Visualizing Second-Order Tensor Fields With Bi-Cubic Interpolation [IN PROGRESS]

David Lovitz

September 12, 2016

1 Introduction

In this paper we focus on second-order symmetric stress tensors. In order to study and better understand the properties of a discrete field of tensors we visualize it by converting it into a vector field. Once converted to a vector field we can study the topology of the vector field and attempt to learn more about the original tensor field.

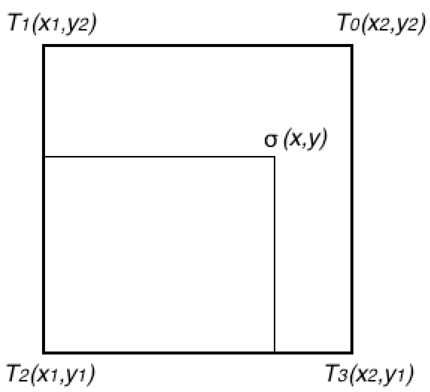
2 Converting a Tensor field into a vector field

2.1 Bi-linear Interpolation

In order to extend tensor data from a discrete grid to a continuous surface we utilize a bi-linear interpolation of the tensors components. bi-linear interpolation utilizes tensor data from 4 neighboring tensors. We define the tensor at a point (x, y) to be:

$$\sigma = T_2(1-x)(1-y) + T_3x(1-y) + T_1(1-x)y + T_0xy \tag{1}$$

The tensor T_0 is located at (x_2, y_2), T_1 is located at (x_1, y_2), T_2 is located at (x_1, y_1), T_3 is located at (x_2, y_1), as shown in figure 1.



2.2 Bi-cubic Interpolation

Bi-cubic interpolation uses tensor information from 16 points to generate a continuous tensor map. We define the value of one of the tensor components at a point (x, y) to be:

$$p(x,y) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} f(0,0) & f(0,1) & f_y(0,0) & f_y(0,1) \\ f(1,0) & f(1,1) & f_y(1,0) & f_y(1,1) \\ f_x(0,0) & f_x(0,1) & f_{xy}(0,0) & f_{xy}(0,1) \\ f_x(1,0) & f_x(1,1) & f_{xy}(1,0) & f_{xy}(1,1) \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{3}$$

Generating the bicubic interpolation is more computationally intensive than bi-linear interpolation however it may be able to provide additional information that can improve the accuracy of separatrices

2.3 Generating Vector field

In order to visualize the tensor field and to convert it into a form that is easier to study we compute the eigenvectors at discrete points within each grid section based on tensors that are calculated from the bi-linear interpolation. The result is a vector field which can be studied using techniques from vector field topology. Since we are assuming all of the tensors are symmetric and 2x2 the eigenvectors are given by:

$$\left\{ \pm \frac{\sqrt{a^2 - 2ac + 4b^2 + c^2} + a - c}{2b}, 1 \right\} \tag{4}$$

When the original tensor is in the form:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{5}$$

At each point within a unit square equation 4 is evaluated using equation 1 to calculate a, b, c. The result is a vector field that contains compressed information about the original tensor field.

3 Vector Field Analysis

3.1 Locating Singularities

A singularity is defined as: *A point x_0 is a singularity if and only if the two eigenvalues of a tensor field T at x_0 are equal.* Topologically singularities are the points at which streamlines cross. To locate singularities we take advantage of the following properties which hold when evaluated at x_0: [1]

$$\sigma_{11} - \sigma_{22} = 0 \tag{6}$$

$$\sigma_{12} = 0 \tag{7}$$

The location of singularities within the vector field is found by solving this system of equations using equation 1 or 2

3.2 Separatrices

We make use of the definition of separatrices from Delmarcelle and Hesselink who define them as the streamlines that separate one sector of the vector field form another. In order to draw separatrices we define the following:

$$\begin{aligned}
\alpha &= \frac{1}{2}\frac{\partial(\sigma_{11} - \sigma_{22})}{\partial x} \\
\beta &= \frac{1}{2}\frac{\partial(\sigma_{11} - \sigma_{22})}{\partial y} \\
\gamma &= \frac{\partial \sigma_{12}}{\partial x} \\
\delta &= \frac{\partial \sigma_{12}}{\partial y}
\end{aligned} \tag{8}$$

