

# Exercise 3

Machine Learning I

2A-1.

Short way:

## Prerequisites

The simplest most high-level way is to use predefined rules for matrix differentiation. The rules for this algebra are laid out in the “Matrix Cookbook” on page 8 and page 10.

The book can be found here:

<https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf>

Some important rules from this book (let  $\mathbf{x}$  be a column vector):

$$\begin{aligned}\frac{\partial \mathbf{x}^T \mathbf{a}}{\partial \mathbf{x}} &= \frac{\partial \mathbf{a}^T \mathbf{x}}{\partial \mathbf{x}} = \mathbf{a} \\ \frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T \mathbf{A} \mathbf{x} &= \mathbf{x}^T (\mathbf{A}^T + \mathbf{A}) \\ \frac{\partial \mathbf{a}^T \mathbf{X}^{-1} \mathbf{b}}{\partial \mathbf{X}} &= -\mathbf{X}^{-T} \mathbf{a} \mathbf{b}^T \mathbf{X}^{-T} \\ \frac{\partial \ln |\det \mathbf{X}|}{\partial \mathbf{X}} &= (\mathbf{X}^{-1})^T = (\mathbf{X}^T)^{-1}.\end{aligned}$$

We have:

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \prod_{n=1}^N (2\pi)^{-\frac{d}{2}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{x}_n - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}_n - \boldsymbol{\mu})} = (2\pi)^{-\frac{dN}{2}} |\boldsymbol{\Sigma}|^{-\frac{N}{2}} e^{-\frac{1}{2} \sum_{n=1}^N (\mathbf{x}_n - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}_n - \boldsymbol{\mu})}$$

As usual, we take the logarithmic transform:

$$\ln L(\boldsymbol{\mu}, \boldsymbol{\Lambda}) = -\frac{dN}{2} \log 2\pi + \frac{N}{2} \log |\boldsymbol{\Lambda}| - \frac{1}{2} \sum_{n=1}^N (\mathbf{x}_n - \boldsymbol{\mu})^T \boldsymbol{\Lambda}(\mathbf{x}_n - \boldsymbol{\mu}).$$

Calculation of the mean:

First, let us calculate out the brackets in the previous exponent:

$$\sum_{n=1}^N (\mathbf{x}_n - \boldsymbol{\mu})^T \boldsymbol{\Lambda}(\mathbf{x}_n - \boldsymbol{\mu}) = \sum_{n=1}^N \mathbf{x}_n^T \boldsymbol{\Lambda} \mathbf{x}_n - \mathbf{x}_n^T \boldsymbol{\Lambda} \boldsymbol{\mu} - \boldsymbol{\mu}^T \boldsymbol{\Lambda} \mathbf{x}_n + \boldsymbol{\mu}^T \boldsymbol{\Lambda} \boldsymbol{\mu}.$$

Now apply the rules of matrix differentiation algebra:

$$\begin{aligned}
 \frac{\partial}{\partial \boldsymbol{\mu}} \ln L(\boldsymbol{\mu}, \boldsymbol{\Lambda}) &= -\frac{1}{2} \frac{\partial}{\partial \boldsymbol{\mu}} \left[ \sum_{n=1}^N \mathbf{x}_n^T \boldsymbol{\Lambda} \mathbf{x}_n - \mathbf{x}_n^T \boldsymbol{\Lambda} \boldsymbol{\mu} - \boldsymbol{\mu}^T \boldsymbol{\Lambda} \mathbf{x}_n + \boldsymbol{\mu}^T \boldsymbol{\Lambda} \boldsymbol{\mu} \right] \\
 &= -\frac{1}{2} \left[ \sum_{n=1}^N -\mathbf{x}_n^T \boldsymbol{\Lambda} - \mathbf{x}_n^T \boldsymbol{\Lambda} + \boldsymbol{\mu}^T \left( \underbrace{\boldsymbol{\Lambda}^T + \boldsymbol{\Lambda}}_{=2\boldsymbol{\Lambda} \text{ due symmetry}} \right) \right] \\
 &= -\frac{1}{2} \left[ \sum_{n=1}^N -2\mathbf{x}_n^T \boldsymbol{\Lambda} + 2\boldsymbol{\mu}^T \boldsymbol{\Lambda} \right] \\
 &= \sum_{n=1}^N [\mathbf{x}_n^T \boldsymbol{\Lambda} - \boldsymbol{\mu}^T \boldsymbol{\Lambda}].
 \end{aligned}$$

