# Math 260 Exam 3 Take Home

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Problems 1-5 are worth 18 points each. Problem 6 is worth 10 points. The test is out of 100 points.

**Definition 1.** Let $V$ be a real or complex vectorspace, $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. A norm on $V$ is a real-valued function $|| \quad || : V \to \mathbf{R}$ such that

1. for any non-zero vector $v \in V$, $||v|| > 0$,
2. for any scalar $\alpha \in \mathbf{F}$, $||\alpha v|| = |\alpha| ||v||$ for all $v \in V$,
3. for any $u, v \in V$ $||u + v|| \leq ||u|| + ||v||$

We call $V$ a normed linear space.

Let $B = e_1, ..., e_n$ be the usual basis for $\mathbf{F}^n$. For instance, we know that $\mathbf{R}^n$ has the usual euclidean norm: for $v \in \mathbf{F}^n$, $v = a_1 e_1 + ... + a_n e_n$, define

$$||v|| = (\sum_{i=1}^{n} a_i^2)^{\frac{1}{2}} \tag{1}$$

**Example 1.**

$$||(1, -1)|| = \sqrt{1^2 + (-1)^2}$$

Clearly $\mathbf{F}^n$ is a normed linear space. You will show that if $V$ is finite dimensional then $\mathscr{L}(V)$ is a normed linear space.

**Definition 2.** Let $V$ be a finite dimensional normed linear space and let $T \in \mathscr{L}(V)$. Define the operator norm of $T$ to be the smallest number $M$ such that $||Tv|| \leq M||v||$ for any $v \in V$. We will write $||T||$ to mean that smallest number $M$, the operator norm.

Notice that the norms in the expression $||Tv|| \leq M||v||$ are the norm that $V$ was born with. That is, this definition only makes sense if $V$ has a norm.

**Problem 1.** Let $B = e_1, ..., e_n$ be an orthonormal basis for $V$ a normed linear space of dimension $n$. Let $T \in \mathscr{L}(V)$. Let $m = Max\{||Te_1||, ||Te_2||, ..., ||Te_n||\}$. That is, $m$ is the length of the longest vector in the list $Te_1, ..., Te_n$. Prove that for any vector $v \in V$, $||Tv|| \leq mn$.

**Problem 2.** Let $B = e_1, ..., e_n$ be an orthonormal basis for $V$ a normed linear space of dimension $n$. Let $T \in \mathscr{L}(V)$. Show that the operator norm of $T$ exists and is finite. (I am asking you to show that taking any $v \in V$, show that there exists a number $K$ such that $||Tv|| \leq$K$||v||$.) Hint: Use the conclusion of the previous problem. Hint: maybe the problem is easier if you assume $||v|| = 1$?

Now that we know the operator norm exists and is finite:

**Problem 3.** Show that the operator norm is a norm (satisfies definition 1) on $\mathscr{L}(V)$ for a finite dimensional normed linear space $V$.

What does the operator norm have to do with the largest eigenvalue?

Let $T$ be an invertable linear operator of a finite dimensional normed linear space $V$.

**Problem 4.** Let $v$ be an eigenvector for $T$ with eigenvalue $\lambda$. Prove that $||Tv|| = |\lambda|||v||$. Prove that $|\lambda| \leq ||T||$.

**Problem 5.** Suppose $T \in \mathscr{L}(V)$ is invertible. Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that $\lambda$ is an eigenvalue of $T$ if and only if $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$.

**Problem 6.** Let $V$ be a normed linear finite dimensional inner product space over $\mathbf{R}$. What can you say about the relationship between the norm on $V$ and the operator norm on $\mathscr{L}(V, \mathbf{R})$? Define the operator norm on $\mathscr{L}(V, \mathbf{R})$ by letting $||\phi||$ for $\phi \in \mathscr{L}(V, \mathbf{R})$ be the smallest number $M$ such that $|\phi(v)| \leq M||v||$ for all $v \in V$.
Hint: The Reisz Representation Theorem gives a nice association: for every $\phi \in \mathscr{L}(V, \mathbf{R})$ there exists a unique $v \in V$ such that $< \ , v >$ is equal to $\phi(\ )$.