Module MAU34403 Quantum mechanics I (Frolov)
Homework Sheet 6
Each set of homework questions is worth 100 marks

Use Mathematica if necessary

Compulsory Questions

Use the results from practice questions

Problem 1.

Let the wave function be written in the "polar" coordinates form

$$\psi(\vec{x},t) = \sqrt{\rho(\vec{x},t)}\, \exp\Big(\frac{\mathrm{i}}{\hbar} S(\vec{x},t)\Big) \tag{0.1}$$

where S is real and $\rho > 0$. Show that the Schrödinger equation splits into the following two real equations

$$\frac{\partial \log\sqrt{\rho}}{\partial t} = -\frac{1}{2m}\Big(\nabla^2(S) + 2\vec{\nabla}(\log\sqrt{\rho})\cdot\vec{\nabla}(S)\Big) \tag{0.2}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\vec{\nabla}(S)\cdot\vec{\nabla}(S) + V - \frac{\hbar^2}{2m}\,\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}} = 0 \tag{0.3}$$

and that the first equation for ρ is equivalent to the continuity equation for the probability density and current.

Problem 2. Prove the formula

$$-e^{\hat{A}}\frac{d}{dt}e^{-\hat{A}} = \frac{d\hat{A}}{dt} + \sum_{n=1}^{\infty}\frac{1}{(n+1)!}\,\underbrace{\Big[\hat{A},\Big[\hat{A},\cdots\Big[}_{n}\hat{A}\,,\,\frac{d\hat{A}}{dt}\Big]\cdots\Big]\Big] \tag{0.4}$$

where $\hat{A}(t)$ is any operator.

Hint. Consider $\hat{F}(\alpha) = -e^{\alpha\hat{A}}\frac{d}{dt}e^{-\alpha\hat{A}}$ and expand it in Taylor series in α.

Problem 3. Prove the formulae below

1. The orbital angular momentum operator is $\vec{L} = \vec{X}\times\vec{P}$. Show that

$$[L^{\vec{m}}, L^{\vec{n}}] = \mathrm{i}\hbar\, L^{\vec{m}\times\vec{n}}\,, \quad [L^{\vec{n}}, \vec{X}] = \mathrm{i}\hbar\, \vec{X}\times\vec{n}\,, \quad [L^{\vec{n}}, \vec{P}] = \mathrm{i}\hbar\, \vec{P}\times\vec{n} \tag{0.5}$$

where

$$L^{\vec{n}} \equiv \vec{n}\cdot\vec{L} = n^x L^x + n^y L^y + n^z L^z\,, \quad \vec{n}^{\,2} = 1 \tag{0.6}$$

2. The rotation operator through an angle ϑ around the direction of the unit vector $\vec{n}$ is

$$R(\vec{\vartheta}) = \exp\Big(-\mathrm{i}\,\vec{\vartheta}\cdot\vec{L}/\hbar\Big) = \exp\Big(-\mathrm{i}\,\vartheta\, L^{\vec{n}}/\hbar\Big)\,, \quad \vec{\vartheta} = \vartheta\,\vec{n}\,, \quad \vec{n}^{\,2} = 1 \tag{0.7}$$

Prove that the rotation operator transforms $\vec{X}$ as follows

$$\vec{X}(\vec{\vartheta}) = R^{\dagger}(\vec{\vartheta})\,\vec{X}\,R(\vec{\vartheta}) = \vec{X}^{\parallel} + \vec{X}^{\perp}\cos\vartheta - \vec{X}^{\perp}\times\vec{n}\,\sin\vartheta \tag{0.8}$$

where $\vec{X}^{\parallel}$ and $\vec{X}^{\perp}$ are the components of the vector $\vec{X}$ parallel and orthogonal to the unit vector $\vec{n}$

$$\vec{X} = \vec{X}^{\parallel} + \vec{X}^{\perp}\,, \quad \vec{X}^{\parallel} = (\vec{X}\cdot\vec{n})\,\vec{n}\,, \quad \vec{X}^{\perp} = \vec{n}\times(\vec{X}\times\vec{n}) \tag{0.9}$$

Practice Questions

Problem 1.

Let the wave function be written in the "polar" coordinates form

$$\psi(\vec{x},t) = \sqrt{\rho(\vec{x},t)}\,\exp\Big(\frac{\mathrm{i}}{\hbar}S(\vec{x},t)\Big) \tag{0.10}$$

where S is real and $\rho > 0$. Show that the Schrödinger equation splits into the following two real equations

$$\frac{\partial \log\sqrt{\rho}}{\partial t} = -\frac{1}{2m}\Big(\nabla^2(S) + 2\vec{\nabla}(\log\sqrt{\rho})\cdot\vec{\nabla}(S)\Big) \tag{0.11}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\vec{\nabla}(S)\cdot\vec{\nabla}(S) + V - \frac{\hbar^2}{2m}\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}} = 0 \tag{0.12}$$

Problem 2. Prove the formula

$$\epsilon^{\hat{A}}\,\hat{B}\,e^{-\hat{A}} = \hat{B} + \sum_{n=1}^{\infty}\frac{1}{n!}\,\underbrace{\Big[\hat{A},\Big[\hat{A},\cdots\Big[}_{n}\hat{A}\,,\,\hat{B}\Big]\cdots\Big]\Big] \tag{0.13}$$

where $\hat{A}$ and $\hat{B}$ are arbitrary operators.

Hint. Consider $\hat{F}(\alpha) = e^{\alpha\hat{A}}\,\hat{B}\,e^{-\alpha\hat{A}}$ and expand it in Taylor series in α.

Problem 3. Prove the formulae below

1. $\vec{X}$ and $\vec{P}$ are the coordinate and momentum operators

$$e^{\mathrm{i}\,\vec{a}\cdot\vec{P}/\hbar}\,\vec{X}\,e^{-\mathrm{i}\,\vec{a}\cdot\vec{P}/\hbar} = \vec{X} + \vec{a} \tag{0.14}$$

2. The orbital angular momentum operator is $\vec{L} = \vec{X}\times\vec{P}$. Show that

$$[L^{\alpha}, X^{\beta}] = \mathrm{i}\,\hbar\,\epsilon^{\alpha\beta\gamma}X^{\gamma}\,, \quad [L^{\alpha}, P^{\beta}] = \mathrm{i}\,\hbar\,\epsilon^{\alpha\beta\gamma}P^{\gamma}\,, \quad [L^{\alpha}, L^{\beta}] = \sum_{\gamma=1}^{3}\mathrm{i}\,\hbar\,\epsilon^{\alpha\beta\gamma}L^{\gamma}\,. \tag{0.15}$$