Math 216 - How to tell if a Symmetric Matrix is Positive Definite

Let M be a symmetric $n \times n$ matrix, and $Q(\mathbf{x}) = \mathbf{x}^* M \mathbf{x}$ its associated quadratic form. We want methods to tell whether M is positive definite, that is, whether $Q(\mathbf{x}) > 0$ for every $\mathbf{x} \neq 0$.

For simplicity, we will assume that M is invertible, so that zero is not an eigenvalue.

1. Necessary conditions

1. In order for M to be positive definite, the diagonal elements must all be positive. Likewise, for M to be negative definite, the diagonal elements must all be negative.

2. The determinant of M is the product of its eigenvalues. If the size n is even and $\det(M) < 0$, then the eigenvalues must have different signs, so M is neither positive nor negative definite. If n is odd and the determinant is negative, the eigenvalues are not all positive. If n is odd and the determinant is positive, the eigenvalues are not all negative.

2. Necessary and sufficient conditions

3. It is always possible, and sometimes easy, to write $Q(\mathbf{x})$ as a sum or difference of squares, which makes positive or negative definiteness transparent.

Let $p(\lambda) = \det(M - \lambda I)$ denote the characteristic polynomial of M. If there is a computer or graphing calculator handy we can approximate the eigenvalues - the roots of $p(\lambda)$ - numerically easily enough. But there is an easier way to determine positive definiteness.

4. Because there are no non-real (complex) roots of $p(\lambda)$, Descartes' Rule of Signs says that the number of positive roots of $p(\lambda)$ is the number of changes of sign in its coefficients. Thus M is positive definite if and only if $p(\lambda)$ has n changes of sign in its coefficients. Likewise, M is negative definite if and only if $p(\lambda)$ has no changes of sign in its coefficients.

5. Here is a method that avoids computing $p(\lambda)$.

If the diagonal of M is all positive, then M is positive definite if and only if the determinants of all the upper left-hand corners are positive.

If the diagonal of M is all negative, then M is negative definite if and only if the determinants of all the upper left-hand corners are alternate in sign.