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An Introduction for Theoretical Physics Students

Differential Forms and Hodge Theory

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Abstract:

Differential forms and exterior product.
Exterior derivative, Poincaré's lemma and de Rham Cohomology.
Hodge star operator and inner product.
Adjoint exterior derivative, Laplacian and Hodge decomposition.
Stokes' theorem.
Various formulations of electromagnetism.
Kalb-Ramond field and p-form electrodynamics.
Chern-Simons theory.

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1 Notations and Conventions

Einstein summation convention is used here: a repeated index is understood as summed over its allowed range. $[\mu_1\mu_2\cdots\mu_p]$ is the total antisymmetrization of the indices, *without* dividing by $p!$, for example $A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$. We use mostly-plus signature for the metric. Let n be the dimension of the manifold $\mathcal{M}$, i.e.

$$\dim \mathcal{M} = n. \tag{1.1}$$

Levi-Civita Symbol and Tensor

For a detailed discussion on this topic, see my two-part notes on *Index Notation.* The Levi-Civita symbol $\hat{\epsilon}_{\mu_1\mu_2\cdots\mu_n}$ or $\hat{\epsilon}^{\mu_1\mu_2\cdots\mu_n}$ is defined in the same way as that in the Euclidean space:

$$\hat{\epsilon}_{12\cdots n} = 1, \qquad \hat{\epsilon}^{12\cdots n} = 1, \tag{1.2}$$

or

$$\hat{\epsilon}_{012\cdots(n-1)} = 1, \qquad \hat{\epsilon}^{012\cdots(n-1)} = 1, \tag{1.3}$$

in the case of Minkowskian spaces, and all other components are related to these components by the parity of permutation of indices. A general expression for all components can be written as

$$\hat{\epsilon}_{\mu_1\mu_2\cdots\mu_n} = \delta^1_{[\mu_1}\delta^2_{\mu_2}\cdots\delta^n_{\mu_n]}, \qquad \hat{\epsilon}^{\mu_1\mu_2\cdots\mu_n} = \delta^{[\mu_1}_1\delta^{\mu_2}_2\cdots\delta^{\mu_n]}_n. \tag{1.4}$$

When the n-dimensional manifold $\mathcal{M}$ is equipped with a (pseudo-)Riemannian metric $g_{\mu\nu}$, we can define the Levi-Civita tensor as

$$\epsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|\det g_{\mu\nu}|}\,\hat{\epsilon}_{\mu_1\mu_2\cdots\mu_n}. \tag{1.5}$$

It is a pseudotensor. We have

$$\epsilon^{\mu_1\mu_2\cdots\mu_n} = \frac{\operatorname{sgn}(\det g_{\mu\nu})}{\sqrt{|\det g_{\mu\nu}|}}\hat{\epsilon}^{\nu_1\nu_2\cdots\nu_n}. \tag{1.6}$$

We have the following identity for contracting m indices of Levi-Civita tensors

$$\epsilon^{\mu_1\cdots\mu_{n-m}\rho_{n-m+1}\cdots\rho_n}\epsilon_{\nu_1\cdots\nu_{n-m}\rho_{n-m+1}\cdots\rho_n} = \operatorname{sgn}(\det g_{\mu\nu})\, m!\,\delta^{\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_{n-m}}_{\nu_{n-m}]}. \tag{1.7}$$

Notice that here it does not matter whether you put $[\cdots]$ around ν_i or μ_i, the effect is the same: when one group of indices is antisymmetrized, the other group is automatically antisymmetrized as well (convince yourself!). In particular

$$\begin{aligned}
\epsilon^{\rho_1\cdots\rho_n}\epsilon_{\rho_1\cdots\rho_n} &= \operatorname{sgn}(\det g_{\mu\nu})\, n!, \\
\epsilon^{\mu\rho_2\cdots\rho_n}\epsilon_{\nu\rho_2\cdots\rho_n} &= \operatorname{sgn}(\det g_{\mu\nu})\,(n-1)!\delta^\mu_\nu, \\
\epsilon^{\mu\lambda\rho_3\cdots\rho_n}\epsilon_{\nu\sigma\rho_3\cdots\rho_n} &= \operatorname{sgn}(\det g_{\mu\nu})\,(n-2)!(\delta^\mu_\nu\delta^\lambda_\sigma - \delta^\mu_\sigma\delta^\lambda_\nu).
\end{aligned} \tag{1.8}$$

Levi-Civita Tensor in 3+1 Dimensional Minkowskian Space

In 3+1 dimensional Minkowskian space with $g_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1)$, $\det g_{\mu\nu} = -1$, we have

$$\epsilon_{\mu\nu\rho\sigma} = \hat{\epsilon}_{\mu\nu\rho\sigma}, \qquad \epsilon^{\mu\nu\rho\sigma} = -\hat{\epsilon}^{\mu\nu\rho\sigma}, \tag{1.9}$$

i.e.

$$\epsilon_{0123} = 1, \qquad \epsilon^{0123} = -1. \tag{1.10}$$

Then

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} &= -24, \\
\epsilon^{\mu\rho\sigma\lambda}\epsilon_{\nu\rho\sigma\lambda} &= -6\delta^{\mu}_{\nu}, \\
\epsilon^{\mu\rho\lambda\eta}\epsilon_{\nu\sigma\lambda\eta} &= -2\left(\delta^{\mu}_{\nu}\delta^{\rho}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\rho}_{\nu}\right), \\
\epsilon^{\mu\rho\lambda\zeta}\epsilon_{\nu\sigma\eta\zeta} &= -\delta^{\mu}_{[\nu}\delta^{\rho}_{\sigma}\delta^{\lambda}_{\eta]}.
\end{aligned} \tag{1.11}$$

2 Differential Forms

Notice that most discussions in this section (other than the ones involving Levi-Civita *tensor*) do not rely on the existence of a metric on the Manifold. For references on topics covered in this section, see Sections 5.4, 5.5 and Chapter 6 in [1], and Chapter 2 in [2].

2.1 Bases for Differential Forms

Among all covariant tensors of rank $(0, p)$ (those with only lower indices, no upper indices) and the spaces they form, there is a subset that is of particular interest: the totally antisymmetric (skew-symmetric) ones. These are called *p-forms*. A scalar by definition is a 0-form.

Differential forms live in the cotangent space of the manifold $\mathcal{M}$. All p-forms form a vector space denoted as $\bigwedge^p(\mathcal{M})$ ($\bigwedge$ is a capital $\wedge$), sometimes denoted as $\Omega^p(\mathcal{M})$. The coordinate basis of $\bigwedge^p(\mathcal{M})$ for p-forms is $\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$, which is totally antisymmetric under permutations of indices. It vanishes if any two indices coincides, e.g. $\mathrm{d}y \wedge \mathrm{d}y \wedge \mathrm{d}z = 0$. For example, the basis for 2-forms is

$$\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} = \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} - \mathrm{d}x^{\nu} \otimes \mathrm{d}x^{\mu} = -\mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu}, \tag{2.1}$$

and $\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\mu} = 0$ (no sum here). Here $\otimes$ is the tensor product. An immediate consequence of the total antisymmetry is that the bases and all p-forms vanish identically for all $p > n$ where n is the dimension of the manifold. Thus for an n-dimensional manifold, the only non-vanishing forms are 0-forms, 1-forms, 2-forms up to n-forms. Due to this antisymmetry, the p-form basis $\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$ has dimension

$$\dim \bigwedge^p(\mathcal{M}) = \binom{n}{p} = \frac{n!}{(n-p)!p!}, \tag{2.2}$$

because all indices μ_1, μ_2, $\cdots$, μ_p must take different values in any non-vanishing expressions. An immediate consequences is that

$$\dim \bigwedge^p (\mathcal{M}) = \dim \bigwedge^{n-p} (\mathcal{M}) \,, \tag{2.3}$$

i.e. a p-form and a $(n-p)$-form have the same dimension, thus there exists a duality map between them. This is the basis for Hodge duality, which will be explored later. Mathematicians say the direct sum of all spaces of forms

$$\bigwedge (\mathcal{M}) = \bigoplus_{p=0}^{n} \bigwedge^p (\mathcal{M}) = \bigwedge^0 (\mathcal{M}) \oplus \bigwedge^1 (\mathcal{M}) \oplus \cdots \oplus \bigwedge^n (\mathcal{M}) \tag{2.4}$$

forms an *exterior algebra*, which is a graded associative algebra. Notice that the total dimension can be calculated using the relation

$$(a+b)^n = \sum_{p=0}^{n} \begin{pmatrix} n \\ p \end{pmatrix} a^p b^{n-p}. \tag{2.5}$$

Letting $a = b = 1$ in this formula, we have

$$\dim \bigwedge (\mathcal{M}) = \sum_{p=0}^{n} \begin{pmatrix} n \\ p \end{pmatrix} = 2^n. \tag{2.6}$$

A special case of the above counting is that $\dim \bigwedge^n (\mathcal{M}) = \dim \bigwedge^0 (\mathcal{M}) = 1$, i.e. n-forms has only one degree of freedom and is thus dual to a scalar. We have seen this in the discussion on Levi-Civita symbol: an n-form is a totally antisymmetric tensor whose number of indices equals the dimension of the manifold, thus they must all be proportional to Levi-Civita symbol; the only degree of freedom is the normalization (equivalently, any one component), which can be thought as a scalar, i.e. dual to a 0-form. The basis of $\bigwedge^n (\mathcal{M})$ can be written in terms of Levi-Civita symbol $\hat{\epsilon}^{\mu_1\mu_2\cdots\mu_n}$ (not the tensor $\epsilon^{\mu_1\mu_2\cdots\mu_n}$!) as

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} = \hat{\epsilon}^{\mu_1\mu_2\cdots\mu_n} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n. \tag{2.7}$$

Contracting this expression with Levi-Civita tensor (1.5)

$$\epsilon_{\mu_1\mu_2\cdots\mu_n} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} = \sqrt{|\det g_{\mu\nu}|}\, \hat{\epsilon}_{\mu_1\mu_2\cdots\mu_n} \hat{\epsilon}^{\mu_1\mu_2\cdots\mu_n} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n,$$

and using $\hat{\epsilon}_{\mu_1\mu_2\cdots\mu_n} \hat{\epsilon}^{\mu_1\mu_2\cdots\mu_n} = n!$, we have

$$\frac{1}{n!} \epsilon_{\mu_1\mu_2\cdots\mu_n} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} = \sqrt{|\det g_{\mu\nu}|}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n = \mathrm{d}V. \tag{2.8}$$

We see that this quantity is nothing but the coordinate-independent *volume element* of the n-dimensional manifold. The volume element in the language of differential forms is not a scalar like $\mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n$, but rather an n-form $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$ as shown here.

2.2 Components and Exterior Product of Forms

A p-form A by definition has components

$$A = \frac{1}{p!} A_{\mu_1\mu_2\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{2.9}$$

Notice the normalization factor $1/p!$. By definition, $A_{\mu_1\mu_2\cdots\mu_p}$ is totally antisymmetric in all its indices, thus has the property

$$A_{\mu_1\mu_2\cdots\mu_p} = \frac{1}{p!} A_{[\mu_1\mu_2\cdots\mu_p]}. \tag{2.10}$$

For example, for a 1-form A, a 2-form F and a 3-form H, their components are

$$\begin{aligned} A &= A_\mu \mathrm{d}x^\mu, \\ F &= \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \\ H &= \frac{1}{6} H_{\mu\nu\rho} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho. \end{aligned} \tag{2.11}$$

The n-form Levi-Civita tensor can be written as

$$\epsilon = \frac{1}{n!} \epsilon_{\mu_1\mu_2\cdots\mu_n} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}. \tag{2.12}$$

Notice that this is precisely the left hand side of (2.8), thus we can identify

$$\epsilon = \mathrm{d}V, \tag{2.13}$$

i.e. the Levi-Civita tensor as a coordinate-independent n-form in an n-dimensional manifold is precisely the invariant volume element of that manifold! This is another important aspect of Levi-Civita tensor. We often think the volume element as a scalar, but it turns out that it is actually an n-form!

As $A_{\mu_1\mu_2\cdots\mu_p}$ is totally antisymmetric, the independent components can be chosen as the ones with $\mu_1 < \mu_2 < \cdots < \mu_p$, the $1/p!$ normalization factor is chosen such that written in terms of these independent components, we will have

$$A = \sum_{\mu_1<\mu_2<\cdots<\mu_p} A_{\mu_1\mu_2\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{2.14}$$

For example, a 2-form F in three-dimensional Euclidean space with coordinates (x, y, z) can be written as

$$\begin{aligned} F =& \frac{1}{2} F_{xy} \mathrm{d}x \wedge \mathrm{d}y + \frac{1}{2} F_{xz} \mathrm{d}x \wedge \mathrm{d}z + \frac{1}{2} F_{yz} \mathrm{d}y \wedge \mathrm{d}z \\ &+ \frac{1}{2} F_{yx} \mathrm{d}y \wedge \mathrm{d}x + \frac{1}{2} F_{zx} \mathrm{d}z \wedge \mathrm{d}x + \frac{1}{2} F_{zy} \mathrm{d}z \wedge \mathrm{d}y. \end{aligned} \tag{2.15}$$

As $F_{yx} = -F_{xy}$, $\mathrm{d}y \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}y$, we have $F_{yx}\mathrm{d}y \wedge \mathrm{d}x = F_{xy}\mathrm{d}x \wedge \mathrm{d}y$, hence the second line equals the first line, and they can be combined to cancel the $\frac{1}{2}$ factor. Thus written in independent components

$$F = F_{xy} \mathrm{d}x \wedge \mathrm{d}y + F_{xz} \mathrm{d}x \wedge \mathrm{d}z + F_{yz} \mathrm{d}y \wedge \mathrm{d}z \tag{2.16}$$

The wedge product on the bases of forms completely dictates how it acts on two forms of different ranks. The wedge product of a p-form $A^{(p)}$ and a q-form $B^{(q)}$ is a $(p+q)$-form

$$C^{(p+q)} = A^{(p)} \wedge B^{(q)}. \tag{2.17}$$

Notice that

$$A^{(p)} \wedge B^{(q)} = (-1)^{pq}\, B^{(q)} \wedge A^{(p)}. \tag{2.18}$$

This can be easily verified by expanding the two forms over their corresponding coordinate bases and see how the two sets of bases commute, this is where the factor $(-1)^{pq}$ from. The above relation implies that for a 1-form A, $A \wedge A = (-1)^{1\cdot 1}\, A \wedge A = -A \wedge A$, hence $A \wedge A = 0$. But for a 2-form F, $F \wedge F = (-1)^{2\cdot 2}\, F \wedge F = F \wedge F$, hence in general $F \wedge F \neq 0$. Self-wedge product of any even-form is in general non-vanishing, while that of any odd-form vanishes. The components of $A \wedge B$ can be calculated as

$$\begin{aligned}
&C^{(p+q)}\\
=&\frac{1}{p!}A_{\mu_1\mu_2\cdots\mu_p}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge \frac{1}{q!}B_{\mu_{p+1}\mu_{p+2}\cdots\mu_{p+q}}\mathrm{d}x^{\mu_{p+1}} \wedge \mathrm{d}x^{\mu_{p+2}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+q}}\\
=&\frac{1}{p!}\frac{1}{q!}A_{\mu_1\mu_2\cdots\mu_p}B_{\mu_{p+1}\mu_{p+2}\cdots\mu_{p+q}}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge \mathrm{d}x^{\mu_{p+1}} \wedge \mathrm{d}x^{\mu_{p+2}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+q}}\\
=&\frac{1}{p!}\frac{1}{q!}\frac{1}{(p+q)!}A_{[\mu_1\mu_2\cdots\mu_p}B_{\mu_{p+1}\mu_{p+2}\cdots\mu_{p+q}]}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge \mathrm{d}x^{\mu_{p+1}} \wedge \mathrm{d}x^{\mu_{p+2}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+q}},
\end{aligned}$$

comparing this with the template

$$C^{(p+q)} = \frac{1}{(p+q)!}C_{\mu_1\mu_2\cdots\mu_{p+q}}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+q}},$$

we can identify the composition of components under wedge product

$$C_{\mu_1\mu_2\cdots\mu_{p+q}} = \frac{1}{p!}\frac{1}{q!}A_{[\mu_1\mu_2\cdots\mu_p}B_{\mu_{p+1}\mu_{p+2}\cdots\mu_{p+q}]}. \tag{2.19}$$

For example, for a 1-form A and a 2-form F (verify these!)

$$\begin{aligned}
(A \wedge F)_{\rho\mu\nu} &= \frac{1}{2}A_{[\rho}F_{\mu\nu]} = A_\rho F_{\mu\nu} + A_\mu F_{\nu\rho} + A_\nu F_{\rho\mu},\\
(F \wedge F)_{\rho\sigma\mu\nu} &= \frac{1}{4}F_{[\rho\sigma}F_{\mu\nu]} = F_{\rho\sigma}F_{\mu\nu} + F_{\rho\mu}F_{\nu\sigma} + F_{\rho\nu}F_{\sigma\mu}.
\end{aligned} \tag{2.20}$$

In the last expression, the right hand side is a cyclic permutation of the last three indices $\sigma\mu\nu$.

2.3 Exterior Derivative

For any p-form A, its exterior derivative is defined as $\mathrm{d} \wedge A$, but often simply written as $\mathrm{d}A$, which is a $(p+1)$-form. Using $\mathrm{d} = \mathrm{d}x^\rho \partial_\rho$, we have

$$\mathrm{d}A = \mathrm{d}x^\rho \partial_\rho \wedge \frac{1}{p!}A_{\mu_1\mu_2\cdots\mu_p}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$$

$$= \frac{1}{p!} \partial_\rho A_{\mu_1 \mu_2 \cdots \mu_p} \mathrm{d}x^\rho \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$$
$$= \frac{1}{p!} \frac{1}{(p+1)!} \partial_{[\rho} A_{\mu_1 \mu_2 \cdots \mu_p]} \mathrm{d}x^\rho \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p},$$

thus we can identify the components of $\mathrm{d}A$ as

$$(\mathrm{d}A)_{\rho\mu_1 \cdots \mu_p} = \frac{1}{p!} \partial_{[\rho} A_{\mu_1 \cdots \mu_p]}. \tag{2.21}$$

In Maxwell's electrodynamics, the potentials form a 1-form A, and the electromagnetic field is its exterior derivative $F = \mathrm{d}A$. This example will be worked out in detail later. The Leibniz rule for exterior derivative is

$$\mathrm{d}\left(A^{(p)} \wedge B^{(q)}\right) = \left(\mathrm{d}A^{(p)}\right) \wedge B^{(q)} + (-1)^p A^{(p)} \wedge \left(\mathrm{d}B^{(q)}\right). \tag{2.22}$$

The $(-1)^p$ factor in the second term comes from the fact that d is a 1-form, and in order to act on $B^{(q)}$, it has to be swapped with the p-form $A^{(p)}$, and swapping two forms results in the factor $(-1)^{1 \cdot p}$.

Starting from acting on 0-forms (i.e. the usual gradient of a scalar), the exterior derivative d maps p-forms to $(p+1)$-forms, i.e.

$$\bigwedge\nolimits^p (\mathcal{M}) \xrightarrow{\mathrm{d}} \bigwedge\nolimits^{p+1} (\mathcal{M}). \tag{2.23}$$

This map terminates at n-forms, which are also called the *top forms*, because $(n+1)$-forms and beyond all vanish identically, i.e. $\mathrm{d}A^{(n)} = 0$ and $\bigwedge^n (\mathcal{M}) \xrightarrow{\mathrm{d}} 0$. All of these may remind you of raising and lowering operators in quantum mechanics. Mathematicians write this series of maps generated by d as the following sequence

$$0 \hookrightarrow \bigwedge\nolimits^0 (\mathcal{M}) \xrightarrow{\mathrm{d}} \bigwedge\nolimits^1 (\mathcal{M}) \xrightarrow{\mathrm{d}} \bigwedge\nolimits^2 (\mathcal{M}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \bigwedge\nolimits^{n-1} (\mathcal{M}) \xrightarrow{\mathrm{d}} \bigwedge\nolimits^n (\mathcal{M}) \xrightarrow{\mathrm{d}} 0 \tag{2.24}$$

which is called the *de Rham complex*. In gauge theory such as electrodynamics, $\bigwedge^0$ is the space of gauge parameters (they are scalars), $\bigwedge^1$ the space of gauge potentials (they are 1-forms), $\bigwedge^2$ the space of field strength (electric and magnetic fields, they form 2-forms), $\bigwedge^3$ the space of equations of motion (i.e. Maxwell's equations, Bianchi identity, which are 3-forms) because it is dual to $\bigwedge^1$ (i.e. vectors, as Maxwell's equations are vector equations). All of these will be discussed in detail later.

