Lecture 3

Why H-infinity Control and H2 Control

Outline:

I. Introduction

II. Norms for signals

III. Norms for systems

IV. Robust stability and Disturbance attenuation

V. Extensions

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I. Introduction

System => Treat as a **<u>Generalized Amplifier</u>**

Q: system gain?

A: system gain $:= \dfrac{\|\text{output} \quad \text{signal}\|}{\|\text{input} \quad \text{signal}\|}$.

Q: (i) How to describe a signal in mathematical terms?
(ii) How to measure the "size" of a signal?

Q: What are the potential engineering applications of this concept?
A: Disturbance attenuation!

II. Norms for signals（scalar case）: different ways to measure the "size" of a signal

Consider deterministic signals as time function $e(t)$, $t \in (-\infty, \infty)$

- 1-norm ：$\| e \|_1 \stackrel{\Delta}{=} \int_{-\infty}^{\infty} | e(t) | \, dt$
- 2-norm ：$\| e \|_2 \stackrel{\Delta}{=} (\int_{-\infty}^{\infty} | e(t) |^2 \, dt)^{\frac{1}{2}}$, i.e., total energy of *e(t)*
- ∞-norm ：$\| e \|_\infty \stackrel{\Delta}{=} \sup_{t \in R} | e(t) |$; ie., peak value of $e(t)$
- power signal：a signal *e(t)* is called a power signal if its average power is finite, i.e., *e(t)* satisfies

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e(t)^2 \, dt < \infty$$

Thus we denote

$$pow(u) \stackrel{\Delta}{=} \left(\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e(t)^2 \, dt \right)^{\frac{1}{2}} \quad \text{(eventual average power)}$$

For stationary stochastic signals [BB91, pp. 75]:

- the root-mean-square(RMS) value of *e(t)* is defined by

$$\| e \|_{rms} \stackrel{\Delta}{=} \left[E(e(t)^2) \right]^{1/2} = (R_e(0))^{1/2} = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} S_e(\omega) d\omega \right]^{1/2}$$

where $R_u(\tau) = E(e(t)e(t+\tau))$ represents the autocorrelation of *e* and

$S_u(\omega) = \int_{-\infty}^{\infty} R_e(\tau) e^{-j\omega\tau} d\tau$ represents its power spectral density (PSD).

Interpretation:

(i) $\| e \|_{rms}$ represents the average value of *e* (or instantaneous power in average).

(ii) The variance of e denoted by $\sigma_u^2 = \|e\|_{rms}^2 = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} S_e(\omega) d\omega \right]$.

III. Norms for systems（scalar case）

Consider LTI causal system $T(s)$.

- H2 norm: Assume $T(s)$ is <u>strictly proper</u>（i.e., $T(\infty)=0$）and <u>stable</u>, then

$$\| T(s) \|_2 := \sup_{\substack{u \in L_2 \\ u \neq 0}} \frac{\|y\|_\infty}{\|u\|_2},$$ u and y represent the input and output, respectively.

or, equivalently,

$$\| T(s) \|_2 := [\ \frac{1}{2\pi} \int_{-\infty}^{\infty} Trace\left(T^*(jw)T(jw)\right) dw\ ]^{\frac{1}{2}} \quad \text{(MIMO case)}$$

$$\| T(s) \|_2 := [\ \frac{1}{2\pi} \int_{-\infty}^{\infty} |T(jw)|^2\, dw\ ]^{\frac{1}{2}} = [\ \int_{-\infty}^{\infty} \left| T_{imp}(t) \right|^2 dt\ ]^{\frac{1}{2}} = \left\| T_{imp}(t) \right\|_2 \quad \text{(SISO case)}$$

where T_{imp} represents the impulse response of the system $T(s)$.

- Computation of H2 norm (State space formula)

  Lemma 4.4 [ZD98]: Given a stable, strictly proper system $G(s) = C(sI - A)^{-1}B$. Then

$$\| T(s) \|_2 = \sqrt{trace(B^* QB)} = \sqrt{trace(CPC^*)}\ .$$

  where Q and P are observability and controllability Grammians that can be obtained from the following Lyapunov equations:

$$A^*Q + QA + C^*C = 0; \qquad AP + PA^* + BB^* = 0$$

Matlab command?

