Summary of Topics Covered in Michaelmas Term: Equations of Mathematical Physics.

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Contents

1 Chapter 1: Diff. Equations

1. Integrating factor.
$$\dot{y}(x) + P(x)y(x) = 0, \quad y(x) = e^{-\int P(x)\mathrm{d}x} \tag{1}$$
2. Bernoulli Equation.
$$\dot{y} + P(x)y = Q(x)y^n \quad \text{(substitute } v = y^{1-n}\text{)} \tag{2}$$
3. Non-linear 1^{st} order O.D.E.'s.
    - If possible seperate,
    $$f(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x,y) \quad \rightarrow \quad p(y)\mathrm{d}y = q(x)\mathrm{d}x \tag{3}$$
    - Exact equations $M + N\dot{y} = 0$
    $$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}, \quad \mathrm{d}f = 0 \rightarrow f = 0 \tag{4}$$
4. 2^{nd} order linear O.D.E.'s
    - Constant coefficient ($y = e^{\lambda x}$, solve)
        - (hom.)
            (a) Real roots ($\lambda_1, \lambda_2 \in \mathbb{R}$): $y_H = \mathrm{A}e^{\lambda_1 x} + \mathrm{B}e^{\lambda_2 x}$
            (b) Complex roots ($\lambda_1, \lambda_2 \in \mathbb{C}$): $y_H = e^{ax}\left(\mathrm{A}\sin(bx) + \mathrm{B}\cos(bx)\right)$
            (c) Repeated roots ($\lambda_1 = \lambda_2 = \lambda$): $y_H = \mathrm{A}e^{\lambda x} + \mathrm{B}xe^{\lambda x}$
        - (non-hom.)
            (a) Solve homogenous, use **ansatz** to solve for $\tilde{y}$
    - Reduction of Order: given $y_1(x)$, find $y_2(x)$ using
    $$y_2 = u(x)y_1 \tag{5}$$
    - Power Series:
    $$\textbf{Ansatz: } y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{6}$$
    - Frobenius Series:
    $$\textbf{Ansatz: } y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \tag{7}$$
        - Find and solve indicial Eq. ($f(r) = 0$, where f is quadratic)
        - Get coefficient relation for a_m's
        - If possible, relate series to actual function (exponential, trigonometric, hyperbolic etc)
5. P.D.E.'s - factorise 1^{st}, 2^{nd} order P.D.E.'s
    - Factorise differential operators
    - Define ε, η variables to reach $\partial_\varepsilon \partial_\eta u = 0$

2 Chapter 2: Vector Calculus

1. Directional derivative:
$$D_{\hat{u}}f = \hat{u} \cdot \vec{\nabla} f, \quad \text{grad} f = \vec{\nabla} f = (f_x, f_y, f_z) \tag{8}$$

2. Vector field derivatives:
$$\text{div} f = \vec{\nabla} \cdot \vec{F} \quad \rightarrow \quad \text{amount of } \vec{F} \text{ leaving surface/region.} \tag{9}$$
$$\text{curl} f = \vec{\nabla} \times \vec{F} \quad \rightarrow \quad \text{circulation of } \vec{F} \text{ on surface/region.} \tag{10}$$

3. Find general expressions for $\vec{\nabla} f$, $\vec{\nabla} \cdot \vec{F}$, $\vec{\nabla} \times \vec{F}$, $\mathrm{d}\vec{r}$, $\mathrm{d}\vec{S}$, $\mathrm{d}V$ in curvilinear coordinates.

4. 3D Levi-Civita symbol:
$$\varepsilon_{ijk} = \begin{cases} 1 & \text{, for even permutations of } ijk \\ -1 & \text{, for odd permutations of } ijk \\ 0 & \text{, for repeats in } ijk \end{cases} \tag{11}$$
$$(\vec{A} \times \vec{B})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} A_j B_k \tag{12}$$

3 Chapter 3: Vector Integral Calculus

1. Line integrals: $\oint f \mathrm{d}\vec{r}$, $\oint \vec{F} \cdot \mathrm{d}\vec{r}$, $\oint \vec{F} \times \mathrm{d}\vec{r}$
   - Integration over curve
   - Parametrise, seperate into multiple curves if needed.

