Asset Pricing with Time-Consistent Mean-variance Portfolio Selection

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Harry Markowitz's Mean-Variance Portfolio Selection

A quantitative idea that investors aim to maximise returns while minimising risk.

For a portfolio, we denote the return and weight on the i^{th} asset by R_i and w_i respectly, and R_P for the return on portfolio. Then the *mean-variance* is referred to *expected returns and risk*.

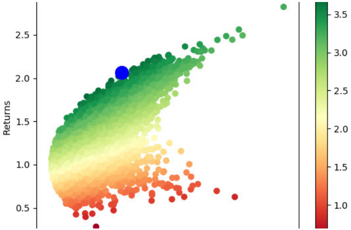
$$E(R_P) = \sum_{i=1}^{N} w_i E(R_i)$$

The variance on a portfolio is

$$var(R_P) = E(R_P - E(R_P))^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j cov(R_i, R_j)$$

Introduction

Markowitz[1] provides a way for investors to optimise their portfolios by taking advantage of diversified investments.
By Monte Carlo simulations and under the mean-variance portfolio selection framework, the investor's problem is to choose the weights w_i to maximise the expected return $E(R)$ and minimise the variance $var(R)$, taking into account their risk tolerance. This often leads to a trade-off, as assets with higher expected returns also tend to have higher risk.



[1] Harry Markowitz. "Portfolio selection". In: *The Journal of Finance* 7.1 (1952), pp. 77–91.

Introduction

Bellman's optimality principle

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.[a]

[a] R.E. Bellman. *Dynamic Programming*. [1957]. Dover, 2003. ISBN: 0-486-42809-5.

Problems faced: Mean-variance criterion does not satisfy Bellman's optimality principle, due to its lack of time consistency.

Time consistency: It refers to the property that an optimal plan made at one point in time remains optimal at any later point in time.

Introduction

Why fails?

1. A strategy may be optimal at the time of decision-making, but it may no longer be optimal in a subsequent period.
2. Future expected returns and variances could be different from the current estimates, then mean-variance trade-off could also change.
3. Mean and variance of the total return cannot be simply broken down into the means and variances of the returns at each stage, this leads to the violation of Bellman's optimality principle.

Solutions: Developing a corresponding multi-period formulation using these time-consistent formulations, based on a general semi-martingale setting.

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General definitions and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of completeness and right-continuity, where $T \in (0, \infty)$ is a fixed and finite time horizon.

Definition (Martingale): A stochastic process X is a martingale if it is (i) $\mathcal{F}_t$-measurable, for all $t \in \{0, ..., T\}$,
(ii) $E[|X_t|] < \infty$, for all $t \in \{0, ..., T\}$,
(iii) $E[X_t|\mathcal{F}_s] = X_s$, P-a.s., for all $s, t \in \{0, ..., T\}$ such that $s < t$.

Definition (Semi-Martingale): A stochastic process $X = \{X_t\}_{t \geq 0}$ is called a semi-martingale if it can be decomposed as a sum of a local martingale M and a finite variation process A, both starting at zero:

$$X_t = M_t + A_t, \quad t \geq 0.$$

Definition (Doob Decomposition): Every integrable process $X = \{X_t\}_{t \geq 0}$ can be decomposed uniquely as $X = M + A$, where $M = \{M_t\}_{t \geq 0}$ is a martingale and $A = \{A_t\}_{t \geq 0}$ is a predictable, finite variation process.

Definition (GTW Decomposition): For a semi-martingale $X = \{X_t\}_{t \geq 0}$, the GTW (Galtchouk-Kunita-Watanabe) decomposition is of the form:

$$X_t = X_0 + M_t + A_t$$

where $M = \{M_t\}_{t \geq 0}$ is a local martingale, $A = \{A_t\}_{t \geq 0}$ is a predictable finite variation process and X_0 is the initial value of X.

Problem setting

Let Θ be the set of trading strategies where

$$\Theta := \Theta_S := \{v \in L(S) | \int vdS \in \mathcal{H}^2(P)\}$$

Let V_t be the total wealth at time t, a trading strategy $v \in \Theta$ is self-financing over the period $[0, T]$ if

$$V_t(V_0, v) = V_0 + \int_0^t v_u dS_u$$

where V_0 is the initial capital. Then the *mean-Variance portfolio selection (MVPS)* with risk aversion γ can be formulated as the problem to

$$\text{maximise } E[V_T(V_0, v)] - \frac{\gamma}{2}\text{Var}[V_T(V_0, v)] \text{ over all } v \in \Theta$$

General definitions and preliminaries

Now we consider two cases, under a discrete time setting.

