notes on Differential Geometry
Noel J. Hicks Chapter 1 Problems

David L. Meretzky

Sunday December 16th, 2018

1 Manifolds

1.1 Manifolds

These notes are for the most part directly from [Hic65].

Definition 1. Let $r > 0$ and $A \subset R^n$ be an open set. A map $f : A \to R$ is called C^r if it is r times continuously differentiable.

Definition 2. Let u_i denote the usual projection maps of the i coordinate of R^n onto R. Let $r > 0$ and $A \subset R^n$ be an open set. A map $f : A \to R^m$ is called C^r if $u_i \circ f$ is C^r for all of the projection maps.

Definition 3. If f is C^r for all $r \in N^+$ we say that f is C^∞. If f is real analytic, we say f is C^ω. If f is simply continuous then f is C^0.

Definition 4. Let M be a set. A chart on M is a pair (ϕ, U) such that U is a subset of M and ϕ is a 1-1 map of U onto an open subset of R^n. We call the sets U coordinate domains.

Definition 5. Two charts, (ϕ, U) and (θ, V) on M are C^r related if $\phi \circ \theta^{-1}$ and $\theta \circ \phi^{-1}$ are C^r on $\theta(U \cap V)$ and $\phi(U \cap V)$ respectively.

Definition 6. A C^r subatlas of a set M is a collection of C^r related charts, $\{(\theta_h, U_h)\}_{h \in H}$, such that

$$\bigcup_{h \in H} U_h = M.$$

Proposition 1. Every subatlas is contained in a maximal subatlas called an atlas.

Proof. Let $\{(\theta_h, U_h)\}_{h \in H}$ be a subatlas. Order the collection of all subatlases by inclusion. This clearly forms a poset. Take any chain which contains $\{(\theta_h, U_h)\}_{h \in H}$. Let (θ_1, U_1) and (θ_2, U_2) be charts in the union. There exists a minimal subatlas somewhere in the chain which contains both charts, therefore they are C^r equivalent. It is clear that the union of all charts in the union is all of M. Thus the union of all subatlases in this chain is again a subatlas. Apply Zorn's Lemma. □

Proposition 2. Every subatlas induces a topology on M and this topology is the same as the topology induced by the maximal subatlas. Let $\{(\phi_h, U_h)\}_{h\in H}$ be a subatlas on M. For all open sets A of R^n, define the topology of M to be the topology with sub-base given by the sets $\phi_h^{-1}(U_h \cap A)$.

Proof. Suppose (ϕ, U) is a chart which is compatible with a subatlas $\{(\phi_h, U_h)\}_{h\in H}$, but is not contained in that atlas, we will show that $(\phi, U) \cup \{(\phi_h, U_h)\}_{h\in H}$ induces the same topology on M as $\{(\phi_h, U_h)\}_{h\in H}$.

Pick an open set A in R^n, we show that $\phi^{-1}(U \cap A)$ was already open in the topology induced by $\{(\phi_h, U_h)\}_{h\in H}$. Note that $U \cap A \cap U_h$ is an open set in R^n for all h. Since $\phi_h \circ \phi^{-1}$ is an open map, $\phi_h \circ \phi^{-1}(U \cap A \cap U_h)$ is open in R^n. It follows that $\phi_h^{-1}(\phi_h \circ \phi^{-1}(U \cap A \cap U_h)) = \phi^{-1}(U \cap A \cap U_h)$ is open in M for all h. Clearly,

$$U \cap A = \bigcup_{h\in H} U \cap A \cap U_h$$

from which it follows that

$$\phi^{-1}(U \cap A) = \bigcup_{h\in H} \phi^{-1}(U \cap A \cap U_h)$$

is open in M. $\square$

Proposition 3. The content of definition 6 does not change if we alter definition 4 so that each θ is a 1-1 map of U onto either an open subset of R^n, an open ball in R^n, or all of R^n. [Spi99]

Proof. By a proper choice of function, for instance a renormalization of arctangent, we can obtain a homeomorphism of an open ball in R to all of R. The equivalence of definition 6 when we use either open balls in R^n or all of R^n follows with a little thought. It remains to show the equivalence in the case of open balls and open sets.

Let $\{(\phi_h, U_h)\}_{h\in H}$ be a subatlas where each $\phi_h(U_h)$ is an open set of R^n. Let $p \in M$, there exists a chart (ϕ, U) such that $p \in U$ and $\phi(U)$ is an open set in R^n containing $\phi(p)$. Pick an open ball B centered around $\phi(p)$ contained entirely in $\phi(U)$. Then $\phi^{-1}(B) \subseteq U$. The collection $\{(\phi_p, \phi_p^{-1}(B_p))\}_{p\in M}$ is a subatlas which generates the same maximal atlas as $\{(\phi_h, U_h)\}_{h\in H}$ since $\phi_h \circ \phi_p^{-1}$ is the C^r identity on B_p. † $\square$

Definition 7. The association to a maximal atlas gives us an equivalence relation and therefore a partition of the subatlases. We call an equivalence class a topological/differential/smooth/analytic structure in the cases C^0, C^r, C^∞, and C^ω respectively.

†Milnor told us that it is not enough to say that $\{(\phi_p, \phi_p^{-1}(B_p))\}_{p\in M}$ and $\{(\phi_h, U_h)\}_{h\in H}$ generate the same topology on M.

Definition 8. A topological/differential/smooth/analytic manifold is a set M together with a structure Σ, of the associated type. †

1.2 Smooth Functions

Let A be a subset of a C^r manifold M.

Definition 9. Let $f : A \to R$. We say that f is C^s if $f \circ \phi^{-1}$ is C^s from $\phi(A \cap U)$ for every C^r chart (ϕ, U).

