Homework 4

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Problem 1

Find an approximation solution for the IVP (here $0 < \epsilon << 1$):

$$\frac{dy}{dt} + \epsilon y^2 - y = 0$$

subject to the initial condition $y(0) = 1$.

If we let $y(t, \epsilon)$ be the solution to the problem we get that

$$y(t, \epsilon) \approx y(t, 0) + \epsilon \frac{\partial y(t, 0)}{\partial \epsilon} + \epsilon^2 \frac{\partial^2 y(t, 0)}{\partial \epsilon^2} + \mathcal{O}(\epsilon^3)$$

for some perturbation ϵ. Now we take the partial with respect to the perturbation ϵ of the original problem to get

$$\frac{\partial}{\partial \epsilon}\left(\frac{\partial}{\partial t} y + \epsilon y^2 - y\right) = 0, \quad \frac{\partial y(0, \epsilon)}{\partial \epsilon} = 1,$$

which can be rewritten as

$$\frac{\partial}{\partial t}\frac{\partial y(t, \epsilon)}{\partial \epsilon} = \frac{\partial y(t, \epsilon)}{\partial \epsilon} - y^2 - 2\epsilon y \frac{\partial y(t, \epsilon)}{\partial \epsilon}.$$

The unperturbed solution to this is $y(t, 0) = e^t$. Now if we define $Y = \partial y(t, 0)/\partial \epsilon$ we get the following differential equation

$$\frac{dY}{dt} = Y - e^{2t}, \quad Y(0) = 0$$

where the general solution to this is $Ae^t - e^{2t}$. Using the initial condition sets $A = 1$, and therefore the solution is

$$Y = e^t - e^{2t}, \quad y \approx e^t + \epsilon(e^t - e^{2t}).$$

Problem 2

I used an online source for guidance with this problem.
Reference: https://www.iist.ac.in/sites/default/files/people/multiplescale.pdf

Find an approximate solution for the IVP (here $0 < \epsilon << 1$):

$$\frac{d^2y}{dt^2} + \epsilon\frac{dy}{dt} + y = 0$$

subject to the initial condition $y(0) = 0$ and $\frac{dy}{dt}(0) = 1$.

To apply the method of multiple scales with introduce the variables

$$T_0 = t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t$$

where they each represent different time scales due to the dampening effecting the amplitude and phase shift of the oscillator. Using the chain rule we get

$$\begin{aligned}
\frac{d}{dt} &= \frac{\partial}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial}{\partial T_2}\frac{\partial T_2}{\partial t} + \mathcal{O}(\epsilon^3) \\
&= \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} \\
\frac{d^2}{dt^2} &= \frac{\partial^2}{\partial T_0^2} + 2\epsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \epsilon^2\left(\frac{\partial^2}{\partial T_0 \partial T_2} + \frac{\partial^2}{\partial T_1^2}\right) + \mathcal{O}(\epsilon^3)
\end{aligned}$$

and thus the perturbed problem becomes

$$\frac{\partial^2 y}{\partial T_0^2} + 2\epsilon\frac{\partial^2 y}{\partial T_0 \partial T_1} + \epsilon^2\left(\frac{\partial^2 y}{\partial T_0 \partial T_2} + \frac{\partial^2 y}{\partial T_1^2}\right) + \epsilon\left(\frac{\partial y}{\partial T_0} + \epsilon\frac{\partial y}{\partial T_1} + \epsilon^2\frac{\partial y}{\partial T_2}\right) + y = 0$$

where we omit terms of $\mathcal{O}(\epsilon^3)$. To get an asymptotic approximation for y in the form

$$\tilde{y}(t) = y_0 + \epsilon y_1 + \epsilon^2 y_2 \approx y(t)$$

we plug $\tilde{y}(t)$ back into the perturbed equation and get

$$\begin{aligned}
\frac{\partial^2 y_0}{\partial T_0^2} + \epsilon\frac{\partial^2 y_1}{\partial T_0^2} + \epsilon^2\frac{\partial^2 y_2}{\partial T_0^2} + 2\epsilon\frac{\partial^2 y_0}{\partial T_0 \partial T_1} + 2\epsilon^2\frac{\partial^2 y_0}{\partial T_0 \partial T_2} + 2\epsilon^2\frac{\partial^2 y_1}{\partial T_0 \partial T_1} + \epsilon^2\frac{\partial y_0}{\partial T_1^2} + \\
2\epsilon\left(\frac{\partial y_0}{\partial T_0} + \epsilon\frac{\partial y_0}{\partial T_1} + \epsilon\frac{\partial y_1}{\partial T_0}\right) + y_0 + \epsilon y_1 + \epsilon^2 y_2 = 0
\end{aligned}$$

