Classical Field Theory

Based on the lectures of Andrei Parnachev
David Lawton

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Contents

1 Lecture: 1

Definition 1.1. Gauss' Law: for any vector field $\vec{E}$,

$$\oint \vec{E} \cdot \mathrm{d}\vec{S} = \iiint_V \nabla \cdot \vec{E} \mathrm{d}V \tag{1}$$

Definition 1.2. Dirac Delta Function $\delta(x)$

$$\delta(x) = \begin{cases} 0 & , x \neq 0 \\ 1 & , x = \infty \end{cases} \tag{2}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0) \tag{3}$$

1.1 Point-like Electric Charges

Consider charged point particle with charge q_i at position $\vec{x_i}$. This particle generates a field $\vec{E}$,

$$\vec{E} = \frac{q_i}{4\pi\epsilon_0} \frac{\vec{x} - \vec{x_i}}{|\vec{x} - \vec{x_i}|^3} \tag{4}$$

The force acting on another charge q_j at $\vec{x}$ is $\vec{F} = q_j \vec{E}$.
The electric field is **linear**

$$\vec{E} = \sum_i \vec{E_i} \tag{5}$$

Suppose continuous distribution of charge density $\rho(x)$. We can imagine a many infinitessimal volume elements $\mathrm{d}V$ at position $\vec{x_i}$ with charge $\mathrm{d}q_i \approx \rho(\vec{x_i})\mathrm{d}V$.

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_i \frac{\vec{x} - \vec{x_i}}{|\vec{x} - \vec{x_i}|^3} \mathrm{d}q_i \tag{6}$$

We then take the limit as $\mathrm{d}V$ becomes infinitessimal, turning the sum into an integral

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\vec{x} - \vec{x'}}{|\vec{x} - \vec{x'}|^3} \rho(\vec{x'}) \mathrm{d}^3 x' \tag{7}$$

Next we will show that the divergence of the electric field of a point charge is zero $\forall \vec{x} \neq \vec{x_i}$. First w.l.o.g. we let $\vec{x_i} = 0$. Then

$$\vec{E} = \frac{q_i}{4\pi\epsilon_0} \frac{\vec{x}}{|\vec{x}|^3}$$

$$\begin{aligned}
\nabla \cdot \vec{E} &= \partial_i E_i \\
&= \frac{q_i}{4\pi\epsilon_0} \left(\frac{\partial}{\partial x} \left[\frac{x}{\sqrt{x^2+y^2+z^2}^3} \right] + x \leftrightarrow y + x \leftrightarrow z \right) \\
&= \frac{q_i}{4\pi\epsilon_0} \left(\left[\frac{1}{\sqrt{x^2+y^2+z^2}^3} - 3\frac{x^2}{\sqrt{x^2+y^2+z^2}^5} \right] + x \leftrightarrow y + x \leftrightarrow z \right) \\
&= \frac{q_i}{4\pi\epsilon_0} \left(\frac{3}{|\vec{x}^3|} - 3\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^5} \right) \\
&= 0 \quad \forall \vec{x} \neq 0 \text{ Undefined for } \vec{x} = 0
\end{aligned}$$

Now construct some surface $\mathfrak{S}$ with some charge $\mathrm{q}_i notcontainedwithinit.Then,$