Set 4: Polynomial Interpolation – Part 1

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Interpolation Topics

1. Interpolation Overview
2. Lagrange Interpolation – Section 8.1
3. Newton Interpolation – Section 8.2
4. Complexity and Barycentric Forms – Sections 8.1, 8.2 and 8.3
5. Conditioning, Errors, and Convergence – Sections 8.1 and 8.2
6. Hermite Interpolation – Section 8.5
7. Piecewise Interpolation and Splines – Section 8.4 and 8.8
8. Multidimensional Interpolation – Section 8.6
9. Rational Interpolation (notes)

References

In addition to the text, the following are useful references for this topic.

1. Isaacson and Keller, Analysis of Numerical Methods, Wiley Press, 1966.
2. Bartle, The Elements of Real Analysis, Wiley, Second Edition, 1976.
3. Higham, Accuracy and Stability of Numerical Algorithms, SIAM, Second Edition, 2002.
4. Dahlquist and Bjorck, Numerical Methods, Prentice-Hall, 1974.
5. Ueberhuber, Numerical Computation, Volume 1, Springer, 1995.

Polynomial Interpolation

- Find $p_n(x) \in \mathbb{P}_n \mid y_i = p_n(x_i) \;\; 0 \leq i \leq n$.
- This is the simplest problem. Others are also of interest.
- $n + 1$ parameters and $n + 1$ constraints
- global, local/piecewise nonsmooth, or local/piecewise smooth polynomial interpolation
- polynomials and their derivatives are cheap to evaluate
- many representations of polynomials, i.e., parameterizations
- efficient interpolation algorithms exist and can be adapted to many circumstances: quadrature, differentiation, integration.
- accuracy achieved and achievable may be a problem
- accuracy considered in terms of the application of the polynomial

Polynomial Interpolation

Lemma 4.1. *Let* x_i, $0 \leq i \leq n$, *have distinct values. If* $p_n(x) \in \mathbb{P}_n$ *and* $q_n(x) \in \mathbb{P}_n$ *are polynomials of degree at most* n *such that* $p_n(x_i) = q_n(x_i), \;\; 0 \leq i \leq n$, *then* $p_n(x) = q_n(x)$.

Proof. $d_n(x) = p_n(x) - q_n(x)$ is a polynomial of degree at most n such that $d_n(x_i) = 0$, $0 \leq i \leq n$, and so has $n + 1$ roots. It follows that $d_n(x) \equiv 0$. $\square$

Polynomial Interpolation

- Find $p_n(x) \in \mathbb{P}_n \mid y_i = p_n(x_i) \;\; 0 \leq i \leq n$.
- Parameterization:

$$p_n(x) = \alpha_0\beta_0(x) + \alpha_1\beta_1(x) + \cdots + \alpha_n\beta_n(x) \quad \alpha_i \in \mathbb{R} \quad \beta_i(x) \in \mathbb{P}_n$$

- $\{\beta_i(x)\}$ chosen to be a basis of $\mathbb{P}_n$ and may or may not depend upon mesh points x_i, $i = 0, \ldots, n$.
- Particular bases are often constructed using insights into polynomials that lend structure to the problem of determining the coefficients.
- The structure often allows the direct construction of the polynomial, i.e., a linear system of equations is not considered explicitly.

Polynomial Interpolation

$$
\begin{aligned}
y_0 &= \alpha_0\beta_0(x_0) + \alpha_1\beta_1(x_0) + \ldots + \alpha_n\beta_n(x_0) \\
y_1 &= \alpha_0\beta_0(x_1) + \alpha_1\beta_1(x_1) + \ldots + \alpha_n\beta_n(x_1) \\
&\ \vdots \\
y_n &= \alpha_0\beta_0(x_n) + \alpha_1\beta_1(x_n) + \ldots + \alpha_n\beta_n(x_n)
\end{aligned}
$$

$$
\begin{pmatrix}
\beta_0(x_0) & \beta_1(x_0) & \ldots & \beta_n(x_0) \\
\beta_0(x_1) & \beta_1(x_1) & \ldots & \beta_n(x_1) \\
\vdots & \vdots & & \vdots \\
\beta_0(x_n) & \beta_1(x_n) & \ldots & \beta_n(x_n)
\end{pmatrix}
\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}
=
\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}
$$

