L6: Group actions

Group actions

Def A <u>group action</u> of a group G on a set X is a map
$G \times X \to X$ written as $(g, x) \mapsto g.x$
such that i) $g_1.(g_2.x) = (g_1 g_2).x$
ii) $e.x = x$

We sometimes write $G \circlearrowright X$.

Recall that a map (of sets) $G \times X \to X$ can equivalently be given by
$$G \to \{ f : X \to X \}$$
$$g \mapsto \rho(g)$$
where $\rho(g)(x) = g.x$

Prop A map $G \times X \to X$ defines a group action if and only if the corresponding map $G \xrightarrow{\rho} \{f : X \to X\}$ is such that $\rho(g) \in S_X \; \forall g$ and $\rho : G \to S_X$ is a group homomorphism.

Pf Note that i) $\iff \rho(g_1)(\rho(g_2)(x)) = \rho(g_1 g_2)(x)$
$\iff \rho(g_1) \circ \rho(g_2) = \rho(g_1 g_2)$
ii) $\iff \rho(e) = id_X$

"$\Rightarrow$": $\rho(g) \circ \rho(g^{-1}) = \rho(g g^{-1}) = \rho(e) = id_X \Rightarrow \rho(g)$ surj
$\rho(g^{-1}) \circ \rho(g) = \ldots = id_X \Rightarrow \rho(g)$ inj
$\hookrightarrow \rho(g) \in S_X$.

"$\Leftarrow$": ✓

□

Examples: 0) trivial action: For any set X we define
$$G \times X \to X$$
$$(g, x) \mapsto x$$

1) defining action of S_X on X: $\quad S_X \times X \to X$
$$(\sigma, x) \mapsto \sigma(x)$$

Claim: This corresponds to $id : S_X \to S_X$ under above.

2) G acts on G by $\rho_\ell : G \times G \to G$ "left regular action"

$(g, x) \mapsto gx$

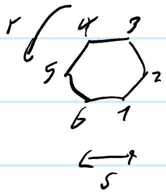
or $\rho_r : G \times G \to G$ "right regular action"

$(g, x) \mapsto xg^{-1}$

$\rho_{ad} : G \times G \to G$ "adjoint action"

$(g, x) \mapsto gxg^{-1}$

<u>Exc</u>: Verify that these are group actions.



3) $D_{2n} \times \{1, \dots, n\} \to \{1, \dots, n\}$

$(r^i, j) \mapsto i+j \mod n$

$(r^i s, j) \mapsto i-j \mod n$