Unit 2.1: Introduction to Circuit Simulation

Simple R-L Circuit

It is often necessary to solve a differential equation in order to determine a voltage or current in an electric circuit. For example, consider the circuit shown in Fig. 2.1.

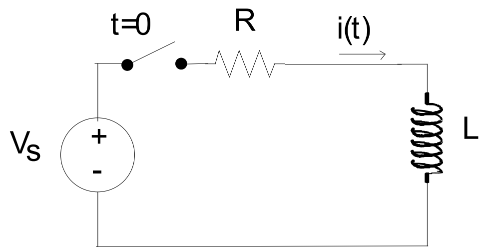

Fig. 2.1 R-L circuit

When the switch closes, the relationship between the current, source voltage and circuit parameters is defined by the differential equation

$$L\frac{di}{dt}+Ri=V_m\cos\left(\omega t+\phi\right) \tag{2.1}$$

For an initial value of current equal to zero, the solution to this differential equation can be shown to be

$$i=\frac{V_m}{\sqrt{R^2+\left(\omega L\right)^2}}\left[\cos\left(\omega t+\phi-\theta\right)-\cos\left(\phi-\theta\right)e^{-\left(\frac{R}{L}\right)t}\right] \tag{2.2}$$

where

$$\theta=\tan^{-1}\left(\frac{\omega L}{R}\right) \tag{2.3}$$

Solutions for simple circuits described by first or second-order differential equations are not difficult to determine by hand. But what happens if we have to analyze a large circuit? In this situation we are forced to use a computer program which uses a numerical solution strategy for solving large sets of differential equations. What we will be looking at in this section is how to set up a computer program to accomplish this task. Hopefully this will give you some insight into how circuit simulation programs such as PSpice operate.

A very simple numerical technique for solving differential equations is based on the Euler integration rule.
Suppose that we wanted to solve a differential equation of the form

$$\frac{dx}{dt}=f\left(x,t\right) \tag{2.4}$$

To find the variable *x(t)* we could take the integral of both sides of the equality

$$x = \int f\left(x,t\right)dt \tag{2.5}$$

and find the unknown variable by solving an integral. If we knew the variable x at time t, we can use the Euler integration rule to find x at time $t+\Delta t$ where

$$x(t+\Delta t) = x(t) + \int_{t}^{t+\Delta t} f\left(x,t\right)dt \cong x(t) + f(x,t)\Delta t \tag{2.6}$$

Note that this is an approximation that is valid when Δt is small.

We can apply Equation (2.6) to find the current for the circuit shown in Fig. 2.1. Since

$$\frac{di}{dt} = -\frac{R}{L}i + \frac{V_m}{L}\cos\left(\omega t + \phi\right) \tag{2.7}$$

then the update equation for finding the current becomes

$$i(t+\Delta t) = i(t) + \Delta t\left(-\frac{R}{L}i(t) + \frac{V_m}{L}\cos\left(\omega t + \phi\right)\right) \tag{2.8}$$

Solving for the current using the Euler rule involves an iterative process which can be implemented on the computer as follows:

1. Determine an appropriate time step Δt
2. Set i(0) equal to the initial current and set t=0
3. Determine the derivative of the current using equation (2.7)
4. Find $i(t+\Delta t)$ using the update formula in (2.8)
5. Set $t = t+\Delta t$, if more iterations are required then go to step 3
6. Quit simulation loop

For the approximation in Equation (2.6) to be valid, the time step Δt needs to be rather small. There is a trade-off between the time step and the number of iterations required to obtain a simulation. Large time steps will result in simulation errors and small time steps will require more iterations. One way to determine if the time step is appropriate is to cut the time-step by two until the solution converges.

Simple R-L-C Circuit

Suppose that we now add a capacitor to give us the circuit shown in Fig. 2.2.

