L7: Subgroups

Subgroups

Def A subset $H \subseteq G$ of a group (G,m) is a subgroup if the restriction of m to $H \times H$ turns H into a group. We write $H \leq G$ in that case.

Rem In particular we ask that $a,b \in H \Rightarrow m(a,b) \in H$.

Prop $H \subseteq G$ is a subgroup $\Leftrightarrow$ i) H is non-empty and
ii) $a,b \in H \Rightarrow ab^{-1} \in H$

Pf "$\Rightarrow$": H a group $\Rightarrow e \in H \Rightarrow H \neq \emptyset$

Part ii) is clear

"$\Leftarrow$": H non-empty i.e. $\exists a \in H \overset{ii)}{\Rightarrow} e = aa^{-1} \in H$. (*)

But then $\forall a \in H \overset{ii)}{\Rightarrow} a^{-1} = ea^{-1} \in H$ (**)

And thus $\forall a,b \in H \Rightarrow a,(b^{-1}) \in H \overset{ii)}{\Rightarrow} ab = a(b^{-1})^{-1} \in H$.

We have shown that m restricts to a map $m_H = m|_{H \times H} : H \times H \to H$

i.e. $a,b \in H \Rightarrow ab \in H$.

It remains to verify that m_H satisfies the three group axioms.

associativity : ✓

unit : (*)

inverse : (**) □

Examples 0) Every group has trivial subgroup $\{e\} \leq G$

improper subgroup $G \leq G$

1) $n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\} \subseteq \mathbb{Z}$ is a subgroup of $(\mathbb{Z},+)$

2) Given $x \in G$ a group. $\langle x \rangle = \{x^n \mid n \in \mathbb{Z}\} \subseteq G$ is a subgroup

3) Let $G \times X \to X$ be a group action, and $s \in X$.
The stabilizer of s $\quad G_s := \{g \in G \mid g.s = s\}$
is a subgroup.

4) Let $\varphi : G \to H$ be a group homomorphism. Then
- $\ker \varphi := \{g \in G \mid \varphi(g) = e\}$
- $\operatorname{im} \varphi := \varphi(G) = \{\varphi(g) \in H \mid g \in G\}$ are subgroups.

Exc Prove 1) - 4)