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0.1 Appendix A: Matrix terminology and properties of the matrix product

Let us recall some matrix terminology

- A matrix with one column is called a column vector, and a matrix with one row is called a row vector. An $n \times n$ matrix is called a square matrix.
- The identity matrix I_n is a square $n \times n$ matrix that consists of ones on the main diagonal, and zeroes everywhere else

$$I_n \equiv \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \quad \Leftrightarrow \quad (I_n)_{ij} = \delta_{ij} \tag{A.1}$$

- The transpose of an $m \times n$ matrix $A = (A_{ia})$ is the $n \times m$ matrix A^t with rows and columns of A interchanged

$$A^t = (A^t_{ai}) \quad \Leftrightarrow \quad (A^t)_{ai} \equiv A^t_{ai} = A_{ia} \tag{A.2}$$

  A matrix is symmetric if $A^t = A$, and it is antisymmetric if $A^t = -A$.

- The complex conjugate $A^* = (A^*_{ia})$ of a matrix $A = (A_{ia})$ consists of the complex conjugates of every elements.

  A matrix is real if $A^* = A$, and it is imaginary if $A^* = -A$

- The hermitian conjugate or adjoint $A^\dagger = (A^\dagger_{ai})$ of a matrix $A = (A_{ia})$ is a transpose complex conjugate

$$A^\dagger = (A^t)^* = (A^*)^t \,, \quad (A^\dagger)_{ai} \equiv A^\dagger_{ai} = A^*_{ia} \tag{A.3}$$

  A matrix is hermitian (or self-adjoint) if $A^\dagger = A$, and it is anti-hermitian (or skew-hermitian) if $A^\dagger = -A$.

- The trace of a square $n \times n$ matrix is the sum of its diagonal elements

$$\mathrm{tr} A = \sum_{i=1}^{n} A_{ii} \tag{A.4}$$

- The determinant of a square $n \times n$ matrix is defined by

$$\det A = \sum_{i_1,\dots,i_n=1}^{n} \epsilon^{i_1 i_2 \cdots i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n} \tag{A.5}$$

The matrix product operation satisfies many properties, in particular

1. The trace of AB is equal to the trace of BA

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) \tag{A.6}$$

and therefore the trace satisfies the cyclic property

$$\mathrm{tr}(A_1 A_2 A_3 \cdots A_d) = \mathrm{tr}(A_2 A_3 \cdots A_d A_1) = \mathrm{tr}(A_3 \cdots A_d A_1 A_2) = \cdots \tag{A.7}$$

2. The determinant of AB is equal to the product of the determinants of A and B

$$\det(AB) = \det(A)\det(B) \tag{A.8}$$

if both A and B are square

3. The inverse A^{-1} of a square matrix A is defined as

$$AA^{-1} = A^{-1}A = I \tag{A.9}$$

4. A matrix is orthogonal if its inverse is equal to its transpose

$$A^t = A^{-1} \quad \Leftrightarrow \quad A^t A = AA^t = I \tag{A.10}$$

5. Two matrices A_1 and A_2 are similar if there is an invertible matrix V such that

$$A_2 = V A_1 V^{-1} \tag{A.11}$$

A_1 and A_2 have the same trace $\mathrm{tr} A_1 = \mathrm{tr} A_2$, the same determinant $\det A_1 = \det A_2$ and the same set of eigenvalues.

6. The transpose, hermitian conjugate and inverse of a product satisfy

$$(AB)^t = B^t A^t\,, \quad (AB)^\dagger = B^\dagger A^\dagger\,, \quad (AB)^{-1} = B^{-1}A^{-1} \tag{A.12}$$

7. The commutator of two square matrices A, B of the same size is

$$[A\,,\,B] \equiv AB - BA \tag{A.13}$$

It is anti-symmetric $[A, B] = -[B, A]$ and satisfies Jacobi's identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = O \tag{A.14}$$

where O denotes the zero matrix.

8. The dot product of column vectors $A = (A_i)$ and $B = (B_i)$ is the product of the row vector $A^\dagger$ and B

$$A^\dagger B = \sum_{i=1}^{m} A_i^* B_i \tag{A.15}$$

   - The length or norm of a column vector A is

$$|A| = \sqrt{A^\dagger A} = \sqrt{\sum_{i=1}^{m} A_i^* A_i} \tag{A.16}$$

- A normalised vector has unit norm.
- Two column vectors are orthogonal or perpendicular if their dot product vanishes.

9. A matrix is unitary if its inverse is equal to its hermitian conjugate

$$U^\dagger = U^{-1} \quad \Leftrightarrow \quad U^\dagger U = U U^\dagger = I \tag{A.17}$$

The rows and columns of a unitary matrix constitute orthonormal sets.

10. Any unitary matrix U can be written in the form

$$U = e^{\mathrm{i}H} = I + \mathrm{i}H - \frac{1}{2}H^2 + \cdots \,, \quad H^\dagger = H \tag{A.18}$$

where H is a hermitian matrix.