- H∞-norm: Assume $T(s)$ is proper and stable, then

$$\| T(s) \|_\infty := \sup_{\substack{u \in L_2 \\ u \neq 0}} \frac{\|y\|_2}{\|u\|_2}$$

or, equivalently,

MIMO case: $\| T(s) \|_\infty := \sup_{\omega \in R \cup \{\infty\}} \bar{\sigma}(T(j\omega))$,

where $\bar{\sigma}(H)$ denotes the largest singular value of the matrix H.

SISO case: $\| T(s) \|_\infty := \sup_{\omega \in R \cup \{\infty\}} | T(j\omega) |$.

The ∞-norm of $T(s)$ equals (i) the peak value on the Bode magnitude plot of T(s), or (ii) the distance in the complex plane from the origin to the farthest point on the Nyquist plot of $T(s)$.

- Computation of H∞-norm: Bisection Algorithm [ZD98, §4.4].

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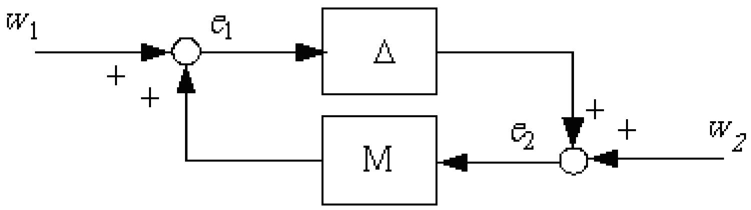
IV. Robust stability and Disturbance attenuation

● Robust stability



M- Δ loop for stability analysis

Small Gain Theorem [ZD98, Theorem 8.1]: Suppose $M \in RH_\infty$ and let $\gamma \succ 0$. Then the interconnected system shown in the above figure is well-posed and internally stable for all $\Delta(s) \in RH_\infty$ with

(a) $\|\Delta(s)\|_\infty \le 1/\gamma$ if and only if $\|M(s)\|_\infty < \gamma$.

(b) $\|\Delta(s)\|_\infty < 1/\gamma$ if and only if $\|M(s)\|_\infty \le \gamma$.

Stability margin optimization problem: the problem of enlarging the stability margin of systems with unstructured or structured uncertainty

I. Unstructured perturbations case : The maximal size of perturbation guaranteed by this approach = $\frac{1}{\|M(s)\|_\infty}$ where $M = F_l(P,K)$, and $\|M(s)\|_\infty$ is the size of the transfer function that the perturbation "sees", which depends on the controller to be determined.

II. Structured perturbations case : This is the case that there are multiple sources of perturbations in the loop => μ theory (robust stability and robust performance)

(i) Controller synthesis framework: $\Delta - P - K$ framework with Δ being block-diagonal.

(ii) The maximal size of perturbation is $\frac{1}{\|M(s)\|_\mu}$ where $M = F_l(P,K)$.

● Disturbance attenuation

Example: (資料來源: 謝坤志 MS thesis, June, 2004)

