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Feedback control synthesis of multiple frequency domain specifications via generalized KYP lemma

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SUMMARY

This paper considers a control synthesis problem for linear systems to meet design specifications in terms of multiple frequency domain inequalities in (semi)finite ranges. Our approach is based on the generalized Kalman–Yakubovich–Popov (GKYP) lemma, and dynamic output feedback controllers of order equal to the plant are considered. A new multiplier expansion is proposed to convert the synthesis condition to a linear matrix inequality (LMI) condition through a standard linearizing change of variables. In a single objective setting, the LMI condition may or may not be conservative, depending on the choice of the basis for the multiplier expansion. We provide a qualification for the basis matrix to yield non-conservative LMI conditions. It is difficult to determine the basis matrix meeting such a qualification in general. However, it is shown that qualified bases can be found for some cases, and that the qualification condition can be used to find reasonable choices of the basis for other cases. The synthesis method is then extended to the multiple objective case where a sufficient condition is given for the existence of a controller to meet all the prescribed specifications. Finally, design examples for an active magnetic bearing are given to illustrate the effectiveness of the proposed design method. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Frequency domain inequalities (FDIs) have played a crucial role in describing design specifications for feedback control synthesis. Due to the infinite dimensionality, however, FDIs

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are not directly useful for rigorous analysis and design of control systems. The Kalman–Yakubovich–Popov (KYP) lemma [1, 2] has been proven to be a powerful tool [3–5] to convert an FDI to a linear matrix inequality (LMI) which is numerically tractable. Many of the state–space theories have been developed with the aid of the KYP lemma in one way or the other. On the other hand, a drawback of the standard KYP lemma is that it does not exactly encompass practical situations. Namely, it characterizes FDIs in the entire frequency range, while practical requirements are usually described by multiple FDIs in (semi)finite ranges; e.g. small sensitivity in a low-frequency range and control roll-off in a high-frequency range. Hence, the standard KYP lemma cannot be applied directly to such FDIs arising from practical requirements.

The prevailing method for adjusting the discrepancy is the so-called weighting functions. The basic idea is to approximate an FDI for a transfer function $H(s)$ in a certain frequency range by another FDI for a weighted transfer function $W(s)H(s)$ in the entire frequency range. Here, $W(s)$ is typically a low/band/high-pass filter, and is called the weighting function. This method has proven useful in practice, but suffers from some disadvantages. Most state space theories, such as H_2 and H_∞ synthesis, yield controllers of the same order as the generalized plant containing weighting functions. Hence, the controller order increases as the weighting functions become more complex. The control designer's challenge then is to find the simplest possible weighting functions that provide reasonable approximations of the original FDIs and yield the desired shapes of the closed-loop frequency responses. However, the design iterations to search for good weighting functions can be tedious and time consuming.

The generalized Kalman–Yakubovich–Popov (GKYP) lemma, recently developed by the authors and their collaborators [6–9], extends the standard KYP lemma to provide an exact (or non-conservative) LMI characterization of FDIs in (semi)finite frequency ranges. It has been demonstrated that the GKYP lemma is effective for several engineering design problems, including open-loop shaping for feedback control synthesis, digital filter design, and structure/control design integration [7, 9, 10]. In particular, these design problems have been reduced exactly to semidefinite programmings or convex optimization problems involving LMIs. The class of 'solved' problems is essentially characterized by the property that the FDIs define convex constraints on the original design parameters. Unfortunately, a fundamental problem in systems and controls—feedback control synthesis with FDI specifications on *closed-loop* transfer functions—does not belong to this class. This is because the FDIs impose non-convex constraints on the denominator coefficients of the controller transfer function in general. If a single FDI is imposed for the entire frequency range, then such synthesis problem can be solved exactly, by seeking feasibility conditions [11, 12], or introducing a change of variables [13, 14]. However, the problem with multiple FDIs in (semi)finite frequency ranges has not been addressed in the literature, and remains essentially open. The objective of this paper is to address this unsolved problem.

In this paper, a GKYP synthesis theory is developed to allow for direct treatment of multiple FDI specifications on closed-loop transfer functions in various frequency ranges without introducing weighting functions. We will develop a multiplier method to render the synthesis conditions convex through a standard linearizing change of variables [15]. In particular, the single-objective setting (i.e. one FDI in a given frequency range) is considered first, and a condition is given for the multiplier basis to yield non-conservative design equations. We discuss how to choose the basis to satisfy the condition exactly for some cases and approximately for other cases. The well-known de Oliveira multiplier [15, 16] turns out to be a special case of our

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result for continuous-time FDIs in the entire frequency range. The synthesis method is then extended, with some conservatism, to the case of multi-objective (i.e. multiple FDI) specifications. Finally, design examples for an active magnetic bearing (AMB) are given to illustrate the effectiveness of the proposed design method.

We use the following notation. For a matrix M, its transpose and complex conjugate transpose are denoted by M^{T} and M^*, respectively. The Hermitian part of a square matrix M is denoted by $\mathrm{He}(M) := M + M^*$. For a Hermitian matrix, $M > (\geqslant) 0$ and $M < (\leqslant) 0$ denote positive (semi)definiteness and negative (semi)definiteness. The symbol $\mathbb{H}_n$ stands for the set of $n \times n$ Hermitian matrices. For matrices Φ and P, $\Phi \otimes P$ means their Kronecker product. The quadratic function of $G \in \mathbb{C}^{n \times m}$ with weight $\Pi \in \mathbb{H}_{n+m}$ is defined by

$$\rho(G, \Pi) := [I_n \;\; G]\Pi[I_n \;\; G]^*$$

Given a positive integer q, let $\mathbb{Z}_\ell$ be the set of positive integers up to ℓ, i.e. $\mathbb{Z}_\ell := \{1, 2, \ldots, \ell\}$.

2. PROBLEM STATEMENT AND FORMULATION

2.1. Problem statement

We consider the feedback system consisting of linear time-invariant n_{p}th order plant $G(\lambda)$ and n_{c}th order controller $K(\lambda)$

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(\lambda) \begin{bmatrix} w \\ u \end{bmatrix}, \quad u = K(\lambda) y$$

where $u(t) \in \mathbb{R}^{n_u}$ is the control input, $y(t) \in \mathbb{R}^{n_y}$ is the measured output, $w(t) \in \mathbb{R}^{n_w}$ and $z(t) \in \mathbb{R}^{n_z}$ are the signals used to describe a performance specification, and λ is the frequency variable ($\lambda = s$ for continuous-time systems and $\lambda = z$ for discrete-time systems). Let the closed-loop transfer function from w to z be denoted by $G(\lambda) \star K(\lambda)$.

The control synthesis problem of our interest is, given the plant $G(\lambda)$, $\Pi \in \mathbb{H}_{n_z + n_w}$, and $\Phi, \Psi \in \mathbb{H}_2$, find a full order ($n_{\mathrm{c}} = n_{\mathrm{p}}$) controller $K(\lambda)$ such that the closed-loop system $H(\lambda) := G(\lambda) \star K(\lambda)$ satisfies

$$\|H(\lambda)\| < \infty, \quad \rho(H(\lambda), \Pi) < 0 \quad \forall \lambda \in \mathbf{\Lambda}(\Phi, \Psi) \tag{1}$$

where

$$\mathbf{\Lambda}(\Phi, \Psi) := \{\lambda \in \mathbb{C} \,|\, \rho(\lambda, \Phi) = 0, \rho(\lambda, \Psi) \geqslant 0\} \tag{2}$$

For technical reasons, we assume $\infty \in \mathbf{\Lambda}$ if $\mathbf{\Lambda}$ is unbounded. For clarity of exposition, we shall restrict our attention to the single-objective control problem described by (1) in the main body of our theoretical developments. However, we will later discuss extensions to a more general problem where there are regional pole constraints and multiple FDI constraints of the form (1).

The first condition in (1) means that $H(\lambda)$ has no poles in $\mathbf{\Lambda}(\Phi, \Psi)$, and is necessary for the second condition to be well posed. The second condition is specified by the three Hermitian matrices Π, Φ, and Ψ. The matrix Π represents frequency properties such as the positive real (or passivity) condition $H(\lambda) + H(\lambda)^* > 0$ and the bounded real (or small gain) condition $\|H(\lambda)\| < \gamma$.

