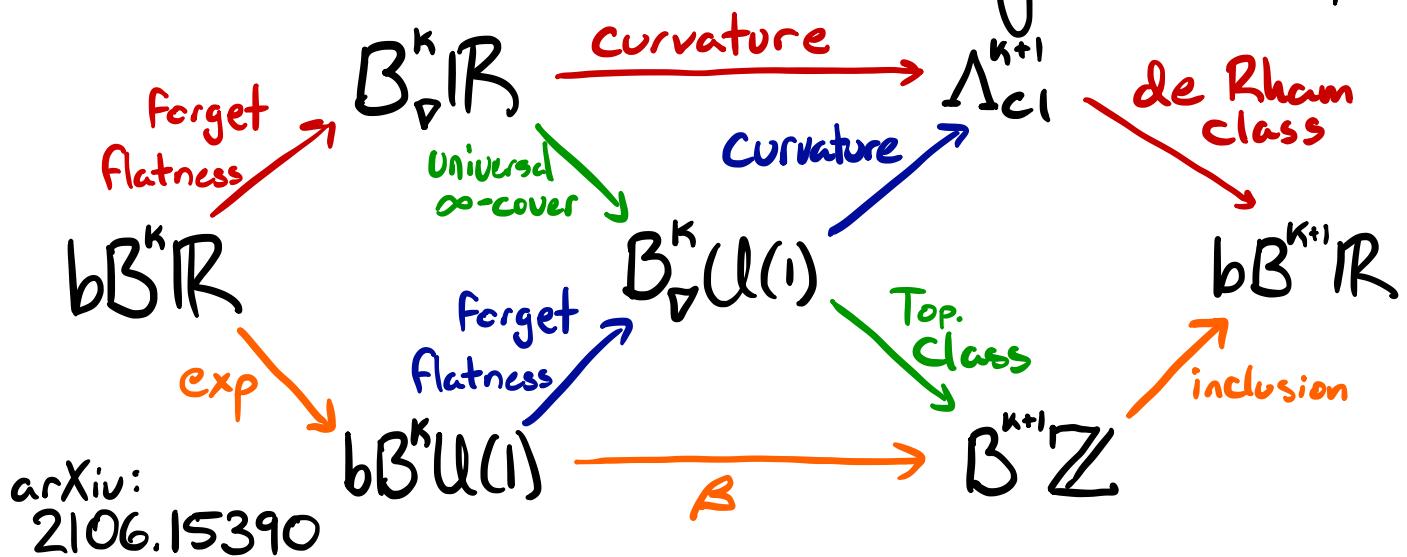


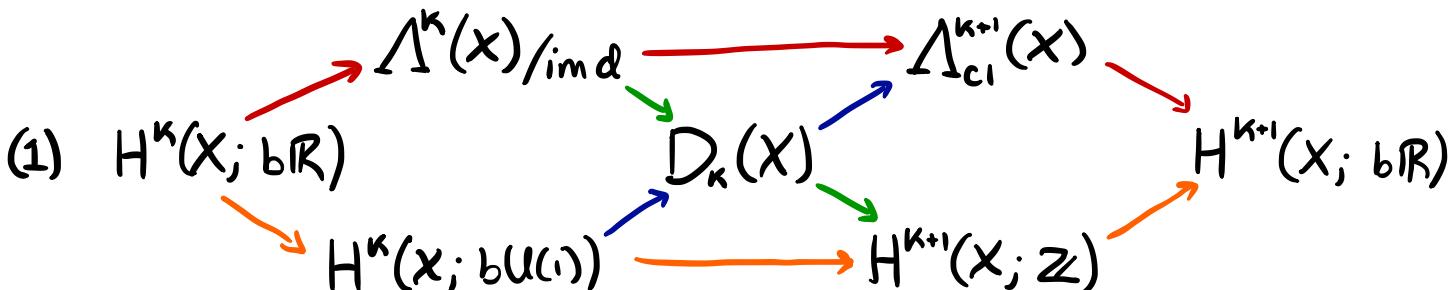
# Modal Fracture of Higher Groups



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at HoTT/UF 2021

A bit of history:



- In 1973, Cheeger and Simons introduced differential characters
- This is Deligne cohomology in the differential geometric setting  
“Ordinary differential cohomology”
- In 2008, Simons and Sullivan showed that the character diagram (1) characterizes ordinary diff. coh.

