Math 260 Exam 2 Solutions

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Sunday November 25th, 2018

In problems 1 and 2 your wording may be different than that of book. I am looking for correctness in the concepts. Problems $1-3$ are worth 70 points. The test is out of 100.

Problem 1. Give definitions for the following terms: (Careful, pay attention to the vector spaces involved, write them explicitly).

1. linear map
2. addition and product of linear maps.
3. null space/kernel of a linear map, prove it is a subspace
4. range of a linear map, prove it is a subspace
5. injectivity of a function
6. surjectivity of a function
7. invertability of a linear map, isomorphism
8. operator
9. define a linear map $[\ \ ]_B : V \to \mathbf{F}^n$
10. define a linear map $[\ \ ]_{B'}^{B} : \mathscr{L}(V, W) \to \mathbf{F}^{m\times n}$

    Definitions from 10/17 notes: For a vector $v \in V$ and a pair of linear transformations $T : V \to W$ and $S : W \to U$

11. define $[T]_{B'}^{B}[v]_B$. How should we denote (write down a symbol for) it?
12. define $[S]_{B''}^{B'}[T]_{B'}^{B}$. How should we denote (write down a symbol for) it? Hint: it may be easier to figure out first how to denote (write down a symbol for) $[T]_{B'}^{B}[v]_B$ and $[S]_{B''}^{B'}[T]_{B'}^{B}$ then figure out how you should define it.

Solution 1. See the text for Definitions 1-8. For definitions 9 and 10 see theorem 1 and 2 of the 10/17 notes. For the defintion of $[T]_{B'}^{B}[v]_B$ see equation (20) of the 10/17 notes. Most of the 10/17 notes are devoted to showing that we can denote $[T]_{B'}^{B}[v]_B$ by $[Tv]_{B'}$.

The definition of $[S]_{B''}^{B'}[T]_{B'}^{B}$ is given as follows: $[T]_{B'}^{B}$ is the matrix who's i^{th} column is given as $[Tv_i]_{B'}$ where the $v_i \in B$. Define $[S]_{B''}^{B'}[T]_{B'}^{B}$ to be the matrix who's i^{th} column is given by $[S]^{B'}_{B''}[Tv_i]_{B'}$ (This expression is defined in equation 20 of the 10/17 notes). That is send each column of $[T]^B_{B'}$ through $[S]^{B'}_{B''}$ one at a time. We can denote $[S]^{B'}_{B''}[T]^B_{B'}$ as $[S \circ T]^B_{B''}$.

Prove the following (what I call the fundamental theorem of linear maps):

Problem 2 (3.5)**.** Suppose $v_1, ..., v_n$ is a basis of V and $w_1, ..., w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, ..., n$

Solution 2. See the text.

Problem 3. State and prove (Axler's) Fundamental Theorem of Linear Maps.

Solution 3. See the text.

Do problems 4-9 to obtain the remaining 30 points:

Problem 4. (5 points) Let E_n be the standard basis for $\mathbf{R}^n$. Suppose T is a linear map from $\mathbf{R}^3 \to \mathbf{R}^2$ and

$$[T]^{E_3}_{E_2} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Write T as a linear map. What vector spaces are the kernel and range of T subspaces of? Find a basis of the kernel. Find a basis of the range. What are the dimensions of these spaces?

Solution 4. The image of the first basis vector e_1 of E_3 is $1e_1 + 4e_2$ in $\mathbf{R}^2$. Thus we may write $T(1, 0, 0) = (1, 4)$. Using the other two columns of the matrix we see that $T(0, 1, 0) = (2, 5)$ and $T(0, 0, 1) = (3, 6)$. Using the linearity of T we can write $T(a, b, c) = aT(1, 0, 0)+bT(0, 1, 0)+cT(0, 0, 1) = a(1, 4)+b(2, 5)+c(3, 6) = (a + 2b + 3c, 4a + 5b + 6c)$. The kernel of T is a subspace of the domain (input space) which is $\mathbf{R}^3$. The range of T is a subspace of the codomain (target or output space) which is $\mathbf{R}^2$. The kernel is the set

$$\{(a, b, c) \in \mathbf{R}^3 | T(a, b, c) = (0, 0)\} =$$

$$\{(a, b, c) \in \mathbf{R}^3 | (a + 2b + 3c, 4a + 5b + 6c) = (0, 0)\}$$

If (a, b, c) is in the kernel, we must have $a+2b+3c = 0$ and that $4a+5b+6c = 0$.

The first equation yeilds $3c = -a-2b$. Plugging this into the second equation we obtain $4a + 5b - 2a - 4b = 0$. Thus $2a + b = 0$. So $-2a = b$ and $3c = -a - 2(-2a) = -a + 4a = 3a$. So $a = c$. Thus

$$\{(a, b, c) \in \mathbf{R}^3 | T(a, b, c) = (0, 0)\} = \{(a, b, c) \in \mathbf{R}^3 | a = c = -b/2)\} = \{(a, -2a, a) \in \mathbf{R}^3\}$$

. This space has dimension 1. It has a single basis vector $(1, -2, 1)$. By the fundamental theorem of Linear Maps (Axler) we have $dim(\mathbf{R}^3) = dim(Ker\ T) + dim(Ran\ T)$ from which we may conclude that $dim(Ran\ T) = 2$ because the equation has the form $3 = 1 + 2$. Since $\mathbf{R}^2$ has dimension 2 we conclude that $Ran\ T$ is all of $\mathbf{R}^2$. A basis for $Ran\ T$ is just E_2.