Notice that if the derivative d is taken in a tensor product way with a rank (p, q) tensor, the outcome is usually not a tensor of rank $(p, q+1)$, because the components do not transform in the standard way under a change of coordinates. But exterior derivative $\mathrm{d}\wedge$ of a p-form is a $(p+1)$-form. Let us illustrate these with a (0,1)-tensor A, i.e. 1-form. Recall that under a change of coordinates $x^\mu \to y^\alpha$, components transform as

$$\partial_\mu = (\partial_\mu y^\alpha)\, \tilde{\partial}_\alpha, \qquad A_\nu = (\partial_\nu y^\beta) \tilde{A}_\beta, \tag{2.25}$$

where $\tilde{\partial}_\alpha = \partial / \partial y^\alpha$. For the tensor product $\mathrm{d} \otimes A$, its components $\partial_\mu A_\nu$ then transform as

$$\partial_\mu A_\nu = (\partial_\mu y^\alpha)\, \tilde{\partial}_\alpha \left[(\partial_\nu y^\beta) \tilde{A}_\beta\right] = (\partial_\mu y^\alpha)\, (\partial_\nu y^\beta) \tilde{\partial}_\alpha \tilde{A}_\beta + (\partial_\mu y^\alpha)\, (\tilde{\partial}_\alpha \partial_\nu y^\beta) \tilde{A}_\beta. \tag{2.26}$$

The first term on the right is the standard tensor transformation rule for (0,2)-tensor $\partial_\mu A_\nu$, but the second term is an extra which violates the standard rule. Thus $\mathrm{d} \otimes A$ is not a tensor, unless the transformation is linear. However, for the exterior derivative $\mathrm{d}A$, which is the antisymmetrization of the tensor derivative $\mathrm{d} \otimes A$, the components transform by the antisymmetrization of the above expression

$$\partial_{[\mu} A_{\nu]} = \left(\partial_{[\mu} y^\alpha\right) (\partial_{\nu]} y^\beta) \tilde{\partial}_\alpha \tilde{A}_\beta + (\partial_\mu y^\alpha) (\tilde{\partial}_\alpha \partial_\nu y^\beta) \tilde{A}_\beta - (\partial_\nu y^\alpha) (\tilde{\partial}_\alpha \partial_\mu y^\beta) \tilde{A}_\beta.$$

Notice that the first term

$$\left(\partial_{[\mu} y^\alpha\right) (\partial_{\nu]} y^\beta) \tilde{\partial}_\alpha \tilde{A}_\beta = (\partial_\mu y^{[\alpha})(\partial_\nu y^{\beta]}) \tilde{\partial}_\alpha \tilde{A}_\beta = (\partial_\mu y^\alpha) (\partial_\nu y^\beta) \tilde{\partial}_{[\alpha} \tilde{A}_{\beta]}$$

agrees with the standard transformation of 2-forms $\partial_{[\mu} A_{\nu]}$. The extra terms vanish in this case due to antisymmetrization

$$\begin{aligned}
&(\partial_\mu y^\alpha) (\tilde{\partial}_\alpha \partial_\nu y^\beta) \tilde{A}_\beta - (\partial_\nu y^\alpha) (\tilde{\partial}_\alpha \partial_\mu y^\beta) \tilde{A}_\beta \\
=&\tilde{A}_\beta \left\{ \left[(\partial_\mu y^\alpha) \tilde{\partial}_\alpha \right] \partial_\nu y^\beta - \left[(\partial_\nu y^\alpha) \tilde{\partial}_\alpha \right] \partial_\mu y^\beta \right\} \\
=&\tilde{A}_\beta (\partial_\mu \partial_\nu y^\beta - \partial_\nu \partial_\mu y^\beta) = \tilde{A}_\beta (\partial_{[\mu} \partial_{\nu]} y^\beta) = 0
\end{aligned}$$

where we have used the chain rule $\partial_\mu = (\partial_\mu y^\alpha) \tilde{\partial}_\alpha$. Then

$$\partial_{[\mu} A_{\nu]} = (\partial_\mu y^\alpha) (\partial_\nu y^\beta) \tilde{\partial}_{[\alpha} \tilde{A}_{\beta]}, \tag{2.27}$$

which satisfies the standard tensor transformation rule for 2-forms, thus $\mathrm{d}A$ is a 2-form. Generalization to higher rank forms is straightforward: the extra terms in the transformation all vanish due to $\partial_{[\mu} \partial_{\nu]} = 0$. This confirms that exterior derivative d maps p-forms to $(p+1)$-forms, i.e. $\bigwedge^p (\mathcal{M}) \xrightarrow{\mathrm{d}} \bigwedge^{p+1} (\mathcal{M})$.

2.4 Poincaré's Lemma, de Rham Cohomology and Topology

An important property of the exterior derivative is the operator identity

$$\mathrm{d}^2 = 0. \tag{2.28}$$

Here d^2 is a short-hand notation for $\mathrm{d} \wedge \mathrm{d} \wedge$, i.e. $\mathrm{d} \wedge \mathrm{d} \wedge \ldots = 0$. This is a consequence of commutativity of partial derivatives $\partial_{[\mu} \partial_{\nu]} = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0$, when applying (2.21) twice. An operator that squares to zero is called *nilpotent*. Here the exterior derivative d is nilpotent.

Now we introduce two concepts:

- If a p-form A satisfies $\mathrm{d}A = 0$, it is called a *closed* p-form.
- If a p-form A can be written *globally* (everywhere on the entire manifold) as $A = \mathrm{d}B$ in terms of a certain $(p-1)$-form B, it is called an *exact* p-form.

Obviously, $\mathrm{d}^2 = 0$ implies an exact form is always closed

$$\begin{array}{ccc} A = \mathrm{d}B & \Rightarrow & \mathrm{d}A = 0 \\ \text{exactness} & \Rightarrow & \text{closedness} \end{array} . \tag{2.29}$$

However, the inverse may not always be true. This is the content of *Poincaré's lemma*:

$$\begin{array}{ccc} \mathrm{d}A = 0 & \Rightarrow & A = \mathrm{d}B \\ \text{closedness} & \Rightarrow & \text{exactness} \end{array} \quad \text{only } \textit{locally}! \tag{2.30}$$

The key word here is "locally". The exact statement of the lemma defines what "locally" means in a mathematically precise way, which will be omitted here. The basic idea is that, whether the statement is true globally (i.e. everywhere on the entire manifold) depends on the *topology* of the manifold. In other words, non-trivial topology of the manifold $\mathcal{M}$ is the obstruction that prevents the statement from being true globally. For trivial topology of $\mathbb{R}^{m,n-m}$, it is globally true, but for non-trivial topology like torus, it is not true. The failure of this statement is a probe of the topology of the manifold $\mathcal{M}$. Therefore, we see how differential forms and exterior derivative can be used to study topology. This subject is called *de Rham cohomology* in mathematics.

This subject has become an increasingly popular subject in theoretical physics over the last few decades due to its close relationship to gauge theory. In Maxwell's electrodynamics, the simplest example of a gauge theory, the electromagnetic field F is a closed 2-form, closed due to Bianchi identity $\mathrm{d}F = 0$. In physics, we say this means we can express the electromagnetic field in terms of the gauge potential 1-form A, i.e. write $F = \mathrm{d}A$. This is basically the statement of Poincaré's lemma. Now we see there is an underlying assumption here: the topology of the manifold is trivial, which is true in this case because physically our space is Minkowskian $\mathbb{R}^{3,1}$, which is indeed topologically trivial. Moreover, if the electromagnetic field vanishes identically $F = 0$, i.e. $\mathrm{d}A = 0$, then we know the gauge potential is of pure gauge form $A = \mathrm{d}\vartheta$, i.e. $A_\mu = \partial_\mu \vartheta$ where ϑ is the gauge parameter (a 0-form). This is again Poincaré's lemma. If we live in a manifold of non-trivial topology like a higher dimensional torus, these statements break down: a vanishing field strength F may be associated with a non-trivial gauge potential A which is not of pure gauge form. The classification of A in this situation is intimately related to certain aspects (e.g. homology) of the topology of the manifold $\mathcal{M}$.

Let us quantify the above story.[1] As we will be talking about cohomology, you may wonder what *homology* is. That is a way to study the topology of a given manifold. We will not discuss it here. You can read Chapter 3 and Section 6.1.1 in [1] for a quick introduction. There is a duality between homology and cohomology, just like the duality between tangent space (vectors) and cotangent space (1-forms). This is the content of *de Rham's theorem*, see Section 6.2.2 in [1]. Now let us talk about cohomology. It is essentially about the exactness and closedness of forms. Mathematicians define the following concepts

[1] Since we will use electromagnetism as an example to illustrate the ideas, it is better for the readers to study the later section about electromagnetism at this point to get familiar with the language, and then come back to read the rest of this section.

- p^{th} *cocycle group* $Z^p(\mathcal{M})$: the space of closed p-forms, i.e.

$$Z^p(\mathcal{M}) = \left\{ A^{(p)} \mid \mathrm{d}A^{(p)} = 0 \right\} = \ker \mathrm{d}_{p+1}. \tag{2.31}$$

  Here *kernel* "ker" means the set of all elements which are mapped to the zero element under a certain map. The map here is d on p-forms, which yields $(p+1)$-forms. $\ker \mathrm{d}_{p+1}$ means all p-forms that are mapped to the zero $(p+1)$-form under d. In the context of electromagnetism:

  - $Z^1(\mathcal{M})$ is the space of gauge potential A which corresponds to vanishing electromagnetic field $F = \mathrm{d}A = 0$.
  - $Z^2(\mathcal{M})$ is the space of electromagnetic field F, as the Bianchi identity $\mathrm{d}F = 0$ has to be satisfied.

- p^{th} *coboundary group* $B^p(\mathcal{M})$: the space of exact p-forms, i.e.

$$B^p(\mathcal{M}) = \left\{ A^{(p)} \mid A^{(p)} = \mathrm{d}B^{(p-1)} \right\} = \operatorname{im} \mathrm{d}_p. \tag{2.32}$$

  Here *image* "im" means the set of all outputs of a set under a map. The map here is d on $(p-1)$-forms, which yields p-forms. $\operatorname{im} \mathrm{d}_p$ means the whole set of p-forms that all $(p-1)$-forms are mapped to under d. Sometimes $B^p(\mathcal{M})$ is also denoted as $\mathrm{d} \bigwedge^{p-1}(\mathcal{M})$. In the context of electromagnetism:

  - $B^1(\mathcal{M})$ is the space of gauge potential A which is of pure gauge form, i.e. $A_\mu = \partial_\mu \vartheta$ where the scalar ϑ is the gauge parameter.
  - $B^2(\mathcal{M})$ is the space of electromagnetic field F for which a gauge potential A exists globally, i.e. all F that can be written in the form $F = \mathrm{d}A$. If $F \notin B^2(\mathcal{M})$, then there is not a gauge potential that exists globally such that $F = \mathrm{d}A$ everywhere.

(2.29) implies that $B^p(\mathcal{M}) \subseteq Z^p(\mathcal{M})$ is always true. A way to quantify the difference of these two groups is their quotient. This is the most important concept here, the p^{th} *de Rham cohomology group* $H^p(\mathcal{M})$, which is the quotient of the above two groups

$$\begin{aligned} H^p(\mathcal{M}) &= Z^p(\mathcal{M}) / B^p(\mathcal{M}) = \ker \mathrm{d}_{p+1} / \operatorname{im} \mathrm{d}_p \\ &= \left\{ A^{(p)} \mid \mathrm{d}A^{(p)} = 0 \text{ and } A^{(p)} \sim A^{(p)} + \mathrm{d}B^{(p-1)} \right\}. \end{aligned} \tag{2.33}$$

Here "$\sim$" means we identify two p-forms to be *equivalent* (also called *cohomologous*) if they differ by an exact form. $H^p(\mathcal{M})$ is the space of *inequivalent* closed p-forms. When the topology of the manifold is trivial, Poincaré's lemma (2.30) holds globally, which means $Z^p(\mathcal{M}) \subseteq B^p(\mathcal{M})$, then $Z^p(\mathcal{M}) = B^p(\mathcal{M})$, thus their quotient is unity which contains only the identity element under addition, that is the zero p-form (i.e. all components are zero). In this case, we can write $H^p(\mathcal{M}) = \{0\}$, meaning all closed p-forms on $\mathcal{M}$ are equivalent to the zero p-form, i.e. they are all exact. This is precisely the statement in Poincaré's lemma (2.30). Thus $H^p(\mathcal{M}) \neq \{0\}$ represents exactly the obstruction that prevents Poincaré's lemma (2.30) from being true globally. By definition, $H^p(\mathcal{M})$ is trivial when $p \leqslant -1$ or $p \geqslant n+1$.

In the context of electromagnetism:

- $H^1(\mathcal{M})$ is the space of gauge-inequivalent potential A which yields vanishing electromagnetic field. We know that in our world, the Minkowskian space $\mathbb{R}^{3,1}$, if the electromagnetic field vanishes, then the gauge potential is of pure gauge form (i.e. exact). This means the gauge potential is equivalent to zero up to a total derivative of the gauge parameter. In the above language, this is written as $H^1\left(\mathbb{R}^{3,1}\right) = \{0\}$. This is because $\mathbb{R}^{3,1}$ has trivial topology. If we consider electromagnetism on a manifold of non-trivial topology, say a torus, then this is not true anymore: a vanishing electromagnetic field can be associated with multiple gauge potentials that are not of pure gauge form, i.e. $H^1(\mathcal{M}) \neq \{0\}$. In this case, there will be multiple gauge-inequivalent potentials that all have vanishing electromagnetic field.

- $H^2(\mathcal{M})$ is the space of electromagnetic field F inequivalent to $\mathrm{d}A$, i.e. $F_{\mu\nu} \not\approx \partial_\mu A_\nu - \partial_\nu A_\mu$. In Minkowskian space $\mathbb{R}^{3,1}$, all electromagnetic field configurations can be written as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in terms of a potential A_μ, thus they are all cohomologous/equivalent to zero electromagnetic field. Again this is the statement of Poincaré's lemma when it holds globally, due to the trivial topology of $\mathbb{R}^{3,1}$. Thus $H^2\left(\mathbb{R}^{3,1}\right) = \{0\}$. If $\mathcal{M}$ has non-trivial topology, electromagnetic field will have a part that cannot be written in the form of $\partial_\mu A_\nu - \partial_\nu A_\mu$, and $H^2(\mathcal{M}) \neq \{0\}$ parameterizes this part. This often shows up in discussions of magnetic monopoles in non-Abelian gauge theory.

For $\mathbb{R}^{m,n-m}$, its topology is trivial, hence according to Poincaré's lemma (2.30), $Z^p\left(\mathbb{R}^{m,n-m}\right) = B^p\left(\mathbb{R}^{m,n-m}\right)$, and the quotient has only the identity element

$$H^p\left(\mathbb{R}^{m,n-m}\right) = \{0\} \qquad (1 \leqslant p \leqslant n) \tag{2.34}$$

Moreover

$$H^0\left(\mathbb{R}^n\right) = \mathbb{R}. \tag{2.35}$$

In general

$$H^0(\mathcal{M}) \cong \mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R} \tag{2.36}$$

depending on the topology of $\mathcal{M}$ (see Section 6.2.1 in [1]). On a circle

$$H^0\left(\mathbb{S}^1\right) = H^1\left(\mathbb{S}^1\right) = \mathbb{R}. \tag{2.37}$$

$H^1\left(\mathbb{S}^1\right) = \mathbb{R}$ means there is a real number worth of gauge inequivalent potentials A on a circle which all yield vanishing field strength $F = \mathrm{d}A = 0$. On a torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$

$$H^0\left(\mathbb{T}^2\right) = \mathbb{R}, \qquad H^1\left(\mathbb{T}^2\right) = \mathbb{R} \oplus \mathbb{R}, \qquad H^2\left(\mathbb{T}^2\right) = \mathbb{R}. \tag{2.38}$$

On a n-sphere

$$H^0\left(\mathbb{S}^n\right) \cong H^n\left(\mathbb{S}^n\right) = \mathbb{R}, \qquad H^p\left(\mathbb{S}^n\right) = \{0\}\,. \qquad (1 \leqslant p \leqslant n-1) \tag{2.39}$$

On a *compact* manifold, the following *Poincaré duality* holds

$$H^p(\mathcal{M}) = H^{n-p}(\mathcal{M}) \,. \tag{2.40}$$

This can already be seen from the above examples.

In the BRST quantization of non-Abelian gauge theory in quantum field theory, you will encounter another example of a nilpotent operator, the BRST charge operator, which squares to zero. The BRST operator is a fermionic operator acting on fermionic fields called ghosts in physicist's language. But from mathematician's perspective, ghosts are differential forms and the BRST operator is an exterior derivative acting on these forms. Therefore, a large part of the discussions here still hold and can be directly transplanted to that context of BRST quantization. This is called *BRST cohomology.* The physical Hilbert space of a quantum non-Abelian gauge theory is made of cohomologically inequivalent classes under the nilpotent BRST operator.

3 Hodge Theory

All discussions in the previous section, other than (2.8), do not reply on the existence of a metric on the manifold. In this section, we will assume that the manifold $\mathcal{M}$ is equipped with a (pseudo-)Riemannian metric $g_{\mu\nu}$, and see what additional structures can arise in the theory. In the so-called Hodge theory, we will see that the metric never appears alone, but disguises itself in the form of Hodge star operator. All familiar operations involving the metric, such as raising and lowering indices and contractions of vectors or 1-forms, are done in the aid of the Hodge star operator. For references on topics covered in this section, see Section 7.9 in [1], and Section 2.4 in [2].

