L24: Finitely generated abelian groups

Finitely generated abelian groups ("linear algebra / $\mathbb{Z}$")

<u>Def</u> An abelian group A is said to be finitely generated if $A = \langle S \rangle$ for a finite set $S \subseteq A$.

$\Leftrightarrow$ Every element $x \in A$ can be written as

$$x = n_1 s_1 + \dots + n_\ell s_\ell \quad \text{where } S = \{s_1, \dots, s_\ell\}, \ n_i \in \mathbb{Z}$$

$\Leftrightarrow$ The group hom $\varphi : \mathbb{Z}^\ell \to A$, $(n_1, \dots, n_\ell) \mapsto \sum n_i s_i$ is surjective.

· finite groups are fin. gen $(S = G)$

<u>Ex</u> · cyclic groups are fin. gen   eg $\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$

· products of fin. gen groups are fin gen eg $\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

$\mathbb{Q}$ is not fin. gen   <u>Exc</u>

<u>Thm</u> (Fundamental thm of fin. gen. abelian groups)

Let A be a fin. gen. abelian group. Then

1) $A \cong \mathbb{Z}^r \times \mathbb{Z}/k_1\mathbb{Z} \times \dots \times \mathbb{Z}/k_\ell\mathbb{Z}$

where · $r \geq 0$ called <u>rank</u>

· $k_i \mid k_{i+1}$ k_i are called invariant factors.

$k_i \geq 2$

2) The numbers $(r, k_1, \dots, k_\ell)$ are uniquely determined by A.

<u>Existence proof</u> (sketch of version 1)

By assumption $\exists \ \varphi : \mathbb{Z}^N \to A$ surjective group hom.

so that $A \cong \mathbb{Z}^N / \ker\varphi$.

Suppose that $\ker\varphi = \langle u_1, \dots, u_k \rangle = \operatorname{im} M$

where $M : \mathbb{Z}^k \to \mathbb{Z}^N$, $M((0, \dots, \underset{j}{1}, \dots, 0)) = u_j$

As in linear algebra: $M \in Mat_{N\times k}(\mathbb{Z})$

"$N \times k$ matrices with integer coefficients"

Recall: Gauss elimination $\to$ $M = \underset{\text{row ops}}{U} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \ddots 0 \end{pmatrix} \underset{\text{column ops}}{V}$ with U, V invertible

"Same" without division (see example) $\rightsquigarrow$ $M = U \begin{pmatrix} k_1 & & & \\ & \ddots & & \\ & & k_\ell & \\ & & & 0 \ddots 0 \end{pmatrix} V$

$\Rightarrow$ after changing basis we can assume $M = \begin{pmatrix} k_1 & & & \\ & \ddots & & \\ & & k_\ell & \\ & & & 0 \ddots 0 \end{pmatrix} \}r$

so that $A \cong \mathbb{Z}^N / \mathrm{im}\, M \cong \mathbb{Z}/k_1\mathbb{Z} \times \dots \times \mathbb{Z}/k_\ell\mathbb{Z} \times \mathbb{Z}^r$

$\exists$ variant !

Rmk $\begin{pmatrix} k_1 & & & \\ & \ddots & & \\ & & k_\ell & \\ & & & 0 \ddots 0 \end{pmatrix}$ is called the Smith normal form of M.

Ex $\cdot$ $\begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 0 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & \\ & 7 \end{pmatrix}$

$\cdot$ $\begin{pmatrix} 10 & 5 \\ 6 & 1 \end{pmatrix} \sim \begin{pmatrix} 4 & 4 \\ 6 & 1 \end{pmatrix} \sim \begin{pmatrix} 4 & 4 \\ 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 0 & 10 \\ \underset{= \gcd(10,6)}{2} & -3 \end{pmatrix}$ row ops

$\sim \begin{pmatrix} 2 & -3 \\ 0 & 10 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 10 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ -20 & 10 \end{pmatrix}$ $1 = \gcd(2,-3)$ column ops

$\sim \begin{pmatrix} 1 & 0 \\ 10 & -20 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -20 \end{pmatrix} \sim \begin{pmatrix} 1 & \\ & 20 \end{pmatrix}$

$\cdot$ $\begin{pmatrix} 3 & \\ & 5 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 \\ & 5 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 \\ & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ -5 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 15 \end{pmatrix}$