# Homework 5

David Miller
MAP5345: Partial Differential Equations 5

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*1. Consider the vector space* $\mathbb{R}^n = \{v = (v_1, \ldots, v_n)$ *such that* $v_1 \in \mathbb{R}, \ldots, v_n \in \mathbb{R}\}$ *and consider the dot product* $u \cdot v = u_1 v_1 + \ldots + u_n v_n$*. Verify that the dot product is an inner product.*

*Proof.* Let $u, v, w$ be in the vector space $\mathbb{R}^n$ and $\alpha$ be some scalar in $\mathbb{R}$. Letting $\langle u, v \rangle$ be the dot product we get

$$\begin{aligned} \langle u+v, w \rangle &= (u_1 + v_1)w_1 + \ldots + (u_n + v_n)w_n \\ &= u_1 w_1 + v_1 w_1 + \ldots + u_n w_n + v_n w_n = \langle u, w \rangle + \langle v, w \rangle \end{aligned}$$

$$\langle \alpha u, v \rangle = \alpha u_1 v_1 + \ldots + \alpha u_n v_n = \alpha(u_1 v_1 + \ldots + u_n v_n) = \alpha \langle u, v \rangle$$

$$\langle u, v \rangle = u_1 v_1 + \ldots + u_n v_n = v_1 u_1 + \ldots + v_n u_n = \langle v, u \rangle$$

$$\langle u, u \rangle = u_1^2 + \ldots + u_n^2 = \begin{cases} > 0 & u_i \neq 0 \text{ for some } i \\ 0 & u_i = 0 \, \forall \, i \end{cases}$$

From this we can see that the dot product is an inner product.

□

*2. Consider the function spaces*

$$\mathcal{F} = \{f : [0, L] \to \mathbb{R} \text{ such that } f, f', f'' \text{ are continuous, and } f(0) = f(L) = 0.\}$$
$$\mathcal{G} = \{g : [0, L] \to \mathbb{R} \text{ such that } g, g', g'' \text{ are continuous, and } g'(0) = g'(L) = 0\}$$

*Show that $\mathcal{F}$ and $\mathcal{G}$ are vector spaces over $\mathbb{R}$ and over $\mathbb{C}$.*

Since $f(x), (x)$ are elements of the fields we define our vector space over, they inherit the axioms of the fields. Let $f_1, f_2, f_3 \in \mathcal{F}$ and $g_1, g_2, g_3 \in \mathcal{G}$ with some constant $\alpha = \beta + i\gamma$ in $\mathbb{C}$ such that $\beta, \gamma \in \mathbb{R}$. Let us redefine the notation for $f, g$

$$[]$$

*3. Consider $\mathcal{F}$ as defined above. For two functions $u(x), v(x) \in \mathcal{F}$, let*

$$\langle u, v \rangle = \int_0^L u(x)v(x)\,dx$$

*Verify that the operation $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{F}$ over $\mathbb{R}$.*

*Proof.* Let $u(x), v(x), w(x) \in \mathcal{F}$ with some scalar in $\mathbb{R}$. Then we have

$$\langle u + v, w \rangle = \int_0^L (u(x) + v(x))w(x)\,dx = \int_0^L u(x)w(x)\,dx + \int_0^L v(x)w(x)\,dx = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle \alpha u, v \rangle = \int_0^L \alpha u(x)v(x)\,dx = \alpha \int_0^L u(x)v(x)\,dx = \alpha \langle u, v \rangle$$

$$\langle u, v \rangle = \int_0^L u(x)v(x)\,dx = \int_0^L v(x)u(x)\,dx = \langle v, u \rangle$$

$$\langle u, u \rangle = \int_0^L u(x)u(x)\,dx = \int_0^L u^2(x)\,dx = \begin{cases} > 0 & u \neq 0 \\ 0 & u = 0 \end{cases}$$

$\square$

*4. Consider the same setup as in 3, but with $\mathcal{F}$ a vector space over $\mathbb{C}$. Let*

$$\langle u, v \rangle = \int_0^L u(x)\overline{v(x)}\, dx$$

*Verify that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{F}$ over $\mathbb{C}$.*

