Module MAU34403 Quantum mechanics I (Frolov)
Homework Sheet 9
Each set of homework questions is worth 100 marks

Use Mathematica if necessary

Compulsory Questions

Use the results from practice questions

Problem 1. Consider the tensor product, $\mathscr{H}^1 \otimes \mathscr{H}^1$, of two spin-1 irreducible representations of $\mathfrak{su}(2)$. Find all highest weight states in $\mathscr{H}^1 \otimes \mathscr{H}^1$, and decompose $\mathscr{H}^1 \otimes \mathscr{H}^1$ into the direct sum of irreducible representations.

Problem 2. Consider a periodic Heisenberg spin-1/2 chain of length 3 described by the Hamiltonian

$$H = J\sum_{i=1}^{3}\sum_{\alpha=1}^{3} S_i^\alpha S_{i+1}^\alpha + \hbar\, B\sum_{i=1}^{3} S_i^z\,, \quad S_4^\beta \equiv S_1^\beta \;\;\forall\;\; \beta \tag{0.1}$$

The Hamiltonian acts in the Hilbert space which is the tensor product of 3 copies of the spin-$1/2$ irreducible representation (spin up-down)

$$\mathscr{H} = \mathscr{H}^{1/2} \otimes \mathscr{H}^{1/2} \otimes \mathscr{H}^{1/2} \tag{0.2}$$

The basis of the 8-dimensional space is given by the vectors

$$|\uparrow\uparrow\uparrow\rangle\,,\quad |\uparrow\uparrow\downarrow\rangle\,,\quad |\uparrow\downarrow\uparrow\rangle\,,\quad |\uparrow\downarrow\downarrow\rangle\,,\quad |\downarrow\uparrow\uparrow\rangle\,,\quad |\downarrow\uparrow\downarrow\rangle\,,\quad |\downarrow\downarrow\uparrow\rangle\,,\quad |\downarrow\downarrow\downarrow\rangle \tag{0.3}$$

which are eigenvectors of S_i^z. The spin-1/2 operator $S_i^\alpha = \hbar\sigma_i^\alpha/2$ acts only on the particle at the i-th site

$$S_1^\alpha = S^\alpha \otimes I \otimes I\,, \quad S_2^\alpha = I \otimes S^\alpha \otimes I\,, \quad S_3^\alpha = I \otimes I \otimes S^\alpha \tag{0.4}$$

(a) For the spin chain of length $L = 3$ express the Hamiltonian in terms of the identity operator, the Casimir operator $\hat{C} = \sum_{\alpha=1}^{3} \mathbb{S}^\alpha\mathbb{S}^\alpha$, $\mathbb{S}^\alpha = \sum_{i=1}^{3} S_i^\alpha$, and the z-component $\mathbb{S}^z$ of the total spin, and find the spectrum of H.

Problem 3. The time-independent Schrödinger equation for a particle in a central field is

$$\Big(-\frac{\hbar^2}{2\mu\, r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{2\mu\, r^2} + V(r)\Big)\varphi_E(r,\theta,\phi) = E\,\varphi_E(r,\theta,\phi) \tag{0.5}$$

The variables are separated as

$$\varphi_E(r,\theta,\phi) = \mathcal{R}_{E\ell m}(r)\, Y_\ell^m(\theta,\phi) \tag{0.6}$$

where $Y_\ell^m(\theta,\phi)$ are the spherical harmonics.

(a) Show that $\mathcal{R}_{E\ell m}(r)$ satisfies the following equation

$$\Big(-\frac{\hbar^2}{2\mu\, r^2}\frac{d}{dr}r^2\frac{d}{dr}+V_{\text{eff}}(r)\Big)\mathcal{R}_{E\ell}(r)=E\,\mathcal{R}_{E\ell}(r)\,,\quad V_{\text{eff}}(r)\equiv V(r)+\frac{\ell(\ell+1)\hbar^2}{2\mu\, r^2} \tag{0.7}$$

(b) Introduce the function $\mathcal{U}_{E\ell}(r)=r\,\mathcal{R}_{E\ell}(r)$ and show that it satisfies the following equation

$$\Big(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+V_{\text{eff}}(r)\Big)\mathcal{U}_{E\ell}(r)=E\,\mathcal{U}_{E\ell}(r) \tag{0.8}$$

(c) Introduce the function $\mathcal{B}(r)=\sqrt{r}\,\mathcal{R}_{E\ell}(r)$ and show that it satisfies the following equation

$$\frac{d^2\mathcal{B}}{dr^2}+\frac{1}{r}\frac{d\mathcal{B}}{dr}+\Big(k^2-\frac{(\ell+\frac{1}{2})^2}{r^2}-\frac{2\mu}{\hbar^2}V(r)\Big)\mathcal{B}=0\,,\quad k\equiv\sqrt{\frac{2\mu}{\hbar^2}E} \tag{0.9}$$