考慮圖 2.2 的標準迴授系統架構。

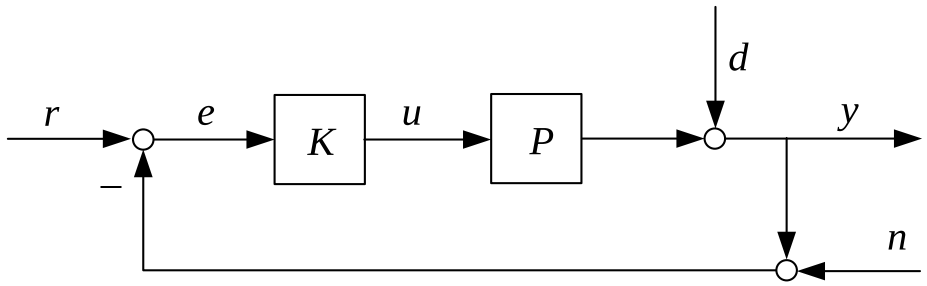


圖 2.2 標準迴授系統架構

其中， P 為受控體；K 為控制器；r 為參考輸入； u 為控制輸入；e 是追蹤誤差；d 是外來干擾； y 是系統的輸出；n 為量測雜訊。

所謂強健性能的控制器 K 設計是要使得追蹤誤差 $e(t)$ 愈小愈好。假設圖 2.2 之 $r = n = 0$， $d(t)$ 訊號屬於某一函數集合

$$D = \left\{ d \middle| \quad d = Wv, \quad v \in H_2, \quad \| v \|_2 \leq 1 \right\}$$

其中，$W(s)$ 稱為權重函數(weighting function)，用以整形外來干擾 $d(t)$。從 $d(s)$ 到 $e(s)$ 間之轉移函數為

$$e(s) = \frac{-d(s)}{1 + K(s)P(s)} = \frac{-W(s)}{1 + K(s)P(s)} v(s) \qquad (2.1)$$

若假設 $S(s)$ 為靈敏度函數(sensitivity function)

$$S(s) = \frac{1}{1 + K(s)P(s)}$$

則(2.1)式改寫成

$$e(s) = \left[-W(s)\ S(s) \right]\ v(s) .$$

把追蹤誤差 $e(t)$ 視為能量訊號來看，由以下的推導可看出 ∞ -norm 是一種「最糟情況的量度」

$$
\begin{aligned}
\| e(s) \|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} | e(jw) |^2 \, dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} | WS(jw) |^2 | v(jw) |^2 \, dw \\
&\le \frac{1}{2\pi}\int_{-\infty}^{\infty} \sup_w \ | WS(jw) |^2 | v(jw) |^2 \, dw \\
&\le \sup_w \ | WS(jw) |^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} | v(jw) |^2 \, dw \\
&\le \sup_w \ | WS(jw) |^2 \ \| v(s) \|_2^2 \\
&\le \sup_w \ | WS(jw) |^2 \\
&= \| WS(s) \|_\infty^2 .
\end{aligned}
$$

$$\Rightarrow \quad \| e(s) \|_2 \le \| WS(s) \|_\infty$$

$$\Rightarrow \quad \sup_{\| v(s) \|_2 \le 1} \| e \|_2 = \| WS \|_\infty \qquad (2.2)$$

(2.2)式的結果可以解釋為：對於所有可能的干擾輸入 $d = Wv$ 且 $\| v \|_2 \le 1$，其產生的追蹤誤差訊號其能量不超過 $\| WS \|_\infty$。因此， 可設計控制器 $K(s)$ 使得

$$\| WS \|_\infty = \left\| \frac{W(s)}{1+K(s)P(s)} \right\|_\infty < \ \varepsilon \qquad (2.3)$$

其中 ε 為一極小正數。由(2.3)式知

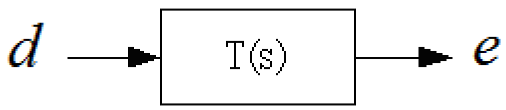
$$\sup_{\| v \|_2 \le 1} \| e \|_2 < \ \varepsilon \ \Rightarrow \ \| e \|_2 < \ \varepsilon, \quad \forall v 滿足 \| v \|_2 \le 1 .$$

(2.4)

亦即只要(2.3)式滿足規格需求，則(2.4)式自動滿足。我們稱(2.2)式為最糟情況的設計，一旦最糟的情況都能滿足規格設計，那麼其他一般的情況下，設計自然都能滿足要求了。這就是 H_∞ 控制器設計抑制外來干擾的基本理念。

The Other cases:

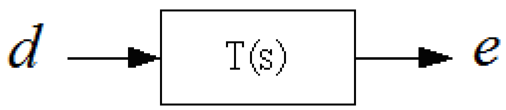
Consider T(s) which is strictly proper and stable



Q：If we know how big the input is, how big is the output going to be？

- Assume d is noise or disturbance of known dynamics and e is the signal of interest