Monomial Form, Existence and Uniqueness

Assume the polynomial, $p_n(x)$, is represented uniquely in terms of monomials, x^i

$$p_n(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$$

i.e., monomials are a basis for the vector space $\mathbb{P}_n$

Consider the constraints

$$\begin{aligned} y_0 &= \alpha_0 + \alpha_1 x_0 + \alpha_2 x_0^2 + \cdots + \alpha_n x_0^n \\ y_1 &= \alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2 + \cdots + \alpha_n x_1^n \\ &\vdots \\ y_n &= \alpha_0 + \alpha_1 x_n + \alpha_2 x_n^2 + \cdots + \alpha_n x_n^n \end{aligned}$$

Monomial Form, Existence and Uniqueness

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & & \vdots & \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^n \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}$$

$$V^T a = y$$

Example

$$(x, y) = \{(1, 10) \;\; (2, 26) \;\; (3, 58) \;\; (4, 112)\}$$

4 distinct x_i implies cubic $p_3(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$

$$\alpha_0 + \alpha_1 * 1 + \alpha_2 * 1 + \alpha_3 * 1 = 10$$

$$\alpha_0 + \alpha_1 * 2 + \alpha_2 * 4 + \alpha_3 * 8 = 26$$

$$\alpha_0 + \alpha_1 * 3 + \alpha_2 * 9 + \alpha_3 * 27 = 58$$

$$\alpha_0 + \alpha_1 * 4 + \alpha_2 * 16 + \alpha_3 * 64 = 112$$

Example

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 26 \\ 58 \\ 112 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\det(V^T) = 12 \text{ and } \kappa_2(V^T) = 1.17 \times 10^3$$

Solution yields: the cubic $p_3(x) = 4 + 3x + 2x^2 + 1x^3$

Complexity: LU factorization $O(n^3)$, Bjorck and Pereyra $O(n^2)$

Monomial Form, Existence and Uniqueness

- The matrix V is called a Vandermonde matrix.
- The determinant has a known closed form

$$\det(V) = \det(V^T) = \prod_{j=0}^{n} \{ \prod_{i=j+1}^{n} (x_i - x_j) \}$$

- Therefore, if the x_i are $n+1$ distinct values there is a unique interpolating polynomial of degree n

Vandermonde Systems

- Vandermonde matrices appear in many numerical problems.
- V derived from $x_j = \omega^j$ for $0 \leq j \leq n-1$ where $\omega = exp(-2\pi i)/n$, i.e., an n-th root of unity, $V/\sqrt{n}$ is a unitary matrix that defines the Discrete Fourier Transform. It is perfectly conditioned.
- Bjorck and Pereyra (Mathematics of Computation 24(112), 1970) have derived and analyzed algorithms to solve $V^T b = c$ and $Vb = c$ requiring $(5n^2)/2$ operations.
- These often produce accurate solutions despite ill-conditioning of V or V^T.
- For interpolation, the algorithm produces the Newton form polynomial as an intermediate result and then converts to the monomial form!

Vandermonde Matrix Conditioning

- Higham (2002) presents a detailed analysis of the conditioning of Vandermonde matrices and the stability of solving related systems.
- V and V^T can be very ill-conditioned.
- The ∞-norm conditioning for $x_i = 1/(i+1)$ increases faster than $n!$, i.e., $\kappa_\infty > n^{n+1}$
- for any real choice of x_i, κ_2 increases at least exponentially and

$$\kappa_2 \geq \left(\frac{2}{n+1}\right)^{1/2} \left(1+\sqrt{2}\right)^{n-1}$$

with equality using equally spaced points on $[0, 1]$.

