Introduction to Complex Analysis

Based on the lectures of Professor Andreea Nicoara
Compiled By Evin Fleming

Contents

Overview

Why should one study complex analysis? Why is real analysis sometimes not sufficient to answer some of our questions? Einstein is often misquoted as saying "a worthy problem is seldom solved within it's original domain of conception." Complex analysis not only answers questions from real analysis but also presents its own.

0.1 Geometry of $\mathbb{C}$

When we convert problems of classical geometry into questions about points on the complex plane, they become much easier. All our favourite geometric objects can be represented. For example A circle of radius r around a can be represented as $\{z \in \mathbb{C} \mid |z-a| = r\}$. Multiplication by i represents a rotation by 90^o

0.2 Functions in $\mathbb{C}$

The concept of "smoothness" in $\mathbb{C}$ is very different from that in $\mathbb{R}$. When we consider a function $f : \mathbb{C} \to \mathbb{C}$, we must look at it's real and imaginary components:

$$z = x + iy,\ f(z) = u(z) + iv(z) = u(x, y) + iv(x, y),\ u : \mathbb{C} \to \mathbb{R},\ v : \mathbb{C} \to \mathbb{R}$$

When we consider the question of differentiability, our condition for it become much more stringent but results in much more powerful properties:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h},\ h \in \mathbb{C}$$

We obtain the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Using the above, we can show that the u, v components of a holomorphic are harmonic i.e they are zero under the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

0.3 Power Series

We call functions that are differentiable in the complex plane holomorphic.

A surprising result is that if $f(z)$ has one derivative then it is has all derivatives. We can represent a holomorphic around a point z_0 by a power series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^n$$

This is the Taylor expansion in $\mathbb{C}$. Hence holomorphic $\implies$ analytic.

In $\mathbb{R}$, $f(x)$ may have derivatives of all order and it's Taylor expansion may still not converge.
Some interesting consequences follow from this:

i) Zeroes of holomorphic functions are isolated (not necessarily true for real valued functions)

ii) Zeroes of holomorphic functions have finite order.

iii) Two holomorphic functions that agree on a line segment agree everywhere on their domain.

0.4 Complex Integration

Cauchy's remarkable theorem tells us that on a domain Ω, the integral of a holomorphic function on a simple closed curve γ equals 0.

$$\int_{\gamma} f(z).dz = 0$$

A consequence of Cauchy's theorem is a super useful computational device, Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z).dz}{z - a}$$

Where the curve γ contains a.
This means that the integral around a pole yields something. This value is called the residue (what is left over).
We obtain many incredibly useful results from residues. It allows us to evaluate real integrals that would be inaccessible otherwise such as

$$\int_{-\infty}^{\infty} \frac{\sin(x.dx)}{x}$$

Why does this work? It works because $\mathbb{C}$ is the smallest algebraically closed field containing $\mathbb{R}$ i.e any polynomial of degree n has exactly n roots. An integral on $\mathbb{R}$ with a polynomial divider will factor completely over $\mathbb{C}$ $\implies$ if we take an integral around the complex roots and along the real axis.

0.5 The Riemann Mapping Theorem

If Ω is a simply connected domain (open region with no holes) such that $\Omega \neq \mathbb{C}$, for any $z_0 \in \Omega$, there exists a unique function $f(z)$ that is holomorphic on Ω such that $f(z_0) = 0$, $f'(z_0) > 0$ and $f(z)$ defines a 1-to-1 mapping between Ω and the disc of radius 1 centred at the origin.

For a holomorphic function with $f'(z_0) \neq 0 \implies f$ is conformal at that point i.e it preserves angles but not necessarily distances. Riemann's theorem implies that all simply-connected regions that are proper subsets of $\mathbb{C}$ are conformally equivalent to the disc of radius 1. This is not true in $\mathbb{R}$ or even $\mathbb{C}^2$

Chapter 1

Arithmetic in $\mathbb{C}$

"The shortest path between any two truths in the real domain is through the complex domain"
- Jacques Hadamard

1.1 Algebraic Operations in $\mathbb{C}$

We can consider the complex plane as a set of ordered pairs (a, b) $a, b \in \mathbb{R}$ in which a represents the real part of the number and b is the imaginary part. Equivalently we can write a complex number z as $z = a + bi$ where $i = \sqrt{-1}$. We define arithmetic on $\mathbb{C}$ in the following way: For $z_1 = a + bi,\ z_2 = c + di$:

$$
\begin{aligned}
&z_1 + z_2 = z_2 + z_2 = (a + c) + (c + d)i \\
&z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i \\
&z_1 \cdot z_2 = z_2 \cdot z_1 = (ac - bd) + (ad + bc)i \\
&\frac{z_1}{z_2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}
\end{aligned}
$$

Example 1.1.1.
Let $z_1 = 1 + 2i$ and $z_2 = -3 + 5i$.

$$
\begin{aligned}
&(1 + 2i) + (-3 + 5i) = (1 - 3) + (2 + 5)i = -2 + 7i \\
&(1 + 2i) - (-3 + 5i) = (1 + 3) + (2 - 5)i = 4 - 3i \\
&(1 + 2i)(-3 + 5i) = -3 + 5i - 6i - 10 = -13 - i \\
&\frac{1 + 2i}{-3 + 5i} = \frac{1 + 2i}{-3 + 5i}\frac{-3 - 5i}{-3 - 5i} = \frac{(1 + 2i)(-3 + 5i)}{(-3 + 5i)(-3 - 5i)} = \frac{7 - 11i}{34} = \frac{7}{34} - \frac{11}{34}i
\end{aligned}
$$

Remark 1.1.2.

$\mathbb{C}$ is a field (a commutative, division ring). It satisfies all the field axioms. The field axioms are as follows: Let $z_1, z_2, z_3 \in \mathbb{C}$

1. Closure under addition: $z_1 + z_2 \in \mathbb{C}$

2. Existence of Additive identity 0: $z_1 + 0 = 0 + z_1 = z_1$
3. Existence of additive inverse: $z_1 + (-z_1) = (-z_1) + z_1 = 0$
4. Closure Under Multiplication: $z_1 \cdot z_2 \in \mathbb{C}$
5. $\mathbb{C}\backslash\{0\}$ has a multiplicative identity 1: $z_1 \cdot 1 = 1 \cdot z_1 = z_1$
6. Every element of $\mathbb{C}\backslash\{0\}$ has a multiplicative inverse: $z_1 = x + iy \implies \frac{1}{z_1} = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$
7. $\mathbb{C}$ is commutative: $z_1 + z_2 = z_2 + z_1,\ z_1 \cdot z_2 = z_2 \cdot z_1$
8. $\mathbb{C}$ has no zero divisors i.e $z_1 \cdot z_2 = 0 \implies z_1 = 0$ or $z_2 = 0$ i.e $\mathbb{C}\backslash\{0\}$ is closed under multiplication.
9. $\mathbb{C}$ is distributive: $z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3,\ (z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$

Remark 1.1.3.

1. $\mathbb{R}$ and $\mathbb{Q}$ are also fields
2. $\mathbb{C} = \mathbb{R}[i]$ i.e $\mathbb{C}$ is obtained by adjoining i to $\mathbb{R}$
3. $\mathbb{R}$ is ordered
4. $\mathbb{C}$ is not ordered (tutorial exercise)

Definition 1.1.4 (The Complex Conjugate and modulus of a complex number)**.**

If $z = a + bi$, the complex conjugate of z, $\bar{z} = a - bi$. The modulus of z, $|z| = \sqrt{a^2 + b^2}$. Note that $|z|^2 = z \cdot \bar{z}$

Complex numbers originally arose as a way of considering solutions to equations that were previous thought to have no solutions such as $x^2 + 1 = 0$. It begs the question, "Does there exist algebraic equations that don't have solutions in $\mathbb{C}$?" The answer is no! $\mathbb{C}$ is the smallest algebraically closed field containing $\mathbb{R}$.

As a consequence of $\mathbb{C}$ being the smallest algebraically closed field containing $\mathbb{R}$ we get the residue theorem. Algebraic geometry is most often done in $\mathbb{C}$ instead of $\mathbb{R}$ (because nasty phenomena if things don't split.)

A seminal result in complex analysis that we require heavier machinery to prove is the following theorem.

Theorem 1.1.5 (The Fundamental Theorem of Algebra)**.** *Let* $a_m x^m + a_{m-1}x^{m-1} + .. + a_1 x + a_0 = 0$ *with* $a_m \neq 0$ *and equation with real coefficients* $a_0, .., a_m$*, then the equation has exactly m complex roots (counted with multiplicity).*

Proof: *"Patience you must have my young padawan. Have patience and all will be revealed"* -Yoda

Proposition 1.1.6 (Useful Identities).

- $\mathrm{Re}\{z\} = \frac{1}{2}(z + \overline{z})$, $\mathrm{Im}\{z\} = \frac{1}{2i}(z - \overline{z})$
- $|z|^2 = |z||\bar{z}| = z\bar{z}$
- $|a_1 a_2..a_n| = |a_1||a_2|...|a_n|$
- $|a_1 + a_2 + ... + a_n| \leqslant |a_1| + |a_2| + .. + |a_n|$ (The Triangle Inequality)
- $|a_1 - a_2| \geqslant \big||a_1| - |a_2|\big|$ (The Reverse Triangle Inequality)
- $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{ab} = \overline{a}.\overline{b}$
- $|a+b|^2 = (a+b)(\overline{a} + \overline{b}) = |a|^2 + 2\mathrm{Re}\{a\overline{b}\} + |b|^2$
- $|a-b|^2 = |a|^2 - 2\mathrm{Re}\{a\overline{b}\} + |b|^2$
- $|a+b|^2 + |a-b|^2 = 2|a|^2 + 2|b|^2$
- $$\left|\sum_{i=1}^{n} a_i b_i\right| \leqslant \left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} |b_i|^2\right)^{\frac{1}{2}}, \quad \text{(Cauchy's Inequality)}$$

1.2 Graphical Representation of $\mathbb{C}$

Courtesy of our own William Rowan Hamilton, we can represent the complex plane as a copy of $\mathbb{R}^2$ in which the vertical axis is the imaginary axis and the horizontal axis is the real axis. We can uniquely define a complex number by specifying it in rectangular coordinates as above ($z = (a, b)$), or with polar coordinates ($z = (r, \theta)$ i.e $a = r\cos\theta$ and $b = r\sin\theta$). We call θ the argument. It is measured counter-clockwise from the real axis. r is the modulus from above. When we consider complex numbers in this way, the operations we perform on them have a nice geometric intuition. The addition of complex numbers amounts to vector addition once the two complex numbers are viewed as vectors. The conjugate of a complex number is the reflection across the real axis.

Question: What about multiplication of complex numbers?

For any complex number z, if arg(z)=θ, then arg(z)=$\theta + 2\pi k$, $k \in \mathbb{Z}$. In other words, the argument is multiple-valued. The argument corresponding to $k = 0$ is known as the principal argument and shall be denoted by Arg(z).
Using Euler's formula makes calculations in polar coordinates more natural.

Proposition 1.2.1 (Euler's Formula).

$$e^{i\theta} = \cos\theta + i\sin\theta$$

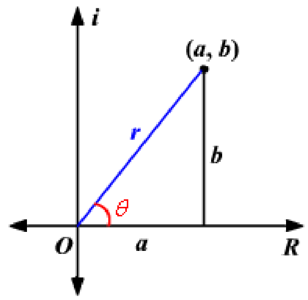

Figure 1.1: Polar vs Rectangular Coordinates

Proof: This shall come when we look at power series in chapter 8.

From Euler's formula, we can see that for $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ then $z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$. This means that when we multiply complex numbers, their arguments are added and their moduli are multiplied.

Note that if $z = re^{i\theta}$, then its multiplicative inverse is $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$.

Theorem 1.2.2 (De Moivre's Formula). *Suppose $z^m = re^{i\theta} = x + iy$, then there are m roots of $x + iy$,*

$$z_1 = |r|^{\frac{1}{m}} e^{i\left(\frac{1}{m}\theta\right)},\ z_2 = |r|^{\frac{1}{m}} e^{i\left(\frac{1}{m}(\theta+2\pi)\right)}, ..., z_m = |r|^{\frac{1}{m}} e^{i\left(\frac{1}{m}(\theta+(m-1)2\pi)\right)}$$

Proof: If $z^m = re^{i\theta}$, then $z^m = re^{i(\theta+k2\pi)}\ \forall\ k \in \mathbb{Z}$ The theorem follows by taking the mth root of both sides, noting that we only attain m unique solutions because:

$$\frac{1}{m}(\theta + m2\pi) = \frac{1}{m}\theta + 2\pi \implies e^{i\left(\frac{1}{m}(\theta+m2\pi)\right)} = e^{i\left(\frac{1}{m}(\theta)+2\pi\right)} = e^{i\left(\frac{\theta}{m}\right)}$$

Example 1.2.3.

Find z such that $z^3 = -8 - 8i$:

$$r = \sqrt{8^2 + 8^2} = \sqrt{128} = 8\sqrt{2},\ \arg(-8-8i) = \frac{3\pi}{4} \text{ in } [0, 2\pi)$$

$$z_1 = \left(8\sqrt{2}\right) e^{i\frac{3\pi}{4}\cdot\frac{1}{3}} = 2\sqrt[6]{2}e^{\frac{\pi}{4}}$$

$$z_2 = \left(8\sqrt{2}\right) e^{i(\frac{3\pi}{4}+2\pi)\cdot\frac{1}{3}} = 2\sqrt[6]{2}e^{i\frac{19\pi}{12}}$$

$$z_3 = 2\sqrt[6]{2}e^{i(\frac{\pi}{4}+\frac{4\pi}{3})}$$

Example 1.2.4.
Of particular interest of the so called roots of unity, the solutions to the equation $z^n = 1$ for fixed n. From De'Moivre's formula, we know that these are:

$$e^{i0},\ e^{i\frac{2\pi}{n}}, e^{2\cdot\frac{2\pi}{n}}, .., e^{(n-1)\cdot\frac{2\pi}{n}}$$

Graphically, these are the vertices of an n-sided regular polygon.
Example 1.2.5.
When we work in $\mathbb{C}$ every polynomial of degree n has n (not necessarily distinct roots). Using the quadratic formula for polynomials of degree 2, we have always find these roots:

$$az^2 + bz + c = 0 \implies z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In $\mathbb{C}$ this expression always makes sense since we allow negative square roots.

1.3 Some Graphical Representations of Regions in $\mathbb{C}$

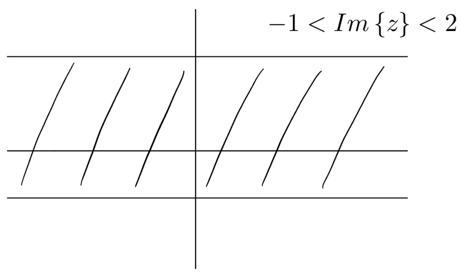

Figure 1.2: A Strip in $\mathbb{C}$

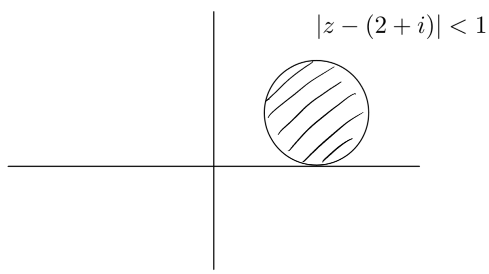

Figure 1.3: A Circle Centred at 2+i

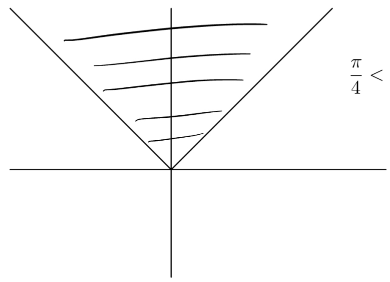

Figure 1.4: A Wedge in $\mathbb{C}$

1.4 The Extended Complex Plane and Stereographic Projection

We establish the so-called extended complex plane by introducing an additional element ∞ with the following properties:

i) $a + \infty = \infty + a = \infty \ \forall a$ finite

ii) $b \cdot \infty = \infty \cdot b = \infty$ for $b \neq 0$ (here we could have $b = \infty$)

iii) $\frac{a}{0} = \infty$ for $a \neq 0$

iv) $\frac{a}{\infty} = 0$ for $a \neq \infty$

Note that we have not defined $\infty + \infty$ or $0 \cdot \infty$
With this construction of the extended complex plane $\mathbb{C} \cup \{\infty\}$ we can introduce a one-to-one correspondence with the unit sphere.

We achieve our one-to-one correspondence by first identifying $\mathbb{C}$ with $\mathbb{R}^2$ embedded in

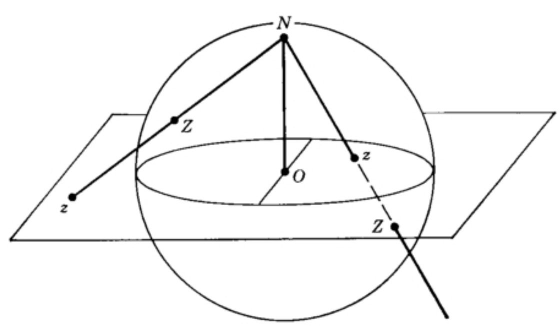

Figure 1.5: Visualisation of Stereographic Projection

$\mathbb{R}^3$. For each $z = x + yi \in \mathbb{C}$, we identify it with the point $(x, y, 0) \in \mathbb{R}^3$. The unit sphere is $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. We consider the line that passes through our point (x_1, x_2, x_3) and the North Pole $(0, 0, 1)$. The point at which the line intersects the the x_1, x_2

plane is the point in $\mathbb{C}$ that our point in the sphere gets mapped to in our correspondence.

$$L(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} t \cdot x_1 \\ t \cdot x_2 \\ 1 + t \cdot (x_3 - 1) \end{pmatrix}$$

Our complex number is the point at which the bottom component of our vector is 0 $\iff 1 + t \cdot (x_3 - 1) = 0 \implies t = \frac{1}{1-x_3}$. Using this we find that:

$$\begin{pmatrix} \frac{x_1}{1-x_3} \\ \frac{x_2}{1-x_3} \\ 0 \end{pmatrix} \to z = \frac{x_1}{1-x_3} + i \frac{x_2}{1-x_3}$$

A tutorial question asks you to find the inverse map.

Remark 1.4.1.

What if $(x_1, x_2, x_3) = (0, 0, 1)$? We call this point a pole (because it corresponds to the north pole). This is the point that corresponds to ∞ in our correspondence.

Definition 1.4.2 (The Riemann Sphere)**.**

In the above context, S^2 is often called the Riemann Sphere. Because there's a one-to-one correspondence between the Riemann Sphere and $\mathbb{C} \cup \{\infty\}$, we often use the terms interchangeably.

Remark 1.4.3.

The Riemann sphere is the easiest example of a Riemann surface (complex 1-dimensional objects one can construct using holomorphic functions.)

Proposition 1.4.4. *Stereographic Projection maps every circle on* S^2 *into either a line or circle on* $\mathbb{C}$*. Great circles through* $(0, 0, 1)$ *always get mapped to lines.*

Figure 1.6: Riemann thinking about spheres

Chapter 2

Point Set Topology

"Point set topology is a disease from which the human race will soon recover"
-Henri Poincare

2.1 Open and Closed Sets

Definition 2.1.1 (Open Ball \ Open Neighbourhood)**.**

$B(z_0, r) = \{z \in \mathbb{C} \mid |z_0 - z| < r,\ r \in \mathbb{R}, r > 0\}$ is called an open ball \ neighbourhood.

Definition 2.1.2 (Open and Closed Sets)**.**

A set X is open if and only if $\forall x \in X \ \exists\, r > 0$ such that $B(x, r) \subset X$. A set Y is closed if and only if its complement $Y^c = \mathbb{C} \backslash Y$ is open.

Example 2.1.3.

$B(0, 1)$ is open while $B(0, 1) \cup \{1\}$ is not. $\{z \in \mathbb{C} \mid |z| \leqslant 1\} = \overline{B}(0, 1)$ is closed.

Remark 2.1.4.

Sets are not like windows, they can be open and closed at the same time (clopen). There are also sets that are neither open nor closed.

Definition 2.1.5 (Limit Point)**.**

x^* is a limit point of a set X is a point such that $\forall \varepsilon > 0,\ [B(x^*, \varepsilon) \cap X] \backslash \{x^*\} \neq \emptyset$. Note that it is not required that $x^* \in X$.

Lemma 2.1.6. *A set is closed if and only if it contains all of its limit points.*

Proof:
" $\Longrightarrow$ "
Suppose A is closed $\iff$ A^c is open. Let x^* be a limit point of A $\iff$ $\forall \varepsilon > 0,\ B(x^*, \varepsilon) \cap A \backslash \{x^*\} \neq \emptyset$. If x^* were an element of A^c there would exist $\varepsilon_0 > 0$ such that $B(x^*, \varepsilon_0) \subset$

$A^c \implies B(x^*, \varepsilon_0) \cap A = \emptyset$ This is a contradiction.
" $\impliedby$ "
Suppose A contains all its limit points. We wish to show that A^c is open. Let $y^* \in A^c \implies y^*$ is not a limit point of A $\implies \exists \varepsilon_1 > 0$ such that $B(y^*, \varepsilon_1) \cap A \backslash \{y^*\} = \emptyset = B(y^*, \varepsilon_1) \cap A \iff B(y^*, \varepsilon_1) \subset A^c$ i.e A^c is open so A is closed $\square$

Definition 2.1.7 (Boundary of a Set)**.**

The boundary of a set X, $\partial X = \{x \in \mathbb{C} \mid \forall\ \varepsilon\ B(x, \varepsilon) \cap X \backslash \{x\} \neq \emptyset\}$ and $B(x, \varepsilon) \cap X^c \backslash \{x\} \neq \emptyset\}$

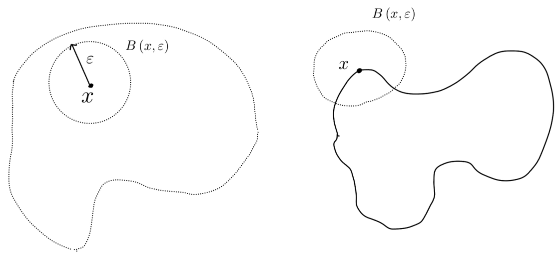

Figure 2.1: An Open Set (left) and the boundary of a set (right)

Definition 2.1.8 (Closure of a Set)**.**

The closure of a set X, $\overline{X} = \partial X \cup X$. It is the smallest closed set that contains X.

