SIAM J. NUMER. ANAL.
Vol. 19, No. 2, April 1982

0036-1429/82/1902-0011 $01.00/0

RESTRICTED RANGE APPROXIMATION BY SPLINES AND VARIATIONAL INEQUALITIES*

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Abstract. Error estimates are obtained for the approximation of nonnegative functions from below by nonnegative splines. The maximum norm results are of the same order as the Jackson-type estimates for unconstrained approximation by splines. The L_2 results are used to estimate the rate of convergence of the Ritz method for certain variational inequalities. The more general problem of approximating functions f in the set $W=\{g \in C[a, b]: l(x) \leqq g(x) \leqq u(x),\ x \in [a, b]\}$ by splines in W is also discussed.

1. Introduction. The aim of this paper is to obtain L_p estimates for the error in approximating functions f, with $l(x) \leqq f(x) \leqq u(x)$ on $[a, b]$, by splines s satisfying the same constraint. The most complete results are obtained in the important special case of approximation of nonnegative functions from below by nonnegative splines.

Given a knot sequence, $\mathbf{t}$: $a=t_0<t_1<t_2<\cdots<t_n=b$, and a positive integer k, denote by $\mathcal{S}(k, \mathbf{t})$ the space of all polynomial splines of order k with knots $\mathbf{t}$. Thus $s \in \mathcal{S}(k, \mathbf{t})$ if and only if $s^{(k-2)}$ is continuous on $[a, b]$ and the restriction of s to each interval $[t_i, t_{i+1})$ is a polynomial of degree $\leqq k-1$. The mesh size of $\mathbf{t}$ is $\delta = \max_i (t_{i+1}-t_i)$. For any nonnegative integer j and $1 \leqq p \leqq \infty$ let $L_p^j[a, b]$ be the space of functions which are j-fold integrals of $L_p[a, b]$ functions. Let $C^j[a, b]$ denote the space of all j times continuously differentiable functions on $[a, b]$. Let $\omega(g, I, h)$ denote the usual maximum norm modulus of continuity of g on the interval I. Let $\|\cdot\|_p = \|\cdot\|_{L_p[a,b]}$ denote the usual L_p-norm.

Among our results are the following:

THEOREM 1. *Let k be a positive integer. There is a constant C, depending on k alone, such that if $\mathbf{t}$: $a=t_0<t_1<\cdots<t_n=b$ is a knot sequence and $f \in C^j[a, b]$, for some $0 \leqq j \leqq k-1$, is a nonnegative function then there exists an $s \in \mathcal{S}(k, \mathbf{t})$ with*

$$0 \leqq s(x) \leqq f(x) \quad \textit{for all } x \in [a, b],$$

and

$$\|(f-s)^{(i)}\|_\infty \leqq C\delta^{j-i}\omega(f^{(j)}, [a, b], \delta) \quad \textit{for all } i=0, 1, \cdots, j.$$

THEOREM 2. *Let $1 \leqq p < \infty$ and k be a positive integer. There is a constant C, depending on k alone, such that if $\mathbf{t}$: $a=t_0<t_1<\cdots<t_n=b$ is a knot sequence and $f \in L_p^{j+1}[a, b]$, for some $0 \leqq j \leqq k-1$, is a nonnegative function then there exists an $s \in \mathcal{S}(k, \mathbf{t})$ with*

$$0 \leqq s(x) \leqq f(x) \quad \textit{for all } x \in [a, b],$$

and

$$\|(f-s)^{(i)}\|_p \leqq C\delta^{j+1-i}\|f^{(j+1)}\|_p \quad \textit{for all } i=0, 1, \cdots, j.$$

The analogous results hold for infinite intervals with, of course, infinite knot sequences.

These theorems are already known when $k=2$. In this piecewise linear case they are results of Mosco and Strang [13] and Strang [16]. Strang [17] also gives an analogue of Theorem 2 for piecewise linear approximations in two variables. Mosco and Strang define the partial order $\leqq$ on functions by $g \leqq h$ if and only if $g(x) \leqq h(x)$ for all

* Received by the editors August 14, 1980, and in revised form April 9, 1981.

