Homework 11

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MAP5345: Partial Differential Equations I

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Problem 1

Consider polar coordinates (r, θ), whic are related to Cartesian coordinates via $x = rcos\theta$ and $y = rsin\theta$.

a) Use the chain rule for partial derivatives to calculate $\frac{\partial}{\partial r}, \frac{\partial^2}{\partial r^2}, \frac{\partial}{\partial \theta}, \frac{\partial^2}{\partial \theta^2}$ in terms of x and y partial derivatives.

The coordinate transformation

$$x \mapsto r\cos(\theta), \quad y \mapsto r\sin(\theta)$$

yields the following

$$u_\theta = u_x x_\theta + u_y y_\theta = -u_x r\sin(\theta) + u_y r\cos(\theta)$$

$$u_r = u_x x_r + u_y y_r = -u_x \sin(\theta) + u_y \cos(\theta)$$

$$\begin{aligned}
u_{\theta\theta} &= -r\cos(\theta)u_x - r\sin(\theta)\partial_\theta u_x - r\sin(\theta)u_y + r\cos(\theta)\partial_\theta u_y \\
&= -r\cos(\theta)u_x - r\sin(\theta)\Big(- u_{xx} r\sin(\theta) + u_{xy} r\cos(\theta)\Big) \\
&- r\sin(\theta)u_y + r\cos(\theta)\Big(- r u_{xy}\sin(\theta) + u_{yy} r\cos(\theta)\Big) \\
&= -r\Big(\cos(\theta)u_x + \sin(\theta)u_y \Big) + r^2\Big(\sin^2(\theta)u_{xx} - 2\cos(\theta)\sin(\theta)u_{xy} + \cos^2(\theta)u_{yy}\Big)
\end{aligned}$$

$$\begin{aligned}
u_{rr} &= \cos(\theta)\partial_r u_x + \sin(\theta)\partial_r u_y \\
&= \cos^2(\theta)u_{xx} + 2\cos(\theta)\sin(\theta)u_{xy} + \sin^2(\theta)u_{yy}
\end{aligned}$$

b) Use your results to show that the Laplacian in polar coordiantes is given by

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Dividing the expression for $u_{\theta\theta}$ yields

$$\frac{1}{r^2}u_{\theta\theta} = -\frac{1}{r}u_r + \sin^2(\theta)u_{xx} - 2\cos(\theta)\sin(\theta)u_{xy} + \cos^2(\theta)u_{yy}$$

Then adding the above to u_{rr} gets us

$$\begin{aligned} & u_{rr} + \frac{1}{r^2}u_{\theta\theta} = -\frac{1}{r}u_r + u_{xx} + u_{yy} \\ \Rightarrow \quad & u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \end{aligned}$$

Problem 2

Using Julia, graph $J_0(x)$, i.e. the Bessel function of the first kind and of order $n = 0$. Choose a reasonable domain to graph the function, so that you can see at least a few roots of J_0. On the same graph, plot the function

$$\sqrt{\frac{2}{\pi x}}\cos(x - \pi/4 - n\pi/2) \tag{1}$$

and verify that the two are asymptotic for $x >> 1$. How many roots do you have to go out for the asymptotic formula to capture the roots pretty accurately? Do the same for $J_1(x)$, the first-kind Bessel function of order $n = 1$. Also, verify the asymptotic formula.