Definition 2.1.9 (Interior of a Set)**.**

The interior of a set $X, X^o := \overline{X} \backslash \partial X$. It is the set of all points x in X such that there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset X$. It is the largest open set contained in X.

Example 2.1.10.

If $X = B(0,1)$, $\overline{X} = \{|z| \leqslant 1\} = \overline{B}(0,1)$, $\partial X = \{|z| = 1\}$

Definition 2.1.11 (Bounded Set)**.**

X is bounded if $\exists\ R \in (0, \infty)$ such that $X \subset B(0, R)$

Definition 2.1.12 (Compactness in $\mathbb{C}$)**.**

In $\mathbb{C}$, X is compact if and only if it is closed and bounded. (This is not the definition of compactness in general, it's a result of the Heine-Borel Theorem that holds in $\mathbb{C}$).

2.2 Connected Sets

Definition 2.2.1 (Disconnected Sets)**.**

X is disconnected if $\exists$ non-empty open sets U, V such that $U \cap V = \emptyset$ and $X = U \cup V$. If there exists no such sets, then X is connected.

Definition 2.2.2 (A Region)**.**

A Region is a connected open set in $\mathbb{C}$.

The above definition of connectedness can be cumbersome to work with, we introduce a slightly stronger form of connectedness that is adequate for most of what we'd like to do.

Definition 2.2.3 (Polygonally Connected Sets)**.**

X is polygonally connected if there exists a polygonal line contained in X, connecting any 2 points in X.

Proposition 2.2.4. $B(z_0, r) = \{w \in \mathbb{C} \mid |z_0 - w| < r\}$ *is polygonally connected in* $\mathbb{C}$.

Proof: Let $x, y \in B(z_0, r)$ Simply choose the polygonal path $[x, z_0], [z_0, y]$

Theorem 2.2.5. *For every open set in* $\mathbb{C}$*, connectedness* $\iff$ *polygonal connectedness.*

Proof:
" $\implies$ "
Let A be a connected, open set in $\mathbb{C}$. Let $x \in A$. Define the following sets:

$$U = \{y \in A \mid \text{ there is no polygonal path between y and x}\}$$

$$V = \{y \in A \mid \text{ y and x are polygonally connected}\}$$

Clearly $A = U \cup V$. U is open because if there is not polygonal path between x and y, then there is no path between x and all the elements in some open ball around y. V is open because if x and y are connected then we can form a path between x and any element in some open ball found y contained in A by adjoining the path from x to y and y to whatever element we want. A is a disjoint union of U and V, both of which are open, hence one of them must be empty by our assumption of connectedness. y is polygonally connected to y so U is non-empty, hence V is empty and A is polygonally connected.
" $\impliedby$ "
Assume A is polygonally connected. Assume A is the union of disjoint open sets U, V. Let $x \in U$, $y \in V$. By polygonal connectedness, there exists a continuous polygonal path $\gamma : [0, 1] \to A = U \cup V$. Since γ is continuous, both $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are open and cover $[0, 1]$, but they are disjoint $\implies$ $[0, 1]$ is not connected. This is a contradiction.

Corollary 2.2.6. *Every region in $\mathbb{C}$ is polygonally connected.*

Proof: Every open set is equal to a union of open balls. We can then apply the previous theorem.

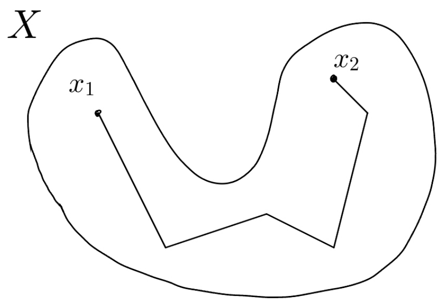

Figure 2.2: A Polygonally Connected Set

Chapter 3

Sequences, Series and Continuous Functions

3.1 Sequences in $\mathbb{C}$

Definition 3.1.1 (A Convergent Sequence in $\mathbb{C}$)**.**

A sequence $\{z_n\}$ is said to be convergent to z^* in $\mathbb{C}$ if and only if $\forall \varepsilon > 0 \; \exists \; N \in \mathbb{N}$ such that $|z_n - z^*| < \varepsilon, \; \forall \; n \geqslant N$
Example 3.1.2.
Let $z_n = z^n$ for some fixed $z \in \mathbb{C}$ such that $|z| < 1$. This series converges to 0 by the properties of geometric sequences.

Proposition 3.1.3. $\{z_m\}$ *converges to* $z \iff \mathsf{Re}\{z_m\}$ *converges to* $\mathsf{Re}\{z\}$ *and* $\mathrm{Im}\{z_m\}$ *converges to* $\mathrm{Im}\{z\} \iff$ *for* $z \neq 0$, $\mathsf{arg}(z_m)$ *converges to* $\mathsf{arg}(z)$ *and* $|z_m|$ *converges to* $|z|$

Proof: We shall prove that the 1st condition is equivalent to the second. The equivalence to the 3rd follows naturally.
" $\implies$ "
Assume $z_m = x_m + iy_m$ converges to $z = x + iy \iff \forall \varepsilon > 0 \exists \; N$ such that $\forall n \geqslant N, \; |z_n - z| < \varepsilon$. $|x_m - x| \leqslant |z_n - z| < \varepsilon$. The imaginary component follows identically.
" $\impliedby$ "
Assume $\{x_m\}$ and $\{y_m\}$ converge to x and y respectively. Choose N such that $|x_m - x|, |y_m - y| < \frac{\varepsilon}{\sqrt{2}}$. $|z_m - z|^2 = |x_m - x|^2 + | + |y_m - y|^2 < \varepsilon^2$

Definition 3.1.4 (Cauchy Sequence)**.**

$\{z_n\}$ is a Cauchy sequence if and only if $\forall \varepsilon > 0, \; \exists \; N \in \mathbb{N}$ such that $\forall n, m \geqslant N, \; |z_n - z_m| < \varepsilon$

Proposition 3.1.5. *Every convergent sequence is Cauchy.*

Proof: Assume $\{z_n\}$ is convergent to $z \iff \forall \varepsilon > 0 \exists \; N \in \mathbb{N}$ such that $|z_n - z| < \frac{\varepsilon}{2} \; \forall n \geqslant N$.
Let $m, n \geqslant N \implies |z_n - z_m| = |z_n - z + z - z_m| \leqslant |z_n - z| + |z_m - z| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \square$

Remark 3.1.6.

In $\mathbb{C}$, every Cauchy sequence is convergent because $\mathbb{R}$ is complete i.e all its Cauchy sequences are convergent. This tells us $\mathbb{C}$ is also complete because we can split each Cauchy sequence into 2 real Cauchy sequences like above.

Definition 3.1.7 (Convergent Series)**.**

$\sum_{k=1}^{\infty} z_k$ converges if the sequence $\{\sum_{k=1}^{n} z_k\}$ converges as $n \to \infty$.

3.2 Continuous Functions

Definition 3.2.1 (Equivalent definitions of continuous functions)**.**

f is continuous at z_0 if and only if the following hold.

1. $\forall \{z_n\} \to z_0,\ \lim_{n\to\infty} f(z_n) = f(\lim_{n\to\infty} z_n) = f(z_0)$
2. $\forall \varepsilon > 0\ \exists\ \delta > 0$ such that $|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$
3. $\forall$ open set V, $f^{-1}(V)$ is also open

Proposition 3.2.2. $f(z) = f(x, y) = u(x, y) + iv(x, y)$ *is continuous if and only if* u *and* v *are continuous.*

Proof: Follows very similarly to the proof of Proposition 3.1.3.

Proposition 3.2.3. *If f and g are continuous on D, then :*

1. *f+g is continuous on D*
2. *f·g is continuous on D*
3. $\frac{f}{g}$ *is continuous on D given that g*$\neq 0$

Theorem 3.2.4. *If* $f : D \to \mathbb{C}$ *is continuous and* $D \subset \mathbb{C}$ *is connected, then* $f(D)$ *is connected. Over* $\mathbb{R}$ *this is the intermediate value theorem.*

Theorem 3.2.5. *If* $f : D \to \mathbb{C}$ *is continuous and* $D \subset \mathbb{C}$ *is compact then* $f(D)$ *is compact.*

Definition 3.2.6 (Uniform Convergence)**.** *A sequence of functions* $\{f_n\}$ *converges to* f *uniformly if and only if* $\forall \varepsilon > 0,\ \exists\ N \in \mathbb{N}$ *such that* $|f_n(z) - f(z)| < \varepsilon,\ \forall\ n \geqslant N\ \forall z$ *in the domain of* f_n

Theorem 3.2.7. *If* $\{f_n\}$ *converges uniformly to* f *and* f_n *is continuous for all n, then f is continuous.*

Proof: Assume f_n converges uniformly to $f \iff \forall \varepsilon > 0, \exists\ N \in \mathbb{N}$ such that for every $n \geqslant N$ $|f_n(x) - f(x)| < \frac{\varepsilon}{3}\ \forall x \in D$. Since f_n is continuous at x, $\exists\ \delta > 0$ such that $|x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$. Then for all $n \geqslant N$

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| \leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f_(x_0)| < \varepsilon \quad \square$$

Remark 3.2.8.

The above theorem does not necessarily hold if convergence is not uniform. Observe the example below. $f_n = x^n$ on $[0,1]$ are continuous functions. If $x \neq 1$, $\lim_{n\to\infty} f_n(x) = 0$. If $x = 1$, $\lim_{n\to\infty} f_n(x) = 1$. Hence:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 \text{ if } x \in [0,1) \\ 1 \text{ if } x = 1 \end{cases}$$

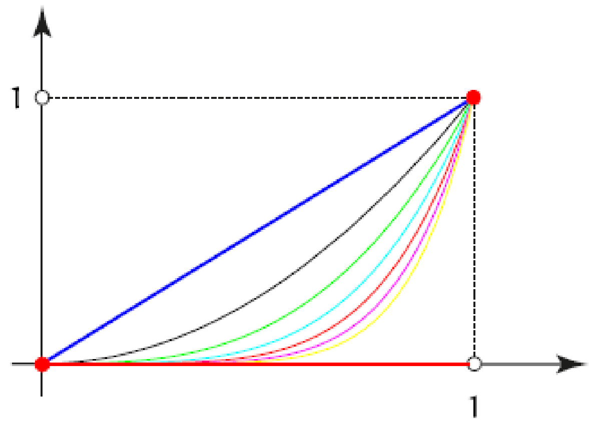

Figure 3.1: A sequence of continuous functions converging to a discontinuous function

Theorem 3.2.9 (Weierstrass M-Test)**.** *If* f_n *is continuous for every n and* $|f_n| \leqslant M_n$, $M_n \in \mathbb{R}$ *and* $\sum_{n=0}^{\infty} M_n$ *converges, then* $\sum_{i=0}^{\infty} f_n$ *converges to a continuous function.*

Proof: We proceed by showing that $\{\sum_{i=0}^{N} f_n\}$ converges uniformly (Convergence itself is immediate). Since $\sum_{n=0}^{\infty} M_n$ exists $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geqslant N$ $|\sum_{n=0}^{\infty} M_n - \sum_{n=0}^{N} M_n| = |\sum_{n=N+1}^{\infty} M_n| < \varepsilon$

$$\left|\sum_{n=0}^{\infty} f_n - \sum_{n=0}^{N} f_n\right| = \left|\sum_{n=N+1}^{\infty} f_n\right| \leqslant \sum_{n=N+1}^{\infty} |f_n| \leqslant \left|\sum_{i=N+1}^{\infty} M_n\right| < \varepsilon$$

Chapter 4

The Cauchy-Riemann Equations and Holomorphic Functions

4.1 Differentiation

Definition 4.1.1 (C^r functions)**.**

A function $f = u + iv : D \to \mathbb{C}$ is of class C^r if both u and v have continuous derivatives up to order r. If u and v are infinitely continuously differentiable, we say f is smooth or of class C^∞.

Lemma 4.1.2. *If* D *is a domain and* $u(x, y) : D \to \mathbb{C}$ $\frac{\partial u}{\partial x} = u_x = 0$ *on* D *and* $\frac{\partial u}{\partial y} = u_y = 0$ *on* D*, then* u *is constant.*

Proof: D is a domain $\implies$ D is polygonally connected i.e $\forall (x_1, y_1) \neq (x_2, y_2),\ (x_i, y_i) \in D$ are connected by a polygonal line. D is open so we can replace this polygonal line by another one consisting of only vertical and horizontal line segments. We then apply the fundamental theorem of calculus.

$$[(x_1, y_1), (x_1', y_1)] \to u(x_1', y_1) - u(x_1, y_1) = \int_{x_1}^{x_1'} \underbrace{\frac{\partial u}{\partial x}(t)}_{=0} .dt$$

The same argument applies to the vertical components.

4.2 The Cauchy Riemann Equations

Definition 4.2.1 (Holomorphic Functions)**.**

$f : D \to \mathbb{C}$ is holomorphic if the following limit exists:

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h},\ h \in \mathbb{C}$$

Remark 4.2.2.

The above condition is stronger than the existence of the partial derivatives of the components of f as the limit can be taken along any direction, not just along the x and y-axes.

Theorem 4.2.3 (Derivation of the Cauchy-Riemann Equations)**.**

Suppose f is holomorphic at z. This means that it does not matter which path the following limit is evaluated on:

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Firstly take h to be purely real. Since each complex number can be written as $z = x + iy$ we can consider f as a function of x and y. Moreover, we can split f into its real and imaginary component functions $f(x, y) = u(x, y) + iv(x, y)$

$$\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{h \to 0} \frac{v(x+h, y) - v(x, y)}{h} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

We then consider if h is purely imaginary, i.e $h \to ih, h \in \mathbb{R}$

$$\lim_{h \to 0} \frac{u(x, y+h) - u(x, y)}{ih} + i \lim_{h \to 0} \frac{v(x, y+h) - v(x, y)}{ih} =$$
$$-i \lim_{h \to 0} \frac{u(x, y+h) - u(x, y)}{h} + \lim_{h \to 0} \frac{v(x, y+h) - v(x, y)}{h} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Since these 2 results must be equal, we can equate their real components and imaginary components respectively

$$\iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These 2 equalities are called the Cauchy-Riemann equations.
Example 4.2.4.
$f(z) = z$ satisfies the Cauchy-Riemann equations while $f(z) = z\bar{z}$ does not.

Theorem 4.2.5. *If* $f : D \to \mathbb{C}$ *is holomorphic on a domain D if and only if it satisfies the Cauchy-Riemann equations on D.*

Proof: In Baks and Newman, it uses the mean-value theorem.

Remark 4.2.6.

A function can satisfy the Cauchy-Riemann equations at a point without being holomorphic. To rectify bad behaviour we need f to be holomorphic in a neighbourhood of z, then the function is differentiable at z.

Lemma 4.2.7. $f : D \to \mathbb{R}$ *with* $D \subset \mathbb{C}$ *is holomorphic if and only if it is constant.*

Proof: $f = u + iv$ being a real valued function implies that $v = 0 \implies v_y = u_x = 0, v_x = -u_y = 0 \implies u$ is a constant function.

Proposition 4.2.8. f *is holomorphic if it is independent of* $\bar{z}$

Proof: Since $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$, by the chain rule:

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial u}{\partial \overline{z}} + i\frac{\partial v}{\partial \overline{z}} = \frac{1}{2}\left(u_x + iv_x + iu_y - v_y\right) = 0$$

Remark 4.2.9.

Using the chain rule for multivariable functions we can define the following operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \ \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Lemma 4.2.10.

1. $\overline{\left(\frac{\partial}{\partial z}\right)} = \frac{\partial}{\partial \overline{z}}$

2. $\left(\frac{\partial (z)}{\partial z}\right) = 1, \ \left(\frac{\partial (\overline{z})}{\partial z}\right) = 0$

3. $\left(\frac{\partial (z)}{\partial \overline{z}}\right) = 0, \ \left(\frac{\partial (\overline{z})}{\partial \overline{z}}\right) = 1$

Proof: Simple computation

Example 4.2.11.

1. $f(z) = z$ is independent of $\overline{z}$ and $C^1 \implies$ holomorphic
2. $f(z) = |z|^2 = z \cdot \overline{z}$ is not independent of $\overline{z} \implies$ not holomorphic.
3. $f(z) = |z| = \sqrt{z \cdot \overline{z}}$ is not independent of $\overline{z} \implies$ not holomorphic.

Chapter 5

Harmonic Functions

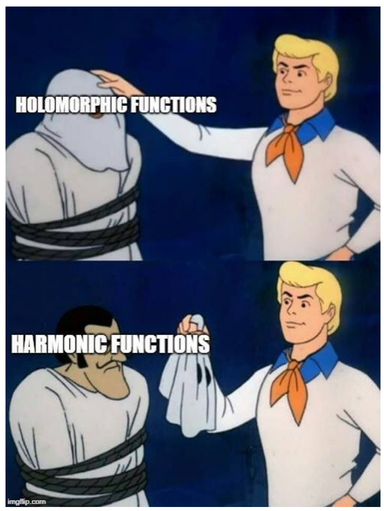

Figure 5.1: Fred demonstrating that the components of holomorphic functions are harmonic

5.1 The Laplacian

Theorem 5.1.1. *If* $f : D \to \mathbb{C}$ *is holomorphic, then* f *is infinitely, continuously, differentiable (of class* C^∞*).*

Proof: Requires the Cauchy-Integral formula.

Definition 5.1.2 (The Laplacian Operator and Harmonic Functions)**.**

The Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. A function f is harmonic if $\Delta f = 0$

Example 5.1.3. Here are some examples of harmonic functions:

- $f(x, y) = c$
- $f(x, y) = ax + by + c$

- $f(x, y) = x^2 - y^2$

Proposition 5.1.4. *Both the real and imaginary components of a holomorphic function are harmonic.*

Proof: Let v_{xy} denote $\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$ and v_{yx} analogously.

$$u_{xx} + u_{yy} = \frac{\partial}{\partial x}(u_x) + \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial x}(v_y) - \frac{\partial}{\partial y}(v_x) = v_{yx} - v_{xy} = 0$$

The final equality holds since holomorphic functions are C^2, therefore their mixed partial derivatives are equal. The proof that v is harmonic is almost identical.

5.2 Simple-Connectedness

Definition 5.2.1 (Simply-Connected Domain)**.**

$D \subset \mathbb{C}$ is simply-connected if any loop can be contracted to a point inside D.

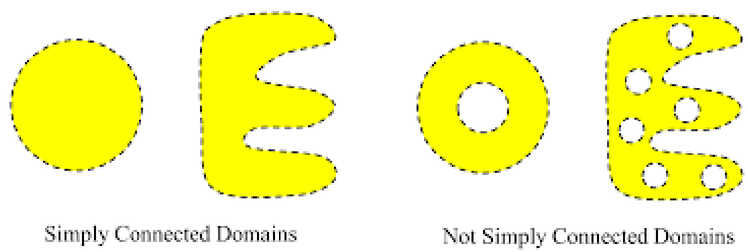

Figure 5.2: Simply-Connected Domains vs Non-Simply-Connected Domains

Example 5.2.2.
The unit disc in $\mathbb{C}$ is simply-connected in $\mathbb{C}$ while an annulus is not simply-connected. Note that both of these are connected. Connected $\not\Rightarrow$ Simply-Connected

5.3 Conjugate Harmonic Functions

Proposition 5.3.1. *If $D \subset \mathbb{C}$ is simply connected and $u : D \to \mathbb{R}$ is a harmonic function, then there exists a unique function (up to the addition of a constant) $v : D \to \mathbb{R}$ such that $f = u + iv$ is holomorphic.*

Proof: Requires Cauchy's theorem (certain integrals to be path independent).

Corollary 5.3.2. *Applying the above proposition to $-if$, we can also find the real component if given the imaginary.*

Example 5.3.3.

Let $u(x,y) = ax + by + c$. u is harmonic. To find its conjugate harmonic function we look at the conditions required by the Cauchy-Riemann equations.

$$v_y = u_x \implies v = \int u_x(x,y).dy + g(x)$$
$$v_x = -u_y \implies v = -\int u_y(x,y).dx + h(y)$$

These integrals are path independent as D is simply connected.

$$\implies \int u_x(x,y).dy + g(x) = -\int u_y(x,y).dx + h(y)$$
$$v = \int a.dy + g(x) = ay + g(x)$$
$$v = -\int b.dx + h(y) = -bx + h(y)$$
$$\implies g(x) = -bx + d,\ h(y) = ay + d \implies v(x,y) = -bx + ay + d$$

Chapter 6

Holomorphic Polynomials

6.1 Properties of Holomorphic Polynomials

Proposition 6.1.1. *If f and g are both holomorphic on D then :*

- $f + g$ *is holomorphic on D*
- $f \cdot g$ *is holomorphic on D*
- *if* $g \neq 0$ *on D, then* $\frac{f}{g}$ *is holomorphic on D*

Proof: The exact same procedure as proving the analogous statements for real-valued differentiable functions.

