Quantum Mechanics 1: Homework 7

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1 Problem 1

A coherent state of a one-dimensional harmonic oscillator is defined to be an eigendstate of the (non-Hermitian) annihilation operator a

$$a\left|\lambda\right\rangle = \alpha\left|\lambda\right\rangle, \quad \langle\lambda|\lambda\rangle = 1, \quad \alpha \in \mathbb{C}. \tag{1}$$

Problem 1.1. Find $|\lambda\rangle$.

Start by recognising that since a is the annihalation operator, the only non-zero eigenstate must be a infinite sum of the eigenstates $|n\rangle$ of the number operator $N = a^\dagger a$, $N\left|n\right\rangle = n\left|n\right\rangle$.

$$\lambda = \sum_{n=0}^{\infty} c_n \left|n\right\rangle, \quad c_n \in \mathbb{C}$$
$$\text{where, } c_n = \langle n|\lambda\rangle$$

We can then rewrite $|\lambda\rangle$ as

$$\left|\lambda\right\rangle = \sum_{n=0}^{\infty} \left|n\right\rangle \langle n|\lambda\rangle$$

but since

$$\left|n\right\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\left|0\right\rangle, \left\langle n\right| = \left\langle 0\right| \frac{a^n}{\sqrt{n!}} \tag{2}$$

we can once again rewrite $|\lambda\rangle$, using Eq. 3 as well,

$$\left|\lambda\right\rangle = \sum_{n=0}^{\infty} \left\langle 0\right| a^n \left|\lambda\right\rangle \left|n\right\rangle = \langle 0|\lambda\rangle \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left|n\right\rangle$$

Now, we simpluse the normalisation of $|\lambda\rangle$ to find $\langle 0|\lambda\rangle = c_0$.

$$\begin{aligned} 1 = \langle\lambda|\lambda\rangle &= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \\ &= |c_0|^2 e^{|\alpha|^2} \end{aligned}$$

$$\therefore \quad c_0 = \exp[-\frac{1}{2}|\alpha|^2]$$

We can then write $|\lambda\rangle$ as an expansion of vectors in fock space.

$$|\lambda\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{3}$$

Problem 1.2. Express $|\lambda\rangle$ in the form $|\lambda\rangle = f(a^\dagger)\,|0\rangle$.

For this, we once again use Eq. 2 to write $|\lambda\rangle$ in terms of $a^\dagger$ operators acting on the vacuum state $|0\rangle$.

$$\begin{aligned} |\lambda\rangle &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (a^\dagger)^n |0\rangle \end{aligned}$$

Therefore, using the exponential series expansion,

$$|\lambda\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle = \exp[\alpha(a^\dagger - \frac{\alpha^*}{2})]\,|0\rangle \tag{4}$$

which is the required form, with $f(a^\dagger) = \exp[\alpha(a^\dagger - \frac{\alpha^*}{2})]$.

Problem 1.3. Prove the minimum uncertainty relation for this state.

We must first recall that we can write the position and momentum operators in terms of annihalation and creation operators.

$$X = \eta(a^\dagger + a), \quad P = \frac{i\hbar}{2\eta}(a^\dagger - a) \tag{5}$$

The minimum uncertainty relation is given by

$$\Delta X \Delta P \geq \frac{\hbar}{2} \tag{6}$$

where ΔX and ΔP are the uncertainties in X, P respectively, defined by

$$\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}, \quad \Delta P = \sqrt{\langle P^2\rangle - \langle P\rangle^2} \tag{7}$$

where the uncertainity is with respect to the state $|\lambda\rangle$. That is to say $\langle X\rangle = \langle\lambda|X|\lambda\rangle$, $\langle X^2\rangle = \langle\lambda|X^2|\lambda\rangle$ for

X, and the same for P. We first calculate $\langle X\rangle$ and then $\langle P\rangle$,

$$\begin{aligned}\langle X\rangle &= \langle\lambda|\eta(a^\dagger + a)|\lambda\rangle \\ &= \eta\left(\langle\lambda|a^\dagger|\lambda\rangle + \langle\lambda|a|\lambda\rangle\right) \\ &= \eta(\alpha^* + \alpha)\end{aligned}$$

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$$\begin{aligned}\langle P\rangle &= \langle\lambda|\frac{i\hbar}{2\eta}(a^\dagger - a)|\lambda\rangle \\ &= \frac{i\hbar}{2\eta}\left(\langle\lambda|a^\dagger|\lambda\rangle - \langle\lambda|a|\lambda\rangle\right) \\ &= \frac{i\hbar}{2\eta}(\alpha^* - \alpha)\end{aligned}$$

