Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

SF/SF JH Maths/TP — Michaelmas Term 2023

MAU22101: Abstract algebra I: Group theory

12/12/2023 RDS-SIM COURT 09.30 - 11.30

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Instructions to Candidates:

Attempt all the questions. All questions are weighted equally.

Materials Permitted for this Examination:

No additional material is permitted except for writing implements.

You may not start this examination until you are instructed to do so by the Invigilator.

1. Let $\phi\colon G \to H$ be a group homomorphism between the groups G and H. Let $G_1 \subset G$ and $H_1 \subset H$ be subsets. Decide whether the following implications hold (by either providing a proof or a counterexample).

   (a) $H_1 \leq H \implies \phi^{-1}(H_1) \leq G$.

   (b) $H_1 \lhd H \implies \phi^{-1}(H_1) \lhd G$.

   (c) $G_1 \lhd G \implies \phi(G_1) \lhd H$.

   (d) $G_1 \lhd G \impliedby \phi(G_1) \lhd H$.

2. Let G be a finite group of order $|G| = pm$ where p is a prime and such that $(p-1, m) = 1$. Suppose that G contains a normal subgroup $H \lhd G$ of order p, that is $|H| = p$. Show that H is contained in the center of G,

$$H \leq Z(G).$$

3. Prove that the additive group of the rational numbers $(\mathbb{Q}, +)$ has no proper subgroup of finite index.

4. Let $G \times X \to X$ be a G-action and let $V \subset X$ be a G-orbit. Given $x, y \in V$ show that there exists $g \in G$ such that

$$\mathrm{Stab}_G(x) = g\mathrm{Stab}_G(y)g^{-1}$$

(i.e. the corresponding stabilizer subgroups are conjugate).

5. Let G be a group of order $|G| = 56$. Show that G is not simple, i.e. it contains a non-trivial, proper *normal* subgroup.

6. Let G be a finite group and let $E = \max\{|x| \mid x \in G\}$ be the order of the element with the largest order.

   - Assume that G is abelian. Show that $x^E = e$ for all $x \in G$.
   - Find a counterexample showing that we cannot drop the assumption that G be abelian in the first part.