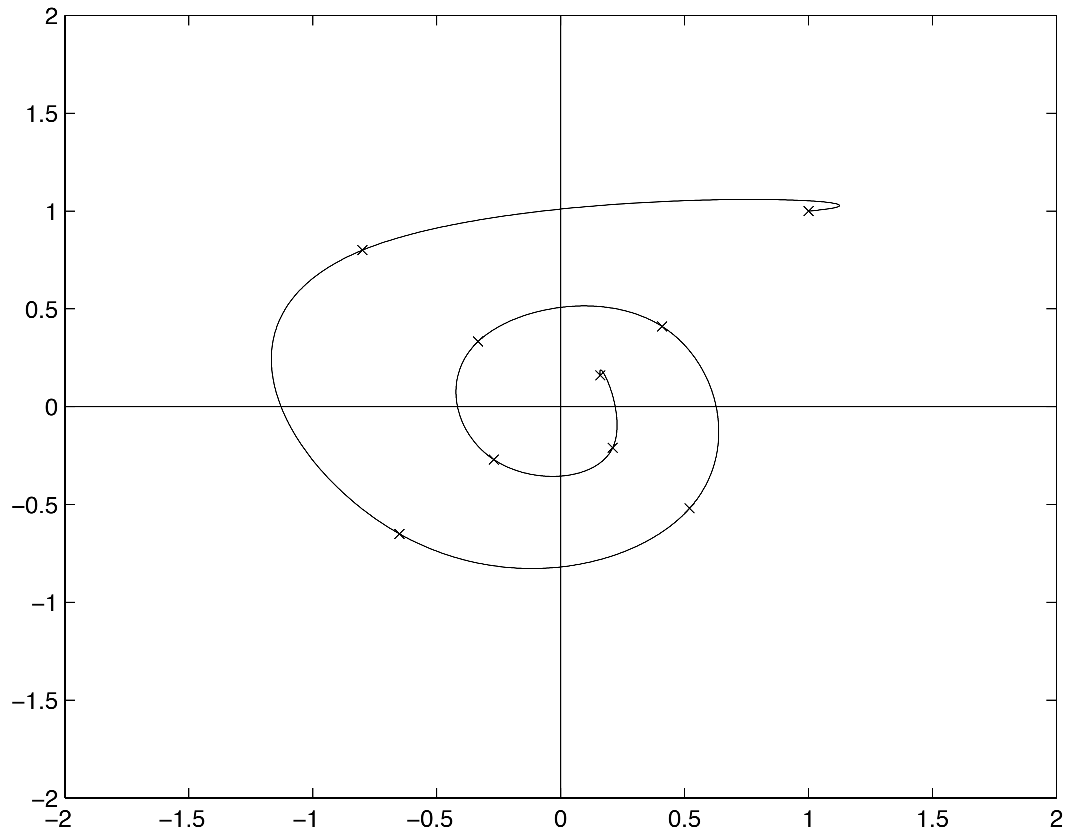
$Ts' = d$ form, first divided difference boundary, uniform

