Notes on Equivariant Stable Homotopy Theory

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1 Unstable Equivariant Homotopy

1.1 Adjoints and Actions

Let G be a finite group.

Definition 1. *A G space is a space X together with a continuous map $\phi : G \times X \to X$ where we denote $\phi(g,x)$ as $g \cdot x$ such that e, the identity of the group satisfies $e \cdot x = x \ \forall x \in X$ and $g \cdot (h \cdot x) = (g \cdot h) \cdot x \ \forall g, h \in G$.*

Definition 2. *An equivariant map or a G-map $f : X \to Y$ is a continuous map such that*

$$f(g \cdot x) = g \cdot f(x)$$

with the notation of definition 1.

$$\begin{tikzcd}
G \times X \arrow[r, "id_G \times f"] \arrow[d, "\phi"'] & G \times Y \arrow[d, "\phi"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Two maps are "G-homotopic" if they are homotopic through G-maps. G acts on $I \times X$ by

$$g(t,x) = (t, gx)$$

. If $\{f_t\}_{t \in I}$ is an I indexed family of maps from X to Y, with $H(x,t) := f_t(x)$ the following diagram must commute.

$$\begin{tikzcd}
(t,x) \arrow[r, "H"] \arrow[d, "g"'] & f_t(x) \arrow[d, "g"] \\
(t,gx) \arrow[r, "H"] & \frac{gf_t(x)}{f_t(gx)}
\end{tikzcd}$$

When we consider Top_*, the category of pointed topological spaces, the base point is fixed under the action of G.

Denote the category G-spaces and G-equivariant maps by **G-Map**. If H is a subgroup of a group G, the inclusion map $i : H \to G$ induces a forgetful functor $i_H^* :$ **G-Map** $\to$ **H-Map** which restricts the action of G on a G space to an H space.

Filling in the gratuitous details: let Y be a G-space, $\phi : G \times Y \to Y$, then $i^*(\phi) := \phi(i(-), -) : H \times Y \to Y$.

This functor has both left and right adjoints. Let X be an H space we define the left and right adjoints as follows:

Definition 3. *Define the functor* $G \times_H - : $ ***H-Map*** $\to$ ***G-Map*** *as* $G \times_H X := G \times X/(gh, x) \sim (g, hx)$*. This is the left adjoint and it is called induction.*

Definition 4. *Define the functor* $Map^H(G, -)$: ***H-Map*** $\to$ ***G-Map*** *to be* $Map^H(G, -) := \{$ *H equivariant maps from* $G \to X\}$

We have the following sequence of ajoints:

$$G \times_H - \dashv i_H^* \dashv Map^H(G, -)$$

The adjoint pair $G \times_H - \dashv i_H^*$ is an example of a free forgetful adjunction. The adjoint pair $i_H^* \dashv Map^H(G, -)$ is slightly more nuanced.

The unit of the adjunction, η, is a morphism in **G-Map** from $X \to Map^H(G, i_H^*(X))$ defined as follows. Note that X is a G-space and therefore has an action $\phi : G \times X \to X$. Define the unit on each element of $x \in X$ to be the curried version of ϕ. Explicitly,

$$\eta_X : x \mapsto \phi(-, x) : G \to X$$

and function $\phi(-, x)$ is clearly still G equivariant and therefore also H equivariant.

The counit of the adjunction, ε, is a morphism in **H-Map** from $i_H^*(Map^H(G, Y)) \to Y$ where Y is an H-space. Define the counit on each element of $f_y \in i_H^*(Map^H(G, Y))$ to be the evaluation at the identity of G. Define,

$$\varepsilon_{i_H^*(Map^H(G,Y))} : f \mapsto f(e)$$

Let ϕ be the H action on the H-space $i_H^*(Map^H(G, Y))$ and let ψ denote the H action on the H-space Y. Note that f is H-equivariant and therefore, $\varepsilon(\phi(h, f)) = \varepsilon(f(-h)) = f(eh) = hf(e) = \psi(h, \varepsilon(f))$. I neglect to write the component of ε here as a subscript because it is so long.

