MAU22101: Exercises Week 9

Problem 1

1. Show that A_4 has a normal subgroup $H \lhd A_4$ isomorphic to $H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
2. Show that $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ has a unique subgroup of order 3.
3. Conclude that $A_4 \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes_\phi \mathbb{Z}/3\mathbb{Z}$ where $\phi\colon \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ is the inclusion of the unique subgroup of order 3.

Problem 2 Finish the proof started in the lecture that A_n is simple for $n \geqslant 5$ as follows. Suppose $N \lhd A_n$ is a non-trivial normal subgroup, in particular it contains a non-trivial element $\sigma \in N$.

1. Depending on the cycle decomposition of σ find a 3-cycle (ijk) such that $\sigma(ijk)\sigma^{-1}(kji)$ is either a 3-cycle or we land in one of the previous cases. In case that,
   (a) σ contains a $\geqslant$ 4-cycle,
   (b) σ contains at least two 3-cycles,
   (c) σ contains one 3-cycle and one 2-cycle,
   (d) σ contains only 2-cycles.
2. Conclude that A_n is simple.

Problem 3 Let G be a group and $S \subset G$ a subset. We define the group generated by S to be

$$\langle S \rangle := \bigcap_{S \subset H \leqslant G} H.$$

Show that for any group K a group homomorphism $\phi\colon \langle S \rangle \to K$ is completely determined by its restriction $\phi|_S\colon S \to K$. That is, show that if we have group homomorphisms $\phi_1, \phi_2\colon \langle S \rangle \to K$ such that

$$\phi_1(s) = \phi_2(s) \quad \forall s \in S,$$

then $\phi_1 = \phi_2$.

Problem 4 Let n be a natural number and define the group

$$G := \langle s_1, \dots, s_n \mid R \rangle$$

where R is the set of relations consisting of

$$\begin{aligned} s_i^2, &\quad \forall i = 1, \dots, n, \\ (s_i s_{i+1})^3, &\quad \forall i = 1, \dots, n-1, \\ (s_i s_j)^2, &\quad \forall i, j, \text{ such that } |i - j| > 1. \end{aligned}$$

- Show that there exists a surjective group homomorphism $\phi\colon G \to S_{n+1}$