11. The dot product of column vectors $A = (A_i)$ and $B = (B_i)$ does not change (is invariant) if one replaces A and B with UA and UB where U is unitary

$$(UA)^\dagger UB = A^\dagger U^\dagger UB = A^\dagger B \tag{A.19}$$

0.2 Appendix B: Groups

Def. A group is a nonempty set G on which there is defined a binary operation $(a, b) \mapsto ab$, called multiplication, satisfying the following properties

- Closure: If a and b belong to G, then ab is also in G.
- Associativity : $a(bc) = (ab)c$ for all $a, b, c \in G$.
- Identity: There is an element $e \in G$ such that $ae = ea = a$ for all a in G.
- Inverse: If $a \in G$, then there is an element $a^{-1} \in G$: $aa^{-1} = a^{-1}a = 1$.

If $ab = ba$ for all $a, b, \in G$ then G is called abelian, and the group multiplication is often denoted as summation: $ab \to a + b = b + a$, and the identity element as 0. Otherwise, it is called either nonabelian or noncommutative.

Examples of discrete groups

1. The set of integers $\mathbb{Z}$ with the binary operation being the usual addition of numbers is an abelian group

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\} \tag{B.1}$$

$ab \to a + b = b + a$, $e \to 0$, $a^{-1} \to -a$.

2. The set $\mathbb{Z}_2$ of two elements e and ω satisfying

$$\omega\omega = e \,, \quad \omega^{-1} = \omega \tag{B.2}$$

$\mathbb{Z}_2$ is a finite group.

E.g, $\{-1, 1\} \cong \mathbb{Z}_2$ with the usual multiplication of numbers.

Reflection group is $\mathbb{Z}_2$.

3. The cyclic group $\mathbb{Z}_n$ consists of n elements

$$e\,,\ \omega\,,\ \omega^2 \equiv \omega\omega\,,\ \omega^3 \equiv \omega\omega\omega\,,\ \ldots\,,\ \omega^{n-1}\,,\ \omega^n = e \tag{B.3}$$

   It is abelian and finite, and generated by ω.

   E.g. $e = 1$, $\omega = \exp(\frac{2\pi \mathrm{i}}{n})$, $\omega^k = \exp(\frac{2\pi \mathrm{i}}{n}k)$ with the usual multiplication of numbers.

4. S^n is the group of permutations of n numbers $1, 2, \ldots, n$. It is nonabelian and it has $n!$ elements. $\mathbb{Z}_n \subset S^3$ is a subgroup. $\mathbb{Z}_2 \cong S^2$.

Lie groups are groups which are manifolds. For example

(a) Real line $\mathbb{R}$ (or complex line $\mathbb{C} \cong \mathbb{R}^2$) is an abelian Lie group with the binary operation being the usual addition of numbers

(b) $U(1)$ is an abelian Lie group. It consists of complex numbers of modulus 1 with the binary operation being the usual multiplication of numbers. As a manifold it is a circle S^1

Examples of matrix groups with the binary operation being the usual multiplication of matrices which are manifolds

1. The **general** linear group $GL(n,\mathbb{C})$ ($GL(n,\mathbb{R})$) consisting of all $n \times n$ complex (real) matrices with non-zero determinant

2. The **special** linear group $SL(n,\mathbb{C})$ ($SL(n,\mathbb{R})$) consisting of all $n \times n$ complex (real) matrices with determinant equal to 1

$$\det A = 1\,, \quad A \in \mathrm{Mat}(n,\mathbb{F})\,, \quad \mathbb{F} = \mathbb{C} \text{ or } \mathbb{R}$$

3. The **orthogonal** group $O(n,\mathbb{C})$ ($O(n,\mathbb{R})$) consisting of all $n \times n$ complex (real) matrices satisfying

$$A^t A = I\,, \quad A \in \mathrm{Mat}(n,\mathbb{F})\,, \quad \mathbb{F} = \mathbb{C} \text{ or } \mathbb{R}$$

4. The **special orthogonal** group $SO(n,\mathbb{C})$ ($SO(n,\mathbb{R})$) consisting of all $n \times n$ complex (real) matrices satisfying

$$A^t A = I\,, \quad \det A = 1\,, \quad A \in \mathrm{Mat}(n,\mathbb{F})\,, \quad \mathbb{F} = \mathbb{C} \text{ or } \mathbb{R}$$

5. The **pseudo-orthogonal** group $O(p,q,\mathbb{C})$ ($O(p,q,\mathbb{R})$) consisting of all $n \times n$, $n = p + q$ complex (real) matrices satisfying

$$A^T \eta A = \eta\,, \quad \eta = \mathrm{diag}\,(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q})\,, \quad A \in \mathrm{Mat}(n,\mathbb{F})\,, \quad \mathbb{F} = \mathbb{C} \text{ or } \mathbb{R}$$

6. The **special pseudo-orthogonal** group $SO(p,q,\mathbb{C})$ ($SO(p,q,\mathbb{R})$) consisting of all $n \times n$, $n = p + q$ complex (real) matrices $A \in \mathrm{Mat}(n,\mathbb{F})$ satisfying

$$A^T \eta A = \eta\,, \quad \det A = 1\,, \quad \eta = \mathrm{diag}\,(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q})$$

7. The **unitary** group $U(n)$ consisting of all $n \times n$ complex matrices satisfying

$$U^\dagger U = I\,, \quad U \in \mathrm{Mat}(n, \mathbb{C})$$

8. The **special unitary** group $SU(n)$ consisting of all $n \times n$ complex matrices satisfying

$$U^\dagger U = I\,, \quad \det U = 1\,, \quad U \in \mathrm{Mat}(n, \mathbb{C})$$