- There is one agent who is maximising his mean-Variance preferences from wealth and also receives a random endowment given by a square-integrable random Variable H. Then the total wealth is given by $V_t(x, v) = x + v\Delta S_t + H$
- There are J agents that are each maximising their mean-Variance preferences given their risk aversion $\gamma^j > 0$ and random endowment H^j. This allows to compute the optimal trading strategy v^j for each agent $j = 1, \dots, J$. We assume that the price is at each time set such that the optimal strategies v^j clear the market, that is, $\sum_{j=1}^{J} \hat{v}^j = 0$ for each $t = 0, \dots, T$. Total wealth is given by $V^j_t(x, v) = x + v^j \Delta S_t + H^j$

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Objective: An agent with risk aversion γ is trying to maximise

$$E[V_T(V_0, v)] - \frac{\gamma}{2}\mathrm{Var}[V_T(V_0, v)] \text{ over all } v \tag{1}$$

where the total wealth at time t is $V_t(x, v) = x + v\Delta S_t + H$

Based on the previous works by Christoph Czichowsky. "Time-consistent mean-variance portfolio selection in discrete and continuous time". In: *Finance and Stochastics* 17.2 (2012), pp. 227–271, to study (1) as a dynamic optimisation problem, one can consider

$$\text{maximise } U_t(v) = E[V_T(V_0, v)|\mathcal{F}_t] - \frac{\gamma}{2}\mathrm{Var}[V_T(V_0, v)|\mathcal{F}_t] \text{ over all } v \in \Theta_t \tag{2}$$

Is (2) time consistent?

Problems: When applying the solution v to equation (1) within the interval $[0, t]$, **we are optimising a particular criterion (such as maximising returns or minimising risk)** for that specific period. In this period, the solution v is the optimal strategy.

However, when we try to find a new optimal strategy by maximising equation (2) over all $v \in \Theta_t(v)$, **this optimisation task considers different information or constraints**, leading to a different optimal strategy for the interval $(t, T]$. Since the conditionally optimal strategy is based on the conditions present in the interval $(t, T]$, which may have changed from the conditions in the initial period $[0, t]$.

The resulting strategy, therefore, differs from v because the optimisation was carried out under different conditions. This reflects the time inconsistency issue in the MVPS problem, where optimal strategies can change over time due to evolving conditions, preferences, or available information.

Mathematical explantion: The time inconsistency of equation (2) arises from the conditional mean-variance criterion, $U_t(\cdot)$, which includes a conditional variance term that depends on future decisions. To see this, we write

$$
\begin{aligned}
U_t(v) &= E[V_T(V_0, v)|\mathcal{F}_t] - \frac{\gamma}{2}\mathrm{Var}[V_T(V_0, v)|\mathcal{F}_t] \\
&= E[V_T(V_0, v)|\mathcal{F}_t] - \frac{\gamma}{2}(E[\mathrm{Var}[V_T(V_0, v)|\mathcal{F}_{t+h}]|\mathcal{F}_t] \\
&+ \mathrm{Var}[E[V_{t+h}(V_0, v) + \int_{t+h}^{T} v dS + H|\mathcal{F}_{t+h}]|\mathcal{F}_t]]) \\
&= E[U_{t+h}(v)|\mathcal{F}_t] - \frac{\gamma}{2}\mathrm{Var}[E[\int_{t+h}^{T} v dS|\mathcal{F}_{t+h}] + V_{t+h}(V_0, v) + H|\mathcal{F}_t]])
\end{aligned}
$$

This incentivises the investor to deviate from the optimal strategy at later times, which violates the principle of time consistency.

One agent settings

Possible Solution: Strotz[2] suggest limiting the choice of strategies to only those which the investor will actually follow, as opposed to considering all possible strategies. In essence, instead of globally optimising for the entire period $(t, T]$, the optimisation is performed locally on infinitesimally small time intervals $(t, t + dt]$, progressing backward from T.

By following this rule, the investor, at any point t in $[0, T]$, chooses the strategy for the interval $(t, t + dt]$ that he determines to be optimal for his criterion $U_t(\cdot)$ at that time. Consequently, he has no incentive to deviate from this locally optimal strategy, leading to time-consistent behaviour.

[2] Robert H. Strotz. "Myopia and inconsistency in dynamic utility maximization". In: *Review of Economic Studies* 23.3 (1956), pp. 165–180.

