

Neural networks with backprop library

Justin Le

The *backprop* library performs backpropagation over a *heterogeneous* system of relationships. It offers both an implicit (ad¹-like) and explicit graph building usage style. Let's use it to build neural networks!

Repository source is on github², and so are the rendered unstable docs³.

```
{-# LANGUAGE DeriveGeneric      #-}
{-# LANGUAGE GADTs              #-}
{-# LANGUAGE LambdaCase        #-}
{-# LANGUAGE RankNTypes        #-}
{-# LANGUAGE ScopedTypeVariables #-}
{-# LANGUAGE StandaloneDeriving #-}
{-# LANGUAGE TypeApplications   #-}
{-# LANGUAGE TypeInType        #-}
{-# LANGUAGE TypeOperators      #-}
{-# LANGUAGE ViewPatterns       #-}
{-# OPTIONS_GHC -fno-warn-orphans #-}

import      Data.Functor
import      Data.Kind
import      Data.Maybe
import      Data.Singletons
import      Data.Singletons.Prelude
import      Data.Singletons.TypeLits
import      Data.Type.Combinator
import      Data.Type.Product
import      GHC.Generics           (Generic)
import      Numeric.Backprop
import      Numeric.Backprop.Iso
import      Numeric.LinearAlgebra.Static hiding (dot)
import      System.Random.MWC
import qualified Generics.SOP      as SOP
```

Ops

First, we define values of `Op` for the operations we want to do. `Ops` are bundles of functions packaged with their heterogeneous gradients. For simple numeric functions, *backprop* can derive `Ops` automatically. But for matrix operations, we have to derive them ourselves.

The types help us with matching up the dimensions, but we still need to be careful that our gradients are calculated correctly.

¹<http://hackage.haskell.org/package/ad>

²<https://github.com/mstksg/backprop>

³<https://mstksg.github.io/backprop>

L and R are matrix and vector types from the great *hmatrix* library.

First, matrix-vector multiplication:

```
matVec
  :: (KnownNat m, KnownNat n)
  => Op '[ L m n, R n ] (R m)
matVec = op2' $ \m v -> ( m #> v
                          , \ (fromMaybe 1 -> g) ->
                            (g `outer` v, tr m #> g)
                          )
```

Now, dot products:

```
dot :: KnownNat n
    => Op '[ R n, R n ] Double
dot = op2' $ \x y -> ( x <.> y
                      , \case Nothing -> (y, x)
                        Just g -> (konst g * y, x * konst g)
                      )
```

Polymorphic functions can be easily turned into Ops with op1/op2 etc., but they can also be run directly on graph nodes.

```
logistic :: Floating a => a -> a
logistic x = 1 / (1 + exp (-x))
```

A Simple Complete Example

At this point, we already have enough to train a simple single-hidden-layer neural network:

```
simpleOp
  :: (KnownNat m, KnownNat n, KnownNat o)
  => R m
  -> BPOp s '[ L n m, R n, L o n, R o ] (R o)
simpleOp inp = implicitly $ \ (w1 :< b1 :< w2 :< b2 :< Ø) ->
  let z = logistic $ liftR2 matVec w1 x + b1
  in  logistic $ liftR2 matVec w2 z + b2
where
  x = constRef inp
```

Here, simpleOp is defined in implicit (non-monadic) style, given a tuple of inputs and returning outputs. Now simpleOp can be “run” with the input vectors and parameters (a L n m, R n, L o n, and R o) and calculate the output of the neural net.

```
runSimple
  :: (KnownNat m, KnownNat n, KnownNat o)
  => R m
  -> Tuple '[ L n m, R n, L o n, R o ]
  -> R o
runSimple inp = evalBPOp (simpleOp inp)
```