9. As a manifold,. $SU(2) \cong S^3$

10. The **pseudo-unitary** group $U(p, q)$ consisting of all $n \times n$, $n = p+q$ complex matrices satisfying

$$A^\dagger \eta A = \eta\,, \quad \eta = \mathrm{diag}\,(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q})\,, \quad A \in \mathrm{Mat}(n, \mathbb{C})$$

11. The **special pseudo-unitary** group $SU(p, q)$ consisting of all $n \times n$, $n = p+q$ complex matrices $A \in \mathrm{Mat}(n, \mathbb{C})$ satisfying

$$A^\dagger \eta A = \eta\,, \quad \eta = \mathrm{diag}\,(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q})\,, \quad \det A = 1$$

0.3 Appendix C: Algebras and representations

We use the fields of real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$.

Algebras

Def. Let $\mathscr{A}$ be a **vector space** over a field $\mathbb{F}$, and let $\mathscr{A}$ be equipped with a **multiplication** (or binary) operation, $\mathscr{A} \times \mathscr{A} \mapsto \mathscr{A}$ denoted by $*$ so that $\forall\, \mathcal{S}, \mathcal{T} \in \mathscr{A}\,,\ \mathcal{S} * \mathcal{T} \in \mathscr{A}$. Then, $\mathscr{A}$ is an **algebra** over $\mathbb{F}$ if $\forall\, \mathcal{S}, \mathcal{T}, \mathcal{U} \in \mathscr{A}$ and $\forall\, a, b \in \mathbb{F}$ ("scalars")

1. $(\mathcal{S} + \mathcal{T}) * \mathcal{U} = \mathcal{S} * \mathcal{U} + \mathcal{T} * \mathcal{U} \quad \leftarrow$ right distributivity
2. $\mathcal{U} * (\mathcal{S} + \mathcal{T}) = \mathcal{U} * \mathcal{S} + \mathcal{U} * \mathcal{T} \quad \leftarrow$ left distributivity
3. $(a\,\mathcal{S}) * (b\,\mathcal{T}) = (ab)(\mathcal{S} * \mathcal{T}) \quad \leftarrow$ compatibility with "scalars"

These three properties mean that the operation is **bilinear**. Therefore, given a basis $\mathcal{E}_i$, $i = 1, \ldots, \dim \mathscr{A}$ of $\mathscr{A}$ the product $*$ is completely determined by the structure constants $a_{ij}^k \in \mathbb{F}$ defined by

$$\mathcal{E}_i * \mathcal{E}_j = \sum_{k=1}^{\dim \mathscr{A}} f_{ij}^k\, \mathcal{E}_k \tag{C.4}$$

Note that the dimension of $\mathscr{A}$ may be infinite.

Def. The center of $\mathscr{A}$ is the subalgebra of elements that commute with all elements of $\mathscr{A}$ and is denoted by

$$\mathscr{Z}(\mathscr{A}) = \{\, \mathcal{Z} \in \mathscr{A} \,|\, \mathcal{Z} * \mathcal{T} = \mathcal{T} * \mathcal{S}\,,\ \forall\, \mathcal{T} \in \mathscr{A}\,\} \tag{C.5}$$

Def. An algebra $\mathscr{A}$ is called

- **commutative** if $\mathcal{S} * \mathcal{T} = \mathcal{T} * \mathcal{S} \quad \forall\ \mathcal{S}, \mathcal{T} \in \mathscr{A}$.
- **unital** if $\exists\, \mathcal{I} \in \mathscr{A}\ :\ \mathcal{I} * \mathcal{S} = \mathcal{S} * \mathcal{I} = \mathcal{S} \quad \forall\ \mathcal{S} \in \mathscr{A}$.

  $\mathcal{I}$ is called a unit or identity element of $\mathscr{A}$
- **associative** if $(\mathcal{S} * \mathcal{T}) * \mathcal{U} = \mathcal{S} * (\mathcal{T} * \mathcal{U}) \quad \forall\ \mathcal{S}, \mathcal{T}, \mathcal{U} \in \mathscr{A}$

Ex 1. The vector space of $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{F})$ with the usual matrix multiplication is a unital associative algebra over $\mathbb{F}$.
Ex 2. The vector space of linear operators acting in a vector space $\mathscr{V}$ over $\mathbb{F}$ with the usual operator multiplication is a unital associative algebra over $\mathbb{F}$ denoted by $\mathrm{End}(\mathscr{V})$.
Ex 3. The vector space of hermitian operators over $\mathbb{R}$ acting in a complex inner product vector space with the following multiplication

$$\hat{X} * \hat{P} = \frac{1}{2}(\hat{X}\hat{P} + \hat{P}\hat{X}) \tag{C.6}$$

is a unital and commutative (but not associative) algebra over $\mathbb{R}$. Indeed, if $\hat{X}^\dagger = \hat{X}$, $\hat{P}^\dagger = \hat{P}$ then

$$(\hat{X} * \hat{P})^\dagger = \frac{1}{2}(\hat{X}\hat{P} + \hat{P}\hat{X})^\dagger = \frac{1}{2}(\hat{P}^\dagger\hat{X}^\dagger + \hat{X}^\dagger\hat{P}^\dagger) = \frac{1}{2}(\hat{P}\hat{X} + \hat{X}\hat{P}) = \hat{X} * \hat{P} \tag{C.7}$$

