What is an LCC? ts1:1

Σ-coproducts

~~Adj~~ $\mathbb{C} \underset{R}{\overset{L}{\rightleftarrows}} D$ $\Rightarrow$

(n-algebraic)

Def: A Cartesian closed category is a cat $\mathbb{C}$ in which all finite products exist and the $\times \dashv [-,-]$.

Def: An (n-algebraic) locally cartesian closed category is a category $\mathbb{C}$ together with a functor

$\mathbb{C}/(-) : \mathbb{C}^{op} \longrightarrow CAT$

$\mathbb{C}/A$ objects $\quad c' \xrightarrow{c} C \quad c^* = \mathbb{C}/_c : \mathbb{C}/_c \longrightarrow \mathbb{C}/_{c'}$

$$\begin{array}{ccc} c^*X & \longrightarrow & X \\ {\scriptstyle c^*f}\downarrow & & \downarrow{\scriptstyle f} \\ c' & \xrightarrow[c]{} & C \end{array}$$

~~Question~~ s.t. $\mathbb{C}/_- \Leftarrow \Pi_c$

Question what's the relationship between locally c.c. and c.c.?

Ex:

what it's name is called	LCC	CC
Set	✓	✓
CAT	✗	✓
GRP	✗	✓
Top	?	?
Top ... os	✓	✓

Exercise: wherever this makes sense

$$\begin{tikzcd} \mathbb{C}/A \arrow[r, "f^*"] \arrow[r, phantom, shift right=3, "\Leftarrow"] & \mathbb{C}/B \arrow[l, bend left=50, "\Sigma_f"] \end{tikzcd}$$

Left adjoint

$B \xrightarrow{f} A$

Show $\Sigma_f \left(\begin{tikzcd} X \arrow[d, "b"] \\ B \end{tikzcd} \right) = \begin{tikzcd} X \arrow[d] \arrow[dr, "fb"] & \\ B \arrow[r, "f"'] & A \end{tikzcd}$

Further Coincidence no. 1

Exercise 1) with canonically chosen pullbacks

Show that the following $[(-)^{op}, SET] \cong SET/(-)$

$SET^{op} \longrightarrow CAT$

(idea below)

2)

$$\begin{tikzcd}[column sep=huge, row sep=huge] {[B^{op}, Set]} \arrow[r, "{[f, Set]}"] \arrow[d, "\cong"'] \arrow[r, phantom, shift left=4, "\Leftarrow"] \arrow[r, phantom, shift right=4, "\Rightarrow"] & {[A^{op}, SET]} \arrow[d, "\cong"] \arrow[l, bend right=50, "Lan_f"'] \arrow[l, bend left=50, "Ran_f"] \\ Set/B \arrow[r] \arrow[r, phantom, shift left=4, "f^* \Leftarrow"] \arrow[r, phantom, shift right=4, "\Rightarrow"] & SET/A \arrow[l, bend right=40] \arrow[l, bend left=40] \end{tikzcd}$$

$A \xrightarrow{f} B$

$$\begin{array}{c} X \\ \downarrow \\ A \end{array} \in \mathrm{Set}/A \cong [A^{op}, \mathrm{Set}] \ni X$$

X is really an A indexed set $X \leftrightsquigarrow (X_a)_{a\in A}$

$$\sum_{a\in A} X_a \xrightarrow{p} A$$
$$(a,x) \longmapsto a$$

$$\boxed{\sum_{b\in B} X = X\times B \quad \longmapsto \quad X\times B \xrightarrow{\pi} B}$$

$$\begin{array}{c} X \\ \downarrow p \\ A \end{array} \longmapsto X_a := p^{-1}\{a\}$$

preimage

in SET/A

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "p"'] & Y \arrow[dl, "q"] \\ A & \end{tikzcd}$$

in $\mathbb{C}/A$ think of objects as A indexed families and maps as fiberwise mappings.

$$\begin{array}{c} X \\ \downarrow p \\ A \end{array} \in \mathbb{C}/A \quad (X_a \mid a \in A)$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "p"'] & Y \arrow[dl, "q"] \\ A & \end{tikzcd} \longleftrightarrow (h_a : X_a \to Y_a \mid a \in A)$$

Exercise Show for any object A $\mathrm{SET}/A \xrightarrow{A^*} \mathrm{Set}/0$ ← initial object

$0 \xrightarrow{!_A} A$ is constant valued on $0 \to 0$.