One agent settings

Main work: In discrete time and 1-dimension Consider a trading period $T \in N$, where trades occur at fixed times $k = 0, 1, \dots, T$. At each time point k, the trader decides on the number of shares, v_{k+1}, to hold over the next time period $(k, k+1]$.

locally mean-variance efficient (LMVE)

Let ψ be a strategy and $k \in \{1, \dots, T\}$. A local perturbation of at time k is any strategy $v \in \Theta$ with $v_j = \psi_j$ for all $j \neq k$. We call a trading strategy $\hat{v} \in \Theta$ LMVE if $U_{k-1}(\hat{v}) \geq U_{k-1}(\hat{v} + \delta \mathbb{1}_k)$ P-a.s, $\forall k \in \{1, \dots, T\}$ and any $\delta \in \Theta$

Lemma 1: A strategy $v \in \Theta$ is LMVE if and only if it satisfies

$$\hat{v}_k = \frac{E[\Delta S_k | \mathcal{F}_{k-1}]}{\gamma \mathrm{Var}[\Delta S_k | \mathcal{F}_{k-1}]} - \frac{\mathrm{Cov}(\Delta S_k, \sum_{i=k+1}^{T} \hat{v}_i \Delta S_i + H | \mathcal{F}_{k-1})}{\mathrm{Var}[\Delta S_k | \mathcal{F}_{k-1}]}$$

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J agents settings: Useful facts

In the sense of Doob decomposition, for some time t, we have

$$S_t = M_t + A_t$$

$$S_t = \sum_i^t (S_i - E[S_i|\mathcal{F}_{i-1}]) + \sum_i^t (E[S_i|\mathcal{F}_{i-1}] - S_{i-1}) + S_0$$

where we set

$$M_t = \sum_i^t (S_i - E[S_i|\mathcal{F}_{i-1}]) + S_0$$

$$A_t = \sum_i^t (E[S_i|\mathcal{F}_{i-1}] - S_{i-1})$$

J agents settings: Useful facts

We can obtain that

$$\Delta A_t = \Delta S_t - \Delta M_t \tag{3}$$

$$\Delta A_t = A_t - A_{t-1} = E[S_t|\mathcal{F}_{t-1}] - S_{t-1} = E[\Delta S_t|\mathcal{F}_{t-1}] \tag{4}$$

Since M is a martingale, we have

$$\begin{aligned} E[\Delta M_t|\mathcal{F}_{t-1}] &= E[\Delta S_t - \Delta A_t|\mathcal{F}_{t-1}] \\ &\overset{(4)}{=} E[\Delta S_t - E[\Delta S_t|\mathcal{F}_{t-1}]|\mathcal{F}_{t-1}] \\ &= E[S_t - E[S_t|\mathcal{F}_{t-1}]|\mathcal{F}_{t-1}] \\ &= 0 \end{aligned} \tag{5}$$

We only consider the MVPS in a small period $(t-1, t]$

$$
\begin{aligned}
f(v^j) &= E[V_t(x, v_t^j)|\mathcal{F}_{t-1}] - \frac{\gamma}{2}\text{Var}[V_t(x, v_t^j)|\mathcal{F}_{t-1}] \\
&= E[V_{t-1} + v_t^j \Delta S_t + H_j] - \frac{\gamma}{2}\text{Var}[V_{t-1} + v_t^j \Delta S_t + H_j]
\end{aligned}
$$

where V_t is the capital at time t, v_t^j is the strategy of j^{th} agent at time t. By finding the first order derivative and setting to 0, we can obtain:

$$
\hat{v}_t^j = \frac{E[\Delta S_t|\mathcal{F}_{t-1}] - \gamma_j \text{Cov}(H_j, S_t|\mathcal{F}_{t-1})}{\gamma_j \text{Var}[S_t|\mathcal{F}_{t-1}]} \tag{6}
$$

J agents settings: equilibrium prices

We assume that the price is at each time set such that the optimal strategies v^j clear the market, i.e $\sum_{j=1}^J \hat{v}^j = 0$ for each t, that is,

$$\sum_{j=1}^{J} \frac{E[\Delta S_t|\mathcal{F}_{t-1}] - \gamma_j \mathrm{Cov}(H_j, S_t|\mathcal{F}_{t-1})}{\gamma_j \mathrm{Var}[S_t|\mathcal{F}_{t-1}]} = 0$$