Corollary 6.1.2. *Polynomials of z are holomorphic.*

Definition 6.1.3 (Degree of a Polynomial)**.**

The degree of a holomorphic polynomial is the minimum positive integer such that $f^{n+1}(z) = 0$

Proposition 6.1.4. *If deg*$(P(z)) = m$ *and deg*$(Q(z)) = n$*, then:*

- $deg(PQ) = m + n$
- $deg(P + Q) \leqslant \max\{n, m\}$

Theorem 6.1.5 (Polynomial Long Division Algorithm)**.** $\forall\ P(z), f(z) \in \mathbb{C}[z]$*. There exists unique polynomials* $h(z), g(z)$ *such that* $deg(g(z) < deg(f(z)$ *and* $P(z) = f(z)h(z) + g(z)$

Theorem 6.1.6 (Fundamental Theorem of Algebra)**.** *Every Polynomial of degree n has exactly n roots counted with mulitplicity.*

We can make a few steps in proving the above statement with the toolkit that we're acquired so far. If we know that $P(z)$, a polynomial of degree n, has a root α_1, we can deduce the existence of the rest of the roots by the division algorithm.
If we let $f(z) = (z - \alpha_1)$ we can apply the division algorithm to get that $P(z) = (z - \alpha_1)h(z) + g(z)$. Since $1 = deg(f(z)) > deg(g(z)) \implies deg(g(z)) = 0 \implies g(z) = C$.

$P(\alpha_1) = (\alpha_1 - \alpha_1)(h(\alpha_1) + g(\alpha_1) \implies g(\alpha_1) = C = 0$. $\deg(h(z)) = n - 1$. We then apply the same procedure to $h(z)$. This produces a factorisation of $P(z)$ into n factors. The one step we're now missing is the guaranteed existence of at least 1 root of each polynomial. This comes later with our discussion of Cauchy's Theorem.

6.2 Rational Functions

Definition 6.2.1 (Rational Function)**.** *Let* $P(z), Q(z)$ *be polynomials.* $R(z) = \frac{P(z)}{Q(z)}$ *is called a rational function.* $R(z) : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$

Remark 6.2.2.

We assume that $P(z)$ and $Q(z)$ have no common factors.

Definition 6.2.3 (Zeroes and Poles of a Rational Function)**.** *By definition,* $R(z_0) = \infty$ *if* $Q(z_0) = 0$. z_0 *is called a pole of* $R(z)$. *If* $R(z_0) = 0$, z_0 *is called a zero of* $R(z)$.

Remark 6.2.4.

How might we define $R(\infty)$? We begin by looking at $R_1(z) = R(\frac{1}{z})$.

$$R(z) = \frac{a_0 + a_1 z + .. + a_n z^n}{b_0 + b_1 z + .. + b_n z^n} \implies R_1(z) = z^{m-n} \frac{a_0 z^n + .. + a_n}{b_0 z^m + .. + b_m}$$

We say that $R(\infty) = R_1(0)$. When does this occur?

Theorem 6.2.5. *A rational function has as many zeros as it has poles on* $\mathbb{C} \cup \{\infty\}$. *The number of zeroes*=max $\deg(P), \deg(Q)$. *This number is called the order of the rational function.*

Proof:

$$F(z) = \frac{P(z)}{Q(z)} = \frac{a_0 + a_1 z + .. + a_n z^n}{b_0 + b_1 z + .. + b_m z^m}$$

$$F_1(z) = F\left(\frac{1}{z}\right) = z^{m-n} \frac{a_0 z^n + a_1 z^{n-1} + .. + a_n}{b_0 z^m + b_1 z^{m-1} + .. + b_m}$$

We then look at the 3 possible cases:

Case 1 ($m > n$): F has a zero of order m-n at ∞. F has m-n+n=m zeroes on $\mathbb{C} \cup \{\infty\}$ since it also has the n zeros guaranteed by the fundamental theorem of algebra applied to P. Hence it has m zeros, it has has m poles because Q has m roots.

Case 2 ($m = n$): $F_1(0) = \frac{a_n}{b_m} \neq 0$. F has m=n poles and zeroes.

Case 3 ($m < n$): F has pole of order n-m at ∞ and F has m zeroes on $\mathbb{C} \cup \{\infty\}$. It also has m poles in $\mathbb{C} \cup \{\infty\}$

Corollary 6.2.6. *Let* $\mathsf{F}(z)$ *be a rational function with* p *zeroes on* $\mathbb{C} \cup \{\infty\}$*. Then* $\forall \mathfrak{a} \in \mathbb{C} \cup \{\infty\}$*, the equation* $\mathsf{F}(z) = \mathfrak{a}$ *has exactly* p *roots.*

Remark 6.2.7.

In the next chapter we look at a particularly important class of rational functions, the so-called Möbius transformations.

Chapter 7

Möbius Transformations

7.1 Classification of Möbius Transformations

Definition 7.1.1 (Möbius Transformation or Linear Fractional Transformation)**.**

A Möbius transformation, also known as a linear fractional transformation, it a rational function of the form:

$$S(z) = \frac{az+b}{cz+d},\ ad - bc \neq 0$$

Corollary 7.1.2.

By the results of the previous chapter, we know that $S(z) = w$ has exactly 1 solution.

Remark 7.1.3.

Many important mappings are included under the umbrella of Möbius transformations:

1. If $c = 0,\ a = d = 1$, we attain a parallel translation $S(z) = z + b\ S(\infty) = \infty,\ \infty$ is a fixed point.
2. If $b = c = 0,\ d = 1, |a| = 1\ S(z) = az$ is a rotation. $S(0) = 0, S(\infty) = \infty$
3. $b = c = d = 0,\ a \in \mathbb{R}^+$, $S(z) = az$ is a homothetic transformation.
4. If $a = d = 0,\ b = c,\ S(z) = \frac{1}{z}$ is called an inversion. $S(1) = 1, S(-1) = -1$

Remark 7.1.4. *Note in the above that a combination of 2. and 3. can used to represent* $S(z) = az = \frac{a}{|a|}|a|z$ *for arbitrary a.*

Remark 7.1.5. *By the above corollary, we can see that any Möbius transformation is a bijection from* $\mathbb{C} \cup \{\infty\}$ *with itself.*

Theorem 7.1.6. *Any Möbius Transformation can be written as a composition of translations,rotations, homothetic transformations and inversions.*

Proof: Posted on Blackboard.

Proposition 7.1.7. *Möbius transformations with* $ad - bc = 1$ *form a group under composition.*

Remark 7.1.8.

We can represent each member, $S(z) = \frac{az+b}{cz+d}$ of the family of the Möbius transformations with $ad - bc = 1$ by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If we wish to compose $S_1(z) = \frac{az+b}{cz+d}$ with $S_2(z) = \frac{ez+f}{gz+h}$ to get $S_1(S_2(z)) = S_1 \circ S_2(z)$, we simply multiply the matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \implies S_1 \circ S_2(z) = \frac{(ae + bg)z + (af + bh)}{(ce + dg)z + (cf + dh)}$$

Theorem 7.1.9. *Let z_0, z_1, z_2 be distinct points in $\mathbb{C} \cup \{\infty\}$. Let w_0, w_1, w_2 also be distinct to each other. Then there exists a unique Möbius transformation $S(z)$ such that $S(z_i) = w_i$*

Proof: Posted on Blackboard.

Theorem 7.1.10. *A Möbius transformation maps circles to circles on $\mathbb{C} \cup \{\infty\}$*

Proof: Posted on Blackboard.

Theorem 7.1.11. *When restricted to $\mathbb{C}$, Möbius transformations map lines to circles or lines and circles to circles or lines.*

Proof: Posted on Blackboard.

Chapter 8

Power Series

"The object of mathematical rigour is to sanction and legitimize the conquests of intuition, and there was never any other object for it."
-Jacques Hadamard (He had a lot of good quotes)

8.1 Convergence of Power Series

Definition 8.1.1 (Power Series)**.**

A power series is a series of the form $f_n(z) = \sum_{i=0}^{n} \alpha_i x^i, \ \alpha_i \in \mathbb{C}$

Remark 8.1.2.

The most general form of power series if $f_n(z) = \sum_{i=1}^{n} \alpha_i (z - z_0)^i$ but without loss of generality we can work with the above simply by translating the plane so that z_0 is at the origin. We say that the power series is centred at z_0

When we consider convergence of power series, we have to consider that z is not constant. Hence convergence of a power series is dependent on what values of z we allow. When we do this we come across this remarkable proposition.

Proposition 8.1.3. *The only possible shapes of the region for convergence for a power series are:*

1. *A point*
2. *A circle*
3. *The whole of* $\mathbb{C}$

The reason for the above becomes apparent when we consider the concept of the radius of convergence.

Definition 8.1.4 (Radius Of Convergence)**.** *The radius of convergence of a power series centred at the origin is the maximum R such that for* $|z| < R$ *the series is convergent.*

In the above proposition, 1. corresponds to R=0, 2. corresponds to $R \in (0, \infty)$ and 3. corresponds to $R = \infty$

We'd like to actually calculate the radius of convergence for power series. Usually the easiest method is to use Hadamard's formula. In order to use this, we must first acquaint ourselves with the limsup.

8.2 limsups and liminfs

Definition 8.2.1 (limsup and liminf of a Sequence)**.**

The limsup of a sequence is defined to be the largest limit point of a series.The liminf is the smallest limit point. If the limit of a sequence exists, the limit, limsup and the liminf are all equal.

Example 8.2.2.

1. $\{(-1)^n\}$ has a limsup of 1, the liminf is -1.
2. 3,5,1,3,5,1.. has limsup 5, the liminf is 1.

Theorem 8.2.3. *If* $\limsup_{n\to\infty} C_n = L$ *(with* $C_n \in \mathbb{R}$*) then:*

1. $\forall N \geqslant 1$ *and* $\varepsilon > 0$, $\exists k \geqslant N$ *such that* $C_k \geqslant L - \varepsilon$ *(there exists elements in the sequence arbitrarily close to L).*
2. $\forall \varepsilon > 0,\ N \geqslant 1$ *such that* $C_k < L + \varepsilon\ \forall k \geqslant N$
3. $\limsup_{n\to\infty} bC_n = bL,\ \forall b \geqslant 0$

Theorem 8.2.4 (Hadamard's Formula)**.** *Let R be the radius of convergence of a power series* $\sum_{n=0}^{\infty} a_n x^n$*, then* $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$

8.3 Abel's Theorem

Theorem 8.3.1 (Abel's Convergence Theorem)**.** *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series, then there exists a radius of convergence* $R \in [0, \infty]$ *with the following properties:*

1. *The series converges absolutely if* $|z| \leqslant \rho$ *where* $\rho \in [0, R)$*. The convergence is uniform.*
2. *The series diverges if* $|z| > R$
3. *If* $|z| < R$*, then* $f(z) = \sum_{n=0}^{\infty} a_n x^n$ *is holomorphic on* $B(0, R)$*. Its derivative is obtained by term by term differentiation. Its derivative has the same radius of convergence.*

Note that the above theorem doesn't tell us anything about when $|z| = R$. This may or may not converge.

Lemma 8.3.2 ($\lim_{m\to\infty} m^{\frac{1}{m}} = 1$)**.**

Proof of Lemma: Let $m^{\frac{1}{m}} = 1 + \delta_m$. We wish to show that $\delta_m \to 0$ as $m \to \infty$.

$$m = (1+\delta_m)^m = 1 + \binom{m}{1}\delta_m + \binom{m}{2}\delta_m^2 + .. > 1 + \binom{m}{2}\delta_m^2 = 1 + \frac{m(m-1)}{2}\delta_m^2$$

$$m > 1 + \frac{m(m-1)}{2}\delta_m^2 \implies m - 1 > \frac{m(m-1)}{2}\delta_m^2$$

Without loss of generality, $m > 1$. We simplify $1 > \frac{m}{2}\delta_m^2 \implies \frac{2}{m} > \delta_m^2 > 0$. By the squeeze theorem, δ_m goes to zero as $m \to \infty$.

Proof of Abel's theorem:
1. If $|z| < R \implies \exists\ \rho \in (|z|, R) \implies \frac{1}{R} < \frac{1}{\rho} \implies \exists \varepsilon > 0$ such that $\frac{1}{R} < \frac{1}{R} + \varepsilon < \frac{1}{\rho}$. Let $L = \frac{1}{R}$ and apply theorem 8.2.3 (2) with $C_n = |a_n|^{\frac{1}{n}} \in \mathbb{R} \implies \exists N \geqslant 1$ such that $\frac{1}{\rho} > |a_n|^{\frac{1}{n}}\ \forall n \geqslant N$.
$\frac{1}{\rho^n} > |a_n|\ \forall n \geqslant N \implies |a_n z^n| = |a_n||z|^n < \left(\frac{|z|}{\rho}\right)^n$ which is a convergent geometric series. Since it is dominated by this convergent series independent of z, it converges uniformly.

2. Choose ρ such that $|z| > \rho > R \implies \frac{1}{\rho} < \frac{1}{R}$. Since $\frac{1}{R} = L$, we can apply theorem 8.2.3(1) $\implies \exists$ arbitrarily high n such that $|a_n|^{\frac{1}{n}} > \frac{1}{\rho} \implies \frac{1}{\rho} = L - \varepsilon,\ |a_n|^{\frac{1}{n}} > \frac{1}{\rho} \implies |a_n| > \left(\frac{1}{\rho}\right)^n \implies |a_n z^n| = |a_n||z|^n > \left(\frac{|z|}{\rho}\right)^n$. $\frac{|z|}{\rho} > 1 \implies \sum_{n=0}^{\infty} |a_n z^n|$ is bounded below by a divergent geometric series, hence it diverges.

3. We look at $f_1(z) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ obtained by term by term differentiation. We wish to show that the derivative our series is indeed $f_1(z)$. We can show that $f_1(z)$ has the same radius of convergence of $f(x)$ because $\lim_{m\to\infty} m^{\frac{1}{m}} = 1$.

$$S_n(z) = \sum_{k=1}^{n-1} a_k x^k,\ R_n(z) = \sum_{k=n}^{\infty} a_k x^k$$

$$\limsup_{n\to\infty} |n a_n|^{\frac{1}{n-1}} = \limsup_{n\to\infty} \left(|n a_n|^{\frac{1}{n}}\right)^{\frac{n}{n-1}} = \limsup_{m\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}$$

To show holomorphicity, choose $z \neq z_0$ with $|z|, |z_0| < \rho < R$

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0) + S_n'(z_0) + \frac{R_n(z) - R_n(z_0)}{z - z_0}$$

$$\frac{R_n(z) - R_n(z_0)}{z - z_0} = \frac{\sum_{k=n}^{\infty} a_k z^k - \sum_{k=m}^{\infty} a_k z_0^k}{z - z_0} = \sum_{k=n}^{\infty} \frac{a_k(z^k - z_0^k)}{z - z_0}$$

$$= \sum_{k=n}^{\infty} a_k (z^{k-1} + z^{k-2} z_0 + .. + z z_0^{k-2} + z_0^{k-1})$$

The last step is justified by the following calculation:

$$(z-z_0)\sum_{n=0}^{k-1} z^{k-(n+1)}z_0^n = \sum_{n}^{k-1} \underbrace{z^{k-n}z_0^n}_{b_n} - \underbrace{z^{k-(n+1)}z_0^{n+1}}_{b_{n+1}} = b_0 - b_k = z^k - z_0^k$$

$$\implies \left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| \leqslant \sum_{k=n}^{\infty} |a_k|k\rho|^{k-1}$$

the above is bounded by the remainder of a convergent power series, hence it is convergent

$\implies \exists N \geqslant 1$ such that $\forall n \geqslant N$ such that $\left|\dfrac{R_n(z)-R_n(z_0)}{z-z_0}\right| < \dfrac{\varepsilon}{3}$

$f_1(z) = \displaystyle\sum_{n=0}^{\infty} na_n z^{n-1}$ converges on $|z| < R$ by the ratio test:

$$\lim_{n\to\infty} \frac{(n+1)|a_{n+1}||z^{n-1+1}|}{(n)|a_n||z^{n-1}|} = \lim_{n\to\infty}\frac{n+1}{n}\cdot\lim_{n\to\infty}\frac{|a_{n+1}||z^n|}{|a_n||z^{n-1}|}$$

The first part of the above is 1 and the second converges because $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ does on $|z| < R$

We proved convergence for $f_1(z) \implies \exists N_2 \geqslant 1$ such that $\forall n \geqslant N_2$

$|S_n'(z_0) - f_1(z_0)| < \dfrac{\varepsilon}{3}$

$S_n(z), S_n'(z)$ polynomials $\implies$ holomorphic and differentiable

$\implies \exists \delta > 0$ such that $\forall\ z$ such that $|z-z_0| < \delta$

$$\left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)\right| < \frac{\varepsilon}{3}$$

$\implies \left|\dfrac{f(z)-f(z_0)}{z-z_0} - f_1(z_0)\right| < \varepsilon$ (By a 3-ε argument)

Example 8.3.3.

1. $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n^3},\ a_n = \frac{1}{n^3} \implies \limsup_{n\to\infty}\left|\frac{1}{n^3}\right|^{\frac{1}{n}} = \left(\limsup_{n\to\infty}\frac{1}{|n|^{\frac{1}{n}}}\right)^3 = 1 = \frac{1}{R}$
2. $\displaystyle\sum_{n=0}^{\infty} n!z^n,\ a_n = n! \implies \limsup_{n\to\infty}(n!)^{\frac{1}{n}} = \infty = \frac{1}{R} \implies R = 0$
3. $\displaystyle\sum_{n=0}^{\infty}\zeta^{n^2}z^n, |\zeta| < 1,\ a_n = \zeta^{n^2} \implies \limsup_{n\to\infty}\left(\zeta^{n^2}\right)^{\frac{1}{n}} = 0 = \frac{1}{R} \implies R = \infty$
4. $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n!},\ a_n = \frac{1}{n!} \implies \limsup_{n\to\infty}\left(\frac{1}{n!}\right)^{\frac{1}{n}} = 0 = \frac{1}{R} \implies R = \infty$

Corollary 8.3.4 (Corollary to part 3 of Abel's theorem)**.** *Power series are infinitely differentiable on* $|z| < R$

Proof: By part 3, we can differentiate as many times as we want and the radius of convergence remains constant.

Corollary 8.3.5. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *have radius of convergence* $R > 0$*. Then* $a_n = \frac{f^{(n)}(0)}{n!}$.

Proof: $f(z) = a_0 + a_1 z + a_2 z^2 .. = \sum_{n=0}^{\infty} a_n z^n$

$$\begin{aligned}
&f(0) = a_0 \\
&f^{(1)}(0) = a_1 \\
&f^{(2)}(0) = 2! a_2 \\
&\vdots \\
&f^{(k)}(0) = k! a_k \implies \frac{f^{(k)}(0)}{k!} = a_k
\end{aligned}$$

Remark 8.3.6. *Power Series are smooth functions. If* $\sum_{n=0}^{\infty} a_n z^n$ *satisfies* $f^{(k)}(z) = 0\ \forall n \geqslant 0,\ \forall |z| > \delta > 0 \implies a_n = 0\ \forall n \geqslant 0 \implies f(z) = 0$.

Proposition 8.3.7. *Let* $\{z_k\}$ *be a sequence of non-zero complex numbers such that* $\lim_{k\to\infty} z_k = 0$ *and let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series. If* $f(z_k) = 0,\ \forall k \in \mathbb{N}$*, then* $f(z) = 0$

Proof: f is continuous at 0 since it is differentiable. $\implies f(0) = \lim_{k\to\infty} f(z_k) = 0 \implies a_0 = 0 = f(0)$

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{k\to\infty} \frac{f(z_k) - f(0)}{z_k - 0} = f'(0) = \sum_{n=1}^{\infty} n a_n z^{n-1}\Big|_{z=0} = a_1 = 0.$$

The limit exists because f(z) is a power series that can be differentiated.
We inductively assume that we have proven that $a_0, a_1, ..., a_{p-1}$ are all zero. We wish to show that $a_p = 0$
Because $a_0 = ... = a_{p-1} = 0$

$$\implies f(z) = \sum_{n=p}^{\infty} a_n z^n \implies a_p = \frac{f(z)}{z^p}\Big|_{z=0} = \sum_{n=p}^{\infty} a_n z^{n-p}\Big|_{z=0} = (a_p + a_{p+1} z + ..)\Big|_{z=0}$$

Look at :

$$\sum_{n=p}^{\infty} a_n z^{n-p} = \sum_{j=0}^{\infty} a_{j+p} z^j$$

From the proof of the assertion 1 of Abel's convergence theorem, we know that $\forall z, |z| < R$, $\exists \rho \; 0 < \rho < R$ such that:

$$\sum_{n=p}^{\infty} |a_n z^n| \leqslant \sum_{n=p}^{\infty} \frac{|z|^n}{\rho^n} \implies \sum_{n=p}^{\infty} |a_n z^{n-p}| \leqslant \sum_{n=p}^{\infty} \frac{|z|^{n-p}}{\rho^n} = \frac{1}{\rho^p} \sum_{n=p}^{\infty} \left(\frac{|z|}{\rho}\right)^{n-p}$$

$$\implies \sum_{n=p}^{\infty} a_n z^{n-p} = f(z) \text{ is convergent}$$

$$\implies \text{ continuous at 0 and differentiable at 0 as well}$$

$$a_p = \sum_{n=p}^{\infty} a_n z^{n-p}\Big|_{z=0} = \lim_{k\to\infty} \frac{f(z_k)}{|z_k|^p} = 0$$

Corollary 8.3.8. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be zero for* $z \in S$ *where* S *is a set with an accumulation point. at 0 and S is inside the domain of convergence of f, then* $f(z) = 0$ *i.e* $a_n = 0$, $\forall n$

Proof: S has an accumulation point at 0. $\exists \{z_k\}$ such that $z_k \neq 0$, $\forall k$ and $\lim_{k\to\infty} z_k = 0$. We apply our proposition to this sequence

Corollary 8.3.9 (Uniqueness of Power Series)**.** *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n \; \forall z \in S$, *where S has an accumulation point at 0 and S is inside both of their domains of convergence. Then* $a_n = b_n \forall n$

Remark 8.3.10. *This corollary gives uniqueness of power series because if* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and we are given that there exists another power series* $g(z)$ *which has the same derivatives as f on a set accumulating at 0* $\implies$ $f(z) = g(z)$, $\forall z$ *in some neighbourhood around the origin.*

Proof:

$$\sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (a_n - b_n) z^n = 0, \; a_n - b_n = 0, \; \forall n \implies a_n = b_n, \; \forall n$$

Chapter 9

Complex Transcendental Functions

We studied polynomials, rational functions, and power series. Next, we introduce the complex exponential and trigonometric functions via power series.

