Set 9: Polynomial Interpolation – Part 5

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Foundations of Computational Math 1

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Convergence of Interpolating Polynomials

- convergence of polynomials
- interpolation strategies
- convergence of interpolation strategies

Convergence on Interval

Approximation by polynomials is motivated by the following theorem:

Theorem 9.1. *(Weierstrass Approximation Theorem) If* $f(x) \in \mathcal{C}^{(0)}[a, b]$ *then* $\forall \epsilon > 0$ $\exists n \in \mathbb{Z}$ *and polynomial* $p_n(x)$ *with degree at most* n *such that*

$$\|f(x) - p_n(x)\|_\infty < \epsilon.$$

This is uniform convergence, i.e., pointwise error at all points in interval is bounded and the bound is going to 0.

Convergence on Interval

- Theorem 9.1 gives no insight into how to choose $p_n(x)$ and does not relate necessarily to an interpolation strategy.
- The result can be derived as a corollary to a constructive theorem due to Bernstein.
- A sequence of polynomials is defined and shown to converge uniformly.

Bernstein Polynomials

Definition 9.1. Let $f(x)$ be a real function defined on $[0,1]$. The n-th Bernstein polynomial for f is

$$
\begin{aligned}
B_n(x) = B_n(x;f) &= \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n} f(\frac{k}{n}) \phi_{n,k}(x) \\
&= \sum_{k=0}^{n} f(x_k) \phi_{n,k}(x) \\
x_k &= k/n
\end{aligned}
$$

Bernstein Polynomials

- Sum of $f(x)$ at uniformly-spaced points.
- The weight $\phi_{n,k}(x)$ is non-negative on $[0, 1]$ and $\sum_{k=0}^{n} \phi_k(x) = 1$.
- The weight $\phi_{n,k}(x)$ can be very small for k where x is far from k/n.
- The weight $\phi_{n,k}(x)$ achieves its maximum on $[0, 1]$ at $x = k/n$.
- The construction is not interpolatory, i.e., $B_n(x_k)$ is not necessarily equal to $f(x_k)$.
- $B_n(x)$ usually interpolates $f(x)$ but where and how often it does is not controlled.

Bernstein Approximation

Theorem 9.2. *If* $f(x) \in \mathcal{C}^{(0)}[0,1]$ *then* $B_n(x)$ *converges uniformly to* $f(x)$ *on* $[0,1]$, *i.e.,*

$$\lim_{n\to\infty} \|f(x) - B_n(x)\|_\infty = 0$$

Proof. See Bartle, Elements of Real Analysis (1976) □

Corollary 9.3. *If, in addition, on* $[0,1]$, $f(x)$ *satisfies the Lipschitz condition* $|f(x) - f(\hat{x})| < \lambda|x - \hat{x}|$ *then*

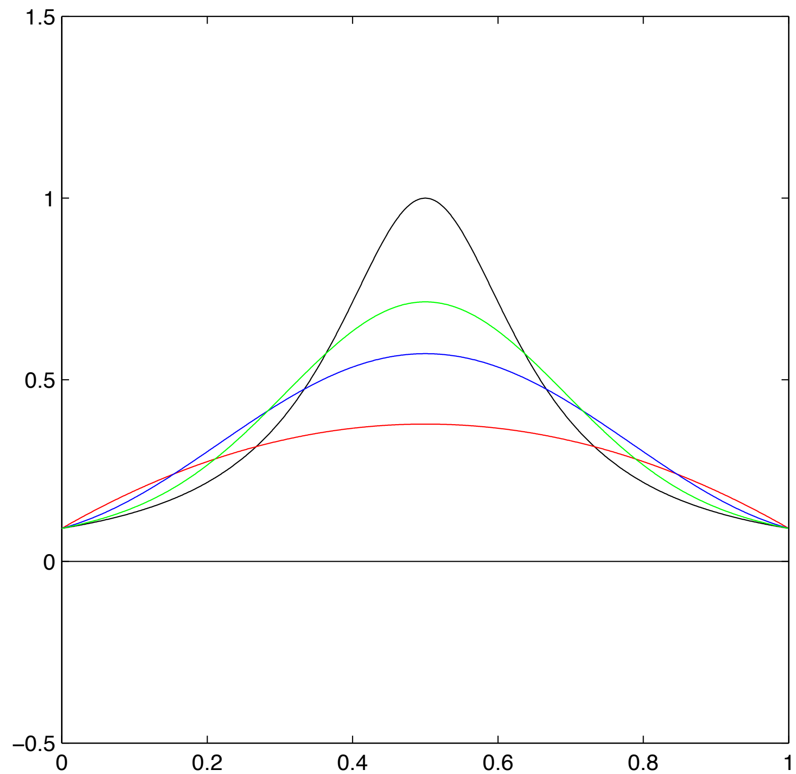
$$\|f(x) - B_n(x)\|_\infty < \frac{9}{4}\lambda n^{-1/2}$$

Proof. See Isaacson and Keller (1966) □

Bernstein Approximation

- Easily updated to apply to $[a, b]$.
- Convergence is much slower than other approximation methods.
- Even if $f(x) \in \mathcal{C}^{(p)}[0, 1]$ with $p \geq 2$ convergence remains relatively slow.
- Useful theoretical result but Bernstein polynomials are not used in practice for this type of approximation.
- Bernstein polynomials are used when "shape" is important.
- This shows that polynomials **can converge uniformly** to a continuous f.