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$x \in [a, b]$. They choose as an approximation any maximal member, s^*, of the set of splines $s \in \mathcal{S}(2, \mathbf{t})$ which satisfy $0 \leqq s \leqq f$. Such an s^* interpolates to f sufficiently often to guarantee the required degree of approximation. Daniel [6] has shown that these maximal spline arguments do not generalize to higher order splines. He constructs for $f(x) = x^3$ and $[a, b] = [0, 1]$ a sequence $\{s_N\}$ of maximal constrained quadratic splines for which $\|f - s_N\|_\infty$ is of the exact order $(\delta_N)^2$ rather than the expected order $(\delta_N)^3$.

Our proofs use "local techniques" and proceed in two steps. Firstly, the function is approximated by a piecewise polynomial satisfying the constraints. Then this piecewise polynomial approximant, which is probably multivalued, is smoothed to a spline. Such a two-step "plan of attack" was first used by Chui, Smith and Ward [4] in connection with monotone approximation by splines. In the case of the problem at hand the piecewise polynomial step follows easily from results of [1] and it is only the smoothing step that requires new arguments.

Theorems like Theorems 1 and 2 do not hold for polynomials. Consider, for example, $f(x) = e^{-1/x^2}$. The only polynomial q satisfying $0 \leqq q(x) \leqq f(x)$ on $[0, 1]$ is the zero polynomial! If, however, we only require $q(x) \leqq f(x)$ (one-sided approximation) then good approximation by polynomials is possible. (Popov [15] surveys results on one-sided approximation by polynomials and splines.) Also Theorem 1 cannot be generalized to L_p norms and L_p moduli of continuity, $1 \leqq p < \infty$. Consider, for example, f_ε, where $f_\varepsilon(x) = 0$ for $x \in [0, \varepsilon]$ and $f_\varepsilon(x) = x - \varepsilon$ for $x \in [\varepsilon, 1]$. Fix the knot sequence $\mathbf{t}$: $0 = t_0 < t_1 < \cdots < t_n = 1$. Then the L_p modulus of continuity $\omega(f'_\varepsilon, [0, 1], \delta)_p \to 0$ as $\varepsilon \to 0^+$ but (consider the first subinterval) the error in the best constrained approximation

$$E_p^*(f_\varepsilon, 2, \mathbf{t}) = \inf \{\|f_\varepsilon - s\|_p : s \in \mathcal{S}(2, \mathbf{t}) \text{ and } 0 \leqq s \leqq f_\varepsilon\}$$

does not converge to zero as $\varepsilon \to 0^+$.

Daniel [6], Mosco and Strang [13] and Strang [16, 17] were motivated to consider questions of restricted range approximation by applications to the approximate solution of variational inequalities and of problems in control theory. We will consider the first application.

Suppose one seeks to minimize some functional

$$I(v) = a(v, v) - 2(f, v)$$

over a closed convex nonempty set $\mathcal{C}$ in a Hilbert space V. Here $f \in V'$. If $a(u, v)$ is a continuous, coercive, symmetric bilinear form on V then (Lions–Stampacchia [10]) there is a unique solution $u \in \mathcal{C}$ characterized by the inequality

$$a(u, u - v) \leqq (f, u - v) \quad \forall v \in \mathcal{C}.$$

The Ritz method for the approximate solution of such a problem is to approximate $\mathcal{C}$ by a finite dimensional set $\mathcal{C}_N$ and minimize I over $\mathcal{C}_N$ rather than $\mathcal{C}$. If u_N minimizes I over $\mathcal{C}_N$ then one is interested in how close u_N is to u. Many error bounds have been found (Brezzi–Hager–Raviart [3], Falk [7], Hager [8], Mittleman [11], Mosco [12], Mosco–Strang [13]). Most involve estimating the distance from u to $\mathcal{C}_N$ and from u_N to $\mathcal{C}$. This is where approximation theorems of the type of Theorem 2 are needed. They fill the role which in unconstrained problems is taken by theorems about the error in interpolation. We give a concrete example of such an application in § 4.

2. Restricted range approximation by polynomials. In this section we prove a lemma concerning restricted range approximation by polynomials. This will be used later to build piecewise polynomials, satisfying the constraints. The lemma may be interpreted as saying that for polynomials of fixed degree, good restricted range approximation is possible whenever restricted range approximation is possible.

