Group Theory - Homework 4

Problem 3. Let $N \leq G$ be a subgroup. Show that the following are equivalent:

1. $N \trianglelefteq G$
2. $gNg^{-1} \subset N$ for all $g \in G$
3. $gNg^{-1} = N$ for all $g \in G$
4. $gN = Ng$ for all $g \in G$
5. For all $g \in G$, there exists $g' \in G$ such that $gN \subset Ng'$
6. $G/H = H\backslash G$, ie. the set of orbits for the right regular and the left regular action coincide

Solution:

Recall by definition, $N \trianglelefteq G$ iff $g^{-1}ng \in H$ $\forall g \in G$ $\forall n \in N$.

① $\Rightarrow$ ②

Take $g^{-1}ng \in gNg^{-1}$, by assumption, $g^{-1}ng \in N$.

② $\Rightarrow$ ③

Take $n \in N$ and $g \in G$, then $g^{-1}ng \in N$, so $\underbrace{g(g^{-1}ng)g^{-1}}_{= n} \in gNg^{-1}$,

so $N \subset gNg^{-1}$.

③ $\Rightarrow$ ④

$gNg^{-1} = N \Rightarrow \underbrace{(gNg^{-1})g}_{= gN} = Ng$

④ $\Rightarrow$ ⑤

we set $g' = g$, then $gN = Ng'$, (so in particular $gN \subset Ng'$

⑤ $\Rightarrow$ ⑥

we want to prove the equality

$$\{gN \mid g\in G\} = \{Ng \mid g\in G\}.$$

Take $g\in G$, then $\exists g'\in G$ such that $gN\subset Ng'$. we know $|gN| = |Ng'|$, so necessarily $gN = Ng'$, and $gN \in N\backslash G$. We've shown

$$G/N \subset N\backslash G.$$

but we know $|G/N| = |N\backslash G|$ $(= |G|/|N|)$, so we're done.

⑥ $\Rightarrow$ ①

Take $n\in N$ and $g\in G$. we know there exists some $g'\in G$ such that

$gN = Ng'$ (because $gH \in G/N = N\backslash G$).

$\Downarrow$

$gN(g')^{-1} = N$, so there exists $m\in N$ such that $n = gm(g')^{-1}$.

Then $g^{-1}ng = g^{-1}(gm(g')^{-1})g = m\underbrace{(g')^{-1}g}_{\in N} \in N$

$\hookrightarrow g^{-1}Ng' = N \Rightarrow g^{-1}g' \in N \Rightarrow \underbrace{(g^{-1}g')^{-1}}_{=(g')^{-1}g} \in N$.

Problem 1 Prove that if $N \trianglelefteq G$ is a normal subgroup then G/N carries a group structure such that the map

$$\begin{aligned}\pi : G &\longrightarrow G/N\\ x &\longmapsto xN\end{aligned}$$

is a group homomorphism.

Solution: First we'll show that the operation

$$\begin{aligned}\cdot : G/N \times G/N &\longrightarrow G/N\\ xN\cdot yN &\longmapsto xyN\end{aligned}$$

is well defined.

i.e. $xN = xnN$ and $yN = ymN$ $\forall n, m \in N$, so we should check

$$xnN \cdot ymN = xN \cdot yN$$

$$\underbrace{xnN \cdot ymN = xnymN}_{\text{By definition}} = \underbrace{xnymN = xn(yN)}_{m \in N} = \underbrace{xn(yN) = xn(Ny)}_{N \triangleleft G} = \underbrace{xn(Ny) = x(Ny)}_{n \in N} = \underbrace{x(Ny) = xyN}_{N \triangleleft G}.$$

Now its trivial to check $\pi : G \to G/N$ is a homomorphism:

$$\pi(xy) = xyN = xN \cdot yN = \pi(x) \cdot \pi(y).$$

See also the subsection "Invariant subgroup, cosets, and the quotient group" of chapter I.2 of Zee's "Group theory for physicists in a Nutshell".

Problem 2. A right action $X \circlearrowleft G$ is a map $\rho : X \times G \to X$ satisfying

i) $\rho(x, e) = x$ for all $x \in X$

ii) $\rho(\rho(x, g), h) = \rho(x, gh)$ for all $g, h \in G$, for all $x \in X$.

