LUMPED PARAMETER LINE MODELING
Part 1: Overhead Line Models

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Introduction
This set of notes will discuss "by-phase" models for representing both overhead and underground cable systems. By-phase representation differs from symmetrical component representation in that the detailed coupling between conductors is properly modeled. This is critical for correctly modeling the unbalanced nature of most transients.

Part 1 of these notes will focus on overhead line models. Cable models to be used for underground circuits will be discussed in Part 2.

Conductor Characteristics
Overhead conductors are normally made out of copper or aluminum since the resistivity of these materials is rather low and the materials are relatively inexpensive (compared to silver). The larger capacity conductors are often constructed of a number of individual strands of wire. In situations requiring better conductor mechanical strength, steel is used as a reinforcement. Fig. 1 shows a cross section of an aluminum conductor, steel reinforced (ACSR) cable in which the aluminum conductors are wound about a steel core. Conductors are characterized by a number of features such as cross-sectional area, number of strands, diameter of conductor, current capacity, geometric mean radius (GMR), and resistance per unit length. These characteristics are found in tables, such as the one shown in Fig. 2a for copper conductors.

For the larger conductors, cross-sectional area is normally given in terms of circular mils (CMIL) where 1 CMIL = 3.14×10^{-6} in^2. A common conductor used in distribution primary circuits has a cross-sectional area of 336,400, or 336 MCMIL, which is often referred to as a 336 conductor size. The smaller size conductors are often described using the American Standard Wire Gage (A.W.G.). In this system a 4/0 or 0000 (pronounced 4 aught) A.W.G. size would be equivalent to a 211,600 CMIL size while a 2 A.W.G. size would be equivalent to 66,400 CMIL. Tables often include both types of numbers.

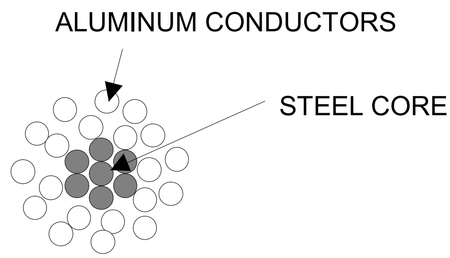

Fig. 1 ACSR Construction

Size, CMIL	*Size, A.W.G.*	*Stranding*	*Outside Diameter, in.*	*Current Capacity, Amps*	*GMR D_S, ft*	*Rdc, 25^oC Ω/mile*	*Rac, 25^oC Ω/mile*	*Rac, 50^oC Ω/mile*
500,000		19	.811	840	.0256	.1170	.1196	.1303
211,600	4/0	19	.528	480	.01668	.276	.278	.303
66,370	2	1	.258	220	.00836	.864	.864	.945

Fig. 2a Conductor Parameters for Copper, Hard Drawn, 97.3% Conductivity

Commercial computer programs for circuit analysis store parameters for common types of phase conductors in equipment tables. An example of a portion of an equipment table from the Milsoft WindMil program is shown in Fig. 2b. Note that each conductor has a special string used to describe it. The Preferred Neutral corresponds to the default neutral which would be used for a typical multi-phase overhead line construction.

```
                                      Overhead Conductor Data

Equipment Database: C:\Milsoft\Examples\EQDB

--------------------------------------------------------------------------------------------------------
                                            --Resistance@--
                                   Amp-     25 C     50 C     GMR     Diameter
Description           Material     acity    (ohms/mile)      (feet)   (inches)    Preferred Neutral
--------------------------------------------------------------------------------------------------------

#1/0 ACSR 6/1         ACSR           230    0.8880   1.1200   0.00446    0.00000    #2 ACSR 6/1
#2/0 ACSR 6/1         ACSR           270    0.7060   0.8950   0.00510    0.00000    #1/0 ACSR 6/1
#3/0 ACSR 6/1         ACSR           300    0.5600   0.7230   0.00600    0.00000    #1/0 ACSR 6/1
#4/0 ACSR 6/1         ACSR           340    0.4450   0.5920   0.00814    0.00000    #1/0 ACSR 6/1
336 MCM ACSR 30/7     ACSR           530    0.2780   0.3060   0.02550    0.00000    #1/0 ACSR 6/1
336 MCM ACSR 26/7     ACSR           530    0.2780   0.3060   0.02440    0.00000    #1/0 ACSR 6/1
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Fig. 2b Phase Conductor Equipment Table

Conductor Resistance

The series resistance of a conductor is expressed in tables in terms of Ohms per unit length, such as Ohms per mile or Ohms per thousand feet. The DC resistance of a wire, based on a uniform current distribution is given by

$$R = \rho \frac{l}{A} \qquad \text{(Ohms)} \tag{1}$$

where

ρ - resistivity of the wire
l - length of the wire
A- cross-sectional area

Since the resistivity of copper is less than that of aluminum, then copper conductors have less resistance per unit length than equivalent-sized aluminum conductors. Of course this is offset by the fact that aluminum is less expensive than copper.

Equation (1) assumes that the current density is uniform over the cross-sectional area of the conductor. When considering AC operation this assumption is no longer true. As the frequency increases, the current density becomes more concentrated towards the outer part of the conductor. This phenomenon is referred to as the skin-effect and it explains why the 60 Hz resistance is higher than the DC resistance by a factor of about 1.03 at 60 Hz. As the frequency increases above 60 Hz, the resistance will continue to increase.

The resistance of a conductor also varies with temperature. This is why conductor parameter tables list conductor resistance at several temperatures, such as 25^{o} C and 50^{o} C. Assuming that the variation of resistance with temperature is linear, it is possible to obtain the resistance at other temperatures by applying interpolation. Normally the resistance at 25^{o} C would be used for light loading conditions while the resistance at 50^{o} C would apply to heavy loading conditions.

The ampacity of a conductor is determined by the cross-sectional area and the resistivity of the material. For underground applications, ampacity is also related to the ability of the cable system to dissipate heat. The ampacity refers to the amount of current a conductor can carry without damaging the conductor. If operated above its rated ampacity, a conductor has a tendency to sag, and if this overcurrent condition is maintained for too long a time period, the sag could become permanent. In some cases, overcurrents have caused conductors to sag until a tree-limb was contacted, causing a short circuit. When running power flow studies on a computer, the computer program is used to flag conductors that are operating close to or above their rated ampacity..

Line Inductance

Definitions

The voltage drop across a line is not only a function of a series resistance, but series and mutual inductance as well. The inductance of a power transmission line is a consequence of the magnetic fields generated by the sinusoidal conductor currents. In general, the voltage drop (induced voltage) associated with a circuit is given by

$$e(t) = \frac{d\lambda(t)}{dt} \quad \text{(Volts)} \tag{2}$$

where e represents the induced voltage and λ the flux linkages for the circuit. For lines the number of turns is simply one. In the case of a linear circuit, there is a constant which relates the voltage to the rate in change in current, i, such that

$$e(t) = L\frac{di(t)}{dt} \tag{3}$$

where L is defined as the inductance, in units of Henries. The inductance can also be defined as the ratio of the total flux linkages to the line current where

$$L = \frac{\lambda}{i} \text{ (Henries)} \tag{4}$$

When the voltages and currents are sinusoidal, phasor analysis can be applied. In this case equations (2) and (3) can be rewritten as

$$V = j\omega\psi = j\omega LI \tag{5}$$

where V represents the phasor voltage, ψ the phasor flux linkages, I the phasor current, and ω is the frequency in radians per second.

Many times when working with multiphase circuits, the total flux linkages will be due to currents flowing in multiple adjacent conductors. In a two conductor circuit, the voltage drop across conductor 1 can be written as

$$V_1 = j\omega\left(\psi_{11} + \psi_{12}\right) = j\omega L_{11}I_1 + j\omega L_{12}I_2 \tag{6}$$

where L_{11} is a self-inductance associated with the voltage drop on conductor 1 due to the current in conductor 1, I_1 , and L_{12} is the mutual inductance associated with the voltage drop on wire 1 due to the current in conductor 2, I_2.

Calculation of Internal Flux Linkages

Consider the single conductor shown in Fig. 3, with radius r_1. The flow of current will generate the flux pattern illustrated in the figure, with a direction conforming to the right-hand rule. Note that the resulting magnetic field exists internal to the conductor as well as external to the conductor. In our analysis we will need to analysis both internal and external effects when calculating the equivalent inductance. Assuming that the conductor is not ferromagnetic, then the relationship between current and flux linkages for a conductor will be linear and superposition may be applied when looking at the effect of multiple conductors.

