Definition 0.1. *Branch Point*

Any point on a Riemann Surface where at least 2 sheets meet is called a branch point.

Example 0.2.

0 is a branch point for any function $f(z) = z^n$ with $n \geqslant 2$. If we look at a curve γ going around a branch point in the domain, the image winds around the branch point of the Riemann surface as many times as there are sheets glued at that branch point.
The order of a branch point =# of sheets -1.

Remark 0.3.

1. A branch point does not have to connect all sheets of a Riemann surface
2. The order of a branch point is not necessarily finite.

Remark 0.4 ($f(z) = z^3$)**.**

1. The same construction works for any other cut in $\mathbb{C}$ that is a ray originating from 0, not just the positive real numbers.
2. The fundamental regions of the Riemann surface for $f(z) = z^3$ give the branches of $w^{\frac{1}{3}}$ (the inverse of $f(z) = z^3$) $\Longrightarrow$ we have 3 such branches. Generally for $f(z) = z^n$, there are n branches of $w^{\frac{1}{n}}$, w=0 branch point of order n-1. (n sheets). The Fundamental regions are wedges centred at zero with angle $\frac{2\pi}{n}$

Cauchy's Theorem

1.1 Statement of Cauchy's theorem

Practically the most important theorem in Complex Analysis.

Theorem 1.2. *Let f be a holomorphic function on a simply connected domain D and let γ be a piece-wise, C^1, simply-closed curve inside D. Then the integral along γ is 0.*

$$\int_\gamma f(z)dz = 0$$

"Cauchy Unfortunately looks like Vladimir Putin but we still love him"- Andreea Nicoara

Figure 1.1: Enter Caption

Break-down of the statement:

- Simply connected: A Topological domain with "no holes", "Topologically trivial"
- Holomorphic functions: Differentiable in $\mathbb{C}$
- Simply-Closed Curves: Given below
- Piecewise C^1: Given below
- $\int_\gamma f(z)dz$: Given below

1.3 Curves in $\mathbb{C}$

Curves in $\mathbb{C}$ are parametrically defined, i.e $\exists$ a continuous map $\gamma : [a, b] \to \mathbb{C},\ a, b \in \mathbb{R}$
$\vec{\gamma}(t) = (x(t), y(t)) \iff x(t) + iy(t) = \vec{\gamma}(t)$

Example 1.4.

1. $\gamma(t) = (cos(t), sin(t)),\ t \in [0, \pi]$

2. $\gamma(t) = e^{2it},\ t \in [0, \pi]$
3. $\gamma(t) = e^{it}; t \in [0, 4\pi]$
4. $\gamma(t) = 1,\ t \in [0, 2\pi]$

3 and 4 are not good for line integrals. What do we need to impose in order to fix the problem?

To fix 4, we want to look at the tangent vector. If $\vec{\gamma}(t) = (x(t), y(t))$, the the tangent vector $T_\gamma(t) = (x'(t), y'(t)) = x'(t) + iy'(t)$. We see that the tangent, we which can also denote by $\dot{\gamma}(t) = (x'(t), y'(t)) = (0, 0)$. We want to only consider piece-wise, continuously differentiable curves. These are curves $\gamma : [a, b] \to \mathbb{C}$ satisfying the following 2 properties:

1. $x(t), y(t)$ are of class C^1 (continuous first order derivatives on some partition $[a, x_1], [x_1, x_2], .., [x_n, b]$ of $[a, b]$. At these points $(x_1, .., x_n)$ for some finite n.
2. At these points $(x_1, .., x_n)$, the derivative may be 0 or undefined but that's okay as long as n is finite and the problems are resticted to these points.

Remark 1.5.

- Condition 1 above allows things like:

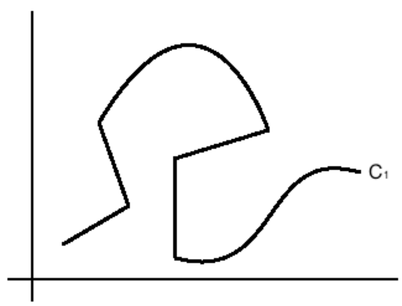

Figure 1.2: Piecewise differentiable curve

- Condition 2) allows examples like 4. Piecewise C^1 rules out things like the Peano Space-Filling Curve (everywhere C^0, but nowhere C^1, mega-nasty, it is generated by iteratively, below are the first 3 iterations)

