Lecture 2:
1.1. Theory: Robert Solow's Economic Growth Model

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last revised: 2019-10-15

Original course by Melissa Dell (Harvard Econ 1342), revised by Brad DeLong, research assistance by Anish Biligiri

<https://github.com/braddelong/public-files/blob/master/econ-135-lecture-2.pptx>

Lecture 2: Solow Theory: Outline

Slides: <https://github.com/braddelong/public-files/blob/master/econ-135-lecture-2.pptx>
Read After: J. Bradford DeLong: Lecture Notes: The Solow Growth Model <https://tinyurl.com/dl-2020-01-18f>
Read After: Partha Dasgupta (2007): Economics: A Very Short Introduction, chapters 5-8 & Epilogue <https://delong.typepad.com/files/dasgupta-economics.pdf>
Do: Assignment 2 (3 pts): Letter to GSI, due Sa Jan 25 9:00 am <https://tinyurl.com/dl-2020-01-12g>

1. **Administration**: Office hours poll

2. **Lecture:** Solow basics

   - <https://www.bradford-delong.com/2020/01/lecture-notes-the-solow-growth-model-the-history-of-economic-growth-econ-135.html> <http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-3-growing.ipynb> <http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-4-using.ipynb>

3. **Review**: ~~Growth patterns~~

4. **Lecture:** Solving the Solow model

   - <https://nbviewer.jupyter.org/github/braddelong/lecture-support-2020/blob/master/lecture-support-solow-2020-01-23.ipynb>

5. **Big Ideas**: Principal takeaways from this class

6. **MOAR** references:

- Robert Solow (1956): A Contribution to the Theory of Economic Growth <http://piketty.pse.ens.fr/files/Solow1956.pdf>
- Moses Abramovitz (1956): Resource and Output Trends in the United States Since 1870 <https://www.nber.org/chapters/c5650.pdf>
- Robert Solow (1957): Technical Change and the Aggregate Production Function <http://www.piketty.pse.ens.fr/files/Solow1957.pdf>
- Moses Abramovitz (1986): Catching Up, Forging Ahead, and Falling Behind <http://www.j-bradford-delong.net/teaching_Folder/Econ_210c_spring_2002/Readings/Abramovitz.pdf>
- Robert Solow (1987): Growth Theory and After <https://www.nobelprize.org/prizes/economic-sciences/1987/solow/lecture/>

Office Hours

Office Hours:

- M 11:15-12:30,
- T 11:15-12:00
- By appointment: email <**delong@econ.berkeley.edu**>

Solow Model Basics

Lecture Notes: <https://www.bradford-delong.com/2020/01/lecture-notes-the-solow-growth-model-the-history-of-economic-growth-econ-135.html>

Let's assume three things about the relationship between an economy's resources and the total output it produces and income it generates:

- Production
- Other parts of the model
- Balanced-growth equilibrium
- Convergence to equilibrium

Solow Model Basics: Production

Let's assume three things about the relationship between an economy's resources and the total output it produces and income it generates:

1. A proportional increase in the economy's capital intensity $\boldsymbol{\kappa}$, measured by the capital stock divided by total production $\kappa = \mathbf{K/Y}$, will carry with it the same (smaller) proportional increase in income and production Y no matter how rich and productive the economy is. A 1% increase in capital intensity will always increase income and production by the same proportional amount θ.
2. If two economies have the same capital intensity, defined as the same capital-output ratio $\boldsymbol{\kappa}$, and have the same level of technology- and organization-driven efficiency-of-labor **E**, then the ratio of their levels of income and output will be equal to the ratio of their labor forces **L**.
3. If two economies have the same capital intensity, defined as the same capital-output ratio $\boldsymbol{\kappa}$, and have the same labor forces **L**, then the ratio of their levels of income and output will be equal to the ratio of their technology- and organization-driven efficiencies-of-labor **E**