Example 1. *Let* G *be the cyclic group of four elements* $\{e, x, x^2, x^3\}$ *and let* H *be the cyclic group of two elements* $\{e, x^2\}$ *sitting as a subgroup inside of* G*. Let the set* $X = \{0, 1, 2, 3\}$*.*

Let the action of H *on* X *be as follows:* e *acts trivially on* X*,* x^2 *interchanges* 0 *and* 2 *and also interchanges* 1 *and* 3*.*

Then $G \times_H X = G \times X/\{(e, 0) \sim (x^2, 2), (e, 1) \sim (x^2, 3), (e, 2) \sim (x^2, 0), (e, 3) \sim (x^2, 1), (x, 0) \sim (x^3, 2), (x, 1) \sim (x^3, 3), (x, 2) \sim (x^3, 0), (x, 3) \sim (x^3, 1)\}$

When H is the trivial subgroup $G \times_H$ simply endows X with the trivial action. Thus $G \times_H$ can be viewed as free over G/H. In fact, this association is given explicitly in the text. Let X be a G space. Then i^*X is an H space. In this case, we obtain the following isomorphism (in this category that means G-homeomorphism) $G/H \times X \cong G \times_H i^*X$. Letting $[g] = gH \in G/H$, define $([g], x) \mapsto (g, g^{-1}x)$ and in the other direction $(g, x) \mapsto ([g], gx)$. These are mutually inverse maps and are well defined on cosets.

The key thing to note here is suppose $h \in H$ and $g_1 h = g_2$, then $g_1 \in [g_2]$, $(g_2, x) \mapsto (g_2, g_2^{-1}x) = (g_1 h, (g_1 h)^{-1}x) = (g_1 h, h^{-1}g^{-1}x) \sim (g_1, hh^{-1}g_1^{-1}x) = (g_1, g_1^{-1}x)$. This proves the map is well defined on cosets.

Example 2. *Let * be the zero object for **Grp**. **G-Map**$(*, X) \cong \{x \in X | g \cdot x = x \ \forall g \in G\}$ the set of G fixed points in X. Denote this set X^G. This isomorphism is natural in X and is given explicitly by $f \to f(*)$.*

Example 3. *The set of H fixed points is naturally isomorphic to **G-Map**$(G/H, X) \cong \{x \in X | h \cdot x = x \ \forall h \in H\} = X^H$.*

For $g \in G$ and $x \in x^H$, gx is then fixed by gHg^{-1}. So g is a homomorphism from X^H to $X^{gHg^{-1}}$. Suppose g is in H, clearly it is in the stabilizer of X^H then the homomorphism that g induces is trivial. Suppose that g is just in the normalizer of H, then g is a homomorphism from X^G to itself but is not neccesarily trivial. Therefore, the Weyl group of H, $N(H)/H$ is isomorphic to the group of automorphisms of X^H.

Any G map $f : X \to Y$ must preserve the fixed point set of H and the action of $N(H)/H$. Let f^H denote the induced map of the fixed point set.

If $K \subseteq H$ then $X^H \subseteq X^K$. Suppose x is fixed by H then it sure is fixed by K.

Which $g \in G$ define an action $X^K \to X^H$? That is, for x fixed by K, when is gx fixed by H?

Precisely when $g^{-1}hg \in K \ \forall h \in H$: Then $hgx = gkg^{-1}gx = gkx = gx$.

Definition 5. *The category $\mathscr{O}$ of canonical orbits of a group G has objects which are transitive G-sets G/K, and G/H and morphisms $\alpha : G/H \to G/K$ given by $\alpha(gH) = g\gamma K$ where $\gamma K \in N(H)/K$.*

That is, α must be well defined on cosets. $\alpha(hH) = h\gamma K$ but also $\alpha(hH) = \alpha(H) = \gamma K$. Therefore $\gamma^{-1}h\gamma \in K \subseteq H$ as desired.