Now we can solve for  $\boldsymbol{\mu}$ :

$$\begin{aligned}
 \sum_{n=1}^N [\mathbf{x}_n^T \boldsymbol{\Lambda} - \boldsymbol{\mu}^T \boldsymbol{\Lambda}] &= 0 \\
 \Leftrightarrow \sum_{n=1}^N \mathbf{x}_n^T \boldsymbol{\Lambda} &= \sum_{n=1}^N \boldsymbol{\mu}^T \boldsymbol{\Lambda} \\
 \Leftrightarrow \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n^T &= \boldsymbol{\mu}^T \\
 \Leftrightarrow \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n &= \boldsymbol{\mu}
 \end{aligned}$$

Calculation of Variance:

This one is more involved. But by just utilizing the rules of the Matrix Cookbook, we quickly get to a solution. Note: Now I use  $\boldsymbol{\Sigma}$  instead of  $\boldsymbol{\Lambda}$  for convenience.

$$\begin{aligned}
 \frac{\partial}{\partial \boldsymbol{\Sigma}} \ln L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) &= -\frac{N}{2} \boldsymbol{\Sigma}^{-T} - \frac{1}{2} \sum_{n=1}^N [-\boldsymbol{\Sigma}^{-T} (\mathbf{x}_n - \boldsymbol{\mu})(\mathbf{x}_n - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-T}] \\
 &= -\frac{N}{2} \underbrace{\boldsymbol{\Sigma}^{-1}}_{\text{symmetry}} - \frac{1}{2} \sum_{n=1}^N \left[ -\underbrace{\boldsymbol{\Sigma}^{-1}}_{\text{symmetry}} (\mathbf{x}_n - \boldsymbol{\mu})(\mathbf{x}_n - \boldsymbol{\mu})^T \underbrace{\boldsymbol{\Sigma}^{-1}}_{\text{symmetry}} \right]
 \end{aligned}$$

Now solve for  $\boldsymbol{\Sigma}^{-1}$ :

$$\begin{aligned}
& -\frac{N}{2} \underbrace{\Sigma^{-1}}_{\text{symmetry}} - \frac{1}{2} \sum_{n=1}^N \left[ -\underbrace{\Sigma^{-1}}_{\text{symmetry}} (\mathbf{x}_n - \boldsymbol{\mu})(\mathbf{x}_n - \boldsymbol{\mu})^T \underbrace{\Sigma^{-1}}_{\text{symmetry}} \right] = 0 \quad | \cdot \Sigma \text{ left } | \cdot 2 \\
\Rightarrow & -N + \sum_{n=1}^N [(\mathbf{x}_n - \boldsymbol{\mu})(\mathbf{x}_n - \boldsymbol{\mu})^T \Sigma^{-1}] = 0 \quad | \cdot \Sigma \text{ right } \\
\Rightarrow & -N\Sigma + \sum_{n=1}^N [(\mathbf{x}_n - \boldsymbol{\mu})(\mathbf{x}_n - \boldsymbol{\mu})^T] = 0 \\
\Rightarrow & \frac{1}{N} \sum_{n=1}^N [(\mathbf{x}_n - \boldsymbol{\mu})(\mathbf{x}_n - \boldsymbol{\mu})^T] = \Sigma
\end{aligned}$$

## 2A-3.

Direct calculation:

$$N(x; \mu_1, \sigma_1^2) N(x; \mu_2, \sigma_2^2) = (2\pi)^{-\frac{1}{2}} \sigma_1 \sigma_2 e^{-\frac{1}{2} \left[ \frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(x-\mu_2)^2}{\sigma_2^2} \right]}.$$

Calculation of the exponent and completing the square:

$$\begin{aligned}
\frac{(x - \mu_1)^2}{\sigma_1^2} + \frac{(x - \mu_2)^2}{\sigma_2^2} &= \frac{x^2 - 2\mu_1 x + \mu_1^2}{\sigma_1^2} + \frac{x^2 - 2\mu_2 x + \mu_2^2}{\sigma_2^2} \\
&= \frac{(\sigma_2^2 + \sigma_1^2)x^2 - 2x(\sigma_2^2\mu_1 + \sigma_1^2\mu_2) + \sigma_2^2\mu_1^2 + \sigma_1^2\mu_2^2}{\sigma_1^2\sigma_2^2} \\
&= \frac{x^2 - 2x \frac{(\sigma_2^2\mu_1 + \sigma_1^2\mu_2)}{(\sigma_2^2 + \sigma_1^2)} + \frac{\sigma_2^2\mu_1^2}{(\sigma_2^2 + \sigma_1^2)} + \frac{\sigma_1^2\mu_2^2}{(\sigma_2^2 + \sigma_1^2)}}{\frac{\sigma_1^2\sigma_2^2}{(\sigma_2^2 + \sigma_1^2)}} \\
&\propto \frac{\left( x - \frac{(\sigma_2^2\mu_1 + \sigma_1^2\mu_2)}{(\sigma_2^2 + \sigma_1^2)} \right)^2}{\frac{\sigma_1^2\sigma_2^2}{(\sigma_2^2 + \sigma_1^2)}}
\end{aligned}$$

