Johns Hopkins Category Theory Seminar
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What is an LCC?

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0.1 What is an LCC?

Definition 1. A cartesian closed category is a category $\mathcal{C}$ in which all finite products exist and the product functor has a right adjoint.

Note that the empty product in a category is a terminal object. The empty product is the limit over the empty diagram. Thus any cartesian closed category has a terminal object. The statement that the product functor has a right adjoint means that it has exponential objects. This says in fancier language that it is closed with respect to it's cartesian monoidal structure.†

Definition 2. A locally cartesian closed category is a category $\mathcal{C}$ together with a functor $\mathcal{C}/(-) : \mathcal{C}^{op} \to \mathrm{CAT}$. On objects A of $\mathcal{C}^{op}$, $\mathcal{C}/A$ is the slice category over A. On morphisms $C' \xrightarrow{c} C$, $\mathcal{C}/c$ is a functor of slice categories. Let $c^* = \mathcal{C}/c : \mathcal{C}/C \to \mathcal{C}/C'$ defined on objects by the left vertical leg of the pullback square:

$$\begin{tikzcd} c^*X \arrow[r] \arrow[d, "c^*f"'] & X \arrow[d, "f"] \\ C' \arrow[r, "c"'] & C \end{tikzcd}$$

Furthermore, $\mathcal{C}/(-)$ has a right adjoint denoted $\Pi_{(-)}$.

We are going to investigate the adjunction:

$$\begin{tikzcd} \mathrm{CAT} \arrow[d, bend left, "\Pi_{(-)}"] \arrow[d, phantom, "\dashv"] \\ \mathcal{C}^{op} \arrow[u, bend left, "\mathcal{C}/(-)"] \end{tikzcd}$$

Example 1. What is the relationship between locally cartesian closed and cartesian closed?

†A cartesian monoidal category is a monoidal category whose monoidal structure is given by the category theoretic product. Closure means that the hom-sets are internal.

Category	LCC	CC
SET	✓	✓
CAT	×	✓
GRP	×	✓
Top	?	?
Topos	✓	✓

Exercise 1. † Given the diagram $B \xrightarrow{f} A$ in $\mathcal{C}$, show that the left adjoint to $f^* : \mathcal{C}/A \to \mathcal{C}/B$ is the functor $\Sigma_f\big(X \xrightarrow{b} B\big) = X \xrightarrow{f \circ b} A$.

Solution 1. Since slice categories are the order of the day, what better way to prove that this is an adjunction than to show that $f^*\Sigma_f\big(X \xrightarrow{b} B\big)/f^*(\mathcal{C}/A)$ is an initial object of $\big(X \xrightarrow{b} B\big)/f^*(\mathcal{C}/A)$. Said another way: for any object $Y \xrightarrow{i} A \in \mathcal{C}/A$ and map h, there exist maps η_X and k such that the whole diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[rr, bend left, "h"] \arrow[dr, "b"'] & f^*\Sigma_f X \arrow[r, "k"] \arrow[d, "f^*\Sigma_f b"] & f^*y \arrow[dl, "f^*i"] \\
& B &
\end{tikzcd}$$

The map η_X exists because the identity map id_X makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "b"'] & \Sigma_f X = X \arrow[d, "\Sigma_f b"] \\
B \arrow[r, "f"'] & A
\end{tikzcd}$$

Therefore we obtain the morphism η_X from the definition of a pullback where we are pulling $\Sigma_f b$ back along f.

†tslil comments that this holds in any category $\mathcal{C}$ which has pullbacks.

$$\begin{tikzcd}
X \arrow[dr, "\eta_X"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "b"'] & & \\
& f^*\Sigma_f X \arrow[r] \arrow[d, "f^*\Sigma_f b"] & \Sigma_f X \arrow[d, "\Sigma_f b"] \\
& B \arrow[r, "f"'] & A
\end{tikzcd}$$

Now to show that the morphism k exists we note that since f^*Y is defined in terms of a pullback we can form the following commuting diagram:

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[dr, "g"'] & f^*Y \arrow[r, "l"] \arrow[d, "f^*i"] & y \arrow[d, "i"] \\
& B \arrow[r, "f"] & A
\end{tikzcd}$$

Since $i \circ l \circ h = f \circ g$, we obtain the diagram where we pull back on

$$\begin{tikzcd}
f^*\Sigma_f X \arrow[r, "l'"] \arrow[d, "f^*\Sigma_f g"'] & \Sigma_f X \arrow[r, "l \circ h"] \arrow[d, "\Sigma_f g"'] & Y \arrow[dl, "i"] \\
B \arrow[r, "f"] & A &
\end{tikzcd}$$

Since the diagram commutes, we obtain a pullback arrow to f^*Y. This completes the verification.

tslil calls the following Fruitful Coincidence Number 1:

Exercise 2. With cannonically chosen pullbacks show the natural isomorphism of functors

$$[(-)^{op}, SET] \cong SET/(-)$$

from $SET^{op} \to$ CAT.