Definition 6. *We define a contravariant functor $F_X : \mathscr{O}^{op} \to$ **Set**, which takes G sets G/H to fixed point sets X^H. Again, $\alpha : G/H \to G/K$ iff $\exists \gamma \in G$ such that $\forall h \in H$, $\gamma^{-1}h\gamma \in K$. We will show that these γ are exactly the elements of G which define a map of fixed point sets $\alpha_* : X^K \to X^H$, thus $F(\alpha) = \alpha_* : X^K \to X^H$.*

Note: $\{\gamma \in G | h = \gamma k \gamma^{-1} \forall h \in H\} = \{\gamma \in G | \gamma^{-1}h\gamma = k \forall h \in H\} = \{\gamma \in G | \gamma^{-1}h\gamma \in K \forall h \in H\}$.

We see that the functor F is a presheaf from in $[\mathscr{O}^{op}, Set]$

1.2 Cells, Spheres, and G-CW complexes

Definition 7. *Define the disk and sphere, $D(V) = \{v \in V | \ ||v|| \leq 1\}$ and $S(V) = \{v \in V | \ ||v|| = 1\}$. Define the representation sphere $S^V = D(V)/S(V)$*

with the quotient topology. This is isomorphic to V_+, the one point compactification of V.

Letting $V = \mathbb{R}^n$, then from now on S^n is going to denote $S^{\mathbb{R}^n}$ with trivial G action.

We are going to define $G - CW$ complexes where the action will take place to have cells of the form $G/H \times D^n$ and $G/H \times S^{n-1}$. This is going to nicely let us use the product hom adjunction along with example 3 to reduce statements about G-CW-cells in a complex X to CW-cells in the fixed points of X under H. $(G/H \times D^n, X) = (D^n, (G/H, X)) = (D^n, X^H)$.

We choose not to use cells of the form $G \times_H D(V)$ and $G \times_H S(V)$ where $D(V)$ and $S(V)$ have an H action because of some non-trivial mathematics that says that the G-CW-complexes of the form $G/H \times D^n$ and $G/H \times S^{n-1}$ can be used to recover G-CW-complexes of the other form.

In particular, we assemble the G-CW-complexes by induction over both the grading by the natural numbers and by the elements of $\mathscr{O}$.

Example 4. *Let $G = S^1$, we consider the sphere as a S^1-CW complex. It has two 0-cells, $S^1/S^1 \times D^0$, which are fixed points corresponding to the north and south poles. In addition it has one 1-cell $S^1/e \times D^1$, This has attaching maps which identify each endpoint of the 1-cell with one of the two 0-cells.*

Example 5. *Let $G = C_2$, we consider the sphere as a C_2-CW complex. The action of C_2 on the sphere will rotate it 180 degrees. It has two 0-cells, $C_2/C_2 \times D^0$, which are fixed points corresponding to the north and south poles. In addition it has one 1-cell $C_2/e \times D^1$, This has attaching maps which identify each endpoint of the 1-cell with one of the two 0-cells. Note: (e, D^1) is the line of longitude going through Greenwich, (g, D^1) is the line of longitude going through Singapore. It also has one 2-cell $C_2/e \times D^1$. Note: (e, D^2) is the Western Hemisphere and (g, D^2) is the Eastern Hemisphere.*

1.3 Dimension and Weak Equivalence

Adams says the first theorem in homotopy theory is the theorem that $\pi_r(S^n) = 0$ for $r < n$. More coloquially, you can't lasso a beachball.

Proposition 1. *If $dimV^H < dimW^H$ for all H, then $[S^V, S^W]^G = 0$.*

In the equivariant sense, $dimV^{()}$ is going to be a function that assigns to a subgroup $H \subseteq G$ the value $dim(V^H)$. So S^V and S^W are functions of subgroups of G.

Definition 8. *The Hurewicz Dimension, $Hur(X)$ is defined to be the greatest n such that $\pi_r(X) = 0$ for all $r < n$.*

For spheres we have $Hur(S^{V^H}) = dim(S^{V^H})$. The following proposition generalizes 2.4

Proposition 2. *If $dim(X^H) < Hur(Y^H)$ for all H, then $[X, Y]^G = 0$.*

Recall in the ordinary case that a map $f : X \to Y$ between path connected spaces is called an n-equivalence if $f_* : \pi_r(X) \to \pi_r(Y)$ is an isomorphism for $r < n$ and epi (right invertible) for $r = n$.

