Group Theory Assignment 6

~ Problem 1 Recall that the centre of a group is defined by

$Z(G) = \{ g \in G \mid gh = hg \ \forall h \in G \}$

a) show that $Z(G)$ is an abelian subgroup of G

b) Show that $Z(G) \trianglelefteq G$

a) Recall that a group H is a subgroup of a group G, $H \leq G$, iff and only if

1) $H \neq \emptyset$ 2) $a,b \in H \Rightarrow a \cdot b \in H$, 3) $a \in H \Rightarrow a^{-1} \in H$

1) By definition of the identity element $e \in G$

$eg = ge = g \ \forall g \in G$

Therefore, $e \in Z(G) \Rightarrow Z(G)$ non-empty.

2) Suppose that $a,b \in Z(G)$, we then have that

$\forall g \in G \quad (ab)g = a(bg) = a(gb) = (ag)b = (ga)b$
$= g(ab)$

Hence, $ab \in Z(G)$

3) Suppose that $c \in Z(G)$, then

$\forall g \in G, \quad cg = gc \rightarrow c^{-1}(cg)c^{-1} = c^{-1}(gc)c^{-1}$
$\rightarrow (c^{-1}c)gc^{-1} = c^{-1}g(cc^{-1})$
$\rightarrow egc^{-1} = c^{-1}ge$
$\rightarrow gc^{-1} = c^{-1}g$

$\Rightarrow c^{-1} \in Z(G) \quad \therefore Z(G) \leq G$

By definition of $Z(G)$, we have that ~~$\forall$~~ $a \in G, b \in Z(G)$

$ab = ba$

In particular, since $Z(G) \leq G$ as above, we then have that all elements of $Z(G)$ commute with $Z(G)$, hence $Z(G)$ is an abelian subgroup of G.

b) Recall the definition of a normal subgroup: N is a normal subgroup of G if and only if:

$\forall g \in G: \; gN = Ng$ (every right coset of N in G is a left coset)

Since $gx = xg$ for each $g \in G$ and $x \in Z(G)$:

$gZ(G) = Z(G)g$

Thus,

$Z(G) \trianglelefteq G$.

~ <u>Problem 2</u> Show that $S_X \cong S_n$ for $n = |X|$

[See Dummit & Foote ex. 1.6.10]

Since the cardinality of X is n, we have a bijection $\theta: \{1,2,\dots,n\} \to X$.

Define the following function

$\varphi: S_n \to S_X$ with $\varphi(\sigma) = \theta \circ \sigma \circ \theta^{-1}$ $\forall \sigma \in$ ~~[illegible]~~ S_n

We want to show that φ is a) well-defined, b) bijective and c) is a group homomorphism.

a) Essentially, we want to show that $\varphi(\sigma)$ is a bijection $X \to X$, given that $\sigma: \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ is a bijection. For any permutation σ of $\{1,2,\dots,n\}$, it is clear that $\varphi(\sigma) = \theta \circ \sigma \circ \theta^{-1}$ is a function from $X \to X$, as $\theta: \{1,2,\dots,n\} \to X$, $\theta^{-1}: X \to \{1,2,\dots,n\}$ (which makes sense as θ is a bijection) and σ is a bijection from $\{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

- Suppose $a,b \in X$ such that $\varphi(\sigma)(a) = \varphi(\sigma)(b)$. We get

$(\theta \circ \sigma \circ \theta^{-1})(a) = (\theta \circ \sigma \circ \theta^{-1})(b)$

$\Rightarrow (\sigma \circ \theta^{-1})(a) = (\sigma \circ \theta^{-1})(b)$, since θ is injective

$\Rightarrow \theta^{-1}(a) = \theta^{-1}(b)$, since σ is injective

$\Rightarrow a = b$ since θ is bijective, hence θ^{-1} is injective

Thus $\varphi(\sigma)$ is an injection

- Let $y \in X$ be arbitrary. Then, we may take

$x = \varphi(\sigma^{-1})(y) = (\theta \circ \sigma^{-1} \circ \theta^{-1})(y)$

so that $\varphi(\sigma)(x) = y$. Thus, $\varphi(\sigma)$ is a bijection, hence $\varphi(\sigma)$ is a permutation of X

b) We want to show that $\varphi: S_n \to S_X$ is a bijection. We define $\psi: S_X \to S_n$ by

$$\psi(\tau) = \theta^{-1} \circ \tau \circ \theta \quad \text{for any } \tau \in S_X.$$

As in (a), ψ is well-defined. Moreover, for any $\sigma \in S_n$

$$(\psi \circ \varphi)(\sigma) = \psi(\theta \circ \sigma \circ \theta^{-1}) = \theta^{-1} \circ \theta \circ \sigma \circ \theta^{-1} \circ \theta = \sigma$$

and for any $\tau \in S_X$

$$(\varphi \circ \psi)(\tau) = \varphi(\theta^{-1} \circ \tau \circ \theta) = \theta \circ \theta^{-1} \circ \tau \circ \theta \circ \theta^{-1} = \tau$$

Therefore, ψ is a two-sided inverse of φ so that φ is a bijection

c) Let $\sigma, \tau \in S_n$. Then,

$$\varphi(\sigma) \circ \varphi(\tau) = \theta \circ \sigma \circ \theta^{-1} \circ \theta \circ \tau \circ \theta^{-1} = \theta \circ \sigma \circ \tau \circ \theta^{-1} = \varphi(\sigma \circ \tau)$$

Therefore, φ is a homomorphism and hence an isomorphism.

