Problem One

- Let G be a group with subset $S \subset G$. We define the normal closure $N(S)$ to be the smallest normal subgroup containing S, that is

$$N(S) \equiv \bigcap_{S \subset N \trianglelefteq G} N$$

1) Show that $N(S) \trianglelefteq G$

2) Show that $N(S)$ is generated by the set $\{gsg^{-1} \mid s \in S, g \in G\}$, that is

$$N(S) = \langle gsg^{-1}, s \in S, g \in G \rangle$$

(Hint: Showing the inclusion $\subset$ amounts to showing that the right hand side is a normal subgroup of G. Use that H is a normal subgroup if $H \subset xHx^{-1}$ $\forall x \in G$).

~ $N(S) \leq G$ as it is the intersection of subgroups of G. Let $n \in N(S)$ and let $g \in G$. Then, $gng^{-1} \in N$ for all $N \trianglelefteq G$ with $S \subseteq N$ ie: $gng^{-1} \in N(S)$. Since n, g are arbitrary elements, it follows that $N(S) \trianglelefteq G$.

~ Define $S_c = \{gsg^{-1} \mid s \in S$ and $g \in G\}$. Then $\langle S_c \rangle \leq G$ by definition. For all $h \in G$, we have

$$\begin{aligned} h\langle S_c \rangle h^{-1} &= \{hgsg^{-1}h^{-1} \mid s \in S \text{ and } g \in G\} \\ &= \{(hg)s(hg)^{-1} \mid s \in S \text{ and } g \in G\} \\ &= \{g'sg'^{-1} \mid s \in S \text{ and } g' \in G\} \\ &= \langle S_c \rangle \end{aligned}$$

Hence $\langle S_c \rangle \trianglelefteq G$. Moreover, $\{1s1^{-1} \mid s \in S\} \subset S_c$, hence $N(S) \subset \langle S_c \rangle$.
If $x \in \langle S_c \rangle$, then $x = gsg^{-1}$ for some $s \in S$ and $g \in G$. But, $s \in N(S)$ and $N(S) \trianglelefteq G$, hence $x \in N(S)$ and it follows that $\langle S_c \rangle \subset N(S)$.
Therefore,

$$N(S) = \langle S_c \rangle$$

Problem 2

→ Show that $G = (\mathbb{Q}, +)$ is not finitely generated

~ Seeking a contradiction, assume that $G = (\mathbb{Q}, +)$ is finitely generated and let $\{v_1, v_2, \dots, v_n\}$ be non-zero generators of $\mathbb{Q}$. Expressing the generators as fractions

$$r_i = a_i/b_i$$

where $a_i, b_i \in \mathbb{Z}$ non-zero b_i, we get that every rational number r can be written as the sum

$$r = c_1 v_1 + \dots + c_n v_n$$

for some integers $c_1, c_2, \dots, c_n$.

Then, we have that

$$r = \frac{m}{b_1 \cdots b_n}$$

where m is an integer (in terms of a_i, c_i)

Let p be prime that does not divide $b_1 \cdots b_n$ and chose $r = \frac{1}{p}$.

Then, we must have that

$$\frac{1}{p} = \frac{m}{b_1 \cdots b_n}$$

for some integer m. We ~~then~~ have that

$$pm = b_1 \cdots b_n$$

$\Rightarrow$ $p \mid b_1 \cdots b_n$ which contradicts our choice of the prime number p.

Thus, $\mathbb{Q}$ cannot be finitely generated.

⇒ Show that $\mathbb{Q}^\times = (\mathbb{Q} \setminus \{0\}, \times)$ is not finitely generated

~ Suppose that $\mathbb{Q}^\times$ is finitely generated and let

$$v_i = a_i/b_i$$

be generators for $i = 1, \dots, n$ $a_i, b_i \in \mathbb{Z}$

Then, every non-zero rational number r can be written as

$$r = r_1^{c_1} \cdots r_n^{c_n} = a_1^{c_1} \cdots a_n^{c_n} / b_1^{c_1} \cdots b_n^{c_n}$$

Let p be a prime number that does not divide $b_1 \cdots b_n$. $r = \frac{1}{p}$.

Then, as above, this leads to a contradiction.

Problem 3.

- Let p be prime and $A = \mathbb{Z}/p^{n_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{n_k}\mathbb{Z}$ for some positive integers $n_1, \dots, n_k$.

1) Show that $pA : \{pa \mid a \in A\}$ is a subgroup of A.
2) Show that $pA \cong \mathbb{Z}/p^{n_1-1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{n_k-1}\mathbb{Z}$
3) Show that $A/pA \cong (\mathbb{Z}/p\mathbb{Z})^k$

~ $0 \in pA$ since $0 = p0 \Rightarrow pA$ is non-empty
If $x \in pA$, then $x = pa$ for some $a \in A$ and $-x = (-p)a = p(-a) \in pA$ since $-a \in A \Rightarrow pA$ closed under inversion.
If $y = pa'$ is another element of pA, then $x + y = pa + pa' = p(a+a') \in A$, since $a + a' \in A \Rightarrow pA$ closed under group operation
Hence, $pA < A$.

