Math 260 Exam 3 Take Home Solutions

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Problems 1-5 are worth 18 points each. Problem 6 is worth 10 points. The test is out of 100 points.

Definition 1. Let V be a real or complex vectorspace, $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. A norm on V is a real-valued function $|| \quad || : V \to \mathbf{R}$ such that

1. for any non-zero vector $v \in V$, $||v|| > 0$,
2. for any scalar $\alpha \in \mathbf{F}$, $||\alpha v|| = |\alpha| ||v||$ for all $v \in V$,
3. for any $u, v \in V$ $||u + v|| \le ||u|| + ||v||$

We call V a normed linear space.

Let $B = e_1, ..., e_n$ be the usual basis for $\mathbf{F}^n$. For instance, we know that $\mathbf{R}^n$ has the usual euclidean norm: for $v \in \mathbf{F}^n$, $v = a_1 e_1 + ... + a_n e_n$, define

$$||v|| = (\sum_{i=1}^{n} a_i^2)^{\frac{1}{2}} \tag{1}$$

Example 1.

$$||(1, -1)|| = \sqrt{1^2 + (-1)^2}$$

Clearly $\mathbf{F}^n$ is a normed linear space. You will show that if V is finite dimensional then $\mathscr{L}(V)$ is a normed linear space.

Definition 2. Let V be a finite dimensional normed linear space and let $T \in \mathscr{L}(V)$. Define the operator norm of T to be the smallest number M such that $||Tv|| \le M||v||$ for any $v \in V$. We will write $||T||$ to mean that smallest number M, the operator norm.

Notice that the norms in the expression $||Tv|| \le M||v||$ are the norm that V was born with. That is, this definition only makes sense if V has a norm.

Problem 1. Let $B = e_1, ..., e_n$ be an orthonormal basis for V a normed linear space of dimension n. Let $T \in \mathscr{L}(V)$. Let $m = Max\{||Te_1||, ||Te_2||, ..., ||Te_n||\}$. That is, m is the length of the longest vector in the list $Te_1, ..., Te_n$. Prove that for any unit-length vector $v \in V$, $||Tv|| \le mn$.

Solution 1. We have that $v = \sum_{i=1}^{n} a_i e_i$ moreover, the requirement that $||v|| = 1$ means that $\sum_{i=1}^{n} a_i^2 = 1$, and thus for each i, $|a_i| \le 1$. We compute,

$$||Tv|| = ||T(\sum_{i=1}^{n} a_i e_i)|| = ||\sum_{i=1}^{n} a_i Te_i|| \le \sum_{i=1}^{n} |a_i| ||Te_i|| \le \sum_{i=1}^{n} ||Te_i|| \le \sum_{i=1}^{n} m = nm. \tag{2}$$

Problem 2. Let $B = e_1, ..., e_n$ be an orthonormal basis for V a normed linear space of dimension n. Let $T \in \mathscr{L}(V)$. Show that the operator norm of T exists and is finite. (I am asking you to show that taking any $v \in V$, show that there exists a number K such that $||Tv|| \leq$K$||v||$.) Hint: Use the conclusion of the previous problem. Hint: maybe the problem is easier if you assume $||v|| = 1$?

Solution 2. Assume that $||v|| = 1$, then $||Tv|| \leq nm = nm||v||$. Letting $K = nm$ we are finished. Now suppose that we have any $u \in V$,

$$||Tu|| = ||u||||(\frac{1}{||u||}Tu)|| = ||u||||T(\frac{u}{||u||})||$$

A vector divided by it's norm is of norm 1. Let $v = \frac{u}{||u||}$. Then $||v|| = 1$. Furthermore,

$$||Tu|| = ||u||||Tv|| \leq ||u||nm||v|| = K||u||$$

where $K = nm||v|| = nm$.

Now that we know the operator norm exists and is finite:

Problem 3. Show that the operator norm is a norm (satisfies definition 1) on $\mathscr{L}(V)$ for a finite dimensional normed linear space V.