Basics: Production Notation

- Y: total income and production
- E: technological and organizational efficiency-of-labor
- L: labor force
- y: per-worker income and production
- κ: the capital-intensity of the economy, as measured by the capital-output ratio K/Y (Greek lower kappa)
- θ: the relative salience in economic growth of capital-intensity vis-a-vis technological and organizational progress (Greek lower theta)
    - if α is the share of income received by capital under the marginal productivity theory of distribution, then $\alpha=\theta/(1+\theta)$, $1-\alpha=1/(1+\theta)$, $\theta = \alpha/(1-\alpha)$

$$(2.1.2)\ Y = \kappa^{\theta} E L \ ; \ (2.1.3)\ y = \kappa^{\theta} E \ ; \ (2.1.1)\ \kappa = \frac{K}{Y}$$

Basics: Production Algebra

Then there is one and only one equation that satisfies those three rules of thumb:

$$Y = \kappa^{\theta} E L$$

And it is also worth writing down

- a version of this equation in per-worker form, where y = Y/L
- the definition of capital intensity κ: κ= K/Y

$$y = \kappa^{\theta} E \qquad\qquad \kappa = \frac{K}{Y}$$

We have just done what economists typically do: take a complex situation, strip things down to some salient piece of it, and then formalize and algebraize that piece in the hope of gaining insight…

Solow Model Basics: Notes

(2.1.2) $Y = \kappa^\theta EL$; (2.1.3) $y = \kappa^\theta E$; (2.1.1) $\kappa = \frac{K}{Y}$

The code in the nbViewer documents is static. But you should also look at:

- <http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-2-basics.ipynb>
- <http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-3-growing.ipynb>
- <http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-4-using.ipynb>

The Rest of the Model: Growth Rates

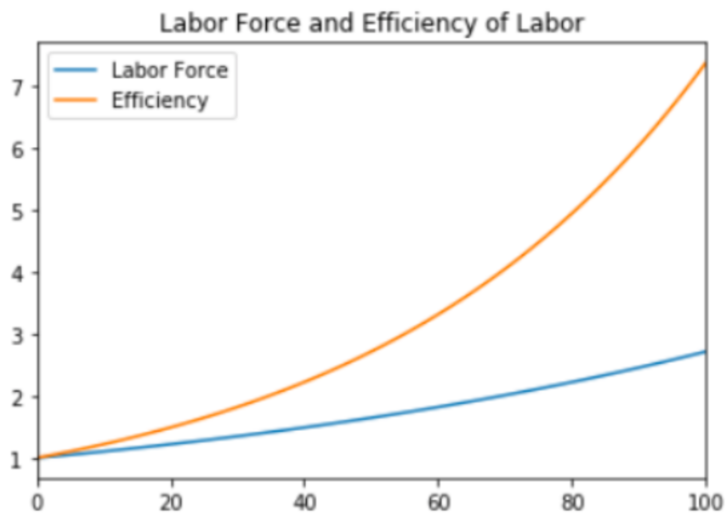
$$\frac{dE}{dt} = gE \qquad \frac{dL}{dt} = g_L L = nL \qquad \frac{dK}{dt} = sY - \delta K = \left(\frac{s}{\kappa} - \delta\right) K$$

Variables change over time:

- growth of labor g_L: proportional at a constant n (for now)
- growth of labor efficiency g_E: proportional at a constant g (for now)
- rate of change of capital: savings minus depreciation
    - growth of capital $g_K = s/\kappa-\delta$
- What do these mean?

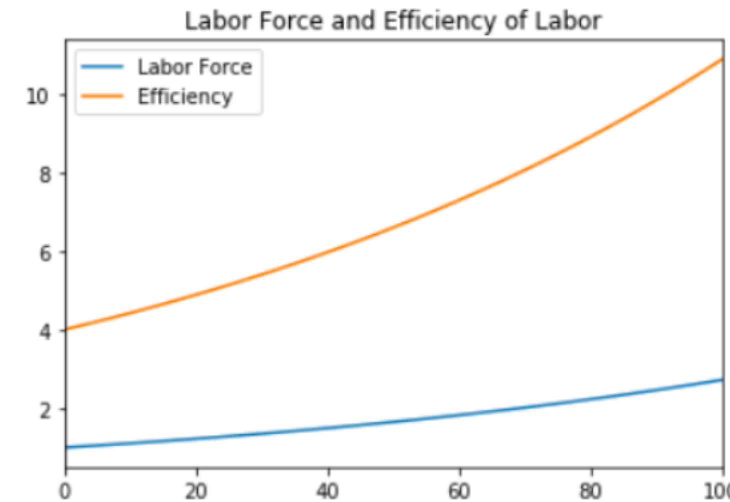
Now let's look at the rate of change of capital-intensity κ as a function of the level of capital-intensity κ, for constant n, g, s, δ, and θ…

