Lecture 4

H-infinity Control Theory

I. Problem Formulations

II. LMI Approach

- Gahinet et al's approach (Elimination Lemma)
- Scherer et al's approach (Change of variables)

III. ARE Approach

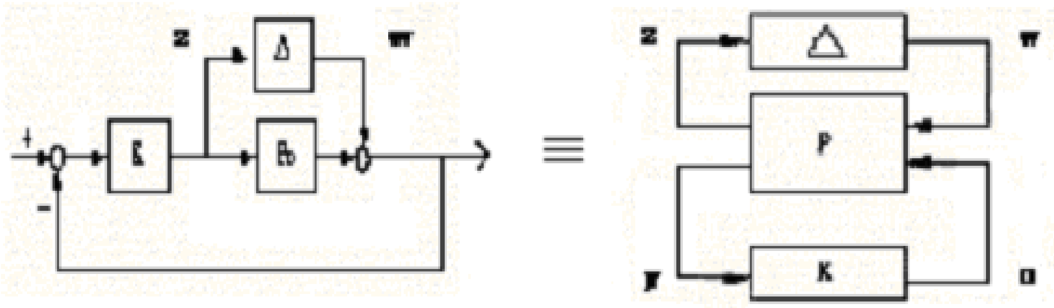
Book by K. Zhou and J. Doyle

Yung-Shan Chou

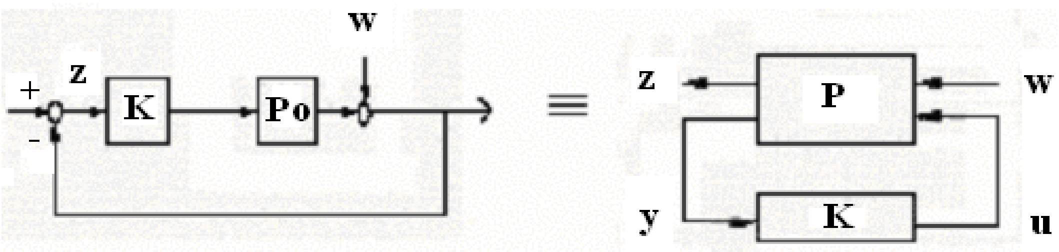
I. Problem Formulations

Examples:

(1) Stability margin optimization problem => H_∞ control



(2) Disturbance attenuation problem => H_2 or H_∞ control



- **Optimal H∞ control problem**: find a stabilizing controller such that $\|T_{zw}\|_\infty$ is minimized.

- **Suboptimal H∞ control problem**: given $\gamma > 0$, find a stabilizing controller such that $\|T_{zw}\|_\infty < \gamma$.

II. LMI Approach

- **Gahinet et al's approach (Elimination Lemma) [GA93]**

  (全階及降階 Hinf 控制器設計：Gahinet 等人的作法)

一般而言，工業上所設計存在的控制器大部份都是簡單的 PID (Proportional plus Integral plus Derivative)控制器，因為在H_∞控制理論問題中，所設計的控制器階數一般與廣義受控體的同階，但廣義受控體的階數往往很高，因而常常需要進行降階設計，其理由是降階控制器的運算量比高階控制器來的少而且也比較容易以硬體實現。

考慮$P-K$架構，假設廣義受控體P為連續時間，線性非時變系統，則$P(s)$為

$$P(s)=\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}+\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(sI-A)^{-1}\begin{pmatrix} B_1 & B_2 \end{pmatrix} \leftrightarrow \left[\begin{array}{c|cc} \mathrm{A} & \mathrm{B}_1 & \mathrm{B}_2 \\ \hline \mathrm{C}_1 & \mathrm{D}_{11} & \mathrm{D}_{12} \\ \mathrm{C}_2 & \mathrm{D}_{21} & \mathrm{D}_{22} \end{array}\right]$$

其中，$A\in\Re^{n\times n}$，$D_{11}\in\Re^{p_1\times m_1}$及$D_{22}\in\Re^{p_2\times m_2}$，並且符合如下假設：

Assumption A1: $(A,\ B_2,\ C_2)$ is stabilizable and detectable.

Assumption A2: $D_{22}=0$.

利用實界引理(Bounded Real Lemma)，給定$\gamma>0$，則存在k階H_∞控制器$K(s)=D_k+C_k(sI-A_k)^{-1}B_k$，使得閉迴路系統$T_{zw}$穩定而且$\|T_{zw}\|_\infty<\gamma$，若且唯若，存在矩陣$X_{cl}\in\Re^{(n+k)\times(n+k)}$，以及控制器參數$A_k, B_k, C_k, D_k$滿足下列矩陣不等式

$$\begin{pmatrix} A_{cl}^T X_{cl}+X_{cl}A_{cl} & X_{cl}B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{pmatrix}<0\ , \qquad X_{cl}>0\ . \qquad (\mathrm{G}.1)$$

其中
$$\left(\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right)=\left[\begin{array}{cc|c} A+B_2D_kC_2 & B_2C_k & B_1+B_2D_kD_{21} \\ B_kC_2 & A_k & B_kD_{21} \\ \hline C_1+D_{12}D_kC_2 & D_{12}C_k & D_{11}+D_{12}D_kD_{21} \end{array}\right]$$

