L14: More group actions

Prop Let $H \leq G$ be a subgroup. Then
$$G \times G/H \to G/H \quad \text{defines a group action.}$$
$$(g, V) \mapsto gV := \{gv \mid v \in V\}$$

Pf •) We show first that the same formula defines a group action
$G \circlearrowright \mathcal{P}(G) := \{V \mid V \text{ a subset of } G\}$.

i) $g_1 \cdot (g_2 \cdot V) = \{g_1 v \mid v \in g_2 V\} = \{g_1 g_2 v \mid v \in V\} = (g_1 g_2) \cdot V$

ii) $e \cdot V = \{ev \mid v \in V\} = V$

•) It remains to show that the above action restricts to $G/H \subseteq \mathcal{P}(G)$, i.e. that $g \cdot V \in G/H$ whenever $V \in G/H$. By definition $V \in G/H \Leftrightarrow \exists g_2 \in G$ s.t. $V = g_2 \cdot H$. But then $g \cdot (g_2 \cdot H) = (g \cdot g_2) \cdot H \in G/H$. □

Def Let G be a group.

- A G-set (X, S) is a tuple consisting of a set X together with a group action $G \times X \to X$.
- A map $\varphi : X_1 \to X_2$ is a <u>homomorphism of G-sets</u> if
$$\varphi(gx) = g\,\varphi(x)$$
- We call φ an isom of G-sets if it is furthermore a bijection.

Recall: • transitive = single orbit

• $\mathrm{Stab}_G(x) = G_x = \{g \in G \mid g \cdot x = x\}$

Prop Let $G \circlearrowright X$ be transitive and $x \in X$. Then there is an isomorphism of G-sets
$$G/\mathrm{Stab}_G(x) \xrightarrow{\cong} X$$
$$[g] \longmapsto g \cdot x$$

Pf Exercise!

Cor (Orbit-Stabilizer formula)
Let $G \circlearrowright X$. Then $G \cdot x \cong G/\mathrm{Stab}_G(x)$
and thus $|G \cdot x| = |G| / |\mathrm{Stab}_G(x)|$ if G is finite

Application

Prop $|S_n| = n!$

Pf $S_n \curvearrowright \{1, 2, \dots, n\} =: X$

We claim that the action is transitive e.g. $X = S_n \cdot x$ for $x = n \in X$.

This is clear ($\sigma_\ell = (n\ \ell)$ satisfies $\sigma(n) = \ell$).

Moreover $\mathrm{Stab}_{S_n}(n) = \{ \sigma : \{1, \dots, n\} \to \{1, \dots, n\} \mid \sigma \text{ bij } \& \ \sigma(n) = n \}$

$\cong S_{n-1}$

restrict σ to $\sigma|_{\{1, 2, \dots, n-1\}}$

But then $n = |X| = \left| \frac{S_n}{S_{n-1}} \right|$

$\Rightarrow |S_n| = n \cdot |S_{n-1}| = \dots = n \cdot (n-1) \cdot \dots \cdot 2 \cdot \underset{1}{\underbrace{|S_1|}}$ $\square$

Thm (Class equation) $G \curvearrowright X$

$$|X| = |Fix_G(X)| + \sum_{i=1}^{\ell} \left| \frac{G}{Stab_G(x_i)} \right|$$

where $Fix_G(X) = \{ x \in X \mid g.x = x \ \forall g \in G \}$

and $x_1, \dots, x_\ell$ are a set of representative for $X/G \setminus Fix_G(X)$

i.e. $X/G = \bigcup_{x \in Fix_G(X)} [x] \cup \bigcup_{i=1}^{\ell} G.x_i$

Pf We have $|X| = \sum_{[x] \in X/G} |G \cdot x|$

$$= \underbrace{\sum_{\substack{[x] \in X/G \\ |G.x| = 1}} 1}_{|Fix_G(X)|} + \sum_{i=1}^{\ell} |G.x_i|$$

$$|G.x_i| = \left| \frac{G}{Stab_G(x_i)} \right|$$