Growth Rates: L & E




```
T = 100

L_0 = 1
n = 0.01
E_0 = 1
g = 0.02
```




```
T = 100

L_0 = 1
n = 0.01
E_0 = 4
g = 0.01
```

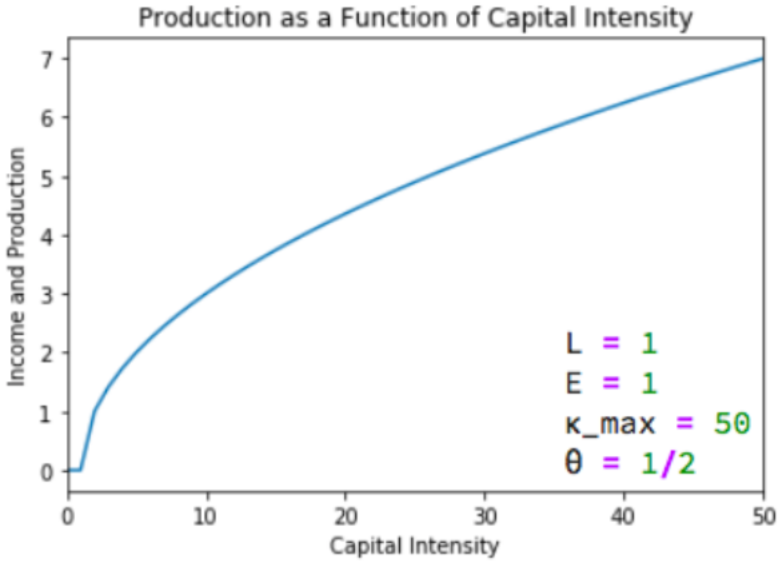

```
T = 100

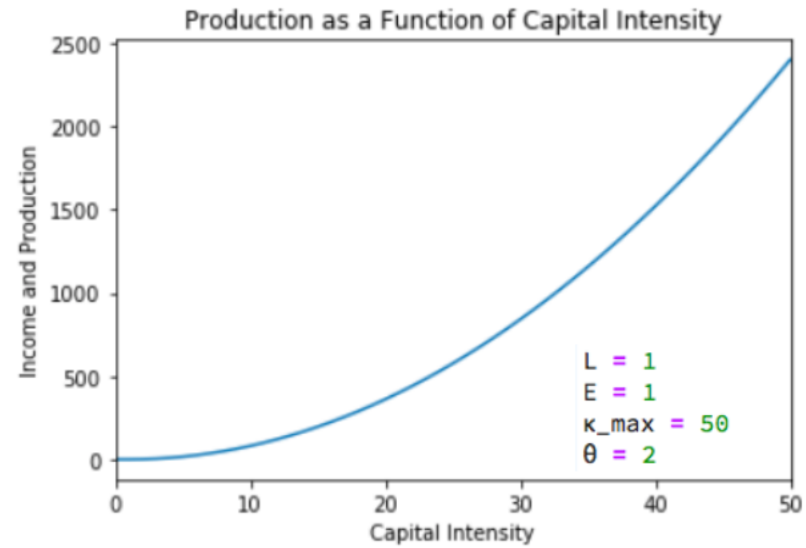
L_0 = 4
n = 0.00
E_0 = 1
g = 0.03
```

