Set 8: Polynomial Interpolation – Part 4

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Foundations of Computational Math 1

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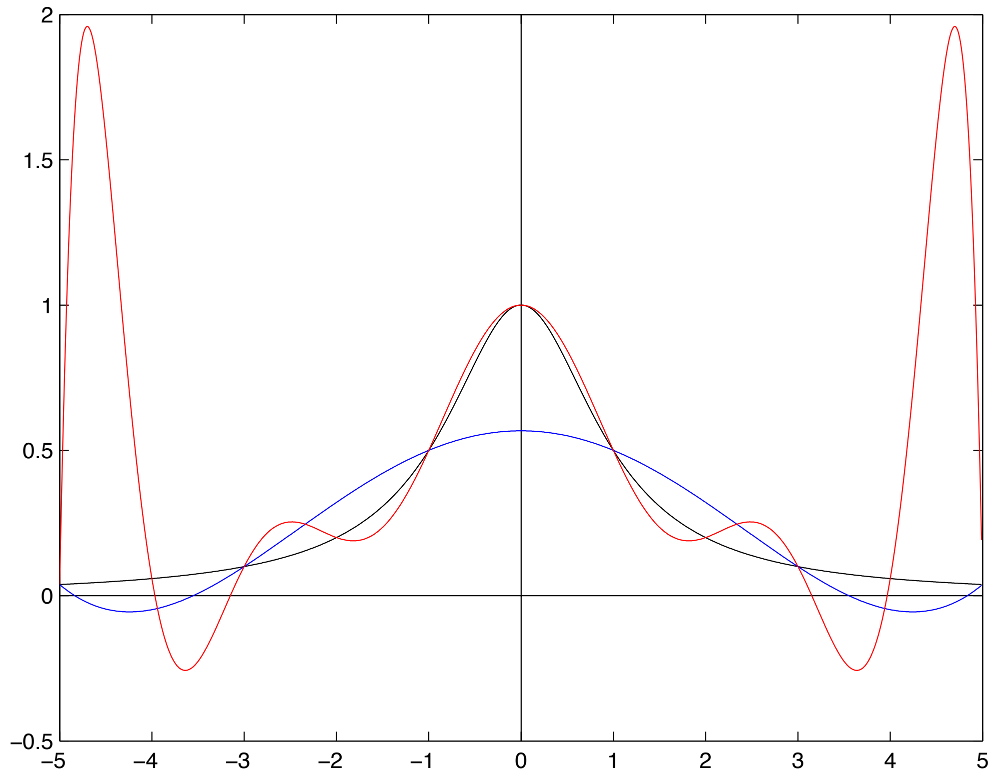
Hermite Interpolation and Osculatory Polynomials



Note. function values OK at points, derivatives are not, sometimes even wrong sign

Approach

Solution:

- specify function values $f(x_i) = y_i$
- specify derivative values $f'(x_i) = y'_i$

Repeat approaches:

- power basis: $p_n(x) = \sum_{i=0}^{n} \alpha_i x^i$
- Newton form: $p_n(x) = \sum_{i=0}^{n} \alpha_i \Omega_i(x)$
- Lagrange form: $p_n(x) = \sum_{i=0}^{n} \Big[y_i \psi_i(x) + y'_i \Psi_i(x) \Big]$

Constrain Derivatives

For example, given 4 constraints construct $p_3(x) = \sum_{i=0}^{3} \alpha_i x^i$:

$$y(0) = p_3(0)$$

$$y'(0) = p_3'(0)$$

$$y(1) = p_3(1)$$

$$y'(1) = p_3'(1)$$

$$\Downarrow$$

$$\alpha_0 = y(0)$$

$$\alpha_1 = y'(0)$$

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = y(1)$$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = y'(1)$$

Example

$$\alpha_0 = y(0)$$

$$\alpha_1 = y'(0)$$

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = y(1)$$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = y'(1)$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \\ y(1) \\ y'(1) \end{pmatrix}$$