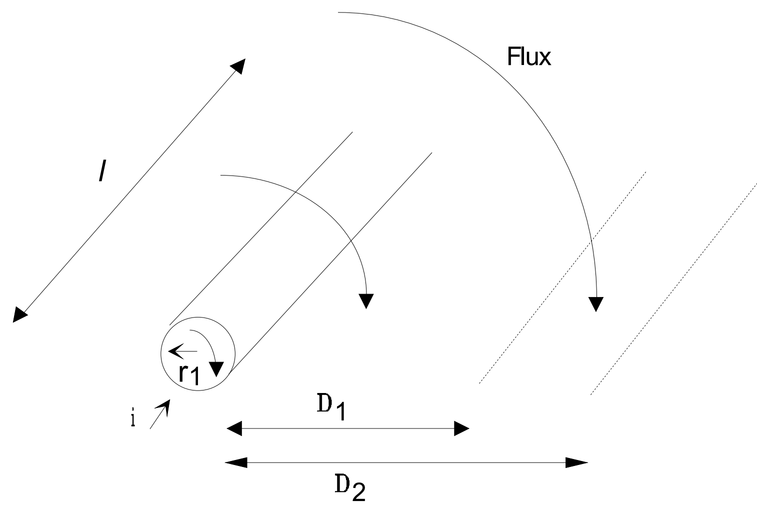

Fig. 3 Single Conductor

The relationship between the current and the magnetic field is defined by Maxwell's mmf law as

$$I = \int_S \vec{J} \cdot d\vec{a} = \oint_C \vec{H} \cdot d\vec{l} \quad \text{(Amperes)} \tag{7}$$

Assuming that the current density, J, is uniform over the cross-sectional area a of the conductor, then it is possible to calculate the magnetic field intensity, H, as a function of the distance from the center of each conductor, r. There will be two cases, one for the field intensity inside the conductor and a second one for the field intensity outside the conductor. We will look at the internal case first. For a closed contour $r \leq r_1$ then equation (7) becomes

$$\frac{\pi r^2}{\pi r_1^2} I = 2\pi r H \tag{8}$$

where H is a constant around a closed contour of radius, r. Solving for the flux density, B

$$B = \frac{\mu_o r}{2\pi r_1^2} I \tag{9}$$

The incremental flux at a distance of *r* from the center of the conductor becomes

$$d\phi = Bda = Bldr = \frac{\mu_o lrI}{2\pi r_1^2} dr \tag{10}$$

which means that the incremental flux linkages are described by

$$d\lambda = \left[\frac{\pi r^2}{\pi r_1^2} \right] d\phi = \frac{\mu_o l r^3 I}{2\pi r_1^4} dr \tag{11}$$

The total internal flux linkages per unit length for the conductor are now found by integrating the incremental flux linkages and dividing by the length of the conductor

$$\lambda_{\text{int}} = \int_0^{r_1} \frac{\mu_o r^3 I}{2\pi r_1^4} \, dr = \frac{\mu_o I}{8\pi} \tag{12}$$

Dividing by the current, *I*, to obtain the inductance per unit length

$$L_{\text{int}} = \frac{\mu_o}{8\pi} \qquad \text{(Henries/meter)} \tag{13}$$

Finally, substituting $\mu_o = 4\pi \times 10^{-7} (H/m)$ the inductance due to the internal flux is

$$L_{\text{int}} = \frac{1}{2} \times 10^{-7} \ \text{(Henries/Meter)} \tag{14}$$

Calculation of External Flux Linkages

To calculate the impact of the flux external to the conductor, consider a closed contour $D_1 \le r \le D_2$ where Equation (7) becomes

$$I = 2\pi r H \tag{15}$$

Again, solving for the flux density, *B*

$$B = \frac{\mu_o}{2\pi r} I \tag{16}$$

The incremental flux external to the conductor then becomes

$$d\phi = Bda = Bldr = \frac{\mu_o lI}{2\pi r} dr \tag{17}$$

The incremental flux linkages for this case are given by

$$d\lambda = d\phi = \frac{\mu_o lI}{2\pi r} dr \tag{18}$$

The external flux per unit length within the contour, located between locations D_1 and D_2, linking the conductor is now found by integrating the incremental flux linkages and dividing by the length

$$\lambda_{D1,D2} = \int_{D_1}^{D_2} \frac{\mu_o I}{2\pi r}\, dr = \frac{\mu_o I}{2\pi} \ln\left(\frac{D_2}{D_1}\right) \tag{19}$$

Dividing by the current, *I*, yields the inductance

$$L_{D1,D2} = \frac{\mu_o}{2\pi} \ln \frac{D_2}{D_1} \qquad \text{(Henries/Meter)} \tag{20}$$

Substituting $\mu_o = 4\pi \times 10^{-7} (H / m)$, then the inductance due to the flux between points D_1 and D_2 is given by

$$L_{D1,D2} = 2 \times 10^{-7} \ln \frac{D_2}{D_1} \qquad \text{(Henries/Meter)} \tag{21}$$

Flux Linkages for Circuits with Multiple Conductors

Suppose that we have an *n* conductor system in which the sum of the currents adds up to zero, as shown in Fig. 4. Point *P* is a reference point in which the distance from this point to the various conductors is given by D_{iP}. The flux linkages of conductor 1 due to the current I_1, and which includes both the internal flux and the external flux between the conductor and point *P,* is given by

$$\lambda_{1P,1} = \left(\frac{1}{2} I_1 + 2 I_1 \ln \frac{D_{1p}}{r_1}\right) 10^{-7} = 2 \times 10^{-7} \left(\frac{1}{4} + \ln \frac{D_{1P}}{r_1}\right) I_1 \tag{22}$$

In order to combine the effects of the internal and external flux, it is possible to make use of the log identities to simplify equation (22)

$$\lambda_{1P,1} = 2\times10^{-7}\left(\ln(e^{\frac{1}{4}}) + \ln\frac{D_{1P}}{r_1}\right)I_1 = 2\times10^{-7}\left(\ln\frac{D_{1p}e^{\frac{1}{4}}}{r_1}\right)I_1 = 2\times10^{-7}\left(\ln\frac{D_{1P}}{r_1 e^{-\frac{1}{4}}}\right)I_1 \qquad (23)$$

The term in the denominator of the log argument is referred to as the geometric mean radius (GMR) and is defined for conductor *i* as

$$GMR_i = r_i e^{-\frac{1}{4}} \qquad (24)$$

Hence the final equation for the flux linkages for conductor 1 due to the current in conductor 1 is given as

$$\lambda_{1P,1} = 2\times10^{-7}\left(\ln\frac{D_{1P}}{GMR_1}\right)I_1 \qquad (25)$$

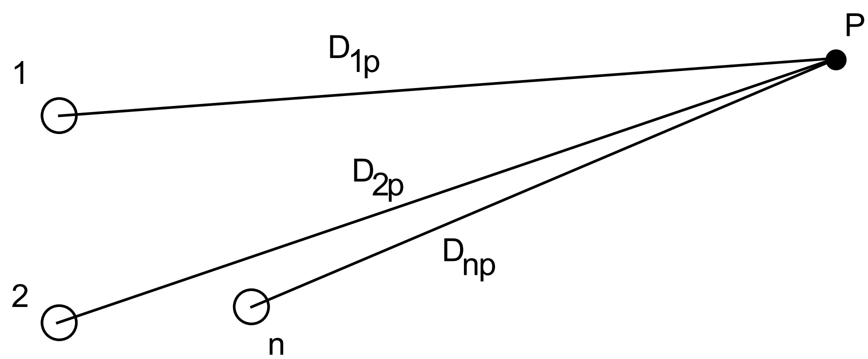

Fig. 4 Multi-Conductor Circuit

The flux linkages with conductor 1 generated by a current in conductor 2, excluding the flux outside point *P*, is found by substituting $D_2=D_{2P}$ and $D_1=D_{12}$ into equation (19) above

$$\lambda_{1P,2} = 2\times10^{-7}\left(\ln\frac{D_{2P}}{D_{12}}\right)I_2 \qquad (26)$$

The total flux linkages with conductor 1 due to the currents flowing in all of the conductors, not including the flux beyond point P, is given by

$$\lambda_{1P} = \lambda_{1P,1} + \lambda_{1P,2} + \ldots + \lambda_{1P,n} = 2\times10^{-7}\left(\ln\frac{D_{1P}}{GMR_1}I_1 + \ln\frac{D_{2P}}{D_{12}}I_2 + \ldots + \ln\frac{D_{nP}}{D_{1n}}I_n\right) \qquad (27)$$

Breaking up the arguments in the log terms results in

$$\lambda_{1P} = 2\times 10^{-7}\begin{pmatrix} \ln\dfrac{1}{GMR_1}I_1 + \ln\dfrac{1}{D_{12}}I_2 + ... + \ln\dfrac{1}{D_{1n}}I_n \\ +(\ln D_{1p})I_1 + (\ln D_{2P})I_2 + ... + (\ln D_{nP})I_n \end{pmatrix} \quad (28)$$

Since all of the current add up to zero, then the current in conductor *n* can be replaced by

$$I_n = -\left(I_1 + I_2 + ... + I_{n-1}\right) \quad (29)$$