Problem 3 Let G be a group and let $p \mid |G|$ be the smallest prime dividing $|G|$. Suppose that G has a subgroup $H \leq G$ of index p. Show that H is normal by

a) Consider the action $G \circlearrowright G/H$ and let K be the kernel of the corresponding group homomorphism $\phi: G \to S_{G/H}$. Show that $K \leq H$

b) Show that the image of ϕ is a group of order p (Hint: use Lagrange (twice) to show that the order of the image is a divisor of $|G|$ and of $p!$)

c) Deduce that $H = K$ and that H is normal.

a) $\phi: G \to \mathrm{Sym}(G/H)$ such that $\phi(g)(xH) = gxH$

where $xH = \{xh \mid h \in H\}$ for $x \in G$

$$\ker\phi = \{g \in G \mid \phi(g) = e_{\mathrm{Sym}(G/H)}\}$$

$$\Rightarrow \ker\phi = \{g \in G \mid \phi(g)(xH) = xH, \ \forall x \in G\}$$

$$= \{g \in G \mid gxH = xH, \ \forall x \in G\} = \{g \in G \mid (x^{-1}gx)H = H, \ \forall x \in G\}$$

$$= \{g \in G \mid x^{-1}gx \in H \ \forall x \in G\} = \{g \in G \mid g \in xHx^{-1} \ \forall x \in G\}$$

$$= \bigcap_{x \in G} xHx^{-1}$$

If $x \in K = \ker\phi$, then $xaH = aH$ for every coset aH. In particular, this is true for the coset H itself: $xH = H$ and so $x \in H$. Therefore $K \subset H$. Recall that ~~the~~ $\ker\phi$ is a (normal) subgroup of G, thus $K \leq H$.

b) Let $|G:H| = p$ and $|H:K| = k$. Then, $|G:K| = |G:H||H:K| = pk$. By the first isomorphism theorem, since H has p left cosets $G/K \cong$ a subgroup of S_p (namely, the image of G under ϕ).

By Lagrange's theorem:

$pk = |G/K|$ divides $p!$ $\Rightarrow k \,|\, p!/p = (p-1)!$

But, $|G/K|$ also divides $|G|$. But, all prime divisors of $(p-1)!$ are less than p, and by minimality of p, every prime divisor of K is greater than or equal to p. Therefore, $K = 1$.

c) Thus, from (b) $|H:K| = 1 = |H/K|$

$\Rightarrow H = K$.

Since $\ker\phi = K$ is a normal subgroup of $G \Rightarrow H \trianglelefteq G$

~ Problem 4 Let G be a finite group of order n. The left-regular representation defines a group homomorphism $\phi: G \to S_n$. For an element $g \in G$, we can consider the cycle decomposition of $\phi(g)$. Show that the cycle decomposition of $\phi(g)$ consists of $|G|/|g|$ cycles of length $|g|$.

Let $g \in G$ and let $H = \langle g \rangle$. Suppose that $|g| = m$. By Lagrange's Theorem, we have that $n = lm$, where $l = |G:H|$. For the right cosets of H in G, we can fix representatives $y_1 \equiv 1, y_2, y_3, \ldots, y_l$. Then, we can list the group elements in the following order

$1, g, \ldots, g^{m-1}, y_2, gy_2, \ldots g^{m-1}y_2 \ldots, y_l, xy_l, \ldots, g^{m-1}y_l$

where $g^m y_i = y_i$ by definition of $\langle g \rangle$.

Left multiplication by x then affects the following rearrangement

$$g, g^2, \ldots, 1, gy_2, g^2y_2, \ldots, y_2, \ldots, gy_l, x^2y_l, \ldots, y_l$$

Hence $\sigma_g \equiv (1\ 2\ 3 \ldots m)(m+1\ m+2 \ldots 2m) \ldots ((l-1)m+1\ (l-1)m+2 \ldots, lm)$ is the cycle decomposition of the permutation associated with x. Therefore, the cycle decomposition consists of $l = |G|/|g|$ cycles of length $|g|$.

Problem 5 Suppose that the center of G has index n. Show that every conjugacy class has at most n elements. (Hint: use the Orbit-Stabilizer theorem).

If $x \in G$, then the conjugacy class of x is the orbit of x when G acts on itself by conjugation:

$\text{Orb}(x) : \{gxg^{-1} \mid g \in G\}$

By the Orbit-Stabilizer theorem

$$|\text{Orb}(x)| = [G : \text{Stab}(x)] = |G|/|\text{Stab}(x)|$$

But, $\text{Stab}(x) = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid gx = xg\} = C_G(x)$ where $C_G(x)$ is the centralizer of x in G. But, by definition of the center $Z(G)$, $Z(G) \subseteq C_G(x)$, hence $Z(G) \leq C_G(x)$.

Therefore, $|Z(G)| \leq |C_G(x)|$. Thus, we have that

$$|\text{Orb}(x)| = |G|/|\text{Stab}(x)|$$
$$= |G|/|C_G(x)| \leq \frac{|G|}{|Z(G)|} = |G : Z(G)| = n$$

ie: Each conjugacy class has at most n elements.