We then calculate $\langle X^2\rangle$ and $\langle P^2\rangle$.

$$\begin{aligned}\langle X^2\rangle &= \langle\lambda|\eta^2(a^\dagger + a)^2|\lambda\rangle \\ &= \eta^2(\langle\lambda|(a^\dagger)^2|\lambda\rangle + \underbrace{\langle\lambda|a^\dagger}_{\langle\lambda|\alpha^*}\underbrace{a|\lambda\rangle}_{\alpha|\lambda\rangle} + \langle\lambda|\underbrace{aa^\dagger}_{1+a^\dagger a}|\lambda\rangle + \langle\lambda|a^2|\lambda\rangle) \\ &= \eta^2((\alpha^*)^2 + |\alpha|^2 + 1 + |\alpha|^2 + \alpha^2)\end{aligned}$$

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$$\begin{aligned}\langle P^2\rangle &= -\langle\lambda|\frac{\hbar^2}{2\eta}(a^\dagger - a)^2|\lambda\rangle \\ &= -\frac{\hbar^2}{2\eta}(\langle\lambda|(a^\dagger)^2|\lambda\rangle - \underbrace{\langle\lambda|a^\dagger}_{\langle\lambda|\alpha^*}\underbrace{a|\lambda\rangle}_{\alpha|\lambda\rangle} - \langle\lambda|\underbrace{aa^\dagger}_{1+a^\dagger a}|\lambda\rangle + \langle\lambda|a^2|\lambda\rangle) \\ &= -\frac{\hbar^2}{2\eta}((\alpha^*)^2 - |\alpha|^2 - 1 - |\alpha^2| + \alpha^2)\end{aligned}$$

Using the earlier results, we can say

$$\langle X^2\rangle = \eta^2 + \langle X\rangle^2\,, \quad \langle P^2\rangle = \frac{\hbar^2}{4\eta^2} + \langle P\rangle^2 \tag{8}$$

which implies, using the ΔX and ΔP definitions and Eq. 7,

$$\Delta X = \eta, \quad \Delta P = \frac{\hbar}{2\eta}$$

therefore $\Delta X \Delta P = \frac{\hbar}{2}$.

Problem 1.4. Find the wave function $\psi_\lambda(x)$ of a coherent state in the coordinate representation.

The wave function $\psi_\lambda(x)$ is defined as

$$\psi_\lambda(x) = \langle x|\lambda\rangle \tag{9}$$

where we recall $|\lambda\rangle$ is given by Eq. 4. We also recall that the ground state wave function $\psi_0(x)$ is a Gaussian,

$$\psi_0(x) = \langle x|0\rangle = \frac{1}{\sqrt{\eta\sqrt{2\pi}}}\exp\left[-\frac{x^2}{4\eta^2}\right] \tag{10}$$

We then calculate $\psi_\lambda(x)$ using Eq. 9 and Eq. 3.

$$\psi_\lambda(x) = \langle x|\lambda\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle x|n\rangle$$

we then use two relations,

$$\langle x|n\rangle = \psi_n(x) \text{ and } \psi_n(x) = \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{x}{\eta\sqrt{2}}\right)\psi_0(x), \tag{11}$$

where H_n is the nth **Hermite polynomial**, to write

$$\psi_\lambda = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\frac{1}{\sqrt{2^n}}H_n\left(\frac{x}{\eta\sqrt{2}}\right)\psi_0(x)$$

We then write the generating function of the Hermite polynomials,

$$e^{2x't-t^2} = \sum_{n=0}^{\infty}\frac{H_n(x')}{n!}t^n \tag{12}$$

in our equation, $x' = \frac{x}{\eta\sqrt{2}}$ and $t = \frac{\alpha}{\sqrt{2}}$. We then have

$$\psi_\lambda(x) = \frac{1}{\sqrt{\eta\sqrt{2\pi}}}\exp\left[-\frac{x^2}{4\eta^2} + \frac{x\alpha}{\eta} - \frac{\alpha^2}{2} - \frac{|\alpha|^2}{2}\right]$$

which can be shown, by completing the square, and factorising to be

$$\psi_\lambda(x) = \frac{1}{\sqrt{\eta\sqrt{2\pi}}}\exp\left[-\frac{1}{2}\left(\frac{x-2\alpha\eta}{\eta\sqrt{2}}\right)^2 - \frac{\alpha}{2}\langle P\rangle\right] \tag{13}$$

Problem 1.5. Write $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)\,|n\rangle$$

Show that the distribution of $|f(n)|^2$ is of the Poisson form. Find the expectation of N, n, and most probable value n_{mp} of n, hence of E.

