Module MAU34403 Quantum mechanics I (Frolov)
Homework Sheet 3
Each set of homework questions is worth 100 marks

Use Mathematica if necessary

Compulsory Questions

Problem 1. The tensor product

Def. The tensor product of two nonzero representations $(\rho, \mathscr{V})$ and $(\sigma, \mathscr{W})$ of the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ of a Lie algebra $\mathcal{G}$, $[\mathcal{E}_i\,,\,\mathcal{E}_j] = \sum_k c_{ij}^k\,\mathcal{E}_k$, is the representation $(\rho\otimes\sigma, \mathscr{V}\otimes\mathscr{W})$ defined on the generators $\mathcal{E}_i$ by

$$(\rho\otimes\sigma)(\mathcal{E}_i) = \rho(\mathcal{E}_i)\otimes I_{\mathscr{W}} + I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_i) \qquad \forall\, i = 1,\ldots,\dim\mathcal{G} \tag{0.1}$$

where $I_{\mathscr{V}} = \rho(\mathcal{I})$ and $I_{\mathscr{W}} = \sigma(\mathcal{I})$ are the identity operators in $\mathscr{V}$ and $\mathscr{W}$, respectively.

(a) Prove that the tensor product is indeed a representation.

(b) Extend the definition of the tensor product of two representations to the tensor product of any number L of representations $(\rho_a, \mathscr{V}_a)$, $a, 1, \ldots, L$

Problem 2. Consider the XXX periodic Heisenberg spin-1/2 chain described by the Hamiltonian

$$H = J\sum_{i=1}^{L}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z) = J\sum_{i=1}^{L}\sum_{\alpha=1}^{3} S_i^\alpha S_{i+1}^\alpha\,, \quad S_{L+1}^\beta \equiv S_1^\beta \;\;\forall\;\beta \tag{0.2}$$

The Hamiltonian acts in the Hilbert space which is the tensor product of L copies of two-dimensional spaces (spin up-down)

$$\mathscr{H} = \underbrace{\mathbb{C}^2\otimes\mathbb{C}^2\otimes\cdots\otimes\mathbb{C}^2}_{L} \tag{0.3}$$

and the spin-1/2 operator $S_i^\alpha = \hbar\sigma_i^\alpha/2$ acts only on the particle at the i-th site

$$S_i^\alpha = I\otimes I\otimes\cdots\otimes I\otimes \underbrace{S^\alpha}_{i-\text{th term}} \otimes I\otimes\cdots\otimes I \tag{0.4}$$

(a) The Casimir operator $\hat{C}\equiv\sum_{\alpha=1}^3\hat{S}^\alpha\hat{S}^\alpha$ of the $\mathfrak{su}(2)$ algebra commutes with all $\hat{S}^\alpha$ and must be proportional to the identity operator for an irreducible representation. Consider the spin chain of length $L=2$ and compute $\hat{C} = \sum_{\alpha=1}^3\mathbb{S}^\alpha\mathbb{S}^\alpha$ where $\mathbb{S}^\alpha$ is the total spin of system

$$\mathbb{S}^\alpha = \sum_{i=1}^{L} S_i^\alpha \tag{0.5}$$

Use the fact that $\hat{C}$ and $\mathbb{S}^3$ commute to find Casimir operator eigenvectors and eigenvalues. Use the bra-ket notation to find the eigenvectors.

(b) For the spin chain of length $L = 2$ express the Hamiltonian in terms of the identity operator and the Casimir operator, and find the spectrum of H.

Problem 3. Let the hermitian operators $\hat{X}$ and $\hat{P}$ satisfy the canonical commutation relation

$$[\hat{X}\,,\ \hat{P}] = \mathrm{i}\,\hbar\,\hat{I} \tag{0.6}$$

Introduce the following two families of unitary operators

$$\hat{U}(a) = e^{-\mathrm{i}\,a\,\hat{P}}\,, \quad \hat{V}(b) = e^{-\mathrm{i}\,b\,\hat{X}}\,, \quad a, b \in \mathbb{R} \tag{0.7}$$

Use the results from practice questions and

(i) Find

$$\hat{X}(a) = \hat{U}(a)\hat{X}\,\hat{U}^{\dagger}(a) \tag{0.8}$$

How would you interpret the operator $\hat{U}(a)$?

