Introduction to Group Theory: Summary

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Contents

1 Lecture 1: Definitions & Examples

Definition 1.1. A **group** is pair (G, m) such that G is a **set** and $m : G \times G \to G$ is a mapping from G to itself s.t.

- G is associative under m, ie. $m(a, (b, c)) = m((a, b), c) \; \forall \; a, b, c \in G$.

- G has a unit, ie. $\exists \; e \in G$ s.t. $m(e, g) = m(g, e) = g \; \forall \; g \in G$.

- Each element of G has an inverse, ie. $\forall \; a \in G, \exists \; b \in G$ s.t. $m(a, b) = m(b, a) = e$.

Remark. We usually write $m(a, b)$ as $a * b$, $a \cdot b$, or ab. Associativity becomes $a(bc) = (ab)c$.
We also write the inverse of element a as a^{-1}.
The notation (G, m) is rewritten simply as G for convenience.

Example.

1. $G = \{e\}$ (Trivial group)

2. $(\mathbb{Z}, +)$, $e = 0$, $a^{-1} = -a$ (Integers under addition)

3. $(\mathbb{Q}, +)$ (Rational numbers under addition)[1]

4. $(\mathbb{Q}^x = \mathbb{Q} \setminus \{0\}, *)$, $e = 1$, $a^{-1} = \frac{1}{a}$

5. $GL(n, \mathbb{R}) = \{n \times n \text{ matrix } A \text{ with entries in } \mathbb{R} | \det A \neq 0\}$, $e = \mathbb{I}$, $A^{-1} = A^{-1}$ (General linear group)

6. $S(X) = \{f : X \times X \to X | f \text{ bijective}\}$, $e = Id_X$, $f^{-1} = f^{-1}$

Definition 1.2. A group is **abelian** if all elements of the set are commutative under the mapping, ie. for group $G = (G, m)$, $ab = ba \; \forall \; a, b \in G$.[2] Note: $a * b$ often written as $a + b$ for abelian groups.

Proposition 1.1. *For any group the following is true.*

1. *The unit is unique*

2. *For each $a \in G$, a^{-1} is uniquely determined.*

3. *$(a^{-1})^{-1} = a$*

4. *$(ab)^{-1} = b^{-1}a^{-1}$*

5. *For any $a_1, ..., a_n$, the value of $a_1 \cdot ... \cdot a_n$ is independent of bracketing.*

Proof. Each numbered proof correspond to the respective number in the proposition.

1. Suppose $e, e\prime$ are both units of group G. Then $e = e'e = e'$. $\square$

2. Given $a \in G$, suppose $\exists \; b_1, b_2 \in G$ s.t. they both satisfy the conditions of the inverse of a. Then $b_1 = b_1 e = b_1(ab_2) = (b_1 a)b_2 = eb_2 = b_2$. $\square$

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[1] For any field F, $(F, +)$ and $(F \setminus \{0\}, *)$ are groups.
[2] 1, 2, 3 abelian. 4, 5 generally non-abelian ($n \geq 2$ in 4, $|X| \geq 3$ in 5.)

3. Let $b = (a^{-1})^{-1}$, therefore $ba^{-1} = e = a^{-1}b$. a satisfies this, and since the inverse is uniquely determined, $a = b = (a^{-1})^{-1}$.

4. $ab(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aa^{-1} = e$. Similar for $(b^{-1}a^{-1})ab$. Therefore $b^{-1}a^{-1}$ satisfies the conditions of the inverse of ab, and is therefore equal to $(ab)^{-1}$ since the inverse is uniquely determined.

5. Proof by induction. Let $f(a_1, ..., a_n)$ be a bracketing of $a_1, ..., a_n$. Define $f(a_1, ..., a_n) = (a_1(...(a_{n-1}a_n)...)) := m_n(a_1, ..., a_n)$.

   Induction on n:
   $n = 1, 2$: $m(a_1) = a_1$, $m_2(a_1, a_2) = m(a_1, a_2)$.
   $n \geq 3$: $f = m(f_1(a_1, ..., a_k), f_2(a_{k+1}, ..., a_n))$.
   By ind. hyp. $f_1 = m_k$, $f_2 = m_{n-k}$.
   It remains to show that $m(m_k, m_{n-k}) = m_n \ \forall \ k$.

   $$
   \begin{aligned}
   k = 1 &: m(a_1, m_{n-1}(a_2, ..., a_n)) = m_n(a_1, ..., a_n). \\
   k > 1 &: m(m_k(a_1, ..., a_k), m_{n-k}(a_{k+1}, ..., a_n)) = m(m(a_1, m_{k-1}(a_2, ..., a_k)), m_{n-k}(a_{k+1}, ..., a_n)). \\
   &= m(a_1, m(m_{k-1}(a_2, ..., a_k), m_{n-k}(a_{k+1}, ..., a_n))) \text{ by associativity.} \\
   &= m(a_1, m_{n-1}(a_2, ..., a_n)) = m_n(a_1, ..., a_n)
   \end{aligned}
   $$

   $\square$

Remark. Either of left or right inverse, uniquely characterise a^{-1}.

Proposition 1.2. *Left and right cancellation hold in any group.*

$$ax = ay \therefore x = y \tag{1}$$
$$xa = ya \therefore x = y \tag{2}$$

Proof. Multiply by a^{-1} from left, right respectively.

$\square$

Remark. Let (G, m), $m : G \times G \to G$ satisfy:

- $m(a, m(b, c)) = m(m(a, b), c)$ (Associativity)

- $\exists \ e \in G$ s.t. $m(e, g) = g$, $\forall \ g \in G$. (Left-unit)

- $\forall \ a \in G$, $\exists \ b \in G$ s.t. $m(b, a) = e$. (Left-inverse)

then (G, m) is a group.

Notation.

$$x^n = x \cdot (x, \ n - 2 \text{ times}) \cdot x, \ x^{-n} = x^{-1} \cdot (x^{-1}, \ n - 2 \text{ times}) \cdot x^{-1} \tag{3}$$
$$n \cdot x = x + x + x + ... + x, \ -n \cdot x = (-x) + (-x) + (-x) + ... + (-x) \text{ (For abelian)} \tag{4}$$

Definition 1.3. The **order** of $x \in G$ is the smallest $n \in \mathbb{Z}^+$ s.t. $x^n = e$. The order is denoted $|x| = n$.

Example.
- $G = \mathbb{C}^\times, x = i, |x| = 4$.

