Monte Carlo Techniques

We know from earlier that the SDE
$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$
with constant r and σ has the solution
$$S_T = S_0 \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right)T + \sigma\phi\sqrt{T}\right\},$$
for some time horizon T; with $\phi \sim N(0,1)$; $W_t \sim N(0,t)$ and can be written $\phi\sqrt{T}$.
It is often more convenient to express in time stepping form
$$S_{t+\delta t} = S_t \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t}\right\}.$$
In general a closed form solution of an arbitrary SDE is difficult if e.g.

1. $r = r(t)$ and $\sigma = \sigma(S,t)$, i.e. the parameters are no longer constant
2. the SDE is complicated.

The need for Monte Carlo requires numerical integration of stochastic differential equations. Previously we considered the Forward **Euler-Maruyama** method. Why did this work?

Consider
$$dX_t = a(X_t,t)\,dt + b(X_t,t)\,dW_t \tag{1}$$
The simplest scheme for solving (1) is using the E-M method. That is
$$\int_{t_n}^{t_{n+1}} dX_s = \int_{t_n}^{t_{n+1}} a(X_s,s)\,ds + \int_{t_n}^{t_{n+1}} b(X_s,s)\,dW_s$$
$$X_{n+1} = X_n + \int_{t_n}^{t_{n+1}} a(X_s,s)\,ds + \int_{t_n}^{t_{n+1}} b(X_s,s)\,dW_s$$
Using the left hand integration rule:
$$\begin{aligned}\int_{t_n}^{t_{n+1}} a(s,X_s)\,ds &\approx a(t_n,X_n)\int_{t_n}^{t_{n+1}} ds = a(t_n,X_n)\,\delta t\\ \int_{t_n}^{t_{n+1}} b(s,X_s)\,ds &\approx b(t_n,X_n)\int_{t_n}^{t_{n+1}} dW_s = b(t_n,X_n)\,\Delta W_n\end{aligned}$$
$$X_{n+1} = X_n + a(t_n,X_n)\,\delta t + b(t_n,X_n)\,\Delta W_n$$
where $\Delta W_n = (W_{n+1} - W_n)$.

The Forward **Euler-Maruyama** method for GBM gives
$$\frac{\delta S_t}{S_t} = \frac{S_{t+\delta t} - S_t}{S_t} \sim r\delta t + \sigma\phi\sqrt{\delta t}$$
i.e
$$S_{t+\delta t} \sim S_t\left(1 + r\delta t + \sigma\phi\sqrt{\delta t}\right).$$

Now do a Taylor series expansion of the exact solution, i.e.

$$e^{\left(r-\frac{1}{2}\sigma^2\right)\delta t+\sigma\phi\sqrt{\delta t}} \sim 1+\left(r-\tfrac{1}{2}\sigma^2\right)\delta t+\sigma\phi\sqrt{\delta t}+\tfrac{1}{2}\sigma^2\phi^2\delta t$$

so we have

$$S_{t+\delta t} \sim S_t\left(1+r\delta t+\sigma\phi\sqrt{\delta t}+\tfrac{1}{2}\sigma^2\left(\phi^2-1\right)\delta t+..\right)$$

which differs from the Euler method at $O(\delta t)$ by the term $\frac{1}{2}\sigma^2\left(\phi^2-1\right)\delta t$. The term

$$\tfrac{1}{2}\left(\phi^2-1\right)\delta t,$$

is called the *Milstein correction.*

This unusual looking effect only arises for SDEs. To see this we do something similar to an earlier section where we considered the case of zero volatility, in the absence of randomness so set $\sigma=0$

$$\frac{dS_t}{S_t}=rdt$$

which has exact solution

$$S_{t+\delta t}=S_t e^{r\delta t}.$$

The Euler approximation is

$$\frac{\delta S_t}{S_t}=\frac{S_{t+\delta t}-S_t}{S_t}\sim r\delta t$$

or written explicitly

$$S_{t+\delta t}=S_t\left(1+r\delta t+...\right)$$

Expanding the exact solution in a series about t we find that

$$\begin{aligned} S_{t+\delta t} &= S_t e^{r\delta t} \\ &= S_t\left(1+r\delta t+O\left(\delta t^2\right)\right) \end{aligned}$$

so both the exact solution and Euler approximation agree to $O\left(\delta t^2\right)$.