~ We show that there exists $\phi : pA \to \mathbb{Z}/p^{n_1-1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{n_k-1}\mathbb{Z}$,
such that $\phi(pa) \mapsto (\overline{px_1}, \overline{px_2}, \dots, \overline{px_n})$ is a bijective homomorphism
$\rightarrow \phi(pa)$ is well-defined as each px_i has a unique residue class modulo p^{n_i-1} (as $\overline{px_i} = px_i \bmod p^{n_i} = x_i \bmod p^{n_i-1}$)
$\Rightarrow$ let $pa, pb \in pA$, then

$$
\begin{aligned}
\cancel{\phi(pa)\phi(pb)} \quad \phi(pa \cdot pb) &= (\overline{p^2 x_1 x_1'}, \overline{p^2 x_2 x_2'}, \dots, \overline{p^2 x_n x_n'}) \\
&= (\overline{px_1}, \overline{px_2}, \dots, \overline{px_k}) \cdot (\overline{px_1'}, \overline{px_2'}, \dots, \overline{px_n'}) \\
&= \phi(pa) \cdot \phi(pb)
\end{aligned}
$$

Hence ϕ is a group homomorphism
$\Rightarrow \phi$ is a surjection since each element in $\mathbb{Z}/p^{n_1-1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{n_k-1}\mathbb{Z}$ has a preimage in pA (that is, since $\phi(pa) = \phi(pb) \Rightarrow a = b$)
$\Rightarrow \phi$ is injective since all distinct elements in pA map to distinct ~~classes~~ residue classes modulo p^{n_i-1}

Hence $pA \cong \mathbb{Z}/p^{n_1-1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{n_k-1}\mathbb{Z}$

- Firstly, we define a group homomorphism $\phi: A \mapsto (\mathbb{Z}/p\mathbb{Z})^k$. In this case, the kernel of ϕ, $\ker\phi = pA$. By the first isomorphism theorem, we then have that $A/pA \cong (\mathbb{Z}/p\mathbb{Z})^k$.

Problem 4

- List all the isomorphism classes of abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$. Write the groups in both the primary factor decomposition and the invariant factor decomposition

~ First, we'll determine the possible Abelian groups of the relevant prime power orders:

Order p^β	Partitions of β	Abelian Groups
2^3	3; 2,1; 1,1,1	$\mathbb{Z}_8$; $\mathbb{Z}_4 \times \mathbb{Z}_2$; $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
3^2	2; 1,1	$\mathbb{Z}_9$; $\mathbb{Z}_3 \times \mathbb{Z}_3$
5	1	$\mathbb{Z}_5$

~ The $3 \cdot 2 \cdot 1 = 6$ possible ~~combinations~~ isomorphism classes for the primary factor decomposition of G are thus

$\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$ $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$

$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

The corresponding isomorphism classes for the invariant factor decomposition of G are

$\mathbb{Z}_{360}$, $\mathbb{Z}_2 \times \mathbb{Z}_{180}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{90}$, $\mathbb{Z}_6 \times \mathbb{Z}_{60}$, $\mathbb{Z}_3 \times \mathbb{Z}_{120}$

$\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{30}$

Problem Five

- Let $A = \mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/45\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$. Find the number of elements of order 2 and the number of subgroups of index 2 in A (in the second part, note that any subgroup of index 2 arises from a group homomorphism $\varphi: A \to \mathbb{Z}/2\mathbb{Z}$ and that any such homomorphism is determined by its value on the generators
$(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$.)

~ If $\mathbb{Z}_n$ is a cyclic group and $d \mid n$, then there is exactly one element of order d in $\mathbb{Z}_n$. Therefore, there is exactly one element of order two in each of $\mathbb{Z}/60\mathbb{Z}$, $\mathbb{Z}/12\mathbb{Z}$, and $\mathbb{Z}/36\mathbb{Z}$ (denote $[30]$, $[6]$ $[18]$ respectively) and by Lagrange's theorem, there is no element of order 2 in $\mathbb{Z}/45\mathbb{Z}$. Let $a = (a_1, a_2, a_3, a_4) \in A$ with $|a| = 2$. Then, $a_2 = 0$ and at least one of $\{a_1, a_3, a_4\}$ must be non-zero and equal to an element of order 2 in the corresponding cyclic subgroup. Therefore, there are $2 \times 1 \times 2 \times 2 - 1 = 7$ elements of order 2 (removed the identity element $a = 0$).

~ Every subgroup of A is normal since A is abelian. But, every normal subgroup is the kernel of some group homomorphism with A as the domain. It then follows by Lagrange's theorem and the First Isomorphism Theorem that if $N < A$ with $|N : A| = 2$, then $N = \ker\varphi$ for some group homomorphism $\varphi: A \to \mathbb{Z}/2\mathbb{Z}$.
This homomorphism is determined by its value on the generators
$(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$
We must have $\varphi((0,1,0,0)) = 0$ (otherwise, we could restrict φ to the subgroup generated by $(0,1,0,0)$ and conclude that $\mathbb{Z}/45\mathbb{Z}$ contains an element of order 2, a contradiction)
Also, $\varphi(e) \neq 0$ for one of $\{(1,0,0,0), (0,0,1,0), (0,0,0,1)\}$ (otherwise $N = A$). As these are the only restrictions on φ, we have ~~that~~
$2 \times 1 \times 2 \times 2 - 1 = 7$ subgroups of index 2 in A.