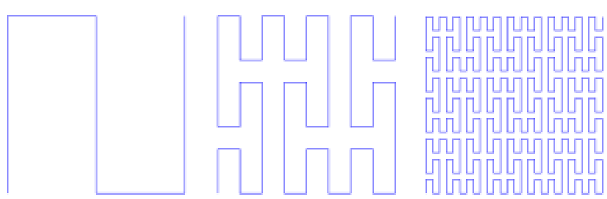

Figure 1.3: First 3 iterations of the Peano space-filling curve

Definition 1.6 (The Trace of a Curve).

If $\gamma : [a, b] \to \mathbb{C}$ is a curve, then the image of $[a, b]$ is called the trace of γ
Note that in the above Examples 1-3 above all have the unit circle as their trace.

Definition 1.7 (Endpoints, Closed Curves and Arcs)**.**

We call $\gamma(a)$ the initial point and $\gamma(b)$ is the terminal point. If they equal each other, γ is called a closed curve. If they don't, then γ is called an arc.

Definition 1.8 (Simple Curves)**.**

Let $\gamma[a, b] \to \mathbb{C}$ be a curve. If $\exists\ c, d \in [a, b]$ such that $\gamma(c) = \gamma(d)$ with $c \neq d$, we say γ has self-intersections. If a curve is not self-intersecting, it is simple.

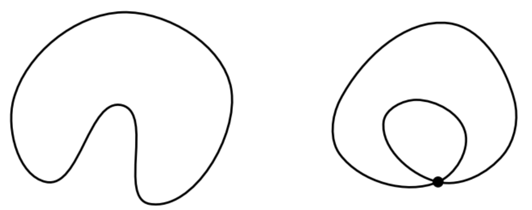

Figure 1.4: Simple closed curves vs non-simple closed curve

Remark 1.9.

Essentially, injectivity holds on simple curves.

Definition 1.10 (Orientation of a Curve)**.**

Curves have orientations, in other words $\gamma : [a, b] \to \mathbb{C}$, i.e it goes from $\gamma(a)$ to $\gamma(b)$ along its trace. Notice that $\alpha : [a, b] \to \mathbb{C}$ defined by $\alpha(t) = \gamma(a + b - t)$ goes from $\gamma(b)$ to $\gamma(a)$ (it reverses orientation).

Remark 1.11.

$T_\gamma(t) = -T_\alpha(t)$

We now wish to define what we mean by $\int_C f(z)dz$

Definition 1.12.

Let $z : [a, b] \to \mathbb{C}$ be a piece-wise C^1 curve γ and let f be a continuous everywhere on $z(t)$, then :

$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt$$

Example 1.13.

$f(z) = z^2$ with $\gamma(t) = e^{it},\ t \in [0, 2\pi]$.

$z(t) = cos(t) + isin(t),\ u(t) = cos(t), z; v(t) = sin(t) \implies \dot{u}(t) = -\sin(t),\ \dot{v}(t) = cos(t)$

$$\int_C f(z)dz = \int_0^{2\pi} (cos(t)+isin(t)(-sin(t)+icos(t)).dt = \int_0^{2\pi} (e^{it})^2 ie^{it}.dt = i\int_0^{2\pi} e^{3it}dt = 0$$

Note that the above is in line with what Cauchy's theorem states.

A natural question is "What if we considered $z(t) = e^{2it},\ t \in [0, \pi]$ instead of $t \in [0, 2\pi]$?" Is this integral well defined? i.e does it not care about which parametrisation of the curve we choose?

Definition 1.14 (Equivalent Parametrisations of Curves)**.**

2 curves $\gamma(t)\ t \in [a, b]$ and $\alpha(t)\ t \in [c, d]$ are equivalent if $\exists$ a C^1 bijection $\lambda(t)[c, d] \to [a, b]$ such that $\lambda(d) = b,\ \lambda(c) = a, a; \lambda'(t) \geqslant 0$ and $\gamma(\lambda(t)) = \alpha(t)$
Note that the postivity of the derivative is imposed to ensure that we're not including any changes of orientation. $z_1(t) = e^{2it}z\ t \in [0, \pi]$ is equivalent to $z_2(t) = e^{it},\ t \in [0, 2\pi]$ with $\lambda(t) = 2t$ is an example.

