Cosets and the Fundamental Isomorphism Theorem

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Let G be a group and a be a fixed element of G. Also let H be a subgroup of G. Define the (right) coset *Ha by*

$$Ha = \{ha : h \in H\}$$

Claim. *If* $a \in Hb$ *then* $Ha = Hb$.

Proof. $a \in Hb$ *means* $a = h_1 b$ *for some* $h_1 \in H$. *Let* $x \in Ha$. *Then* $x = h_2 a = h_2(h_1 b) = (h_2 h_1)b \in Hb$. *Therefore* $Ha \subseteq Hb$. *Suppose* $y \in Hb$. *Then* $y = h_3 b$ *for some* $h_3 \in H$. *But* $b = h_1^{-1}a$ *so* $y = h_3(h_1^{-1}a) = (h_3 h_1^{-1})a \in Ha$. *Therefore* $Hb \subseteq Ha$. *We conclude* $Ha = Hb$.

Theorem 1. *Suppose G is a group and H is a subgroup of G. Then*

$$Ha = Hb \text{ iff } ab^{-1} \in H.$$

Proof. ($\rightarrow$): *Suppose* $Ha = Hb$. *Since* $a \in Ha$ *we know that* $a \in Hb$, *so* $a = hb$ *for some* $h \in H$. *Therefore* $ab^{-1} = h \in H$.

($\leftarrow$): *Suppose* $ab^{-1} \in H$. *Then* $ab^{-1} = h$ *for some* $h \in H$. *Therefore* $a = hb$ *for some* $h \in H$, *which gives* $a \in Hb$ *and, in consequence,* $Ha = Hb$ *as desired.* ■

Theorem 2. *Let* $f: G \rightarrow H$ *be a homomorphism with kernel K. then* $f(a) = f(b)$ *if and only if* $Ka = Kb$.

Proof. $f(a) = f(b) \leftrightarrow f(a)f(b)^{-1} = e \leftrightarrow f(ab^{-1}) = e \leftrightarrow ab^{-1} \in K \leftrightarrow Ka = Kb.$ ■

The Fundamental Isomorphism Theorem. *Let* $f: G \rightarrow H$ *be a homomorphism of G* onto *H. If K is the kernel of f then* $H \cong G/K$.

Proof. To show that G/K *is isomorphic to H we must look for an isomorphism from* G/K *to H which matches each coset Kx with the element* $f(x)$*; call this function* ϕ. *Thus,* ϕ *is defined by the identity*

$$\phi(Kx) := f(x)$$

Note: if $Ka = Kb$ then $f(a) = f(b)$, which implies $\phi(Ka) = \phi(Kb)$ and therefore ϕ is well-defined.

ϕ is injective: if $\phi(Ka) = \phi(Kb)$ then $f(a) = f(b)$ so $Ka = Kb$ by Theorem 2.

ϕ is surjective since every element of H is of the form $f(x) = \phi(Kx)$.

Finally $\phi(KaKb) = \phi(Kab) = f(ab) = f(a)f(b) = \phi(Ka)\,\phi(Kb)$.

Therefore ϕ is an isomorphism from G/K onto H. ■

Exercise 1. For each $x \in \mathbb{R}$, it is conventional to write $\operatorname{cis} x = \cos x + i \sin x$.
Prove that $\operatorname{cis}(x+y) = (\operatorname{cis} x)(\operatorname{cis} y)$

Solution. $\operatorname{cis}(x+y) = \cos(x+y) + i\sin(x+y)$

$$= \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \sin y\cos x)$$

$$= \cos x\cos y - \sin x\sin y + i\sin x\cos y + i\sin y\cos x$$

$(\operatorname{cis} x)(\operatorname{cis} y) = (\cos x + i\sin x)(\cos y + i\sin y)$

$$= \cos x\cos y + i\sin y\cos x + i\sin x\cos y - \sin x\sin y$$

$$= \cos x\cos y - \sin x\sin y + i\sin x\cos y + i\sin y\cos x$$

$$= \operatorname{cis}(x+y)$$

Exercise 2. Let T designate the set $\{\operatorname{cis} x : x \in \mathbb{R}\}$, that is, the set of all the complex numbers lying on the unit circle, with operation multiplication. Use part 1 to prove that T is a group. (T is called the circle group.)

Solution: (i) closure: $t_1 \in T \wedge t_2 \in T \Rightarrow t_1 = \operatorname{cis}(x_1)$ some $x_1 \in \mathbb{R}$
$t_2 = \operatorname{cis}(x_2)$ some $x_2 \in \mathbb{R}$.

