# Math 260
# 10/17 Notes Continued
# Introduction to Chapter 5

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In chapter 5 we will begin a systematic study of how certain bases are more useful than others when working with specific operators. As an introduction to this chapter we will revisit some important examples.

## Differentiation with different bases

We saw in chapter 3 that the application of a linear map $T$ on a vector $v \in V$ gives us a unique vector $Tv \in W$ regardless of how we express $v$ in terms of a basis for $V$. Here is the answer to the exercise at the end of the 10/17/18 notes:

Let $B_1 = 1, x, x^2$ and $B_2 = 1, (x-3), (x-3)^2$ be bases for $\mathcal{P}_2(\mathbf{F}, x)$. It is easy to check that the matricies associated to the differentiation operator $D \in \mathscr{L}(\mathcal{P}_2(\mathbf{F}, x))$ [1] with respect to these two bases are the same. That is,

$$[D]_{B_1}^{B_1} = [D]_{B_2}^{B_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The entries of column 1 of the matrix are the image of the first basis vector 1 under the differentiation map represented in terms of the bases, That is $[D1]_{B_1} = [D1]_{B_2}$. We check that in either basis the results are the same:

$$D(1) = 0 = 0(1) + 0x + 0x^2 = 0(1) + 0(x-3) + 0(x-3)^2.$$

Taking the second basis vector of $B_1$, $x$, and representing the image $Dx$ in terms of the basis $B_1$, we have $Dx = 1(1) + 0x + 0x^2$. Thus the associated column vector, $[Dx]_{B_1}$ matches the second column of $[D]_{B_1}^{B_1}$. Similarly, representing $D(x-3)$ with respect to $B_2$, we have $D(x-3) = 1(1) + 0(x-3) + 0(x-3)^2$ and therefore $[D(x-3)]_{B_2}$ also matches the second column.

Let us also check that $[Dx^2]_{B_1} = [D(x-3)^2]_{B_2}$ and that this column vector matches the third column of the matrix. With respect to $B_1$, $Dx^2 = 0(1) + 2x + 0x^2$. With respect to $B_2$ $D(x-3)^2 = 0(1) + 2(x-3) + 0(x-3)^2$. Thus the third columns match and the matricies are the same as desired.

Now lets show that the vector $p(x) = 2 + 9x + 5x^2$ looks different with respect to these two bases.

[1] recall that the extra arguement $x$ of $\mathcal{P}_2(\mathbf{F}, x)$ signifies that we are talking about the vectorspace of polynomials of degree at most 2 with coefficients in $\mathbf{F}$ in indeterminate $x$

Clearly,

$$[p(x)]_{B_1} = \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix}$$

however representing $p(x)$ with respect to $B_2$ is slightly more complicated.

Only $(x-3)^2$ has an $x^2$ term so for now, $p(x) = a(1) + b(x-3) + 5(x-3)^2$. Now we may solve for $b$. We know that $p(x)$ must have 9 as the coefficient for the $x$ term. Expanding $5(x-3)^2 = 5x^2 - 30x + 45$ so $b = 39$. We now have $p(x) = 2 + 9x + 5x^2 = a(1) + 39(x-3) + 5(x-3)^2 = a - 39(3) + 45 + 9x + 5x^2$. Thus $2 = a - 117 + 45$. So $a = 74$ and $p(x) = 74(1) + 39(x-3) + 5(x-3)^2$. Thus

$$[p(x)]_{B_2} = \begin{pmatrix} 74 \\ 39 \\ 5 \end{pmatrix}$$

Performing the matrix multiplication, $[D]_{B_1}^{B_1}[p(x)]_{B_1} = [D(p(x))]_{B_1}$,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \\ 0 \end{pmatrix}$$

We have that in terms of $B_1$ $D(p(x)) = 9(1) + 10x + 0x^2$.

We also have that with respect to the second basis, $[D]_{B_2}^{B_2}[p(x)]_{B_2} = [D(p(x))]_{B_2}$,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 74 \\ 39 \\ 5 \end{pmatrix} = \begin{pmatrix} 39 \\ 10 \\ 0 \end{pmatrix}$$

So in terms of $B_2$ $D(p(x)) = 39(1) + 10(x-3) + 0(x-3)^2$. Checking this is correct: $10x - 30 + 39 = 10x + 9$ as desired.