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These properties are expressed, respectively, as

$$\rho(H(\lambda), \Pi_{\mathrm{pr}}) = [I \;\; H(\lambda)] \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I \\ H(\lambda)^* \end{bmatrix} < 0, \quad \Pi_{\mathrm{pr}} := \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$$

$$\rho(H(\lambda), \Pi_{\mathrm{br}}) = [I \;\; H(\lambda)] \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ H(\lambda)^* \end{bmatrix} < 0, \quad \Pi_{\mathrm{br}} := \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}$$

On the other hand, a pair of Φ and Ψ provides a (semi)finite frequency range as follows.

For the continuous-time case, define

$$\Omega_{\mathrm{c}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Psi_{\mathrm{c}} = \tau \begin{bmatrix} -\varpi_1 \varpi_2 & \mathrm{j}\varpi_{\mathrm{c}} \\ -\mathrm{j}\varpi_{\mathrm{c}} & -1 \end{bmatrix}$$

where

$$\tau = \pm 1, \quad \varpi_1, \varpi_2 \in \mathbb{R}, \quad \varpi_1 < \varpi_2, \quad \varpi_{\mathrm{c}} := (\varpi_2 + \varpi_1)/2$$

Then $\mathbf{\Lambda}(\Omega_{\mathrm{c}}, 0)$ and $\mathbf{\Lambda}(0, \Omega_{\mathrm{c}})$ are the imaginary axis and the closed right half plane, respectively. Note that $\mathbf{\Lambda}(0, \Psi_{\mathrm{c}})$ is the inside (when $\tau = 1$) or outside (when $\tau = -1$) of the circle with the centre at $\mathrm{j}\varpi_{\mathrm{c}}$, passing through the points $\mathrm{j}\varpi_1$ and $\mathrm{j}\varpi_2$. It is then easily seen that

$$\mathbf{\Lambda}(\Omega_{\mathrm{c}}, \Psi_{\mathrm{c}}) = \{\mathrm{j}\omega : \omega \in \mathbb{R}, \tau(\omega - \varpi_1)(\omega - \varpi_2) \leqslant 0\}$$

which is the frequency interval $\varpi_1 \leqslant \omega \leqslant \varpi_2$ when $\tau = 1$, or the frequency range $\omega \leqslant \varpi_1$ or $\varpi_2 \leqslant \omega$ when $\tau = -1$.

For the discrete-time case, define

$$\Omega_{\mathrm{d}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Psi_{\mathrm{d}} = \begin{bmatrix} -2\cos\vartheta_0 & \mathrm{e}^{\mathrm{j}\vartheta_{\mathrm{c}}} \\ \mathrm{e}^{-\mathrm{j}\vartheta_{\mathrm{c}}} & 0 \end{bmatrix}$$

where

$$\vartheta_1, \vartheta_2 \in \mathbb{R}, \quad 0 < \vartheta_2 - \vartheta_1 \leqslant 2\pi, \quad \vartheta_{\mathrm{c}} := (\vartheta_2 + \vartheta_1)/2, \quad \vartheta_0 := (\vartheta_2 - \vartheta_1)/2$$

Then $\mathbf{\Lambda}(\Omega_{\mathrm{d}}, 0)$ and $\mathbf{\Lambda}(0, \Omega_{\mathrm{d}})$ are the unit circle and outside of the unit circle, respectively. Note that $\mathbf{\Lambda}(0, \Psi_{\mathrm{d}})$ is the half plane above (when $\sin\vartheta_{\mathrm{c}} > 0$) or below (when $\sin\vartheta_{\mathrm{c}} < 0$) the straight line passing through the points $\mathrm{e}^{\mathrm{j}\vartheta_1}$ and $\mathrm{e}^{\mathrm{j}\vartheta_2}$. It then follows that

$$\mathbf{\Lambda}(\Omega_{\mathrm{d}}, \Psi_{\mathrm{d}}) = \{\mathrm{e}^{\mathrm{j}\theta} : \theta \in \mathbb{R}, (\theta - \vartheta_1)(\theta - \vartheta_2) \leqslant 0\}$$

which is the frequency interval $\vartheta_1 \leqslant \theta \leqslant \vartheta_2$.

Throughout the paper, we shall impose the following for tractability and practicality:

Assumption 1

(a) The lower right $n_w \times n_w$ block matrix of Π is positive semidefinite, i.e. $\Pi_{22} \geqslant 0$ where

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \in \mathbb{H}_{n_z + n_w}, \quad \Pi_{22} \in \mathbb{H}_{n_w}$$

(b) The pair (Φ, Ψ) is chosen so that the set $\mathbf{\Lambda}(\Phi, \Psi)$ is not empty, nor a single point, nor the entire complex plane.

Item (a) ensures that the feasible set for the closed-loop transfer function $H(\lambda)$ is convex, and does not exclude such important specifications as the small gain and passivity. Item (b) excludes the trivial cases and ensures that $\mathbf{\Lambda}(\Phi, \Psi)$ is one of the following: (i) straight line or circle; (ii) half plane; or inside or outside of a circle; (iii) intersection of (i) and (ii). See [9] for the details and a precise characterization of (Φ, Ψ) satisfying (b).

2.2. State space formulation via a dual GKYP lemma

Let state space realizations of the plant $G(\lambda)$ and the controller $K(\lambda)$ be given by

$$\begin{bmatrix} \lambda x \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} \lambda x_c \\ u \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix} \tag{4}$$

where $x(t) \in \mathbb{R}^{n_p}$, $x_c(t) \in \mathbb{R}^{n_c}$, and all the coefficient matrices are real. The closed-loop system $H(\lambda) := G(\lambda) \star K(\lambda)$ is described by the following state space realization:

$$\begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{bmatrix} = \left[\begin{array}{cc|c} A + B_2 D_c C_2 & B_2 C_c & B_1 + B_2 D_c D_{21} \\ B_c C_2 & A_c & B_c D_{21} \\ \hline C_1 + D_{12} D_c C_2 & D_{12} C_c & D_{11} + D_{12} D_c D_{21} \end{array}\right] \tag{5}$$

where the state vector is $[x^{\mathsf{T}} \;\; x_c^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^n$ with $n := n_p + n_c$.

Consider the transfer function $H(\lambda)$ specified by (5). Note that $\|H(\lambda)\| < \infty$ is equivalent to $\det(\lambda I - \mathsf{A}) \neq 0$, provided (5) is a minimal realization. The following result presents a dual version of the GKYP lemma in [9] that characterizes an FDI in a (semi)finite frequency range in terms of LMIs.

Theorem 1

Let $\Phi, \Psi \in \mathbb{H}_2$, $\Pi \in \mathbb{H}_{n_z + n_w}$, and $H(\lambda)$ in (5) be given and consider $\mathbf{\Lambda}(\Phi, \Psi)$ defined by (2). Suppose Assumption 1 holds. The following statements are equivalent.

(i) $\det(\lambda I - \mathsf{A}) \neq 0$ and $\rho(H(\lambda), \Pi) < 0$ hold for all $\lambda \in \mathbf{\Lambda}(\Phi, \Psi)$.

(ii) There exist $P = P^*$ and $Q = Q^* > 0$ such that

$$F \Omega F^* < 0, \quad F := \begin{bmatrix} I & \mathsf{A} & 0 & \mathsf{B} \\ 0 & \mathsf{C} & I & \mathsf{D} \end{bmatrix}, \quad \Omega := \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix} \tag{6}$$

Proof

Define

$$H_*(\lambda) := H(\lambda^*)^* = \mathsf{B}^* (\lambda I - \mathsf{A}^*)^{-1} \mathsf{C}^* + \mathsf{D}^*$$

Note that $\rho(H(\lambda), \Pi) < 0$ holds for all $\lambda \in \Lambda(\Phi, \Psi)$ if and only if $\sigma(H_*(\lambda), J(n_z, n_w)\Pi J(n_w, n_z)) < 0$ holds for all $\lambda \in \mathbb{C}$ such that $\sigma(\lambda, J(1,1)\Phi J(1,1)) = 0$ and $\sigma(\lambda, J(1,1)\Psi J(1,1)) \geqslant 0$, where

$$\sigma(G, \Theta) := \begin{bmatrix} G \\ I \end{bmatrix}^* \Theta \begin{bmatrix} G \\ I \end{bmatrix}, \quad J(n,m) := \begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix}$$

The result then follows from Theorem 3 and its remark in [9]. □

With the result of Theorem 1, the synthesis problem can be formulated as the search for the parameters $Q > 0$, P, and $K(s)$ satisfying (6) where the state space matrices are defined by $G(\lambda)\star K(\lambda) = \mathsf{C}(\lambda I - \mathsf{A})^{-1}\mathsf{B} + \mathsf{D}$. This formulation is not directly useful for the control synthesis because the resulting condition is not convex due to the product terms between P, Q, and the controller parameters. We shall develop a multiplier method to re-parametrize the condition so that the problem becomes convex.

