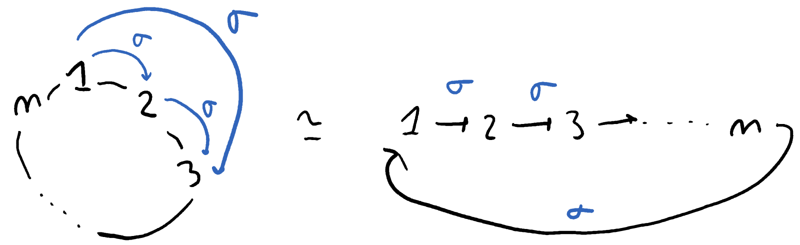
Group Theory - Homework 2

Problem 2. Show that the order of an element in S_n equals the lcm of the lengths of the cycles in its cycle decomposition.

Claim 1. The order of a cycle is equal to its length.

Proof. First we consider $\sigma = (1\ 2 \cdots m) \in S_n$. Notice that if $i \in \{1, \dots, m\}$,

$$\sigma^i(1) = 1+i \text{ if } i+1 \leq m, \qquad \sigma^i(1) = i+1 \pmod m \text{ if } i+1 > m.$$



In general, for $k \in \{1, \dots, m\}$,

$$\sigma^i(k) = \begin{cases} i+k & \text{if } i+k \leq m \\ i+k \pmod m & \text{if } i+k > m. \end{cases}$$

This is problem 10 in p. 33 of D&F.

(You can prove this by induction on i.)

This implies that $|\sigma| = m$, as $\sigma^m(k) = k+m \pmod m = k$ for all $k \in \{1, \dots, m\}$.
For an arbitrary cycle of length m

$$\alpha := (a_1\ a_2 \cdots a_m)$$

you could prove $\alpha^i(a_k) = a_{k+i \pmod m}$, or better yet, you could show $\alpha = \gamma^{-1}(1\ 2 \cdots m)\gamma$, where γ is the permutation

$$\gamma = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ 1 & 2 & & m \end{pmatrix} \quad (\text{i.e. } \gamma(a_i) = i).$$

Since α is a conjugate of σ, they both have the same order. ∎

Claim 2. Take $x, y \in G$. If x and y commute, then $(xy)^n = x^n y^n$ for all $n \in \mathbb{Z}$. → This is problem 24 of Sec. 1 in D&F.

i.e. $xy = xy$ Remember this isn't true in general!

For positive n,

$$\left.\begin{aligned} (xy)^n &= (xy)(xy)\cdots(xy) \\ &= x^n y^n \end{aligned}\right\}$$

This is hardly a proof. Use induction if you're not convinced, and then consider negative n!

Now we're ready to tackle the problem. Let

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$$

be the cycle decomposition of σ (i.e. each σ_i is a cycle, and they are pair-wise disjoint). We've just shown $|\sigma_i| =$ length of σ_i. If $L := \operatorname{lcm}_i(|\sigma_i|)$, then clearly

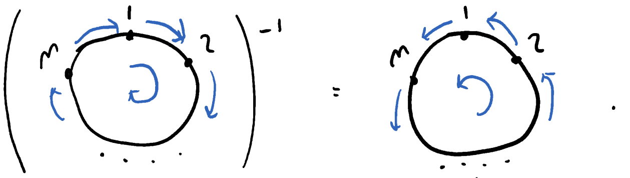
$$\sigma_i^L = 1 \quad \text{for all } i.$$

Then

$$\begin{aligned} \sigma^L &= (\sigma_1 \cdots \sigma_m)^L \\ &= \sigma_1^L \cdots \sigma_m^L \\ &= 1 \cdots 1 = 1. \end{aligned}$$

disjoint cycles commute!

Could you argue, by contradiction, that if $1 < H < L$ then $\sigma^H \neq 1$? At some point you'll need to observe a cycle and its inverse are never disjoint; for example



$\parallel$ $(1\ 2\ \cdots\ m)^{-1}$ $\qquad$ $\parallel$ $(m\ m{-}1\ \cdots\ 2\ 1)$

Problem 1. Let $\sigma = (1\ 2 \cdots \ell) \in S_n$ be an ℓ-cycle where $\ell = 2k$ is even. Find the cycle decomposition of σ^k.