Alternatively, we can define simpleOp in explicit monadic style, were we specify our graph nodes explicitly. The results should be the same.

```
simpleOpExplicit
  :: (KnownNat m, KnownNat n, KnownNat o)
```

```

=> R m
-> BPOp s '[ L n m, R n, L o n, R o ] (R o)
simpleOpExplicit inp = withInps $ \(w1 :< b1 :< w2 :< b2 :< Ø) -> do
  -- First layer
  y1 <- matVec -$ (w1 :< x1 :< Ø)
  let x2 = logistic (y1 + b1)
  -- Second layer
  y2 <- matVec -$ (w2 :< x2 :< Ø)
  return $ logistic (y2 + b2)
where
  x1 = constRef inp

```

Now, for the magic of *backprop*: the library can now take advantage of the implicit (or explicit) graph and use it to do backpropagation, too!

```

simpleGrad
  :: forall m n o. (KnownNat m, KnownNat n, KnownNat o)
=> R m
-> R o
-> Tuple '[ L n m, R n, L o n, R o ]
-> Tuple '[ L n m, R n, L o n, R o ]
simpleGrad inp targ params = gradBPOp opError params
where
  opError :: BPOp s '[ L n m, R n, L o n, R o ] Double
  opError = do
    res <- simpleOp inp
    -- we explicitly bind err to prevent recomputation
    err <- bindRef $ res - t
    dot -$ (err :< err :< Ø)
  where
    t = constRef targ

```

The result is the gradient of the input tuple's components, with respect to the `Double` result of `opError` (the squared error). We can then use this gradient to do gradient descent.

With Parameter Containers

This method doesn't quite scale, because we might want to make networks with multiple layers and parameterize networks by layers. Let's make some basic container data types to help us organize our types, including a recursive `Network` type that lets us chain multiple layers.

```

data Layer :: Nat -> Nat -> Type where
  Layer :: { _lWeights :: L m n
            , _lBiases  :: R m
            }
          -> Layer n m
  deriving (Show, Generic)

data Network :: Nat -> [Nat] -> Nat -> Type where
  NØ    :: !(Layer a b) -> Network a '[] b
  (:&)  :: !(Layer a b) -> Network b bs c -> Network a (b ': bs) c

```

A `Layer n m` is a layer taking an `n`-vector and returning an `m`-vector. A `Network a '[b, c, d] e`

would be a Network that takes in an a-vector and outputs an e-vector, with hidden layers of sizes b, c, and d.

Isomorphisms

The *backprop* library lets you apply operations on “parts” of data types (like on the weights and biases of a Layer) by using Iso’s (isomorphisms), like the ones from the *lens* library. The library doesn’t depend on lens, but it can use the Isos from the library and also custom-defined ones.

First, we can auto-generate isomorphisms using the *generics-sop* library:

```
instance SOP.Generic (Layer n m)
```

And then can create isomorphisms by hand for the two Network constructors:

```
netExternal :: Iso' (Network a '[] b) (Tuple '[Layer a b])
netExternal = iso (\case NØ x      -> x ::< Ø)
                  (\case I x :< Ø -> NØ x      )

netInternal :: Iso' (Network a (b '[: bs) c) (Tuple '[Layer a b, Network b bs c])
netInternal = iso (\case x :& xs      -> x ::< xs ::< Ø)
                  (\case I x :< I xs :< Ø -> x :& xs      )
```

An Iso' a (Tuple as) means that an a can really just be seen as a tuple of as.