Solution 2. We need to show that on an object A of SET, there is an equivalence of small categories between $[A^{op}, SET]$ and SET/A. Furthermore we need to show that this equivalence is natural in A. For any object A of SET, define functors F_A and G_A as follows:

$$\begin{tikzcd}
{[A^{op}, SET]} \arrow[d, bend left, "F_A"] \arrow[d, phantom, "\dashv"] \\
SET/A \arrow[u, bend left, "G_A"]
\end{tikzcd}$$

Let $X, Y \in SET$ and $g : X \to Y$. Denote a functor from A to X in $[A^{op}, SET]$ by just $\{X_a\}_{a \in A}$, an A indexed set. Similarly, we abuse notation to let g mean the morphism in this category which is a natural transformation from X to Y, a cofiber-wise mapping. In SET/A we have $g_* : Y/A \to X/A$ by precomposing the morphism from Y to A with g.

Thinking of functors from A to X in this way, we use the following fact to construct the functor F_A: note that in the category of sets, the coproduct of a set X over an indexing set B is exactly the product $B \times X$. That is, $\Sigma_{b\in B} X = B \times X$.

Define F_A by taking the disjoint union, that is the coproduct of $\{X_a\}_{a\in A}$. Thus $F_A(\{X_a\}_{a\in A}) = \Sigma_{a\in A} X_a$. In the case where every $a \in A$ goes to the same set X, we have $\Sigma_{a\in A} X_a = A \times X$. This is not true for every functor $X \in [A, SET]$. In the special case we couple this product with the usual projection map we obtain a map over A, $\pi_1 : A \times X \to A$. Otherwise, define the map over A to be the map $\pi : \Sigma_{a\in A} X_a \to A$ by $\pi(X_a) = \pi(X, a) = a$.

Define G_A which takes in a map over A, $\begin{tikzcd} X \arrow[dr, "p"] & \\ & A \end{tikzcd}$ and returns an A indexed set $\{X_a\}_{a\in A}$.

Do this by defining $G_A(\begin{tikzcd} X \arrow[dr, "p"] & \\ & A \end{tikzcd}) = \{X_a\}_{a\in A}$ where $X_a = p^{-1}(a)$, the preimage under p of all $a \in A$.

Firstly we see that the counit of the equivalence is equality. We have

$$G_{F_A(A)} F_A(\{X_a\}_{a\in A}) = \{X_a\}_{a\in A} \tag{1}$$

in the functor category.

The unit is also equality. This map, going between $\begin{tikzcd} X \arrow[dr, "p"] & \\ & A \end{tikzcd}$ and

$$F_{G_A(A)} G_A(\begin{tikzcd} X \arrow[dr, "p"] & \\ & A \end{tikzcd}) = \begin{tikzcd} \Sigma_{a\in A} p^{-1}(a) \arrow[dr, "\pi"] & \\ & A \end{tikzcd} = \begin{tikzcd} \Sigma_{a\in A} X_a \arrow[dr, "\pi"] & \\ & A \end{tikzcd}$$

is slightly harder to see.

Firstly, p is defined for all of X. Secondly each pair of sets X_a and $X_{a'}$ must be disjoint. because otherwise, there exists an $x \in X_a \cap X_{a'}$ and therefore, $p(x) = a$ and $p(x) = a'$. Since $a \neq a'$ this contradicts the well definition of p. Thus the X_a form a partition of X. The coproduct is just their union which is all of X. The naturality of the counit in p follows easily because the counit is a fiberwise operation, maps over A still must respect the disjointedness of the fibers.