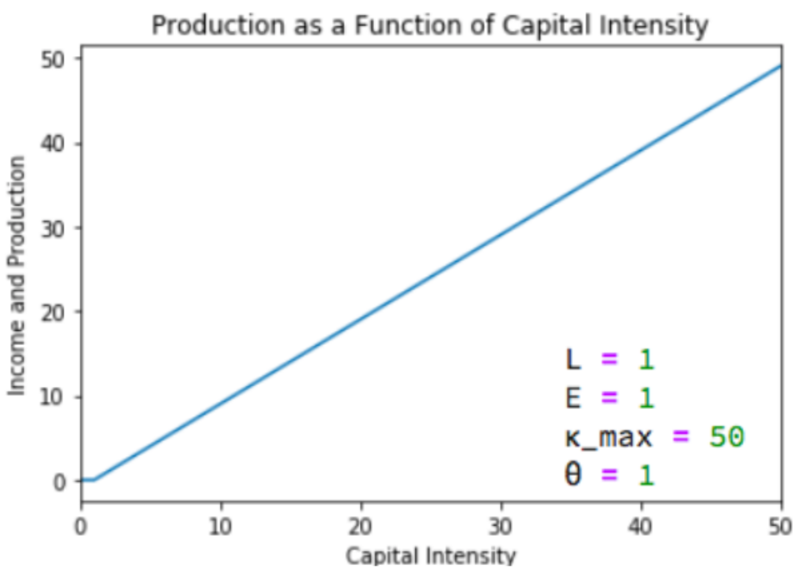

```
T = 100

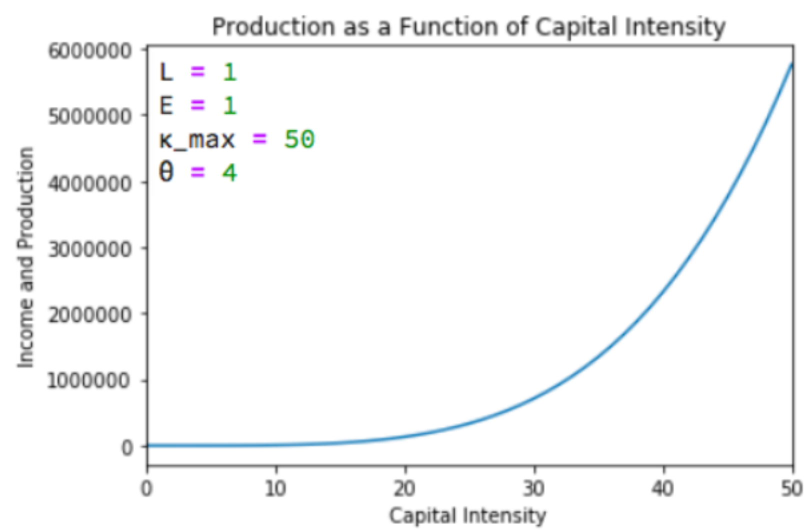
L_0 = 4
n = 0.02
E_0 = 1
g = 0.00
```

