L16: Sylow's Theorem

Sylow's theorem

Def Let G be a group and p a prime.

1) A group H of order p^α for $\alpha \geq 1$ is called a p-group. If $H \leq G$ it is called a p-subgroup.
2) If $|G| = p^\alpha m$, $p \nmid m$ then a subgroup $H \leq G$ is called a Sylow p-subgroup if $|H| = p^\alpha$.
3) $n_p = \#\{\text{sylow } p\text{-subgroups of } G\}$

Thm (Sylow's theorem) Let G be a finite group and p a prime. Then

1) There exists a Sylow p-subgroup.
2) Let Q be a p-subgroup and P a Sylow p-subgroup, then $\exists g \in G$ s.t. $Q \leq gPg^{-1}$.
3) $$\left|G/N_G(P)\right| = n_p \equiv 1 \pmod p$$

Cor
- All Sylow p-subgroups are conjugate
- $|G| = p^\alpha m$: $n_p \mid m$ & $n_p \equiv 1 \pmod p$

Thm [Cauchy] Let G be a finite group & $p \mid |G|$. Then G has an element of order p.

Proof of 1) Induction over $|G|$. Write $|G| = p^\alpha m$. We can assume $\alpha \geq 1$. Consider the class equation $|G| = Z(G) + \sum_{i=1}^{\ell} |G/C_G(g_i)|$

Case $p^\alpha \mid |C_G(g_i)|$ for some i : But then as $|C_G(g_i)| < |G|$ we get a Sylow p-subgroup by induction.

Suppose $p^\alpha \nmid C_G(g_i)$ for any i. By Lagrange we have that $p \mid |G/C_G(g_i)|$ for all i. But then also $p \mid Z(G)$ (as $p \mid |G|$). Since $Z(G)$ is abelian we can apply a previous theorem to get $H \leq Z(G)$ s.t. $|H| = p$. We now apply the inductive hypothesis to G/H to obtain $K \leq G/H$ s.t. $|K| = p^{\alpha-1}$. Let P be the preimage of K under $\pi : G \to G/H$, i.e. $P = \pi^{-1}(K)$. But then $P/H \cong K$ and thus $|P| = |H| \cdot |K| = p \cdot p^{\alpha-1} = p^\alpha$. $\square$

Proof of ii) and iii) Let $P = P_1$ be a Sylow p-subgroup and denote by

$$\mathcal{P} := \{ gPg^{-1} \mid g \in G \} = \{ P_1, P_2, \dots, P_r \}$$

the set of subgroups conjugate to P_1.

Since $G \circlearrowright \mathcal{P}$ so does every subgroup $H \leq G$ and we can consider the corresponding class equation

a) $H = G$: By definition the action is transitive and we get

$$r = |\mathcal{P}| = |G / \mathrm{Stab}_G P|$$

But $\mathrm{Stab}_G P_1 = \{ g \in G \mid gPg^{-1} = P \} = N_G(P)$

As $P \leq N_G(P)$ we obtain $r \mid m$ and in part $p \nmid r$.

b) $H = Q$ for Q a p-subgroup

$$r = |\mathrm{Fix}_Q(\mathcal{P})| + \sum_{\substack{i \text{ s.t.} \\ P_i \notin \mathrm{Fix}_Q(\mathcal{P})}} \left| \frac{Q}{\mathrm{Stab}_Q(P_i')} \right|$$

Since $p \mid \left| \frac{Q}{\mathrm{Stab}_Q(P_i')} \right|$ but $p \nmid r$ we get $\mathrm{Fix}_Q(\mathcal{P}) \neq \phi$,

i.e. $\exists P_i$ s.t. $Q = \mathrm{Stab}_Q(P_1') = N_Q(P_1') \leq N_G(P_1')$.

But then $P_1 \cdot Q \leq G$ s.t. $\dfrac{P_1 \cdot Q}{P_1} = \dfrac{Q}{P_1 \cap Q}$

and from $|P_1 \cdot Q| = |P_1| \cdot \left| \frac{Q}{P_1 \cap Q} \right|$ we see that

- $P_1 \cdot Q$ is a Sylow p-subgroup
- $P_1 \cap Q = Q \implies Q \leq P_1$. This proves ii)

c) $H = P$ for P a Sylow p-subgroup

As above we obtain that $P_i \in \mathrm{Fix}_P(\mathcal{P})$

$\Rightarrow P \leq P_i$. But this shows that $P = P_i$.

In part $\mathrm{Fix}_P(\mathcal{P}) = \{P\}$ and as above

$$r = 1 + \sum \left| \frac{P}{\mathrm{Stab}_Q(P_i')} \right|$$

$$p \mid ? \qquad .$$

$\square$