Contents

1 Charged particle in an electromagnetic field

Consider a (nonrelativistic) spinless particle of charge q in an arbitrary electromagnetic field. The Hamiltonian describing the motion of such a particle is

$$H = \frac{(\vec{P} - q\vec{A})^2}{2m} + q\,\Phi \tag{1}$$

- $\vec{A}$ and Φ are external magnetic vector potential and electric scalar potential, respectively.
- They in general depend on the particle's position operator $\vec{X}$, and time t, e.g. $\vec{A} = \vec{A}(\vec{X}, t)$.
- The magnetic $\vec{B}$ and electric $\vec{E}$ fields are expressed in terms of $\vec{A}$ and Φ as

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \text{and} \qquad \vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A}$$

1.1 Charged particle in a stationary uniform magnetic field

Consider the motion of a charged particle in a stationary uniform magnetic field $\vec{B} = (0, 0, B)$.

- Choose $\vec{A} = \frac{B}{2}(-y, x, 0)$ and $\Phi = 0$
- The Hamiltonian of a spinless particle takes the form

$$H = \frac{(\vec{P} - q\vec{A})^2}{2m} = \frac{(P^x + \frac{q}{2}BY)^2}{2m} + \frac{(P^y - \frac{q}{2}BX)^2}{2m} + \frac{(P^z)^2}{2m} \tag{2}$$

- The Hamiltonian commutes with
  - (i) P^z which generates translations along the z-axis
  - (ii) $L^z = XP^y - YP^x$ which generates rotations about the z-axis
  - (iii) $P^x - \frac{q}{2}BY$ which generates translations along the x-axes and shifts of P^y
  - (iv) $P^y + \frac{q}{2}BX$ which generates translations along the y-axes and shifts of P^x
- The existence of these conserved operators is due to the invariance of the properties of the system in the stationary uniform magnetic field under space translations and rotations.

1.2 Hamiltonian and separation of variables

- $P^x - \frac{q}{2}BY$ and $P^y + \frac{q}{2}BX$ do not commute with each other

$$[P^x - \frac{q}{2}BY\ ,\ P^y + \frac{q}{2}BX] = -\mathrm{i}\,\hbar\, qB \tag{3}$$

- Up to the $-qB$ factor it is the canonical commutation relation
- Perform the canonical transformation

$$\begin{aligned} \Pi_x &= P^x + \frac{q}{2}BY\ , \quad \mathcal{X} = \frac{X}{2} - \frac{P^y}{qB} \\ \Pi_y &= P^x - \frac{q}{2}BY\ , \quad \mathcal{Y} = \frac{X}{2} + \frac{P^y}{qB} \end{aligned} \tag{4}$$

- The Hamiltonian takes the form

$$H = \frac{\Pi_x^2 + q^2B^2\mathcal{X}^2}{2m} + \frac{(P^z)^2}{2m} \tag{5}$$

(i) $\frac{\Pi_x^2+q^2B^2\mathcal{X}^2}{2m}$ is the Hamiltonian of a one-dimensional harmonic oscillator of mass m and frequency

$$\omega = \frac{|qB|}{m} \tag{6}$$

which is called the **Larmor frequency**, and in the case of an electron the **cyclotron frequency**.

(ii) $\frac{(P^z)^2}{2m}$ is the Hamiltonian of a free particle in one dimension

(iii) $\mathcal{Y}$ and Π_y do not appear in the Hamiltonian, and therefore the spectrum is infinitely degenerate.

- Introduce the creation and annihilation operators

$$a = \frac{1}{2\eta}\mathcal{X} + \mathrm{i}\,\frac{\eta}{\hbar}\,\Pi_x\,,\;\; a^\dagger = \frac{1}{2\eta}\mathcal{X} - \mathrm{i}\,\frac{\eta}{\hbar}\,\Pi_x\,,\quad b = \frac{1}{2\eta}\mathcal{Y} + \mathrm{i}\,\frac{\eta}{\hbar}\,\Pi_y\,,\;\; b^\dagger = \frac{1}{2\eta}\mathcal{Y} - \mathrm{i}\,\frac{\eta}{\hbar}\,\Pi_y\,,\quad \eta = \sqrt{\frac{\hbar}{2m\omega}} \tag{7}$$