We verify the naturality of this isomorphism of functors as follows:

Let $f : B \to A$. Let X be an object over A.

$$G_A(\begin{tikzcd} X \arrow[dr, "p"] & \\ & A \end{tikzcd}) = \{X_a\}_{a\in A}$$

Applying $[f, SET]$ to $\{X_a\}_{a\in A}$, we obtain $\{X_fb\}_{b\in B}$. Meanwhile, pulling back the above diagram on f we obtain

$$\begin{tikzcd} f^*X \arrow[r] \arrow[d, "f^*p"'] & X \arrow[d, "p"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

applying G_B to the left vertical leg we obtain:

$$G_B(\begin{tikzcd} f^*X \arrow[dr, "f^*p"] & \\ & B \end{tikzcd}) = \{X_{fb}\}_{b\in B}$$

as desired.

Exercise 3. Show for any object A the cannonical map $SET/A \to SET/\emptyset$ is constant valued. Note: $\emptyset$ is initial in SET.

Solution 3. The morphism $c : \emptyset \to A$ induces a morphism $SET/A \to SET/\emptyset$ by pulling back along the inital object. Pick an object $\begin{tikzcd} X \arrow[dr, "p"] & \\ & A \end{tikzcd}$ of SET/A. Form the pullback along $\emptyset \to A$.

$$\begin{tikzcd} c^*X \arrow[r] \arrow[d, "c^*p"'] & X \arrow[d, "p"] \\ \emptyset \arrow[r, "c"'] & A \end{tikzcd}$$

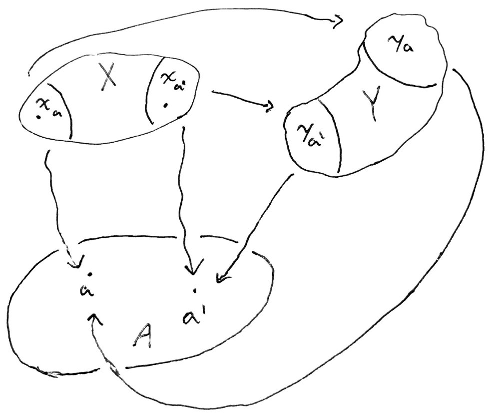
where $c^*X = \{(z, x) \in \emptyset \times X | c(z) = p(x)\}$ However, $\emptyset \times X = \emptyset$. Thus $c^*X = \emptyset$ for all X. Thus the map $SET/A \to SET/\emptyset$ has constant value $\begin{tikzcd} \emptyset \arrow[dr, "id"] & \\ & \emptyset \end{tikzcd}$ and since this agrees with the pullback, we can denote this map c^*.

0.2 What is an LCC?

In $\mathcal{C}/A$ we want to think of objects as A indexed families and maps as fiberwise mappings. Let p and q be maps over A and h a morphism between them. Then h is equivalently given by a family of fiberwise mappings

$$(h_a : X_a \to Y_a | a \in A)$$

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & A & \end{tikzcd}$$



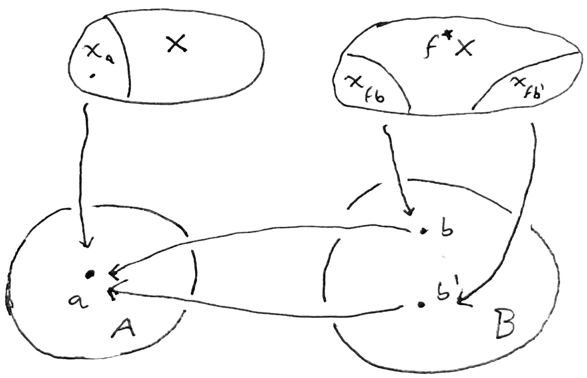
Lets look again at what the map $f^* : SET/A \to SET/B$ induced from $f : B \to A$ does fiberwise.

Note that $f^*\left(X \xrightarrow{p} A\right)$ is defined by the following pullback diagram:

$$\begin{tikzcd} \{(b,x) \in B \times X | fb = px\} \arrow[r] \arrow[d, "f^*p"'] & X \arrow[d, "p"] \\ B \arrow[r, "f"'] & A \end{tikzcd} \tag{1}$$

In the notation of the category $[A^{op}, SET]$, we are asking for what $f^*(X_a | a \in A)$ is in $[B^{op}, SET]$. To do this we must apply the functor $G_B\left(f^*X \xrightarrow{f^*p} B\right)$. Thus we must take the preimage $f^*p^{-1}(b)$ for each $b \in B$. For each $b \in B$ the preimage in $\{(b,x) \in B \times X | fb = px\}$ is given by a b-indexed subset of X such that for all x in this subset, $px = fb$. Therefore, since we denote the subset of X which maps to a as X_a, denote these b-indexed subsets as X_{fb}.