II

With these pictures in mind using SET lets look at f^*

$$\begin{tikzcd} B \arrow[d, "f"] & SET/A \arrow[r, "\cong"] \arrow[d, "f^*"'] & {[A^{op}, SET]} \arrow[d, "{[f, Set]}"] & \ni X \arrow[d, mapsto] \\ A & SET/B \arrow[r, "\cong"'] & {[B^{op}, SET]} & ([f,Set]X)_b \equiv X_{fb} \end{tikzcd}$$

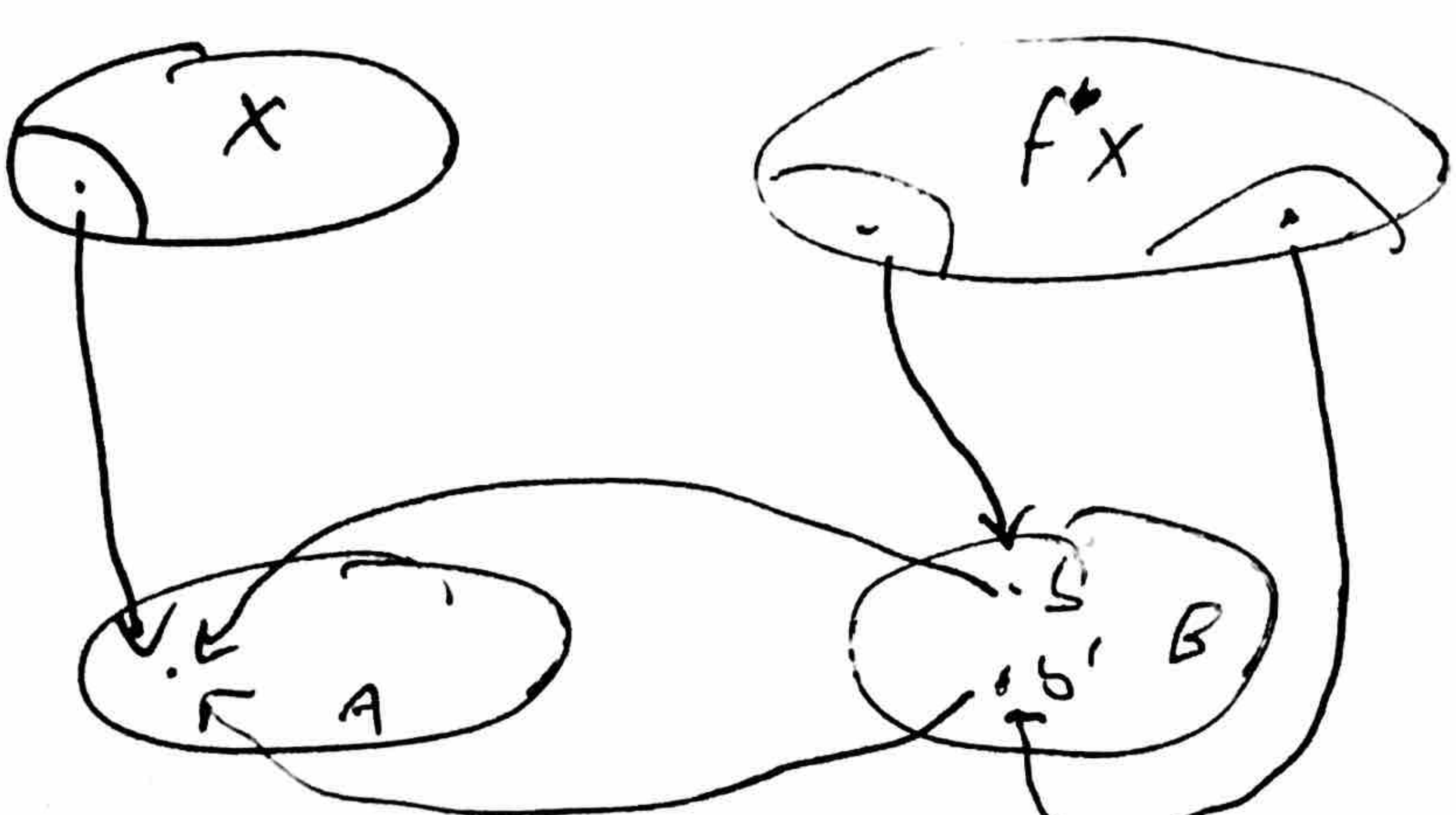
$f^*(X \to A)$

$$\begin{tikzcd} \{(b,x) \in B \times X \mid fb = px\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

$\pi^{-1}\{b\} = X_{fb}$

$f^*(X_a \mid a \in X) :\equiv (X_{fb} \mid b \in B)$

$f^*(h_a : X_a \to Y_a \mid a \in A)$
$\equiv (h_{fb} : X_{fb} \to Y_{fb} \mid b \in B)$



Look for some External proof.

