Math 260 Exercises 1.B Solutions

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Problem 1. *Prove that* $-(-v) = v$ *for every* $v \in V$

Solution 1. *By the definition, it is clear that* $-(-v)$ *is the additive inverse for* $(-v)$, $-(-v) + (-v) = 0$, *however,* $v + (-v) = 0$ *so* v *is also an additive inverse for* $(-v)$. *Thus by the uniqueness of additive inverses* ***1.26*** *the result holds.*

Problem 2. *Suppose* $a \in \mathbb{F}$, $v \in V$ *and* $av = 0$. *Prove that* $a = 0$ *or* $v = 0$.

Solution 2. *Suppose* $v \neq 0 \in V$, *and suppose that* $a \neq 0$. *Then there exists a multiplicative inverse for* a, a^{-1} *such that* $a^{-1}a = 1$. *Then* $a^{-1}(av) = (a^{-1}a)v = 1v = v$ *Thus* $v = a^{-1}(av) = a^{-1}(0) = 0$ *where the last equality holds because a number times the* 0 *vector is* 0. *This is a contradiction. Therefore* $a = 0$.
Suppose $a \neq 0 in \mathbb{F}$, *then there exists a multiplicative inverse for* a, a^{-1} *such that* $a^{-1}a = 1$. *Then* $a^{-1}(av) = (a^{-1}a)v = 1v = v$. *Thus* $v = a^{-1}(av) = a^{-1}(0) = 0$ *where the last equality holds because a number times the* 0 *vector is* 0. *Thus* $v = 0$

Problem 3. *Suppose* $v, w \in V$. *Explain why there exists a unique* $x \in V$ *such that* $v + 3x = w$.

Solution 3. *There exists a unique additive identity for* v, $(-v)$. *Let* $x = (1/3)(w + (-v))$ *then* $v + (3)(1/3)(w + (-v)) = v + w - v = w$.

Problem 4. *The empty set is not a vectorspace. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?*

Solution 4. *The empty set does not have an additive identity.*

Problem 5. *Show that in the definition of a vectorspace (1.19), the additive inverse condition can be replaced with the condition that* $0v = 0$ *for all* $v \in V$. *Here* 0 *on the left side is the number* 0, *and the* 0 *on the right side is the additive identity of* V.

Solution 5. *Let* $0 = 0v = (1-1)v = 1v + (-1)v = v + (-1)v$. *Therefore,* $(-1)v$ *is an additive inverse for* v.

Problem 6. *Let* ∞ *and* $-\infty$ *denote two distinct objects, neither of which is in* $\mathbb{R}$. *Define an addition and scalar multiplication on* $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ *as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for* $t \in \mathbb{R}$ *define* $t\infty = -\infty$ *if* $t < 0$, $t\infty = 0$ *if* $t = 0$ *and* $t\infty = \infty$ *for* $t > 0$. *Also* $t + \infty = \infty + t = \infty$, $t + (-\infty) = (-\infty) + t = -\infty$.

$\infty + \infty = \infty$, $(-\infty) + (-\infty) = (-\infty)$, $0 = (-\infty) + \infty$.
Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vectorspace over $\mathbb{R}$? Explain.

Solution 6. *note that for any $t \in \mathbb{R}$, and any $s \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$, $t + s = (t+0)+s = (t+(-\infty+\infty))+s = ((t+-\infty)+\infty)+s = (-\infty+\infty)+s = 0+s = s$. Thus t is an additive identity for $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. Therefore, since $t \neq 0$ we fail to have uniqueness of additive identities. Thus $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ cannot be a vectorspace. You can follow this back further. What other properties does it not satisfy?*