Spiral Data



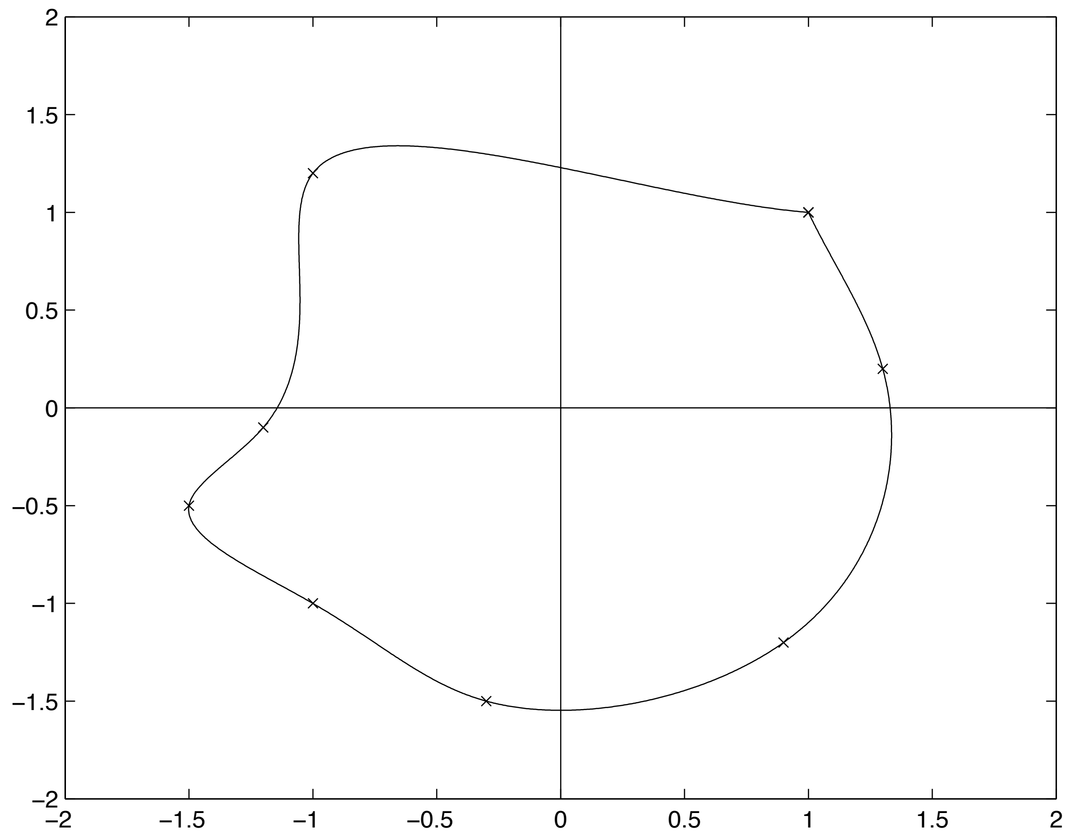
$Ts' = d$ form, negative first divided difference boundary, arclength

Spiral Data



$Ts' = d$ form, negative first divided difference boundary, uniform

Closed Curve Data



$Ts' = d$ form, first divided difference boundary, arclength. Where is (x_0, y_0)?

Non-interpolating Curve

- In graphics, the points may not correspond to points on a curve.
- They may be 'anchor points' defined by a user to produce a particular shape.
- Points may be dragged interactively to affect shape
- Points may be added to affect shape.
- Parametric interpolation is no longer appropriate.
- Bezier curves or B-spline curves