Finally, the identity operator is hermitian, and

$$\hat{I} * \hat{X} = \frac{1}{2}(\hat{I}\hat{X} + \hat{X}\hat{I}) = \hat{X} \tag{C.8}$$

In quantum mechanics observables are represented by hermitian operators, and the algebra of observables is exactly the one we have just defined.
Ex 4. The Heisenberg algebra $\mathfrak{H}$ (also called the Weyl algebra) is a unital associative algebra over $\mathbb{C}$ generated by elements $\mathcal{I}$, $\mathcal{X}$, $\mathcal{P}$ that satisfy the relation

$$[\mathcal{X}\,,\,\mathcal{P}] \equiv \mathcal{X} * \mathcal{P} - \mathcal{P} * \mathcal{X} = \mathrm{i}\,\hbar\mathcal{I} \tag{C.9}$$

It means that the vectors of $\mathfrak{H}$ are linear combinations of the identity element $\mathcal{I}$ and all words made of $\mathcal{X}$ and $\mathcal{P}$

$$\mathcal{I}\,,\ \ \mathcal{Z}_i\,,\ \ \mathcal{Z}_i\mathcal{Z}_j\,,\ \ \mathcal{Z}_i\mathcal{Z}_j\mathcal{Z}_k\,,\ \ldots\ ,\ \mathcal{Z}_{i_1}\cdots\mathcal{Z}_{i_n}\,,\ \ldots \tag{C.10}$$

where $\mathcal{Z}_1 = \mathcal{X}$, $\mathcal{Z}_2 = \mathcal{P}$. The multiplication is defined in a natural way by "gluing" words

$$(\mathcal{Z}_{i_1}\cdots\mathcal{Z}_{i_k}) * (\mathcal{Z}_{j_1}\cdots\mathcal{Z}_{j_n}) = \mathcal{Z}_{i_1}\cdots\mathcal{Z}_{i_k}\mathcal{Z}_{j_1}\cdots\mathcal{Z}_{j_n} \tag{C.11}$$

and the Heisenberg algebra commutation relation (C.9) is taken into account by identifying vectors $\mathcal{T}_1$ and $\mathcal{T}_2$ if

$$\mathcal{T}_2 - \mathcal{T}_1 = \sum_a \mathcal{V}_a * (\,[\mathcal{X}\,,\,\mathcal{P}] - \mathrm{i}\,\hbar\mathcal{I}\,) * \mathcal{W}_a \tag{C.12}$$

for some elements $\mathcal{V}_a, \mathcal{W}_a$ of $\mathfrak{H}$.

Because of this identification the words (C.10) are linear dependent in $\mathfrak{H}$, and a basis of $\mathfrak{H}$ can be given by ordered words, e.g

$$\underbrace{\mathcal{X}\cdots\mathcal{X}}_{k}\underbrace{\mathcal{P}\cdots\mathcal{P}}_{n} = \mathcal{X}^k\mathcal{P}^n \tag{C.13}$$

or

$$\underbrace{\mathcal{P}\cdots\mathcal{P}}_{k}\underbrace{\mathcal{X}\cdots\mathcal{X}}_{n} = \mathcal{P}^k\mathcal{X}^n \tag{C.14}$$

For example,

$$\mathcal{X}\mathcal{P}\mathcal{X}\mathcal{P} = \mathcal{X}\mathcal{X}\mathcal{P}\mathcal{P} + \mathcal{X}[\mathcal{P},\mathcal{X}]\mathcal{P} = \mathcal{X}\mathcal{X}\mathcal{P}\mathcal{P} - \mathrm{i}\,\hbar\,\mathcal{X}\mathcal{P} - \mathcal{X}(\,[\mathcal{X}\,,\,\mathcal{P}] - \mathrm{i}\,\hbar\,\mathcal{I}\,)\mathcal{P} \sim \mathcal{X}^2\mathcal{P}^2 - \mathrm{i}\,\hbar\,\mathcal{X}\mathcal{P} \tag{C.15}$$

If we denote

$$\mathcal{E}_{k,n} \equiv \mathcal{X}^k\mathcal{P}^n \tag{C.16}$$

then the equation above takes the form

$$\mathcal{X}\mathcal{P} * \mathcal{X}\mathcal{P} = \mathcal{E}_{1,1} * \mathcal{E}_{1,1} = \mathcal{E}_{2,2} - \mathrm{i}\,\hbar\,\mathcal{E}_{1,1} \tag{C.17}$$

Thus, if one works with a basis then the multiplication becomes nontrivial. To find all structure constants with respect to the basis $\mathcal{E}_{k,n}$ one has to compute the commutator $[\mathcal{P}^k\,,\,\mathcal{X}^n]$ for arbitrary positive integers k, n.

