Homework 1 Foundations of Computational Math 1 Fall 2017

Problem 1.1

This problem considers three basic vector norms: $\|.\|_1$,$\|.\|_2$,$\|.\|_\infty$.

1.1.a. Prove that $\|.\|_1$ is a vector norm.

1.1.b. Prove that $\|.\|_\infty$ is a vector norm.

1.1.c. Consider $\|.\|_2$.

(i) Show that $\|.\|_2$ is definite.

(ii) Show that $\|.\|_2$ is homogeneous.

(iii) Show that for $\|.\|_2$ the triangle inequality follows from the Cauchy inequality $|x^H y| \leq \|x\|_2\|y\|_2$.

(iv) Assume you have two vectors x and y such that $\|x\|_2 = \|y\|_2 = 1$ and $x^H y = |x^H y|$, prove the Cauchy inequality holds for x and y.

(v) Assume you have two arbitrary vectors $\tilde{x}$ and $\tilde{y}$. Show that there exists x and y that satisfy the conditions of part (iv) and $\tilde{x} = \alpha x$ and $\tilde{y} = \beta y$ where α and β are scalars.

(vi) Show the Cauchy inequality holds for two arbitrary vectors $\tilde{x}$ and $\tilde{y}$.

Problem 1.2

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear function, i.e.,

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$$

.

1.2.a. Suppose you are given a routine that returns $F(x)$ given any $x \in \mathbb{R}^n$. How would you use this routine to determine a matrix $A \in \mathbb{R}^{m\times n}$ such that $F(x) = Ax$ for all $x \in \mathbb{R}^n$?

1.2.b. Show A is unique.

Problem 1.3

Let $y \in \mathbb{R}^m$ and $\|y\|$ be any vector norm defined on $\mathbb{R}^m$. Let $x \in \mathbb{R}^n$ and A be an $m \times n$ matrix with $m > n$.

1.3.a. Show that the function $f(x) = \|Ax\|$ is a vector norm on $\mathbb{R}^n$ if and only if A has full column rank, i.e., $rank(A) = n$.

1.3.b. Suppose we choose $f(x)$ from part (1.3.a) to be $f(x) = \|Ax\|_2$. What condition on A guarantees that $f(x) = \|x\|_2$ for any vector $x \in \mathbb{R}^n$?

Problem 1.4

Theorem 1. *If $\mathcal{V}$ is a real vector space with a norm $\|v\|$ that satisfies the parallelogram law*

$$\forall x, \quad y \in \mathcal{V}, \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{1}$$

then the function

$$f(x,y) = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$$

is an inner product on $\mathcal{V}$ and $f(x,x) = \|x\|^2$.

This problem proves this theorem by a series of lemmas. Prove each of the following lemmas and then prove the theorem.

Lemma 2. $\forall x \in \mathcal{V}$

$$f(x,x) = \|x\|^2$$

Lemma 3. $\forall x, \quad y \in \mathcal{V}$ *$f(x,x)$ is definite and $f(x,y) = f(y,x)$, i.e., (f is symmetric)*

Lemma 4. *The following two "cosine laws" hold $\forall x, \quad y \in \mathcal{V}$:*

$$2f(x,y) = \|x+y\|^2 - \|x\|^2 - \|y\|^2 \tag{2}$$

$$2f(x,y) = -\|x-y\|^2 + \|x\|^2 + \|y\|^2 \tag{3}$$

Lemma 5. $\forall x, \quad y \in \mathcal{V}$:

$$|f(x,y)| \leq \|x\|\|y\| \tag{4}$$

$$f(x,y) = \gamma\|x\|\|y\|, \quad sign(\gamma) = sign(f(x,y)), \quad 0 \leq |\gamma| \leq 1 \tag{5}$$

Lemma 6. $\forall x, \quad y, \quad z \in \mathcal{V}$:

$$f(x+z,y) = f(x,y) + f(z,y)$$

Lemma 7. $\forall x, \quad y \in \mathcal{V}, \quad \alpha \in \mathbb{R}$

$$f(\alpha x, y) = \alpha f(x,y)$$

Problem 1.5

1.5.a. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ be nonsingular matrices. Show $(AB)^{-1} = B^{-1}A^{-1}$.

1.5.b. Suppose $A \in \mathbb{R}^{m\times n}$ with $m > n$ and let $M \in \mathbb{R}^{n\times n}$ be a nonsingular square matrix. Show that $\mathcal{R}(A) = \mathcal{R}(AM)$ where $\mathcal{R}(\dot{})$ denotes the range of a matrix.

Problem 1.6

Consider the matrix

$$L = \begin{pmatrix} \lambda_{11} & 0 & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{pmatrix}$$

Suppose that $\lambda_{11} \neq 0$, $\lambda_{33} \neq 0$, $\lambda_{44} \neq 0$ but $\lambda_{22} = 0$.

1.6.a. Show that L is singular.

1.6.b. Determine a basis for the nullspace $\mathcal{N}(L)$.

Problem 1.7

Suppose $A \in \mathbb{C}^{m\times n}$ and let the matrix B be **any submatrix of** A. Show that $\|B\|_p \leq \|A\|_p$.

Problem 1.8

Suppose that $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ and let $E = uv^T$.

1.8.a. Show that $\|E\|_F = \|E\|_2 = \|u\|_2\|v\|_2$.

1.8.b. Show that $\|E\|_\infty = \|u\|_\infty\|v\|_1$.

Problem 1.9

Let $\mathcal{S}_1 \subset \mathbb{R}^n$ and $\mathcal{S}_2 \subset \mathbb{R}^n$ be two subspaces of $\mathbb{R}^n$.

1.9.a. Suppose $x_1 \in \mathcal{S}_1$, $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. $x_2 \in \mathcal{S}_2$, and $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. Show that x_1 and x_2 are linearly independent.

1.9.b. Suppose $x_1 \in \mathcal{S}_1$, $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. $x_2 \in \mathcal{S}_2$, and $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. Also, suppose that $x_3 \in \mathcal{S}_1 \cap \mathcal{S}_2$ and $x_3 \neq 0$, i.e., the intersection is not empty. Show that x_1, x_2 and x_3 are linearly independent.

Problem 1.10

Suppose $A \in \mathbb{C}^{m \times n}$. Consider the matrix norm $\|A\|$ induced by the two vector 1-norms $\|x\|_1$ and $\|y\|_1$ for $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ respectively,

$$\|A\| = \max_{\|x\|_1=1} \|Ax\|_1.$$

Is this induced norm the same as the matrix 1-norm defined by

$$\|A\|_1 = \max_{1 \le i \le n} \|Ae_i\|_1?$$

If so prove it. If not give counterexample to disprove it.

Problem 1.11

Consider the definition of the matrix norm $\|A\| = \max_{i,j} |\alpha_{i,j}|$ where $e_i^T A e_j = \alpha_{i,j}$.

1.11.a. Show that this defines a matrix norm.

1.11.b. Show that the matrix norm is not consistent.