2. Surface integrals:
   - Parametric equation for S, $(\vec{r}(u, v))$
   - $\mathrm{d}\vec{s} = \hat{n} \mathrm{d}s = \left(\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\right) (|\vec{r}_u \times \vec{r}_v| \mathrm{d}u \mathrm{d}v) = (\vec{r}_u \times \vec{r}_v) \mathrm{d}u \mathrm{d}v$

3. Volume Integrals:
$$\iiint_R f(x, y, z) \mathrm{d}V \tag{13}$$

4. Theorems:
   - Green's Theorem:
$$\oint_{C=\delta A} (P\mathrm{d}x + Q\mathrm{d}y = \iint_A (\partial_x Q - \partial_y P) \mathrm{d}x \mathrm{d}y \tag{14}$$
   - Stokes' Theorem:
$$\iint_D (\vec{\nabla} \times \vec{F}) \cdot \mathrm{d}\vec{s} = \oint_{C=\delta D} \vec{F} \cdot \mathrm{d}\vec{r} \tag{15}$$
   - Divergence Theorem:
$$\iiint_R (\vec{\nabla} \cdot \vec{F}) \mathrm{d}V = \oiint_{S=\delta R} \vec{F} \cdot \mathrm{d}\vec{s} \tag{16}$$

4 Chapter 4: Fourier Series

1. **Important**: A function $f(x)$ is periodic if $f(x) = f(x+nL)$, where L is the fundamental period.

2. A function is even if $f(x) = f(-x)$, and odd if $f(x) = -f(-x)$. Important to remember, symmetric integral of a odd function is zero.
$$\int_{-a}^{a} f_{\text{odd}}(x)\mathrm{d}x = 0 \tag{17}$$
Also important and very useful to remember the following.
$$\begin{aligned} f_{\text{odd}} \times g_{\text{odd}} &= h_{\text{even}} \\ f_{\text{odd}} \times g_{\text{even}} &= h_{\text{odd}} \\ f_{\text{even}} \times g_{\text{even}} &= h_{\text{even}} \end{aligned}$$

3. Integrals of trigonometric functions:
$$\int_{-\,L/2}^{L/2} \cos\left(\frac{2\pi m x}{L}\right)\mathrm{d}x = \begin{cases} L & \text{, if } m = 0 \\ 0 & \text{, if } m \neq 0 \end{cases}, \quad \int_{-\,L/2}^{L/2} \sin\left(\frac{2\pi m x}{L}\right)\mathrm{d}x = 0 \tag{18}$$
Sine function is odd, Cosine is even. Leads to
$$\int_{-\,L/2}^{L/2} \sin\left(\frac{2\pi m x}{L}\right)\cos\left(\frac{2\pi n x}{L}\right)\mathrm{d}x = 0 \tag{19}$$
$$\int_{-\,L/2}^{\frac{L}{2}} \cos\left(\frac{2\pi m x}{L}\right)\cos\left(\frac{2\pi n x}{L}\right)\mathrm{d}x = \int_{-\,L/2}^{L/2} \sin\left(\frac{2\pi m x}{L}\right)\sin\left(\frac{2\pi n x}{L}\right)\mathrm{d}x = \begin{cases} L/2 & \text{, if } m = n \\ 0 & \text{, if } m \neq n \end{cases} \tag{20}$$