Ex 5. Lie algebras

Def. A **Lie algebra** is a vector space $\mathcal{G}$ over a field $\mathbb{F}$ with a bilinear operation $[\cdot,\cdot]$: $\mathcal{G}\times\mathcal{G}\mapsto\mathcal{G}$ which is called a commutator or a Lie bracket, such that the following axioms are satisfied:

- It is skew symmetric: $[\mathcal{J},\mathcal{J}] = \mathcal{O}$ which implies $[\mathcal{J},\mathcal{K}] = -[\mathcal{K},\mathcal{J}]$ for all $\mathcal{J},\mathcal{K}\in\mathcal{G}$
- It satisfies the Jacobi Identity: $[\mathcal{J},[\mathcal{K},\mathcal{L}]] + [\mathcal{K},[\mathcal{L},\mathcal{J}]] + [\mathcal{L},[\mathcal{J},\mathcal{K}]] = \mathcal{O}$

  where $\mathcal{O}$ is the zero vector of $\mathcal{G}$.

Clearly, a Lie algebra is in general a non-associative algebra with the multiplication $*$ given by the bracket $[\cdot,\cdot]$. A Lie algebra $\mathcal{G}$ is called abelian if $[\mathcal{J},\mathcal{K}] = \mathcal{O}$ for all $\mathcal{J},\mathcal{K}\in\mathcal{G}$.

Given a basis $\mathcal{E}_i$, $i = 1,\ldots,\dim\mathcal{G}$ of $\mathcal{G}$ its Lie algebra structure is determined by commutators of the basis vectors

$$[\mathcal{E}_i\,,\,\mathcal{E}_j] = \sum_{k=1}^{\dim\mathcal{G}} c_{ij}^k\,\mathcal{E}_k \tag{C.18}$$

Here $c_{ij}^k\in\mathbb{F}$ are called the structure constants of the Lie algebra $\mathcal{G}$.

Examples

(i) The Heisenberg-Lie algebra $\mathfrak{h}_n$ is a $2n+1$ dimensional Lie algebra over $\mathbb{C}$ whose basis vectors (generators) $\mathcal{X}_i\,,\,\mathcal{P}_i\,,\,i = 1,\ldots,n$ and $\mathcal{C}$ satisfy the following commutation relations

$$[\mathcal{X}_i\,,\,\mathcal{P}_j] = \mathcal{C}\delta_{ij}\,,\quad [\mathcal{C}\,,\,\mathcal{X}_i] = \mathcal{O}\,,\quad [\mathcal{C}\,,\,\mathcal{P}_i] = \mathcal{O}\,,\quad \forall\; i,j = 1,\ldots,n \tag{C.19}$$

The element $\mathcal{C}$ is central because it commutes with all elements of the Lie algebra.

(ii) Any associative algebra $\mathscr{A}$ is a Lie algebra with the commutator

$$[\mathcal{S},\mathcal{T}] \equiv \mathcal{S}*\mathcal{T} - \mathcal{T}*\mathcal{S}\,,\quad \forall\,\mathcal{S},\mathcal{T}\in\mathscr{A}$$

The structure constants are related as $c_{ij}^k = f_{ij}^k - f_{ji}^k$.

(iii) In particular, the vector space of $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{F})$ is the Lie algebra (of the group $GL(n, \mathbb{F})$) denoted by $\mathfrak{gl}(n, \mathbb{F})$. Any nondegenerate matrix $G \in GL(n, \mathbb{F})$ close enough to the identity matrix can be represented as $G = \exp(A)$, $A \in \mathfrak{gl}(n, \mathbb{F})$.

It has many Lie subalgebras which do not originate from associative algebras.

All matrix algebras below are Lie subalgebras of $\mathfrak{gl}(n, \mathbb{F})$ with the commutator given by the usual matrix commutator

$$[A, B] \equiv AB - BA \tag{C.20}$$

(iv) The Lie algebra $\mathfrak{sl}(n)$ over $\mathbb{C}$ (or $\mathbb{R}$) is the vector space of $n \times n$ **traceless matrices**,

$\mathrm{tr} A = 0 \ \ \forall\, A \in \mathfrak{sl}(n)$. If $A \in \mathfrak{sl}(n)$ then $SL(n) \ni G = e^A$, $\det G = 1$.

(v) The Lie algebra $\mathfrak{so}(n)$ over $\mathbb{C}$ (or $\mathbb{R}$) is the vector space of $n \times n$ **anti-symmetric matrices**, $A^t = -A \ \ \forall\, A \in \mathfrak{so}(n)$. If $A \in so(n)$ then $SO(n) \ni G = e^A$ is special orthogonal that is $G^t G = I_n$, $\det G = 1$.

(vi) The Lie algebra $\mathfrak{u}(n)$ over $\mathbb{R}$ is the vector space of $n \times n$ **anti-hermitian matrices**,

$A^\dagger = -A \ \ \forall\, A \in \mathfrak{u}(n)$. If $A \in \mathfrak{u}(n)$ then $U(n) \ni G = e^A$ is unitary that is $G^\dagger G = I_n$.

(vii) The Lie algebra $\mathfrak{su}(n)$ over $\mathbb{R}$ is the vector space of $n \times n$ **traceless anti-hermitian matrices**, $A^\dagger = -A\,,\ \mathrm{tr} A = 0 \ \ \forall\, A \in \mathfrak{su}(n)$. If $A \in su(n)$ then $SU(n) \ni G = e^A$ is special unitary that is $G^\dagger G = I_n$, $\det G = 1$.