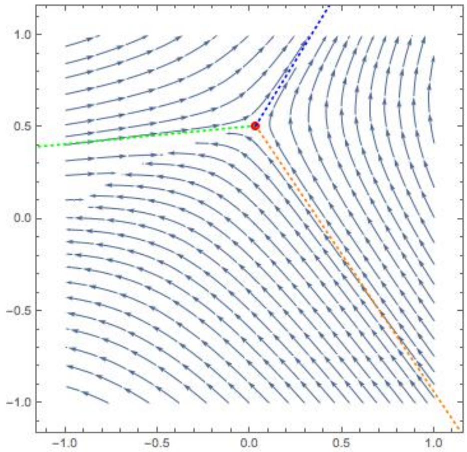
If we let θ_k be the angle between a separatrix s_k and the x-axis, It is shown in [2] that $\tan\theta_k = x_k$ where x_k is a root of the equation:

$$\delta x^3 + (\gamma + 2\beta)x^2 + (2\alpha - \delta)x - \gamma = 0 \tag{9}$$

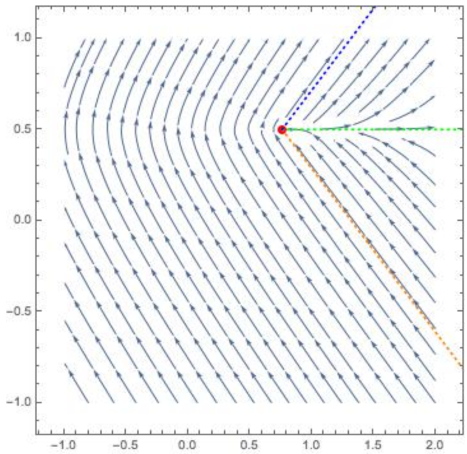
Where $\alpha, \beta, \gamma, \delta$ are evaluated at a singularity point. It follows that since equation 8 is third order that there are a maximum of three separatrices at each singularity point. Taking the arctan of each of the roots returns values between $-\frac{\Pi}{2}$ and $\frac{\Pi}{2}$, these angles will only be correct for one of the eigenvector fields, depending on whether $+$ or $-$ is chosen in equation 4. To check this we set a small distance ϵ to traverse out from the singularity along the angle returned by arctan. We then check the angle of the eigenvector field at this point and compare it to the angle returned by arctan if they are different we add π to the angle.

3.3 Bi-linear Interpolated Vector Field Examples

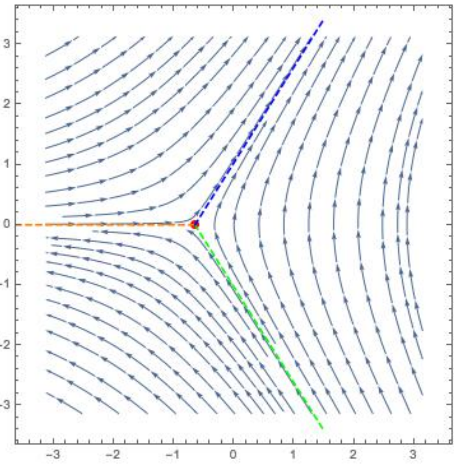
Here are some example vector fields along with the numerical input used to create them. Each singularity point is shown with a red dot and separatrices are drawn in colored dashed lines.




```
T0 = {{(28.68712), (-10.2041)}, {(-10.2041), (-0.18743)}};
T1 = {{(1.062496), (-10.2041)}, {(-10.2041), (2.019996)}};
T2 = {{(1.062496), (10.2041)}, {(10.20412), (2.019996)}};
T3 = {{(28.68712), (10.20412)}, {(20.20412), (-0.18743)}};
```

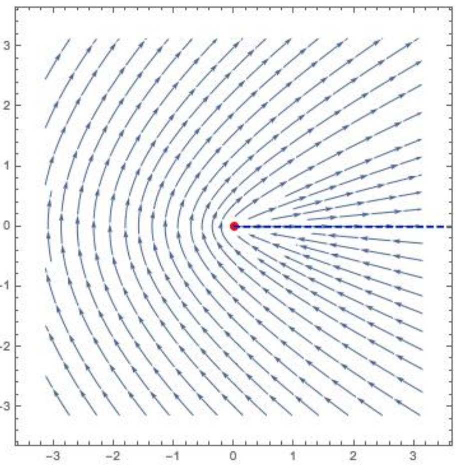



```
T0 = {{(1.062496), (-10.2041)}, {(-10.2041), (2.019996)}};
T1 = {{(-1.062496), (-10.2041)}, {(-10.2041), (-4.10248)}};
T2 = {{(-1.062496), (10.2041)}, {(10.2041), (-4.10248)}};
T3 = {{(1.062496), (10.2041)}, {(10.2041), (2.019996)}};
```



Input is the following equations for the tensor components:

$$
\begin{aligned}
a &= \cos x \\
b &= -y \\
c &= -2x
\end{aligned}
\tag{10}
$$



Input is the following equations for the tensor components:

$$
\begin{aligned}
a &= -3x^2 \\
b &= -y \\
c &= -2x^2
\end{aligned}
\tag{11}
$$

Bi-linear interpolation is accurate when used with symbolic equations for the tensor components. When used with numerical input the accuracy decreases when larger domains are used and the separatrices begin to cross the vector field.

