Set 6: Repeated Linear Interpolation

Kyle A. Gallivan

Department of Mathematics

Florida State University

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Repeated Linear Interpolation

The unique interpolating polynomial, here in Lagrange form,

$$p_n(x) = \sum_{i=0}^{n} y_i \, \ell_i^{(n)}(x)$$

can be evaluated without the initial $O(n^2)$ evaluation of parameters by exploiting simple formulas for combining interpolating polynomials.

We assume we have stored only (x_i, y_i) and we want to evaluate $p_n(x)$ for some value of x.

Basic Merging of Interpolation

Lemma 6.1. *Suppose a mesh of points is given by a set of points* $\mathcal{S}$ *containing* n *points and two additional distinct points* x_p *and* x_q*. Let* $p(x)$ *and* $q(x)$ *be interpolating polynomials of degree* $n-1$ *so that* $\forall x_i \in \mathcal{S},\ \ p(x_i) = q(x_i)$ *and let the values of* $p(x_p)$ *and* $q(x_q)$ *be given. The interpolating polynomial of degree* n *that interpolates the values at* x_p*,* x_q *and any mesh point in* $\mathcal{S}$ *is*

$$g(t) = \frac{(x_q - x)}{(x_q - x_p)} p(x) + \frac{(x_p - x)}{(x_p - x_q)} q(x).$$

Basic Merging of Interpolation

- The coefficient of x^n is the same definition of the divided difference that yielded the recursion that defines the divided difference table.
- This is essentially the same form of update used to produce that recursion.

$$g(t) = \frac{(x_q - x)}{(x_q - x_p)} p(x) + \frac{(x_p - x)}{(x_p - x_q)} q(x)$$

$$= \frac{(x_q - x + x_p - x_p)}{(x_q - x_p)} p(x) + \frac{(x_p - x)}{(x_p - x_q)} q(x)$$

$$= p(x) + \frac{(x - x_p)}{(x_p - x_q)} \left[p(x) - q(x) \right]$$

Aitken Form Example

Given $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$ and a value of x, evaluate $p_3(x)$.

Use notation

$$
\begin{gathered}
Q_{0123}(x) = p_3(x) \\
Q_i(x) = y_i \quad \text{interpolates} \quad (x_i, y_i) \\
Q_{01}(x) \quad \text{interpolates} \quad (x_0, y_0), (x_1, y_1) \\
Q_{13}(x) \quad \text{interpolates} \quad (x_1, y_1), (x_3, y_3) \\
\vdots
\end{gathered}
$$

Aitken Form Example

Step 0: $Q_0 = y_0, Q_1 = y_1, Q_2 = y_2, Q_3 = y_3$

Step 1: evaluate each expression for the value of x

$$Q_{01} = \frac{(x_1 - x)}{(x_1 - x_0)} Q_0 - \frac{(x_0 - x)}{(x_1 - x_0)} Q_1$$

$$Q_{02} = \frac{(x_2 - x)}{(x_2 - x_0)} Q_0 - \frac{(x_0 - x)}{(x_2 - x_0)} Q_2$$

$$Q_{03} = \frac{(x_3 - x)}{(x_3 - x_0)} Q_0 - \frac{(x_0 - x)}{(x_3 - x_0)} Q_3$$

Aitken Form Example

Step 2: evaluate each expression for the value of x

$$Q_{012} = \frac{(x_2 - x)}{(x_2 - x_1)} Q_{01} - \frac{(x_1 - x)}{(x_2 - x_1)} Q_{02}$$

$$Q_{013} = \frac{(x_3 - x)}{(x_3 - x_1)} Q_{01} - \frac{(x_1 - x)}{(x_3 - x_1)} Q_{03}$$

Aitken Form Example

Step 3: evaluate each expression for the value of x

$$Q_{0123} = \frac{(x_3 - x)}{(x_3 - x_2)} Q_{012} - \frac{(x_2 - x)}{(x_3 - x_2)} Q_{013}$$

- $Q_{0123} = p_3(x)$
- repeated linear function evaluation, two unshared points define weights
- recurrence easily generalizes
- $O(n)$ storage required
- $O(n^2)$ computations required for each evaluation.
- other recurrences possible, e.g., Neville's form

Aitken Form Example

Aitken pattern for $n = 5$

$$
\begin{array}{llllll}
f_0 = Q_0 & & & & & \\
f_1 = Q_1 & Q_{0,1} & & & & \\
f_2 = Q_2 & Q_{0,2} & Q_{0,1,2} & & & \\
f_3 = Q_3 & Q_{0,3} & Q_{0,1,3} & Q_{0,1,2,3} & & \\
f_4 = Q_4 & Q_{0,4} & Q_{0,1,4} & Q_{0,1,2,4} & Q_{0,1,2,3,4} & \\
f_5 = Q_5 & Q_{0,5} & Q_{0,1,5} & Q_{0,1,2,5} & Q_{0,1,2,3,5} & Q_{0,1,2,3,4,5}
\end{array}
$$

where linear combination is used to produce $Q_{0,\ldots,i,k}(x)$ from $Q_{0,\ldots,i}(x)$ and $Q_{0,\ldots,i-1,k}(x)$.

Neville Form Example

Step 0: $N_0 = y_0, N_1 = y_1, N_2 = y_2, N_3 = y_3$

Step 1: evaluate each expression for the value of x

$$
\begin{aligned}
N_{01} &= \frac{(x_1 - x)}{(x_1 - x_0)} N_0 - \frac{(x_0 - x)}{(x_1 - x_0)} N_1 \\
N_{12} &= \frac{(x_2 - x)}{(x_2 - x_1)} N_1 - \frac{(x_1 - x)}{(x_2 - x_1)} N_2 \\
N_{13} &= \frac{(x_3 - x)}{(x_3 - x_1)} N_1 - \frac{(x_1 - x)}{(x_3 - x_1)} N_3
\end{aligned}
$$

Neville Form Example

Step 2: evaluate each expression for the value of x

$$N_{012} = \frac{(x_2 - x)}{(x_2 - x_0)} N_{01} - \frac{(x_0 - x)}{(x_0 - x_2)} N_{12}$$

$$N_{123} = \frac{(x_3 - x)}{(x_3 - x_1)} N_{12} - \frac{(x_1 - x)}{(x_3 - x_1)} N_{23}$$

Neville Form Example

Step 3: evaluate each expression for the value of x

$$N_{0123} = \frac{(x_3 - x)}{(x_3 - x_0)} N_{012} - \frac{(x_0 - x)}{(x_3 - x_0)} N_{123}$$

$$N_{0123} = p_3(x)$$

Neville Form Example

Neville pattern for $n = 5$

$$
\begin{array}{llllll}
f_0 = N_0 & & & & & \\
f_1 = N_1 & N_{0,1} & & & & \\
f_2 = N_2 & N_{1,2} & N_{0,1,2} & & & \\
f_3 = N_3 & N_{2,3} & N_{1,2,3} & N_{0,1,2,3} & & \\
f_4 = N_4 & N_{3,4} & N_{2,3,4} & N_{1,2,3,4} & N_{0,1,2,3,4} & \\
f_5 = N_5 & N_{4,5} & N_{3,4,5} & N_{2,3,4,5} & N_{1,2,3,4,5} & N_{0,1,2,3,4,5}
\end{array}
$$

where linear combination is used to produce $N_{i,\ldots,i+k}(x)$ from $N_{i,\ldots,i+k-1}(x)$ and $N_{i+1,\ldots,i+k}(x)$.