where we omit terms of $\mathcal{O}(\epsilon^3)$. For the sake of brevity only first-order approximation will be done, but second order is done in a similar fashion. Gathering terms we get

$$\begin{aligned}
\mathcal{O}(1) &: \frac{\partial^2 y_0}{\partial T_0^2} + y_0 = 0 \\
\mathcal{O}(\epsilon) &: \frac{\partial^2 y_1}{\partial T_0^2} + y_1 + 2\frac{\partial^2 y_0}{\partial T_0 \partial T_1} + 2\frac{\partial y_0}{\partial T_0} = 0
\end{aligned}$$

The initial conditions are $y_0 = 1 \Rightarrow \frac{\partial y_0}{\partial T_0} = 0$ and $y_1 = 0 \Rightarrow \frac{\partial y_1}{\partial T_0} = -\frac{\partial y_0}{\partial T_1}$ for $T_0 = T_1 = 0$. The general solution and its derivatives becomes

$$\begin{aligned} y_0 &= A(T_1)cos(T_0) + B(T_0)sin(T_0) \\ \frac{\partial y_0}{\partial T_0} &= -A(T_1)sin(T_0) + B(T_0)cos(T_0) \\ \frac{\partial^2}{\partial T_0 \partial T_1} &= -sin(T_0)\frac{\partial A}{\partial T_1} + cos(T_0)\frac{\partial B}{\partial T_1} \end{aligned}$$

Plugging this back in yields

$$\frac{\partial^2 y_1}{\partial T_0^2} + y_1 = 2\left(\frac{\partial A}{\partial T_1} + A\right)sin(T_0) - s\left(\frac{\partial B}{\partial T_1} + B\right)cos(T_0)$$

The coefficients of $cos(T_0)$ and $sin(T_0)$ must vanish so we end up getting

$$\frac{\partial A}{\partial T_1} + A = 0, \quad \frac{\partial B}{\partial T_1} + B = 0 \quad \Rightarrow \quad A = ae^{-T_1}, \quad B = be^{-T_1}$$

Substituting this back in we get

$$y_0 = ae^{-T_1}cos(T_0) + be^{-T_1}sin(T_0)$$

where, from initial conditions, we get $a = 1$ and $b = 0$. Therefore we obtain the perturbed solution

$$x = e^{-T_1}cos(T_0) + \mathcal{O}(\epsilon) \quad \rightarrow \quad x = e^{-\epsilon t}cos(t) + \mathcal{O}(\epsilon)$$

which us valid for time of order $1/\epsilon$.

Problem 3

NOTE: For the sake of brevity, some algebra is omitted in the writeup of Problem 3.

Find an approximate formula for each root of the following algebraic equations (here $0 < \epsilon << 1$*):*

(a) $xe^{-x} = \epsilon$

For the regularly perturbed root $x = 0$ we let

$$x \approx x_0 + \epsilon x_1 + \epsilon^2 x_2$$

Plugging back into the original system we get

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2) - \epsilon(1 + x_0 + \epsilon x_1 + \epsilon^2 x_2)$$

where we used the Taylor series expansion for e^x. Collecting terms we get

$$\begin{aligned} \mathcal{O}(1) &: x - 0 = 0 \\ \mathcal{O}(\epsilon) &: x_1 - 1 = 0 \Rightarrow x_1 = 1 \\ \mathcal{O}(\epsilon^2) &: x_2 - x_1 = 0 \Rightarrow x_2 = 1 \end{aligned}$$

where we get

$$x = \epsilon + \epsilon^2$$

for the root $x = 0$. Now considering the other perturbed roots, we take the log of both sides to get

$$x = log(x) + log(\frac{1}{\epsilon})$$

where x is the perturbed root. We can treat the perturbed root as a fixed point and apply fixed point iteration to determine x

$$\begin{aligned} x_1 &= log(\frac{1}{\epsilon}) \\ x_2 &= log(\frac{1}{\epsilon}) + log(log(\frac{1}{\epsilon})) \\ x_3 &= log(\frac{1}{\epsilon}) + log(log(\frac{1}{\epsilon}) + log(log(\frac{1}{\epsilon}))) \\ &= log(\frac{1}{\epsilon}) + log(log(\frac{1}{\epsilon})) + \frac{log(log(\frac{1}{\epsilon}))}{log(\frac{1}{\epsilon})} + \mathcal{O}\left(\left(\frac{log(log(\frac{1}{\epsilon}))}{log(\frac{1}{\epsilon})}\right)^2\right) \end{aligned}$$

As $\epsilon \to 0$ we get that the expression converges.