- $G = \mathrm{GL}(2, \mathbb{R}), x = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, |x| = 6$

2 Lecture 2: Integers Modulo n and the Quaternion Group

2.1 Integers Modulo n: $\mathbb{Z}/n\mathbb{Z}$

Definition 2.1. Let $a, b \in \mathbb{Z}$. We say a, b have the same residue mod n, and write $a \equiv b$ (mod n) if $\exists \, k \in \mathbb{Z}$ s.t. $a - b = k \cdot n$.

Given $a \in \mathbb{Z}$ denote by

$$\bar{a} = \{b \in \mathbb{Z} | b \equiv a \ (\text{mod n})\}$$
$$= \{a + kn \in \mathbb{Z} | k \in \mathbb{Z}\} \subseteq \mathbb{Z}$$

and define

$$\mathbb{Z}/n\mathbb{Z} = \{\bar{a} \subseteq \mathbb{Z} | a \in \mathbb{Z}\} \tag{5}$$

Lemma 2.1. • $a \equiv b \ (mod \ n) \ \Leftrightarrow \ \bar{a} = \bar{b}$

  • $\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, ..., \overline{n-1}\}$

Proof.

$$a \equiv b \ (\text{mod n}) \ \Rightarrow \ a = b + k \cdot n$$
$$\Rightarrow \ b = a - k \cdot n = a + l \cdot n, \ l \in \mathbb{Z}$$
$$\Rightarrow \ b = a \ (\text{mod n})$$

$$\mathbb{Z}/n\mathbb{Z} = \{\bar{a} \subseteq \mathbb{Z} | a \in \mathbb{Z}\}, \ \bar{a} = \{a + kn \in \mathbb{Z} | k \in \mathbb{Z}\}$$
$$\Rightarrow \ \forall \, a < n, \ \bar{a} = \{a + kn \in \mathbb{Z} | k \in \mathbb{Z}\},$$
$$\forall \, a \geq n, \ \bar{a} = \{n + b + kn \in \mathbb{Z} | k \in \mathbb{Z}, a = n + b\}$$
$$= \{b + (k+1)n | k \in \mathbb{Z}\} = \bar{b}$$

$$\Rightarrow \ \forall \, a \geq n, \ \bar{a} = \overline{a - n}$$
$$\Rightarrow \ \mathbb{Z}/n\mathbb{Z} = \{\bar{a} \subseteq \mathbb{Z} | a \in \mathbb{Z}, \ a < n\} = \{\bar{0}, \bar{1}, ..., \overline{n-1}\}$$

$\square$

Proposition 2.2. *The assignment $m(\bar{a}, \bar{b}) = \overline{a + b}$ is well defined, and $(\mathbb{Z}/n\mathbb{Z}, m)$ is an abelian group.*

Proof. Let $\bar{a}_1 = \bar{a}_2$, $\bar{b}_1 = \bar{b}_2$, this implies that $a_1 \equiv a_2$ (mod n), $b_1 \equiv b_2$ (mod n). It is then necessary that $a_1 + b_1 = a_2 + b_2$ (mod n). Thus $\overline{a_1 + b_1} = \overline{a_2 + b_2}$, and so m is well defined.

It remains to show that $(\mathbb{Z}/n\mathbb{Z}, m)$ is a group. Therefore we must show that it is associative, and contains a left-unit, and left-inverse.

Let $G = (\mathbb{Z}/n\mathbb{Z}, m)$. For $a, b, c \in G$,

$$m(a, b) = a + b, \ m(b, c) = b + c$$
$$\Rightarrow m(a, m(b, c)) = a + b + c = m(m(a, b), c),$$

therefore G is associative.

To show the existence of the left-unit, we must show there exists $e \in G$ s.t. $e + g = g \ \forall \, g \in G$. For $m(\bar{a}, \bar{b}) = \overline{a + b} = \bar{b}$, where $\bar{a}, \bar{b} \in G$, it is clear from this that $a = k \cdot n$ for some $k \in \mathbb{Z}$. Therefore $a = 0$ (mod n). This implies that $\bar{a} = \bar{0}$. Therefore $\bar{0}$ is the left-unit of G (and consequently right-unit as abelian).

To show the existence of the left-inverse, we must show that $\forall \, a \in G, \ \exists \, b \in G$ s.t. $m(a, b) = e$. If

$m(\bar{a}', \bar{b}') = \bar{0}$, then $a' = -b'$ (mod n). Therefore, $a' = n - b'$ (mod n), which implies that $\bar{a} = \overline{n - b}'$. Hence, $\forall \, \bar{g} \in G, \ \bar{g}^{-1} = \overline{n - g}$.

$G = (\mathbb{Z}/n\mathbb{Z}, m)$ then satisfies all the required conditions of a group. To show that G is abelian, one must only note that $a + b = b + a \ \forall \ a, b \in \mathbb{Z}$ which implies that $\overline{a + b} = \overline{b + a}$, and consequently $m(\bar{a}, \bar{b}) = m(\bar{b}, \bar{a}) \ \forall \bar{a}, \bar{b} \in G$.

$\square$

Notation. We write $a = \bar{a}$, for example, in $\mathbb{Z}/5\mathbb{Z}$, we write $2 + 3 = 0$.

Lemma 2.3. $1 \in \mathbb{Z}/n\mathbb{Z}$ *has order* n.

Proof.

$$n \cdot 1 = n = 0$$
$$k \cdot 1 = k \neq 0, \text{ for } 0 < k < n$$

$\square$

2.2 Quaternion Group

Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with $m : Q_8 \times Q_8 \to Q_8$ given by:

$$i^2 = j^2 = k^2 = -1$$
$$ij = k, \ ki = j, \ jk = i$$
$$ji = -k, \ ik = -j, \ kj = -i$$

where signs manipulate as expected.

Proposition 2.4. (Q_8, m) *is a group.*

Proof. Simple to show associativity, left-unit, left-inverse. Not done here.

$\square$

3   Lecture 3: Generators-Relations

Given a set $r_1, r_2, r_3, ..., r_l$ of words (relations) in $g_1^{\pm}, g_2^{\pm}, ..., g_k^{\pm}$ (generators). We can define a group

$$G = \langle g_1, ..., g_k | r_1, ..., r_l \rangle \tag{6}$$

This is called the presentation of a group. We will define the group more precisely later.

