

# Lecture 2:

## 1.1. Theory: Robert Solow's Economic Growth Model

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last revised: 2019-10-15

Original course by Melissa Dell (Harvard Econ 1342), revised by Brad DeLong, research assistance by Anish Biligiri

<<https://github.com/braddelong/public-files/blob/master/econ-135-lecture-2.pptx>>

# Lecture 2: Solow Theory: Outline

**Slides:** <<https://github.com/braddelong/public-files/blob/master/econ-135-lecture-2.pptx>>

**Read After:** J. Bradford DeLong: Lecture Notes: The Solow Growth Model <<https://tinyurl.com/dl-2020-01-18f>>

**Read After:** Partha Dasgupta (2007): Economics: A Very Short Introduction, chapters 5-8 & Epilogue <<https://delong.typepad.com/files/dasgupta-economics.pdf>>

**Do:** Assignment 2 (3 pts): Letter to GSI, due Sa Jan 25 9:00 am <<https://tinyurl.com/dl-2020-01-12g>>

1. **Administration:** Office hours poll

2. **Lecture:** Solow basics

- <<https://www.bradford-delong.com/2020/01/lecture-notes-the-solow-growth-model-the-history-of-economic-growth-econ-135.html>> <<http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-3-growing.ipynb>> <<http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-4-using.ipynb>>

3. **Review:** Growth patterns

4. **Lecture:** Solving the Solow model

- <<https://nbviewer.jupyter.org/github/braddelong/lecture-support-2020/blob/master/lecture-support-solow-2020-01-23.ipynb>>

5. **Big Ideas:** Principal takeaways from this class

6. **MOAR** references:

- Robert Solow (1956): A Contribution to the Theory of Economic Growth <<http://piketty.pse.ens.fr/files/Solow1956.pdf>>
- Moses Abramovitz (1956): Resource and Output Trends in the United States Since 1870 <<https://www.nber.org/chapters/c5650.pdf>>
- Robert Solow (1957): Technical Change and the Aggregate Production Function <<http://www.piketty.pse.ens.fr/files/Solow1957.pdf>>
- Moses Abramovitz (1986): Catching Up, Forging Ahead, and Falling Behind <[http://www.j-bradford-delong.net/teaching\\_Folder/Econ\\_210c\\_spring\\_2002/Readings/Abramovitz.pdf](http://www.j-bradford-delong.net/teaching_Folder/Econ_210c_spring_2002/Readings/Abramovitz.pdf)>
- Robert Solow (1987): Growth Theory and After <<https://www.nobelprize.org/prizes/economic-sciences/1987/solow/lecture/>>

# Office Hours

## Office Hours:

- M 11:15-12:30,
- T 11:15-12:00
- By appointment: email <[delong@econ.berkeley.edu](mailto:delong@econ.berkeley.edu)>

# Solow Model Basics

**Lecture Notes:** <<https://www.bradford-delong.com/2020/01/lecture-notes-the-solow-growth-model-the-history-of-economic-growth-econ-135.html>>

**Let's assume three things about the relationship between an economy's resources and the total output it produces and income it generates:**

- Production
- Other parts of the model
- Balanced-growth equilibrium
- Convergence to equilibrium

# Solow Model Basics: Production

**Let's assume three things about the relationship between an economy's resources and the total output it produces and income it generates:**

1. A proportional increase in the economy's capital intensity  $\kappa$ , measured by the capital stock divided by total production  $\kappa = \mathbf{K}/\mathbf{Y}$ , will carry with it the same (smaller) proportional increase in income and production  $\mathbf{Y}$  no matter how rich and productive the economy is. A 1% increase in capital intensity will always increase income and production by the same proportional amount  $\theta$ .
2. If two economies have the same capital intensity, defined as the same capital-output ratio  $\kappa$ , and have the same level of technology- and organization-driven efficiency-of-labor  $\mathbf{E}$ , then the ratio of their levels of income and output will be equal to the ratio of their labor forces  $\mathbf{L}$ .
3. If two economies have the same capital intensity, defined as the same capital-output ratio  $\kappa$ , and have the same labor forces  $\mathbf{L}$ , then the ratio of their levels of income and output will be equal to the ratio of their technology- and organization-driven efficiencies-of-labor  $\mathbf{E}$ .