Because σ is a cycle of length $2k$, we know from the previous solution that

$$\sigma^k(i) = \begin{cases} i+k & \text{for } i \in \{1, \dots, k\} \\ i+k \pmod{2k} & \text{for } i \in \{k+1, \dots, 2k\} \end{cases}$$

Remember we compute cycle decompositions algorithmically:

What's the image of 1? $\longrightarrow$ $\sigma^k(1) = 1+k$

We close the cycle containing 1 $\longrightarrow$ $\sigma^k(1+k) = 1+k+k \pmod{2k} = 1$

So far $\sigma^k = (1\ \ k+1)(2 \cdots$

Repeat for smallest num. remaining $\longrightarrow$ $\sigma^k(2) = 2+k$, $\sigma^k(2+k) = 2 \pmod{2k}$

So far $\sigma^k = (1\ \ k+1)(2\ \ k+2)(3 \cdots)$

Eventually...

$\sigma^k(k) = 2k$, $\sigma^k(2k) = 3k \pmod{2k} = k$.

And finally $\sigma^k = (1\ \ k+1)(2\ \ k+2) \cdots (k\ \ 2k)$

Problem 3. Let G be a group. Show that the three maps $\rho_\ell, \rho_r, \rho_{ad} : G\times G \longrightarrow G$ defined by

$$\rho_\ell(g,x) = gx$$
$$\rho_r(g,x) = xg^{-1}$$
$$\rho_{ad}(g,x) = gxg^{-1}$$

define group actions of G on G.

We say $\rho : G\times M \longrightarrow M$ is a group action of G on M if ① $\rho(g,\rho(h,x)) = \rho(gh,x)$ $\forall g,h\in G$, $\forall x\in M$
② $\rho(e,x) = x$ $\forall x \in M$.

We'll just check ① and ② for each map:

- ρ_ℓ : ① $\rho_\ell(g,\rho_\ell(h,x)) = \rho_\ell(g,hx) = ghx = \rho_\ell(gh,x)$
  ② $\rho_\ell(e,x) = ex = x$ $\forall x\in G$.
- ρ_r : ① $\rho_r(g,\rho_r(h,x)) = \rho_r(g,xh^{-1}) = xh^{-1}g^{-1} = x(gh)^{-1} = \rho_r(gh,x)$
  ② $\rho_r(e,x) = xe^{-1} = xe = x$ $\forall x\in G$.
- ρ_{ad} : ① $\rho_{ad}(g,\rho_{ad}(h,x)) = \rho_{ad}(g,hxh^{-1}) = ghxh^{-1}g^{-1} = ghx(gh)^{-1} = \rho_{ad}(gh,x)$
  ② $\rho_{ad}(e,x) = exe^{-1} = e\cdot x\cdot e = x$ $\forall x \in G$.

Problem 4. Let $g\in G$, and define the map $\varphi : G\to G$ by $\varphi(x) = gxg^{-1}$. Show that φ is a group-isomorphism.

A map $(G,\cdot) \xrightarrow{f} (H,*)$ is an isomorphism iff ① $f(a\cdot b) = f(a) * f(b)$ $\forall a,b\in G$
② f is bijective.

Notice φ has the inverse $\psi : G\to G$ given by $\psi(x) = g^{-1}xg$;

$$\psi\varphi(x) = \psi(gxg^{-1}) = g^{-1}gxgg^{-1} = exe = x.$$

As for ①,

$$\varphi(xy) = gxyg^{-1} = gxeyg^{-1} = gx(g^{-1}g)yg^{-1} = \varphi(x)\varphi(y).$$

Problem 5. Let G be a group. Show that the formula

$$\mathbb{Z}/2\mathbb{Z}\times G \longrightarrow G$$
$$(0,g) \longmapsto g$$
$$(1,g) \longmapsto g^{-1}$$

defines a group action of $\mathbb{Z}/2\mathbb{Z}$ on G.

$\mathbb{Z}/2\mathbb{Z}$ is the group

	0	1
0	0	1
1	1	0

. The only non trivial relation is

$$(1,(1,g)) = (1,g^{-1}) = g = (0,g) = (1+1,g).$$

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