Proposition 1.15.

Equivalence of curves is an equivalence relation (relexive, symmetric and transitive). To prove it's reflexive, we simple use the identity map, for symmetry, we take the inverse of our bijection and to show transitivity, we simply compose the bijections. We write $\gamma \sim \alpha$

Proposition 1.16.

Let γ and α be 2 curves with the same trace and let f be a continuous function on this trace. If $\gamma \sim \alpha$ then:

$$\int_\gamma f = \int_\alpha f$$

i.e the line integral is well defined.

Proof: $\gamma \sim \alpha \implies \exists\, \lambda(t)$ such that $\gamma(\lambda(t)) = \alpha(t) \implies \alpha'(t) = \gamma'(t)\lambda'(t)$
$\alpha : [c, d] \to \mathbb{C},\ \gamma : [a, b] \to \mathbb{C}$.

$$\int_\alpha f = \int_c^d f(\alpha(t))\alpha'(t).dt = \int_c^d f(\gamma(\lambda(t))\gamma'(\lambda(t))\lambda'(t).dt = \int_a^b f(\gamma(\tau)\gamma'(\tau)d\tau = \int_\gamma f \quad \square$$

Proposition 1.17 (Properties of line integrals)**.**

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 curve and let $f, g \in C^1$

1. $\int_\gamma f + g = \int_\gamma f + \int_\gamma g$
2. $\int_\gamma cf = c\int_\gamma f$

3. If $-\gamma$ is γ with the opposite orientation, then $\int_{-\gamma} f = -\int_\gamma f$

Lemma 1.18.

Let $f : [a, b] \to \mathbb{C}$ be a continuous function, then:

$$\left|\int_a^b f(t).dt\right| \leqslant \int_a^b |f(t)|.dt$$

Proof: On blackboard

Corollary 1.19.

Let $\gamma; [a, b] \to \mathbb{C}$ be a piecewise C^1 curve of length L and let f be a continuous along γ with $|f(z)| \leqslant M$, then:

$$\left|\int_\gamma f\right| \leqslant L.M$$

Proof: With $\gamma(t) = (x(t), y(t))$

$$\begin{aligned}\left|\int_\gamma f\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\ &\leqslant M\int_a^b |\gamma'(t)|.dt = M\int_a^b \sqrt{(x'(t)^2 + (y'(t))^2}dt = ML \qquad \square\end{aligned}$$

We wish to look at

$$\int_\gamma f dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b f(\gamma(t))(x'(t)+iy'(t))dt = \int_a^b f(\gamma(t))\underbrace{x'(t)dt}_{dx} + \int_a^b f(\gamma(t))\underbrace{iy'(t)dt}_{dy}$$

More generally, write $\int_\gamma p dx + \int_\gamma q dy$ (Most general integral along a curve).

What is the best condition that this expression above must satisfy? The answer is that it can be independent of γ but dependent on only the endpoints $\gamma(a)$, $\gamma(b)$. The next question is what condition should p and q satisfy for this to be the case? The answer is the following theorem:

Theorem 1.20.

Let D be a domain and $\gamma \subset D$. The line intregral $\int_\gamma p.dx + \int_\gamma q.dy$ depends only on $\gamma(a)$ and $\gamma(b) \iff \exists\, U(x, y) : D \to \mathbb{C}$ such that $U_x = p$ and $U_y = q$

Proof:
" $\Longleftarrow$ " by the fundamental theorem of calculus:

$$\int_\gamma p.dx + \int_\gamma q.dy = \int_\gamma U_x.dx + \int_\gamma U_y.dy = \int_a^b U_x x'(t)dt + \int_a^b U_y y'(t).dt$$

$$= \int_a^b \frac{\partial}{\partial t}(U(x(t), y(t)).dt = U(x(b), y(b)) - U(x(a), y(a))$$

The conclusion is that the integral depends only on the endpoints.

" $\implies$ " Let $(x(a), y(a)) = (x_0, y_0)$ and $(x(b), y(b)) = (x, y)$. By our assumption, the integral only depends on these points. Hence we can replace γ with a polygonal line $\tilde{\gamma}$ whose components are either vertical or horizontal. Recall that open and connected in $\mathbb{C} \to$ polygonally connected. The rest of this theorem shall be proven after reading week.