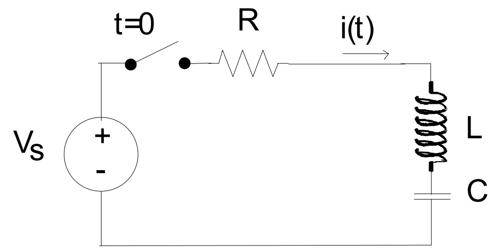

Fig. 2.2 R-L-C circuit

The equation which describes this circuit is now given by

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt+v_c\left(0\right)=V_m\cos\left(\omega t+\phi\right) \tag{2.9}$$

If we take the derivative of both sides of the equality and divide by L then

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=-\frac{\omega V_m}{L}\sin\left(\omega t+\phi\right) \tag{2.10}$$

It is more convenient to write a computer routine to solve a number of coupled first-order differential equations than one higher-order differential equation. To get an alternate formulation let us write two first-order differential equations which describe this circuit using the capacitor voltage and inductor current as variables. For this simple loop circuit

$$\frac{dv_C}{dt}=\frac{1}{C}i_L \tag{2.11}$$

$$\frac{di_L}{dt}=-\frac{R}{L}i_L-\frac{1}{L}v_C+\frac{V_m}{L}\cos\left(\omega t+\phi\right) \tag{2.12}$$

The Euler updates are now defined by

$$v_C(t+\Delta t)=v_C(t)+\Delta t\left(\frac{1}{C}i_L\left(t\right)\right) \tag{2.13}$$

$$i_L(t+\Delta t)=i_L(t)+\Delta t\left(-\frac{R}{L}i_L\left(t\right)-\frac{1}{L}v_C\left(t\right)+\frac{V_m}{L}\cos\left(\omega t+\phi\right)\right) \tag{2.14}$$

We can now apply the same algorithm described above for the R-L circuit with a few slight modifications:

1. Determine an appropriate time step Δt
2. Set i(0) equal to the initial inductor current, v(0) to initial capacitor voltage, and set t=0
3. Determine the derivative of the current and the voltage using equations (2.11) and (2.12)
4. Find $i(t+\Delta t)$ and $v(t+\Delta t)$ using the update formulas in (2.13) and (2.14)
5. Set $t= t+\Delta t$, if more iterations are required then go to step 3
6. Quit simulation loop

The Euler simulation technique won't work as well for complex circuits since this approach does not accurately represent the interaction between the voltage and current variables. It is difficult to redo the solution with a higher-order Taylor expansion since the higher order derivatives would become too complicated. A methodology has been developed for single-step procedures, which use only first-order derivative terms, but have a high degree of accuracy. These types of algorithms developed for solving differential equations are called Runge-Kutta methods.

One example of a low-order Runge-Kutta algorithm involves the update equation

$$x(t+\Delta t) = x(t) + \frac{\Delta t}{2}\Big(f\big(x(t),t\big) + f\big(\overline{x}(t+\Delta t), t+\Delta t\big)\Big) \tag{2.15}$$

where

$$\overline{x}(t+\Delta t) = x(t) + f\big(x(t),t\big)\Delta t \tag{2.16}$$

Equation (2.16) is first used to predict $x(t+\Delta t)$, then an improved estimate is obtained through equation (2.15). These equations can be applied iteratively until convergence is reached. This pair of equations is often referred to as a predictor-corrector set.

Examples

1. Write a Matlab program based on the Euler update to find the current for the circuit shown in Fig. 2.1. Assume that R=40 Ohms, L=75 mH, the voltage magnitude is 100 V, the frequency is 400 radians/sec and the voltage angle is 60 degrees. Plot the simulation result against the solution given in Equations (2.2) and (2.3). The plot should correspond to 3 cycles of the voltage forcing function

```
clear all;
% part 1, problem 1
R=input('Input the resistance in ohms: ');
L=input('Input the inductance in henrys: ');
Vm=input('Input the Voltage Magnitude: ');
w=input('Input the ang. freq : ');
ang=input('Input the Voltage angle: ');

phi=ang/180*pi;
theta=atan(w*L/R);

delta=10^(-4);
Ieuler(1)=0;
Icalc(1)=0;
```

```
tottime=3*2*pi/w;

for step=1:tottime/delta;

 t=step*delta;

 Ieuler(step+1)= Ieuler(step) + delta*(-R/L*Ieuler(step) + Vm/L*cos(w*t
+ phi));
 Icalc(step +1)= Vm/sqrt(R^2+(w*L)^2) * (cos(w*t+phi-theta) - cos(phi-
theta)*exp(-R/L*t));

end;

%dump output
versus=0:delta*w:6*pi;

plot(versus,Ieuler); hold on;
plot(versus,Icalc,'r-.');

title('Current Waveforms for the RL Circuit');xlabel('wt');
axis([0 6*pi -3 3]);
legend('Euler Iteration','Calculated Current');
```

