Module MAU34403 Quantum mechanics I (Frolov)
Homework Sheet 7
Each set of homework questions is worth 100 marks

Use Mathematica if necessary

Compulsory Questions

Use the results from practice questions

Problem 1. A coherent state of a one-dimensional harmonic oscillator is defined to be an eigenstate of the (non-Hermitian) annihilation operator a

$$a\,|\lambda\rangle = \lambda\,|\lambda\rangle\,, \quad \langle\lambda|\lambda\rangle = 1 \tag{0.1}$$

where α is in general a complex number.

(a) Find $|\lambda\rangle$

(b) Express $|\lambda\rangle$ in the form $|\lambda\rangle = f(a^\dagger)|0\rangle$

(c) Prove the minimum uncertainty relation for such a state.

(d) Find the wave function $\psi_\lambda(x)$ of a coherent state in the coordinate representation

(e) Write $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)\,|n\rangle \tag{0.2}$$

Show that the distribution of $|f(n)|^2$ with respect to n is of the Poisson form. Find the expectation value $\bar{n}$ of $N = a^\dagger a$. Find the most probable value $n_{\rm mp}$ of n, hence of E.

Problem 2. Consider a particle in the following potential

$$V(x) = \begin{cases} -\nu\delta(x) & \text{for } x < a \\ +\infty & \text{for } x > a \end{cases} \tag{0.3}$$

where $a > 0\,,\ \nu > 0$.

(a) Find the wave function for a scattering state. Do not normalise it.

(b) Find wave functions for bound states, and a quantisation condition for the bound state spectrum. Do not normalise the wave functions.

(c) Show that the energy quantisation condition can be written in the form

$$\frac{z}{W} = 1 - e^{-z}\,, \tag{0.4}$$

where z and W have to be identified.

Sketch plots of the left and right hand sides of the energy quantisation condition for $W = 1/2$ and $W = 2$.

Prove that there may exist at most only one bound state that is the ground state of the system. Find the values of W for which there is the ground state.

(d) Find the ground state energy E_0 in the limit $W \to \infty$. Explain the result obtained.

Practice Questions

Problem 1. Consider a particle in the potential of a rectangular well

$$V(x) = \begin{cases} V_L & \text{for } |x| > a \\ V_{\min} & \text{for } |x| < a \end{cases} \tag{0.5}$$

1. Find the energy quantisation condition for odd parity states
2. Show that the energy quantisation condition can be written in the form

$$-\cot z = \sqrt{\frac{W^2}{z^2} - 1} \tag{0.6}$$

where z and W have to be identified.

Sketch plots of the left and right hand sides of the energy quantisation condition.

Find the values of W for which there are n odd parity bound states.

Problem 2. Consider a particle in the following potential

$$V(x) = \begin{cases} -\nu\delta(x) & \text{for } x < a \\ +\infty & \text{for } x > a \end{cases} \tag{0.7}$$

where $a > 0\,,\ \nu > 0$.

Find the wave function for a scattering state. Do not normalise it.

Problem 3. Scattering in one dimension

(a) Consider the following four functions

$$\varphi_\alpha^\pm(E,x) \equiv \sqrt{\frac{m}{2\pi\,\hbar^2}}\,\frac{e^{\pm \mathrm{i}\,k_\alpha\,x}}{\sqrt{k_\alpha}}\,, \quad k_\alpha \equiv \frac{\sqrt{2m(E - V_\alpha)}}{\hbar}\,, \quad \alpha = L, R \tag{0.8}$$

Prove that they are normalised as

$$\int dx\, \bar{\varphi}_L^a(E_1, x)\varphi_L^b(E_2, x) = \delta_{ab}\delta(E_1 - E_2)\,, \quad a, b = +, -\,, \tag{0.9}$$

where $\bar{\varphi}$ denotes the complex conjugate of φ, and φ_R^a satisfy the same normalisations.

Hint. Use the formula

$$\delta(f(x) - f(y)) = \frac{1}{f'(x)}\delta(x - y)\,, \quad f(x) \neq f(y) \text{ for } x \neq y \tag{0.10}$$

(b) Prove that the **Wronskian**

$$W(f_1, f_2) \equiv f_1 f_2' - f_1' f_2 \tag{0.11}$$

of any two functions satisfying the time-independent Schrödinger equation does not depend on x.

(c) Let

$$\varphi_L(E, x) = \begin{cases} \varphi_L^+(E, x) + S_{LL}\varphi_L^-(E, x) & \text{for } x < -a \\ S_{RL}\varphi_R^+(E, x) & \text{for } x > a \\ A_{ML}\varphi_1(x) + B_{ML}\varphi_2(x) & \text{for } |x| < a \end{cases} \tag{0.12}$$

and

$$\varphi_R(E, x) = \begin{cases} S_{LR}\varphi_L^-(E, x) & \text{for } x < -a \\ S_{RR}\varphi_R^+(E, x) + \varphi_R^-(E, x) & \text{for } x > a \\ A_{MR}\varphi_1(x) + B_{MR}\varphi_2(x) & \text{for } |x| < a \end{cases} \tag{0.13}$$

Consider the four pairs (φ_L, φ_L^*), (φ_R, φ_R^*), (φ_L, φ_R^*) and (φ_L, φ_R), and compute their Wronskians for $x < -a$ and $x > a$.