Gahinet 等人[1]利用變數消減引理(Elimination Lemma)，將上述(G.1)式的限制條件轉換為等價的條件如下(**註**: (1) 1^{st} ineq of (G1) 等價轉換為 (G.2) and (G.3). (2) 2^{nd} ineq of (G.1) 等價轉換為(G..4) and (G.5) (K.Zhou book , pp.272, Lemma 14.2))。

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【定理 3.1】[GA93]若 Assumptions A1,A2 滿足，給定一正數 $\gamma>0$。存在 k 階 H_∞ 控制器 $(k\le n)$，使得閉迴路系統 T 穩定，而且 $\|T_{zw}\|_\infty<\gamma$，若且唯若，存在 $n\times n$ 對稱矩陣 R 和 S ，滿足下列條件

$$\left(\begin{array}{c|c} N_R & 0 \\ \hline 0 & I \end{array}\right)^T\left(\begin{array}{cc|c} AR+RA^T & RC_1^T & B_1 \\ C_1R & -\gamma I & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma I \end{array}\right)\left(\begin{array}{c|c} N_R & 0 \\ \hline 0 & I \end{array}\right)<0 \qquad (G.2)$$

$$\left(\begin{array}{c|c} N_S & 0 \\ \hline 0 & I \end{array}\right)^T\left(\begin{array}{cc|c} A^TS+SA & SB_1 & C_1^T \\ B_1^TS & -\gamma I & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma I \end{array}\right)\left(\begin{array}{c|c} N_S & 0 \\ \hline 0 & I \end{array}\right)<0 \qquad (G.3)$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix}\ge 0 \qquad (G.4)$$

$$\mathrm{Rank}(I-RS)\le k. \qquad (G.5)$$

其中，N_R 和 N_S 分別是 (B_2^T,D_{12}^T) 和 (C_2,D_{21}) 零空間(null space)的正交基底，I 代表單位矩陣。

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全階(full order)Hinf 控制器設計: 對於全階控制器 $(k=n)$ 的情況，限制條件(G..5)自動滿足，所以只需求解(G.2)式~(G.4)式即可。

全階控制器合成演算法如下：

Step1: Solve LMIs (G.2), (G.3), (G.4) for a pair of R and S.

Step2: Choose $N = I_n$ and $M = I_n - RS$.

Step3: Solve the following equation for X_{cl}

$$X_{cl}\begin{pmatrix} R & I \\ M^T & 0 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & N^T \end{pmatrix}.$$

Step 4: Substituting X_{cl} into (G.1). Solve the resulting LMI for the

controller parameters A_k, B_k, C_k, D_k.

$$\text{Recall (G1): } \begin{pmatrix} A_{cl}^T X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{pmatrix} < 0, \quad X_{cl} > 0.$$

$$\left(\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right) = \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array}\right].$$

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降階(Reduced-order)Hinf 控制器設計: 但是當$0 < k < n$時，因為限制條件(G.5)式在參數空間(R, S)是非凸的(non-convex)，這使得(R, S)的求解非常困難。因此，求解降階控制器的關鍵就在於如何找到一組參數(R, S)滿足(G.2)式~(G.4)式，而且符合(G.5)式的矩陣秩(matrix rank)條件。這問題可視為求解如下的最小秩問題：

$$\begin{aligned} &\underset{R,\, S>0}{\text{minimize}} \quad \text{Rank}\begin{pmatrix} R & I_n \\ I_n & S \end{pmatrix} \\ &\text{subject to} \quad \text{LMI's (G.2), (G.3), (G.4).} \end{aligned}$$

文獻[GA93,GI94,G93]將上述最佳化問題轉變成另一種形式的數學問題如下。定義目標函數

$$\Psi(R,S) := \sum_{i=1}^{n-k} \lambda_i \begin{pmatrix} R & I_n \\ I_n & S \end{pmatrix}$$

其中，$\lambda_1(\cdot) \le \cdots \le \lambda_{n-k}(\cdot)$ 代表對稱矩陣 $\begin{pmatrix} R & I_n \\ I_n & S \end{pmatrix}$ 的 $(n-k)$ 個最小特徵值。

於是，降階控制器問題就轉化為如下非凸最佳化問題：

$$\begin{array}{ll} \underset{R,\,S}{\text{minimize}} & \Psi(R, S) \\ \text{subject to} & \text{LMIs}\,(G.2), (G.3), (G.4). \end{array} \qquad (\,*\,)$$

降階控制器合成演算法如下：

步驟 1： 求解問題(*)之 R and S。

步驟 2：利用奇異值分解(singular value decomposition, SVD)求得行全秩(full-column-rank)矩陣 $M, N \in \Re^{n\times k}$，使得 $MN^T = I - RS$， i.e., assume $I - RS = U\Sigma V^T = U\begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix}V^T$, then we choose $M = U\begin{pmatrix} \Sigma_1 \\ 0_{(n-k)\times k} \end{pmatrix}$ and $N = V\begin{pmatrix} I_k \\ 0_{(n-k)\times k} \end{pmatrix}$。

步驟 3： 求解下面線性方程式的 X_{cl}

$$X_{cl}\begin{pmatrix} R & I_n \\ M^T & 0 \end{pmatrix} = \begin{pmatrix} I_n & S \\ 0 & N^T \end{pmatrix}.$$

步驟 4： 將 X_{cl} 代入(G.1)式，解 LMI，求得控制器參數 A_k, B_k, C_k, D_k。

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- **Scherer et al's approach (Change of variables)[SGC97]**

  Let

$$P\begin{cases}\dot{x}=Ax+B_1w+B_2u\\ z=C_1x+D_{11}w+D_{12}u\\ y=C_2x+D_{21}w+D_{22}u\end{cases} \qquad K\quad\begin{cases}\dot{x}_k=A_kx_k+B_ky\\ u=C_kx_k+D_ky\end{cases}$$

Assumption A1: $(A,\ B_2,\ C_2)$ is stabilizable and detectable.