Definition 3. In $[B^{op}, SET]$, $f^*(X_a | a \in X) = (X_{fb} | b \in B)$. Where $X_{fb} = \{x \in X | fb = px\}$.



The action of f^* on maps h over A written fiberwise is defined as follows.

Definition 4. In $[B^{op}, SET]$,

$$f^*(h_a : X_a \to Y_a | a \in A) = (h_{fb} : X_{fb} \to Y_{fb} | b \in B)$$

This is to say that f^* induces a map of the pullbacks $f^*h : f^*X \to f^*Y$ and if this map f^*h is going to still be a map over B, then it must agree on the fibers. Since the pullback defines the fibers as X_{fb} and Y_{fb}, this means that f^*h is a family of maps $X_{fb} \to Y_{fb}$.

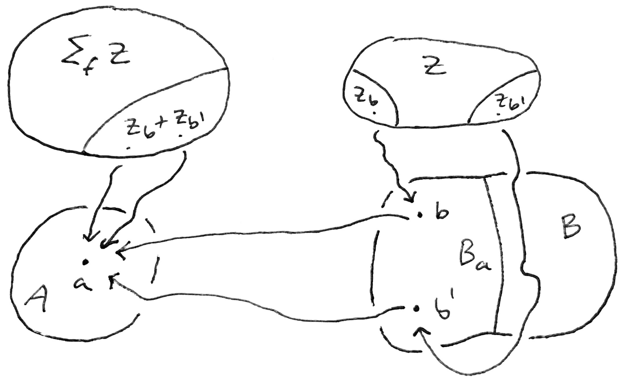
Continuing, recall the left adjoint to f^*, Σ_f defined to be the post composition along f of maps over B. Now we examine what it looks like in fiber notation across the equivalence of categories.

Let $q : Z \to B$ be a map over B, that is, $(Z_b | b \in B)$ where $Z_b = q^{-1}(b)$ and $B_a = f^{-1}(a)$.

$$\begin{tikzcd}
Z \arrow[d, "q"'] \arrow[dr, "\Sigma_f q"] & \\
B \arrow[r, "f"'] & A
\end{tikzcd}$$

We will need to work out the preimages of each element of A to obtain the representation of $\Sigma_f q$ in $[A^{op}, SET]$. Compute, $(\Sigma_f q)^{-1}(a) = (fq)^{-1}(a) = q^{-1}f^{-1}(a) = q^{-1}B_a = \Sigma_{b \in f^{-1}(a)} q^{-1}(b)$ where the final Σ is the coproduct. Notice that $q^{-1}B_a = \{q^{-1}(b_1), ..., q^{-1}(b_n)\} = \{Z_{b_1}, ..., Z_{b_n}\} = \Sigma_{b \in f^{-1}(a)} q^{-1}(b)$ is the definition in SET of the coproduct.

Definition 5. In $[A^{op}, SET]$ define $\Sigma_f(Z_b|b \in B) = (\Sigma_{b\in B_a} Z_b | a \in A)$. Note that in the notation of the previous paragraph $Z_b = q^{-1}(b)$ and $B_a = f^{-1}(a)$. †



Exercise 4. Prove $\Sigma_f(h_b : Z_b \to Y_b | b \in B) = \Sigma_{b\in B} h_b : \Sigma_{b\in B_a} Z_b \to \Sigma_{b\in B_a} Y_b$ for all $a \in A$.

Solution 4. For each $a \in A$, we need to make a map $\Sigma_{b\in B_a} Z_b \to \Sigma_{b\in B_a} Y_b$. For each $b \in B_A$ we have a map from $Z_b \to Y_b \hookrightarrow \Sigma_{b\in B_a} Y_b$, Thus there exists a map $\Sigma_{b\in B} h_b : \Sigma_{b\in B_a} Z_b \to \Sigma_{b\in B_a} Y_b$ for all $a \in A$. Note that on $\Sigma_{b\in B_a} Z_b$, $\Sigma_{b\in B} h_b = \Sigma_{b\in B_a} h_b$.