Definition 10. If N is a C^k manifold and $f : A \to N$ continuous, we say that f is C^s if for every real valued C^s g, with open domain B, $g \circ f$ is C^s on $A \cap f^{-1}(B)$.

This definition is really saying that f is C^s, if pulling back along f gives a morphism $f^* : C^s(N, R) \to C^s(M, R)$ defined appropriately. Furthermore, Hicks makes note of the fact that r, k and s are independent. Thus, as a special case, f pulls back the charts on N if and only if $s \leq k$.

There is a local version of this structure condition.

Definition 11. Let f be N-valued with domain not neccesarily open. We say that f is C^s at p, a point in the domain of f, if there exists an open neighborhood U of p such that $f|_U$ is C^s in the sense of definition 10.

Hicks notes that if f is C^s at every point of it's domain, then the domain of f is open.

The following theorem of Whitney gives us reason to specialize to the case of C^∞ structures.

Proposition 4. Every C^r atlas for $r \geq 1$ contains a C^∞ atlas.

Some subcollection of charts are all C^∞ related and furthermore, are themselves maximal.

Problem 1. The map $f : A \to N$ is C^∞ on A iff f is C^∞ pointwise on A.

Solution 1. Suppose $f : A \to N$ is C^∞ on A let p be a point of A. Since A is open we let it be the requisite neighborhood.

Suppose that f is C^∞ pointwise on A. Take any $g : B \to R$ be C^∞ on B open in N. This says for each $p \in A$ there exists a neighborhood V_p such that for every chart (ϕ, U) then $g \circ f|_{V_p} \circ \phi^{-1}$ is C^∞ on $\phi(V_p \cap f^{-1}(B) \cap U)$. Furthermore, $V_p \subset A$. We want to show that f is C^s on A in the sense of definition 10.

†For now we hold off on the requirement that M be second countable and Hausdorff.

Clearly, $\cup_{p \in A} V_p = A$. Since f restricted to any V_p of A is C^s, f is C^s on A.

Problem 2. If $f : A \to N$ is C^∞ on A then f restricted to any open subset U is still C^∞.

Solution 2. Take every point of $U \cap A$. By problem 1 there exists a neighborhood U_p on which $f|_{U_p}$ is C^∞. Replace each neighborhood U_p with the intersection $U \cap U_p$. Then definition 11 is satisfied. By probelm 1, the result follows.

Problem 3. Let U_h be a set of open sets who's union is A in M and let $f_h : U_h \to N$ be C^∞. Let f be a function such that $f|_{U_h} = f_h$ for each h. Prove f is C^∞ on A.

Solution 3. This follows directly from the argument given in the reverse direction of problem 1. At any point $p \in A$, there exists an h such that $p \in U_h$. Thus replace the V_p in the proof of problem 1 with the U_h.

Problem 4. Let $A \subseteq R^n$. Let $f : A \to R^k$ be C^∞. Let $B \subseteq R^k$ be an open subset with a C^∞ function $g : B \to R$. Then $g \circ f$ is C^∞ on $A \cap f^{-1}(B)$.

Solution 4. Note that R^n is itself a manifold with C^∞ structure determined by the single chart (id, R^n). The result follows by the fact that definition 10 is satisfied.

Problem 5. If $f : A \to N$ is C^∞ on $A \subseteq M$, and (ϕ, U) is a chart on M, then $f \circ \phi^{-1}$ is C^∞ on $\phi(A \cap U)$.

Solution 5.

Problem 6. Let P be a C^∞ s-manifold. If $F : A \to N$ is C^∞ on $A \subseteq M$ and $g : B \to P$ is C^∞ on an open subset $B \subseteq N$ then $g \circ f$ is C^∞ on $A \cap f^{-1}(B)$.

Solution 6.

Problem 7. The map $f : A \to N$ is C^∞ on $A \subseteq M$ iff for every coordinate pair (ϕ, U) in a subatlas on N, the functions $x_i \circ f$ are C^∞ on $A \cap f^{-1}(U)$, for $i = 1, ..., d$ and $x_i = u_i \circ \phi$.

Solution 7. Suppose that $f : A \to N$ is C^∞ on $A \subseteq M$, if we can show that each x_i is C^∞ on U the result in the first direction follows. Clearly, ϕ is C^∞ and thus by definition 2 $u_i \circ \phi$ is smooth.

Definition 12. Let $C^\infty(A, N)$ denote the set of C^∞ functions mapping an open set A in a manifold M into a manifold N.

1.3 Vectors and vector fields

Definition 13. Let m be a point of R^n. If X_m is a euclidean vector with tail at m, and f is a C^∞ function defined in a neighborhood of m, define $X_m f = X_m \cdot (\nabla f)_m$ where $(\nabla f)_m$ is the gradient vector field of f at m.

Proposition 5. It follows from the definition of the dot product that

1. $X_m(af + bg) = aX_m f + bX_m g$
2. $X_m(fg) = f(m)X_m g + g(m)X_m f$

Proof.

$$\begin{aligned} X_m(af + bg) &= X_m \cdot (\nabla(af + bg))_m \\ &= aX_m \cdot (\nabla f)_m + bX_m \cdot (\nabla g)_m \\ &= aX_m f + bX_m g \end{aligned}$$

$$\begin{aligned} X_m(fg) &= X_m \cdot (\nabla(fg))_m \\ &= X_m \cdot (f(m)(\nabla g)_m + g(m)(\nabla f)_m \\ &= f(m)X_m g + g(m)X_m f \end{aligned}$$

□

References

[Hic65] N.J. Hicks. *notes on Differential Geometry.* Van Nostrand Mathematical Studies, Princeton, NJ, 1965.

[Spi99] M. Spivak. *A comprehensive introduction to Differential Geometry.* Publish or Perish Inc., Houston, TX, 1999.