Distributed Parameter Line Modeling – Single Phase

Transmission Line Equations

The lumped-parameter representation of a transmission line is sufficient for low frequency calculations. However for higher frequency events, such as those associated with lightning strikes or switching, we will not obtain the proper results. In these situations we will have to model how the inductance and capacitance is actually distributed along the line. For this reason, this next type of model will be referred to as a "distributed-parameter model" as opposed to a lumped-line model.

In order to come up with a model, we will assume that a line is composed of small differential segments of the type shown below in Fig. 6, where Δx is the length of the segment. We are neglecting the resistance, which will give us a lossless line model. The inductance L and capacitance C are given in terms of Henries/meter and Farads/meter. The voltage drop across this element is given by

$$-\Delta v = L\Delta x \frac{\partial i}{\partial t} \tag{1}$$

In the limit as Δx approaches zero, we can rewrite this equation as

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} \tag{2}$$

where we are using partial derivatives in this formulation since voltage and current are functions of both position along the line segment <u>and</u> time.

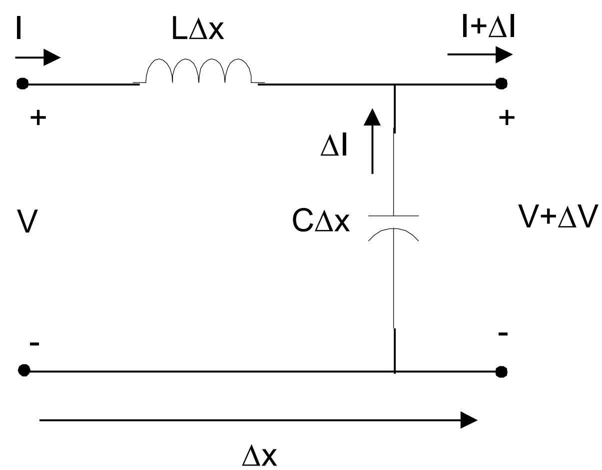

Fig. 6 Line Segment Model

The relationship between the capacitor charging can be approximated by

$$-\Delta i = C\Delta x \frac{\partial v}{\partial t} \tag{3}$$

Again in the limit as Δx approaches zero, we get

$$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \tag{4}$$

We can now eliminate the current as a variable by taking the partial of equation (2) and equation (4) with respect to x and t respectively and substituting one into the other as follows:

From (2) $$\frac{\partial^2 v}{\partial x^2} = -L\frac{\partial i^2}{\partial x \partial t} \tag{5}$$

From (4) $$\frac{\partial^2 i}{\partial x \partial t} = -C\frac{\partial^2 v}{\partial t^2} \tag{6}$$

And finally
$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} \tag{7}$$

Similarly for the current, starting with (4) and (2):
$$\frac{\partial^2 i}{\partial x^2} = -C\frac{\partial^2 v}{\partial x \partial t} \tag{8}$$
$$\frac{\partial^2 v}{\partial x \partial t} = -L\frac{\partial^2 i}{\partial t^2} \tag{9}$$
which leads to
$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \tag{10}$$

Equations (7) and (10) are the two relationships which define voltage and current on a lossless transmission line as a function of distance and time. The solution for current will have the form

$$i(x,t) = f_1(x - \upsilon t) + f_2(x + \upsilon t) \tag{11}$$

where
$$\upsilon = \frac{1}{\sqrt{LC}} \tag{12}$$
We can verify this by substituting (11) into (10) and show that the equality holds. If the solution for current has this particular form then if the voltage and current are related by (2) then
$$\begin{aligned}\frac{\partial v}{\partial x} &= -L\frac{\partial i}{\partial t} = -L\left[\frac{\partial f_1(x-\upsilon t)}{\partial t} + \frac{\partial f_2(x+\upsilon t)}{\partial t}\right] \\ &= L\upsilon\left[f_1'(x-\upsilon t) - f_2'(x+\upsilon t)\right]\end{aligned} \tag{13}$$
and integrating both side with respect to x
$$\begin{aligned}v &= L\upsilon\left[f_1(x-\upsilon t) - f_2(x+\upsilon t)\right] \\ &= Z_o f_1(x-\upsilon t) - Z_o f_2(x+\upsilon t)\end{aligned} \tag{14}$$