LEMMA 2.1. *Let j be a nonnegative integer, l and u be two extended real-valued functions and*

$$W=\{g\in C[a,b]:\ l(x)\leqq g(x)\leqq u(x)\ on\ [a,b]\}.$$

Suppose $f\in C^j[a,b]\cap W$ and $\pi_j\cap W$ is nonempty. Then there is a $q\in\pi_j\cap W$ such that

$$\|(f-q)^{(i)}\|_\infty\leqq(b-a)^{j-i}\omega(f^{(j)},[a,b],b-a),\qquad i=0,1,\cdots,j. \tag{2.1}$$

Lemma 2.1 follows easily from the following lemma [1, Lemma 3.3]:

LEMMA 2.2. *Let j be a nonnegative integer, l and u be two extended real-valued functions, and*

$$W=\{g\in C[0,1]:\ l(x)\leqq g(x)\leqq u(x)\ on\ [0,1]\}.$$

Suppose $f\in C^j[0,1]\cap W$ and $\pi_j\cap W$ is nonempty. Then there is at least one $q\in\pi_j\cap W$ which interpolates to $f\,j+1$ times in a Hermite sense. That is, there exist points $0\leqq z_1<\cdots<z_m=1$ and positive integers d_i so that

$$q^{(k)}(z_i)=f^{(k)}(z_i),\qquad k=0,\cdots,d_i-1,$$

and

$$\sum_{i=1}^m d_i=j+1.$$

Proof of Lemma 2.1. It is sufficient to prove the lemma when $[a,b]=[0,1]$ since the general case follows by change of variables. From Lemma 2.2 and Rolle's theorem there exist points $\xi_0,\cdots,\xi_j$ in $[0,1]$ such that

$$f^{(i)}(\xi_i)=q^{(i)}(\xi_i)\qquad i=0,1,\cdots,j.$$

Then for each $i=0,1,\cdots,j-1$

$$\|(f-q)^{(i)}\|_\infty=\left\|\int_{\xi_i}^x(f-q)^{(i+1)}(t)\,dt\right\|_\infty\leqq\|(f-q)^{(i+1)}\|_\infty$$

so that arguing inductively, using that $q^{(j)}(x)$ is constant, we find

$$\|(f-q)^{(i)}\|_\infty\leqq\omega(f^{(j)},[0,1],1),\qquad i=j,j-1,\cdots,0.$$

This establishes (2.1). □

3. Restricted range approximation by splines. In this section we will prove theorems concerning restricted range approximation by splines, special cases of which are Theorems 1 and 2.

The following notation will be used throughout this section. $C_1, C_2,\cdots$ will denote constants depending only on k. Given a strictly increasing knot sequence $\mathbf{t}=\{t_i\}$, let $\{N_{i,j}\}$ be the normalized B-splines of order j with knots $\mathbf{t}$. The normalization is so that $\sum_{e=i-j+1}^{i}N_{e,j}(x)=1$ for all $x\in[t_i,t_{i+1})$. Also define the mesh length of $\mathbf{t}$ to be $\delta=\max_i(t_{i+1}-t_i)$ and the minimum subinterval length to be $m=\min_i(t_{i+1}-t_i)$.

Let $k\geqq2$ be an integer and $d=2(k-1)^2$. Given a partition $\mathbf{t}$: $a=t_0<t_1<\cdots<t_N=b$ of $[a,b]$ define M as the integer part of N/d. If $M\leqq2$ define $I_0=[a,b]$. Otherwise define points $x_i=t_{id}$, $0\leqq i\leqq M$, $x_{M+1}=t_N$, and intervals $I_i=[x_i,x_{i+2}]$, $0\leqq i\leqq M-1$. We will prove:

THEOREM 3. *Let $k\geqq2$ be an integer. There exists a constant C, depending only on k, with the following property. Let l,f, and u be extended real-valued functions with*

