Math 260 Exercises 3.A Solutions

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Wednesday October 24th, 2018

Problem 1. Suppose $T \in \mathscr{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{(j,k)} \in \mathbf{F}$ for $j = 1, ..., m$ and $k = 1, ..., n$ such that

$$T(x_1, .., x_n) = (A_{(1,1)}x_1 + ... + A_{(1,n)}x_n, ..., A_{(m,1)}x_1 + ... + A_{(m,n)}x_n)$$

for every $(x_1, ..., x_n) \in \mathbf{F}^n$.

Solution 1. Keep track of which vectors are which in vectorspace, particularly the basis vectors.

Let $E = (1, 0, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, ..., 1)$ be the usual basis (of length n) for $\mathbf{F}^n$. Then for each vector in the basis we can apply T and obtain a list of n vectors $T(1, 0, 0, ..., 0)$, $T(0, 1, 0, ..., 0)$,...,$T(0, 0, ..., 1)$ in $\mathbf{F}^m$. Each of these vectors in $\mathbf{F}^m$ can be expressed in terms of the standard basis for $\mathbf{F}^m$ as follows:

$$T(1, 0, 0, ..., 0) = A_{(1,1)}(1, 0, 0, ..., 0) + A_{(2,1)}(0, 1, 0, ..., 0) + ... + A_{(m,1)}(0, 0, ..., 1)$$

$$T(0, 1, 0, ..., 0) = A_{(1,2)}(1, 0, 0, ..., 0) + A_{(2,2)}(0, 1, 0, ..., 0) + ... + A_{(m,2)}(0, 0, ..., 1)$$

$$T(0, 0, 1, ..., 0) = A_{(1,3)}(1, 0, 0, ..., 0) + A_{(2,3)}(0, 1, 0, ..., 0) + ... + A_{(m,3)}(0, 0, ..., 1)$$

and so on

$$T(0, 0, 0, ..., 1) = A_{(1,n)}(1, 0, 0, ..., 0) + A_{(2,n)}(0, 1, 0, ..., 0) + ... + A_{(m,n)}(0, 0, ..., 1)$$

Note: in each of the sums above, there are exactly m summands.

Expressing $(x_1, ..., x_n)$ in terms of the basis for $\mathbf{F}^n$,

$$(x_1, ..., x_n) = x_1(1, 0, 0, ..., 0) + x_2(0, 1, 0, ..., 0) + ... + x_n(0, 0, ..., 1)$$

and applying the linear transformation T we obtain a vector in $\mathbf{F}^m$:

$$\begin{aligned} T(x_1, ..., x_n) = x_1 T(1, 0, 0, ..., 0)+ \\ x_2 T(0, 1, 0, ..., 0)+ \\ ... \\ +x_n T(0, 0, ..., 1) \end{aligned}$$

representing the image of each basis vector under T as a linear combination of the basis for $\mathbf{F}^m$ as above, we obtain the rather large sum:

$$\begin{aligned} T(x_1, ..., x_n) = x_1(A_{(1,1)}(1, 0, 0, ..., 0) + A_{(2,1)}(0, 1, 0, ..., 0) + ... + A_{(m,1)}(0, 0, ..., 1))+ \\ x_2(A_{(1,2)}(1, 0, 0, ..., 0) + A_{(2,2)}(0, 1, 0, ..., 0) + ... + A_{(m,2)}(0, 0, ..., 1))+ \\ ... \\ +x_n(A_{(1,n)}(1, 0, 0, ..., 0) + A_{(2,n)}(0, 1, 0, ..., 0) + ... + A_{(m,n)}(0, 0, ..., 1)) \end{aligned}$$

Gathering like terms it is easy to see that this is exactly the form desired in the statement of the problem.

Problem 2. Suppose $T \in \mathscr{L}(V, W)$ and $v_1, ..., v_m$ is a list of vectors in V such that $Tv_1, ..., Tv_m$ is a linearly independent list in W. Prove that $v_1, ..., v_m$ is linearly independent.

Solution 2. Let $a_1, ..., a_m$ be scalars such that $a_1v_1 + ... + a_mv_m = 0$. We need to show that $a_1 = ... = a_m = 0$. Apply T,

$$T(a_1v_1 + ... + a_mv_m) = T(0) = 0$$

but by linearity

$$a_1Tv_1 + ... + a_mTv_m = 0$$

since $Tv_1, ..., Tv_m$ is a linearly independent list in W then $a_1 = ... = a_m = 0$.

Problem 3. Show that every linear map from a 1-dimensional space to itself is multiplication by some scalar. Prove that if $dimV = 1$ and $T \in \mathscr{L}(V)$, then there exists $\lambda \in \mathscr{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution 3. Since $dimV = 1$, the length of any basis is 1. Pick a basis for V, it consists of a list of a single non-zero vector. Call this vector u. Pick any non-zero vector $v \in V$. Then v can be expressed as a linear combination of the basis, $v = cu$ for $c \neq 0 \in \mathbf{F}$. Applying T to u we obtain a vector Tu in V. Tu can be expressed in terms of a basis for V as $Tu = \lambda u$ for some $\lambda \in \mathbf{F}$. Multiplying both sides by c we obtain $cTu = c\lambda u$, consequently using the linearity of T, $Tcu = \lambda cu$, substituting in the equation $v = cu$, we obtain the result $Tv = \lambda v$.