3. OUTPUT FEEDBACK SYNTHESIS

3.1. Multiplier expansion

We shall introduce a multiplier through the projection lemma [12] to obtain a sufficient condition for (6). The condition is also necessary, eliminating the potential conservatism in the design, if the multiplier basis satisfies a certain inequality.

Lemma 1

Let matrices $\Omega \in \mathbb{H}_{2n+n_z+n_w}$ and $R \in \mathbb{C}^{n\times(2n+n_z)}$, and the state space realization of $H(\lambda)$ in (5) be given. Define F by (6) and

$$F_A := \begin{bmatrix} \mathsf{A} \\ -I \\ \mathsf{C} \end{bmatrix}, \quad F_B := \begin{bmatrix} \mathsf{B} \\ 0 \\ \mathsf{D} \end{bmatrix}$$

The following statements are equivalent.

(i) The following conditions hold:

$$F\Omega F^* < 0$$

$$R^{\perp *}[I \;\; F_B]\Omega[I \;\; F_B]^* R^{\perp} < 0 \tag{7}$$

(ii) There exists $W \in \mathbb{C}^{n\times n}$ such that

$$[I \;\; F_B]\Omega[I \;\; F_B]^* < F_A W R + (F_A W R)^* \tag{8}$$

Proof

Note that F in (6) can be written as

$$F = \begin{bmatrix} I & \mathsf{A} & 0 \\ 0 & \mathsf{C} & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & \mathsf{B} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & \mathsf{D} \end{bmatrix} = F_A^{\perp *}[I \;\; F_B]$$

Then the result simply follows from the projection lemma [12,17]. □

The multiplier-expanded equation (8) will be used as a basis for synthesis. In particular, the equation will be equivalently converted to an LMI synthesis condition in the next section. Hence, conservatism associated with (8) needs to be carefully analysed. For an arbitrary R, (8) gives a sufficient condition for (6). On the other hand, (6) and (8) become equivalent if R is chosen to satisfy (7). Thus, condition (7) precisely captures the gap or conservatism between the synthesis condition (8) and the original design objective.

For synthesis, it is desired that matrix R be chosen to satisfy (7). However, condition (7) involves the yet unknown controller parameters and hence has to be properly interpreted to give a condition useful for synthesis:

Condition 1

Condition (7) holds for some $P, Q \in \mathbb{H}_n$, and matrices (A_c, B_c, C_c, D_c) satisfying $Q > 0$ and (6), where $(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$ are defined by (5).

This condition is independent of the unknown parameters P, Q and (A_c, B_c, C_c, D_c), and thus can be used to fix R *before* the control design. With R satisfying Condition 1, there exists a controller that meets the specification (1) if and only if there exist matrices $P, Q \in \mathbb{H}_n$, $W \in \mathbb{C}^{n\times n}$, and (A_c, B_c, C_c, D_c) such that $Q > 0$ and (8) hold. We will show how to solve the synthesis problem (8) in the next section. How to choose an appropriate R will be addressed in the section that follows.

Finally, we give a remark on the relation between the multiplier expansion described in this section and the one used in our prior work on the static gain synthesis [18]. In particular, the former can be considered as a partial expansion, and a further expansion of the quadratic term of F_B in (8) via the Schur complement would yield the full multiplier expansion in [18] that avoids direct product of F_B and Π. More specifically, it can be shown that (8) is equivalent to

$$\begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix} < \mathrm{He} \begin{bmatrix} F_A & F_B \\ 0 & -I \end{bmatrix} \begin{bmatrix} WR & 0 \\ -\Pi_{21} J(0, n_z) & -\Pi_{22} \end{bmatrix}$$

provided $\Pi_{22} > 0$. This condition shows that a certain structure can be imposed on the multiplier in [18] without loss of generality if the specification is convex, i.e. $\Pi_{22} > 0$.

3.2. Reduction to LMIs

The synthesis problem described by (8) is non-convex due to the product term between the multiplier W and the controller parameters. Below, we show that the change of variable introduced by de Oliveira *et al.* [15] works perfectly to convert the problem to an LMI problem, provided R satisfies an additional structural constraint.

Let X, Y, U and V be defined by

$$W = \begin{bmatrix} X & * \\ U & * \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} Y & V \\ * & * \end{bmatrix}^*$$

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Note that, given any $X, Y, U, V \in \mathbb{C}^{n_p \times n_p}$ with U and V invertible, the blanks '$*$' can be filled to satisfy the above two equalities for some W. In particular, we have

$$\begin{bmatrix} X & (I - XY^*)V^{-*} \\ U & -UY^*V^{-*} \end{bmatrix} \begin{bmatrix} Y^* & (I - Y^*X)U^{-1} \\ V^* & -V^*XU^{-1} \end{bmatrix} = I$$

Let us define the new variables

$$\begin{bmatrix} M & G \\ H & L \end{bmatrix} := \begin{bmatrix} YAX & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} V & YB_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} U & 0 \\ C_2X & I \end{bmatrix} \tag{9}$$

$$\mathscr{P} := FPF^*, \quad \mathscr{Q} := FQF^*, \quad Z := YX + VU \tag{10}$$

Then, by the congruence transformation

$$T := \begin{bmatrix} I & 0 \\ Y & V \end{bmatrix}, \quad \mathscr{T} := \mathrm{diag}(T, T, I_{n_z}) \tag{11}$$

we have

$$[\mathscr{A} \;\; \mathscr{B}] := [\mathscr{T} F_A W T^* \;\; \mathscr{T} F_B] = \left[\begin{array}{cc|c} AX + B_2H & A + B_2LC_2 & B_1 + B_2LD_{21} \\ M & YA + GC_2 & YB_1 + GD_{21} \\ -X & -I & 0 \\ -Z & -Y & 0 \\ C_1X + D_{12}H & C_1 + D_{12}LC_2 & D_{11} + D_{12}LD_{21} \end{array}\right] \tag{12}$$

which is affine in the new variables. Now, suppose R in (8) has been chosen to satisfy the following (we will discuss how to choose such R later).

Condition 2
There exists a fixed matrix $\mathscr{R} \in \mathbb{C}^{n \times (2n+n_z)}$ satisfying $R\mathscr{T}^* = T^*\mathscr{R}$ for all matrices $Y, V \in \mathbb{C}^{n_p \times n_p}$, where T and $\mathscr{T}$ are defined in (11).

Then, through the congruence transformation of (8) by $\mathscr{F}$, we obtain the following result.

Lemma 2
Consider the plant $G(\lambda)$ in (3) and the controller $K(\lambda)$ in (4) with $n_c = n_p$, and let $P, Q \in \mathbb{H}_n$, $R \in \mathbb{C}^{n \times (2n+n_z)}$, $\Phi, \Psi \in \mathbb{H}_2$, and $\Pi \in \mathbb{H}_{n_w+n_z}$ be given where $n := 2n_p$. Suppose R satisfies Condition 2. Then the following statements are equivalent.

(i) There exist matrices $P, Q \in \mathbb{H}_n$, a multiplier $W \in \mathbb{R}^{n \times n}$ and a controller (A_c, B_c, C_c, D_c) such that (8) is satisfied, where $(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$ are defined in (5).