$l(x) \leqq f(x) \leqq u(x)$ on $[a, b]$. Suppose $f \in C^j[a, b]$, for some $0 \leqq j \leqq k-1$, and to each interval I_e, defined above, there corresponds a polynomial $p_e \in \pi_j$ such that

$$l(x) \leqq p_e(x) \leqq u(x) \quad \text{for } x \in I_e. \tag{3.1}$$

Then there exists an $s \in \mathcal{S}(k, \mathbf{t})$ with

$$l(x) \leqq s(x) \leqq u(x) \quad \text{for } x \in [a, b], \tag{3.2}$$

and

$$\|(f-s)^{(i)}\|_{L_\infty[a,b]} \leqq C\left(\frac{\delta}{m}\right)^i \delta^{j-i}\omega(f^{(j)}, [a, b], \delta) \quad \text{for } i = 0, 1, \cdots, j. \tag{3.3}$$

THEOREM 4. *Let $1 \leqq p < \infty$ and $k \geqq 2$ be an integer. There exists a constant C, depending only on k, with the following property. Let l, f, and u be extended real valued functions with $l(x) \leqq f(x) \leqq u(x)$ on $[a, b]$. Suppose $f \in L_p^{j+1}[a, b]$, for some $0 \leqq j \leqq k-1$, and to each interval I_e, defined above, there corresponds a polynomial $p_e \in \pi_j$ such that*

$$l(x) \leqq p_e(x) \leqq u(x) \quad \text{for } x \in I_e. \tag{3.4}$$

Then there exists an $s \in \mathcal{S}(k, \mathbf{t})$ with

$$l(x) \leqq s(x) \leqq u(x) \quad \text{for } x \in [a, b], \tag{3.5}$$

and

$$\|(f-s)^{(i)}\|_{L_p[a,b]} \leqq C\left(\frac{\delta}{m}\right)^i \delta^{j+1-i}\|f^{(j+1)}\|_{L_p[a,b]} \quad \text{for } i = 0, 1, \cdots, j. \tag{3.6}$$

For $k = 1$, that is, approximation by piecewise constant functions, Theorem 1 is obvious. For $k \geqq 2$, Theorem 1 is an easy corollary of Theorem 3. Firstly, we can remove the dependence on (δ/m) by selecting from the sequence $\mathbf{t}$ a subsequence $\hat{\mathbf{t}}$: $a = \hat{t}_0 < \hat{t}_1 < \cdots < \hat{t}_r = b$ which is quasi-uniform. It is possible to do this so that

$$\tfrac{1}{2}\delta \leqq \min_i (\hat{t}_{i+1} - \hat{t}_i) \leqq \max_i (\hat{t}_{i+1} - \hat{t}_i) \leqq 2\delta,$$

where $\delta = \max_i (t_{i+1} - t_i)$ (see, for example, de Boor [2, pp. 197–198]). Secondly, since the zero polynomial satisfies all the inequalities (3.1) when $l(x) = 0$, Theorem 3 applies to the approximation of nonnegative functions, f, by splines $s \in \mathcal{S}(k, \hat{\mathbf{t}}) \subset \mathcal{S}(k, \mathbf{t})$ with

$$0 = l(x) \leqq s(x) \leqq f(x) = u(x) \quad \text{for } x \in [a, b].$$

Similar arguments may be used to derive Theorem 2 from Theorem 4.

The hypothesis contained in inequalities (3.1) and (3.4) is a statement about the smoothness of functions that can be fitted between l and u. The author does not know if it can be weakened. However, (3.1) cannot be replaced by the mere existence of an $s \in \mathcal{S}(k, \mathbf{t})$ with $l(x) \leqq s(x) \leqq u(x)$ on $[a, b]$. This follows from the counterexample of Daniel [6], discussed in the introduction (choose l as the maximal spline and $u = f = x^3$). The hypothesis is easily checked when constraints are applied only at a finite number of points, for example at the knots. Of course, the hypothesis is trivially satisfied when l or u belongs to π_j, as in Theorems 1 and 2.

We will need a generalization of a lemma first proven by Chui, Smith and Ward. The proof is the one due to C. de Boor which appeared in [5].

