L19: Groups of order 30

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Claim Let G be a group of order 30, then G has a normal subgroup $H \trianglelefteq G$ of order 15.

Pf Want $H = P_3 P_5$ for P_3, P_5 Sylow 3,5-subgroups, but for this we need $P_3 \leq N_G(P_5)$ or vice versa. If either P_3 or P_5 is normal we are done.

$n_3 = 1$ or $n_5 = 1$

$n_5 \in \{1, 2, 3, 6\}$ $n_5 \equiv 1 \mod 5$ $n_5 \in \{1, 6\}$

$n_3 = \{1, 2, 5, 10\}$ $n_3 \equiv 1 \mod 3$ $n_3 \in \{1, 10\}$

Only remaining case: $n_5 = 6$, $n_3 = 10$.

Note that distinct subgroups of order 5 intersect only at e (Lagrange).

We count
- $6 \cdot (5-1)$ elements of order 5 $= 24$
- $10 \cdot (3-1)$ elements of order 3 $= 20$

$\overline{\overline{44}} > 30$ ↯

Note $H \leq G$ is of index 2, hence normal (was exercise: pf $G/H = \{H, G \setminus H\}$, $H \backslash G = \{H, G \setminus H\}$)

and $H = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ ($\cong \mathbb{Z}/15\mathbb{Z}$ see below)

As above we obtain

$$G \cong H \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$$

For $\varphi: \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}) \overset{\text{Exc}}{\cong} (\mathbb{Z}/3\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times$

$\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

Either φ is trivial ($\Rightarrow G$ abelian, $G \cong \mathbb{Z}/30\mathbb{Z}$)

or given by an elt of order 2 in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$: $(1,0) \rightsquigarrow G_{30}'$

$(0,2) \rightsquigarrow G_{30}''$

$(1,2) \rightsquigarrow D_{30}$

$\Rightarrow \exists$ at most 4 groups of order 30 up to isomorphism (exactly: exercise)