F Community assignments
M Community 'centres'
X Attribute Matrix

The likelihood of observing G represented by adjacency matrix A:

$$L_G = \sum_{u,v} A_{uv} \log(P_{uv}) + (1 - A_{uv}) \log(1 - P_{uv}) \tag{1}$$

We write the likelihood as a function of a row F_u of the community membership matrix F.

$$L_G(F_u) = \sum_{v} A_{uv} \log(P_{uv}(F_u)) + (1 - A_{uv}) \log(1 - P_{uv}(F_u)) \tag{2}$$

We compute observation probabilities with a sigmoid function

$$P_{uv} = \frac{1}{1 + exp(\frac{h_{uv}(F_u) - R_u}{2T})} \tag{3}$$

$$= \sigma\left(\frac{R_u - h_{uv}(F_u)}{2T}\right) \tag{4}$$

The current radius of the hyperbolic disk R_u is given by

$$R_u = r_u - 2\log\left|\frac{2T(1 - \exp(-(1-\beta)\log(u))}{\sin(T\pi)m(1-\beta)}\right| \tag{5}$$

Hyperbolic distance between nodes u and v is given by

$$h_{uv}(F_u) = \operatorname{arccosh}\left(\cosh r_u \cosh r_v - \sinh r_u \sinh r_v \cos(\Delta\theta_{uv}(F_u))\right) \tag{6}$$

The change in angles is given by the hyperbolic law of cosines:

$$\Delta\theta_{uv}(F_u) = \pi - |\pi - |\theta_u(F_u) - \theta_v|| \tag{7}$$

We are defining theta to be a weighted mean of the centres of the communities that u is assigned to.

$$\theta_u(F_u) = \frac{F_u M}{F_u \mathbf{1}} = \frac{\sum_c F_{uc} M_c}{\sum_c F_{uc}} \tag{8}$$

In this section we compute the partial derivative of the likelihood with respect to F_u.

$$\frac{\partial L_G(F_u)}{\partial F_u} = \sum_v \left[\frac{A_{uv}}{P_{uv}(F_u)} - \frac{1 - A_{uv}}{1 - P_{uv}(F_u)} \right] \frac{\partial P_{uv}(F_u)}{\partial F_u} \tag{9}$$

The partial derivative of P_{uv}:

$$\frac{\partial P_{uv}(F_u)}{\partial F_u} = -P_{uv}(1 - P_{uv}) \frac{1}{2T} \frac{\partial h_{uv}(F_u)}{\partial F_u} \tag{10}$$

The partial derivative of hyperbolic distance:
let:

$$x = \cosh r_u \cosh r_v - \sinh r_u \sinh r_v \cos(\Delta\theta_{uv}(F_u) \tag{11}$$

then

$$\frac{\partial h_{uv}(F_u)}{\partial F_u} = \frac{1}{\sqrt{x^2 - 1}} \sinh r_u \sinh r_v \sin(\Delta\theta_{uv}(F_u)) \frac{\partial \Delta\theta_{uv}(F_u)}{\partial F_u} \tag{12}$$

Angle difference:

$$\frac{\partial \Delta\theta_{uv}(F_u)}{\partial F_u} = \mathrm{Sign}(\pi - |\theta_u(F_u) - \theta_v|) * \mathrm{Sign}(\theta_u(F_u) - \theta_v) * \frac{\partial \theta_u(F_u)}{\partial F_u} \tag{13}$$

Finally θ_u (using the quotient rule):

$$\frac{\partial \theta_u(F_u)}{\partial F_u} = \frac{(F_u \mathbf{1})(\mathbf{1}^{\mathrm{T}} M) - (F_u M)(\mathbf{1}^{\mathrm{T}} \mathbf{1})}{(F_u \mathbf{1})^2} \tag{14}$$

We repeat this process for a given community centre M_c, first by writing the likelihood as a function of M_c.

$$L_G(M_c) = \sum_{uv} A_{uv} \log(P_{uv}(M_c)) + (1 - A_{uv}) \log(1 - P_{uv}(M_c)) \tag{15}$$

$$P_{uv}(M_c) = \frac{1}{1 + exp(\frac{h_{uv}(M_c) - R_u}{2T})} \tag{16}$$
$$= \sigma\left(\frac{R_u - h_{uv}(M_c)}{2T}\right) \tag{17}$$

$$h_{uv}(M_c) = \text{arcosh}\Bigg(\cosh r_u \cosh r_v - \sinh r_u \sinh r_v \cos(\Delta\theta_{uv}(M_c))\Bigg) \tag{18}$$

$$\Delta\theta_{uv}(M_c) = \pi - |\pi - |\theta_u(M_c) - \theta_v(M_c)|| \tag{19}$$

$$\theta_u(M_c) = \frac{\sum_{c'} F_{uc'} M_{c'}}{\sum_{c'} F_{uc'}} = \frac{\sum_{c' \neq c} F_{uc'} M_{c'}}{\sum_{c'} F_{uc'}} + \frac{F_{uc} M_c}{\sum_{c'} F_{uc'}} \tag{20}$$

θ_v is computed as θ_u.

