L21: Simple groups/Jordan-Hölder

Simple groups / the alternating group

<u>Def</u> A group G is <u>simple</u> if the only normal subgroups are $\{e\}$ and G.

<u>Ex</u> $G = \mathbb{Z}/p\mathbb{Z}$ for p prime is simple.

<u>Thm</u> (Jordan-Hölder) Let G be a finite group. Then there exits

$$\{e\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \dots \triangleleft G_{n-1} \triangleleft G_n = G$$

"composition series"

s.t. G_i/G_{i-1} is simple $\forall i$.

Moreover, the groups $G_1/G_0, G_2/G_1, \dots, G_n/G_{n-1}$ are uniquely determined by G up to reordering. ← "composition factors"

<u>Pf</u> Existence: Induction on $|G|$: $|G| = 1$: ✓

Let G_{n-1} be a maximal normal proper ($\neq G$) subgroup of G. We claim G/G_{n-1} is simple: Recall: normal subgroups of G/G_{n-1} correspond to normal subgroups of G containing G_{n-1}.

By maximality there are only G_{n-1} or G, which correspond to $\{e\}$ and G/G_{n-1}.

By induction hypothesis G_{n-1} has a composition series.

Uniqueness: Induction on $|G|$: $|G| = 1$: ✓

Let $\{G_i\}_{i=1,\dots,n}$, $\{G_i'\}_{i=1,\dots,m}$ two composition series for G.

If $G_{n-1} = G'_{m-1}$ the statement follows by induction.

Otherwise, note that $G_{n-1} \cdot G'_{m-1} \triangleleft G$ (Exercise)

and obtain $G_{n-1} \triangleleft G_{n-1} \cdot G'_{m-1} \triangleleft G$ and hence $G_{n-1} \cdot G'_{m-1} = G$ as above.

$$\begin{array}{lcccc}
 & \{e\} = G_0 \triangleleft \dots \triangleleft G_{n-2} \triangleleft & & & \\
 & & G_{n-1} & & \\
\{e\} = H_0 \triangleleft \dots \triangleleft H_{k-2} \triangleleft H_{k-1} \triangleleft & G_{n-1} \cap G'_{m-1} & & \triangleleft & G \\
 & & G'_{m-1} & & \\
 & \{e\} = G'_0 \triangleleft \dots \triangleleft G'_{m-2} \triangleleft & & &
\end{array}$$

Let $S_1, \dots, S_k$ be the composition factors of $G_{n-1} \cap G'_{m-1}$.

By induction $\left(S_1, \dots, S_k, \dfrac{G_{n-1}}{G_{n-1} \cap G'_{m-1}}\right) \cong \left(G_1, \dfrac{G_2}{G_1}, \dots, \dfrac{G_{n-1}}{G_{n-2}}\right)$

$\dfrac{G_{n-1}}{G_{n-1} \cap G'_{m-1}} \cong \dfrac{G}{G'_{m-1}}$

same up to isom & permutation

$$\Rightarrow \left(S_1, \dots, S_k, \frac{G_1}{G_{n-1}'}, \frac{G_1}{G_{n-1}}\right) \simeq \left(G_1, \frac{G_2}{G_1}, \dots, \frac{G_{n-1}}{G_{n-2}}, \frac{G}{G_{n-1}}\right)$$

Same argument with $G_i \rightsquigarrow G_i'$ gives the result. $\square$

<u>Rem</u> Classification of finite simple groups (proof ~10'000 pages)

<u>Thm</u> Every finite simple group is isom. to

- cyclic group of prime order
- alternating group A_n for $n \geq 5$

} infinite families

group of Lie type

Or one of 26 "sporadic groups"