Problem 5. (5 points) Let $T : \mathbf{R}^5 \to \mathbf{R}^3$ be a linear transformation whose kernel is of dimension 3. What is the dimension of the range? What does the set of points in the range look like geometrically? Hint: there are only 4 possible things that it could look like.

Solution 5. By the fundamental theorem of Linear Maps (Axler) we have $dim(\mathbf{R}^3) = dim(Ker\ T) + dim(Ran\ T)$ from which we may conclude that $dim(Ran\ T) = 2$ because the equation has the form $5 = 3 + 2$. The Range is a two dimensional subspace of $\mathbf{R}^3$. Geometrically the range looks like a plane through the origin.

Problem 6. (5 points) Show that every linear map from a 1-dimensional space to itself is multiplication by some scalar. More precisely, prove that if $dimV = 1$ and $T \in \mathscr{L}(V, V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution 6. See homework answers to $3A$, specifically problem 7.

Problem 7. (5 points) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ by $T(x, y, z) = (x + y + z, 0, 0)$ find a basis for the kernel of T. What is the dimension of the range? What is the dimension of the kernel?

Solution 7. The kernel of T is given by the set

$$\{(x, y, z) \in \mathbf{R}^3 | T(x, y, z) = (x + y + z, 0, 0) = (0, 0, 0)\}$$

If (x, y, z) is in the kernel, we must have $x + y + z = 0$. So $z = -x - y$. Thus

$$KerT = \{(x, y, -x - y) \in \mathbf{R}^3\} = \{(x, 0, -x) + (0, y, -y) | x, y \in \mathbf{R}\}$$

. This is a two dimensional space with basis $(1, 0, -1)$ and $(0, 1, -1)$ (see test 1 solutions for a direct proof). Clearly, $dim(Ran\ T) = 1$ since only the e_1 coordinate is ever non-zero in $(x + y + z, 0, 0)$. We can check using the fundamental theorem of Linear Maps (Axler), $dim(\mathbf{R}^3) = dim(Ker\ T) + dim(Ran\ T)$ that $dim(Ran\ T) = 1$ and $dim(Ker\ T) = 2$ checks out because the equation has the form $3 = 2 + 1$.

Problem 8. (5 points)Let $T : \mathbf{R}^3 \to \mathbf{R}^3$. Suppose

$$[T]_{B'}^{B} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

where $B = v_1, v_2, v_3$ and $B' = w_1, w_2, w_3$ are both bases for $\mathbf{R}^3$. Apply the base change formula to obtain

$$[T]_{C'}^{C} = \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$$

Write the base change matricies out. Begin by figuring out out what C and C' are.

Solution 8. The base change formula is $[I]_{C'}^{B'}[T]_{B'}^{B}[I]_{B}^{C} = [T]_{C'}^{C}$.

The columns of the matrix $[T]_{B'}^{B}$ tell us that $Tv_1 = aw_1 + dw_2 + gw_3$, $Tv_2 = bw_1 + ew_2 + hw_3$, and $Tv_3 = cw_1 + fw_2 + iw_3$. Notice that the rows of $[T]_{C'}^{C}$ are just reorderings of the rows of $[T]_{B'}^{B}$. So if $B' = w_2, w_3, w_1$ instead of $B' = w_1, w_2, w_3$ the matrix $[T]_{B'}^{B}$ would have the form

$$\begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$$

This is what change of basis is for. Multiplying $[T]_{B'}^{B}$ on the left by the matrix $[I]_{C'}^{B'}$ will change the basis B' into C'. Choose $C' = w_2, w_3, w_1$. Once this is done we dont need to even change the basis B. Let $C = B$.

Check yourself

$$[I]_{C'}^{B'} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$[I]_{B}^{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Furthermore, check that the change of basis formula works.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$$

Problem 9. (5 points) Use Problem 2, that is, apply Theorem (3.5) to show that if two finite dimensional vector spaces V and W have the same dimension, then they must be isomorphic.

Solution 9. See the second half of 3.59 of the text in section 3D.

Problem 10 (Bonus 5 points)**.** Suppose V and W are of dimension n and m respectively, pick a vector $v \in V$. Define

$$E_v = \{T \in \mathscr{L}(V, W) | Tv = 0\}$$

that is, E_v is the set of linear transformations which send v to $0 \in W$. Show that E_v is a subspace of $\mathscr{L}(V, W)$. Suppose $v \neq 0$ what is the dimension of E_v?

Solution 10. It is very straightforward to show that E_v is a subspace of $\mathscr{L}(V, W)$. Finding the dimension of E_v requires more thought. I will sketch two ways of discovering this.

Sketch solution 1:

Find a linear map $\phi_v : \mathscr{L}(V, W) \to W$ who's kernel is exactly E_v. Then apply the Fundamental Theorem of Linear Maps (Axler).

Sketch solution 2:

This is a longer proof which does not use the fundamental theorem of linear maps.

Let B be a basis of V which contains v. Let B' be a basis of W. Define a basis C of $\mathscr{L}(V, W)$ to be the $n \times m$ linear maps $L_{(i,j)}$ which take the i^{th} vector of B to the j^{th} vector of B', and take all other vectors of B to 0, for $i \in 1, ..., n$ and $j \in 1, ..., m$. It is easy to show this is a basis.

Then show that the basis vectors (linear maps) in C which take v to 0 form a basis of E_v. This is not hard. Just show that their span contains E_v and that E_v contains their span. Then count how many basis vectors (linear maps) in C take v to 0? This is the dimension of E_v.