Elements of G are words (combinations) of $g_1^{\pm}, g_2^{\pm}, ..., g_l^{\pm}$ under the equivalence relation given by

- removing/adding $g_i g_i^{-1}, g_i^{-1} g_i, e$,

- replacing an occurence of r_i with e.

Example. Dihedral Group

$$D_{2n} = \langle r, s | r^n = s^2 = (sr)^2 = e \rangle \tag{7}$$

Let w try to enumerate all the elements of D_{2n}: If f is any word in $r^{\pm}, s^{\pm}$, use $r^{-1} = r^{n-1}$ and $s^{-1} = s$ to get a word in r, s. Since $s^2 = e$, we can assume

$$f = r^{i_1} s r^{i_2} s...s r^{i_l}, \; i_j > 0 \tag{8}$$

and then use $sr = (sr)^{-1} = r^{-1} s^{-1} = r^{n-1}$ to move the terms around and reach the form $f = sr^i$ or $f = r^i$.

$$\Rightarrow D_{2n} = \{e, r, ..., r^{n-1}, s, rs, ..., r^{n-1} s\} \tag{9}$$

These elements are not necessarily distinct.

D_{2n} is the group of symmetries on a regular n-gon. D_{2n} can be realised as

$$r = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \; \theta = \frac{2\pi}{n}, \; s = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

Remark.

- $G = \langle \text{gen.} | \text{rel.} \rangle$ is always a group.

- Generally, it is difficult to decide for $x \in G$, if $x = e$.

4   Lecture 4: Symmetric Group

The symmetric group is the group of bijective maps from a set of n elements to the its These map between permutations of these n elements.

$$S_n = \{\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\} | \sigma \text{ bij.}\} \tag{11}$$

Each element of S_n can be written in the form of the permutation it maps the original set to, ie. $(\sigma(1), ..., \sigma(n))$.

$$\sigma = \begin{pmatrix} 2 & 1 & 3 \end{pmatrix} \in S_3 \text{ or } \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3. \tag{12}$$

as well as this these maps can be decomposed into cycles. For example,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 45 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} \in S_5$$

$$1 \to 3 \to 5 \to 1, \; 2 \to 4 \to 2$$

$$\Rightarrow \sigma = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 4 \end{pmatrix}$$

this is called cycle decomposition.

Definition 4.1. Given $a_1, a_2, ..., a_l \in \{1, ..., n\}$, all distinct, we define an **'l-cycle'**: $S_n \ni \sigma := \begin{pmatrix} a_1 & ... & a_l \end{pmatrix}^3$ by the formula

$$\sigma(x) = \begin{cases} a_{j+1} & , \text{ if } x = a_j \\ x & , \text{ else} \end{cases} \tag{13}$$

Lemma 4.1. *Let $\sigma = \begin{pmatrix} a_1 & a_2 & ... & a_l \end{pmatrix}$ and $\tau = \begin{pmatrix} b_1 & b_2 & ... & b_k \end{pmatrix}$ be such that $\{a_1, a_2, ..., a_l\} \wedge \{b_1, b_2, ..., b_k\} = \emptyset$. Then $\sigma \cdot \tau = \tau \cdot \sigma$.*

Proof. Since the two sets, $A = \{a_1, a_2, ..., a_l\}, B = \{b_1, b_2, ..., b_k\}$ are disjoint, $\sigma(b_i) = b_i, \tau(a_i) = a_i$ and $\sigma(x) = \tau(x) = x \; \forall \; x$ not in A, B. Therefore it follows that $\sigma \cdot \tau(b_i) = \tau \cdot \sigma(b_i) = b_{i+1}, \sigma \cdot \tau(a_i) = \tau \cdot \sigma(a_i) = a_{i+1}$, and $\sigma \cdot \tau(x) = \tau \cdot \sigma(x) = x \; \forall \; x$ not in A, B. Therefore $\forall \; g \in \{1, ..., n\}, \tau \cdot \sigma = \sigma \cdot \tau$. $\square$

Proposition 4.2. *Every $\sigma \in S_n$ admits a decomposition into disjoint cycles.*

Proof. For some $\sigma \in S_n$, and $i \in \{1, ..., n\}$ be s.t. $i = \min\{j \in \{1, ..., n\} | \sigma(j) \neq j\}$. For some l_1, l_2, we have $\sigma^{l_1}(i) = \sigma^{l_2}(i)$. Therefore, $\sigma^{l_1 - l_2}(i) = i$, and w.l.o.g. $l = l_1 - l_2 > 0$. Set $\sigma = \begin{pmatrix} i & f(i) & ... & f^{l-1}(i) \end{pmatrix}$.
The proof is continued by replacing σ with $\sigma_1^{-1} \cdot \sigma$, and $i = \min\{j \in \{1, ..., n\} | \sigma(j) \neq j\}$ with $i = \min\{j \in \{1, ..., n\} \min\{j \in \{1, ..., n\} | \sigma_1^{-1} \cdot \sigma(j) \neq j\}$ and repeating. $\square$

Remark. Not every product of cycles is a cycles decomposition. ie. If the cycles are not disjoint.

$^3 \begin{pmatrix} a_1 & a_2 & ... & a_l \end{pmatrix} = \begin{pmatrix} a_2 & ... & a_l & a_1 \end{pmatrix}$

5 Lecture 5: The Category of Groups.

Consider $G = \{e, a, b, c\}$ with multiplication.

$$
\begin{array}{c|cccc}
 & e & a & b & c \\
\hline
e & e & a & b & c \\
a & a & e & c & b \\
b & b & c & a & e \\
c & c & b & e & a \\
\end{array}
\tag{14}
$$

Note that if we assign $a = 2, b = 1, c = 3$ this 'is' (isomorphic to, this will be defined later) $\mathbb{Z}/4\mathbb{Z}$.

Definition 5.1. Let G, H be groups. A group **homomorphism** is a map $\varphi : G \to H$ s.t.

$$
\varphi(a \cdot_G b) = \varphi(a) \cdot_H \varphi(b)
\tag{15}
$$

($\cdot_G, \cdot_H$ denote the binary maps of G, H respectively.)

Definition 5.2. If group homomorphism φ is a bijection, then it is called a **group isomorphism**. In this case G, H are **isomorphic**.