## A bit more history:

- Bunke, Nikolaus, & Völkl (2012) construct diff. coh. theories as sheaves of spectra on smooth manifolds

$$\begin{array}{ccccc}
 & \mathcal{A}(\hat{E})^k(X) & \longrightarrow & \mathcal{Z}(\hat{E})^k(X) & \\
 & \searrow & & \nearrow & \\
 H^{k-1}(X; Z(\hat{E})) & & \hat{E}^k(X) & & H^k(X; Z(\hat{E})) \\
 & \swarrow & & \nearrow & \\
 & H^k(X; S(\hat{E})) & \longrightarrow & H^k(X; U(E)) &
 \end{array}$$

- Schreiber (2013) shows how these hexagons arise from any adjoint modality / comodality

$$\begin{array}{ccccc}
 & \Pi_{dR}\hat{E} & \xrightarrow{d} & \flat_{dR}\hat{E} & \\
 \nearrow & \downarrow & & \nearrow & \downarrow \\
 \Pi_{dR}\flat\hat{E} & & \hat{E} & & \Pi\flat_{dR}\hat{E} \\
 \downarrow & \nearrow & \downarrow & \nearrow & \\
 \flat\hat{E} & \longrightarrow & \Pi\hat{E} & &
 \end{array}$$

## Modal Fracture of Higher Groups:

Thm: For any crisp higher group  $G$ , (Unstable version of Schreiber §4.1.2)

$$\begin{array}{ccccc}
 & \overset{\infty}{\widehat{G}} & \xrightarrow{\Theta} & g & \\
 b\overset{\infty}{G} \xrightarrow{(-)\flat} & \downarrow \pi & \xrightarrow{\Theta} & \xrightarrow{(-)\sharp} & sg = b\beta\overset{\infty}{G} \\
 & bG & \xrightarrow{(-)\flat} & \xrightarrow{\Theta} & \xrightarrow{(-)\sharp} \\
 & b\pi & & & s\theta
 \end{array}$$

Where

- $\overset{\infty}{\widehat{G}} \xrightarrow{\pi} G$  is the universal  $\infty$ -cover
- $G \xrightarrow{\Theta} g$  is the infinitesimal remainder

And

- Both squares are pullbacks.

- The top and bottom sequences are fiber sequences.
- $sg = b\beta\overset{\infty}{G}$

# Cohesive HoTT - Crispness and $b$ -comodality

(Shulman)

$$\Delta \vdash \Gamma \vdash a : A$$

Add **crisp variables** to express discontinuous dependence

$$x :: A$$

Crisp terms:  $\Delta \vdash \cdot \vdash a : A$  have only crisp variables.

Comodality  $b$ :  $bA$  is inductively generated by crisp  $a :: A$ .

$$\frac{\Delta \vdash \cdot \vdash A : \text{Type}}{\Delta \vdash \Gamma \vdash bA : \text{Type}}$$

$$\frac{\Delta \vdash \cdot \vdash a : A}{\Delta \vdash \Gamma \vdash a^b : bA}$$

$$\Delta \vdash \Gamma, x : bA \vdash C : \text{Type}$$

$$\Delta \vdash \Gamma \vdash a : bA$$

$$\Delta, x :: A \vdash \Gamma \vdash c : C(x^b)$$

$$\frac{}{\Delta \vdash \Gamma \vdash \text{let } x^b \equiv a \text{ in } c : C(a)}$$

$$(\text{let } x^b \equiv a^b \text{ in } c \equiv c(a))$$

Counit:  $(-)_b : bA \rightarrow A$

$$a^b \mapsto a$$

$$u \mapsto \text{let } a^b \equiv u \text{ in } a.$$

The shape modality  $\int$

The "shape" modality  $\int$  reflects into discrete types.

- In differential geometry,  $\int \equiv \text{Loc}_R$  takes the homotopy type.
- In simplicial cohesion,  $\int \equiv \text{Loc}_\Delta$  is the geometric realization.  
e.g.  $S^n = \{x : R^{n+1} \mid |x| = 1\}$ , then  $\int S^n = S^n$  ... higher inductive.

Axiom: For a crisp type  $X$

$$bX \xrightarrow[-]{(-)_b} X \quad \text{iff} \quad X \xrightarrow[\sim]{(-)^s} \int X$$

In either case, we say  $X$  is discrete

Thm (Shulman): For crisp  $X$  and  $Y$

$$b(\int X \rightarrow Y) \simeq b(X \rightarrow bX)$$

The universal  $\infty$ -cover and infinitesimal remainder:

Def: A higher group is a type  $G$  equipped with a pointed, 0-connected type  $BG$  (called the "delooping") with

$$G = \Omega BG$$

Def: Let  $G$  be a crisp higher group

- The universal  $\infty$ -cover is the fiber

$$\begin{array}{ccccc} \tilde{G} & \xrightarrow{\pi} & G & \xrightarrow{(-)^d} & SG \\ & \curvearrowright & & & \\ B\tilde{G} & \longrightarrow & BG & \longrightarrow & SBG \end{array}$$

