Math 260 Exam 2

David L. Meretzky

Monday November 19th, 2018

In problems 1 and 2 your wording may be different than that of book. I am looking for correctness in the concepts. Problems $1-3$ are worth 70 points. The test is out of 100.

Problem 1. Give definitions for the following terms: (Careful, pay attention to the vector spaces involved, write them explicitly).

1. linear map
2. addition and product of linear maps.
3. null space/kernel of a linear map, prove it is a subspace
4. range of a linear map, prove it is a subspace
5. injectivity of a function
6. surjectivity of a function
7. invertability of a linear map, isomorphism
8. operator
9. define a linear map $[\ \ ]_B : V \to \mathbf{F}^n$
10. define a linear map $[\ \ ]_{B'}^{B} : \mathscr{L}(V,W) \to \mathbf{F}^{m\times n}$

    Definitions from 10/17 notes: For a vector $v \in V$ and a pair of linear transformations $T : V \to W$ and $S : W \to U$

11. define $[T]_{B'}^{B}[v]_B$. How should we denote (write down a symbol for) it?
12. define $[S]_{B''}^{B'}[T]_{B'}^{B}$. How should we denote (write down a symbol for) it? Hint: it may be easier to figure out first how to denote (write down a symbol for) $[T]_{B'}^{B}[v]_B$ and $[S]_{B''}^{B'}[T]_{B'}^{B}$ then figure out how you should define it.

Prove the following (what I call the fundamental theorem of linear maps):

Problem 2 (3.5)**.** Suppose $v_1, ..., v_n$ is a basis of V and $w_1, ..., w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, ..., n$

Problem 3. State and prove (Axler's) Fundamental Theorem of Linear Maps.

Do problems 4-9 to obtain the remaining 30 points:

Problem 4. (5 points) Let E_n be the standard basis for $\mathbf{R}^n$. Suppose T is a linear map from $\mathbf{R}^3 \to \mathbf{R}^2$ and

$$[T]_{E_2}^{E_3} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Write T as a linear map. What vector spaces are the kernel and range of T subspaces of? Find a basis of the kernel. Find a basis of the range. What are the dimensions of these spaces?

Problem 5. (5 points) Let $T : \mathbf{R}^5 \to \mathbf{R}^3$ be a linear transformation whose kernel is of dimension 3. What is the dimension of the range? What does the set of points in the range look like geometrically? Hint: there are only 4 possible things that it could look like.

Problem 6. (5 points) Show that every linear map from a 1-dimensional space to itself is multiplication by some scalar. More precisely, prove that if $dim V = 1$ and $T \in \mathscr{L}(V, V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Problem 7. (5 points) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ by $T(x, y, z) = (x + y + z, 0, 0)$ find a basis for the kernel of T. What is the dimension of the range? What is the dimension of the kernel?

Problem 8. (5 points)Let $T : \mathbf{R}^3 \to \mathbf{R}^3$. Suppose

$$[T]_{B'}^{B} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

where $B = v_1, v_2, v_3$ and $B' = w_1, w_2, w_3$ are both bases for $\mathbf{R}^3$. Apply the base change formula to obtain

$$[T]_{C'}^{C} = \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$$

Write the base change matricies out. Begin by figuring out out what C and C' are.

Problem 9. (5 points) Use Problem 2, that is, apply Theorem (3.5) to show that if two finite dimensional vector spaces V and W have the same dimension, then they must be isomorphic.

Problem 10 (Bonus 5 points)**.** Suppose V and W are of dimension n and m respectively, pick a vector $v \in V$. Define

$$E_v = \{T \in \mathscr{L}(V, W) | Tv = 0\}$$

that is, E_v is the set of linear transformations which send v to $0 \in W$. Show that E_v is a subspace of $\mathscr{L}(V, W)$. Suppose $v \neq 0$ what is the dimension of E_v?