Program 1 Foundations of Computational Math 1 Fall 2017

Due date: 11:59PM on Monday, 9 October 2016

Consider a polynomial

$$p_n(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

$p_n(x)$ can be evaluated using Horner's rule (written here with the dependence on the formal argument x more explicitly shown)

$c_n(x) = \alpha_n$
for $i = n-1 : -1 : 0$
  $c_i(x) = x c_{i+1}(x) + \alpha_i$
end

$p_n(x) = c_0(x)$

If the roots of the polynomial are known we can use a recurrence based on

$$p_n(x) = \alpha_n(x - \rho_1)\cdots(x - \rho_n) \tag{1}$$

given by:

$d_0 = \alpha_n$
for $i = 1 : n$
  $d_i = d_{i-1} * (x - \rho_i)$
end

$p_n(x) = d_n$

This algorithm can be shown to compute $p_n(x)$ to high relative accuracy (Higham 2002 Accuracy and Stability of Numerical Algorithms, Second Edition). Specifically,

$$d_n = p_n(x)(1 + \mu), \qquad |\mu| \leq \gamma_{2n+1}$$

where $\gamma_k = ku/(1 - ku)$ and u is the unit roundoff of the floating point system used.

1.1

An error analysis of Horner's rule shows that the computed value of the polynomial satisfies

$$\hat{c}_0 = (1 + \theta_1)\alpha_0 + (1 + \theta_3)\alpha_1 x + \cdots + (1 + \theta_{2n-1})\alpha_{n-1}x^{n-1} + (1 + \theta_{2n})\alpha_n x^n \tag{2}$$

where $|\theta_k| \leq \gamma_k$ (Higham 2002 Accuracy and Stability of Numerical Algorithms, Second Edition). The pattern on the subscript is odd numbers, i.e., increment of 2, until the last which is even, i.e., last increment is 1.

Let

$$\tilde{p}_n(x) = |\alpha_0| + |\alpha_1|x + \cdots + |\alpha_n|x^n.$$

1. Show that

$$\frac{|p_n(x) - \hat{c}_0|}{|p_n(x)|} \leq \gamma_{2n} \frac{\tilde{p}_n(|x|)}{|p_n(x)|} \tag{3}$$

   and therefore

$$\kappa_{rel} = \frac{\tilde{p}(|x|)}{|p(x)|}$$

   is a relative condition number for perturbations to the coefficients bounded by γ_{2n}.

2. Is Equation(2) and the associated bound a backward stability bound?

3. Find examples of conditions on $p_n(x)$ and x that guarantee that $p_n(x)$ is perfectly conditioned with respect to perturbations to the coefficients of the monomial form of $p_n(x)$.

4. When would you expect the value of $p_n(x)$ to be relatively very sensitive with respect to perturbations to the coefficients?

1.2

Equation (3) also yields an a priori bound on the forward error

$$|p_n(x) - \hat{c}_0|$$

that can be computed along with evaluating $p_n(x)$ with Horner's rule.

Write a code that evaluates $p_n(x)$ and the forward error bound using Horner's rule as well as the product form (1). Your code should allow the selection of single or double precision evaluation of each of these.

1. Apply the code to the polynomial

$$\begin{aligned} p_9(x) &= (x-2)^9 \\ &= x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512 \end{aligned}$$

   to evaluate $p_9(x)$ via Horner's rule and the a priori bound on forward error at several hundred points in the interval $[1.91, 2.1]$ in single precision. Use $p_9(x)$ evaluated using the product form in double precision as "exact" for the purposes of this exercise. Plot the product form values across the interval and use the forward error bound to plot curves above and below the "exact" product form curve to show where the computed values must lie. (Recall, for IEEE single precision $u \approx 5.9 \times 10^{-8}$ and for IEEE double precision $u \approx 1.1 \times 10^{-16}$.) Also plot the values of $p_9(x)$ computed with Horner's rule and verify the correctness of the a priori bounding curves. Comment on the tightness of the bounds and the computed values of $p_9(x)$ and quantify your conclusions, i.e., do not simply appeal to the picture.

2. Repeat the previous procedure with $p_9(x)$ evaluated using the product form in single precision as the "exact" value of $p_9(x)$. Are the conclusions significantly changed?

3. Is the fact that the a priori error bound is evaluated in single precision significant? Would you expect evaluating it in double precision to be a good idea?

1.3

The computed value on step i of Horner's rule satisfies

$$(1+\epsilon_i)\hat{c}_i = x\hat{c}_{i+1}(1+\delta_i) + \alpha_i, \quad |\delta_i| \leq u, \quad |\epsilon_i| \leq u$$

Define $\hat{c}_i = c_i + e_i$ with $e_n = 0$ and c_i the exact value of the parameter in Horner's rule evaluated in exact arithmetic. Show that

$$e_i = xe_{i+1} + x\hat{c}_{i+1}\delta_i - \epsilon_i\hat{c}_i$$

$$|e_i| \leq u\beta_i$$

$$\beta_i = |x|\beta_{i+1} + |x||\hat{c}_{i+1}| + |\hat{c}_i|, \quad \beta_n = 0$$

and therefore we have the bound

$$|p_n(x) - \hat{c}_0| \leq u\beta_0$$

This bound is called a running error bound for Horner's rule and can also be easily incorporated into the code for simulatneous evaluation with the values above (Higham 2002 Accuracy and Stability of Numerical Algorithms, Second Edition).

1. Add the computation of the running error bound to the code above (single and double precision should be supported) and compare this bound's prediction with those of the a priori bound above when Horner's rule and the two error bounds are evaluated in single precision and the "exact" value of $p_9(x)$ is evaluated in double precision.

2. Plot, quantify and discuss your observations.

3. Propose two other polynomials and repeat the comparisons of a priori versus running error bounds. Make sure you discuss why and how you generated the polynomials.

Submission of Results

Expected results comprise:

- A document describing your solutions as prescribed in the notes on writing up a programming solution posted on the class website.
- The source code, makefiles, and instructions on how to compile and execute your code including the Math Department machine used, if applicable.
- Code documentation should be included in each routine.
- All text files that do not contain code or makefiles must be PDF files. **Do not send Microsoft word files of any type**.

These results should be submitted by 11:59 PM on the due date. Submission of results is to be done via FSU Dropbox at `https://dropbox.fsu.edu`. Drop the files off for Zhifeng Deng, email `zdeng@math.fsu.edu`. You should login to FSU Dropbox using your MyFSU login. If for some reason you cannot use FSU Dropbox please email the files to Zhifeng at the email address above.