Vandermonde Matrix Conditioning

- complex roots of unity, i.e., the Fourier matrix $V/\sqrt{n}$, is the only choice that is perfectly conditioned.
- If $0 \leq x_0 \leq x_1 \leq \cdots \leq x_n$ then the computed solution, $\hat{a}$ to $V^T a = y$ using a generalization of Bjorck and Pereyra's algorithm satisfies the (amazing) component-wise bound

$$|y - V^T \hat{a}| \leq (n(n+4)u + O(u^2))|V^T||\hat{a}|$$

  where u is unit roundoff. (Higham 2002)
- So the ill-conditioning does not necessarily preclude an accurate solution when the correct algorithm is chosen and there are some constraints on the x_i.

Alternative Forms

- $p_n(x)$ is unique for distinct x_i
- V^T tends to get increasingly ill-conditioned, $\kappa \sim 10^n$
- other representations can be chosen to construct $p_n(x)$
- Two use a basis that depends on the mesh and yields significant structure in the construction/linear system:
  1. Lagrange form
  2. Newton form
- As noted, using Bjorck and Pereyra yields the Newton form as an intermediate step so the Vandermonde approach and the monomial form are of limited practical use.

Lagrange Form

Given (x_i, y_i) $0 \le i \le n$ with distinct x_i values.

Consider a polynomial $m_0(x) \in \mathbb{P}_n$ defined by

$$m_0(x) = (x - x_1)(x - x_2) \cdots (x - x_{n-1})(x - x_n)$$

$$m_0(x_0) = (x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_{n-1})(x_0 - x_n)$$

$$\ell_0(x) = \frac{m_0(x)}{m_0(x_0)} = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x = x_j \end{cases}$$

Lagrange Form

Define

$$m_i(x) = \prod_{j=0, j\neq i}^{n} (x - x_j) \text{ and } \ell_i(x) = \frac{m_i(x)}{m_i(x_i)} = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{if } x = x_j \end{cases}$$

Each $m_i(x)$ solves a very simple and fundamental interpolation problem on the given mesh.

Lagrange Form

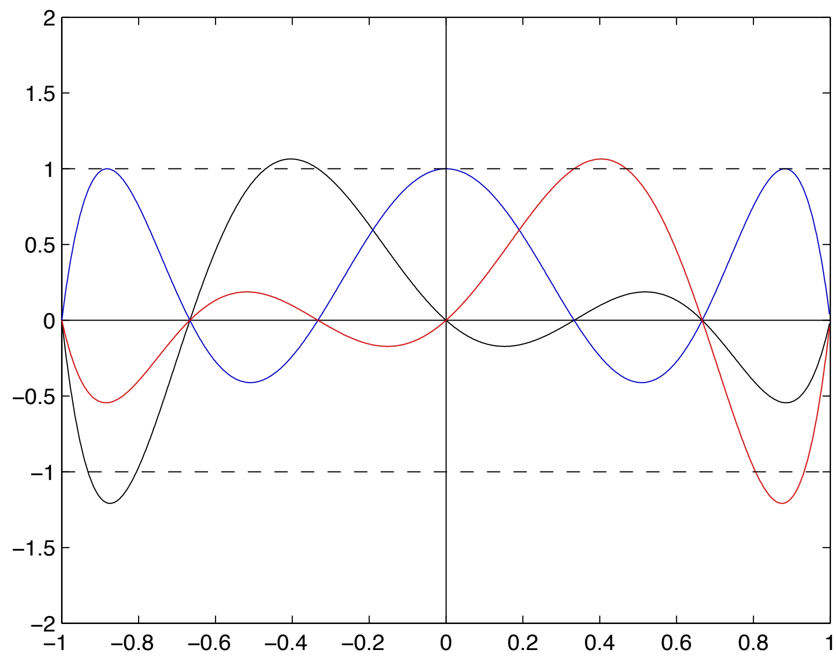
The solutions to the simple interpolation problems are easily combined to solve the interpolation problem that defines $p_n(x)$.