Proposition 5.1. *Let $\varphi : G \to H$ be a group homomorphism. Then*

1. *$\varphi(e_G) = e_H$*

2. *$\varphi(a^{-1}) = \varphi(a)^{-1}$*

Proof. From the definition of a group homomorphism:

$$
\begin{aligned}
\varphi(e_G \cdot_G b) &= \varphi(e_G) \cdot_H \varphi(b) \\
&= \varphi(b)
\end{aligned}
$$

This implies

$$
\varphi(b) = \varphi(e_G) \cdot_H \varphi(b)
$$

which can only be true if

$$
\varphi(e_G) = e_H
\tag{16}
$$

proving the first condition. The second condition begins in a similar way, from the definition of a group homomorphism we know

$$
\varphi(a \cdot_G a^{-1}) = \varphi(a) \cdot_H \varphi(a^{-1})
$$

but

$$
\begin{aligned}
\varphi(a \cdot_G a^{-1}) &= \varphi(e_G) = e_H \\
&\Rightarrow \varphi(a) \cdot_H \varphi(a^{-1}) = e_H \\
&\Rightarrow \varphi(a^{-1}) = \varphi(a)^{-1}
\end{aligned}
$$

$\square$

Definition 5.3. $|G|$ is called the **order** of G. This is defined as the number of elements in the group for a finite group (group in which the underlying set is finite) or infinity for a non-finite group.

Proposition 5.2. *Let G be a group of order 2. Then $G \cong \mathbb{Z}/2\mathbb{Z}$ (Isomorphic)*

Proof. G group $\Rightarrow \exists\, e \in G$, and $a \in G$ s.t $a \neq e$. Define map $\varphi : \mathbb{Z}/n\mathbb{Z} \to G$.

$$0 \mapsto e$$
$$1 \mapsto a$$

We must then check that $\varphi(x + y) = \varphi(x)\varphi(y)\ \forall\ ,y \in F/n\mathbb{Z}$:

x	y	
0	0	Trivially true
1	1	$\varphi(1+1) = \varphi(0) = e$
		$\varphi(1)\varphi(1) = a^2$
		Only true if $a^2 = e$, which must be true since $a^2 = a$ implies $a = e$.
0	1	True since $\varphi(1) = a$, $e\varphi(1) = a$
1	0	True since $\varphi(1) = a$, $e\varphi(1) = a$

$\square$

Proposition 5.3.

1. *Let $f : H \to K$, $g : G \to H$ be group homomorphisms, then so is $f \circ g$.*

2. *Let $f : H \to K$ be a group isomorphism, then so is f^{-1}.*

Proof.

1. $(f \circ g)(ab) = f(g(ab) = f(g(a)g(b)) = f(g(a))f(g(b)) = (f \circ g)(a)(f \circ g)(b)$
   Therefore $f \circ g$ satisfies the condition of a group homomorphism.

2. $f(f^{-1}(ab)) = ab = f(f^{-1}(a))f(f^{-1}(b))$, then since f is injective, $f(p) = f(q)$ implies $p = q$. Therefore $f^{-1}(ab) = f^{-1}(a)f^{-1}(b)$, and f^{-1} is then a group homomorphism.

$\square$

6   Lecture 6: Group Actions

Definition 6.1. A **group action** of a group G on a set X is a map $G \times X \to X$ written as $(g, x) \mapsto g.x$ s.t.

- $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

- $e \cdot x = x$

We sometimes write $G \circlearrowleft X$.

A map of sets $G \times X \to X$ can be equivalently given by

$$G \to^{\rho} \{f : X \to X\}$$
$$g \mapsto \rho(g)$$

where $\rho(g)(x) = g \cdot x$.

Proposition 6.1. *A map $G \times X \to X$ defines a group action iff[4] the corresponding map $G \to^{\rho} \{f : X \to X\}$ is s.t. $\rho(g) \in S_X \; \forall \; g \in G$ and $\rho : G \to S_X$ is a group homomorphism.[5]*

Proof.

$$\rho(g_1)(\rho(g_2)(x)) = \rho(g_1 g_2)(x) \;\Leftrightarrow\; \rho(g_1) \circ \rho(g_2) = \rho(g_1 g_2)$$
$$\Leftrightarrow\; \rho(e) = \mathrm{id}_X$$

$$\text{`` }\Rightarrow\text{ '' : } \rho(g) \circ \rho(g^{-1}) = \rho(gg^{-1}) = \rho(e) = \mathrm{id}_X \;\Rightarrow \rho(g) \text{ surj.}[6]$$
$$\rho(g^{-1}) \circ \rho(g) = \rho(g^{-1}g) = \mathrm{id}_X \;\Rightarrow \rho(g) \text{ inj.}[7]$$

This implies that $\rho(g) \in S_X$, as the requirement is that the map is bijective. Since the first line, at the start of the proof, is true for any for any group action, this implies that $\rho : G \to S_X$ is a group homomorphism. Then, since all steps taken are reversible, the same process can be taken in reverse to show the bijectivity of $\rho(g)$, and ρ being a group homomorphism, implies that the map $G \times X \to X$ is a group action, proving `` $\Leftarrow$ ''.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Example.

1. Trivial action: For any X, we define $G \times X \to X$ s.t. $(g, x) \mapsto x$.

2. Defining action of S_X on X: $S_X \times X \to X$, $(\sigma, x) \mapsto \sigma(x)$.
   We can claim that the map ρ from above is in this scenario $\mathrm{id} : S_X \to S_X$.

3. G acting on itself by

   $\rho_l : G \times G \to G \qquad (g, x) \mapsto gx \qquad$ Called **left regular action**.

   $\rho_r : G \times G \to G \qquad (g, x) \mapsto xg^{-1} \qquad$ Called **right regular action**.   Verification of each as a group

   $\rho_{adj.} : G \times G \to G \quad (g, x) \mapsto gxg^{-1} \qquad$ Called **adjoint action**.

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[4]If and only if

[5]S_X is the symmetric group on set X

[7]Maps are surjective if and only if a right inverse exists for each element. Maps are injective if and only if a left inverse exists for each element.

action:

ρ_l :

We can let the map corresponding to $\rho_l, \phi : G \to \{f : X \to X\}$ be s.t. $\phi(g)(x) = gx$.
Then $\phi(g_1)(\phi(g_2)(x)) = \phi(g_1)(g_2x) = g_1g_2x = \phi(g_1g_2)(x)$.
$\Rightarrow \phi(g_1) \circ \phi(g_2) = \phi(g_1g_2), \phi(e) = e$
As in the above proof, this implies both left and right inverses therefore $\rho_l \in S_X$

ρ_r :

Analagous to ρ_l

$\rho_{adj.}$:

Similar to ρ_l, define $\phi(g)(x) = gxg^{-1}$. Then $\phi(g_1)(\phi(g_2)(x)) = \phi(g_1)(g_2xg_2^{-1})$
$= g_1g_2xg_2^{-1}g_1^{-1} = (g_1g_2)x(g_1g_2)^{-1} = \phi(g_1g_2)(x)$.
$\Rightarrow \phi(g_1) \circ \phi(g_2) = \phi(g_1g_2), \phi(e) = e$
Same conclusion can be drawn as in the case of ρ_l.

