MAU22101: Exercises Week 8

Problem 1 Given group N and H together with a group homomorphism $\varphi\colon H \to \mathrm{Aut}(N)$ we defined the semidirect product $N \rtimes_\varphi H$ to be $N \times H$ with the group multiplication given by the formula

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1 \cdot \varphi(h_1)(n_2), h_1 \cdot h_2).$$

1. Show that this indeed defines a group.
2. Show that $N \rtimes H$ contains N as a normal subgroup and H as a subgroup.
3. Give a formula for the adjoint action in terms of φ.

Problem 2 Let N, H be groups and let $\varphi_1, \varphi_2\colon H \to \mathrm{Aut}(N)$ be two group homomorphisms. Suppose that

$$N \rtimes_{\varphi_1} H \cong N \rtimes_{\varphi_2} H.$$

Also suppose that there does not exist any group homomorphism $\psi\colon N \to H$ except for the trival one (i.e. sending $N \ni n \mapsto e \in H$). Show that in that case there exist

$$F \in \mathrm{Aut}(N), \quad G \in \mathrm{Aut}(H),$$

such that

$$\varphi_1(h) = F \circ \varphi_2(G(h)) \circ F^{-1}.$$

Problem 3 We saw in the lecture that any group G of order $|G| = 30$ is either abelian or isomorphic to

$$\mathbb{Z}/15\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z},$$

where $\varphi\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/15\mathbb{Z})$ is inclusion of an element of order 2 of $\mathrm{Aut}(\mathbb{Z}/15\mathbb{Z})$. Use the previous problem to show that different elements of order 2 give non-isomorphic groups.

Problem 4 Prove that a group of order $351 = 3^3 \cdot 13$ has a normal Sylow p-subgroup for some prime p dividing its order.

Problem 5 Show that

1. $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \cong \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \times \mathrm{Aut}(\mathbb{Z}/m\mathbb{Z})$ whenever $(m, n) = 1$
2. $|(\mathbb{Z}/p^\alpha\mathbb{Z})^\times| = p^{\alpha-1}(p-1)$ where p is a prime number and $\alpha \in \mathbb{N}$.
3. Conclude that

$$|(\mathbb{Z}/n\mathbb{Z})^\times| = p_1^{\alpha_1-1}(p_1-1) \cdot \cdots \cdot p_k^{\alpha_k-1}(p_k-1)$$

where $n = p_1^{\alpha_1} \cdot \cdots \cdot p_k^{\alpha_k}$ is the prime factorization of n (i.e. $p_1, \ldots, p_k$ are pairwise distinct primes). (Remark: The function $\varphi(n) := |(\mathbb{Z}/n\mathbb{Z})^\times|$ is called *Euler's phi function*).

Problem 6

1. Find all elements of order 2 in $\mathrm{Aut}(\mathbb{Z}/15\mathbb{Z}) \cong \mathbb{Z}/15\mathbb{Z}^\times$ by checking all the elements.
2. Write down explicitly the group isomorphism $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$.
3. Find all elements of order 2 in $\mathbb{Z}/3\mathbb{Z}^\times$ and $\mathbb{Z}/5\mathbb{Z}^\times$.
4. Use the previous two parts to check your answer in the first part (or rather give an alternate calculation).