Module MAU34403 Quantum mechanics I (Frolov)
Homework Sheet 8
Each set of homework questions is worth 100 marks

Use Mathematica if necessary

Compulsory Questions

Use the results from practice questions

Problem 1. The S-matrix allows one to express the outgoing amplitudes, B_L and A_R in terms of the incoming amplitudes A_L and B_R. It is sometimes more convenient to use the transfer matrix, M, which allows one to express the amplitudes to the right of the potential, A_R and B_R, in terms of those to the left, A_L and B_L

$$\begin{pmatrix} A_R \\ B_R \end{pmatrix} = \begin{pmatrix} M_{AA} & M_{AB} \\ M_{BA} & M_{BB} \end{pmatrix} \begin{pmatrix} A_L \\ B_L \end{pmatrix} \tag{0.1}$$

(a) Find the M-matrix for scattering from a single delta-function potential at point a

$$V(x) = \nu\,\delta(x-a) \tag{0.2}$$

(b) By the method from practice questions, find the M-matrix for scattering from the double-delta function

$$V(x) = \nu\,\delta(x+a) + \nu\,\delta(x-a) \tag{0.3}$$

What is the transmission coefficient for this potential?

Problem 2. The motion of two particles in one dimension is described by the Hamiltonian

$$H = \frac{P_1^2}{2m} - \frac{2P_1P_2}{m_{12}} + \frac{2P_2^2}{m} - k_1\,X_1X_2 + V_1(X_1 + \frac{X_2}{2}) + V_2(X_2 - 2X_1)\,, \quad m>0\,,\ m_{12}>0\,, \tag{0.4}$$

where the potentials V_1 and V_2 are given by

$$V_1(X) = \begin{cases} +\infty & \text{for } X<0 \\ k_1\,X^2/2 & \text{for } 0<X<a \\ +\infty & \text{for } X>a \end{cases}\,,$$
$$V_2(X) = \begin{cases} +\infty & \text{for } X<0 \\ k_2\,X^2/2 & \text{for } X>0 \end{cases}\,. \tag{0.5}$$

(a) Introduce the following new coordinates Y_1 and Y_2

$$Y_1 = X_1 + \frac{X_2}{2}\,, \quad Y_2 = X_2 - 2X_1\,. \tag{0.6}$$

Assume their conjugate momenta Π_1, Π_2 are linear functions of P_1, P_2, and find them. Check that they satisfy the canonical commutation relations. How many independent relations do you need to check?

Express the Hamiltonian in terms of the new coordinates and momenta. In classical mechanics for which values of m_{12}, m, k_1 and k_2 is the energy of the system positive unless it is at rest?

(b) Separate the variables and find the eigenvalues of the Hamiltonian for values of m_{12}, m, k_1 and k_2 from the previous question.

(c) Find the normalised ground state wave function.

Problem 3. Consider the orbital angular momentum in the coordinate representation

$$\vec{L} = -\mathrm{i}\hbar\,\vec{x}\times\vec{\nabla} \quad \Leftrightarrow \quad L_\alpha = -\mathrm{i}\hbar\sum_{\beta,\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}x_\beta\frac{\partial}{\partial x_\gamma}\,, \quad x_1\equiv x\,,\ x_2\equiv y\,,\ x_3\equiv z\,,$$
$$L_\pm \equiv L_x \pm \mathrm{i}\,L_y \tag{0.7}$$

and introduce the spherical coordinates

$$x = r\cos\phi\sin\theta\,,\quad y = r\sin\phi\sin\theta\,,\quad z = r\cos\theta\,,\quad r\geq 0\,,\ \ 0\leq\theta\leq\pi\,,\ \ 0\leq\phi\leq 2\pi \tag{0.8}$$

(a) Show that $L_\pm$ and $\vec{L}^{\,2}$ in terms of the spherical coordinates are given by

$$L_+ = \hbar\,e^{\mathrm{i}\phi}\Big(\frac{\partial}{\partial\theta} + \mathrm{i}\,\cot\theta\frac{\partial}{\partial\phi}\Big)\,,\quad L_- = \hbar\,e^{-\mathrm{i}\phi}\Big(-\frac{\partial}{\partial\theta} + \mathrm{i}\,\cot\theta\frac{\partial}{\partial\phi}\Big) \tag{0.9}$$

$$\vec{L}^{\,2} = -\hbar^2\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Big) \tag{0.10}$$