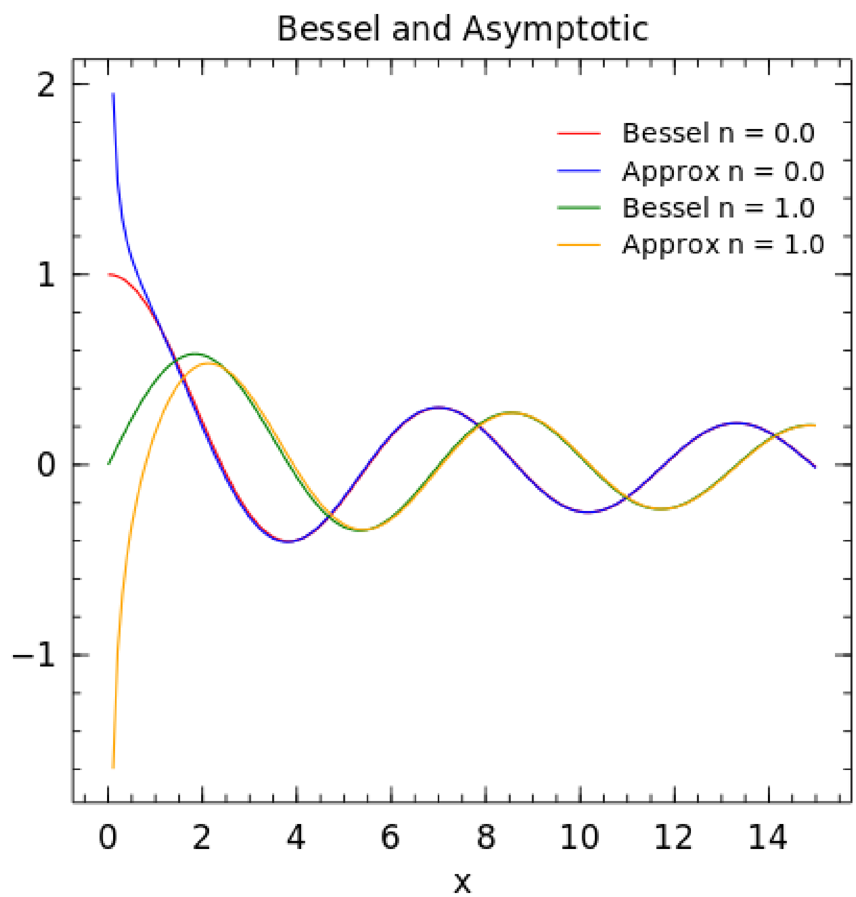

Figure 1: Bessel Function plotted against its asymptotic formula.

Problem 3

For a drum head with given wave-speed c and radius a, can you determine the fundamental frequency, and thus the 'pitch' of the sound that drum will produce? For a realistic drum with $c = 400$ m/s and $a = 0.5$, what is the fundamental frequency in Hz? Also find the first harmonic.

The fundamental frequency will be the first root of J_0, the $0-th$ order Bessel function, times the wave speed c. Similarly the $n-th$ harmonic will be the $n+1-th$ root of J_0. From this we get

Harmonic	**Frequency (Hz)**
0 (Fundamental)	1923.8608
1	4416.0624
2	6922.9816
3	9433.2272
4	11944.7336
$\vdots$	$\vdots$

Problem 4

Consider the Bessel ODE derived in class. Put this ODE into the form of a Sturm-Liouville problem. What is the weighted inner product for which the functions $R_m(r) = J_0(\sqrt{\lambda}r)$ are orthogonal?

The Sturm-Liouville equation

$$(p(x)u'(x))' - q(x)u(x) = -\lambda m(x)u(x)$$

has inner product (orthogonal functions)

$$\langle f, g\rangle = \int m(x)f(x)\overline{g(x)}\,dx$$

The ODE we have in class can be multiplied by r to obtain

$$\begin{aligned} R''(r) + \frac{1}{r}R'(r) + R(r)\left(\lambda - \frac{n^2}{r^2}\right) &= 0 \\ rR''(r) + R'(r) + rR(r)\left(\lambda - \frac{n^2}{r^2}\right) &= 0 \qquad \text{(multiply by } r\text{)} \\ \Rightarrow \quad u = R(r), \quad p = r, \quad q = \frac{n^2}{r}, \quad m &= r \end{aligned}$$

Therefore r is the weight that makes the Bessel functions orthogonal. The resulting inner product for the Bessel functions is

$$\langle R_i, R_j\rangle = 2\pi \int rR_iR_j\,dr$$

where the 2π comes from the integral about the θ domain $[0, 2\pi]$.

Problem 5

Consider radial vibrations of a circular drum of radius $a = 0.5$ m. The medium's speed of sound given by $c = 400$ m/s.

a) Now, suppose that the initial displacement is $u_0 = a^2 - r^2$ and there is no initial velocity. Write don the exact solution to the IBVP.