1. Show a map $\rho : X \times G \to X$ is a right action iff the map $\rho_\ell : G \times X \to X$ defined by $\rho_\ell(g, x) = \rho(x, g^{-1})$ is a left action.

Solution:

$\Rightarrow$) Take $\rho : X \times G \to X$ a right action, then

i) $\rho_\ell(e, x) = \rho(x, e^{-1}) = \rho(x, e) = x$.

ii) $\rho_\ell(g, \rho_\ell(h, x)) = \rho(\rho(x, h^{-1}), g^{-1}) = \rho(x, h^{-1}g^{-1}) = \rho(x, (gh)^{-1}) = \rho_\ell(gh, x)$.

$\Leftarrow$) Completely analogous (I'm sorry).

2. Show that the set of orbits of ρ and ρ_ℓ are the same, ie

$$G \backslash X := \{\rho_\ell(G,x) \mid x \in X\} = \{\rho(x,G) \mid x \in X\} =: X/G.$$

Solution: we'll actually show $\rho_\ell(G,x) = \rho(x,G)$.

$\subseteq$) If $\rho_\ell(g,x) \in \rho_\ell(G,x)$, then $\rho_\ell(g,x) = \rho(x,g^{-1}) \in \rho(x,G)$

$\supseteq$) If $\rho(x,g) \in \rho(x,G)$, then $\rho(x,g) = \rho_\ell(g^{-1},x) \in \rho_\ell(G,x)$.

3. Write down the formulas for the right actions corresponding to three left actions $G \curvearrowright G$ (left-/right-regular and adjoint).

Left actions	Right actions
Left-regular $g \cdot h = gh$	$h \cdot g = g^{-1} \cdot h = g^{-1}h$.
Right-regular $g \cdot h = hg^{-1}$	$h \cdot g = g^{-1} \cdot h = hg$
Adjoint $g \cdot h = ghg^{-1}$	$h \cdot g = g^{-1} \cdot h = g^{-1}hg$.

Problem 4. Let $H \leq G$ be a subgroup of G. We define the normalizer of H in G by

$$N_G(A) := \{g \in G \mid gHg^{-1} = H\}$$

- Show that $N_G(H)$ is a subgroup of G, and that H is a normal subgroup of $N_G(H)$.

Solution: Take $x, y \in N_G(H)$, then

$$xyH(xy)^{-1} = x(yHy^{-1})x^{-1} \underset{y \in N_G(H)}{=} xHx^{-1} \underset{x \in N_G(H)}{=} H,$$

so $N_G(H)$ is closed w.r.t the operation in G, and clearly $e \in N_G(H)$, so $N_G(H) \leq G$. Clearly $H \trianglelefteq N_G(H)$.

- Let $K \leq G$ be another subgroup and suppose that $K \subset N_G(H)$. Show that $HK := \{hk \mid h \in H, k \in K\}$ is a subgroup of G and that $H \trianglelefteq HK$

Solution: First we'll show HK is a subgroup:

- $e \in HK$ (as $e \in H$ and $e \in K$)
- If $h_1, h_2 \in H$ and $k_1, k_2 \in K$,

$$h_1k_1 \cdot h_2k_2 \underset{k_1H = Hk_1}{=} \underbrace{h_1h_ak_1k_2}_{\in HK} \quad \text{for some } h_a \in H$$

Now we'll show $H \trianglelefteq HK$. Take $x \in H$ and $hk \in HK$, then

$$hk \cdot x \cdot (hk)^{-1} = h(\underbrace{kxk^{-1}}_{H\text{, as } k \in N_G(H)})h^{-1} \in H.$$

Problem 5. Let $N \trianglelefteq G$ be a normal subgroup of a finite group G and suppose that $(|N|, |G/N|) = 1$. Prove that N is the unique subgroup of G of order $|N|$. (Hint: Given a subgroup $K \leq G$ of order $|N|$, consider its image under the map $G \to G/N$).

Take $K \leq G$ of order $|K| = |N|$. Consider $\pi: K \to G/N$.