Recall from calculus that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

9.1 The Exponential Function

Definition 9.1.1 (The Exponential Function)**.**

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!}, \quad (0! = 1! = 1)$$

Theorem 9.1.2 (Euler's Formula)**.**

$$e^{i\theta} = \cos\theta + i\sin\theta, \; e^{-i\theta} = \cos\theta - i\sin\theta$$

Proof: If we substitute $i\theta$ into the definition of e^x we get:

$$e^{i\theta} = 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + .. = 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + .. = \underbrace{\left(1 - \frac{\theta^2}{2!} + ..\right)}_{\cos(\theta)} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + ..\right)}_{\sin(\theta)}$$

Corollary 9.1.3. *One can expression* sin *and* cos *as the following:*

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \; \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Remark 9.1.4.

The exponential function converges for all values of $z \in \mathbb{C} \iff$ it's radius of convergence is ∞.

Theorem 9.1.5 (Properties of the Exponential)**.**

1. $(e^z)' = e^z$

2. $e^{a+b} = e^a.e^b$

3. $e^z \neq 0 \forall\, z \in \mathbb{C}$

4. *If* $z = x + iy,\ e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y))$

Proof:

1. $(e^z)' = \left(\sum_{n=o}^{\infty} \frac{z^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n'=0}^{\infty} \frac{z^{n'}}{n'!} = e^z$

2. Let $a = z$ and $b = c - z$ so $a + b = c$
   c is a constant. $(e^a e^b)' = (e^z e^{c-z})' = e^z e^{c-z} + (-1)e^z e^{c-z} = e^z e^{c-z} - e^z e^{c-z} = 0 \implies e^a e^b = c'$, where c' is a constant. To determine it, set $z = 0 \implies e^0 e^c = c' = e^c \implies e^c = c' \implies e^a e^b = e^{a+b}$

3. $e^z e^{-z} = e^{z-z} = 1$, $1 \in \mathbb{C}$ and $\mathbb{C}$ is a field $\implies$ it has no zero divisors $\implies e^z \neq 0,\ e^{-z} \neq 0,\ \forall z$

4. Euler's formula and the observation that it holds on $\mathbb{R}$, a set with an accumulation point at 0.

Corollary 9.1.6.

1. $\cos^2(z) + \sin^2(z) = 1$

2. $(\cos(z))' = -\sin(z),\ (\sin(z))' = \cos(z)$

3. $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b),\ \sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$

Proof:

1. $\cos^2(z) + \sin^2(z) = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 = 1$

2. $(\cos(z))' = \frac{ie^{iz}-ie^{-iz}}{2} = \frac{-e^{iz}+e^{-iz}}{2i} = -\sin(z)$, $(\sin(z))'$ is analogous

3. Routine calculation using Euler's formula.

Example 9.1.7.

$$\begin{aligned}
\tan(z) &= \frac{\sin(z)}{\cos(z)} = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}, \\
\cot(z) &= \frac{1}{\tan(z)} = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \\
\sec(z) &= \frac{1}{\cos(z)} = \frac{2}{e^{iz} + e^{-iz}} \\
\csc(z) &= \frac{1}{\sin(z)} = \frac{2i}{e^{iz} - e^{-iz}}
\end{aligned}$$

Remark 9.1.8.

All of these transcendental functions in $\mathbb{C}$ are expressed in terms of the exponential e^z so there is actually only one transcendental function in complex analysis, namely e^z.

Figure 9.1:

9.2 The Logarithm

To study the inverses of all of the transcendental functions we introduced, we need to look at the inverse of e^z, namely the complex logarithm.

Remark 9.2.1 (Derivation of the Logarithm)**.**

$$\text{Set } e^z = w \iff e^x e^{iy} = w \implies \begin{cases} e^x = |w| \implies x = \ln(|w|) \text{ (uniquely determined)} \\ e^{iy} = \frac{w}{|w|} \text{ (infinitely many solutions)} \end{cases},$$

$\mathrm{Log}(w) := \ln(|w|) + i.\arg(w)$

Remark 9.2.2.

To make this formula well-defined, we need to specify the interval of length 2π where $\arg(w)$ takes values $\implies$ the values of $\log(w)$ differ from one another by multiples of $2\pi i$.

Definition 9.2.3 (Branch of the logarithm)**.**

Each such set of values corresponding to an interval of length 2π is called a branch of the logarithm.

Question: What is the standard way to do this?

We let $\arg(w) \in (-\pi, \pi)$.

Definition 9.2.4. *Principle branch of the Logarithm*

The branch of the logarithm corresponding to $(-\pi, \pi]$ is called the principal branch of the logarithm. **Question:** Why this choice?

Note that $\exists w_0 \in \mathbb{C}$ such that w_0 does not have a logarithm. Indeed, $\log 0$ does not exist because $e^z \neq 0\ \forall z \in \mathbb{C}$ as we showed before and $\arg 0$ does not exist, so we take 0 out.

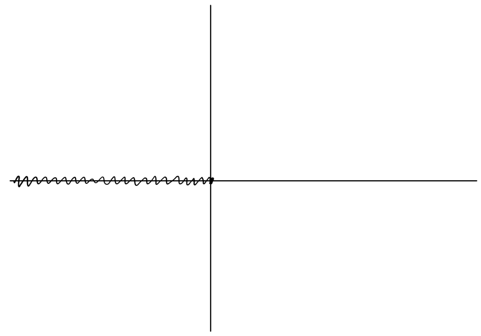

Figure 9.2: Standard branch cut for the Logarithm

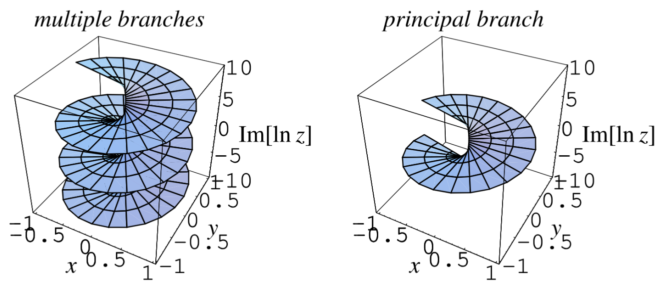

Figure 9.3: Branches for the logarithm visualised

Remark 9.2.5.

$\text{Log}(w)$ is discontinuous on the negative real axis and at 0 $\implies$ we define $\log w$ by making a cut like we cut a piece of paper.

Lemma 9.2.6. $\forall z_1, z_2 \in \mathbb{C}\backslash 0,\ \mathit{log}(z_1 z_2) = \mathit{log}(z_1) + \mathit{log}(z_2)$

Remark 9.2.7.

$\arg(z_1 + z_2) = \arg(z_1) + \arg(z_2)$ This implies that we can leave a branch of the logarithm by multiplying. This means that all computations are to be made in the infinite set of values defining log and arg.

Example 9.2.8.

$$z_1 = -1 + i,\ \mathsf{arg}(z_1) = \frac{3\pi}{4} \text{ (principal value)},\ |z_1| = \sqrt{2}$$
$$z_2 = i,\ \mathsf{arg}(z_2) = \frac{\pi}{2},\ |z_2| = 1$$
$$z_1 \cdot z_2 = -1 - i,\ \mathsf{arg}(z_1 \cdot z_2) = \frac{5\pi}{4} = -\frac{3\pi}{4} \ \mathsf{mod}(2\pi)$$

Remark 9.2.9.

For $x \in \mathbb{R}^+$, $x = e^{\ln(x)}$. In $\mathbb{C}$ when is $a^b = e^{b\mathrm{Log}(a)}$ unique?
Case 1: $a \in \mathbb{R}^+ \implies \mathrm{Log}(a) = \ln(|a|)$ is unique

Case 2: $b \in \mathbb{Z} \implies a^b = \begin{cases} a^b & b > 0 \\ \frac{1}{a^{-b}} & \text{if } b < 0 \\ 1 & \text{if b=0} \end{cases}$

a^b has finitely many values but is not uniquely determined when $b = \frac{p}{q} \in \mathbb{Q}$, then $a^b = a^{\frac{p}{q}}$ has q values. Otherwise it has infinitely many values.

Example 9.2.10.

$$a^b = (i)^i = e^{i\log(i)}, \; \log(i) = \ln(|i|) + i\left(\frac{\pi}{2} + 2\pi k\right) = 0 + i(\frac{\pi}{2} + 2\pi k)$$
$$e^{i(i(\frac{\pi}{2}+2\pi k))} = e^{-\frac{\pi}{2}-2\pi k} \text{ has infinitely many values}$$

Remark 9.2.11.

Using the above definition of the Log and trig functions, one can easily find the inverse trig functions.

Example 9.2.12.

$$\begin{aligned} &\sin(z) = w \iff \frac{e^{iz} - e^{-iz}}{2i} = w \\ &2iw = e^{iz} - e^{-iz} \iff 2ie^{iz}w = e^{2i} - 1 \\ &\iff e^{2iz} - (2iw)e^{iz} - 1 = 0 \\ &\implies e^{iz} = \frac{2iw \pm \sqrt{-4w^2 - 4(-1)}}{2} = iw \pm \sqrt{1 - w^2} \\ &\implies z = \frac{1}{i}\log(iw \pm \sqrt{1 - w^2}) \end{aligned}$$

Chapter 10

Riemann Surfaces

"When you were kids you played with legos, now you are playing with images of the complex plane with certain cuts made to it"

.

-Andreea Nicoara

10.1 Visualising Complex Functions

What does the image of the strip $\mathbb{R} \times (-\pi, \pi)$ look like under the exponential $e^z = e^x(\cos\theta + i\sin\theta)$?

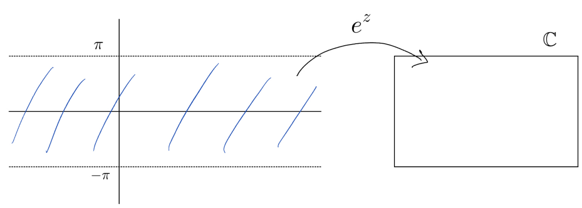

Figure 10.1: Image of a Strip under the Exponetial

Definition 10.1.1 (Fundamental Region of $f(z)$)**.**

A region who's image under f is the whole complex plane.

Remark 10.1.2.

$\mathbb{R} \times (-\pi, \pi),\ \mathbb{R} \times (\pi, 3\pi), \mathbb{R} \times (-3\pi, \pi)$ are all fundamental regions of the exponential.

Remark 10.1.3.

Each fundamental region with some boundary corresponds to a branch of the inverse function (in the case of the exponential, the logarithm)

Remark 10.1.4.

What happens if we want to cut along any other ray originating at 0 and going off to infinity. Is it possible? The answer is yes. The argument of the ray gives where to cut the y axis into strips in the domain plane (where to begin your 2π interval).
It's easier to visualise them as sheets. All sheets of this Riemann surfaces meet. The point at which two sheets meet is called a branch point. Any point on the Riemann surface where at least 2 sheets meet is called a branch point. 0 is a branch point for $f(z) = z^3$ or any other $f(z) = z^n$ with $n \in \mathbb{N}\backslash\{0, 1\}$ In the above, the 3 wedges centred at 0 fundamental regions.

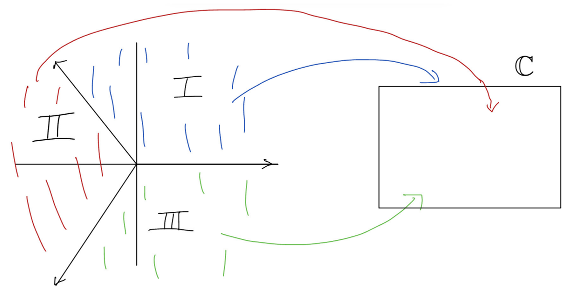

Figure 10.2: Visualisation of 0 as a branch point of $f(z) = z^3$

Definition 10.1.5 (Branch Point)**.**

Any point on a Riemann Surface where at least 2 sheets meet is called a branch point.
Example 10.1.6.
0 is a branch point for any function $f(z) = z^n$ with $n \geqslant 2$. If we look at a curve γ going around a branch point in the domain, the image winds around the branch point of the Riemann surface as many times as there are sheets glued at that branch point.
The order of a branch point =# of sheets -1.

Remark 10.1.7.

1. A branch point does not have to connect all sheets of a Riemann surface
2. The order of a branch point is not necessarily finite.

Remark 10.1.8 ($f(z) = z^3$)**.**

1. The same construction works for any other cut in $\mathbb{C}$ that is a ray originating from 0, not just the positive real numbers.
2. The fundamental regions of the Riemann surface for $f(z) = z^3$ give the branches of $w^{\frac{1}{3}}$ (the inverse of $f(z) = z^3$) $\implies$ we have 3 such branches. Generally for $f(z) = z^n$, there are n branches of $w^{\frac{1}{n}}$, w=0 branch point of order n-1. (n sheets). The Fundamental regions are wedges centred at zero with angle $\frac{2\pi}{n}$

Chapter 11

Cauchy's Theorem

11.1 Statement of Cauchy's theorem

Practically the most important theorem in Complex Analysis.

Theorem 11.1.1. *Let* f *be a holomorphic function on a simply connected domain D and let* γ *be a piece-wise,* C^1*, simply-closed curve inside D. Then the integral along* γ *is 0.*

$$\int_\gamma f(z)dz = 0$$

"Cauchy Unfortunately looks like Vladimir Putin but we still love him"- Andreea Nicoara

Figure 11.1: Augustin Cauchy (left) and Vladimir Putin (right)

Break-down of the statement:

- Simply connected: A Topological domain with "no holes", "Topologically trivial"
- Holomorphic functions: Differentiable in $\mathbb{C}$
- Simply-Closed Curves: Given below
- Piecewise C^1: Given below
- $\int_\gamma f(z)dz$: Given below

11.2 Curves in $\mathbb{C}$

Curves in $\mathbb{C}$ are parameterically defined, i.e $\exists$ a continuous map $\gamma : [a, b] \to \mathbb{C},\ a, b \in \mathbb{R}$
$\vec{\gamma}(t) = (x(t), y(t)) \iff x(t) + iy(t) = \vec{\gamma}(t)$
Example 11.2.1.

1. $\gamma(t) = (cos(t), sin(t)),\ t \in [0, \pi]$
2. $\gamma(t) = e^{2it},\ t \in [0, \pi]$
3. $\gamma(t) = e^{it}; t \in [0, 4\pi]$
4. $\gamma(t) = 1,\ t \in [0, 2\pi]$

3 and 4 are not good for line integrals. What do we need to impose in order to fix the problem?

To fix 4, we want to look at the tangent vector. If $\vec{\gamma}(t) = (x(t), y(t))$, the the tangent vector $T_\gamma(t) = (x'(t), y'(t)) = x'(t) + iy'(t)$. We see that the tangent, we which can also denote by $\dot{\gamma}(t) = (x'(t), y'(t)) = (0, 0)$. We want to only consider piece-wise, continuously differentiable curves. These are curves $\gamma : [a, b] \to \mathbb{C}$ satisfying the following 2 properties:

1. $x(t), y(t)$ are of class C^1 (continuous first order derivatives on some partition $[a, x_1], [x_1, x_2], .., [x_n, b]$ of $[a, b]$. At these points $(x_1, .., x_n)$ for some finite n.
2. At these points $(x_1, .., x_n)$, the derivative may be 0 or undefined but that's okay as long as n is finite and the problems are restricted to these points.

Remark 11.2.2.

- Condition 1 above allows things like:

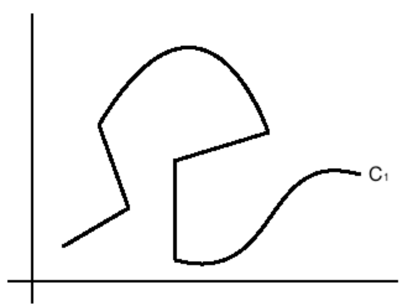

Figure 11.2: A Piecewise Differentiable Curve

- Condition 2) allows examples like 4. Piecewise C^1 rules out things like the Peano Space-Filling Curve (everywhere C^0, but nowhere C^1, mega-nasty, it is generated iteratively, below are the first 3 iterations)

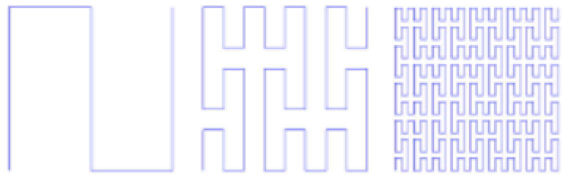

Figure 11.3: The first 3 iterations of the Peano space-filling curve

Definition 11.2.3 (The Trace of a Curve)**.**

If $\gamma : [a, b] \to \mathbb{C}$ is a curve, then the image of $[a, b]$ is called the trace of γ
Note that in the above Examples 1-3 above all have the unit circle as their trace.

Definition 11.2.4 (Endpoints, Closed Curves and Arcs)**.**

We call $\gamma(a)$ the initial point and $\gamma(b)$ is the terminal point. If they equal each other, γ is called a closed curve. If they don't, then γ is called an arc.

Definition 11.2.5 (Simple Curves)**.**

Let $\gamma[a, b] \to \mathbb{C}$ be a curve. If $\exists\ c, d \in [a, b]$ such that $\gamma(c) = \gamma(d)$ with $c \neq d$, we say γ has self-intersections. If a curve is not self-intersecting, it is simple.

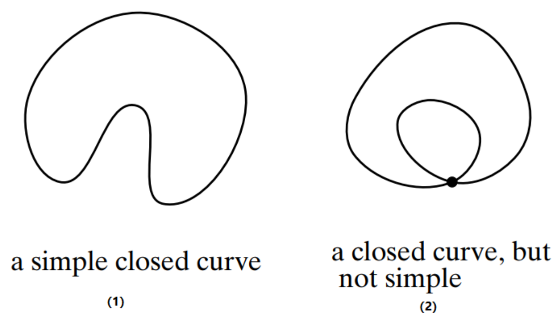

Figure 11.4:

Remark 11.2.6.

Essentially, injectivity holds on simple curves.

Definition 11.2.7 (Orientation of a Curve)**.**

Curves have orientations, in other words $\gamma : [a, b] \to \mathbb{C}$, i.e it goes from $\gamma(a)$ to $\gamma(b)$ along its trace. Notice that $\alpha : [a, b] \to \mathbb{C}$ defined by $\alpha(t) = \gamma(a + b - t)$ goes from $\gamma(b)$ to $\gamma(a)$ (it reverses orientation).

Remark 11.2.8.

$T_\gamma(t) = -T_\alpha(t)$

We now wish to define what we mean by $\int_C f(z)dz$

Definition 11.2.9.

Let $z : [a, b] \to \mathbb{C}$ be a piece-wise C^1 curve γ and let f be a continuous everywhere on $z(t)$, then :

$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt$$

Example 11.2.10.
$f(z) + z^2$ with $\gamma(t) = e^{it},\ t \in [0, 2\pi]$.

$z(t) = cos(t) + isin(t),\ u(t) = cos(t), z; v(t) = sin(t) \implies \dot{u}(t) = -\sin(t),\ \dot{v}(t) = cos(t)$

$$\int_C f(z)dz = \int_0^{2\pi}(cos(t)+isin(t)(-sin(t)+icos(t)).dt = \int_0^{2\pi}(e^{it})^2ie^{it}.dt = i\int_0^{2\pi}e^{3it}dt = 0$$

Note that the above is in line with what Cauchy's theorem states.

A natural question is "What if we considered $z(t) = e^{2it},\ t \in [0, \pi]$ instead of $t \in [0, 2\pi]$?" Is this integral well defined? i.e does it not care about which parametrisation of the curve we choose?

Definition 11.2.11 (Equivalent Parameterisation of Curves)**.**

2 curves $\gamma(t)$ $t \in [a, b]$ and $\alpha(t)$ $t \in [c, d]$ are equivalent if $\exists$ a C^1 bijection $\lambda(t)[c, d] \to [a, b]$ such that $\lambda(d) = b,\ \lambda(c) = a, a; \lambda'(t) \geqslant 0$ and $\gamma(\lambda(t)) = \alpha(t)$
Note that the positivity of the derivative is imposed to ensure that we're not including any changes of orientation. $z_1(t) = e^{2it}z$ $t \in [0, \pi]$ is equivalent to $z_2(t) = e^{it},\ t \in [0, 2\pi]$ with $\lambda(t) = 2t$ is an example.

Proposition 11.2.12.

Equivalence of curves is an equivalence relation (reflexive, symmetric and transitive). To prove it's reflexive, we simple use the identity map, for symmetry, we take the inverse of our bijection and to show transitivity, we simply compose the bijections. We write $\gamma \sim \alpha$

Proposition 11.2.13.

Let γ and α be 2 curves with the same trace and let f be a continuous function on this trace. If $\gamma \sim \alpha$ then:

$$\int_\gamma f = \int_\alpha f$$

i.e the line integral is well defined.

Proof: $\gamma \sim \alpha \implies \exists\, \lambda(t)$ such that $\gamma(\lambda(t)) = \alpha(t) \implies \alpha'(t) = \gamma'(t)\lambda'(t)$
$\alpha : [c, d] \to \mathbb{C},\ \gamma : [a, b] \to \mathbb{C}$.

$$\int_\alpha f = \int_c^d f(\alpha(t))\alpha'(t).dt = \int_c^d f(\gamma(\lambda(t))\gamma'(\lambda(t))\lambda'(t).dt = \int_a^b f(\gamma(\tau))\gamma'(\tau)d\tau = \int_\gamma f \text{ where } \tau = \lambda(t)$$

Proposition 11.2.14 (Properties of line integrals)**.**

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 curve and let $f, g \in C^1$

1. $\int_\gamma f + g = \int_\gamma f + \int_\gamma g$
2. $\int_\gamma cf = c\int_\gamma f$
3. If $-\gamma$ is γ with the opposite orientation, then $\int_{-\gamma} f = -\int_\gamma f$

Lemma 11.2.15.

Let $f : [a, b] \to \mathbb{C}$ be a continuous function, then:

$$\left|\int_a^b f(t).dt\right| \leqslant \int_a^b |f(t)|\,.dt$$

Proof: Posted on Blackboard.