Recall Eq. 3, which is in the correct form:

$$|\lambda\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle$$

Clearly $|f(n)|^2 = |e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$. Therefore

$$|f(n)|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{14}$$

which is of the Poisson form ($\rho = e^-\lambda\frac{\lambda^n}{n!}$), with $\lambda = |\alpha|^2$.

We now find the expectation value n of N, which by substituting Eq. 3 is given by

$$\begin{aligned}\langle N\rangle = \langle\lambda|N|\lambda\rangle &= \sum_{n=0}^{\infty}\langle n|e^{-|\alpha|^2}\frac{\alpha^2 n}{n!}n|n\rangle \\ &= e^{-|\alpha|^2}\sum_{n=1}^{\infty}\frac{|\alpha|^{2n}}{n!}n \quad n=1 \text{ starting value, since 0th term vanishes} \\ &= |\alpha|^2 e^{-|\alpha|^2}\sum_{n=1}^{\infty}\frac{|\alpha|^{2(n-1)}}{(n-1)!}\end{aligned}$$

Once again using the exponential series expansion, we find

$$\langle N\rangle = |\alpha|^2 \tag{15}$$

The most probable value n_{mp} of n is the mode of the Poisson distribution, which is the floor of the mean. Therefore $n_{mp} \approxeq |\alpha|^2$.

2 Problem 2

Consider a particle in the potential $V(x)$,

$$V(x) = \begin{cases} -\nu\delta(x) & \text{if } x < a \\ +\infty & \text{if } x \geq a \end{cases} \tag{16}$$

where $\nu > 0$, $a > 0$.

Problem 2.1. Find the wavefunctions for a scattering state. Do not normalise it.

We begin by splitting the wave function into three distinct regions,

$$\psi(x) = \begin{cases} \psi_L(x) & \text{if } x < 0 \\ \psi_M(x) & \text{if } 0 < x < a \\ \psi_R(x) & \text{if } x > a \end{cases} \tag{17}$$

clearly, $\psi_R(x) = 0$ since the potential is infinite, and $\psi_L(x)$, $\psi_M(x)$ are given by the Schrödinger equation,

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0, \quad k^2 = \frac{2m}{\hbar^2}E, \quad x < 0 \tag{18}$$

since we are considering a scattering state, $E > 0$. The general solution to Eq. 18 is

$$\psi_L(x) = A_L \exp(ikx) + B_L \exp(-ikx), \quad \psi_M(x) = A_M \exp(ikx) + B_M \exp(-ikx) \tag{19}$$

where A_L, B_L, A_M, B_M are constants, which we determine using the boundary conditions. First, we use the continuity of the wave function at $x = 0, a$.

$$\begin{aligned} \psi_M(a) &= 0 \\ \psi_L(0) &= \psi_M(0) \end{aligned}$$

which gives us

$$\begin{aligned} A_M \exp(ika) &= -B_M \exp(-ika) \\ B_M &= -A_M \exp(2ika) \end{aligned}$$

and so, by defining $A = 2iA_M \exp(ika)$, we have

$$\psi_M(x) = A(\frac{\exp(ik(x-a)) - \exp(-ik(x-a))}{2i}) = -A\sin(k(x-a)) \tag{20}$$

We then use the continuity of the derivative of the wave function at $x = 0$.

$$A_L + B_L = A\sin(ka)$$

which implies

$$\psi_L(x) = 2A_L i \sin(kx) + A\sin(ka)\exp(-ikx) \tag{21}$$

Then, we use integration of the Schrödinger equation across the discontinuity at $x = 0$ to find the relation between A_L and A.

$$\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon} \frac{d^2\psi(x)}{dx^2}dx + \int_{-\epsilon}^{\epsilon} \nu\delta(x)\psi(x)dx = -E\int_{-\epsilon}^{\epsilon} \psi(x)dx \tag{22}$$