Exercise 5. Find the unit and counit of $\Sigma_f \dashv f^*$.

$$\begin{tikzcd} {[A^{op}, SET]} \arrow[d, bend left, "f^*"] \\ {[B^{op}, SET]} \arrow[u, bend left, "\Sigma_f"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

In the following diagram, as above, we do not distinguish between f^* as a map from SET/A to SET/B and it's transpose over the equivalence from solution 2. That is to say, we denote $[f, SET] : [A^{op}, SET] \to [B^{op}, SET]$ by f^* also. Similarly, we do not distinguish Σ_f from $Lan_{[f,SET]}$, the left adjoint to $[f, SET]$.

†We are defining the preimage of a set to be the coproduct of the preimages of the points in that set.

$$
\begin{tikzcd}
{[A^{op}, SET]} \arrow[rr, "f^*" description] \arrow[d, bend left, "F_A"] \arrow[d, phantom, "\cong"] & & {[B^{op}, SET]} \arrow[ll, bend right, "\Sigma_f"'] \arrow[ll, bend left] \arrow[d, bend left, "F_B"] \arrow[d, phantom, "\cong"] \arrow[ll, phantom, shift right=3, "\perp"] \arrow[ll, phantom, shift left=3, "\perp"] \\
SET/A \arrow[rr, "f^*" description] \arrow[u, bend left, "G_A"] & & SET/B \arrow[ll, bend right, "\Sigma_f"'] \arrow[ll, bend left] \arrow[u, bend left, "G_B"] \arrow[ll, phantom, shift right=3, "\perp"] \arrow[ll, phantom, shift left=3, "\perp"]
\end{tikzcd}
\tag{2}
$$

Solution 5. Combining definition 3 and definition 5 we obtain the data for the unit and counit.

To obtain the unit we need to find a map

$$(Z_b|b \in B) \to f^*\Sigma_f(Z_b|b \in B)$$

which is natural in B. We compute

$$f^*\Sigma_f(Z_b|b \in B) = f^*(\Sigma_{b\in B_a} Z_b|a \in A) = (\Sigma_{b\in B_{fb'}} Z_b|b' \in B)$$

For every $b' \in B$ take every $b \in B_{fb'}$ and form the coproduct of the Z_b. Since $b' \in B_{fb'}$ for each b', Z_b' is always a member of the coproduct for that index. Reindexing $(Z_b|b \in B)$, we have for each $(Z_{b'}|b' \in B) \to (\Sigma_{b\in B_{fb'}} Z_b|b' \in B)$ simply include $Z_{b'} \hookrightarrow \Sigma_{b\in B_{fb'}} Z_b$ because $Z_{b'}$ is one of the terms in the coproduct.

To find the counit of the adjunction, for a given object $(X_a|a \in X)$ over A we need to find a natural map, from $\Sigma_f f^*(X_a|a \in X) \to (X_a|a \in X)$. We compute,

$$\Sigma_f f^*(X_a|a \in X) = \Sigma_f(X_{fb}|b \in B) = (\Sigma_{b\in B_a} X_{fb}|a \in A) = (\Sigma_{b\in B_a} X_a|a \in A)$$

If $b \in B_a$, then $fb = a$ which gives us the last equality above. Since then for all such b we have $X_a \hookrightarrow X_a$, we also have for all a, $\Sigma_{b\in B_a} X_{fb} \to X_a$. Alternatively, $(\Sigma_{b\in B_a} X_a|a \in A) = (B_a \times X_a|a \in A)$. Thus there is a natural map $(\varepsilon_{A_a} : \Sigma_{b\in B_a} X_a \to X_a|a \in X)$ over A. This is the counit of the adjunction.

Exercise 6. Show that if this adjunction and a terminal object exist then the category $\mathcal{C}$ has products. Define $(-) \times B$ to be the composite map

$$\mathcal{C} \xrightarrow{\cong} \mathcal{C}/\mathbf{1} \xrightarrow{\mathcal{C}/B^*} \mathcal{C}/B \xrightarrow{\Sigma_k} \mathcal{C}/\mathbf{1} \xrightarrow{\cong} \mathcal{C}$$

Solution 6. By examining the above problem closely, we see that this is simply the counit described above in the case where $A = \mathbf{1}$ in diagram (1). We have that

$$X_{fb} = \{x \in X|px = fb\}$$

but in every case, $px = fb$ because they are both equal to the single element of $\mathbf{1}$. Thus $X_{fb} = X$ for every b. Similarly, $B_a = B$ because there is again just a single element of B. Thus the quantity $(\Sigma_{b\in B_a} X_{fb} | a \in \mathbf{1}) = \Sigma_{b\in B} X = X \times B$.

