Math 260 Exam 1 Take Home

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Bonus 10 points

Let S be the unit sphere in $\mathbb{R}^3$. The set of points of the sphere does not make up a vectorspace because vector addition is not commutative.

In this bonus problem we will describe a way to define a vectorspace for each point of the sphere. For any point $x \in S$ we will define a vector space T_x, the tangent plane at a point of the sphere. This is a plane in $\mathbb{R}^3$ tangent to the sphere at the point in question.

Let $\gamma(t)$ be a curve which runs along the surface of the sphere. Let t be a time parameter in the range from $(-1, 1)$. Thus a curve is a function $\gamma(t) : (-1, 1) \to S$. The derivative of $\gamma(t)$ at a time $t = \alpha$ is the velocity vector of the curve at the time $\alpha \in (-1, 1)$. For instance we denote the velocity vector at $t = 0$, $\gamma'(0)$.

For a point $x \in S$ define the vector space T_x as follows:

Consider the collection of all curves $\gamma(t) : (-1, 1) \to S$ such that $\gamma(0) = x$. Some of these curves have the same velocity vector at the point x, i.e. there exist two curves γ_1, γ_2 such that $\gamma_1'(0) = \gamma_2'(0)$. A vector in T_x is going to be the collection of all curves which have a particular velocity vector at the point $t = 0$. For instance, pick a curve γ with velocity vector at $t = 0$ given by $\gamma'(0)$. Let $[\gamma]$ denote the collection of all curves which have the same velocity vector at x. This is a vector in T_x. Note that if $\gamma_1'(0) = \gamma_2'(0)$ then $[\gamma_1] = [\gamma_2]$.

The next question is how do we define vector space operations? How do we define addition of two collections of functions? What should we mean by $[\gamma] + [\delta]$? What do we mean by multiplication by a scalar $\lambda[\gamma]$? What is the zero vector?

Definition 1 (Addition in T_x)**.** Define $[\gamma] + [\delta]$ as follows: pick any $\gamma \in [\gamma]$ and any $\delta \in [\delta]$ and take the sum of their velocity vectors $\gamma'(0) + \delta'(0)$. Then pick any curve $\tau : (-1, 1) \to S$ such that $\tau(0) = x$ and $\tau'(0) = \gamma'(0) + \delta'(0)$. Define $[\gamma] + [\delta] = [\tau]$. We can also write $[\gamma] + [\delta]$ as $[\gamma + \delta]$.

Problem 1. Show that this definition of addition is independent of whichever vectors are choosen out of the collections $[\gamma]$ and $[\delta]$ i.e. pick $\gamma_1, \gamma_2 \in [\gamma]$, (note $[\gamma_1] = [\gamma_2] = [\gamma]$), also pick $\delta_1, \delta_2 \in [\delta]$ show that $[\gamma_1 + \delta_1] = [\gamma_1 + \delta_2] = [\gamma_2 + \delta_1] = [\gamma_2 + \delta_2]$.

Problem 2. Define the 0 vector and define multiplication by a scalar λ. You will also need to show that these definitions are independent of whichever vectors are chosen out of $[\lambda\gamma]$ and $[0]$.

Problem 3. Verify that T_x is a vectorspace.

Recall our discussion of the notation $\mathbf{F}^S$ meaning the collection of functions from S to $\mathbf{F}$.

Problem 4. Prove that for positive integers a, b, and c, $a^{b+c} = a^b \cdot a^c$ and $a^{b^c} = a^{bc}$. Along the way you will need to prove that certain collections of functions are in 1-1 correspondence.