Then let $\epsilon \to 0$ to find

$$\left.\frac{d\psi}{dx}\right|_{0^-}^{0^+} + \frac{2m\nu}{\hbar^2}\psi(0) = 0$$

since the last term vanishes as $\psi(x)$ is continuous at $x = 0$. We then fill in $\psi'_L(0)$ and $\psi'_M(0)$ to find

$$\begin{aligned} \psi'_M(0) - \psi'_L(0) &= \frac{2m\nu}{\hbar^2}\psi_M(0) \\ kA\cos(ka) + 2iA_Lk - iAk\sin(ka) &= \frac{2m\nu}{\hbar^2}(A\sin(ka)) \\ \Rightarrow A_L &= \frac{A}{2ik}\left(\frac{2m\nu}{\hbar^2}\sin(ka) + ik\sin(ka) - k\cos(ka)\right) \end{aligned}$$

$\psi_L(x)$ can then be written as

$$\begin{aligned} \psi_L(x) &= \frac{A}{k}\left(\frac{2m\nu}{\hbar^2}\sin(ka) + ik\sin(ka) - k\cos(ka)\right)\sin(kx) + A\sin(ka)(\cos(kx) - i\sin(kx)) \\ &= \frac{A}{k}\left(\frac{2m\nu}{\hbar^2}\sin(ka)\sin(kx) + ik\sin(ka)\sin(kx) - k\cos(ka)\sin(kx) - iAk\sin(ka)\sin(kx) + iA\sin(ka)\cos(kx)\right) \\ &= A\left[\frac{2m\nu}{\hbar^2}\sin(ka)\sin(kx) + \sin(ka)\cos(kx) - k\cos(ka)\sin(kx)\right] + A\sin(ka) \\ &= A\frac{2m\nu}{\hbar^2}\sin(ka)\sin(kx) - A\sin(k(x-a)) \end{aligned}$$

Therefore, the wave function for a scattering state of a potential described by Eq. 16 is

$$\psi(x) = \begin{cases} A\frac{2m\nu}{\hbar^2}\sin(ka)\sin(kx) - A\sin(k(x-a)) & \text{if } x < a \\ -A\sin(k(x-a)) & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases} \tag{23}$$

Problem 2.2. Find wave function for bound states, and a quantisation condition for the bound state spectrum. Do not normalise it.

Once again we split the wave function into three distinct regions,

$$\psi(x) = \begin{cases} \psi_L(x) & \text{if } x < 0 \\ \psi_M(x) & \text{if } 0 < x < a \\ \psi_R(x) = 0 & \text{if } x > a \end{cases} \tag{24}$$

and states for which $E < 0$ are bound states. We can also write $E' = -E$ for simplicity. The Schrödinger equation is then solved by the general solution

$$\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}, \quad \kappa^2 = \frac{2m}{\hbar^2}E' \tag{25}$$

We then annalyse the behaviour at the limits of each region, and the continuity of the wave function at $x = 0, a$.

$$\psi_L(-\infty) = 0 \quad \Rightarrow B_L = 0$$

$$\begin{aligned} \psi_M(a) &= 0 \\ \psi_L(0) &= \psi_M(0) \end{aligned}$$

which gives us

$$\begin{aligned} A_M\exp(\kappa a) &= -B_M\exp(-\kappa a) \\ \Rightarrow \psi_M(x) &= C(\exp(\kappa(x-a)) - \exp(-\kappa(x-a))), \quad C = A_Me^{\kappa a} \\ &= D\sinh(\kappa(x-a)) \end{aligned}$$

using the first condition, and then

$$A = -De^{-\kappa x}\sinh(\kappa a)$$

using the second condition. To find the quantisation condition, we integrate the Schrödinger equation across the discontinuity at $x = 0$, as in Eq. 22.

$$\begin{aligned}
\psi'_M(0) - \psi'_L(0) &= -\frac{2m\nu}{\hbar^2}\psi(0) \\
\kappa D\cosh(\kappa a) + \kappa D\sinh(\kappa a) &= \frac{2m\nu}{\hbar^2}D\sinh(\kappa a) \\
\Rightarrow \kappa e^{\kappa a} &= \frac{m\nu}{\hbar^2}(e^{\kappa a} - e^{-\kappa a}) \\
\frac{\kappa}{1 - e^{-2\kappa a}} &= \frac{m\nu}{\hbar^2} \\
\Rightarrow e^{2\kappa a}\left(1 - \frac{\kappa\hbar^2}{m\nu}\right) &= 1
\end{aligned}$$

This leads us to the quantisation condition,

$$e^{2\kappa a} = \frac{m\nu}{m\nu - \hbar\sqrt{2mE'}} \tag{26}$$

Problem 2.3. Show that the energy quantisation condition can be written in the form

$$\frac{z}{W} = 1 - e^{-z} \tag{27}$$

where z, W are yet to be determined.
Sketch plots of the left and right hand sides of the energy quantisation condition for $W = 1/2, 2$. Prove that there exist at most only one bound state that is the ground state of the system, find values of W for which there is a ground state.

