Linear Algebra's Applications in Graph Theory

Jeff Coismain

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Graph theory is a sub-field of mathematics that specializes in studying and working with a data structure called a graph. A graph, $G = (V, E)$, is a set of vertices V, along with a set of edges E, where the set of edges represent associations between a pair of vertices (including a vertex and itself), and that association is referred to as an *adjacency* (denoted as j∼k, which reads j is adjacent to k). If a set of vertices has no adjacency, that set is said to be an independent set of vertices. The number of adjacencies a vertex possesses is referred to as it's *degree*, and so each vertex in an independent set has a degree of zero. (Skiena 2012) On the other hand, if *all* graph's vertices has an adjacency to all other vertices in the graph, that graph is said to be *connected*.

As one can imagine, graphs can be used to model practically any type of relationship, and there are also several types of graphs with which to accomplish this (e.g. trees, cyclic, directed, colored, etc.). Each of these specialized graphs have their own unique characteristics that allow for a special use or feature. For example, in a colored graph, each vertex has a *color* which is an additional part of its state that must be taken into account since it may be useful to partition the vertices according to their color. This is particularly useful in scheduling problems where vertices may represent meetings for example, and an adjacency between them represents an overlapping time slot.

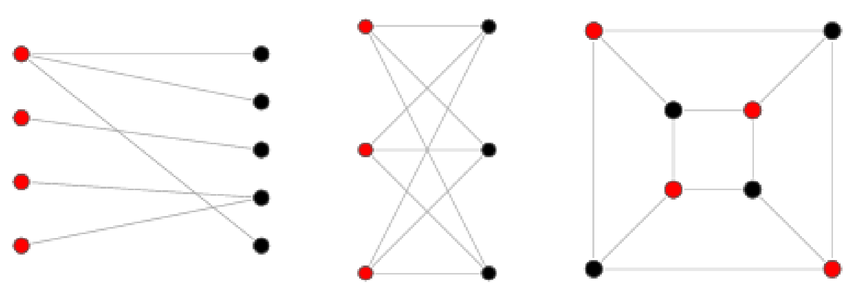

Figure 1: Some simple colored graphs

A graph that is able to partition it's set of vertices V into separate groups v_1, and v_2 such that no vertex $v \in V_1$ shares an adjacency with another vertex in V_1, no vertex $v \in V_2$ shares an adjacency with another vertex group is said to be *bipartite*. (Tucker 2012)

Graphs also have different representations, one in particular is the *adjacency matrix*. A graph G with a set of vertices $V = v_1, v_2, ..., v_n$ can be represented by an $n \times n$ binary matrix A:

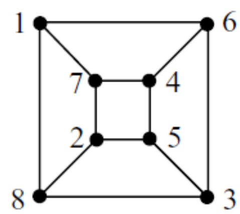

Figure 2: Some simple colored graphs

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(Beineke 2004)

In the above matrix A, each row j and column k represents a vertex, where each element $A_{j,k}$ represents an adjacency between vertex j and vertex k(1 for the affirmative, 0 for no adjacency). For example, the matrix A above represents a graph with an order of 8(the amount of vertices it possesses), where *vertex 1* (the first row) has an adjacency with *vertex 6*(the sixth column), *vertex 7* (the seventh column), and *vertex 8* (the eighth column).

$$A_{j,k} = \begin{cases} 1 & \text{if } j{\sim}k \\ 0 & \text{otherwise} \end{cases}$$

A graph's adjacency matrix also has eigenvalues, where the set of all of a graph's eigenvalues is called it's *spectrum*. The following theorems are below are with regards to an adjacency matrix's eigenvalues and are of major importance in the study of spectral graph theory:

Principle Axis Theorem: If $\mathbf{A}$ is a real symmetric matrix of order n, then then A has n real eigenvalues and a corresponding set of orthonormal eignevectors. (Beineke, 2004)

If U is a matrix with an orthonormal set of eigenvectors in it's columns, then U^t and $U^tAU = D$, where D is the diagonal matrix where it's trace consists of the corresponding eigenvalues. (Beineke, 2004)

Diagonalization Theorem: If A is a real symmetric matrix, then there exists a matrix U such that $U^TAU = D$. Additionally, the minimum polynomial is

$$\prod(x - \lambda_i)$$

...where the product is applied over the distinct eigenvalues. (Beineke, 2004)

Perron-Frobenius Theorem: If A is a non-negative matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, then $|\lambda_1| \leq |\lambda_k|$, for $k = 1, 2, ..., n$, and the eigenvalue λ_1 has an eigenvector with all it's values > 0. If A is indecomposable, then the eigenvalue λ_1 is simple ($\lambda_1 > \lambda_2$), and the eigenvector's values are all > 0. (Beineke, 2004)

For many scheduling problems, the solutions actually comes down to determining whether or not the graph is bipartite. The following theorem shows another application of eigenvalues and their corresponding adjacency matrices with regards to making this determination:

Theorem 5.2: A graph is bipartite if and only if its spectrum is symmetric about 0.

Suppose v_i and v_j are non-adjacent vertices and x is an eigenvector with eigenvalue λ. If σ is the sum of the labels on the neighbours of v_i and v_j, then: $\lambda x_i = (Ax)_i = \Sigma = (Ax)_j = \lambda x_j$, and so $\lambda(x_i - x_j) = 0$.