Substituting for I_n in the last expression for flux linkages leads to

$$\lambda_{1P} = 2\times 10^{-7}\begin{pmatrix} \ln\dfrac{1}{GMR_1}I_1 + \ln\dfrac{1}{D_{12}}I_2 + ... + \ln\dfrac{1}{D_{1(n-1)}}I_{n-1} + \ln\dfrac{1}{D_{1n}}I_n \\ +(\ln D_{1p})I_1 + (\ln D_{2P})I_2 + ... + (\ln D_{(n-1)P})I_{n-1} \\ +(\ln D_{nP})(-I_1 - I_2 - ... - I_{n-1}) \end{pmatrix} \quad (30)$$

which when equating the coefficients for the currents becomes

$$\lambda_{1P} = 2\times 10^{-7}\begin{pmatrix} \ln\dfrac{1}{GMR_1}I_1 + \ln\dfrac{1}{D_{12}}I_2 + ... + \ln\dfrac{1}{D_{1(n-1)}}I_{n-1} + \ln\dfrac{1}{D_{1n}}I_n \\ +(\ln\dfrac{D_{1p}}{D_{np}})I_1 + (\ln\dfrac{D_{2P}}{D_{nP}})I_2 + ... + (\ln\dfrac{D_{(n-1)P}}{D_{nP}})I_{n-1} \end{pmatrix} \quad (31)$$

As the point P is moved further and further away from the conductors, the ratio D_{iP} / D_{nP} approaches one and the second set of log terms, $\ln\dfrac{D_{ip}}{D_{np}}$, all approach zero. When P is an infinite distance away from the set of conductors, the flux linkages we are calculating now includes the total internal and external flux linkages. Hence, provided the sum of the conductor currents all add up to zero, one can show that the total flux linkages for conductor 1 are given by

$$\lambda_{1P} = 2\times 10^{-7}\left(\ln\frac{1}{GMR_1}I_1 + \ln\frac{1}{D_{12}}I_2 + ... + \ln\frac{1}{D_{1n}}I_n\right) \quad (32)$$

In the phasor domain, for sinusoidal currents, the induced voltage per unit length across conductor 1 is given by

$$V_1 = j\omega\psi_{1P} = j\omega 2\times 10^{-7}\left(\ln\frac{1}{GMR_1}I_1 + \ln\frac{1}{D_{12}}I_2 + ... + \ln\frac{1}{D_{1n}}I_n\right)\text{(Volts/meter)} \quad (33)$$

Where the coefficients for the currents can also be described by self and mutual reactances

$$V_1 = jX_{11}I_1 + jX_{12}I_2 + \ldots + jX_{1n}I_n \tag{34}$$

where

$$X_{11} = 2\times 10^{-7}\omega \ln \frac{1}{GMR_1} \quad \text{(Ohms/meter)} \tag{35}$$

$$X_{12} = 2\times 10^{-7}\omega \ln \frac{1}{D_{12}} \quad \text{(Ohms/meter)} \tag{36}$$

...

$$X_{1n} = 2\times 10^{-7}\omega \ln \frac{1}{D_{1n}} \quad \text{(Ohms/meter)} \tag{37}$$

In general, for a frequency of 60 Hz, and using the unit conversion that 1609 meters is equivalent to 1 mile, the equations for self and mutual inductance become

$$X_{ii} = .1213 \ln \frac{1}{GMR_i} \quad \text{(Ohms/mile)} \tag{38}$$

$$X_{ij} = .1213 \ln \frac{1}{D_{ij}} \quad , \quad i \neq j \text{ (Ohms/mile)} \tag{39}$$

Single-Phase Circuit Analysis

As an application of the equations for reactance derived above, consider the single-phase secondary circuit shown in Fig. 5 below

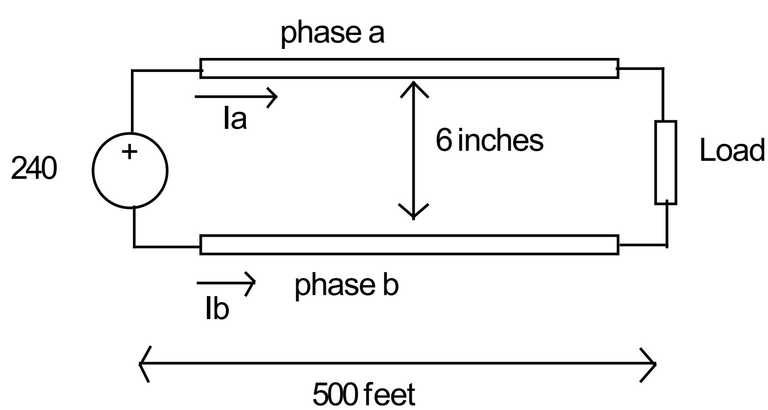

Fig. 5 Single-Phase Circuit

Both phases are constructed of 2 AWG copper conductors spaced 6 inches apart. The distance between the source and load is 500 feet. Assume that the load is resistive and consumes 40 kW at 240 volts.

First, calculate the various impedances. For 2 AWG conductors, from Fig. 2, GMR=0.00836 ft. and R=0.945 Ohms/mile at 50º C (using the higher of the two resistances since we are approaching the rated current capacity). The total impedance is given by multiplying the impedance per unit length by the length. Since both conductors will have the same resistance, then

$$R_{aa} = R_{bb} = 0.945 \times \left(\frac{500}{5280} \right) = 0.0895\, Ohms \tag{40}$$

Also since both conductors have the same GMR then

$$X_{aa} = X_{bb} = 0.1213 \ln \left(\frac{1}{.00836} \right) \times \left(\frac{500}{5280} \right) = 0.0550\, Ohms \tag{41}$$

And finally since the distance from conductor a to b is the same as the distance from conductor b to a then

$$X_{ab} = X_{ba} = 0.1213 \ln \left(\frac{1}{6/12} \right) \times \left(\frac{500}{5280} \right) = 0.0080\, Ohms \tag{42}$$

In order to calculate a voltage drop, we will also need an impedance for the load, which can be approximated as

$$Z_{load} = \frac{240^2}{40,000} = 1.44\, Ohms \tag{43}$$

The voltage across the load can now be written as

$$V_{load} = V_{source} - V_a + V_b = V_{source} - \left[(R_a + jX_{aa}) I_a + jX_{ab} I_b \right] + \left[\left(R_b + jX_{bb} \right) I_b + jX_{ba} I_a \right] \tag{44}$$

where for the single-phase loop, $I_b = -I_a$, and the above equation can be simplified to

$$\begin{aligned} V_{load} &= V_{source} - \left[(R_a + jX_{aa}) I_a - jX_{ab} I_a \right] + \left[\left(R_b + jX_{bb} \right) \left(-I_a \right) + jX_{ba} I_a \right] \\ &= V_{source} - (R_a + R_b + jX_{aa} + jX_{bb} - j2X_{ab}) I_a \end{aligned} \tag{45}$$

If both conductors are of the same type

$$V_{load} = V_{source} - 2(R_a + j(X_{aa} - X_{ab}))I_a \tag{46}$$

To include the load constraint, let

$$I_a = V_{load} / Z_{load} \tag{47}$$

so that

$$V_{load} = V_{source} - 2(R_a + j(X_{aa} - X_{ab}))\left(\frac{V_{load}}{Z_{load}}\right) \tag{48}$$

which becomes

$$V_{load} = \left(\frac{Z_{load}}{2(R_a + j(X_{aa} - X_{ab})) + Z_{load}}\right)V_{source} \tag{49}$$

Substituting in the numbers

$$V_{load} = \left(\frac{1.44}{2(0.0895 + j(0.055 - 0.008)) + 1.44}\right)240 = 213\angle -3.3^o \ Volts \tag{50}$$

Ungrounded Three-Wire Circuit Analysis

Power circuits generally consist of three-phase feeders. Consider the scenario shown in Fig. 6 which involves three phases without a neutral or a ground return. Since the circuit has no neutral return, then by definition

$$I_a + I_b + I_c = 0 \tag{51}$$

Neglecting resistance, voltage drops across each phase are given by the following equations

$$\begin{aligned} V_{aa'} &= jX_{aa}I_a + jX_{ab}I_b + jX_{ac}I_c \\ V_{bb'} &= jX_{ba}I_a + jX_{bb}I_b + jX_{bc}I_c \\ V_{cc'} &= jX_{ca}I_a + jX_{cb}I_b + jX_{cc}I_c \end{aligned} \tag{52}$$

where $X_{ab} = X_{ba}, X_{bc} = X_{cb}, X_{ac} = X_{ca}$.