(viii) The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3, \mathbb{R})$ are isomorphic. Indeed, the Pauli matrices divided by $2\mathrm{i}$ form a basis of $\mathfrak{su}(2)$ and satisfy the commutation relations (**??**)

$$\left[\frac{\sigma^\alpha}{2\,\mathrm{i}}\,,\,\frac{\sigma^\beta}{2\,\mathrm{i}}\right] = \epsilon^{\alpha\beta\gamma}\frac{\sigma^\gamma}{2\,\mathrm{i}} \tag{C.21}$$

The matrices

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad T^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\,, \quad T^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{C.22}$$

form a basis of $\mathfrak{so}(3)$ and also satisfy the commutation relations (**??**)

$$\left[T^\alpha\,,\,T^\beta\right] = \epsilon^{\alpha\beta\gamma}T^\gamma \tag{C.23}$$

The one-to-one map $\sigma^\alpha/2\mathrm{i} \leftrightarrow T^\alpha$ provides the isomorphism.

Ex 6. A universal enveloping algebra of a Lie algebra $\mathcal{G}$ over $\mathbb{F}$ with basis elements $\mathcal{E}_i$ satisfying the commutation relations

$$[\mathcal{E}_i\,,\,\mathcal{E}_j] = \sum_{k=1}^{\dim\mathcal{G}} c_{ij}^k\,\mathcal{E}_k \tag{C.24}$$

is a unital associative algebra $\mathcal{U}(\mathcal{G})$ over $\mathbb{F}$ generated by elements $\mathcal{I}$, $\mathcal{E}_i$, $i = 1, \ldots, \dim\mathcal{G}$ that satisfy the relations (**??**).

It means that the vectors of $\mathfrak{H}$ are linear combinations of the identity element $\mathcal{I}$ and all words made of $\mathcal{E}_i$'s

$$\mathcal{I}\,,\;\; \mathcal{E}_i\,,\;\; \mathcal{E}_i\mathcal{E}_j\,,\;\; \mathcal{E}_i\mathcal{E}_j\mathcal{E}_k\,,\;\ldots\;,\;\mathcal{E}_{i_1}\cdots\mathcal{E}_{i_n}\,,\;\ldots \tag{C.25}$$

The multiplication is defined in a natural way by "gluing" words

$$(\mathcal{E}_{i_1}\cdots\mathcal{E}_{i_k}) * (\mathcal{E}_{j_1}\cdots\mathcal{E}_{j_n}) = \mathcal{E}_{i_1}\cdots\mathcal{E}_{i_k}\mathcal{E}_{j_1}\cdots\mathcal{E}_{j_n} \tag{C.26}$$

and the Lie algebra commutation relations (**??**) are taken into account by identifying vectors $\mathcal{T}_1$ and $\mathcal{T}_2$ if

$$\mathcal{T}_2 - \mathcal{T}_1 = \sum_{a,ij} \mathcal{V}_a^{ij} * \Big(\; [\mathcal{E}_i\,,\,\mathcal{E}_j] - \sum_{k=1}^{\dim\mathcal{G}} c_{ij}^k\,\mathcal{E}_k \Big) * \mathcal{W}_a^{ij} \tag{C.27}$$

for some elements $\mathcal{V}_a^{ij}, \mathcal{W}_a^{ij}$ of $\mathcal{U}(\mathcal{G})$.

Ex 7. A unital associative algebra $\mathscr{A}$ over $\mathbb{F}$ can be generated by elements $\mathcal{I}\,,\,\mathcal{Z}_1\,,\,\mathcal{Z}_2\,,\ldots\,,\mathcal{Z}_N$ that satisfy the defining relations

$$\mathcal{F}_\alpha(\mathcal{Z}_1,\ldots,\mathcal{Z}_N) = \mathcal{O}\,, \quad \alpha = 1,\ldots,M \tag{C.28}$$

where $\mathcal{O}$ is the zero vector of $\mathscr{A}$.

It means that the vectors of $\mathscr{A}$ are linear combinations of the identity element $\mathcal{I}$ and all words made of $\mathcal{Z}_i$'s

$$\mathcal{I}\,,\;\; \mathcal{Z}_i\,,\;\; \mathcal{Z}_i\mathcal{Z}_j\,,\;\; \mathcal{Z}_i\mathcal{Z}_j\mathcal{Z}_k\,,\;\ldots\;,\;\mathcal{Z}_{i_1}\cdots\mathcal{Z}_{i_n}\,,\;\ldots \tag{C.29}$$

The multiplication is defined in a natural way by "gluing" words

$$(\mathcal{Z}_{i_1}\cdots\mathcal{Z}_{i_k}) * (\mathcal{Z}_{j_1}\cdots\mathcal{Z}_{j_n}) = \mathcal{Z}_{i_1}\cdots\mathcal{Z}_{i_k}\mathcal{Z}_{j_1}\cdots\mathcal{Z}_{j_n} \tag{C.30}$$

and the relations (C.28) are taken into account by identifying vectors $\mathcal{T}_1$ and $\mathcal{T}_2$ if

$$\mathcal{T}_2 - \mathcal{T}_1 = \sum_{a,\beta} \mathcal{V}_a^{\beta} * \mathcal{F}_\beta * \mathcal{W}_a^{\beta} \tag{C.31}$$

for some elements $\mathcal{V}_a^{\beta}, \mathcal{W}_a^{\beta}$ of $\mathfrak{H}$.

