notes on Differential Geometry
Noel J. Hicks Chapter 1 Problems

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1 Manifolds

1.1 Manifolds

These notes are for the most part directly from [Hic65].

Definition 1. Let $r > 0$ and $A \subset R^n$ be an open set. A map $f : A \to R$ is called C^r if it is r times continuously differentiable.

Definition 2. Let u_i denote the usual projection maps of the i coordinate of R^n onto R. Let $r > 0$ and $A \subset R^n$ be an open set. A map $f : A \to R^m$ is called C^r if $u_i \circ f$ is C^r for all of the projection maps.

Definition 3. If f is C^r for all $r \in N^+$ we say that f is C^∞. If f is real analytic, we say f is C^ω. If f is simply continuous then f is C^0.

Definition 4. Let M be a set. A chart on M is a pair (ϕ, U) such that U is a subset of M and ϕ is a 1-1 map of U onto an open subset of R^n. We call the sets U coordinate domains.

Definition 5. Two charts, (ϕ, U) and (θ, V) on M are C^r related if $\phi \circ \theta^{-1}$ and $\theta \circ \phi^{-1}$ are C^r on $\theta(U \cap V)$ and $\phi(U \cap V)$ respectively. Note that by definition 1, $\theta(U \cap V)$ and $\phi(U \cap V)$ must be open.

Definition 6. A C^r subatlas of a set M is a collection of C^r related charts, $\{(\theta_h, U_h)\}_{h \in H}$, such that

$$\bigcup_{h \in H} U_h = M.$$

Proposition 1. Every subatlas is contained in a maximal subatlas called an atlas.

Proof. Let $\{(\theta_h, U_h)\}_{h \in H}$ be a subatlas. Order the collection of all subatlases by inclusion. This clearly forms a poset. Take any chain which contains $\{(\theta_h, U_h)\}_{h \in H}$. Let (θ_1, U_1) and (θ_2, U_2) be charts in the union. There exists a minimal subatlas somewhere in the chain which contains both charts, therefore they are C^r equivalent. It is clear that the union of all charts in the union is all of M. Thus the union of all subatlases in this chain is again a subatlas. Apply Zorn's Lemma. $\square$

Proposition 2. Every subatlas induces a topology on M. More specifically, let $\{(\phi_h, U_h)\}_{h\in H}$ be a subatlas on M. Let A be a subset of M. Define the induced topology on M as follows. We say that A is open in M if for any chart (ϕ_h, U_h), $\phi_h(A\cap U_h)$ is open in $\mathbb{R}^n$. This is the coarsest topology which makes the charts continuous. Note that by the comment in definition 5, each coordinate domain is open in M.

Proof. Let (ϕ, U) be any chart. Clearly, $\phi(\varnothing \cap U) = \varnothing$ which is open in $\mathbb{R}^n$. Similarly, $\phi(M \cap U) = \phi(U)$ is open in $\mathbb{R}^n$. Therefore, M and $\varnothing$ are both open. Suppose A and B are open. Since each chart is injective, $\phi(U \cap A \cap B) = \phi(U\cap A)\cap\phi(U\cap B)$ thus $A\cap B$ is open. Let $\{A_i\}_{i\in I}$ be an arbitrary collection of open sets. We have $\phi((\cup_i A_i)\cap U) = \cup_i \phi(A_i \cap U)$, a union of open sets. Thus the collection of open sets is closed under arbitrary union and finite intersection. □

Proposition 3. The topology induced by a subatlas is the same as the topology induced by the maximal subatlas.

Proof. Suppose (ϕ, U) is a chart which is compatible with a subatlas $\{(\phi_h, U_h)\}_{h\in H}$, but is not contained in that atlas, we will show that $(\phi, U) \cup \{(\phi_h, U_h)\}_{h\in H}$ induces the same topology on M as $\{(\phi_h, U_h)\}_{h\in H}$.

Clearly the topology induced by $(\phi, U) \cup \{(\phi_h, U_h)\}_{h\in H}$ will be contained in the topology induced by $\{(\phi_h, U_h)\}_{h\in H}$ so it just remains to show that if A is open with respect to $\{(\phi_h, U_h)\}_{h\in H}$, then it is still open with respect to $(\phi, U) \cup \{(\phi_h, U_h)\}_{h\in H}$. That is, we must show that $\phi(A \cap U)$ is open in $\mathbb{R}^n$.