Salience of Capital: θ

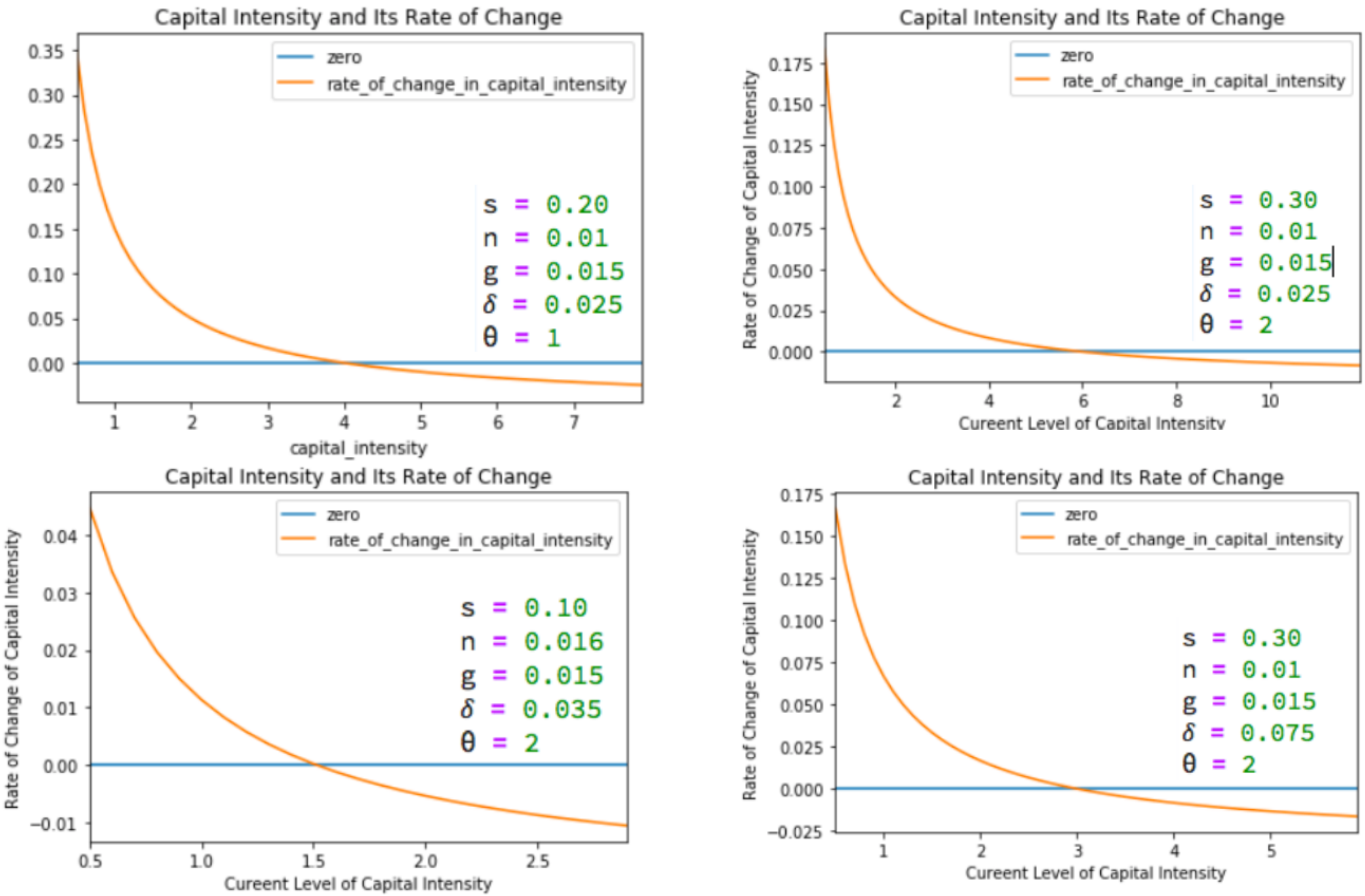






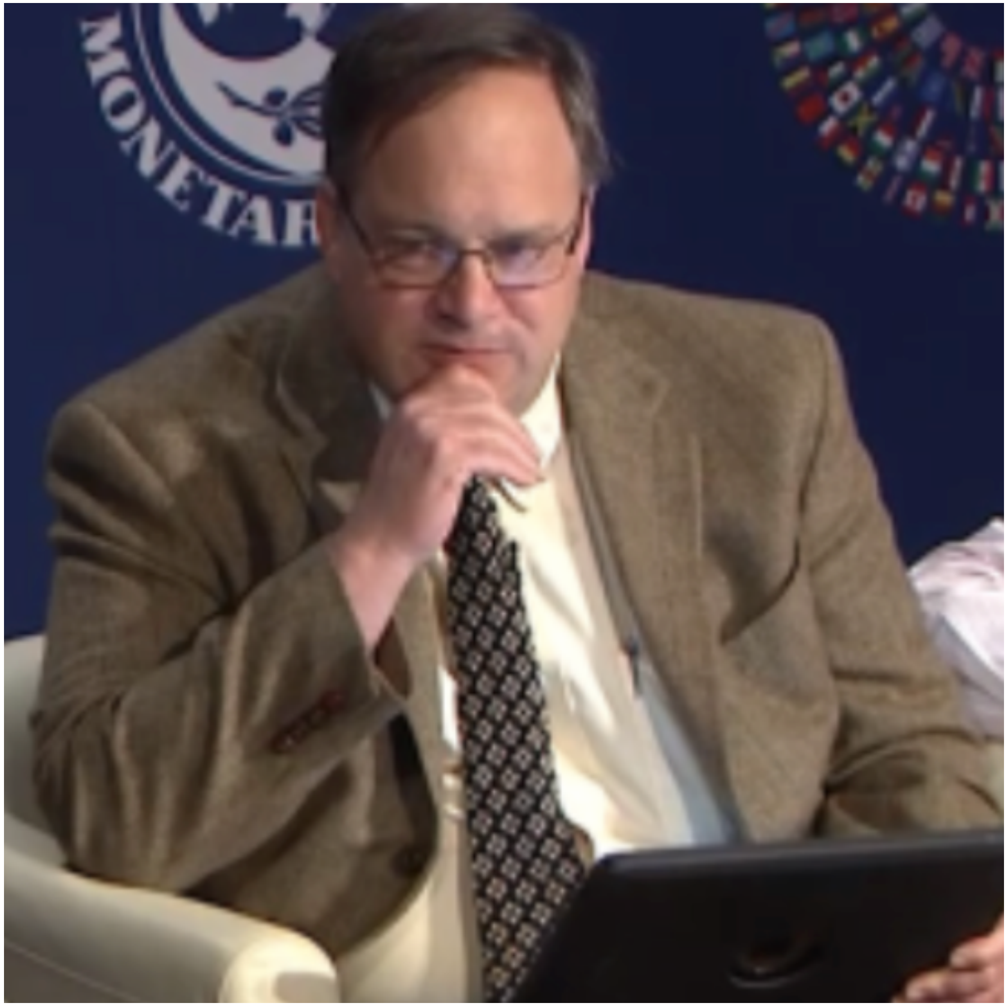


Change in Capital-Intensity κ as a Function of Its Level