Although we got the same answer, the computations with one of these two bases was much more difficult. An even more awful supposition: suppose we had used the basis $B_3 = 1,\ x,\ \frac{1}{2}(3x^2 - 1)$ [2]. You should check that $[D]_{B_3}^{B_3} \neq [D]_{B_1}^{B_1} = [D]_{B_2}^{B_2}$. Furthermore, check that $B_3$ is a basis. Find $[D]_{B_3}^{B_3}$ and $[p(x)]_{B_3}$. When you are finished, see the section **Answers**.

Carry out the multiplication $[D]_{B_3}^{B_3}[p(x)]_{B_3}$ to obtain $[D(p(x))]_{B_3}$. Show that unbracketing this expression also gives us the correct answer for $D(p(x))$.

Different bases result in different matrix representations of the same differentiation operator $D$. However, matrix multiplication always gives us the correct response. For any basis $B$, $[D]_B^B[p]_B = [Dp]_B$:

[2] https://en.wikipedia.org/wiki/Legendre_polynomials The legendre polynomails are a special basis for $\mathcal{P}(\mathbf{F}, x)$. In chapter 6 we will have a notion of angle between two vectors. We will see that the vectors of $B_3$ stick out at "right angles" to one another, whatever that means.

$$
\begin{tikzcd}
p \arrow[r, "D"] \arrow[d, "{[\ \ ]_B}"'] & Dp \arrow[d, "{[\ \ ]_B}"] \\
{[p]_B} \arrow[r, "{[D]_B^B}"] & \dfrac{[Dp]_B}{[D]_B^B[p]_B}
\end{tikzcd}
$$

## Changing Basis

Let $B_1$ and $B_2$ be bases for a vector space $V$. Let $v \in V$. Let $I$ denote the identity operator on $V$. Using matricies,

$$[I]_{B_2}^{B_1}[v]_{B_1} = [Iv]_{B_2}$$

However since $Iv = v$, we have $[Iv]_{B_2} = [v]_{B_2}$. So we have the change of basis formula for vectors:

$$[I]_{B_2}^{B_1}[v]_{B_1} = [v]_{B_2}$$

The result is similar for matricies.

Let $T$ be an operator on $V$. Let $A$, $B$, $C$, and $D$ be bases for $V$. If we have the matrix of $T$ with respect to $B$ and $C$, $[T]_B^C$, and we would like to see it with respect to $A$ and $D$, we use the following arguement. Recall that matrix multiplication was defined to preserve composition of linear maps:

$$[I]_D^C[T]_C^B = [I \circ T]_D^B = [T]_D^B$$

since $I \circ T = T$. Similarly $T = T \circ I$ gives us via precomposing,

$$[I]_D^C[T]_C^B[I]_B^A = [I \circ T]_D^B[I]_B^A = [I \circ T \circ I]_D^A = [T]_D^A$$

so the change of basis formula for matricies is then:

$$[I]_D^C[T]_C^B[I]_B^A = [T]_D^A$$

In general $A = D$ and $B = C$. Although it is possible to represent an operator from $V$ to $V$ using two different bases we will avoid this usually.

You should check the following computations using the data from the running example in the previous section:

1. $$[I]_{B_1}^{B_2}[D]_{B_2}^{B_2}[I]_{B_2}^{B_1}[p]_{B_1} = [Dv]_{B_1}$$

2. $$[I]_{B_1}^{B_2}[D]_{B_2}^{B_2}[I]_{B_2}^{B_1} = [D]_{B_1}^{B_1}$$

3. $$[I]_{B_2}^{B_1}[D]_{B_1}^{B_1}[I]_{B_1}^{B_2} = [D]_{B_2}^{B_2}$$

4.

$$[I]_{B_3}^{B_2}[D]_{B_2}^{B_2}[I]_{B_2}^{B_3} = [D]_{B_3}^{B_3}$$

5.

$$[I]_{B_2}^{B_3}[D]_{B_3}^{B_3}[I]_{B_3}^{B_2} = [D]_{B_2}^{B_2}$$

## Answers

1.

$$[D]_{B_3}^{B_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

2.

$$[p(x)]_{B_3} = \begin{pmatrix} 11/3 \\ 9 \\ 10/3 \end{pmatrix}$$