Differential Equations

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Abstract

For this assignment, I will be defining and analyzing mulitiple topics in ODE, such as: Differential Operators, Linear Differential Operators (LDO), Solutions and the Wronskian.

Differential Operator

We will begin by defining what an **operator** is. An **operator** is a function whose domain is a set of functions. A **Differential Operator** is a function of the differentiation operator, it is helpful and primarily used for notation.

There are **two** types of differential operators:

- Linear Operators
- Non-linear operators (Ex: Schwarizan Derivative)

A differential operator is denoted **D**.

Dx = x', here we are applying **D** to the function f(x) to get the derivative.

Examples:

- **D**(x^5)= $5x^4$
- **D**(cosx) = -sinx

We can also use **D** as a polynomial differential operator.

Examples:

- y"+4y'+4y=0 can be written as $\mathbf{D}^2$y+4**D**y+4y=0
- (**D**+7)(**D**-3)=0 can be written as $\mathbf{D}^2$+4**D**-21=0

Linear Differential Operator

Linear Differential Operator (LDO) - The set of derivatives that work on a function to provide a solution.

A linear operator, denoted **L**, is an operator whose constants c_1, c_2 and any functions x_1 and x_2 is **L**[$c_1x_1+c_2$+x_2]= c_1x_1 + c_2+x_2.

Example: L[x]= x"+px'+qx can also be written as **L** = $\mathbf{D}^2$+p**D**+qI, hence, suggesting **L** is linear.
Proof:

$$\begin{aligned} L[c_1x_1 + c_2x_2] &= (c_1x_1 + c_2x_2)'' + p(c_1x_1 + c_2x_2)' + q(c_1x_1 + c_2x_2) \\ &= c_1x_1'' + c_2x_2'' + c_1px_1' + c_2px_2' + c_1qx_1 + c_2qx_2 \\ &= c_1(x_1'' + px_1' + qx_1) + c_2(x_2'' + px_2' + qx_2) \\ &= c_1L[x_1] + c_2L[x_2] \end{aligned}$$

Linear Differential Operator of degree n - "A polynomial in D of degree n whose coefficients are continuous functions of t"
Consider the ODE -

$$L = a_n(t)D^n + a_n + 1(t)D^{n-1} + ... + a_1(t)D[x] + a_0(t)I$$
$$L = a_n(t)D^n[x] + a_n + 1(t)D^{n-1}[x] + ... + a_1 D[x] + a_0 x$$
$$= a_n x^{(n)} + a_n + 1x^{(n-1)} + ... + a_1 x' + a_0 x$$

Properties of differential operators:

- **Sum** - Let p(D) and q(D) be polynomial operators such that for any function u
$$[\mathrm{p(D)+q(D)]u = p(D)u+q(D)u}$$
- **Linearity** - Let u_1 and u_2 be functions and c_1 a constant such that
$$\mathrm{p(D)(c_1u_1+c_2u_2) = c_1p(D)u_1+c_2(D)u_2}$$
- **Multiplication** - Let p(D) = g(D)h(D) be polynomials in D such that
$$\mathrm{p(D)u = g(D)h(D)u)}$$
- **Substitution** - $\mathrm{p(D)e}^{ax} = \mathrm{p(a)e}^{ax}$
Proof:
$$\mathrm{De}^{ax} = \mathrm{ae}^{ax} \Rightarrow D^2\mathrm{e}^{ax} = \mathrm{a}^2\mathrm{e}^{ax}, \ ... \ ,\mathrm{D}^k\mathrm{e}^{ax} = \mathrm{a}^k\mathrm{e}^{ax}$$
$$(\mathrm{D}^n+\mathrm{c}_1\mathrm{D}^{n-1}+ \ ... \ +\mathrm{c}_1)\mathrm{e}^{ax} = (\mathrm{a}_n+\mathrm{c}_1\mathrm{a}^{n-1}+ \ ... \ +\mathrm{c}_n)\mathrm{e}^{ax}$$
- **Exponential Shift** - For functions: $\mathrm{x}^k\mathrm{e}^{ax}$ and x^ksinax
$$\mathrm{p(D)e}^{ax}\mathrm{u} = \mathrm{e}^{ax}\mathrm{p(D+a)u}$$
Proof:
$$\text{When p(D) = D} \Rightarrow De^{ax} = \mathrm{e}^{ax}\mathrm{Du(x)+ae}^{ax}\mathrm{u(x)} = \mathrm{e}^{ax}\mathrm{(D+a)u(x)}$$
$$\mathrm{D}^2\mathrm{e}^{ax}\mathrm{u} = \mathrm{D(De}^{ax}\mathrm{u)} = \mathrm{D(e}^{ax}\mathrm{(D+a)u)}$$
$$= \mathrm{e}^{ax}\mathrm{(D+a)((D+a)u} \Rightarrow e^{ax}\mathrm{(D+a)}^2 \text{ by mult. rule}$$

Example: Find $\mathrm{D}^3\mathrm{e}^{-x}$sinx
Solution:
$$\text{Using bullet 5, we get } \mathrm{D}^3\mathrm{e}^{-x}\mathrm{sinx} = \mathrm{e}^{-x}\mathrm{(D\text{-}1)}^3\mathrm{sinx} = \mathrm{e}^{-x}(\mathrm{D}^3\text{-}3\mathrm{D}^2+3\mathrm{D}\text{-}1)\mathrm{sinx}$$
$$= \mathrm{e}^{-x}(2\mathrm{cosx}+2\mathrm{sinx})$$
$$\mathrm{D}^2\mathrm{sinx} = \text{-sinx and } \mathrm{D}^3\mathrm{sinx} = \text{-cosx}$$

