Math 260 Exercises 1.C Solutions

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Problem 1. *For each of the following subsets of $\mathbb{F}^3$, determine whether it is a sub-space for $\mathbb{F}^3$*

1. $\{(x_1, x_2, x_3) \in \mathbb{F}^3 | x_1 + 2x_2 + 3x_3 = 0\}$
2. $\{(x_1, x_2, x_3) \in \mathbb{F}^3 | x_1 + 2x_2 + 3x_3 = 4\}$
3. $\{(x_1, x_2, x_3) \in \mathbb{F}^3 | x_1 x_2 x_3 = 0\}$
4. $\{(x_1, x_2, x_3) \in \mathbb{F}^3 | x_1 = 5x_3\}$

Solution 1.

1. *Yes, as you should check.*
2. *No, as you should check.*
3. *No, notice that* $(1, 0, 1)$ *and* $(0, 1, 0)$ *are in this set but their sum is not:* $(1, 0, 1) + (0, 1, 0) = (1, 1, 1)$.
4. *Yes, as you should check.*

Problem 2. *Show that the set of differentiable real-valued functions* f *on the interval* $(-4, 4)$ *such that* $f'(-1) = 3f(2)$ *is a subspace of* $\mathbb{R}^{(-4,4)}$.

Solution 2. *See class notes on example* 1.35 *part d. For* f *and* g *satisfying the differential equations* $f'(-1) = 3f(2)$ *and* $g'(-1) = 3g(2)$*, the linearity of the derivative says that* $f'(-1) + g'(-1) = (f + g)'(-1)$*. Similarly,* $3f(2) + 3g(2) = 3(g + f)(2)$ *therefore* $f + g$ *satisfies the differential equation and therefore is in the subspace. Show that this set satisfies the* 2 *remaining conditions.*

Problem 3. *Is* $\mathbb{R}^2$ *a subspace of the complex vectorspace* $\mathbb{C}^2$*?*

Solution 3. *No. We shall have more to say on this later. For now, take note of a few things. As we said in class,* $\mathbb{R}^2$ *strictly speaking is not a subspace of* $\mathbb{R}^3$*. Think about their definitions. The subspace* W *of* $\mathbb{R}^3$ *which has vectors of the form* $(x_1, x_2, 0)$ *where* x_1 *and* x_2 *are in* $\mathbb{R}$ *looks enough like* $\mathbb{R}^2$*. That we call it* $\mathbb{R}^2$ *in* $\mathbb{R}^3$.

The other problem (one which was pointed out to me by a student) is that scalar multiplication is defined differently for $\mathbb{C}^2$ *and* $\mathbb{R}^2$*. They are vectorspaces with scalars coming from different fields. One can show that there are no sub-spaces of* $\mathbb{C}^2$ *which "look like"* $\mathbb{R}^2$ *in the way that* W *looks like* $\mathbb{R}^2$.

Problem 4. *Give an example of a nonempty subset* U *of* $\mathbb{R}^2$ *such that* U *is closed under addition and under taking additive inverses, but is not a subspace* *of* $\mathbb{R}^2$

Give an example of a nonempty subset U *of* $\mathbb{R}^2$ *such that* U *is closed under scalar multiplication but is not a subspace of* $\mathbb{R}^2$

Solution 4. *"I will not deprive you of the pleasure of discovering the answer for yourself." See if you can use some graphical intuition. What do all subspaces of* R^2 *look like? Lines or the full plane or the trivial subspace. See if you can find some other shapes which satisfy additive properties, multiplicative properties.*

Problem 5. *Suppose* U_1 *and* U_2 *are subspaces of* V*. Prove that the intersection* $U_1 \cap U_2$ *is a subspace of* V

Solution 5. *Suppose* $u \in U_1 \cap U_2$ *and* $w \in U_1 \cap U_2$*. Then* $u, w \in U_1$ *and therefore* $u + w \in U_1$*. Similarly,* $u, w \in U_2$ *and therefore* $u + w \in U_2$*. So* $u + w \in U_1$ *and* $u + w \in U_2$*. Therefore* $u + w \in U_1 \cap U_2$*. So* $U_1 \cap U_2$ *is closed under addition. Verify the other* 2 *properties from* 1.34*.*

Problem 6. *Suppose*

$$U = \{(x, x, y, y) \in \mathbb{F}^4 | x, y \in \mathbb{F}\}.$$

Find a subspace W *of* $\mathbb{F}^4$ *such that* $\mathbb{F}^4 = U \oplus W$

Solution 6. *Going back to the definition, we need to find a description of a subspace* W *such that given any vector in* $\mathbb{F}^4$*, that vector can be decomposed uniquely as a sum of a vector in* U *and a vector in* W*.*