Solution 3. We must show that for $T \in \mathscr{L}(V)$ such that $T \neq 0 \in \mathscr{L}(V)$, $||T|| > 0$. Since $T \neq 0$, there exists at least one vector $v \in V$ such that $Tv \neq 0 \in V$. Thus

$$0 < ||Tv|| \leq ||T||||v||.$$

Since $Tv \neq 0$ it follows that $||v|| > 0$ and since $0 < ||T||||v||$ we must have that $||T|| > 0$ as desired.

Next we must show that for $T \in \mathscr{L}(V)$, $||\alpha T|| = |\alpha|||T||$
Given any $v \in V$ by the definition of the operator norm for the operator αT we have

$$||\alpha Tv|| \leq ||\alpha T||||v||.$$

However,

$$||\alpha Tv|| = |\alpha|||Tv|| \leq |\alpha|||T||||v||.$$

Thus, $||\alpha T|| \leq |\alpha|||T||$.

To show the reverse inequality, note that for $\alpha \neq 0$:

$$||Tv|| = ||\alpha T\frac{1}{\alpha}v|| \leq ||\alpha T||||\frac{1}{\alpha}v|| = |\frac{1}{\alpha}|||\alpha T||||v||$$

from which it follows that

$$||T|| \leq |\frac{1}{\alpha}|||\alpha T||$$

and thus

$$|\alpha|||T|| \leq ||\alpha T||.$$

The desired equality

$$|\alpha|||T|| = ||\alpha T||$$

is obtained. The case $\alpha = 0$ is immediate.

For any $T, S \in \mathscr{L}(V)$ and any $v \in V$, we compute

$$||(S+T)(v)|| = ||Sv+Tv|| \leq ||Sv||+||Tv|| \leq ||S||||v||+||T||||v|| = (||S||+||T||)||v||$$

from which it follows that

$$||S + T|| \leq ||S|| + ||T||$$

where the second inequality above comes from the triangle inequality for the norm on V.

Let T be an invertable linear operator of a finite dimensional normed linear space V.

Problem 4. Let v be an eigenvector for T with eigenvalue λ. Prove that $||Tv|| = |\lambda|||v||$. Prove that $|\lambda| \leq ||T||$.

Solution 4. We have $||Tv|| = ||\lambda v|| = |\lambda|||v||$. Now compute

$$|\lambda|||v|| = ||Tv|| \leq ||T||||v||$$

from which it follows that $|\lambda| \leq ||T||$.

Problem 5. Suppose $T \in \mathscr{L}(V)$ is invertible. Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

Solution 5. Let v be an eigenvector for λ. Then $T^{-1}Tv = T^{-1}\lambda v = \lambda T^{-1}v$. But since $T^{-1}Tv = v$, we must have that $\lambda T^{-1}v = v$ from which it follows that $T^{-1}v = \frac{1}{\lambda}v$. Since all steps of the proof hold with equality, the result holds in both directions.

Problem 6. Let V be a finite dimensional inner product space over $\mathbf{R}$. What can you say about the relationship between the induced norm on V and the operator norm on $\mathscr{L}(V, \mathbf{R})$? Define the operator norm on $\mathscr{L}(V, \mathbf{R})$ by letting $||\phi||$ for $\phi \in \mathscr{L}(V, \mathbf{R})$ be the smallest number M such that $|\phi(v)| \leq M||v||$ for all $v \in V$.
Hint: The Reisz Representation Theorem gives a nice association: for every $\phi \in \mathscr{L}(V, \mathbf{R})$ there exists a unique $v \in V$ such that $\langle\ \ , v\rangle$ is equal to $\phi(\ \ )$.

Solution 6. Pick any ϕ. There exists a unique $v \in V$ such that $\langle\ \ , v\rangle$ is equal to $\phi(\ \ )$. Now for any $u \in V$, we compute

$$|\phi(u)| = |\langle u, v\rangle| \leq ||u||||v||$$

by the The Reisz Representation Theorem and the Cauchy-Schwarz Inequality. It must follow that

$$||\phi|| \leq ||v||.$$

To show the reverse inequality note that

$$|\phi(v)| = |\langle v, v\rangle| = ||v||^2 \leq ||\phi||||v||$$

from which it follows that

$$||v|| \leq ||\phi||$$

and furthermore that

$$||\phi|| = ||v||.$$