Set 0: Overview

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Foundations of Computational Math 1
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Overview

- Numerical Considerations
  - Finite representation of numbers
  - Finite precision arithmetic
  - Numerical stability: effects of discretization, algorithm design, and finite precision on the relationship between the computed solution and the true solution
- Basic Problems
  - Interpolation of a function
  - Approximation of a function

Overview

- Application of Techniques
  - Numerical differentiation of a function
  - Numerical quadrature (evaluation of a definite integral)
  - Numerical integration (solution of an ODE or PDE as IVP or BVP)
- Analysis of efficiency and effectiveness of algorithms
  - computational cost
  - storage cost
  - numerical robustness

Interpolation and Approximation

- Given a function $f(x)$ in some form:
  - symbolically/analytically
  - discrete values of f and possibly some of its derivatives
- Given assumptions about $f(x)$
  - smooth, number of continuous derivatives, periodic etc.
  - bounds on certain derivatives
- Given a class of functions $\mathcal{S}$ and a parameterization.
- Find $g \in \mathcal{S}$ that satisfies some constraints, e.g.,
  - g agrees with f and/or its derivatives at specified points
  - g is closest to f using some metric

Discrete Function Values

For some problems the starting point data comprises discrete values (x_i, y_i) for $0 \leq i \leq n$ where both are known or just x_i are known.

1. Tables of functions: each y_i is the result of extensive computations
2. Sampled functions: $y_i = f(x_i)$
   - x_i dictated by someone else, e.g., financial market reporting
   - x_i chosen relative to knowledge or assumptions about $f(x)$, e.g., sampling of images or time-varying signals
3. x_i may be known and y_i an unknown: assumptions on $f(x)$ are used to generate relationships between several y_i to integrate ODEs or PDEs.

Discrete Values

Some questions given discrete values (x_i, y_i) for $0 \leq i \leq n$

1. Can values of y be approximated for values of x not in the table?
2. Can the quality of the approximation for values of x not in the table be characterized and/or estimated?
3. Can derivatives or integrals of $f(x)$ be approximated from the data in the table?
4. What is the conditioning of the problem?
5. Can these tasks be done efficiently and stably?

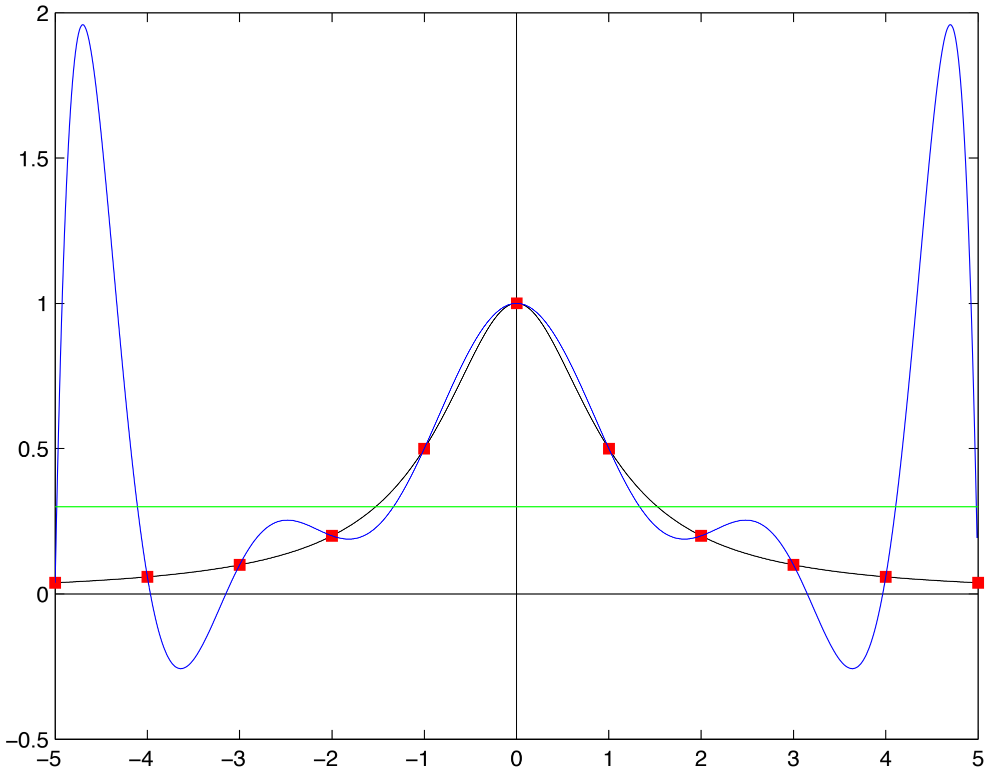
Approximation

- A class of functions must be selected for the "approximating function" $p(x)$
- A class of functions must be selected for the "approximated function" $f(x)$ in order to analyze the accuracy achieved and achievable by any given technique.
- Typical approximating function classes:
  - polynomials of degree n or less, $\mathbb{P}_n$
  - rational functions $p(x) = N(x)/D(x)$
  - real exponentials, e.g., for a parameterized probability distribution
  - trigonometric or complex exponential, e.g., Fourier approximation

Approximation

- The approximation $p(x)$ will have a parameterization in terms of some number of parameters k.
- Fundamental problems:
  1. Exact data, $y_i = f(x_i)$ and $k = n \rightarrow$ interpolation
  2. Exact data, $y_i = f(x_i)$ and $k < n \rightarrow$approximation via optimization
  3. Inexact data, $y_i = f(x_i) + \epsilon_i \rightarrow$ approximation via optimization
- interpolation finds n parameters of $p(x)$ given n constraints, e.g., $y_i = p(x_i)$
- approximation finds n parameters of $p(x)$ by solving an optimization problem for a cost function and possibly constraints, both expressed in terms of (x_i, y_i), e.g., projection on to a subspace of functions.

Interpolation and Approximation



Interpolation and Approximation

- black line is $f(x)$
- red points are $f(x_i)$ for known x_i
- blue line is an interpolating polynomial of degree 10 constructed using red points
- green line is an approximating constant function
- note the approximating function may interpolate $f(x)$ but the points are not necessarily known