$\mathrm{SET}/B \xrightarrow{\Sigma_f} \mathrm{SET}/A$

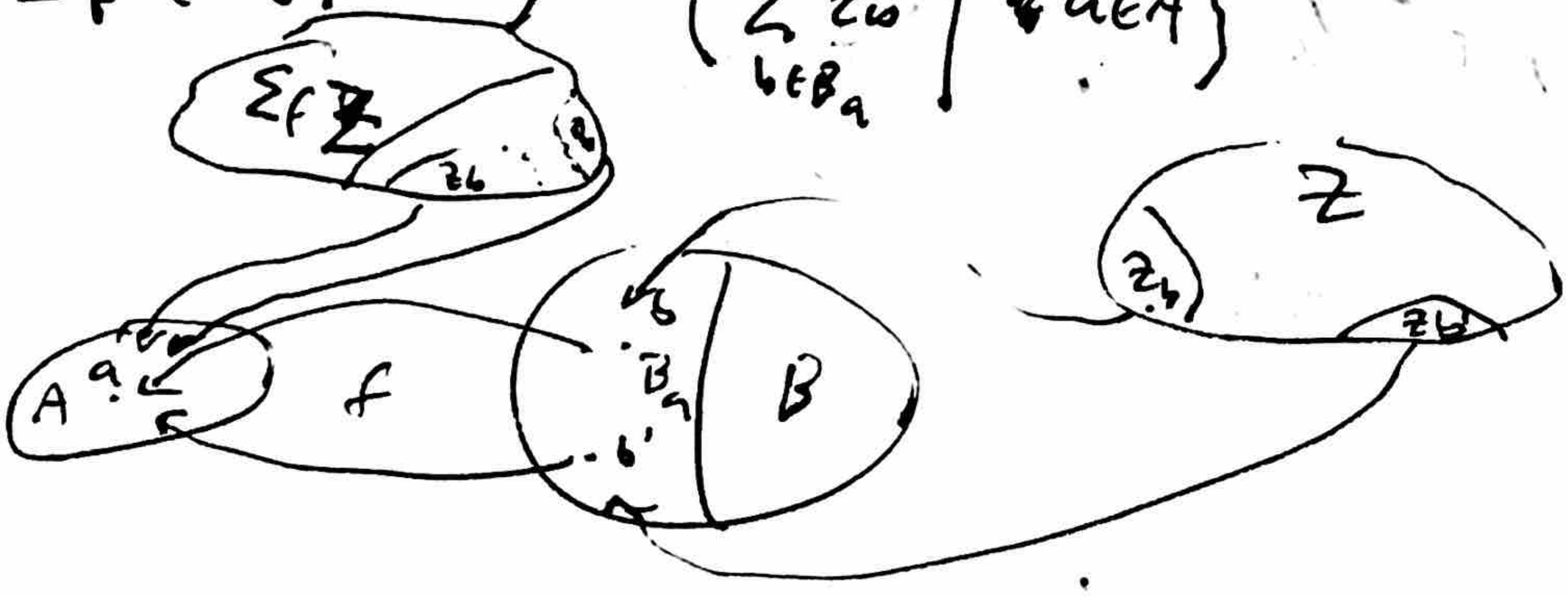
$$\begin{array}{l} Z \\ \ \ \searrow q \\ \ \ \ \ \ B \end{array}$$

$$(Z_b \mid b \in B) \longmapsto \begin{array}{l} Z \\ \ \ \searrow fq \\ \ \ \ \ \ A \end{array} \rightsquigarrow [A^{op}, \mathrm{SET}]$$

Need to work out preimages $fq^{\leftarrow}\{a\} = q^{\leftarrow} f^{\leftarrow}\{a\} = \sum_{b \in f^{\leftarrow}\{a\}} q^{\leftarrow}\{b\}$

preimage of $\{a\}$

$$\Sigma_f (Z_b \mid b \in B) := \left(\sum_{b \in B_a} Z_b \;\middle|\; a \in A \right)$$



<u>Ex</u>: prove in Set $\Sigma_f (h_b : Z_b \to Y_b \mid b \in B) \equiv \left(\sum_{b \in B} h_b : \sum_{b \in B_a} Z_b \to \sum_{b \in B_a} Y_b \;\middle|\; a \in A \right)$

2) unit and counit of

$\Sigma_f \dashv f^*$?