Since (ϕ, U) is a chart which is compatible with the subatlas, U is open in M under the previous topology. Since A is open in M under the previous topology as well, we have that $\phi_h(A \cap U \cap U_h)$ is open for all charts. Since $\phi \circ \phi_h$ is bicontinuous, $\phi(A \cap U \cap U_h) = \phi \circ \phi_h(\phi_h(A \cap U \cap U_h))$ is open for all charts. Noting that

$$\phi(A \cap U) = \phi(A \cap U \cap M) = \phi(A \cap U \cap (\cup_h U_h)) = \cup_h \phi(A \cap U \cap U_h)$$

we see that $\phi(A \cap U)$ is the union of open sets and is therefore open. □

Proposition 4. The content of definition 6 does not change if we alter definition 4 so that each θ is a 1-1 map of U onto either an open subset of R^n, an open ball in R^n, or all of R^n. [Spi99]

Proof. By a proper choice of function, for instance a renormalization of arctangent, we can obtain a homeomorphism of an open ball in R to all of R. The equivalence of definition 6 when we use either open balls in R^n or all of R^n follows with a little thought. It remains to show the equivalence in the case of open balls and open sets.

Let $\{(\phi_h, U_h)\}_{h\in H}$ be a subatlas where each $\phi_h(U_h)$ is an open set of R^n. Let

$p \in M$, there exists a chart (ϕ, U) such that $p \in U$ and $\phi(U)$ is an open set in R^n containing $\phi(p)$. Pick an open ball B centered around $\phi(p)$ contained entirely in $\phi(U)$. Then $\phi^{-1}(B) \subseteq U$. The collection $\{(\phi_p, \phi_p^{-1}(B_p))\}_{p\in M}$ is a subatlas which generates the same maximal atlas as $\{(\phi_h, U_h)\}_{h\in H}$ since $\phi_h \circ \phi_p^{-1}$ is the C^r identity on B_p. † $\square$

Definition 7. The association to a maximal atlas gives us an equivalence relation and therefore a partition of the subatlases. We call an equivalence class a topological/differential/smooth/analytic structure in the cases C^0, C^r, C^∞, and C^ω respectively.

Definition 8. A topological/differential/smooth/analytic manifold is a set M together with a structure Σ, of the associated type. †

1.2 Smooth Functions

Let A be a subset of a C^r manifold M.

Definition 9. Let $f : A \to R$. We say that f is C^s if $f \circ \phi^{-1}$ is C^s from $\phi(A \cap U)$ for every C^r chart (ϕ, U).

Definition 10. If N is a C^k manifold and $f : A \to N$ continuous, we say that f is C^s if for every real valued C^s g, with open domain B, $g \circ f$ is C^s on $A \cap f^{-1}(B)$.

This definition is really saying that f is C^s, if pulling back along f gives a morphism $f^* : C^s(N,R) \to C^s(M,R)$ defined appropriately. Furthermore, Hicks makes note of the fact that r, k and s are independent. Thus, as a special case, f pulls back the charts on N if and only if $s \leq k$.

There is a local version of this structure condition.

Definition 11. Let f be N-valued with domain not neccesarily open. We say that f is C^s at p, a point in the domain of f, if there exists an open neighborhood U of p such that $f|_U$ is C^s in the sense of definition 10.

Hicks notes that if f is C^s at every point of it's domain, then the domain of f is open.

The following theorem of Whitney gives us reason to specialize to the case of C^∞ structures.

Proposition 5. Every C^r atlas for $r \geq 1$ contains a C^∞ atlas.

†Milnor told us that it is not enough to say that $\{(\phi_p, \phi_p^{-1}(B_p))\}_{p\in M}$ and $\{(\phi_h, U_h)\}_{h\in H}$ generate the same topology on M.

†For now we hold off on the requirement that M be second countable and Hausdorff.

Some subcollection of charts are all C^∞ related and furthermore, are themselves maximal.

Problem 1. The map $f : A \to N$ is C^∞ on A iff f is C^∞ pointwise on A.

Solution 1. Suppose $f : A \to N$ is C^∞ on A let p be a point of A. Since A is open we let it be the requisite neighborhood.

Suppose that f is C^∞ pointwise on A. Take any $g : B \to R$ be C^∞ on B open in N. This says for each $p \in A$ there exists a neighborhood V_p such that for every chart (ϕ, U) then $g \circ f|_{V_p} \circ \phi^{-1}$ is C^∞ on $\phi(V_p \cap f^{-1}(B) \cap U)$. Furthermore, $V_p \subset A$. We want to show that f is C^s on A in the sense of definition 10.