$\therefore\ t_1 t_2 = (\operatorname{cis} x_1)(\operatorname{cis} x_2) = \operatorname{cis}(x_1 + x_2) \in T$

②

ociativity ~~t_1~~, Suppose $t_1, t_2, t_3 \in T$. then

~~$t_1 t_2 t_3$~~ $t_1 = \text{cis}\, x_1$, $t_2 = \text{cis}\, x_2$, $t_3 = \text{cis}\, x_3$. Since $x_1, x_2, x_3 \in \mathbb{R}$.

$$t_1(t_2 t_3) = \text{cis}\, x_1 \left[(\text{cis}\, x_2)(\text{cis}\, x_3) \right]$$

$$= \text{cis}\, x_1 \left[\text{cis}\,(x_2 + x_3) \right]$$

$$= \text{cis}\,(x_1 + (x_2 + x_3))$$

$$(t_1 t_2) t_3 = \left[(\text{cis}\, x_1)(\text{cis}\, x_2) \right] \text{cis}\, x_3.$$

$$= \left[\text{cis}\,(x_1 + x_2) \right] \text{cis}\, x_3$$

$$= \text{cis}\,((x_1 + x_2) + x_3)$$

$$= \text{cis}\,(x_1 + (x_2 + x_3))$$

$$= t_1(t_2 t_3)$$

We conclude that T is a group.

$\exists\, 1 = \cos 0 + i \sin 0 \in T$

identity: $\forall\, t_1 \in T$ ~~$\exists\, 1 = \cos 0$ $\exists$ cis $1 = \text{cis}(0) \in T$~~

$$\text{s.t. } 1 t_1 = 1(\cos x_1 + i \sin x_1) = 1 \cos x_1 + 1 i \sin x_1$$

$$= \cos x_1 + i \sin x_1$$

$$= t_1$$

inverses: $t_1 \in T \rightarrow t_1 = \text{cis}\, x_1 \in T \rightarrow t_1^{-1} = \text{cis}\,(-x_1) \in T$

$$\rightarrow t_1 t_1^{-1} = (\text{cis}\, x_1)(\text{cis}(-x_1)) = \text{cis}\,(x_1 - x_1) = \text{cis}\,(0) = 1$$

Prove that $f(x) = \text{cis}\, x$ is a homomorphism from $\mathbb{R}$ onto T

$f(x+y) = \text{cis}(x+y) = \text{cis}\, x \; \text{cis}\, y$.

$\therefore$ f preserves the structure.

$t_1 \in T \to t_1 = \text{cis}\, x_1$ for some $x_1 \in \mathbb{R}$

$\to t_1 = f(x_1)$ for some $x_1 \in \mathbb{R}$

$\therefore$ f is onto.

Exercise 4 Prove that $\ker f = \{2n\pi : n \in \mathbb{Z}\} = \langle 2\pi \rangle$

Solution: $\ker f = \{x \in \mathbb{R} : f(x) = 1\}$

$= \{x \in \mathbb{R} : \text{cis}\, x = 1\}$

$= \{x \in \mathbb{R} : x = 2n\pi \text{ for some } n \in \mathbb{Z}\}$

$= \{2n\pi : n \in \mathbb{Z}\} = \langle 2\pi \rangle$

Exercise 5. Use the Fundamental Isomorphism thm. to conclude that $T \cong \mathbb{R}/\langle 2\pi \rangle$

Solution: since $f(x) = \text{cis}\, x$ is a homomorphism from $\mathbb{R}$ onto T with kernel $\langle 2\pi \rangle$, we conclude that.

$T \cong \mathbb{R}/\langle 2\pi \rangle$

) Prove that $g(x) = \text{cis}\, 2\pi x$ is a homomorphism from $\mathbb{R}$ onto T, with kernel $\mathbb{Z}$.

$T = \{\text{cis}\, x ; x \in \mathbb{R}\}$.

Solution: $g(x+y) = \text{cis}\,(2\pi(x+y))$

$= \text{cis}\,(2\pi x + 2\pi y)$

$= (\text{cis}\, 2\pi x)(\text{cis}\, 2\pi y)$

$= g(x)\, g(y)$

$\therefore$ g is a homomorphism

$t = \text{cis}\, y$, some $y \in \mathbb{R}$.

g is onto : $t \in T$ ~~no~~ $\exists\, t' \in \mathbb{R}$ s.t. $g(t') = t$?

$g(t') = t \rightarrow \text{cis}\, 2\pi t' = t$

$\rightarrow \text{cis}\, 2\pi t' = \text{cis}\, y$

$\rightarrow 2\pi t' = y + 2k\pi$

$t' = k + \dfrac{y}{2\pi} \in \mathbb{R}$ $\therefore g$ is onto.

$\text{Ker}\, g = \{x \in \mathbb{R} : \cancel{\text{cis}\, 2\pi x}\ g(x) = 1\}$.

$= \{x \in \mathbb{R} : \text{cis}\, 2\pi x = 1\}$

$= \{x \in \mathbb{R} : \text{cis}\, 2\pi x = \text{cis}\, 0\}$.

$\Rightarrow \{x \in \mathbb{R} : 2\pi x = 0 + 2k\pi$ for some $k \in \mathbb{Z}\}$.

$= \{x \in \mathbb{R} : x = k,$ some $k \in \mathbb{Z}\}$.

$= \mathbb{Z}$.

Exercise 7 Conclude that $T \cong \mathbb{R}/\mathbb{Z}$

Since $g(x) = \operatorname{cis} 2\pi x$ is a homomorphism from $\mathbb{R}$ onto T and $\ker g = \mathbb{Z}$, we conclude that

$$T \cong \mathbb{R}/\mathbb{Z}$$

$\square$