Assumption A2: $D_{22}=0$.

Note: For simplicity, it is usually assumed that $D_{22}=0$. For the case that $D_{22}\neq 0$, see [ZD98, pp.293], i.e., suppose $K(s)$ is a controller for P with D_{22} set to zero. Then the controller for $D_{22}\neq 0$ is $K(s)[I+D_{22}K(s)]^{-1}$.

Let

$$P(s)\ \leftrightarrow\left[\begin{array}{c|cc}A & B_1 & B_2\\ \hline C_1 & D_{11} & D_{12}\\ C_2 & D_{21} & 0\end{array}\right]$$

where $A\in\mathbb{R}^{n\times n}$, $B_1\in\mathbb{R}^{n\times m_1}$, $B_2\in\mathbb{R}^{n\times m_2}$, $C_1\in\mathbb{R}^{p_1\times n}$, $C_2\in\mathbb{R}^{p_2\times n}$, $D_{11}\in\mathbb{R}^{p_1\times m_1}$, $D_{12}\in\mathbb{R}^{p_1\times m_2}$, $D_{21}\in\mathbb{R}^{p_2\times m_1}$ and

$$K(s)\leftrightarrow\left[\begin{array}{c|c}A_k & B_k\\ \hline C_k & D_k\end{array}\right]$$

where $A_k\in\mathbb{R}^{k\times k}$, $B_k\in\mathbb{R}^{k\times p_2}$, $C_k\in\mathbb{R}^{m_2\times k}$, $D_k\in\mathbb{R}^{m_2\times p_2}$.

The w到z的閉迴路系統T_{zw}可表為

$$T_{zw}\begin{cases}\dot{x}_{cl}=A_{cl}x_{cl}+B_{cl}w\\ z=C_{cl}x_{cl}+D_{cl}w\end{cases}$$

其中$x_{cl}=\begin{bmatrix}x\\x_k\end{bmatrix}$，而

$$T_{zw}=\left(\begin{array}{c|c}A_{cl}&B_{cl}\\\hline C_{cl}&D_{cl}\end{array}\right)=\left[\begin{array}{cc|c}A+B_2D_kC_2&B_2C_k&B_1+B_2D_kD_{21}\\B_kC_2&A_k&B_kD_{21}\\\hline C_1+D_{12}D_kC_2&D_{12}C_k&D_{11}+D_{12}D_kD_{21}\end{array}\right]$$

-<u>選擇通道</u>

訊號w到z各通道的性能要求有時不盡相同，所以可定義有興趣的、欲加以控制的閉迴路子系統T_{ij}如下(註：論文中記為T_j)

$$T_{ij}(s):=L_iT_{zw}(s)R_j$$

其中L_i與R_j是用來選取不同性能要求的輸出/輸入通道。因此，從w_j到z_i的閉迴路系統T_{ij}，其狀態空間矩陣可表示如下：

$$T_{ij}=L_iT_{zw}R_j=\left[\begin{array}{c|c}A_{cl}&B_{cl}R_j\\\hline L_iC_{cl}&L_iD_{cl}R_j\end{array}\right]=\left[\begin{array}{cc|c}A+B_2D_kC_2&B_2C_k&(B_1+B_2D_kD_{21})R_j\\B_kC_2&A_k&(B_kD_{21})R_j\\\hline L_i(C_1+D_{12}D_kC_2)&L_i(D_{12}C_k)&L_i(D_{11}+D_{12}D_kD_{21})R_j\end{array}\right]$$

$$=\left[\begin{array}{cc|c}A+B_2D_kC_2&B_2C_k&(B_1R_j)+B_2D_k(D_{21}R_j)\\B_kC_2&A_k&B_k(D_{21}R_j)\\\hline (L_iC_1)+(L_iD_{12})D_kC_2&(L_iD_{12})C_k&(L_iD_{11}R_j)+(L_iD_{12})D_k(D_{21}R_j)\end{array}\right]$$

與T_{zw}之數學式比較

$$T_{zw}=\left(\begin{array}{c|c}A_{cl}&B_{cl}\\\hline C_{cl}&D_{cl}\end{array}\right)=\left[\begin{array}{cc|c}A+B_2D_kC_2&B_2C_k&B_1+B_2D_kD_{21}\\B_kC_2&A_k&B_kD_{21}\\\hline C_1+D_{12}D_kC_2&D_{12}C_k&D_{11}+D_{12}D_kD_{21}\end{array}\right]$$