4. $D_{2n} \times \{1, ..., n\} \to \{1, ..., n\}$:

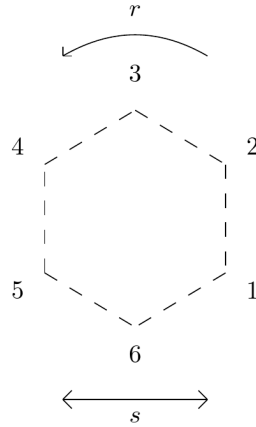$$(r^i, j) \mapsto i + j \bmod n$$
$$(r^i s, j) \mapsto i - j \bmod n$$

Figure 1: Illustration of the action of the D_{12} group on the set $\{1, ..., n\}$. The element s of D_{2n} reflects across the vertical, and the element r rotates by $\frac{2\pi}{n}$ in the anti-clockwise direction.

7 Lecture 7: Subgroups

Definition 7.1. A subset $H \leq G$ of a group (G, m) is a **subgroup** if the restriction of m to $H \times H$ turns H into a group. We write $H \leq G$ in that case.

Remark. To specify, H is required to have contain the unit, inverses and be associative, it is also required to have closure under m. That is to say $\forall\, a, b \in H, m(a, b) \in H$.

Proposition 7.1. $H \subseteq G$ *is a subgroup iff*

- H *is non-empty.*

- *if $a, b \in H$, $ab^{-1} \in H$.*

Proof.

" $\Rightarrow$ " : $H \leq G \Rightarrow \exists\, e \in H$ s.t. $ea = a \;\forall\, a \in H$.

$\therefore H$ non-empty.

$H \leq G \Rightarrow \forall\, b \in H, \exists\, c \in H$ s.t. $bc = e \; (c = b^{-1})$

$\therefore \forall\, a, b \in H, m(a, b^{-1}) = ab^{-1} \in H$ due to closure under m

" $\Leftarrow$ " : As a subset of G, associativity implied.

H non-empty $\therefore \exists\, h \in H$. Second condition implies $aa^{-1} \in H \Rightarrow e \in H$. (unit exists)

w.l.o.g. we can impose that the subset H is not the trivial group, ie. $H \neq (\{h = e\}, m)$,

$\Rightarrow \exists\, h \in H$ s.t. $h \neq e. \Rightarrow eh^{-1} = h^{-1} \in H$. (inverses exist)

for any 2 elements $a, b \in H, ab^{-1} \in H$ and $b^{-1} \Rightarrow a, b^{-1} \in H$ so $ab \in H$. (closure)

$\therefore H \leq G$.

$\square$

Example.

1. Every group G has a trivial subgroup, $(\{e\}, m) \leq G$, and an improper subgroup, $G \leq G$.

2. $n\mathbb{Z} = \{nk | k \in \mathbb{Z}\} \subseteq \mathbb{Z}$ is a subgroup of $(\mathbb{Z}, +)$.

3. Given $x \in G$, $\langle x \rangle = \{x^n | n \in \mathbb{Z}\} \subseteq G$ is the subgroup of G generated by x.

4. Let $G \times X \to X$ be a group action, and $s \in X$. The **stabilizer of** s, $G_s := \{g \in G | g \cdot s = s\}$, forms a subgroup of G.

5. Let $\varphi : G \to H$ be a group homomorphism. Then

   - $\ker \varphi := \{g \in G | \varphi(g) = e\}$
   - $\operatorname{im} \varphi := \varphi(G) = \{\varphi(g) \in H | g \in G\}$

   are both subgroups, of G and H respectively.

8   Lecture 8: Cyclic Groups and Subgroups.

8.1   Cyclic Groups

Definition 8.1. A group H is **cyclic** if it can be generated by a single element. That is to say $\exists \; x \in H$ s.t. $H = \langle x \rangle = \{x^n | n \in \mathbb{Z}\}$.

Proposition 8.1. *If $H = \langle x \rangle$, then $|H| = |x|$ [8]. Also*

1. *if $|H| = n < \infty$, then the elements $e, x, x^2, ..., x^{n-1}$ are all distinct and $H = \{e, x, ..., x^{n-1}\}$.*

2. *if $|H| = \infty$ then $H = \{..., x^{-2}, x^{-1}, e, x, x^2, ...\}$ and $x^a \neq x^b$ if $a \neq b$.*

Proof.

$|x| = n :$

We first show $H = \langle x \rangle \subseteq \{e, x, ..., x^{n-1}\}$.

Let $x^m \in H$, $m = kn + r$, $0 \leq r \leq n - 1$

$x^m = x^{kn+r} = x^n \cdot x^n \cdot ... \cdot x^n \cdot x^r = e \cdot x^r = x^r$

$\Rightarrow \; H = \langle x \rangle = \{x^r | 0 \leq r \leq n - 1\} = \{e, x, ..., x^{n-1}\}$.

To show $x^i \neq x^j \; \forall \; i, j$ s.t. $0 \leq i, j \leq n - 1$,

assume $x^i = x^j$, $\Rightarrow x^{i-j} = e$. But, $0 \leq i - j \leq n - 1$, contradicting $|x| = n$

$|x| = \infty :$

$x^i \neq x^j \; \forall \; i \neq j$ shown as in the first case.

$\square$

Proposition 8.2. *Let $H = \langle x \rangle$, then*

- *if $|H| = n$, there exists a group isomorphism defined by*

$$\varphi : \mathbb{Z}/n\mathbb{Z} \to H \tag{17}$$

$$k \mapsto x^k \tag{18}$$

- *and if $|H| = \infty$, there exists group isomorphism defined by*

$$\varphi : \mathbb{Z} \to H$$

$$k \mapsto x^k$$

Proof.

