Module MAU34403 Quantum mechanics I (Frolov)
Homework Sheet 5
Each set of homework questions is worth 100 marks

Use Mathematica if necessary

Compulsory Questions

Use the results from practice questions

Problem 1.
Consider a conservative system of N particles with the Hamiltonian

$$H = \sum_{a=1}^{N} \frac{\vec{P}_a^{\,2}}{2m_a} + V(\vec{X}_1, \vec{X}_2, \dots, \vec{X}_N) \tag{0.1}$$

where H, $\vec{P}$, $\vec{X}$ are operators. Here and in what follows we remove hats from operators to simplify the notations.

Take the expectation values of both sides of the Heisenberg equations for $\vec{X}_a$ and $\vec{P}_a$, and derive equations for the expectation values of $\vec{X}_a$ and $\vec{P}_a$. Exclude $\vec{P}_a$ and get Newton's equations for the expectation values of $\vec{X}_a$. Are they the same as in classical mechanics? This is the Ehrenfest theorem.

Problem 2. Consider a particle of spin 1/2 at rest in a uniform magnetic field which points in the z-direction: $\vec{B} = (0, 0, B)$. The Hamiltonian of the particles is

$$H = -\gamma\, \vec{B} \cdot \vec{S} = -\frac{\gamma\, B\, \hbar}{2} \sigma^z \tag{0.2}$$

(a) Use $U(t)$ to find how the spins S^α depend on time in the Heisenberg picture.

(b) Find the expectation values of S^α at time t in the state $|\psi\rangle$

$$|\psi\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}\,|\downarrow\rangle \tag{0.3}$$

Use any picture you like. How would you interpret the results obtained?

Problem 3. Consider the following state vector

$$|\psi\rangle = \int dx\, \psi(x)\, |x\rangle \tag{0.4}$$

where $\psi(x)$ is a Gaussian wave packet in the coordinate space at time $t = 0$

$$\psi(x) = \frac{1}{\sqrt{\sqrt{\pi}\Delta}} \exp\Big(\frac{\mathrm{i}}{\hbar} k(x-a) - \frac{(x-a)^2}{2\Delta^2}\Big) \tag{0.5}$$

(a) Let the particle with this wave function be free, that is let the Hamiltonian be $H = \frac{P^2}{2m}$.

Find the time evolution of the Gaussian wave packets in the coordinate and momentum spaces. Compute the probability densities of the wave packets in the coordinate and momentum spaces, and check that they are normalised to 1.

Hint. First find $\psi(p,t)$, and then use it to find $\psi(x,t)$.

(b) Find the expectation values of and the uncertainty in the coordinate X and the momentum P with respect to $]\psi(t)\rangle$, and comment the results obtained.

Can one have both uncertainties ΔX and ΔP to be very small for substantial period of time?

Practice Questions

Problem 1.

Consider a conservative system of N particles with the Hamiltonian

$$H = \sum_{a=1}^{N} \frac{\vec{P}_a^{\,2}}{2m_a} + V(\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_N) \tag{0.6}$$

where H, $\vec{P}$, $\vec{X}$ are operators. Here and in what follows we remove hats from operators to simplify the notations.

Find the Heisenberg equations for $\vec{X}_a$ and $\vec{P}_a$ (it requires to compute the commutators of H with $\vec{X}_a$ and $\vec{P}_a$)

Problem 2. Consider a particle of spin 1/2 at rest in a uniform magnetic field which points in the z-direction: $\vec{B} = (0,0,B)$. The Hamiltonian of the particles is

$$H = -\gamma\,\vec{B}\cdot\vec{S} = -\frac{\gamma\,B\,\hbar}{2}\sigma^z \tag{0.7}$$

(a) Find the evolution operator $U(t) \equiv U(t,t_0)$.

(b) Use $U(t)$ to find how a state vector

$$|\psi\rangle = \cos\frac{\theta}{2}\,|\!\uparrow\rangle + \sin\frac{\theta}{2}\,|\!\downarrow\rangle \tag{0.8}$$

depends on time in the Schrödinger picture.

Problem 3. Consider the following state vector

$$|\psi\rangle = \int dx\,\psi(x)\,|x\rangle \tag{0.9}$$

where $\psi(x)$ is a Gaussian wave packet in the coordinate space at time $t = 0$

$$\psi(x) = \frac{1}{\sqrt{\sqrt{\pi}\Delta}} \exp\Big(\frac{\mathrm{i}}{\hbar}k(x-a) - \frac{(x-a)^2}{2\Delta^2}\Big) \tag{0.10}$$

(a) Find the Gaussian wave packet $\psi(p)$ in the momentum space

(b) Let the particle with this wave function be free, that is let the Hamiltonian be $H = \frac{P^2}{2m}$.

Find the time evolution of the Gaussian wave packets in the coordinate and momentum spaces.