L23: Free groups

Def Let G be a group, $S \subseteq G$ a subset. We say that G is generated by S if the only subgroup of G containing S is G itself.
We write $G = \langle S \rangle$
More generally we define $\langle S \rangle$, the subgroup generated by S, to be the smallest subgroup containing S.

Rmk smallest subgroup

top down

$$\langle S \rangle = \bigcap_{\substack{H \leq G \\ S \subseteq H}} H \overset{\text{Exc}}{\leq} G$$

subgroup

bottom up

$$\langle S \rangle = \left\{ \begin{array}{l} \text{all elts that can be written as products} \\ \text{of finitely many elts of } S \text{ or } S^{-1} \end{array} \right\}$$

$$= \bigcup_{n \geq 0} S^{(n)}$$

where $S^{0} = S \cup S^{-1}$

$$S^{(n)} = S^{(n-1)} \cdot (S \cup S^{-1})$$

$$\| $$

$$\{ xy \in G \mid x \in S^{(n-1)},\ y \in S \text{ or } y^{-1} \in S \}$$

Exc: The two definitions coincide.

Ex • S_n is generated by $\{ (ij) \mid i \neq j \}$
• A_n, $n \geq 5$ is gen by $\{ (ijk) \}$

Prop $G = \langle S \rangle \iff$ For any group H a group hom $\varphi : G \to H$ is completely determined by the map of sets $\varphi|_S : S \to H$

Pf Exc

Def A group G with a subset $S \subseteq G$ is said to be free (on S) if for any group H and map of sets $\varphi : S \to H$ there exists a unique group hom $\bar{\varphi} : G \to H$ s.t. $\bar{\varphi}|_S = \varphi$.

$$\mathrm{Hom}_{\text{Group}}(G, H) \cong \mathrm{Hom}_{\text{Set}}(S, H)$$

Thm For any set S there exists a free group on S, we denote it by F_S.

Pf (sketch) Construct $G = \{$ sequence of elements in $\{\pm 1\} \times S \}$ that are reduced (no elements $(+,x), (-,x)$ are adjacent)

define group structure : concatenate & reduce

Show it is a group : cumbersome

<u>Gen & relations revisited</u>

Given a set S and a subset $R \subseteq F_S$ we define

$$\langle S \mid G \rangle := F_S / N(R) \quad \text{where} \quad N(R) := \bigcap_{R \subseteq N \trianglelefteq F_S} N.$$

Prop $G = \langle S \mid R \rangle$ has the following universal property. Given any group K together with a map $\varphi : S \to K$ s.t. the corresponding group hom $\overline{\varphi} : F_S \to K$ satisfies $\overline{\varphi}(R) \subseteq \{e\}$ there exists a unique group hom $\overline{\varphi} : G \to K$.

Pf Exc. (unpacking def).

Similarly we can define <u>free abelian groups</u>.

Prop $G = \mathbb{Z}^r$ is the free abelian group on r generators. I.e. given any other abelian group B together with a map $\varphi : \{1, \dots, r\} \to B$, there exists a unique group hom $\overline{\varphi} : \mathbb{Z}^r \to B$ s.t. $\overline{\varphi}(\underbrace{(0, \dots, 1, \dots, 0)}_{e_i}) = \varphi(i)$

Pf <u>Exc</u>