- φ well defined:
  Finite order: $k \equiv l \pmod{n} \Rightarrow k = l + mn$, $m \in \mathbb{Z}$.
  Then $x^k = x^{l+mn} = x^l (x^n)^m = x^l$. No equivalent terms in $\mathbb{Z}$ for non-finite order.

- φ group homomorphism:
  $\varphi(k_1 + k_2) = x^{k_1 + k_2} = x^{k_1} \cdot x^{k_2} = \varphi(k_1) \cdot \varphi(k_2)$

---

[8]The order of element $x \in G$ can also be defined as the order of the subgroup which it generates.

- φ bijective:
  Follows from previous proposition. Also provable through existence of left, right inverses.

  $\square$

8.2 Subgroups of Cyclic Groups

Proposition 8.3. *Let $G = \langle x \rangle$ be cyclic and $H \leq G$ be a subgroup. Then H is also cyclic.*

Proof. Trivial if $H = \{e\}$. Assume not the case, and let $l = \min\{m \in \mathbb{Z}^{>0} | x^m \in H\}$. Then it is obvious that $\langle x^l \rangle \subseteq H$ (closure of subgroup). We then let $x^k \in H$ be an arbitrary element of H. We can write $k = l \cdot k' + r$, $0 \leq r \leq l - 1$, then $x^r = x^{k - lk'} = x^k (x^l)^{k'} \in H$ (closure of H). But $r < l$, and l is the minimum greater than 0. Hence, $r = 0, x^r = e$, and $x^k = (x^l)^{k'} \in \langle x^l \rangle$. Thus, $H \subseteq \langle x^l \rangle$, but $\langle x^l \rangle \subseteq H$, and so $H = \langle x^l \rangle$. $\qquad\square$

Proposition 8.4. *Let $G = \langle x \rangle$ be an infinite cyclic group ($|G| = \infty$). Then the assignment $n \mapsto \langle x^n \rangle$ defines a bijection from $\mathbb{N}$ and subgroups of G.*

Proof. By Prop. 8.2.1, each subgroup of G is of the form $\langle x^n \rangle$, $n \in \mathbb{Z}$. Since $\langle x^{-n} \rangle = \langle x^n \rangle$, we can assume $n \in \mathbb{N}$ (hence, map is surjective). Suppose $\langle x^n \rangle = \langle x^m \rangle$, then

$$x^n = x^{km}$$
$$\Rightarrow n = km$$

Similarly $m = k'n$. Then $n = kk'n$, but $k, k' \in \mathbb{N}$, so $k = k' = 1$ and $m = n$ (map is then injective). Then since the outlined map is both injective and surjective, it is bijective. $\qquad\square$

Remark. From Thm. 8.1.1, $G = \langle x \rangle$ s.t. $|G| = \infty$ is isomorphic to $\mathbb{Z}$. Since each element x^n is mapped to n, each subgroup $\langle x^n \rangle$ is mapped to the corresponding subgroup generated by n, $\{..., -n-n, -n, 0, n, n+n, ...\}$ which is written as $n\mathbb{Z} = \{n \cdot k | k \in \mathbb{Z}\}$.

9 Lecture 9: Euclidean Algorithm

Definition 9.1. We say m is a divisor of n if $\exists\, k \in \mathbb{Z}$ s.t. $n = km$ and write $m|n$ in that case.

Example.

- $1|n\ \forall n \in \mathbb{Z}$

- $d|m,\ d|n\ \Rightarrow\ d|m \pm n$

- $n|0,\ \forall n$

- $d|n\ \Rightarrow\ |d| \le |n|,\ if n \ne 0$

- $n|n,\ \forall n$

Definition 9.2. For $m, n \in \mathbb{Z}$ we define the greatest common divisor, $\gcd(m,n) := (m,n) := \max\{d \in \mathbb{Z}^{>0}|\ d|m,\ d|n\}$, we set $(0,0) = 0$.

Lemma 9.1.

1. $(m,n) = (n,m)$

2. $(m,n) = (m, n + am)\ \forall\, a \in \mathbb{Z}$

3. $(m,n) = (r,n),\ \forall\, r \equiv m\ (\text{mod } n)$

4. $(m,0) = |m|$

Proof.   1. Trivial, from definition of g.c.d., changing order does not change set of common denominators.

2. It is sufficient for this proof to show that the sets $A = \{d \in \mathbb{Z}^{>0}|\ d|m,\ d|n\}$ and $B = \{d \in \mathbb{Z}^{>0} \mid d|m,\ d|n + am\}$, for any $n \in \mathbb{Z}$, are equal.

   For any $d \in A$, $d|m$, $d|n$. We can then write $m = dp$, $n = dq$. It follows that $m + an = d(p + aq)$, and so $d|m + an$. Thus, $A \subseteq B$.
   For any $d \in B$, an analogous argument shows that $B \subseteq A$. Therefore $A = B$, and so $\max\{d \in \mathbb{Z}^{>0}|\ d|m,\ d|n\} = \max\{d \in \mathbb{Z}^{>0}|\ d|m,\ d|n + am\}$, and so $(m,n) = (m, n + am)\ \forall\, n \in \mathbb{Z}$.

3. Since $r \equiv m\ (\text{mod } n)$, $(r,n) = (m + an, n) = (m,n)$ by the 2nd part of Lemma 9.1.

4. For $d \in \mathbb{Z}$ s.t. $d|m$, it must be that $d|0$ also (0 divisible by all integers). Therefore, $\{d \in \mathbb{Z} \mid d|m\} \subseteq \{d \in \mathbb{Z} \mid d|0\}$ and so $(m,0) = \max\{d \in \mathbb{Z} \mid d|m,\ d|0\} = \max\{d \in \mathbb{Z} \mid d|m\} = |m|$.
   $\square$

Proposition 9.2 (Euclidean Algorithm). *Let $m, n \in \mathbb{Z}$, then $\exists\, a, b$ s.t*

$$(m,n) = am + bn \tag{19}$$

    The procedure of Euclid's algorithm to find (m,n) for $m, n \in \mathbb{Z}$, where we can presume $m > n$ w.l.o.g. (swapping), involves subtracting the maximum multiples of n from m s.t the remainder is non negative . This makes use of the second part of Lemma 9.1. The procedure is repeated until one of the remainders is zero, in which case the g.c.d. is $(m,n) = (r_{n-1}, 0) = |r_{n-1}|$. $r_{n-1}|$ is the remainder produced after $n - 1$ iterations of the procedure.

