MAU22101: Exercises Week 5

Problem 1 Let $G \times X \to X$ be a transitive G-action and let $x \in X$. Show that there is an isomorphism of G-sets

$$\begin{aligned} \phi\colon G/\mathrm{Stab}_G(x) &\to X \\ [g] &\mapsto g.x, \end{aligned}$$

where $\mathrm{Stab}_G(x) := \{g \in G \mid g.x = x\}$ is the stabilizer subgroup of G. That is, show that

i) ϕ is well-defined,

ii) ϕ is a homorphism of G-sets,

iii) ϕ is a bijection.

Problem 2 Let $G \times X \to X$ be a G-action and let $V \subset X$ be a G-orbit. Given $x, y \in V$ show that there exists $g \in G$ such that

$$\mathrm{Stab}_G(x) = g\mathrm{Stab}_G(y)g^{-1}$$

(i.e. the corresponding stabilizer subgroups are conjugate).

Problem 3 Let $N \lhd G$ be a normal subgroup and let $\pi\colon G \to G/N$ be the canonical projection map $\pi(x) = [x]$. Show that there is a one-to-one correspondence

$$\begin{aligned} \{\text{subgroups of } G/N\} &\longleftrightarrow \{\text{subgroups of } G \text{ containing } N\} \\ H &\mapsto \pi^{-1}(H) \\ K/N &\mapsfrom K. \end{aligned}$$

Moreover, show that $\pi^{-1}(K/N) = KN$ for any subgroup $K \leqslant G$ (not necessarily containing N).

Problem 4 Prove that the additive group of rational numbers $(\mathbb{Q}, +)$ has no proper subgroups of finite index.

Problem 5 Prove Fermat's little theorem that for $a \in \mathbb{Z}$ and a prime p we have

$$a^p \equiv a \pmod{p}.$$

(Hint: use Lagrange's theorem in the group $(\mathbb{Z}/p\mathbb{Z})^\times$.)