2. Write a Matlab program based on the Euler update to find the current for the circuit shown in Fig. 2.2. Assume that R=40 Ohms, L=75 mH, C=10 μF, the voltage magnitude is 100 V, the frequency is 400 radians/sec and the voltage angle is 60 degrees. Use an initial capacitor voltage of 0. Plot the simulation result for 3 cycles of the voltage forcing function

```
clear all;
% part 1, problem 2
R=input('Input the resistance in ohms: ');
L=input('Input the inductance in henrys: ');
C=input('Input the capacitance in farads: ');
Vm=input('Input the Voltage Magnitude: ');
w=input('Input the ang. freq : ');
ang=input('Input the Voltage angle: ');

phi=60/180*pi;

delta=5.5e-7;% time step
Vcap(1)=0;
Iind(1)=0;
tottime=3*2*pi/w;

for step=1:tottime/delta;

 t=step*delta;

 Vcap(step+1)= Vcap(step) + delta*(Iind(step)/C);
 Iind(step+1)= Iind(step) + delta*(-R/L*Iind(step) - Vcap(step)/L +
Vm/L*cos(w*t + phi));
end;
```

```
versus=0:delta*w:6*pi;
plot(versus,Iind);
title('Inductor Current Waveform for the RLC Circuit');xlabel('wt');
```

3. Use the Matlab function ODE45 to redo the simulation specified in exercise 2.

```
clear all;
% part 1, problem 3
global R;
global L;
global C;
global Vm;
global w;
global ang;

R=input('Input the resistance in ohms: ');
L=input('Input the inductance in henrys: ');
C=input('Input the capacitance in farads: ');
Vm=input('Input the Voltage Magnitude: ');
w=input('Input the ang. freq : ');
ang=input('Input the Voltage angle: ');

tottime=3*2*pi/w;

[tout,yout] = ODE45('funcp1', [0 tottime], [0 0]);
figure(2),plot(tout*w, yout(:,2));
title('Inductor Current Waveform for the RLC Circuit');xlabel('wt');


function i_prime=funcp1(t,i)

%set of first-order differential equations for RLC circuit
global R;
global L;
global C;
global Vm;
global w;
global ang;

phi=ang/180*pi;

% i1=iL and i2=diL/dt
% i=[i1 i2]
i_prime(1,1)= (1/C)*i(2); % diL/dt
i_prime(2,1)= -R/L*i(2) -1/(L)*i(1) + Vm/L*cos(w*t + phi); % diL^2 /
dt^2
```

Formatting Differential Equations for Circuit Analysis

One technique that can be used to formulate a set of differential equations is the method of state variable analysis. The state variables normally selected (not always) are capacitor voltages and inductor currents. Once these state variable voltages and currents are determined, then any other circuit quantity can be calculated.

As an example of how to formulate a problem in terms of differential equations, look at the circuit given in Fig. 2.3. Let us select the capacitor voltage v_C and the inductor current i_L as the state variables to solve for. Next we
Write a set of equations that relate the first derivatives of the state variables to the circuit state variables and the circuit sources using Kirchoff voltage and current law principles.