Case 1: assume *d* is an impulsive disturbance



Then

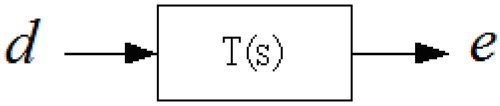
$$\|e\|_2 = \|T(s)\|_2$$
$$\|e\|_\infty = \|T_{imp}\|_\infty$$
$$pow(e) = 0$$

Conclusions:

(1) Minimizing the energy of e due to impulsive disturbances can be amount to minimizing $\|T(s)\|_2$.

(2) Minimizing the peak value of e due to impulsive disturbances can be amount to minimizing $\|T_{imp}(t)\|_\infty$ (where $T_{imp}(t)$ denotes the impulse response of $T(s)$).

Case 2: assume d is a sinusoidal disturbance with unknown frequency



Then

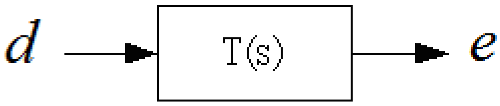
$$\| e \|_2 = \infty$$
$$\| e \|_\infty = \sup_{t \ge 0} \left| \left| T(j\omega^*) \right| \sin(\omega^* t + \phi) \right| = \left| T(j\omega^*) \right| \le \| T \|_\infty$$
$$pow(e) = \frac{1}{\sqrt{2}} \left| T(j\omega^*) \right| \le \frac{1}{\sqrt{2}} \|T\|_\infty$$

Conclusions:

(1) Minimizing the peak value or power of e due to sinusoidal disturbance with unknown frequency can be amount to minimizing $\| T(s) \|_\infty$

- More commonly, the dynamics of the disturbance signals will not be known a priori, there are three ways to characterize the signals:

Case 1: assume *e* is a class of bounded-energy signals (we know only with the upper bound $\| e \|_2$)



Then

$$\| e \|_2 \le \| T(s) \|_\infty \| d \|_2$$
$$\| e \|_\infty \le \| T(s) \|_2 \| d \|_2$$
$$pow(e) = 0$$

Case2: assume d is a class of bounded-peak signals (we know only with the upper bound $\| d \|_\infty$), e.g., unit step input

$$d \longrightarrow \boxed{T(s)} \longrightarrow e$$

Then

$$\|e\|_2 \le \quad \infty \quad \| d \|_\infty$$
$$\| e \|_\infty \le \| T(s) \|_1 \| d \|_\infty$$
$$pow(e) \le \| T(s) \|_\infty \| d \|_\infty$$

Case3: assume e is a class of bounded-power signals (we know only

pow(e)<∞), e.g., sine wave

$$d \longrightarrow \boxed{T(s)} \longrightarrow e$$

Then

$$\| e \|_2 \le \quad \infty \quad pow(d)$$
$$\| e \|_\infty \le \quad \infty \quad pow(d)$$
$$pow(e) \le \| T(s) \|_\infty \; pow(d)$$

- For stationary stochastic signals d [BB91, pp.110],

The output spectral density $S_e(\omega) = |T(j\omega)|^2 S_d(\omega)$.

Hence

$$\|e\|_{rms} = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} | T(j\omega) |^2 S_d(\omega)\, d\omega \right]^{1/2}$$

If $S_d(\omega) \approx I$ for those frequencies for which $T(j\omega)$ is significant, i.e., white noise with unity power spectral density (or unit covariance), then

$$\|e\|_{rms} \approx \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} | T(j\omega) |^2 d\omega \right]^{\frac{1}{2}} = H_2 \text{ norm of the system } T(s)$$

i.e., variance of e denoted by

$$\sigma_e^2 = \|e\|_{rms}^2 \approx \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} | T(j\omega) |^2 d\omega \right] = \|T(s)\|_2^2$$

On the other hand, if we remove the unity power spectral density assumption, then

$$\|e\|_{rms} \le \|T(s)\|_\infty \|d\|_{rms}, \text{ i.e.,} \quad \sigma_e^2 \le \|T(s)\|_\infty^2 \, \sigma_d^2$$

