Group Theory Assignment 7

~ Problem 1 Let G be ~~a~~ group of order 330. Show that G has a normal subgroup $H \lhd G$ of order $|H| = 11$

Recall the Sylow theorems:

1) For every prime factor p with multiplicity n of the order of a finite group G, there exists a Sylow p-subgroup of G, of order p^n.
2) Given a finite group G and a prime number p, all Sylow p-groups of G are conjugate to each other. That is, if H and K are Sylow p-subgroups of G, then there exists an element $g \in G$ with $g^{-1}Hg = K$
3) Let p be a prime factor ~~with~~ with multiplicity n of the order of a finite group G, so that the order of G can be written as $p^n m$, where $n > 0$ and p does not divide m. Let n_p be the number of Sylow p-subgroups of G. Then the following hold:
   - n_p divides m, which is the index of the Sylow p-subgroup in G
   - $n_p \equiv 1 \mod p$
   - $n_p = |G : N_G(P)|$, where P is any Sylow p-subgroup of G and N_G denotes the normalizer.

So, the prime factorization of $330 = \overbrace{2 \times 3 \times 5}^{m=30} \times 11$. Hence, the number n_{11} of Sylow 11-subgroups of G divides $\frac{330}{11} = 30$. Furthermore, $n_{11} \equiv 1 \mod 11 \Leftrightarrow n_{11} = 11k + 1$ for some $k \in \mathbb{N}^{<0}$. Therefore, $k = 0$ and $n_{11} = 1$. As a consequence of Theorem 2, since $n_{11} = 1$, the unique Sylow 11-subgroup H is conjugate to itself: $g \in G$ with $g^{-1}Hg = H$. Hence, $H \trianglelefteq G$.

~ Problem 2 Exhibit all Sylow 3-subgroups of S_4

We have that $|S_4| = 4! = 24 = 3 \times 2^3$. The number n_3 of Sylow 3-subgroups therefore divides $2^3 = 8$ and $n_3 \equiv 1 \mod 3 \Leftrightarrow 3k+1$ for some $k \in \mathbb{N}^{\leq 0}$
Hence, $k = 0$ or $1 \Rightarrow n_3 = 1$ or 4.
But, every 3-cycle in S_4 generates subgroups of order 3, which are not equal if these cycles are disjoint (eg: $\langle(123)\rangle \neq \langle(124)\rangle$. Hence, $n_3 = 4$. Moreover, there are $\frac{4 \times 3 \times 2}{3} = 8$ distinct 3-cycles in S_4.
Hence, each Sylow 3-subgroup contains $8/4 = 2$ distinct 3-cycles.
If we denote the identity of S_4 by e, then the Sylow 3-subgroups are

$$\langle(123)\rangle = \{e, (123), (132)\}$$
$$\langle(124)\rangle = \{e, (124), (142)\}$$
$$\langle(134)\rangle = \{e, (134), (143)\}$$
$$\langle(234)\rangle = \{e, (234), (243)\}$$

~ Problem 3 Find the number of Sylow 2- and Sylow p-subgroups of D_{2p}

We have $|D_{2p}| = 2p$. Therefore, $n_2 \mid p$ and it follows that either $n_2 = 1$ or $n_2 = p$. Consider the usual presentation of D_{2p}:

$$D_{2p} = \langle r, s \mid r^p = s^2 = 1, rs = sr^{-1} \rangle$$

If $x \in D_{2p}$, then $x = s^a r^b$ for some $a \in \{0,1\}$ and $b \in \{0, 1, \ldots, p-1\}$.
If $a = 0$, then $|x| = p$, since $\langle r \rangle$ is of prime order p. If $a = 1$, then $x \neq 1$ and

$$x^2 = sr^b sr^b = s^2 r^{-b} r^b = 1$$

$\Rightarrow |x| = 2$. Therefore, the elements of order 2 are the elements of the form sr^b for $b \in \{0, 1, 2, \ldots, p-1\}$. Hence, $n_2 = p$.

The number of Sylow p-subgroups divides 2 and hence is either 1 or 2. But $\langle r \rangle$ is a Sylow p-subgroup. Let $x = s^a r^b$ be any element of D_{2p} and let r^c be any element of $\langle r \rangle$. Then,

$$x r^c x^{-1} = s^a r^b r^c r^{-b} s^{-a} = s^a r^c s^{-a}$$

If $a = 0$, then $x r^c x^{-1} \in \langle r \rangle$.

If $a = 1$, then $x r^c x^{-1} = s r^c s^{-1} = s r^c s = s^2 r^{-c} = r^{-c} \in \langle r \rangle$

Hence, $\langle r \rangle \triangleleft D_{2p}$ ie: $n_p = 1$.