# Basics: Production Notation

- Y: total income and production
- E: technological and organizational efficiency-of-labor
- L: labor force
- y: per-worker income and production
- $\kappa$ : the capital-intensity of the economy, as measured by the capital-output ratio  $K/Y$  (Greek lower kappa)
- $\theta$ : the relative salience in economic growth of capital-intensity vis-a-vis technological and organizational progress (Greek lower theta)
  - if  $\alpha$  is the share of income received by capital under the marginal productivity theory of distribution, then  $\alpha=\theta/(1+\theta)$ ,  $1-\alpha=1/(1+\theta)$ ,  $\theta = \alpha/(1-\alpha)$

$$(2.1.2) \ Y = \kappa^\theta EL ; (2.1.3) \ y = \kappa^\theta E ; (2.1.1) \ \kappa = \frac{K}{Y}$$

# Basics: Production Algebra

Then there is one and only one equation that satisfies those three rules of thumb:

$$Y = \kappa^{\theta} EL$$

And it is also worth writing down

- a version of this equation in per-worker form, where  $y = Y/L$
- the definition of capital intensity  $\kappa$ :  $\kappa = K/Y$

$$y = \kappa^{\theta} E$$

$$\kappa = \frac{K}{Y}$$

We have just done what economists typically do: take a complex situation, strip things down to some salient piece of it, and then formalize and algebraize that piece in the hope of gaining insight...

# Solow Model Basics: Notes

$$(2.1.2) Y = \kappa^\theta EL ; (2.1.3) y = \kappa^\theta E ; (2.1.1) \kappa = \frac{K}{Y}$$

The code in the nbViewer documents is static. But you should also look at:

- <<http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-2-basics.ipynb>>
- <<http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-3-growing.ipynb>>
- <<http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-4-using.ipynb>>



# The Rest of the Model: Growth Rates

$$\frac{dE}{dt} = gE$$

$$\frac{dL}{dt} = g_L L = nL$$

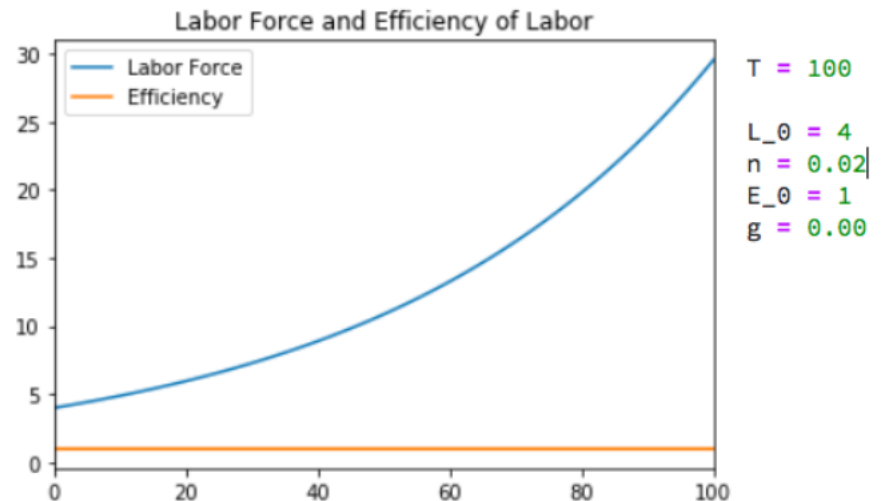
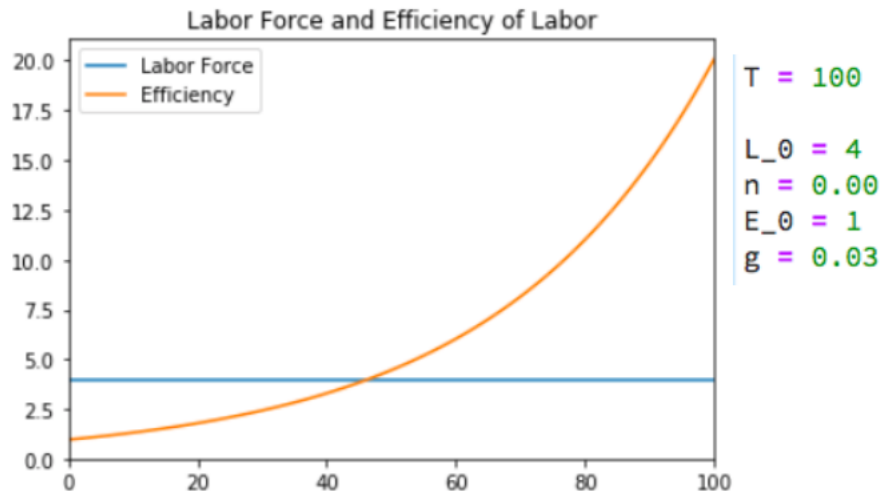
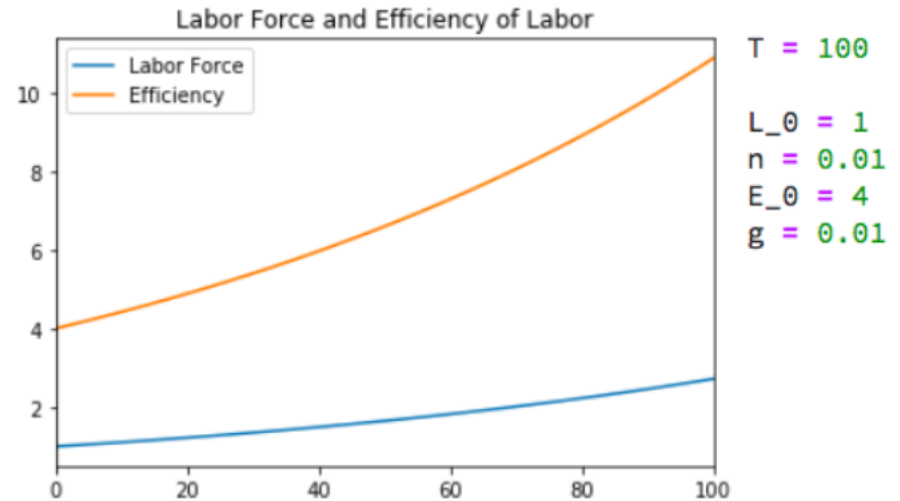
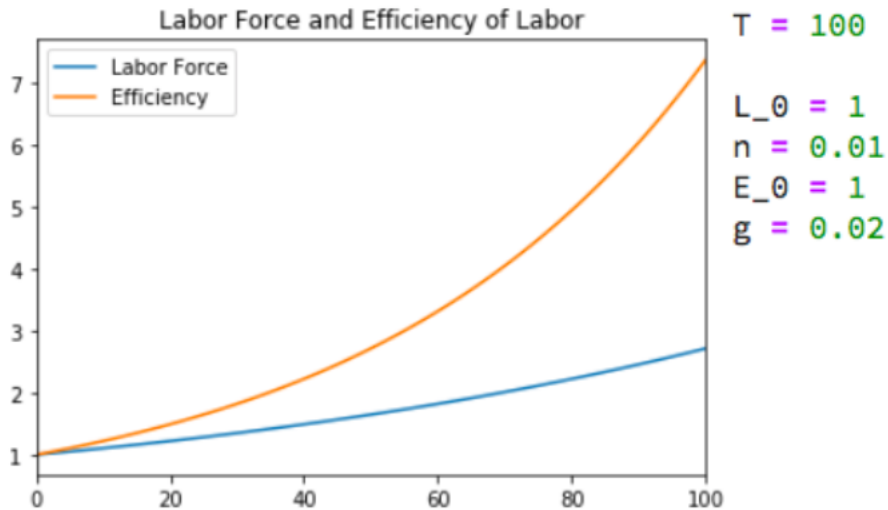
$$\frac{dK}{dt} = sY - \delta K = \left( \frac{s}{\kappa} - \delta \right) K$$

