MAU22101: Exercises Week 4

Problem 1 Complete the proof of the statement that if $N \lhd G$ is a normal subgroup then G/N carries a group structure such that the map

$$\begin{aligned} \pi\colon G &\longrightarrow G/N \\ x &\longmapsto xN \end{aligned}$$

is a group homomorphism.

Problem 2 A *right action* $X \circlearrowleft G$ is a map $\rho\colon X \times G \to X$ satisfying

i) $\rho(x, e) = x$

ii) $\rho(\rho(x, g), h) = \rho(x, gh)$.

1. Show a map $\rho\colon X \times G \to X$ is a right action if and only if the map $\rho_l\colon G \times X \to X$ defined by $\rho_l(g, x) = \rho(x, g^{-1})$ is a left action.
2. Show that the set of orbits for ρ and ρ_l are the same, i.e.

$$_G\backslash^X := \{\rho(G, x) \mid x \in X\} = \{\rho_l(x, G) \mid x \in X\} =: {}^X/_G.$$

3. Write down the formulas for the right actions corresponding to three left actions $G \circlearrowright G$ (left-/right-regular and adjoint).

Problem 3 Let $N \leqslant G$ be a subgroup. Show that the following are equivalent

1. $N \lhd G$
2. $gNg^{-1} \subset N$ for all $g \in G$
3. $gNg^{-1} = N$ for all $g \in G$
4. $gN = Ng$ for all $g \in G$
5. For all $g \in G$ there exists $g' \in G$ such that $gN \subset Ng'$
6. $G/H = H\backslash G$, i.e. the set or orbits for the right regular and the left regular action coincide.

Problem 4 Let $H \leqslant G$ be a subgroup of G. We define the *normalizer of H in G* by

$$N_G(A) := \{g \in G \mid gHg^{-1} = H\}.$$

- Show that $N_G(H)$ is a subgroup of G, and that H is a normal subgroup of $N_G(H)$.
- Let $K \leqslant G$ be another subgroup and suppose that $K \subset N_G(H)$. Show that $HK := \{hk \mid h \in H, k \in K\}$ is a subgroup of G and that $H \lhd HK$.

Problem 5 Let $N \lhd G$ be a normal subgroup of a finite group G and suppose that $(|N|, |G/N|) = 1$. Prove that N is the unique subgroup of G of order $|N|$. (Hint: Given a subgroup $K \leqslant G$ of order $|N|$ consider its image under the map $G \to G/N$).

Problem 6 Let $N \lhd G$ be a normal subgroup of G, moreover, suppose that N is abelian. Show that the adjoint action induces a group homomorphism

$$\phi\colon G/N \to \mathrm{Aut}(N),$$

defined by $\phi([x])(n) = xnx^{-1}$.

Deduce that if G is a group of order pq, where p and q are both primes such that $p \nmid (q-1)$, and G has a normal subgroup N of order q, then G is abelian.