Module MAU34403 Quantum mechanics I (Frolov)
Homework Sheet 4
Each set of homework questions is worth 100 marks

Use Mathematica if necessary

The XXX periodic Heisenberg spin-1/2 chain of length 2 is described by the Hamiltonian

$$H = \frac{3}{4}J + \frac{J}{\hbar^2}\sum_{\alpha=1}^{3} S_1^\alpha S_2^\alpha \tag{0.1}$$

The Hamiltonian acts in the Hilbert space which is the tensor product of 2 copies of two-dimensional spaces (spin up-down)

$$\mathscr{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \tag{0.2}$$

and the spin-1/2 operator $S_i^\alpha = \hbar\sigma_i^\alpha/2$ acts only on the particle at the i-th site

$$S_1^\alpha = S^\alpha \otimes I\,, \quad S_2^\alpha = I \otimes S^\alpha \tag{0.3}$$

The Hamiltonian commutes with the total spin operator

$$\mathbb{S}^\alpha = S_1^\alpha + S_2^\alpha = S^\alpha \otimes I + I \otimes S^\alpha \tag{0.4}$$

From problem 2 of HW3 we know that the orthonormal vectors

$$|e_1\rangle \equiv \,|\uparrow\uparrow\rangle\,, \quad |e_0\rangle \equiv \,\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\,, \quad |e_{-1}\rangle \equiv \,|\downarrow\downarrow\rangle\,, \tag{0.5}$$

are eigenvectors of $\hat{H}$ with the eigenvalue $E_1 = J$ while the vector

$$|f\rangle \equiv \,\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{0.6}$$

is an eigenvector of $\hat{H}$ with the eigenvalue $E_0 = 0$. These vectors are also eigenvectors of $\mathbb{S}^3$ with eigenvalues $s_1 = \hbar$, $s_0 = 0$ and $s_{-1} = -\hbar$.

Compulsory Questions

Problem 1. Spin-1 representation

Introduce the operators $\mathbb{S}^\pm = \mathbb{S}^1 \pm \mathrm{i}\,\mathbb{S}^2$, find commutation relations between $\mathbb{S}^+$, $\mathbb{S}^-$ and $\mathbb{S}^3$, and find how $\mathbb{S}^\pm$ act on $|e_m\rangle$, $m = 1, 0, -1$. How would you interpret their action?

Problem 2. Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{13}}\Big(\,(2+\mathrm{i})|\uparrow\uparrow\rangle - (2-\mathrm{i})|\uparrow\downarrow\rangle + \mathrm{i}\,|\downarrow\uparrow\rangle - (1-\mathrm{i})|\downarrow\downarrow\rangle\,\Big) \tag{0.7}$$

(a) If the result of a measurement is s_0, what is the state of the system after it?

(b) What is the probability to measure first E_1 and immediately after s_0?

What is the probability to measure first s_0 and immediately after E_1?

Why are these probabilities equal?

Problem 3. Consider the same state as in Problem 2

$$|\psi\rangle = \frac{1}{\sqrt{13}}\Big((2+\mathrm{i})|\uparrow\uparrow\rangle - (2-\mathrm{i})|\uparrow\downarrow\rangle + \mathrm{i}\,|\downarrow\uparrow\rangle - (1-\mathrm{i})|\downarrow\downarrow\rangle \Big) \tag{0.8}$$

Check that the general uncertainty relation

$$\Delta\hat{A}^2\Delta\hat{B}^2 \geq \Big(\frac{1}{2}\langle[\hat{A}\,,\,\hat{B}]_+\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\Big)^2 - \frac{1}{4}\langle[\hat{A}\,,\,\hat{B}]\rangle^2 \tag{0.9}$$

holds for

(i) $\mathbb{S}^3$ and H

(ii) $\mathbb{S}^3$ and $\mathbb{S}^1$

Practice Questions

Problem 1. Spin-1 representation

Find how the total spin operators $\mathbb{S}^\alpha$ act on $|e_m\rangle$, $m = 1, 0, -1$.

Problem 2. Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{13}}\Big((2+\mathrm{i})|\uparrow\uparrow\rangle - (2-\mathrm{i})|\uparrow\downarrow\rangle + \mathrm{i}\,|\downarrow\uparrow\rangle - (1-\mathrm{i})|\downarrow\downarrow\rangle \Big) \tag{0.10}$$

(a) Expand $|\psi\rangle$ over the basis f and $|e_m\rangle$, $m = 1, 0, -1$.

Find the probabilities to measure E_0 and E_1, and s_1, s_0 and s_{-1}.

(b) If the result of a measurement is E_1, what is the state of the system after it?

(c) Find the expectation values of and the uncertainty in the Hamiltonian H and the total spin operators $\mathbb{S}^\alpha$ with respect to $|\psi\rangle$