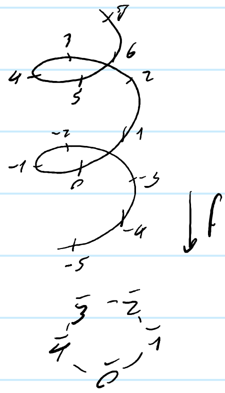
Normal subgroups



Ex $\mathbb{Z} \xrightarrow{f} \mathbb{Z}/5\mathbb{Z}$ is a group homomorphism
$\ell \mapsto \bar{\ell}$

- $\ker f = 5\mathbb{Z} \leq \mathbb{Z}$
- $f^{-1}(\{\bar{\ell}\}) = \{\ell + 5k \mid k \in \mathbb{Z}\} \subseteq \mathbb{Z}$
  $\qquad\qquad\qquad\qquad \overset{\parallel}{\ell + 5\mathbb{Z}}$

Def Let $H \leq G$ be a subgroup. Given $g \in G$ we call the set $gH := \{gh \mid h \in H\}$ a <u>left coset</u>
$Hg := \{hg \mid h \in H\}$ a <u>right coset</u>.
We define $G/H = \{gH \mid g \in G\}$
$H\backslash G = \{Hg \mid g \in G\}$

Notation $g \in G$ we write $[g] := gH \in G/H$

Lemma $xH = yH \Leftrightarrow x^{-1}y \in H$ $(\Leftrightarrow y^{-1}x \in H)$

Pf: "$\Rightarrow$" $xH = yH \Rightarrow y \cdot e \in yH = xH \Rightarrow \exists h \in H$ s.t. $y = x \cdot h$
$\Rightarrow x^{-1}y = h \in H$

"$\Leftarrow$": $xH \supseteq yH$: $yh = x\underbrace{x^{-1}y}_{\in H}h \in xH$

$\subseteq$: same

Q When is G/H a group? (s.t. $G \to G/H$, $x \mapsto [x]$ is a group hom.)

Need $[x] \cdot [y] := [xy]$. Is this well-defined?
From above $[x] = [xh_1]$, $[y] = [yh_2]$ for any $h_1, h_2 \in H$,
thus we require $[xy] = [x] \cdot [y] = [xh_1] \cdot [yh_2] = [xh_1yh_2]$
$\Rightarrow \underbrace{(xy)^{-1}xh_1yh_2}_{y^{-1}h_1yh_2} \in H$
$y^{-1}h_1yh_2 \in H \qquad \Rightarrow y^{-1}h_1y \in H \quad \forall h_1 \in H$

Def A subgroup $N \leq G$ is called <u>normal</u>, denoted $N \trianglelefteq G$, if
$g^{-1}hg \in H \quad \forall g \in G,\ h \in N$.

Thm Let $N \triangleleft G$ be a normal subgroup. Then G/H carries a group structure s.t. $G \to G/N$ is a group homomorphism.

Pf Exercise

Example S_3 : $H = \langle (1\,2) \rangle = \{e, (1\,2)\}$

$N = \langle (1\,2\,3) \rangle = \{e, (1\,2\,3), \overset{(1\,2\,3)^2}{\overset{\shortparallel}{(3\,2\,1)}}\}$

$H \leq S_3$ but $H \ntrianglelefteq S_3$

$N \trianglelefteq S_3 \qquad S_3/N = \{N, (1\,2)N\}$

$= \{\underbrace{\{e, (1\,2\,3), (3\,2\,1)\}}_{e_{S_3/N}}, \{(1\,2), (2\,3), (1\,3)\}\}$

$\overset{\text{isom.}}{\cong} \mathbb{Z}/2\mathbb{Z}.$

Rem $N \triangleleft G \overset{\text{Exercise}}{\iff} G/H = H\backslash G$

Thm (univ. property) Let $N \triangleleft G$. Suppose we are given a group homomorphism $\varphi : G \to K$ s.t. $\varphi(N) = \{e\}$. Then there exists a unique group homomorphism $\bar{\varphi} : G/N \to K$ s.t.
$\bar{\varphi}([x]) = \varphi(x)$.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & K \\ G/N \arrow[ur, "\bar{\varphi}"'] & \end{tikzcd}$$

Pf • $\bar{\varphi}$ is well-defined: $\varphi([xh]) = \varphi(xh) = \varphi(x)\varphi(h) = \varphi(x) \cdot e = \varphi(x) = \varphi([x])$
• $\bar{\varphi}$ unique: ✓
• $\bar{\varphi}$ group hom: ✓

Thm Let $\varphi : G \to H$ be a group homomorphism. Then $K = \ker \varphi$ is a normal subgroup. Moreover, the induced homomorphism $\bar{\varphi} : G/K \to H$ defines a group isomorphism

$$G/_{\ker \varphi} \cong \operatorname{im} \varphi.$$

Pf · K is normal: Let $k \in K$ and $g \in G$, then

$$\varphi(gkg^{-1}) = \varphi(g)\underbrace{\varphi(k)}_{=e}\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = e$$

Hence $gKg^{-1} \subseteq K$.

· $\bar{\varphi}$ is injective: $\bar{\varphi}([x]) = e$

$\Rightarrow \varphi(x) = e \Rightarrow x \in K$

$\Rightarrow [x] = e_{G/K}$ $\square$

Slogan "normal subgroups are kernels"

Pf "$\Leftarrow$": ✓

"$\Rightarrow$" $\ker(G \to G/N) = \{x \in G \mid \underset{xN = N}{[x] = e_{G/N}}\} = N.$

Prop $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$

Pf We define $\varphi: \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ by $\varphi(1) = \bar{1}$. It is surjective

$\Rightarrow \mathbb{Z}/\ker\varphi \overset{\bar{\varphi}}{\cong} \mathbb{Z}/n\mathbb{Z}$

It remains to determine $\ker\varphi = \{k \in \mathbb{Z} \mid \bar{k} = \bar{0}\}$

$= n\mathbb{Z}$ $\square$