The general solution of the PDE under rotational invariance is

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos(\sqrt{\lambda_n} ct) + B_n \sin(\sqrt{\lambda} ct)) J_0(\sqrt{\lambda_n} r)$$

but under zero initial velocity we get $B_n = 0$ and are left with

$$u(r,t) = \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n} ct) J_0(\sqrt{\lambda_n} r)$$

Using projection to determine A_n we arrive at

$$A_n = \frac{\langle a^2 - r^2, J_0(\sqrt{\lambda_n} r)\rangle}{\langle J_0(\sqrt{\lambda_n} r), J_0(\sqrt{\lambda_n} r)\rangle} = \frac{\int_0^{0.5} (a^2 - r^2) J_0(\sqrt{\lambda_n} r) r\, dr}{\int_0^{0.5} J_0(\sqrt{\lambda_n} r) J_0(\sqrt{\lambda_n} r) r\, dr}$$

Therefore the exact solution is

$$\sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n} ct) J_0(\sqrt{\lambda_n} r), \quad A_n = \frac{\int_0^{0.5} (a^2 - r^2) J_0(\sqrt{\lambda_n} r) r\, dr}{\int_0^{0.5} J_0(\sqrt{\lambda_n} r) J_0(\sqrt{\lambda_n} r) r\, dr}$$

b) Perform the projection numerically, using the trapezoidal rule, in order to find a numerical approximation to the first three coefficients in your solution.

Using the trapezoidal rule to perform quadrature we get the follwoing coefficients

$$A_1 \approx 0.0375909, \quad A_2 \approx 0.0194997, \quad A_3 \approx 0.0047067$$

c) Visualize a few different time-values of this approximate solution in Julia.

Decay Coefficients

Find the lower bounds of the following functions

Theorem 1. If $f_{ext} \in C^n$ then there exists a M such that $c_k \leq \frac{M}{|k|^n}$

Theorem 2. If $f_{ext} \in L^2$ and if $c_k = \frac{\langle f, e^{ikx} \rangle}{\langle e^{ikx}, e^{ikx} \rangle}$ and there exists M, $\epsilon > 0$ such that $|c_k| \leq \frac{M}{|k|^{1+n+\epsilon}}$ then $f_{ext} \in C^n$

a) $|x|$

We know that $f_{ext} \in C^0$ and therefore $f_{ext} \notin C^1$ and the decay rate is bounded below by 0. From Theorem 2 we then know that $\nexists M$ or ϵ such that $|c_k| \leq \frac{M}{|k|^{2+\epsilon}}$ and therefore the decay rate is bounded above by 2.

b) x

We know that f_{ext} is piecewise polynomial and therefor $f_{ext} \notin C^0$ and the decay rate is bounded below by some constant M. From Theorem 2 we then know that $\nexists M$ or ϵ such that $|c_k| \leq \frac{M}{|k|^{1+\epsilon}}$ and therefore is bounded above by 1.

c) $\pi^2 - x^2$

Since $f_{ext} \in C^0$ the proof follows similarly from part (a). Therefore the decay rate is bounded below by 0 and above by 2.

d) $\sqrt{\pi^2 - x^2}$

Since $f_{ext} \in C^0$ the proof follows similarly from part (a). Therefore the decay rate is bounded below by 0 and above by 2.

e) $x(\pi^2 - x^2)$

We know that $f_{ext} \in C^1$ and therefore $f_{ext} \notin C^2$ and the decay rate is bounded below by 1. From Theorem 2 we then know that $\nexists M$ or ϵ such that $|c_k| \leq \frac{M}{|k|^{3+\epsilon}}$ and therefore the decay rate is bounded above by 3.

f) $(\pi^2 - x^2)^2$

We know that $f_{ext} \in C^2$ and therefore $f_{ext} \notin C^3$ and the decay rate is bounded below by 2. From Theorem 2 we then know that $\nexists M$ or ϵ such that $|c_k| \leq \frac{M}{|k|^{4+\epsilon}}$ and therefore the decay rate is bounded above by 4.