Math 260 Exercises 2.A Solutions

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Problem 1. Suppose v_1, v_2, v_3, and v_4 spans V. Prove that the list $v_1 - v_2$, $v_2 - v_3$, $v_3 - v_4$,v_4 also spans V.

Solution 1. Let v be any vector in V. Then there exist scalars α_1, α_2, α_3, and α_4, such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$. However $v_3 = (v_3 - v_4) + v_4$. Similarly, $v_2 = (v_2 - v_3) + (v_3 - v_4) + v_4$ and $v_1 = (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4$.

Making these substitutions we obtain that $v = \alpha_1((v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4) + \alpha_2((v_2 - v_3) + (v_3 - v_4) + v_4) + \alpha_3((v_3 - v_4) + v_4) + \alpha_4 v_4$. Collecting like terms we have that $v = \alpha_1(v_1 - v_2) + (\alpha_1 + \alpha_2)(v_2 - v_3) + (\alpha_1 + \alpha_2 + \alpha_3)(v_3 - v_4) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)v_4$ as desired.

Problem 2. Find a number t such that $(3, 1, 4)$, $(2, -3, 5)$, $(5, 9, t)$ is not linearly independent in $\mathbf{R}^3$.

Solution 2. It suffices to find t such that there exist $a, b, c \in \mathbf{R}$ non-zero such that $a(3, 1, 4) + b(2, -3, 5) + c(5, 9, t) = 0$. We have the following 3 systems of equations $3a + 2b + 5c = 0$, $a - 3b + 9c = 0$, and $4a + 5b + tc = 0$.

Solving for c in equation 1. We have $c = \frac{-3a-2b}{5}$. Plugging this into equation 2 we obtain $a - 3b + \frac{9(-3a-2b)}{5} = 0$. Expanding this we obtain $\frac{5a}{5} - \frac{15b}{15} - \frac{27a}{5} - \frac{54b}{15} = -66a - 42b = 11a + 7b = 0$ thus $a = -\frac{7b}{11}$. Equation 3 then becomes $-\frac{28b}{11} + \frac{55b}{11} + t\frac{\frac{21b}{11} - 2b}{5} = \frac{135b}{55} + t\frac{b}{55} = 0$. If t is equal to -135, then there are non-zero coefficients as desired.

Problem 3. Prove or give counterexample: If $v_1, v_2, ...v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, ...\lambda v_m$ is linearly independent.

Proof. Suppose that $a_1, ...a_m \in \mathbf{F}$ such that $a_1 \lambda v_1 + ...a_m \lambda v_m = 0$. Then $(a_1 \lambda)v_1 + ...(a_m \lambda)v_m = 0$, so each $a_i \lambda = 0$. Since $\lambda \neq 0$, $a_i = 0$ as desired. □

Problem 4. Prove or give counterexample: If $v_1...v_m$ and $w_1...w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, ...v_m + w_m$ is linearly independent.

Solution 3. Counterexample: Let $v_1 = (1, 0)$ and let $v_2 = (0, 1)$ and let $w_1 = (0, 1)$ and $w_2 = (1, 0)$. Then $v_1 + w_1 = (1, 1)$ and $v_2 + w_2 = (1, 1)$. This is clearly a dependent list.

Problem 5. Suppose $v_1...v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w...v_m + w$ is linearly dependent then $w \in Span(v_1...v_m)$.

Proof. There exist non-zero coefficients $a_1, ...a_m \in \mathbf{F}$ such that $a_1(v_1 + w) + ...a_m(v_m + w) = 0$. Then factoring out w we have $(a_1 + ... + a_m)w + (a_1v_1 + ...a_mv_m) = 0$ and $a_1v_1 + ...a_mv_m = -(a_1 + ... + a_m)w$. Since $v_1...v_m$ is linearly independent and not all of the $a_1, ...a_m \in \mathbf{F}$ are zero, $-(a_1 + ... + a_m)w \neq 0$. Thus $-(a_1 + ... + a_m) \neq 0$. We can then divide both sides of the equation by $-(a_1 + ... + a_m)$ and we see that w is in the span of $v_1...v_m$. □

Problem 6. Suppose $v_1...v_m$ is linearly independent in V and $w \in V$. Show that $v_1...v_m, w$ is linearly independent if and only if $w \notin Span(v_1..v_m)$.

Proof. Note that w cannot be the 0 vector

Suppose $v_1...v_m, w$ is linearly independent. Suppose $w \in Span(v_1..v_m)$. Then there exist non-zero coefficients $a_1...a_m$ such that $a_1v_1 + ... + a_mv_m = 1 \cdot w$. The coefficients must be non-zero otherwise w is the zero vector. Then $a_1v_1 + ... + a_mv_m - 1 \cdot w = 0$. But since $v_1...v_m, w$ is linearly independent then the coefficients must all be 0. This is a contradiction. Therefore $w \notin Span(v_1..v_m)$.

Suppose now that $w \notin Span(v_1..v_m)$. Suppose that $v_1...v_m, w$ is linearly dependent. This violates the Linear Dependence Lemma. The result follows by contradiction. □