Definition 9. *Let n be a function which assigns to each subgroup $H \subseteq G$ a value $n(H)$ which may be an integer or ∞, such that conjugate subgroups have the same value, $n(H) = n(gHg^{-1})$, then a G-map $f : X \to Y$ is an n-equivalence if $f^H : X^H \to Y^H$ is an ordinary $n(H)$-equivalence for each H.*

Example 6. *For example, $f : S^V \to S^W$ is an n-equivalence if for each subgroup $H \subset G$, $f^H : S^{V^H} \to S^{W^H}$ is an $n(H)$-equivalence. That is, $f_*^H : \pi_r(S^{V^H}) \to \pi_r(S^{W^H})$ is an isomorphism of groups for $r < n(H)$ and is epi for $r = n(H)$.*

The f^H here are going to be the adjoint conjugates of G-maps $f : X \to Y$, under the chain of adjoints $(G/H \times D^n, X) = (D^n, (G/H, X)) = (D^n, X^H)$.

Theorem 1 (G-Whitehead)**.** *Let W be a G-CW-complex and let $f : X \to Y$ be a G-map which is an n-equivalence. Then the induced map $f_* : [W, X]^G \to [W, Y]^G$ is onto if $dim(W^H) \leq n(H)$ for all H, and is one-to-one if $dim(W^H) < n(H)$ for all H.*

1.4 Notes on Suspension Theory

In **Top**, If X is locally compact and Hausdorff, $hom(X \times I, Y) \cong hom(I, Y^X)$. In **Top**$_*$, homotopies between based spaces are required to be based maps. A map $h : X \times I \to Y$ is a based homotopy means that $h(x_0, t) = y_0$ for all $t \in I$.

Definition 10. *Non-equivariantly, for each non-negative integer n, we have a homotopy functor $\pi_n = [S^n, -]$ from **Top**$_*$ to **Set**, called the n^{th} homotopy group.*

The forgetful functor $U : \mathbf{Top}_*$ to **Top** has a left adjoint + defined on objects $X \mapsto X \coprod \{*\}$, and on morphisms by mapping the extra point of $X \coprod \{*\}$ to the extra point of $Y \coprod \{*\}$ if $f : X \to Y$. Note that U preserves limits and + preserves colimits.

Definition 11. *For pointed spaces (X, x_0) and (Y, y_0) define the wedge product $X \vee Y$ to be the quotient of the coproduct of the spaces modulo the basepoints, $X \coprod Y/\{x_0 \sim y_0\}$. This can be realized as the following pushout diagram:*

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd}$$

Definition 12. *The smash product of two pointed spaces* (X, x_0) *and* (Y, y_0) *is defined to be* $X \wedge Y = X \times Y / X \vee Y$

Definition 13. *The smash-hom adjunction is the version of the product-hom adjunction for* $\boldsymbol{Top}_*$ *and is given as follows:*

$$X \wedge - \dashv (-)^X$$

Both S^1 and $(-)^{S^1}$ define functors as a special case and are written as Σ and Ω respectively. We call these the reduced suspension functor and the loop space functor. Passing to basepoint preserving homotopy classes of morphisms, we obtain for every pair of pointed spaces X and Y

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

We can define these functors less categorically as follows:

Definition 14. *The join of a two spaces* X *and* Y *in* ***Top*** *is denoted* $X * Y$ *and is defined to be* $(X \times Y \times I)/\{\{x_1, y, 0\} \sim \{x_2, y, 0\}, \{x, y_1, 1\} \sim \{x, y_2, 1\}\}$

Definition 15. *The cone of a space* X *in* ***Top*** *is defined to be the quotient space* $(X \times I)/(X \times \{0\}) = X \times I/\{\{x_1, 0\} \sim \{x_2, 0\}\}$*. The cone of* X *is written* CX*.*

Example 7. *The join of the one point space* $\{*\} = D^0$ *with a topological space* X *is precisely the cone of* X*.*
$\{*\} * X = (X \times \{*\} \times I)/\{\{x_1, *, 0\} \sim \{x_2, *, 0\}, \{x, *, 1\} \sim \{x, *, 1\}\} \cong (X \times I)/\{\{x_1, 0\} \sim \{x_2, 0\}\} = CX$*.*

