Examen Final Analisis Aplicado

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1) a) sean A p.d., $P_1, \dots, P_l$ vectores conjugados de A. ($P_i^T A P_j = 0$). P.D. P_i, P_j l.i.

supongamos $P_i = \sum_{\substack{j=1 \\ i \neq j}}^{k} a_j P_j$ para $a_j \in \mathbb{R}$.

$\Rightarrow P_i^T A P_i = P_i^T A \left(\sum a_j P_j \right) = \sum_{j \neq i} a_j \left(P_i^T A P_j \right) = 0$ pues $i \neq j$

pero como A es p.d. $P_i^T A P_i > 0$!

$\therefore$ P_i no es c.l. de $\{P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_l\}$ $\forall i$

y $\{P_1, \dots, P_l\}$ es un conjunto l.i.

b) P.D. grad. conj. converge en a lo más n pasos

como $P_1, \dots, P_l$ son l.i., generan $\mathbb{R}^l$, entonces para X^* sol. de $Ax = b$, tenemos que

$X^* - X_0 = \alpha_0 P_0 + \dots + \alpha_l P_l$ para algunos $\alpha_i \in \mathbb{R}$

$\Rightarrow P_K^T A (X^* - X_0) = P_K^T A (\alpha_0 P_0 + \dots + \alpha_l P_l)$

$= \alpha_0 P_K^T A P_0 + \dots + \alpha_l P_K^T A P_l = \alpha_K P_K^T A P_K + 0$

$\Rightarrow \boxed{\alpha_K = \dfrac{P_K^T A (X^* - X_0)}{P_K^T A P_K}}$

I

ahora, por el grad. conjugado

$$x_{k+1} = x_0 + \alpha_0 p_0 + \dots + \alpha_k p_k$$

$$\Rightarrow p_{k+1}^T A x_{k+1} = p_{k+1}^T A(x_0 + \dots + \alpha_k p_k) = p_{k+1}^T A x_0 + \dots + \alpha_k p_{k+1}^T A p_k$$

$$= p_{k+1}^T A x_0 \Rightarrow p_{k+1}^T A(x_{k+1} - x_0) = 0$$

entonces $p_{k+1}^T A(x^* - x_0) = p_{k+1}^T A(x^* - x_{k+1} + x_{k+1} - x_0)$

$$= p_{k+1}^T A(x^* - x_{k+1}) + \cancel{p_{k+1}^T A(x_{k+1} - x_0)} = p_{k+1}^T A(x^* - x_{k+1})$$

$$= p_{k+1}^T A x^* - p_{k+1}^T A x_{k+1} = p_{k+1}^T(b - A x_{k+1}) = p_{k+1}^T r_{k+1}$$

$$= r_{k+1}^T p_k$$

$\therefore$ si usamos $\alpha_k = \dfrac{r_k^T p_k}{p_k^T A p_k}$ se cumple que

$x^* = x_0 + \alpha_0 p_0 + \dots + \alpha_l p_l$ y tenemos que llegar a la solución en máximo l pasos, pues usamos $x_{k+1} = x_k + \alpha_k p_k$

2) a) 2ª cond. de Wolfe fuerte implica

$$s_k^T y_k > 0$$

la condición nos dice que $\nabla f_{k+1}^T s_k \geq c_2 \nabla f_k^T s_k$

como $\nabla f_{k+1}^T = (y_k + \nabla f_k)^T$

$$\Rightarrow (y_k + \nabla f_k)^T s_k \geq c_2 \nabla f_k^T s_k$$

$$\Rightarrow y_k^T s_k \geq c_2 \nabla f_k^T s_k - \nabla f_k^T s_k$$

$$\Rightarrow y_k^T s_k \geq (c_2 - 1) \nabla f_k^T s_k$$

como $c_2 < 1 \Rightarrow c_2 - 1 < 0$

como ∇f_k es dirección de descenso, $\Rightarrow \nabla f_k^T s_k < 0$

$$\therefore (c_2 - 1) \nabla f_k^T s_k > 0$$

$$\therefore \boxed{y_k^T s_k \geq 0}$$

2) b) B_{K+1} y H_{K+1} son inversas $\quad \rho_K = \frac{1}{Y_K^T S_K}$

$B_K H_K = H_K B_K = I,$

sup.