LEMMA 3.1. *Let* $\mathbf{t}=\{t_i\}_{i=-\infty}^{\infty}$ *be a strictly increasing knot sequence. Let* p *be a polynomial of degree* $\leqq k-1$ *and for* $0\leqq j\leqq k-1$ *let*

$$p^{(j)}(x)=\sum_{i=-\infty}^{\infty}\alpha_i^{(j)}N_{i,k-j}.$$

Then the sequence $\{\alpha_i^{(j)}\}_{i=-\infty}^{\infty}$ *has at most* $k-j-1$ *sign changes.*

Proof. A well-known analogue of Rolle's theorem states that if $\{a_i\}$ has n sign changes then $\{\Delta a_i\}$ has no fewer than $n-1$ sign changes. Now for every j, $k-1\geqq j>0$, $\alpha_i^{(j)}$ is a positive multiple of $(\alpha_i^{(j-1)}-\alpha_{i-1}^{(j-1)})$. Hence, by the discrete Rolle's theorem, $\{\alpha_i^{(j-1)}\}$ has at most one more sign change than $\{\alpha_i^{(j)}\}$. The lemma follows since $\{\alpha_i^{(k-1)}\}$ is constant. □

The "smoothing lemma" below concerns the smoothing of overlapping polynomial pieces to C^{k-2} splines. A similar lemma was first established by the author in the special case of $l=0$, $u=f$, and equally spaced knots. Professor R. A. DeVore then gave a simpler proof of the special case. The proof below is a modification of his proof.

LEMMA 3.2. *Let* $k\geqq 2$ *be an integer and* $d=2(k-1)^2$. *There exists a constant* C, *depending only on* k, *with the following property. Let* $\mathbf{t}=\{t_i\}_{i=-\infty}^{\infty}$ *be a strictly increasing knot sequence with* $t_0=a$ *and* $t_d=b$. *Let* $f\in C^j[a,b]$, *for some* $0\leqq j\leqq k-1$, *and* f_1, f_2 *be two polynomials of degree* $\leqq k-1$ *approximating* f *with*

$$\|f-f_i\|_{L_\infty[a,b]}\leqq \delta^j\varepsilon \quad \text{for } i=1,2,$$

where $\varepsilon\geqq\omega(f^{(j)},[a,b],\delta)$. *Then there exists a spline* $s\in\mathcal{S}(k,\mathbf{t})$ *such that*

$$s(x) \text{ is a number between } f_1(x) \text{ and } f_2(x) \text{ for } x\in[a,b], \tag{3.7}$$

$$s=f_1 \quad \text{on } (-\infty,a], \qquad s=f_2 \quad \text{on } [b,\infty) \tag{3.8}$$

and

$$\|(f-s)^{(i)}\|_{L_\infty[a,b]}\leqq C\left(\frac{\delta}{m}\right)^i\delta^{j-i}\varepsilon, \quad \text{for } i=0,1,\cdots,j. \tag{3.9}$$

Remark. Since s has breakpoints only in $[a,b]$, the spacing of the knots outside $[a,b]$ is unimportant. In particular the lemma is true with $\delta=\max_{0\leqq i<d}(t_{i+1}-t_i)$ and $m=\min_{0\leqq i<d}(t_{i+1}-t_i)$.

Proof. Let $f_1=\sum\alpha_iN_{i,k}$, $f_2=\sum\beta_iN_{i,k}$ and $(f_1-f_2)=\sum(\alpha_i-\beta_i)N_{i,k}$. Then f_1-f_2 is a polynomial of degree $\leqq k-1$ and by Lemma 3.1 $\{(\alpha_i-\beta_i)\}_{i=-\infty}^{\infty}$ has at most $k-1$ sign changes. Hence among the $(2k-3)(k-1)+(2k-2)\cdot 1$ indices in the interval $[1-k,d)$ there must be at least one segment $I=\{l-(k-1),\cdots,l+k-2\}$ of $2k-2$ indices i on which $(\alpha_i-\beta_i)$ does not change sign. Define $s=\sum\gamma_iN_{i,k}$ where

$$\gamma_i=\begin{cases}\alpha_i, & i<l,\\ \beta_i, & i\geqq l.\end{cases}$$

Then $s=f_1$ on $(-\infty,t_l]$ and $s=f_2$ on $[t_{l+k-1},\infty)$. On (t_l,t_{l+k-1}) we have two cases. If $(\alpha_i-\beta_i)\geqq 0$, $i\in I$, then for $x\in(t_l,t_{l+k-1})$ we have

$$f_2(x)=\sum\beta_iN_{i,k}(x)\leqq s(x)=\sum\gamma_iN_{i,k}\leqq\sum\alpha_iN_{i,k}(x)=f_1(x).$$