Let

$$\begin{aligned}
\sigma^2 &= \frac{\sigma_1^2\sigma_2^2}{(\sigma_2^2 + \sigma_1^2)} \\
\mu &= \frac{(\sigma_2^2\mu_1 + \sigma_1^2\mu_2)}{(\sigma_2^2 + \sigma_1^2)}
\end{aligned}$$

Now insert into the original equation:

$$\begin{aligned}
N(x; \mu_1, \sigma_1^2)N(x; \mu_2, \sigma_2^2) &\propto (2\pi)^{-\frac{1}{2}}\sigma_1\sigma_2 e^{-\frac{1}{2}\left[\frac{(x-\mu)^2}{\sigma^2}\right]} \\
&\propto (2\pi\sigma^2)^{-\frac{1}{2}}e^{-\frac{1}{2}\left[\frac{(x-\mu)^2}{\sigma^2}\right]} \\
&= N\left(\frac{(\sigma_2^2\mu_1 + \sigma_1^2\mu_2)}{(\sigma_2^2 + \sigma_1^2)}, \frac{\sigma_1^2\sigma_2^2}{(\sigma_2^2 + \sigma_1^2)}\right)
\end{aligned}$$

## 2 A-4.

As usual, we are taking the partial derivative  $\frac{\partial SE(\mathbf{w})}{\partial w_i}$ :

$$\frac{\partial}{\partial w_i} 0.5 \sum_{n=1}^N r_n (w^T \phi(x_n) - t_n)^2 = \sum_{n=1}^N \phi_i(x_n) r_n (w^T \phi(x_n) - t_n).$$

If we tried to solve this for  $w_i$ , we would encounter dependencies on the other  $w_j$ 's. This is an indicator that it would make sense to convert the equation into matrix form and treat the entire derivative as a solution to a system of  $d$  equations.

Converting each term step by step leads to:

$$\sum_{n=1}^N \phi_i(x_n) r_n (w^T \phi(x_n) - t_n) = (r_1 \phi_i(x_1), \dots, r_n \phi_i(x_n)) [(\mathbf{w}^T \Phi^T)^T - \mathbf{t}].$$

where

$$\Phi = \begin{pmatrix} \phi_1(x_1) & \dots & \phi_n(x_1) \\ \dots & \dots & \dots \\ \phi_1(x_n) & \dots & \phi_n(x_n) \end{pmatrix}$$

is the design matrix. For the entire vector, this would lead to:

$$\frac{\partial SE(\mathbf{w})}{\partial \mathbf{w}} = \begin{pmatrix} r_1 \phi_1(x_1) & \dots & r_n \phi_1(x_n) \\ \dots & \dots & \dots \\ r_1 \phi_d(x_1) & \dots & r_n \phi_d(x_n) \end{pmatrix} [(\mathbf{w}^T \Phi^T)^T - \mathbf{t}] = \mathbf{0}.$$

Solving for  $\mathbf{w}$  and utilizing the Penrose inverse  $A^+$ :

$$\begin{aligned}
\mathbf{w} &= \left[ \begin{pmatrix} r_1 \phi_1(x_1) & \dots & r_n \phi_1(x_n) \\ \dots & \dots & \dots \\ r_1 \phi_d(x_1) & \dots & r_n \phi_d(x_n) \end{pmatrix} \Phi \right]^+ \begin{pmatrix} r_1 \phi_1(x_1) & \dots & r_n \phi_1(x_n) \\ \dots & \dots & \dots \\ r_1 \phi_d(x_1) & \dots & r_n \phi_d(x_n) \end{pmatrix} \mathbf{t} \\
&= \left( \sum_{n=1}^N r_n \phi(x_n) \phi(x_n)^T \right)^+ \left( \sum_{n=1}^N r_n t_n \phi(x_n) \right).
\end{aligned}$$