the quantisation condition Eq. 26 can be easily manipulated to the required form by defining $z = 2\kappa a$ and $W = \frac{2m\nu a}{\hbar^2}$, and doing some simple algebra as below,

$$\begin{aligned}
\frac{\kappa}{1 - e^{-z}} &= \frac{m\nu}{\hbar^2} \\
\Rightarrow \quad \frac{\kappa\hbar^2}{m\nu} &= 1 - e^{-z} \\
\Rightarrow \quad \frac{z\hbar^2}{2am\nu} &= 1 - e^{-z} \\
\Rightarrow \quad \frac{z}{W} &= 1 - e^{-z}
\end{aligned}$$

thusly illustrating Eq. 27.

We then sketch the left and right hand sides of the energy quantisation condition for $W = 1/2, 2$, in figure 1.

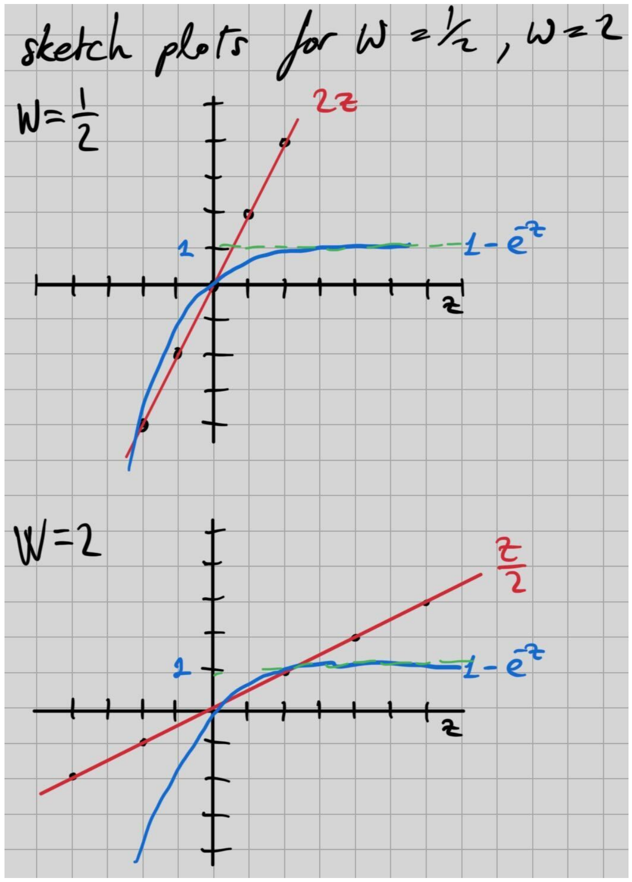

Figure 1: Plot of the left and right hand sides of the energy quantisation condition for $W = 1/2, 2$. Note that for $W = 1/2$, there are no solutions as there are no non zero intersections of the two curves for $z > 0$. Conversely, for $W = 2$, there is one solution.

Since both sides of the equation pass through the origin, and the right hand side has a constantly decreasing derivative, it is clear that there are solutions only for W such that

$$\frac{1}{W} = \frac{d}{dz}\left(\frac{z}{W}\right) < \left[\frac{d}{dz}\left(1 - e^{-z}\right)\right]_{z=0} = 1$$

this implies that there exists a bound state only for $W > 1$. It is also implied, since linear and exponential functions can only intersect twice, that there is only one state, as one intersection is always at the origin.

Problem 2.4. Find the ground state energy E_0 in the limit $W \to \infty$. Explain the result obtained.

for $W \to \infty$, clearly there are only two possible z-values. The first is $z = 0$, which is not a solution, since only $z > 0$ values are relevant. The second is $z = W \to \infty$, which is a solution, as

$$1 = \lim_{W \to \infty} \frac{W}{W} = 1 - e^{-z} = 1 - e^{-W} = 1$$

The result obtained implies

$$\sqrt{\frac{2mE'}{\hbar^2}} = \kappa = \frac{m\nu}{\hbar^2}$$

which then leads to the conclusion,

$$E = -E' = \frac{m\nu^2}{2\hbar^2} \tag{28}$$

this goes to infinity if $W \to \infty$ because of $\nu \to \infty$ (infinitely deep delta function), but if $a \to \infty$ (no wall), the energy $E = E'$ depends only on the strength of the delta function.