Example

$$y(0) = 3, \;\; y'(0) = 2$$

$$y(1) = 6, \;\; y'(1) = 1$$

$$\Downarrow$$

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ -3 \end{pmatrix}$$

$$p_3(x) = -3x^3 + 4x^2 + 2x + 3, \;\; p_3'(x) = -9x^2 + 8x + 2$$

$$p_3(0) = 3, \;\; p_3'(0) = 2, \;\; p_3(1) = 6, \;\; p_3'(1) = 1$$

Monomial Form – Hermite Interpolation

$$p_d(x_i) = y_i \text{ and } p'_d(x_i) = y'_i \quad 0 \leq i \leq n$$

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^d \\ 0 & 1 & 2x_0 & \dots & dx_0^{d-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^d \\ 0 & 1 & 2x_1 & \dots & dx_1^{d-1} \\ \vdots & \vdots & & \vdots & \\ 1 & x_n & x_n^2 & \dots & x_n^d \\ 0 & 1 & 2x_n & \dots & dx_n^{d-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{d-1} \\ \alpha_d \end{pmatrix} = \begin{pmatrix} y_0 \\ y'_0 \\ y_1 \\ y'_1 \\ \vdots \\ y_n \\ y'_n \end{pmatrix}$$

$$V^T a = y \text{ and } d = 2n+1$$

Monomial Form – Hermite Interpolation

- V is a confluent Vandermonde matrix.
- The confluent columns corrrespond to derivative value constraints.
- V is nonsingular if the values defining the "nonconfluent" columns are distinct.
- Existence and uniqueness of the Hermite interpolating polynomial of degree $2n+1$ follows.

Newton Form – Hermite Interpolation

The easiest way to compute a Hermite interpolating polynomial is to use the Newton form. The question is what do we do with divided differences of the form $y[x_i, x_i]$?

$$y[x_i, x_i] = \lim_{x_j \to x_i} y[x_i, x_j]$$

$$= \lim_{x_j \to x_i} \frac{y(x_j) - y(x_i)}{x_j - x_i}$$

$$= y'(x_i)$$

This can be used to define the necessary form of the divided difference table.

Newton Form – Hermite Interpolation

Given, $n = 2$, $(x_0, y_0), (x_0, y_0'), (x_1, y_1), (x_1, y_1')$, we create the table for

$$\hat{n} = 3, \quad (\hat{x}_0, y_0), (\hat{x}_1, y_0), (\hat{x}_1, y_1), (\hat{x}_2, y_1')$$

and use derivative values for divided differences with repeated values of $\hat{x}_i$, e.g., $y[\hat{x}_0, \hat{x}_1] = y[x_0, x_0]$.

i	0	1	2	3
$\hat{x}_i$	x_0	x_0	x_1	x_1
f_i	y_0	y_0	y_1	y_1
$y[*,*]$	–	$y[x_0, x_0] = y_0'$	$y[x_0, x_1]$	$y[x_1, x_1] = y_1'$
$y[*,*,*]$	–	–	$y[x_0, x_0, x_1]$	$y[x_0, x_1, x_1]$
$y[*,*,*,*]$	–	–	–	$y[x_0, x_0, x_1, x_1]$

Newton Form – Hermite Interpolation

Using the Newton form in terms of $\hat{x}_i$ first

$$
\begin{aligned}
H_3(x) = y_0 &+ (x-\hat{x}_0)y[\hat{x}_0,\hat{x}_1] \\
&+ (x-\hat{x}_0)(x-\hat{x}_1)y[\hat{x}_0,\hat{x}_1,\hat{x}_2] \\
&+ (x-\hat{x}_0)(x-\hat{x}_1)(x-\hat{x}_2)y[\hat{x}_0,\hat{x}_1,\hat{x}_2,\hat{x}_3]
\end{aligned}
$$

Now substitute differences and derivatives knowing

$$\hat{x}_0 = \hat{x}_1 = x_0 \quad \text{and} \quad \hat{x}_2 = \hat{x}_3 = x_1$$

Newton Form – Hermite Interpolation

$$
\begin{aligned}
H_3(x) &= y_0 + (x - x_0)y[x_0, x_0] \\
&\quad + (x - x_0)^2 y[x_0, x_0, x_1] \\
&\quad + (x - x_0)^2 (x - x_1) y[x_0, x_0, x_1, x_1] \\
&= y_0 + (x - x_0)y_0' + (x - x_0)^2 y[x_0, x_0, x_1] \\
&\quad + (x - x_0)^2 (x - x_1) y[x_0, x_0, x_1, x_1]
\end{aligned}
$$

where the remaining divided differences are defined as in the table.

Note as before, other paths through the table can be used.

It is left as an exercise to consider the Smoktunowicz et al. algorithm for computing the divided differences for the Hermite polynomial.