The left adjoint Π_f

Suppose now we have $f : B \to A$ in $\mathcal{C}$, an LCC. What is $\Pi_f(Z \xrightarrow{q} B)$?

Again, moving upward across the equivalence of categories shown in diagram (2), we think of $Z \xrightarrow{q} B$ as $(Z_b | b \in B)$. The adjunction we are examining takes the form

$$\mathcal{C}/A(X \xrightarrow{p} A, \Pi_f(Z \xrightarrow{q} B)) \cong \mathcal{C}/B(f^*(X \xrightarrow{p} A), Z \xrightarrow{q} B)$$

Using the right side of the adjunction we will investigate what form Π_f will take. Moving across the equivalence, $f^*(X \xrightarrow{p} A) = f^*(X_a | a \in A) = (X_{fb} | b \in B)$.

Maps $(h_b : X_{fb} \to Z_b | b \in B)$ are maps $(\overline{h}_a : X_a \to \Pi_f(Z)_a | a \in A)$.

Example 2. Let $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$, $X = \{x_1, x_2, x_3, x_4\}$, and $Z = \{z_1, z_2, z_3\}$. Define $p : X \to A$ by $p(x_1) = p(x_2) = a_1$ and $p(x_3) = p(x_4) = a_2$. Define $q : Z \to B$ by $q(z_1) = b_1$, $q(z_2) = b_2$, and $q(z_3) = b_3$.

In the context of this example, one can check that

$$f^*X = \{(x_1, b_1), (x_1, b_2), (x_2, b_1), (x_2, b_2), (x_3, b_3), (x_4, b_3)\}$$

alternatively,

$$f^*X = \{X_{fb_1}, X_{fb_2}, X_{fb_3}\}$$

where $X_{fb1} = \{x_1, x_2\}$. Thus we can write

$$f^*X = \{\{x_1, x_2\}_{b_1}, \{x_1, x_2\}_{b_2}, \{x_3, x_4\}_{b_3}\}$$

An example of a map h over B, $(h_b : X_{fb} \to Z_b | b \in B)$ would be a collection of maps

$$h_{b_1} : \{x_1, x_2\}_{b_1} \to \{z_1\}$$
$$h_{b_2} : \{x_1, x_2\}_{b_2} \to \{z_2\}$$

$$h_{b_3} : \{x_3, x_4\}_{b_3} \to \{z_3\}$$

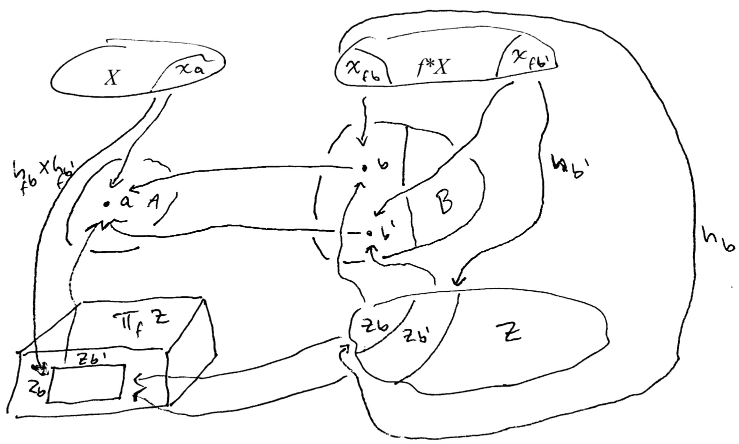
Such maps over B must correspond to maps $(\overline{h}_a : X_a \to \Pi_f(Z)_a | a \in A)$.

$$\overline{h}_{a_1} : \{x_1, x_2\} \to \Pi_f(Z)_{a_1}$$

$$\overline{h}_{a_2} : \{x_3, x_4\} \to \Pi_f(Z)_{a_2}$$

If there exist maps h_{b_1} and h_{b_2} out of X_{fb_1} and X_{fb_2} which are copies, then there exists a map from $\{x_1, x_2\} \to a_1 \times a_2$.

Definition 6. Define $\Pi_f(Z \xrightarrow{q} B) = (\Pi_{b \in B_a} Z_b | a \in A)$.