Note that the voltage is related to the current by an impedance where

$$Z_o = L\upsilon = \frac{L}{\sqrt{LC}} = \sqrt{\frac{L}{C}} \tag{15}$$

Next we need to discuss what f_1 and f_2 represent. If we were to plot $f_1(x-\upsilon t)$ at times corresponding to t=0 and t=τ, then we would have the two waveforms $f_1(x)$ and $f_1(x-\upsilon\tau)$ as shown in Fig. 7, assuming that the waveform had a triangular shape. It appears from this diagram that $f_1(x-\upsilon t)$ represents a waveform traveling to the right at a velocity of $\upsilon = \frac{1}{\sqrt{LC}}$, in the direction of increasing x. Similarly it can be shown that $f_2(x+\upsilon\tau)$ represents a traveling wave moving to the left in the direction of decreasing x. We would say in this case that $f_1(x-\upsilon\tau)$ represents a forward traveling wave and $f_2(x+\upsilon\tau)$ represents a backward traveling wave. It is also interesting to compare equation (11) to (14) which shows that the forward traveling wave components of the voltage and current differ by a positive impedance while the backward traveling wave components of the voltage and current differ by a negative impedance. The solution also suggests that on a loss-free line, that the corresponding voltage and current have the same waveshape.

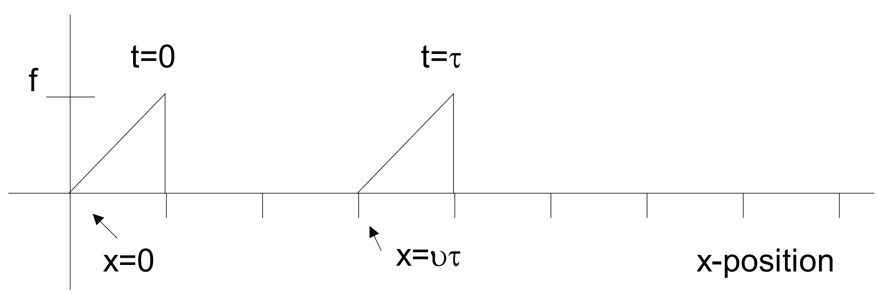

Fig. **7** Traveling Waveform

Reflection and Refraction Coefficients

When a traveling wave on a transmission line hits a discontinuity, due to a connection to a line with a different impedance, then the energy will be divided into a reflected wave and a refracted wave. Superimposing the forward and reflected waveform gives the total line voltage and current. We can compute reflection and refraction coefficients based on the scenario shown in Fig. 8, where cables A and B have different surge impedances. For the forward traveling wave (to the right), the relationship between voltage and current magnitude will be given by:

$$I_1 = \frac{V_1}{Z_A} \tag{16}$$

At the discontinuity between the two cables we will get a reflected wave with magnitude V_2 and a refracted wave with magnitude V_3, where

$$I_2 = -\frac{V_2}{Z_A} \quad ; \quad I_3 = \frac{V_3}{Z_B} \tag{17}$$

The subscript 1 will refer to the forward traveling incident waveform, while subscripts 2 and 3 will refer to the reflected and refracted waveforms.

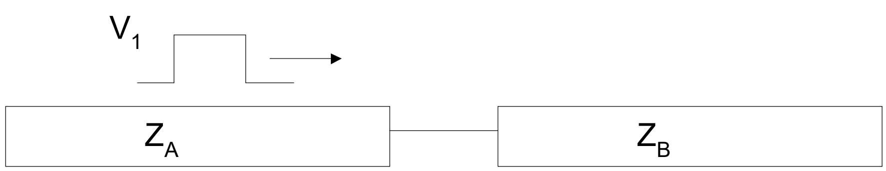

Fig. 8 Discontinuity Caused by Cable Connection

In order for voltage and current to be continuous at the intersections between the two cables we have the following two boundary conditions

$$V_1 + V_2 = V_3 \tag{18}$$

$$I_1 + I_2 = I_3 \tag{19}$$

If we substitute (16) and (17) into (19) we get

$$\frac{V_1}{Z_A} - \frac{V_2}{Z_A} = \frac{V_3}{Z_B} \tag{20}$$

Substituting (18) into (20)

$$\frac{V_1}{Z_A} - \frac{V_2}{Z_A} = \frac{V_1 + V_2}{Z_B} = \frac{V_1}{Z_B} + \frac{V_2}{Z_B} \tag{21}$$

Equating the coefficients for the voltage terms

$$\left(\frac{1}{Z_A}-\frac{1}{Z_B}\right)V_1=\left(\frac{Z_A-Z_B}{Z_AZ_B}\right)V_1=\left(\frac{1}{Z_A}+\frac{1}{Z_B}\right)V_2=\left(\frac{Z_A+Z_B}{Z_AZ_B}\right)V_2 \tag{22}$$

where the ratio between the reflected and incident voltage is called the reflection coefficient and is defined by

$$\Gamma_a=\frac{V_2}{V_1}=\left(\frac{Z_B-Z_A}{Z_B+Z_A}\right) \tag{23}$$

For various values of cable impedances this reflection coefficient can vary between –1 and 1.