可知受控體作了更改：

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \text{(受控體全部通道)}$$

更改為

$$\left[\begin{array}{c|cc} A & B_1 R_j & B_2 \\ \hline L_i C_1 & L_i D_{11} R_j & L_i D_{12} \\ C_2 & D_{21} R_j & D_{22} \end{array}\right] \text{(受控體第}(i,j)\text{通道)，如同} \left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & L_i & 0 \\ 0 & 0 & I \end{array}\right]\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]\left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & R_j & 0 \\ 0 & 0 & I \end{array}\right]$$

故 $T_{ij} = L_i T_{zw} R_j$ 相當於受控體第 (i,j) 通道與控制器之閉回路矩陣。

結論：

（i） 多通道控制問題:

$$\underline{\text{受控體第}}(i_a, j_a)\underline{\text{通道}} \leftrightarrow \left[\begin{array}{c|cc} A & B_1 R_{j_a} & B_2 \\ \hline L_{i_a} C_1 & L_{i_a} D_{11} R_{j_a} & L_{i_a} D_{12} \\ C_2 & D_{21} R_{j_a} & D_{22} \end{array}\right] := \left[\begin{array}{c|cc} A_a & B_{1a} & B_{2a} \\ \hline C_{1a} & D_{12a} & D_{12a} \\ C_{2a} & D_{21a} & D_{22a} \end{array}\right]$$

$$\underline{\text{受控體第}}(i_b, j_b)\underline{\text{通道}} \leftrightarrow \left[\begin{array}{c|cc} A & B_1 R_{j_b} & B_2 \\ \hline L_{i_b} C_1 & L_{i_b} D_{11} R_{j_b} & L_{i_b} D_{12} \\ C_2 & D_{21} R_{j_b} & D_{22} \end{array}\right] := \left[\begin{array}{c|cc} A_b & B_{1b} & B_{2b} \\ \hline C_{1b} & D_{12b} & D_{12b} \\ C_{2b} & D_{21b} & D_{22b} \end{array}\right]$$

（ii） Scherer et al 1997 論文中

$$= \left[\begin{array}{cc|c} A + B_2 \boldsymbol{D_k} C_2 & B_2 \boldsymbol{C_k} & B_{\alpha j} + B_2 \boldsymbol{D_k} F_{\alpha j} \\ \boldsymbol{B_k} C_2 & \boldsymbol{A_k} & \boldsymbol{B_k} F_{\alpha j} \\ \hline C_{\alpha i} + E_{\alpha i} \boldsymbol{D_k} C_2 & E_{\alpha i} \boldsymbol{C_k} & D_{\alpha ij} + E_{\alpha i} \boldsymbol{D_k} F_{\alpha j} \end{array}\right]$$

其中 $B_{\alpha j} \equiv B_1 R_j$(論文記為 B_j)，$C_{\alpha i} \equiv L_i C_1$(論文中記為 C_j)，

$D_{\alpha ij} \equiv L_i D_{11} R_j$(論文中記為 D_j)，$E_{\alpha i} \equiv L_i D_{12}$(論文中記為 E_j)，

$F_{\alpha j} \equiv D_{21} R_j$(論文中記為 F_j)。

See **Appendix 1** for an example.

-<u>找出系統性能分析式</u>

Bounded Real Lemma (H-infinity Performance) [SGC97, pp.898]:

Given a LTI system (A, B, C, D) and a positive value γ. The system is stable (i.e., matrix A is stable) and its H_∞ norm is smaller than γ if and only if there exists a $n \times n$ symmetric matrix P_∞ such that the following linear matrix inequalities hold.

$$P_\infty > 0 \ \text{ and } \begin{bmatrix} A^T P_\infty + P_\infty A & P_\infty B & C^T \\ B^T P_\infty & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0 \qquad \text{(S.1)}$$

Derive synthesis LMIs for T_{ij}, i.e., the (i,j)th channel of T_{zw}:

1. Replace (A, B, C, D) in the analysis LMI (S.1) with the realization of the closed-loop map T_{ij}, i.e., $(A_{cl}, B_{clj}, C_{clj}, D_{clj})$ (Note: (i) They are affine in the controller parameters (A_k, B_k, C_k, D_k). (ii) The resulting inequalities are bilinear matrix inequalities (BMIs)!).

2. Partition the auxiliary variable P_∞ into four blocks

$$P_\infty = \begin{pmatrix} Y & N \\ N^T & * \end{pmatrix}, \qquad P_\infty^{-1} = \begin{pmatrix} X & M \\ M^T & * \end{pmatrix} \tag{S.2}$$

Define matrices

$$P_\infty T_1 = T_2 \quad \textit{with} \quad T_1 := \begin{pmatrix} X & I \\ M^T & 0 \end{pmatrix}, \qquad T_2 := \begin{pmatrix} I & Y \\ 0 & N^T \end{pmatrix} \tag{S.3}$$

Q: How to choose T_1, T_2 based on P_∞ and P_∞^{-1} ?