Example

$$x_0 = 1, \;\; y_0 = 3, \;\; y_0' = 2,$$

$$x_1 = 2, \;\; y_1 = 6, \;\; y_1' = 1$$

$$y[x_0, x_0] = 2, \;\; y[x_0, x_1] = 3, \;\; y[x_1, x_1] = 1$$

$$y[x_0, x_0, x_1] = 1, \;\; y[x_0, x_1, x_1] = -2$$

$$y[x_0, x_0, x_1, x_1] = -3$$

$$H_3(x) = 3 + 2(x-1) + 1(x-1)^2 - 3(x-1)^2(x-2)$$

$$H_3(x) = 8 - 15x + 13x^2 - 3x^3$$

Lagrange Form – Hermite Interpolation

Constraints

$$p(x_0) = y_0, \;\; p'(x_0) = y_0'$$
$$p(x_1) = y_1, \;\; p'(x_1) = y_1'$$
$$p(x_2) = y_2, \;\; p'(x_2) = y_2'$$
$$\vdots$$
$$p(x_n) = y_n, \;\; p'(x_n) = y_n'$$

$2n + 2$ conditions $\;\rightarrow\;$ degree of $p(x)$ is $2n + 1$

Lagrange Form – Hermite Interpolation

Constraints on basis functions

$$p(x) = \sum_{i=0}^{n} \Big[y_i \psi_i(x) + y_i' \Psi_i(x) \Big] \text{ and } p'(x) = \sum_{i=0}^{n} \Big[y_i \psi_i'(x) + y_i' \Psi_i'(x) \Big]$$

$$\delta_{ii} = 1 \quad \delta_{ij} = 0 \text{ for } i \neq j, \quad 0 \leq i, j \leq n$$

$$\psi_i(x_j) = \delta_{ij}, \qquad \Psi_i(x_j) = 0 \rightarrow p(x_i) = y_i$$

$$\psi_i'(x_j) = 0, \qquad \Psi_i'(x_j) = \delta_{ij} \rightarrow p'(x_i) = y_i'$$

Lagrange Form – Hermite Interpolation

$$\psi_i(x) = \ell_i^2(x)\Big[1 - 2\ell_i'(x_i)(x - x_i)\Big]$$

$$\psi_i(x_j) = \delta_{ij} \text{ as desired}$$

$$\psi_i'(x) = 2\ell_i'(x)\ell_i(x)\Big[1 - 2\ell_i'(x_i)(x - x_i)\Big] - 2\ell_i'(x_i)\ell_i^2(x)$$

$$\psi_i'(x_j) = 0, \quad \text{as desired}$$

$$\psi_i'(x_i) = 2\ell_i'(x_i) \times 1\Big[1 - 0\Big] - 2\ell_i'(x_i) \times 1 = 0 \quad \text{as desired}$$

Derivation of $\psi_i(x)$

$\psi_i(x)$ has degree $2n+1$ and has double roots at x_j, $i \neq j$

$\ell_i^2(x)$ has degree $2n$ with

$$\ell_i^2(x_j) = \delta_{ij} \quad n+1 \text{ conditions}$$

$$\left[\ell_i^2(x_j)\right]' = 0 \quad i \neq j \quad \text{but also} \quad \left[\ell_i^2(x_i)\right]' \neq 0 \quad \text{generally}$$

We have a free degree so consider a linear function $g(x)$ and take

$$\psi_i(x) = \ell_i^2(x) g(x)$$

Check conditions and determine $g(x)$.

Derivation of $\psi_i(x)$

We have

$$\psi_i(x_j) = \ell_i^2(x_j)g(x_j) = 0 \quad i \neq j$$

$$g(x_i) = 1 \rightarrow \psi_i(x_i) = \ell_i^2(x_i)g(x_i) = 1$$

$$\therefore \quad \text{take the form } g(x) = 1 + \beta(x - x_1)$$

$$\psi_i(x) = \ell_i^2(x)(1 + \beta(x - x_i))$$

Derivation of $\psi_i(x)$

$$\psi_i(x) = \ell_i^2(x)\,[1 + \beta(x - x_i)]$$

$$\psi_i'(x) = \beta\ell_i^2(x) + 2\,[1 + \beta(x - x_i)]\,\ell_i(x)\ell_i'(x)$$

$$\psi_i'(x_j) = 0 \quad i \neq j$$

So β must be chosen to satisfy $\psi_i'(x_i) = 0$.

