Group Theory - Homework 3

Problem 1. Let $G \times S \to S$ be a group action and take $s \in X$. Show that the stabilizer of the element s,

$$G_s := \{ g \in G \mid g.s = s \}$$

is a subgroup of G.

Recall $H \subset G$ is a subgroup if

1) $e \in H$,
2) If $x, y \in H$, then $xy \in H$.
3) If $x \in H$, then $x^{-1} \in H$

ie. If we restrict the group operations to H, then H is a group.

Let's just check these properties:

1) $e.s = s$ (by definition), so $e \in G_s$.

2) If $x, y \in G_s$, then $xy.s \underset{\cdot \text{ is a group action}}{=} x.(y.s) \underset{y \in G_s}{=} x.s \underset{x \in G_s}{=} s$, so $xy \in G_s$.

3) If x, G_s, then $x^{-1}.s \underset{x \in G_s}{=} x^{-1}.(x.s) = x^{-1}x.s = e.s = s$

Problem 2. Let $\phi : G \to H$ be a group homomorphism. Show that the subsets

$$\ker(\phi) = \{ g \in G \mid \phi(g) = e_H \}$$
$$\operatorname{Im}(\phi) = \{ \phi(g) \in H \mid g \in G \}$$

are subgroups of G and H, respectively.

Let's check $\ker(\phi)$ is a subgroup.

1) $\phi(e_G) = e_H$ (because ϕ is a hom.)

2) If $x, y \in \ker(\phi)$, then $\phi(xy) \underset{\phi \text{ is hom.}}{=} \phi(x) \cdot \phi(y) = e_H \cdot e_H = e_H$.

3) If $x \in \ker(\phi)$, $\phi(x^{-1}) \underset{\phi \text{ is hom.}}{=} \phi(x)^{-1} = e_H^{-1} = e_H$

Now let's see $\operatorname{Im}(\phi)$ is a subgroup:

1) $\phi(e_G) = e_H$, so $e_H \in \operatorname{Im}(\phi)$.

2) If $a, b \in \operatorname{Im}(\phi)$, then $a = \phi(x)$, $b = \phi(y)$ for some $x, y \in G$, so

$$ab = \phi(x)\phi(y) = \phi(xy) \in \operatorname{Im}(\phi)$$

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3) If $a \in Im(\phi)$, write $a = \phi(x)$ for $x \in G$, then

$$a^{-1} = \varphi(x)^{-1} = \phi(x^{-1}) \in Im(\phi).$$

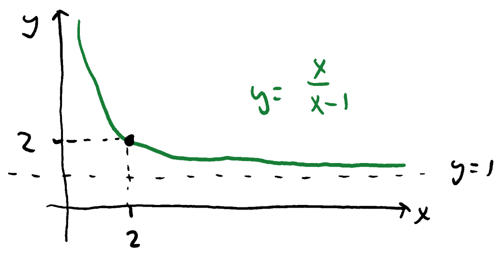
Problem 3. Let G be a group of order $|G| = n > 2$. Show that G cannot have a subgroup H of order $|H| = n-1$.

We know by Lagrange's Theorem that whenever H is a subgroup of G, then

$$|H| \mid |G|.$$

We just need to observe that whenever $n > 2$, it's never true that $n-1 \mid n$.

You could prove that $2 > \frac{n}{n-1} > 1 \quad \forall n > 2$ by induction, or you could convince yourself by looking at the graph



Problem 4. • Prove that if H and K are subgroups of G, then so is their intersection $K \cap H$.

• Prove that the intersection of a non-empty family of subgroups of G is again a subgroup.

• Take $K \cap H$ for H, K subgroups of G.

1) $\underbrace{e \in H}_{H \text{ is a subgroup}}$ and $\underbrace{e \in K}_{K \text{ is a subgroup}} \Rightarrow e \in K \cap H$

2) Given $x, y \in K \cap H$ then $xy \in K$ and $xy \in H$, so $xy \in K \cap H$

K is a subgrp. H is a subgrp.