(b) $x^3 - x + \epsilon = 0$

Unperturbed Roots: $x^3 - x = 0 \Rightarrow x = 0, \pm 1$
Taylor Expanding these roots we get

$$\begin{aligned}
x_{+1} &= 1 + \epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3) \\
x_{-1} &= -1 + \epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3) \\
x_0 &= \epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3)
\end{aligned}$$

Plugging back into the perturbed problem we get the following

$$\begin{aligned}
x_{+1}^3 - x_{+1} - \epsilon &= (1 + \epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3))^3 - (1 + \epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3)) + \epsilon \\
&= (1 + 3\epsilon x_1 + \epsilon^2(3x_1^2 + 3x_2 + \mathcal{O}(\epsilon^3))) - (1 + \epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3)) + \epsilon \\
\mathcal{O}(1) &: 1 - 1 = 0 \\
\mathcal{O}(\epsilon) &: 3x_1 - x_1 + 1 = 0 \Rightarrow x_1 = -\frac{1}{2} \\
\mathcal{O}(\epsilon^2) &: 3x_1^2 + 3x_2 - x_2 = 0 \Rightarrow x_2 = -\frac{3}{8}
\end{aligned}$$

$$\begin{aligned}
x_{-1}^3 - x_{-1} - \epsilon &= (-1 + \epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3))^3 - (-1 + \epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3)) + \epsilon \\
&= (-1 + 3\epsilon x_1 + \epsilon^2(3x_2 - 3x_1^2 + \mathcal{O}(\epsilon^3))) - (-1 + \epsilon x_1 + \epsilon x_2 + \mathcal{O}(\epsilon^3)) + \epsilon \\
\mathcal{O}(1) &: -1 + 1 = 0 \\
\mathcal{O}(\epsilon) &: 3x_1 - x_1 + 1 = 0 \Rightarrow x_1 = -\frac{1}{2} \\
\mathcal{O}(\epsilon^2) &: -3x_1^2 + 3x_2 - x_2 = 0 \Rightarrow x_2 = \frac{3}{8}
\end{aligned}$$

$$\begin{aligned}
x_0^3 - x_0 - \epsilon &= (\epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3))^3 - (\epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3)) + \epsilon \\
&= \mathcal{O}(\epsilon^3) - (\epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3)) + \epsilon \\
\mathcal{O}(1) &: \text{ none} \\
\mathcal{O}(\epsilon) &: -x_1 + 1 = 0 \Rightarrow x_1 = 1 \\
\mathcal{O}(\epsilon^2) &: x_2 = 0
\end{aligned}$$

Putting all this together we get the approximation root formulas

$$\begin{aligned}
x_{+1} &= 1 - \frac{\epsilon}{2} - \frac{3\epsilon^2}{8} \\
x_{-1} &= -1 - \frac{\epsilon}{2} + \frac{3\epsilon^2}{8} \\
x_0 &= \epsilon
\end{aligned}$$

(c) $\epsilon x^3 - x + 1 = 0$

We have a singularly perturbed system so we will apply scaling by letting $x = \delta y$ to get $\epsilon\delta^3 y^3 - \delta y + 1 = 0$. Now we must balance

$$\begin{aligned}
&\text{Balance I and II}: \ \epsilon\delta^3 \sim \delta \Rightarrow \delta \sim \frac{1}{\sqrt{\epsilon}} \\
&\text{Balance I and III}: \ \epsilon\delta^3 \sim 1 \Rightarrow \delta \sim \frac{1}{\sqrt[3]{\epsilon}} \\
&\text{Balance II and III}: \ \delta \sim 1
\end{aligned}$$

where balancing I and II is what we want since we arrive at the equation $y^3 - y + \sqrt{\epsilon} = 0$. Using this formula we get the Taylor Series expansions

$$\begin{aligned}
x_{+1} &= 1 + \sqrt{\epsilon}x_1 + \epsilon x_2 \\
x_{-1} &= -1 + \sqrt{\epsilon}x_1 + \epsilon x_2 \\
x_0 &= \sqrt{\epsilon}x_1 + \epsilon x_2
\end{aligned}$$

