MAU22101: Exercises Week 6

Problem 1 Recall that the *center* of a group is defined by

$$Z(G) := \{g \in G \mid gh = hg \text{ for all } h \in G\}.$$

- Show that $Z(G)$ is an abelian subgroup of G.
- Show that $Z(G) \lhd G$.

Problem 2 Show that $S_X \cong S_n$ for $n = |X|$.

Problem 3 Let G be a group and let $p \mid |G|$ be the smallest prime dividing $|G|$. Suppose that G has a subgroup $H \leqslant G$ of index p. Show that H is normal as follows.

- Consider the action $G \circlearrowright G/H$ and let K be the kernel of the corresponding group homomorphism $\phi\colon G \to S_{G/H}$. Show that $H \leqslant K$.
- Show that the image of ϕ is a group of order p. (Hint: Use Lagrange (twice) to show that the order of the image is a divisor of $|G|$ and of $p!$.)
- Deduce that $H = K$ and that H is normal.

Problem 4 Let G be a finite group of order n. The left-regular representation defines a group homomorphism $\phi\colon G \to S_n$. For an element $g \in G$ we can consider the cycle decomposition of $\phi(g)$. Show that the cycle decomposition of $\phi(g)$ consists of $G/|g|$ cycles of length $|g|$.

Problem 5 Suppose that the center of G has index n. Show that every conjugacy class has at most n elements. (Hint: Use the orbit-stabilizer theorem.)