4. **Important**: Fourier Series, any periodic function can be expanded into the form
$$f(t) = a_0 + \sum_{n=1}^{\infty}\left((a_n\cos\left(\frac{2\pi n t}{T}\right) + b_n\sin\left(\frac{2\pi n t}{T}\right)\right) \tag{21}$$
where the coefficients a_0, a_m, b_m are defined as
$$a_0 = \frac{1}{L}\int_{-\,L/2}^{L/2} f(t)\mathrm{d}t \tag{22}$$
$$a_m = \frac{1}{L}\int_{-\,L/2}^{L/2} f(t)\cos\left(\frac{2\pi m t}{T}\right)\mathrm{d}t, \quad b_m = \frac{1}{L}\int_{-\,L/2}^{L/2} f(t)\sin\left(\frac{2\pi m t}{T}\right)\mathrm{d}t \tag{23}$$

5. Integrating a periodic function over its period L for any piece of f(x) returns the same result.
$$\int_{0}^{L} f(x)\mathrm{d}x = \int_{\alpha}^{\alpha+L} f(x)\mathrm{d}x \tag{24}$$
This is especially useful for piece-wise functions which not defined symmetrically around zero (eg. function defined between 0, 2π).

6. From Eqs. 17, 22, 23, if $f(x)$ is even, $b_m = 0$, and if $f(x)$ is odd, $a_m = a_0 = 0$.

7. **Important**: $\mathbb{C}$ Fourier Series.

$$\frac{1}{L}\int_{-L/2}^{L/2} \exp\left[\frac{2\pi i(m-n)}{L}\right] \mathrm{d}x = \delta_m n \tag{25}$$

which leads to

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left[\frac{2\pi n x i}{L}\right] \tag{26}$$

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) \exp\left[\frac{2\pi n x i}{L}\right] \mathrm{d}x \tag{27}$$

8. Parceval's Theorem

$$\frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2 \mathrm{d}x = \sum_{n=-\infty}^{\infty} |c_n|^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{28}$$

5 Chapter 5: Fourier Transform and Dirac Delta

1. **Important**: Fourier Transform. Fourier transform is analogous to, and derived from $\mathbb{C}$ Fourier series expansion as $L \to \infty$ (non-periodic function). It is defined as

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx} \mathrm{d}x, \quad k = \frac{2\pi n}{L} \tag{29}$$

although the constant preceding the integral varies based on convention, as long as a compensating change is made to the back transform, which is defined as

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \mathrm{d}k \tag{30}$$

2. Properties of the F.T.:

- Suppose $f(x)$ is $\mathbb{C}$ valued.
  - $f(x)$ is called Hermitian if
  $$f(x)^* = f(-x) \tag{31}$$
  - $f(x)$ is called Anti-Hermitian if
  $$f(x)^* = -f(-x) \tag{32}$$
  - Properties: If $f(x)$ is Hermitian, then $\mathfrak{Re}[f(x)]$ is even, $\mathfrak{Im}[f(x)]$ is odd. If $f(x)$ is Anti-Hermitian, then $\mathfrak{Re}[f(x)]$ is odd, $\mathfrak{Im}[f(x)]$ is even.
- If $f(x) \in \mathbb{R}$, then $\tilde{f}(k)^* = \tilde{f}(-k)$
- If $f(x)$ is Hermitian, then $\tilde{f}(k)$ is real, $\tilde{f}(k)^* = \tilde{f}(k)$
- If $f(x)$ is Anti-Hermitian, then $\tilde{f}(k)$ is fully imaginary, $\tilde{f}^* = -\tilde{f}(k)$
- F.T. is linear (follows properties of integrals).

$$\mathcal{F}(\alpha f(x) + \beta g(x)) = \alpha\mathcal{F}(f(x)) + \beta\mathcal{F}(g(x)) \tag{33}$$

- If $x \to x + a$, (Translation)
$$\mathcal{F}(f(x+a)) = e^{ika}\mathcal{F}(f(x)) \tag{34}$$

- If $x \to ax$,

$$\mathcal{F}(f(ax)) = \frac{1}{|a|}\tilde{f}\left({}^{k}\!/\!{}_{a}\right) \tag{35}$$

- Multiplication by $e^{\alpha x}$,

$$\mathcal{F}(e^{\alpha x} f(x)) = \tilde{f}(k + ix) \tag{36}$$

- **Important**: Differentiation.

$$\mathcal{F}(f^{(n)}(x)) = (ik)^n \tilde{f}(k) \tag{37}$$

- $f(x)$, $\tilde{f}(k)$ symmetry. If $\tilde{f}(k)$ is the F.T. of $f(x)$, then $\frac{1}{2\pi}f(-k)$ is the F.T. of $\tilde{f}(x)$

3. Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-ax^2}\mathrm{d}x = \sqrt{\frac{\pi}{a}} \tag{38}$$

- From Eq. 38, and Gaussian integration, we can show the following are true.