3.1 Hodge Star Operator

(2.3) shows that there is a duality between p-forms and $(n-p)$-forms. When the manifold $\mathcal{M}$ is equipped with a metric, hence a Levi-Civita *tensor*, the *Hodge star* operator is defined by its action on the p-form basis which maps it to the $(n-p)$-form basis

$$* \left(\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\right) = \frac{1}{(n-p)!} \epsilon^{\mu_1\mu_2\cdots\mu_p}{}_{\mu_{p+1}\mu_{p+2}\cdots\mu_n} \mathrm{d}x^{\mu_{p+1}} \wedge \mathrm{d}x^{\mu_{p+2}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}, \tag{3.1}$$

where $\epsilon^{\mu_1\mu_2\cdots\mu_p}{}_{\mu_{p+1}\mu_{p+2}\cdots\mu_n}$ is the Levi-Civita tensor (1.5) with first p indices raised by the inverse metric. For 0-forms, $p=0$, the basis is simply 1, then the above definition reads

$$*1 = \frac{1}{n!} \epsilon_{\mu_1\mu_2\cdots\mu_n} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} = \epsilon = \mathrm{d}V \tag{3.2}$$

where (2.8) is used. Thus 1 is dual to the Levi-Civita tensor, i.e. dual to the volume element. The Hodge dual of a p-form A is then

$$*A = * \left(\frac{1}{p!} A_{\mu_1\mu_2\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \right)$$

$$
\begin{aligned}
&= \frac{1}{p!} A_{\mu_1\mu_2\cdots\mu_p} * \left(\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\right) \\
&= \frac{1}{p!} A_{\mu_1\mu_2\cdots\mu_p} \frac{1}{(n-p)!} \epsilon^{\mu_1\mu_2\cdots\mu_p}{}_{\mu_{p+1}\mu_{p+2}\cdots\mu_n} \mathrm{d}x^{\mu_{p+1}} \wedge \mathrm{d}x^{\mu_{p+2}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \\
&= \frac{1}{(n-p)!} \left(\frac{1}{p!} A_{\mu_1\mu_2\cdots\mu_p} \epsilon^{\mu_1\mu_2\cdots\mu_p}{}_{\mu_{p+1}\mu_{p+2}\cdots\mu_n}\right) \mathrm{d}x^{\mu_{p+1}} \wedge \mathrm{d}x^{\mu_{p+2}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n},
\end{aligned}
$$

i.e.

$$
(*A)_{\mu_1\mu_2\cdots\mu_{n-p}} = \frac{1}{p!} A_{\nu_1\nu_2\cdots\nu_p} \epsilon^{\nu_1\nu_2\cdots\nu_p}{}_{\mu_1\mu_2\cdots\mu_{n-p}}, \tag{3.3}
$$

which can also be written as

$$
(*A)_{\mu_1\mu_2\cdots\mu_{n-p}} = \frac{(-1)^{p(n-p)}}{p!} \epsilon_{\mu_1\mu_2\cdots\mu_{n-p}\mu_{n-p+1}\cdots\mu_n} A^{\mu_{n-p+1}\mu_{n-p+2}\cdots\mu_n}. \tag{3.4}
$$

Applying (3.3) twice

$$
\begin{aligned}
(**A)_{\mu_1\mu_2\cdots\mu_p} &= \frac{1}{(n-p)!} (*A)_{\nu_1\nu_2\cdots\nu_{n-p}} \epsilon^{\nu_1\nu_2\cdots\nu_{n-p}}{}_{\mu_1\mu_2\cdots\mu_p} \\
&= \frac{1}{(n-p)!} \frac{1}{p!} A_{\rho_1\rho_2\cdots\rho_p} \epsilon^{\rho_1\rho_2\cdots\rho_p}{}_{\nu_1\nu_2\cdots\nu_{n-p}} \epsilon^{\nu_1\nu_2\cdots\nu_{n-p}}{}_{\mu_1\mu_2\cdots\mu_p} \\
&= \frac{(-1)^{p(n-p)}}{p!\,(n-p)!} A_{\rho_1\rho_2\cdots\rho_p} \left(\epsilon^{\rho_1\rho_2\cdots\rho_p\nu_1\nu_2\cdots\nu_{n-p}} \epsilon_{\mu_1\mu_2\cdots\mu_p\nu_1\nu_2\cdots\nu_{n-p}}\right) \\
&= \frac{(-1)^{p(n-p)}}{p!\,(n-p)!} A_{\rho_1\rho_2\cdots\rho_p} \left[\operatorname{sgn}\left(\det g_{\mu\nu}\right)(n-p)!\,\delta^{[\rho_1}_{\mu_1}\delta^{\rho_1}_{\mu_1}\cdots\delta^{\rho_1]}_{\mu_1}\right] \\
&= \operatorname{sgn}\left(\det g_{\mu\nu}\right) \frac{(-1)^{p(n-p)}}{p!} A_{\rho_1\rho_2\cdots\rho_p} \delta^{[\rho_1}_{\mu_1}\delta^{\rho_1}_{\mu_1}\cdots\delta^{\rho_1]}_{\mu_1} \\
&= \operatorname{sgn}\left(\det g_{\mu\nu}\right)(-1)^{p(n-p)} A_{\mu_1\mu_2\cdots\mu_p},
\end{aligned}
$$

where at the end we have used (1.7) and $A_{\rho_1\rho_2\cdots\rho_p}$ is totally antisymmetric, thus

$$
A_{\rho_1\rho_2\cdots\rho_p} \delta^{[\rho_1}_{\mu_1}\delta^{\rho_1}_{\mu_1}\cdots\delta^{\rho_1]}_{\mu_1} = p! A_{\rho_1\rho_2\cdots\rho_p} \delta^{\rho_1}_{\mu_1}\delta^{\rho_1}_{\mu_1}\cdots\delta^{\rho_1}_{\mu_1} = p! A_{\mu_1\mu_2\cdots\mu_p}. \tag{3.5}
$$

Therefore $**A$ maps back to A up to a sign, i.e. for a p-form A

$$
**A = \operatorname{sgn}\left(\det g_{\mu\nu}\right)(-1)^{p(n-p)} A. \tag{3.6}
$$

Hodge Star in Two-Dimensional Euclidean Space

In two-dimensional Euclidean space with coordinates (x, y), we have

$$
*\mathrm{d}x = \mathrm{d}y, \qquad *\mathrm{d}y = -\mathrm{d}x, \tag{3.7}
$$

and

$$
*1 = \mathrm{d}x \wedge \mathrm{d}y, \qquad *\left(\mathrm{d}x \wedge \mathrm{d}y\right) = 1. \tag{3.8}
$$

A 1-form $v = v_x \mathrm{d}x + v_y \mathrm{d}y$ is dual to

$$
v = v_x\left(\mathrm{d}x\right) + v_y\left(*\mathrm{d}y\right) = -v_y \mathrm{d}x + v_x \mathrm{d}y, \tag{3.9}
$$

thus (v_x, v_y) is mapped to $(-v_y, v_x)$, i.e. v_i is dual to $-\epsilon_{ij}v_j$. Furthermore, for two 1-forms a and b,

$$a \wedge b = (a_x b_y - a_y b_x)\, \mathrm{d}x \wedge \mathrm{d}y, \tag{3.10}$$

and

$$*\,(a \wedge b) = (a_x b_y - a_y b_x) * (\mathrm{d}x \wedge \mathrm{d}y) = a_x b_y - a_y b_x. \tag{3.11}$$

The cross product of two vectors $\vec{a} \times \vec{b} = a_x b_y - a_y b_x$ in two dimensions is a pseudoscalar. In particular, the curl of any vector is a pseudoscalar. For example, magnetic field in two spatial dimensions is a pseudoscalar $B = \nabla \times \vec{A} = \partial_x A_y - \partial_y A_x$.

Hodge Star in Three-Dimensional Euclidean Space

In three-dimensional Euclidean space with coordinates (x, y, z), we always have $** = 1$ on any p-form as the number $p\,(3-p)$ is always even. We have

$$*\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z, \qquad *\mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}x, \qquad *\mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y, \tag{3.12}$$

and

$$*\,(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}z, \qquad *\,(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x, \qquad *\,(\mathrm{d}z \wedge \mathrm{d}x) = \mathrm{d}y, \tag{3.13}$$

and

$$*1 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, \qquad *\,(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = 1. \tag{3.14}$$

For a 1-form $v = v_x \mathrm{d}x + v_y \mathrm{d}y + v_z \mathrm{d}z$, its Hodge dual is

$$*v = v_x\,(*\mathrm{d}x) + v_y\,(*\mathrm{d}y) + v_z\,(*\mathrm{d}z) = v_x \mathrm{d}y \wedge \mathrm{d}z + v_y \mathrm{d}z \wedge \mathrm{d}x + v_z \mathrm{d}x \wedge \mathrm{d}y. \tag{3.15}$$

Comparing with

$$*v = \frac{1}{2}\,(*v)_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j = (*v)_{xy}\, \mathrm{d}x \wedge \mathrm{d}y + (*v)_{yz}\, \mathrm{d}y \wedge \mathrm{d}z + (*v)_{zx}\, \mathrm{d}z \wedge \mathrm{d}x, \tag{3.16}$$

we can identify $(*v)_{xy} = -\,(*v)_{yx} = v_z$, $(*v)_{yz} = -\,(*v)_{zy} = v_x$ and $(*v)_{zx} = -\,(*v)_{xz} = v_y$, or in matrix form

$$(*v)_{ij} = \begin{pmatrix} 0 & v_z & -v_y \\ -v_z & 0 & v_x \\ v_y & -v_x & 0 \end{pmatrix}. \tag{3.17}$$

Thus a vector is dual to a 2-form, or an antisymmetric matrix. For any two 1-forms $a = a_x \mathrm{d}x + a_y \mathrm{d}y + a_z \mathrm{d}z$ and $b = b_x \mathrm{d}x + b_y \mathrm{d}y + b_z \mathrm{d}z$, we have

$$a \wedge b = (a_x b_y - a_y b_x)\, \mathrm{d}x \wedge \mathrm{d}y + (a_y b_z - a_z b_y)\, \mathrm{d}y \wedge \mathrm{d}z + (a_z b_x - a_x b_z)\, \mathrm{d}z \wedge \mathrm{d}x \tag{3.18}$$

and

$$\begin{aligned} *\,(a \wedge b) &= (a_x b_y - a_y b_x) * (\mathrm{d}x \wedge \mathrm{d}y) + (a_y b_z - a_z b_y) * (\mathrm{d}y \wedge \mathrm{d}z) + (a_z b_x - a_x b_z) * (\mathrm{d}z \wedge \mathrm{d}x) \\ &= (a_y b_z - a_z b_y)\, \mathrm{d}x + (a_z b_x - a_x b_z)\, \mathrm{d}y + (a_x b_y - a_y b_x)\, \mathrm{d}z, \end{aligned} \tag{3.19}$$

i.e.

$$*\,(a \wedge b) = \vec{a} \times \vec{b}, \qquad *(\vec{a} \times \vec{b}) = a \wedge b. \tag{3.20}$$

Hodge Star in 3+1 Dimensional Minkowskian Space

In 3+1 dimensional Minkowskian space with coordinates $\left(x^0 = ct, x, y, z\right)$, we have

$$\begin{aligned}
*\mathrm{d}x^0 &= -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, & *\left(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\right) &= -\mathrm{d}x^0, \\
*\mathrm{d}x &= -\mathrm{d}x^0 \wedge \mathrm{d}y \wedge \mathrm{d}z, & *\left(\mathrm{d}x^0 \wedge \mathrm{d}y \wedge \mathrm{d}z\right) &= -\mathrm{d}x, \\
*\mathrm{d}y &= \mathrm{d}x^0 \wedge \mathrm{d}x \wedge \mathrm{d}z, & *\left(\mathrm{d}x^0 \wedge \mathrm{d}x \wedge \mathrm{d}z\right) &= \mathrm{d}y, \\
*\mathrm{d}z &= -\mathrm{d}x^0 \wedge \mathrm{d}x \wedge \mathrm{d}y, & *\left(\mathrm{d}x^0 \wedge \mathrm{d}x \wedge \mathrm{d}y\right) &= -\mathrm{d}z,
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
*\left(\mathrm{d}x^0 \wedge \mathrm{d}x\right) &= -\mathrm{d}y \wedge \mathrm{d}z, & *\left(\mathrm{d}x \wedge \mathrm{d}y\right) &= \mathrm{d}x^0 \wedge \mathrm{d}z, \\
*\left(\mathrm{d}x^0 \wedge \mathrm{d}y\right) &= \mathrm{d}x \wedge \mathrm{d}z, & *\left(\mathrm{d}x \wedge \mathrm{d}z\right) &= -\mathrm{d}x^0 \wedge \mathrm{d}y, \\
*\left(\mathrm{d}x^0 \wedge \mathrm{d}z\right) &= -\mathrm{d}x \wedge \mathrm{d}y, & *\left(\mathrm{d}y \wedge \mathrm{d}z\right) &= \mathrm{d}x^0 \wedge \mathrm{d}x,
\end{aligned} \tag{3.22}$$

and

$$*1 = \mathrm{d}x^0 \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, \qquad *\left(\mathrm{d}x^0 \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\right) = -1. \tag{3.23}$$

For a 2-form

$$\begin{aligned}
F =& \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
=& F_{0x} \mathrm{d}x^0 \wedge \mathrm{d}x + F_{0y} \mathrm{d}x^0 \wedge \mathrm{d}y + F_{0z} \mathrm{d}x^0 \wedge \mathrm{d}z \\
&+ F_{xy} \mathrm{d}x \wedge \mathrm{d}y + F_{yz} \mathrm{d}y \wedge \mathrm{d}z + F_{zx} \mathrm{d}z \wedge \mathrm{d}x,
\end{aligned} \tag{3.24}$$

its Hodge dual is

$$\begin{aligned}
*F =& F_{0x} *\left(\mathrm{d}x^0 \wedge \mathrm{d}x\right) + F_{0y} *\left(\mathrm{d}x^0 \wedge \mathrm{d}y\right) + F_{0z} *\left(\mathrm{d}x^0 \wedge \mathrm{d}z\right) \\
&+ F_{xy} *\left(\mathrm{d}x \wedge \mathrm{d}y\right) + F_{yz} *\left(\mathrm{d}y \wedge \mathrm{d}z\right) + F_{zx} *\left(\mathrm{d}z \wedge \mathrm{d}x\right) \\
=& - F_{0x} \mathrm{d}y \wedge \mathrm{d}z + F_{0y} \mathrm{d}x \wedge \mathrm{d}z - F_{0z} \mathrm{d}x \wedge \mathrm{d}y \\
&+ F_{xy} \mathrm{d}x^0 \wedge \mathrm{d}z + F_{yz} \mathrm{d}x^0 \wedge \mathrm{d}x + F_{zx} \mathrm{d}x^0 \wedge \mathrm{d}y.
\end{aligned} \tag{3.25}$$

Thus we can identify

$$\begin{aligned}
(*F)_{0x} &= F_{yz}, & (*F)_{xy} &= F_{z0}, \\
(*F)_{0y} &= F_{zx}, & (*F)_{yz} &= F_{x0}, \\
(*F)_{0z} &= F_{xy}, & (*F)_{zx} &= F_{y0},
\end{aligned} \tag{3.26}$$

i.e.

$$(*F)_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \tag{3.27}$$

which can be written as

$$(*F)_{0i} = \frac{1}{2} \epsilon_{ijk} F_{jk}, \qquad (*F)_{ij} = \frac{1}{2} \epsilon_{ijk} F_{k0}. \tag{3.28}$$

Notice that

$$* * F = -F. \tag{3.29}$$

In four dimensions, there are 2-forms that are invariant under the action of $*$. Using

$$* \left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) = \frac{1}{2}\epsilon^{\mu\nu}{}_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma, \tag{3.30}$$

we define

$$\omega_{\mp}^{\mu\nu} = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \pm i * \left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \pm \frac{i}{2}\epsilon^{\mu\nu}{}_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma. \tag{3.31}$$

Then

$$\begin{aligned}
*\omega_{\mp}^{\mu\nu} &= * \left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) \pm \frac{i}{2}\epsilon^{\mu\nu}{}_{\rho\sigma} * \left(\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\right) \\
&= * \left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) \pm \frac{i}{4}\epsilon^{\mu\nu}{}_{\rho\sigma}\epsilon^{\rho\sigma}{}_{\eta\lambda}\mathrm{d}x^\eta \wedge \mathrm{d}x^\lambda \\
&= * \left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) \pm \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\eta\lambda\rho\sigma}\mathrm{d}x^\eta \wedge \mathrm{d}x^\lambda \\
&= * \left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) \pm \frac{i}{4}(-2\delta^\mu_{[\eta}\delta^\nu_{\lambda]})\mathrm{d}x^\eta \wedge \mathrm{d}x^\lambda \\
&= * \left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) \mp i\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \mp i \left[\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \pm i * \left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right)\right] = \mp i\omega_{\mp}^{\mu\nu},
\end{aligned}$$

i.e.

$$*\omega_{\pm}^{\mu\nu} = \pm i\omega_{\pm}^{\mu\nu}. \tag{3.32}$$

We say $\omega_{\pm}^{\mu\nu}$ are *self-dual* (SD) and *anti-self-dual* (ASD) 2-form bases. Similarly, let us define the self-dual and anti-self-dual 2-forms from F

$$F_\pm = F \mp i * F, \tag{3.33}$$

then

$$*F_\pm = *F \mp i * *F = *F \pm iF = \pm i\left(F \mp i * F\right),$$

i.e.

$$*F_\pm = \pm iF_\pm. \tag{3.34}$$

F can be decomposed into the sum of SD and ASD parts

$$F = \frac{1}{2}\left(F_+ + F_-\right). \tag{3.35}$$

Notice that $F_\pm$ are complex.

3.2 Inner Product

For two p-forms A and B, $*B$ is a $(n-p)$-form, then $A \wedge *B$ is an n-form, which is proportional to the volume element or Levi-Civita tensor ϵ, and it is dual to a scalar which is precisely the inner product of A and B. We will show this now. We have

$$A \wedge *B = \left(\frac{1}{p!}A_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\right) \wedge * \left(\frac{1}{p!}B_{\nu_1\cdots\nu_p}\mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_p}\right)$$

$$
\begin{aligned}
&= \frac{1}{(p!)^2} A_{\mu_1\cdots\mu_p} B_{\nu_1\cdots\nu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge * \left(\mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_p}\right) \\
&= \frac{1}{(p!)^2} A_{\mu_1\cdots\mu_p} B_{\nu_1\cdots\nu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge \frac{1}{(n-p)!} \epsilon^{\nu_1\cdots\nu_p}{}_{\mu_{p+1}\mu_{p+2}\cdots\mu_n} \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \\
&= \frac{1}{(n-p)!\,(p!)^2} \epsilon_{\nu_1\cdots\nu_p\mu_{p+1}\mu_{p+2}\cdots\mu_n} A_{\mu_1\cdots\mu_p} B^{\nu_1\cdots\nu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}.
\end{aligned}
$$

Using (2.7) to rewrite $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} = \hat{\epsilon}^{\mu_1\cdots\mu_n} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, and then (1.5), we can write the above expression as

$$
\begin{aligned}
A \wedge *B &= \frac{1}{(n-p)!\,(p!)^2} \epsilon_{\nu_1\cdots\nu_p\mu_{p+1}\mu_{p+2}\cdots\mu_n} A_{\mu_1\cdots\mu_p} B^{\nu_1\cdots\nu_p} \hat{\epsilon}^{\mu_1\cdots\mu_n} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \frac{\sqrt{|\det g_{\mu\nu}|}}{(n-p)!\,(p!)^2} \left(\hat{\epsilon}_{\nu_1\cdots\nu_p\mu_{p+1}\mu_{p+2}\cdots\mu_n} \hat{\epsilon}^{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_n}\right) A_{\mu_1\cdots\mu_p} B^{\nu_1\cdots\nu_p} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \\
&= \frac{\sqrt{|\det g_{\mu\nu}|}}{(n-p)!\,(p!)^2} (n-p)!\, \delta^{\mu_1}_{[\nu_1} \delta^{\mu_2}_{\nu_2} \cdots \delta^{\mu_p}_{\nu_p]} A_{\mu_1\cdots\mu_p} B^{\nu_1\cdots\nu_p} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \frac{\sqrt{|\det g_{\mu\nu}|}}{(p!)^2} p!\, \delta^{\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} \cdots \delta^{\mu_p}_{\nu_p} A_{\mu_1\cdots\mu_p} B^{\nu_1\cdots\nu_p} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \frac{1}{p!} A_{\mu_1\cdots\mu_p} B^{\mu_1\cdots\mu_p} \sqrt{|\det g_{\mu\nu}|} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.
\end{aligned}
$$

Using (2.8) to identify $\sqrt{|\det g_{\mu\nu}|} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \mathrm{d}V = \epsilon$ as the volume element/Levi-Civita symbol, we have

$$
A \wedge *B = \frac{1}{p!} A_{\mu_1\cdots\mu_p} B^{\mu_1\cdots\mu_p} \epsilon, \tag{3.36}
$$

and

$$
\langle A, B \rangle = \int A \wedge *B = \frac{1}{p!} \int A_{\mu_1\cdots\mu_p} B^{\mu_1\cdots\mu_p} \mathrm{d}V. \tag{3.37}
$$

