All I Know Is I Know Nothing:

Hyperbolic Geometry and Hyperboloids

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Linear Algebra—Extra Credit

Hyperbolic Geometry and Hyperboloids

Immanuel Kant best articulated my experience thus far with non-Euclidean geometries and particularly hyperbolic geometry: “Whereas the beautiful is limited, the sublime is limitless, so that the mind in the presence of the sublime, attempting to imagine what it cannot, has pain in the failure but pleasure in contemplating the immensity of the attempt”. My former astronomy professor and research supervisor would likely scoff, as he chided me for using synonyms in my report documenting my analysis of hundreds of cepheids (to study diffuse interstellar bands). “Imagination is the enemy of the precise, Gabe!” Yet obscure and amazing subjects such as these have motivated me over and over and over again to expand my understanding, and geometry is visual, colorful, and messy, at least one-quarter art. This paper will likely testify more to my confusion than anything, but forms a necessary step toward my ultimate understanding.

Hyperbolic Geometry

“A triangle sum is not always equal to 180°.” WHAT?!! “Some geometries do not…” Wait! GEOMETR**IES**? Ahem. “Some geometries do not follow Euclid’s axiom that given a line and a point, there exists a line, a single line, through the point and parallel to the original line.” This reality-shattering moment led to an exploration of hyperbolic geometry. How could it not when such geometry looks a lot like this! (see coral reef

below). I discovered a number of fascinating divergences from Euclidean geometry. Hyperbolic triangles ALWAYS sum to less than 180°. The converse of the Alternate Interior Angle Theorem does not hold. Rectangles cease to exist there (two parallel lines have at most one common perpendicular line). Parallel lines are NOT everywhere equidistant. For a given line and a point not on that line, there exist AT LEAST TWO lines passing through the point parallel to the given line. And parallelism is NOT transitive. For example, with some sets of three lines, two are parallel to a third but NOT each other. MIND BLOWN!

Physical Representations of Hyperboloids

The image (of anything, everything) always serves as my first inspiration, so I will start my discussion of hyperboloids there. A hyperboloid of two sheets resembles a pair of bowls, one oriented "up" and one "down". This figure depicts a Minkowski space, a four-dimensional spacetime continuum that preserves distances in ways Euclidean space cannot (as Euclidean experiences time dilation and length contraction). This formulation also represents time differently than other spacetime continua, as the time term is negative: $-t^2 + x^2 + y^2 + z^2$ Once, scholars considered space and time as independent quantities. "An event could be given coordinates $(x_1, . . . , x_{n+1}) \in R_{n+1}$, with the coordinate x_{n+1} representing time, and the only reasonable metric was the Euclidean metric with the positive definite square-norm $x_1{}^2 + \cdot \cdot \cdot + x_{n+1}{}^2$. The above image depicts hyperbolic space as, at least in the hyperboloid of two sheets model, the union of disjoint sets (upper half plane and lower half plane) and a line, or boundary circle, which restricts hyperbolic space, but does not belong to it.

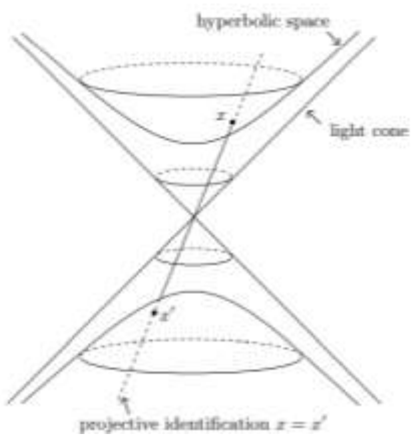


Fig. 1: Minkowski Space

Simplified Quadrics

Quadrics (generalizations of conic sections—parabolas, hyperbolas, and ellipses) are all surfaces that can be expressed as a second degree polynomial in x, y and z. Dr.