If there are **no** relations $\mathcal{F}_\alpha$ then $\mathscr{A}$ is called a **free** algebra.

Represenions

Def. A representation of a unital associative algebra $\mathscr{A}$ (also called a left $\mathscr{A}$-module) denoted by $(\rho, \mathscr{V})$ is a vector space $\mathscr{V}$ together with a homomorphism of algebras $\rho:\;\mathscr{A}\mapsto \mathrm{End}(\mathscr{V})$, i.e., a linear map preserving the multiplication and unit

$$\rho:\;\mathscr{A}\mapsto \mathrm{End}(\mathscr{V})\,, \quad \mathscr{A}\ni\mathcal{T}\mapsto\rho(\mathcal{T})\in\mathrm{End}(\mathscr{V})\,, \quad \rho(\mathcal{S}*\mathcal{T}) = \rho(\mathcal{S})\rho(\mathcal{T}) \tag{C.32}$$

Clearly, $\rho(\mathscr{A})$ is a subalgebra of the algebra of operators acting in $\mathscr{V}$.

In what follows by an algebra $\mathscr{A}$ we will mean a unital associative algebra.

Examples.

(i) $\mathscr{V} = \mathcal{O}$

(ii) $\mathscr{V} = \mathscr{A}$ and $\rho : \mathscr{A} \mapsto \mathrm{End}(\mathscr{A})$ is defined as follows: $\rho(\mathcal{T})$ is the operator of left multiplication by $\mathcal{T}$, so that $\rho(\mathcal{T})\mathcal{V} = \mathcal{T} * \mathcal{V}$. This representation is called the regular representation of $\mathscr{A}$.

(iii) $\mathscr{A} = \mathfrak{H}$ is the Heisenberg algebra, $\mathscr{V} = L^2(\mathbb{R})$ and $\rho : \mathfrak{H} \mapsto \mathrm{End}(L^2(\mathbb{R}))$ is defined as follows

$$\rho(\mathcal{X})\psi(x) = x\,\psi(x)\,, \quad \rho(\mathcal{P})\psi(x) = -\mathrm{i}\,\hbar\,\frac{d}{dx}\psi(x)\,, \quad \psi(x) \in L^2(\mathbb{R}) \tag{C.33}$$

Then, $\rho(\mathcal{T})$ is found from the requirement that ρ is an algebra homomorphism, e.g.

$$\rho(\mathcal{X}\mathcal{P}) = \rho(\mathcal{X})\rho(\mathcal{P})\,, \quad \rho(\mathcal{X}\mathcal{P})\psi(x) = -\mathrm{i}\,\hbar\,x\,\frac{d}{dx}\psi(x) \tag{C.34}$$

(iv) $\mathscr{A} = \mathcal{U}(\mathfrak{su}(2))$ is the universal enveloping algebra of $\mathfrak{su}(2)$ generated by $\mathcal{I}$, $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$ satisfying the commutation relations $\left[\mathcal{J}^\alpha\,,\,\mathcal{J}^\beta\right] = \epsilon^{\alpha\beta\gamma}\mathcal{J}^\gamma\,,$
$\mathscr{V} = \mathbb{C}$ and $\rho : \mathcal{U}(\mathfrak{su}(2)) \mapsto \mathrm{End}(\mathbb{C})$ is defined as follows $\rho(\mathcal{J}^\alpha) = 0$, $\rho(\mathcal{I}) = 1$.
It is the spin-0 representation of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$.

(v) $\mathscr{A} = \mathcal{U}(\mathfrak{su}(2))$ is the universal enveloping algebra of $\mathfrak{su}(2)$, $\mathscr{V} = \mathbb{C}^2$ and $\rho : \mathcal{U}(\mathfrak{su}(2)) \mapsto \mathrm{End}(\mathbb{C}^2)$ is defined as follows $\rho(\mathcal{J}^\alpha) = \sigma^\alpha/2\mathrm{i}$.
It is the spin 1/2 representation of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$.

(vi) $\mathscr{A} = \mathcal{U}(\mathfrak{su}(2))$ is the universal enveloping algebra of $\mathfrak{su}(2)$, $\mathscr{V} = \mathbb{C}^3$ and $\sigma : \mathcal{U}(\mathfrak{su}(2)) \mapsto \mathrm{End}(\mathbb{C}^3)$ is defined by $\sigma(\mathcal{J}^\alpha) = T^\alpha$ where T^α are given by (C.22).
It is the spin 1 representation of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$.

Def. A subrepresentation of a representation $\mathscr{V}$ of an algebra $\mathscr{A}$ is a subspace $\mathscr{W} \subset \mathscr{V}$ which is invariant under all the operators $\rho(\mathcal{T}) : \mathscr{V} \mapsto \mathscr{V}$, $\mathcal{T} \in \mathscr{A}$, that is $\rho(\mathcal{T})\mathscr{W} \subset \mathscr{W}\ \forall\,\mathcal{T}$

For example, $\mathcal{O}$ and $\mathscr{V}$ are always subrepresentations.