Plugging this back into the equation we get

$$\begin{aligned}
y_{+1}^3 - y_{+1} + \sqrt{\epsilon} &= (1 + \sqrt{\epsilon}x_1 + \epsilon x_2)^3 - (1 + \sqrt{\epsilon}x_1 + \epsilon x_2) + \sqrt{\epsilon} \\
&= (1 + 3\sqrt{\epsilon}x_1 + \epsilon(3x_2 + 3x_1^2) + \mathcal{O}(\epsilon^{3/2})) - (1 + \sqrt{\epsilon}x_1 + \epsilon x_2 + \mathcal{O}(\epsilon^{3/2})) + \sqrt{\epsilon} \\
\mathcal{O}(1) &: 1 - 1 = 0 \\
\mathcal{O}(\sqrt{\epsilon}) &: x_1 + 1 = 0 \Rightarrow x_1 = -1 \\
\mathcal{O}(\epsilon) &: 2x_2 + 3x_1^2 = 0 \Rightarrow x_2 = -\frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
y_{-1}^3 - y_{-1} + \sqrt{\epsilon} &= (-1 + \sqrt{\epsilon}x_1 + \epsilon x_2)^3 - (-1 + \sqrt{\epsilon}x_1 + \epsilon x_2) + \sqrt{\epsilon} \\
&= (1 + 3\sqrt{\epsilon}x_1 + \epsilon(3x_2 - 3x_1^2) + \mathcal{O}(\epsilon^{3/2})) - (1 + \sqrt{\epsilon}x_1 + \epsilon x_2 + \mathcal{O}(\epsilon^{3/2})) + \sqrt{\epsilon} \\
\mathcal{O}(1) &: 1 - 1 = 0 \\
\mathcal{O}(\sqrt{\epsilon}) &: x_1 + 1 = 0 \Rightarrow x_1 = -1 \\
\mathcal{O}(\epsilon) &: 2x_2 - 3x_1^2 = 0 \Rightarrow x_2 = \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
y_0^3 - y_0 + \sqrt{\epsilon} &= (\sqrt{\epsilon}x_1 + \epsilon x_2 + \mathcal{O}(\epsilon^{3/2}))^3 - (\sqrt{\epsilon}x_1 + \epsilon x_2 + \mathcal{O}(\epsilon^{3/2})) + \sqrt{\epsilon} \\
&= \sqrt{\epsilon}(-x_1 + 1) - \epsilon x_2 + \epsilon^{3/2}x_1 + \epsilon^2(x_1^2 x_2 + x_1 x_2) \\
\mathcal{O}(1) &: \text{none} \\
\mathcal{O}(\sqrt{\epsilon}) &: -x_1 + 1 = 0 \Rightarrow x_1 = 1 \\
\mathcal{O}(\epsilon) &: x_2 = 0 \Rightarrow x_2 = 0
\end{aligned}$$

Putting all this together we get the following

$$\begin{aligned}
x_{+1} &= \delta y_{+1} = \epsilon^{-1/2} - \frac{1}{2} - \frac{3}{8}\epsilon^{1/2} \\
x_{-1} &= \delta y_{-1} = \epsilon^{-1/2} + \frac{1}{2} + \frac{3}{8}\epsilon^{1/2} \\
x_0 &= \delta y_0 = 1 + \epsilon
\end{aligned}$$

(d) $(1-\epsilon)x^2 - 2x + 1 = 0$

Trying to do our regular Taylor series expansion, we let

$$x \approx x_0 + \epsilon x_1 + \epsilon^2 x_2$$

we end up with

$$(1-\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2)^2 - 2(x_0 + \epsilon x_1 + \epsilon^2 x_2) + 1$$

Collecting terms we get

$$\begin{aligned}
\mathcal{O}(1) &: x_0^2 - 2x_0 + 1 = 0 \Rightarrow x_0 = 1 \\
\mathcal{O}(\epsilon) &: 2x_0x_1 + x_0^2 - 2x_1 = 0 \Rightarrow 1 = 0
\end{aligned}$$

where we arrive at a contradiction. This is due to the repeated roots, but if we look at the roots with respect to the perturbation we get

$$x = \frac{1 \pm \sqrt{\epsilon}}{1-\epsilon}$$

where we see that it scales with respect to $\sqrt{\epsilon}$. Expanding with respect to this we get

$$(1-\epsilon)(x_0 + \sqrt{\epsilon}x_1 + \epsilon x_2)^2 - 2(x_0 + \sqrt{\epsilon}x_1 + \epsilon x_2) + 1 = 0$$