Bernstein Convergence



$f(x) = 1/(1+10x^2)\ \ -1 \leq x \leq 1$ shifted to $[0,1]$ – black, $B_3(x)$ – red, $B_6(x)$ – blue, $B_{15}(x)$ – green

Convergence of Interpolating Polynomials

Definition 9.2. An interpolating strategy is defined by a sequence, X, of sets of nodes $X_n = \{x_0^{(n)}, \ldots, x_n^{(n)}\}$.

- The sets X_n are chosen independently of any particular $f(x)$.
- Each X_n defines an interpolatory polynomial, $p_n(x)$, of degree n such that given an $f(x)$, $p_n(x_i^{(n)}) = f(x_i^{(n)})$ for $0 \le i \le n$.

Uniform interpolation:

$$X_n = \{x_i^{(n)} = x_0 + ih, \quad h = (b-a)/n\}$$

Chebyshev interpolation:

$$X_n = \{x_j^{(n)} = \cos(\frac{2j+1}{n+1}\frac{\pi}{2})\}$$

Convergence of Interpolating Polynomials

The convergence of

$$\|f(x) - p_n(x)\|_\infty$$

on a closed interval $[a, b]$ for $f(x) \in \mathcal{C}^{(0)}[a, b]$ is complicated.

The result depends on

- the choice of X,
- the class of functions $f(x)$ that may be more constrained than $\mathcal{C}^{(0)}[a, b]$

Runge's Phenomenon

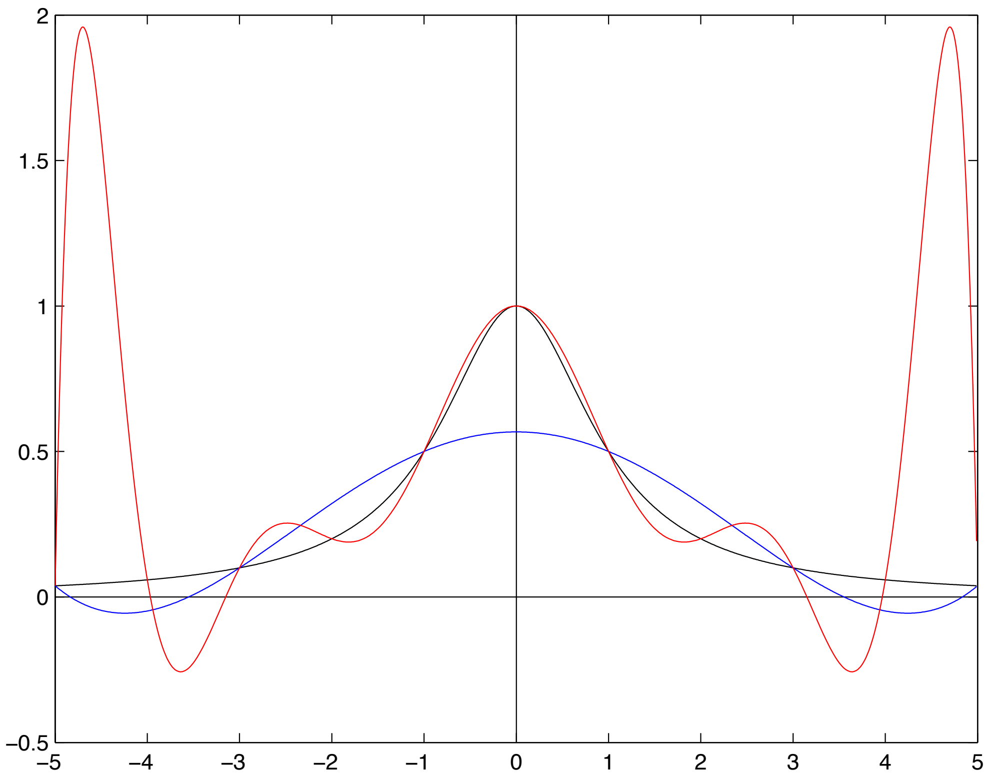
Let $I = [-5, 5]$ and define $x_j^{(n)} = -5 + jh_n$ with $h_n = 10/n$ and $0 \leq j \leq n$. The sets $X_n = \{x_0^{(n)}, \ldots, x_n^{(n)}\}$ define a sequence, X, of sets of nodes each of which define an interpolatory polynomial, $p_n(x)$, of degree n. It can be shown that

$$\lim_{n\to\infty} \|f(x) - p_n(x)\|_\infty$$

does not converge on I for $f(x) = 1/(1 + x^2)$.