Proof. For $m, n \in \mathbb{Z}$, to find (m, n), we use Lemma 9.1. First we can assume $m \geq n$, then $(m, n) = (n, r_1)$ where $r_1 \equiv m \mod n$. Since $0 \leq r_1 \leq |n|$, we can replace n with $r_2 \equiv n \mod |r_1|$. This is repeated until $r_{l+1} = 0 = r_{l-1} \mod r_l$. Then

$$(m, n) = (r_l, r_{l+1}) = (r_l, 0) = |r_l|$$

Since $r_{i+1} + k_{i+1} r_i = r_{i-1}$,

$$\begin{pmatrix} r_i \\ r_{i+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & k_{i+1} \end{pmatrix}}_{A_i} \begin{pmatrix} r_{i-1} \\ r_i \end{pmatrix}$$

Thus

$$\begin{pmatrix} r_l \\ 0 \end{pmatrix} = A_l \cdot A_{l-1} \cdot \ldots \cdot A_1 \cdot A_0 \cdot \begin{pmatrix} m \\ n \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} m \\ n \end{pmatrix}$$

$\square$

10   Lecture 10: Subgroups of Finite Cyclic Groups

Proposition 10.1. *Let $G = \langle x \rangle$ be a cyclic group of order $|G| = n$. Then*

    1. $\langle x^l \rangle = \langle x^{(l,n)} \rangle$

    2. $|\langle x^l \rangle| = \frac{n}{(l,n)}$

In particular there is a one-to-one correspondence

$$A = \{d \in \mathbb{Z}^{>0} \mid d|n\} \;\leftrightarrow\; B = \{H \subseteq G | H \leq G\}$$
$$d \in A \mapsto \langle x^d \rangle$$

Proof.

    1. It is required to show that the two groups are subsets of each other and so are equivalent.

$$\text{`` } \subseteq \text{ ''} \quad d = (l, n) \Rightarrow \exists k \in \mathbb{Z}^{>0} \text{ s.t. } l = kd.$$
$$x^l = x^{kd} = (x^d)^k \in \langle x^d \rangle.$$
$$\Rightarrow \langle x^l \rangle \subseteq \langle x^d \rangle.$$

$$\text{`` } \supseteq \text{ ''} \quad \text{By proposition 9.2, } d = al + bn.$$
$$x^d = (x^l)^a \cdot (x^n)^b = (x^l)^a \in \langle x^l \rangle$$
$$\Rightarrow \langle x^d \rangle \subseteq \langle x^l \rangle$$

$$\Rightarrow \langle x^d \rangle = \langle x^l \rangle$$

    2. For some $\langle x^l \rangle$, we can assume $l = d \cdot k$, where $d = (l, n)$, by 1. The order of the group generated by $\langle x^d \rangle$ is the smallest $k \in \mathbb{N}$ s.t. $(x^d)^k = e$ or $dk = mn$, for $m \in \mathbb{Z}$. Since $d|n$, we can let $m = 1$ w.l.o.g., and so $k = \frac{n}{d} = \frac{n}{(l,n)}$.

Define

$$\varphi : \{d \in \mathbb{Z}^{>0} \mid d|n\} \to \{H \subseteq G | H \leq G\}$$
$$d \mapsto \langle x^d \rangle$$

Each subgroup of G is cyclic, ie. $\forall \, H \leq G$, $H = \langle x^l \rangle$, $l \in \mathbb{Z}$. By proposition 10.1, 1., $\langle x^l \rangle = \langle x^{(l,n)} \rangle$. Since $(l, n)|n$, $(l, n) \in \{d \in \mathbb{Z}^{>0} \mid d|n\}$ and so φ is surjective.

Suppose $\varphi(d_1) = \varphi(d_2)$, then by proposition 10.1, 2., $\frac{n}{d_1} = \frac{n}{d_2}$. Therefore $d_1 = d_2$. Thus, φ is injective. $\quad\square$

Remark. It is also shown that group G has a unique subgroup of order $\frac{|G|}{d}$ for each pos. divisor d of $|G|$.

Corollary 10.1.1. *For $x \in G$, $\langle x^k \rangle = \langle x \rangle$, then $(k, |x|) = 1$.*

Proof.

$$|\langle x^k \rangle| = |\langle x \rangle| = |x|$$
$$= \frac{|x|}{(k, |x|)} \quad \text{by Prop. 10.1}$$

It follows that $(k, |x|) = 1$.

$\quad\square$

Definition 10.1. Let G be a group. We define the **group of automorphisms** of G to be

$$\text{Aut}(G) = \{\varphi : G \to G \mid \varphi \text{ a group isomorphism}\} \tag{20}$$

Lemma 10.2. $\text{Aut}(G) \leq S_G$, *in particular,* $\text{Aut}(G)$ *is a group under composition.*

Proof. The symmetric group is defined in Eq. 11. The symmetric group on G is thereby the group of bijective maps from G to itself. This does not require the maps to be homomorphisms, which is the requirement which the group of automorphisms of G places on these maps.

Let $\varphi \in \text{Aut}(G)$, then φ is a group isomorphism, $\varphi : G \to G$. Then φ is bijective, and thus, $\varphi \in S_G$. Hence, $\text{Aut}(G) \subseteq S_G$.

It is then required to show $\text{Aut}(G)$ is a group under composition. The unit of S_G is id, since $\text{id}_G(g_1)\text{id}_G(g_2) = g_1 g_2$ and id_G is bijective, id_G is a group isomorphism. Thus $\text{id}_G \in \text{Aut}(G)$.

Since each $\varphi \in \text{Aut}(G)$ is bijective we can assume the existence of $\varphi \in S_G$ (bijective). Then since φ, $\varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) \; \forall \; g_1, g_2 \in G$ we can say $\varphi^{-1}(\varphi(g_1)\varphi(g_2)) = g_1 g_2 = \varphi^{-1}(\varphi(g_1))\varphi^{-1}(\varphi(g_2))$. Since $\varphi(g) \in G$, it follows that φ^{-1} is a group isomorphism, and so $\varphi \in \text{Aut}(G)$.

For $\varphi_1, \varphi_2 \in \text{Aut}(G)$, $\varphi_1(g_1 g_2) = \varphi_1(g_1)\varphi_1(g_2)$ and so $\varphi_2(\varphi_1(g_1 g_2)) = \varphi_2(\varphi_1(g_1))\varphi_2(\varphi_1(g_2))$, then $\varphi_2 \circ \varphi_1 \in \text{Aut}(G)$.