This relation is the definition of the inner product $\langle\ ,\ \rangle$. The inner product is symmetric

$$
A \wedge *B = B \wedge *A. \tag{3.38}
$$

3.3 Adjoint Exterior Derivative and Integration by Parts

For a p-form A, $*A$ is a $(n-p)$-form, $\mathrm{d} * A$ is a $(n-p+1)$-form, then $*\mathrm{d} * A$ is a $(p-1)$-form. Let us calculate its components. First, (3.3) yields

$$
(*A)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!} A_{\nu_1\cdots\nu_p} \epsilon^{\nu_1\cdots\nu_p}{}_{\mu_1\cdots\mu_{n-p}} = \frac{(-1)^{p(n-p)}}{p!} A^{\nu_1\cdots\nu_p} \epsilon_{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p}.
$$

Then applying (2.21)

$$
(\mathrm{d} * A)_{\rho\mu_1\cdots\mu_{n-p}} = \frac{(-1)^{p(n-p)}}{(n-p)!p!} \partial_{[\rho} A^{\nu_1\cdots\nu_p} \epsilon_{\mu_1\cdots\mu_{n-p}]\nu_1\cdots\nu_p}.
$$

Now applying (3.3) to this $(n-p+1)$-form to get the $(p-1)$-form

$$
(*\mathrm{d} * A)_{\mu_1\cdots\mu_{p-1}} = \frac{1}{(n-p+1)!} \epsilon^{\rho\sigma_1\cdots\sigma_{n-p}}{}_{\mu_1\cdots\mu_{p-1}} \left[\frac{(-1)^{p(n-p)}}{(n-p)!p!} \partial_{[\rho} A^{\nu_1\cdots\nu_p} \epsilon_{\sigma_1\cdots\sigma_{n-p}]\nu_1\cdots\nu_p}\right]
$$

$$
\begin{aligned}
&= \frac{(-1)^{p(n-p)}}{(n-p)!p!} \epsilon^{\rho\sigma_1\cdots\sigma_{n-p}}{}_{\mu_1\cdots\mu_{p-1}} \partial_\rho \left(A^{\nu_1\cdots\nu_p} \epsilon_{\sigma_1\cdots\sigma_{n-p}\nu_1\cdots\nu_p} \right) \\
&= \frac{(-1)^{(p+1)(n-p)}}{(n-p)!p!} \epsilon^{\sigma_1\cdots\sigma_{n-p}\rho}{}_{\mu_1\cdots\mu_{p-1}} \partial_\rho \left(A^{\nu_1\cdots\nu_p} \epsilon_{\sigma_1\cdots\sigma_{n-p}\nu_1\cdots\nu_p} \right).
\end{aligned}
$$

Notice that $(p+1)\,p$ is always an even number, thus $(-1)^{(p+1)(-p)} = 1$, and $(-1)^{(p+1)(n-p)} = (-1)^{n(p+1)}$. Then using (1.5) and (1.6)

$$
\begin{aligned}
(*\mathrm{d} * A)^{\mu_1\cdots\mu_{p-1}} &= \frac{(-1)^{n(p+1)}}{(n-p)!p!} \epsilon^{\sigma_1\cdots\sigma_{n-p}\rho\mu_1\cdots\mu_{p-1}} \partial_\rho \left(A^{\nu_1\nu_2\cdots\nu_p} \epsilon_{\sigma_1\cdots\sigma_{n-p}\nu_1\cdots\nu_p} \right) \\
&= \frac{(-1)^{n(p+1)}}{(n-p)!p!} \frac{\operatorname{sgn}(\det g_{\mu\nu})}{\sqrt{|\det g_{\mu\nu}|}} \hat{\epsilon}^{\sigma_1\cdots\sigma_{n-p}\rho\mu_1\cdots\mu_{p-1}} \partial_\rho \left(A^{\nu_1\cdots\nu_p} \sqrt{|\det g_{\mu\nu}|} \right) \hat{\epsilon}_{\sigma_1\cdots\sigma_{n-p}\nu_1\cdots\nu_p}.
\end{aligned}
$$

The first $n-p$ indices of the Levi-Civita symbols are contracted

$$
\hat{\epsilon}^{\sigma_1\cdots\sigma_{n-p}\nu_1\cdots\nu_p} \hat{\epsilon}_{\sigma_1\cdots\sigma_{n-p}\mu_1\cdots\mu_{p-1}\rho} = (n-p)!\delta^{\rho}_{[\nu_1}\delta^{\mu_1}_{\nu_2}\cdots\delta^{\mu_{p-1}}_{\nu_p]}.
$$

Then

$$
\begin{aligned}
(*\mathrm{d} * A)^{\mu_1\cdots\mu_{p-1}} &= \frac{(-1)^{n(p+1)}}{p!} \frac{\operatorname{sgn}(\det g_{\mu\nu})}{\sqrt{|\det g_{\mu\nu}|}} \delta^{\rho}_{[\nu_1}\delta^{\mu_1}_{\nu_2}\cdots\delta^{\mu_{p-1}}_{\nu_p]} \partial_\rho \left(\sqrt{|\det g_{\mu\nu}|} A^{\nu_1\cdots\nu_p} \right) \\
&= (-1)^{n(p+1)} \frac{\operatorname{sgn}(\det g_{\mu\nu})}{\sqrt{|\det g_{\mu\nu}|}} \delta^{\rho}_{\nu_1}\delta^{\mu_1}_{\nu_2}\cdots\delta^{\mu_{p-1}}_{\nu_p} \partial_\rho \left(\sqrt{|\det g_{\mu\nu}|} A^{\nu_1\cdots\nu_p} \right) \\
&= (-1)^{n(p+1)} \frac{\operatorname{sgn}(\det g_{\mu\nu})}{\sqrt{|\det g_{\mu\nu}|}} \partial_\rho \left(\sqrt{|\det g_{\mu\nu}|} A^{\rho\mu_1\cdots\mu_{p-1}} \right)
\end{aligned}
$$

Now lowering the open indices, we have

$$
(*\mathrm{d} * A)_{\mu_1\cdots\mu_{p-1}} = (-1)^{n(p+1)} \frac{\operatorname{sgn}(\det g_{\mu\nu})}{\sqrt{|\det g_{\mu\nu}|}} g_{\mu_1\nu_1}\cdots g_{\mu_{p-1}\nu_{p-1}} \partial_\rho \left(\sqrt{|\det g_{\mu\nu}|} A^{\rho\nu_1\cdots\nu_{p-1}} \right), \tag{3.39}
$$

where

$$
A^{\rho\nu_1\cdots\nu_{p-1}} = g^{\rho\sigma} g^{\nu_1\lambda_1}\cdots g^{\nu_{p-1}\lambda_{p-1}} A_{\sigma\lambda_1\cdots\lambda_{p-1}}. \tag{3.40}
$$

Under the inner product (3.37), the *adjoint exterior derivative* operator δ (sometimes denoted as $\mathrm{d}^\dagger$ in analogy to quantum mechanics) is defined as

$$
\langle \mathrm{d}A, B \rangle = \langle A, \delta B \rangle + \text{boundary terms}. \tag{3.41}
$$

This means under integration by parts, d becomes δ. Here let B be a p-form, then A is a $(p-1)$-form. To see the relation between δ and $*\mathrm{d}*$, we start with

$$
\mathrm{d}\,(A \wedge *B) = \mathrm{d}A \wedge *B + (-1)^{p-1} A \wedge \mathrm{d} * B.
$$

Here $\mathrm{d} * B$ is a $(n-p+1)$-form, then (3.6) applied to $\mathrm{d} * B$ can be written as

$$
\mathrm{d} * B = \operatorname{sgn}(\det g_{\mu\nu})\,(-1)^{(p-1)(n-p+1)} * * \mathrm{d} * B.
$$

Notice that $(p-1)\,p$ is an even number, thus $(-1)^{-(p-1)p}=1$. Substitute this in, we have

$$\mathrm{d}\left(A\wedge *B\right)=\mathrm{d}A\wedge *B+\mathrm{sgn}\left(\det g_{\mu\nu}\right)(-1)^{n(p-1)}\,A\wedge **\,\mathrm{d}*B.$$

Integrate it, the left hand side is a total derivative, which yields boundary terms. Thus we have

$$\int \mathrm{d}A\wedge *B=\int A\wedge *\left[-\mathrm{sgn}\left(\det g_{\mu\nu}\right)(-1)^{n(p-1)}*\mathrm{d}*\right]B+\text{boundary terms}.$$

Thus those inside $[\cdots]$ shall be identified as δ

$$\delta=-\mathrm{sgn}\left(\det g_{\mu\nu}\right)(-1)^{n(p+1)}*\mathrm{d}*\,. \tag{3.42}$$

Notice that we have used $(-1)^{n(p-1)}=(-1)^{n(p+1)}$ here. In components, the action of δ is

$$(\delta A)_{\mu_1\cdots\mu_{p-1}}=-\frac{1}{\sqrt{|\det g_{\mu\nu}|}}g_{\mu_1\nu_1}\cdots g_{\mu_{p-1}\nu_{p-1}}\partial_\rho\left(\sqrt{|\det g_{\mu\nu}|}A^{\rho\nu_1\cdots\nu_{p-1}}\right). \tag{3.43}$$

We see the effect of δ or $*\mathrm{d}*$ acting on a form is to contract ∂_ρ with one index of the form. It does not matter which one, because they are all antisymmetric thus all come equal. Therefore, $-\delta$ can be viewed as the generalization of the *divergence*, as opposed to d which is a generalization of the *curl*.

One immediate consequence of (3.42) is that δ is nilpotent as well

$$\delta^2=0. \tag{3.44}$$

This can be easily seen because $\delta\sim *\mathrm{d}*$ and $**\sim 1$, thus $\delta^2\sim *\mathrm{d}**\mathrm{d}*\sim *\mathrm{d}^2*=0$ follows immediately from $\mathrm{d}^2=0$. Furthermore, for a p-form A, because $\mathrm{d}*A$ is a $(n-p+1)$-form, it is only non-vanishing if $p\geqslant 1$, thus δ acting on a 0-form vanishes identically. This is not hard to understand because the divergence of a scalar is not defined. Therefore we have the following sequence

$$0\xleftarrow{\delta}\bigwedge\nolimits^0(\mathcal{M})\xleftarrow{\delta}\bigwedge\nolimits^1(\mathcal{M})\xleftarrow{\delta}\bigwedge\nolimits^2(\mathcal{M})\xleftarrow{\delta}\cdots\xleftarrow{\delta}\bigwedge\nolimits^{n-1}(\mathcal{M})\xleftarrow{\delta}\bigwedge\nolimits^{n}(\mathcal{M})\hookleftarrow 0. \tag{3.45}$$

3.4 Laplacian

The Laplacian is defined as

$$\Delta=(\mathrm{d}+\delta)^2=\delta\mathrm{d}+\mathrm{d}\delta. \tag{3.46}$$

This is obviously *self-adjoint*

$$\langle\Delta A,B\rangle=\langle A,\Delta B\rangle+\text{boundary terms}, \tag{3.47}$$

as $\mathrm{d}+\delta=\mathrm{d}+\mathrm{d}^\dagger$ is: $\left(\mathrm{d}+\mathrm{d}^\dagger\right)^\dagger=\mathrm{d}^\dagger+\mathrm{d}$.

The component expression for Δ is in general lengthy. Here we only work out two simplest cases which are the most useful ones. First, if $A=\phi$ is a 0-form (scalar), $\delta\phi=0$ thus $\Delta\phi=\delta\mathrm{d}\phi$. From $(\mathrm{d}\phi)_\mu=\partial_\mu\phi$, we have

$$\Delta\phi=-\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right),\qquad g=\det g_{\mu\nu}. \tag{3.48}$$

This is indeed the familiar expression for Laplacian in curved space.

If A is a 1-form, then

$$\begin{aligned} (\mathrm{d}A)_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ \delta A &= -\frac{1}{\sqrt{|g|}}\partial_\rho\left(\sqrt{|g|}g^{\rho\sigma}A_\sigma\right), \end{aligned} \tag{3.49}$$

and

$$\begin{aligned} (\delta\mathrm{d}A)_\mu &= -\frac{1}{\sqrt{|g|}}g_{\mu\nu}\partial_\rho\left(\sqrt{|g|}g^{\rho\sigma}g^{\nu\lambda}\partial_{[\sigma}A_{\lambda]}\right), \\ (\mathrm{d}\delta A)_\mu &= -\frac{1}{\sqrt{|g|}}\partial_\mu\partial_\rho\left(\sqrt{|g|}g^{\rho\sigma}A_\sigma\right). \end{aligned} \tag{3.50}$$

These are still not very illuminating. To further simplify them, let us assume the manifold is *flat*, thus $|g| = |\det g_{\mu\nu}| = 1$ and the metric can be pulled out of the derivatives. Now we have

$$\begin{aligned} (\mathrm{d}A)_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, & (\delta\mathrm{d}A)_\mu &= -\partial^2 A_\mu + \partial_\mu\left(\partial\cdot A\right), \\ \delta A &= -\partial\cdot A, & (\mathrm{d}\delta A)_\mu &= -\partial_\mu\left(\partial\cdot A\right), \end{aligned} \tag{3.51}$$

where $\partial^2 = g^{\rho\sigma}\partial_\rho\partial_\sigma$ and $\partial\cdot A = g^{\rho\sigma}\partial_\rho A_\sigma$, then (in flat space)

$$(\Delta A)_\mu = -\partial^2 A_\mu. \tag{3.52}$$

In these expressions, we recognize $-\delta A$ as the Lorentz gauge condition, $\delta\mathrm{d}A$ as part of Maxwell's equations, and ΔA as the wave equation for vector potential.

For general p-forms in *flat space*

$$\begin{aligned} (\mathrm{d}A)_{\rho\mu_1\cdots\mu_p} &= \frac{1}{p!}\partial_{[\rho}A_{\mu_1\cdots\mu_p]}, \\ (\delta A)_{\mu_1\cdots\mu_{p-1}} &= -g^{\rho\sigma}\partial_\rho A_{\sigma\mu_1\cdots\mu_{p-1}}. \end{aligned} \tag{3.53}$$

Then

$$\begin{aligned} (\delta\mathrm{d}A)_{\mu_1\cdots\mu_p} &= -\frac{1}{p!}g^{\rho\sigma}\partial_\rho\partial_{[\sigma}A_{\mu_1\cdots\mu_p]}, \\ (\mathrm{d}\delta A)_{\mu_1\cdots\mu_p} &= \frac{(-1)^p}{(p-1)!}g^{\rho\sigma}\partial_\rho\partial_{[\mu_1}A_{\mu_2\cdots\mu_p]\sigma}. \end{aligned} \tag{3.54}$$

Writing the first line as

$$\begin{aligned} (\delta\mathrm{d}A)_{\mu_1\cdots\mu_p} &= -\frac{1}{p!}g^{\rho\sigma}\partial_\rho\partial_\sigma A_{[\mu_1\cdots\mu_p]} - \frac{(-1)^p\,p}{p!}g^{\rho\sigma}\partial_\rho\partial_{[\mu_1}A_{\mu_2\cdots\mu_p]\sigma} \\ &= -g^{\rho\sigma}\partial_\rho\partial_\sigma A_{\mu_1\cdots\mu_p} - \frac{(-1)^p}{(p-1)!}g^{\rho\sigma}\partial_\rho\partial_{[\mu_1}A_{\mu_2\cdots\mu_p]\sigma}, \end{aligned}$$

we see the second term is precisely the opposite of $(\mathrm{d}\delta A)_{\mu_1\cdots\mu_p}$, thus adding them together, we have (in flat space)

$$(\Delta A)_{\mu_1\cdots\mu_p} = -g^{\rho\sigma}\partial_\rho\partial_\sigma A_{\mu_1\cdots\mu_p}. \tag{3.55}$$

Hence we conclude that *in flat space* we always have

$$\Delta = -\partial^2 = -g^{\mu\nu}\partial_\mu\partial_\nu \tag{3.56}$$

acting on any forms. If you want to obtain a compact expression for Δ in curved space, it is better to write the expression in terms of the metric-compatible covariant derivative (Levi-Civita connection) defined in Riemannian geometry.

3.5 Hodge Decomposition

Similar to $\mathrm{d}^2 = 0$, we have $\delta^2 = 0$ here. This means we can define a similar set of notions similar to closedness and exactness.

- If a p-form A satisfies $\delta A = 0$, it is called a *coclosed* p-form. Such a form has vanishing divergence.
- If a p-form A can be written *globally* as $A = \delta B$ in terms of a certain $(p+1)$-form B, it is called a *coexact* p-form. Such a form is the divergence of another form. The space of all coexact p-forms is denoted as $\delta \bigwedge^{p+1}(\mathcal{M})$.
- If a p-form A satisfies $\Delta A = 0$, it is called a *harmonic* p-form. The space of all harmonic p-forms is denoted as $\mathrm{Harm}^p(\mathcal{M})$.

A theorem says that a form is harmonic if and only if it is both closed and coclosed, i.e.

$$\Delta A = 0 \qquad \Leftrightarrow \qquad \mathrm{d}A = 0 \ \text{ and } \ \delta A = 0. \tag{3.57}$$

The *Hodge decomposition theorem* says that, on a compact orientable Riemannian manifold $\mathcal{M}$ without a boundary, any p-form $\omega^{(p)}$ can be uniquely decomposed globally as a sum of an exact form $\mathrm{d}\alpha^{(p-1)}$, a coexact form $\delta\beta^{(p+1)}$ and a harmonic form $\gamma^{(p)}$

$$\omega^{(p)} = \mathrm{d}\alpha^{(p-1)} + \delta\beta^{(p+1)} + \gamma^{(p)}, \qquad \Delta\gamma^{(p)} = 0, \tag{3.58}$$

i.e.

$$\bigwedge^p(\mathcal{M}) = \mathrm{d}\bigwedge^{p-1}(\mathcal{M}) \oplus \delta\bigwedge^{p+1}(\mathcal{M}) \oplus \mathrm{Harm}^p(\mathcal{M}). \tag{3.59}$$

Furthermore, in this case, *Hodge's theorem* says that there is an isomorphism between the p^{th} de Rham cohomology group and the p^{th} harmonic space

$$H^p(\mathcal{M}) \cong \mathrm{Harm}^p(\mathcal{M}), \tag{3.60}$$

and their dimension is the Betti number

$$b^p = \dim H^p(\mathcal{M}) = \dim \mathrm{Harm}^p(\mathcal{M}). \tag{3.61}$$

4 Stokes' Theorem

The generalization of Stokes' theorem to arbitrary dimensions can be written in a compact way using differential forms

$$\int_{\mathcal{S}} \mathrm{d}\omega = \int_{\partial\mathcal{S}} \omega, \tag{4.1}$$

where $\mathcal{S}$ a p-dimensional hypersurface embedded in the n-dimensional manifold $\mathcal{M}$, whose $(p-1)$-dimensional boundary is denoted as $\partial\mathcal{S}$, and ω is a $(p-1)$-form. *p-forms can only be integrated over p-dimensional hypersurfaces.* In the above equation, on the left we have a p-form $\mathrm{d}\omega$ integrated over the p-dimensional hypersurface $\mathcal{S}$, while on the right we have a $(p-1)$-form ω integrated over the $(p-1)$-dimensional boundary of $\mathcal{S}$. This simple compact expression encodes many familiar identities in vector calculus. We will see a few examples below.

SIDENOTE: Sometimes mathematicians may write the theorem in a more abstract way as

$$\int_{\mathcal{S}} \Omega = \int_{\varphi(\mathcal{S})} \left(\varphi^{-1}\right)^* \Omega. \tag{4.2}$$

By matching with the previous expression, you can identify $\Omega = \mathrm{d}\omega$, and $\varphi = \partial$. Then you can identify $\left(\varphi^{-1}\right)^* = \mathrm{d}^{-1}$ and $\varphi^* = \mathrm{d}$. The "$*$" here is the notation for *pullback* in differential geometry. You can kind of identify $\partial^* = \mathrm{d}$ in this notation, which is not too surprising.