Define

$$p_n(x) = \sum_{i=0}^{n} y_i \; \ell_i(x) \rightarrow p_n(x_i) = y_i \;\; 0 \leq i \leq n$$

Since the interpolating polynomial is unique this is the same $p_n(x)$ as before.

Characteristic Polynomials



$n = 6$ on $[-1, 1]$, equidistant, $\ell_2(x)$ – black, $\ell_3(x)$ – blue, $\ell_4(x)$ – red

Characteristic Polynomials

- basis for $\mathbb{P}_n$, i.e., given distinct x_i, $\forall p_n(x) \in \mathbb{P}_n, \;\; p_n(x) = \sum_{i=0}^{n} \alpha_i \ell_i(x)$ with unique $\alpha_0, \; \ldots \; , \alpha_n$.
- note asymmetric forms
- magnitude is not bounded by 1
- $\ell_i'(x_i)$ is not necessarily 0 (compare location of peak of $\ell_i(x)$ to x_i where $\ell_i(x_i) = 1$)
- matrix form of the interpolation problem is trivial, i.e., $\alpha_i = y_i$ is easily seen.

Example

Let $n = 3$, $(x, y) = \left\{(1,10)\ \ (2,26)\ \ (3,58)\ \ (4,112)\right\}$

$$p_3(x) = 4 + 3x + 2x^2 + 1x^3$$

$$\begin{aligned}
p_3(x) =& 10 \times \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + 26 \times \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} \\
&+ 58 \times \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + 112 \times \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} \\
=& -\frac{5}{3}(x-2)(x-3)(x-4) + 13(x-1)(x-3)(x-4) \\
&- 29(x-1)(x-2)(x-4) + \frac{56}{3}(x-1)(x-2)(x-3)
\end{aligned}$$

Newton Form of Polynomial

Constant: $p_0(x) = y_0$

Point–slope form of line:

$$y(x) = y_0 + m(x - x_0)$$

$$y(x_1) = y_1 = y_0 + m(x_1 - x_0)$$

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

$$\Downarrow$$

$$p_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$p_1(x_0) = y_0 \text{ and } p_1(x_1) = y_1$$

Newton Form of Polynomial

Define the first divided difference:

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

The Netwon polynomial of degree 1 is then defined as:

$$p_1(x) = y_0 + y[x_0, x_1](x - x_0)$$

Note the form

$$p_1(x) = p_0(x) + q_1(x)$$

Consider the general case of increasing degree of interpolating $p_{n-1}(x)$:

$$p_n(x) = p_{n-1}(x) + q_n(x)$$

Incrementing Newton Form of Polynomial

$$p_n(x) = p_{n-1}(x) + q_n(x) \rightarrow q_n(x) = p_n(x) - p_{n-1}(x)$$

$$q_n(x_i) = p_n(x_i) - p_{n-1}(x_i) = 0 \quad \text{for } 0 \leq i \leq n-1$$

$$\therefore \quad q_n(x) = \alpha_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) = \alpha_n \omega_n(x)$$

$$p_n(x_n) = y_n = p_{n-1}(x_n) + \alpha_n \omega_n(x_n)$$

$$\Downarrow$$

$$\alpha_n = \frac{y_n - p_{n-1}(x_n)}{\omega_n(x_n)} \equiv y[x_0, \ldots, x_n]$$

Example

Let $n = 3$, $(x, y) = \big\{(1, 10) \;\; (2, 26) \;\; (3, 58) \;\; (4, 112)\big\}$