Computing partial gradients for M_c:

$$\frac{\partial L_G(M_c)}{\partial M_c} = \sum_{uv} \left[\frac{A_{uv}}{P_{uv}(M_c)} - \frac{1 - A_{uv}}{1 - P_{uv}(M_c)}\right] \frac{\partial P_{uv}(M_c)}{\partial M_c} \tag{21}$$

$$\frac{\partial P_{uv}(M_c)}{\partial M_c} = -P_{uv}(M_c)(1 - P_{uv}(M_c)) \frac{1}{2T} \frac{\partial h_{uv}(M_c)}{\partial M_c} \tag{22}$$

$$\frac{\partial h_{uv}(M_c)}{\partial M_c} = \frac{1}{\sqrt{x^2 - 1}} \sinh r_u \sinh r_v \sin(\Delta\theta_{uv}(M_c)) \frac{\partial \Delta\theta_{uv}(M_c)}{\partial M_c} \tag{23}$$

$$\frac{\partial \Delta\theta_{uv}(M_c)}{\partial M_c} = \text{sign}(\pi - |\theta_u(M_c) - \theta_v(M_c)|)\text{sign}(\theta_u(M_c) - \theta_v(M_c))\left(\frac{\partial \theta_u(M_c)}{\partial M_c} - \frac{\partial \theta_v(M_c)}{\partial M_c}\right) \tag{24}$$

$$\frac{\partial \theta_u(M_c)}{\partial M_c} = \frac{F_{uc}}{\sum_{c'} F_{uc'}} \tag{25}$$

Again, θ_v is computed as θ_u.

The likelihood of observing each attribute X_{uk}:

$$L_X = \sum_{u,k} X_{uk} \log(Q_{uk}) + (1 - X_{uk}) \log(1 - Q_{uk}) \tag{26}$$

where

$$Q_{uk} = \frac{1}{1 + \exp(-\sum_c W_{kc} F_{uc})} \tag{27}$$

$$X_{uk} \sim \text{Bernoulli}(Q_{uk}) \tag{28}$$

The likelihood written as a function of F_u:

$$L_X(F_u) = \sum_{u,k} X_{uk} \log(Q_{uk}(F_u)) + (1 - X_{uk}) \log(1 - Q_{uk}(F_u)) \tag{29}$$

Trivially, we have that:

$$\frac{\partial L_X(F_u)}{\partial F_u} = \sum_k (X_{uk} - Q_{uk}) W_{kc} \tag{30}$$

As a function of W_{kc}:

$$L_X(W_{kc}) = \sum_{u,k} X_{uk} \log(Q_{uk}(W_{kc})) + (1 - X_{uk}) \log(1 - Q_{uk} \tag{31}$$

Again:

$$\frac{\partial L_X(W_{kc})}{\partial W_{kc}} = (X_{uk} - Q_{uk}) F_{uc} \tag{32}$$

Giving us the update equation for F_u as:

$$F_u^{\text{new}} = \max\left(0, F_u^{\text{old}} + \alpha\left(\frac{\partial L_G(F_u)}{\partial F_u} + \frac{\partial L_X(F_u)}{\partial F_u}\right)\right) \tag{33}$$

$$\frac{\partial L_G(F_u)}{\partial F_u} = -\sum_v \frac{1}{2T}\left(A_{uv} - P_{uv}\right) \tag{34}$$

$$\frac{1}{\sqrt{x^2-1}} \sinh r_u \sinh r_v \sin(\Delta\theta_{uv}(F_{uc})) * \tag{35}$$

$$\operatorname{sign}(\pi - |\theta_u(F_{uc}) - \theta_v|) * \operatorname{sign}(\theta_u(F_{uc}) - \theta_v) \tag{36}$$

$$\frac{(F_u\mathbf{1})(\mathbf{1}^{\mathrm{T}}M) - (F_uM)(\mathbf{1}^{\mathrm{T}}\mathbf{1})}{(F_u\mathbf{1})^2} \tag{37}$$

$$\frac{\partial L_X(F_u)}{\partial F_u} = \sum_k (X_{uk} - Q_{uk})W_{kc} \tag{38}$$

Giving us the update equation for M_c as:

$$M_c^{\text{new}} = M_c^{\text{old}} + \alpha\frac{\partial L_G(M_c)}{\partial M_c} \tag{39}$$

$$= M_c^{\text{old}} - \alpha\sum_{uv}\frac{1}{2T}\left(A_{uv} - P_{uv}\right) \tag{40}$$

$$\frac{1}{\sqrt{x^2-1}} \sinh r_u \sinh r_v \sin(\Delta\theta_{uv}(M_c)) * \tag{41}$$

$$\operatorname{sign}(\pi - |\theta_u(M_c) - \theta_v(M_c)|) * \operatorname{sign}(\theta_u(M_c) - \theta_v(M_c)) \tag{42}$$

$$\left(\frac{F_{uc}}{\sum_{c'} F_{uc'}} - \frac{F_{vc}}{\sum_{c'} F_{vc'}}\right) \tag{43}$$

Giving us the update equation for W_{kc} as:

$$W_{kc}^{\text{new}} = W_{kc}^{\text{old}} + \alpha\sum_u \frac{\partial F_X(W_{kc})}{\partial W_{kc}} - \lambda\operatorname{sign}(W_{kc}) \tag{44}$$

$$= W_{kc}^{\text{old}} + \alpha\sum_u (X_{uk} - Q_{uk})F_{uc} - \lambda\operatorname{sign}(W_{kc}) \tag{45}$$