Idea: It comes from the following observations!

a. Introduce more variables into the analysis LMIs such that

$$T_1^T P_\infty T_1 > 0$$

$$\begin{pmatrix} T_1^T A_{cl}^T P_\infty T_1 + T_1^T P_\infty A_{cl} T_1 & T_1^T P_\infty B_{clj} & T_1^T C_{clj}^T \\ B_{clj}^T P T_1 & -\gamma I & D_{clj}^T \\ C_{clj} T_1 & D_{clj} & -\gamma I \end{pmatrix} < 0$$

b. 定義變數 T_1, T_2

First, let $P_\infty = \begin{pmatrix} Y & N \\ N^T & * \end{pmatrix}, \qquad P_\infty^{-1} = \begin{pmatrix} X & M \\ M^T & * \end{pmatrix}$

Need $P_\infty T_1$ (borrowed from the identity

$$P_\infty P_\infty^{-1} = \begin{pmatrix} Y & N \\ N^T & * \end{pmatrix} \begin{pmatrix} X & M \\ M^T & * \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}).$$

Define

$$T_1 := \begin{pmatrix} X & I \\ M^T & 0 \end{pmatrix}$$

Then we get $P_\infty T_1 = \begin{pmatrix} Y & N \\ N^T & * \end{pmatrix} \begin{pmatrix} X & I \\ M^T & 0 \end{pmatrix} = \begin{pmatrix} I & Y \\ 0 & N^T \end{pmatrix} =: T_2$

3. Perform congruence transformation to (S.1) :

(i) applying congruence transformation T_1 to the first LMI of (S.1) we obtain

$$T_1^T P_\infty T_1 > 0 \qquad (*1)$$

(ii) applying congruence transformation diag(T_1,I,I) to the second LMI of (S.1) we obtain

$$\begin{pmatrix} T_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}^T \begin{bmatrix} A_{cl}^T P_\infty + P_\infty A_{cl} & P_\infty B_{clj} & C_{clj}^T \\ B_{clj}^T P_\infty & -\gamma I & D_{clj}^T \\ C_{clj} & D_{clj} & -\gamma I \end{bmatrix} \begin{pmatrix} T_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} < 0$$

$$i.e., \begin{pmatrix} T_1^T A_{cl}^T P_\infty T_1 + T_1^T P_\infty A_{cl} T_1 & T_1^T P_\infty B_{clj} & T_1^T C_{clj}^T \\ B_{clj}^T P T_1 & -\gamma I & D_{clj}^T \\ C_{clj} T_1 & D_{clj} & -\gamma I \end{pmatrix} < 0 \cdots\cdots\cdots\cdots\cdots (*2)$$

4. Change of variables: introduce the new variables $(X,Y,M,N,\hat{A},\hat{B},\hat{C},\hat{D})$ via the original variables ($P_\infty, A_k, B_k, C_k, D_k$): Define

$$\begin{cases} \hat{\boldsymbol{A}} := NA_k M^T + NB_k C_2 \boldsymbol{X} + \boldsymbol{Y} B_2 C_k M^T \\ \qquad + \boldsymbol{Y}(A + B_2 D_k C_2)\boldsymbol{X} \\ \hat{\boldsymbol{B}} := NB_k + YB_2 D_k \\ \hat{\boldsymbol{C}} := C_k M^T + D_k C_2 \boldsymbol{X} \\ \hat{\boldsymbol{D}} := D_k \end{cases}$$

(see [4] eqs.(33),(34), and (35))

In (*2)

At (1,1) : $T_1^T P_\infty A_{cl} T_1 = T_1^T P_\infty^{\ T} A_{cl} T_1 = (P_\infty T_1)^T A_{cl} T_1 = T_2^T A_{cl} T_1$

$$= \begin{pmatrix} I & Y \\ 0 & N^T \end{pmatrix}^T \begin{pmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{pmatrix} \begin{pmatrix} X & I \\ M^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ Y & N \end{pmatrix} \begin{pmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{pmatrix} \begin{pmatrix} X & I \\ M^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A + B_2 D_k C_2 & B_2 C_k \\ Y\left(A + B_2 D_k C_2\right) + N\left(B_k C_2\right) & Y\left(B_2 C_k\right) + N\left(A_k\right) \end{pmatrix} \begin{pmatrix} X & I \\ M^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} AX + B_2 D_k C_2 X + B_2 C_k M^T & A + B_2 D_k C_2 \\ YAX + YB_2 D_k C_2 X + NB_k C_2 X + YB_2 C_k M^T + NA_k M^T & YA + YB_2 D_k C_2 + NB_k C_2 \end{pmatrix}$$

$$= \begin{pmatrix} AX + B_2\left(D_k C_2 X + C_k M^T\right) & A + B_2\left(D_k\right) C_2 \\ Y\left(A + B_2 D_k C_2\right) X + NB_k C_2 X + YB_2 C_k M^T + NA_k M^T & YA + \left(YB_2 D_k + NB_k\right) C_2 \end{pmatrix}$$

$$= \begin{pmatrix} AX + B_2 \widehat{C} & A + B_2 \widehat{D} C_2 \\ \widehat{A} & YA + \widehat{B} C_2 \end{pmatrix}$$

$(1,2) = T_1^T P_\infty B_{clj} = T_2^T B_{clj}$

$$= \begin{pmatrix} I & Y \\ 0 & N^T \end{pmatrix}^T \begin{pmatrix} B_1 + B_2 D_k D_{21} \\ B_k D_{21} \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ Y & N^T \end{pmatrix} \begin{pmatrix} B_1 + B_2 D_k D_{21} \\ B_k D_{21} \end{pmatrix}$$

$$= \begin{pmatrix} B_1 + B_2\left(D_k\right) D_{21} \\ YB_1 + \left(YB_2 D_k + N^T B_k\right) D_{21} \end{pmatrix}$$

$$= \begin{pmatrix} B_j + B_2 \widehat{D} F_j \\ YB_j + \widehat{B} F_j \end{pmatrix}$$