Corollary 11.2.16.

Let $\gamma; [a, b] \to \mathbb{C}$ be a piecewise C^1 curve of length L and let f be a continuous along γ with $|f(z)| \leqslant M$, then:

$$\left|\int_\gamma f\right| \leqslant L.M$$

Proof: With $\gamma(t) = (x(t), y(t))$

$$\begin{aligned}\left|\int_\gamma f\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\ &\leqslant M\int_a^b |\gamma'(t)|.dt = M\int_a^b \sqrt{(x'(t)^2 + (y'(t))^2}dt = ML \qquad \square\end{aligned}$$

We wish to look at

$$\int_\gamma f dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b f(\gamma(t))(x'(t)+iy'(t))dt = \int_a^b f(\gamma(t))\underbrace{x'(t)dt}_{dx} + \int_a^b f(\gamma(t))\underbrace{iy'(t)dt}_{dy}$$

More generally, write $\int_\gamma p dx + \int_\gamma q dy$ (Most general integral along a curve).

What is the best condition that this expression above must satisfy? The answer is that it can be independent of γ but dependent on only the endpoints $\gamma(a)$, $\gamma(b)$. The next question is what condition should p and q satisfy for this to be the case? The answer is the following theorem:

Theorem 11.2.17.

Let D be a domain and $\gamma \subset D$. The line integral $\int_\gamma p.dx + \int_\gamma q.dy$ depends only on $\gamma(a)$ and $\gamma(b) \iff \exists\, U(x,y) : D \to \mathbb{C}$ such that $U_x = p$ and $U_y = q$

Proof:
" $\Longleftarrow$ " by the fundamental theorem of calculus:

$$\int_\gamma p.dx + \int_\gamma q.dy = \int_\gamma U_x.dx + \int_\gamma U_y.dy = \int_a^b U_x x'(t)dt + \int_a^b U_y y'(t).dt$$

$$= \int_a^b \frac{\partial}{\partial t}(U(x(t), y(t)).dt = U(x(b), y(b)) - U(x(a), y(a))$$

The conclusion is that the integral depends only on the endpoints.

" $\Longrightarrow$ " Let $(x(a), y(a)) = (x_0, y_0)$ and $(x(b), y(b)) = (x, y)$. By our assumption, the integral only depends on these points. Hence we can replace γ with a polygonal line $\tilde{\gamma}$ whose components are either vertical or horizontal. This is predicated on the result stating that open and connected in $\mathbb{C} \to$ polygonally connected. Define $U(x,y) = \int_\gamma p.dx + \int_\gamma qd.y$. This is well defined since the integral only depends on the endpoints. We can decompose this into the respective line segments i.e $U(x,y) = \int_{\gamma_1} p.dx + \int_{\gamma_1} q.dy + .. + \int_{\gamma_n} p.dx + \int_{\gamma_n} q.dy$ with γ_i either vertical or horizontal. We consider 2 cases:

1. Last segment γ_n is horizontal $\implies \int_{\gamma_n} q.dy = 0$ and $p = U_x$
2. Last segment γ_n is vertical $\implies \int_{\gamma_n} p.dx = 0$ and $q = U_y$.

We continue this argument inductively. This proves the result.

How does this relate to holomorphicity? We return to

$$\int_\gamma f. \underbrace{dz}_{dx+idy} = \int_\gamma f.dx + \int_\gamma if.dy$$

If we let $p = f$ and $q = if$ by the previous theorem we get that $f(z)dx + if(z)dy$ is an exact differential (its integral only depends on the endpoints) if $\exists\, F(z)$ on D such that $F_x = f$ and $F_y = if$. $\iff -iF_y = f \implies F_x = -iF_y$ but since by the definition of $\int_\gamma f$, f has to be continuous $\implies F$ is holomorphic (by our previous result that said that if the Cauchy Riemann equations are satisfied in a domain and the partial derivatives are continuous, then that implies that the function is holomorphic).

Proposition 11.2.18. *The integral* $\int_\gamma f.dz$ *depends only on the endpoints if and only if f is the derivative of a holomorphic function in a domain D with* $\gamma \subset D$.

Corollary 11.2.19. $\int_\gamma (z-a)^n dz = 0 \; \forall n \geqslant 0$ *and* γ *a closed piecewise* C^1 *curve*

Proof: $f(z) = (z-a)^n \implies F(z) = \frac{(z-a)^{n+1}}{n+1}$ satisfies $F'(z) = f(z)$ and is holomorphic everywhere $\implies \int_\gamma (z-a)^n.dz$ only depends on the endpoints but γ is a closed curve by assumption. $\implies \int_\gamma f = 0$

Remark 11.2.20.

We have no information so far about the case where $n < 0$ (It is not always true in that case).

11.3 Proof Of Cauchy's Theorem

"If you don't remember it you're going to knock your head against a wall...that's the price of not doing things properly..but there are plenty of walls"
-Andreea Nicoara (Discussing why it's important to understand rather than memorise)

We have a certain "plan of attack" to prove Cauchy's theorem:

1. We prove it for a rectangle. This is known as Goursat's Theorem. The proof is posted on Blackboard.

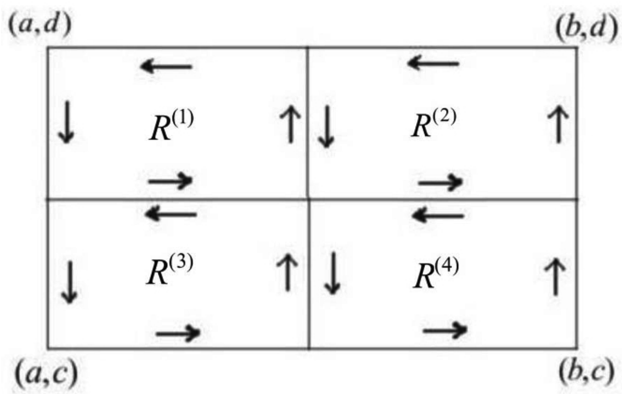

Figure 11.5: Visualisation of the Proof of Goursat's Theorem

   If $f(z)$ is holomorphic on R then $\int_{\partial R} f.dx = 0$ .Goursat noticed that the hypothesis that $f'(z)$ should be continuous is superfluous and gave a bisection argument

2. We prove it for a disc by noticing that one can always inscribe a rectangle due to convexity of the disc.

3. We extend Cauchy's theorem to every simply connected domain D.

Theorem 11.3.1 (Cauchy-Goursat Theorem)**.** *Let f be holomorphic on D, where D is a simply connected domain. Then* $\forall$ *simple, closed contour* $C \subset D$*, the integral along* C *is 0.* $\int_C f = 0$

Theorem 11.3.2 (Weak Cauchy's Theorem)**.** *If f is holomorphic on a simply connected domain D, and f' is continuous on D, then for every simple closed contour* $C \subset D$, $\int_C f = 0$

Proof: Since f' is continuous on D, we can apply Green's theorem. $f = u + iv$, u,v and their partial derivatives are continuous on D. Let E be the region bounded by C.

$$\int_C f(z).dz = \int_C (u+iv)(dx+idy) = \int_C u.dx+u.dy+iv.dx-v.dy = \int_C (u.dx-v.dy)+i\int_C (v.dx+u.dy)$$

We then apply Green's theorem

$$= \int\int_E (-v_x - u_y) dx \wedge dy + i \int\int_E (-v_y + u_x) dx \wedge dy$$

The above is 0 by the Cauchy-Riemann equations.

Remark 11.3.3.

Note that the original theorem is weaker because it considered continuity of the partial derivatives.

Figure 11.6: Goursat thinking about rectangles

Theorem 11.3.4 (Cauchy's Theorem on a Disc)**.** *Let* $D = \{z \in \mathbb{C} \mid |z - z_0| < r\}$*. If f is holomorphic on D, then* $\int_\gamma f(z).dz = 0 \; \forall$ *closed curves* γ *in D.*

Proof: Let $z = x + iy \in D$. Then $(x - x_0)^2 + (y - y_0)^2 < r^2 \implies (x_0, y), (x, y_0) \in D$. Consider the path α consisting of the line segment going from (x_0, y_0) to (x, y_0) and $(x, y_0$ to (x, y). $\alpha \subset D$ since D is convex.
Define $F(z) = \int_\alpha f(z).dz = \int_\alpha f(z).dx + i\int_\alpha f(z).dy$. ($F_y = if$ by the fundamental theorem of calculus.) Now consider the path β consisting of the segment (x, y) to (x_0, y) and (x_0, y) to (x_0, y_0). $\beta \subset D$. $\alpha \cup \beta \subset \partial R$ where R is the rectangle with our four named points as vertices.

$$\int_{\alpha\cup\beta} f(z).dz = \int_\alpha f(z).dz + \int_\beta f(z).dz = \int_{\partial R} f(z).dz = 0 \implies \int_\alpha f(z).dz = \int_{-\beta} f(z).dz$$

If we look at

$$\frac{\partial}{\partial x}\int_{-\beta} f(z).dz = f(z)$$

The last line segment of $-\beta$ is horizontal $\implies$ $f(z)dx + if(z).dy$ is an exact differential (implying it's path independent) $\implies \int_{\gamma} f(z).dz = 0$ for any closed curve γ in D.

Remark 11.3.5.

We don't try to prove the general theorem for simply connected domains but it is a jazzed up version of what we've already done. The additional parts are topological in nature.

Chapter 12

Cauchy Representation Theorem

We continue looking at some of Cauchy's greatest hits with a theorem that tells us that each holomorphic function can be represented as an integral.

12.1 Winding Numbers

Essentially what a winding number of a point with respect to a curve, is the number of times that the curve loops around the point anti-clockwise.

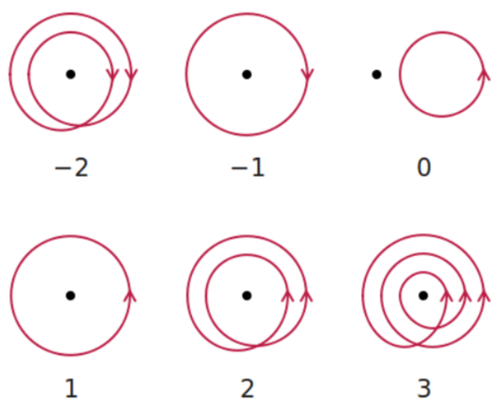

Figure 12.1: winding numbers of a point with respect to various curves

We need to increase the rigour of our definition of a winding number of a point with respect to γ.

Definition 12.1.1 (Winding Number of a Point with Respect to Curve)**.**

Let $\gamma \subset \mathbb{C}$ be a piecewise C^1 curve and let $a \in \mathbb{C}$, $a \notin \gamma$, the winding number of a with respect to γ,

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

Example 12.1.2.

$\gamma(t) = re^{it} + a,\ t \in [0, 2\pi]$
$\gamma'(t) = rie^{it}.$

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{rie^{it}.dt}{re^{it}+a-a} = \frac{1}{2\pi i}\int_0^{2\pi} i.dt = 1$$

Remark 12.1.3.

$n(\gamma, a) = -n(-\gamma, a)$

12.2 Cauchy Representation Formula

We wish to weaken the hypothesis of Cauchy's theorem from f being holomorphic to $\exists \xi \in R$ (a rectangle) such that $\lim_{z\to\xi} f(z)(z-\xi) = 0,\ \xi \notin \partial R,\ f(z)$ holomorphic everywhere else.

Theorem 12.2.1 (Cauchy's theorem with one well-behaved singularity in a rectangle)**.** *Let f be holomorphic on* $R\backslash\{\xi\}$ *and let f satisfy* $\lim_{z\to\xi} f(z)(z-\xi) = 0$*, where* $\xi \notin \partial R$*, then* $\int_{\partial R} f(z).dz = 0$

Proof: Since f is holomorphic on $R\backslash\{\xi\}$, by the previous version of Cauchy's theorem $\int_{\partial R} f = \int_{\partial R_0} f$. Let $\varepsilon > 0$, since $\lim_{z\to\xi} f(z)(z-\xi) = 0$. We choose R_0 so small that $|f(z)(z-\xi)| < \varepsilon$ on ∂R_0 i.e $|f(z)| < \frac{\varepsilon}{|z-\xi|} \iff |f(z)|z-\xi| < \varepsilon$. R_0 has side length $\ell \implies$ the length of ∂R_0 is 4ℓ. $\frac{\ell}{2} \leqslant |t-\xi| \leqslant \frac{\ell\sqrt{2}}{2} \implies |f(z)| < \frac{\varepsilon}{|t-\xi|} \leqslant \varepsilon.\frac{2}{\ell}$

$$\left|\int_{\partial R_0} f\right| \leqslant \int_{\partial R_0} |f| < \frac{2\varepsilon.4\ell}{\ell} = 8\varepsilon \implies \left|\int_{\partial R_0} f\right| < 8\varepsilon$$

Since ε was arbitrary $\int_{\partial R} f = 0$

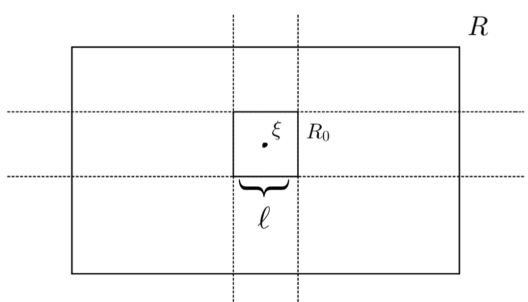

Figure 12.2: Proof of Cauchy's Theorem with a Singularity

Theorem 12.2.2 (Cauchy's theorem with one well-behaved singularity in a disc)**.** *Let D be a disc and let* $\xi \in D$*. If f is holomorphic on* $D\backslash\{\xi\}$ *and f satisfies* $\lim_{z\to\xi} f(z)(z-\xi) = 0$*, then* $\int_\gamma f = 0\ \forall$ *closed curves* γ *such that* $\xi \notin \gamma$

Proof: The proof is analogous to how we proved Cauchy's theorem on a disc using Cauchy's theorem on a rectangle. We apply a similar philosophy to the previous theorem.

Theorem 12.2.3 (Cauchy's Representation Formula)**.** *Let* f *be holomorphic on a disc D and let* γ *be any closed curve in D.* $\forall a \notin \gamma$*, then*

$$n(\gamma, a) \cdot f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z).dz}{z-a}$$

Proof: Set $F(z) = \frac{f(z)-f(a)}{z-a}$. Notice that F is holomorphic on D except at a. At a it satisfies $\lim_{z\to a}(z-a)F(z) = 0$ (because f is continuous). We apply our previous theorem to $F(z)$ to conclude that $\int_\gamma F = 0$

$$\iff \int_\gamma \frac{f(z)-f(a)}{z-a} dz = 0 \iff \int_\gamma \frac{f(z).dz}{z-a} - \int_\gamma \frac{f(a).dz}{z-a} = 0$$

$$\iff \int_\gamma \frac{f(z)}{z-a}.dz = \int_\gamma \frac{f(a)}{z-a}.dz = f(a) \int_\gamma \frac{dz}{z-a}$$

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z).dz}{z-a} = f(a) \cdot \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = f(a) \cdot n(\gamma, a)$$

Remark 12.2.4.

By convention we usually take γ to be a simply closed curve with a contained in its interior. Then $n(\gamma, a) = 1$. We then get that:

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z).dz}{z-a}$$

This is called the Cauchy representation of f because it allows us to represent its value at any point by an integral over a closed curve. If we integrate with respect to ξ, and let $z = a$, we get that:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)d\xi}{\xi - z}$$

We now aim to show that holomorphic functions are infinitely differentiable on some domain Ω. If we let z be any point in $\Omega \implies \exists$ a disc D such that $D \subset \Omega$ with the inclusion being strict. If we take a simply closed curve γ contained in D with z in its interior, we get that

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi).d\xi}{\xi - z}$$

On the right hand side we see that we could differentiate with respect to z under the integral sign. We would get

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{d}{dz} \left(\frac{f(\xi)}{\xi - z} \right) d\xi = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^2} d\xi$$

Inductively we can see that:

$$f^n(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$$

This means f has continuous derivatives of all orders. The justification of the above is posted on Blackboard.

Theorem 12.2.5. *Let f be holomorphic on a domain Ω. $\forall z \in \Omega$ f is continuous and has derivatives of all orders.*

Corollary 12.2.6. *Let f be holomorphic on Ω, then all of its derivatives are holomorphic on Ω*

12.3 Morera and Liouville's Theorem

This is a further corollary of the fact that holomorphic functions are infinitely differentiable. It is an almost converse to Cauchy's theorem.

Theorem 12.3.1 (Morera's Theorem)**.** *If $f(z)$ is defined and continuous in Ω and if $\int_\gamma f = 0$ for all closed curves in Ω, then f is analytic in Ω.*

Proof:$\int_\gamma f.dz = 0$ for all closed curves γ in $\Omega \implies \int_\gamma f.dz$ depends only on the endpoints of α if α is an arc with $\alpha : [a, b] \to \Omega$. If $\alpha(a) = P$ and $\alpha(b) = Q \iff$ f is the derivative of a holomorphic function F on $\Omega \implies f$ is holomorphic by a by corollary 12.2.6.

Theorem 12.3.2 (Liouville's Theorem)**.** *A function which is holomorphic and bounded in the whole of $\mathbb{C}$ must be constant.*

Definition 12.3.3 (Entire Functions)**.**

A function holomorphic on the whole of $\mathbb{C}$ is called entire.

Before we prove this theorem, we require the following lemma:

Lemma 12.3.4 (Cauchy's Estimate)**.** *Let γ be a circle of radius r around $a \in \mathbb{C}$. If f is a function holomorphic on domain D such that $\gamma \subset D$ and if $|f(z)| \leqslant M$ everywhere on $\mathbb{C}$, then $|f^n(a)| \leqslant \frac{Mn!}{r^n}$*

Proof of lemma: The form of the bound suggests using the formula for $f^n(a)$ coming from the Cauchy integral representation for f. We know that

$$f^n(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi$$

In this case $\gamma(t) = a + re^{it},\ t \in [0, 2\pi]$. We substitute $\xi = a + re^{it} \implies d\xi = rie^{it}$.

$$f^n(a) = \frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(a + re^{it})}{(re^{it})^{n+1}} \cdot rie^{it}.dt \implies |f^n(a)| = \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(a + re^{it}}{(re^{it})^n}.dt\right| \leqslant \frac{n!}{2\pi}M\int_0^{2\pi}\frac{dt}{r^n} = \frac{Mn!}{r^n}$$

Proof of Liouville's Theorem: We apply Cauchy's estimate for n=1. f is bounded by some constant M i.e $|f(z)| \leqslant M \ \forall z \in \mathbb{C}$. $\forall z_0 \in \mathbb{C}$, let $\gamma_{z_0,r}$ be a circle of radius r around z_0. By Cauchy's estimate $|f'(z_0)| \leqslant \frac{M}{r}$. As $r \to \infty$ $|f'(z_0)| \to 0$ i.e $f'(z_0) = 0$. Since z_0 was arbitrary, $f'(z) = 0 \forall z \implies f$ is constant. Now we can finally complete our proof of the the fundamental theorem of algebra.

Theorem 12.3.5 (The Fundamental Theorem of Algebra Vol.3 (The threequel))**.** *If $p(z)$ is a polynomial of degree $d > 0$, then it has at least 1 root.*

Proof: Assume $p(z) \neq 0 \forall z \in \mathbb{C}$. Let $f(z) = \frac{1}{p(z)}$. Clearly f is entire. We also know that $p(z) = a_d z^d + .. + a_0$ with $a_d \neq 0$. As $z \to \infty$ z^d dominates the rest of the terms. $p(z) \to \infty \implies f(z) \to 0 \implies \exists\, M$ such that $|f(z)| \leqslant M \implies f(z)$ is constant, this implies that $p(z)$ is also constant which is a contradiction.

12.4 Power Series Expansions of Holomorphic Functions

Recall that the 3rd part of Abel's theorem told us that if the radius of convergence of a power series $\sum_j a_j (z - z_0)^j$ is R, then $f(z) = \sum_j a_j (z - z_0)^j$ is holomorphic in the disc of radius R centred at z_0. We now prove the converse of this using the Cauchy representation formula.

Theorem 12.4.1. *Let $f(z)$ be holomorphic on some domain Ω. $\exists\, z_0$ such that $\{z \in \mathbb{C} \mid |z - z_0| < r_{z_0}\}$ and a power series expansion $\sum_j \frac{f^j(z_0)}{j!}(z - z_0)^j$ which converges uniformly to f on any compact subset of the above ball of radius r_{z_0}.*

Proof: Let $z_0 \in \Omega$, $\exists\, r_0$ such that $B(z_0, r_0) \subset \Omega$. Without loss of generality let $z_0 := 0$. $\forall\, K \subset B(0, r_0)$ where K is compact. Since K is compact, $\exists\, r_2 > 0$ such that $K \subset B(0, r_2)$ and $r_2 < r_0$. We choose r_1 such that $r_2 < r_1 < r_0$. Now let $|z| < r_2$ and let $|\zeta| = r_1$ and apply the Cauchy representation formula to f:

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta| = r_1} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{|\zeta| = r_1} f(\zeta) \cdot \frac{1}{\zeta} \cdot \frac{1}{1 - \frac{z}{\zeta}} d\zeta = \frac{1}{2\pi i} \int_{|\zeta| = r_1} f(\zeta) \cdot \frac{1}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n d\zeta$$

Since $\left|\frac{z}{\zeta}\right| < \frac{r_2}{r_1} < 1$, the above is a convergent geometric series. From this we get uniform convergence.

$$f(z) = \int_{|\zeta| = r_1} f(\zeta) \frac{1}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n d\zeta = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{|\zeta| = r_1} f(\zeta) \cdot \frac{1}{\zeta} \cdot \frac{z^n}{\zeta^n} = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{|\zeta| = r_1} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta\right)}_{= \frac{f^{(n)}(0)}{n!}} z^n$$

$f(z) = \sum_{n=0}^{\infty} \left(\frac{f^{(n)}(0)}{n!}\right) z^n$ i.e f has a convergent power series representation on $B(0, r_2) \subset K$

Remark 12.4.2.