Def. A representation $\mathscr{V} \neq \mathcal{O}$ of $\mathscr{A}$ is irreducible if the only subrepresentations of $\mathscr{V}$ are $\mathcal{O}$ and $\mathscr{V}$.

For example, the representation of $\mathfrak{H}$, and $\mathcal{U}(\mathfrak{su}(2))$ discussed above are irreducible.

Def. The direct sum of two nonzero representations $(\rho, \mathscr{V})$ and $(\sigma, \mathscr{W})$ of an algebra $\mathscr{A}$ is the representation $(\rho \oplus \sigma, \mathscr{V} \oplus \mathscr{W})$ defined by

$$(\rho \oplus \sigma)(\mathcal{T}) = \rho(\mathcal{T}) \oplus \sigma(\mathcal{T}) \tag{C.35}$$

For example, the direct sum of the spin 1/2 and spin 1 representations of $\mathcal{U}(\mathfrak{su}(2))$ is given by

$$\begin{gathered}
(\rho \oplus \sigma)(\mathcal{J}^1) = \begin{pmatrix} 0 & 1/2\mathrm{i} & 0 & 0 & 0 \\ 1/2\mathrm{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^1/2\mathrm{i} & O_{23} \\ O_{32} & T^1 \end{pmatrix}, \\
(\rho \oplus \sigma)(\mathcal{J}^2) = \begin{pmatrix} \sigma^2/2\mathrm{i} & O_{23} \\ O_{32} & T^2 \end{pmatrix}, \quad (\rho \oplus \sigma)(\mathcal{J}^3) = \begin{pmatrix} \sigma^3/2\mathrm{i} & O_{23} \\ O_{32} & T^3 \end{pmatrix}.
\end{gathered} \tag{C.36}$$

where O_{mn} is the $m \times n$ zero matrix.

Def. The tensor product of two nonzero representations $(\rho, \mathscr{V})$ and $(\sigma, \mathscr{W})$ of the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ of a Lie algebra $\mathcal{G}$, $[\mathcal{E}_i\,,\,\mathcal{E}_j] = \sum_k c_{ij}^k\,\mathcal{E}_k$, is the representation $(\rho \otimes \sigma, \mathscr{V} \otimes \mathscr{W})$ defined on the generators $\mathcal{E}_i$ by

$$(\rho \otimes \sigma)(\mathcal{E}_i) = \rho(\mathcal{E}_i) \otimes I_{\mathscr{W}} + I_{\mathscr{V}} \otimes \sigma(\mathcal{E}_i) \qquad \forall\, i = 1, \dots, \dim \mathcal{G} \tag{C.37}$$

where $I_{\mathscr{V}} = \rho(\mathcal{I})$ and $I_{\mathscr{W}} = \sigma(\mathcal{I})$ are the identity operators in $\mathscr{V}$ and $\mathscr{W}$, respectively.

For example, the tensor product of two spin 1/2 representations of $\mathcal{U}(\mathfrak{su}(2))$ is given by

$$\begin{aligned}(\rho \otimes \rho)(\mathcal{J}^1) &= \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2\mathrm{i}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}\end{aligned} \tag{C.38}$$

and so on. The resulting four-dimensional representation is not irreducible and can be written as the direct sum of spin-0 and spin 1 representations.

Def. Two representations $(\rho_1, \mathscr{V}_1)$ and $(\rho_2, \mathscr{V}_2)$ of $\mathscr{A}$ are called equivalent, or isomorphic, if there is an invertible linear operator $\Phi : \mathscr{V}_1 \mapsto \mathscr{V}_2$ which commutes with the action of $\mathscr{A}$, i.e.,

$$\Phi\,\rho_1(\mathcal{T}) = \rho_2(\mathcal{T})\Phi \quad \Leftrightarrow \quad \rho_2(\mathcal{T}) = \Phi\,\rho_1(\mathcal{T})\Phi^{-1} \tag{C.39}$$

For example the standard spin 1/2 representation of $\mathfrak{su}(2)$: $\rho_1(\mathcal{J}^\alpha) = \frac{\sigma^\alpha}{2\mathrm{i}}$ is equivalent to the following one

$$\rho_2(\mathcal{J}^1) = \frac{\sigma^2}{2\mathrm{i}}\,, \quad \rho_2(\mathcal{J}^2) = \frac{\sigma^3}{2\mathrm{i}}\,, \quad \rho_2(\mathcal{J}^3) = \frac{\sigma^1}{2\mathrm{i}} \tag{C.40}$$

with the matrix Φ equal to

$$\Phi = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} & \frac{1}{2} + \frac{i}{2} \\ -\frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \end{pmatrix} \tag{C.41}$$

Schur's Lemma. Any two irreducible representations of $\mathscr{A}$ of the same dimension are equivalent.

If $\mathscr{V}_1 = \mathscr{V}_2 = \mathscr{V}$, and $\mathscr{V}$ is an inner product vector space, and Φ is a unitary operator, then the two representations $(\rho_1, \mathscr{V})$ and $(\rho_2, \mathscr{V})$ of $\mathscr{A}$ are called unitarily equivalent.