(b) Show that $\vec{\nabla}^2$ in terms of spherical coordinates is given by

$$\vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Big) \tag{0.11}$$

Practice Questions

Problem 1. The S-matrix allows one to express the outgoing amplitudes, B_L and A_R in terms of the incoming amplitudes A_L and B_R. It is sometimes more convenient to use the transfer matrix, M, which allows one to express the amplitudes to the right of the potential, A_R and B_R, in terms of those to the left, A_L and B_L

$$\begin{pmatrix} A_R \\ B_R \end{pmatrix} = \begin{pmatrix} M_{AA} & M_{AB} \\ M_{BA} & M_{BB} \end{pmatrix}\begin{pmatrix} A_L \\ B_L \end{pmatrix} \tag{0.12}$$

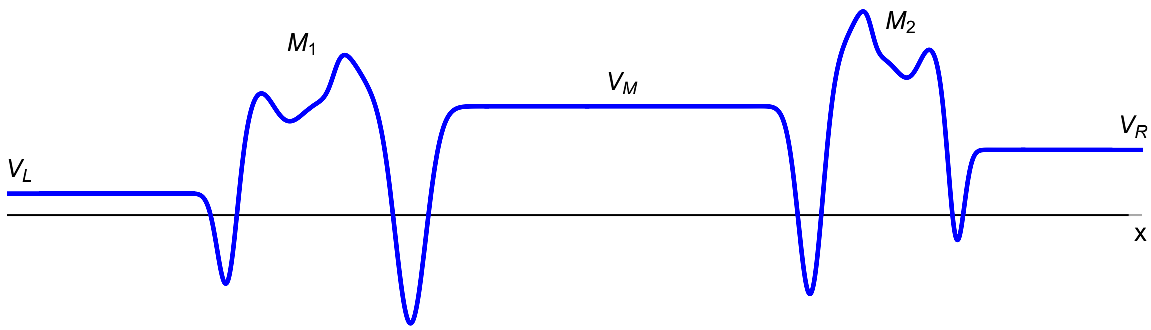

Figure 1: Potential which consists of two isolated pieces. V_L, V_R, and V_M are constants.

(a) Find the four elements of the transfer matrix, in terms of the elements of the S-matrix, and vice-versa. Use that the S-matrix is symmetric and unitary to simplify the expressions. Express the reflection and transmission coefficients R and T in terms of elements of the M-matrix.

(b) Suppose you have a potential consisting of two isolated pieces, see Figure 1. Show that the M-matrix for the combination is the product of the two M-matrices for each section separately:

$$M = M_2 M_1 \tag{0.13}$$

Problem 2. Consider the spherical coordinates

$$x = r\cos\phi\sin\theta\,,\quad y = r\sin\phi\sin\theta\,,\quad z = r\cos\theta\,,\quad r \geq 0\,,\ 0 \leq \theta \leq \pi\,,\ 0 \leq \phi \leq 2\pi \tag{0.14}$$

(a) Check that

$$\begin{aligned} p_x &= \cos\phi\sin\theta\, p_r - \frac{\sin\phi}{r\sin\theta}\, p_\phi + \frac{1}{r}\cos\phi\cos\theta\, p_\theta \\ p_y &= \sin\phi\sin\theta\, p_r + \frac{\cos\phi}{r\sin\theta}\, p_\phi + \frac{1}{r}\sin\phi\cos\theta\, p_\theta \\ p_z &= \cos\theta\, p_r - \frac{1}{r}\sin\theta\, p_\theta \end{aligned} \tag{0.15}$$

together with (0.14) gives a classical canonical transformation.

(b) Check that

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\phi\sin\theta\frac{\partial}{\partial r} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{1}{r}\cos\phi\cos\theta\frac{\partial}{\partial\theta}\,, \\ \frac{\partial}{\partial y} &= \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{1}{r}\sin\phi\cos\theta\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial z} &= \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta} \end{aligned} \tag{0.16}$$

(c) Show that the orbital angular momentum components L_α in terms of the spherical coordinates are given by

$$\begin{aligned} L_z &= -\mathrm{i}\,\hbar\,\frac{\partial}{\partial\phi} \\ L_x &= -\mathrm{i}\,\hbar\Big(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\Big) \\ L_y &= -\mathrm{i}\,\hbar\Big(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\Big) \end{aligned} \tag{0.17}$$