Proof. See Isaacson and Keller (1966) $\square$

Runge's Phenomenon



$f(x) = 1/(1 + x^2)$ – black, $p_5(x)$ – blue, $p_{10}(x)$ – red

Runge's Phenomenon

- The divergence occurs near the endpoints of the interval.
- This is typical behavior so keep order low to be effective with uniformly spaced points.
- Non-uniform points more dense near endpoints are needed for better interpolation strategies, e.g., Chebyshev.

Convergence of Interpolating Polynomials

For each degree n we can define the "best" polynomial approximation:

Definition 9.3. Let $p_n^*(x) \in \mathbb{P}_n$ be such that
$E_n^* = \|f(x) - p_n^*(x)\|_\infty \leq \|f(x) - q_n(x)\|_\infty \;\; \forall q_n(x) \in \mathbb{P}_n$.

This approximation will be discussed in much more detail later.

Convergence of Interpolating Polynomials

Lemma. *Let the sequence X define an interpolating strategy, and let the Lebesgue constant be*

$$\Lambda_n(X) = \|\sum_{j=0}^{n} |\ell_j^{(n)}(x)|\|_\infty$$

for the set of nodes $X_n = \{x_0^{(n)}, \ldots, x_n^{(n)}\}$ where $\ell_j^{(n)}(x)$ are the Lagrange characteristic functions associated with X_n.

If $f(x) \in \mathcal{C}^{(0)}[a, b]$ then

$$E_n^* \leq \|f(x) - p_n(x)\|_\infty \leq (1 + \Lambda_n(X))E_n^*$$

for $n = 0, 1, \ldots$.

Convergence of Interpolating Polynomials

- A small Lebesgue constant $\Lambda_n(X)$ guarantees good ∞ norm approximation of $f(x)$ for the associated $p_n(x)$.
- Bounding the Lebesgue constant $\Lambda_n(X)$ is a key task when analyzing an interpolating strategy.
- Erdos (1961) showed $\forall X \;\; \exists C > 0$ such that

$$\Lambda_n(X) > \frac{2}{\pi} \log(n+1) - C \;\; n = 0, 1, \ldots$$

  so $\Lambda_n(X) \to \infty$.

- Natonson (1965) showed for equally spaced nodes

$$\Lambda_n(X) \approx \frac{2^{n+1}}{en \log n}$$

Convergence of Interpolating Polynomials

- The error bound predicted by the Lebesgue constant is not achieved for all $f(x) \in \mathcal{C}^{(0)}[a, b]$.
- A particular strategy may work well with a particular f or some particular class of f
- Unfortunately, no interpolating strategy, X, converges for all $f(x) \in \mathcal{C}^{(0)}[a, b]$.

Convergence of Interpolating Polynomials

Theorem 9.4. *(Faber 1914) Given an interpolating strategy defined by any sequence of node sets X on $[a, b]$, $\exists\ f(x) \in \mathcal{C}^{(0)}[a, b]$ such that $\|f(x) - p_n(x)\|_\infty$ does not converge.*

Summary

- (Bernstein) $B_n(x)$ converge uniformly for all $f(x) \in \mathcal{C}^{(0)}[a, b]$ but not an interpolating strategy since the number and position of points where they agree with $f(x)$ depend on $f(x)$.
- (Faber) No $p_n(x)$ defined by an X converges for all $f(x) \in \mathcal{C}^{(0)}[a, b]$.
- (Bernstein) and (Brutman, Passow) interpolant for $|x|$ on $[-1, 1]$ diverges almost everywhere for a variety of well-known node sets.
- For an interpolating strategy to converge uniformly:
  - the class of $f(x)$ is more restrictive than $\mathcal{C}^{(0)}[a, b]$,
  - the nodes in $X_n = \{x_0^{(n)}, \ldots, x_n^{(n)}\}$ are chosen carefully

Uniform Convergence of Interpolating Polynomials

Theorem 9.5. *Let $I = [-1, 1]$ and let the interpolating strategy be defined by the sets $X_n = \{x_0^{(n)}, \dots, x_n^{(n)}\}$ given by the Chebyshev zeros*

$$x_j^{(n)} = \cos\left(\frac{2j+1}{n+1}\frac{\pi}{2}\right) \quad 0 \le j \le n.$$

- *If $f(x) \in \mathcal{C}^{(2)}[I]$ then $\|f(x) - p_n(x)\|_\infty$ converges uniformly on I.*
- *If $f(x) \in \mathcal{C}^{(0)}[I]$ satisfies the Lipschitz condition $|f(x) - f(\hat{x})| < \lambda |x - \hat{x}|$ then $\|f(x) - p_n(x)\|_\infty$ converges uniformly on I.*

Proof. See Isaacson and Keller (1966), Ueberhuber (1995) $\square$

We will discuss this interpolation strategy in more detail later.