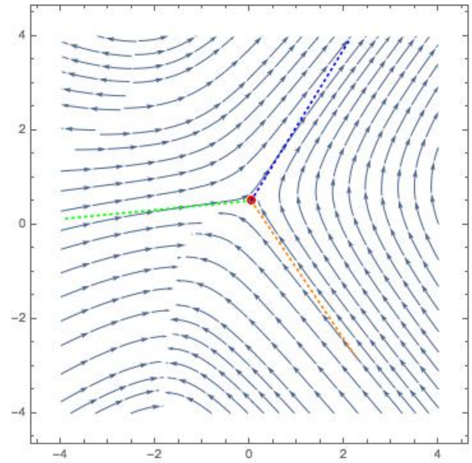


Figure 1: A Bector field generated from numerical inputs with the domain expanded by a factor of 4.

3.4 Bi-cubic Interpolated Vector Field Examples

Separatrices are found by approximating the vector field around each singularity by a Taylor series. This first example only utilizes the first term of the Taylor series which leads to large inaccuracies as the separatrix extends away from the singularity.

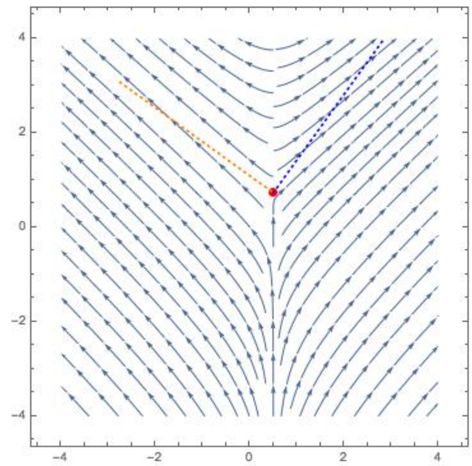


Figure 2: Separatrices drawn utilizing only the first term of a Taylor expansion, based on numerical input of 16 tensors.

3.5 Improving Separatrix Accuracy

As shown by Delmarcelle in [3] the angle (θ) between a separatrix and the x-axis is given by the following:

$$\tan 2\theta = \frac{2T_{12}}{T_{11} - T_{22}} \tag{12}$$

Which is evaluated at each singularity within the domain. In order to calculate this we approximate the tensor components using a Taylor expansion:

$$\begin{aligned} \frac{T_{11} - T_{22}}{2} &\approx \alpha(x - x_0) + \beta(y - y_0) + ... \\ T_{12} &\approx \gamma(x - x_0) + \delta(y - y_0) + ... \end{aligned} \tag{13}$$

Where $\alpha, \beta, \gamma, \delta$ are defined as above in equation 8. The ... represents additional terms from the Taylor series. In order to improve the accuracy of the angle we calculate additional terms of the Taylor expansion by utilizing a bi-cubic interpolation instead of the bi-linear interpolation used by Delmarcelle. The Taylor expansions are defined by the sums [3]:

$$\begin{aligned} \frac{T_{11} - T_{22}}{2} &= \sum_{k \geq 0}^{\infty} P_{m_p+k}(x - x_0, y - y_0) \\ T_{12} &= \sum_{k \geq 0}^{\infty} Q_{m_q+k}(x - x_0, y - y_0) \end{aligned} \tag{14}$$

$P_m(x, y)$ and $Q_m(x, y)$ are homogeneous polynomials of degree m. $P_m(x, y)$ and $Q_m(x, y)$ are defined by:

$$\begin{aligned} P_m(x, y) &= \sum_{i=0}^{m} p_i^{(m)} x^{m-i} y^i \\ P_m(x, y) &= \sum_{i=0}^{m} q_i^{(m)} x^{m-i} y^i \end{aligned} \tag{15}$$

The coefficients $p_i^{(m)}$ and $q_i^{(m)}$ are calculated from the bi-cubic interpolation and are given by the following definitions evaluated at the singularity:

$$\begin{aligned} p_i^{(m)} &= \frac{1}{m!} \binom{m}{i} \frac{1}{2} \frac{\partial (T_{11} - T_{22})^m}{\partial x^{m-i} \partial y^i} \\ q_i^{(m)} &= \frac{1}{m!} \binom{m}{i} \frac{1}{2} \frac{\partial T_{12}^m}{\partial x^{m-i} \partial y^i} \end{aligned} \tag{16}$$

References

[1] Thierry Delmarcelle, Lambertus Hesselink *The Topology of Symmetric, Second-Order Tensor Fields.* Proceedings IEEE Visualization pp. 140-148, 1994.

[2] Thierry Delmarcelle *The Visualization of Second-Order Tensor Feilds.* PhD. thesis, Stanford University, 1994.

[3] Thierry Delmarcelle *The Visualization of second-order tensor fields.* NASA Technical Documents, 1995.