- The Hamiltonian becomes

$$H = \hbar\,\omega\,(a^\dagger a + \frac{1}{2}) + \frac{P_z^2}{2m} \tag{8}$$

- Its eigenvectors are

$$|n, n_b, p_z\rangle \equiv |n\rangle \otimes |n_b\rangle \otimes |p_z\rangle\,,\quad a^\dagger a\,|n\rangle = n\,|n\rangle\,,\quad b^\dagger b\,|n_b\rangle = n_b\,|n_b\rangle\,,\quad P_z\,|p_z\rangle = p_z\,|p_z\rangle \tag{9}$$

1.3 Spectrum of the Hamiltonian

- The spectrum of a charged particle in a uniform magnetic field is

$$E_{n,n_b,z} = \hbar\,\omega\,(n + \frac{1}{2}) + \frac{p_z^2}{2m} \tag{10}$$

(i) The nonnegative integer n counts an infinite set of **Landau levels**

(ii) n_b labels the infinite degeneracy of the Landau levels

(iii) An electron moving from one Landau level to the next emits or absorbs photons.

(iv) The overall spectrum is continuous because the particle can move freely parallel to $\vec{B}$.

(v) If one bounds this motion for example by adding $kZ^2/2$ to H then the spectrum is discrete.

1.4 The physical meaning of $b^\dagger$

- The physical meaning of $a^\dagger$ is clear. It moves the particle from one Landau level to the next one up.
- What does $b^\dagger$ do?
- L^z in terms of the ladder operators

$$\begin{aligned} L^z = XP^y - YP^x &= (\mathcal{X}+\mathcal{Y})\frac{qB}{2}(\mathcal{Y}-\mathcal{X}) - \frac{1}{qB}(\Pi_x - \Pi_y)\frac{1}{2}(\Pi_x + \Pi_y) \\ &= \frac{m}{qB}\frac{\Pi_y^2 + q^2B^2\mathcal{Y}^2}{2m} - \frac{m}{qB}\frac{\Pi_x^2 + q^2B^2\mathcal{X}^2}{2m} = \mathfrak{s}(qB)\,\hbar\,(b^\dagger b - a^\dagger a) \end{aligned} \tag{11}$$

  where $\mathfrak{s}(qB)$ is the sign of qB.

- $|n, n_b, p_z\rangle$ are eigenvectors of L^z

$$L^z\,|n, n_b, p_z\rangle = \mathfrak{s}(qB)\,\hbar\,(n_b - n)|n, n_b, p_z\rangle \tag{12}$$

  (i) The spectrum of L^z is quantised, and in units of $\hbar$ it can only take integer values $\hbar\,\ell\,,\ \ell \in \mathbb{Z}$.

  (ii) $b^\dagger$ depending on the sign of qB either increases, $qB > 0$, or decreases $qB < 0$ the angular momentum by $\hbar$.

  (iii) Use

$$\ell = \mathfrak{s}(qB)\,(n_b - n) \Leftrightarrow n_b = n + \mathfrak{s}(qB)\,\ell$$

  to label the infinite degeneracy of Landau levels

$$a^\dagger a\,|n, \ell, p_z\rangle = n\,|n, \ell, p_z\rangle\,, \quad L^z\,|n, \ell, p_z\rangle = \hbar\,\ell\,|n, \ell, p_z\rangle\,, \quad P_z\,|n, \ell, p_z\rangle = p_z\,|n, \ell, p_z\rangle \tag{13}$$

1.5 Expectation values and uncertainties

- Let the particle be constrained to move in the xy-plane, and let it be in the state

$$|\psi\rangle = \frac{1}{4}\Big(|0,0\rangle - 2|1,0\rangle + \mathrm{i}\,|2,0\rangle + 3|0,1\rangle - \mathrm{i}\,|1,1\rangle\Big) \tag{14}$$

where

$$N\,|n,\ell\rangle = a^\dagger a\,|n,\ell\rangle = n\,|n,\ell\rangle\,, \quad L^z\,|n,\ell\rangle = \hbar\,\ell\,|n,\ell\rangle \tag{15}$$