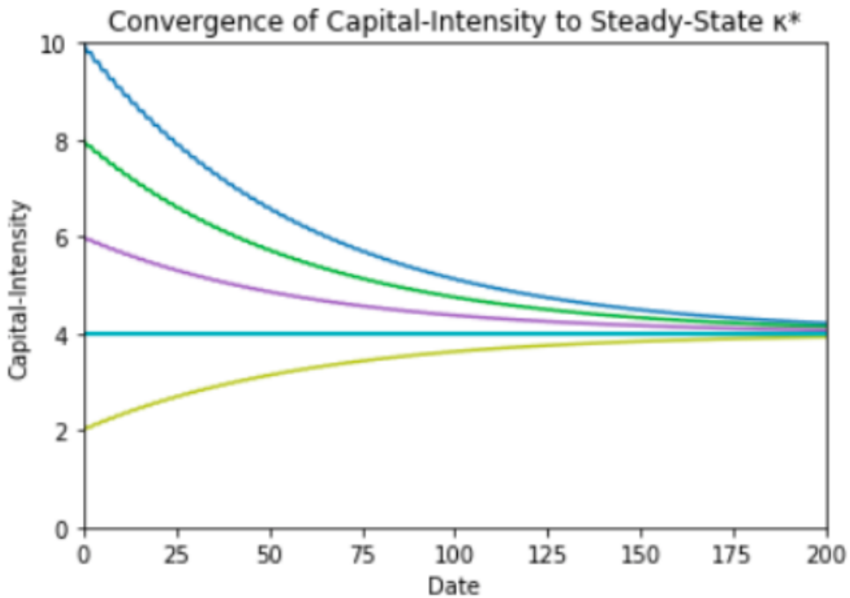


Catch Our Breath…

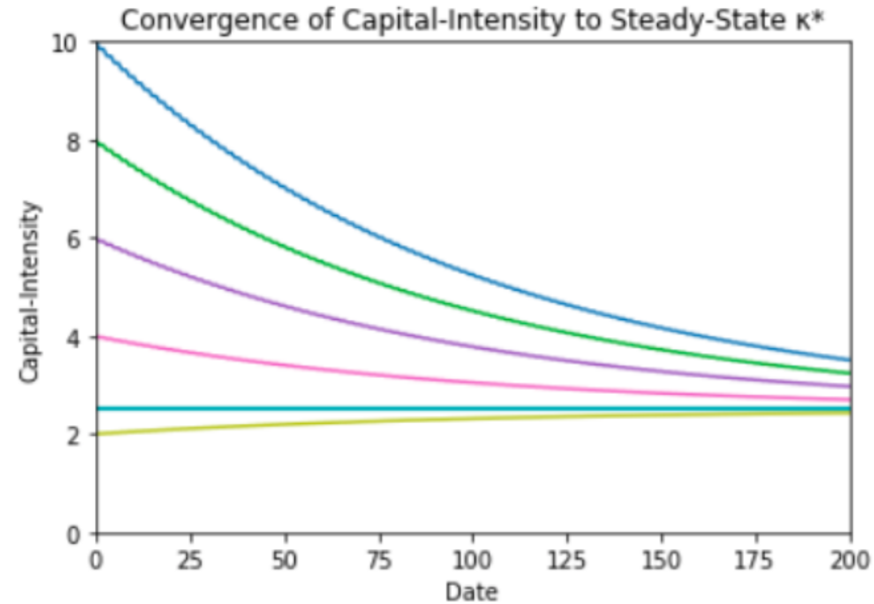
- Ask a couple of questions?
- Make a couple of comments?
- Any more readings to recommend?