$$B_{K+1} = (I - \rho_K y_K s_K^T) B_K (I - \rho_K s_K y_K^T) + \rho_K y_K y_K^T$$

$$H_{K+1} = (I - \rho_K s_K y_K^T) H_K (I - \rho_K y_K s_K^T) + \rho_K s_K s_K^T$$

$$\Rightarrow B_{K+1} H_{K+1} = (I - \rho_K y_K s_K^T) B_K (I - \rho_K s_K y_K^T)(I - \rho_K s_K y_K^T) H_K \, *$$

$$* (I - \rho_K y_K s_K^T) + \rho_K y_K y_K^T (I - \rho_K y_K s_K^T) H_K (I - \rho_K s_K y_K^T)$$

$$+ \rho_K s_K s_K^T (I - \rho_K s_K y_K^T) B_K (I - \rho_K y_K s_K^T) + \rho_K s_K s_K^T$$

$$+ \rho_K y_K y_K^T$$

veamos que $(I - \rho_K s_K y_K^T)(I - \rho_K s_K y_K^T) = I - \rho_K s_K y_K^T -$

$$- \rho_K s_K y_K^T + \rho_K s_K y_K^T \rho_K s_K y_K^T = I - 2\rho_K s_K y_K^T$$

$$+ \frac{s_K \cancel{(y_K^T s_K)} y_K^T}{\cancel{(y_K^T s_K)} y_K^T s_K} = I - 2\rho_K s_K y_K^T + \rho_K y_K^T s_K$$

$$= I - \rho_K s_K y_K^T$$

como, $(I - \rho_K s_K y_K^T)^2 = (I - \rho_K s_K y_K^T)$, la matriz es una proyección

$$(I - \rho_K y_K s_K^T) B_K (I - \rho_K s_K y_K^T) H_K (I - \rho_K y_K s_K^T)$$

$$= (B_K - \rho_K y_K s_K^T B_K)(H_K - \rho_K s_K y_K^T H_K)(I - \rho_K y_K s_K^T)$$

$$= B_K H_K - \rho_K B_K s_K y_K^T H_K - \rho_K y_K s_K^T B_K H_K + \rho_K^2 y_K s_K^T B_K s_K y_K^T H_K)(I - \rho_K$$

veamos que $(\rho_k y_k y_k^T)(I - \rho_k s_k y_k^T)$

$= \rho_k y_k y_k^T - \rho_k^2 y_k(\cancel{y_k^T s_k}) y_k^T = \rho_k (y_k y_k^T - y_k y_k^T) = \bar{0}$

y veamos que $(\rho_k s_k s_k^T)(I - \rho_k y_k s_k^T)$

$= \rho_k s_k s_k^T - \rho_k^2 s_k(\cancel{s_k^T y_k}) s_k^T = \rho_k (s_k s_k^T - s_k s_k^T) = \bar{0}$

Entonces

$B_{k+1} H_{k+1} = (I - \rho_k y_k s_k^T) B_k (I - \rho_k s_k y_k^T) H_k (I - \rho_k y_k s_k^T)$

$+ \rho_k^2 y_k(\cancel{y_k^T s_k}) s_k^T = (I - \rho_k y_k s_k^T) B_k (I - \rho_k s_k y_k^T) *$

$* H_k (I - \rho_k y_k s_k^T) + (\rho_k y_k s_k^T)$

$$B_{k+1} s_k = y_k$$

$$B_{k+1} H_k = B_{k+1}\left[(I - \rho_k s_k y_k^T) H_k (I - \rho_k y_k s_k^T) + \rho s_k s_k^T\right]$$

$$= B_{k+1}(I - \rho_k s_k y_k^T) H_k (I - \rho_k y_k s_k^T) + B_{k+1} \rho s_k s_k^T$$

$$= \left[B_{k+1} - \rho_k (B_{k+1} s_k) y_k^T\right] H_k (I - \rho_k y_k s_k^T) + \rho y_k s_k^T$$

$$= (B_{k+1} - \rho_k y_k y_k^T) H_k (I - \rho_k y_k s_k^T) + \rho y_k s_k^T$$

$$= (I - \rho_k y_k s_k^T) B_k (I - \rho_k s_k y_k^T) H_k (I - \rho_k y_k s_k^T) + \rho y_k s_k^T$$