(i) Check that it has norm 1

$$\langle\psi|\psi\rangle = \frac{1}{16}(1+4+1+9+1) = 1 \tag{16}$$

(ii) Find the probabilities to measure $n=0$, $n=1$ and $n=2$, and $\ell=0$ and $\ell=1$

$$P(n=0) = \frac{1}{16}(1+9) = \frac{5}{8}\,, \quad P(n=1) = \frac{1}{16}(4+1) = \frac{5}{16}\,, \quad P(n=2) = \frac{1}{16} \tag{17}$$

$$P(\ell=0) = \frac{1}{16}(1+4+1) = \frac{3}{8}\,, \quad P(\ell=1) = \frac{1}{16}(9+1) = \frac{5}{8} \tag{18}$$

(iii) If the result of a measurement is $n=0$, what is the state of the system after it?

$$|\psi_{n=0}\rangle = \frac{1}{\sqrt{10}}\Big(|0,0\rangle + 3|0,1\rangle\Big) \tag{19}$$

If the result of a measurement is $\ell=0$, what is the state of the system after it?

$$|\psi_{\ell=0}\rangle = \frac{1}{\sqrt{6}}\Big(|0,0\rangle - 2|1,0\rangle + \mathrm{i}\,|2,0\rangle\Big) \tag{20}$$

(iv) What is the probability to measure first $n = 0$ and immediately after $\ell = 0$?

The probability to find $\ell = 0$ by measuring $|\psi_{n=0}\rangle$ is $|\langle 0, 0|\psi_{n=0}\rangle|^2 = 1/10$.

The probabilities multiply, so

$$P(n = 0 \to \ell = 0) = P(n = 0)|\langle 0, 0|\psi_{n=0}\rangle|^2 = \frac{5}{80} = \frac{1}{16} \tag{21}$$

What is the probability to measure first $\ell = 0$ and immediately after $n = 0$?

$$P(\ell = 0 \to n = 0) = P(\ell = 0)|\langle 0, 0|\psi_{\ell=0}\rangle|^2 = \frac{3}{8} \cdot \frac{1}{6} = \frac{1}{16} \tag{22}$$

(v) Find the expectation values of and the uncertainty in N and $l_z \equiv L_z/\hbar$ wrt $]\psi\rangle$

$$\langle N\rangle = P(n = 0)0 + P(n = 1)1 + P(n = 1)2 = \frac{5}{16} + \frac{2}{16} = \frac{7}{16} \tag{23}$$

$$\begin{aligned} \Delta N &= \sqrt{P(n = 0)(0 - \langle N\rangle)^2 + P(n = 1)(1 - \langle N\rangle)^2 + P(n = 2)(2 - \langle N\rangle)^2} \\ &= \sqrt{\frac{5}{8}\frac{7^2}{16^2} + \frac{5}{16}(1 - \frac{7}{16})^2 + \frac{1}{16}(2 - \frac{7}{16})^2} = \frac{\sqrt{95}}{16} \approx 0.609175 \end{aligned} \tag{24}$$

$$\langle l_z\rangle = P(\ell = 0)0 + P(\ell = 1) = \frac{5}{8} \tag{25}$$

$$\Delta l_z = \sqrt{P(\ell = 0)(0 - \frac{5}{8})^2 + P(\ell = 1)(1 - \frac{5}{8})^2} = \frac{\sqrt{15}}{8} \approx 0.484123 \tag{26}$$

(vi) Check that the general uncertainty relation

$$\Delta\hat{A}^2\Delta\hat{B}^2 \geq \Big(\frac{1}{2}\langle[\hat{A}\,,\,\hat{B}]_+\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\Big)^2 - \frac{1}{4}\langle[\hat{A}\,,\,\hat{B}]\rangle^2 \tag{27}$$

holds for N and $l_z \equiv L_z/\hbar$

Answer. Since N and l_z commute the rhs of the general uncertainty relation gives

$$\Big(\langle l_z\,N\rangle - \langle l_z\rangle\langle N\rangle\Big)^2 \tag{28}$$