3) Given $x \in K \cap H$, then $x^{-1} \in K$ and $x^{-1} \in H$, so $x^{-1} \in K \cap H$

• Let $\{K_i\}_{i \in A}$ be a (not necessarily countable) family of subgroups of G, and define

$$K := \bigcap_{i \in A} K_i$$ (A is just a set for my indices, it could be anything!)

1) Each K_i is a subgroup, so $e \in K_i \ \forall i$, meaning $e \in K$.

- If $x,y \in K$, then $x,y \in K_i$ $\forall i \in A$. It follows that, because each K_i is a subgroup.

$$xy \in K_i \ \forall i \in A \ (\text{ie } xy \in K).$$

- If $x \in K$, then $x \in K_i$ $\forall i \in A$, then $x^{-1} \in K_i$ $\forall i \in A$, then $x^{-1} \in K$

Each K_i is a subgroup.

Problem 6. Let A be an Abelian group. Prove that the set $H = \{a \in A \mid a^2 = e\}$ is a subgroup. Find an example of a non-Abelian group where this fails.

So again, $H = \{a \in A \mid a^2 = e\}$. Let's check this is a subgroup (for Abelian A!)

1) $e \in H$, as $e^2 = e$.

2) If $x,y \in H$, then $(xy)^2 = x(yx)y \underbrace{=}_{A \text{ is Abelian}} xxyy = x^2y^2 \underbrace{=}_{x,y \in H} e \cdot e = e$

3) If $x \in H$, then $(x^{-1})^2 = (x^2)^{-1} = e^{-1} = e$

For a counte-example, conside $G = S_3 = \{Id, (12), (13), (23), (123), (132)\}$

You can check that $H = \{Id, (12), (13), (23)\}$, but this isn't a subgroup, as

$$\underset{\uparrow \atop H}{(12)}\underset{\uparrow \atop H}{(13)} = (132) \notin H$$

S_3 is the smallest non-Abelian group there is (up to isomorphism). Keep it in mind for counte-examples!

Problem 7. Let $m,n \in \mathbb{Z}^{>0}$ be positive integers. It follows from the classification of subgroups that

$$m\mathbb{Z} \cap n\mathbb{Z} = k\mathbb{Z}$$

for some positive integer k. Convince yourself that $k = \operatorname{lcm}(m,n)$, and show that

$$k = \frac{mn}{\gcd(m,n)}$$

Hint: we can write $\gcd(m,n) = am + bn$ for some integers a, b.

- First note that $k \in m\mathbb{Z}$ and $k \in n\mathbb{Z}$, so $m \mid k$ and $n \mid k$.

If $m \mid L$ and $n \mid L$, (so L a common multiple of m and n), then

$L \in m\mathbb{Z} \cap n\mathbb{Z} = k\mathbb{Z} \Rightarrow k \mid L$

This all means $k = \operatorname{lcm}(m,n)$

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Now notice

$$m\mathbb{Z} \ni m \cdot \underbrace{\frac{n}{\gcd(m,n)}}_{\in\,\mathbb{Z}} = n \cdot \underbrace{\frac{m}{\gcd(m,n)}}_{\in\,\mathbb{Z}} \in n\mathbb{Z}$$

So $(mn)/\gcd(m,n) \in m\mathbb{Z} \cap n\mathbb{Z} = k\mathbb{Z}$, so there must exist some $k' \in \mathbb{Z}$ such that

$$k' \cdot k = \frac{mn}{\gcd(m,n)}$$

Now's when we use the hint:

$$\frac{1}{k'} = \frac{1}{k' \cdot k} \cdot k = \frac{\gcd(m,n)}{mn} \cdot k = \frac{am+bn}{mn} \cdot k$$

$$= \underbrace{\underbrace{\frac{k}{n}}_{\in\,\mathbb{Z}} \cdot a + \underbrace{\frac{k}{m}}_{\in\,\mathbb{Z}} \cdot b}_{\in\,\mathbb{Z}, \text{ as } a,b \in \mathbb{Z}}$$

So since $1/k' \in \mathbb{Z}$, necessarily $k' = 1$

(The sign is fixed because k and $\gcd(m,n)$ are positive).