Problem 4. Energy spectrum of a hydrogen-like ion is determined by the radial Schrödinger equation (0.8)

$$\Big(\frac{d^2}{dr^2}+\frac{2\mu Z e^2}{\hbar^2}\frac{1}{r}-\frac{\ell(\ell+1)}{r^2}\Big)\mathcal{U}_{E\ell}(r)=\kappa^2\,\mathcal{U}_{E\ell}(r)\,,\quad \kappa\equiv\sqrt{-\frac{2\mu}{\hbar^2}E} \tag{0.10}$$

(a) Look for a solution of (0.21) in terms of a power series in ρ

$$\mathcal{W}=\sum_{k=0}^{\infty}w_k\rho^k \tag{0.11}$$

Substitute the series in (0.19), and show that the coefficients w_k satisfy the recursion relation

$$w_{k+1}=\frac{k+\ell+1-\rho_0}{(k+1)(k+2\ell+2)}\,w_k \tag{0.12}$$

(b) Find $\langle X\rangle$, $\langle X^2\rangle$, $\langle P_x\rangle$ and $\langle P_x^2\rangle$ for an electron in the ground state of a hydrogen atom, and verify the uncertainty relation for this state.

Practice Questions

Problem 1. Consider a periodic Heisenberg spin-1/2 chain of length 3 described by the Hamiltonian

$$H=J\sum_{i=1}^{3}\sum_{\alpha=1}^{3}S_i^\alpha S_{i+1}^\alpha+\hbar\,B\sum_{i=1}^{3}S_i^z\,,\quad S_4^\beta\equiv S_1^\beta\ \ \forall\ \beta \tag{0.13}$$

The Hamiltonian acts in the Hilbert space which is the tensor product of 3 copies of the spin-$1/2$ irreducible representation (spin up-down)

$$\mathscr{H}=\mathscr{H}^{1/2}\otimes\mathscr{H}^{1/2}\otimes\mathscr{H}^{1/2} \tag{0.14}$$

The basis of the 8-dimensional space is given by the vectors

$$|\uparrow\uparrow\uparrow\rangle\,,\quad |\uparrow\uparrow\downarrow\rangle\,,\quad |\uparrow\downarrow\uparrow\rangle\,,\quad |\uparrow\downarrow\downarrow\rangle\,,\quad |\downarrow\uparrow\uparrow\rangle\,,\quad |\downarrow\uparrow\downarrow\rangle\,,\quad |\downarrow\downarrow\uparrow\rangle\,,\quad |\downarrow\downarrow\downarrow\rangle \tag{0.15}$$

which are eigenvectors of S_i^z. The spin-1/2 operator $S_i^\alpha=\hbar\sigma_i^\alpha/2$ acts only on the particle at the i-th site

$$S_1^\alpha=S^\alpha\otimes I\otimes I\,,\quad S_2^\alpha=I\otimes S^\alpha\otimes I\,,\quad S_3^\alpha=I\otimes I\otimes S^\alpha \tag{0.16}$$

(a) Find all highest weight states in $\mathscr{H}$, and decompose $\mathscr{H}$ into the direct sum of irreducible representations of $\mathfrak{su}(2)$, and express the basis vectors of these representations in terms of the eigenvectors of S_i^z.

Problem 2. Energy spectrum of a hydrogen-like ion is determined by the radial Schrödinger equation (0.8)

$$\Big(\frac{d^2}{dr^2} + \frac{2\mu Ze^2}{\hbar^2}\frac{1}{r} - \frac{\ell(\ell+1)}{r^2}\Big)\mathcal{U}_{E\ell}(r) = \kappa^2\,\mathcal{U}_{E\ell}(r)\,, \quad \kappa \equiv \sqrt{-\frac{2\mu}{\hbar^2}E} \tag{0.17}$$

(a) Show that in terms of the dimensionless variable ρ and constant ρ_0

$$\rho \equiv 2\kappa\, r\,, \quad \rho_0 \equiv \frac{\mu\, Ze^2}{\hbar^2\kappa} = \gamma \tag{0.18}$$

it takes the form

$$\Big(\frac{d^2}{d\rho^2} + \frac{\rho_0}{\rho} - \frac{\ell(\ell+1)}{\rho^2} - \frac{1}{4}\Big)\mathcal{U}(\rho) = 0\,, \quad \mathcal{U}(\rho) = \mathcal{U}_{E\ell}(\rho/\kappa) \tag{0.19}$$

(b) Introduce the following function $\mathcal{W}$

$$\mathcal{W}(\rho) = \rho^{-\ell-1}\, e^{\rho/2}\,\mathcal{U}(\rho)\,, \quad \mathcal{W}(0) = w_0 = \text{const}\,, \quad \mathcal{W}(\rho) \to \rho^{-\ell-1+\gamma} \;\text{ as }\; \rho\to\infty \tag{0.20}$$

and show that it satisfies the equation

$$\rho\,\frac{d^2\mathcal{W}}{d\rho^2} + (2\ell+2-\rho)\frac{d\mathcal{W}}{d\rho} + (\rho_0-\ell-1)\mathcal{W} = 0 \tag{0.21}$$