Recall the monomial form $p_3(x) = 4 + 3x + 2x^2 + 1x^3$

$$p_0(x) = y_0 = 10$$

$$p_1(x) = y_0 + \frac{26 - p_0(2)}{(2-1)}(x-1) = 10 + 16(x-1)$$

$$\begin{aligned} p_2(x) &= p_1(x) + \frac{58 - p_1(3)}{(3-2)(3-1)}(x-1)(x-2) \\ &= 10 + 16(x-1) + 8(x-1)(x-2) \end{aligned}$$

$$\begin{aligned} p_3(x) &= p_2(x) + \frac{112 - p_2(4)}{(4-3)(4-2)(4-1)}(x-1)(x-2)(x-3) \\ &= 10 + 16(x-1) + 8(x-1)(x-2) + 1(x-1)(x-2)(x-3) \end{aligned}$$

Recursive Form

Given (x_i, y_i), for $0 \le i \le n$, let

- $\omega_{k:k+s}(x) = (x - x_k)(x - x_{k+1}) \cdots (x - x_{k+s})$
- $p_{n-1}(x)$ interpolates $y_0, \ldots, y_{n-1}$ and $q_{n-1}(x)$ interpolates $y_1, \ldots, y_n$
- $p_n(x)$ interpolates $y_0, \ldots, y_n$

Lemma.

$$p_n(x) = q_{n-1}(x) + \frac{x - x_n}{x_n - x_0}\Big[q_{n-1}(x) - p_{n-1}(x)\Big]$$

Recursive Form

Proof. Check interpolation conditions for $1 \le i \le n-1$ and the endpoints.

$$
\begin{aligned}
p_n(x_i) &= q_{n-1}(x_i) + \frac{x_i - x_n}{x_n - x_0}\Big[q_{n-1}(x_i) - p_{n-1}(x_i)\Big] \\
&= q_{n-1}(x_i) + \frac{x_i - x_n}{x_n - x_0} \times 0 = y_i \\
p_n(x_n) &= q_{n-1}(x_n) + \frac{x_n - x_n}{x_n - x_0}\Big[q_{n-1}(x_n) - p_{n-1}(x_n)\Big] \\
&= q_{n-1}(x_n) + 0 = y_n \\
p_n(x_0) &= q_{n-1}(x_0) + \frac{x_0 - x_n}{x_n - x_0}\Big[q_{n-1}(x_0) - p_{n-1}(x_0)\Big] \\
&= q_{n-1}(x_0) - q_{n-1}(x_0) + p_{n-1}(x_0) = y_0
\end{aligned}
$$

$\square$

Recursive Form

$$
\begin{aligned}
p_{n-1}(x) &= \sum_{i=0}^{n-1} \omega_{0:i-1}(x) y[x_0, \ldots, x_i] \\
&= y[x_0, \ldots, x_{n-1}] x^{n-1} + r_{n-2}(x)
\end{aligned}
$$

$$
\begin{aligned}
q_{n-1}(x) &= \sum_{i=1}^{n} \omega_{1:i-1}(x) y[x_1, \ldots, x_i] \\
&= y[x_1, \ldots, x_n] x^{n-1} + \tilde{r}_{n-2}(x)
\end{aligned}
$$

$$
\begin{aligned}
p_n(x) &= \sum_{i=0}^{n} \omega_{0:i-1}(x) y[x_0, \ldots, x_i] \\
&= y[x_0, \ldots, x_n] x^n + \hat{r}_{n-1}(x)
\end{aligned}
$$

Recursive Form

Compare leading term coeffiecients:

$$p_{n-1}(x) = y[x_0, \ldots, x_{n-1}]x^{n-1} + r_{n-2}(x)$$

$$q_{n-1}(x) = y[x_1, \ldots, x_n]x^{n-1} + \tilde{r}_{n-2}(x)$$

$$p_n(x) = y[x_0, \ldots, x_n]x^n + \hat{r}_{n-1}(x)$$

$$p_n(x) = q_{n-1}(x) + \frac{x - x_n}{x_n - x_0}\Big[q_{n-1}(x) - p_{n-1}(x)\Big]$$

$$= \frac{x^n}{x_n - x_0}\Big[y[x_1, \ldots, x_n] - y[x_0, \ldots, x_{n-1}]\Big] + \text{lower order terms}$$

$$\therefore y[x_0, \ldots, x_n] = \frac{y[x_1, \ldots, x_n] - y[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