Summary:

$$d \longrightarrow \boxed{T(s)} \longrightarrow e$$

Table 2.1 Output norms and pow for two inputs

	$d(t)=\delta(t)$	$d(t)=\sin(\omega^* t)$
$\|e\|_2$	$\|T(s)\|_2$	∞
$\|e\|_\infty$	$\|T(s)\|_\infty$	$\lvert T(j\omega^*)\rvert$
$pow(e)$	0	$\frac{1}{\sqrt{2}}\lvert T(j\omega^*)\rvert$

Table 2.2 System Gains

	$\|d\|_2$	$\|d\|_\infty$	$pow(d)$
$\|e\|_2$	$\|T(s)\|_\infty$	∞	∞
$\|e\|_\infty$	$\|T(s)\|_2$	$\|T_{imp}\|_1$	∞
$pow(e)$	0	$\le \|T(s)\|_\infty$	$\|T(s)\|_\infty$

抑制干擾問題(reduce the effect of the signal d on the signal e)，何時用 H_2 or H_∞ control:

Minimize the energy of e due to impulsive disturbance d => H_2 control
Minimize the peak value of e due to bounded energy disturbance d => H_2 control
Minimize the RMS value of e due to white noise u with unit spectral density (or unit covariance) => H_2 control

Minimize the peak value/power of e due to sinusoidal disturbance d => H_∞ control
Minimize the energy of e due to bounded energy disturbance d => H_∞ control
Minimize the power of e due to bounded-peak disturbance d => H_∞ control
Minimize the power of e due to bounded-power disturbance d => H_∞ control
Minimize the RMS value of e due to white noise d => H_∞ control

• Minimize the peak value of *e* due to impulsive disturbance *d*:

minimize $\| T_{imp} \|_{\infty} := \sup_{t \in R} \left| T_{imp}(t) \right|$ where T_{imp} denotes the impulse response of $T(s)$.

• Minimize the peak value of *e* due to bounded-peak disturbance *d*:

minimize $\| T_{imp} \|_{1} := \int_{-\infty}^{\infty} \left| T_{imp}(t) \right| dt$. (MIT research)

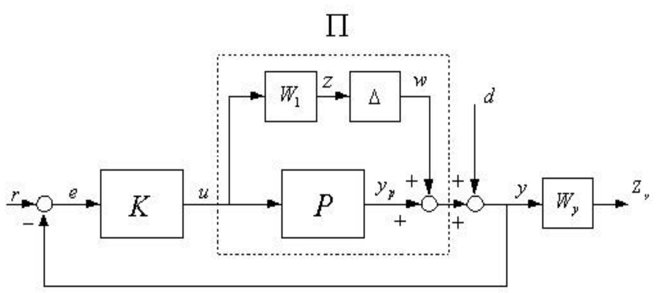
I. Extensions

Example: Servo control + mixed H_2 / H_∞ performance

Problem: Consider the following control system configuration. Find a controller such that the following design objectives are satisfied:
(1) closed-loop stability;
(2) perfect asymptotically tracking ability subject to step input, i.e.,

$e(\infty) \to 0$;

(3) robustness properties (against uncertainty Δ and external disturbance d).



To achieve tracking objective: introduce internal model $\frac{s+a}{s}$ where $a > 0$,

i.e., set $K(s) = \frac{s+a}{s} K_1(s)$. Then solve either one of the following robust control problems.

• Problem 1: determine K_1 to

minimize $\|T_{zw}\|_\infty$ (optimize stability margin) subject to $\| T_{ed} \|_i < \gamma$, $i = 2, \infty$ (disturbance attenuation rate is at least ..)

- Problem 2: determine K_1 to

minimize $\|T_{ed}\|_i$ ($i = 2, \infty$) (optimize disturbance attenuation)

subject to $\| T_{zw} \|_\infty < \gamma$ (stability margin is at least …)

- Problem 3: determine K_1 to

minimize $\alpha_1 \| T_{zw} \|_\infty + \alpha_2 \|T_{ed}\|_2$ where α_1 and α_2 are given positive numbers.

