**Module MAU34403 Quantum mechanics I (Frolov)**
**Homework Sheet 2**
Each set of homework questions is worth $100$ marks

Use Mathematica if necessary

## Compulsory Questions

**Problem 1.** Consider two sets of creation and annihilation operators whose action on basis vectors is defined by

$$\hat{a}|n\rangle = \sqrt{n}\, e^{\mathrm{i}\phi_{n-1}}|n-1\rangle\,, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\, e^{-\mathrm{i}\phi_n}|n+1\rangle$$
$$a|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{0.1}$$

Find a unitary operator which relates the two sets, i.e. an operator such that

$$\hat{a} = \hat{U} a \hat{U}^\dagger\,, \quad \hat{a}^\dagger = \hat{U} a^\dagger \hat{U}^\dagger\,, \tag{0.2}$$

and write it in the form $\hat{U} = \sum_{i,j=0}^{\infty} U_{ij}|i\rangle\langle j|$

**Problem 2.** Let $\hat{a}$ and $\hat{a}^\dagger$ satisfy the canonical commutation relations $[\hat{a}\,,\,\hat{a}^\dagger] = \hat{I}$

Consider the following anti-hermitian operators

$$\hat{L}_1 = \frac{\mathrm{i}}{2}\Big(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\Big)\,, \quad \hat{L}_2 = \mathrm{i}\,\hat{a}^\dagger\hat{a} + \frac{\mathrm{i}}{2}\hat{I}\,, \quad \hat{L}_3 = \frac{1}{2}\Big(\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger\Big) \tag{0.3}$$

The operators $\hat{L}_i$ provide a unitary (by anti-hermitian operators) representation of the $sl(2,\mathbb{R})$ algebra in the Fock space created by $\hat{a}^\dagger$ from the vacuum state $|0\rangle$.

A. Show that $\hat{L}_i$ are anti-hermitian operators.

B. Find how $\hat{L}_i$ act on the basis vectors $|n\rangle$.

C. Is the representation irreducible? If not, can it be written as the direct sum of irreducible representations?

**Problem 3.** Consider the following operator

$$\hat{C} = \hat{L}_1^2 - \hat{L}_2^2 + \hat{L}_3^2 \tag{0.4}$$

Any operator made of $\hat{a}$ and $\hat{a}^\dagger$ can be reduced to the form where all creation operators are to the left of the annihilation operators, e.g.

$$\hat{a}\hat{a}\hat{a}^\dagger = \hat{a}(\hat{a}^\dagger\hat{a} + \hat{I}) = (\hat{a}^\dagger\hat{a} + \hat{I})\hat{a} + \hat{a} = \hat{a}^\dagger\hat{a}\hat{a} + 2\hat{a} \tag{0.5}$$

It is called the normal-ordered form.

Write $\hat{L}_i^2$ in the normal-ordered form for each $i = 1, 2, 3$, and use it to show that $\hat{C} = -\frac{3}{4}\hat{I}$.

**Problem 4.** Consider the following three families of unitary operators

$$\hat{U}_k(t) = e^{-\mathrm{i}\, t\, \hat{H}_k}\,, \quad t \in \mathbb{R} \tag{0.6}$$

where the hermitian operators are given by

$$\hat{H}_1 = \frac{1}{2}\Big(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\Big)\,, \quad \hat{H}_2 = \hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{I}\,, \quad \hat{H}_3 = \frac{\mathrm{i}}{2}\Big(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}\Big) \tag{0.7}$$

Show that the operators

$$\hat{b}_1(t) = \hat{U}_1(t)\,\hat{a}\,\hat{U}_1^\dagger(t)\,, \quad \hat{b}_1^\dagger(t) = \hat{U}_1(t)\,\hat{a}^\dagger\,\hat{U}_1^\dagger(t) \tag{0.8}$$

are expressed through $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{b}_1(t) = c_1(t)\,\hat{a} + d_1(t)\,\hat{a}^\dagger\,, \quad \hat{b}_1^\dagger(t) = c_1^*(t)\,\hat{a}^\dagger + d_1^*(t)\,\hat{a} \tag{0.9}$$

and find how $c_1(t)$ and $d_1(t)$ are expressed through $t$. Use the formula to check explicitly that $\hat{b}_1(t)$ and $\hat{b}_1^\dagger(t)$ satisfy the canonical commutation relation. It is an example of the Bogoliubov transformation.

