MAU22101: Exercises Week 3

Problem 1 Let $G \times X \to X$ be a group action and let $s \in X$. Show that the stabilizer of the element s,

$$G_s := \{g \in G \mid g.s = s\}$$

is a subgroup of G.

Problem 2 Let $\phi\colon G \to H$ be a group homomorphism. Show that the two subsets

- $\ker(\phi) := \{g \in G \mid \phi(g) = e\} \subset G$
- $\mathrm{im}(\phi) := \{\phi(g) \in H \mid g \in G\} \subset H$

are subgroups of G and H, respectively.

Problem 3 Let G be a group of order $|G| = n > 2$. Show that G cannot have a subgroup H of order $|H| = n - 1$.

Problem 4

- Prove that if H and K are subgroups of G, then so is their intersection $H \cap K$.
- Prove that the intersection of an arbitrary nonempty collection of subgroups of G is again a subgroup of G (do not assume that the collection is countable).

Problem 5 Let $m, n \in \mathbb{Z}^{>0}$ be positive integers. It follows from the classification of subgroups of $\mathbb{Z}$ that $m\mathbb{Z} \cap n\mathbb{Z} = k\mathbb{Z}$ for some positive integer k. Convince yourself that k is the least common multiple of m and n and show that

$$k = \frac{mn}{\gcd(m,n)}.$$

(Hint: Write $\gcd(m,n) = am + bn$ for some integers a and b.)

Problem 6 Let A be an abelian group. Prove that the set $H := \{a \in A \mid a^2 = e\}$ is a subgroup. Find an example of a non-abelian group where this fails.