L22: The alternating group A_n

<u>The alternating group A_n</u>

<u>Recall</u>: Every element $\sigma \in S_n$ has a cycle decomposition

$$\sigma = (a_1 .. a_{k_1})(a_{k_1+1} .. a_{k_2}) \dots (a_{k_l} .. a_n)$$

<u>Prop</u> i) $(a_1 \dots a_n)(a_n .. a_{n+m}) = (a_1 \dots a_{n+m})$ if $a_1, \dots, a_{n+m}$ are all distinct.

ii) Every element $\sigma \in S_n$ can be written as a product of 2-cycles aka transpositions.

iii) $\sigma (a_1 \dots a_n) \sigma^{-1} = (\sigma(a_1) \dots \sigma(a_n))$

iv) σ and τ are conjugate $\iff$ σ and τ have the same cycle type (i.e. # of k-cycles for each k).

<u>Thm</u> There exists a group homomorphism $\varepsilon: S_n \to \{\pm 1\} \cong \mathbb{Z}^\times \cong \mathbb{Z}/2\mathbb{Z}$ "sign of a permutation" uniquely determined by $\varepsilon((1\,2)) = -1$.

Moreover i) $\varepsilon((a_1 .. a_l)) = (-1)^{l-1}$

ii) $\varepsilon(\tau) = (-1)^k$ if τ can be written as a product of k transpositions.

<u>Pf</u> i) $\underset{\varepsilon((\sigma(1)\ \sigma(2)))}{\overset{\shortparallel}{\varepsilon(\sigma(1\,2)\sigma^{-1})}} = \varepsilon(\sigma)\,\varepsilon((1\,2))\,\varepsilon(\sigma^{-1}) = -1$

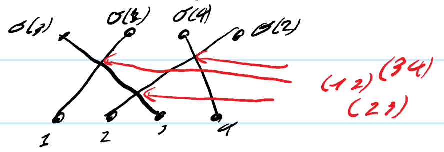
Hence we obtain $\varepsilon((i\,j)) = -1$

With this ii) follows immediately and i) follows from writing

$$(a_1 .. a_l) = (a_1\, a_2)(a_2\, a_3) .. (a_{l-1}\, a_l)$$

This also shows uniqueness.

<u>Construction</u>: Motivation: How to write σ as product of transpositions:



$\sigma = (3\,4) \circ (1\,2) \circ (2\,3)$

$\varepsilon(\sigma) = (-1)^{\#\text{ intersections}}$

transpositions $\leftrightarrow$ intersection points $\leftrightarrow$ i,j s.t. $\sigma(i) - \sigma(j)$ and $i - j$ have opposite sign

$$\varepsilon(\sigma) = \prod_{i<j} \frac{\sigma(j) - \sigma(i)}{j - i} \in \mathbb{R} \setminus \{0\}$$

$$\varepsilon(\sigma \circ \tau) = \prod_{i<j} \frac{\sigma(\tau(j)) - \sigma(\tau(i))}{\tau(j) - \tau(i)} \cdot \prod_{i<j} \frac{\tau(j) - \tau(i)}{j - i}$$

$$\bullet\ \varepsilon(\sigma\circ\tau) = \underbrace{\prod_{i<j} \frac{\sigma(\tau(i)) - \sigma(\tau(j))}{\tau(i) - \tau(j)}}_{} \cdot \underbrace{\prod_{i<j} \frac{\tau(i) - \tau(j)}{i - j}}_{\varepsilon(\tau)}$$

$$\prod_{\substack{k,l \\ \tau^{-1}(k) < \tau^{-1}(l)}} \underbrace{\frac{\sigma(k) - \sigma(l)}{k - l}}_{=:\ \varepsilon_{kl}} \qquad \text{Note } \varepsilon_{kl} = \varepsilon_{lk}$$

$$\| $$

$$\prod_{\{k,l\} \subseteq \{1,\dots,n\}} \varepsilon_{kl} = \prod_{k<l} \varepsilon_{kl} = \varepsilon(\sigma)$$

$\varepsilon((12)) = -1$ Exercise

We obtain ii) $\varepsilon(\sigma) = (-1)^k$ and thus $\varepsilon(\sigma) \in \{\pm 1\}$.