$$L\frac{di_L}{dt} = v_C \tag{2.17}$$

$$C\frac{dv_C}{dt} = \frac{V_S - v_C}{R} - i_L \tag{2.18}$$

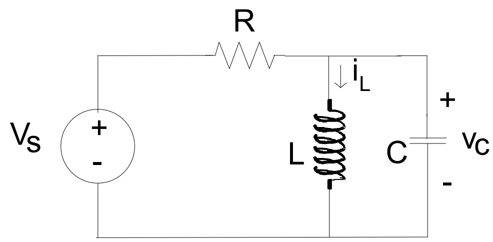

Fig. 2.3 R-L-C circuit

Rewriting these equations in the state variable format, with the derivatives on the left hand side of the equality results in

$$\frac{di_L}{dt} = \frac{1}{L}v_C \tag{2.19}$$

$$\frac{dv_C}{dt} = -\frac{1}{C}i_L - \frac{1}{RC}v_C + \frac{1}{RC}V_S \tag{2.20}$$

The circuit equations are now in a format amenable to computer simulation.

Circuit Topology

In order to write a set of state variable equations for a more complex circuit, we will need a systematic way of formulating a set of independent equations. This can be accomplished by making use of circuit topology theory. To apply topology concepts, we start out by constructing the *graph* for a given circuit, in which all the elements in the circuit are replaced with lines. The lines in this graph which interconnect the circuit nodes are referred to as *branches.*

As an example consider the circuit shown in Fig. 2.4a. When the elements are replaced by branches we get the resulting graph shown in Fig. 2.4b. A circuit *tree* is defined by constructing a set of branches that connect to every node in the graph without forming loops. An example of a tree for the graph shown in Fig 2.4b. is illustrated in Fig. 2.4c, where the branches of the tree are indicated by the thicker lines. There are a number of different ways to form this tree, so this represents only one of a number of tree structure possibilities. Once the tree structure has been determined, then the branches that are left are referred to as the *links*. The set of links is also called the *cotree*.

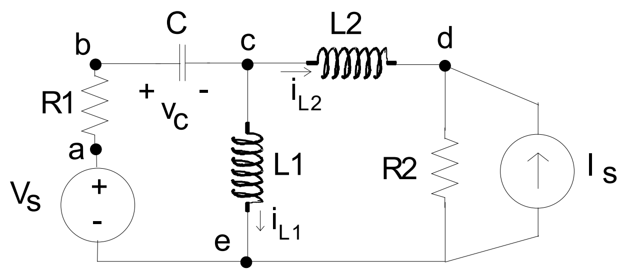

Fig. 2.4a Example Circuit

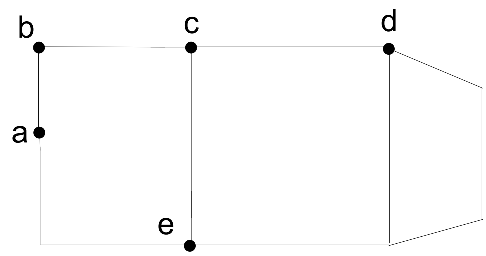

Fig. 2.4b Graph for Example Circuit

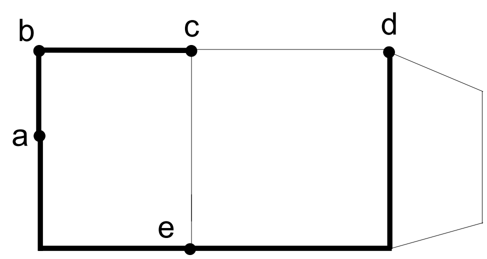

Fig. 2.4c Tree Structure for Circuit Graph

We can use these topology concepts to write a set of state variable equations by applying the following steps:

(1) Set up a tree which contains all of the capacitors in the circuit and exclude the inductors.
(2) The state variables to be chosen will be the capacitor voltages in the tree branches and the inductor currents which correspond to the tree links.
(3) Write a Kirchoff current law equation for each capacitor in a tree branch in terms of state variables and inputs.
(4) Write a Kirchoff voltage law equation for each inductor in a link in terms of state variables and inputs.
(5) Manipulate the equations until they appear in the standard form, with the derivatives of the state variables appearing on the left-hand side and the state variables and inputs appearring on the right-hand side.