**Variables change over time:**

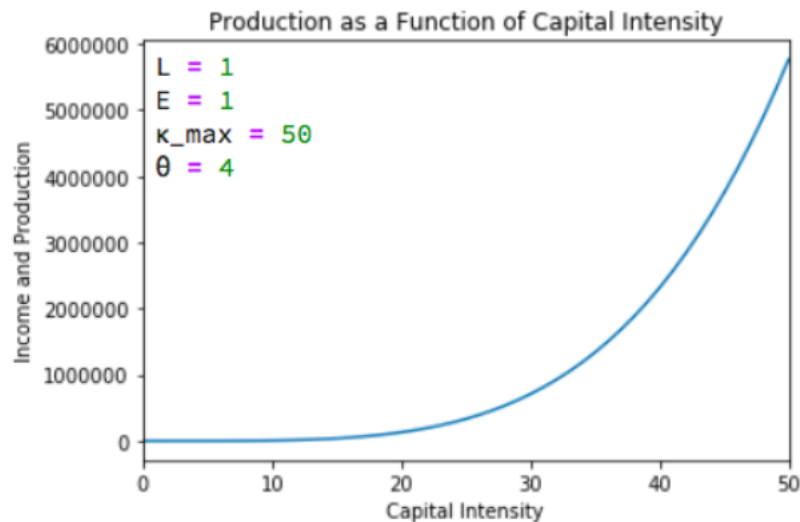
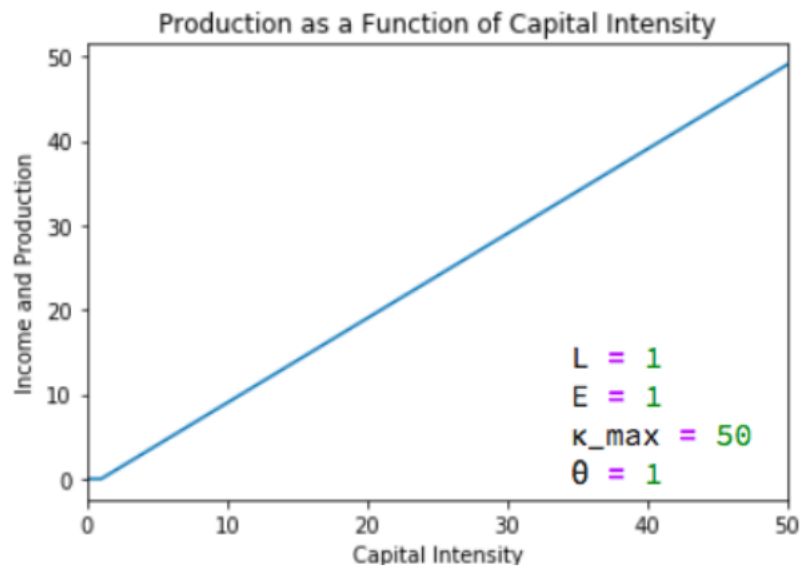
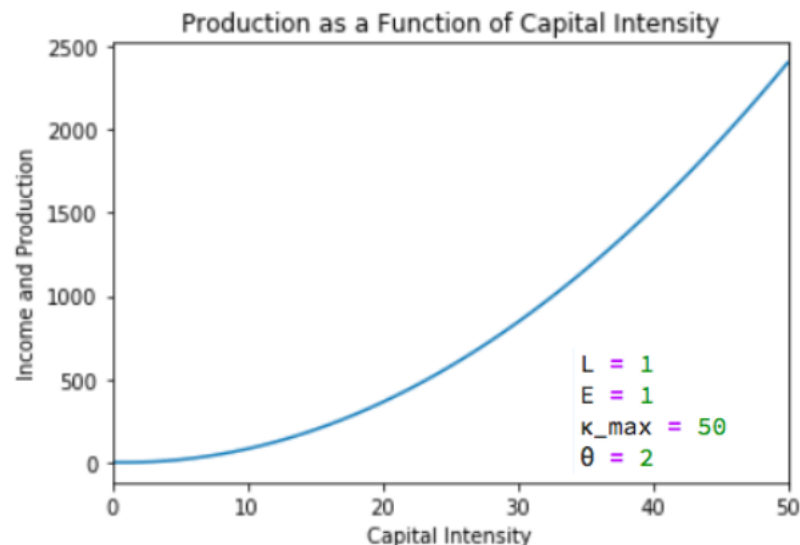
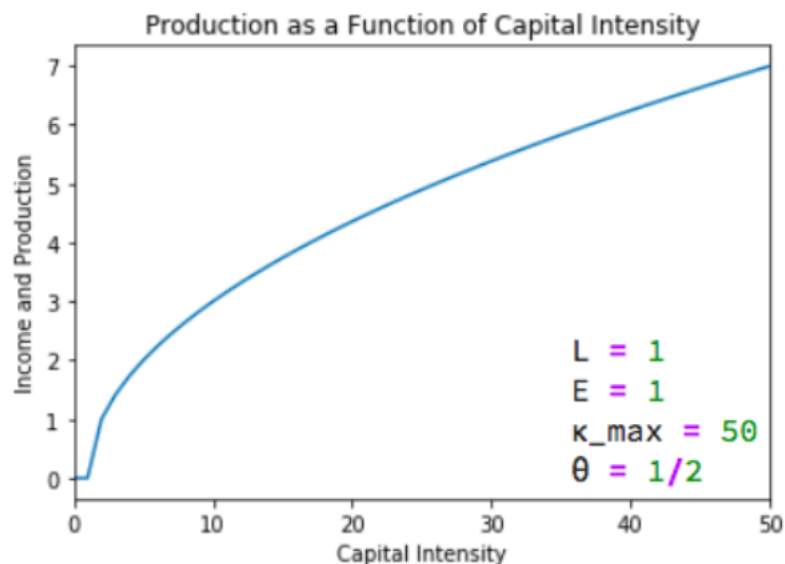
- growth of labor  $g_L$ : proportional at a constant  $n$  (for now)
- growth of labor efficiency  $g_E$ : proportional at a constant  $g$  (for now)
- rate of change of capital: savings minus depreciation
  - growth of capital  $g_K = s/\kappa - \delta$
- What do these mean?