Fundamental Theorem of Calculus

This is the most trivial example of Stokes' theorem. Consider in one dimension, if $\mathcal{S} = [a,b] \subset \mathcal{M} = \mathbb{R}$ and $\omega = f$ is a 0-form, then $\mathrm{d}f = f'(x)\,\mathrm{d}x$ and $\partial\mathcal{S} = \{a\} \cup \{b\}$, and the theorem yields the fundamental theorem of calculus

$$\int_a^b f'(x)\,\mathrm{d}x = f(x)\Big|_a^b. \tag{4.3}$$

Fundamental Theorem for Curls

Let $\mathcal{S} = \Sigma$ be a two-dimensional surface embedded in the three-dimensional Euclidean space $\mathcal{M} = \mathbb{R}^3$, with one-dimensional boundary $\mathcal{C} = \partial\Sigma$. This boundary $\mathcal{C}$ is always a union of closed loops. Let A be the 1-form (say the vector potential in electromagnetism)

$$\omega = A = A_i \mathrm{d}x^i = A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z = \vec{A} \cdot \mathrm{d}\vec{x},$$

then the 2-form $\mathrm{d}\omega = \mathrm{d}A$ is given by

$$\begin{aligned} \mathrm{d}A &= \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y} + \mathrm{d}z\frac{\partial}{\partial z}\right) \wedge \left(A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z\right) \\ &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

This expression can be repackaged in a more familiar way. Let us define the vector $\vec{B}$ (the magnetic field) as

$$\vec{B} = \nabla \times \vec{A}, \qquad B^i = \epsilon^{ijk} \partial_j A_k, \tag{4.4}$$

i.e.

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \qquad B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \qquad B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}.$$

The area element vector $\mathrm{d}\vec{\Sigma}$ on the two-dimensional surface Σ can be defined as the Hodge dual of the 2-form basis on the surface Σ according to (3.3)

$$\mathrm{d}\Sigma_i = \frac{1}{2}\epsilon_{ijk}\mathrm{d}x^j \wedge \mathrm{d}x^k, \tag{4.5}$$

i.e.

$$\mathrm{d}\Sigma_x = \mathrm{d}y \wedge \mathrm{d}z, \qquad \mathrm{d}\Sigma_y = \mathrm{d}z \wedge \mathrm{d}x, \qquad \mathrm{d}\Sigma_z = \mathrm{d}x \wedge \mathrm{d}y.$$

Then the above expression for $\mathrm{d}A$ can be written as

$$\mathrm{d}A = \vec{B} \cdot \mathrm{d}\vec{\Sigma} = \left(\nabla \times \vec{A}\right) \cdot \mathrm{d}\vec{\Sigma}. \tag{4.6}$$

In fact, this relation can be directly derived in index notation using $\partial_{[i}A_{j]} = \epsilon_{ijk}B^k$

$$\begin{aligned} \mathrm{d}A &= \mathrm{d}x^i\partial_i \wedge A_j\mathrm{d}x_j = \partial_i A_j \mathrm{d}x^i \wedge \mathrm{d}x_j = \frac{1}{2}\partial_{[i}A_{j]}\mathrm{d}x^i \wedge \mathrm{d}x_j \\ &= \frac{1}{2}\epsilon_{ijk}B^k \mathrm{d}x^i \wedge \mathrm{d}x_j = B^k\left(\frac{1}{2}\epsilon_{kij}\mathrm{d}x^i \wedge \mathrm{d}x_j\right). \end{aligned}$$

We identify the last expression as $\vec{B} \cdot \mathrm{d}\vec{\Sigma}$. This tells us how to identify the area element (4.5). Then Stokes' theorem $\int_\Sigma \mathrm{d}A = \int_{\partial\Sigma} A$ reads

$$\iint_\Sigma \left(\nabla \times \vec{A}\right) \cdot \mathrm{d}\vec{\Sigma} = \oint_{\mathcal{C}} \vec{A} \cdot \mathrm{d}\vec{x}. \tag{4.7}$$

In electromagnetism, this tells us two equivalent ways to compute the magnetic flux Φ_m penetrating the area Σ whose boundary is $\mathcal{C}$

$$\Phi_\mathrm{m} = \iint_\Sigma \vec{B} \cdot \mathrm{d}\vec{\Sigma} = \oint_{\mathcal{C}} \vec{A} \cdot \mathrm{d}\vec{x}. \tag{4.8}$$

Gauss Law (Divergence Theorem)

Still in three-dimensional Euclidean space, let $\mathcal{S} = \mathcal{V} \subset \mathcal{M} = \mathbb{R}^3$ be a three-dimensional volume whose boundary is a two-dimensional surface $\Sigma = \partial\mathcal{V}$. Still, the boundary surface Σ is a union of closed surfaces. Let $\omega = E$ be a 2-form

$$\omega = E = \frac{1}{2}E_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j. \tag{4.9}$$

The 2-form is Hodge dual to a vector[2] $\vec{E} = *E$ according to (3.3)

$$E^i = \frac{1}{2}\epsilon^{ijk}E_{jk}. \tag{4.10}$$

[2]Which is also a 1-form, as there is no difference between vector and 1-form in Euclidean space.

In electromagnetism, this is the electric field $\vec{E}$. Using the inverse relation $E_{ij} = \epsilon_{ijk} E^k$

$$E = \frac{1}{2} E_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j = \frac{1}{2} \epsilon_{ijk} E^k \mathrm{d}x^i \wedge \mathrm{d}x^j = E^k \left(\frac{1}{2} \epsilon_{kij} \mathrm{d}x^i \wedge \mathrm{d}x^j \right),$$

which according to (4.5) this 2-form can be written as

$$E = \vec{E} \cdot \mathrm{d}\vec{\Sigma}. \tag{4.11}$$

On the other hand, using $** = 1$ on any forms in three dimensions, we can write

$$\mathrm{d}E = * (*\mathrm{d}*) (*E) = * (*\mathrm{d}*) \vec{E}.$$

Applying (3.42) on 1-form $\vec{E}$, we recognize $\delta = - * \mathrm{d}*$, then

$$\mathrm{d}E = - * \delta \vec{E}. \tag{4.12}$$

$\delta \vec{E}$ is a 0-form, the divergence of the 1-form $\vec{E}$: according to (3.43), $\delta \vec{E} = -\nabla \cdot \vec{E}$, thus

$$\mathrm{d}E = * \left(\nabla \cdot \vec{E} \right). \tag{4.13}$$

As $\nabla \cdot \vec{E}$ is a scalar, which can be thought as $\nabla \cdot \vec{E}$ multiplying 1, the 0-form basis, and $*$ acting on 1 yields the volume element according to (3.2)

$$\mathrm{d}E = \left(\nabla \cdot \vec{E} \right) * 1 = \left(\nabla \cdot \vec{E} \right) \mathrm{d}V.$$

The nice thing about the above derivation is that it is coordinate/component-free, which is what mathematicians like and what differential forms are intended for. You can arrive at the same conclusion by directly calculating the components

$$\begin{aligned}
\mathrm{d}E &= \mathrm{d}x^k \partial_k \wedge \frac{1}{2} E_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j = \frac{1}{2} \partial_k E_{ij} \mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j \\
&= \frac{1}{2} \partial_k \left(\hat{\epsilon}_{ijl} E^l \right) \left(\hat{\epsilon}^{kij} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \right) = \frac{1}{2} \left(\hat{\epsilon}_{lij} \hat{\epsilon}^{kij} \right) \left(\partial_k E^l \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= \frac{1}{2} \left(2\delta^k_l \right) \left(\partial_k E^l \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \left(\partial_k E^k \right) \mathrm{d}V,
\end{aligned}$$

where we have used $\hat{\epsilon}_{lij} \hat{\epsilon}^{kij} = 2\delta^k_l$. Putting everything together, Stokes' theorem in this case $\int_{\mathcal{V}} \mathrm{d}E = \int_{\partial \mathcal{V}} E$ yields the Gauss law (divergence theorem)

$$\iiint_{\mathcal{V}} \left(\nabla \cdot \vec{E} \right) \mathrm{d}V = \oiint_{\Sigma} \vec{E} \cdot \mathrm{d}\vec{\Sigma}. \tag{4.14}$$

In electromagnetism, $\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$ is the electric charge density, where ε_0 is the vacuum electric permittivity. Thus the above relation tells us two equivalent ways to compute the total electric charge Q_{e} enclosed in the volume $\mathcal{V}$

$$\frac{Q_{\mathrm{e}}}{\varepsilon_0} = \iiint_{\mathcal{V}} \left(\nabla \cdot \vec{E} \right) \mathrm{d}V = \oiint_{\Sigma} \vec{E} \cdot \mathrm{d}\vec{\Sigma}. \tag{4.15}$$

5 Volume and Area Elements in Spherical Coordinates

As a simple example, we illustrate how the volume element (3.2) and area element (4.5) work in spherical coordinates in $\mathcal{M} = \mathbb{R}^3$.

For the volume element, we start with

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned} \tag{5.1}$$

The 1-form basis is given by

$$\begin{aligned} \mathrm{d}x &= \sin\theta\cos\phi\mathrm{d}r + r\cos\theta\cos\phi\mathrm{d}\theta - r\sin\theta\sin\phi\mathrm{d}\phi, \\ \mathrm{d}y &= \sin\theta\sin\phi\mathrm{d}r + r\cos\theta\sin\phi\mathrm{d}\theta + r\sin\theta\cos\phi\mathrm{d}\phi, \\ \mathrm{d}z &= \cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta. \end{aligned} \tag{5.2}$$

The 2-form basis can be computed from them using $\mathrm{d}r\wedge\mathrm{d}r = \mathrm{d}\theta\wedge\mathrm{d}\theta = \mathrm{d}\phi\wedge\mathrm{d}\phi = 0$ and $\mathrm{d}r\wedge\mathrm{d}\theta = -\mathrm{d}\theta\wedge\mathrm{d}r$ etc. We have

$$\begin{aligned} \mathrm{d}x\wedge\mathrm{d}y &= r\sin^2\theta\mathrm{d}r\wedge\mathrm{d}\phi + r^2\sin\theta\cos\theta\mathrm{d}\theta\wedge\mathrm{d}\phi, \\ \mathrm{d}y\wedge\mathrm{d}z &= -r\sin\phi\mathrm{d}r\wedge\mathrm{d}\theta - r\sin\theta\cos\theta\cos\phi\mathrm{d}r\wedge\mathrm{d}\phi + r^2\sin^2\theta\cos\phi\mathrm{d}\theta\wedge\mathrm{d}\phi, \\ \mathrm{d}z\wedge\mathrm{d}x &= r\cos\phi\mathrm{d}r\wedge\mathrm{d}\theta - r\sin\theta\cos\theta\sin\phi\mathrm{d}r\wedge\mathrm{d}\phi + r^2\sin^2\theta\sin\phi\mathrm{d}\theta\wedge\mathrm{d}\phi. \end{aligned} \tag{5.3}$$

By calculating $(\mathrm{d}x\wedge\mathrm{d}y)\wedge\mathrm{d}z$ and using (3.2), the volume 3-form is

$$\mathrm{d}V = \mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = r^2\sin\theta\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi, \tag{5.4}$$

where recognize the prefactor $r^2\sin\theta$ as the measure in spherical coordinates, i.e. $\sqrt{\det g_{\mu\nu}}$.

Now let us work out the area element on the unit sphere. For the unit sphere, we set $r = 1$ and $\mathrm{d}r = 0$, then

$$\begin{aligned} \mathrm{d}x\wedge\mathrm{d}y &= \sin\theta\cos\theta\mathrm{d}\theta\wedge\mathrm{d}\phi, \\ \mathrm{d}y\wedge\mathrm{d}z &= \sin^2\theta\cos\phi\mathrm{d}\theta\wedge\mathrm{d}\phi, \\ \mathrm{d}z\wedge\mathrm{d}x &= \sin^2\theta\sin\phi\mathrm{d}\theta\wedge\mathrm{d}\phi. \end{aligned}$$

Using (4.5), i.e. identifying $\mathrm{d}\Sigma_x = \mathrm{d}y\wedge\mathrm{d}z$, $\mathrm{d}\Sigma_y = \mathrm{d}z\wedge\mathrm{d}x$, $\mathrm{d}\Sigma_z = \mathrm{d}x\wedge\mathrm{d}y$, we have the three components of the area element vector $\mathrm{d}\vec{\Sigma}$

$$\begin{aligned} \mathrm{d}\Sigma_x &= \sin^2\theta\cos\phi\mathrm{d}\theta\mathrm{d}\phi, \\ \mathrm{d}\Sigma_y &= \sin^2\theta\sin\phi\mathrm{d}\theta\mathrm{d}\phi, \\ \mathrm{d}\Sigma_z &= \sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}\phi. \end{aligned} \tag{5.5}$$

Factoring out the common factor

$$\mathrm{d}\Omega_2 = \sin\theta\mathrm{d}\theta\mathrm{d}\phi, \tag{5.6}$$

which is the area element on the unit sphere, the above components can be written in a vector form as

$$\begin{pmatrix} \mathrm{d}\Sigma_x \\ \mathrm{d}\Sigma_y \\ \mathrm{d}\Sigma_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \mathrm{d}\Omega_2, \tag{5.7}$$

i.e.

$$\mathrm{d}\vec{\Sigma} = \hat{r}\mathrm{d}\Omega_2, \tag{5.8}$$

where $\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is the unit vector in the radial direction, i.e. the out-going unit normal vector on the unit sphere.

6 Various Formulations of Electromagnetism

We will see that the theory of differential forms and Hodge duality is a natural language to formulate gauge theories in physics. Maxwell's electrodynamics is one of the simplest examples of gauge theory. We will see how it can be formulated in differential forms in a very concise way.

In this section, we consider Minkowskian space of various dimensions, with metric $g_{\mu\nu} = (-1, 1, 1, \cdots)$. Here $\mu, \nu = 0, 1, 2, \cdots$ are the spacetime indices, $i, j = 1, 2, \cdots$ the spatial indices, and $x^0 = ct$.

6.1 Maxwell Theory in Differential Forms

Let us first show that Maxwell's electrodynamics can be formulated in the most concise manner in terms of differential forms. The electric and magnetic fields together form the field strength 2-form F. Two of four Maxwell's equations, the sourceless ones, can be written as the Bianchi identity

$$\mathrm{d}F = 0. \tag{6.1}$$

This means F is closed. As the spacetime has trivial topology of $\mathbb{R}^{3,1}$, Poincaré's lemma applies. Hence F is exact, i.e. there exists a 1-form A globally such that

$$F = \mathrm{d}A. \tag{6.2}$$

This 1-form A is the gauge potential. The gauge transformation is to change A by an exact part

$$A \to A' = A + \mathrm{d}\vartheta. \tag{6.3}$$

where the gauge parameter ϑ is a 0-form. $\mathrm{d}^2 = 0$ immediately implies the field strength F (electric and magnetic fields) is gauge invariant

$$F \to F' = F. \tag{6.4}$$

Let the charge current J be a 1-form. Then the action is given by inner products

$$S = -\frac{1}{2\mu_0}\langle F, F\rangle + \langle A, J\rangle = \int \left(-\frac{1}{2\mu_0} F \wedge *F + A \wedge *J\right). \tag{6.5}$$

Here μ_0 is the vacuum magnetic permeability (not to be confused with the index μ!). The other two Maxwell's equations, the ones with source, can be obtained by a variation of this action with respect to A. In order to distinguish from the adjoint exterior derivative δ in (3.42), we denote the variation of A as $\hat{\delta}A$. Notice that the inner product is symmetric, thus

$$\hat{\delta}\langle F, F\rangle = 2\langle\hat{\delta}F, F\rangle = 2\langle \mathrm{d}\hat{\delta}A, F\rangle = 2\langle\hat{\delta}A, \delta F\rangle,$$

where in the last step we integrate by parts using (3.41). Thus

$$\hat{\delta}S = -\frac{1}{2\mu_0}2\langle\hat{\delta}A, \delta F\rangle + \langle\hat{\delta}A, J\rangle = -\frac{1}{\mu_0}\langle\hat{\delta}A, \delta F - \mu_0 J\rangle.$$

At the minimum of the action $\hat{\delta}S = 0$, as $\hat{\delta}A$ is arbitrary, the other argument must vanish, thus we obtain

$$\delta F = \mu_0 J. \tag{6.6}$$

This is the other half of Maxwell's equations written in compact form. Furthermore, acting δ on this equation, the nilpotency $\delta^2 = 0$ implies the conservation of charge current J

$$\delta J = 0. \tag{6.7}$$

Notice that (6.6) is a 1-form equation. Using (3.42), $\delta F = (-1)^n * \mathrm{d} * F$, (6.6) can be written as

$$(-1)^n * *\mathrm{d} * F = *\mu_0 J.$$

Acting by $*$, using (3.6) for 3-form, $* * (\mathrm{d} * F) = (-1)^n\, \mathrm{d} * F$, it can be written as

$$\mathrm{d} * F = *\mu_0 J. \tag{6.8}$$

Thus (6.6) can also be written as a 3-form equation. Alternatively, in terms of the potential A, (6.6) becomes

$$\delta\mathrm{d}A = \Delta A - \mathrm{d}\delta A = \mu_0 J. \tag{6.9}$$

Lorentz gauge condition is simply $\delta A = 0$, then the equation becomes the wave equation $\Delta A = \mu_0 J$.

6.2 Maxwell Theory in Covariant Index Notation

Now we translate the above formalism in differential forms into the covariant index notation.

The gauge potential 1-form has components A_μ

$$A = A_\mu \mathrm{d}x^\mu. \tag{6.10}$$

The gauge transformation is

$$A'_\mu = A_\mu + \partial_\mu \vartheta. \tag{6.11}$$

For the field strength 2-form F, using $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, we can write

$$F = \mathrm{d}A = (\mathrm{d}x^\mu \partial_\mu) \wedge (A_\nu \mathrm{d}x^\nu) = (\partial_\mu A_\nu)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

$$\begin{aligned}
&= \frac{1}{2}\left(\partial_\mu A_\nu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \frac{1}{2}\left(\partial_\nu A_\mu\right) \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \\
&= \frac{1}{2}\left(\partial_\mu A_\nu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu - \frac{1}{2}\left(\partial_\nu A_\mu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu ,
\end{aligned}$$

thus we have

$$F = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2} \partial_{[\mu} A_{\nu]} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu , \tag{6.12}$$

i.e. the components of F are

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{6.13}$$

This of course agrees with (2.21). The invariance of field strength is from

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \vartheta - \partial_\nu \partial_\mu \vartheta = F_{\mu\nu} . \tag{6.14}$$

For Bianchi identity, let us first write

$$\mathrm{d}F = \left(\mathrm{d}x^\rho \partial_\rho\right) \wedge \left(\frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) = \frac{1}{2}\left(\partial_\rho F_{\mu\nu}\right) \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu .$$

We split the right hand side into three equal terms, and relabel the dummy indices in a cyclic way

$$\mathrm{d}F = \frac{1}{6}\left(\partial_\rho F_{\mu\nu}\right) \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \frac{1}{6}\left(\partial_\mu F_{\nu\rho}\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho + \frac{1}{6}\left(\partial_\nu F_{\rho\mu}\right) \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu .$$

Using the antisymmetry of wedge product

$$\begin{aligned}
\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho &= -\mathrm{d}x^\mu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\nu = \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu , \\
\mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu &= -\mathrm{d}x^\rho \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu = \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu ,
\end{aligned}$$

it can be recombined as

$$\mathrm{d}F = \frac{1}{6}\left(\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu}\right) \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu . \tag{6.15}$$

As $F_{\mu\nu}$ is antisymmetric, $F_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} - F_{\nu\mu}\right)$, it can be further written as

$$\mathrm{d}F = \frac{1}{12} \partial_{[\rho} F_{\mu\nu]} \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu . \tag{6.16}$$

This again agrees with (2.10) and (2.21). As $\mathrm{d}F$ is a 3-form, it is Hodge dual to a $(n-3)$-form. Using

$$* \left(\mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) = \frac{1}{(n-3)!} \epsilon^{\rho\mu\nu}{}_{\sigma_1 \cdots \sigma_{n-3}} \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_{n-3}} ,$$

we have

$$* \left(\mathrm{d}F\right) = \frac{1}{2\,(n-3)!} \epsilon^{\rho\mu\nu}{}_{\sigma_1 \cdots \sigma_{n-3}} \partial_\rho F_{\mu\nu} \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_{n-3}} . \tag{6.17}$$

Thus there are three equivalent ways to write the component form of Bianchi identity

$$\begin{aligned}\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} &= 0,\\ \partial_{[\rho} F_{\mu\nu]} &= 0,\\ \epsilon^{\sigma_1\cdots\sigma_{n-3}\rho\mu\nu}\partial_\rho F_{\mu\nu} &= 0.\end{aligned} \tag{6.18}$$

Using (3.43), $(\delta F)_\nu = -\partial^\mu F_{\mu\nu}$. Then the other half of Maxwell's equations (6.6) has components $-\partial^\mu F_{\mu\nu} = \mu_0 J_\nu$, or

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu. \tag{6.19}$$

Applying (3.43) on J, $\delta J = -\partial^\mu J_\mu$, thus the conservation equation $\delta J = 0$ can be written as

$$\partial_\mu J^\mu = 0. \tag{6.20}$$

In terms of gauge potential A_μ, (6.19) becomes

$$\Delta A^\mu - \partial^\mu (\partial \cdot A) = -\mu_0 J^\mu, \tag{6.21}$$

where $\Delta = \partial_\mu \partial^\mu$ and $\partial \cdot A = \partial_\mu A^\mu$. Under the Lorentz gauge condition $\delta A = -\partial^\mu A_\mu = 0$, the above equation becomes the wave equation $\Delta A^\mu = -\mu_0 J^\mu$.