To obtain the relationship between the magnitude of the refracted and the incident wave we again substitute (18) into (20), but this time looking at the relationship between V_3 and V_1.

$$\frac{V_1}{Z_A}-\frac{V_3-V_1}{Z_A}=\frac{V_3}{Z_B} \tag{24}$$

Equating the coefficients for the voltage terms again

$$\left(\frac{2}{Z_A}\right)V_1=\left(\frac{1}{Z_B}+\frac{1}{Z_A}\right)V_3=\left(\frac{Z_A+Z_B}{Z_AZ_B}\right)V_3 \tag{25}$$

If we take the ratio of the refracted to the incident voltage we get a refraction coefficient defined by

$$\Gamma_b=\frac{V_3}{V_1}=\left(\frac{2Z_B}{Z_B+Z_A}\right) \tag{26}$$

For various values of impedance this refraction coefficient can vary between 0 and 2.

Suppose that we have a step function voltage applied to a line. A voltage waveform will propagate down the line as shown in Fig. 9. When this step function hits the discontinuity a reflection will be generated with an amplitude of $V_2=\Gamma_a V_1$. This reflection will be a backward traveling wave which when superimposed with the forward traveling wave will give us the total voltage as a function of time and position on the first transmission line. A refracted wave will be injected into the second transmission line with an amplitude given by $V_3=\Gamma_b V_1$. When this refracted wave traveling in the forward direction hits a new discontinuity, then this will produce another reflected waveform.

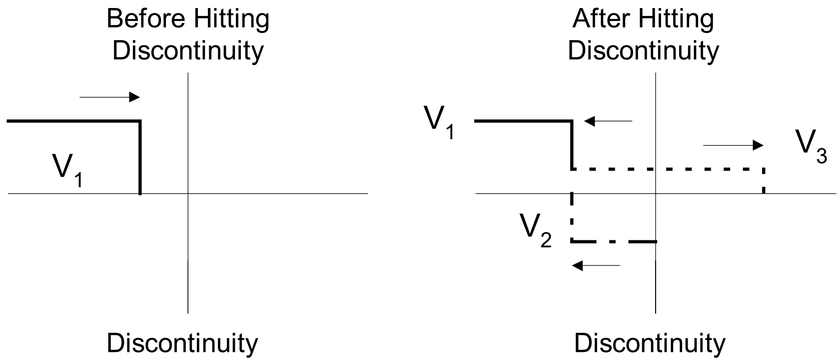

Fig. 9 Voltage Waveforms at Discontinuity

As a quick example of how these coefficients can be applied to a power system, consider the case of an overhead line connected to an underground cable. The overhead cable has an impedance of Z_A= 400 Ohms and the underground cable has an impedance of Z_B= 50 Ohms. Suppose that a surge with an amplitude of 300 kV is traveling down the overhead line, then how much of this voltage will get into the underground cable? This can be determined by looking at the refraction coefficent:

$$\Gamma_b = \left(\frac{2Z_B}{Z_B + Z_A} \right) = \frac{2 \times 50}{50 + 400} = 0.22 \tag{27}$$

Multiplying the magnitude of the incident waveform times this refraction coefficient gives us 0.22x300kV=66kV. An overvoltage on an overhead circuit might result in a flashover, which would not necessarily result in permanent damage. However, an overvoltage on an underground cable would likely damage the insulation and possibly destroy the cable.

It is also possible that instead of having a second transmission line, we have an open circuit instead. In this case the impedance $Z_B = \infty$ which will give us a reflection coefficient of $\Gamma_A = 1$ and a refraction coefficient of $\Gamma_B = 2$. This means that an incident wave traveling down the line will have a reflected wave of the same magnitude, which basically gives us a voltage doubling effect. This is the reason why underground cables normally need surge arrestors at the ends of the circuit.