3) Show that $(\Sigma \dashv *)$ + terminal object $\longrightarrow$ $\times$ functor exists. have products

$$\mathbb{C} \xrightarrow{\cong} \mathbb{C}/1 \xrightarrow{/A^*} \mathbb{C}/A \xrightarrow{\Sigma_A} \mathbb{C}/1 \cong \mathbb{C}$$

is $(-) \times A$

Suppose now we have $f : B \to A$ of $\mathbb{C}$ an LCC what is $\Pi_f(Z \xrightarrow{q} B)$?

right adjoint

$$\mathbb{C}/A\left(X \xrightarrow{f} A,\ \Pi_f\left(Z \xrightarrow{q} B\right)\right)$$

think of $Z \xrightarrow{q} B$ as $(Z_b \mid b \in B)$

$$\cong \mathbb{C}/B\left(f^*\left(X \xrightarrow{p} A\right),\ Z \xrightarrow{q} B\right)$$

"i.e." elts of $\Pi_f\left(Z \xrightarrow{q} B\right)$ are maps (not elts)

$$\begin{tikzcd} f^*\left(X \xrightarrow{p} A\right) \arrow[r] \arrow[dr, "f^*p"'] & Z \arrow[d, "q"] \\ & B \end{tikzcd}$$

$X \xrightarrow{p} A \leftrightarrow (X_a \mid a \in A)$, $f^*(X_a \mid a \in A) \equiv (X_{fb} \mid b \in B)$

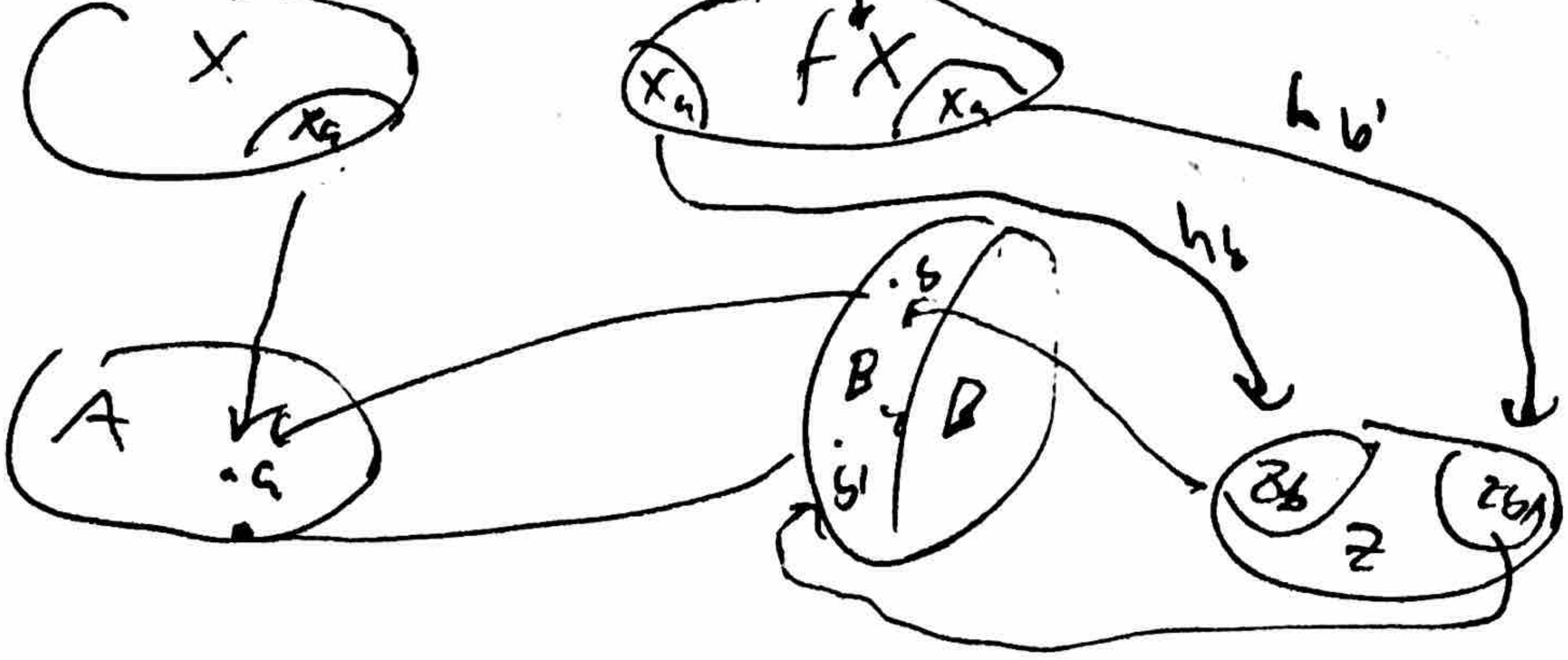
$\to (h_b : X_{fb} \to Z_b \mid b \in B)$ are maps.