**Now let's look at the rate of change of capital-intensity  $\kappa$  as a function of the level of capital-intensity  $\kappa$ , for constant  $n$ ,  $g$ ,  $s$ ,  $\delta$ , and  $\theta$ ...**

# Growth Rates: L & E

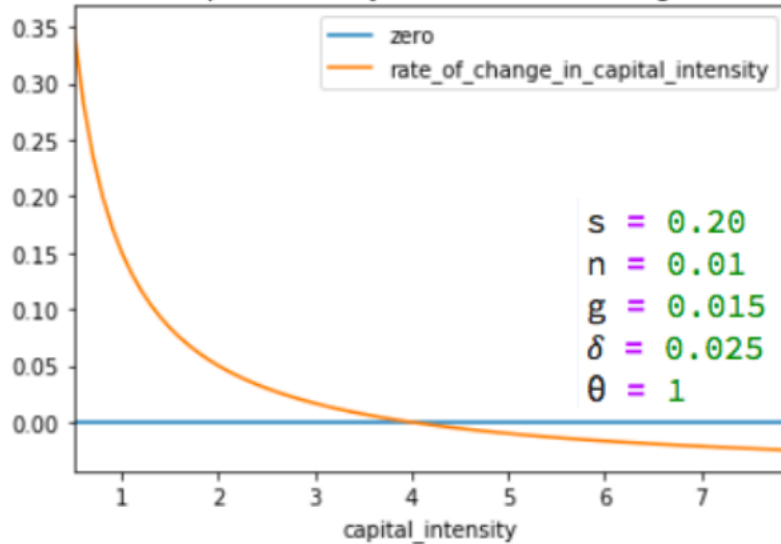


# Salience of Capital: $\theta$

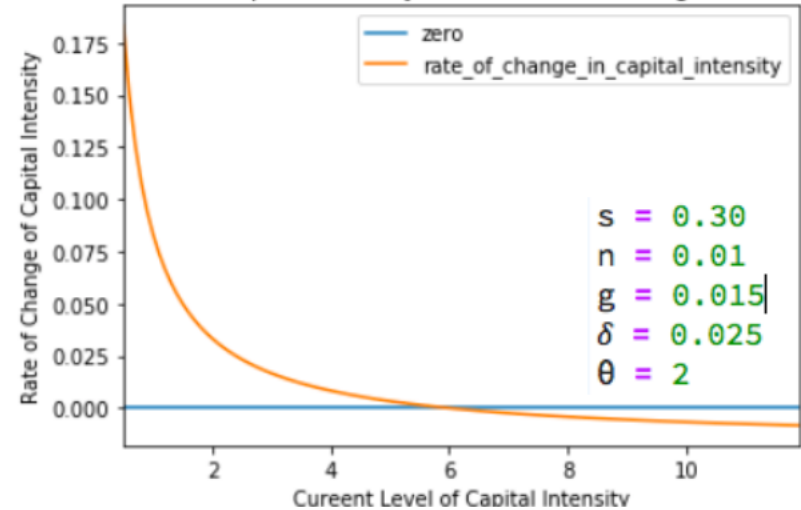


# Change in Capital-Intensity $\kappa$ as a Function of Its Level

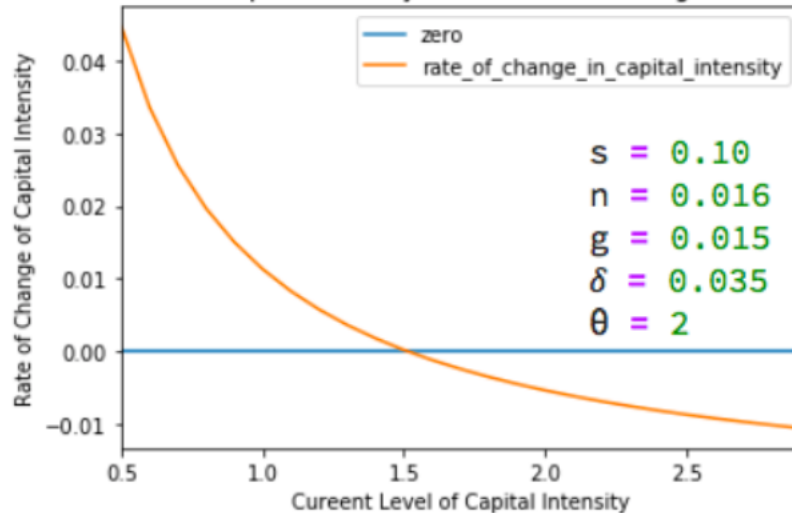
Capital Intensity and Its Rate of Change