Hence we obtain

$$S_{t-1} = E[S_t|\mathcal{F}_{t-1}] - \mathrm{Cov}\left(\frac{\sum_{j=1}^J H_j}{\sum_{j=1}^J \gamma_j^{-1}}, S_t|\mathcal{F}_{t-1}\right) \tag{7}$$

Take equation (7) to back to equation (6), we obtain

$$\hat{v}_t^j = \frac{\mathrm{Cov}\left(\frac{\sum_{j=1}^J H_j}{\sum_{j=1}^J \gamma_j^{-1}} - \gamma_j H_j, S_t|\mathcal{F}_{t-1}\right)}{\gamma_j \mathrm{Var}[S_t|\mathcal{F}_{t-1}]} \tag{8}$$

J agents settings: minimal martingale measure

We will provide a probability measure which can take equilibrium prices to a martingale.

$$S_{t-1} = E[S_t|\mathcal{F}_{t-1}] - \mathrm{Cov}(\frac{\sum_{j=1}^J H_j}{\sum_{j=1}^J \gamma_j^{-1}}, S_t|\mathcal{F}_{t-1})$$

let

$$H = \frac{\sum_{j=1}^J H_j}{\sum_{j=1}^J \gamma_j^{-1}} \text{ and } \hat{H}_t = E[H|\mathcal{F}_t]$$

Then

$$S_t = E[S_{t+1}|\mathcal{F}_t] - \mathrm{Cov}(H, S_{t+1}|\mathcal{F}_t) \tag{9}$$

It can be implied that

$$E[\hat{H}_t|\mathcal{F}_{t-1}] = E[E[H|\mathcal{F}_t]|\mathcal{F}_{t-1}] = E[H|\mathcal{F}_{t-1}] = \hat{H}_{t-1}$$

that is, $\hat{H}_t$ is a P-martingale.

J agents settings: minimal martingale measure

Let

$$\hat{Z}_t = \prod_{s=1}^{t}(1 - \Delta \hat{H}_s) \text{ and } \hat{Z}_T = \frac{d\hat{P}}{dP}$$

We claim that $\hat{P}$ is the minimal martingale measure[3] and

$$E^{\hat{P}}[S_{t+1}|\mathcal{F}_t] = S_t$$

Proof:
Some useful properties.

$$\text{Cov}(\hat{H}_{t+1} - H, S_{t+1}|\mathcal{F}_t) = 0 \tag{10}$$

$$E[\Delta \hat{H}_{t+1} S_{t+1}|\mathcal{F}_t] = \text{Cov}(\hat{H}_{t+1}, S_{t+1}|\mathcal{F}_t) \tag{11}$$

[3] **schweizer1995minimal**.

$$
\begin{aligned}
E^{\hat{P}}[S_{t+1}|\mathcal{F}_t] &\overset{Bayes}{=} E[\frac{\hat{Z}_T}{\hat{Z}_t}S_{t+1}|\mathcal{F}_t] \overset{tp}{=} E[E[\frac{\hat{Z}_T}{\hat{Z}_t}S_{t+1}|\mathcal{F}_{t+1}]|\mathcal{F}_t] \\
&= E[E[\frac{\hat{Z}_{t+1}}{\hat{Z}_t}S_{t+1}|\mathcal{F}_{t+1}]|\mathcal{F}_t] = E[\frac{\hat{Z}_{t+1}}{\hat{Z}_t}S_{t+1}|\mathcal{F}_t] \\
&= E[(1-\Delta\hat{H}_{t+1})S_{t+1}|\mathcal{F}_t] = E[S_{t+1}|\mathcal{F}_t] - E[\Delta\hat{H}_{t+1}S_{t+1}|\mathcal{F}_t]
\end{aligned}
$$

Take equation (9), (10) and (11) to the above equation, we get

$$
\begin{aligned}
E^{\hat{P}}[S_{t+1}|\mathcal{F}_t] &= S_t + \mathrm{Cov}(H, S_{t+1}|\mathcal{F}_t) - E[\Delta\hat{H}_{t+1}S_{t+1}|\mathcal{F}_t] \\
&= S_t + \mathrm{Cov}(\hat{H}_t, S_{t+1}|\mathcal{F}_t) - \mathrm{Cov}(\hat{H}_t, S_{t+1}|\mathcal{F}_t) \\
&= S_t
\end{aligned}
$$

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Future works

1. Prove the Lemma 1 (LMVE strategy is equivalent to v_k
2. Using Doob decomposition and GTW decomposition to simplify v_k.
3. For J agents case, simplify the results
4. Consider the quadratic transaction costs.

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