L26: Primary Component Decomposition

Towards uniqueness:

Def Let A be an abelian group, p a prime. We set
- $\mathrm{Tor}(A) := \{a \in A \mid na = 0 \text{ for some } n > 0\}$ "torsion part"
- $A_p := \{a \in A \mid p^\alpha a = 0 \text{ for some } \alpha > 0\}$ "p-primary component".

Lemma $\mathrm{Tor}(A) \leq A$, $A_p \leq \mathrm{Tor}(A) \leq A$.

Cor Let A be a f.g. ab group. Then $A/\mathrm{Tor}(A) \cong \mathbb{Z}^r$ for r appearing as rank in its invariant factor decomp.

Cor The rank r is well-defined.

Pf Suppose $\mathbb{Z}^{r_1} \cong A/\mathrm{Tor}(A) \cong \mathbb{Z}^{r_2}$. Then $\mathbb{Z}^{k_1} \cong \mathbb{Z}^{k_2}$, i.e. there exist $k_1 \times k_2$ and $k_2 \times k_1$ matrices U, V st. $UV = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ and $VU = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

but then $k_1 = k_2$. $\square$

Recall
- $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ for $(m,n) = 1$
- $A = \mathbb{Z}/k_1\mathbb{Z} \times \mathbb{Z}/k_2\mathbb{Z} \times \dots \times \mathbb{Z}/k_\ell\mathbb{Z}$

$$\begin{array}{lll}
\cong & \mathbb{Z}/p_1^{n_1^1}\mathbb{Z} \times \mathbb{Z}/p_1^{n_1^2}\mathbb{Z} \times \dots \times \mathbb{Z}/p_1^{n_1^\ell}\mathbb{Z} & \cong A_{p_1} \\
\times & \mathbb{Z}/p_2^{n_2^1}\mathbb{Z} \times \dots \times \mathbb{Z}/p_2^{n_2^\ell}\mathbb{Z} & \times A_{p_2} \\
& \vdots & \vdots \\
& \mathbb{Z}/p_M^{n_M^1}\mathbb{Z} \times \dots \times \mathbb{Z}/p_M^{n_M^\ell}\mathbb{Z} & A_{p_M}
\end{array}$$

where $k_i = p_1^{n_1^i} \cdots p_M^{n_M^i}$ are prime decomp of $k_1, \dots, k_\ell$

To show well-definedness of k_i we can show well-definedness of $n_i^1 \leq n_i^2 \leq \dots \leq n_i^\ell$ for each p_i separately.

We have also shown

Thm (primary factor decomposition) Let A be a f.g. ab group. $A \cong A_{p_1} \times \dots \times A_{p_n} \times \mathbb{Z}^r$ where $p_1, \dots, p_r$ are the primes st. $A_p \neq \{e\}$.

Prop Suppose $A = \mathbb{Z}/p^{n_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{n_\ell}\mathbb{Z} \cong \mathbb{Z}/p^{m_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{m_k}\mathbb{Z}$
where $1 \le n_1 \le n_2 \le \dots \le n_\ell$, $1 \le m_1 \le \dots \le m_k$.
Then $n_i = m_i$ (and $\ell = k$).

Pf We use that $pA = \{pa \in A \mid a \in A\} \cong \mathbb{Z}/p^{n_i - 1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{n_\ell - 1}\mathbb{Z}$
where $n_i > 1$

and $A/pA \cong (\mathbb{Z}/p\mathbb{Z})^\ell$ **Exc!**

Iterating the argument we obtain

$$p^{n-1}A / p^n A \cong (\mathbb{Z}/p\mathbb{Z})^{\#\{i \mid n_i \geq n\}}$$

and thus $|p^{n-1}A/p^n A| = p^{\#\{i \mid n_i \geq n\}}$

We can thus read off $\#\{i \mid n_i \geq n\}$ for all n
and obtain $\#\{i \mid n_i \geq n\} = \#\{i \mid m_i \geq n\}$ $\forall n$
$\overset{\text{Exc}}{\Longrightarrow}$ $n_i = m_i$ $\square$

Ex ·) Find all isomorphism classes of abelian group of order 18:
$18 = 2 \cdot 3^2 \rightsquigarrow A = A_2 \times A_3$
Possibilities for A_2: $A_2 = \mathbb{Z}/2\mathbb{Z}$
A_3: $A_3 = \mathbb{Z}/3^{n_1}\mathbb{Z} \times \dots \times \mathbb{Z}/3^{n_k}\mathbb{Z}$
ways of writing $3^2 = 3^{n_1} \cdots 3^{n_k}$ $n_1 \le \dots \le n_k$
i.e. ways of writing $2 = n_1 + \dots + n_k$
$2 = 2$
$= 1 + 1$

$\rightsquigarrow$ · $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \cong \mathbb{Z}/18\mathbb{Z}$
· $\underbrace{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}}_{\text{primary decomp}} \cong \underbrace{\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}}_{\text{invariant factor decomp}}$

·) $|A| = 8 = 2^3$:
$3 = 3$ $\mathbb{Z}/8\mathbb{Z}$
$= 1 + 2$ $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
$= 1 + 1 + 1$ $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$