Math 260 Exercises 1.A Solutions

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Problem 1. *Suppose a and b are real numbers, not both 0. Find real numbers c and d such that $\frac{1}{(a+bi)} = c + di$.*

Solution 1. *Notice that we can multiply the denominator by it's conjugate:* $\frac{1}{(a+bi)} * 1 = \frac{1}{(a+bi)} * \frac{(a-bi)}{(a-bi)} = \frac{(a-bi)}{(a^2+b^2)} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$. *Let* $c = \frac{a}{a^2+b^2}$ *and* $d = -\frac{b}{a^2+b^2}$.

This problem shows that every complex number has a multiplicative inverse. We can then be sure that our definition of division for complex numbers from class will hold, that is for $\alpha \in \mathbb{C}$ where $\alpha \neq 0$. This is exactly the requirement that for $\alpha = a + bi$, a and b are real numbers which are not both 0.

Problem 2. *Verify properties of $\mathbb{C}$:*

Note from class: Think of λ as a scalar or number like something in $\mathbb{F}$ and think of α, β, and γ as vectors in $\mathbb{F}^n$, (in this case the vectors are actually in $\mathbb{C}^n$) for $n = 1$.

Show that for all α, β, γ λ in $\mathbb{C}$.

1. $\alpha + \beta = \beta + \alpha$ *commutativity of addition*
2. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ *associativity of addition*
3. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ *commutativity of multiplication*
4. *Show that for every α there is a β so that $\alpha + \beta = 0$*
5. *Show that for every α there is a β so that $\alpha\beta = 1$*
6. $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$

Solution 2. *Let $\alpha = a + bi$, $\beta = c + di$ $\gamma = (e + fi)$ and $\lambda = (h + ki)$*

1. $\alpha + \beta = (a + bi) + (c + di)$ *by definition of α and β*
   $= (a + c) + (b + d)i$ *by definition of addition of complex numbers*
   $= (c + a) + (d + b)i$ *by commutativity of addition for $\mathbb{R}$*
   $= (c + di) + (a + bi)$ *by definition of addition of complex numbers*
   $= \beta + \alpha$ *by definition of α and β*

2. $(\alpha + \beta) + \gamma = ((a + bi) + (c + di)) + (e + fi)$ *by definition of α,β, and γ*
   $= ((a+c)+(b+d)i)+(e+fi)$ *by definition of addition of complex numbers*
   $= ((a + c + e) + (b + d + f)i)$ *by definition of addition of complex numbers*
   $= (a+bi)+((c+e)+(d+f)i)$ *by definition of addition of complex numbers*
   $= (a+bi)+((c+di)+(e+fi))$ *by definition of addition of complex numbers*
   $= \alpha + (\beta + \gamma)$ *by definition of α,β, and γ*

3. $(\alpha\beta)\gamma = ((a+bi)(c+di))(e+fi)$ *by definition of* α,β, *and* γ
$= ((ac-bd)+(ad+bc)i)(e+fi)$ *by definition of multiplication of complex numbers*
$= ((ac-bd)e-(ad+bc)f)+((ad+bc)e+(ac-bd)f)i$ *by definition of multiplication of complex numbers*
$= (ace-bde-adf+bcf)+(ade+bce+acf-bdf)i$ *by defintion of associativity of real numbers*
$= (a+bi)((ce-df)+(cf+de)i)$ *by definition of multiplication of complex numbers*
$= (a+bi)((c+di)(e+fi))$ *by definition of multiplication of complex numbers*
$= \alpha(\beta\gamma)$ *by definition of* α,β, *and* γ

4. *for* $\alpha = (a+bi)$ *since for* a,$b \in \mathbb{R}$ *there exist unique additive inveses* $(-a)$ *and* $(-b)$ *define* $\beta = ((-a)+(-b)i)$. *Then* $\alpha+\beta = (a+bi)+((-a)+(-b)i) = (a+(-a))+(b+(-b))i = 0+0i$.

5. *define* β *as in problem 1.*

6. $\lambda(\alpha+\beta) = (h+ki)((a+bi)+(c+di))$ *by definition of* α, β, *and* λ
$= (h+ki)((a+c)+(b+d)i)$ *by definition of addition of complex numbers*
$= (h(a+c)-k(b+d))+(h(b+d)+k(a+c))i$ *by definition of multiplication of complex numbers*
$= (ha+hc-kb-kd)+(hb+hd+ka+kc)i$ *by distributivity of real numbers*
$= (ha-kb)+(hb+ka)i+(hc-kd)+(hd+kc)i$ *by definition of addition of complex numbers*
$= (h+ki)(a+bi)+(h+ki)(c+di)$ *by definition of multiplication of complex numbers*
$= \lambda\alpha+\lambda\beta$ *by definition of* α, β, *and* λ

I will not usualy do the proofs in such detail but it is important to see how to do proofs like this the first time around.

Problem 3. *Find* $x \in \mathbb{R}^4$ *such that* $(4,-3,1,7)+2x = (5,9,-6,8)$.