Definition 16. *The suspension of a space* X *in* ***Top*** *is defined to be the quotient space* $(X \times I)/(X \times \{0\}, X \times \{1\}) = X \times I/\{x_1, 0\} \sim \{x_2, 0\}, \{x_1, 1\} \sim \{x_2, 1\}\}$*. The suspension of* X *is denoted* SX*.*

Example 8. *The join of the two point space* $\{*\} \coprod \{*\} = S^0$ *with a topological space* X *is precisely the suspension of* X*.*
$SX = S^0 * X$*.*

Definition 17. *The reduced suspension of a pointed space* $X \in \boldsymbol{Top}_*$ *denoted* ΣX *is defined to be* $SX/(\{x_0\} \times I) = SX/\{\{x_0, t\} \sim \{x_0, s\}\}$ *where* x_0 *is the basepoint of* X *and* $s, t \in I$*.*

Example 9. *Now it is verified that these two constructions of* ΣX *are indeed the same. That is,* $S^1 \wedge X = SX/(\{x_0\} \times I)$*. By definition,* $SX/(\{x_0\} \times I) = ((X \times I)/(X \times \{0\}, X \times \{1\}))/(\{x_0\} \times I) = (S^1 \times X)/(X \times \{0\}, \{x_0\} \times S^1) = S^1 \times X/S^1 \vee X = S^1 \wedge X$*.*

Proposition 3. *The 0 sphere, S^0, is the unit when $\wedge$ makes the category of compactly generated hausdorff spaces into a symmetric monoidal category.*

Proof. Compute $S^0 \wedge X = S^0 \times X / S^0 \vee X = X \coprod X / * \coprod X = X$. $\square$

Proposition 4. *As right and left adjoints respectively to the inclusion functor, the one point compactification preserves products, Stone-ech compactification preserves coproducts. Thefore $X^* \wedge Y^* = (X \times Y)^*$ where * denotes the one-point compactification. Similarly, $X' \vee Y' = (X \coprod Y)'$ where $'$ denotes the Stone-ech compactification.*

Proposition 5. *The product $S^1 \wedge S^n \cong S^{n+1}$.*

Proof. The base case holds because $S^1 \wedge S^0 = S^1$. Notice that for $S^1 \wedge S^n$ is the same thing as the wedge of the one point compactifications $\mathbb{R}^* \wedge \mathbb{R}^{n*}$. By the previous proposition this is equal to $\mathbb{R}^{n+1*}$.
Less categorically, $S^1 \wedge S^n = (I \times S^n)/(\{0 \times S^n\}, \{1 \times S^n\}, \{I \times x_0\}) = (I \times S^n)/(\{0 \times S^n\}, \{1 \times S^n\}) = S^{n+1}$. $\square$

Proposition 6. *Let X be a pointed space. Then $\pi_n(X) = \pi_{n-1}(\Omega X)$*

Proof. By definition, $\pi_n(X) = [S^n, X] = [S \wedge S^{n-1}, X] = [S^{n-1}, \Omega X] = \pi_{n-1}(\Omega X)$ $\square$

1.5 The G-suspension Theorem

For the unreduced suspension of G-spaces without base-point, the join $S(V) * X$ is used. For a reduced suspension of G-spaces with base-point the smash product $S^V \wedge X$ is used. Adam's defines the action of G on these products the same way:

$$g(x, y) = (gx, gy)$$

Adams claims that the naturality of the projection from $X * Y$ down to $X \wedge S^1 \wedge Y$ is natural and therefore the proofs for the equivariant case go through using G-Whitehead, provided that 3 conditions following the example hold:

Example 10. *Note that in the non-equivariant case it is easy to see the existance of a projection in the following example. Letting $X = S^0$ then $X * Y = SY$ which has a projection down to $\Sigma Y = S^1 \wedge Y = S^0 \wedge S^1 \wedge Y$.*

1. The restriction of the comparison map to a fixed-point-set is another instance of the same comparison map, for instance

$$X^H * Y^H \to X^H \wedge S^1 \wedge Y^H.$$

2. The comparison map is classically a weak equivalence.

3. The G-spaces involved are G-CW-complexes.

Proposition 7. *The projection we are looking for has kernel $\{x_0, y_0, I\}$. The proof of this follows the arguement given in example* 9. *Naturality follows by preservation of basepoint.*