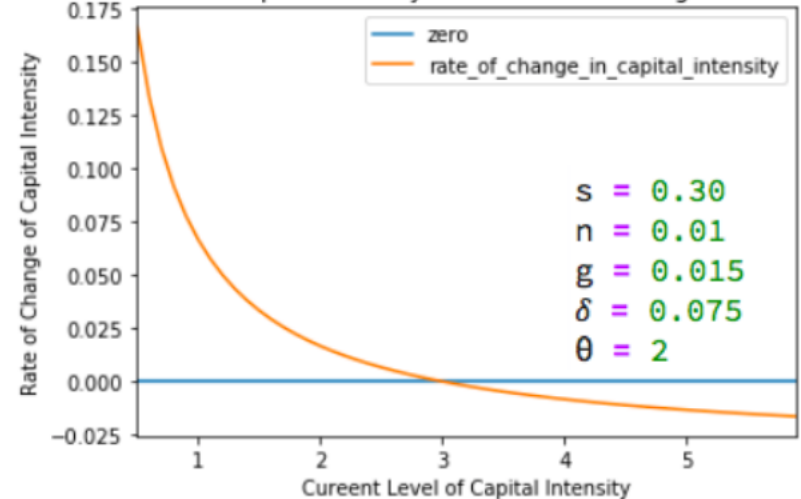
Capital Intensity and Its Rate of Change



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Capital Intensity and Its Rate of Change

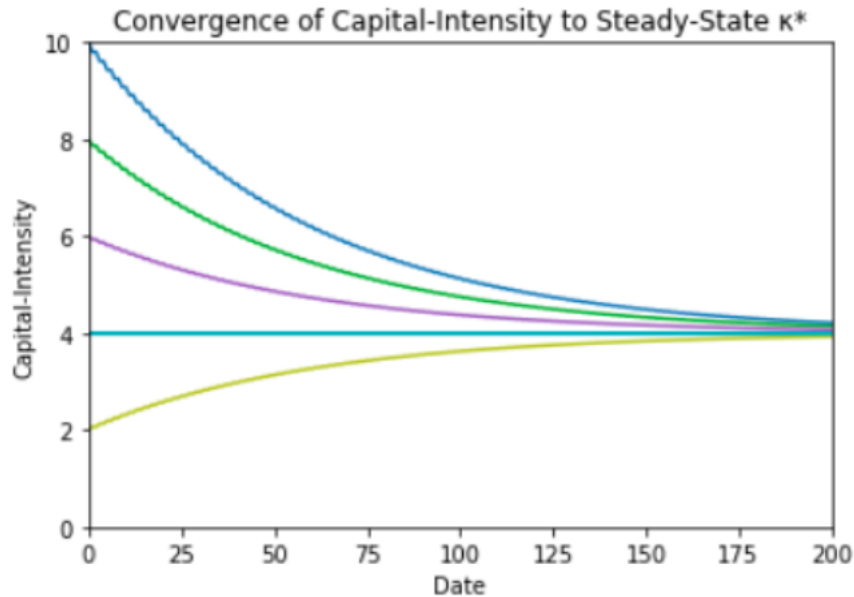


# Catch Our Breath...

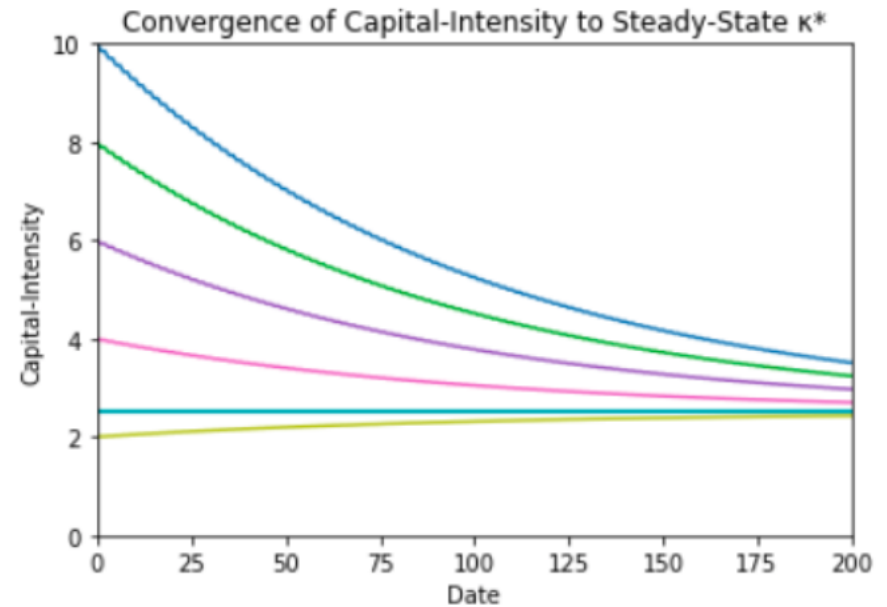
- Ask a couple of questions?
- Make a couple of comments?
- Any more readings to recommend?



