Math 260 Exercises 2.B Solutions

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Problem 1. Verify all the assertions in Example 2.28.

Solution 1. Omitted.

Problem 2. Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \textit{ and } x_3 = 7x_4\}$$

1. Find a basis of U.
2. Extend the basis to a basis of $\mathbf{R}^5$.
3. Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$

Solution 2.

1. The list $\{(3,1,0,0,0),(0,0,7,1,0),(0,0,0,0,1)\}$ suffices. It is easily seen to be linearly independent.

   Suppose we take any vector $(x_1, x_2, x_3, x_4, x_5) \in U$. Then

   $$(x_1, x_2, x_3, x_4, x_5) = x_2(3,1,0,0,0) + x_4(0,0,7,1,0) + x_5(0,0,0,0,1)$$

   shows that the list spans U. Thus it is a basis.

2. append the standard basis $\{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1)\}$ to the end of the list. Clearly $(0,0,0,0,1)$ is already in the list and therefore is already in it's span. We may remove it. Similarly, remove more of the vectors until we are left with a linearly independent list following the procedure of the proof of 2.33.

3. Taking the span of the first and third standard basis vectors gives the desired subspace W.

Problem 3. Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Solution 3. The key thing to note here is that degree is simply the largest power present in the polynomial. Therefore, $x^3 + x^2$ has degree 3. A counter example is given by the list $1 + x^3$, $x + x^3$, $x^2 + x^3$, x^3.

Problem 4. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, ...u_m$ is a basis for U and $w_1, ..., w_n$ is a basis of W. Prove that

$$u_1, ..., u_m, w_1, ..., w_n$$

is a basis of V.

Solution 4. Note that because U and W form a direct sum their intersection is $\{0\}$. The u's form a basis and are therefore linearly independent. We will add the w's one at a time to the list of u's.

Step 1

Adding w_1 we obtain

$$u_1, ..., u_m, w_1$$

which is still independent by the following application of the Linear Dependence Lemma. Suppose the list were dependent. By the lemma, one of the elements would be in the span of the previous ones. Since

$$u_1, ..., u_m$$

is independent, w_1 must be the element in the span of the u's. This however would contradict the fact that U and W has intersection $\{0\}$.

Step j

Adding w_j to the list obtained in step $j-1$ we obtain the list

$$u_1, ..., u_m, w_1, ...w_{j-1}, w_j$$

which is clearly still linear independent because w_j cannot be in the span of the previous elements. Suppose

$$w_j = \alpha_1 u_1 + ... + \alpha_m u_m + \beta_1 w_1 + ... + \beta_{j-1} w_{j-1}$$

then

$$w_j - (\beta_1 w_1 + ... + \beta_{j-1} w_{j-1}) = \alpha_1 u_1 + ... + \alpha_m u_m$$

which implies since the intersection of U and W is the 0 vector that

$$w_j - (\beta_1 w_1 + ... + \beta_{j-1} w_{j-1}) = 0.$$

The equation above contradicts the linear independence of the w's and therefore

$$u_1, ..., u_m, w_1, ...w_{j-1}, w_j$$

is linearly independent.

We may repeat step j until all of the w's are added.

Note that any vector in V can be written as a sum of a vector in U plus a vector in W. It follows that the list created at the final step spans V. Since it is also linearly independent, the list is a basis.