Module MAU34403 Quantum mechanics I (Frolov)
Homework Sheet 7
Each set of homework questions is worth 100 marks

Use Mathematica if necessary

Compulsory Questions

Use the results from practice questions

Problem 1. A coherent state of a one-dimensional harmonic oscillator is defined to be an eigenstate of the (non-Hermitian) annihilation operator a

$$a\,|\lambda\rangle = \lambda\,|\lambda\rangle\,, \quad \langle\lambda|\lambda\rangle = 1 \tag{0.1}$$

where α is in general a complex number.

(a) Find $|\lambda\rangle$

(b) Express $|\lambda\rangle$ in the form $|\lambda\rangle = f(a^\dagger)|0\rangle$

(c) Prove the minimum uncertainty relation for such a state.

(d) Find the wave function $\psi_\lambda(x)$ of a coherent state in the coordinate representation

(e) Write $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)\,|n\rangle \tag{0.2}$$

Show that the distribution of $|f(n)|^2$ with respect to n is of the Poisson form. Find the expectation value $\bar{n}$ of $N = a^\dagger a$. Find the most probable value $n_{\rm mp}$ of n, hence of E.

Problem 2. Consider a particle in the following potential

$$V(x) = -\nu\,\delta(x+a) - \nu\,\delta(x-a)\,, \tag{0.3}$$

where $\nu > 0$, $a > 0$. It is a double delta-function well.

(a) Sketch the following potential

$$V_\epsilon(x) = -\nu\,\delta_\epsilon(x+a) - \nu\,\delta_\epsilon(x-a)\,. \tag{0.4}$$

Here $\epsilon/a \ll 1$, and $\delta_\epsilon(x)$ is a regularised delta-function

$$\delta_\epsilon(x) = \frac{1}{2\epsilon}\Big(\theta(x+\epsilon) - \theta(x-\epsilon)\Big)\,, \tag{0.5}$$

where $\theta(x)$ is the Heaviside function.

(b) Find the energy quantisation condition for the even parity bound states of the particle in the potential $V(x)$. Show that there is only one bound state. Find the normalised ground state wave function. Solve the energy quantisation condition numerically for $m = 1$, $\hbar = 1$, $\nu = 2$, $a = 0.1$, for $m = 1$, $\hbar = 1$, $\nu = 2$, $a = 1$, and for $m = 1$, $\hbar = 1$, $\nu = 2$, $a = 10$, and plot the wave functions. Comment on the pictures.

(c) Find the energy quantisation condition for the odd parity bound states of the particle in the potential $V(x)$. Show that there may exist only one odd parity bound state, and find the values of ν for which it exists. Find the normalised excited bound state wave function. Solve the energy quantisation condition numerically for $m = 1$, $\hbar = 1$, $\nu = 2$, $a = 0.51$, for $m = 1$, $\hbar = 1$, $\nu = 2$, $a = 1$, and for $m = 1$, $\hbar = 1$, $\nu = 2$, $a = 10$, and plot the wave functions. Comment on the pictures.

(d) For $m = 1$, $\hbar = 1$, $\nu = 1$, $a = 1$, plot the ground state wave function and the odd bound state wave function in the same figure, and then plot the function

$$\frac{1}{\sqrt{2}}\Big(\psi_{\rm e}(x) + \psi_0(x)\Big) \tag{0.6}$$

in another one. Do the same for $m = 1$, $\hbar = 1$, $\nu = 1$, $a = 10$. Comment on the pictures.

(e) In the limit of large separation, $2a$, between the wells, obtain a simple formula for the splitting ΔE between the excited (odd parity) energy level, $E_{\rm o}$, and the ground state (even parity) energy level, $E_{\rm e}$.

Practice Questions

Problem 1. Consider a particle in the potential of a rectangular well

$$V(x) = \begin{cases} V_L & \text{for } |x| > a \\ V_{\rm min} & \text{for } |x| < a \end{cases} \tag{0.7}$$

1. Find the energy quantisation condition for odd parity states
2. Show that the energy quantisation condition can be written in the form

$$-\cot z = \sqrt{\frac{W^2}{z^2} - 1} \tag{0.8}$$

where z and W have to be identified.

Sketch plots of the left and right hand sides of the energy quantisation condition.

Find the values of W for which there are n odd parity bound states.

Problem 2. Consider a particle in the following potential

$$V(x) = -\nu\,\delta(x+a) - \nu\,\delta(x-a)\,, \tag{0.9}$$

where $\nu > 0$, $a > 0$. It is a double delta-function well.

Find the energy quantisation condition for the even parity bound states of the particle in the potential $V(x)$. Show that there is only one bound state. Find the normalised ground state wave function. Solve the energy quantisation condition numerically for $m = 1$, $\hbar = 1$, $\nu = 2$, $a = 0.1$, for $m = 1$, $\hbar = 1$, $\nu = 2$, $a = 1$, and for $m = 1$, $\hbar = 1$, $\nu = 2$, $a = 10$, and plot the wave functions. Comment on the pictures.

Problem 3. Scattering in one dimension

(a) Consider the following four functions

$$\varphi_\alpha^\pm(E,x) \equiv \sqrt{\frac{m}{2\pi\,\hbar^2}}\,\frac{e^{\pm i\,k_\alpha\,x}}{\sqrt{k_\alpha}}\,, \quad k_\alpha \equiv \frac{\sqrt{2m(E-V_\alpha)}}{\hbar}\,, \quad \alpha = L, R \tag{0.10}$$

Prove that they are normalised as

$$\int dx\,\bar\varphi_L^a(E_1,x)\varphi_L^b(E_2,x) = \delta_{ab}\delta(E_1-E_2)\,, \quad a,b = +,-\,, \tag{0.11}$$

where $\bar\varphi$ denotes the complex conjugate of φ, and φ_R^a satisfy the same normalisations.

Hint. Use the formula

$$\delta(f(x)-f(y)) = \frac{1}{f'(x)}\delta(x-y)\,, \quad f(x) \neq f(y) \text{ for } x \neq y \tag{0.12}$$

(b) Prove that the **Wronskian**

$$W(f_1,f_2) \equiv f_1 f_2' - f_1' f_2 \tag{0.13}$$

of any two functions satisfying the time-independent Schrödinger equation does not depend on x.

(c) Let

$$\varphi_L(E,x) = \begin{cases} \varphi_L^+(E,x) + S_{LL}\varphi_L^-(E,x) & \text{for } x < -a \\ S_{RL}\varphi_R^+(E,x) & \text{for } x > a \\ A_{ML}\varphi_1(x) + B_{ML}\varphi_2(x) & \text{for } |x| < a \end{cases} \tag{0.14}$$

and

$$\varphi_R(E,x) = \begin{cases} S_{LR}\varphi_L^-(E,x) & \text{for } x < -a \\ S_{RR}\varphi_R^+(E,x) + \varphi_R^-(E,x) & \text{for } x > a \\ A_{MR}\varphi_1(x) + B_{MR}\varphi_2(x) & \text{for } |x| < a \end{cases} \tag{0.15}$$

Consider the four pairs (φ_L, φ_L^*), (φ_R, φ_R^*), (φ_L, φ_R^*) and (φ_L, φ_R), and compute their Wronskians for $x < -a$ and $x > a$.