Clearly, $\cup_{p \in A} V_p = A$. Since f restricted to any V_p of A is C^s, f is C^s on A.

Problem 2. If $f : A \to N$ is C^∞ on A then f restricted to any open subset U is still C^∞.

Solution 2. Take every point of $U \cap A$. By problem 1 there exists a neighborhood U_p on which $f|_{U_p}$ is C^∞. Replace each neighborhood U_p with the intersection $U \cap U_p$. Then definition 11 is satisfied. By probelm 1, the result follows.

Problem 3. Let U_h be a set of open sets who's union is A in M and let $f_h : U_h \to N$ be C^∞. Let f be a function such that $f|_{U_h} = f_h$ for each h. Prove f is C^∞ on A.

Solution 3. This follows directly from the argument given in the reverse direction of problem 1. At any point $p \in A$, there exists an h such that $p \in U_h$. Thus replace the V_p in the proof of problem 1 with the U_h.

Problem 4. Let $A \subseteq R^n$. Let $f : A \to R^k$ be C^∞. Let $B \subseteq R^k$ be an open subset with a C^∞ function $g : B \to R$. Then $g \circ f$ is C^∞ on $A \cap f^{-1}(B)$.

Solution 4. Note that R^n is itself a manifold with C^∞ structure determined by the single chart (id, R^n). The result follows by the fact that definition 10 is satisfied.

Problem 5. If $f : A \to N$ is C^∞ on $A \subseteq M$, and (ϕ, U) is a chart on M, then $f \circ \phi^{-1}$ is C^∞ on $\phi(A \cap U)$.

Solution 5.

Problem 6. Let P be a C^∞ s-manifold. If $F : A \to N$ is C^∞ on $A \subseteq M$ and $g : B \to P$ is C^∞ on an open subset $B \subseteq N$ then $g \circ f$ is C^∞ on $A \cap f^{-1}(B)$.

Solution 6.

Problem 7. The map $f : A \to N$ is C^∞ on $A \subseteq M$ iff for every coordinate pair (ϕ, U) in a subatlas on N, the functions $x_i \circ f$ are C^∞ on $A \cap f^{-1}(U)$, for $i = 1, ..., d$ and $x_i = u_i \circ \phi$.

Solution 7. Suppose that $f : A \to N$ is C^∞ on $A \subseteq M$. By definition 2, ϕ is C^∞ and thus $u_i \circ \phi$ is smooth. Thus, each x_i is C^∞ on U the result in the first direction follows.

Suppose for every coordinate pair (ϕ, U) in a subatlas on N, the functions $x_i \circ f$ are C^∞ on $A \cap f^{-1}(U)$. Let $g : B \to R$.

Definition 12. Let $C^\infty(A, N)$ denote the set of C^∞ functions mapping an open set A in a manifold M into a manifold N.

1.3 Vectors and vector fields

Definition 13. Let m be a point of R^n. If X_m is a euclidean vector with tail at m, and f is a C^∞ function defined in a neighborhood of m, define $X_m f = X_m \cdot (\nabla f)_m$ where $(\nabla f)_m$ is the gradient vector field of f at m.

Proposition 6. It follows from the definition of the dot product that

1. $X_m(af + bg) = aX_m f + bX_m g$
2. $X_m(fg) = f(m)X_m g + g(m)X_m f$

Proof.

$$\begin{aligned} X_m(af + bg) &= X_m \cdot (\nabla(af + bg))_m \\ &= aX_m \cdot (\nabla f)_m + bX_m \cdot (\nabla g)_m \\ &= aX_m f + bX_m g \end{aligned}$$

$$\begin{aligned} X_m(fg) &= X_m \cdot (\nabla(fg))_m \\ &= X_m \cdot (f(m)(\nabla g)_m + g(m)(\nabla f)_m \\ &= f(m)X_m g + g(m)X_m f \end{aligned}$$

□

References

[Hic65] N.J. Hicks. *notes on Differential Geometry*. Van Nostrand Mathematical Studies, Princeton, NJ, 1965.

[Spi99] M. Spivak. *A comprehensive introduction to Differential Geometry*. Publish or Perish Inc., Houston, TX, 1999.