L4: Symmetric group

Symmetric group

$S_n = \{\sigma : \{1, 2, \dots, n\} \to \{1, \dots, n\} \mid \sigma \text{ bijection}\}$

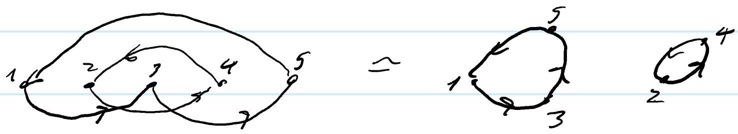
"Symmetric group on n elements."

an element $\sigma \in S_n$ can be given by the list $(\sigma(1), \dots, \sigma(n))$

e.g. $\sigma = (2, 1, 3) \in S_3$

sometimes written as $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$

Ex $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} \in S_5$



we write $\sigma = (1\,3\,5)(2\,4)$ cycle decomposition

Def Given $a_1, \dots, a_\ell \in \{1, \dots, n\}$ all distinct we define

$S_n \ni \sigma =: (a_1 \cdots a_\ell)$ "an ℓ-cycle"

by the formula $\sigma(x) = \begin{cases} a_{j+1} & \text{if } x = a_j \\ x & \text{else} \end{cases}$

Rem $(a_1 \cdots a_\ell) = (a_2 \cdots a_\ell\, a_1)$

Lemma Let $\sigma = (a_1 \cdots a_\ell)$ and $\tau = (b_1 \cdots b_m)$ be such that

$\{a_1, \dots, a_\ell\} \cap \{b_1, \dots, b_m\} = \emptyset$ "disjoint".

Then $\sigma \cdot \tau = \tau \cdot \sigma$

Pf Exc.

Prop Every $\sigma \in S_n$ admits a decomposition into disjoint cycles,

i.e. $\exists\, \sigma_1, \dots, \sigma_m \in S_n$ disjoint cycles s.t.

$\sigma = \sigma_1 \cdot \ldots \cdot \sigma_m$

Pf (sketch) Let $i = \min\{j \mid \sigma(j) \neq j\}$

for some ℓ_1, ℓ_2 we have $\sigma^{\ell_1}(i) = \sigma^{\ell_2}(i)$

$\Rightarrow \sigma^{\ell_1 - \ell_2}(i) = i$ without loss of generality

$\ell = \ell_1 - \ell_2 > 0$

set $\sigma_i = (i \ f(i) \cdots f^{l-1}(i))$
replace σ with $\sigma_1^{-1} \cdot \sigma$ and repeat.

$\square$

Rmk Not every product of cycles is a cycle decomp.
e.g. $(1\,2\,3)(3\,4) = (1\,2\,3\,4)$
$(1\,2\,3)(4\,5)(3\,4) = (1\,2\,3\,5\,4)$