Let $S^V : [X,Y]^G \to [S^V \wedge X, S^V \wedge Y]^G$ by $f \mapsto id_{S^V} \wedge f$. Adams says this is a (1-1) correspondence under suitable conditions. Let $\Omega^V(Z)$ be the space of pointed maps Z^{S^V} with G action given by

$$(g\omega)(s) = g(\omega(g^{-1}s))$$

It suffices to show that the unit of the adjunction

$$Y \to \Omega^V(S^V \wedge Y)$$

is an n-equivalence.

The $n = n(H)$ function must have the following properties:

1. For each subgroup $H \subset G$ such that $V^H > 0$ ($dim > 0$) we have $n(H) \leq 2Hur(Y^H) - 1$
2. For each pair of subgroups $K \subset H \subset G$ such that $V^K > V^H$ we have that $n(H) \leq Hur(Y^K) - 1$.

Theorem 2. *If the above conditions are satisfied, the map*

$$Y \to \Omega^V(S^V \wedge Y)$$

is an n-equivalence.

By G-Whitehead, the function S^V (equivariant suspension) is onto if X is a G-CW-complex and $dim(X^H) \leq n(H) - 1$ for each H.

1.6 Allowable Representations

We choose some class of "allowable representations" of G so that this class is closed under sums and isomorphisms. There is an ordering on the allowable representations writing $W \geq V$ if $W \cong U \oplus W$ for some U. Let X be a finite-dimensional G-CW-complex and Y a G-space.

Theorem 3. *There exists an allowable $W_O = W_O(X)$ such that for any allowable $W \geq W_O$ and any allowable V, the map*

$$S^V : [S^W \wedge X, S^W \wedge Y]^G \to [S^V \wedge S^W \wedge X, S^V \wedge S^W \wedge Y]^G$$

is a 1-1 correspondence. Additionally, this holds for any subcomplex or subdivision of $S^W \wedge X$.

The result follows from the previous theorem provided we can satisfy the following inequalities on the dimensions.

1. If for some H there is an allowable V with $V^H > O$ then
$$dimW^H + dimX^H \leq 2dimW^H - 2.$$
If there is any non-zero V^H by putting enough copies of it into W we can increase this $dimW^H$ until the inequality is satisfied.

2. If for some $K \subset H$ there is an allowable V with $V^K > V^H$, then
$$dimW^H + dimX^H \leq dimW^K - 2$$
. By putting sufficiently many copies of V into W we can increase $dimW^K - dimW^H$. This holds for all larger W.

1.7 The G-analogue of the Spanier-Whitehead Category

Let $\mathcal{C}$ be a category with objects given by allowable representations V of G and morphisms given by inner-product-preserving R-linear G-maps.

Given two G-spaces X,Y with basepoints, we wish to define a functor which takes an object V of $\mathcal{C}$ and gives us the set $[S^V \wedge X, S^V \wedge Y]^G$.

For any morphism $i : V \to W$ in $\mathcal{C}$, we first use i to identify W with $U \oplus V$ where U is the orthogonal complement of the image of $i(V)$ under the inner product. We now associate to i the following composite function:

$$\begin{tikzcd}
{[S^V \wedge X, S^V \wedge Y]^G} \arrow[r, "S^U"] & {[S^U \wedge S^V \wedge X, S^U \wedge S^V \wedge Y]^G} \arrow[d, leftrightarrow] \\
& {[S^W \wedge X, S^W \wedge Y]^G;}
\end{tikzcd}$$

So we now have a **Set** valued functor. Denote it F. Clearly, $\mathcal{C}$ has finite coproducts and products.

Now we wish to check that the functor has the following equalizing property:

that is, if $f, g : U \to V$ in $\mathcal{C}$ then there is a further morphism $h : V \to W$ such that $F(hf) = F(hg)$.