When $(\alpha_i-\beta_i)\leqq 0$, $i\in I$, then

$$f_1(x)\leqq s(x)\leqq f_2(x), \qquad x\in(t_l,t_{l+k-1}).$$

This proves (3.7) and (3.8) and only the error estimates (3.9) remain.

Since at each point x

$$|(f-s)(x)| \leqq \max_{i\in\{1,2\}} |(f-f_i)(x)|,$$

we conclude

$$\|f-s\|_{L_\infty[a,b]} \leqq \delta^j \varepsilon.$$

The estimates for $\|(f-s)^{(i)}\|_\infty$ follow by the standard device of approximating f (and its derivatives) by a piecewise polynomial interpolant p (and its derivatives), and using Markov's inequality to estimate $\|(s-p)^{(i)}\|_\infty$. $\square$

We are now ready to prove Theorems 3 and 4.

Proof of Theorem 3. Recall that $\mathbf{t}$: $a=t_0<t_1<\cdots<t_N=b$ is a partition of $[a,b]$. Also $d=2(k-1)^2$ and M is the integer part of N/d. If $M\leqq 2$, $I_0=[a,b]$. Otherwise $x_i=t_{id}$, $0\leqq i\leqq M$, $x_{M+1}=t_N$ and $I_i=[x_i, x_{i+2}]$, $0\leqq i\leqq M-1$.

For $M\leqq 2 (N<3d)$ the spline result follows easily from the polynomial result of Lemma 2.1. Hence, we assume in what follows, $M>2$.

By applying Lemma 2.1 on each of the intervals $I_0, I_1, \cdots, I_{M-1}$ we find polynomials f_e of degree $\leqq j$, satisfying

$$l(x)\leqq f_e(x)\leqq u(x) \quad \text{for } x\in I_e, \tag{3.10}$$

and

$$\|(f-f_e)^{(i)}\|_{L_\infty(I_e)} \leqq |I_e|^{j-i}\omega(f^{(j)}, I_e, |I_e|), \qquad 0\leqq i\leqq j. \tag{3.11}$$

Here $|I_e|$ denotes the length of the interval I_e and does not exceed $2d\delta$. Hence from (3.11) and the properties of $\omega(\cdot,\cdot,\cdot)$ we obtain

$$\|(f-f_e)^{(i)}\|_{L_\infty(I_e)} \leqq C_1\delta^{j-i}\omega(f^{(j)}, I_e, \delta), \qquad 0\leqq i\leqq j, \quad 0\leqq e\leqq M-1. \tag{3.12}$$

We proceed to smooth our multivalued approximation to C^{k-2}. For each $0\leqq e\leqq M-2$ smooth the polynomials f_e and f_{e+1} together on $I_e\cap I_{e+1}=[x_{e+1}, x_{e+2}]$ using Lemma 3.2 and estimate (3.12). Call the resulting element of $\mathcal{S}(k,\mathbf{t})$ s_e. We define the piecewise polynomial s by taking $s=f_0$ on $[x_0,x_1]$; $s=s_e$ on $[x_{e+1},x_{e+2}]$, $0\leqq e\leqq M-2$; and $s=f_{M-1}$ on $[x_M, x_{M+1}]$.

Since $l(x)\leqq f_e(x)\leqq u(x)$ on I_e ((3.10)), and for each x, $s_e(x)$ lies between $f_e(x)$ and $f_{e+1}(x)$, we see $l(x)\leqq s(x)\leqq u(x)$ on $[a,b]$. Since $s_{e-1}=f_e$ for $x\geqq x_{e+1}$ and $s_e=f_e$ for $x\leqq x_{e+1}$, the spline pieces join up in a globally C^{k-2} manner. Thus $s\in\mathcal{S}(k,\mathbf{t})$.

