L15: Groups acting on themselves

Groups acting on themselves

Recall we had three "interesting" actions $G \circlearrowright G$ left/right-regular & adjoint

Thm [Cayley's thm] Any finite group is isom to a subgroup of S_n (for $n = |G|$).

Pf Let $G \circlearrowright G$ be the left-regular action. It corresponds to a group-hom. $\rho: G \to S_G \cong S_n$

But ρ is injective: $\rho(g) = id_G \Leftrightarrow \underbrace{\rho(g)(h)}_{gh} = h \ \forall h \overset{h=e}{\Rightarrow} g = e$. $\square$

Lemma X a finite set. Then $S_X \cong S_n$ for $n = |X|$.

Pf Choose a bijection $\varphi: \{1, 2, \dots, n\} \to X$. Then $\begin{array}{l}\psi: S_X \to S_n \\ \sigma \mapsto \varphi^{-1} \circ \sigma \circ \varphi\end{array}$ defines the required isomorphism.

Adjoint action

Recall $G \overset{S_{ad}}{\circlearrowright} G$ by $g \cdot_{ad} h = ghg^{-1}$

Def • Let G be a group. The center of G is

$$Z(G) = \{g \in G \mid gx = xg \ \forall x \in G\} = \mathrm{Fix}_G^{ad}(G)$$

• Let $x \in G$ we call $C_G(x) = \{g \in G \mid gxg^{-1} = x\} = \mathrm{Stab}_G^{ad}(x)$ the centralizer of x in G.

• Two elements $g_1, g_2 \in G$ are conjugate if $G \cdot_{ad} g_1 = G \cdot_{ad} g_2$

• The orbits of S_{ad} are called conjugacy classes.

Lemma $Z(G) \trianglelefteq G$ & $Z(G)$ is abelian

Pf Exercise

Thm (Class equation)

$$|G| = |Z(G)| + \sum_{i=1}^{l} \left| \frac{G}{C_G(g_i)} \right|$$

where $g_1, \dots, g_l$ is a set of representatives of conjugacy classes not contained in the center.

Thm If G has order p^n for $n \geq 1$ and p prime, then $Z(G) \neq \{e\}$.

Pf Suppose $Z(G) = \{e\}$. But then
$$|G| = 1 + \sum_{i=1}^{\ell} \left|\frac{G}{C_G(g_i)}\right|$$

- p divides $|G|$
- Since $\left|\frac{G}{C_G(g_i)}\right| = \frac{|G|}{|C_G(g_i)|} = \frac{p^n}{|C_G(g_i)|}$ and $C_G(g_i) \neq G$

we get that $p \mid \left|\frac{G}{C_G(g_i)}\right| \quad \forall i$

But then p also divides $1 = |G| - \sum_{i=1}^{\ell} \left|\frac{G}{C_G(g_i)}\right|$ $\lightning$ $\square$

Cor If $|G| = p^2$ then G is abelian.

Pf By above $Z(G) \neq \{e\}$ hence $|Z(G)| \in \{p, p^2\}$.

If $|Z(G)| = p^2$ we are done ($Z(G) = G$).

Otherwise we obtain that $G/Z(G)$ is cyclic. $(\cong \mathbb{Z}/p\mathbb{Z})$

Let $y \in G$ be any element s.t. $[e] \neq [y] \in G/Z(G)$ i.e. a generator.

Claim Any $x \in G$ can be written as $x = y^i z$ for some $i \in \mathbb{Z}$ and $z \in Z(G)$

Pf We have $[x] = [y]^i = [y^i]$ for some i, but then $y^{-i} x \in Z(G)$ i.e. $\exists z \in Z(G)$ st $y^{-i} x = z$.

Claim G is abelian
$$y^i z_1 \cdot y^j z_2 = y^i y^j z_1 z_2 = y^{i+j} z_2 z_1 = y^j z_2 y^i z_1 \quad \square$$

Rmk With more work we can show
$$G \cong \mathbb{Z}_{p^2}$$
$$G \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

Aside: If G, H are groups we can turn $G \times H$ into a group via $(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 h_2)$