L18: Semi-direct products

<u>Groups of order pq, $p<q$, p & q prime</u>

Let G be a group of order $|G| = pq$

$n_q \in \{1, p\}$ & $n_q \equiv 1 \mod q$ $\Rightarrow$ $n_q = 1$

prev assignment: if $p \nmid q-1$ $\Rightarrow$ G is abelian

Thus G has a normal subgroup $Q \trianglelefteq G$, $|Q| = q$

Let P be a Sylow p-subgroup. Then $P \cap Q = \{e\}$

and thus $|PQ| \underset{\uparrow \text{2nd isom thm}}{=} |Q| \cdot \left|\frac{P}{P \cap Q}\right| = q \cdot p \Rightarrow PQ = G$

Define $F: Q \times P \to G$

$(x, y) \mapsto xy$

same proof as for direct product shows F is a bijection, but maybe not a group homomorphism.

$F(q_1, p_1) \cdot F(q_2, p_2) = q_1 p_1 q_2 p_2 = q_1 \underset{\in Q}{(p_1 q_2 p_1^{-1})} \cdot p_1 p_2$

$= F(q_1 \cdot p_1 q_2 p_1^{-1}, p_1 p_2)$

Recall that the adjoint action defines a group homomorphism

$\varphi: P \to \mathrm{Aut}(Q)$ $\varphi(x)(y) = xyx^{-1}$.

$F(q_1 \varphi(p_1)(q_2), p_1 p_2)$

<u>Construction</u> (semi-direct product)

Let P, Q be groups and $\varphi: P \to \mathrm{Aut}(Q)$ a group homomorphism.

We define $Q \rtimes_\varphi P := (Q \times P, m_\varphi)$ where

$m_\varphi((q_1, p_1), (q_2, p_2)) := (q_1 \cdot_Q \varphi(p_1)(q_2), p_1 \cdot_P p_2)$

It is called the <u>semi-direct product</u> of P and Q.

<u>Prop</u> $Q \rtimes_\varphi P$ is a group

<u>Pf</u> Exercise

We have shown

<u>Prop</u> Let G be a group of order $|G| = pq$, $p<q$ p, q prime. Then

G is isomorphic to $\mathbb{Z}/q\mathbb{Z} \rtimes_\varphi \mathbb{Z}/p\mathbb{Z}$

for some homomorphism $\varphi: \mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/q\mathbb{Z}) \cong \mathbb{Z}/q\mathbb{Z}^\times$

We have also shown

<u>Thm</u> (Characterization of semi-direct products)

Let G be a group and suppose $N \trianglelefteq G$, $H \leq G$ s.t.

- $NH = G$
- $N \cap H = \{e\}$

Then $G \cong N \rtimes_\varphi H$

where $\varphi: H \to \mathrm{Aut}(N)$ is the adjoint action.

<u>Pf</u> Exercise

How many $\varphi: \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}^\times$ are there?

- $p \nmid q-1 \Rightarrow \varphi$ is trivial, i.e. $\varphi(x) = e \ \forall x \in \mathbb{Z}/p\mathbb{Z} \Rightarrow G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$
- $p \mid q-1 \Rightarrow$ Cauchy's thm $\Rightarrow \mathbb{Z}/q\mathbb{Z}^\times$ has a subgroup $\mathbb{Z}/p\mathbb{Z}$

  $\Rightarrow$ there exists at least one non-abelian group

  $G = \mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$

<u>Fact</u> (proof maybe later) q prime

$(\mathbb{Z}/q\mathbb{Z})^\times$ is cyclic (of order $q-1$)

Using fact $\Rightarrow$ there exists exactly one (up to isomorphism) non-abelian group G of order pq, $p \mid q-1$