Let us look at how this algorithm can be applied to the circuit shown in Fig. 2.4a. The tree shown in Fig. 2.4c was already constructed so that the capacitors were included as tree branches while the inductors remained in the links.
For this circuit the state variables will be the capacitor voltage v_C, and the inductor currents i_{L1} and i_{L2}. The inputs for this circuit are the voltage source value v_S and the current source value i_S. Writing a Kirchoff current law equation for the capacitor gives us:

$$C\frac{dv_C}{dt} = i_{L1} + i_{L2} \tag{2.21}$$

Writing Kirchoff voltage laws for the two inductors results in:

$$L_1\frac{di_{L1}}{dt} = v_S - R_1\left(i_{L1} + i_{L2}\right) - v_C \tag{2.22}$$

$$L_2\frac{di_{L2}}{dt} = v_S - R_1\left(i_{L1} + i_{L2}\right) - v_C - R_2\left(i_{L2} + i_S\right) \tag{2.23}$$

Note how equations (2.22) and (2.23) relate to the loops that correspond to each inductor's link.
We finally rearrange the terms so that the state variable derivatives all appear on the left-hand side.

$$\begin{aligned}
\frac{dv_C}{dt} &= (0)v_C + \left(\frac{1}{C}\right)i_{L1} + \left(\frac{1}{C}\right)i_{L2} + (0)v_S + (0)i_S \\
\frac{di_{L1}}{dt} &= \left(-\frac{1}{L_1}\right)v_C + \left(-\frac{R_1}{L_1}\right)i_{L1} + \left(-\frac{R_1}{L_1}\right)i_{L2} + \left(\frac{1}{L_1}\right)v_S + (0)i_S \\
\frac{di_{L2}}{dt} &= \left(-\frac{1}{L_2}\right)v_C + \left(-\frac{R_1}{L_2}\right)i_{L1} + \left(-\frac{R_1 + R_2}{L_2}\right)i_{L2} + \left(\frac{1}{L_2}\right)v_S + \left(-\frac{R_2}{L_2}\right)i_S
\end{aligned} \tag{2.24}$$

Now these equations are in a form that can be solved in a computer program using a numerical integration technique.

Still More Examples

4. Write a Matlab simulation program based on the function ODE45 for the circuit shown in Fig. 2.5.

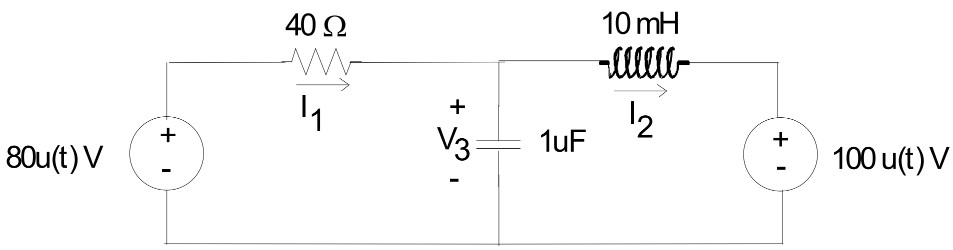

Fig. 2.5 Test Circuit

```
% Main Matlab routine
clear all;
%Part 2, problem 1.
global R;
global L;
global C;

R=input('Input the resistance in ohms: ');
L=input('Input the inductance in henrys: ');
C=input('Input the capacitance in farads: ');

tottime=0.4e-3;

[tout,yout] = ODE45('funcp2', [0 tottime], [0 0]);
figure(1),plot(tout, yout(:,1));
title('Capacitor Voltage Waveform for the RLC Circuit');xlabel('t');

figure(2),plot(tout, yout(:,2),tout, (80-yout(:,1))/R);
title('Inductor & Resistor Current and  Waveforms for the RLC
Circuit');xlabel('t');
legend('i_L', 'i_R');


% Function used by integration routine
function Xprime=funcp2(t,X)

%set of first-order differential equations for RLC circuit
global R;
global L;
global C;

% X1=Vc and X2=IL
% i=[Vc IL]
Xprime(1,1)= -X(1)/(R*C) - X(2)/C + 80/(R*C); % dVc/dt
Xprime(2,1)= X(1)/L - 100/L;                            % diL / dt
```