Collect terms again we get

$$\begin{aligned}
\mathcal{O}1 &: x_0^2 - 2x_0 + 1 = 0 \Rightarrow x_0 = 1 \\
\mathcal{O}(\sqrt{\epsilon}) &: 2x_0x_1 - 2x_1 = 0 \Rightarrow \text{ Inconclusive} \\
\mathcal{O}(\epsilon) &: -x_0^2 + x_1^2 + 2x_0x_2 - 2x_2 = 0 \Rightarrow x_1 = \pm 1
\end{aligned}$$

From this we have that the perturbed roots are

$$x_{\pm 1} = 1 \pm \sqrt{\epsilon} + \mathcal{O}(\epsilon)$$

(e) $\epsilon(x^2+x)+1=0$

This system is singularly perturbed so we must do scaling ($x=\delta y$) and balancing

$$\epsilon\delta^2 y^2+\epsilon\delta y+1=0$$

Balancing term I with term II we get $\delta\sim 1$ which leads to the same problem as we originally had. Balancing term I and III we get $\delta\sim\frac{1}{\sqrt{\epsilon}}$ which works out. We end up getting

$$y^2+\sqrt{\epsilon}y+1=0$$

It is important to note if we try balancing term II and III we get $\delta\sim\frac{1}{\epsilon}$ which leads to another singular perturbed system. Applying the usual expansion (w.r.t to $\sqrt{\epsilon}$ instead of ϵ) we get

$$(y_0+\sqrt{\epsilon}y_1+\epsilon y_2)^2+\sqrt{\epsilon}(y_0+\sqrt{\epsilon}y_1+\epsilon y_2)+1$$

Collecting terms we get

$$\begin{aligned}\mathcal{O}(1): y_0^2+1=0 &\Rightarrow y_0=\pm i\\ \mathcal{O}(\sqrt{\epsilon}): 2y_0y_1+y_0=0 &\Rightarrow y_1=\frac{1}{2}\\ \mathcal{O}(\epsilon): y_1^2+2y_0y_2+y_1=0 &\Rightarrow y_2=\frac{i}{8}\end{aligned}$$

Therefore fore the perturbed roots we get

$$y_\pm=\pm i-\frac{1}{2}\sqrt{\epsilon}\pm\frac{i}{8}+\mathcal{O}(\epsilon^{3/2})$$

(f) $x^2 - 1 = \epsilon x$

Applying our regular Taylor series expansion we get

$$x \approx x_0 + \epsilon x_1 + \epsilon^2 x_2$$

and plugging back into the equation

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2)^2 - \epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2) - 1$$

Collecting terms we get

$$\mathcal{O}(1) : x_0^2 - 1 = 0 \Rightarrow x_0 = \pm 1$$
$$\mathcal{O}(\epsilon) : 2x_0 x_1 - x_0 = 0 \Rightarrow x_1 = \frac{1}{2}$$
$$\mathcal{O}(\epsilon^2) : x_1^2 + 2x_0 x_2 - x_1 = 0 \Rightarrow x_2 = \pm\frac{1}{8}$$

Therefore the perturbed roots are

$$x_{\pm 1} = \pm 1 + \frac{1}{2}\epsilon \pm \frac{1}{8}\epsilon^2$$

(g) $x^2 - 1 = \epsilon e^x$

Taking the log of both sides we get

$$log(x^2 - 1) = \log(\epsilon) + x \quad \Rightarrow \quad x = log(x^2 - 1) + log(\frac{1}{\epsilon})$$

for which we can apply fixed point iteration. Taking when only the ϵ term is present to be our initial guess we get

$$\begin{aligned}
x_1 &= log(\frac{1}{\epsilon}) \\
x_2 &= log(log(\frac{1}{\epsilon})^2 - 1) + log(\frac{1}{\epsilon})
\end{aligned}$$

We can see from the above that if we keep expanding we get $\mathcal{O}(log(\frac{1}{\epsilon})^3)$. The above iteration converges as $\epsilon \to 0$.

(h) $x^2 - 4 = \epsilon ln\, x$

Just like question 3(a), we will use fixed point iteration

$$\begin{aligned}
x_n &= 2 + \frac{\epsilon ln(x)}{x+2} \Rightarrow \\
x_1 &= 2 \\
x_2 &= 2 + \frac{\epsilon ln(2)}{4} \\
x_3 &= 2 + \frac{\epsilon ln(2 + \frac{\epsilon ln(2)}{4})}{4 + \frac{\epsilon ln(2)}{4}} \\
&= 2 + \frac{\epsilon}{4}(1 - \frac{\epsilon ln(2)}{16})(ln(2) + ln(1 + \frac{\epsilon ln(2)}{8})
\end{aligned}$$

The other perturbed root is near zero. We can see that the above iteration converges as $\epsilon \to 0$.