The Milstein correction can be thought of as being a stochastic effect (a result of Itô's lemma in a sense).

Milstein Integration

We approximate the solution of the SDE

$$dG_t=A\left(G_t,t\right)dt+B\left(G_t,t\right)dW_t$$

which is compact form for

$$G_{t+\delta t}=G_t+\int_t^{t+\delta t}A\left(G_s,s\right)ds+\int_t^{t+\delta t}B\left(G_s,s\right)dW_s,$$

by

$$G_{t+\delta t}\sim G_t+A\left(G_t,t\right)\delta t+B\left(G_t,t\right)\sqrt{\delta t}\phi+B\left(G_t,t\right)\frac{\partial}{\partial G_t}B\left(G_t,t\right)\cdot\tfrac{1}{2}\left(\phi^2-1\right)\delta t.$$

Note: We use the same value of the random number $\phi\sim N(0,1)$ in both of the expressions

$$B\left(G_t,t\right)\sqrt{\delta t}\phi$$

and

$$B(G_t,t)\frac{\partial}{\partial G_t}B(G_t,t)\cdot\tfrac{1}{2}\left(\phi^2-1\right)\delta t.$$

The error of the Milstein scheme is $O(\delta t)$ which makes it better than the Euler-Maruyama method which is $O\left(\delta t^{1/2}\right)$. The Milstein makes use of Itô's lemma to increase the accuracy of the approximation by adding the second order term.

Some texts express the scheme in difference form. So a SDE written

$$dY_t = A(Y_t,t)\,dt + B(Y_t,t)\,dW_t$$

can be discretized as

$$Y_{i+1} = Y_i + A\Delta t + B\Delta W_t + \tfrac{1}{2}B\frac{\partial B}{\partial Y_i}\left((\Delta W_t)^2-\Delta t\right)$$

Applying Milstein to the earlier example of GBM

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where

$$\begin{aligned} A(S_t,t) &= rS_t \\ B(S_t,t) &= \sigma S_t \end{aligned}$$

gives

$$\begin{aligned} S_{t+\delta t} &\sim S_t + rS_t\delta t + \sigma S_t\sqrt{\delta t}\phi + \sigma S_t\frac{\partial}{\partial S_t}\sigma S_t\cdot\tfrac{1}{2}\sigma^2\left(\phi^2-1\right)\delta t \\ &= S_t\left(1 + r\delta t + \sigma\phi\sqrt{\delta t} + \tfrac{1}{2}\sigma^2\left(\phi^2-1\right)\delta t\right) \end{aligned}$$

As another example, the CIR model for the spot rate is

$$dr_t = (\eta - \gamma r_t)\,dt + \sqrt{\alpha r_t}dW_t.$$

So identifying

$$\begin{aligned} A(r_t,t) &= \eta - \gamma r_t \\ B(r_t,t) &= \sqrt{\alpha r_t} \end{aligned}$$

and substituting into the Milstein scheme gives

$$\begin{aligned} r_{t+\delta t} &\sim r_t + (\eta-\gamma r_t)\,\delta t + \sqrt{\alpha r_t\delta t}\phi + \sqrt{\alpha r_t}\frac{\partial}{\partial r_t}\sqrt{\alpha r_t}\cdot\tfrac{1}{2}\left(\phi^2-1\right)\delta t \\ &= r_t + (\eta-\gamma r_t)\,\delta t + \sqrt{\alpha r_t\delta t}\phi + \tfrac{1}{4}\alpha\left(\phi^2-1\right)\delta t. \end{aligned}$$

To conclude, a SDE for the process Y_t

$$dY_t = A(Y_t,t)\,dt + B(Y_t,t)\,dW_t$$

can be discretized using Milstein as

$$Y_{i+1} = Y_i + A\delta t + B\phi\sqrt{\delta t} + \tfrac{1}{2}B\frac{\partial B}{\partial Y_i}\left(\phi^2-1\right)\delta t,$$

where $\frac{1}{2}\left(\phi^2-1\right)\delta t$ is the **Milstein correction term.** The same random number $\phi\sim N(0,1)$ is used per time-step.

Monte-Carlo methods are centred on evaluating definite integrals as expectations (or averages). Before studying this in greater detail, we consider the simple problem of estimating expectations of functions of uniformly distributed random numbers.

