Math 260 Final Exam

David L. Meretzky

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Problem 1. (30 points) Give definitions for the following terms:

1. linear combination of a list of vectors
2. span of a list of vectors
3. linear independence of a list of vectors
4. linear map
5. kernel of a linear map
6. range of a linear map
7. injectivity of a function
8. surjectivity of a function
9. $[\ \ ]_B : V \to \mathbf{F}^n$
10. invariant subspace
11. eigenvalue
12. eigenvector

Do any 7 of the following problems, each worth 10 points.

Problem 2. Define the dot product on $\mathbf{R}^n$. Show it is an inner product.

Problem 3. Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -3 \\ 3 & 2 & 5 \end{pmatrix}$$

Problem 4. Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$kerT = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 | x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove T is surjective.

Problem 5. Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$ show that $v_1, ..., v_m, w$ is linearly independent in V if and only if $w \notin span(v_1, ..., v_m)$.

Problem 6. Prove that eigenvectors associated to distinct eigenvalues must be linearly independent. That is, let $v_1, ..., v_n$ be eigenvectors with eigenvalues $\lambda_1, ..., \lambda_n$ such that $\lambda_1 \neq ... \neq \lambda_n$. Show $v_1, ..., v_n$ are linearly independent.

Problem 7. Let $T \in \mathscr{L}(V)$ prove that T has at most $dim(V)$ distinct eigenvalues. Hint: Use the solution to the previous problem.

Problem 8. Prove $(1, 2)$, $(3, 5)$ is a basis for $\mathbf{F}^2$.

Problem 9. Theorem 5.26. Give three equivalent conditions conditions under which an operator $T \in \mathscr{L}(V)$ is upper triangular with respect to a basis $B = v_1, ..., v_n$.

Problem 10. Suppose the matrix of a linear operator T is upper triangular with respect to some basis. Prove that the eigenvalues of T are precisely the entries on the diagonal of that upper triangular matrix.

Problem 11. Perform the Gram-Schmidt Procedure on the following list of independent vectors: $(2, 1, 1)$, $(1, 2, 2)$, $(-2, -2, 1)$. Use the usual dot product in $\mathbf{R}^3$.

Problem 12. Give an example of a linear map $T : \mathbf{R}^4 \to \mathbf{R}^4$ such that $Ran\, T = Ker\, T$.