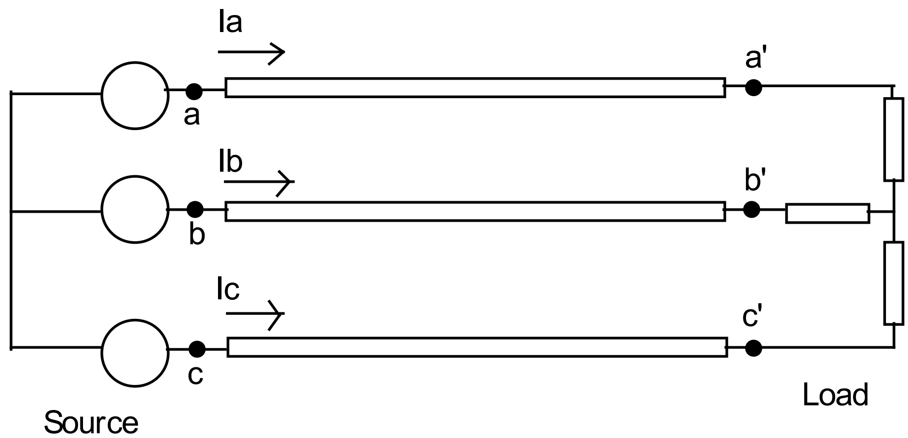

Fig. 6 Three-Wire Circuit

In general, the mutual reactances X_{ab}, X_{ac} and X_{bc} are not the same, since the distances between the different combinations of phase conductors are not the same. However, let us suppose for the sake of example that the self-reactances were all the same and that the mutual reactances were all the same, that is

$$
\begin{aligned}
X_s &= X_{aa} = X_{bb} = X_{cc} \\
X_m &= X_{ab} = X_{bc} = X_{ac}
\end{aligned}
\tag{53}
$$

Then the equations for the three-phase voltage drops would become

$$
\begin{aligned}
V_{aa'} &= jX_sI_a + jX_mI_b + jX_mI_c = jX_sI_a + jX_m(I_b + I_c) \\
V_{bb'} &= jX_mI_a + jX_sI_b + jX_mI_c = jX_sI_b + jX_m(I_a + I_c) \\
V_{cc'} &= jX_mI_a + jX_mI_b + jX_sI_c = jX_sI_c + jX_m(I_a + I_b)
\end{aligned}
\tag{54}
$$

Finally, making use of the fact that the currents add up to zero results in

$$
\begin{aligned}
V_{aa'} &= jX_sI_a + jX_m(-I_a) = j(X_s - X_m)I_a = jX_1I_a \\
V_{bb'} &= jX_sI_b + jX_m(-I_b) = j(X_s - X_m)I_b = jX_1I_b \\
V_{cc'} &= jX_sI_c + jX_m(-I_c) = j(X_s - X_m)I_c = jX_1I_c
\end{aligned}
\tag{55}
$$

where $X_1 = X_s - X_m$ is referred to as the positive sequence reactance. Note that the voltage drop across each phase is only a function of that phase's current.

Many times when we analyze three-phase circuits, we use per-phase analysis to perform the calculations. In doing per-phase analysis we assume that the three-phase circuit is balanced and that the mutual reactances all have the same values. The line impedance we are using in this case is the positive sequence reactance. If we run into a situation where the individual phase mutual reactances are not equal, then how should one calculate the mutual reactance? The normal procedure is to use an average. If

$$X_m = \frac{X_{ab}+X_{bc}+X_{ac}}{3} = .1213\frac{\ln\frac{1}{D_{ab}}+\ln\frac{1}{D_{bc}}+\ln\frac{1}{D_{ac}}}{3}$$
$$= .1213\ln\left(\frac{1}{D_{ab}D_{bc}D_{ac}}\right)^{\frac{1}{3}} = .1213\ln\frac{1}{(D_{ab}D_{bc}D_{ac})^{\frac{1}{3}}} \qquad (56)$$

then letting the denominator of the log expression be represented by the Geometric Mean Distance (GMD), where

$$GMD = \left(D_{ab}D_{bc}D_{ca}\right)^{\frac{1}{3}} \qquad (57)$$

one can see that

$$X_m = 0.1213\ln\frac{1}{GMD} \qquad (58)$$

This leads to a convenient equation for calculating the positive sequence impedance

$$X_1 = X_s - X_m = 0.1213\ln\frac{1}{GMR} - 0.1213\ln\frac{1}{GMD} = 0.1213\ln\frac{GMD}{GMR}\ \text{(Ohms/mile)} \qquad (59)$$

Equation (59) is often used to calculate the impedance used for balanced three-phase circuit analysis.

Analysis of Four-Wire Circuits with a Neutral Return

Most three-phase circuits include a neutral return conductor, as shown in Fig. 7. The sum of the currents still add up to zero, that is

$$I_a + I_b + I_c + I_n = 0 \qquad (60)$$

Neglecting resistance, voltage drops across each phase are given by the following equations

$$\begin{aligned}
V_{aa'} &= jX_{aa}I_a + jX_{ab}I_b + jX_{ac}I_c + jX_{an}I_n \\
V_{bb'} &= jX_{ba}I_a + jX_{bb}I_b + jX_{bc}I_c + jX_{bn}I_n \\
V_{cc'} &= jX_{ca}I_a + jX_{cb}I_b + jX_{cc}I_c + jX_{cn}I_n \\
V_{nn'} &= jX_{na}I_a + jX_{nb}I_b + jX_{nc}I_c + jX_{nn}I_n
\end{aligned} \qquad (61)$$

Substituting for the neutral current, where

$$I_n = -I_a - I_b - I_c \qquad (62)$$

then

$$\begin{aligned} V_{aa'} &= j(X_{aa} - X_{an})I_a + j(X_{ab} - X_{an})I_b + j(X_{ac} - X_{an})I_c \\ V_{bb'} &= j(X_{ba} - X_{bn})I_a + j(X_{bb} - X_{bn})I_b + j(X_{bc} - X_{bn})I_c \\ V_{cc'} &= j(X_{ca} - X_{cn})I_a + j(X_{cb} - X_{cn})I_b + j(X_{cc} - X_{cn})I_c \end{aligned} \tag{63}$$

Note that the neutral wire impedances can be combined with the a,b,c impedances in order to eliminate the neutral return current and voltage drop from the equations needed to solve for the voltage drop for phases a,b and c.