Using (3.37), the action can be written as

$$S = \int \mathrm{d}^n x \left(-\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu \right). \tag{6.22}$$

The first term gives the Lagrangian density for the electromagnetic field

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}. \tag{6.23}$$

6.3 Maxwell Theory in Conventional Vector Form

Now we rewrite the above covariant formalism in the more familiar spatial vector form by splitting components of all forms into the temporal ones and spatial ones. The space is Euclidean thus we do not distinguish vectors and 1-forms. All spatial vectors are denoted by "$\vec{\ }$". $x^0 = ct$. The spatial gradient operator is ∇, whose components are ∂_i. Then the components of d is

$$\partial_\mu = (\partial_0, \nabla) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla \right). \tag{6.24}$$

The speed of light in vacuum is given by

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}, \tag{6.25}$$

where ε_0 the vacuum electric permittivity.

The charge current is written as

$$J^\mu = \left(c\rho, \vec{j} \right), \qquad J_\mu = \left(-c\rho, \vec{j} \right), \tag{6.26}$$

where the temporal component ρ is identified with the electric charge density, and the spatial part $\vec{j}$ is identified with the electric current density. The conservation law reads

$$\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0. \tag{6.27}$$

The gauge potential is written as

$$A^\mu = \left(\frac{1}{c}\phi, \vec{A}\right), \qquad A_\mu = \left(-\frac{1}{c}\phi, \vec{A}\right), \tag{6.28}$$

where ϕ is the electric (scalar) potential and $\vec{A}$ the vector potential. Then

$$A_\mu J^\mu = \left(-\frac{1}{c}\phi, A_i\right)\begin{pmatrix} c\rho \\ j^i \end{pmatrix} = -\rho\phi + \vec{j}\cdot\vec{A}. \tag{6.29}$$

For the field strength, $F_{00} = 0$ due to antisymmetry. For $\mu = i$ and $\nu = 0$, we have

$$F_{i0} = \partial_i A_0 - \partial_0 A_i = \partial_i\left(-\frac{1}{c}\phi\right) - \left(\frac{1}{c}\frac{\partial}{\partial t}\right)A_i = -\frac{1}{c}\partial_i\phi - \frac{1}{c}\frac{\partial A_i}{\partial t} = -F_{0i}.$$

Comparing this with the definition of electric field

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{6.30}$$

we can identify

$$F_{i0} = -F_{0i} = \frac{1}{c}E_i. \tag{6.31}$$

Therefore, the field strength 2-form has components

$$F_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{c}\vec{E} \\ \frac{1}{c}\vec{E} & F_{ij} \end{pmatrix}. \tag{6.32}$$

The spatial part of the field strength

$$F_{ij} = \partial_i A_j - \partial_j A_i \tag{6.33}$$

can be identified with the magnetic field, but its nature (scalar, vector etc) varies with dimensions. We will show how this work in two and three spatial dimensions, and write the remaining equations for each cases.

6.4 Electrodynamics in 3+1 Dimensions

In 3+1 dimensions with coordinates $\left(x^0 = ct, x, y, z\right)$, magnetic field is a pseudovector, just like angular momentum, which can be identified with the spatial part of the field strength 2-form as

$$B^i = \frac{1}{2}\epsilon^{ijk}F_{jk} = \epsilon^{ijk}\partial_j A_k, \qquad F_{ij} = \epsilon_{ijk}B^k, \tag{6.34}$$

and $B_i = B^i$. In vector notation this is just $\vec{B} = \nabla \times \vec{A}$. In component

$$B_x = F_{yz} = \partial_y A_z - \partial_z A_y,$$

$$B_y = F_{zx} = \partial_z A_x - \partial_x A_z, \tag{6.35}$$
$$B_z = F_{xy} = \partial_x A_y - \partial_y A_x.$$

In matrix form

$$F_{ij} = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}. \tag{6.36}$$

Then the full field strength tensor is

$$F_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & B_z & -B_y \\ \frac{1}{c}E_y & -B_z & 0 & B_x \\ \frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{6.37}$$

Let us raise the first index[3]

$$F^{\mu}{}_{\nu} = \eta^{\mu\rho} F_{\rho\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & B_z & -B_y \\ \frac{1}{c}E_y & -B_z & 0 & B_x \\ \frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix},$$

i.e.

$$F^{\mu}{}_{\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & B_z & -B_y \\ \frac{1}{c}E_y & -B_z & 0 & B_x \\ \frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{6.38}$$

Then raise the second index

$$F^{\mu\nu} = F^{\mu}{}_{\rho} \eta^{\rho\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & B_z & -B_y \\ \frac{1}{c}E_y & -B_z & 0 & B_x \\ \frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

i.e.

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & B_z & -B_y \\ -\frac{1}{c}E_y & -B_z & 0 & B_x \\ -\frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{6.39}$$

The Lagrangian density can be calculated as

$$\begin{aligned} F^{\mu\nu} F_{\mu\nu} &= -F^{\mu\nu} F_{\nu\mu} = -F^{\mu}{}_{\nu} F^{\nu}{}_{\mu} \\ &= -\mathrm{tr}\left[\begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & B_z & -B_y \\ \frac{1}{c}E_y & -B_z & 0 & B_x \\ \frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & B_z & -B_y \\ \frac{1}{c}E_y & -B_z & 0 & B_x \\ \frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix} \right] \end{aligned}$$

[3] Recall that multiplication by an elementary matrix from the left corresponds to a row operation, while that from the right corresponds to a column operation.

$$
\begin{aligned}
&= -\mathrm{tr}\begin{pmatrix} \frac{1}{c^2}\left(E_x^2+E_y^2+E_z^2\right) & * & * & * \\ * & \frac{1}{c^2}E_x^2 - B_z^2 - B_y^2 & * & * \\ * & * & \frac{1}{c^2}E_y^2 - B_z^2 - B_x^2 & * \\ * & * & * & \frac{1}{c^2}E_z^2 - B_y^2 - B_x^2 \end{pmatrix} \\
&= -\left[\frac{1}{c^2}\left(E_x^2+E_y^2+E_z^2\right) + \frac{1}{c^2}x - B_z^2 - B_y^2 + \frac{1}{c^2}E_y^2 - B_z^2 - B_x^2 + \frac{1}{c^2}E_z^2 - B_y^2 - B_x^2\right] \\
&= -\frac{2}{c^2}\left(E_x^2+E_y^2+E_z^2\right) + 2\left(B_x^2+B_y^2+B_z^2\right),
\end{aligned}
$$

thus

$$\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = -\frac{\vec{E}^2}{c^2} + \vec{B}^2. \tag{6.40}$$

This shows that $-\frac{1}{c^2}\vec{E}^2 + \vec{B}^2$ is a scalar that is invariant under Lorentz transformation. The action is

$$S = \int \mathrm{d}^4x \left(\frac{1}{2}\varepsilon_0\vec{E}^2 - \frac{1}{2\mu_0}\vec{B}^2 - \rho\phi + \vec{j}\cdot\vec{A}\right). \tag{6.41}$$

For the Bianchi identity $\partial_{[\rho}F_{\mu\nu]} = 2\left(\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu}\right) = 0$, ρ, μ and ν can take any three of the four values 0, x, y and z. For $\rho,\mu,\nu = x,y,z$, we have

$$\partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy} = \partial_x B_x + \partial_y B_y + \partial_z B_z = \nabla\cdot\vec{B},$$

which yields the magnetic Gauss law

$$\nabla\cdot\vec{B} = 0. \tag{6.42}$$

For $\rho,\mu,\nu = 0,x,y$, $\rho,\mu,\nu = 0,x,z$ and $\rho,\mu,\nu = 0,y,z$ respectively, we have

$$
\begin{aligned}
\partial_0 F_{xy} + \partial_x F_{y0} + \partial_y F_{0x} &= \frac{1}{c}\frac{\partial}{\partial t}B_z + \frac{\partial}{\partial x}\left(\frac{1}{c}E_y\right) + \frac{\partial}{\partial y}\left(-\frac{1}{c}E_x\right) = \frac{1}{c}\left(\frac{\partial\vec{B}}{\partial t} + \nabla\times\vec{E}\right)_z, \\
\partial_0 F_{xz} + \partial_x F_{z0} + \partial_z F_{0x} &= \frac{1}{c}\frac{\partial}{\partial t}\left(-B_y\right) + \frac{\partial}{\partial x}\left(\frac{1}{c}E_z\right) + \frac{\partial}{\partial z}\left(-\frac{1}{c}E_x\right) = -\frac{1}{c}\left(\frac{\partial\vec{B}}{\partial t} + \nabla\times\vec{E}\right)_y, \\
\partial_0 F_{yz} + \partial_y F_{z0} + \partial_z F_{0y} &= \frac{1}{c}\frac{\partial}{\partial t}B_x + \frac{\partial}{\partial y}\left(\frac{1}{c}E_z\right) + \frac{\partial}{\partial z}\left(-\frac{1}{c}E_y\right) = \frac{1}{c}\left(\frac{\partial\vec{B}}{\partial t} + \nabla\times\vec{E}\right)_x,
\end{aligned}
$$

thus we obtain Faraday's law of induction

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0. \tag{6.43}$$

Using

$$\partial_\mu F^{\mu\nu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & B_z & -B_y \\ -\frac{1}{c}E_y & -B_z & 0 & B_x \\ -\frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$
= \begin{pmatrix} -\frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \\ \frac{1}{c^2}\frac{\partial E_x}{\partial t} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} \\ \frac{1}{c^2}\frac{\partial E_y}{\partial t} + \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} \\ \frac{1}{c^2}\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -\frac{1}{c}\nabla\cdot\vec{E} \\ \left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \nabla\times\vec{B}\right)_x \\ \left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \nabla\times\vec{B}\right)_y \\ \left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \nabla\times\vec{B}\right)_z \end{pmatrix}^{\mathrm{T}},
$$

the temporal component of (6.19) yields the electric Gauss law

$$
\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \tag{6.44}
$$

and the spatial components of (6.19) yield Ampère's law with Maxwell's addition

$$
\nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0\vec{j}. \tag{6.45}
$$

We see Bianchi identity produces the two sourceless Maxwell's equations, while (6.19) produces the other two Maxwell's equations with sources.

6.5 Electromagnetic Duality in 3+1 Dimensions

In 3+1 dimensions, the Hodge dual of the field strength 2-form F is also a 2-form, which is defined as the dual field strength

$$
\tilde{F} = *F, \qquad F = -*\tilde{F}. \tag{6.46}
$$

The corresponding dual electric and magnetic fields $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ are defined in the standard way from $\tilde{F}$

$$
\frac{1}{c}\tilde{E}_i = \tilde{F}_{i0}, \qquad \tilde{B}^i = \frac{1}{2}\hat{\epsilon}^{ijk}\tilde{F}_{jk}. \tag{6.47}
$$

From

$$
\begin{aligned}
F = &-\frac{1}{c}E_x\mathrm{d}x^0\wedge\mathrm{d}x - \frac{1}{c}E_y\mathrm{d}x^0\wedge\mathrm{d}y - \frac{1}{c}E_z\mathrm{d}x^0\wedge\mathrm{d}z \\
&+ B_y\mathrm{d}x\wedge\mathrm{d}y + B_x\mathrm{d}y\wedge\mathrm{d}z + B_y\mathrm{d}z\wedge\mathrm{d}x,
\end{aligned} \tag{6.48}
$$

we have

$$
\begin{aligned}
\tilde{F} = &-\frac{1}{c}E_x * \left(\mathrm{d}x^0\wedge\mathrm{d}x\right) - \frac{1}{c}E_y * \left(\mathrm{d}x^0\wedge\mathrm{d}y\right) - \frac{1}{c}E_z * \left(\mathrm{d}x^0\wedge\mathrm{d}z\right) \\
&+ B_y * (\mathrm{d}x\wedge\mathrm{d}y) + B_x * (\mathrm{d}y\wedge\mathrm{d}z) + B_y * (\mathrm{d}z\wedge\mathrm{d}x) \\
=&\frac{1}{c}E_x\mathrm{d}y\wedge\mathrm{d}z - \frac{1}{c}E_y\mathrm{d}x\wedge\mathrm{d}z + \frac{1}{c}E_z\mathrm{d}x\wedge\mathrm{d}y \\
&+ B_y\mathrm{d}x^0\wedge\mathrm{d}z + B_x\mathrm{d}x^0\wedge\mathrm{d}x + B_y\mathrm{d}x^0\wedge\mathrm{d}y,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\tilde{F} =& B_x\mathrm{d}x^0\wedge\mathrm{d}x + B_y\mathrm{d}x^0\wedge\mathrm{d}z + B_y\mathrm{d}x^0\wedge\mathrm{d}y \\
&+ \frac{1}{c}E_z\mathrm{d}x\wedge\mathrm{d}y + \frac{1}{c}E_x\mathrm{d}y\wedge\mathrm{d}z + \frac{1}{c}E_y\mathrm{d}z\wedge\mathrm{d}x.
\end{aligned} \tag{6.49}
$$

Thus we can identify

$$
\tilde{\vec{E}} = -c\vec{B}, \qquad \tilde{\vec{B}} = \frac{1}{c}\vec{E}. \tag{6.50}
$$

These can also be identified from

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{6.51}$$

as

$$\begin{aligned}
\tilde{E}_i &= c\,(*F)_{i0} = \frac{c}{2}\epsilon_{i0jk}F^{jk} = -\frac{c}{2}\epsilon_{0ijk}F^{jk} = -c\left(\frac{1}{2}\epsilon_{ijk}F^{jk}\right) = -cB_i,\\
\tilde{B}^i &= \frac{1}{2}\epsilon^{ijk}\,(*F)_{jk} = \frac{1}{2}\epsilon^{ijk}\left(\frac{1}{2}\epsilon_{jk\rho\sigma}F^{\rho\sigma}\right) = \frac{1}{2}\epsilon^{ijk}\left(\epsilon_{jkl0}F^{l0}\right) = \frac{1}{2}\epsilon^{ijk}\left(\epsilon_{0ljk}F_{l0}\right)\\
&= \frac{1}{2}\epsilon^{ijk}\epsilon_{ljk}\left(\frac{1}{c}E_l\right) = \frac{1}{2}2\delta^i_l\left(\frac{1}{c}E_l\right) = \frac{1}{c}E^i.
\end{aligned}$$

Thus we have

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & \frac{1}{c}E_z & -\frac{1}{c}E_y \\ -B_y & -\frac{1}{c}E_z & 0 & \frac{1}{c}E_x \\ -B_z & \frac{1}{c}E_y & -\frac{1}{c}E_x & 0 \end{pmatrix}. \tag{6.52}$$

Under this duality, electric and magnetic fields swap roles. The Bianchi identity and the other two Maxwell's equations also swap roles. We have

$$\mathrm{d}F = 0 \qquad \Rightarrow \qquad \mathrm{d}*\tilde{F} = 0 \qquad \text{or} \qquad \delta\tilde{F} = 0. \tag{6.53}$$

In components

$$\partial_{[\rho}F_{\mu\nu]} = 0 \qquad \Rightarrow \qquad \partial_\mu\tilde{F}^{\mu\nu} = 0. \tag{6.54}$$

We see the Bianchi identity now looks like the other two Maxwell's equations, but without source. We also have

$$\mathrm{d}*F = *\mu_0 J \qquad \Rightarrow \qquad \mathrm{d}\tilde{F} = *\mu_0 J. \tag{6.55}$$

In components

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \qquad \Rightarrow \qquad \partial_{[\rho}\tilde{F}_{\mu\nu]} = -\mu_0\epsilon_{\rho\mu\nu\sigma}J^\sigma. \tag{6.56}$$

The other two Maxwell's equations now look like the Bianchi identity, but with a non-vanishing source. The vector forms of the equations can be obtained by the substitutions $\vec{E} = c\tilde{\vec{B}}$ and $\vec{B} = -\frac{1}{c}\tilde{\vec{E}}$

$$\nabla\cdot\vec{B} = 0, \qquad \Rightarrow \qquad \nabla\cdot\tilde{\vec{E}} = 0, \tag{6.57}$$

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0, \qquad \Rightarrow \qquad \nabla\times\tilde{\vec{B}} - \frac{1}{c^2}\frac{\partial\tilde{\vec{E}}}{\partial t} = 0. \tag{6.58}$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \qquad \Rightarrow \qquad \nabla\cdot\tilde{\vec{B}} = \mu_0 c\rho, \tag{6.59}$$

$$\nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{j}, \qquad \Rightarrow \qquad \nabla\times\tilde{\vec{E}} + \frac{\partial\tilde{\vec{B}}}{\partial t} = -\frac{1}{\varepsilon_0 c}\vec{j}, \tag{6.60}$$

We see clearly from these equations that the electric and magnetic fields swap roles. In particular, without the charge sources ρ and $\vec{j}$, the four Maxwell's equations would be invariant under the substitution

$$\vec{E} \to c\vec{B}, \qquad \vec{B} \to -\frac{1}{c}\vec{E}. \tag{6.61}$$

This can be viewed as a discrete symmetry operation on Maxwell's equations. The source-less Maxwell's equations do enjoy this symmetry. This is called the *electromagnetic duality*, which suggests that electric and magnetic fields behave in a similar way in the absence of charged sources. Notice that the Maxwell Lagrangian $\mathcal{L}_{\text{Maxwell}}$ flips sign. This duality symmetry is broken by the coupling to sources J^μ, because we only have electric charges and currents. If the two equations from the Bianchi identity also have source terms, those would be the magnetic charge density ρ_{m} and magnetic current $\vec{j}_{\text{m}}$. They would swap roles with the electric charge density and current ρ and $\vec{j}$ under the duality transformation to make the full Maxwell's equations invariant. But as an experimental fact (so far), magnetic charges (often called *magnetic monopoles*) do not exist. Thus the full Maxwell's equations as we currently know do not have this electromagnetic duality symmetry. In 1931, Dirac found that *if* magnetic monopoles exist, quantum mechanics requires that the unit magnetic charge, often denoted as g in this context (not to be confused with other g in this note), has to be quantized together with the unit electric charge e, according to the *Dirac quantization condition* $eg = 2\pi\hbar n$ ($n \in \mathbb{Z}$). This *would* explain why electric charges are quantized as we know. The modern incarnation of electromagnetic duality in theoretical physics, in particular in supersymmetric gauge theory and string theory, is called *Montonen–Olive duality* or *S-duality* (for a short review, see [3]). Electromagnetic duality can only exist in 3+1 dimensions, as only in this case can the field strength 2-form F be Hodge dual to another 2-form. Equivalently, the magnetic field is only a (pseudo)vector in 3+1 dimensions, a prerequisite for the existence of this duality. In other dimensions, magnetic field is not even a vector. For example, we will see below that in 2+1 dimensions magnetic field is a (pseudo)scalar, thus there is no way it can be swapped with the electric field, which is always a vector in any dimensions.