Running a network

Now, we can write the BPOp that represents running the network and getting a result. We pass in a Sing bs (a singleton list of the hidden layer sizes) so that we can “pattern match” on the list and handle the different network constructors differently.

```
netOp
  :: forall s a bs c. (KnownNat a, KnownNat c)
  => Sing bs
  -> BPOp s '[ R a, Network a bs c ] (R c)
netOp sbs = go sbs
where
  go :: forall d es. KnownNat d
    => Sing es
    -> BPOp s '[ R d, Network d es c ] (R c)
  go = \case
    SNil -> withInps $ \(x :< n :< Ø) -> do
      -- peek into the NØ using netExternal iso
      l :< Ø <- netExternal #<~ n
      -- run the 'layerOp' BP, with x and l as inputs
      layerOp ~$ (x :< l :< Ø)
    SNat `SCons` ses -> withInps $ \(x :< n :< Ø) -> withSingI ses $ do
      -- peek into the (:&) using the netInternal iso
      l :< n' :< Ø <- netInternal #<~ n
      -- run the 'layerOp' BP, with x and l as inputs
      z <- layerOp ~$ (x :< l :< Ø)
      -- run the 'go ses' BP, with z and n as inputs
      go ses ~$ (z :< n' :< Ø)
  layerOp
    :: forall d e. (KnownNat d, KnownNat e)
```

```

=> BPOp s '[ R d, Layer d e ] (R e)
layerOp = withInps $ \(x :< l :< ∅) -> do
  -- peek into the layer using the gTuple iso, auto-generated with SOP.Generic
  w :< b :< ∅ <- gTuple #<~ l
  y <- matVec -$ (w :< x :< ∅)
  return $ logistic (y + b)

```

There's some singletons work going on here, but it's fairly standard singletons stuff. From *backprop* specifically, `(#<~)` lets you “split” an input ref with the given iso, and `(~$)` lets you “run” an BP within an BP, by plugging in its inputs.

Note that this library doesn't support truly pattern matching on GADTs, and that we had to pass in `Sing bs` as a reference to the structure of our networks.

Gradient Descent

Now we can do simple gradient descent. Defining an error function:

```

errOp
  :: KnownNat m
  => R m
  -> BRef s rs (R m)
  -> BPOp s rs Double
errOp targ r = do
  err <- bindRef $ r - t
  dot -$ (err :< err :< ∅)
  where
    t = constRef targ

```

And now, we can use *backprop* to generate the gradient, and shift the *Network*! Things are made a bit cleaner from the fact that `Network a bs c` has a `Num` instance, so we can use `(-)` and `(*)` etc.

```

train
  :: (KnownNat a, SingI bs, KnownNat c)
  => Double
  -> R a
  -> R c
  -> Network a bs c
  -> Network a bs c
train r x t n = case backprop (errOp t =<< netOp sing) (x ::< n ::< ∅) of
  (_, _ :< I g :< ∅) -> n - (realToFrac r * g)

```

(`::<`) is cons and `∅` is nil for tuples.)

Main

`main`, which will train on sample data sets, is still in progress! Right now it just generates a random network using the *mwc-random* library and prints each internal layer.

```

main :: IO ()
main = withSystemRandom $ \g -> do
  n <- uniform @(Network 4 '[3,2] 1) g
  void $ traverseNetwork sing (\l -> l <$ print l) n