Alternatively: $\sigma \in S_n \to A_\sigma \in Mat_{n \times n}$

$A_\sigma(e_i) = e_{\sigma(i)}$ $\quad e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$

$$A_\sigma = \begin{array}{r} \sigma(2) \\ \sigma(1) \end{array} \begin{pmatrix} 0 & 0 & \\ \vdots & 1 & \\ 1 & \vdots & \dots \\ 0 & 0 & \end{pmatrix}$$

$$\varepsilon(\sigma) := \det(A_\sigma).$$

Def $A_n := \ker(\varepsilon : S_n \to \{\pm 1\})$ "the alternating group"

Lemma $|A_n| = \frac{n!}{2}$, $n \geq 2$

Prop Let $\sigma \in S_n$. Then

$\sigma \in A_n \overset{i)}{\Longleftrightarrow}$ σ can be written as an even product of transpositions

possibly empty

$\overset{ii)}{\Longleftrightarrow}$ σ can be written as a product of 3-cycles.

Pf i) follows directly from properties of ε.

$\overset{ii)}{\Rightarrow}$ write σ as product of evenly many transpositions

$\sigma = (i\,j)(k\,l) \cdot \ldots \cdot \tau_{2n}$

We write $(i\,j)(k\,l)$ as product of 3-cycles:

- If $(i\,j) = (k\,l)$ then $(i\,j)(k\,l) = e$
- If $|\{i,j\} \cap \{k,l\}| = 1$ say $j = l$, $i \neq k$: $(i\,j)(j\,k) = (i\,j\,k)$ ($i \neq k$)
- If $\{i,j\} \cap \{k,l\} = \emptyset$ $(i\,j)(k\,l) = (i\,j)(j\,k)(k\,j)(k\,l)$

$= (i\,j\,k)(j\,k\,l)$

Ex
- $A_1 = A_2 = \{e\}$
- $A_3 = \langle (1\,2\,3) \rangle \cong \mathbb{Z}/3\mathbb{Z}$

A_4 $|A_4| = \frac{4!}{2} = 12 = 3 \cdot 4$

Claim $n_2 = 1$: $\left\{ \begin{matrix} e \\ (1\,2)(3\,4) \\ (1\,3)(2\,4) \\ (1\,4)(2\,3) \end{matrix} \right\}$ is a normal Sylow 2-subgroup

$\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Pf Exercise

$\rightsquigarrow$ $A_4 \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes_\varphi \mathbb{Z}/3\mathbb{Z}$

where φ is determined by the unique (Exc) subgroup of $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ of order 3.

Lemma In A_n, $n \geq 5$ all 3-cycles are conjugate

Pf Let (ijk) be any 3-cycle. Define $\sigma \in S_n$ by $\sigma(1) = i$, $\sigma(2) = j$, $\sigma(3) = k$

and arbitrary on $4, \dots, n$. Then $\sigma(123)\sigma^{-1} = (ijk)$.

If $\sigma \in A_n$ we are done. Else replace $\sigma \rightsquigarrow \sigma(45)$.

Thm A_n is simple for $n \geq 5$.

Pf (sketch) Let $N \trianglelefteq A_n$ be non-trivial. We have to show that $N = A_n$

Note that since every elt in A_n can be written as products of 3-cycles & by above (a normal subgroup contains entire conjugacy classes) it suffices to show N contains a 3-cycles.

Cases: i) $\exists\ \sigma \in N$ has cycle decomp with a ≥ 4-cycle

$\sigma = (ijkl\dots)\dots$

$$\text{Then} \quad \underbrace{\sigma(ijk)\sigma^{-1}\ (ijk)^{-1}}_{\in N} = \overset{(\sigma(i)\ \sigma(j)\ \sigma(k))}{\overset{\shortparallel}{(jkl)}}(kji) = (ilj)$$

ii) $\sigma = (ijk)(lmn)\cdot\dots$ Exc

iii) $\sigma = (ijk)(lm)\cdot\dots$ Exc

iv) $\sigma = (ij)(kl)\cdot\dots$ Exc

$\square$