Motivating Example: Estimate $\theta = \mathbb{E}\left[e^{U^2}\right]$, where $U \sim U(0,1)$.

We note that $\mathbb{E}\left[e^{U^2}\right]$ can be expressed in integral form, i.e.

$$\mathbb{E}\left[e^{U^2}\right] = \int_0^1 e^{x^2} p(x)\, dx$$

where $p(x)$ is the density function of a $U(0,1)$

$$p(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

hence

$$\mathbb{E}\left[e^{U^2}\right] = \int_0^1 e^{x^2} dx.$$

This integral does not have an analytical solution. The theme of this section is to consider solving numerically, using simulations. We use the Monte Carlo simulation procedures:

1. Generate a sequence $U_1, U_2, ..., U_n \sim U(0,1)$ where U_i are i.i.d (independent and identically distributed)
2. Compute $Y_i = e^{U_i^2}$ $(i = 1, ..., n)$
3. Estimate θ by

$$\begin{aligned} \widehat{\theta}_n &\equiv \frac{1}{n}\sum_{i=1}^{n} Y_i \\ &= \frac{1}{n}\sum_{i=1}^{n} e^{U_i^2} \end{aligned}$$

   i.e. use the sample mean of the $e^{U_i^2}$ terms.

Monte Carlo Integration

When a closed form solution for evaluating an integral is not available, numerical techniques are used. The purpose of Monte Carlo schemes is to use simulation methods to approximate integrals in the form of expectations.

Suppose $f(\cdot)$ is some function such that $f:[0,1]\rightarrow\mathbb{R}$. The basic problem is to evaluate the integral

$$I=\int_0^1 f(x)\,dx$$

i.e. diagram

Consider e.g. the earlier problem $f(x)=e^{x^2}$, for which an analytical solution cannotbe obtained.

Note that if $U\sim U(0,1)$ then

$$\mathbb{E}[f(U)]=\int_0^1 f(u)\,p(u)\,du$$

where the density $p(u)$ of a uniformly distributed random variable $U(0,1)$ is given earlier. Hence

$$\begin{aligned}\mathbb{E}[f(U)] &= \int_0^1 f(u)\,p(u)\,du\\ &= I.\end{aligned}$$

So the problem of estimating I becomes equivalent to the exercise of estimating $\mathbb{E}[f(U)]$ where $U\sim U(0,1)$.

Very often we will be concerned with an arbitrary domain, other than $[0,1]$. This simply means that the initial part of the problem will involve seeking a transformation that converts $[a,b]$ to the domain $[0,1]$. We consider two fundamental cases.

1. Let $f(\cdot)$ be a function s.t. $f:[a,b]\rightarrow\mathbb{R}$ where $-\infty<a<b<\infty$. The problem is to evaluate the integral

$$I=\int_a^b f(x)\,dx.$$

In this case consider the following substitution

$$y=\frac{x-a}{b-a}$$

which gives $dy=dx/(b-a)$. This gives

$$\begin{aligned}I &= (b-a)\int_0^1 f(y\times(b-a)+a)\,dy\\ &= (b-a)\,\mathbb{E}[f(U\times(b-a)+a)]\end{aligned}$$

where $U\sim U(0,1)$. Hence I has been expressed as the product of a constant and expected value of a function of a $U(0,1)$ random number; the latter can be estimated by simulation.

2. Let $g(\cdot)$ be some function s.t. $g:[0,\infty)\rightarrow\mathbb{R}$ where $-\infty<a<b<\infty$. The problem is to evaluate the integral

$$I=\int_0^\infty g(x)\,dx,$$

provided $I < \infty$. So this is the area under the curve $g(x)$ between 0 and ∞. In this case use the following substitution

$$y = \frac{1}{1+x}$$

which is equivalent to $x = -1 + \frac{1}{y}$. This gives

$$\begin{aligned} dy &= -dx/(1+x)^2 \\ &= -y^2 dx. \end{aligned}$$

The resulting problem is

$$\begin{aligned} I &= \int_0^1 \frac{g\left(\frac{1}{y} - 1\right)}{y^2} dy \\ &= \mathbb{E}\left[\frac{g\left(-1 + \frac{1}{U}\right)}{U^2}\right] \end{aligned}$$

where $U \sim U(0,1)$. Hence I has again been expressed as the expected value of a function of a $U(0,1)$ random number; to be estimated by simulation.