Bezier Curves

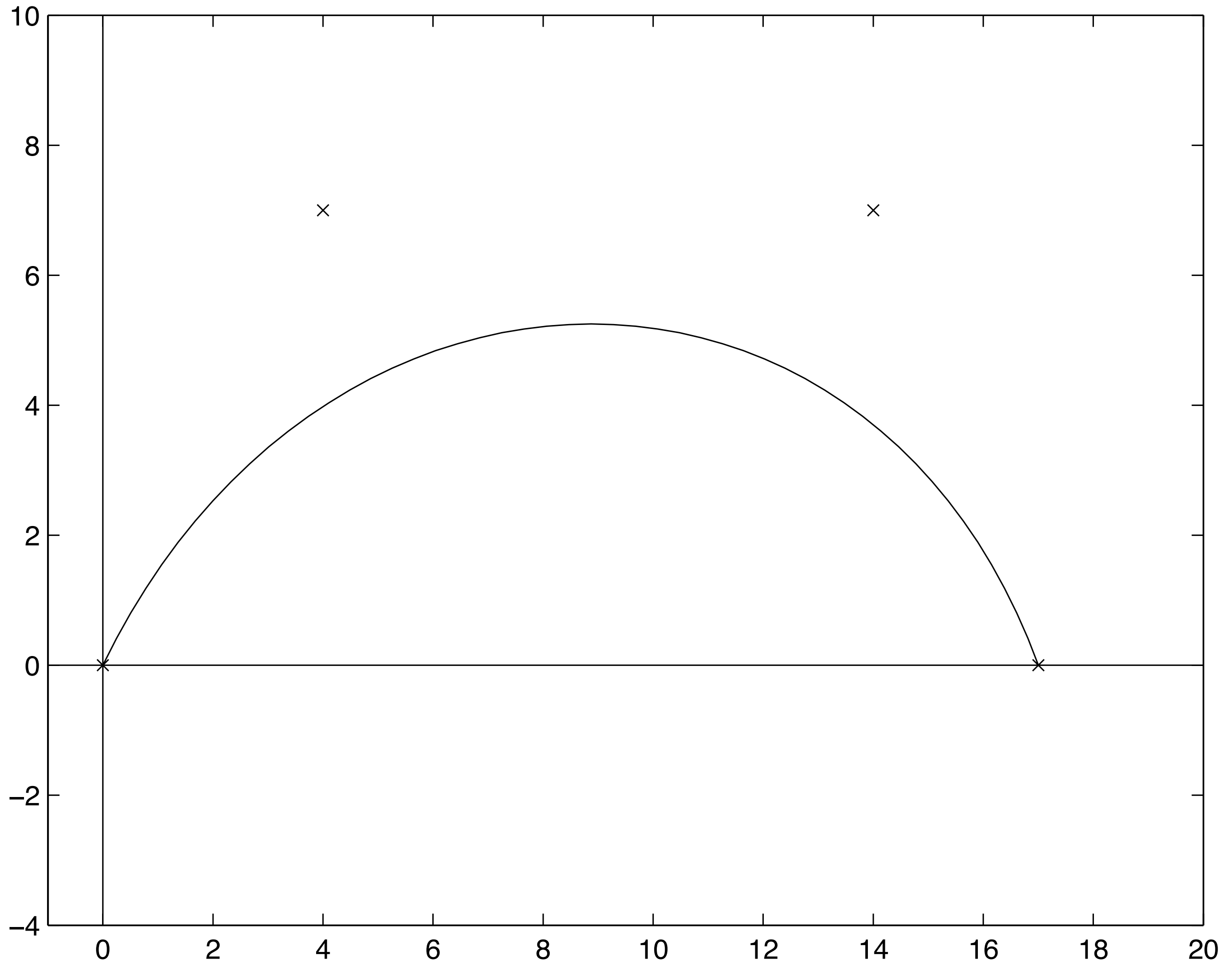
Recall the Bernstein basis polynomials on $0 \le t \le 1$,