Let v *be any vector in* $\mathbb{F}^4$*. By definition* $v = (\alpha, \beta, \gamma, \delta)$ *where* $\alpha, \beta, \gamma, \delta \in \mathbb{F}$*. We would like to find a unique way to write* v *as a sum of a vector in* U *and a vector in* W*, written* $v = u + w$ *where* $u \in U$ *and* $w \in W$*. Since* $u \in U$ *it will be of the form shown above. In particular we could let* $u = (\alpha, \alpha, \gamma, \gamma)$ *Expanding* $v = u + w$*:*

$$(\alpha, \beta, \gamma, \delta) = (\alpha, \alpha, \gamma, \gamma) + w$$

subtracting the vector $(\alpha, \alpha, \gamma, \gamma)$ *from both sides, we obtain*

$$(\alpha, \beta, \gamma, \delta) - (\alpha, \alpha, \gamma, \gamma) = (0, \beta - \alpha, 0, \delta - \gamma) = w.$$

Define $W = \{(0, \beta - \alpha, 0, \delta - \gamma) | \alpha, \beta, \gamma, \delta \in \mathbb{F}\}$*.*

It remains to check that for every $(\alpha, \beta, \gamma, \delta) \in \mathbb{F}^4$ *it has a unique decomposition as a sum of two vectors, one from* U *and one from* W*.*

Letting $(p, q, r, s) \in \mathbb{F}^4$*,* $(p, q, r, s) = u + w$*. We see that* $(p, q, r, s) = (p, p, r, r) + (0, q - p, 0, s - r)$ *where* $u = (p, p, r, r)$ *and* $w = (0, q - p, 0, s - r)$*. Thus* $U + W = \mathbb{F}^4$*. It remains to show that* $U + W$ *is a direct sum.*

To show $U \oplus W$, we need to show the decomposition $v = u + w$ is unique. Note that the first component of w must be 0 (because otherwise it wouldn't be of the form of vectors in W). This means that the first component of u must be p. The first component of u must be the same as the second component, so the second component of u must be p. This determines uniquely the first and second elements of u and w.

Similarly, the third component of w must be 0 so the third component of u must be r. The third component of u must be the same as the fourth component. This determines uniquely the third and fourth elements of w.

For yourself you should make sure that you can verify that W is subspace.

Problem 7. *Suppose*

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 | x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$

Solution 7. *Let $(\alpha, \beta, \gamma, \delta, \varepsilon) \in \mathbb{F}^5$. We are trying to find a decomposition like so:*

$$(\alpha, \beta, \gamma, \delta, \varepsilon) = (x, y, x + y, x - y, 2x) + w$$

Let x be α and y be β and subtract to solve for w.

$$(\alpha, \beta, \gamma, \delta, \varepsilon) - (\alpha, \beta, \alpha + \beta, \alpha - \beta, 2\alpha) = (0, 0, \gamma - (\alpha + \beta), \delta - (\alpha - \beta), \varepsilon - 2\alpha) = w$$

Then let $W = \{(0, 0, \gamma - (\alpha + \beta), \delta - (\alpha - \beta), \varepsilon - 2\alpha) | \alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{F}\}$. I will leave it to you to check that $U + W = \mathbb{F}^5$ and then check that $U \oplus W = \mathbb{F}^5$. In english: check that every vector in $\mathbb{F}^5$ can be written in this way. Then check that this way is unique. For yourself, check W is a subspace.

Problem 8. *Suppose*

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 | x, y \in \mathbb{F}\}.$$

Find three subspaces W_1, W_2, and W_3 of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution 8. *Let W be the same subspace as in the answer to the previous problem, the problem will be solved if we can find W_1, W_2, and W_3 such that $W = W_1 \oplus W_2 \oplus W_3$.*

Conjecture that $W_1 = \{(0, 0, x, 0, 0) | x \in \mathbb{F}\}$, $W_2 = \{(0, 0, 0, y, 0) | y \in \mathbb{F}\}$ and $W_3 = \{(0, 0, 0, 0, z) | z \in \mathbb{F}\}$. It is easy to show that $W_1 \oplus W_2 \oplus W_3 = \{(0, 0, x, y, z) | x, y, z \in \mathbb{F}\}$. Now we need to verify that $W = W_1 \oplus W_2 \oplus W_3$.

(We could also solve the problem by proving that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$. This is a simpler way. To show this is a direct sum note that the intersection of

U and $W_1 \oplus W_2 \oplus W_3$ is only the 0 vector. There are a few more things to verify.)

First we show that W is contained in $\{(0,0,x,y,z)|x,y,z \in \mathbb{F}\}$. For any $(0,0,\gamma-(\alpha+\beta),\delta-(\alpha-\beta),\varepsilon-2\alpha) \in W$, just let $x = \gamma-(\alpha+\beta)$, let $y = \delta-(\alpha-\beta)$ and let $z = \varepsilon - 2\alpha$.

To show that $\{(0,0,x,y,z)|x,y,z \in \mathbb{F}\}$ is contained in W is slightly harder. Let $(0,0,x,y,z) \in \{(0,0,x,y,z)|x,y,z \in \mathbb{F}\}$. To show that $(0,0,x,y,z) \in W$ we need to find α, β, γ, δ, and ε such that $\gamma-(\alpha+\beta) = x$, $\delta-(\alpha-\beta) = y$ and $\varepsilon - 2\alpha = z$. Let α and β both equal 0. Let $\gamma = x$, $\delta = y$, and $\varepsilon = z$. Thus, $(0,0,x,y,z) \in W$, $\{(0,0,x,y,z)|x,y,z \in \mathbb{F}\}$ is contained in W, and the proof is complete.