L3: Generators-Relations

Generators - Relations : dihedral group

Given a set $r_1, r_2, \dots, r_\ell$ of words in $g_1^{\pm 1}, \dots, g_k^{\pm 1}$ we can define (we will do this more precisely later) a group

$$G = \langle \underset{\substack{\uparrow \\ \text{generators}}}{g_1, \dots, g_k} \mid \underset{\substack{\uparrow \\ \text{relations}}}{r_1, \dots, r_\ell} \rangle \qquad \text{"presentation of } G\text{"}$$

Elements of G are words in $g_1^{\pm 1}, \dots, g_k^{\pm 1}$ under the equiv relation given by
- removing / adding $\cdot\, g_i g_i^{-1}$, $\cdot\, g_i^{-1} g_i$, $\cdot\, e$
- replacing an occurrence of r_i by e.

<u>Ex</u> $\quad D_{2n} = \langle r, s \mid r^n = s^2 = (sr)^2 = e \rangle$

Let us try to enumerate all the elements of D_{2n}:
If f is any word in $r^{\pm 1}, s^{\pm 1}$ use $r^{-1} = r^{n-1}$ and $s^{-1} = s$ to get a word in r, s. Since $s^2 = e$ can assume

$$f = r^{i_1} s\, r^{i_2} s \cdot \ldots \cdot s\, r^{i_\ell} \qquad i_j \geq 0$$

use $sr = (sr)^{-1} = r^{-1} s^{-1} = r^{n-1} s$ to "shuffle" all s past r to obtain $f = r^i s$ or $f = r^i$.

$\Rightarrow D_{2n} = \{ e, r, \dots, r^{n-1}, s, rs, \dots, r^{n-1}s \}$

We do not know if they are all distinct.

D_{2n} is the group of symmetries of a regular n-gon.
can realize D_{2n} as $r = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, $\theta = \frac{2\pi}{n}$

$$s = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

<u>Rmk</u> $\cdot$ $G = \langle \text{gen} \mid \text{rel} \rangle$ is always a group.
$\cdot$ Generally it is hard to decide for $x \in G$ if $x = e$.