We note how powerful the Cauchy representation formula is. It gives a power series expansion for f and the coefficients in terms of the derivatives of f.

Corollary 12.4.3. *Let f,g be holomorphic functions on Ω. If we have a sequence $\{z_j\}$ such that* $\lim_{j\to\infty} z_j = z_0 \in \Omega$ *and if* $f(z_j) = g(z_j)$*, then f and g are identical on* Ω*.*

Proof: We want to expand f and g in terms of their power series around z_0, say f on $B(z_0, r_1)$ and g on $B(z_0, r_2)$. Let $r = min\{r_1, r_2\}$. Let $\{z_{j_k}\} \subset \{z_j\} \cap B(z_0, r)$ be a sub-sequence such that f and g both have convergent power series expansions. The functions agree at each point in the sequence. Since the limit of the sequence is z_0 they agree everywhere on $B(z_0, r)$ by a previous theorem about power series.
By repeating this argument, we show that $f = g$ everywhere on Ω because Ω is a domain which implies that it's path connected. This means $\forall\ z_0 \in \Omega$, $\exists$ a path γ between z and z_0. We take a small neighbourhood in Ω along this path and repeat the argument above. Since the path is compact in Ω finitely many neighbourhoods cover it

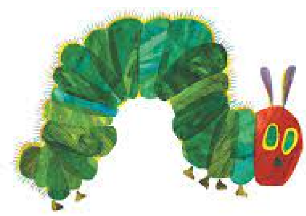

Figure 12.3: Visualisation of Proof

Remark 12.4.4.

This argument involving a path will be used again later. It is the basis for what is called "analytic continuation". It is a useful technique for constructing holomorphic function.

Chapter 13

Zeroes and Singularities of Mappings

13.1 Local Representation around Zeroes

Proposition 13.1.1 (Local Representation of a Holomorphic Function near a Zero)**.** *Let f be holomorphic on Ω and let $z_0 \in \Omega$. If $f(z_0) = 0$ but $f \neq 0$, then z_0 is a zero of finite order d i.e $f(z) = (z - z_0)^d g(z)$ on $B(0, r)$ where $g(z_0) \neq 0$*

Proof: The fact that the function is not identically zero means that $\forall r > 0 \ \exists \ z_1 \in \Omega$ such that $|z_1 - z_0| < r$ and $f(z_1) \neq 0$. Assume for the sake of contradiction that f vanishes to infinite order at z_0 (all its derivative vanish at z_0. Let us consider the power series expansion for f around z_0 which converges on $B(z_0, r) \subset \Omega$ for some r. If all the derivatives vanish, then our convergent power series expansion is identically equal to 0 $\implies$ $f = 0$ in this open ball. This is a contradiction since we know the existence of z_1 such that $f(z_1) \neq 0$. Since the order of vanishing of f at z_0 is finite $\implies$ $\exists$ a minimal k such that $f^{(k)}(z_0) \neq 0$. We now wish to see how d and k are related.

$$f(z) = (z - z_0)^k \left(\frac{f^k(z_0)}{k!} + \frac{f^{k+1}(z_0)}{(k+1)!}(z - z_0) + .. \right)$$

We shall call the part inside the bracket $g(z)$. This is a convergent power series on $B(z_0, r)$ since the power series converges. Moreover $g(z_0) \neq 0$ because $f^k(z_0) \neq 0$. Hence d=k.

Example 13.1.2.

$f(z) = \sin(z),\ f(0) = 0$. What is the order of this 0 at 0?

$f(z) = z - \frac{z^3}{3!} - \frac{z^5}{5!} + .. = z\left(1 - \frac{2}{3!} + ..\right)$. The order of the lowest non-vanishing derivative at 0 is 1.

Proposition 13.1.3. *Zeroes of a holomorphic function f, such that f is not identically zero on a domain Ω are isolated*

Proof: Isolated means that $\nexists$ a sequence $\{z_n\}$ in Omega such that $\lim_{n\to\infty} z_n = z,\ f(z_n) = 0\ \forall z_n \in\ \Omega$. Assume that our zero is not isolated $\implies$ f coincides with $g = 0$ on a set with accumulation point in Ω $\implies$ $f = 0$

Remark 13.1.4.

The moral of the story is that with respect to zeroes, holomorphic functions behave like polynomials i.e the zeroes are isolated and of finite order. This does not mean however that the number of zeroes has to be finite.

13.2 Classification of Singularities

We have 3 types of singularity:

1. Removal Singularities
2. Poles Singularities (non-removable)
3. Essential Singularities (non-removable, "The worst type, or maybe the most interesting type")

Theorem 13.2.1. *Let* Ω *be a domain with* $a \in \Omega$ *and let f be a holomorphic function on* $\Omega \backslash \{a\}$*.* $\exists$ $F(z)$ *that is holomorphic on* Ω *such thar* $F(z) = f(z)$ *on* $\Omega \backslash \{a\}$ $\iff$ $\lim_{z \to a} f(z)(z-a) = 0$*. If such F exists, then it is unique*

Definition 13.2.2 (Removable Singularity)**.**

If f fulfills the above hypotheses, then f is said to have a removable singularity at a since $\exists$ F that's holomorphic on Ω that extends f to a.

Proof of Theorem:
" $\implies$ " if $\lim_{z\to a} f(z)(z-a) = \lim_{z\to a} F(z)(z-a) = 0$ since F is holomorphic and therefore continuous.
" $\impliedby$ " Let γ be a circle of radius r in Ω containing a. If $z_0 \neq 0$ and $z_0 \in \gamma$. Let $G(z) = \frac{f(z)-f(z_0)}{z-z_0}$. By the same argument given in the proof of the Cauchy representation formula, $\lim_{z\to z_0} G(z)(z-z_0) = 0$ so

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z_0}, \text{ define } F(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} \text{ inside } \gamma$$
$$\text{then } F(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - a}$$

We claim that F is holomorphic everywhere in γ. Apart from $z = a$, $F(z) = f(z)$ and both f and F are holomorphic. If we look at $\lim_{z\to a} F(z)(z-a) = \lim_{z\to a} f(z)(z-a) = 0$
By Cauchy's theorem (with well behaved singularity) applied to the disc contained inside γ, $\int_\gamma F(z).dz = 0$ By Morera's theorem in the same disc, $F(z)$ is holomorphic inside γ.
To prove uniqueness, we know F is holomorphic, this implies it is continuous on Ω and also at a. $F(a) = \lim_{z\to a} F(z) - \lim_{z\to a} f(z)$ which exists. F and f only differ at a which is determined by this limit. Thus we have shown that F is unique.

Definition 13.2.3 (Pole of a function)**.**

A complex valued function $f : \Omega \to \mathbb{C} \cup \{\infty\}$ is said to have a pole at $z_0 \in \Omega$ if $\lim_{z \to z_0} f(z) = \infty$ and f is holomorphic on a punctured disc around z_0.

Our goal now is to show that the order of the pole is finite.

13.3 Meromorphic functions

"True rigour is productive, being distinguished in this from another rigour which is purely formal and tiresome, casting a shadow over the problems it touches"
-Emilé Picard

Definition 13.3.1 (Meromorphic Functions)**.**

A function f which is holomorphic on a domain Ω that's holomorphic except for some poles is called meromorphic.

Remark 13.3.2.

Holomorphic functions are trivially meromorphic. Rational functions are the simplest non-trivial examples of meromorphic functions.

Theorem 13.3.3. *If $f : \Omega \to \mathbb{C}$ is holomorphic on a punctured disc around z_0. Then $\exists$ a holomorphic g on some smaller disc contained in this disc and $\exists$ a finite d such that*

$$f(z) = \frac{g(z)}{(z - z_0)^d} \text{ and } g(z_0) \neq 0$$

Proof: Posted on Blackboard.

Corollary 13.3.4. *Poles of meromorphic functions are isolated.*

Proof: In the representation $f(z) = \frac{g(z)}{(z-z_0)^d}$ $g(z_0) \neq 0$ and is holomorphic inside a disc around $z_0 \implies$ no other poles inside this disc.

Remark 13.3.5.

The most general form of a meromorphic function is $f(z) = \frac{g(z)}{h(z)}$ where g and h are holomorphic with h not the 0 map. If g and h have a common zero, then this zero is a removable singularity for f.

Example 13.3.6.

1. $f(z) = \frac{\sin(z)}{\cos(z)} = \tan(z)$
2. Rational functions

Proposition 13.3.7. *Let f and g be 2 meromorphic functions such that g is not the 0 map. Then:*

- *f+g is meromorphic*

- $f \cdot g$ *is meromorphic*
- $\frac{f}{g}$ *is meromorphic.*

Remark 13.3.8.

$\mathcal{O}$ is the ring of holomorphic functions and $\mathcal{M}$ is the field of meromorphic functions.

If $f \in \mathcal{M}$ at a point z_0, we have 2 possibilities:

1. f is holomorphic at $z_0 \implies \exists$ local convergent Taylor expansion
2. f has a pole at $z_0 \implies f(z) = \frac{g(z)}{(z-z_0)^d} \iff g(z) = f(z)(z-z_0)^d$ is holomorphic with a convergent Taylor expansion near z_0.

$$g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \implies f(z) = \underbrace{\frac{a_0}{(z-z_0)^d} + .. + \frac{a_{d-1}}{z-z_0}}_{\text{The principal part}} + a_d + a_{d+1}(z-z_0) + ..$$

The part up to and not including a_d is called the principal part of f. The rest is called the holomorphic part. The principal part describes the pole $\implies$ this will be the part that will give us residues since the holomorphic part integrates to 0 by Cauchy's theorem. The expansion above is a generalisation of a Taylor expansion called a Laurent expansion (i.e it allows negative powers). The Laurent expansion of f looks like:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

Definition 13.3.9 (Essential Singularities)**.** *An essential singularity of a function is a point for which the expansion about that point has infinitely many terms in the principal part.*

Example 13.3.10.

$$f(z) = e^{\frac{1}{z}} \text{ at } z_0 = 0$$

The above has no limit in $\mathbb{C} \cup \{\infty\}$. It has different behaviour depending on the direction from which we approach z_0.
Along the positive real axis, we get that $f(z) \to \infty$. Along the negative real axis we get $f(z) \to 0$. Along the imaginary axis, we have $|f(z)| \to 1$ (all the points on the unit circle).

Theorem 13.3.11 (Casorati-Weierstrass Theorem)**.** *A holomorphic function comes arbitrarily close to every value in* $\mathbb{C} \cup \{\infty\}$ *in a neighbourhood of an essential singularity a.*

Proof: Assume it does not hold i.e $\exists\, w_0$ and $\varepsilon > 0$ such that $|f(z) - w_0| > \varepsilon\ \forall z$ in a punctured neighbourhood of radius δ around a.
Consider $g(z) = \frac{1}{f(z)-w_0}$ which ought to be holomorphic and bounded in the disc with $|g(z)| < \frac{1}{\varepsilon}$. Let $\tilde{g}(z)$ be the extension of g. We have two cases for $\tilde{g}$:

1. $\tilde{g}(a) \neq 0 \implies f(z) - w_0$ has a finite limit. If $w_0 = \infty$, $f(z) - w_0$ has a limit point at a. If $w_0 \in \mathbb{C}$ then $f(z) - w_0$ has a removable singularity at a. Both of these are contradictions.

2. $\tilde{g}(a) = 0 \implies \lim_{z \to a} \left|\frac{1}{g(z)}\right| = \lim_{z \to a} |f(z) - w_0| = \infty$. If $w_0 \neq \infty$ then $f(z) - w_0$ has a pole at a. If $w_0 = \infty \implies f(z)$ stays away from $\infty \implies f(z)$ is bounded in a neighbourhood of a, then a is a removable singularity. Both of these cases are contradictions.

Theorem 13.3.12 (Picard's Theorem). *Let $a \in \mathbb{C}$ be an isolated essential singularity of f. Then away from a, f assumes every complex number infinitely often with at most one exception*

Proof: Way too difficult for this module

Remark 13.3.13.

This theorem only applies to $\mathbb{C}$. No mention of the Riemann sphere. f is then allowed to miss 2 values.

Theorem 13.3.14 (Picard's Little Theorem for Holomorphic Functions). *If an entire function omits 2 points $w_1, w_2 \in \mathbb{C}$ then f is constant*

Proof: Way too difficult for this module

Theorem 13.3.15 (Picard's Little Theorem for Meromorphic Functions). *If a meromorphic function omits 3 values $w_1, w_2, w_3 \in \mathbb{C}$ then f is constant.*

Proof: We use Picard's little theorem for holomorphic functions. Let $g(z) = \frac{1}{f(z)-w_1} \in \mathcal{O}$. $g(z)$ omits $\frac{1}{w_2-w_1}$ and $\frac{1}{w_3-w_1} \implies$ g is constant with $g(z) = k$
We get 2 cases:

1. k=0, then $f(z) - w_1 = \infty$ with w_1 finite $\implies g(z) = \infty$ which is a contradiction

2. $k \neq 0$, then $k = \frac{1}{f(z)-w_1} \implies f(z) = w_1 + \frac{1}{k}$

Remark 13.3.16. *More general ideas about value distribution come from an area of mathematics called Nevalinna theory.*

Figure 13.1: Picard

13.4 The Argument Principle

We wish to further develop our idea of a winding number. We have yet to verify that the winding number is necessarily an integer.

Theorem 13.4.1. *For any closed curve γ and $a \notin \gamma$, the winding number of a with respect to γ is a integer*

Let γ given by $z(t)$ with $t \in [0,1]$ and $n(a,\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$. Define the following function:

$$h(\tau) = \int_0^\tau \frac{z'(t).dt}{z(t)-a},\ \tau \in [0,1]$$

By the fundamental theorem of calculus: $h'(\tau) = \frac{z'(\tau)}{z(\tau)-a}$. Consider $e^{-h(\tau)}(z(\tau)-a)$ and take its derivative with respect to τ:

$$\frac{d}{dt}[e^{-h(\tau)}(z(\tau)-a)] = -h'(\tau)e^{-h(\tau)}(z(\tau)-a)+e^{-h(\tau)}z'(\tau) = -z'(\tau)e^{-h(\tau)}+z'(\tau)e^{-h(\tau)} = 0$$

This implies that $e^{-h(\tau)}(z(\tau)-a)$ is constant a C. If we let $\tau = 0$, the

$$h(0) = \int_0^0 \frac{z'(t).dt}{z(t)-a} = 0,\ z(0)-a = C \implies e^{h(\tau)} = \frac{z(\tau)-a}{z(0)-a}$$

If we set $\tau = 1$, then $z(1) = z(0)$ since γ is closed $\implies e^{h(1)} = 1 \implies h(1) = 2\pi k i = \int_\gamma \frac{dz}{z-a}$

Theorem 13.4.2 (The Argument Principle for Meromorphic Functions)**.** *Let $a_1, .., a_p$ be the zeroes of a meromorphic function in a disc D with orders $d_1, .., d_p$ respectively. Let $b_1, .., b_m$ be the poles of f in D with orders $k_1, .., k_p$ respectively. For any curve γ that does not pass through any a_i or b_i then :*

$$\sum_{j=1}^{p} d_j n(\gamma, a) - \sum_{\ell=1}^{m} k_\ell n(\gamma, b_\ell) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z).dz}{f(z)}$$

Remark 13.4.3.

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z).dz}{f(z)}$$

Counts the zeroes and poles insider γ since it is equal to the sum of the orders of the zeroes inside γ-the sum of the orders of the poles inside γ. If γ is a simple closed curve in D (since all winding numbers are either 0 if outside or 1 if inside)

Proof of Theorem: $p, m < \infty$. Then by piecing together representations near zeroes and poles.

$$f(z) = \frac{\prod_{j=1}^{p}(z-a_j)^{d_j}}{\prod_{\ell=1}^{m}(z-b_\ell)^{k_\ell}} g(z) \text{ where g is holomorphic and non-zero in D}$$

We claim that:

$$\frac{f'(z)}{f(z)} = \frac{d_1}{z-a_1} + .. + \frac{d_p}{z-a_p} + \frac{(-k_1)}{z-b_1} + .. + \frac{(-k_m)}{z-b_m} + \frac{g'(z)}{g(z)}$$

To prove the claim we simply use linearity of the derivative alongside the product rule.
Case 1: $p = 1, m = 0$ one zero, no poles

$$\begin{aligned}
f(z) &= (z-a_1)^{d_1} g(z) \\
f'(z) &= d_1(z-a_1)^{d_1-1} g(z) + (z-a_1)^{d_1} g'(z) \\
\frac{f'(z)}{f(z)} &= \frac{d_1}{z-a_1} + \frac{g'(z)}{g(z)}
\end{aligned}$$

Case 2: $p = 0, m = 1$ no zeroes, one pole

$$\begin{aligned}
f(z) &= \frac{g(z)}{(z-b_1)} k_1 \\
f'(z) &= \frac{-k_1}{(z-b_1)^{k_1+1}} g(z) + \frac{g'(z)}{z-b_1} k_1 \\
\frac{f'(z)}{f(z)} &= \frac{-k_1}{z-k_1} + \frac{g'(z)}{g(z)}
\end{aligned}$$

We've proven the claim so we can now compute the integral.

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z).dz}{f(z)} = d_1 \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{.dz}{z-a_1}}_{n(\gamma,a_1)} +.. + d_p \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a_p} - \frac{1}{2\pi i}\int_\gamma \frac{.dz}{z-b_1} + ..+$$

$$(-k_m)\frac{1}{2\pi i}\int_\gamma \frac{.dz}{z-b_m} + \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{g'(z).dz}{g(z)}}_{=0}$$

Corollary 13.4.4. *If f is holomorphic on a disc D and has zeroes $a_1, .., a_p$, not necessarily distinct with each having order 1, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z).dz}{f(z)} = \sum_{j=1}^{p} n(\gamma, a_j) \text{ on every closed curve } \gamma$$

Proof: We observe that:

$$\frac{d}{dz}[\log(f(z))] = \frac{f'(z)}{f(z)}$$

If γ is given by a parameter $t \in [0,1]$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z).dz}{f(z)} = \frac{\log(\gamma(1)) - \log(\gamma(0))}{2\pi i} = \frac{1}{2\pi}(arg(\gamma(1)) - arg(\gamma(0)))$$

The last equality holds because of the following properties of the logarithm and closed curves.

$$\log(\gamma(1)) = \ln|\gamma(1)| + i\arg(\gamma(1))$$
$$\log(\gamma(0)) \ln|\gamma(0)| + i\arg(\gamma(0))$$
$$|\gamma(1)| = |\gamma(0)|$$

To see how many times $f(z)$ assumes the value a instead of 0, we look at:

$$\frac{1}{2\pi i}\underbrace{\int_\gamma \frac{f'(z).dz}{f(z)-a}}_{\frac{d}{dz}(\log(f(z)-a)} = \frac{1}{2\pi i}\int_{f(\gamma)} \frac{dw}{w-a} = n(f(\gamma), a) = n(f(\gamma), b) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z).dz}{f(z)-b}$$

The above holds whenever both a and b are contained inside $f(\gamma)$ or are both outside of it. Note that $f(\gamma)$ is also a closed curve.

Corollary 13.4.5. *(Open Mapping Theorem) A non-constant holomorphic function maps open sets to open sets.*

Proof: For any z_0 inside Ω domain of f. Let $f(z_0) = a$ and let γ be a circle around z_0 such that it's inside Ω and $f(z) - a$ only has one zero inside γ. This is true because holomorphic functions have isolated zeroes and we can take γ close enough to z_0 to make that the case. Look at $f(\gamma) = \Gamma$ closed curve. Since γ is compact and f is holomorphic hence continuous $\implies f(\gamma) = \Gamma$ compact $\implies \exists\ r_a > 0$ such that the disc of radius r_a centered at a $B(a, r_a) \subset f(\Omega)$ and this disc has no intersection with Γ, $\{B(a, r_a)\} \cap \Gamma = \emptyset$
Now we want to look at $f(z) - a = g(z)$. $f(z_0) = a \implies$ g has a zero of order p at z_0 ($p \geqslant 1$ here). By the above remark, for any $b \in B(a, r_a)$, its winding number with respect to Γ and that of a with respect to Γ are the same, $n(\Gamma, b) = n(\Gamma, a)$. That means $f(z) - b$ also has p roots $z_1, .., z_p$ inside γ for some $\varepsilon > 0$ i.e $f(z_i) = b \implies b \in f(\Omega)\ \forall b \in B(a, r_a)$ hence the image of every open set is open since this construction holds for every point a in the image.

Remark 13.4.6.

Note the distinction between a continuous map and an open map. A continuous map means that preimages of open sets are open while for an open map, the images of open sets are open.