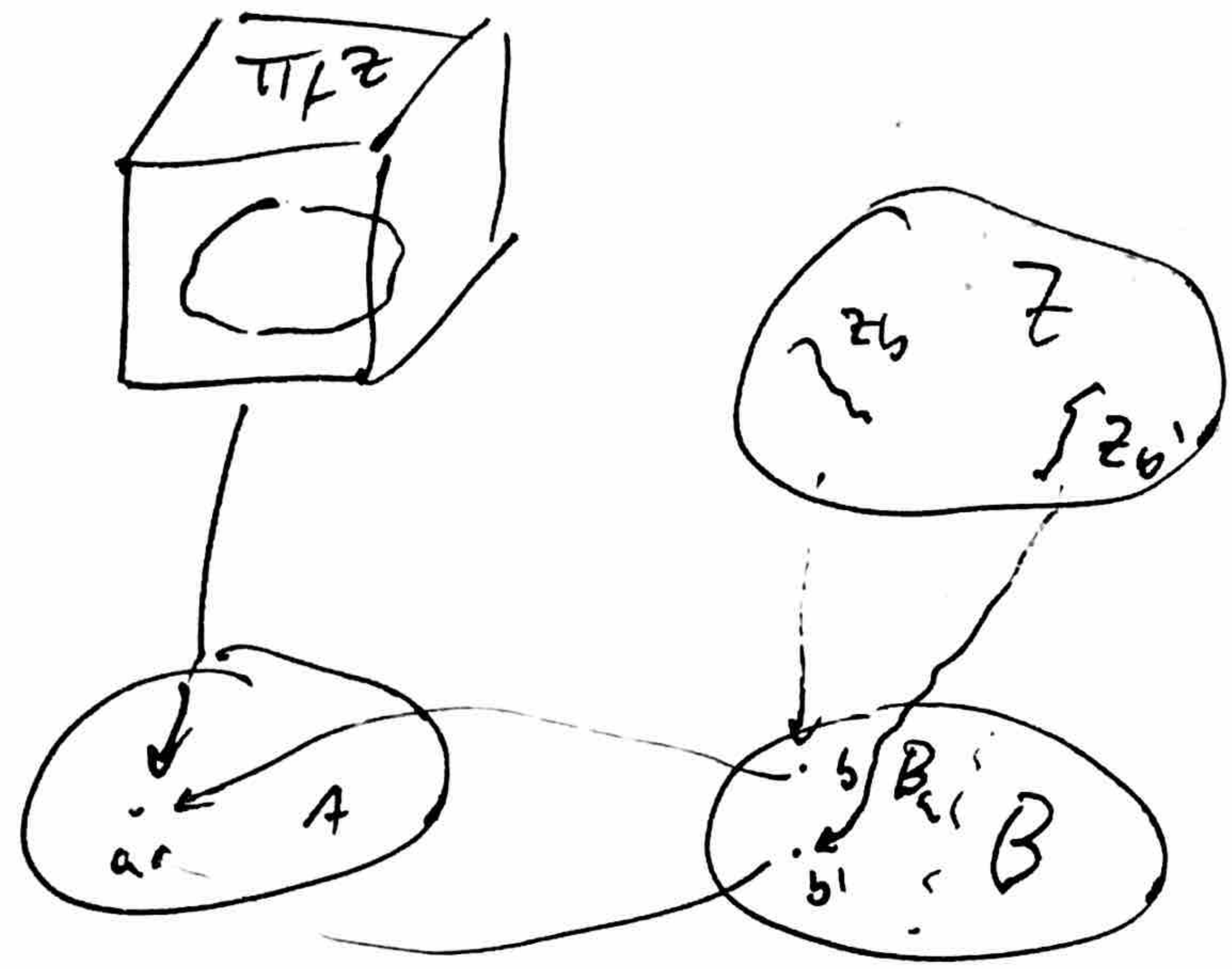
$b, b' \in B_a$

$$\begin{tikzcd} & Z_b \\ X_a \arrow[ur, "h_b"] \arrow[dr, "h_b"'] & \\ & Z_{b'} \end{tikzcd}$$

"$X_{fb} = X_{fb'}$"

$\hookrightarrow X_a \longrightarrow \begin{matrix} Z_b \\ \times \\ Z_{b'} \end{matrix} \leftrightsquigarrow$ is an element of $Z_b \times Z_{b'}$





$\Pi_f (z_b \mid b \in B) :\equiv \left(\prod_{b \in B_a} z_b \,\middle|\, a \in A\right) \qquad B \xrightarrow{f} A$

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how to read $\prod_{b \in B_a} z_b$?

