**Module MAU34403 Quantum mechanics I (Frolov)**
**Homework Sheet 1**
Each set of homework questions is worth 100 marks

Use Mathematica if necessary

**Def.** A group is a nonempty set $G$ on which there is defined a binary operation $(a, b) \mapsto ab$, called multiplication, satisfying the following properties

(i) Closure: If $a$ and $b$ belong to $G$, then $ab$ is also in $G$.

(ii) Associativity : $a(bc) = (ab)c$ for all $a, b, c \in G$.

(iii) Identity: There is an element $e \in G$ such that $ae = ea = a$ for all a in $G$.

(iv) Inverse: If $a \in G$, then there is an element $a^{-1} \in G$: $aa^{-1} = a^{-1}a = 1$.

**Def.** A Lie algebra is a vector space $\mathcal{G}$ over $\mathbb{C}$ (or $\mathbb{R}$) with a bilinear operation $[\cdot, \cdot]$: $\mathcal{G} \times \mathcal{G} \mapsto \mathcal{G}$ which is called a commutator or a Lie bracket, such that the following axioms are satisfied:

(i) It is skew symmetric: $[\mathcal{J}, \mathcal{J}] = \mathcal{O}$ which implies $[\mathcal{J}, \mathcal{K}] = -[\mathcal{K}, \mathcal{J}]$ for all $\mathcal{J}, \mathcal{K} \in \mathcal{G}$

(ii) It satisfies the Jacobi Identity: $[\mathcal{J}, [\mathcal{K}, \mathcal{L}]] + [\mathcal{K}, [\mathcal{L}, \mathcal{J}]] + [\mathcal{L}, [\mathcal{J}, \mathcal{K}]] = \mathcal{O}$
where $\mathcal{O}$ is the zero vector of $\mathcal{G}$.

Given a basis $\mathcal{E}_i$, $i = 1, \ldots, \dim \mathcal{G}$ of $\mathcal{G}$ its Lie algebra structure is determined by commutators of the basis vectors

$$[\mathcal{E}_i\,,\,\mathcal{E}_j] = \sum_{k=1}^{\dim \mathcal{G}} c_{ij}^k\,\mathcal{E}_k \tag{0.1}$$

Here $c_{ij}^k \in \mathbb{C}$ (or $c_{ij}^k \in \mathbb{R}$ if $\mathcal{G}$ over $\mathbb{R}$) are called the structure constants of the Lie algebra $\mathcal{G}$.

## Compulsory Questions

**Problem 1.** Matrices. Find the structure constants of the algebra Mat$(n, \mathbb{C})$ with respect to the basis matrices $E_{ab}$, i.e. compute

$$E_{ab}E_{cd} = \sum_{ij} f_{ab,cd}^{ij}\,E_{ij} \tag{0.2}$$

A. Express $f_{ab,cd}^{ij}$ as a product of the Kronecker deltas.

B. List all nonzero structure constants $f_{ab,cd}^{ij}$ of the algebra Mat$(2, \mathbb{C})$.

**Problem 2.** Groups. Prove that the set of all unitary $n \times n$ matrices with unit determinant forms a group. It is denoted by $SU(n)$.

**Problem 3.** Lie algebras. Prove that the space of all traceless anti-hermitian $n \times n$ matrices is a Lie algebra over $\mathbb{R}$ with the Lie bracket given by the matrix commutator. It is denoted by $\mathfrak{su}(n)$. What is the dimension of $\mathfrak{su}(n)$?

**Problem 4.** Bras and kets

A. Consider an $n \times n$ matrix $S$ which acts on the canonical basis vectors as follows

$$S|1\rangle = |2\rangle\,,\ S|2\rangle = |3\rangle\,,\ \cdots\ S|n-1\rangle = |n\rangle\,,\ S|n\rangle = |1\rangle \tag{0.3}$$

(i) Use bras and kets to find the matrix.
Write it in the form $S = \sum_{i,j=1}^{n} S_{ij}|i\rangle\langle j|$ and as a table for $n = 4$.

(ii) Check explicitly that it is unitary for any $n$ by using bras and kets.

B. Consider an $n \times n$ diagonal matrix $Q = \mathrm{diag}\,(1, q, q^2, \ldots, q^{n-1})$.

(i) If $Q$ is unitary which condition does $q$ satisfy?

(ii) If $Q$ is traceless which condition does $q$ satisfy?

(iii) Can $Q$ be unitary and traceless?

(iv) Write it in the form $Q = \sum_{i,j=1}^{n} Q_{ij}|i\rangle\langle j|$.

C. Compute $SQ$ and $QS$ by using bras and kets.

D. Can $q$ be chosen so that $SQ = r\,QS$ where $r \in \mathbb{C}$, and if yes how is $r$ related to $q$?

## Practice Questions

**Problem 1.** Matrices

(a) Prove that $\Re\,\mathrm{tr}(AH) = 0$ if $A$ is anti-hermitian, and $H$ is hermitian.
Here $\Re\, z$ is the real part of $z$. What can you say about $A$ and $H$ if they both are real matrices?

(b) Prove that the rows and columns of a unitary matrix constitute orthonormal sets.

(c) Prove that the determinant of a unitary matrix is a complex number of modulus 1.

**Problem 2.** Groups. Prove that the set of all complex orthogonal $n \times n$ matrices with unit determinant forms a group. It is denoted by $SO(n, \mathbb{C})$.

**Problem 3.** Lie algebras

(a) The commutator of two square matrices $A, B$ of the same size is

$$[A\,,\ B] \equiv AB - BA \tag{0.4}$$

Prove that it satisfies Jacobi's identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = O \tag{0.5}$$

where $O$ denotes the zero matrix.

(b) Prove that the space of all complex anti-symmetric $n \times n$ matrices is a Lie algebra over $\mathbb{C}$ with the Lie bracket given by the matrix commutator. It is denoted by $\mathfrak{so}(n, \mathbb{C})$. What is the dimension of $\mathfrak{so}(n, \mathbb{C})$?

**Problem 4.** Bras and kets

A. Compute products $\sigma^\alpha \sigma^\beta$ of the Pauli matrices by using bras and kets.

B. Consider an $n \times n$ matrix $S$ which acts on the canonical basis vectors as follows

$$S|1\rangle = |2\rangle\,,\ S|2\rangle = |3\rangle\,,\ \cdots\ S|n-1\rangle = |n\rangle\,,\ S|n\rangle = |1\rangle \tag{0.6}$$

(i) Use bras and kets to find the matrix.
Write it in the form $S = \sum_{i,j=1}^{n} S_{ij} |i\rangle\langle j|$ and as a table for $n = 4$.

(ii) Check explicitly that it is unitary for any $n$ by using bras and kets.