*Proof.* Let $u(x), v(x), w(x) \in \mathcal{F}$ with some scalar $\alpha = \beta + i\gamma$ in $\mathbb{C}$. Then we have

$$\langle u+v, w \rangle = \int_0^L \Big(u(x) + v(x)\Big)\overline{w(x)}\, dx = \int_0^L u(x)\overline{w(x)}\, dx + \int_0^L v(x)\overline{w(x)}\, dx = \langle u, w \rangle + \langle v, w \rangle$$

$$\begin{aligned}\langle u, v+w \rangle &= \int_0^L u(x)\Big(\overline{v(x) + w(x)}\Big)\, dx = \int_0^L u(x)\Big(\overline{v(x)} + \overline{w(x)}\Big)\, dx \\ &= \int_0^L u(x)\overline{v(x)}\, dx + \int_0^L u(x)\overline{w(x)}\, dx = \langle u, v \rangle + \langle u, w \rangle\end{aligned}$$

$$\langle \alpha u, v \rangle = \int_0^L \alpha u(x)\overline{v(x)}\, dx = \alpha \int_0^L u(x)\overline{v(x)} = \alpha \langle u, v \rangle$$

$$\langle u, \alpha v \rangle = \int_0^L u(x)\overline{\alpha v(x)}\, dx = \overline{\alpha} \int_0^L u(x)\overline{v(x)}\, dx = \overline{\alpha} \langle u, v \rangle$$

$$\overline{\langle v, u \rangle} = \int_0^L \overline{v(x)\overline{u(x)}}\, dx = \int_0^L \overline{v(x)} u(x)\, dx = \int_0^L u(x)\overline{v(x)}\, dx = \langle u, v \rangle$$

$$\langle u, u \rangle = \int_0^L u(x)\overline{u(x)}\, dx = \int_0^L \mathcal{R}(u(x))^2 + \mathcal{I}(u(x))^2\, dx = \begin{cases} > 0 & u(x) \neq 0 \\ = 0 & u(x) = 0 \end{cases}$$

where we use $\mathcal{R}(u(x)), \mathcal{I}(u(x))$ as the real and imaginary part of $u(x)$, respectively. $\square$

*5. Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space $V$. For any $v \in V$, let $\|v\| = \sqrt{\langle v, v \rangle}$. Prove that $\|\cdot\|$ is a norm.*

*Proof.* Let $v, w \in V$ then we have

$$\begin{aligned}
\|v\| &= \sqrt{\langle v, v \rangle} = \begin{cases} > 0 & v \neq 0 \\ 0 & v = 0 \end{cases} \\
\|kv\| &= \sqrt{\langle kv, kv \rangle} = \sqrt{k \langle v, kv \rangle} = \sqrt{k^2 \langle v, v \rangle} = |k| \sqrt{\langle v, v \rangle} = |k| \langle v, v \rangle \\
\|v + w\| &=
\end{aligned}$$

$\square$

*6. Consider the inner product for complex-valued functions on the interval $(-L, L)$,*

$$\langle f, g\rangle = \int_{-L}^{L} f(x)\overline{g(x)}\,dx$$

*a) Consider the functions $X_n(x) = e^{in\pi x/L}$. Prove that $X_n$ and $X_m$ are orthogonal for all integers $n, m$ such that $n \neq m$.*

*Proof.*

$$\begin{aligned}
\langle X_m, X_n\rangle &= \int_{-L}^{L} e^{im\pi x/L} e^{in\pi x/L}\,dx \\
&= \int_{-L}^{L} cos((m+n)\pi x/L) + isin((m+n)\pi x/L)\,dx \\
&= \left(\frac{L}{(m+n)\pi}\underbrace{sin((m+n)\pi x/L)}_{=0} - \frac{L}{(m+n)\pi}cos((m+n)\pi x/L)\right)\Bigg|_{-L}^{L} \\
&= \frac{L}{(m+n)\pi}(cos(-(m+n)\pi) - cos((m+n)\pi)) = 0
\end{aligned}$$

