L13: The three isomorphism theorems

The three isomorphism theorems

Thm 1 $\varphi: G \to H$ a group hom $\Rightarrow$ $G/\ker\varphi \cong \operatorname{im}\varphi$.

Def Let $B \leq G$, we define the normalizer of B in G to be
$$N_G(B) = \{g \in G \mid gBg^{-1} = B\}.$$

Lemma $B \trianglelefteq N_G(B) \leq G$

Thm 2 Let $A, B \leq G$ and suppose $A \leq N_G(B)$. Then
$$\frac{AB}{B} \cong \frac{A}{A \cap B}$$
where $AB := \{ab \in G \mid a \in A, b \in B\}$.

In particular i) $AB \leq G$

ii) $B \trianglelefteq AB$

iii) $A \cap B \trianglelefteq A$

Pf i) $a_1 b_1 \cdot a_2 b_2 = a_1 a_2 \underbrace{a_2^{-1} b_1 a_2}_{\in B} b_2 \in AB$

$$(ab)^{-1} = b^{-1}a^{-1} = a^{-1}\underbrace{aba^{-1}}_{\in B}$$

ii) $AB \subseteq N_G(B) \cdot N_G(B) = N_G(B)$

Define $\varphi: A \to \frac{AB}{B}$ by $\varphi(a) = [a]$.

Since φ is surj. $([ab] = abB = [a])$ we get
$$A/\ker\varphi \cong \frac{AB}{B}$$

But $\ker\varphi \ni a \Leftrightarrow [a] = [e] \Leftrightarrow a \in B$

hence $\ker\varphi = A \cap B$, which also shows $A \cap B \trianglelefteq A$.

Thm 3 Suppose we are given $H, K \trianglelefteq G$ s.t. $H \subseteq K$. Then
$$\frac{G/H}{K/H} \cong G/K$$

In part $K/H \trianglelefteq G/H$.

Pf Start with $\psi_1 : G \xrightarrow{\text{surj}} G/K$ since $H \leq K = \ker \psi_1$ we obtain

$\psi_2 : G/H \to G/K$ still surj.

$\ker \psi_2 \ni [x]_{G/H} \iff x \in K \iff [x] \in K/H$ which implies the thm. $\square$