Then,

$$\begin{aligned}\langle l_z\,N\rangle &= \langle\psi|l_z\,N\,\frac{1}{4}\Big(|0,0\rangle - 2|1,0\rangle + \mathrm{i}\,|2,0\rangle + 3|0,1\rangle - \mathrm{i}\,|1,1\rangle\Big)\\ &= \langle\psi|l_z\,\frac{1}{4}\Big(-2|1,0\rangle + 2\mathrm{i}\,|2,0\rangle - \mathrm{i}\,|1,1\rangle\Big) = \langle\psi|\frac{1}{4}\Big(-\mathrm{i}\,|1,1\rangle\Big) = \frac{1}{16}\end{aligned} \tag{29}$$

Thus,

$$\Big(\langle l_z N\rangle - \langle l_z\rangle\langle N\rangle\Big)^2 = (\frac{1}{16} - \frac{5}{8}\,\frac{7}{16})^2 \approx 0.0444946 \tag{30}$$

$$\Delta N^2 = \frac{95}{16^2}\,, \quad (\Delta l_z)^2 = \frac{15}{64}\,, \quad \Delta l_z^2(\Delta N)^2 \approx 0.0869751 \tag{31}$$

and the inequality holds.

1.6 The ground state wave function

- The wave function of the ground state $|0, 0, p_z\rangle$ is annihilated by both a and b.

  (i) a and b in terms of the original coordinate and momenta operators

$$a = \frac{1}{2\eta}(\frac{X}{2} - \frac{P^y}{qB}) + \mathrm{i}\,\frac{\eta}{\hbar}\,(P^x + \frac{q}{2}BY)\,, \quad b = \frac{1}{2\eta}(\frac{X}{2} + \frac{P^y}{qB}) + \mathrm{i}\,\frac{\eta}{\hbar}\,(P^x - \frac{q}{2}BY) \tag{32}$$

  (ii) Since $|qB| = m\omega = \frac{\hbar}{2\eta^2}$, we get

$$\begin{aligned} a + b &= \frac{1}{2\eta}X + \mathrm{i}\,\frac{2\eta}{\hbar}\,P^x = 2\eta(\frac{\mathrm{i}}{\hbar}P^x + \frac{1}{4\eta^2}X) \\ a - b &= -\frac{P^y}{qB\eta} + \mathrm{i}\,\frac{\eta}{\hbar}qBY = \mathrm{i}\,\frac{\hbar}{qB\eta}(\frac{\mathrm{i}}{\hbar}P^y + \frac{1}{4\eta^2}Y) \end{aligned} \tag{33}$$

  (iii) Thus, the ground-state wave function

$$\psi_0(x, y, z) = \langle x, y, z|0, 0, p_z\rangle \tag{34}$$

  satisfies

$$\begin{aligned} (\frac{\mathrm{i}}{\hbar}P^x + \frac{1}{4\eta^2}X)\psi_0(x, y, z) &= (\frac{\partial}{\partial x} + \frac{1}{4\eta^2}x)\psi_0(x, y, z) = 0 \\ (\frac{\mathrm{i}}{\hbar}P^y + \frac{1}{4\eta^2}Y)\psi_0(x, y, z) &= (\frac{\partial}{\partial y} + \frac{1}{4\eta^2}y)\psi_0(x, y, z) = 0 \end{aligned} \tag{35}$$

(iv) It is

$$\psi_0(x,y,z) = \frac{1}{2r_B\sqrt{\pi}}\, e^{-\frac{x^2+y^2}{4r_B^2}}\, \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{\mathrm{i}}{\hbar}p_z\, z} \tag{36}$$

where

$$r_B \equiv \sqrt{2}\,\eta = \sqrt{\frac{\hbar}{m\omega}} = \sqrt{\frac{\hbar}{|qB|}} \tag{37}$$

- In CM the trajectory of a particle in a magnetic field is a circular helix of radius

$$r = \frac{v_\perp}{\omega} = \sqrt{\frac{2E_\perp}{m\,\omega^2}} \tag{38}$$

(i) $v_\perp = \sqrt{v_x^2 + v_y^2}$ is the speed of the particle in the xy-plane

(ii) $E_\perp$ is the corresponding kinetic energy.