- we know $\pi(K) \leq G/N$, so $|\pi(K)| \mid |G/N|$.
- we also know $K/\ker\pi \cong \pi(K)$, so $\underset{\overset{\shortparallel}{|N|}}{|K|} = |\pi(K)| \cdot |\ker \pi|$

Then $|\pi(K)| \mid |G/N|$ and $|\pi(K)| \mid |N|$, so necessarily $\pi(K) = \{\overset{\overset{e_{G/N}}{\shortparallel}}{e}\} \subset G/N$.

This means $K \subset N$ (remember $\ker(G \to G/N) = N$), so because $|K| = |N|$, we're done.

Important ingredients of the proof:

- If $\varphi: G \to H$ is a homomorphism, then
  - a) $\operatorname{Im}\varphi \leq H$
  - b) $G/\ker\varphi \cong \operatorname{Im}\varphi$
- If $H \leq G$, then $|H| \mid |G|$

<u>Problem 6.</u> Let $N \trianglelefteq G$ be a normal subgroup of G, and suppose N is Abelian. Show that the adjoint action induces a group homomorphism

$$\phi: G/N \to \operatorname{Aut}(N),$$

defined by $\phi([x])(n) = xnx^{-1}$.

Deduce that if G is a group of order pq, where p and q are both primes such that $p \nmid (q-1)$, and G has a normal subgroup N of order q, then G is Abelian.

Solution:

We use the commutativity of N to check $\phi: G/N \to \mathrm{Aut}(N)$ is well defined. Take $x \in G$, $m, n \in N$, then

$$\phi([xm])(n) = xmn(xm)^{-1} = xmnm^{-1}x^{-1}$$
$$= xmm^{-1}nx^{-1} = xnx^{-1} = \phi([x])(n),$$

So $\phi([x])$ doesn't depend on the representative of $[x]$ we choose (remember $[x] := xN$.

ϕ is a homomorphism: take $x, y \in G$, $n \in N$, then

$$\phi([x]\cdot[y])(n) = \phi([xy])(n) = xyn(xy)^{-1}$$
$$= xyny^{-1}x^{-1}$$
$$= \phi([x])(yny^{-1})$$
$$= \phi([x]) \circ \phi([y])(n)$$

(Remember $\mathrm{Aut}(N)$ is a group w.r.t. composition).

Take $|G| = pq$, for p, q primes with $p \nmid (q-1)$. Suppose $N \trianglelefteq G$ with $|N| = q$.

Claim. N is a finite group of prime order q, so $N \cong \mathbb{Z}_q$. In particular, N is Abelian.

Proof: Take $x \in N \setminus \{e\}$, then $|\langle x \rangle| \mid q$ and $|\langle x \rangle| \neq 1$, so necessarily $|\langle x \rangle| = q$ (i.e. any non-identity element in N is a generator). ■

Now consider $\phi: G/N \to \mathrm{Aut}(N)$ same as before,

$p = |G/N| = |\ker \phi| |\mathrm{Im}\, \phi|$, so necessarily $|\mathrm{Im}\, \phi| \in \{1, p\}$.

How can we use the information $p \nmid (q-1)$?

Claim. $|Aut(N)| = q-1$.

Proof. N is cyclic, say $N = \langle x \rangle$. Any $f \in Aut(N)$ is completely determined by $f(x)$ (ie $f(x^a) = f(x)^a \ \forall a \in \mathbb{Z}$). We have $q-1$ choices for $f(x)$:

$$N = \{\underbrace{x, x^2, \ldots, x^{q-1}}, e\}$$

we can choose $f(x)$ from these folks. ($f(x) := e$ doesn't define an invertible function). ⊠

Because $Im\,\phi \leq Aut(N)$. $|Im\,\phi| \mid q-1$, so necessarily $Im\,\phi = \{Id : N \to N\}$.

ie. Given any $x \in G$, $n \in N$

$$\underset{\overset{\shortparallel}{n}}{\phi([x])(n)} = x n x^{-1} \Rightarrow xn = nx$$

So N is Abelian, and everything else in G commutes with N. We still need one final observation.

Take $z \in G - H$, then necessarily $|\langle z \rangle| = p$. So we're in the situation of problem 4. $\langle z \rangle \leq G$, $\langle z \rangle \subset N_G(N) \Rightarrow \langle z \rangle N \leq G$.

Because $|\langle z \rangle N| = pq$, we're done ($\langle z \rangle N$ is clearly commutative).

ie $\langle z \rangle N = G$.