Math 260 Exercises 5.A Solutions

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Problem 1. Suppose $T \in \mathscr{L}(V)$ and U is a subspace of V. Prove that if $U \subseteq null\ T$ then U is T invariant. Prove that if $Ran\ T \subseteq U$ then U is T invariant.

Solution 1. Let $u \in U$. Suppose $U \subseteq null\ T$. Then $Tu = 0 \in U$ since U is a subspace and therefore contains 0. Now suppose that $Ran\ T \subseteq U$, then $Tu \in Ran\ T \subseteq U$ so again, $Tu \in U$.

Problem 2. Suppose $T \in \mathscr{L}(\mathbf{R}^2)$ defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Solution 2. Suppose that (x, y) is an eigenvector for T with eigenvalue λ. Then $T(x, y) = \lambda(x, y)$. Thus $(-3y, x) = \lambda(x, y)$. So $-3y = \lambda x$ and $x = \lambda y$. Thus $-3y = \lambda^2 y$. It follows that $-3 = \lambda^2$. So $\lambda = \pm\sqrt{-3} = \pm i\sqrt{3}$. Since $\pm i\sqrt{3}$ does not lie in $\mathbf{R}$ T has no eigenvectors.

Problem 3. Suppose $T \in \mathscr{L}(\mathbf{F}^2)$ defined by $T(w, z) = (z, w)$. Find the eigenvalues of T.

Solution 3. Suppose that (w, z) is an eigenvector for T with eigenvalue λ. Then $T(w, z) = \lambda(w, z)$. Thus $(z, w) = \lambda(w, z)$. So $z = \lambda w$ and $w = \lambda z$. Thus $z = \lambda^2 z$. It follows that $1 = \lambda^2$. So $\lambda = \pm 1$. Clearly $(1, 1)$ is an eigenvector with eigenvalue 1. Suppose $\lambda = -1$, then if (w, z) is an eigenvector we have, $(z, w) = -1(w, z)$. Thus $z = -w$ and $w = -z$. It follows that $(1, -1)$ is an eigenvector with eigenvalue -1. Thus the eigenvectors are those non-zero vectors in the span of the vectors $(1, 1)$ and $(1, -1)$.

Problem 4. Define $T \in \mathscr{L}(F^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvectors and eigenvalues.

Solution 4. Clearly, all non-zero vectors in the span of $(0, 0, 1)$ are eigenvectors with eigenvalue 5. Also all non-zero vectors in the span of $(1, 0, 0)$ are eigenvectors with eigenvalue 0. Since eigenvectors with distinct eigenvalues are linearly independent. If there was another eigenvector it would have to be $(0, 1, 0)$. The equation $\lambda(0, 1, 0) = (2, 0, 0)$ has no solutions. Thus there are no more eigenvectors with eigenvalues different than 0 and 5. But $dim(Ker(T - 0I)) = dim(Ker(T - 5I)) = 1$, so there are no more eigenvectors with eigenvalue 0 or 5.

Problem 5. Define $T \in \mathscr{L}(\mathbf{F}^n)$ by $T(x_1, ..., x_n) = (x_1, 2x_2, 3x_3, ..., nx_n)$. Find all eigenvalues and eigenvectors. Find all invariant subspaces of T.

Solution 5. Let $E = e_1, ..., e_n$ be the standard basis. It is easy to see that each e_i is an eigenvector with eigenvalue i. Clearly the span of each e_i is an invariant subspace. I claim that every subspace U of V of the form $span(e_i) \oplus span(e_j) \oplus ... \oplus span(e_l)$ is T invariant. Let $u \in span(e_i) \oplus span(e_j) \oplus ... \oplus span(e_l)$ for some collection $e_i, e_j, ..., e_l$ within E. We see that $Tu = a_i T e_i + ... + a_l T e_l = i a_i e_i + ... + l a_l e_l \in span(e_i) \oplus span(e_j) \oplus ... \oplus span(e_l)$. Note however, that $e_1 + e_2$ is not an eigenvector.

Problem 6. Find all eigenvalues and eigenvectors of the differentiation operator on $\mathcal{P}(\mathbf{F}, x)$.

Solution 6. Suppose $p(x) = a_0 + a_1 x + ... a_n x^n$ such that $Dp(x) = \lambda p(x)$. The coefficient of the x^n term in $Dp(x)$ is 0. Thus $a_n \lambda = 0$ so $\lambda = 0$. Thus $\lambda p(x) = Dp(x)$ iff $p(x) = c$ in which case $p(x)$ is a non-zero constant polynomial with eigenvalue 0.

Note: if we allow extend the definition of $\mathcal{P}(\mathbf{F}, x)$ to include trigonometric functions, we have $De^{\lambda x} = \lambda e^{\lambda x}$. Does D^2 have eigenvectors? Does D^3, or D^4?