(iii) When $E_\perp = \hbar\omega/2$ the radius r agrees with the uncertainty r_B of the Gaussian wave packet (36).

1.7 Excited states wave functions

- Applying $a^\dagger$ and $b^\dagger$ to the ground state, we can get all wave functions. Let $qB > 0$

(i) $a^\dagger$ and $b^\dagger$ are ($r_B \equiv \sqrt{2}\,\eta = \sqrt{\frac{\hbar}{|qB|}}$)

$$\frac{a^\dagger + b^\dagger}{\sqrt{2}} = r_B(-\frac{\mathrm{i}}{\hbar}P^x + \frac{1}{2r_B^2}X)\,, \qquad \frac{a^\dagger - b^\dagger}{\sqrt{2}}\,\mathrm{i} = r_B(-\frac{\mathrm{i}}{\hbar}P^y + \frac{1}{2r_B^2}Y) \tag{39}$$

(ii) Introduce

$$A_+^\dagger \equiv \frac{a^\dagger + b^\dagger}{\sqrt{2}}\,, \quad A_-^\dagger \equiv \frac{a^\dagger - b^\dagger}{\sqrt{2}}\,\mathrm{i}\,, \qquad a^\dagger = \frac{A_+^\dagger - \mathrm{i}\,A_-^\dagger}{\sqrt{2}}\,, \quad b^\dagger \equiv \frac{A_+^\dagger + \mathrm{i}\,A_-^\dagger}{\sqrt{2}} \tag{40}$$

(iii) The wave functions

$$\begin{aligned} \phi_{k_+k_-}(x,y,z) &= \langle x,y,z|(A_+^\dagger)^{k_+}(A_-^\dagger)^{k_-}|0,0,p_z\rangle \\ &= \frac{1}{\sqrt{2^{k_+}\,k_+!}}H_{k_+}(\frac{x}{\sqrt{2}\,r_B})\,\frac{1}{\sqrt{2^{k_-}\,k_-!}}H_{k_-}(\frac{y}{\sqrt{2}\,r_B})\,\psi_0(x,y,z) \end{aligned} \tag{41}$$

form a basis but they are not eigenfunctions of H and L_z

(iv) The eigenfunctions of H and L_z

$$\psi_{n\ell}(x,y,z) = \langle x,y,z|(a^\dagger)^n(b^\dagger)^\ell|0,0,p_z\rangle \tag{42}$$

(v) For example

$$\psi_{01}(x,y,z) = \frac{x + \mathrm{i}\,y}{\sqrt{2}\,r_B}\,\psi_0(x,y,z)\,, \quad \psi_{10}(x,y,z) = \frac{x - \mathrm{i}\,y}{\sqrt{2}\,r_B}\,\psi_0(x,y,z) \tag{43}$$

2 Particle in an infinitely deep well with a delta-function barrier in the middle

Consider a particle in the following potential

$$V(x) = \begin{cases} \nu\,\delta(x) & \text{for } \ |x| < a \\ +\infty & \text{for } \ |x| > a \end{cases} \tag{44}$$

where $\nu > 0$. It is an infinitely deep well with a delta-function barrier in the middle.

2.1 Quantisation conditions

- Find the quantisation conditions for the spectrum of the Hamiltonian

  (i) The energy spectrum is found by solving TISE

  $$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = -\frac{\hbar^2}{2m}\psi''(x) + \nu\delta(x)\psi(0) = E\psi(x)\,, \quad \psi(-a) = \psi(a) = 0\,, \tag{45}$$

  and ψ is continuous everywhere, in particular at $x = 0$.

  (ii) Since $E > 0$, the general solution of the equation for $x < 0$ and $x > 0$ is

  $$\psi(x) = A\cos kx + B\sin kx\,, \quad k \equiv \sqrt{\frac{2mE}{\hbar^2}} \tag{46}$$

  (iii) Since the potential is even, eigenfunctions are either even or odd functions of x.

(iv) Consider first even functions

$$\psi_{\rm e}(x) = (A\cos kx + B\sin kx)\theta(-x) + (A\cos kx - B\sin kx)\theta(x) \tag{47}$$

where $\theta(x)$ is the Heaviside function.