Lattice Diagrams

A methodology referred to as lattice (or pulse-bounce) diagrams has been developed to track the forward and backward traveling waveforms on a transmission system. A lattice diagram, shown in Fig. 10, has a vertical time scale which is usually set up in units corresponding to the time it takes a waveform to travel from one end of the line to the next. This time factor is defined by

$$\tau = \frac{line\ length}{velocity} = \frac{l}{1/\sqrt{LC}} = l\sqrt{LC} \tag{28}$$

The horizontal axis of the lattice diagram corresponds to position. Using this tool, one can determine the voltage (or current) as a function of time and position. At a given time and position the voltage (current) is found by summing up all the waves that have passed in both directions up to that particular time. If this line had losses, then the waveshapes would be both attenuated and distorted. However for this lossless case the waveshapes maintain their integrity.

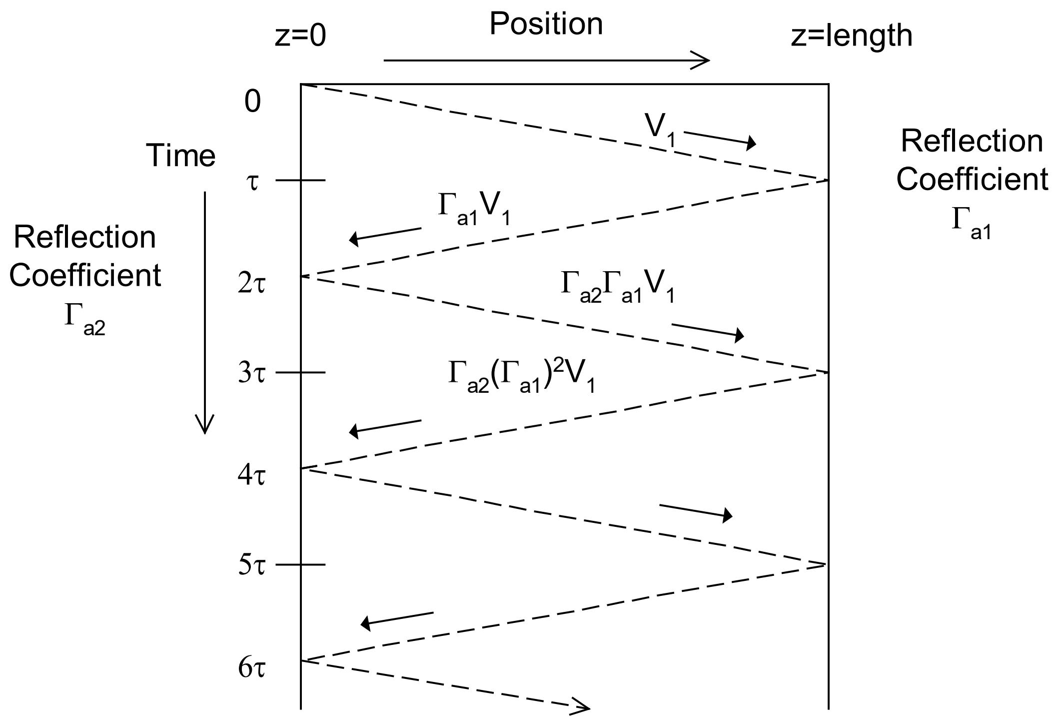

Fig. 10 Lattice Diagram

Sources that inject a waveform into a cable typically have an output impedance. If the voltage is represented by V_s and the source impedance is given by R_s, then for a cable of surge impedance Z_o, the incident waveform magnitude is

$$V_1 = \frac{Z_o}{R_s + Z_o} V_s \tag{29}$$

The source impedance also factors into the computation of the reflection coefficient at the source end of the circuit, where

$$\Gamma_a = \left(\frac{R_s - Z_o}{Z_o + R_s} \right) \tag{30}$$

Sometime resistance is purposefully inserted at the source end of the circuit. This is done to reduce the initial amount of voltage which is injected into the transmission line when it is energized. This element inserted to accomplish this is called a closing resistor and it would have the effect of minimizing transient overvoltages.