It remains only to show the error estimates. From (3.12) and inequality (3.9) of Lemma 3.2,

$$\|(f-s)^{(i)}\|_{L_\infty(I_e\cap I_{e+1})} \leqq C_2\left(\frac{\delta}{m}\right)^i \delta^{j-i}\omega(f^{(j)}, I_e\cup I_{e+1}, \delta), \tag{3.13}$$

for $0\leqq i\leqq j$ and $0\leqq e\leqq M-2$. On $[x_0,x_1]$ and $[x_M, x_{M+1}]$ we have the (stronger) estimate (3.12). Hence

$$\|(s-f)^{(i)}\|_{L_\infty[a,b]} \leqq C_3\left(\frac{\delta}{m}\right)^i \delta^{j-i}\omega(f^{(j)}, [a,b], \delta)$$

for $0\leqq i\leqq j$, as required. $\square$

Proof of Theorem 4. The proof of Theorem 4 is identical with that of Theorem 3 up to inequality (3.13). To complete the proof one must relate L_∞ and L_p norms locally and take sums.

It will be convenient to define $x_{-1}=x_0=a$, and $x_{M+2}=x_{M+1}=b$. Now for $0\leqq i\leqq j$, $0\leqq e\leqq M$

$$\text{(3.14)}\quad \begin{aligned} \|(s-f)^{(i)}\|_{L_p[x_e,x_{e+1}]} &\leqq \left(\int_{x_e}^{x_{e+1}} \|(s-f)^{(i)}\|^p_{L_\infty[x_e,x_{e+1}]}\,d\nu\right)^{1/p} \\ &\leqq C_4\left[\left(\frac{\delta}{m}\right)^i \delta^{j-i}\omega(f^{(j)},[x_{e-1},x_{e+2}],\delta)\right]\delta^{1/p} \end{aligned}$$

by (3.12), (3.13) and since $x_{e+1}-x_e\leqq d\delta$. Since $f\in L_p^{j+1}[a,b]$ and $x_{e+2}-x_{e-1}\leqq 3\,d\delta$ an application of Hölder's inequality yields

$$\text{(3.15)}\quad \omega(f^{(j)},[x_{e-1},x_{e+2}],3\,d\delta)\leqq \int_{x_{e-1}}^{x_{e+2}} 1\cdot|f^{(j+1)}(\nu)|\,d\nu \leqq C_5\delta^{1-1/p}\|f^{(j+1)}\|_{L_p[x_{e-1},x_{e+2}]}.$$

Substituting (3.15) into (3.14),

$$\|(s-f)^{(i)}\|_{L_p[x_e,x_{e+1}]}\leqq C_6\left(\frac{\delta}{m}\right)^i \delta^{j+1-i}\|f^{(j+1)}\|_{L_p[x_{e-1},x_{e+2}]},$$

$0\leqq i\leqq j$, $0\leqq e\leqq M$. Finally taking pth powers, summing (using that each subinterval $[x_l, x_{l+1}]$ is contained in at most 3 subintervals $[x_{e-1}, x_{e+2}]$) and then taking pth roots we find

$$\|(s-f)^{(i)}\|_{L_p[a,b]}\leqq C_7\left(\frac{\delta}{m}\right)^i \delta^{j+1-i}\|f^{(j+1)}\|_{L_p[a,b]},\qquad 0\leqq i\leqq j. \qquad \square$$

Remark. Theorem 3 remains true if we replace (3.3) by a local estimate. For example, if $x\in[x_e, x_{e+1}]$ then the same proof shows

$$|(f-s)^{(i)}(x)|\leqq C\left(\frac{\delta_e}{m_e}\right)^i \delta_e^{j-i}\,\omega(f^{(j)},[x_{e-1},x_{e+2}],\delta_e),$$

where

$$\delta_e=\max_{\{i:\,x_{e-1}\leqq t_i<t_{i+1}\leqq x_{e+2}\}}(t_{i+1}-t_i),\qquad m_e=\min_{\{i:\,x_{e-1}\leqq t_i<t_{i+1}\leqq x_{e+2}\}}(t_{i+1}-t_i).$$

Thus for constrained approximation, just as for unconstrained approximation, accuracy can often be improved by judicious knot placement.