Solution 3. *Add the additive inverse of* $(4,-3,1,7)$ *to both sides of the equation. Then perform scalar multiplication by* $1/2$. *We find that* $x = (1/2, 6, -7/2, 1/2)$.

Problem 4. *Explain why there does not exist* $\lambda \in \mathbb{C}$ *such that* $\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$.

Solution 4. *By defintion* $\lambda(2-3i, 5+4i, -6+7i) = (\lambda(2-3i), \lambda(5+4i), \lambda(-6+7i))$. *Suppose there was such a* λ, *then since a list is determined by it's components and their orders* $\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$ *if and only if* $\lambda(2-3i) = 12-5i$, $\lambda(5+4i) = 7+22i$ *and* $\lambda(-6+7i) = -32-9i$. *Let* $\lambda = h+ki$. *Then* $\lambda(2-3i) = (h+ki)(2-3i) = (2h+3k)+(-3h+2k)i = 12-5i$, $\lambda(5+4i) = (h+ki)(5+4i) = (5h-4k)+(4h+5k)i = 7+22i$ *and* $\lambda(-6+7i) = (h+ki)(-6+7i) = (-6h-7k)+(7h-6k)i = -32-9i$.

We also now have that $(2h+3k)+(-3h+2k)i = 12-5i$ *if and only if* $2h+3k = 12$ *and* $-3h+2k = -5$*. Similarly,* $(5h-4k)+(4h+5k)i = 7+22i$ *if and only if* $5h-4k = 7$ *and* $4h+5k = 22$*. Also* $(-6h-7k)+(7h-6k)i = -32-9i$ *if and only if* $(-6h-7k) = -32$ *and* $(7h-6k) = -9$*.*

Solving for h *from* $2h+3k = 12$ *one obtains* $h = \frac{12-3k}{2}$*. Plugging this result into* $-3h+2k = -5$ *we get that* $-3h+2k = -3*\frac{12-3k}{2}+2k = (13/2)k-18 = -5$ *means* $k = 2$ *and* $h = 3$*. Now still* $5h-4k = 7$ *and* $4h+5k = 22$ *must hold as well as the third pair of relations* $(-6h-7k) = -32$ *and* $(7h-6k) = -9$*. Plugging in our values for* h *and* k *we get that* $5(3)-4(2) = 7$ *holds and* $4(3)+5(2) = 22$ *hold. In the last relation,* $(-6(3)-7(2)) = -32$ *but* $(7(3)-6(2)) = 9 \neq -9$*. Our possible solutions* h *and* k *do not satisfy all three relations. Later we will prove that this cannot be the case. In part, Linear Algebra is the study of systems of linear equations.*

Problem 5. *Verify properties of* $\mathbb{F}^n$*: Let* $x, y, z \in \mathbb{F}^n$ *be vectors and* $a, b \in \mathbb{F}$ *be scalars*

1. *show associativity of addition* $(x+y)+z = x+(y+z)$
2. *show associativity of scalar multiplication* $(ab)x = a(bx)$
3. *show there is a multiplicative identity* $1x = x$
4. *show distributivity of scalar multiplication (vector side)* $a(x+y) = ax+ay$
5. *show distributivity of scalar multiplication (scalar side)* $(a+b)x = ax+bx$

Solution 5.

1. $(x+y)+z = ((x_1,...,x_n)+(y_1,...,y_n))+(z_1,...,z_n)$ *by definition of* $x, y, z \in \mathbb{F}^n$
   $= (x_1+y_1,...,x_n+y_n)+(z_1,...,z_n)$ *by definition of vector addition in* $\mathbb{F}^n$
   $= ((x_1+y_1)+z_1,...,(x_n+y_n)+z_n)$ *by definition of vector addition in* $\mathbb{F}^n$
   $= (x_1+(y_1+z_1),...,x_n+(y_n+z_n))$ *by definition of associativity in* $\mathbb{F}$
   $= (x_1,...,x_n)+(y_1+z_1,...,y_n+z_n)$ *by definition of vector addition in* $\mathbb{F}^n$
   $= (x_1,...,x_n)+((y_1,...,y_n)+(z_1,...,z_n))$ *by definition of vector addition in* $\mathbb{F}^n$
   $= x+(y+z)$ *by definition of* $x, y, z \in \mathbb{F}^n$

2. *This follows through a similar argument with the critical step being* $(ab)x_i = a(bx_i)$ *because multiplication in* $\mathbb{F}$ *is associative.*

3. $1x = 1(x_1,...,x_n) = (1x_1,...,1x_n) = (x_1,...,x_n) = x$.

4. $a(x+y) = a((x_1,..,x_n)+(y_1,...y_n)) = a(x_1+y_1,...,x_2+y_2) = (a(x_1+y_1),...,a(x_n+y_n)) = (ax_1+ay_1,...,ax_n+ay_n) = (ax_1,..,ax_n)+(ay_1,...ay_n) = a(x_1,..,x_n)+a(y_1,...y_n) = ax+ay$.

5. *The scalar side follows from a similar argument namely, the critical step again is distributivity in* $\mathbb{F}$*. Make sure you can justify every step in parts 2-5.*