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(ii) There exist $\mathscr{P}, \mathscr{Q} \in \mathbb{H}_n$ and real matrices X, Y, Z, M, G, H, L satisfying

$$[I \ \ \mathscr{B}] \begin{bmatrix} \Phi \otimes \mathscr{P} + \Psi \otimes \mathscr{Q} & 0 \\ 0 & \Pi \end{bmatrix} \begin{bmatrix} I \\ \mathscr{B}^* \end{bmatrix} < \mathscr{A}\mathscr{R} + (\mathscr{A}\mathscr{R})^* \tag{13}$$

where $\mathscr{A}$ and $\mathscr{B}$ are defined by (12).

Moreover, the parameters $(P, Q, A_c, B_c, C_c, D_c, W)$ and $(\mathscr{P}, \mathscr{Q}, M, G, H, L, X, Y, Z)$ are related through the bijective mapping defined by (9) and (10).

As a direct consequence of Theorem 1 and Lemmas 1 and 2, we have the following result.

Theorem 2

Consider the plant $G(\lambda)$ of order n_p in (3). Let $R \in \mathbb{C}^{n\times(2n+n_z)}$, $\Phi, \Psi \in \mathbb{H}_2$, and $\Pi \in \mathbb{H}_{n_z+n_w}$ be given where $n := 2n_p$. Suppose Assumption 1 holds and R satisfies Conditions 1 and 2. Then the following statements are equivalent.

(i) There exists a dynamic output feedback controller $K(\lambda)$ in (4) with $n_c = n_p$ satisfying the specification in (1).
(ii) There exist $\mathscr{P}, \mathscr{Q} \in \mathbb{H}_n$ and real matrices X, Y, Z, M, G, H, L satisfying $\mathscr{Q} > 0$ and (13) where $\mathscr{A}$ and $\mathscr{B}$ are defined by (12).

Moreover, (ii) implies (i) for any choice of R. If statement (ii) is true, the parameters (A_c, B_c, C_c, D_c) of controller $K(\lambda)$ in statement (i) can be calculated by solving (9) and (10).

Since $\Pi_{22} \geqslant 0$, the condition in (13) can be made linear in $\mathscr{B}$ via the Schur complement. The resulting equation is an LMI in terms of variables $X, Y, Z, M, G, H, L, \mathscr{P}$, and $\mathscr{Q}$. Once we solve the LMI with the additional condition $\mathscr{Q} > 0$, the controller parameters can be recovered as follows. First, let U and V be any factor such that $VU = Z - YX$ where non-singularity of $Z - YX$ can be assumed without loss of generality due to the strictness of the LMIs. The controller parameters can then be obtained by solving (9) for (A_c, B_c, C_c, D_c).

4. NON-CONSERVATIVE/REASONABLE CHOICES OF R

In this section, we would like to choose R such that feasibility of (13) and $\mathscr{Q} > 0$ is necessary and sufficient for the existence of a controller (4) that meets the specification (1).

4.1. A general approach

In view of Theorem 2, such R can be characterized by Conditions 1 and 2. It can readily be verified that R satisfies Condition 2 if and only if it has the following structure:

$$R = \begin{bmatrix} r_1 I & 0 & r_2 I & 0 & \Gamma \\ 0 & r_1 I & 0 & r_2 I & 0 \end{bmatrix} \in \mathbb{C}^{2n_p\times(4n_p+n_z)} \tag{14}$$

where $r_1, r_2 \in \mathbb{C}$ and $\Gamma \in \mathbb{C}^{n_p\times n_z}$. In this case, we have $\mathscr{R} = R$. On the other hand, the set of R satisfying Condition 1 does not seem to have a simple parametrization, and it turns out to be difficult in general to find R satisfying both conditions exactly.

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However, one can find such R for some special cases, and the conditions can be used to find reasonable (but potentially conservative) choices of R for other cases. We will show in the next subsection how to choose R satisfying Conditions 1 and 2 for the case where the frequency range is not restricted and closed-loop stability is required, i.e. $\mathbf{\Lambda}(\Phi, \Psi)$ is either the closed right half plane or outside of the unit circle. The subsection that follows will suggest reasonable choices of R for the general restricted frequency case. Below, we shall describe our idea that leads to such results.

We would like to find R with the structure (14) satisfying Condition 1. For tractability, we consider a condition that guarantees Condition 1:

Condition 3
Inequality (7) is satisfied for all B_c, D_c, and $P, Q \in \mathbb{H}_n$ such that $Q > 0$.

This condition appears to be much stronger than Condition 1 and there may be no R satisfying the condition. However, we may find a reasonable choice for R by trying to meet Condition 3.

We first claim the following.

Lemma 3
Let R be given by (14) and suppose it satisfies Condition 3 and $|r_1|^2 + |r_2|^2 \neq 0$. Then $\Gamma = 0$.

Proof
Note that the null space of R is given by

$$R^{\perp} = \begin{bmatrix} r_2 I & \bar{r}_1 \Gamma \\ -r_1 I & \bar{r}_2 \Gamma \\ 0 & -r_0 I \end{bmatrix}$$

where $r_0 := |r_1|^2 + |r_2|^2$. Also note that Condition 3 implies that the term associated with P and Q on the left-hand side of (7) is negative (semi)definite for all Hermitian P and $Q > 0$. Hence we have

$$\tilde{\Gamma}^*[(N\Phi N^*) \otimes P + (N\Psi N^*) \otimes Q]\tilde{\Gamma} \leqslant 0, \quad \tilde{\Gamma} := \begin{bmatrix} I & 0 \\ 0 & \Gamma \end{bmatrix}, \quad N := \begin{bmatrix} \bar{r}_2 & -\bar{r}_1 \\ r_1 & r_2 \end{bmatrix} \tag{15}$$

for all P and $Q > 0$. We now claim that $\Gamma \neq 0$ implies that $\Phi = 0$ and $\Psi \leqslant 0$. First, note that N is non-singular because $\det(N) = r_0 > 0$. If $\Phi \neq 0$, then there exist vectors u and v such that

$$a := u^*(N\Phi N^*)u \neq 0, \quad w := \Gamma v, \quad \|w\| = 1$$

When $P = (\mu/a)I$, we have

$$z^*\tilde{\Gamma}^*[(N\Phi N^*) \otimes P]\tilde{\Gamma} z = (u^* N\Phi N^* u) \otimes (w^* P w) = \mu, \quad z := \begin{bmatrix} u_1 w \\ u_2 v \end{bmatrix}$$

Hence if $\mu > 0$ is sufficiently large, then (15) is violated and therefore Φ must be zero. Similarly, if Ψ has a strictly positive eigenvalue, then there exist vectors u and v such that

$$a := u^*(N\Psi N^*)u > 0, \quad w := \Gamma v, \quad \|w\| = 1$$

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When $Q = (\mu/a)I$, we have

$$z^*\tilde{\Gamma}^*[(N\Psi N^*) \otimes Q]\tilde{\Gamma}z = (u^*N\Phi N^*u) \otimes (w^*Qw) = \mu, \quad z := \begin{bmatrix} u_1 w \\ u_2 v \end{bmatrix}$$

Again, (15) is violated for sufficiently large $\mu > 0$ and we conclude that $\Psi \leqslant 0$. Thus, the set of frequency variables $\Lambda(\Phi, \Psi)$ is a single point or empty, violating Assumption 1. □

When R is given by (14) with $\Gamma = 0$, condition (7) reduces to

$$\begin{bmatrix} (\eta^*\Phi\eta)P + (\eta^*\Psi\eta)Q & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \bar{\eta}_1 \mathsf{B} \\ I & \mathsf{D} \end{bmatrix} \Pi \begin{bmatrix} 0 & \bar{\eta}_1 \mathsf{B} \\ I & \mathsf{D} \end{bmatrix}^* < 0 \tag{16}$$

where $\eta = [\bar{\eta}_1 \ \ \bar{\eta}_2]^*$ is a vector such that $[r_1 \ \ r_2]\eta = 0$, e.g. $\eta = [\bar{r}_2 \ \ -\bar{r}_1]^*$. We would like to choose R (or equivalently η) so that this inequality holds for all P and $Q > 0$. There may be no such choice, and we take the following heuristic approach; choose η to be the solution of the following optimization problem:

$$\min_{\eta} \eta^*\Psi\eta \quad \text{s.t.} \ \eta^*\Phi\eta = 0, \quad |\eta_1| \leqslant \gamma \tag{17}$$

Since we have no information regarding the inertia of P, we impose the constraint $\eta^*\Phi\eta = 0$ to minimize the worst-case effect of P. Since feasible Q is positive definite, we try to minimize its coefficient $\eta^*\Psi\eta$ so that (16) is more likely to be satisfied. The second constraint $|\eta_1| \leqslant \gamma$ is imposed so that the first term in (16) becomes as negative as possible while keeping the magnitude of the second term small. Clearly, the direction of the optimizer (and hence R) is independent of the value of γ, and we set $\gamma = 1$. We shall solve the optimization problem and determine R for specific cases in the subsections below.