How shall we read $\Pi_{b \in B_a} Z_b$? We can read it as either,

1. products of sets
2. sequences of values indexed by B_a
3. functions $f : B_a \to (Z_b | b \in B)$

Example 3. In the case $B_a = B$ for every $a \in A$, and $Z_b = Z$ for all $b \in B$ we have $\Pi_f(Z \xrightarrow{q} B) = (\Pi_{b \in B_a} Z_b | a \in A) = (\Pi_{b \in B} Z | a \in A) = (Z^B | a \in A)$.

$$\begin{tikzcd}
{[B^{op}, SET]} \arrow[d, bend left, "\Pi_f"] \\
{[A^{op}, SET]} \arrow[u, bend left, "f^*"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

What is the unit and counit of the adjunction $f^* \dashv \Pi_f$? The unit is going to be an A indexed family of maps:

$\eta_{(X_a|a\in A)} : (X_a|a \in A) \to \Pi_f f^*(X_a|a \in A) = \Pi_f(X_{fb}|b \in B) = (\Pi_{b\in B_a} X_{fb}|a \in A)$

which can be written more succinctly as

$$(\eta_a : X_a \to \Pi_{b\in B_a} X_{fb}|a \in A)$$

because $b \in B_a$ ensures that $fb = a$ we may write the definition as follows

Definition 7. We define the unit of the adjunction $f^* \dashv \Pi_f$ to be the A indexed family of maps $(\eta_a : X_a \to \Pi_{b\in B_a} X_a|a \in A)$ where these maps are given in multiple contexts as either

1. the diagonal (product of sets)
2. constant sequence (sequences of values)
3. constant function (functions $f : B_a \to (X_a|a \in A)$)

Example 4. In order to find the counit we compute the following

$$f^*\Pi_f(Z_b|b \in B) = f^*(\Pi_{b'\in B_a} Z_{b'}|a \in A) = (\Pi_{b'\in B_{fb}} Z_{b'}|b \in B)$$

A map from $f^*\Pi_f(Z_b|b \in B)$ to $(Z_b|b \in B)$ is given by the b^{th} projection, the b^{th} term or the evaluation of the choice function at b.

Definition 8. We define the counit of the adjunction $f^* \dashv \Pi_f$ to be the B indexed family of maps $(\varepsilon_b : \Pi_{b'\in B_{fb}} Z_{b'} \to Z_b|b \in B)$ where these maps are given in multiple contexts as either

1. b^{th} the projection (product of sets)
2. b^{th} term (sequences of values)
3. evaluation at b (functions $f : B_{fb} \to (Z_b|b \in B)$)

Exercise 7. What are the traingle equalities saying?

Applying f^* to the following diagram $(\eta_a : X_a \to \Pi_{b\in B_a} X_{fb}|a \in A)$, we obtain,

$$(f^*\eta_b : X_{fb} \to \Pi_{b'\in B_{fb}} X_{fb'}|b \in B)$$

Applying ε_{f^*X} we obtain,

$$((id_{f^*})_b = (\varepsilon_{f^*X} \circ f^* \circ \eta_{f^*X})_b : X_{fb} \to \Pi_{b'\in B_{fb}} X_{fb'} \to X_{fb}|b \in B)$$

Roughly this is saying, for each b, the inclusion of X_{fb} into a B_{fb} indexed power of X_{fb} followed by projection onto a single copy of X_{fb} again is the identity.

To find the second triangle equality, we apply the unit η at $\Pi_f(Z_b|b \in b) = (\Pi_{b' \in B_a} Z_{b'}|a \in A)$ to obtain

$$(\eta_a : \Pi_{b \in B_a} Z_b \to \Pi_{b \in B_a}(\Pi_{b' \in B_{fb}} Z_{b'})|a \in A)$$

Appling Π_f to the counit $(\varepsilon_b : \Pi_{b' \in B_{fb}} Z_{b'} \to Z_b|b \in B)$ we obtain a map $(\Pi_f \circ \varepsilon_a : \Pi_{b \in B_a} \Pi_{b' \in B_{fb}} Z_{b'} \to \Pi_{b \in B_a} Z_b|a \in A)$. The composition of these maps modulo reindexing yeilds a map equal to the identity at $\Pi_f(Z)$

$$((id_{\Pi_f(Z)})_a = (\Pi_f \circ \varepsilon \circ \eta_{\Pi_f(Z)})_a : \Pi_{b \in B_a} Z_b \to \Pi_{b \in B_a}(\Pi_{b' \in B_{fb}} Z_{b'}) \to \Pi_{b \in B_a} Z_b|a \in A)$$

We can interpret this as the statement, for each $a \in A$, the product of the Z_b where $b \in B_a$ can be included into the $(\Pi_{b \in B_a} Z_b)^{B_a}$ as the diagonal map. Then for each $b' \in B_a$, project the b'th term of $(\Pi_{b \in B_a} Z_b)^{B_a}$ onto $Z_{b'}$. Thus we obtain a map into $\Pi_{b' \in B_a} Z'_b$.

