Homework 3 Foundations of Computational Math 1 Fall 2017

Problem 3.1

Consider the data points

$$(x, y) = \{(0, 2), \ \ (0.5, 5), \ \ (1, 8)\}$$

Write the interpolating polynomial in both Lagrange and Newton form for the given data.

Problem 3.2

Use this divided difference table for this problem.

i	0		1		2		3		4		5
x_i	-1		0		2		4		5		6
f_i	13		2		-14		18		67		91
$f[-,-]$		-11		-8		16		49		24	
$f[-,-,-]$			1		6		11		$-25/2$		
$f[-,-,-,-]$				1		1		$-47/8$			
$f[-,-,-,-,-]$					0		$-55/48$				
$f[-,-,-,-,-,-]$						$-55/336$					

3.2.a

Use the divided difference information about the unknown function $f(x)$ and consider the unique polynomial, denoted $p_{1,5}(x)$, that interpolates the data given by pairs (x_1, f_1), (x_2, f_2), (x_3, f_3), (x_4, f_4) , and (x_5, f_5). Use two different sets of divided differences to express $p_{1,5}(x)$ in two distinct forms.

3.2.b

What is the significance of the value of 0 for $f[x_0, x_1, x_2, x_3, x_4]$?

3.2.c

Denote by $p_{0,4}(x)$, the unique polynomial, that interpolates the data given by pairs (x_0, f_0), (x_1, f_1), (x_2, f_2), (x_3, f_3), and (x_4, f_4) and recall the definition of $p_{1,5}(x)$ from part (a). Use the divided difference information about the unknown function $f(x)$ to derive error estimates for $f(x) - p_{1,5}(x)$ and $f(x) - p_{0,4}(x)$ for any $x_0 \leq x \leq x_5$.

Problem 3.3

Assume you are given distinct points $x_0, \ldots, x_n$ and, $p_n(x)$, the interpolating polynomial defined by those points for a function f.

3.3.a. If $p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$ is the Lagrange form show that

$$\sum_{i=0}^{n} \ell_i(x) = 1$$

3.3.b. Assume $x \neq x_i$ for $0 \leq i \leq n$ and show that the divided difference $f[x_0, \ldots, x_n, x]$ satisfies

$$f[x_0, \ldots, x_n, x] = \sum_{i=0}^{n} \frac{f[x, x_i]}{\prod_{j=0, j\neq i}^{n}(x_i - x_j)}$$

Problem 3.4

Text exercise 8.10.1 on page 375

Problem 3.5

Text exercise 8.10.3 on page 376

Problem 3.6

Text exercise 8.10.8 on page 376

Problem 3.7

Text exercise 8.10.4 on page 376

Problem 3.8

Let $p_n(x)$ be the unique polynomial that interpolates the data

$$(x_0, y_0), \ldots, (x_n, y_n)$$

Suppose that we assume the form

$$p_n(x) = \alpha_0 + \alpha_1(x - x_0) + \cdots + \alpha_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

and let

$$a = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix} \quad y = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$$

3.8.a. Show that the constraints yield a linear system of equations

$$La = y$$

where L is lower triangular.

3.8.b. Show that the linear system yields a recurrence for the α_i that is equivalent to one of the standard definitions of the divided differences and therefore this is the Newton form of $p_n(x)$.

3.8.c. Show that

$$y[x_0, \dots, x_n] = \sum_{i=0}^{n} \frac{y_i}{\omega'_{n+1}(x_i)}, \quad \text{where } \omega_{k+1} = (x - x_0) \dots (x - x_k)$$

and express the result in terms of the vectors a and y and some matrix. Relate the matrix to L in the expression $La = y$ proved earlier.

Problem 3.9

Consider a polynomial

$$p_n(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

$p_n(\gamma)$ can be evaluated using Horner's rule (written here with the dependence on the formal argument x more explicitly shown)

$c_n(x) = \alpha_n$
for $i = n-1 : -1 : 0$
$\qquad c_i(x) = x c_{i+1}(x) + \alpha_i$
end

$p_n(x) = c_0(x)$

Note that when evaluating $x = \gamma$ the algorithm produces $n+1$ constants $c_0(\gamma), \dots, c_n(\gamma)$ one of which is equal to $p_n(\gamma)$.

3.9.a

Suppose that Horner's rule is applied to evaluate $p_n(\gamma)$ and that the constants $c_0(\gamma), \dots, c_n(\gamma)$ are saved. Show that

$$p_n(x) = (x - \gamma) q(x) + p_n(\gamma)$$
$$q(x) = c_1(\gamma) + c_2(\gamma) x + \cdots + c_n(\gamma) x^{n-1}$$

3.9.b

Suppose that Horner's rule, with labeling modified appropriately, is applied to evaluate $p_n(\gamma)$ and that the constants $c_0^{(1)}(\gamma), \ldots, c_n^{(1)}(\gamma)$ are saved to define $p_n(\gamma) - c_0^{(1)}(\gamma)$ and $q_{(1)}(x) = c_1^{(1)}(\gamma) + c_2^{(1)}(\gamma)x + \cdots + c_n^{(1)}(\gamma)x^{n-1}$. Suppose further that Horner's rule is applied to evaluate $q_{(1)}(\gamma)$ and that the constants $c_1^{(2)}(\gamma), \ldots, c_n^{(2)}(\gamma)$ are saved to define $q_{(1)}(\gamma) = c_1^{(2)}(\gamma)$ and $q_{(2)}(x) = c_2^{(2)}(\gamma) + c_3^{(2)}(\gamma)x + \cdots + c_n^{(2)}(\gamma)x^{n-2}$. This can continue until Horner's rule is applied to evaluate $q_{(n)}(\gamma) = c_n^{(n)}(\gamma)$ and $q_{(n+1)}(x) = 0$, i.e., there are no constants other than $c_n^{(n)}(\gamma)$ produced.

Show that

$$
\begin{aligned}
q_{(1)}(\gamma) &= p_n'(\gamma) \\
q_{(2)}(\gamma) &= p_n''(\gamma)/2 \\
q_{(3)}(\gamma) &= p_n'''(\gamma)/3! \\
&\vdots \\
q_{(n-1)}(\gamma) &= p_n^{(n-1)}(\gamma)/(n-1)! \\
q_{(n)}(\gamma) &= p_n^{(n)}(\gamma)/n!
\end{aligned}
$$

and therefore form the coefficients of the Taylor form of $p_n(x)$

$$p_n(x) = p_n(\gamma) + (x-\gamma)p_n'(\gamma) + \frac{(x-\gamma)^2}{2}p_n''(\gamma) + \frac{(x-\gamma)^3}{3!}p_n'''(\gamma) \cdots + \frac{(x-\gamma)^{n-1}}{(n-1)!}p_n^{(n-1)}(\gamma) + \frac{(x-\gamma)^n}{n!}p_n^{(n)}(\gamma)$$