(trade-off between stability margin and disturbance attenuation)

- Problem 4: determine K_1 to

minimize $\alpha_1 \| T_{zw} \|_\infty^2 + \alpha_2 \|T_{ed}\|_2^2$ where α_1 and α_2 are given positive numbers. (Matlab LMI toolbox p. 5-12)

(trade-off between stability margin and disturbance attenuation)

The resulting controller $K(s) = \frac{s+a}{s} K_1(s)$.

References

[DFT92] J.C. Doyle, B.A. Francis and A.R. Tannenbaum, Feedback Control Theory, Macmillan Publishing Company, Inc. 1992. (Chapter 2)
[BB91] S.P. Boyd and C.H. Barratt, Linear Controller Design: Limits of Performance, Prentice Hall, 1991.

[ZD98] K. Zhou and J. C. Doyle, Essentials of Robust Control, Prentice Hall, 1998.

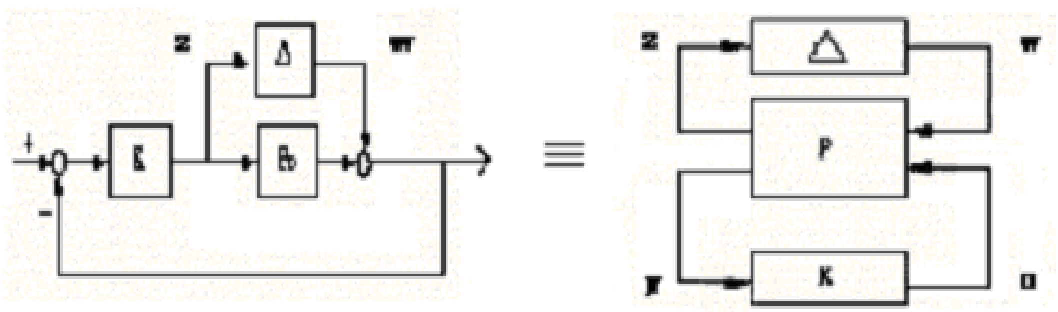
Example: Given a continuous-time generalized plant P(s), compute an internally stabilizing controller K(s) that minimizes the closed-loop H_∞ performance.

[gopt,K] = hinflmi(P,r)

[gopt,K,X1,X2,Y1,Y2] = hinflmi(P,r,g,tol,options)

Input	
P	plant SYSTEM matrix (see LTISYS)
r	1x2 vector specifying the dimensions of D22. That is, R(1) = nbr of measurements R(2) = nbr of controls
g	user-specified target for the closed-loop performance. Set g=0 to compute gopt, and set g=GAMMA to test whether the performance GAMMA is achievable.
tol	relative accuracy required on GOPT (default=1e-2)
options	optional 3-entry vector of control parameters for the numerical computations. (see document for details)
Output	
gopt	best H-infinity performance
K	central H-infinity controller for gamma = GOPT
X1,X2,..	X = X2/X1 and Y = Y2/Y1 are solutions of the two H-infinity Riccati inequalities for gamma = GOPT. Equivalently, R = X1 and S = Y1 are solutions of the characteristic LMIs since X2=Y2=GOPT*eye .
Note: See also HINFRIC, HINFMIX, HINFGS in MATLAB.	

% Program for the Stability margin optimization problem




```
clear; clc;

nump=[2 1]; denp=[1 3 -3]; [ap,bp,cp,dp]=tf2ss(nump,denp); % nominal plant P0

A=ap;
B1=[0;0];
B2=bp;
C1=[0 0];
C2=-cp;
D11=0;
D12=1;
D21=-1;
D22=-dp;

P=ltisys(A,[B1 B2],[C1;C2],[D11 D12;D21 D22]);
[gopt,K]=hinflmi(P,[1 1])    % alternative command: [gopt_ric,K_ric]=hinfric(P,[1 1])
```

% Implication: stability margin is $\frac{1}{gopt}$.