- The boundary condition gives

$$A\cos ka - B\sin ka = 0 \tag{48}$$

- The function is obviously continuous at $x = 0$, and therefore, everywhere.
- We need to satisfy the Schrödinger equation at $x = 0$. Computing the derivatives, we get

$$\begin{aligned} \psi'_{\rm e}(x) &= -k(A\sin kx - B\cos kx)\theta(-x) - k(A\sin kx + B\cos kx)\theta(x) \\ \psi''_{\rm e}(x) &= -k^2\psi(x) - 2kB\,\delta(x) \end{aligned} \tag{49}$$

- Thus,

$$\frac{\hbar^2 kB}{m} + \nu A = 0 \quad \Rightarrow \quad B = -\frac{\nu\, m}{\hbar^2 k}\,A \tag{50}$$

- Substituting B into (48), we get the quantisation conditions for the eigenvalues of even eigenfunctions

$$\cos ka + \frac{\nu\, m}{\hbar^2 k}\sin ka = 0 \quad \Rightarrow \quad -\cot ka = \frac{\nu\, ma}{\hbar^2}\,\frac{1}{ka} \tag{51}$$

(v) Consider now odd functions

$$\psi_0(x) = (A\cos kx + B\sin kx)\theta(-x) - (A\cos kx - B\sin kx)\theta(x) \tag{52}$$

– The continuity condition at $x = 0$ gives $A = 0$.

– Thus

$$\psi_0(x) = B\sin kx \tag{53}$$

– From $\psi(a) = 0$ and the wave function normalisation condition we get

$$\psi_{2n-1}(x) = \frac{1}{\sqrt{a}}\sin k_{2n-1}x\,, \quad k_{2n-1} \equiv \frac{n\pi}{a}\,, \quad E_{2n-1} = \frac{\hbar^2}{2m}n^2\frac{\pi^2}{a^2} \tag{54}$$

– Thus, the delta-function barrier does not change the odd functions of the infinitely deep well.

2.2 Weak and strong coupling expansions

- Denote the energy eigenvalues by E_n, $n = 0, 1, 2, \ldots$, $E_n < E_{n+1}$.

(i) Assume that for small ν the energy levels can be expanded in a Tailor series in ν

$$E_n = E_n^{(0)} + E_n^{(1)}\nu + E_n^{(2)}\nu^2 + \cdots \tag{55}$$

Find $E_n^{(0)}$ and $E_n^{(1)}$, and comment on the results obtained.

– Clearly odd energy levels are the same as for the infinitely deep well

$$E_{2n-1} = \frac{\hbar^2}{2m}n^2\frac{\pi^2}{a^2} \quad \Rightarrow \quad E_{2n-1}^{(0)} = \frac{\hbar^2}{2m}n^2\frac{\pi^2}{a^2}\,, \quad E_{2n-1}^{(1)}\nu = 0\,, \quad n = 1, 2, \ldots \tag{56}$$

– For $\nu = 0$ the even energy levels are also the same as for the infinitely deep well

$$E_{2n}^{(0)} = \frac{\hbar^2}{8m}(2n+1)^2\frac{\pi^2}{a^2}\,, \quad n = 0, 1, 2, \ldots \tag{57}$$

– To find $E_{2n}^{(1)}$ we use the quantisation conditions for the eigenvalues of even eigenfunctions

$$-\cot k_{2n}a = \frac{\nu\, ma}{\hbar^2}\,\frac{1}{k_{2n}a}\,, \quad k_{2n} = k_{2n}^{(0)} + k_{2n}^{(1)}\,\nu + \cdots\,, \quad k_{2n}^{(0)}a = \frac{\pi}{2} + \pi n \tag{58}$$

– We get

$$k_{2n}^{(1)} = \frac{m}{\hbar^2}\,\frac{1}{k_{2n}^{(0)}a} = \frac{m}{\hbar^2}\,\frac{1}{\frac{\pi}{2}+\pi n} \tag{59}$$

– Thus,

$$\begin{aligned} E_{2n} &= \frac{\hbar^2}{2m}k_{2n}^2 \approx \frac{\hbar^2}{2m}(k_{2n}^{(0)} + k_{2n}^{(1)}\nu)^2 \approx \frac{\hbar^2}{8m}(2n+1)^2\frac{\pi^2}{a^2} + \frac{\hbar^2}{m}k_{2n}^{(0)}k_{2n}^{(1)}\nu \\ &= \frac{\hbar^2}{8m}(2n+1)^2\frac{\pi^2}{a^2} + \frac{1}{a}\,\nu \quad \Rightarrow \quad E_{2n}^{(1)} = \frac{1}{a} \end{aligned} \tag{60}$$

– The first correction in ν to the energy levels is positive and independent of n.