4.2. Case 1: the entire frequency range

We consider the case $\Phi = 0$ and $\Psi = \Omega_c$ or Ω_d so that $\Lambda(\Phi, \Psi)$ is the instability region on the complex plane for the continuous-time or discrete-time setting. In this case, the specification in (1) requires the closed-loop stability in addition to the FDI on the entire frequency range. The variable P then disappears from Equations (6) and (7), and the former becomes a standard LMI that arises in the classical KYP lemma (e.g. [5]).

For the discrete-time case with the entire frequency range, we have

$$\Phi = 0, \quad \Psi = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Phi \otimes P + \Psi \otimes Q = \begin{bmatrix} -Q & 0 \\ 0 & Q \end{bmatrix}$$

and the optimization problem in (17) becomes

$$\min_{\eta} |\eta_2|^2 - |\eta_1|^2 \quad \text{s.t.} \ |\eta_1| \leqslant 1$$

It can readily be verified that a solution to this problem is given by $\eta = [1 \ \ 0]^*$. All the other solutions are scalar multiple of this vector and hence leads to the same R

$$R := [0 \ \ I \ \ 0] \in \mathbb{R}^{n \times (2n+n_z)}$$

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It turns out that this R satisfies Condition 1. To see this, fix a solution to (6) and note that

$$\Sigma_1 := \begin{bmatrix} -Q & 0 \\ 0 & 0 \end{bmatrix} + \Upsilon \leqslant \begin{bmatrix} I & \mathsf{A} \\ 0 & \mathsf{C} \end{bmatrix} \begin{bmatrix} -Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & \mathsf{A} \\ 0 & \mathsf{C} \end{bmatrix}^* + \Upsilon =: \Sigma_2$$

which holds due to $Q > 0$, where

$$\Upsilon := \begin{bmatrix} 0 & \mathsf{B} \\ I & \mathsf{D} \end{bmatrix} \Pi \begin{bmatrix} 0 & \mathsf{B} \\ I & \mathsf{D} \end{bmatrix}^*$$

Inequality (6) is described by $\Sigma_2 < 0$ and hence we have $\Sigma_1 < 0$ which is exactly the condition in (7). Hence (6) implies (7), indicating satisfaction of Condition 1. Thus, the above choice of R gives a non-conservative synthesis condition. When specialized to the discrete-time H_∞ norm condition, our condition (8) with this R reduces to the condition by de Oliveira *et al.* [15].

For the continuous-time case with the entire frequency range, we have

$$\Phi = 0, \quad \Psi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Phi \otimes P + \Psi \otimes Q = \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix}$$

and the optimization problem in (17) becomes

$$\min_{\eta} \eta_1 \bar{\eta}_2 + \bar{\eta}_1 \eta_2 \quad \text{s.t. } |\eta_1| \leqslant 1$$

Clearly, the optimal value is unbounded and not attained. A suboptimal solution which yields an arbitrarily small objective function value is given by $\eta = [1 \ -1/\varepsilon]^*$ with sufficiently small $\varepsilon > 0$. Hence a reasonable choice of R is

$$R := [I \ \ \varepsilon I \ \ 0] \in \mathbb{R}^{n \times (2n + n_z)}$$

Again, it turns out that this R satisfies Condition 1. First, note that (7) becomes

$$\begin{bmatrix} -(2/\varepsilon)Q & 0 \\ 0 & 0 \end{bmatrix} + \Upsilon < 0$$

This condition holds for sufficiently small ε if and only if $\sigma(\mathsf{D}, \Pi) < 0$, which is implied by the lower right block of (6). Thus, (6) implies (7), rendering condition (13) non-conservative. The resulting analysis condition (8) is similar to, but different from, the continuous-time H_∞ norm condition by Shaked [19].

The results in this section can be summarized in Table I where $\varepsilon > 0$ is a sufficiently small number.

Table I. Nonconservative choices of R (entire frequency range).

$\Lambda(\Phi, \Psi)$	$\{s \in \mathbb{C} \mid s + s^* \geqslant 0\}$	$\{z \in \mathbb{C} \mid \lvert z \rvert \geqslant 1\}$
R	$[I \ \ \varepsilon I \ \ 0]$	$[0 \ \ I \ \ 0]$

4.3. Case 2: the restricted frequency range

Consider the discrete-time case with a restricted frequency range where Φ and Ψ are given by

$$\Phi_\mathrm{d} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Psi_\mathrm{d} = \begin{bmatrix} -2\cos\vartheta_0 & \mathrm{e}^{\mathrm{j}\vartheta_\mathrm{c}} \\ \mathrm{e}^{-\mathrm{j}\vartheta_\mathrm{c}} & 0 \end{bmatrix}$$

To solve the optimization problem in (17), first note that $\eta^*\Phi_\mathrm{d}\eta = 0$ if and only if $|\eta_1| = |\eta_2|$. Due to Assumption 1, Ψ_d must have a negative eigenvalue. Hence the optimal solution of (17) is strictly negative, and the optimizer η is non-zero. This implies that η_1 and η_2 are both non-zero. In this case, we can assume without loss of generality that the optimizer is given by $\eta_1 = 1$ and $\eta_2 = \mathrm{e}^{\mathrm{j}\phi}$ for some real ϕ. Note that

$$\eta^*\Psi_\mathrm{d}\eta = \begin{bmatrix} 1 \\ \mathrm{e}^{\mathrm{j}\phi} \end{bmatrix}^* \begin{bmatrix} -2\cos\vartheta_0 & \mathrm{e}^{\mathrm{j}\vartheta_\mathrm{c}} \\ \mathrm{e}^{-\mathrm{j}\vartheta_\mathrm{c}} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \mathrm{e}^{\mathrm{j}\phi} \end{bmatrix} = 2(\cos(\vartheta_\mathrm{c} + \phi) - \cos\vartheta_0)$$

Hence $\eta^*\Psi_\mathrm{d}\eta$ is minimum when $\vartheta_\mathrm{c} + \phi = \pi$. Thus, we have $\eta_1 = 1$ and $\eta_2 = -\mathrm{e}^{-\mathrm{j}\vartheta_\mathrm{c}}$, leading to

$$R = [I \;\; \mathrm{e}^{\mathrm{j}\vartheta_\mathrm{c}} I \;\; 0]$$

Next we consider the continuous-time case where Φ and Ψ are given by

$$\Phi_\mathrm{c} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Psi_\mathrm{c} = \tau \begin{bmatrix} -\varpi_1\varpi_2 & \mathrm{j}\varpi_\mathrm{c} \\ -\mathrm{j}\varpi_\mathrm{c} & -1 \end{bmatrix}$$

The following lemma is useful.

Lemma 4

Let $x, y \in \mathbb{C}$ be given. The following statements are equivalent.

(i) $x\bar{y} + \bar{x}y = 0$
(ii) There exist $\alpha, \beta \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $x = \alpha z$ and $y = \mathrm{j}\beta z$.