$(1,3) = C_{clj} T_1$

$$= \left(C_1 + D_{12} D_k C_2 \quad D_{12} C_k\right) \begin{pmatrix} X & I \\ M^T & 0 \end{pmatrix}$$

$$= \left(C_1 X + D_{12} D_k C_2 X + D_{12} C_k M^T \quad C_1 + D_{12} D_k C_2\right)$$

$$= \left(C_j X + E_j\left(C_k M^T + D_k C_2 X\right) \quad C_j + E_j D_k C_2\right)$$

$$= \left(C_j X + E_j \widehat{C} \quad C_j + E_j \widehat{D} C_2\right)$$

In (*1):

$$At\ (1,1):\ T_1^T P_\infty T_1 = T_1^T T_2$$

$$= \begin{pmatrix} X & XY + M^T N^T \\ I & IY \end{pmatrix}$$

$$= \begin{pmatrix} X & I \\ I & Y \end{pmatrix}$$

(*1) and (*2) become (S.4) and (S.5), respectively.

****

Theorem [4, pp.904]: The closed-loop system T is stable and the channel map T_{ij} satisfies $\|T_{ij}\|_\infty < \gamma$ if and only if there exist matrices $X = X^T \in \mathbb{R}^{n\times n}$, $Y = Y^T \in \mathbb{R}^{n\times n}$, $\hat{A} \in \mathbb{R}^{n\times n}$, $\hat{B} \in \mathbb{R}^{n\times p_2}$, $\hat{C} \in \mathbb{R}^{m_2\times n}$, $\hat{D} \in \mathbb{R}^{m_2\times p_2}$ satisfying the following linear matrix inequalities

$$\begin{bmatrix} \mathbf{X} & I \\ I & \mathbf{Y} \end{bmatrix} > 0 \tag{S.4}$$

$$\begin{bmatrix} A\mathbf{X} + \mathbf{X}A^T + B_2\hat{C} + (B_2\hat{C})^T & * & * & * \\ \hat{\mathbf{A}} + (A + B_2\hat{\mathbf{D}}C_2)^T & A^T\mathbf{Y} + \mathbf{Y}A + \hat{\mathbf{B}}C_2 + (\hat{\mathbf{B}}C_2)^T & * & * \\ (B_j + B_2\hat{\mathbf{D}}F_j)^T & (\mathbf{Y}B_j + \hat{\mathbf{B}}F_j)^T & -\gamma I & * \\ C_j\mathbf{X} + E_j\hat{\mathbf{C}} & C_j + E_j\hat{\mathbf{D}}C_2 & D_j + E_j\hat{\mathbf{D}}F_j & -\gamma I \end{bmatrix} < 0 \tag{S.5}$$

****

Optimal H∞ control problem：

$$\min_{\text{all stabilizing controllers } K} \left\| T_{ij} \right\|_\infty \Leftrightarrow \min_{X,Y,\hat{A},\hat{B},\hat{C},\hat{D},\gamma} \gamma$$

subject to (S.4), (S.5)

Algorithm for obtaining an optimal **H∞** controller:

1. Get the generalized plant $P(s) \leftrightarrow \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$. Assume $D_{22}=0$.

2. Determine the system matrices (B_j, C_j, D_j, E_j, F_j) for the selected

   channel from w_j to z_j, i.e., 選通道後之受控體資料如下

$$\left[\begin{array}{c|cc} \mathrm{A} & \mathrm{B}_1 R_j & \mathrm{B}_2 \\ \hline L_i \mathrm{C}_1 & L_i \mathrm{D}_{11} R_j & L_i \mathrm{D}_{12} \\ \mathrm{C}_2 & \mathrm{D}_{21} R_j & \mathrm{D}_{22} \end{array}\right] =: \left[\begin{array}{c|cc} \mathrm{A} & B_{\alpha j} & \mathrm{B}_2 \\ \hline C_{\alpha i} & D_{\alpha i j} & E_{\alpha i} \\ \mathrm{C}_2 & F_{\alpha j} & \mathrm{D}_{22} \end{array}\right]$$

3. Solve the linear objective minimization problem over the LMI

   constraints (S.4) and (S.5) for X 、Y 、$\hat{A}$ 、$\hat{B}$ 、$\hat{C}$ 、$\hat{D}$, and γ, where

   $X = X^T \in \mathbb{R}^{n\times n}, Y = Y^T \in \mathbb{R}^{n\times n}, \hat{A} \in \mathbb{R}^{n\times n}, \hat{B} \in \mathbb{R}^{n\times p_2}, \hat{C} \in \mathbb{R}^{m_2\times n}, \hat{D} \in \mathbb{R}^{m_2\times p_2}$

   (use matlab command mincx)

4. Find $n\times n$ matrices M and N satisfying

   $MN^T = I - XY$ (e.g., choose $N = I$, then $M = I - XY$)

5. The system parameters (A_k, B_k, C_k, D_k) of the controller K can be

   calculated via the formulas listed below.

$$\boldsymbol{D_K} = \hat{D}$$

$$\boldsymbol{C_K} = \left(\hat{C} - \hat{D}C_2X\right)M^{-T}$$

$$\boldsymbol{B_K} = N^{-1}\left(\hat{B} - YB_2D_K\right)$$

$$\boldsymbol{A_K} = N^{-1}\left[\hat{A} - NB_KC_2X - YB_2C_KM^T - Y\left(A + B_2D_KC_2\right)X\right]M^{-T}$$

III. ARE Approach

以下之H_∞控制器設計方法乃摘錄自[ZD98], Section 14.6 General H_∞ Solutions. For a simplified version, see Section 14.2 [ZD98] or DGKF's paper [DGKF89].

