MAU22101: Solutions Week 1

Problem 1 Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z} \setminus \{0\}\}$. Show that G is a group under multiplication.

Problem 2 Find the order of each element in $\mathbb{Z}/12\mathbb{Z}$.

Problem 3 Let G be a group and $u, v \in G$ elements. Show that the elements uv and vu have the same order, i.e. $|uv| = |vu|$.

Problem 4 Prove that if $x^2 = e$ for all $x \in G$, then G is abelian.

Problem 5 Let (G, m) be a tuple consisting of a set G together with a map $m\colon G \times G \to G$ written as $m(a, b) = a \cdot b$ satisfying

i) The map m is *associative*, i.e. $a \cdot (b \cdot c)) = (a \cdot b) \cdot c$ for all $a, b, c \in G$.

ii) G has a *left-unit*, i.e. there exists $e \in G$ such that $e \cdot a = a$ for all $a \in G$.

iii) There are *left-inverses*, i.e. for every $a \in G$ there exists a $b \in G$ such that $b \cdot a = e$.

Show that in that case G is a group as follows.

1. Show that left-cancellation holds in G, i.e. $a \cdot u = a \cdot v \implies u = v$.
2. Show that if b is a left-inverse then it is also a right-inverse to a. (Hint: Show that $(a \cdot b)^2 = a \cdot b$ and be inspired by the first part.).
3. Show that e is also a right-unit. (Hint: Compute $a \cdot e \cdot a$.)
4. Conclude that G is a group.

Problem 6 Show that $\langle a, b \mid a^2 = b^2 = (ab)^n = e \rangle$ gives a presentation for D_{2n} in terms of the two generators $a = s$ and $b = rs$. (Show that the relations for r and s follow from the relations for a and b and vice versa.)