Often in the context of electromagnetic duality (its modern incarnation actually), you will see an additional term added to the action besides the Maxwell term $F^{\mu\nu}F_{\mu\nu}$. It is just called the θ-term because its coefficient is often denoted by θ. Based on just spacetime and gauge symmetries, we see that in 3+1 dimensions there exists another gauge invariant scalar which is quadratic in the field strength 2-form F

$$S_\theta = -\frac{\theta e^2}{16\pi^2\hbar}\int F \wedge F. \tag{6.62}$$

Because it is in terms of F, it is automatically gauge invariant. $F \wedge F$ is a 4-form, and *only* in 3+1 dimensions is it proportional to the volume element (3.2) and Hodge dual to a scalar, thus the above construction of an action term is possible. In components, we can write

$$\begin{aligned}
F \wedge F &= \left(\frac{1}{2}F_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\right) \wedge \left(\frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) \\
&= \frac{1}{4}F_{\rho\sigma}F_{\mu\nu}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \frac{1}{4}\hat{\epsilon}^{\rho\sigma\mu\nu}F_{\rho\sigma}F_{\mu\nu}\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3,
\end{aligned}$$

where in the last step we have used (2.7) $\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu = \hat{\epsilon}^{\rho\sigma\mu\nu}\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$. Thus

$$S_\theta = -\frac{\theta e^2}{64\pi^2\hbar}\int \mathrm{d}^4x\hat{\epsilon}^{\rho\sigma\mu\nu}F_{\rho\sigma}F_{\mu\nu}. \tag{6.63}$$

Notice that there is no metric in this term, thus even when the manifold is not equipped with a metric, we can still write it down. Independence of metric implies this term is not sensitive to the measurement of length, i.e. is invariant under deformations as long as the topology is preserved. For this reason, such a term is said to be *topological*. Using

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = -\frac{1}{2}\hat{\epsilon}^{\mu\nu\rho\sigma}F_{\rho\sigma}, \qquad \hat{\epsilon}^{\rho\sigma\mu\nu}F_{\rho\sigma}F_{\mu\nu} = -2\tilde{F}^{\mu\nu}F_{\mu\nu},$$

it can also be written as

$$S_\theta = \frac{\theta e^2}{32\pi^2\hbar}\int \mathrm{d}^4x\tilde{F}^{\mu\nu}F_{\mu\nu}. \tag{6.64}$$

Furthermore, we can write it as

$$\begin{aligned}\hat{\epsilon}^{\rho\sigma\mu\nu}F_{\rho\sigma}F_{\mu\nu} &= 4\hat{\epsilon}^{0ijk}F_{0i}F_{jk} = 4\hat{\epsilon}^{ijk}\left(-\frac{1}{c}E_i\right)\left(\hat{\epsilon}_{jkl}B^l\right)\\ &= -\frac{4}{c}\left(\hat{\epsilon}^{ijk}\hat{\epsilon}_{ljk}\right)E_iB^l = -\frac{4}{c}\left(2\delta^i_l\right)E_iB^l = -\frac{8}{c}E_iB^i,\end{aligned}$$

i.e.

$$\hat{\epsilon}^{\rho\sigma\mu\nu}F_{\rho\sigma}F_{\mu\nu} = -\frac{8}{c}\vec{E}\cdot\vec{B}, \qquad \tilde{F}^{\mu\nu}F_{\mu\nu} = \frac{4}{c}\vec{E}\cdot\vec{B}, \tag{6.65}$$

Then

$$S_\theta = \frac{\theta e^2}{8\pi^2\hbar c}\int \mathrm{d}^4x\vec{E}\cdot\vec{B} = \frac{\theta}{2\pi}\alpha\int \mathrm{d}^4x\vec{E}\cdot\vec{B}, \tag{6.66}$$

where $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$ is the fine structure constant.

One feature of the θ-term is that it is actually a boundary term, despite it can be written as $\int \mathrm{d}^4x\vec{E}\cdot\vec{B}$. To see this, we use

$$\mathrm{d}\left(A\wedge\mathrm{d}A\right) = \mathrm{d}A\wedge\mathrm{d}A - A\wedge\mathrm{d}^2A = F\wedge F \tag{6.67}$$

where we have used $\mathrm{d}^2 = 0$. This shows that $F\wedge F$ is actually a total derivative, thus its integral is a boundary term

$$\int_{\mathcal{M}} F\wedge F = \int_{\mathcal{M}} \mathrm{d}\left(A\wedge\mathrm{d}A\right) = \int_{\partial\mathcal{M}} A\wedge\mathrm{d}A. \tag{6.68}$$

Now it is written as a 3-form $A\wedge\mathrm{d}A$ integrated over the boundary of $\mathcal{M}$ which is a three-dimensional manifold. This 3-form is called (Abelian) Chern-Simons 3-form. It defines a different type of gauge theory on 3-manifolds, known as topological quantum field theory of Schwarz type in physics. This will be discussed later.

6.6 Electrodynamics in 2+1 Dimensions

In 2+1 dimensions, i.e. two spatial dimensions, the magnetic field is a pseudoscalar because F_{ij} has only one independent component. It is defined as

$$B = \frac{1}{2}\epsilon^{ij}F_{ij} = \epsilon^{ij}\partial_i A_j = \partial_x A_y - \partial_y A_x. \tag{6.69}$$

This expression is often written as $B = \nabla \times \vec{A}$ in two spatial dimensions. Here $\nabla \times \vec{A}$ by definition has a single component $\partial_x A_y - \partial_y A_x$. *In two spatial dimensions, cross product of two vectors, as well as the curl of a vector, are pseudoscalars, not pseudovectors like in three dimensions.* F_{ij} can be written as

$$F_{ij} = B\epsilon_{ij} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}. \tag{6.70}$$

The full field strength tensor is then

$$F_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y \\ \frac{1}{c}E_x & 0 & B \\ \frac{1}{c}E_y & -B & 0 \end{pmatrix}. \tag{6.71}$$

Let us raise the first index:

$$F^{\mu}{}_{\nu} = \eta^{\mu\rho}F_{\rho\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y \\ \frac{1}{c}E_x & 0 & B \\ \frac{1}{c}E_y & -B & 0 \end{pmatrix},$$

i.e.

$$F^{\mu}{}_{\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y \\ \frac{1}{c}E_x & 0 & B \\ \frac{1}{c}E_y & -B & 0 \end{pmatrix}. \tag{6.72}$$

Then we raise the second index

$$F^{\mu\nu} = F^{\mu}{}_{\rho}\eta^{\rho\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y \\ \frac{1}{c}E_x & 0 & B \\ \frac{1}{c}E_y & -B & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

i.e.

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y \\ -\frac{1}{c}E_x & 0 & B \\ -\frac{1}{c}E_y & -B & 0 \end{pmatrix}. \tag{6.73}$$

The Lagrangian density can be calculated as

$$F^{\mu\nu}F_{\mu\nu} = -F^{\mu\nu}F_{\nu\mu} = -F^{\mu}{}_{\nu}F^{\nu}{}_{\mu} = -\mathrm{tr}\left[\begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y \\ \frac{1}{c}E_x & 0 & B \\ \frac{1}{c}E_y & -B & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y \\ \frac{1}{c}E_x & 0 & B \\ \frac{1}{c}E_y & -B & 0 \end{pmatrix}\right]$$

$$
\begin{aligned}
&= -\mathrm{tr}\begin{pmatrix} \frac{1}{c^2}\left(E_x^2+E_y^2\right) & * & * \\ * & \frac{1}{c^2}E_x^2 - B^2 & * \\ * & * & \frac{1}{c^2}E_y^2 - B^2 \end{pmatrix} \\
&= -\left[\frac{1}{c^2}\left(E_x^2+E_y^2\right) + \frac{1}{c^2}E_x^2 - B^2 + \frac{1}{c^2}E_y^2 - B^2\right] = -\frac{2}{c^2}\left(E_x^2+E_y^2\right) + 2B^2,
\end{aligned}
$$

thus

$$
\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = -\frac{\vec{E}^2}{c^2} + B^2, \tag{6.74}
$$

and

$$
S = \int \mathrm{d}^3x \left(\frac{1}{2}\varepsilon_0\vec{E}^2 - \frac{1}{2\mu_0}B^2 - \rho\phi + \vec{j}\cdot\vec{A}\right). \tag{6.75}
$$

These scalar quantities are the same as in 3+1 dimensions, except that $\vec{B}^2$ is replaced by B^2.

The Bianchi identity (6.16) is a 3-form, i.e. has three indices that are totally antisymmetric. The number of totally antisymmetric indices equals the number of the dimension of the space in this case, thus it is unique up to an overall normalization and is proportional to the Levi-Civita tensor in this space $\partial_{[\rho}F_{\mu\nu]} \sim \epsilon_{\rho\mu\nu}$. In other words, $\mathrm{d}F$ is Hodge dual to a scalar. Thus there is only one single component for the Bianchi identity, where all three indices take different values:

$$
\begin{aligned}
\partial_{[0}F_{xy]} &= \partial_0 F_{xy} + \partial_x F_{y0} + \partial_y F_{0x} = \frac{1}{c}\frac{\partial}{\partial t}B + \frac{\partial}{\partial x}\left(\frac{1}{c}E_y\right) + \frac{\partial}{\partial y}\left(-\frac{1}{c}E_x\right) \\
&= \frac{1}{c}\left(\frac{\partial B}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = \frac{1}{c}\left(\frac{\partial B}{\partial t} + \nabla\times\vec{E}\right).
\end{aligned}
$$

Thus the Bianchi identity $\mathrm{d}F = 0$ in 2+1 dimensions is just Faraday's law of induction

$$
\nabla\times\vec{E} + \frac{\partial B}{\partial t} = 0, \tag{6.76}
$$

where $\nabla\times\vec{E} = \partial_x E_y - \partial_y E_x$, the curl of the electric field, is a pseudoscalar. There is no Gauss law for magnetic field, because B is a scalar, it makes no sense to talk about its divergence. Using

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y \\ -\frac{1}{c}E_x & 0 & B \\ -\frac{1}{c}E_y & -B & 0 \end{pmatrix} \\
&= \left(-\frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right), \frac{1}{c^2}\frac{\partial E_x}{\partial t} - \frac{\partial B}{\partial y}, \frac{1}{c^2}\frac{\partial E_y}{\partial t} + \frac{\partial B}{\partial x}\right),
\end{aligned}
$$

Maxwell's equations (6.19) in 2+1 dimensions read

$$
\begin{aligned}
\nabla\cdot\vec{E} &= \frac{1}{\varepsilon_0}\rho, \\
\frac{\partial B}{\partial y} - \frac{1}{c^2}\frac{\partial E_x}{\partial t} &= \mu_0 j_x, \\
-\frac{\partial B}{\partial x} - \frac{1}{c^2}\frac{\partial E_y}{\partial t} &= \mu_0 j_y.
\end{aligned} \tag{6.77}
$$

7 Kalb-Ramond Field and p-Form Electrodynamics

7.1 Point Charges Coupled to Electromagnetic Field

For simplicity, in this section we set the speed of light $c = 1$. In the above we have seen that Maxwell's electrodynamics couples to matter via the action term

$$S_{\text{coupling}} = \langle A, J \rangle = \int A \wedge *J = \int \mathrm{d}^n x A_\mu J^\mu. \tag{7.1}$$

Here

$$J^\mu = (\rho, \vec{j}) \tag{7.2}$$

is the charge current of the matter. Elementary particles that couple to electrodynamics, such as electrons and muons, are point particles. Imagine a moving point particle of charge e, its worldline is a one-dimensional curve $\mathcal{C}$ parameterized by its proper time τ. The parametric equation of the worldline $\mathcal{C}$ is given by $x^\mu = X^\mu(\tau)$, and the velocity vector is given by

$$u^\mu = \left.\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right|_{x=X(\tau)} = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = (\gamma, \gamma\vec{v}), \tag{7.3}$$

where $\vec{v} = \mathrm{d}\vec{x}/\mathrm{d}t$ is the spatial velocity on the worldline and $\gamma = 1/\sqrt{1-\vec{v}^2}$. The charge current is localized on its worldline, meaning it vanishes outside $\mathcal{C}$. Thus we can write this kind of charge current in a covariant way as

$$J^\mu = e u^\mu \delta^{(n)}(x - X(\tau)), \tag{7.4}$$

where $\delta^{(n)}(x - X) = \delta\left(x^0 - X^0(\tau)\right)\delta\left(x^1 - X^1(\tau)\right)\cdots\delta\left(x^{n-1} - X^{n-1}(\tau)\right)$. By comparing the above two expressions for J^μ, we can identify the components

$$\rho = \gamma e \delta^{(n)}(x - X(\tau)), \qquad \vec{j} = \gamma e \vec{v} \delta^{(n)}(x - X(\tau)). \tag{7.5}$$

These are the generalization of the non-relativistic expressions for point charge density and current $\rho = e\delta^{(3)}(\vec{x} - \vec{X}(t))$ and $\vec{j} = e\vec{v}\delta^{(3)}(\vec{x} - \vec{X}(t))$. Now the coupling term in the action can be written as

$$S_{\text{coupling}} = e \int \mathrm{d}^n x \delta^{(n)}(x - X(\tau)) A_\mu u^\mu. \tag{7.6}$$

The δ-function integral has only non-vanishing support on the worldline, thus is equivalent to a contour integral on the worldline

$$\int \mathrm{d}^n x \delta^{(n)}(x - X(\tau)) \ldots = \int_{\mathcal{C}} \mathrm{d}\tau \ldots. \tag{7.7}$$

Then

$$S_{\text{coupling}} = e \int_{\mathcal{C}} \mathrm{d}\tau A_\mu u^\mu = e \int_{\mathcal{C}} \mathrm{d}\tau A_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = e \int_{\mathcal{C}} A_\mu \mathrm{d}x^\mu = e \int_{\mathcal{C}} A. \tag{7.8}$$

Thus the coupling to a point charge is simply the integral of the gauge potential 1-form A over its one-dimensional worldline. We see here that the differential form language is very natural in dealing with the coupling between electromagnetic field and matter. The naturalness comes from the matching of the following three facts:

- Mathematically the integral $\int_\Sigma A$ of a p-form A is only defined over a p-dimensional hypersurface Σ.
- The worldline of a point charge $\Sigma = \mathcal{C}$ is one-dimensional (i.e. $p = 1$). Equivalently, the charge current J is a 1-form.
- The gauge potential A of the electromagnetic field that a point charge couples to is a 1-form (i.e. $p = 1$).

Here the theory of differential forms tells us that $\int_\mathcal{C} A$ is the only viable mathematical structure one can write down for such a coupling in order to guarantee covariance.

7.2 Strings Coupled to Kalb-Ramond Field

The naturalness of the differential form language allows us to easily generalize to charge sources of other dimensions and write down a gauge theory whose elementary sources are not point charges. For example, consider a charged string. It has one spatial dimension. Thus its worldsheet Σ is two-dimensional. To preserve the aforementioned mathematical structure, it has to couple to a 2-form gauge potential B, therefore the coupling term will be

$$S_{\text{coupling}} = \int_\Sigma B. \tag{7.9}$$

This action is an integral over the two-dimensional worldsheet. In analogy, we can write down the corresponding integral over the spacetime

$$S_{\text{coupling}} = \langle B, J\rangle = \int B \wedge *J = \frac{1}{2}\int \mathrm{d}^n x B_{\mu\nu} J^{\mu\nu}, \tag{7.10}$$

where now the charge current J is a 2-form, matching the fact that the string worldsheet is two-dimensional. In the same spirit as Maxwell's theory, the field strength is now a 3-form H

$$H = \mathrm{d}B, \qquad H_{\rho\mu\nu} = \frac{1}{2}\partial_{[\rho} B_{\mu\nu]}, \tag{7.11}$$

which satisfies Bianchi identity as a consequence of $\mathrm{d}^2 = 0$

$$\mathrm{d}H = 0, \qquad \partial_{[\sigma} H_{\rho\mu\nu]} = 0. \tag{7.12}$$

The gauge transformation is generated by a 1-form gauge parameter α

$$B' = B + \mathrm{d}\alpha, \qquad B'_{\mu\nu} = B_{\mu\nu} + \partial_{[\mu}\alpha_{\nu]}, \tag{7.13}$$

which ensures the invariance of the field strength $H' = H$ again as a consequence of $\mathrm{d}^2 = 0$. The generalization of the Maxwell action is simply $\langle H, H\rangle$. Thus the full action is

$$\begin{aligned} S &= -\frac{1}{2}\langle H, H\rangle + \langle B, J\rangle = \int \left(-\frac{1}{2} H \wedge *H + B \wedge *J\right) \\ &= \int \mathrm{d}^n x \left(-\frac{1}{12} H_{\rho\mu\nu} H^{\rho\mu\nu} + \frac{1}{2} B_{\mu\nu} J^{\mu\nu}\right). \end{aligned} \tag{7.14}$$

Variation of the action with respect to the gauge potential B

$$\hat{\delta}S = -\langle \hat{\delta}H, H\rangle + \langle \hat{\delta}B, J\rangle = -\langle \mathrm{d}\hat{\delta}B, H\rangle + \langle \hat{\delta}B, J\rangle = -\langle \hat{\delta}B, \delta H\rangle + \langle \hat{\delta}B, J\rangle = -\langle \hat{\delta}B, \delta H - J\rangle$$

yields the equation of motion

$$\delta H = J, \qquad -\partial_\rho H^{\rho\mu\nu} = J^{\mu\nu}. \tag{7.15}$$

Acting δ on the equation of motion, $\delta^2 = 0$ yields the conservation of the charge current 2-form

$$\delta J = 0, \qquad \partial_\mu J^{\mu\nu} = 0. \tag{7.16}$$

The above 2-form gauge potential B is known as the *Kalb-Ramond field* in the literature, first proposed by Michael Kalb and Pierre Ramond in 1974. It is the equivalent of the 1-form electric potential A in Maxwell's electrodynamics where charges are primarily point-like. Here the elementary charged objects are string-like, hence the potential they couple to is naturally a 2-form and the field strength tensor is a 3-form H, the equivalent of the 2-form electromagnetic field F in Maxwell's electrodynamics. In string theory, Kalb-Ramond field is one of the three types of massless excitations of closed strings, the other two being the spacetime metric and a scalar field called dilaton.

7.3 Charged Branes and p-Form Electrodynamics

In string theory, there are higher dimensional objects called branes. D-branes are the extended objects where open strings end ("D" here stands for Dirichlet boundary conditions imposed on open strings). A Dp-brane has p spatial dimensions. A D0-brane is a point, a D1-brane is like a line, a D2-brane is like a plane, a D3-brane is like a volume etc.