1) Products of sets
2) Sequences of values indexed by B_a
3) Functions $f : (b : B_a) \to z_b$

In the case $B_a = B$ for every a and moreover that $z_b \equiv Z$ for every b

$B \times A \xrightarrow{f : \pi_2} A \quad \rightsquigarrow \Pi_f (Z \mid b \in B) \equiv \left(\prod_{b \in B} Z \,\middle|\, a \in A\right) = \left(Z^B \,\middle|\, a \in A\right)$

what is the unit and counit of $\Pi_f \Rightarrow f^*$?

$\eta^{(X_a \mid a \in X)} : (X_a \mid a \in A) \to \Pi_f f^* (X_a \mid a \in A) \equiv \Pi_f (X_{fb} \mid b \in B) \equiv \left(\prod_{b \in B} X_{fb} \,\middle|\, a \in A\right)$

$\eta_a^{(X_a \mid a \in A)} : X_a \longrightarrow \prod_{b \in B_a} X_a \mid a \in A$

what is? $\begin{cases} \text{products} \longrightarrow \text{diagonal} \\ \text{sequences} \longrightarrow \text{Constant seq} \\ \text{functions} \longrightarrow \text{Const function.} \end{cases}$

$f^* \Pi_f (\cancel{Z_b} \mid b \in B)$

$\equiv f^* \left(\prod_{b' \in B_a} Z_{b'} \mid a \in B \right)$

$\equiv \left(\prod_{b' \in B_{fb}} Z_{b'} \mid b \in B \right) \xrightarrow{\varepsilon^{(Z_b \mid b \in B)}} (Z_b \mid b \in B)$

$\varepsilon_{b'} : \prod_{b' \in B_{fb}} Z_{b'} \to Z_b \mid b \in B)$

Products $\longrightarrow$ proj

Sequences $\longrightarrow$ b'th term

Functions $\longrightarrow$ evaluations at b.

Excercise ~~the~~ Find what triangle equalities ~~are~~ saying?

Lemma if $\mathbb{C}$ is an LCC then in $\mathbb{C}/A$ for $B \xrightarrow{f} A$ the composite $\mathbb{C}/A \xrightarrow{f^*} \mathbb{C}/B \xrightarrow{\Sigma_f} \mathbb{C}/A$ is $(-) \times (B \xrightarrow{f} A)$

Starting to feel Locally CC is CC in each slice

proof: $\Sigma_f f^* (Y_a \mid a \in A) \equiv \Sigma_f (Y_{fb} \mid b \in B) \equiv \left(\sum_{b \in B_a} Y_{fb} \mid a \in A \right) = \left(\sum_{b \in B_a} Y_a \mid a \in A \right)$

$= (Y_a \times B_a \mid a \in A)$. Excercise Complete proof + externalize.

Fibrations are LCC in $\mathbf{CAT}$.

Lemma if $\mathbb{C}$ is LCC then $\mathbb{C}/A$ $(-) \times (B \xrightarrow{f} A)$ has a Right Adjoint.

proof:
$$\begin{tikzcd} \mathbb{C}/A \arrow[r, shift left=2, "f^*"] \arrow[r, phantom, "\Rightarrow"] & \mathbb{C}/B \arrow[l, shift left=2, "\Pi_f"] \arrow[r, shift left=2, "\Sigma_f"] \arrow[r, phantom, "\Rightarrow"] & \mathbb{C}/A \arrow[l, shift left=2, "f^*"] \end{tikzcd}$$

$\Pi_f f^* (Y_a \mid a \in A) \equiv \Pi_f (Y_{fb} \mid b \in B) \equiv \left(\prod_{b \in B_a} Y_{fb} \mid a \in A \right) = \left(\prod_{b \in B_a} Y_a \mid a \in A \right)$

$= (Y_a^{B_a} \mid a \in A)$ both come out fibrewise in the liver there is extern-al object.