L17: Applications of Sylow's theorem

Applications

Cor $n_p = 1 \iff$ There exists a normal Sylow p-subgroup.

Pf "$\Rightarrow$" Let P be the unique Sylow p-subgroup. For any $g \in G$, gPg^{-1} is another Sylow p-subgroup, hence $gPg^{-1} = P$ and thus $P \trianglelefteq G$.

"$\Leftarrow$" Let P be a normal Sylow p-subgroup. Let Q be any other Sylow p-subgroup. Then $\exists g \in G$ s.t. $Q = gPg^{-1} \underset{P \text{ normal}}{=} P$ $\square$

Groups of order 6: Let G be a group of order $6 = 2 \cdot 3$

n_2: $n_2 \mid 3$ & $n_2 \equiv 1 \pmod 2$

$\hookrightarrow$ $n_2 \in \{1, 3\}$

n_3: $n_3 \mid 2$ & $n_3 \equiv 1 \pmod 3$

$\hookrightarrow$ $n_3 \in \{1, 2\}$ $\Rightarrow$ $n_3 = 1$

Case $n_2 = 1$: The Sylow 2-subgroup is normal

Let $K, H \trianglelefteq G$ be the Sylow 2 and 3 subgroups.

As $K \cap H \leq K$ and $K \cap H \leq H$ we get $K \cap K = \{e\}$ by Lagrange.

By the theorem below we get $\underset{\mathbb{Z}/2\mathbb{Z}}{\underset{\cong}{K}} \times \underset{\mathbb{Z}/3\mathbb{Z}}{\underset{\cong}{H}} \cong G$

Case $n_2 = 3$: Let $X = \{\text{Sylow 2-subgroups}\}$

Then $G \circlearrowright X$ transitively & $|X| = 3$

We get $\varphi: G \to S_X \cong S_3$

Note that $\ker \varphi \leq \underset{N_G(x)}{\underset{\shortparallel}{\mathrm{Stab}_G(x)}}$ for all $x \in X$

And $n_2 = \left|\frac{G}{N_G(x)}\right|$

Thus $|N_G(x)| = 2$.

If $\ker \varphi \neq \{e\} \Rightarrow \ker \varphi = N_G(x) \geq x$

and thus $\ker \varphi = x$ $\forall x \in X$

Hence $\ker \varphi = \{e\}$ & φ is an isomorphism.

<u>Prop</u> (characterization of direct products)

Let G be a group and $H, K \trianglelefteq G$ normal subgroups such that i) $HK = G$

ii) $H \cap K = \{e\}$

Then $H \times K \cong G$.

<u>Pf</u> Define $\varphi : H \times K \to G$.

$(h, k) \mapsto hk$

<u>Claim</u> φ is a bijection

<u>Pf</u> φ surj: i)

φ inj: $\varphi(h_1, k_1) = \varphi(h_2, k_2) \Rightarrow h_1k_1 = h_2k_2$

$\Rightarrow h_2^{-1}h_1 = k_2k_1^{-1} \in H \cap K \underset{ii)}{=} \{e\}$

$\Rightarrow h_2^{-1}h_1 = k_2k_1^{-1} = e$

$\Rightarrow h_1 = h_2, \; k_1 = k_2$

<u>Claim</u> $hk = kh \;\; \forall h \in H, k \in K$

<u>Pf</u> $H \trianglelefteq G \Rightarrow kh = hk'$ for some $k' \in K$ (namely $k' = h^{-1}kh$)

$K \trianglelefteq G \Rightarrow kh = h'k$ for some $h' \in H$ ($h' = khk^{-1}$)

$\Rightarrow \varphi(h', k) = kh = \varphi(h, k') \underset{\text{above claim}}{\Rightarrow} h' = h, \; k = k'$

I.e. $kh = hk$.

<u>Claim</u> φ is a group hom.

<u>Pf</u> $\varphi(h_1, k_1)\varphi(h_2, k_2) = h_1k_1h_2k_2 = h_1h_2k_1k_2 = \varphi(h_1h_2, k_1k_2)$

$\square$