L5: The category of groups

The category of groups

Consider $G = \{e, a, b, c\}$ with multiplication

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

Observation if we write $a=2$, $b=1$, $c=3$ this "is" $\mathbb{Z}/4\mathbb{Z}$.
isomorphic

Def ·) Let G and H be groups. A group <u>homomorphism</u> is a map $\varphi : G \to H$ s.t.

$$\varphi(a \cdot_G b) = \varphi(a) \cdot_H \varphi(b)$$

·) If φ is furthermore a bijection we call it a group <u>isomorphism</u>. In that case we say G and H are <u>isomorphic</u>.

Prop Let $\varphi : G \to H$ be a group homomorphism. Then

i) $\varphi(e_G) = e_H$

ii) $\varphi(a^{-1}) = \varphi(a)^{-1}$

Pf Exercise.

Def $|G|$ is called the <u>order</u> of G.

Prop Let G be a group of order 2. Then G is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Pf G group $\Rightarrow$ $e \in G$ and $a \in G$ s.t. $a \neq e$.

Define $\varphi : \mathbb{Z}/2\mathbb{Z} \to G$

$$0 \mapsto e$$
$$1 \mapsto a$$

we check that $\varphi(x+y) = \varphi(x)\varphi(y)$:

for $x=0$ or $y=0$: ✓

$x=1$ and $y=1$: $\varphi(1+1) = \varphi(0) = e$

$\varphi(1)\varphi(1) = a^2$

but $a^2 = e$ or $a^2 = a$

$\underset{\text{cancellation}}{\Rightarrow}$ $a = e$ ↯ $\square$

<u>Prop</u> i) Let $f: H \to K$ and $g: G \to H$ be group homomorphisms then so is $f \circ g$.

ii) Let $f: H \to K$ be a group isomorphism, then so is f^{-1}.

<u>Pf</u> i) $(f \circ g)(ab) = f(g(ab)) = f(g(a)g(b)) = f(g(a))f(g(b))$

$= (f\circ g)(a)\,(f \circ g)(b)$

ii) $f(f^{-1}(ab)) = ab = f(f^{-1}(a)) \cdot f(f^{-1}(b))$

$= f(f^{-1}(a) \cdot f^{-1}(b))$

$\underset{\text{inj.}}{\overset{f}{\Rightarrow}}$ $f^{-1}(ab) = f^{-1}(a)\,f^{-1}(b)$ $\square$