Proof

The fact that (ii) implies (i) can be verified by direct substitution. To show the converse, suppose (i) holds and let $x = x_r + \mathrm{j}x_i$ with $x_r, x_i \in \mathbb{R}$ and $y = y_r + \mathrm{j}y_i$ with $y_r, y_i \in \mathbb{R}$. Then

$$x\bar{y} + \bar{x}y = 2(x_r y_r + x_i y_i) = 2\det\begin{bmatrix} x_r & -y_i \\ x_i & y_r \end{bmatrix} = 0$$

This implies the existence of $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} x_r & -y_i \\ x_i & y_r \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0, \quad \begin{bmatrix} a \\ b \end{bmatrix} \neq 0$$

The first equality is equivalent to $ax + \mathrm{j}by = 0$. If $a = 0$, then $b \neq 0$ and hence $y = 0$. In this case, (ii) holds with $\alpha := 1$, $\beta := 0$, and $z := x$. If $a \neq 0$, then (ii) holds with $\alpha := -b/a$, $\beta := -1$, and $z := \mathrm{j}y$. Thus (i) implies (ii). □

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Now consider the problem in (17). First note that

$$\eta^*\Phi_c\eta = 0 \quad \Leftrightarrow \quad \eta = \begin{bmatrix} \alpha \\ j\beta \end{bmatrix} z$$

by Lemma 4. Then the problem becomes

$$\max_{\xi\in\mathbb{R}^2} \tau\xi^{\mathsf{T}}\tilde{\Psi}_c\xi \quad \text{s.t. } |\xi_1|\leqslant 1, \quad \tilde{\Psi}_c := \begin{bmatrix} \varpi_1\varpi_2 & \varpi_c \\ \varpi_c & 1 \end{bmatrix}$$

with $\xi := |z|[\alpha \ \beta]^{\mathsf{T}}$. If $\tau > 0$, we should maximize

$$\xi^{\mathsf{T}}\tilde{\Psi}_c\xi = \xi_2^2 + 2\varpi_c\xi_1\xi_2 + \varpi_1\varpi_2\xi_1^2$$

The maximum is not attained and can be approached by ξ such that ξ_2 is sufficiently large and $|\xi_1|\leqslant 1$. In the limit, ξ is considered proportional to $[0 \ 1]^{\mathsf{T}}$. Thus,

$$\eta \sim \begin{bmatrix} 0 \\ j \end{bmatrix}, \quad R = [I \ 0 \ 0]$$

If $\tau < 0$, then we should minimize

$$\xi^{\mathsf{T}}\tilde{\Psi}_c\xi = (\xi_2 + \varpi_c\xi_1)^2 - (\varpi_1 - \varpi_2)^2\xi_1^2/4$$

subject to $|\xi_1|\leqslant 1$. Since $\varpi_1 \neq \varpi_2$, the solution is given by

$$\xi = \begin{bmatrix} 1 \\ -\varpi_c \end{bmatrix}, \quad \eta \sim \begin{bmatrix} 1 \\ -j\varpi_c \end{bmatrix}, \quad R = [j\varpi_c I \ \ I \ \ 0]$$

Table II summarizes these cases where (ϖ_1, ϖ_2) and $(\vartheta_1, \vartheta_2)$ are real scalars specifying the frequency ranges and satisfy $\varpi_1 < \varpi_2$ and $0 < \vartheta_2 - \vartheta_1 < 2\pi$, respectively, and $\varpi_c := (\varpi_1 + \varpi_2)/2$ and $\vartheta_c := (\vartheta_1 + \vartheta_2)/2$.

In the case of the FDI in the entire frequency range, we have seen that the solution of (17) provides R satisfying Conditions 1 and 2, leading to non-conservative synthesis conditions. Unfortunately, this does not seem to be the case in general if we consider the FDI in a restricted frequency range. However, for the special case of the continuous-time frequency range $\varpi_1 \leqslant \omega \leqslant \varpi_2$, the choice of R given above turns out to give an exact synthesis condition.

Proposition 1

Suppose $\Phi_{22} = 0$ and $\Psi_{22} < 0$ hold and let $R := [I \ 0 \ 0]$. If the condition in (13) is infeasible, then there is no controller (4) that meets the specification (1) with D_c satisfying $\rho(\mathsf{D}, \Pi) < 0$.

Proof

Suppose such a controller exists. Note that Conditions 1 and 2 hold because (16) becomes $\mathrm{diag}(\Psi_{22}Q, \rho(\mathsf{D}, \Pi)) < 0$ which is satisfied for the controller. Consequently, feasibility of

Table II. Reasonable choices of R (restricted frequency range).

$\Lambda(\Phi, \Psi)$	$\{j\omega \mid \varpi_1 \leqslant \omega \leqslant \varpi_2\}$	$\{j\omega \mid \omega \leqslant \varpi_1 \text{ or } \varpi_2 \leqslant \omega\}$	$\{e^{j\theta} \mid \vartheta_1 \leqslant \theta \leqslant \vartheta_2\}$
R	$[I \ 0 \ 0]$	$[j\varpi_c I \ I \ 0]$	$[I \ e^{j\vartheta_c} I \ 0]$

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condition (13) becomes necessary and sufficient for the existence of a controller that meets the specification (1). By supposition, such a controller exists and hence condition (13) must be feasible. □

As mentioned before, the proposition captures the case of the frequency range $\varpi \leqslant \omega \leqslant \varpi_2$ in the continuous-time setting. Suppose D_c has been fixed so that $\rho(\mathsf{D}, \Pi) < 0$ holds. A typical situation would be the case where a strictly proper controller is to be designed to meet a small gain requirement for a system with $D_{11} = 0$. Then, (13) with $R := [I\ 0\ 0]$ provides a *necessary and sufficient condition* for the existence of such controller.

Another observation from (16) is that, if there is no controller high-frequency gain D_c that makes $\rho(\mathsf{D}, \Pi)$ negative definite, the LMI synthesis condition in Theorem 2 will always be infeasible for the heuristic choices of R in Table II even when a feasible controller exists. A typical and important case is the sensitivity minimization in the low-frequency range. In this case, $\Pi = \mathrm{diag}(-\gamma I, I)$, $D_{11} = I$, and D_{12} or D_{21} is zero so that the gain bound γ cannot be less than 1 to ensure $\rho(\mathsf{D}, \Pi) < 0$. For this type of designs, we need to choose non-zero Γ in (14). However, it is not clear at the moment how to choose an appropriate Γ. An alternative would be to put a low-pass filter with sufficiently high cut-off frequency in series with the sensitivity transfer function so that $\rho(\mathsf{D}, \Pi) < 0$ is automatically satisfied.

5. EXTENSIONS

5.1. *Multi-objective control*

We consider the following control problem: find $K(\lambda)$ such that

$$\|H_k(\lambda)\| < \infty, \quad \rho(H_k(\lambda), \Pi_k) < 0 \quad \forall \lambda \in \mathbf{\Lambda}(\Phi_k, \Psi_k) \tag{18}$$

holds for all $k \in \mathbb{Z}_\ell$ where (Φ_k, Ψ_k, Π_k) defines a frequency domain specification to be achieved for the closed-loop system $H_k(\lambda) := G_k(\lambda) \star K(\lambda)$. The transfer function $G_k(\lambda)$ is given and represents a plant with a selected disturbance-performance (i.e. w-z) channel for multi-objective control. In particular, $G_k(\lambda)$ is specified by (3) with different state space matrices B_1, C_1, D_{11}, D_{12}, and D_{21} for different k. The generalized plants $G_k(\lambda)$ are assumed to share the same actuators and sensors so that matrices A, B_2, and C_2 are independent of k. In this case, $\|H_k(\lambda)\| < \infty$ $(k \in \mathbb{Z}_\ell)$ are replaced by $\det(\lambda I - \mathsf{A}) \neq 0$ where A is independent of k.

The following result provides a sufficient condition for feasibility of the above control problem. The result can be obtained from Theorem 2 in a straightforward manner, and hence its proof is omitted.

Corollary 1

Let $R_k \in \mathbb{C}^{n \times (2n+n_z)}$, $\Phi_k, \Psi_k \in \mathbb{H}_2$, $\Pi_k \in \mathbb{H}_{n_w+n_z}$, and systems $G_k(\lambda)$ as in (3) be given where $k \in \mathbb{Z}_\ell$. Assume that all $G_k(\lambda)$ share the same sensors/actuators and thus matrices A, B_2, and C_2 are common for all k. Suppose R_k satisfies Condition 2 and let $\mathscr{R}_k$ be the matrix satisfying $R\mathscr{T}^* = T^*\mathscr{R}$ with $R = R_k$. Then, there exists a dynamic feedback controller $K(\lambda)$ such that the frequency domain specifications (18) are satisfied for all $k \in \mathbb{Z}_\ell$ if there exist matrices

X, Y, Z, M, G, H, L, and $\mathscr{P}_k, \mathscr{Q}_k \in \mathbb{H}_n$ such that $\mathscr{Q}_k > 0$ and

$$\begin{bmatrix} I & \mathscr{B}_k \end{bmatrix} \begin{bmatrix} \Phi_k \otimes \mathscr{P}_k + \Psi \otimes \mathscr{Q}_k & 0 \\ 0 & \Pi_k \end{bmatrix} \begin{bmatrix} I \\ \mathscr{B}_k^* \end{bmatrix} < \mathscr{A}_k \mathscr{R}_k + (\mathscr{A}_k \mathscr{R}_k)^* \tag{19}$$

holds for all $k \in \mathbb{Z}_\ell$, where $\mathscr{A}_k$ and $\mathscr{B}_k$ are defined as in (12) in terms of the state space vertex matrices of the plant. In this case, one such controller is given by (4) where the parameters are calculated from (9) and (10).