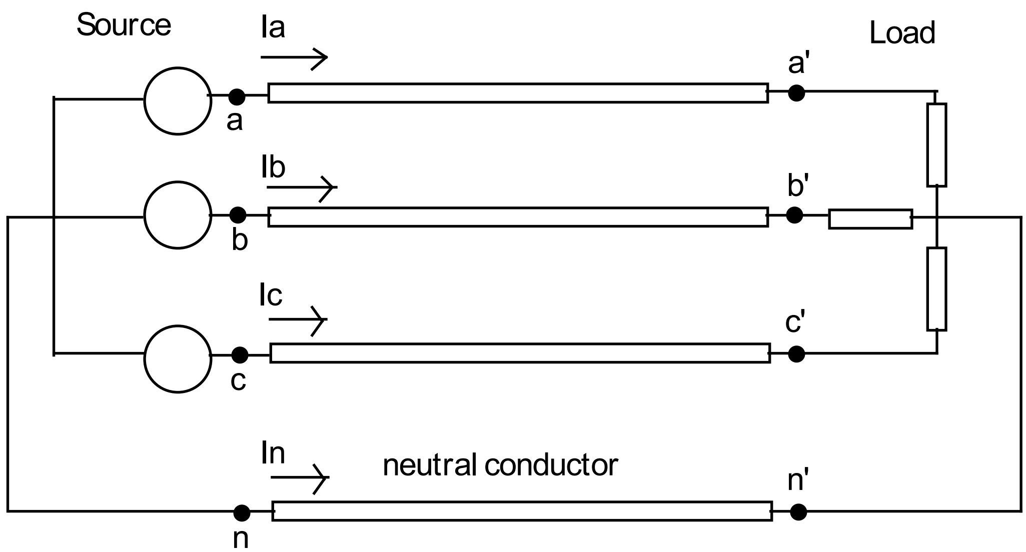

Fig. 7 Four-Wire Circuit

Modeling the Effect of an Earth Path Return

It is common practice to ground to earth along the length of the feeder circuit, at transformers and at customer locations. Hence for imbalanced circuits, it is possible for earth to be one of the paths for current flow back to the source. Usually this earth return current is small during steady-state conditions. However during faults the earth return current can rise to substantial levels. The calculation of the inductance of the earth path and the mutual inductance between the earth path and the overhead conductors involves a set of rather complex equations. This section will briefly review the work of Carson and a set of equations commonly referred to as Carson's corrections for earth return. The modeling approach we will use is based on the assumption that soil resistivity is in the range of 50-500 Ohm-meters, that the application is for low frequencies (around 60 Hz) and that normal overhead configurations are being considered.

Suppose that we have the situation shown in Fig. 8 for which the earth ground is part of the circuit path. Carson showed that the earth return can be modeled as a single conductor, or a "dirt pipe", which has an equivalent GMR given by the symbol d_e , a distance between the overhead conductor and the equivalent earth conductor which is denoted by $D_e^{'}$, as illustrated in Fig. 9, and a resistance per unit length given by

$$R_e = \frac{\omega \mu_o}{8} \; (Ohms \, / \, meter) = 0.0953 \; (Ohms \, / \, mile) \tag{64}$$

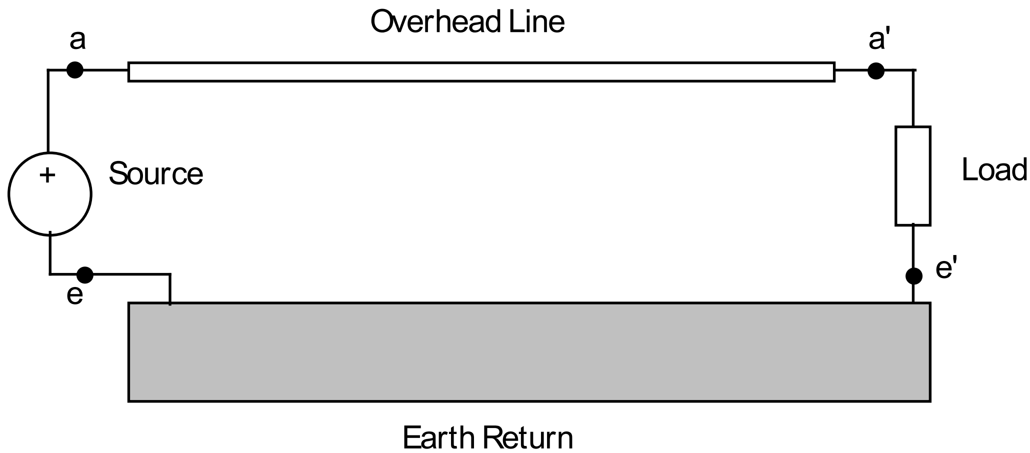

Fig. 8 Circuit with an Earth Return

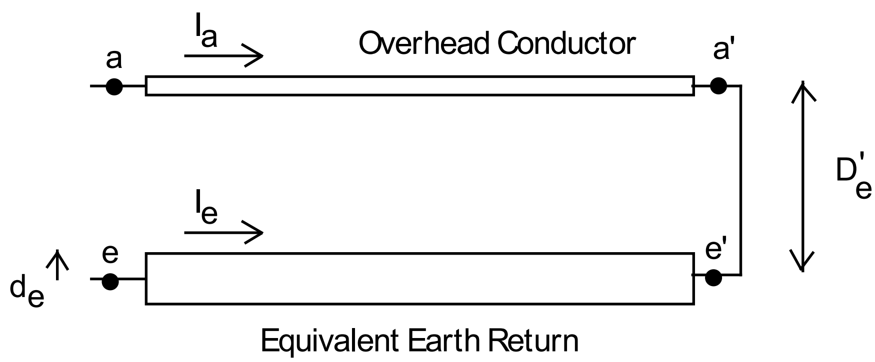

Fig. 9 Equivalent Model for an Earth Return

The voltage drop across the overhead conductor and earth return is given by

$$\begin{aligned} V_{aa'} + V_{e'e} &= (R_a + jX_{aa})I_a + jX_{ae}I_e - (R_e + jX_{ee})I_e - jX_{ea}I_a \\ &= [(R_a + R_e) + j(X_{aa} + X_{ee} - 2X_{ae})]I_a \end{aligned} \tag{65}$$

which makes use of the fact that the earth current is equal in magnitude but opposite in polarity to the phase current. If we substitute in for the reactance in terms of the log expressions involving geometry, we can obtain the voltage drop per unit length

$$\begin{aligned} V_{aa'} + V_{e'e} &= \left[(R_a + R_e) + j.1213(\ln \frac{1}{GMR_a} + \ln \frac{1}{d_e} - 2\ln \frac{1}{D_e^{'}}) \right] I_a \\ &= \left[\left(R_a + R_e\right) + j.1213 \ln \left(\frac{D_e^{'2}}{GMR_a d_e} \right) \right] I_a \end{aligned} \quad (66)$$

The ratio of $D_e^{'2} / d_e$ occurs so often that it is also referred to as

$$D_e = \frac{D_e^{'2}}{d_e} \quad (67)$$

where for the purpose of computing inductive reactance, the earth path can be thought of as an equivalent conductor with a GMR=1 ft. and a distance to the overhead conductor of

$$D_e^{'} = \sqrt{D_e} \quad (68)$$

Using this, the voltage drop per unit length can be simplified to

$$V_{aa'} + V_{e'e} = \left[\left(R_a + 0.0953\right) + j0.1213 \ln \left(\frac{D_e}{GMR_a} \right) \right] I_a \text{ (Volts/mile)} \quad (69)$$

Carson determined that

$$D_e = 2160 \sqrt{\frac{\rho}{f}} \quad (ft.) \quad (70)$$

where ρ is the soil resistivity in Ohm-meters and f is the frequency in Hertz. Note that the loop voltage drop can be found in terms of the phase current only.

Analysis of Four-Wire Grounded Circuits

Finally, let us consider a typical three-phase circuit that has a neutral wire and is also grounded as shown in Fig. 10. For unbalanced conditions, the return current, denoted by I_g, is divided up between the neutral and earth ground path.