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}\mathrm{d}x = e^{b^2/4a}\sqrt{\frac{\pi}{a}} \tag{39}$$

$$\int_{-\infty}^{\infty} xe^{-ax^2}\mathrm{d}x = 0 \tag{40}$$

$$\int_{-\infty}^{\infty} x^{2k+1}e^{-ax^2}\mathrm{d}x = 0, \quad k \in \mathbb{Z} \tag{41}$$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\mathrm{d}x = \frac{1}{2a}\sqrt{\frac{\pi}{a}} \tag{42}$$

$$\int_{0}^{\infty} xe^{-ax^2}\mathrm{d}x = \frac{1}{2a} \tag{43}$$

4. Dirac Delta function, $\delta(x)$. Continuous variable analogue of Kronicker Delta derived from Heaviside step function $H(x)$

- Definition of Dirac delta function.

$$\delta(x) = \frac{\mathrm{d}H}{\mathrm{d}x} = \begin{cases} 0 & \text{, if } x > 0 \\ \infty & \text{, if } x = 0 \\ 0 & \text{, if } x < 0 \end{cases} \tag{44}$$

$$\therefore \quad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1 \tag{45}$$

- Properties of Dirac delta function.

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0) \tag{46}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-c)\mathrm{d}x = f(c) \tag{47}$$

$$\int_{-\infty}^{\infty} f(x)\dot{\delta}(x)\mathrm{d}x = -\dot{f}(0) \tag{48}$$

$$\int_{-\infty}^{\infty} f(x)\delta(ax)\mathrm{d}x = \frac{1}{|a|}f(0) \tag{49}$$

$$\int_{-\infty}^{\infty} g(x)\delta(f(x))\mathrm{d}x = \sum_i \frac{g(x_i)}{|\dot{f}(x_i)|} \tag{50}$$

- **Important**: If the root of δ is outside the interval of integration, result is zero.

$$\int_a^b \delta(x-c)f(x)\mathrm{d}x = \begin{cases} f(c) & \text{, if } c \in [a,b] \\ 0 & \text{, otherwise} \end{cases} \tag{51}$$

5. F.T. with Dirac delta.
   - Convolution of two real valued functions is defined as

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}y, \text{ Note: } f * g(x) = g * f(x) = h(x) \tag{52}$$

   - F.T. of h(x) reults in

$$\tilde{h}(k) = \mathcal{F}(f * g(x)) = 2\pi \tilde{f}(k)\tilde{g}(x) \tag{53}$$

   Letting $g(x) = \delta(x)$, and using Eq. 52

$$f * \delta(x) = f(x) \tag{54}$$

   Then using Eq. 53,

$$\tilde{\delta}(x) = \frac{1}{2\pi} \tag{55}$$

$$\therefore \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\mathrm{d}x \tag{56}$$

6. Plancheral's Theorem: supposing F.T. of $f(x)$ is well-defined.

$$\int_{-\infty}^{\infty} |f(x)|^2\mathrm{d}x = 2\pi \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\mathrm{d}k \tag{57}$$

7. Existence of F.T.: Fourier Transform of $f(x)$ exists if:
   (a) If $f(x)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x < \infty \tag{58}$$

   (b) If $f(x)$ has a finite number of extrema and discontinuities.
8. F.T. and O.D.E.'s: Recall Eq. 37. Using this, F.T. changes differential equations to algebraic ones. Examples in lecture notes.