This result provides a sufficient condition for the existence of a dynamic feedback controller that achieves the multiple FDI specifications in (18). The condition can be rewritten as LMIs via the Schur complement due to $\Pi_{22} \geqslant 0$, and can be solved numerically. The associated degree of conservatism is dependent upon the choices of R_k, and some reasonable values are given in the previous section. It should be noted that this formulation does not assume common 'Lyapunov matrices' $(\mathscr{P}, \mathscr{Q})$ as in the quadratic stability literature [20] or in the more recent multi-objective control [13, 14], but rather, $(\mathscr{P}, \mathscr{Q})$ can be interpreted as 'parameter dependent' as discussed in [15, 21–23]. Thus, we can expect reduced conservatism when compared with these existing techniques for multi-objective robust control. It should be emphasized, however, that the main contribution of this paper is not the conservatism reduction but the synthesis method to meet FDI specifications in (semi)finite frequency ranges, which have not been addressed in the literature.

5.2. Regional pole constraints

The design specifications in (18) encompass frequency domain shaping of closed-loop transfer functions. However, the closed-loop stability has not been captured, and hence one may wish to include a stability constraint, or more generally, regional pole constraints, as an additional design specification. The following lemma gives a basic result for an eigenvalue characterization.

Lemma 5
Let $A \in \mathbb{C}^{n \times n}$ and $\Phi \in \mathbb{H}_2$ be given. Suppose $\det(\Phi) < 0$. Then the following statements are equivalent.

(i) Each eigenvalue λ of A satisfies $\rho(\lambda, \Phi) < 0$.
(ii) There exists $P = P^* > 0$ such that $\rho(A, \Phi \otimes P) < 0$.
(iii) There exist W and $P = P^* > 0$ such that

$$\Phi \otimes P < \mathrm{He} \begin{bmatrix} A \\ -I \end{bmatrix} W [-qI \;\; pI]$$

where $r := [p \;\; q]^{\mathsf{T}} \in \mathbb{C}^2$ is an arbitrary fixed vector satisfying $r^* \Phi r < 0$.

Proof
The equivalence (i) $\Leftrightarrow$ (ii) has been shown in [24, 25]. The equivalence (ii) $\Leftrightarrow$ (iii) follows from the projection lemma. □

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The condition in (iii) can be used to give additional constraints in the design equations discussed in the previous sections. In particular, we replace A with the closed-loop matrix A in (5), apply a congruence transformation by $\mathrm{diag}(T, T)$, and use the change of variables in (12). As a result, we add the following constraint to the design:

$$\Phi \otimes \mathscr{P} < \mathrm{He}(J\mathscr{A}S), \quad J := [I_{2n} \;\; 0], \quad S := [-qI \;\; pI] \tag{20}$$

In the multiobjective control, $\mathscr{A}$ depends on k in general but $J\mathscr{A}$ will always be independent of k, justifying the omission of the subscript k. On the other hand, multiple inequalities of the same form, but with different Φ and S, can be added to enforce regional pole constraints expressed as the intersection of half planes and circles.

6. DESIGN EXAMPLE

The main objective of this section is to illustrate the design procedure proposed in this paper. In particular, we will design a controller using statement (ii) of Theorem 2 with the heuristic choices of R described in Table II. The design method is potentially conservative and another aim of this example is to show that the degree of conservatism can be small enough for some applications to allow for direct design of controllers to meet multiple specifications in different frequency ranges.

We consider the control of an AMB. With a constant biasing, the normalized dynamics of an AMB, from the voltage input to the displacement output, can be described by

$$P(s) = \frac{1}{s^3 + \kappa s^2 - \kappa}$$

where κ ranges between about 0.3 and 3 for physically reasonable AMB designs [26]. Below, we take $\kappa = 0.5$. In this case, $P(s)$ has an unstable real pole at $s = 0.657$ and an oscillatory mode at natural frequency $\omega = 0.872$ with damping $\zeta = 0.6663$.

The problem is to design a stabilizing controller $K(s)$ to meet the following specifications:

$$\begin{aligned} |P(\mathrm{j}\omega)S(\mathrm{j}\omega)| &< \gamma_0 \quad \forall |\omega| \leqslant \varpi_0 \\ |K(\mathrm{j}\omega)S(\mathrm{j}\omega)| &< \gamma_1 \quad \forall |\omega| \geqslant \varpi_1 \\ |K(\mathrm{j}\omega)P(\mathrm{j}\omega)S(\mathrm{j}\omega)| &< \gamma_2 \quad \forall |\omega| \geqslant \varpi_2 \end{aligned} \tag{21}$$

where $S := 1/(1 - PK)$ is the sensitivity function. These three specifications address position regulation against input-port disturbance, sensitivity of the control input to the sensor noise, and robustness against the multiplicative plant uncertainty.

We set the parameter values

$$\varpi_0 = 0.5, \quad \varpi_1 = 3, \quad \varpi_2 = 0.8, \quad \gamma_1 = 4, \quad \gamma_2 = 10$$

and minimize γ_0 subject to the above constraints over a set of stabilizing full order controllers using Theorem 2 and (20) with $p = -q = 1$. The optimal value is found to be $\gamma_0 = 4.45$ and the controller is

$$K(s) = -\frac{2.2207(s + 5.975)(s^2 + 1.234s + 0.9334)}{(s + 3.369)(s^2 + 2.235s + 4.492)}$$

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The resulting close-loop frequency responses are plotted in Figure 1, where the specification bound for the solid curve is indicated by the shaded region with a solid boundary, and similarly for the dashed and dash-dotted curves. We see that the bounds on $|PS|$ and $|KS|$ are fairly tight, suggesting effectiveness of the design method.

On the other hand, the bound on $|KPS|$ is not as tight as those on $|PS|$ and $|KS|$. The design would be difficult if the resulting frequency response is insensitive to the change in the bound specification. If the result is sensitive, however, frequency shaping can still be done even when the bound is conservative, by iteratively revising the design specification. To illustrate this point, let us consider the case where the specification on $|KPS|$ is relaxed to $\varpi_2 = 1$. In this case, the optimal value of $\gamma_0 = 2.97$ is achieved by

$$K(s) = -\frac{2.6793(s + 2.911)(s^2 + 1.122s + 0.9738)}{(s + 1.705)(s^2 + 2.104s + 4.242)}$$

The resulting frequency responses are plotted in Figure 2. We see that the slight change introduced to the specification yielded a significant change in the closed-loop responses. Due to this high sensitivity, we can tune the specification to meet the original goal. For example, if we want to minimize γ_0 subject to (21), we may adjust ϖ_2 between 0.8 and 1 so that the constraint on $|KPS|$ in (21) becomes tight. In fact, choosing $\varpi_2 = 0.913$ gives the peak value $\|KPS\|_\infty = 9.97$ at $\omega = 0.825$, while achieving $\gamma_0 = 3.56$. Figure 3 shows how the optimal performance bound γ_0 and the peak value of $|KPS|$ change as ϖ_2 is adjusted. As expected, γ_0 is monotonically decreasing with respect to ϖ_2. The peak value increases when ϖ_2 varies around the natural frequency $\omega = 0.872$ and then reaches a plateau as ϖ_2 becomes sufficiently large. When ϖ_2 becomes smaller than the natural frequency, the frequency at which the peak value occurs becomes smaller than and moves away from the natural frequency. In summary, the multiobjective GKYP synthesis can effectively shape the closed-loop transfer functions in various frequency ranges.