Derivation of $\psi_i(x)$

$$\psi_i(x) = \ell_i^2(x)\left[1 + \beta(x - x_i)\right]$$

$$\psi_i'(x) = \beta\ell_i^2(x) + 2\left[1 + \beta(x - x_i)\right]\ell_i(x)\ell_i'(x)$$

$$\psi_i'(x_i) = \beta + 2\ell_i'(x_i)$$

$$\therefore \beta = -2\ell_i'(x_i) \rightarrow \psi_i'(x_i) = 0$$

Lagrange Form – Hermite Interpolation

$$\Psi_i(x) = \ell_i^2(x)(x - x_i)$$

$$\Psi_i(x_j) = 0, 0 \leq i, j \leq n \quad \text{as desired}$$

$$\Psi_i'(x) = \ell_i^2(x) + 2\ell_i'(x)\ell_i(x)(x - x_i)$$

$$\Psi_i'(x_j) = \delta_{ij}, \quad i \neq j \quad \text{as desired}$$

Lagrange Form – Hermite Interpolation

Theorem 8.1. *Given the constraints,* $0 \le i \le n$*,*

$$H_d(x_i) = y_i, \;\; H'_d(x_i) = y'_i, \;\; x_i \in [a,b], \;\; x_i \neq x_j$$

The unique Hermite interpolation polynomial of degree $d = 2n+1$ *is*

$$H_d(x) = \sum_{i=0}^{n} \Big[y_i \psi_i(x) + y'_i \Psi_i(x) \Big]$$

$$\psi_i(x) = \ell_i^2(x) \Big[1 - 2\ell'_i(x_i)(x - x_i) \Big]$$

$$\Psi_i(x) = \ell_i^2(x)(x - x_i)$$

Further, if $y(x) \in \mathcal{C}^{(d+1)}$ *defines the* y_i *and* y'_i *then* $\exists \xi \in [a,b]$ *such that*

$$y(x) - H_d(x) = \frac{y^{(d+1)}(\xi)}{(d+1)!} \prod_{i=0}^{n} (x - x_i)^2$$

Lagrange Form – Hermite Interpolation

- Construction of the Hermite interpolant requires computing the $m_i(x_i)$ values as before for the Lagrange form.
- Construction of the Hermite interpolant requires computing the $\ell_i'(x_i)$ which requires $m_i'(x_i)$ values.
- $O(n^2)$ incremental construction via recurrences like the forms of Lagrange.
- Complexity of evaluation of the Hermite interpolant is left as an exercise.

Example

$$\psi_i(x) = \ell_i^2(x)\Big[1 - 2\ell_i'(x_i)(x - x_i)\Big], \quad \Psi_i(x) = \ell_i^2(x)(x - x_i)$$

$$x_0 = 1, \;\; y_0 = 3, \;\; y_0' = 2,$$

$$x_1 = 2, \;\; y_1 = 6, \;\; y_1' = 1$$

$$\psi_0(x) = (x-2)^2(2x-1), \quad \psi_1(x) = (x-1)^2(5-2x)$$

$$\Psi_0(x) = (x-2)^2(x-1), \quad \Psi_1(x) = (x-1)^2(x-2)$$

$$H_3(x) = 3(x-2)^2(2x-1) + 2(x-1)(x-2)^2$$

$$+6(x-1)^2(5-2x) + (x-1)^2(x-2)$$

$$H_3(x) = 8 - 15x + 13x^2 - 3x^3$$

$$H_3'(x) = -15 + 26x - 9x^2$$

Example

$$x_0 = 1, \;\; y_0 = 3, \;\; y_0' = 2,$$

$$x_1 = 2, \;\; y_1 = 6, \;\; y_1' = 1$$

$$H_3(x) = 8 - 15x + 13x^2 - 3x^3$$

$$H_3(1) = 8 - 15 + 13 - 3 = 3$$

$$H_3(2) = 8 - 30 + 52 - 24 = 6$$

$$H_3'(x) = -15 + 26x - 9x^2$$

$$H_3'(1) = -15 + 26 - 9 = 2$$

$$H_3'(2) = -15 + 52 - 36 = 1$$

Osculating Polynomial

Definition 8.1. Let $x_i \in [a, b],\ 0 \leq i \leq n$ be distinct points, $m_i \in \mathbb{Z}^+,\ 0 \leq i \leq n$, and $f(x) \in \mathcal{C}^{(m)}[a.b]$ with $m = \max_i m_i$. The unique osculating polynmial, $p_d(x)$, interpolating $f(x)$ satisfies

$$\frac{d^k p}{dx^k}(x_i) = \frac{d^k f}{dx^k}(x_i)$$

$0 \leq i \leq n$ and $0 \leq k \leq m_i$

$$d + 1 = \sum_{i=0}^{n} (m_i + 1)$$

Osculating Polynomial

Cases of Osculating Polynomials:

- $n = 0$: Taylor polynomial of degree m_0 at x_0.
- $\forall i \;\; m_i = 0$: Lagrange/Newton interpolating polynomial
- $\forall i \;\; m_i = 1$: Hermite interpolating polynomial
- General case: Hermite-Birkoff interpolating polynomial (see text p. 349 for basis functions)