Kris Green of St. John Fisher College simplified the equations for hyperboloids to help her Vector Calculus students better understand the anatomy of a hyperboloid, including useful and quite beautiful pictures. She urged students to view a hyperboloid as a function G of three variables $G(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}$. and to examine "contours," what she called "level surfaces" of the form $G(x,y,z) = k$. The constant k determines the surface's shape. If $k > 0$, the function produces a hyperboloid of one sheet. This object is also what emerges from rotating two hyperbolas around an axis.

Let a, b, and c equal 1. Then $x^2 + y^2 - z^2 = 1$ or $x^2 + y^2 = z^2 + 1$.

The "contours" are circles. The smallest circle corresponds to z = 0, giving it a radius of one. As z increases or decreases, the circles grow, in a symmetric way. Because the z term is squared, equal and opposite values produce identical contours.

When k = -1, the equation changes: $x^2 + y^2 = z^2 - 1$.

For this, z = 1 or -1 corresponds to points at the origin. The sum of two squares must always be greater than or equal to 0, so $z^2 - 1$ must also be greater than or equal to 0, meaning that circles cannot exist when the absolute value of z is less than 1.

When $k = 0$, we have $x^2 + y^2 = z^2$.

All the contours are circles, forming a cone. Fig. 1, the representation of Minkowski space, shows the cone sitting between the hyperboloid of one sheet and of two.

a) b) c)

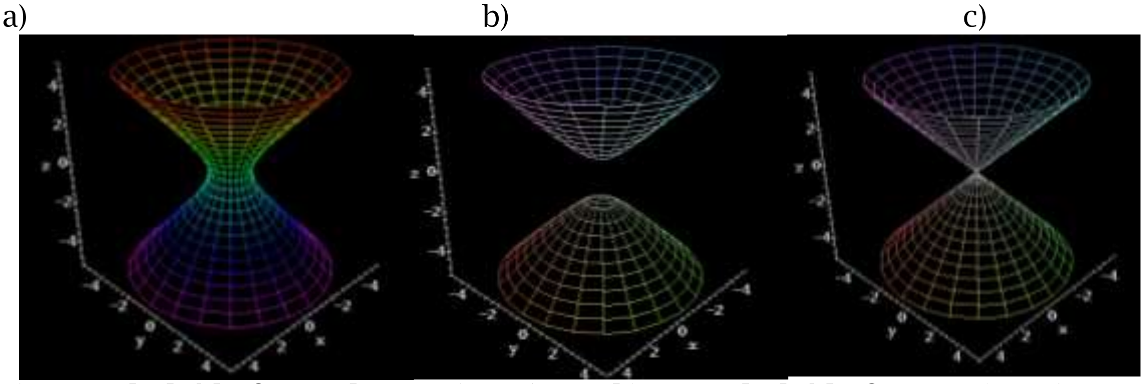


Fig. 2: a) Hyperboloid of one sheet, G(x,y,z) = 1; b) Hyperboloid of two, G(x,y,z) = -1, and c) Elliptical cone

Hyperboloids and Eigenvalues

After exploring the hyperboloid's physical appearance and basic makeup, I wanted to connect it to Linear Algebra.

Let x and v be vectors in vector space V.

Then a hyperboloid, with center v, has equation $(x-v)^T A\ (x-v) = 1$.

A's eigenvectors provide orientation and direction. Its eigenvalues are the reciprocals of the squares of the semi-axes:

$1/a^2$, $1/b^2$ and $1/c^2$.

For the one sheet, we find that the hyperboloid has two positive and one negative eigenvalue. Alternately, the two sheet has one positive and two negative eigenvalues.

Intersections

The intersection of a quadric surface and a plane is called a trace. A ***hyperboloid of one sheet*** has a trace of an ellipse parallel to the xy plane and traces of hyperbolas parallel to the xz and yz planes. We can determine the axis by looking at the variable whose coefficient is negative. A ***hyperboloid of two sheets*** has a trace of an ellipse parallel to the xy plane and traces of hyperbolas parallel to the xz and yz planes. The hyperboloid of two sheets does not REALLY have a trace in the xy plane, but its projection does. We can determine the axis by looking at the variable whose coefficient is positive. An ***elliptical cone*** has a trace of an ellipse parallel to the xy plane and traces of hyperbolas parallel to the xz and yz planes. We can determine the axis by looking at the variable whose coefficient is negative.