(ii) Find $\hat{Z}$ in

$$\hat{U}(a)\hat{V}(b) = e^{\hat{Z}} \tag{0.9}$$

(iii) Find $\hat{W}$ in

$$\hat{U}(a)\hat{V}(b) = \hat{W}\,\hat{V}(b)\hat{U}(a) \tag{0.10}$$

Practice Questions

Problem 1. The tensor product

Def. The tensor product of two nonzero representations $(\rho, \mathscr{V})$ and $(\sigma, \mathscr{W})$ of the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ of a Lie algebra $\mathcal{G}$, $[\mathcal{E}_i\,,\ \mathcal{E}_j] = \sum_k c_{ij}^k\,\mathcal{E}_k$, is the representation $(\rho\otimes\sigma, \mathscr{V}\otimes\mathscr{W})$ defined on the generators $\mathcal{E}_i$ by

$$(\rho\otimes\sigma)(\mathcal{E}_i) = \rho(\mathcal{E}_i)\otimes I_{\mathscr{W}} + I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_i) \qquad \forall\, i = 1, \ldots, \dim\mathcal{G} \tag{0.11}$$

where $I_{\mathscr{V}} = \rho(\mathcal{I})$ and $I_{\mathscr{W}} = \sigma(\mathcal{I})$ are the identity operators in $\mathscr{V}$ and $\mathscr{W}$, respectively.

(a) Show that for all i, j

$$\begin{aligned}
&[\rho(\mathcal{E}_i)\otimes I_{\mathscr{W}}\,,\ \rho(\mathcal{E}_j)\otimes I_{\mathscr{W}}] = [\rho(\mathcal{E}_i)\,,\ \rho(\mathcal{E}_j)]\otimes I_{\mathscr{W}}\\
&[I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_i)\,,\ I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_j)] = I_{\mathscr{V}}\otimes[\sigma(\mathcal{E}_i)\,,\ \sigma(\mathcal{E}_j)]\\
&[\rho(\mathcal{E}_i)\otimes I_{\mathscr{W}}\,,\ I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_j)] = 0 = [I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_i)\,,\ \rho(\mathcal{E}_j)\otimes I_{\mathscr{W}}]
\end{aligned} \tag{0.12}$$

(b) Extend the definition of the tensor product of two representations to the tensor product of any number L of representations $(\rho_a, \mathscr{V}_a)$, $a, 1, \ldots, L$

Problem 2. Consider the XXX periodic Heisenberg spin-1/2 chain described by the Hamiltonian

$$H = J\sum_{i=1}^{L}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z) = J\sum_{i=1}^{L}\sum_{\alpha=1}^{3} S_i^\alpha S_{i+1}^\alpha\,, \quad S_{L+1}^\beta \equiv S_1^\beta \;\forall\; \beta \tag{0.13}$$

The Hamiltonian acts in the Hilbert space which is the tensor product of L copies of two-dimensional spaces (spin up-down)

$$\mathscr{H} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L} \tag{0.14}$$

and the spin-1/2 operator $S_i^\alpha = \hbar\sigma_i^\alpha/2$ acts only on the particle at the i-th site

$$S_i^\alpha = I \otimes I \otimes \cdots \otimes I \otimes \underbrace{S^\alpha}_{i-\text{th term}} \otimes I \otimes \cdots \otimes I \tag{0.15}$$

(a) Show that the total spin of system

$$\mathbb{S}^\alpha = \sum_{i=1}^{L} S_i^\alpha \tag{0.16}$$

commutes with each term $\sum_{\beta=1}^{3} S_j^\beta S_{j+1}^\beta$, $j = 1, \ldots, L$ in the Hamiltonian, and therefore, with the Hamiltonian itself. Note that $\mathbb{S}^\alpha$ acts in the tensor product of L copies of spin operators at each site, and $\Sigma^\alpha \equiv \mathbb{S}^\alpha/\mathrm{i}\hbar$ satisfies the $\mathfrak{su}(2)$ algebra commutation relations

(b) For the spin chain of length $L = 2$ express the Hamiltonian in terms of the identity operator and the Casimir operator $\hat{C} = \sum_{\alpha=1}^{3} \mathbb{S}^\alpha \mathbb{S}^\alpha$.

Problem 3. Let three operators $\hat{A}$, $\hat{B}$ and $\hat{C}$ satisfy the following Lie algebra relations

$$[\hat{A}\,,\, \hat{B}] = \hat{C}\,, \quad [\hat{C}\,,\, \hat{A}] = [\hat{C}\,,\, \hat{B}] = \hat{O}\,, \tag{0.17}$$

where $\hat{O}$ is the zero operator which will be denoted as 0 in what follows.

(a) Show that

$$\hat{B}(t) = e^{t\hat{A}}\hat{B}\,e^{-t\hat{A}} \tag{0.18}$$

is a linear function of $\hat{A}$, $\hat{B}$ and $\hat{C}$, and find it. See problem 4 of HW2 for a hint.

(b) Show that

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{Z}} \tag{0.19}$$

where $\hat{Z}$ s a linear function of $\hat{A}$, $\hat{B}$ and $\hat{C}$, and find it.

Hint. Consider instead $\hat{F}(t) \equiv e^{t\hat{A}}e^{t\hat{B}}$, and derive a differential equation for $\hat{F}(t)$.

(c) Use the result above to find a relation of the form

$$e^{\hat{A}}e^{\hat{B}} = \hat{W}e^{\hat{B}}e^{\hat{A}} \tag{0.20}$$

where $\hat{W}$ should be determined.