Solving the Model




```
κ_max = 10
κ = κ_max
for i in range(5):
    cg = κ_convergence_graph(κ_0=κ, s = 0.20, n = 0.01,
           g = 0.015, δ = 0.025, θ = 1/2, T = 200)
    cg.draw()
    κ = κ-2
```

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Balanced-Growth Equilibrium: Steady-State Capital-Intensity κ*

$$(1.16)\ \kappa^* = \frac{s}{n+g+\delta}$$

This κ^* we define as the *steady-state balanced-growth equilibrium* value of capital-intensity in the Solow growth model. If the capital-intensity $\kappa = \kappa^*$, then it is constant, and the economy is in balanced growth, with Y and K growing at the rate n+g, E and y growing at the rate g, and L growing at the rate n.

Along the Balanced-Growth Path

Everything except κ—which is constant—grows at a constant proportional rate: either n, or g, or n+g;

- Labor force L grows at n
- Income per worker y and the efficiency of labor E grow at g
- Total income Y and the capital stock K grow at n+g

$$E_t^* = e^{gt} E_0$$

$$L_t^* = e^{nt} L_0$$

$$Y_t^* = (\kappa^*)^\theta E_t L_t = (\kappa^*)^\theta e^{gt} E_0 e^{nt} L_0 = (s/(n+g+\delta))^\theta e^{gt} E_0 e^{nt} L_0$$

$$K_t^* = \kappa^* Y_t^* = (s/(n+g+\delta))^{(1+\theta)} e^{gt} E_0 e^{nt} L_0$$

$$y_t^* = (\kappa^*)^\theta E_t = (\kappa^*)^\theta e^{gt} E_0 = (s/(n+g+\delta))^\theta e^{gt} E_0$$

Convergence to Steady-State Capital-Intensity

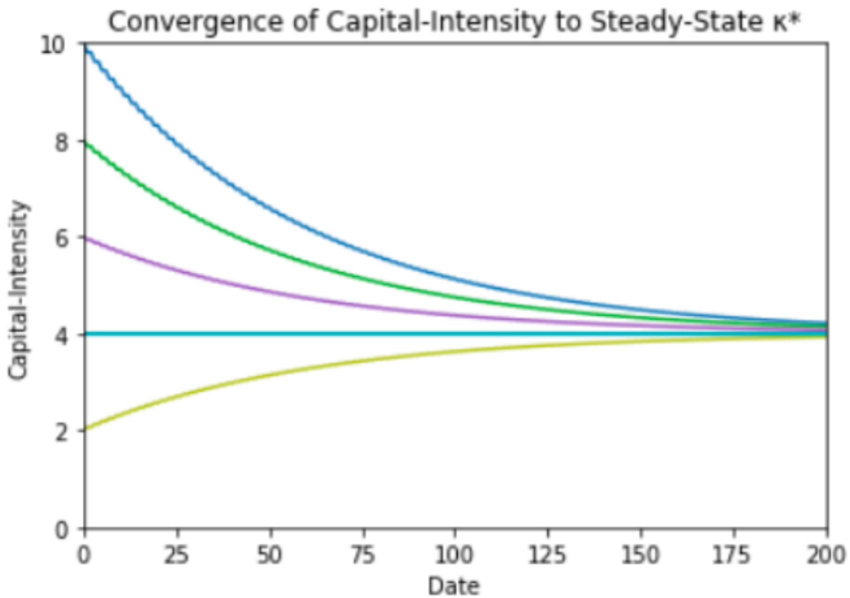
$$(1.18)\ \frac{d\kappa}{dt} = -\frac{n+g+\delta}{1+\theta}(\kappa - \kappa^*)$$