- The infinitesimal remainder is the fiber

$$\begin{array}{ccccc} bG & \xrightarrow{(-)_b} & G & \xrightarrow{\theta} & g \\ & \curvearrowright & & & \\ bBG & \longrightarrow & BG & & \end{array}$$

### The Pullback Squares

Prop: For  $G$  a crisp higher group, is a pullback.

$$\begin{array}{ccccc} & & \tilde{G} & & \\ & \nearrow (-)_b & & \searrow \pi & \\ b\tilde{G} & \xrightarrow{b\pi} & bG & \xrightarrow{(-)_b} & G \end{array}$$

Proof: The fiber  $\text{fib}_{\pi}(g_b)$  is identifiable with  $\Omega SG$ , which is crisply discrete.  $\square$

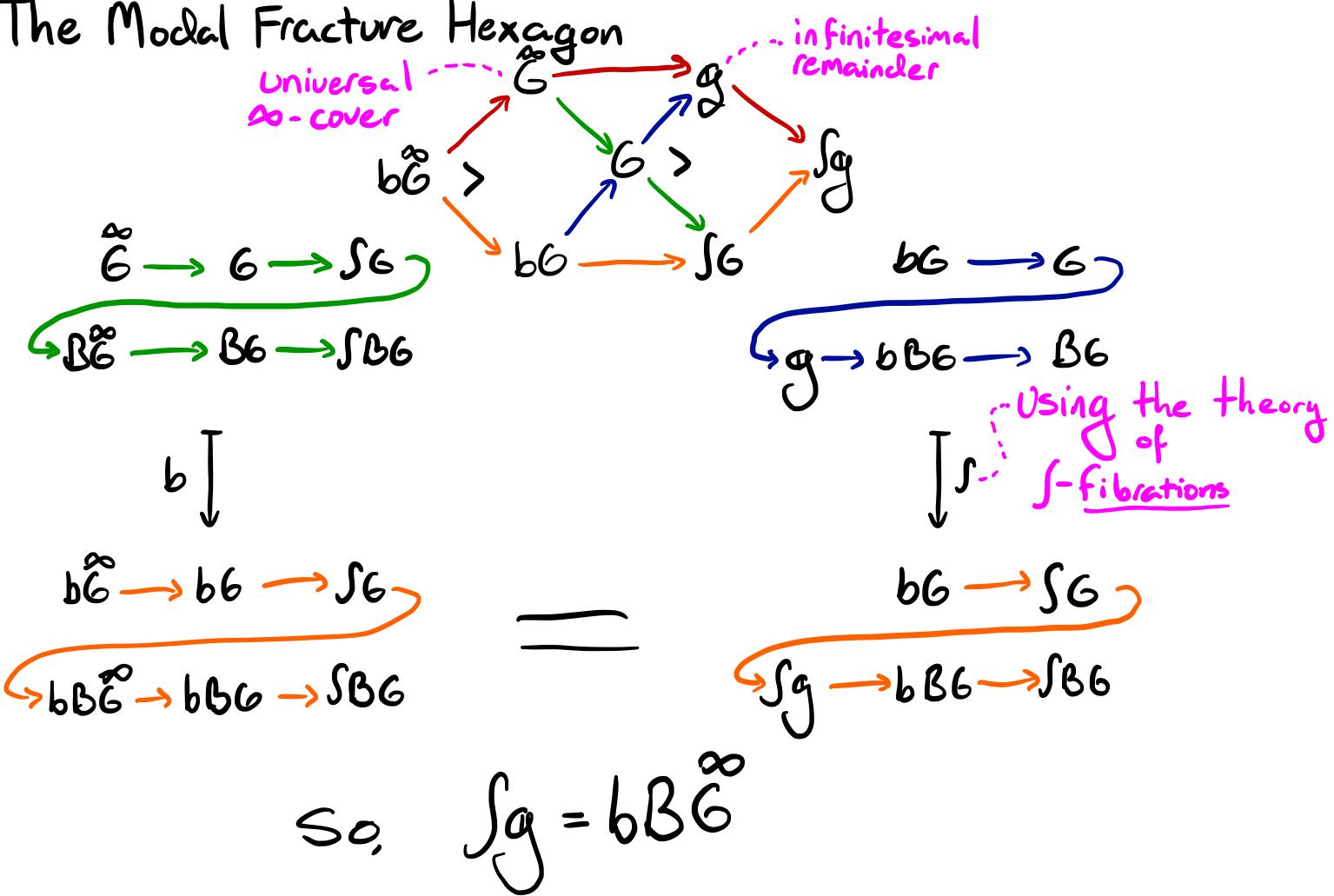
Prop: For  $G$  a crisp higher group, is a pullback.

$$\begin{array}{ccccc} & & g & & \\ & \nearrow \theta & & \searrow (-)^s & \\ G & \xrightarrow{(-)^s} & SG & \xrightarrow{s\theta} & SG \\ & \searrow (-)^s & & \nearrow s\theta & \\ & & SG & & \end{array}$$

Proof: The fibers of  $\theta$  are identifiable with  $bG$ , which is crisply discrete.