13.5 Applications of the Argument Principle

Theorem 13.5.1 (Rouché's Theorem)**.** *Let f and g be holomorphic inside and on a closed curve γ. Suppose also that $|g(z)| < |f(z)|$ on γ. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeroes inside γ.*

Proof: Let:

$$N = \frac{1}{2\pi i}\int_\gamma \frac{f'(z) + g'(z)}{f(z) + g(z)}.dz$$

This is the number of zeroes of $f(z)+g(z)$ inside γ be the argument principle. We then define $\varphi(z) = \frac{g(z)}{f(z)}$.

$$\begin{aligned}
&\frac{f'(z)+g'(z)}{f(z)+g(z)} = \frac{\frac{f'(z)}{f(z)} + \frac{g'(z)}{f(z)}}{1+\varphi(z)} = \frac{f'(z)}{f(z)} \cdot \frac{1}{1+\varphi(z)} + \frac{g'(z)}{f(z)} \cdot \frac{1}{1+\varphi(z)} \\
&= \frac{f'(z)}{f(z)} \cdot \frac{1+\varphi(z)}{1+\varphi(z)} - \frac{\varphi(z)}{1+\varphi(z)} \frac{f'(z)}{f(z)} + \frac{g'(z)}{f(z)} \cdot \frac{1}{1+\varphi(z)} \\
&= \frac{f'(z)}{f(z)} + \frac{1}{1+\varphi(z)} \left(\frac{g'(z)}{f(z)} - \frac{f'(z)}{f(z)} \cdot \frac{g(z)}{f(z)} \right) \\
&= \frac{f'(z)}{f(z)} + \frac{1}{1+\varphi(z)} \left(\frac{f(z)g'(z) - f'(z)g(z)}{(f(z))^2} \right) \\
&= \frac{f'(z)}{f(z)} + \frac{1}{1+\varphi(z)} \cdot \frac{d}{dz} \left(\frac{g(z)}{f(z)} \right) \\
&N = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)+g'(z)}{f(z)+g(z)}.dz = \frac{1}{2\pi i} \int_\gamma \frac{f'(z).dz}{f(z)} + \frac{1}{2\pi i} \int_\gamma \frac{\varphi'(z).dz}{1+\varphi(z)}
\end{aligned}$$

We claim that:

$$\frac{1}{2\pi i} \int_\gamma \frac{\varphi'(z).dz}{1+\varphi(z)} = 0$$

To show this, we use our assumption:

$$\begin{aligned}
&|g(z)| < |f(z)| \implies |\varphi(z)| < 1 \implies \frac{1}{1+\varphi(z)} = \sum_{n=0}^{\infty} (-1)^n (\varphi(z))^n \\
&\implies \frac{\varphi'(z)}{1+\varphi(z)} = \sum_{n=0}^{\infty} (-1)^n (\varphi(z))^n \varphi'(z) \\
&\implies \frac{1}{2\pi i} \int_\gamma \frac{\varphi'(z).dz}{1+\varphi(z)} = \frac{1}{2\pi i} \int \sum_{n=0}^{\infty} (-1)^n (\varphi(z))^n \varphi'(z).dz \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (-1)^n \int (\varphi(z))^n \varphi'(z).dz = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (-1)^n \frac{(\varphi(z))^{n+1}}{n+1} \Big|_{z(0)}^{z(1)} = 0,
\end{aligned}$$

where we parameterise the curve by $t \in [0,1]$ and thus have $z(0) = z(1)$.
Another way to see that

$$\frac{1}{2\pi i} \int_\gamma \frac{\varphi'(z).dz}{1+\varphi(z)} = 0$$

is to notice that $\left|\frac{g(z)}{f(z)}\right| < 1 \implies \varphi(z)$ never reaches the value -1, hence $\frac{\varphi'(z)}{1+\varphi(z)}$ is analytic, and we can apply Cauchy's theorem. This means that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)+g'(z)}{f(z)+g(z)}.dz = \frac{1}{2\pi i} \int_\gamma \frac{f'(z).dz}{f(z)}.$$

Theorem 13.5.2 (Hurwitz's Theorem). *If the sequence $\{f_n\}$ consists of holomorphic functions on a domain Ω such that each f_n is non-zero on Ω and $\{f_n\} \to f$ uniformly on each compact subset of Ω then $f \equiv 0$ or f has no zeroes.*

Proof: $\{f_n\} \to f$ uniformly on compact subsets of Ω. This means we can exchange the integral and the limit. Let γ be any closed curve in Ω.

$$\int_\gamma f(z).dz = \int_\gamma \lim_{n\to\infty} f_n(z).dz = \lim_{n\to\infty} \int_\gamma f_n(z).dz = 0.$$

By Morera's theorem, f is holomorphic.
Assume $\exists\, z_0 \in \Omega$ such that $f(z_0) = 0$ but $f \not\equiv 0$ so $\exists\, \delta > 0$ such that in the punctured disc of radius δ around z_0 $f(z) \neq 0$ and $|f(z)| \geqslant M$ on $\{z \mid |z - z_0| = \delta\}$ (true because the zeroes of a holomorphic function are isolated). $\overline{B}(z_0, \delta)$ is a compact subset of Ω. Because $\overline{B}(z_0, \delta)$ is compact, $\exists\, n_0 \geqslant 1$ such that $|f_n(z) - f(z)| \leqslant \frac{M}{2}$ $\forall n \geqslant n_0$, but then $|f_n(z) - f(z)| \leqslant \frac{M}{2} < M \leqslant |f(z)| \implies$ by Rouché's theorem, then $f_n(z) - f(z) + f(z) = f_n(z)$ has the same number of zeroes as f inside γ which is 1. This is a contradiction since $f_n \neq 0$ on Ω. It follows that $f \equiv 0$ on Ω or f has no zeroes (the two conditions that invalidate the argument given above).
Example 13.5.3.
Let $f_n(z) = \frac{e^z}{n}$, $f_n(z) \neq 0$ because e^z has no zero in $\mathbb{C}$. But $\lim_{n\to\infty} f_n(z) \equiv 0$ on compact subsets of $\mathbb{C}$ since given any compact subset of $\mathbb{C}$, we can take n large enough.

13.6 The Maximum Principle

Theorem 13.6.1 (The Maximum Principle (Negative Formulation)). *Let f be holomorphic and non-constant on domain Ω. Then $|f(z)|$ has no maximum in Ω.*

Remark 13.6.2.

1. $\mathbb{C}$ is not an ordered field $\implies$ the only possible Maximum Principle is for $|f(z)|$
2. We present 2 proofs and both are important.

Proof 1 (Corollary of the Open Mapping Theorem): $\forall z_0 \in \Omega,\ f(z_0) = w_0$. f is holomorphic and non-constant $\implies$ $f(\Omega)$ is open so $\exists \varepsilon$ such that $B(w_0, \varepsilon) \subset f(\Omega)$. $\exists\, w$ such that $|w| > |w_0|$ but $w \in f(\Omega) \implies \exists z \in \Omega$ such that $f(z) = w \implies |f(z)| > |f(z_0)| \implies$ there cannot be a maximum for $|f(z)|$ since z_0 was arbitrary.

Proof 2 (Corollary of the Cauchy Representation Theorem): $\forall z_0 \in \Omega$. Let $\delta > 0$ such that $B(z_0, \delta) \subset \Omega$. Let $\zeta(t) = z_0 + \delta e^{it}$ $t \in [0, 2\pi]$ gives the circle γ of radius δ around z_0 Then $d\zeta = i\delta e^{it}.dt$.

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta).d\zeta}{\zeta - z_0} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + \delta e^{it}) i\delta e^{it}.dt}{z_0 + \delta e^{it} - z_0} = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \delta e^{it}).dt$$

$$\implies |f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \delta e^{it}).dt\right| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \delta e^{it})|.dt$$

If $f(z_0)|$ is a maximum, then $|f(z_0)| \geqslant |f(z_0 + \delta e^{it})|$. We consider 2 cases:

1. If $\exists t'$ such that $|f(z_0)| > |f(z_0 + \delta e^{it'})|$ this means $\exists\ \varepsilon > 0$ such that $\forall t$ satisfying $|t - t'| < \varepsilon$, $|f(z_0)| > |f(z_0 + \delta e^{it})|$ is strict inequality

$$\implies |f(z_0)| > \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \delta e^{it})|.dt$$

   This is a contradiction.

2. If $|f(z_0)| = |f(z_0 + \delta e^{it})|\ \forall\ t \in [0, 2\pi]\ \forall\ \delta > 0$ such that $\overline{B}(z_0, \delta) \subset \Omega \implies f \equiv C$. This is also a contradiction.

$\implies |f(z_0)|$ cannot be a maximum.

Remark 13.6.3.

- Note that $f \not\equiv C$ comes up in both proofs.
- We have the following Mean Value Theorem for holomorphic functions:

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \delta e^{it}).dt$$

  If $\overline{B}(z_0, \delta) \subset \Omega$ .Note that this is a mean value property for holomorphic functions on circles as the one on lines does not exist. This was a tutorial exercise.
- $f = u + iv$ holomorphic $\implies u, v$ harmonic, then harmonic functions also have the mean value property on circles.

Corollary 13.6.4 (Minimum Modulus Principle)**.** *If f is a non-constant analytic function in a region D, then no point $z \in D$ can be a relative minimum unless* $f(z) = 0$

Proof: Suppose $f(z) \neq 0$ and consider $g = \frac{1}{f}$. If z were a minimum point for f, it would be a maximum point of g. Hence g would be constant in D contrary to our assumption.

Theorem 13.6.5 (Positive Formulation of the Maximum Principle)**.** *If $f(z)$ is defined and continuous on a closed and bounded set E and holomorphic on E^o. Then* $\max\{|f(z)|\}$ *is attained on ∂E unless* $f \equiv C$.

Proof: E is closed and bounded $\implies$ E is compact in $\mathbb{C} \implies |f(z)|$ is continuous on $E \implies \exists$ a maximum of $|f(z)|$ on E. Assume $\exists\ z_0 \in E^o$ which is a maximum for $|f(z)| \implies f \equiv C$ by the other formulation of the Maximum Principle $\implies$ max of $|f(z)|$ attained on ∂E since $E = E^o \cup \partial E$ (because E is closed)

Lemma 13.6.6 (Schwarz lemma)**.** *If f is holomorphic on* $B(0, 1)$ *and satisfies that* $|f(z)| \leqslant 1$ *and* $f(0) = 0$, *then* $|f(z)| \leqslant |z|$ *and* $|f'(0)| \leqslant 1$. *If* $|f(z)| = |z|$ *for some* $z \neq 0$ *or if* $|f'(0)| = 1$, *then* $f(z) = c \cdot z$ *with* $|c| = 1$.

Proof (”very cute”) : Consider

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

g is continuous on $B(0,1)$ and holomorphic on $B(0,1)\backslash\{0\} \implies$ removable singularity at 0. By the Maximum Principle on $\overline{B}(0,r)$ with $r \in (0,1)$

$$|g(z)| \leqslant \frac{1}{r} \text{ on } \overline{B}(0,r)$$

We let $r \to 1$, then $|g(z)| \leqslant 1$ on $B(0,1)$. Then $|f'(0)| \leqslant 1$ and $|f(z)/z| \leqslant 1$ if $z \neq 0$ i.e $|f(z)| \leqslant |z|$ if $z \neq 0$.
If $z = 0$, then $|f(z)| \leqslant 0$ as well because $f(0) = 0 \implies |f(z)| \leqslant |z|$ on $B(0,1)$ and $|f'(0)| \leqslant 1$.

If $\exists\ z \in B(0,1)$ such that $|g(z)| = 1$, then either $z = 0$ so $|g(0)| = |f'(0)| = 1$ or $z \neq 0$ i.e $f(z)/z = c$ with $|c| = 1$. Note that we can write any c such that $|c| = 1$ as $c = e^{i\theta}$ for $\theta \in [0, 2\pi]$, so in this case f is a rotation.

Remark 13.6.7.

The Schwarz lemma describes holomorphic maps from the unit ball to the unit ball that map zero to itself.

Chapter 14

Residue Theorem

14.1 Residues

The Residue Theorem is the most important theorem in complex analysis with regards to applications to other fields. It gives values to integrals that are impossible (or very hard) to compute using calculus techniques (no elementary anti-derivatives). For example:

$$\int_0^\infty \frac{\sin(x)}{x}.dx$$

The idea is to look at

$$\frac{1}{2\pi i}\int_\gamma f(z).dz,$$

where γ denotes a closed curve and f is meromorphic. If f is holomorphic, then the integral is zero by Cauchy's theorem. We'd like to look at some γ that encloses certain singularities of f.
We expect

$$\frac{1}{2\pi i}\int_\gamma f(z).dz = \text{ the sum of some values at each of the singularities of f inside } \gamma$$

We call these values residues, i.e. what is left over after Cauchy's theorem integrates to zero the holomorphic part.

Definition 14.1.1 (Residue)**.**

The residue of f at some $z = a$, $\mathrm{Res}_{z=a}(f)$ is a complex number R such that $f - \frac{R}{z-a}$ has a single-valued anti-derivative in $B(a, \delta)\backslash\{a\}$ for some $\delta > 0$.

Example 14.1.2.
The anti-derivative of $(z - a)^m$ for $m \neq -1$ is $(z - a)^{m+1}/(m + 1)$. For $m = -1$, the anti-derivative is $\log(z - a)$, which is not single-valued. Consider if f is meromorphic. If f has a removable singularity at $z = a$, then $\mathrm{Res}_{z=a} f = 0$ since $\exists$ holomorphic extension $\tilde{f}$ that coincides with f everywhere except for a.

$$\frac{1}{2\pi i}\int_\gamma f(z).dz = \frac{1}{2\pi i}\int_\gamma \tilde{f}(z).dz = 0$$

by Cauchy's theorem.
With an essential singularity, there isn't any general statement we can make. In select cases, looking at $f - R/(z-a)$ might help.
For poles at $z = a$, we get a residue here. Recall that a pole has finite order. If the pole has order 1, then we can find the Laurent expansion of f around a and represent it as:

$$f(z) = \frac{A_{-1}}{z-a} + A_0 + A_1(z-a) + ..$$

We want to take out $R/(z-a)$ and get something single valued. Hence R must be A_{-1}.

Q: How do we find the coefficient in general?

A: $R = \lim_{z\to a} f(z)\cdot(z-a) = \lim_{z\to a}(A_1 + A_0(z-a) + ..)) = A_{-1}$
Example 14.1.3.

$$\mathsf{Res}_{z=1}\left(\frac{e^z}{z^2-1}\right) = \mathsf{Res}_{z=1}\left(\frac{e^z}{(z+1)(z-1)}\right) = \lim_{z\to 1}\frac{e^z(z-1)}{(z+1)(z-1)} = \frac{e^1}{2}$$

If a pole at $z = a$ has order $k > 1$

$$\implies f(z) = \frac{A_{-k}}{(z-a)^k} + ... + \frac{A_{-1}}{z-a} + A_0 + A_1(z-a) + ..$$

To find the residue, we should observe:

$$f(z)(z-a)^k = A_{-k} + A_{-k+1}(z-a) + ...$$

If we differentiate $k-1$ times. Then:

$$\left(\frac{d}{dz}\right)^{k-1}\left(f(z)(z-a)^k\right) = (k-1!)A_{-1} + k!A_0(z-a) + ...$$

$$\implies R = A_{-1} = \frac{1}{(k-1)!}(\frac{d}{dz})^{k-1}\left(f(z)(z-a)^k\right)\Big|_{z=a} = \mathsf{Res}_{z=a}(f(z))$$

Example 14.1.4.

$$\mathsf{Res}_{z=3}\left(\frac{2z+3}{(z^2-9)^2}\right) = \mathsf{Res}_{z=3}\left(\frac{(2z+3)}{(z-3)^2(z+3)^2}\right)$$
$$= \frac{1}{(2-1)!}\left(\frac{d}{dz}\right)^{2-1}\left(\frac{2z+3}{(z+3)^2}\right)\Big|_{z=3} = \frac{2(z+3)-2(2z+3)}{(z+3)^3}\Big|_{z=3} = \frac{-6}{6^3} = -\frac{1}{36}$$

14.2 The Residue Theorem

Theorem 14.2.1 (The Residue Theorem)**.** *Let f be meromorphic in a domain Ω and holomorphic on Ω except for isolated singularities $a_1, .., a_m$. If γ is a closed curve in Ω which* *does not pass through any a_j, then:*

$$\int_{\gamma} f(z).dz = 2\pi i \sum_{j=1}^{m} n(\gamma, a_j) \mathrm{Res}_{z=a} f(z)$$

Proof: Without loss of generality, we can assume that only one singularity a is inside γ. a is a pole of order k $\implies$ the Laurent expansion of f near a is:

$$f(z) = \frac{A_{-k}}{(z-a)^k} + \frac{A_{-k+1}}{(z-a)^{k-1}} .. + A_0 + A_1(z-a)..$$

$$\frac{A_{-l}}{(z-a)^l} \text{ has anti-derivative } \frac{A_{-l}(z-a)^{-l+1}}{-l+1} \text{ for } l \neq -1$$

Recall the theorem that states that

$$\int_{\gamma} g(z).dz$$

with g continuous on γ depends only on the end points if and only if it is the anti-derivative of some function on a domain Ω' containing γ.
If we apply this by setting $\Omega' = \Omega \backslash \{a\}$

$$\int_{\gamma} \frac{A_{-l}}{(z-a)^l} dz = 0$$

if $2 \leqslant l \leqslant k$ since γ is closed. Let $R = \mathrm{Res}_{z=a}(f(z)) = A_{-1}$. By the theorem above

$$\int_{\gamma} \left(f - \frac{R}{(z-a)} \right) .dz = 0 \implies \int_{\gamma} f(z).dz = \int_{\gamma} \frac{R}{(z-a)} .dz = R \int_{\gamma} \frac{1}{z-a} = R \cdot n(\gamma, a) 2\pi i$$

Remark 14.2.2.

Notice the difference from $\log(z-a)$. Even if we look on $\Omega \backslash \{a\}$ as we go around a once, we pick up a factor of $2\pi \implies$ not a single-valued branch of an analytic function.

$$\int \frac{dz}{z-a} = \log(z-a) = \log(z-a) + i\arg(z-a)$$

14.3 Applications of the Residue Theorem

Example 14.3.1.

$$\int_{|z|=3/2} \frac{e^z.dz}{(z-1)^2 z(z-2)}$$

$z = 1$ is a double pole inside $B(0, 3/2)$

$$\implies \mathrm{Res}_{z=1}(f) = \frac{1}{(2-1)!} \frac{d}{dz} \left(\frac{e^z}{z(z-2)} \right) \Big|_{z=1} = -e$$

$z = 0$ is a single pole inside $B(0, 3/2)$. $n(\gamma, 1) = 1, n(\gamma, 0) = 1, n(\gamma, 2) = 0$.
$\mathrm{Res}_{z=0}(f(z)) = \frac{1}{2}$. $z = 2$ is a simple pole but its winding number is 0 so it doesn't count.
By the Residue Theorem,

$$\int_{|z|=3/2} \frac{e^z.dz}{(z-1)^2 z(z-2)} = 2\pi i \mathrm{Res}_{z=1} f(z) + 2\pi i \mathrm{Res}_{z=0} f(z) = 2\pi i \left(-e - \frac{1}{2}\right).$$

This is the basic story. We need more complicated applications of the Residue Theorem, however, to handle things like $\displaystyle\int_0^\infty \frac{\sin(x)}{x}.dx$.

Let us look at various types of integrals and see how to handle each one. For a more exhaustive list, take a look at Ahlfors' description posted on Blackboard:

Type 1: $\int_0^{2\pi} R(\cos(\theta), \sin(\theta)).d\theta$ where R is rational

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta)).d\theta$$

$$z = e^{i\theta} = \cos(\theta) + i\sin(\theta) \implies \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$

$$\implies \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i}$$

$$dz = ie^{i\theta} d\theta \implies d\theta = \frac{e^{-i\theta}}{i} dz = \frac{dz}{iz}$$

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta)).d\theta = \int_{|z|=1} R\left(\frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i}\right) \frac{dz}{iz}$$

Example 14.3.2.

$$\int_0^{2\pi} \frac{d\theta}{2+\sin(\theta)} = \int_{|z|=1} \frac{1}{2+\frac{z-\frac{1}{z}}{2i}} \frac{dz}{iz} = \int_{|z|=1} \frac{2z}{4iz+z^2-1} \frac{dz}{z}$$
$$=2\int_{|z|=1} \frac{dz}{z^2+4iz-1} = 2\int_{|z|=1} \frac{dz}{(z-i(-2+\sqrt{3})(z-i(-2-\sqrt{3})}$$

$$\{|z|=1\} = \gamma,\ n(\gamma, i(-2+\sqrt{3})) = 0,\ n(\gamma, i(-2-\sqrt{3})) = 1$$
$$\implies \int_0^{2\pi} \frac{d\theta}{2+\sin(\theta)} = 2\left(2\pi i \mathrm{Res}_{z=i(-2+\sqrt{3})} \frac{1}{(z+2i+i\sqrt{3})(z+2i-i\sqrt{3})}\right)$$
$$= 4\pi i \lim_{z\to i(-2+\sqrt{3})} \frac{(z+2i-i\sqrt{3})}{(z+2i+i\sqrt{3})(z+2i-i\sqrt{3})} = \frac{2\pi}{\sqrt{3}}$$

Type 2: $\int_{-\infty}^{\infty} R(x).dx$ where R is a rationally function with no poles on the real axis

Suppose $R(x)$ has no poles on the real axis where $R(x)$ is a rational function. We use a semicircle contour consisting of a piece C_1 on the real axis and a semicircle C_2 in the upper half plane.

$$\int_{-\infty}^{\infty} R(x).dx = \int_{-\infty}^{\infty} \frac{P(x).dx}{Q(x)}$$

$$2\pi i \sum_a \mathrm{Res}_{z=a}(R(z)) = \int_{-\rho}^{\rho} R(x).dx + \int_{C_2} R(z).dz \text{ (where } a \text{ is a pole in the upper half space)}$$

Let $\rho \to \infty$, we want to show $\int_{C_2} R(z).dz \to 0$.

Our requirement for convergence is that $\mathrm{dep}(Q) - \deg(P) \geqslant 2$. If the difference in degree is 1, then:

$$\frac{P}{Q} \sim \frac{1}{x} \text{ (whose integral is divergent)}$$

If the difference n is 2 or more then:

$$\frac{P}{Q} \sim \frac{1}{x^n} \text{ (whose integral is convergent)}$$

As $\rho \to \infty$, $\left|\frac{P}{Q}\right| \leqslant \frac{1}{\rho^2}$

$$\implies \left|\int_{C_2} R(z).dz\right| \leqslant \int_{C_2} |R(z)|.dz \leqslant \int_{C_2} \frac{1}{\rho^2}.dz = \frac{1}{\rho^2}\pi\rho = \frac{\pi}{\rho} \to 0$$

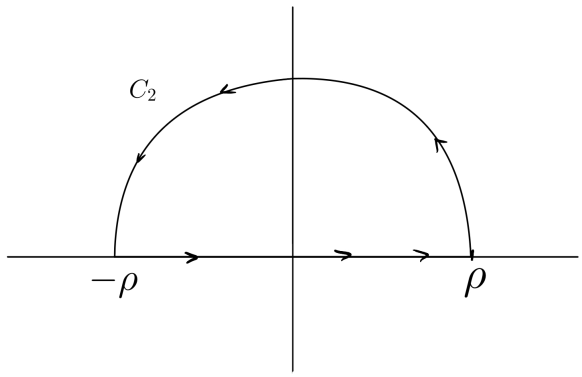

Figure 14.1: General contour for the above type of integral

Example 14.3.3.