Adams says it is easy to reduce to the case where f is an automorphism of V and $g = 1$. One can see via counter-example that it is not sufficient to take $h = 1$ that is, the composite

$$S^V \wedge X \xrightarrow{f^{-1}\wedge 1} S^V \wedge X \xrightarrow{\phi} S^V \wedge Y \xrightarrow{f\wedge 1} S^V \wedge Y$$

need not be G-homotopic to ϕ. However, we can take h to be the injection of V as the second factor in $V \oplus V$. Clearly we have $hf = (1 \oplus f)h$. On the left hand side we are applying f to V and then including it in the sum on the right we are first including V then acting on only the second factor. However $1 \oplus f$ is homotopic to $f \oplus 1$. For, $f \oplus 1$ we have that $(f \oplus 1)h$ and h induce the same function.

We may then pass to the limit and define

$$\{X, Y\}^G = \varinjlim_{V \in \mathcal{C}} [S^V \wedge X, S^V \wedge Y]^G.$$

Proposition 8. *If $dimV^H < dimW^H$ for all H, then*

$$\{S^V, S^W\}^G = 0$$

Proposition 9. *If $dim(X^H) \leq Hur(Y^H) - 1$ for all H, then*

$$\{X, Y\}^G = 0$$

In each case we are taking the limit of sets which are trivial.

Proposition 10. *If X is a finite-dimensional G-CW-complex, then the limit $\{X, Y\}^G$ is attained by $[S^W \wedge X, S^W \wedge Y]^G$ for all sufficiently large W.*

For later use we need to assure ourselves that this category is really "stable" that is, for two G-spaces X, Y and an allowable representation U. For each object V of $\mathcal{C}$ we have a function

$$[S^V \wedge X, S^V \wedge X]^G \xrightarrow{Susp_V} [S^V \wedge X \wedge S^U, S^V \wedge Y \wedge S^U]^G$$

which takes f to $f \wedge 1_U$

Lemma 1. *If X is a finite-dimensional G-CW-complex then passing to the limit of allowable representations we obtain a 1-1 correspondence*

$$\{X, Y\}^G \xrightarrow{Susp} \{X \wedge S^U, Y \wedge S^U\}^G$$

By the previous proposition, for each $Susp_V$ there is a sufficiently large representation which makes each map a 1-1 correspondence.

In order to make the sets $\{X, Y\}^G$ into groups we need a "suspension coordinate" on which G acts trivially. Now assume that trivial representations are allowable. Then the sets $\{X, Y\}^G$ become additive groups, i.e. hom sets of a preadditive category.

Theorem 4. *Suppose X is a finite G-CW-complex and Y is a G-space for which each fixed-point-set Y^H is an ordinary CW-complex with finitely many cells of each dimension. Then $\{X, Y\}^G$ is a finitely generated abelian group.*

Adam's says that the crucial point is because of proposition 10 there is some finite extension of V to W such that $\{X, Y\}^G = [S^W \wedge X, S^W \wedge Y]^G$. To prove that this is finitely generated is a proper adaptation of the usual inductive methods with the methods in section 1.2.

1.8 Pre-additive, Additive, Pre-abelian, and Abelian Categories

Definition 18. *A category is preadditive if each hom set has the structure of an abelian group and each composition $mor(x, y) \times mor(y, z) \to mor(x, z)$ is bilinear. Since every hom set is non-empty, and is an element of* ***Grp*** *the* 0 *map is present in each hom set.*

Proposition 11. *If a preadditive category has an initial or a final object then it has a zero object.*

Proposition 12. *If a preadditive category has either finite products or coproducts then it has both and these objects coincide. A category is called additive if it is preadditive and has finite products and coproducts.*

Definition 19. *A pre-abelian category is an additive category such that every morphism has a kernel and a cokernel. Equivalently it has all finite limits and colimits. Finite limits is equivalent to finite products, given by additivity and equalizers guaranteed as the kernel of $f - g$.*

Proposition 13. *In pre-abelian category every morphism $f : A \to B$ has a canonical decomposition*

$$A \xrightarrow{p} coker(kerf) \xrightarrow{\overline{f}} ker(cokerf) \xrightarrow{i} B$$

Definition 20. *If $\overline{f}$ in the above proposition is always an isomorphism then the pre-abelian category is called abelian. This is equivalent to the category being ballanced, the bijections are isomorphisms.*