*Hint.* Differentiate the relations (0.8) with respect to $t$, and derive a system of first-order linear differential equations on $\hat{b}_1(t)$ and $\hat{b}_1^\dagger(t)$ which can be easily solved.

## Practice Questions

**Problem 1.** Consider two sets of creation and annihilation operators whose action on basis vectors is defined by

$$\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}\, e^{\mathrm{i}\phi_{n-1}}|n-1\rangle\,, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\, e^{-\mathrm{i}\phi_n}|n+1\rangle \\ a|n\rangle &= \sqrt{n}\,|n-1\rangle\,, \quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{aligned} \tag{0.10}$$

Find a unitary operator which relates the two sets, i.e. an operator such that

$$\hat{a} = \hat{U}a\hat{U}^\dagger\,, \quad \hat{a}^\dagger = \hat{U}a^\dagger\hat{U}^\dagger\,, \tag{0.11}$$

and write it in the form $\hat{U} = \sum_{i,j=1}^\infty U_{ij}|i\rangle\langle j|$

**Problem 2.** Let $\hat{a}$ and $\hat{a}^\dagger$ satisfy the canonical commutation relations $[\hat{a}\,,\,\hat{a}^\dagger] = \hat{I}$

(a) Consider the following anti-hermitian operators

$$\hat{L}_1 = \frac{\mathrm{i}}{2}\Big(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\Big)\,, \quad \hat{L}_2 = \mathrm{i}\,\hat{a}^\dagger\hat{a} + \frac{\mathrm{i}}{2}\hat{I}\,, \quad \hat{L}_3 = \frac{1}{2}\Big(\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger\Big) \tag{0.12}$$

Compute commutators between the operators, and show that the operators form a real Lie algebra.

(b) The $sl(2,\mathbb{R})$ algebra is the space of real traceless matrices. Choose as its basis the matrices

$$L_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad L_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \quad L_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{0.13}$$

and verify that they satisfy the same commutation relations as $\hat{L}_i$ do.

**Problem 3.** Consider the following operator

$$\hat{C} = \hat{L}_1^2 - \hat{L}_2^2 + \hat{L}_3^2 \tag{0.14}$$

Use the commutation relations to show that $\hat{C}$ commutes with any $\hat{L}_i$.

An operator commuting with all elements of a Lie algebra is called a Casimir operator.

**Problem 4.** Consider the following three families of unitary operators

$$\hat{U}_k(t) = e^{-\mathrm{i}\, t\, \hat{H}_k}\,, \quad t \in \mathbb{R} \tag{0.15}$$

where the hermitian operators are given by

$$\hat{H}_1 = \frac{1}{2}\Big(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\Big)\,, \quad \hat{H}_2 = \hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{I}\,, \quad \hat{H}_3 = \frac{\mathrm{i}}{2}\Big(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}\Big) \tag{0.16}$$

(a) Show that the operators

$$\hat{b}_2(t) = \hat{U}_2(t)\,\hat{a}\,\hat{U}_2^\dagger(t)\,, \quad \hat{b}_2^\dagger(t) = \hat{U}_2(t)\,\hat{a}^\dagger\,\hat{U}_2^\dagger(t) \tag{0.17}$$

are expressed through $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{b}_2(t) = c_2(t)\,\hat{a} + d_2(t)\,\hat{a}^\dagger\,, \quad \hat{b}_2^\dagger(t) = c_2^*(t)\,\hat{a}^\dagger + d_2^*(t)\,\hat{a} \tag{0.18}$$

and find how $c_2(t)$ and $d_2(t)$ are expressed through $t$. Use the formula to check explicitly that $\hat{b}_2(t)$ and $\hat{b}_2^\dagger(t)$ satisfy the canonical commutation relation. It is an example of the Bogoliubov transformation.

*Hint.* Differentiate the relations (0.17) with respect to $t$, and derive a system of first-order linear differential equations on $\hat{b}_2(t)$ and $\hat{b}_2^\dagger(t)$ which can be easily solved.

(b) Do the same for $\hat{b}_3(t) = \hat{U}_3(t)\,\hat{a}\,\hat{U}_3^\dagger(t)\,,\ \hat{b}_3^\dagger(t) = \hat{U}_3(t)\,\hat{a}^\dagger\,\hat{U}_3^\dagger(t)$