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5} = 2\pi i \sum_{a} \mathsf{Res}_{z=a}\left(\frac{1}{z^2+2z+5}\right), \text{ where a is the pole in the upper half plane}$$

$$\frac{1}{z^2+2z+5} = \frac{1}{(z-(-1+2i))(z-(-1-2i))}, \; z=-1+2i \text{ is the only pole in our region}$$

$$\mathsf{Res}_{z=-1+2i}\left(\frac{1}{z^2+2z+5}\right) = \frac{1}{4i}$$

$$\implies \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5} = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}$$

Remark 14.3.4.

Note that we get the same result from choosing the lower semicircle as our contour.

Type 3: $\int_{-\infty}^{\infty} R(x)e^{ix}.dx$ where R has no poles on the real axis.
We can now consider an extension of the above. Assume once again that $\deg(Q) - \deg(P) \geqslant 2$ and that $R(x)$ has no poles on the real axis.

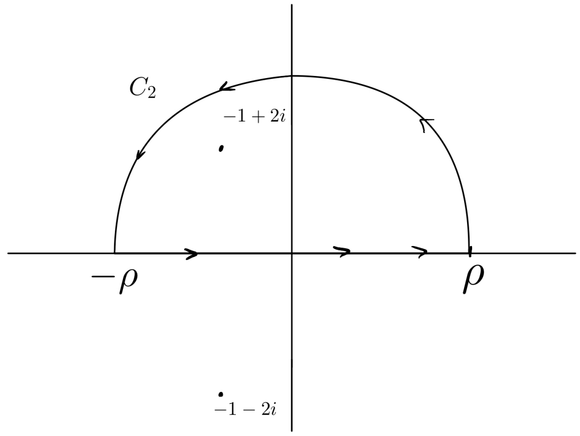

Figure 14.2: The contour of the integral above

$$\int_{\infty}^{\infty} \mathsf{R}(x)e^{ix}dx = \int_{\infty}^{\infty} \mathsf{R}(x)\cos(x).dx + i\int_{-\infty}^{\infty} \mathsf{R}(x)\sin(x).dx$$

We use the same trick as above to show:

$\displaystyle\int_{\infty}^{\infty} \mathsf{R}(x)e^{ix}dx = 2\pi i \sum_{a} \mathrm{Res}_{z=a}(\mathsf{R}(z))$ where a is a pole in the upper half plane

Consider: $|\mathsf{R}(z)e^{iz}| = |\mathsf{R}(z)||e^{iz}| = |\mathsf{R}(z)||e^{ix-y}| = |\mathsf{R}(z)|e^{-y}| \leqslant |\mathsf{R}(z)|$ on the upper half plane

$$\implies \int_{C_2} |\mathsf{R}(z)e^{iz}|.dz = \int_{C_2} |\mathsf{R}(z)||e^{ix-y}|.dz \leqslant \pi\rho\frac{1}{\rho^2}e^{-y} < \frac{\pi}{\rho} \to 0$$

since $0 < e^{-y} \leqslant 1$ on the upper half plane

Type 4: Principal Value Integrals

Even if we have divergent integrals, we can still assign them meaning by means of contour integration. This is very important in applications.

$$\int_{-\infty}^{\infty} \frac{2x+1}{x^2+2x+2}dx \quad \text{diverges as } \deg(Q) - \deg(P) < 2$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x-1)(x^2+4)} \quad \text{has poles on the real axis}$$

Definition 14.3.5 (Cauchy Principal Value)**.**

The Cauchy principal value (PV or pr. v.) of an integral is interpreting

$$\int_{-\infty}^{\infty} f(x).dx \quad \text{as} \quad \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x).dx,$$

where we are only considering symmetric limits.

Remark 14.3.6.

One of the tutorial problems in tutorial 11 shows how taking symmetric limits versus approaching infinity on the negative and the positive sides in non-symmetric ways changes the result.

Example 14.3.7.

$$\int_{-\infty}^{\infty} \frac{dx}{(x-1)(x^2+4)} \text{ this integral has a pole on the real axis}$$

We consider a similar contour to the integral above with a slight modification.

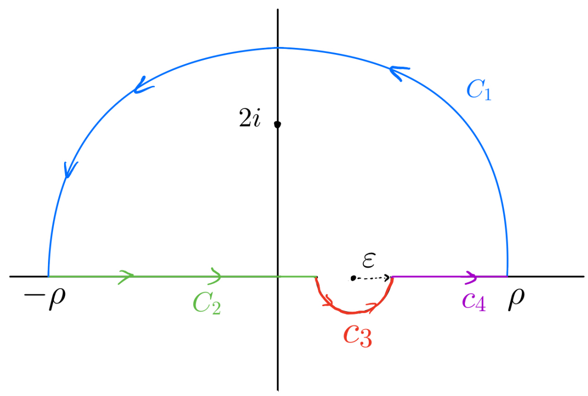

Figure 14.3: Our contour γ around the pole at 1

We complexify our integral

$$\frac{1}{(z-1)(z^2+4)},\ z = \pm 2i \text{ on } \gamma$$

$$\mathsf{Res}_{z=2i}\left(\frac{1}{(z-1)(z^2+4)}\right) = \lim_{z\to 2i}\frac{(z-2i)}{(z-1)(z^2+4)} = \frac{1}{-8-4i}$$

$$\int_{C_3}\frac{dz}{(z-1)(z^2+4)},\ C_3 \text{ given by } z(t) = 1+\varepsilon e^{it}, t\in[\pi, 2\pi], dz = i\varepsilon e^{it}$$

$$\int_{\pi}^{2\pi}\frac{\varepsilon i e^{it}.dt}{(1+\varepsilon e^{it}-1)((1+\varepsilon e^{it})^2+4)} = i\int_{\pi}^{2\pi}\frac{dt}{(1+\varepsilon e^{it})^2+4)}$$

$$\longrightarrow i\int_{\pi}^{2\pi}\frac{dt}{5} \text{ as } \varepsilon\to 0$$

$$2\pi i\left(\frac{1}{2}\mathsf{Res}_{z=1}\frac{1}{(z-1)(z^2+4)}\right) = \frac{1}{5}i\pi$$

$$\implies \mathsf{PV}\left(\int_{-\infty}^{\infty}\frac{dx}{(x-1)(x^2+4)}\right) = 2\pi i\mathsf{Res}_{z=2i}R(z) + 2\pi i\left(\frac{1}{2}\mathsf{Res}_{z=1}R(z)\right)$$

Lemma 14.3.8. *For any simple pole, a contour that is a semi-circular arc of radius ε around the pole shrinks to 0 and picks up $\frac{1}{2}$ the residue for the pole*

Example 14.3.9.
Expanding on the ideas of the previous example we can now tackle our old enemy...

$$\int_{-\infty}^{\infty}\frac{\sin(x).dx}{x}$$

The semicircle would be really bad because $\deg(Q)-\deg(P)\geqslant 2$ is violated so we wouldn't be able to make that go to 0 $\implies$ use a triangular contour instead. We look at:

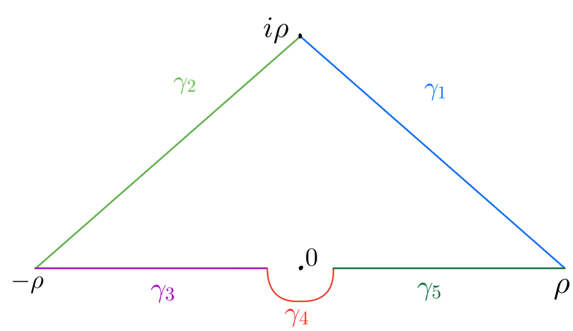

Figure 14.4: Our Triangular Contour

$$\int_{\gamma}\frac{e^{iz}.dz}{z},\ \gamma = \gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4\cup\gamma_5$$

$$\mathsf{PV}\left(\int_{-\infty}^{\infty}\frac{e^{iz}.dz}{z}\right) = \mathsf{PV}\left(\int_{-\infty}^{\infty}\frac{\cos(x).dx}{x} + i\int_{-\infty}^{\infty}\frac{\sin(x).dx}{x}\right)$$

$\frac{\cos(x)}{x}$ is an odd function so it integrates to 0.

$$\int_\gamma \frac{e^{iz}.dz}{z} = 2\pi i \frac{1}{2} \mathrm{Res}_{z=0}\left(\frac{e^{iz}}{z}\right) = \pi i$$

We claim that:

$$\int_{\gamma_1} \frac{e^{iz}.dz}{z} \text{ and } \int_{\gamma_2} \frac{e^{iz}.dz}{z} \to 0 \text{ as } \rho \to \infty$$

Proof: We look at γ_1 which is the line segment from ρ to $i\rho$ i.e the segment of $\rho = y + x$. (γ_2 is similar, with $\rho = y - x$), $y + x = \rho \implies y = \rho - x$.

$$z = x + iy = x + i(\rho - x),\ dz = dx - idx = (1 - i)dx$$

$$\int_{y+x=\rho} \frac{e^{iz}.dz}{z} = \int_0^\rho \frac{e^{i(x+i(\rho-x)(1-i)}.dx}{x + i(\rho - x)}$$

$$\left|\int_{y+x=\rho} \frac{e^{iz}.dz}{z}\right| \leqslant |1-i| \int_0^\rho \frac{e^{-(\rho-x)}.dx}{|x+i(\rho-x)|} \leqslant \frac{2\sqrt{2}}{\rho\sqrt{2}} \int_0^\rho e^{-\rho+x}.dx = \frac{2}{\rho}e^{-\rho} \int_0^\rho e^x.dx = \frac{2}{\rho} - \frac{e^{-\rho}}{\rho} \to 0$$

This means that:

$$\mathsf{PV}\left(\int_{-\infty}^{\infty} \frac{\sin(x).dx}{x}\right) = \mathrm{Im}\{\pi i\} = \pi = 2\int_0^\infty \frac{\sin(x).dx}{x} \iff \int_0^\infty \frac{\sin(x).dx}{x} = \frac{\pi}{2}$$

The beast is slain.

Chapter 15

Analytic Continuation, Conformal Mappings

15.1 Analytic Continuation

The basic idea of analytic continuation is that if we're given a function f that is holomorphic on a given domain Ω, we can sometimes find a holomorphic extension of f, $\tilde{\mathsf{f}}$ defined on some domain $\tilde{\Omega}$ with $\Omega \subset \tilde{\Omega}$

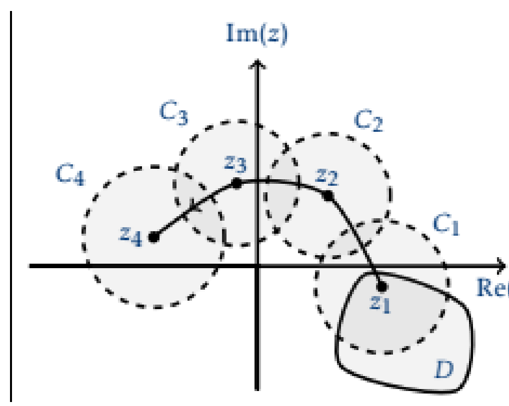

Figure 15.1: We can "patch together" domains to extend our holomorphic function

Remark 15.1.1.

Recall that if f, g are holomorphic on Ω_1, Ω_2 respectively and ff $\exists\ \{z_n\} \to z_0$ such that $z_0 \in \Omega_1 \cap \Omega_2$, $\{z_n\} \subset \Omega_1 \cap \Omega_2$ and $\mathsf{f}(z_j) = \mathsf{g}(z_j)\ \forall\ j$ then $\mathsf{f} \equiv \mathsf{g}$ on $\Omega_1 \cup \Omega_2$

Here is an application of analytic continuation.

Theorem 15.1.2 (Schwarz Reflection Principle)**.** *Let f be holomorphic on a domain* Ω *in the upper-half space. Let* $\partial\Omega$ *intersect the real axis in a simple segment L. Let f be continuous on* $\Omega \cup \mathsf{L}$ *and take real values on L. If* Ω^* *is the reflection of* Ω *with respect to the real axis, then f can be continued holomorphically across L into* Ω^* *by taking* $\mathsf{f}(z) = \overline{\mathsf{f}(\overline{z})}$.

Proof: If f is continuous on L and real-valued there ($L \subset$ real axis) $\implies f(z) = \overline{f(\overline{z})}$. We look at the power series expansions around an arbitrary real value in L. This means $f(z), \overline{f(\overline{z})}$ holomorphic $\implies$ they represent the same function on $\Omega \cup \Omega^*$

$$\tilde{f}(z) = \begin{cases} f(z) & \text{on } \Omega \\ \overline{f(\overline{z})} & \text{on } \Omega^* \end{cases}$$

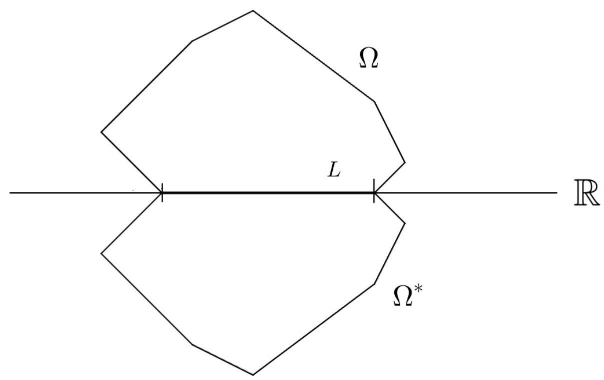

Figure 15.2:

The bigger fish that we want to fry today is conformality.

15.2 Conformal Mappings

Definition 15.2.1 (Conformal Map)**.**

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is conformal if it preserves angles.

Definition 15.2.2 (Isometry)**.**

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry if it preserves both distances between points and angles.

The relationship between holomorphic mappings and conformality is super important.

Definition 15.2.3.

Let γ_1, γ_2 be two C^1 curves that intersect at z_0. The angle from γ_1 to γ_2 at z_0 $\measuredangle(\gamma_1, \gamma_2)$ is the angle measured counterclockwise from $\gamma_1'(z_0)$ (the tangent vector of γ_1 at z_0) to $\gamma_2'(z_0)$

Lemma 15.2.4. *Let $\gamma(t), t \in [0,1]$ be a C^1 curve with $z_0 = \gamma(0)$ and let f be a holomorphic function at z_0. Then the tangent to the curve $f(\gamma(t))$ at $f(z_0)$ is given by $(f \circ \gamma)'(0) = f'(z_0)\gamma'(0)$*

Proof:

$$\frac{f(\gamma(t))-f(\gamma(0))}{t}=\frac{f(\gamma(t))-f(\gamma(0))}{\gamma(t)-\gamma(0)}\frac{\gamma(t)-\gamma(0)}{t}\to(f\circ\gamma)'(0) \text{ as } t\to 0$$

We can make the assumption that $\gamma(t)-\gamma(0)\neq 0$ because for t small enough, there are no self-intersections since $\gamma'(0)\neq 0$ except at finitely many points.

Lemma 15.2.5. *Let f be holomorphic in a neighbourhood of a point z_0. If $f'(z_0)\neq 0$, then $\exists\ \delta>0$ such that $f(z)$ is injective on $B(z_0,\delta)$*

Proof: Let $f(z_0)=w_0$. Look at $g(z)=f(z)-w_0$. $g(z)$ has a zero at z_0 but $g'(z)=f'(z_0)\neq 0 \implies g(z)$ has a simple zero at $z_0 \implies \exists\ \delta>0$ such that $g(z)\neq 0$ on $B(z_0,\delta) \implies$ by the Open Mapping Theorem applied to f, all values in $B(z_0,\delta)$ get to be assumed once by f $\implies$ f is injective.

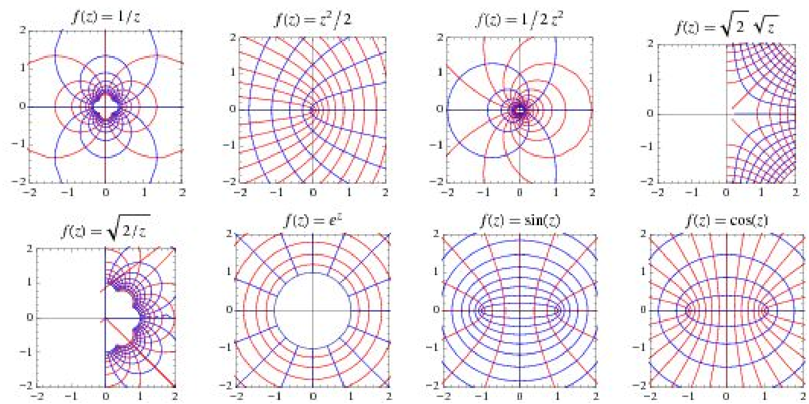

Figure 15.3: We will see that holomorphic functions with non-zero derivative are conformal

Definition 15.2.6 (Univalent Function)**.**

If f is injective in a small neighbourhood of each point in its domain, then f is called locally 1-1 (univalent). If f is holomorphic on a domain Ω and $f'(z)\neq 0$ on $\Omega \implies$ f is univalent.

Theorem 15.2.7. *If $f(z)$ is holomorphic at z_0 and $f'(z_0)\neq 0$, then f is conformal at z_0.*

Proof: Let γ_1,γ_2 be curves such that $\gamma_1(0)=\gamma_2(0)=z_0$ and $\gamma_1'(0)\neq 0,\gamma_2'(0)\neq 0$. By Lemma 1, $(f\circ\gamma_1)'(0)=f'(z_0)\gamma_1'(0)$, $(f\circ\gamma_2)'(0)=f'(z_0)\gamma_2'(0)$. Hence both $\gamma_1'(0)$ and $\gamma_2'(0)$ are multiplied by the same nonzero complex number $f'(z_0) \implies f'(z_0)=Re^{i\theta} \implies$ both $\gamma_1'(0)$ and $\gamma_2'(0)$ as vectors are rotated counter-clockwise through angle $\theta \implies \measuredangle(\gamma_1(0),\gamma_2(0))=\measuredangle(f\circ\gamma_1(0),f\circ\gamma_2(0))\ \forall\gamma_1,\gamma_2$.

More generally:

Theorem 15.2.8. *A mapping is conformal $\iff$ f is holomorphic and univalent*

" $\Longleftarrow$ "
Proven Above
" $\Longrightarrow$ "
Not as interesting, left out (exercise for ambitious reader)

15.3 Interpretation in terms of the Jacobian

$f(z) = u(x, y) + iv(x, y)$
$f : \mathbb{C} \to \mathbb{C} \iff f : \mathbb{R}^2 \to \mathbb{R}^2$. The Jacobian matrix of f, Df is the following:

$$Df(z) = \begin{pmatrix} u_x(z) & v_x(z) \\ u_y(z) & v_y(z) \end{pmatrix} = \begin{pmatrix} u_x(z) & -u_y(z) \\ u_y(z) & u_x(z) \end{pmatrix},$$

where we obtained the latter by applying the Cauchy-Riemann equations. Let $f(z) = Re^{i\theta} \implies$ get a rotation of θ in the image. Conformality comes into the picture here as a consequence of the Cauchy-Riemann equations. Dilation by $\sqrt{|\det(Df(z)|} = \sqrt{u_x^2 + u_y^2}$. If $f'(z) = 0 \implies$ multiplication by the zero matrix, rule out.
Example 15.3.1.

1. $f(z) = z + a$ is a conformal mapping and an isometry everywhere.
2. $f(z) = az$ is conformal if $f'(z) = a \neq 0$. It is an isometry if $|a| = 1$ (rotation)
3. $f(z) = z^n$, $f'(z) = nz^{n-1}$ if $n \geqslant 2$, $f'(0) = 0 \implies$ not conformal at 0. Angles are multiplied by n

Definition 15.3.2. *Two domains Ω_1, Ω_2 are called conformally equivalent if $\exists\, f : \Omega_1 \to \Omega_2$ such that f is bijection and f is conformal.*

Remark 15.3.3.

This is an equivalence relation.

15.4 The Riemann Mapping Theorem

This is the second most important theorem in complex analysis. Riemann thought he had proved this result in his doctoral thesis in 1851, but his proof had a gap. The first correct proof was given in 1900 using potential theory, but it is Carathéodory's proof from 1912 that is now standard.

Theorem 15.4.1. *Let Ω be any simply-connected domain in $\mathbb{C}$ such that Ω is a proper subset of $\mathbb{C}$ and let $z_0 \in \Omega$, then there exists a unique conformal mapping $f : \Omega \to B(0, 1)$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Proof: Too difficult for this module, but covered in Topics in Complex Analysis I

Remark 15.4.2.

1. The Riemann Mapping Theorem says that all simply-connected domains in $\mathbb{C}$ that aren't equal to $\mathbb{C}$ are conformally equivalent (only one such equivalence class)
2. It's not true for $\mathbb{C}^n$ for $n > 1$ e.g $\{|z_1|^2 + |z_2|^2 < 1\}$ not conformally equivalent to $\{|z_1| < 1, |z_2| < 1\}$ (Proven by Poincaré in 1905 by showing that the groups of self-maps of these two domains are not isomorphic as groups).
3. Note the normalisation. $f(z_0) = 0$ and $f'(z_0) > 0$

Example 15.4.3.
Below is the Cayley transform, which maps the upper half plane into the unit ball:

$$f(z) = \frac{z - i}{z + i}$$

f is holomorphic on the upper half-plane since the pole only exists at $z = -i$; it is meromorphic on $\mathbb{C}$.

$$f'(z) = \frac{2i}{(z+i)^2} \neq 0 \;\forall\, z \in \text{ the upper half plane}$$

To get the normalisation in the Riemann Mapping Theorem, we use $f(z) = \frac{z-i}{z+i} \cdot i$

$$f'(z) = \frac{-2}{(z+i)^2},\; f'(i) = \frac{1}{2}$$

And they all lived happily ever after,
The End.