MAU22101: Exercises Week 10

Problem 1 Let G be a group with a subset $S \subset G$. We define the *normal closure* $N(S)$ to be the smallest normal subgroup containing S, that is

$$N(S) := \bigcap_{S \subset N \lhd G} N.$$

1. Show that $N(S) \lhd G$.
2. Show that $N(S)$ is generated by the set $\{gsg^{-1} \mid s \in S,\ g \in G\}$, that is

$$N(S) = \langle gsg^{-1}, s \in S,\ g \in G \rangle.$$

(Hint: Showing the inclusion $\subset$ amounts to showing that the right hand side is a normal subgroup of G. Use that H is a normal subgroup if $H \subset xHx^{-1}$ for all $x \in G$.)

Problem 2 Show that $G = (\mathbb{Q}, +)$ is not finitely generated.

Problem 3 Let p be a prime and $A = \mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_k}\mathbb{Z}$ for some positive integers $n_1, \ldots, n_k$.

1. Show that $pA := \{pa \mid a \in A\}$ is a subgroup of A.
2. Show that $pA \cong \mathbb{Z}/p^{n_1-1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_k-1}\mathbb{Z}$.
3. Show that $A/pA \cong (\mathbb{Z}/p\mathbb{Z})^k$.

Problem 4 List all the isomorphism classes of abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$. Write the groups in both, the primary factor decomposition and the invariant factor decomposition.

Problem 5 Let $A = \mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/45\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$. Find the number of elements of order 2 and the number of subgroups of index 2 in A. (Hint: For the second part, note that any subgroup of index 2 arises from a group homomorphism $\phi\colon A \to \mathbb{Z}/2\mathbb{Z}$ and that any such homomorphism is determined by its value on the generators

$$(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1).)$$