Example

$$D^k f_i \quad 1 \leq k \leq n, \quad 0 \leq i \leq n-k$$

i	0	1	2	3
x_i	1	2	3	4
f_i	10	26	58	112
$D^1 f_{i-1}$	–	$D^1 f_0 = 16$	$D^1 f_1 = 32$	$D^1 f_2 = 54$
$D^2 f_{i-2}$	–	–	$D^2 f_0 = 8$	$D^2 f_1 = 11$
$D^3 f_{i-3}$	–	–	–	$D^3 f_0 = 1$

where $D^k f_i = f[x_i, \ldots, x_{i+k}] = \dfrac{D^{k-1} f_{i+1} - D^{k-1} f_i}{(x_{i+k} - x_i)}$

$$p_3(x) = f_0 + D^1 f_0 \omega_1(x) + D^2 f_0 \omega_2(x) + D^3 f_0 \omega_3(x)$$

$$p_3(x) = 10 + 16(x-1) + 8(x-1)(x-2) + 1(x-1)(x-2)(x-3)$$

Newton and Lagrange Forms

- Assume the distinct meshpoints $x_0, \ldots, x_n$ are given.
- The Lagrange polynomials $\ell_i(x)$, for $i = 0, \ldots, n$, and the Newton polynomials $\omega_0(x) = 1$ and $\omega_i(x) = (x - x_0)(x - x_1) \cdots (x - x_{i-1})$, for $i = 1, \ldots, n$ are both bases for $\mathbb{P}_n$.
- Both bases yield linear equations that relate the data to the coefficients defining the interpolating polynomial $p_n(x)$.
- These equations are highly structured due to the constructions used in the derivations of the forms.

Equidistant Differences

Suppose $x_i = x_{i-1} + h$ for $h > 0$ and $i = 0, \ldots$.

$$\Delta f(x) = f(x+h) - f(x)$$

$$\nabla f(x) = f(x) - f(x-h)$$

$$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$$

$$\Delta^{m+1} f(x) = \Delta^m f(x+h) - \Delta^m f(x)$$

$$f[x_0, x_1] = \frac{\Delta f(x_0)}{h}$$

$$f[x_0, \ldots, x_k] = \frac{\Delta^k f(x_0)}{k! h^k}$$

Equidistant Basic Formulae

$$x = x_0 + sh \rightarrow x - x_i = (s - i)h$$

$$\prod_{i=0}^{n}(x - x_i) = h^{n+1}\prod_{i=0}^{n}(s - i) = \pi_n(s)h^{n+1}$$

$\pi_k(s)$ is called the factorial polynomial. We can therefore generalize the binomial coefficient:

$$\binom{s}{k} \equiv \frac{\prod_{i=0}^{k-1}(s - i)}{k!} = \frac{\pi_{k-1}(s)}{k!}$$

Equidistant Newton Polynomial

$$
\begin{aligned}
& f_0 + f[x_0, x_1](x - x_0) + \ldots + f[x_0, \ldots, x_n](x - x_0) \cdots (x - x_{n-1}) \\
&= f_0 + \frac{\Delta f_0}{h}(x - x_0) + \ldots + \frac{\Delta^n f_0}{n! h^n}(x - x_0) \cdots (x - x_{n-1}) \\
&= f_0 + \frac{\Delta f_0}{h}\pi_0(s) h + \ldots + \frac{\Delta^n f_0}{n! h^n}\pi_{n-1}(s) h^n \\
&= f_0 + \frac{\Delta f_0}{1!}\pi_0(s) + \ldots + \frac{\Delta^n f_0}{n!}\pi_{n-1}(s) \\
&= f_0 + \binom{s}{1}\Delta f_0 + \ldots + \binom{s}{i}\Delta^i f_0 + \ldots + \binom{s}{n}\Delta^n f_0
\end{aligned}
$$