Therefore $\text{Aut}(G)$ is a group under composition.

$\square$

Proposition 10.3. *The group* $\text{Aut}(\mathbb{Z}/n\mathbb{Z})$ *is isomorphic to the multiplicative group of integers modulo n.*

$$\text{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}^\times := \{k \in \mathbb{Z}/n\mathbb{Z} \mid (k, n) = 1\} \tag{21}$$

Where $\mathbb{Z}/n\mathbb{Z}^\times$ *has group multiplication defined by* $\overline{k_1} \cdot \overline{k_2} = \overline{k_1 \cdot k_2}$, *and unit* $\overline{1}$.

Proof. Define map

$$\Psi : \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}^\times$$
$$\varphi \mapsto \varphi(1)$$

- Ψ is well defined:

$$\langle 1 \rangle = \mathbb{Z}/n\mathbb{Z} \Rightarrow \langle \varphi(1) \rangle = \mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle$$
$$\Rightarrow (n, \varphi(1)) = 1, \text{ by Cor. 10.1.1}$$
$$\Rightarrow \varphi(1) \in \mathbb{Z}/n\mathbb{Z}^\times$$

  The first step here relies on the fact that $n = |1| = |\varphi(1)|$ which can be shown from the requirements of a automorphism. It is then easy to show that each automorphism is mapped to an element of $\mathbb{Z}/n\mathbb{Z}^\times$.

- Ψ is injective:
  Assume $\Psi(\varphi_1) = \Psi(\varphi_2)$. Then $\varphi_1(1) = \varphi_2(1)$. Then

$$\varphi_1(k) = k\varphi_1(1) = k\varphi_2(1) = \varphi_2(k).$$

  Therefore $\varphi_1 \equiv \varphi_2$ and so Ψ is injective.

- Ψ is surjective: Let $l \in \mathbb{Z}/n\mathbb{Z}^\times$, then as shown previously, $\langle l \rangle = \langle 1 \rangle = \mathbb{Z}/n\mathbb{Z}$, and $|l| = n$. Hence $\varphi_l(k) := \overline{kl}$ defines a group isomorphism from $\mathbb{Z}/n\mathbb{Z}$ to $\langle l \rangle = \mathbb{Z}/n\mathbb{Z}$. Consuquently it is shown that

$$\forall \, l \in \mathbb{Z}/n\mathbb{Z}^\times, \; \exists \; \varphi_l \in \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \text{ s.t. } \Psi(\varphi_l) = l$$

and so Ψ is surjective.

- Ψ is a group homomorphism: Since $\varphi \in \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$, $\varphi(k) = \underbrace{\varphi(1) + \varphi(1) + \ldots + \varphi(1)}_{k \text{ times}} = k\varphi(1)$.

$$\Psi(\varphi_1 \circ \varphi_2) = \varphi_1 \circ \varphi_2(1) = \varphi_1(\varphi_2(1)) = \varphi_2(1)\varphi_1(1)$$
$$= \Psi(\varphi_1) \cdot \Psi(\varphi_2)$$

Therefore Ψ is a group isomorphism.

$\square$

Remark. We have also shown that if $(k,n) = 1$ then $\exists k' \in \mathbb{Z}$ s.t. $kk' \equiv 1 \pmod{n}$.
This is resulting from the Euclidean algorithm, in Eq. 9.2.

$$\exists a, b \in \mathbb{Z} \text{ s.t. } (k,n) = ak + bn = 1$$
$$\Rightarrow ak \equiv 1 \pmod{n}$$

11   Lecture 11: Normal Subgroups

Example. Define a group homomorphism

$$f : \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$$
$$k \mapsto \overline{k}$$

For this homomorphism, the kernel and preimage can be written as

$$\ker f = 5\mathbb{Z} \leq \mathbb{Z},$$
$$f^{-1}(\{\overline{k}\}) = \{l + 5k | k \in \mathbb{Z} \leq \mathbb{Z}$$

Definition 11.1. Let $H \leq G$. Given $g \in G$ we define the **left coset** of H in G to be

$$gH := \{gh | h \in H\} \tag{22}$$

and the **right coset** of H in G to be

$$Hg := \{hg | h \in H\} \tag{23}$$

Definition 11.2. We can from the previous definition now define the **quotient groups** of G by H to be

$$G/H := \{gH | g \in G\} \tag{24}$$
$$H \backslash G := \{Hg | g \in G\} \tag{25}$$

Notation. For $g \in G$ we write $[g] := gH \in G/H$

Lemma 11.1. *For $x, y \in G$, $H \leq G$,*

$$xH = yH$$
$$\Leftrightarrow x^{-1}y \in H$$
$$\Leftrightarrow y^{-1}x \in H$$

Proof.

$$" \Rightarrow " \ xH = yH$$
$$\Rightarrow \ y \cdot e \in yH = xH, \ \Rightarrow \ y \in xH$$
$$\exists h \in H \text{ s.t. } y = xh$$
$$\Rightarrow x^{-1}y = h \in H, \text{ consequently, } (x^{-1}y)^{-1} = y^{-1}x \in H$$
$$" \Leftarrow " \ x^{-1}y \in H$$
$$\Rightarrow x^{-1}y \in H$$
$$\Rightarrow h \in H, yh = x \underbrace{x^{-1}yh}_{\in H} \in xH$$
$$\text{Similarly, } y^{-1}x \in H \Rightarrow x \in yH$$

$\square$

Next, we ask when is G/H a group?
The answer is that it is a group when the map defined by

$$\phi : G \to G/H$$
$$g \mapsto [g]$$

is a group homorphism. We then require the multiplication of cosets defined by

$$[x] \cdot [y] := [xy]$$

to be well defined.

For $x \in G$, [x]=[xh] for any $h \in H$. We then let $[x] = [xh_1]$ and $[y] = [yh_2]$, $h_1, h_2 \in H$. We then require $[xy] = [xh_1yh_2]$.

Then, from the previous lemma, $(xy)^{-1}xh_1yh_2 \in H$, and $(xy)^{-1}xh_1yh_2 = y^{-1}x^{-1}xh_1yh_2 = y^{-1}h_1yh_2$. Then since $y^{-1}h_1yh_2 \in H$, $y^{-1}h_1y \in H \ \forall \ h_1 \in H$.