If we have knowledge of the initial level of an economy's capital-intensity—$\kappa = \kappa_0$ at some initial moment we index as zero—and if n, g, s, δ, and θ are constant, it immediately follows that at every time $t > 0$:

$$(1.21)\ \kappa_t = \kappa^* + e^{-[(n+g+\delta)/(1+\theta)]t}(\kappa_0 - \kappa^*)$$

(1.18) holds always, for that moment's values of n, g, δ, θ , and s, whatever they may be. (1.21) holds only while n, g, δ, θ , and s are constant. If any of them change, you then have to recalibrate and recompute, with a new initial value of κ equal to its value when the model's parameters jumped, and a new and different value of κ^*.

Convergence to κ*

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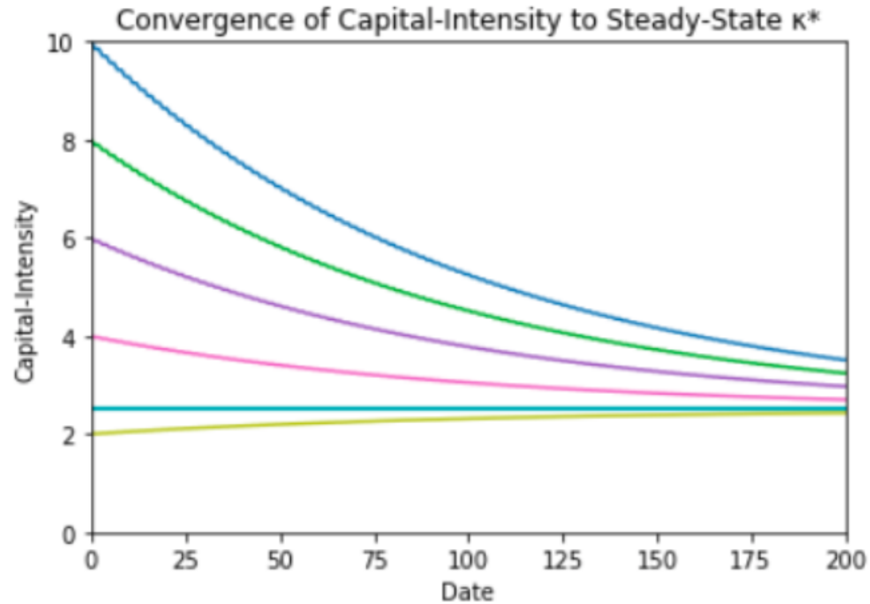


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    κ = κ-2
```

Other Variables in the Model

$$E_t = e^{gt} E_0$$

$$L_t = e^{nt} L_0$$

$$Y_t = (\kappa_t)^{\theta} E_t L_t$$

$$K_t = \kappa_t Y_t$$

$$y_t = (\kappa_t)^{\theta} E_t$$

Catch Our Breath…

- Ask a couple of questions?
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- Any more readings to recommend?

Big Ideas: Lecture 2: Solow Theory

Takeaways from this lecture:

- Math is a **language**!
- Three assumptions about the production function immediately get us to:
  - $Y = \kappa^{\theta} E L$
- Plus:
  - constant: $d\ln(L)/dt = n$; $d\ln(E)/dt = g$; $d\ln(K)/dt = s/\kappa - \delta$
- Results in:
  - $\kappa^* = s/(n+g+\delta)$; $d\kappa/dt = -(n+g+\delta)/(1+\theta)$
  - $y = \kappa^{\theta} E$; $K = \kappa Y$
  - convergence to and then growth along a steady-state balanced-growth path associated with κ^*.

Basic References

Robert Solow (1956): A Contribution to the Theory of Economic Growth <http://piketty.pse.ens.fr/files/Solow1956.pdf>

Moses Abramovitz (1956): Resource and Output Trends in the United States Since 1870 <https://www.nber.org/chapters/c5650.pdf>

Robert Solow (1957): Technical Change and the Aggregate Production Function <http://www.piketty.pse.ens.fr/files/Solow1957.pdf>

Moses Abramovitz (1986): Catching Up, Forging Ahead, and Falling Behind <http://www.j-bradford-delong.net/teaching_Folder/Econ_210c_spring_2002/Readings/Abramovitz.pdf>

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Notes