Lemma 1. Let $\mathcal{C}$ be an LCC. Let $B \xrightarrow{f} A$. Then in $\mathcal{C}/A$, the composite

$$\mathcal{C}/A \xrightarrow{f^*} \mathcal{C}/B \xrightarrow{\Sigma_f} \mathcal{C}/A$$

is the product $(-) \times \left(\begin{tikzcd} B \arrow[dr, "f"] & \\ & A \end{tikzcd}\right)$.

Proof. internal

$$\Sigma_f f^*(X_a|a \in A) = \Sigma_f(X_{fb}|b \in B) = (\Sigma_{b \in B_a} X_{fb}|a \in A) =$$

$$(\Sigma_{b \in B_a} X_a|a \in A) = (X_a \times B_a|a \in A)$$

Given an object $(Y_a|a \in A)$ and maps $(h_a : Y_a \to B_a|a \in A)$ and $(k_a : Y_a \to X_a|a \in A)$ we have a unique map $(l_a : Y_a \to X_a \times B_a|a \in A)$ where $(l_a = h_a \times k_a|a \in A)$. The obvious projections exist. $\square$

Proof. external
Using the diagram below, let h and k be maps over A. Since $f \circ k = i = p \circ h$, we have $f \circ k = p \circ h$ so we obtain a map l from Y to f^*X because it is a pullback. Since the entire diagram commutes, $f^*X = \Sigma_f f^*X$ is the product with projections ε_X and f^*p.

$$\begin{tikzcd}
Y \arrow[dr, "l"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] \arrow[ddrr, bend right, "i"] & & \\
& \Sigma_f f^*X = f^*X \arrow[r, "\varepsilon_X"] \arrow[d, "f^*p"] \arrow[dr, "\Sigma_f f^*p"] & X \arrow[d, "p"] \\
& B \arrow[r, "f"'] & A
\end{tikzcd}$$

□

Lemma 2. If $\mathcal{C}$ is LCC then $\mathcal{C}/A$ has product $(-) \times (B \xrightarrow{f} A)$ has a right adjoint.

Proof. We may compose the following adjunctions

$$\begin{tikzcd} \mathcal{C}/A \arrow[r, bend left, "f^*"] \arrow[r, phantom, "\perp"] & \mathcal{C}/B \arrow[l, bend left, "\Pi_f"] \arrow[r, bend left, "\Sigma_f"] \arrow[r, phantom, "\perp"] & \mathcal{C}/A \arrow[l, bend left, "f^*"] \end{tikzcd}$$

to obtain

$$\begin{tikzcd} \mathcal{C}/A \arrow[r, bend left, "\Sigma_f \circ f^*"] \arrow[r, phantom, "\perp"] & \mathcal{C}/A \arrow[l, bend left, "\Pi_f \circ f^*"] \end{tikzcd}$$

However, by lemma 1, $\Sigma_f \circ f^* = (-) \times (B \xrightarrow{f} A)$. We compute the value of the right adjoint,

$$\Pi_f f^*(Y_a | a \in A) = \Pi_f(Y_{fb} | b \in B)$$
$$= (\Pi_{b \in B_a} Y_{fb} | a \in A) = (\Pi_{b \in B_a} Y_a | a \in A) = (Y_a^{B_a} | a \in A)$$

and see that indeed it is the fiberwise exponential.

$$\begin{tikzcd} \mathcal{C}/A \arrow[d, bend left, "(-)^{B \xrightarrow{f} A}"] \arrow[d, phantom, "\dashv"] \\ \mathcal{C}/A \arrow[u, bend left, "(-) \times (B \xrightarrow{f} A)"] \end{tikzcd}$$

□

This completes the proof of the following theorem.

Theorem 1. If $\mathcal{C}$ is *LCC* then $\mathcal{C}/A$ is *CC* for any object A of $\mathcal{C}$.