# Solving the Model



```
κ_max = 10
κ = κ_max
for i in range(5):
    cg = κ_convergence_graph(κ_0=κ, s = 0.20, n = 0.01,
                             g = 0.015, δ = 0.025, θ = 1/2, T = 200)
    cg.draw()
    κ = κ-2
```



```
κ_max = 10
κ = κ_max
for i in range(5):
    cg = κ_convergence_graph(κ_0=κ, s = 0.15, n = 0.02,
                             g = 0.015, δ = 0.025, θ = 2, T = 200)
    cg.draw()
    κ = κ-2
```

# Balanced-Growth Equilibrium: Steady-State Capital-Intensity $\kappa^*$

$$(1.16) \quad \kappa^* = \frac{s}{n+g+\delta}$$

This  $\kappa^*$  we define as the *steady-state balanced-growth equilibrium* value of capital-intensity in the Solow growth model. If the capital-intensity  $\kappa = \kappa^*$ , then it is constant, and the economy is in balanced growth, with  $Y$  and  $K$  growing at the rate  $n+g$ ,  $E$  and  $y$  growing at the rate  $g$ , and  $L$  growing at the rate  $n$ .

# Along the Balanced-Growth Path

**Everything except  $\kappa$ —which is constant—grows at a constant proportional rate: either  $n$ , or  $g$ , or  $n+g$ ;**

- Labor force  $L$  grows at  $n$
- Income per worker  $y$  and the efficiency of labor  $E$  grow at  $g$
- Total income  $Y$  and the capital stock  $K$  grow at  $n+g$

$$E_t^* = e^{gt} E_0$$

$$L_t^* = e^{nt} L_0$$

$$Y_t^* = (\kappa^*)^\theta E_t L_t = (\kappa^*)^\theta e^{gt} E_0 e^{nt} L_0 = (s/(n + g + \delta))^\theta e^{gt} E_0 e^{nt} L_0$$

$$K_t^* = \kappa^* Y_t^* = (s/(n + g + \delta))^{(1+\theta)} e^{gt} E_0 e^{nt} L_0$$

$$y_t^* = (\kappa^*)^\theta E_t = (\kappa^*)^\theta e^{gt} E_0 = (s/(n + g + \delta))^\theta e^{gt} E_0$$



# Convergence to Steady-State Capital-Intensity

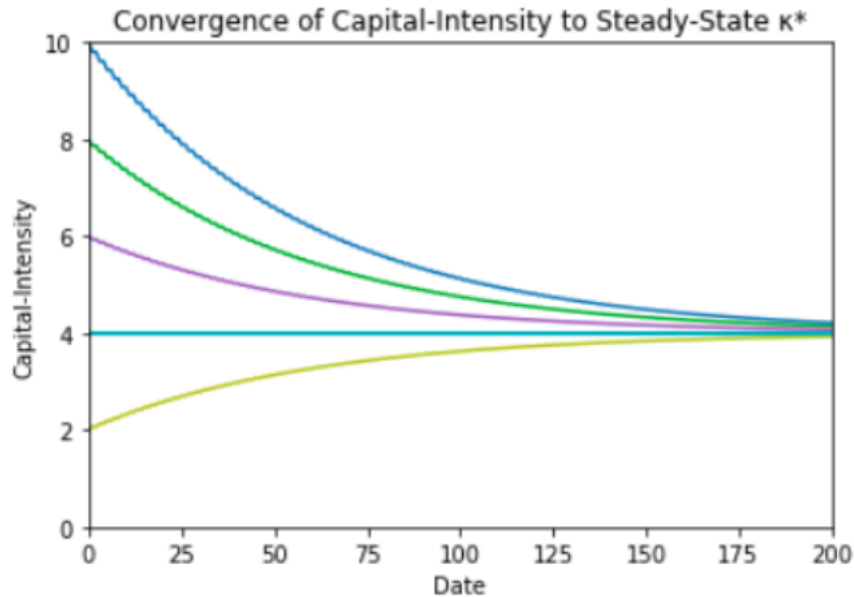
$$(1.18) \quad \frac{d\kappa}{dt} = -\frac{n+g+\delta}{1+\theta}(\kappa - \kappa^*)$$

If we have knowledge of the initial level of an economy's capital-intensity— $\kappa = \kappa_0$  at some initial moment we index as zero—and if  $n$ ,  $g$ ,  $s$ ,  $\delta$ , and  $\theta$  are constant, it immediately follows that at every time  $t > 0$ :

$$(1.21) \quad \kappa_t = \kappa^* + e^{-[(n+g+\delta)/(1+\theta)]t}(\kappa_0 - \kappa^*)$$