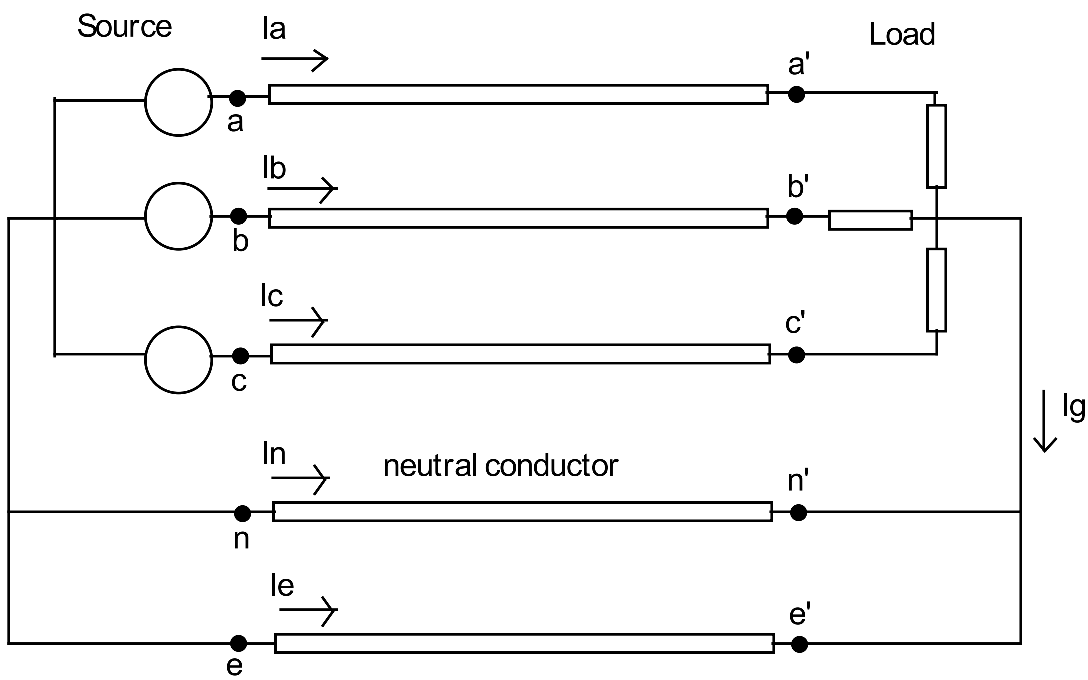

Fig. 10 Four-Wire, Grounded Circuit

To analyze this configuration, first write the equations for the various conductor voltage drops, where

$$
\begin{aligned}
V_{aa'} &= \left(R_a + jX_{aa}\right)I_a + \left(jX_{ab}\right)I_b + \left(jX_{ac}\right)I_c + \left(jX_{an}\right)I_n + \left(jX_{ae}\right)I_e \\
V_{bb'} &= \left(jX_{ba}\right)I_a + \left(R_b + jX_{bb}\right)I_b + \left(jX_{bc}\right)I_c + \left(jX_{bn}\right)I_n + \left(jX_{be}\right)I_e \\
V_{cc'} &= \left(jX_{ca}\right)I_a + \left(jX_{cb}\right)I_b + \left(R_c + jX_{cc}\right)I_c + \left(jX_{cn}\right)I_n + \left(jX_{ce}\right)I_e \\
V_{nn'} &= \left(jX_{na}\right)I_a + \left(jX_{nb}\right)I_b + \left(jX_{nc}\right)I_c + \left(R_n + jX_{nn}\right)I_n + \left(jX_{ne}\right)I_e \\
V_{ee'} &= \left(jX_{ea}\right)I_a + \left(jX_{eb}\right)I_b + \left(jX_{ec}\right)I_c + \left(jX_{en}\right)I_n + \left(R_e + jX_{ne}\right)I_e
\end{aligned}
\qquad (71)
$$

Next, subtract the fifth equation for the voltage drop across the earth ground path from the first four equations

$$
\begin{aligned}
V_{aa'} - V_{ee'} &= \left(R_a + j\left(X_{aa} - X_{ea}\right)\right)I_a + j\left(X_{ab} - X_{eb}\right)I_b + j\left(X_{ac} - X_{ec}\right)I_c + \left(jX_{an} - X_{en}\right)I_n + \left(-R_e + j(X_{ae} - X_{ee})\right)I_e \\
V_{bb'} - V_{ee'} &= j\left(X_{ba} - X_{ea}\right)I_a + \left(R_b + j\left(X_{bb} - X_{eb}\right)\right)I_b + j\left(X_{bc} - X_{ec}\right)I_c + j\left(X_{bn} - X_{en}\right)I_n + \left(-R_e + j\left(X_{be} - X_{ee}\right)\right)I_e \\
V_{cc'} - V_{ee'} &= j\left(X_{ca} - X_{ea}\right)I_a + j\left(X_{cb} - X_{eb}\right)I_b + \left(R_c + j\left(X_{cc} - X_{ec}\right)\right)I_c + j\left(X_{cn} - X_{en}\right)I_n + j\left(X_{ce} - X_{ee}\right)I_e \\
V_{nn'} - V_{ee'} &= j\left(X_{na} - X_{ea}\right)I_a + j\left(X_{nb} - X_{eb}\right)I_b + j\left(X_{nc} - X_{ec}\right)I_c + \left(R_n + j\left(X_{nn} - X_{en}\right)\right)I_n + j\left(X_{ne} - X_{ee}\right)I_e
\end{aligned}
$$

(72)

This expression is further simplified by making use of the fact that the currents sum up to zero and

$$
I_e = -I_a - I_b - I_c - I_n \qquad (73)
$$

Substituting equation (73) into equation (72) and equating coefficient in the expressions for voltage drop lead to

$$\begin{aligned}
V_{aa'} - V_{ee'} &= Z'_{aa} I_a + Z'_{ab} I_b + Z'_{ac} I_c + Z'_{an} I_n \\
V_{bb'} - V_{ee'} &= Z'_{ba} I_a + Z'_{bb} I_b + Z'_{bc} I_c + Z'_{bn} I_n \\
V_{cc'} - V_{ee'} &= Z'_{ca} I_a + Z'_{cb} I_b + Z'_{cc} I_c + Z'_{cn} I_n \\
V_{nn'} - V_{ee'} &= Z'_{na} I_a + Z'_{nb} I_b + Z'_{nc} I_c + Z'_{nn} I_n
\end{aligned} \tag{74}$$

where the diagonal terms are given by

$$\begin{aligned}
Z'_{ii} &= \left(R_i + R_e\right) + j\left(X_{ii} + X_{ee} - 2X_{ie}\right) \\
&= \left(R_i + R_e\right) + j0.1213\left(\ln\frac{1}{GMR_i} + \ln\frac{1}{d_e} - 2\ln\frac{1}{D'_e}\right) \\
&= \left(R_i + 0.0953\right) + j0.1213\ln\frac{D_e}{GMR_i} \quad (Ohms\ /\ mile)
\end{aligned} \tag{75}$$

and the off-diagonal terms are given by

$$\begin{aligned}
Z'_{ij} &= R_e + j\left(X_{ij} + X_{ee} - 2X_{je}\right) \\
&= R_e + j0.1213\left(\ln\frac{1}{D_{ij}} + \ln\frac{1}{d_e} - 2\ln\frac{1}{D'_e}\right) \\
&= 0.0953 + j0.1213\ln\frac{D_e}{D_{ij}} \quad (Ohms\ /\ mile)
\end{aligned} \tag{76}$$

If the neutral conductor and the equivalent earth ground conductor are connected in parallel as shown in Fig. 10, then the voltage $V_{nn'} - V_{ee'} = 0$ and the fourth equation in (74) can be solved for current where

$$I_n = \frac{1}{Z'_{nn}}\left(-Z'_{na} I_a - Z'_{nb} I_b - Z'_{nc} I_c\right) \tag{77}$$

Making this substitution back into the first three equations of (74) gives us the final form for the line equations where the equivalent voltage drops for phases a, b, and c are given by

$$\begin{aligned}
V_a &= Z_{aa} I_a + Z_{ab} I_b + Z_{ac} I_c \\
V_b &= Z_{ab} I_a + Z_{bb} I_b + Z_{bc} I_c \\
V_c &= Z_{ac} I_a + Z_{bc} I_b + Z_{cc} I_c
\end{aligned} \tag{78}$$

where

$$Z_{ij} = Z'_{ij} - \frac{Z'_{in} Z'_{nj}}{Z'_{nn}} \tag{79}$$

The manipulation of the impedances in order to eliminate the neutral current is also referred to as Kron reduction. Note that the primed impedances are for the four-conductor circuit model which exists before the neutral wire has been reduced out of the equations.

Impact of Nonideal Grounds

The tie between the neutral wire, load neutral and earth is not ideal. In fact there will exist a ground resistance, R_g , which has a finite value in Ohms, which will be a function of the soil conditions and the grounding method used. Sources also have a ground impedance, but that value is typically very low. Locations for grounding impedances are illustrated in Fig. 11 below. Note that if the grounding resistance is large and the circuit is imbalanced, it may be necessary to simulate this circuit using a four-conductor model and not reduce out the impact of the neutral wire.