□

*b) Find the $L_2$-norm of each of the functions.*

$$\langle X_n, X_n\rangle = \left(\int_{-L}^{L} e^{in\pi x/L}e^{-in\pi x/L}\,dx\right)^{1/2} = \left(\int_{-L}^{L} 1\,dx\right)^{1/2} = \left(x\Big|_{-L}^{L}\right)^{1/2} = \sqrt{2L}$$

*7. Consider the vector space of all differentiable functions on a fixed interval. Define*

$$D[f] = \frac{df}{dx}$$

*Show that $D$ is a linear operator. What is the target space?*

*8. Let $V$ be a vector space and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be liner operators for $V$ to itself.*
*a) Prove hat $a\mathcal{L}_1 + b\mathcal{L}_2$ is also a linear operator for any $a, b \in \mathbb{R}$.*

*Proof.* Let $uv \in V$ and $\mathcal{L}' = a\mathcal{L}_1 + b\mathcal{L}_2$. We then have

$$\begin{aligned}
\mathcal{L}'(k_1 u + k_2 v) &= a\mathcal{L}_1(k_1 u + k_2 v) + b\mathcal{L}_2(k_1 u + k_2 v) \\
&= a\mathcal{L}_1(k_1 u) + a\mathcal{L}_1(k_2 v) + b\mathcal{L}_1(k_1 u) + b\mathcal{L}_2(k_2 v) \\
&= ak_1\mathcal{L}_1(u) + ak_2\mathcal{L}_1(v) + bk_1\mathcal{L}_2(u) + bk_2\mathcal{L}_1(v) \\
&= k_1\mathcal{L}'(u) + k_2\mathcal{L}'(v)
\end{aligned}$$

□

*b) Prove that $\mathcal{L}_1 \circ \mathcal{L}_2$ is also a linear operator.*
*c) Consider the operator*

$$\mathcal{L}[f] = f'''(x) + 2f'(x) - f(x)$$

*Using your results from problem 7 and 8, write a simple proof to show that $\mathcal{L}$ is a linear operator.*

*9. Consider the space of all polynomials written with real coefficients.*
*a) Show that, with suitable definitions of vector addition and scalar multiplication, this is a vector space.*

*Proof.* Let $p_n(x) = a_0 + a_1x + \ldots a_{n-1}x^{n-1} + a_nx^n$ and $q_m(x) = b_0 + b_1x + \ldots + b_{m-1}x^{m-1} + b_mx^m$ be in the space of all polynomials $\mathbb{P}$. We can then define polynomial addition and multiplication as

$$p_n(x) + q_m(x) = a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + a_nx^n + b_0 + b_1x + \ldots + b_{m-1}x^{m-1} + b_mx^m$$
$$kp_n(x) = ka_0 + ka_1x + \ldots + ka_{n-1}x^{n-1} + ka_nx^n, \quad k \in \mathbb{R}$$

Let $k_1, k_2$ be in $\mathbb{R}$. Then we have

$$k_1p_n(x) = k_1a_0 + k_1a_1x + \ldots + k_1a_{n-1}x^{n-1} + k_1a_nx^n \in \mathbb{P}$$

$$\begin{aligned} k_1(p_n(x) + q_m(x)) &= k_1a_0 + k_1a_1x + \ldots + k_1a_{n-1}x^{n-1} + k_1a_nx^n \\ &\quad + k_1b_0 + k_1b_1x + \ldots + k_1b_{m-1}x^{m-1} + k_1b_mx^m = k_1p_n(x) + k_1q_m(x) \end{aligned}$$

$$(k_1 + k_2)p_n(x) = (k_1 + k_2)a_0 + (k_1 + k_2)a_1x + \ldots + (k_1 + k_2)a_{n-1}x^{n-1} + (k_1 + k_2)a_nx^n$$

□

*b) Can you define an inner product on this space? Prove that it is an inner product.*
*c) Can you find a basis for this space? Is the basisi finite or infinite?*

*10. Consider the vector space $V = \mathbb{R}^3$. Consider an arbitrary linear operator mapping $V$ to itself, $\mathcal{L} : V \to V$. Show that such a linear operator can be represented by matrix multiplication.*