(i) $x^2 - 2\epsilon x - \epsilon = 0$

Trying to do our regular Taylor series expansion, we let

$$x \approx x_0 + \epsilon x_1 + \epsilon^2 x_2$$

we end up with

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2)^2 - 2\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2) - \epsilon$$

Collecting terms we get

$$\begin{aligned} \mathcal{O}(1) &: x_0^2 = 0 \Rightarrow x_0 = 0 \\ \mathcal{O}(\epsilon) &: 2x_0 x_1 + -2x_0 - 1 \Rightarrow -1 = 0 \end{aligned}$$

where we arrive at a contradiction. This is due to the repeated roots, but if we look at the roots with respect to the perturbation we get

$$x = 1 \pm 2\sqrt{\epsilon^2 + \epsilon} \approx 1 \pm 2\sqrt{\epsilon}$$

where we see that it scales with respect to $\sqrt{\epsilon}$. Expanding with respect to this we get

$$(x_0 + \sqrt{\epsilon}x_1 + \epsilon x_2)^2 - 2\epsilon(x_0 + \sqrt{\epsilon}x_1 + \epsilon x_2) - \epsilon = 0$$

Collect terms again we get

$$\begin{aligned} \mathcal{O}1 &: x_0^2 = 0 \Rightarrow x_0 = 0 \\ \mathcal{O}(\sqrt{\epsilon}) &: 2\sqrt{\epsilon}x_0 x_1 = 0 \Rightarrow \text{ Inconclusive} \\ \mathcal{O}(\epsilon) &: 2\epsilon x_0 x_2 + x_1^2 + 2x_0 - 1 = 0 \Rightarrow x_1 = \pm 1 \\ \mathcal{O}(\epsilon^{3/2}) &: 2x_1 x_2 - 2x_1 = 0 \Rightarrow x_2 = 1 \end{aligned}$$

From this we have that the perturbed roots are

$$x = \pm\sqrt{\epsilon} + \epsilon$$

(j) $\epsilon x^5 + x^2 - 2x + 1 = 0$

This is singularly perturbed so we apply scaling by letting $x = \delta y$ and we get

$$\epsilon\delta^5 y^5 + \delta^2 y^2 - 2\delta y + 1 = 0$$

Balancing the term I against term II we get $\delta \sim e^{-1/3}$. It is important to note that when balancing term I against term II and term III we get

$$2^{5/4}y^5 + \frac{\sqrt{2}y^2}{\epsilon^{1/4}} + 2^{5/4} + \epsilon^{1/4} = 0, \quad y^5 + \epsilon^{-2/5}y^2 - 2\epsilon^{-1/5}y + 1 = 0$$

which blow up at small ϵ. Also trying to balancing the other terms against each other leads to δ being proportional to some scalar, which just leads to the original problem essentially. Now, plugging back in and expanding we get

$$(y_0 + \epsilon^{1/3}y_1 + \epsilon^{2/3}y_2)^5 + (y_0 + \epsilon^{1/3}y_1 + \epsilon^{2/3}y_2)^2 - 2\epsilon^{1/3}(y_0 + \epsilon^{1/3}y_1 + \epsilon^{2/3}y_2) + 1$$

Collecting terms we get

$$\begin{aligned}
\mathcal{O}(1) &: y_0^5 + y_2 = 0 \Rightarrow y_0 = 0, -1 \text{ (we will use } y_0 = 0) \\
\mathcal{O}(\epsilon^{1/3}) &: y_0^4 y_1 + 2y_0y_1 - 2y_0 = 0 \Rightarrow y_1 = -\frac{2}{3} \\
\mathcal{O}(\epsilon^{2/3}) &: 5y_0^4 y_2 + 10y_1^2 y_0^3 + 2y_0y_1 + y_1^2 + y_1 + 1 \Rightarrow y_2 = \frac{11}{9}
\end{aligned}$$

Using this and plugging back into $x = \delta y$ we get that the perturbed root is

$$x = \delta y = \frac{1}{e^{1/3}}\left(-1 - \frac{2}{3}\epsilon^{1/3} + \frac{11}{9}\epsilon^{2/3} \right) = -\frac{1}{\epsilon^{1/3}} - \frac{2}{3} + \frac{11}{9}\epsilon^{1/3}$$