$$\phi_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

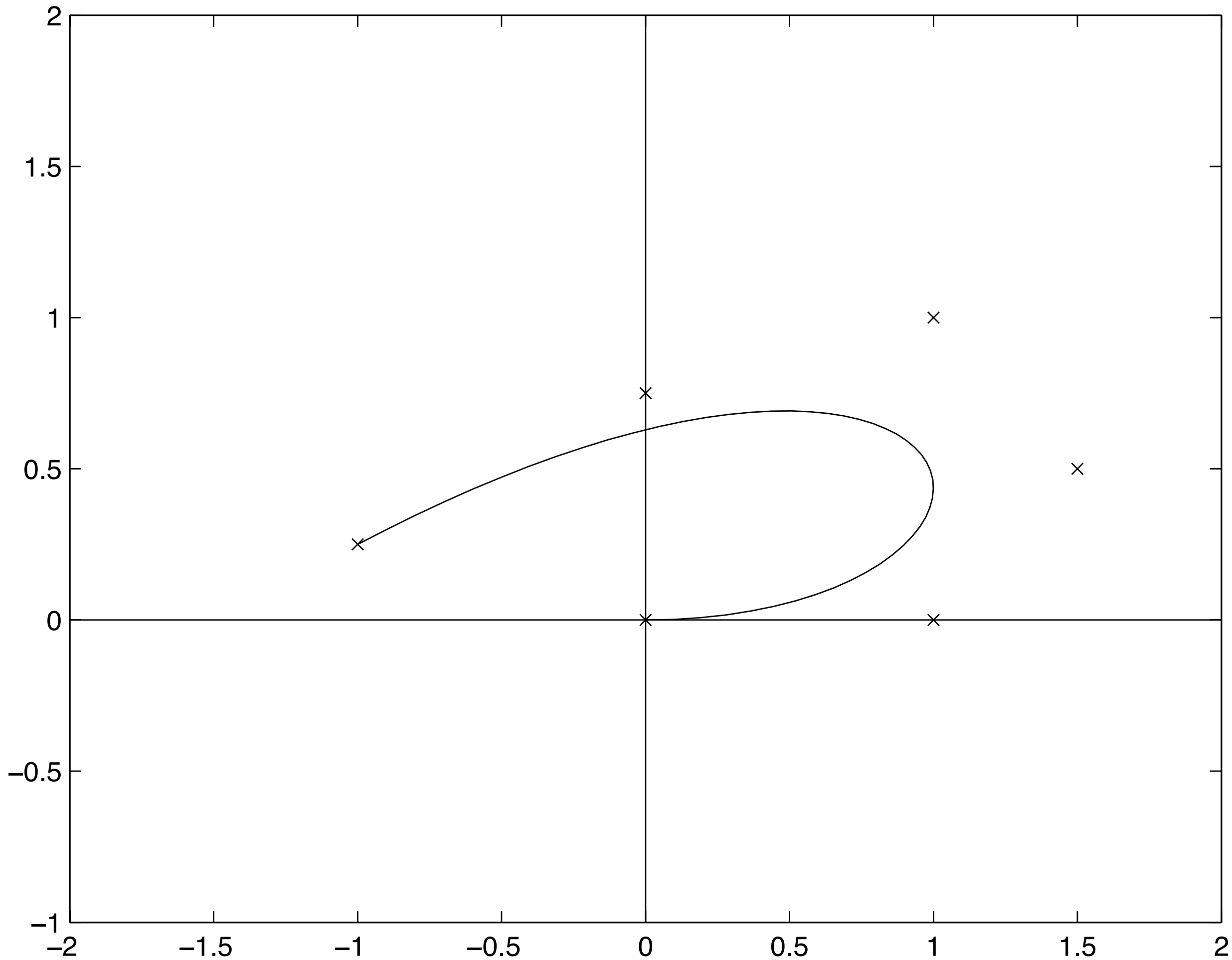
The Bezier curve uses $p_i = (x_i, y_i)$, $0 \le i \le n$, as weights to form $(x(t), y(t))$.