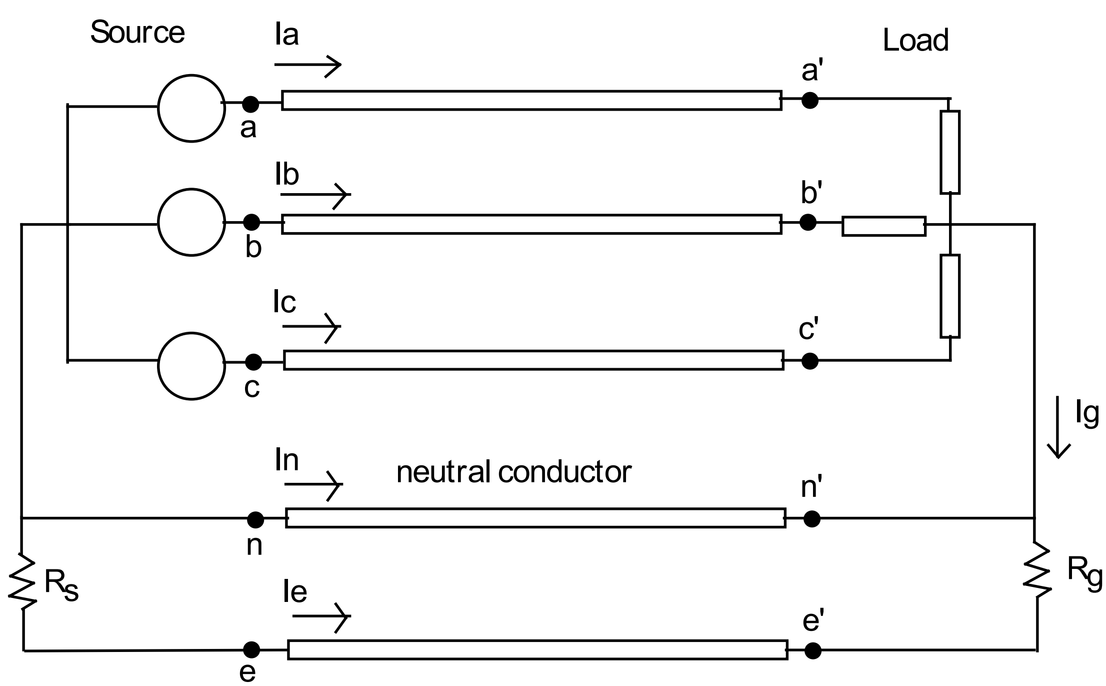

Fig. 11 Four-Wire, Grounded Circuit with Nonideal Grounding

Example Calculations for a Four-Wire Grounded Feeder

Consider an overhead feeder for a three-phase, four-wire, grounded circuit, for which you want to calculate the impedances per unit length. The phase conductors have resistances of 0.306 Ohms per mile with a GMR of 0.0244 feet. The neutral wire has a resistance of 0.592 Ohms per mile and a GMR of 0.00814 feet. The spacing between conductor pairs is as follows

Dab= 3.0 ft. Dbc=4.0 ft. Dac=7.0 ft.
Dan=5.657 ft. Dbn=4.1231 ft. Dcn=5.0 ft.

Assume that the ground has a soil resistivity of 100 Ohm-meters and that the calculations are for 60 Hz, hence

$$D_e = 2160\sqrt{\frac{100}{60}} = 2788.5 \ ft.$$

Applying the formulas for calculating the four-wire impedance matrix

$$\begin{aligned} Z'_{aa} &= R_a + R_e + j0.1213\ln\frac{D_e}{GMR_a} = 0.306 + 0.0953 + j0.1213\ln\frac{2788.5}{.0244} \\ &= 0.4013 + j1.4127 \ (Ohms \ / \ mile) \end{aligned}$$

and

$$\begin{aligned} Z'_{ab} &= R_e + j0.1213\ln\frac{D_e}{D_{ab}} = 0.0953 + j0.1213\ln\frac{2788.5}{3.0} \\ &= 0.0953 + j0.8290 \ (Ohms \ / \ mile) \end{aligned}$$

which when applied to the other terms leads to

$$\left[Z_{abcn}'\right] = \begin{bmatrix} 0.401 + j1.413 & 0.095 + j0.829 & 0.095 + j0.727 & 0.095 + j0.752 \\ 0.095 + j0.829 & 0.401 + j1.413 & 0.095 + j0.795 & 0.095 + j0.791 \\ 0.095 + j0.727 & 0.095 + j0.795 & 0.401 + j1.413 & 0.095 + j0.767 \\ 0.095 + j0.752 & 0.095 + j0.791 & 0.095 + j0.767 & 0.687 + j1.546 \end{bmatrix}$$

Kron reducing out the effect of the neutral wire results in the 3 x 3 impedance matrix where

$$Z_{aa} = Z'_{aa} - \frac{Z'_{an}Z'_{na}}{Z'_{nn}} = .401 + j1.413 - \frac{(.095 + j.752)(.095 + j.752)}{.687 + j1.546} = .458 + j1.078$$

and

$$Z_{ab} = Z'_{ab} - \frac{Z'_{an}Z'_{nb}}{Z'_{nn}} = .095 + j.829 - \frac{(.095 + j.752)(.095 + j.791)}{.687 + j1.546} = .157 + j.478$$

leads to

$$\left[Z_{abc}\right] = \begin{bmatrix} 0.458 + j1.078 & 0.157 + j.478 & 0.153 + j0.385 \\ 0.157 + j0.478 & 0.468 + j1.044 & 0.159 + j0.436 \\ 0.153 + j0.385 & 0.159 + j0.436 & 0.461 + j1.065 \end{bmatrix} \qquad \text{(Ohms/mile)}$$

If one had a balanced load and wanted to do a per-phase calculation, a positive sequence quantity could be obtained by using averages to obtain a self and mutual impedance

$$Z_s = \frac{(0.458 + j1.078) + (0.468 + j1.044) + (0.461 + j1.065)}{3} = 0.462 + j1.062$$
$$Z_m = \frac{(0.157 + j0.478) + (0.153 + j0.385) + (0.159 + j0.436)}{3} = 0.156 + j0.433$$

and using these terms to calculate

$$Z_1 = Z_s - Z_m = 0.306 + j0.630 \qquad \text{(Ohms/mile)}$$

Line Capacitance
A transmission or distribution line is also characterized by capacitance to ground and capacitive coupling between the phases. The shunt capacitance is a consequence of the electric fields created by the line voltages. Capacitive effects are often neglected in short line models for overhead conductors in which the line length is under 50 miles. However for overhead lines with a length greater than 50 miles or for cables, the effect of the shunt capacitance should be included. It is this capacitance which causes the charging current which is observed on unloaded lines. Line capacitance can also become a factor at harmonic frequencies and when electrical transients are being analyzed.

Definitions
The charging current associated with a transmission line is a function of the voltage between phase to ground as well as voltage differences between phases. This charging current is modeled by equivalent capacitances in the line model. The capacitance of a power transmission line is a consequence of the electric fields generated by the sinusoidal conductor voltages. In general, the charging current associated with a circuit is given by:

$$i(t) = \frac{dq(t)}{dt} \qquad \text{(Amperes)} \tag{80}$$

where i represents the current and q the electric charge for a given phase. In the case of a linear circuit, there is a constant which relates the current to the rate in change in voltage, v, such that:

$$i(t) = C\frac{dv(t)}{dt} \tag{81}$$

where C is defined as the capacitance, in units of Farads. The capacitance can also be defined as the ratio of the total charge to the voltage where:

$$C = \frac{q}{V} \text{ (Farads)} \tag{82}$$

When the voltages and currents are sinusoidal, phasor analysis can be applied. In this case equations (80) and (81) can be written as:

$$I = j\omega Q = j\omega CV \tag{83}$$

where I represents the phasor current, Q the phasor conductor charge, V the phasor voltage, and ω is the frequency in radians per second.

Many times when working with multiphase circuits, the total charge will be due to voltages potentials between multiple adjacent conductors. In a two conductor circuit, the capacitive current associated with conductor 1 can be written as

$$I_1 = j\omega(Q_{11} + Q_{12}) = j\omega C_{11}V_1 + j\omega C_{12}V_2 \tag{84}$$

where C_{11} is a self-capacitance associated with the charging current on conductor 1 due to the voltage on conductor 1, V_1 , and C_{12} is the mutual capacitance associated with the charging current on conductor 1 due to the voltage on conductor 2, V_2.