```

Appendix: Boilerplate

And now for some typeclass instances and boilerplates unrelated to the *backprop* library that makes our custom types easier to use.

```
instance KnownNat n => Variate (R n) where
  uniform g = randomVector <$> uniform g <*> pure Uniform
  uniformR (l, h) g = (\x -> x * (h - l) + l) <$> uniform g

instance (KnownNat m, KnownNat n) => Variate (L m n) where
  uniform g = uniformSample <$> uniform g <*> pure 0 <*> pure 1
  uniformR (l, h) g = (\x -> x * (h - l) + l) <$> uniform g

instance (KnownNat n, KnownNat m) => Variate (Layer n m) where
  uniform g = subtract 1 . (* 2) <$> (Layer <$> uniform g <*> uniform g)
  uniformR (l, h) g = (\x -> x * (h - l) + l) <$> uniform g

instance (KnownNat m, KnownNat n) => Num (Layer n m) where
  Layer w1 b1 + Layer w2 b2 = Layer (w1 + w2) (b1 + b2)
  Layer w1 b1 - Layer w2 b2 = Layer (w1 - w2) (b1 - b2)
  Layer w1 b1 * Layer w2 b2 = Layer (w1 * w2) (b1 * b2)
  abs (Layer w b) = Layer (abs w) (abs b)
  signum (Layer w b) = Layer (signum w) (signum b)
  negate (Layer w b) = Layer (negate w) (negate b)
  fromInteger x = Layer (fromInteger x) (fromInteger x)

instance (KnownNat m, KnownNat n) => Fractional (Layer n m) where
  Layer w1 b1 / Layer w2 b2 = Layer (w1 / w2) (b1 / b2)
  recip (Layer w b) = Layer (recip w) (recip b)
  fromRational x = Layer (fromRational x) (fromRational x)

instance (KnownNat a, SingI bs, KnownNat c) => Variate (Network a bs c) where
  uniform g = genNet sing (uniform g)
  uniformR (l, h) g = (\x -> x * (h - l) + l) <$> uniform g

genNet
  :: forall f a bs c. (Applicative f, KnownNat a, KnownNat c)
  => Sing bs
  -> (forall d e. (KnownNat d, KnownNat e) => f (Layer d e))
  -> f (Network a bs c)

genNet sbs f = go sbs
where
  go :: forall d es. KnownNat d => Sing es -> f (Network d es c)
  go = \case
    SNil -> N0 <$> f
    SNat `SCons` ses -> (:&) <$> f <*> go ses

mapNetwork0
  :: forall a bs c. (KnownNat a, KnownNat c)
  => Sing bs
  -> (forall d e. (KnownNat d, KnownNat e) => Layer d e)
  -> Network a bs c

mapNetwork0 sbs f = getI $ genNet sbs (I f)
```

```

traverseNetwork
  :: forall a bs c f. (KnownNat a, KnownNat c, Applicative f)
  => Sing bs
  -> (forall d e. (KnownNat d, KnownNat e) => Layer d e -> f (Layer d e))
  -> Network a bs c
  -> f (Network a bs c)
traverseNetwork sbs f = go sbs
  where
    go :: forall d es. KnownNat d => Sing es -> Network d es c -> f (Network d es c)
    go = \case
      SNil -> \case
        N0 x -> N0 <$> f x
      SNat `SCons` ses -> \case
        x :& xs -> (:&) <$> f x <*> go ses xs

mapNetwork1
  :: forall a bs c. (KnownNat a, KnownNat c)
  => Sing bs
  -> (forall d e. (KnownNat d, KnownNat e) => Layer d e -> Layer d e)
  -> Network a bs c
  -> Network a bs c
mapNetwork1 sbs f = getI . traverseNetwork sbs (I . f)

mapNetwork2
  :: forall a bs c. (KnownNat a, KnownNat c)
  => Sing bs
  -> (forall d e. (KnownNat d, KnownNat e) => Layer d e -> Layer d e -> Layer d e)
  -> Network a bs c
  -> Network a bs c
  -> Network a bs c
mapNetwork2 sbs f = go sbs
  where
    go :: forall d es. KnownNat d => Sing es -> Network d es c -> Network d es c -> Network d es c
    go = \case
      SNil -> \case
        N0 x -> \case
          N0 y -> N0 (f x y)
      SNat `SCons` ses -> \case
        x :& xs -> \case
          y :& ys -> f x y :& go ses xs ys

instance (KnownNat a, SingI bs, KnownNat c) => Num (Network a bs c) where
  (+)      = mapNetwork2 sing (+)
  (-)      = mapNetwork2 sing (-)
  (*)      = mapNetwork2 sing (*)
  negate   = mapNetwork1 sing negate
  abs      = mapNetwork1 sing abs
  signum   = mapNetwork1 sing signum
  fromInteger x = mapNetwork0 sing (fromInteger x)

instance (KnownNat a, SingI bs, KnownNat c) => Fractional (Network a bs c) where
  (/)      = mapNetwork2 sing (/)
  recip    = mapNetwork1 sing recip

```

```
fromRational x = mapNetwork0 sing (fromRational x)
```