$$\mathcal{B}_n(t) = \sum_{k=0}^{n} p_k \phi_{n,k}(t)$$

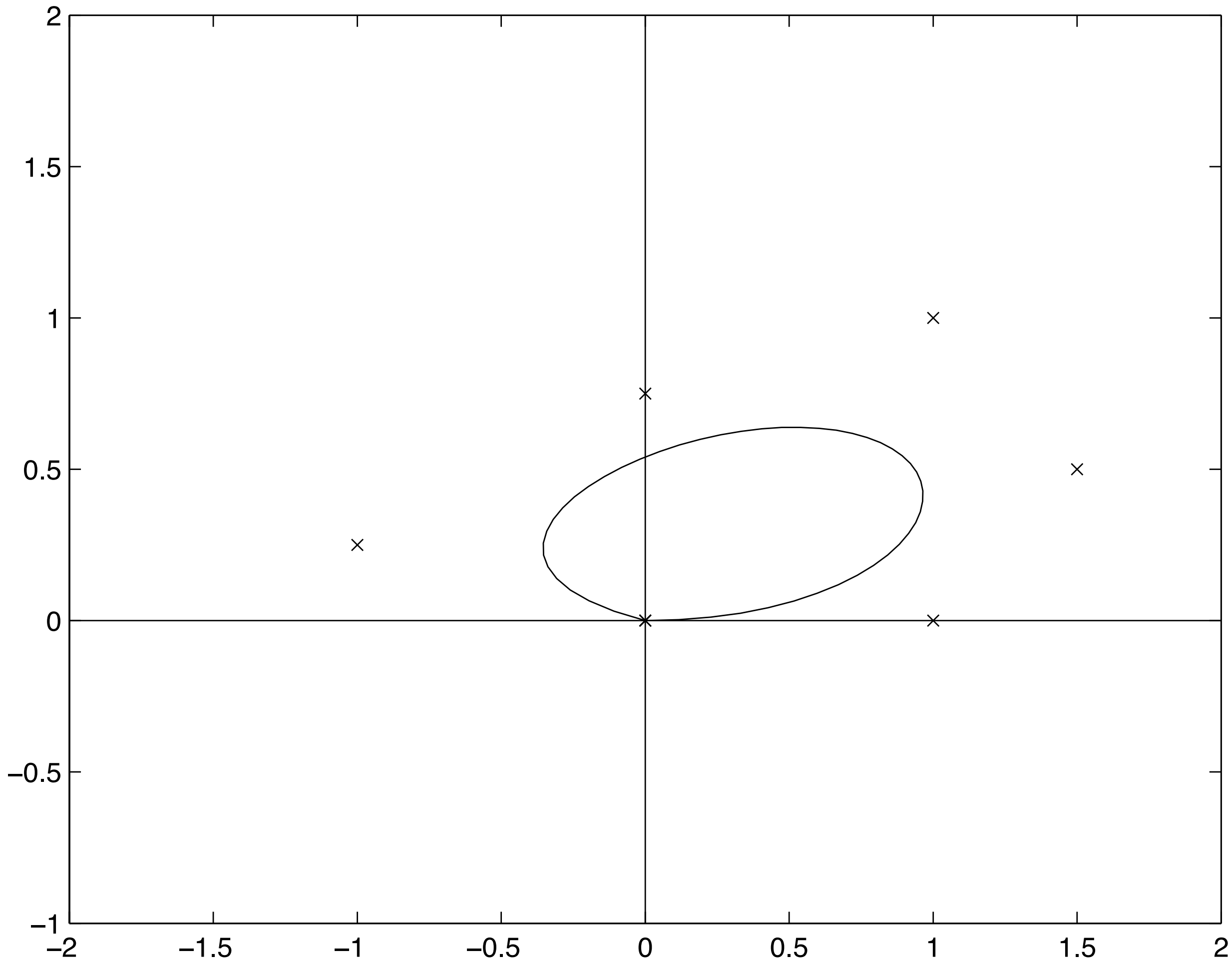
Bezier Curve



Bezier Curve



Bezier Curve



DeCasteljau's Algorithm

Due to the recursive properties of the basis functions, we have an elegant characterization of $\mathcal{B}_n(t)$.

$$
\begin{gathered}
p_{1,i} = (1-t)p_i + tp_{i+1}, \;\; 0 \le i \le n-1 \\
p_{2,i} = (1-t)p_{1,i} + tp_{1,i+1}, \;\; 0 \le i \le n-2 \\
\vdots \\
p_{n,0} = (1-t)p_{n-1,0} + tp_{n-1,1} \\
\mathcal{B}_n(t) = p_{n,0}
\end{gathered}
$$

B-spline Curves

- The Bezier curve is often replaced by the B-spline curve that uses $p_i = (x_i, y_i)$, $0 \le i \le n$, as weights to form $(x(t), y(t))$.
- Two sets of points of interest:
  1. the control points, p_i, $0 \le i \le n$
  2. the knots t_i, $0 \le i \le m$, in the parameter t that define the splines
- Bernstein basis functions replaced by B-splines of degree d.
- It has the form

$$C(t) = \sum_{i=0}^{n} p_i B_{d,i}(t)$$

B-spline Curves

- $B_{d,i}(t)$ involves knots $t_i, \;\; t_{i+1}, \ldots, t_{i+d}, \;\; t_{i+d+1}$
- need $n+d+2$ knots to define $B_{d,i}(t)$, $0 \leq i \leq n$ and, in general, must have $m = n+d+1$
- knots can be simple or have multiplicity k

$$t_i = t_{i+1} = \cdots = t_{i+k-1}$$

- manipulating multiplicity affects shape
- first and last knot with multiplicity $d+1$ clamps curve to first and last point.
- repeating knots and control points closes the curve
- control points and knots can be set separately