考慮$P-K$架構，假設廣義受控體P為連續時間，線性非時變系統，則$P(s)$為

$$P(s)=\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}+\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(sI-A)^{-1}\begin{pmatrix} B_1 & B_2 \end{pmatrix} \leftrightarrow \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$

其中，$A\in\mathbb{R}^{n\times n}$，$D_{11}\in\mathbb{R}^{p_1\times m_1}$及$D_{22}\in\mathbb{R}^{p_2\times m_2}$，並且符合如下假設：

(A1) (A, B_2)為可穩定(stabilizable)且(C_2, A)為可檢測(detectable)。

(A2) $D_{12}=\begin{bmatrix} 0 \\ I \end{bmatrix}$ 且$D_{21}=\begin{bmatrix} 0 & I \end{bmatrix}$；

(A3) G_{12}在虛軸上沒有不變零點。即，對於所有實數ω，下列矩陣為行全秩(full column rank)

$$\begin{bmatrix} A-j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}\quad ;$$

(A4) G_{21}在虛軸上沒有不變零點。即，對於所有實數ω，下列矩陣為列全秩(full row rank)

$$\begin{bmatrix} A-j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}\quad 。$$

定義下面矩陣

$$R := D_{1\bullet}^{*} D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{其中}, \; D_{1\bullet} := \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$$

$$\tilde{R} := D_{\bullet 1} D_{\bullet 1}^{*} - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{其中}, \; D_{\bullet 1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$$

定義以下的漢密爾頓(Hamiltonian)矩陣H_∞和J_∞

$$H_\infty := \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^* C_1 & B^* \end{bmatrix}$$

$$J_\infty := \begin{bmatrix} A^* & 0 \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} C^* \\ -B_1 D_{\bullet 1}^* \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1} B_1^* & C \end{bmatrix}$$

令$X_\infty := \mathrm{Ric}(H_\infty)$，$Y_\infty := \mathrm{Ric}(J_\infty)$。亦即$X_\infty$和$Y_\infty$分別代表以下兩 Riccati 方程式之解。

$$X_\infty A + A^* X_\infty + X_\infty \left(B_1 B_1^* / \gamma^2 - B_2 B_2^* \right) X_\infty + C_1^* C_1 = 0. \qquad \text{(Z.1)}$$

$$A Y_\infty + Y_\infty A^* + Y_\infty \left(C_1^* C_1 / \gamma^2 - C_2^* C_2 \right) Y_\infty + B_1 B_1^* = 0 . \qquad \text{(Z.2)}$$

然後定義

$$F := \begin{bmatrix} F_{11\infty} \\ F_{12\infty} \\ F_{2\infty} \end{bmatrix} := -R^{-1} \left[D_{1\bullet}^* C_1 + B^* X_\infty \right]$$

$$L := \begin{bmatrix} L_{11\infty} & L_{12\infty} & L_{2\infty} \end{bmatrix} := -\left[B_1 D_{\bullet 1}^* + Y_\infty C^* \right] \tilde{R}^{-1}$$

將D, F及L排列如下

$$\left[\begin{array}{c|c} & F' \\ \hline L' & D \end{array}\right] = \left[\begin{array}{c|ccc} & F_{11\infty}^{*} & F_{12\infty}^{*} & F_{2\infty}^{*} \\ \hline L_{11\infty}^{*} & D_{1111} & D_{1112} & 0 \\ L_{12\infty}^{*} & D_{1121} & D_{1122} & I \\ L_{2\infty}^{*} & 0 & I & 0 \end{array}\right]$$

根據以上的符號定義來敘述定理 2.5，H_∞ 控制器的存在條件及設計公式。

定理 ([ZD98], pp. 289-290)： 假設 G 滿足**假設**(A1)(A2)(A3)(A4)。

(a) 存在一控制器 K 使閉迴路系統內部穩定且滿足 $\|T_{zw}\|_\infty < \gamma$，若且唯若，

(i) $\gamma > \max\left(\overline{\sigma}\left[D_{1111} \quad D_{1112}\right], \overline{\sigma}\left[D_{1111}^{*} \quad D_{1121}^{*}\right]\right)$；

(ii) 方程式(Z.1)有解 X_∞ 而且 $X_\infty \geq 0$；

(iii) 方程式(Z.2)有解 Y_∞ 而且 $Y_\infty \geq 0$；

(iv) $\rho(X_\infty Y_\infty) < \gamma^2$。

(b) 當(a)中條件均成立時，所有滿足問題要求的 H_∞ 控制器可表為

$$K := F_l(M_\infty, Q) = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21} \text{（如圖）},$$