A charged D$(p-1)$-brane couples to a p-form potential $C^{(p)}$ whose field strength is a $(p+1)$-form

$$G^{(p+1)} = \mathrm{d}C^{(p)}. \tag{7.17}$$

The gauge transformation is generated by a $(p-1)$-form $\beta^{(p-1)}$

$$C'^{(p)} = C^{(p)} + \mathrm{d}\beta^{(p-1)}. \tag{7.18}$$

The dynamics of these fields is a straightforward generalization of the above action for Kalb-Ramond field

$$S = -\frac{1}{2}\langle G, G\rangle + \langle C, J\rangle = \int \left(-\frac{1}{2}G\wedge *G + C\wedge *J\right), \tag{7.19}$$

where now the charge current of the D$(p-1)$-brane J is a p-form. This is known as *p-form electrodynamics*. Such a p-form gauge field is called *Ramond–Ramond field* (*RR field*) in string theory. On the contrary, the Kalb-Ramond 2-form B that couples to elementary strings is called *NS–NS field*, named after André Neveu and John Schwarz.

8 Chern-Simons Theory: A Topological Field Theory

In the discussion of the θ-term (6.62), we have seen that it is a boundary term whose integrand is the Chern-Simons 3-form (6.68). Although it appears as a boundary term of a four-dimensional theory, it can exist on its own as a well defined field theory in three-dimensional spaces. Now let us look at this field theory defined on three-dimensional manifolds.

8.1 Abelian Chern-Simons Theory

The Abelian Chern-Simons theory on a three-dimensional manifold $\mathcal{M}$ is given by the action

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int_{\mathcal{M}} A \wedge \mathrm{d}A, \tag{8.1}$$

where the real constant k is called the *level* of the theory. Here A is still the 1-form gauge potential, and $F = \mathrm{d}A$ the 2-form field strength, thus our identification of its components with electric field $\vec{E}$ and magnetic field B (remember it is a pseudoscalar in two spatial dimensions) as given in (6.71) still holds. The 3-form $A \wedge \mathrm{d}A$ is the Abelian version of the *Chern-Simons 3-form*. To obtain the equation of motion, we have

$$\hat{\delta} S_{\mathrm{CS}} = \frac{k}{2\pi} \int_{\mathcal{M}} \hat{\delta} A \wedge \mathrm{d}A + \text{boundary terms},$$

thus the equation of motion is simply the vanishing of field strength

$$F = 0, \tag{8.2}$$

i.e. $\vec{E} = 0$ and $B = 0$. Therefore, on a manifold of trivial topology, such as 2+1 dimensional Minkowskian space $\mathbb{R}^{2,1}$, $A = 0$ and the theory is totally trivial, with identically vanishing electric and magnetic fields. This makes this type of gauge theory very different from electrodynamics in 2+1 dimensions, in which you can have non-trivial electric and magnetic field configurations in vacuum. Also notice that by definition, just like the θ-term in (6.62), S_{CS} is independent of the metric, thus the theory is insensitive to the measurement of distance on the manifold. Therefore, Chern-Simons theory is invariant under local deformations of the manifold. It is only sensitive to the topology of the manifold. This is why it is a *topological field theory*. It is the simplest example of this kind. The theory is only non-trivial when the topology of $\mathcal{M}$ is non-trivial.

Under a gauge transformation $\hat{\delta} A = \mathrm{d}\vartheta$, the field strength $\mathrm{d}A$ is invariant, then the action changes by

$$\hat{\delta} S_{\mathrm{CS}} = \frac{k}{4\pi} \int_{\mathcal{M}} \hat{\delta} A \wedge \mathrm{d}A = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}\vartheta \wedge \mathrm{d}A = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}\left(\vartheta \mathrm{d}A\right) = \frac{k}{4\pi} \int_{\partial\mathcal{M}} \vartheta \mathrm{d}A. \tag{8.3}$$

If the manifold has no boundary $\partial\mathcal{M} = \varnothing$, then the right hand side vanishes and Chern-Simons theory is gauge invariant. However, if there is a boundary, then it is not gauge invariant; to make it gauge invariant (which every gauge theory shall be), a boundary term has to be added to offset the right hand side of the above equation under gauge transformation. The boundary term itself is a rich and interesting story which is beyond the scope of the notes.

In components, using (2.7), we can write

$$\begin{aligned} A \wedge \mathrm{d}A &= A_\rho \mathrm{d}x^\rho \wedge \mathrm{d}\left(A_\nu \mathrm{d}x^\nu\right) = A_\rho \mathrm{d}x^\rho \wedge \left(\partial_\mu A_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) \\ &= A_\rho \partial_\mu A_\nu \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \hat{\epsilon}^{\rho\mu\nu} A_\rho \partial_\mu A_\nu \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \end{aligned}$$

Therefore, the action can be written as

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}^3 x \hat{\epsilon}^{\rho\mu\nu} A_\rho \partial_\mu A_\nu. \tag{8.4}$$

When $\mathcal{M} = \mathbb{R}^{2,1}$ is the Minkowskian space, we can further write

$$\begin{aligned} \hat{\epsilon}^{\rho\mu\nu} A_\rho \partial_\mu A_\nu &= \hat{\epsilon}^{0ij} A_0 \partial_i A_j + \hat{\epsilon}^{i0j} A_i \partial_0 A_j + \hat{\epsilon}^{ij0} A_i \partial_j A_0 \\ &= A_0 \hat{\epsilon}^{ij} \partial_i A_j - \hat{\epsilon}^{ij} A_i \partial_0 A_j + \hat{\epsilon}^{ij} A_i \partial_j A_0. \end{aligned}$$

Using $B = \hat{\epsilon}^{ij} \partial_i A_j$ and $F_{i0} = \partial_i A_0 - \partial_0 A_i = \frac{1}{c} E_i$, we can rewrite it as

$$\hat{\epsilon}^{\rho\mu\nu} A_\rho \partial_\mu A_\nu = A_0 B + \frac{1}{c} \hat{\epsilon}^{ij} A_i E_j = A_0 B + \frac{1}{c} \vec{A} \times \vec{E}. \tag{8.5}$$

Alternatively, the last term can be written as

$$\hat{\epsilon}^{ij} A_i \partial_j A_0 = \partial_j \left(\hat{\epsilon}^{ij} A_i A_0\right) - A_0 \hat{\epsilon}^{ij} \partial_j A_i = -\partial_i \left(\hat{\epsilon}^{ij} A_j A_0\right) + A_0 \hat{\epsilon}^{ij} \partial_i A_j.$$

Then

$$\hat{\epsilon}^{\rho\mu\nu} A_\rho \partial_\mu A_\nu = 2 A_0 B - \hat{\epsilon}^{ij} A_i \partial_0 A_j - \partial_i \left(\hat{\epsilon}^{ij} A_j A_0\right). \tag{8.6}$$

Thus the action can be written as

$$\begin{aligned} S_{\mathrm{CS}} &= \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}^3 x \left(A_0 B + \frac{1}{c} \vec{A} \times \vec{E}\right) \\ &= \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}^3 x \left(2 A_0 B - \hat{\epsilon}^{ij} A_i \partial_0 A_j\right) + \text{boundary terms}. \end{aligned} \tag{8.7}$$

We see the action in components is very different than that of the electrodynamics in 2+1 dimensions.

Although Chern-Simons theory alone can be boring in Minkowskian space, it is not if it lives on a manifold of non-trivial topology. It is even more interesting when it is turned into a quantum field theory on such a manifold. This is the subject of *topological quantum field theory*, a very active research area lying at the intersection of theoretical physics and mathematics, due to Witten's ground-breaking work in 1980s [4]. For an early review, see [5]. Another way to make it more interesting is to couple it to matters, i.e. adding additional terms to the action, such as Maxwell action or Schrödinger action. The latter case arises as the low energy effective theory of fractional quantum Hall effect, which has been another very active research area in condensed matter physics since the late 1980s. For an introduction of this aspect of Chern-Simons theory, see [6]. A general introduction to Chern-Simons theory in physics can be found in [7].

8.2 Chern-Simons Theory and Linking Number

To get a taste of the topological aspect of the Chern-Simons theory, let us consider the above Chern-Simons theory in three dimensional Euclidean space. Now the manifold $\mathcal{M}$ over which the Chern-Simons 3-form is integrated is $\mathbb{R}^3$, instead of $\mathbb{R}^{2,1}$, i.e. no time. This is similar to magnetostatics. Furthermore, we couple it to a current J^μ through an $A_\mu J^\mu$ term in the action. Thus the action can be written as

$$S_{\mathrm{CS}} = \int \left(\frac{k}{4\pi} \epsilon^{ijk} A_i \partial_j A_k + A_i J^i \right) \mathrm{d}^3\vec{r} = \int \left(\frac{k}{4\pi} \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{J} \right) \mathrm{d}^3\vec{r}, \tag{8.8}$$

where $B^i = \epsilon^{ijk}\partial_j A_k$, i.e. $\vec{B} = \nabla \times \vec{A}$. The equation of motion obtained by varying $\vec{A}$ is

$$\frac{k}{2\pi}\vec{B} + \vec{J} = 0. \tag{8.9}$$

Thus $\vec{B} \sim \vec{J}$ everywhere! Physically, this means locally the magnetic flux lines ($\vec{B}$ field) is attached to the current $\vec{J}$. This phenomenon is known as "*flux attachment*". The 2+1 dimensional (Minkowskian) version is an important component in our understanding of the fractional quantum Hall effect, which is the archetype of topological phases studied in condensed matter physics. As usual, to solve the equation, we need to fix a gauge first. Let us choose the Coulomb gauge. Thus we have two equations for the vector potential

$$\begin{aligned} \nabla \cdot \vec{A} &= 0, \\ \nabla \times \vec{A} &= -\frac{2\pi}{k}\vec{J}. \end{aligned} \tag{8.10}$$

Maxwellian Magnetostatics in 3+1 Dimensions

To solve the above equations, let us first recall the Maxwellian magnetostatics in 3+1 dimensions, the one we have learned in undergraduate electromagnetism module. Setting ∂_t and $\vec{E}$ to zero in the full Maxwell's equations, we obtain the equations for magnetostatics in three spatial dimensions

$$\begin{aligned} \nabla \cdot \vec{B} &= 0, \\ \nabla \times \vec{B} &= \mu_0 \vec{J}. \end{aligned} \tag{8.11}$$

Recall how we solve them in our electromagnetism module: introducing the vector potential as $\vec{B} = \nabla \times \vec{A}$, using the identity

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A},$$

and employing the Coulomb gauge, we end up with the following two equations

$$\begin{aligned} \nabla \cdot \vec{A} &= 0, \\ \nabla^2 \vec{A} &= -\mu_0 \vec{J}. \end{aligned} \tag{8.12}$$

Each of the three components of the second equation is now a Poisson equation. Using the Green's function/propagator for the Laplace operator in three dimensions

$$\nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi\delta^{(3)}(\vec{r} - \vec{r}'), \tag{8.13}$$

we can write down the solution as (here we assume there is no boundary, i.e. infinite space)

$$\vec{A}\,(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}'. \tag{8.14}$$

Acting ∇^2 on it (acting or $\vec{r}$, not on $\vec{r}'$), and using (8.13), it is straightforward to verify that it solves the second equation in (8.12). Calculating the divergence of it

$$\begin{aligned}
\nabla \cdot \vec{A}\,(\vec{r}) &= \frac{\mu_0}{4\pi} \int \left(\nabla_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|} \right) \cdot \vec{J}(\vec{r}') \mathrm{d}^3\vec{r}' \\
&= \frac{\mu_0}{4\pi} \int \left(-\nabla_{\vec{r}'} \frac{1}{|\vec{r}-\vec{r}'|} \right) \cdot \vec{J}(\vec{r}') \mathrm{d}^3\vec{r}' \\
&= \frac{\mu_0}{4\pi} \int \frac{1}{|\vec{r}-\vec{r}'|} \left[\nabla_{\vec{r}'} \cdot \vec{J}(\vec{r}') \right] \mathrm{d}^3\vec{r}',
\end{aligned}$$

where we have used the translational invariance of the propagator to replace $\nabla_{\vec{r}}$ by $-\nabla_{\vec{r}'}$ in the second step, and integrated by parts in the last step. Recall that the current $\vec{J}$, whose associated density is ρ, satisfies the continuity equation

$$\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0,$$

where in the static case $\partial_t = 0$, it reduces to $\nabla \cdot \vec{J} = 0$. Thus the above expression for $\nabla \cdot \vec{A}$ vanishes, hence the first equation in (8.12) is also satisfied. Thus (8.14) solves (8.12). Calculating the curl of it, we obtain $\vec{B}$

$$\vec{B}\,(\vec{r}) = \frac{\mu_0}{4\pi} \int \left(\nabla_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|} \right) \times \vec{J}(\vec{r}') \mathrm{d}^3\vec{r}'.$$

We can still replace $\nabla_{\vec{r}}$ by $-\nabla_{\vec{r}'}$ and then integrate by parts, or directly evaluate the above gradient. Thus we have two equivalent expressions for $\vec{B}$

$$\vec{B}\,(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\nabla \times \vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}' = -\frac{\mu_0}{4\pi} \int \frac{(\vec{r}-\vec{r}') \times \vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|^3} \mathrm{d}^3\vec{r}'. \tag{8.15}$$

Using this result we can obtain the interaction potential energy between two currents. Recall that the coupling between a current and a magnetic field adds a term $\int \vec{J} \cdot \vec{A} \mathrm{d}^3\vec{r}$ to the Lagrangian, which contributes $-\int \vec{J} \cdot \vec{A} \mathrm{d}^3\vec{r}$ to the Hamiltonian (potential energy), just like the electric one $\int \rho\phi \mathrm{d}^3\vec{r}$, but with a minus sign. A current $\vec{J}_1$'s potential energy under the vector potential $\vec{A}_2$ produced by another current $\vec{J}_2$ given by (8.14) is thus $H_{\text{int}} = -\int \vec{J}_1 \cdot \vec{A}_2 \mathrm{d}^3\vec{r}$, which is

$$H_{\text{int}} = -\frac{\mu_0}{4\pi} \iint \frac{\vec{J}_1(\vec{r}_1) \cdot \vec{J}_2(\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|} \mathrm{d}^3\vec{r}_1 \mathrm{d}^3\vec{r}_2. \tag{8.16}$$

This is very similar to the Coulomb potential for electrostatics

$$H_{\text{int}} = \frac{1}{4\pi\varepsilon_0} \iint \frac{\rho_1(\vec{r}_1)\rho_2(\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|} \mathrm{d}^3\vec{r}_1 \mathrm{d}^3\vec{r}_2. \tag{8.17}$$

As Coulomb potential implies like charges repel and opposite charges attract (*screening* effect), due to the *minus* sign, H_{int} for currents implies the opposite: parallel currents attract and anti-parallel currents repel (*anti-screening* effect).

Chern-Simons Magnetostatics and Gauss Linking Integral

Going back to the Chern-Simons theory, we notice the similarity between the Chern-Simons equations (8.10) and the Maxwellian equations (8.11), both in three dimensional Euclidean space, with the identification $\vec{A} \leftrightarrow \vec{B}$ and $-\frac{2\pi}{k} \leftrightarrow \mu_0$. Therefore, substituting these into (8.15), we immediately obtain the solution to (8.10)

$$\vec{A}(\vec{r}) = \frac{1}{2k}\int \frac{(\vec{r}-\vec{r}')\times\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|^3}\mathrm{d}^3\vec{r}'. \tag{8.18}$$

Now let us evaluate the on-shell[4] Chern-Simons action: plugging the equation of motion $\vec{B} = -\frac{2\pi}{k}\vec{J}$ into the Chern-Simons action to replace $\vec{B}$ by $\vec{J}$, we obtain

$$S_{\mathrm{CS}} = \frac{1}{2}\int \vec{J}\cdot\vec{A}\mathrm{d}^3\vec{r}.$$

Plugging in the solution (8.18), it can be rewritten as

$$S_{\mathrm{CS}} = -\frac{1}{4k}\iint \frac{(\vec{r}-\vec{r}')\cdot\left[\vec{J}(\vec{r})\times\vec{J}(\vec{r}')\right]}{|\vec{r}-\vec{r}'|^3}\mathrm{d}^3\vec{r}\mathrm{d}^3\vec{r}'. \tag{8.20}$$

For a single line current along a contour $\mathcal{C}$ parameterized by the orbit equation $\vec{r}(\lambda)$, the current can be written as

$$\vec{J}(\vec{r}) = e\frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda}\delta^{(2)}\left(\vec{r}-\vec{r}(\lambda)\right), \tag{8.21}$$

where the two dimensional δ-function is in the transverse plane (at a fixed λ) whose intersection with the contour $\mathcal{C}$ is a point located at $\vec{r}(\lambda)$. Often in physics this is written in a more familiar form as $\vec{J}(\vec{r}) = e\vec{v}\delta^{(2)}\left(\vec{r}-\vec{r}(\lambda)\right)$, where e is the charge, and the role of velocity $\vec{v}$ is played by $\frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda}$, the tangent vector of the contour $\mathcal{C}$, here. Accordingly, we can decompose the volume element into the product of that of the transverse directions $\mathrm{d}^2\vec{r}_\perp$ and that of the tangent direction $\mathrm{d}\lambda$: $\mathrm{d}^3\vec{r} = \mathrm{d}^2\vec{r}_\perp\mathrm{d}\lambda$, and the transverse dimensions can be easily integrated out due to the δ-function

$$\int \mathrm{d}^3\vec{r}\vec{J}(\vec{r})\ldots = \int\mathrm{d}\lambda\int\mathrm{d}^2\vec{r}_\perp e\frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda}\delta^{(2)}\left(\vec{r}-\vec{r}(\lambda)\right)\ldots = e\int_{\mathcal{C}}\mathrm{d}\lambda\frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda}\ldots = e\int_{\mathcal{C}}\mathrm{d}\vec{r}\ldots, \tag{8.22}$$

thus a volume integral over a line current becomes a contour integral. Now let us consider that the line current is composed of two interwoven loops $\mathcal{C}_1$ and $\mathcal{C}_2$, with unit charge $e=1$. Thus we can write

$$\int \mathrm{d}^3\vec{r}\vec{J}(\vec{r})\ldots = \oint_{\mathcal{C}_1}\mathrm{d}\vec{r}_1\ldots + \oint_{\mathcal{C}_2}\mathrm{d}\vec{r}_2\ldots.$$

[4]In quantum field theory, the on-shell action is a result of integrating out the gauge field $\vec{A}$

$$\mathcal{Z}[\vec{J}] = \int\mathcal{D}\vec{A}e^{iS[\vec{A};\vec{J}]}. \tag{8.19}$$

Because the action is at most quadratic in $\vec{A}$ (i.e. a Gaussian integral), the net effect of the path integral is the same as imposing the equation of motion of $\vec{A}$, which is equivalent to evaluating the integral using the steepest descent method.

Now the on-shell Chern-Simons action becomes

$$S_{\mathrm{CS}} = -\frac{1}{2k} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \cdot \left(\mathrm{d}^3\vec{r}_1 \times \mathrm{d}^3\vec{r}_2\right), \tag{8.23}$$

where we have dropped the ill-defined self-linking terms $\oint_{\mathcal{C}_1} \oint_{\mathcal{C}_1} \ldots + \oint_{\mathcal{C}_2} \oint_{\mathcal{C}_2} \ldots$. We recognize the Gauss linking integral[5] of two interwoven loops $\mathcal{C}_1$ and $\mathcal{C}_2$ as

$$\Gamma(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \cdot \left(\mathrm{d}^3\vec{r}_1 \times \mathrm{d}^3\vec{r}_2\right). \tag{8.24}$$

Thus the value of the on-shell Chern-Simons action can be expressed in terms of the linking number of the two loops

$$S_{\mathrm{CS}} = -\frac{2\pi}{k} \Gamma(\mathcal{C}_1, \mathcal{C}_2). \tag{8.25}$$

The linking number is a topological invariant in the sense that it is only determined by the braiding of the two loops, and is invariant under local deformations of the contours. This is one of the simplest examples showing the connection between *quantum* Chern-Simons theory and topology. This result was first derived by Polyakov in 1988 [8], which was further generalized in Witten's seminal work [4] the following year that eventually lead to his Fields medal in 1994.

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