– This can be explained by noting that it can be computed as

$$E_{2n}^{(1)}\nu = \int dx\,|\psi_{2n}^{(0)}(x)|^2\nu\delta(x) = |\psi_{2n}^{(0)}(0)|^2\nu = \frac{\nu}{a} \tag{61}$$

where $\psi_{2n}^{(0)}$ is the even eigenfunction at $\nu = 0$.

(ii) Assume that for large ν the energy levels can be expanded in a Tailor series in $\frac{1}{\nu}$

$$E_n = E_n^{(\infty)} + E_n^{(-1)}\frac{1}{\nu} + E_n^{(-2)}\frac{1}{\nu^2} + \cdots \tag{62}$$

Find $E_n^{(\infty)}$ and $E_n^{(-1)}$, and comment on the results obtained.

- We write the quantisation conditions for the eigenvalues of even eigenfunctions in the form

$$\tan k_{2n}a = -\frac{\hbar^2}{\nu\, ma}\, k_{2n}a\,, \quad k_{2n} = k_{2n}^{(\infty)} + k_{2n}^{(-1)}\,\frac{1}{\nu} + \cdots\,, \quad k_{2n}^{(\infty)}a = \pi n\,, \quad n = 1, 2, \ldots \tag{63}$$

- We get

$$k_{2n}^{(-1)} = -\frac{\hbar^2}{ma}\, k_{2n}^{(\infty)} = -\frac{\hbar^2}{ma^2}\,\pi n \tag{64}$$

- Thus,

$$\begin{aligned} E_{2n} &= \frac{\hbar^2}{2m}k_{2n}^2 \approx \frac{\hbar^2}{2m}(k_{2n}^{(\infty)} + k_{2n}^{(-1)}\frac{1}{\nu})^2 \approx \frac{\hbar^2}{2m}n^2\frac{\pi^2}{a^2} + \frac{\hbar^2}{m}k_{2n}^{(\infty)}k_{2n}^{(-1)}\frac{1}{\nu} \\ &= \frac{\hbar^2}{2m}n^2\frac{\pi^2}{a^2} - \frac{\hbar^4 n^2\pi^2}{m^2a^3}\,\frac{1}{\nu} \quad \Rightarrow \quad E_{2n}^{(\infty)} = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}n^2\,, \quad E_{2n}^{(-1)} = -\frac{\hbar^4\pi^2 n^2}{m^2a^3} \end{aligned} \tag{65}$$

- At $\nu = \infty$ the delta-function barrier becomes impenetrable, and the leading order spectrum coincides with the spectrum of a particle in an infinitely deep well of width a.

2.3 Normalised even eigenfunctions

- Find the normalised even eigenfunctions of the Hamiltonian, and use the quantisation conditions to simplify them as much as you can.
  - We've found that
$$\psi_{\rm e}(x) = A\Big((\cos kx - \frac{\nu\, m}{\hbar^2 k}\sin kx)\theta(-x) + (\cos kx + \frac{\nu\, m}{\hbar^2 k}\sin kx)\theta(x)\Big) \tag{66}$$
  - To find A we compute
$$\int dx\, \psi_{\rm e}(x)^2 = \frac{A^2\left((a^2k^2 - w^4)\sin(2ak) + 2ak\left(a^2k^2 - w^2\cos(2ak) + w^4 + w^2\right)\right)}{2a^2k^3} = 1 \tag{67}$$
    where
$$w^2 \equiv \frac{\nu\, m\, a}{\hbar^2} > 0 \tag{68}$$
  - Thus,
$$A^2 = \frac{2a^2k^3}{(a^2k^2 - w^4)\sin(2ak) + 2ak\left(a^2k^2 - w^2\cos(2ak) + w^4 + w^2\right)} \tag{69}$$
  - To simplify this expression we use
$$\cos(2ak) = 1 - \frac{2}{\cot^2(ak) + 1}\,, \quad \sin(2ak) = \frac{2\cot(ak)}{\cot^2(ak) + 1} \tag{70}$$
  - Together with the quantisation condition it gives
$$A^2 = \frac{ak^2}{a^2k^2 + w^4 + w^2} \quad \Rightarrow \quad A = \frac{1}{\sqrt{a}}\frac{1}{\sqrt{1 + \frac{w^2+w^4}{a^2k^2}}} \tag{71}$$