其中 $Q \in RH_\infty$ 且 $\|Q\|_\infty < \gamma$；

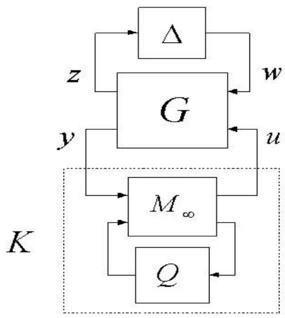

圖　H_∞ 控制器

而

$$M_\infty := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right]$$

其中 $\hat{D}_{11} := -D_{1121}D_{1111}^*\left(\gamma^2 I - D_{1111}D_{1111}^*\right)^{-1} D_{1112} - D_{1122}$

又，$\hat{D}_{12} \in R^{m_2 \times m_2}$ 與 $\hat{D}_{21} \in R^{p_2 \times p_2}$ 為滿足以下的任意矩陣，

$$\hat{D}_{12}\hat{D}_{12}^* := I - D_{1121}\left(\gamma^2 I - D_{1111}^* D_{1111}\right)^{-1} D_{1121}^*$$

$$\hat{D}_{21}^*\hat{D}_{21} := I - D_{1112}^*\left(\gamma^2 I - D_{1111}D_{1111}^*\right)^{-1} D_{1112}$$

又，

$$\begin{aligned} \hat{B}_2 &= Z_\infty (B_2 + L_{12\infty})\hat{D}_{12} \\ \hat{C}_2 &= -\hat{D}_{21}(C_2 + F_{12\infty}) \\ \hat{B}_1 &= -Z_\infty L_{2\infty} + \hat{B}_2 \hat{D}_{12}^{-1}\hat{D}_{11} \\ \hat{C}_1 &= F_{2\infty} + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 \\ \hat{A} &= A + BF + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 \end{aligned}$$

其中

$$Z_\infty = \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1}$$ 。

****

由上述定理可知，若對於所給予之正數γ，(a)中(i)到(iv)之條件皆成立，那麼就可利用條件(ii)和(iii)中所得的 Riccati 方程式的解，X_∞及Y_∞，根據(b)中的公式來找出所有滿足H_∞控制問題要求，亦即內部穩定及 $\|T_{zw}\|_\infty < \gamma$ 的控制器。

Note: The above results can be extended to more general cases. See [ZD98] Section 14.7.

Matlab work:

Different ways to solve H_∞ Control Problem

(1) directly use matlab commands: hinflmi, hinfric

(2) write LMI programs (LMIs by Scherer et al or Gahinet et al) and use LMI solver(feasp, mincx)

Appendix 1

Q: How to 選通道(channel)?

Ans: 假設輸入 w 的維度為 3x1，輸出 z 的維度為 3x1

轉移函數 T_{zw} 的維度為 3x3

$$T_{zw}=\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix},\ T_{zw} \Leftrightarrow \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] = \left[\begin{array}{cc|c} A+B_2D_KC_2 & B_2C_k & B_1+B_2D_kD_{21} \\ B_KC_2 & A_{2k} & B_kD_{21} \\ \hline C_1+D_{12}D_kC_2 & D_{12}C_k & D_{11}+D_{12}D_kD_{21} \end{array}\right]$$

$$T_{ij}=L_i \mathrm{T} R_j$$

$$T_{11}=[1\ \ 0\ \ 0]\mathrm{T}_{zw}\begin{bmatrix}1\\0\\0\end{bmatrix}=[1\ \ 0\ \ 0]\left[C_{cl}(sI-A_{cl})^{-1}B_{cl}+D_{cl}\right]\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$=[1\ \ 0\ \ 0]C_{cl}(sI-A_{cl})^{-1}B_{cl}\begin{bmatrix}1\\0\\0\end{bmatrix}+[1\ \ 0\ \ 0]D_{cl}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$=LC_{cl}(sI-A_{cl})^{-1}B_{cl}R+LD_{cl}R$$

$$\boxed{T_{11}=\left[\begin{array}{c|c} A_{cl} & B_{cl}\begin{bmatrix}1\\0\\0\end{bmatrix} \\ \hline [1\ \ 0\ \ 0]C_{cl} & [1\ \ 0\ \ 0]D_{cl}\begin{bmatrix}1\\0\\0\end{bmatrix} \end{array}\right]=\left[\begin{array}{c|c} A_{cl} & B_{cl}R \\ \hline LC_{cl} & LD_{cl}R \end{array}\right]}$$

$$B_{cl}R = \begin{bmatrix} B_1 + B_2 D_k D_{21} \\ B_k D_{21} \end{bmatrix} R = \begin{bmatrix} B_1 R + B_2 D_k D_{21} R \\ B_k D_{21} R \end{bmatrix}$$

$$T_{ij} = \left[\begin{array}{cc|c} A + B_2 D_K C_2 & B_2 C_k & B_1 R_j + B_2 D_k D_{21} R_j \\ B_K C_2 & A_{2k} & B_k D_{21} R_j \\ \hline L_i C_1 + L_i D_{12} D_k C_2 & L_i D_{12} C_k & L_i D_{11} R_j + L_i D_{12} D_k D_{21} R_j \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A + B_2 D_K C_2 & B_2 C_k & B_{aj} + B_2 D_k F_{aj} \\ B_K C_2 & A_{2k} & B_k F_{aj} \\ \hline C_{ai} + E_{ai} D_k C_2 & E_{ai} C_k & D_{aij} + E_{ai} D_k F_{aj} \end{array}\right]$$

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