EMTP Models

From the solution to the partial differential equations that describe how voltage and current vary on a lossless transmission line, we saw that:

$$i(x,t) = f_1(x - \upsilon t) + f_2(x + \upsilon t) \tag{31}$$

$$v(x,t) = Z_o f_1(x - \upsilon t) - Z_o f_2(x + \upsilon t) \tag{32}$$

where

$$\upsilon = \frac{1}{\sqrt{LC}} \tag{33}$$

$$Z_o = \sqrt{\frac{L}{C}} \tag{34}$$

If we multiply (31) by Z_o and then add the result to (32) we get

$$v(x,t) + Z_o i(x,t) = 2 Z_o f_1(x - \upsilon t) \tag{35}$$

This equation tells us that the sum of the voltage plus impedance times the current is constant with respect to an observer traveling down the transmission line in the direction of increasing x at a velocity υ. Suppose that we are now trying to model the transmission line shown in Fig. 11. The time it takes for a waveform to propagate down the line is given by τ. The term $v + Z_o i$ as seen by an observer at node k at t=0 must be the same as seen by the observer at node m at t= τ.

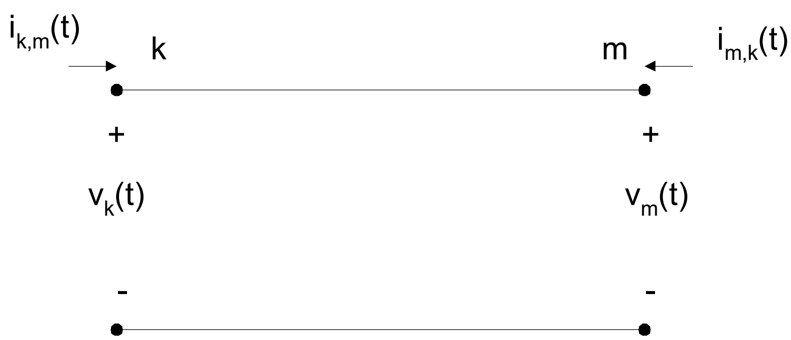

Fig. 11 Transmission Line

We can take this relationship and use it to build an ATP line model, where

$$v_m(t - \tau) + Z_o i_{m,k}(t - \tau) = v_k(t) + Z_o\left(-i_{k,m}(t)\right) \tag{36}$$

Solving for the current at node k, we could also state

$$i_{k,m}(t) = \frac{1}{Z_o} v_k(t) - \frac{1}{Z_o} v_m(t-\tau) - i_{m,k}(t-\tau) = \frac{1}{Z_o} v_k(t) + I_k(t-\tau) \tag{37}$$

$$I_k(t-\tau) = -\frac{1}{Z_o} v_m(t-\tau) - i_{m,k}(t-\tau) \tag{38}$$

Alternately for waveforms traveling in the opposite direction

$$v_k(t-\tau) + Z_o i_{k,m}(t-\tau) = v_m(t) + Z_o\left(-i_{m,k}(t)\right) \tag{39}$$

$$i_{m,k}(t) = \frac{1}{Z_o} v_m(t) - \frac{1}{Z_o} v_k(t-\tau) - i_{k,m}(t-\tau) = \frac{1}{Z_o} v_m(t) + I_m(t-\tau) \tag{40}$$

$$I_m(t-\tau) = -\frac{1}{Z_o} v_k(t-\tau) - i_{k,m}(t-\tau) \tag{41}$$

Equations (37), (38), (40) and (41) are formulated in the same format we used for the network modeling described earlier in the semester. The relationship between voltage and current is given in terms of an impedance and a parallel current source. These equations can also be illustrated in the equivalent model shown in Fig. 12.

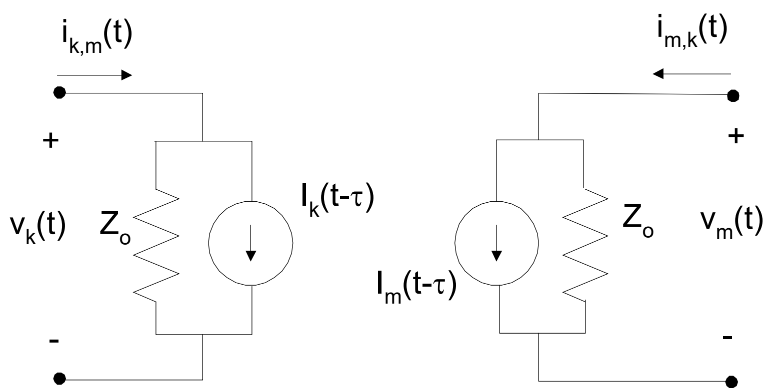

Fig. 12 EMTP Line Model