By the "good fibrations" trick, (Thm 6.1 of "Good fibrations...") this implies that  $\theta$  is an  $\infty$ -cover.  $\square$

# The Modal Fracture Hexagon



## Ordinary Differential Cohomology

Assumption: We have "form classifiers"  $\Lambda^k$ , abelian groups and an exact sequence  $0 \rightarrow bR \rightarrow R \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \rightarrow \dots$

Where  $S\Lambda^k = *$

Define  $\Lambda_{cl}^k := \text{Ker}(\Lambda^k \xrightarrow{d} \Lambda^{k+1})$ , and assume  $b\Lambda_{cl}^k = *$

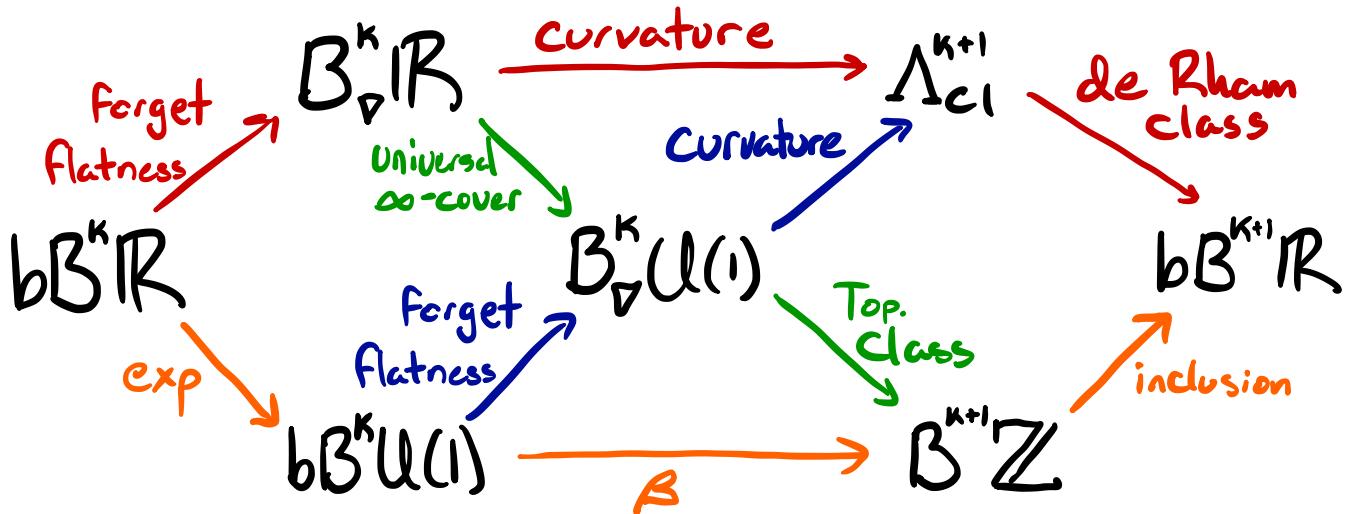
Thm:  $\int \Lambda_{cl}^k = bB^k R$  (the de Rham class)

Def: The classifiers for circle  $k$ -gerbes with connection is

the pullback:

$$\begin{array}{ccc} B_{\nabla}^k U(1) & \longrightarrow & \Lambda_{cl}^{k+1} \\ \downarrow & & \downarrow (-)^* \\ B^{k+1} \mathbb{Z} & \longrightarrow & bB^{k+1} R \end{array}$$

Thm: The modal fracture hexagon of  $B_{\triangleright}^k U(1)$  is



References:

Thank You!

David Jaz Myers:

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- Good fibrations through the modal prism (arXiv: 1908.08034)

Urs Schreiber:

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Mike Shulman:

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- Classifying Types (arXiv: 1906.09435)
- Modal Descent (arXiv: 2003.09713)

Felix Cherubini

- Cohesive Covering Theory

James Simons and Dennis Sullivan:

- Axiomatic Characterization of Ordinary Differential Cohomology (arXiv: math/0701077)