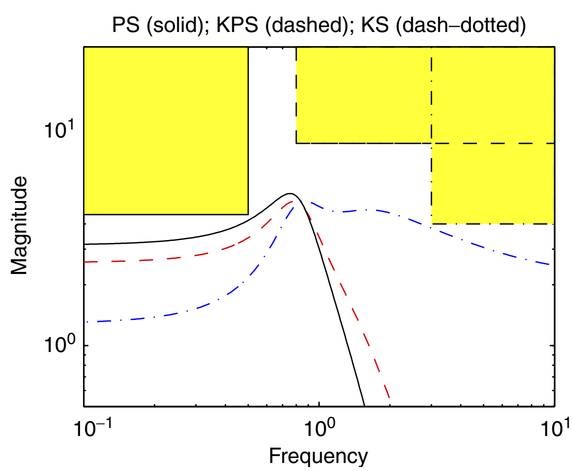

Figure 1. Design result (Case 1: $\varpi_2 = 0.8$).

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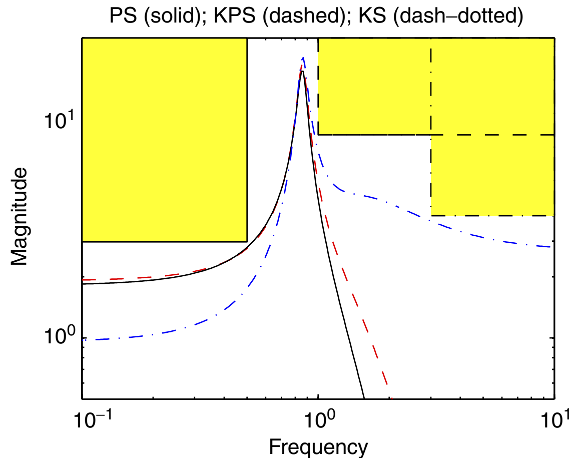

Figure 2. Design result (Case 2: $\varpi_2 = 1$).

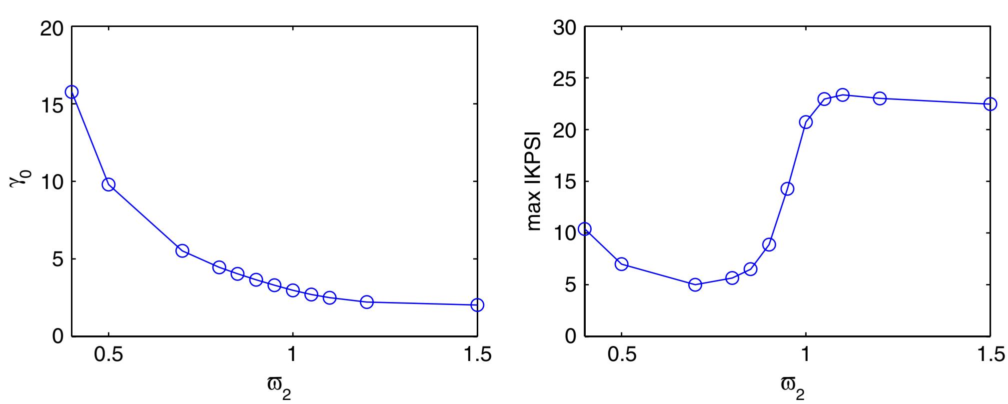

Figure 3. Optimal bound γ_0 and the peak value of $|KPS|$ as functions of ϖ_2.

7. CONCLUSION

We have developed a method for synthesizing dynamic output feedback controllers to achieve multiple FDI specifications in (semi)finite frequency ranges. A sufficient condition for existence of feasible controllers is given in terms of LMIs, and some special cases, where the condition becomes also necessary, are discussed. An example of the AMB illustrated the proposed design method and demonstrated its effectiveness.

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REFERENCES

1. Yakubovich VA. Solution of certain matrix equations appearing in automatic control. *Doklady Akademii Nauk SSSR* 1962; **143**(6):1304–1307.
2. Kalman RE. Lyapunov functions for the problem of Lur'e in automatic control. *Proceedings of the National Academy of Sciences* 1963; **49**(2):201–205.
3. Anderson BDO. A system theory criterion for positive real matrices. *SIAM Journal on Control* 1967; **5**(2):171–182.
4. Willems JC. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Transactions on Automatic Control* 1971; **16**:621–634.
5. Rantzer A. On the Kalman–Yakubovich–Popov lemma. *Systems and Control Letters* 1996; **28**(1):7–10.
6. Iwasaki T, Meinsma G, Fu M. Generalized *S*-procedure and finite frequency KYP lemma. *Mathematical Problems in Engineering* 2000; **6**:305–320 http://www.hindawi.com/GetArticle.aspx?pii = S1024123X00001368
7. Iwasaki T, Hara S, Yamauchi H. Dynamical system design from a control perspective: finite frequency positive-realness approach. *IEEE Transactions on Automatic Control* 2003; **48**(8):1337–1354.
8. Iwasaki T, Hara S, Fradkov AL. Time domain interpretations of frequency domain inequalities on (semi)finite ranges. *Systems and Control Letters* 2005; **54**(7):681–691.
9. Iwasaki T, Hara S. Generalized KYP lemma: unified frequency domain inequalities with design applications. *IEEE Transactions on Automatic Control* 2005; **50**(1):41–59.
10. Hara S, Iwasaki T, Shiokata D. Robust PID control using generalized KYP synthesis. *IEEE Control Systems Magazine* 2006; **26**(1):80–91.
11. Doyle JC, Glover K, Khargonekar PP, Francis BA. State-space solutions to standard H_2 and H_∞ control problems. *IEEE Transactions on Automatic Control* 1989; **34**(8):831–847.
12. Iwasaki T, Skelton RE. All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas. *Automatica* 1994; **30**(8):1307–1317.
13. Scherer C, Gahinet P, Chilali M. Multiobjective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control* 1997; **42**(7):896–911.
14. Masubuchi I, Ohara A, Suda N. LMI-based controller synthesis: a unified formulations and solution. *International Journal of Robust and Nonlinear Control* 1998; **8**:669–686.
15. de Oliveira MC, Geromel JC, Bernussou J. Extended H_2 and H_∞ characterizations and controller parametrizations for discrete-time systems. *International Journal on Control* 2002; **75**(9):666–679.
16. de Oliveira MC, Bernussou J, Geromel JC. A new discrete-time robust stability condition. *Systems and Control Letters* 1999; **37**:261–265.
17. Gahinet P, Apkarian P. A linear matrix inequality approach to H_∞ control. *International Journal of Robust and Nonlinear Control* 1994; **4**:421–448.
18. Iwasaki T, Hara S. Robust control synthesis with general frequency domain specifications: static gain feedback case. *Proceedings of the American Control Conference* 2004; **5**:4613–4618.
19. Shaked U. Improved LMI representations for the analysis and the design of continuous-time systems with polytopic type uncertainty. *IEEE Transactions on Automatic Control* 2001; **46**(4):652–656.
20. Barmish BR. Necessary and sufficient conditions for quadratic stabilizability of an uncertain linear system. *Journal of Optimization Theory and Applications* 1985; **46**(4):399–408.
21. Daafouz J, Bernussou J. Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties. *Systems and Control Letters* 2001; **43**:355–359.
22. Geromel JC, de Oliveira MC, Bernussou J. Robust filtering of discrete-time linear systems with parameter dependent Lyapunov functions. *SIAM Journal of Control and Optimization* 2002; **41**(3):700–711.
23. Apkarian P, Tuan HD, Bernussou J. Continuous-time analysis, eigenstructure assignment, and h_2 synthesis with enhanced linear matrix inequalities characterizations. *IEEE Transactions on Automatic Control* 2001; **46**(12): 1941–1946.
24. Henrion D, Meinsma G. Rank-one LMIs and Lyapunov's inequality. *IEEE Transactions on Automatic Control* 2001; **46**(8):1285–1288.
25. Peaucelle D, Arzelier D. Robust performance analysis with LMI-based methods for real parametric uncertainty via parameter-dependent Lyapunov functions. *IEEE Transactions on Automatic Control* 2001; **46**(4):624–630.
26. Maslen E, Montie D, Iwasaki T. Robustness limitations in self-sensing magnetic bearing. *ASME Dynamic Systems, Measurement and Control* 2006; **128**(2):197–203.

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