Calculation of Capacitive Reactance

To calculate the capacitance associated with a single conductor, consider the electric field produced by the wire with a charge of q as shown in Fig 12, which is assumed to be far above the earth and other conductors. The electric field due to the charged conductor can be calculated using Gauss's law

$$q = \oint_S \vec{D} \cdot d\vec{a} \tag{85}$$

where,

D = Electric Flux Density (Coulombs/m^2)
q = Charge (Coulombs)
a = Surface Area

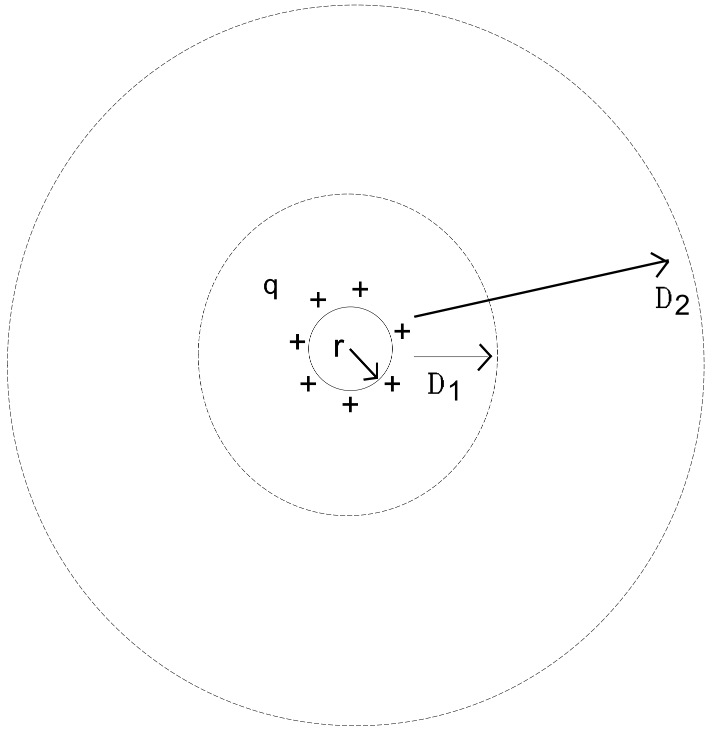

Fig. 12 Electric Field around a Conductor

Assuming that the charge is uniformly distributed over the surface of the conductor, then due to the symmetry of this problem, the electric flux density D is directed radially outward from the conductor. This flux density will be constant on the surface of a cylinder of length l around the conductor with a radius x. In this case, the total charge and flux density are related by:

$$q = D2\pi\, xl \quad \text{(Coulombs)} \tag{86}$$

The electric field at a distance x from the center of the conductor is then given by:

$$E = \frac{D}{\varepsilon} = \frac{q}{\varepsilon 2\pi\, xl} \quad \text{(Volts/Meter)} \tag{87}$$

where the permittivity of free space is defined as $\varepsilon_o = 8.85 \times 10^{-12}$ Farads/Meter.

The voltage difference between any two points in space as a function of the distances from the center of the conductor is now given by:

$$v_{12} = \int_{D_1}^{D_2} E dx = \int_{D_1}^{D_2} \frac{q}{\varepsilon 2\pi\, x} dx = \frac{q}{\varepsilon 2\pi} \ln\left(\frac{D_2}{D_1}\right) \quad \text{(Volts)} \tag{88}$$

where q in this expression represents a line charge per meter.

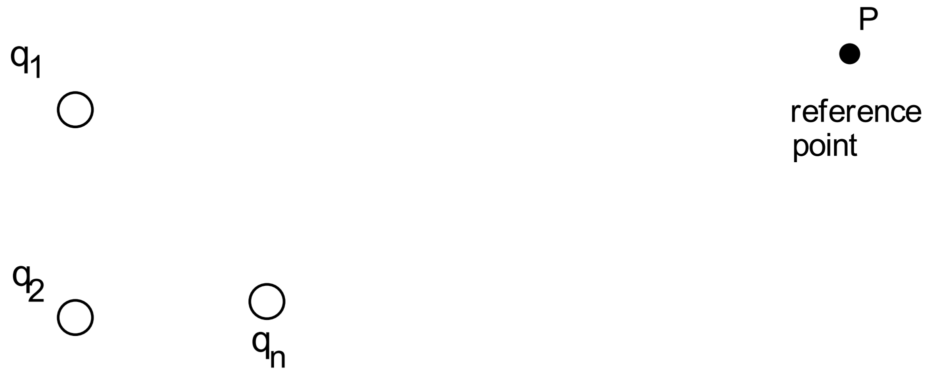

Fig. 13 Multi-Conductor Circuit

For multiple conductors in free space, then since this problem is linear, superposition can be applied. That is, the impact of the first conductor can be superimposed with the impact of the second, third, etc. Suppose that we have a multiconductor system with distances between conductors given by D_{ij} as shown in Fig. 13. The point labeled p in the diagram is a reference point, not a conductor. Also assume that the distance to an earth ground is such that the ground has no effect on the electric field distribution. The potential difference in voltage for wire 1 with respect to point p is given by:

$$v_{1p} = \frac{q_1}{\varepsilon 2\pi} \ln\left(\frac{D_{1p}}{r_1}\right) + \frac{q_2}{\varepsilon 2\pi} \ln\left(\frac{D_{2p}}{D_{12}}\right) + \ldots + \frac{q_n}{\varepsilon 2\pi} \ln\left(\frac{D_{np}}{D_{1n}}\right) \qquad \text{(Volts/meter)} \quad (89)$$

In general, for conductor i, this can be written as:

$$v_{ip} = \frac{1}{2\pi\varepsilon} \sum_{j=1}^{n} q_j \ln\left(\frac{D_{jp}}{D_{ij}}\right) \qquad (90)$$

where

D_{ij} - Distance between conductors i and j, for i≠j
D_{ii} - Radius of conductor i
D_{ip} - Distance between between conductor i and point p.

Rewriting this expression for voltage by splitting it up into two parts,

$$v_{ip} = \frac{1}{2\pi\varepsilon} \sum_{j=1}^{n} q_j \ln\left(\frac{1}{D_{ij}}\right) + \frac{1}{2\pi\varepsilon} \sum_{j=1}^{n} q_j \ln D_{jp} \qquad (91)$$

Note in the limit that as point p is moved at a position an infinite distance from the phase conductors that

$$D_{1p} = D_{2p} = .. = D_{np} \tag{92}$$

For an n conductor system, the charge must sum up to zero, that is

$$\sum_{j=1}^{n} q_j = 0 \tag{93}$$

Hence the second term in the voltage expression drops out and we have an absolute voltage given by:

$$v_i = \frac{1}{2\pi\varepsilon}\sum_{j=1}^{n} q_j \ln\left(\frac{1}{D_{ij}}\right) \tag{94}$$

In the phasor domain, differentiating both sides of the equation gives us

$$V_i = \frac{1}{j\omega 2\pi\varepsilon}\sum_{j=1}^{n} \ln\left(\frac{1}{D_{ij}}\right) I'_j \tag{95}$$

where the prime is used to denote a capacitive current. The capacitive reactance can now be defined as

$$X'_{ij} = \frac{1}{\omega 2\pi\varepsilon}\ln\left(\frac{1}{D_{ij}}\right) \quad \text{(Ohm-meters)} \tag{96}$$

$$X'_{ii} = \frac{1}{\omega 2\pi\varepsilon}\ln\left(\frac{1}{r_i}\right) \quad \text{(Ohm-meters)} \tag{97}$$

Note that in order to obtain the capacitance matrix , the X' matrix needs to be inverted and divided by the frequency in radians/second. One simply can not invert each term separately.

Impact of Earth Plane

If the phase conductors are close to the earth, then the earth will have an impact on the electric field distribution. To approximate the impact, assume that the earth can be modeled as a perfectly conducting medium. The field distribution can be incorporated by using the theory of images. That is, to model a highly conducting ground, use an image conductor on the opposite side of the earth plane, but with an equal but opposite charge.

Let the distance between an actual conductor and an image conductor be represented by D' (primed distance). The voltage given in equation () can now be expressed as

$$v_i = \frac{1}{2\pi\varepsilon}\sum_{j=1}^{n} q_j \ln\left(\frac{1}{D_{ij}}\right) - \frac{1}{2\pi\varepsilon}\sum_{j=1}^{n} q_j \ln\left(\frac{1}{D_{ij}^{'}}\right) = \frac{1}{2\pi\varepsilon}\sum_{j=1}^{n} q_j \ln\left(\frac{D_{ij}^{'}}{D_{ij}}\right) \quad (98)$$

where the capacitive reactance which includes the impact of an earth plane is given by

$$X_{ij}^{'} = \frac{1}{\omega 2\pi\varepsilon}\ln\left(\frac{D_{ij}^{'}}{D_{ij}}\right) \quad \text{(Ohm-meters)} \quad (99)$$

$$X_{ii}^{'} = \frac{1}{\omega 2\pi\varepsilon}\ln\left(\frac{D_{ii}^{'}}{r_i}\right) \quad \text{(Ohm-meters)} \quad (100)$$