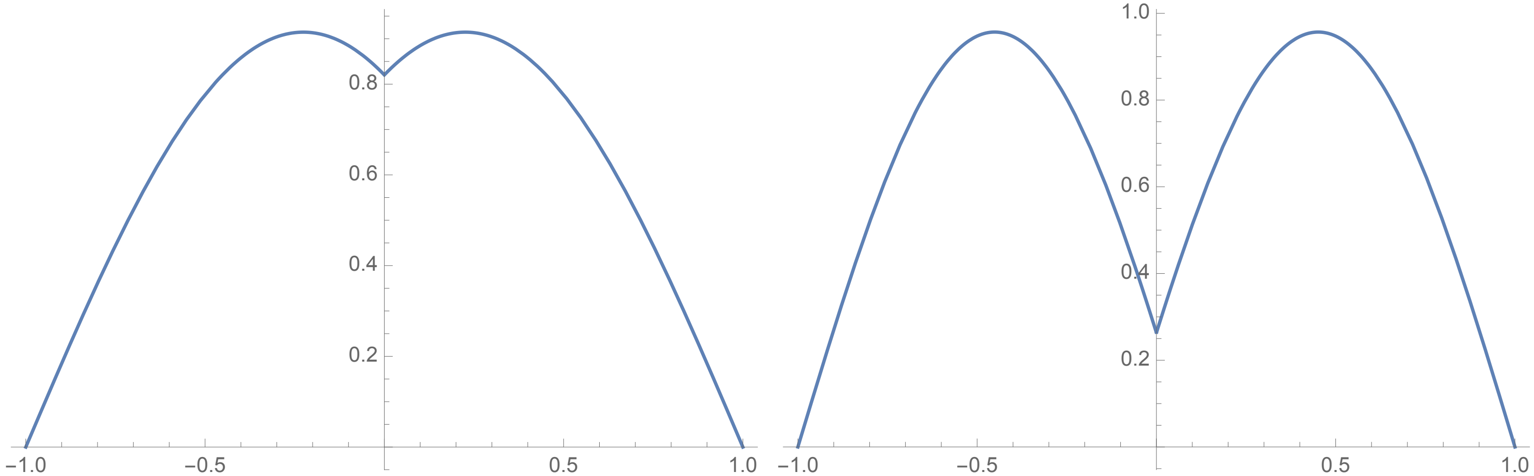

Figure 1: Left: the ground state wave function for $\nu = 1$. Right: the ground state wave function for $\nu = 10$.

- Set $a = 1, m = 1, \hbar = 1$, find the ground state energy for $\nu = 1$ and $\nu = 10$ numerically, and plot the ground state wave functions for these values of ν.
  - Solving the quantisation condition
$$-\cot ka = \frac{\nu\, ma}{\hbar^2}\,\frac{1}{ka} \tag{72}$$
for $a = 1, m = 1, \hbar = 1$, and $\nu = 1$ and $\nu = 10$, we find the ground state energy
$$\begin{aligned} k|_{\nu=1} &= 2.02876 \quad \Rightarrow \quad E|_{\nu=1} = \frac{1}{2}k|_{\nu=1}^2 = 2.05793 \\ k|_{\nu=10} &= 2.86277 \quad \Rightarrow \quad E|_{\nu=10} = \frac{1}{2}k|_{\nu=10}^2 = 4.09773 \end{aligned} \tag{73}$$
  - Plots of the ground state wave functions for $\nu = 1$ and $\nu = 10$ are shown on figure 1