4. Application: Error estimates for Ritz approximations to solutions of variational inequalities. We consider a constrained variational problem of the type mentioned in the introduction. Let $(f, v)=\int_0^1 f(x)v(x)\,dx$, $k\geqq 1$,

$$V=H_0^k[0,1]=\{v\in L_2^k[0,1]:4v^{(j)}(0)=v^{(j)}(1)=0,\ j=0,\cdots,k-1\},$$

and

$$a(u,v)=\int_0^1\sum_{j=0}^k p_j(x)u^{(j)}(x)v^{(j)}(x)\,dx$$

where

$$p_j\in L_\infty[0,1],\qquad j=0,\cdots,k,$$

$$p_k(x)\geqq\min_{[0,1]}p_k(x)>0,\qquad x\in[0,1],$$

and

$$p_j(x) \geqq 0, \qquad x \in [0, 1], \quad j = 0, \cdots, k-1.$$

The assumptions guarantee the existence of positive constants M and γ so that

$$\gamma\|v\|_{H^k}^2 \leqq a(v, v) \quad \text{and} \quad |a(u, v)| \leqq M\|u\|_{H^k}\|v\|_{H^k},$$

for all $u, v \in H_0^k$. As usual the Sobolev space norm is defined by

$$\|u\|_{H^k}^2 = \sum_{i=0}^{k} \|u^{(i)}\|_2^2.$$

We suppose $f \in L_2[0, 1]$, $m \in H_0^k$ and set

$$\mathscr{C} = \{v \in H_0^k : v(x) \geqq m(x) \text{ on } [0, 1]\}.$$

We seek that $u \in \mathscr{C}$ which minimizes

$$I(v) = a(v, v) - 2(f, v).$$

Let $N \geqq 1$ be an integer, $h = 1/N$ and $t_j = jh$, $j = 0, 1, \cdots, N$. Let $\mathscr{S}_h = \mathscr{S}(i, \mathbf{t}) \cap H_0^k[0, 1]$, $i > k$, and $m_h \in \mathscr{S}_h$ be an approximation to m. Let $\mathscr{C}_h = \{v_h \in \mathscr{S}_h : v_h(x) \geqq m_h(x) \text{ on } [0, 1]\}$. The Ritz approximation to u is the function $u_h \in \mathscr{C}_h$ which minimizes $I(v)$ over $\mathscr{C}_h$. We then have the following lemma of Mosco–Strang [13] and Mosco [12]. They state it only for second order variational inequalities ($k = 1$) but the same proof yields:

LEMMA 4.1. *Let $U = u - m$ and $V_h \in \mathscr{S}(i, \mathbf{t})$ satisfy $0 \leqq V_h(x) \leqq U(x)$ for $x \in [0, 1]$. Then*

$$\|u - u_h\|_{H^k} \leqq \frac{M}{\gamma}[\|m - m_h\|_{H^k} + \|U - V_h\|_{H^k}].$$

Suppose $m \in H^i[0, 1] \cap H_0^k[0, 1]$ and $u \in H^i[0, 1] \cap H_0^k[0, 1]$. Then by choosing $m_h \in \mathscr{S}_h$ as a suitable interpolant or quasi-interpolant to m, and choosing V_h as the approximation to U whose existence is guaranteed by Theorem 2, we have

$$\|u - u_h\|_{H^k} \leqq Ch^{i-k}(\|u^{(i)}\|_2 + \|m^{(i)}\|_2)$$

where C is a constant depending only on i, M and γ. This estimate is of the same order as that obtained in the absence of constraints.

Remarks. For $k = 1$ it is known that suitable smoothness hypotheses on the data will guarantee $u \in H^2[0, 1]$ but not $u \in C^2[0, 1]$ (Lewy–Stampacchia [8]). Also for $k = 1$, $i = 2$ Natterer [13] has shown how to obtain optimal L_2, rather than energy norm, estimates for $u - u_h$. See also Mosco [11].

Acknowledgments. I am grateful to Professor J. W. Daniel for interesting me in this problem and to Professor R. A. DeVore for his contribution to Lemma 3.2.

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