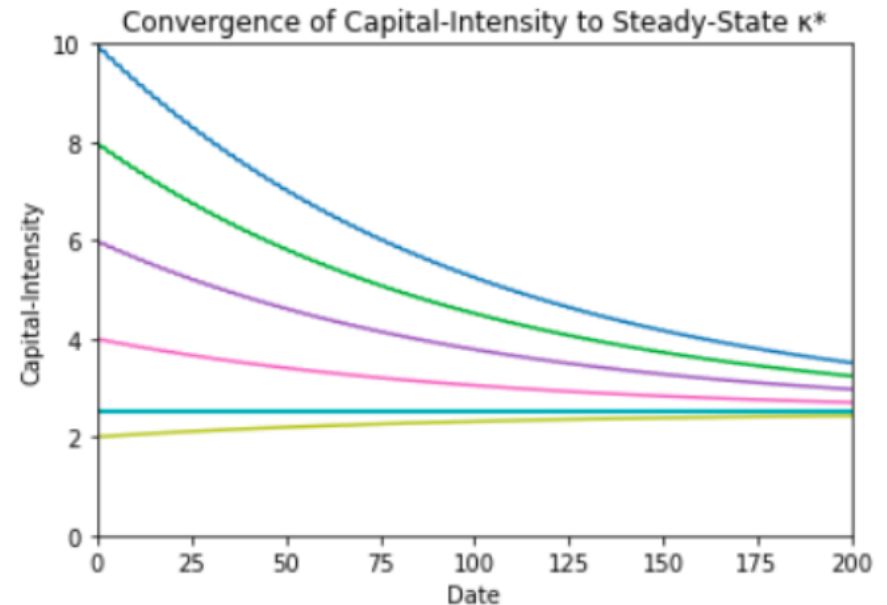
(1.18) holds always, for that moment's values of  $n$ ,  $g$ ,  $\delta$ ,  $\theta$ , and  $s$ , whatever they may be. (1.21) holds only while  $n$ ,  $g$ ,  $\delta$ ,  $\theta$ , and  $s$  are constant. If any of them change, you then have to recalibrate and recompute, with a new initial value of  $\kappa$  equal to its value when the model's parameters jumped, and a new and different value of  $\kappa^*$ .

# Convergence to $\kappa^*$



```

κ_max = 10
κ = κ_max
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    cg.draw()
    κ = κ-2
    
```



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                             g = 0.015, δ = 0.025, θ = 2, T = 200)
    cg.draw()
    κ = κ-2
    
```

# Other Variables in the Model

$$E_t = e^{gt} E_0$$

$$L_t = e^{nt} L_0$$

$$Y_t = (\kappa_t)^\theta E_t L_t$$

$$K_t = \kappa_t Y_t$$

$$y_t = (\kappa_t)^\theta E_t$$

# Catch Our Breath...

- Ask a couple of questions?
- Make a couple of comments?
- Any more readings to recommend?



# Big Ideas: Lecture 2: Solow Theory

## Takeaways from this lecture:

- Math is a **language**!
- Three assumptions about the production function immediately get us to:
  - $Y = \kappa^\theta EL$
- Plus:
  - constant:  $d\ln(L)/dt = n$ ;  $d\ln(E)/dt = g$ ;  $d\ln(K)/dt = s/\kappa - \delta$
- Results in:
  - $\kappa^* = s/(n+g+\delta)$ ;  $d\kappa/dt = -(n+g+\delta)/(1+\theta)$
  - $y = \kappa^\theta E$ ;  $K = \kappa Y$
  - convergence to and then growth along a steady-state balanced-growth path associated with  $\kappa^*$ .

# Basic References

Robert Solow (1956): A Contribution to the Theory of Economic Growth <<http://piketty.pse.ens.fr/files/Solow1956.pdf>>

Moses Abramovitz (1956): Resource and Output Trends in the United States Since 1870 <<https://www.nber.org/chapters/c5650.pdf>>

Robert Solow (1957): Technical Change and the Aggregate Production Function <<http://www.piketty.pse.ens.fr/files/Solow1957.pdf>>

Moses Abramovitz (1986): Catching Up, Forging Ahead, and Falling Behind <[http://www.j-bradford-delong.net/teaching\\_Folder/Econ\\_210c\\_spring\\_2002/Readings/Abramovitz.pdf](http://www.j-bradford-delong.net/teaching_Folder/Econ_210c_spring_2002/Readings/Abramovitz.pdf)>

Robert Solow (1987): Growth Theory and After <<https://www.nobelprize.org/prizes/economic-sciences/1987/solow/lecture/>>

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