Idea:
$$\frac{d^m x}{dt^m} \Rightarrow D^m x$$
,

Take m^{th} derivative of whatever you're working with

$$\frac{d^3 x}{dt^3} \Rightarrow D^3 x$$

Example: $3 \cdot \frac{d^2x}{dt^2}$- t $\cdot \frac{dx}{dt} = t^2$
Operator Notation:
$$3\mathrm{D}^2\mathrm{x\text{-}tDx} = \mathrm{t}^2$$
$$(3\mathrm{D}^2\text{-tD})\mathrm{x} = \mathrm{t}^2$$
$\mathbf{(3D^2\text{-}tD)x}$ is the diff. operator "L"

Solutions of Differential Equations

A **solution** to an ODE is any function y(t) that satisfies the solution.

There are **two** types of ODE solutions:

- General
- Particular

A **general solution** involves x arbitrary elements, it is also known as the **complete solution**.

Theorem: "If and are continuous over the open interval I and is never zero on I,then the linear homogeneous equation (2) has two linearly independent solutions and on I. Moreover, if and are any two linearly independent solutions of Equation (2), then the general solution is given"

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \text{ where } c_1 \text{ and } c_2 \text{ are constants}$$

Theorem: If r_1 and r_2 are two real and unequal roots to $ar^2+br+c = 0$, then

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \text{ is the general solution to ay" + by' + cy = 0}$$

Theorem: If r is the only (repeated) real root to the equation, $ar^2+br+c = 0$, then

$$y = c_1 e^{rx} + c_2 x e^{rx}, \text{ is the general solution to ay" + by' + cy = 0}$$

Example: Find the general solution for the following ODE:

$$dy+8xdx = 0$$

Solution:

$$dy = -8xdx$$
$$\int dy = -\int 8xdx$$
$$y = \frac{-8}{2}x^2 + K$$

Example: Find the general solution for the following ODE:

$$x^2y'\text{-}7xy'+16y = 0$$

Solution:

$$r(r-1) - 7r + 16 = 0$$
$$r^2 - 8r + 16 = 0$$
$$(r-4)^2 = 0$$

A **particular solution** is acquired when particular values (constants) are chosen to satisfy the general solution.

Theorem: The particular solution satisfying the initial condition $y(x_0) = y_0$ is the solution $y = y(x)$ whose value is y_0 when $x = x_0$.

Theorem: The general solution to the nonhomogeneous differential equation has the from $y = y_c + y_p$, where the complementary solution y_c is the general solution to the associated homogeneous equation and y_p is any **particular solution** to the nonhomogeneous equation.

Example: Find the particular solution of dy=-8xdx at y(0)=3

Solution:

$$dy = -8xdx$$
$$\int dy = -\int 8xdx$$
$$y = \frac{-8}{2}\ x^2 + K$$
$$3 = \frac{-8}{2}\ (0)^2 + K$$
$$3 = K$$
$$y = \frac{-8}{2}\ x^2 + 3$$
$$r = 4$$
$$y = c_1x^4 + c_2x^4lnx$$

Example: Find the particular solution for

$$\text{y"-4y'-12y} = 3\text{e}^{5t}$$

Solution:

$$y'' - 4y' - 12y = 0$$
$$r^2 - 4r - 12 = (r-6)(r+2) = 0$$
$$r_1 = -2, r_2 = 6$$
$$y_c(t) = c_1e^{-2t} + c_2e^{6t} \Rightarrow Complimentary$$
$$y_p(t) = Ae^{5t}$$
$$25Ae^{5t} - 4(5Ae^{5t}) - 12(Ae^{5t}) = 3e^{5t}$$
$$-7Ae^{5t} = 3e^{5t}$$
$$-7A = 3$$
$$A = \frac{-3}{7}$$
$$Y_p(t) = \frac{-3}{7}\ e^{5t} \Rightarrow Particular$$

Wronskian

The **Wronskian** of two differentiable functions f and g is W(f,g) = fg'-gf'

$$W(f_1, ..., f_n) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & f_3'(x) & \dots & f_n'(x) \\ \dots & \dots & \dots & \dots & \dots \\ f_1^{n-1}(x) & f_2^{n-1}(x) & f_3^{n-1}(x) & \dots & f_n^{n-1}(x) \end{vmatrix}$$

Theorem: Let f and g be differentiable on [a,b], if the Wronskian is nonzero for some t_0 in [a,b] then f and g are linearly independent on [a,b]. If f and g are linearly dependent then the Wronskian is zero in [a.b].

Example: Find the Wronskian of e^{-2t}, te^{-2t}

Solution:

$$W(e^{-2t}, te^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2e^{-2t} \end{vmatrix}$$

$$e^{-2t}(e^{-2t} - 2te^{-2t}) - te^{-2t}(-2e^{-2t})$$
$$e^{-4t} - 2te^{-4t} + 2te^{-4t}$$
$$e^{-4t} \neq 0$$
$$Linearly Independent$$

Sources

Differential Operator/Linear Differential Operator

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Solutions (General and Particular)

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Wronskian

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