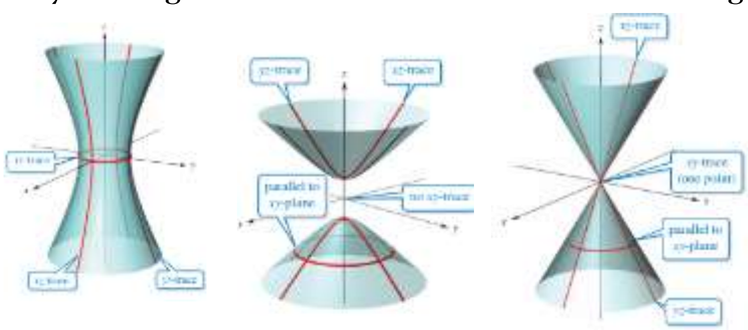
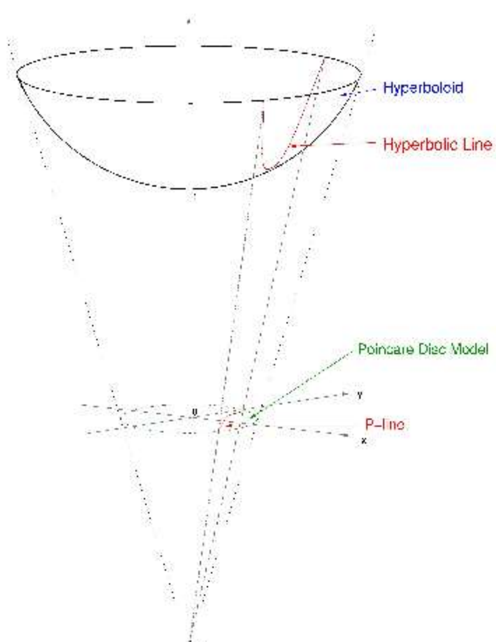


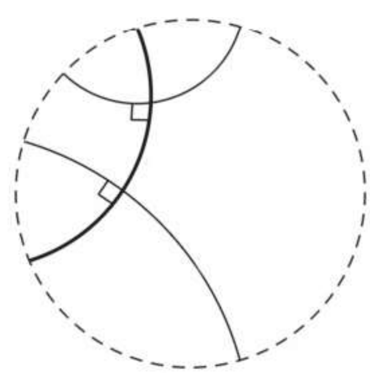
Fig. 3: Traces of a hyperboloid of one sheet, a hyperboloid of two, and c) an elliptical cone

Bektas delineated an algorithm for intersecting hyperboloids and planes using various rotations and for finding the intersection curves.

Conclusion

Below is a projection of the upper plane of a two sheeted hyperboloid to form what mathematicians call a Poincare disc. I love this image because it demonstrates what “lines” look like in hyperbolic geometry, as well as what passes for right angles. I know that this paper merely scratched the surface of what I wanted to write about (I didn’t even mention tangent planes or derivatives), but I remember so little of topology from my undergraduate years. All I know is I know nothing. That’s my conclusion. But this research has left me with an insanely strong desire to change that, especially after learning that hyperbolic geometry is useful for relativistic situations. Back when I was an astronomy major, the physics I enjoyed most was modern (e.g., quantum mechanics).





References

Bektas, S. (2017). On the intersection of a hyperboloid and a plane. *International Journal of Discrete Mathematics*, 2(2), 38-42. doi: 10.11648/j.dmath.20170202.12

Green, K. (1998). *Hyperboloids and cones*. Retrieved from http://citadel.sjfc.edu/faculty/kgreen/vector/block1/plane/node11.html