Notes on Some Set Theoretic Constructions from a Categorical Perspective

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These notes are based on the first chapter of Asu Vaisman's Dover book Cohomology and Differential Forms.

1 Categories

Definition 1. A category \mathcal{C} consists of a class of objects, $Ob(\mathcal{C})$, and for each pair of objects A and B, a class of morphisms Mor(A, B). For each object there is an identity morphism $1_A \in Mor(A, A)$. Lastly, for each triple of objects we have a a binary operation called composition

$$\circ: Mor(A,B) \times Mor(B,C) \to Mor(A,C)$$

If each collection of morphisms is a set, we say that the category is *locally small*. If the class of objects is a set, we say that the category is *small*.

- **Ex 1.** The class of sets forms a category **Set** whose morphisms are functions.
- **Ex 2.** The set of integers forms a category with morphisms given by divisibility |. For example, Mor(4,8) is a singleton because 4 divides 8, however, Mor(4,9) is just \emptyset . Each integer divides itself and lastly, we can check that composition is well defined as follows: if a|b and b|c then a|c.
- **Ex 3.** In general, groups, rings, or any other algebraic objects form a category. The class of groups is a category **Grp** whose morphisms are group homomorphisms.
- **Ex 4.** Every group can itself be viewed as a category with one object, whose morphisms are group elements.

2 Injectivity and Surjectivity

Definition 2. Let A, B and X be objects of a cateogry C. A morphism $u: A \to B$ is said to be injective if for any object X, the map $u: Mor(X, A) \to Mor(X, B)$ defined as $u(v) = u \circ v$ is injective.

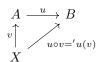


Figure 1.

Proposition 1. We will now verify that this definition agrees with our usual definition of injectivity in the category **Set**. Let u be a set map which satisfies the above property. Let $a_1, a_2 \in A$ such that $a_1 \neq a_2$. Let v_1, v_2 be two morphisms (both constant functions) in Mor(X, A) defied as follows: $v_1 : X \to A$ such that $\forall x \in X, v_1(x) = a_1$ and $v_2 : X \to A$ such that $\forall x \in X, v_2(x) = a_2$ then because $v_1 \neq v_2$, we have $u_1(v_1) \neq u_2(v_2)$ and consequently, $u_1(v_2) \neq u_2(v_3)$.

Suppose now that u is injective in the usual sense. Let $v_1 \neq v_2$. This means there exists an element $x \in X$ such that $v_1(x) \neq v_2(x)$; furthermore $u(v_1(x)) \neq u(v_2(x))$ by the injectivity of u, and thus $u(v_1) \neq u(v_2)$.

Definition 3. A morphism $u: A \to B$ is said to be surjective, if for any object X, the map $u': Mor(B, X) \to Mor(A, X)$ defined on $v \in Mor(B, X)$ as $u'(v) = v \circ u$ is injective.

Figure 2.

Proposition 2. We will now verify that this definition agrees with our usual definition of surjectivity in the category **Set**. Suppose that the above condition holds. Then for any $b \in B$ define a pair of functions $v_1, v_2 \in Mor(B, X)$ which disagree on b, but agree on every other element of B. Then since $u'(v_1) \neq u'(v_2)$, there must be an element $a \in A$ such that u(a) = b, if there were not such an element, then we obtain $u'(v_1) = u'(v_2)$, a contradiction.

Suppose now that u is injective. Then for any other set X take $v_1, v_2 \in Mor(B, X)$ such that $v_1 \neq v_2$. This means that there exists an element $b \in B$ for which $v_1(b) \neq v_2(b)$. Since u is surjective, there exists an element $a \in A$ such that u(a) = b. It follows that $v_1(u(a)) \neq v_2(u(a))$ which shows the injectivity of u'.

A morphism which is both injective and surjective is said to be *bijective*.

3 Isomorphisms and Balance

Definition 4. In a category, two objects A and B are said to be isomorphic if there exist morphisms $f:A\to B$ and $g:B\to A$ so that $g\circ f=1_A$ and $f\circ g=1_B$.

Exercise 2. Prove that the composition of injections, surjections, and isomorphisms are again injections, surjections and isomorphisms.

A category in which the bijections are exactly the isomorphisms is said to be balanced.

Ex 5. When considered as a set map, the inclusion of the integers in the rationals is obviously not surjective. However, in the category of rings, Ring, it is a simple matter to show that this inclusion does define a surjection in Ring.

Exercise 3. Is Set balanced? Is Ring balanced?

4 Initial, Final, and Zero Objects

Very often, when investigating some mathematical object we look at its subobjects or quotient objects. Similarly when looking at a map between two objects it is useful to understand some of the subobjects involved and the way in which the morphism behaves on those subobjects.

We now introduce a pair of definitions and a pair of categories which will give us the proper setting to discuss these constructions.

Definition 5. Given a category C if there is an object $X \in ob(C)$ such that for every other object $C \in ob(C)$, the class Mor(X, C) is a singleton, we say that X is an *initial object* for the category C. That is, there is exactly one morphism from X to every other object in the category.

Definition 6. Given a category \mathcal{C} if there is an object $X \in ob(\mathcal{C})$ such that for every other object $C \in ob(\mathcal{C})$, the class Mor(C, X) is a singleton, we say that X is a *final object* for the category \mathcal{C} . That is, there is exactly one morphism to X from every other object in the category.

Proposition 3. If a category \mathcal{C} has an initial object X, it is unique up to isomorphism. That is, if there is any other object Y which is also initial, it must be isomorphic to X. To see this, note that there is a unique morphism from X to Y and from Y to X. Composing these morphisms we obtain a morphism from X to X or from Y to Y depending on the order of composition. However, Mor(X,X) and Mor(Y,Y) are both singletons and by the definition of a category must contain 1_X and 1_Y respectively. This does not leave much choice for what the compositions are.

Proposition 4. If a category C has a final object X, it is unique up to isomorphism. The proof is similar to the one given in proposition 3.

Definition 7. If a category has both an initial and a final object and they are equal, then that object is called a *zero object* for the category. Note that if a category has a zero object * then between any two objects A and B there is a morphism from A to B which passes through * and is denoted $0_{A,B}$. It is

defined to be the composite map $A \to * \to B$ which exists because * is both initial and final.

5 Subobjects and Quotient Objects

Definition 8. Given an object A in a category C, define the category C_A as follows. The objects of this category are morphisms in C going into A. Morphisms in this category are given by precomposing the morphisms going into A with other morphisms of C. Similarly define a category C^A who's objects are morphisms in C coming out of A and who's morphisms are given by post composition.

Let X and Y be objects of C and let $f: X \to A$ and $g: Y \to A$ be morphisms of C. Notice also that f and g are objects of C_A . Let $u: Y \to X$ be a morphism of C. If $f \circ u = g$, then u is a morphism in C_A . This is shown in the figure below:

Figure 3.

Exercise 3. Verify that C_A and C^A are a categories. For which category would it be more fruitful to discuss an initial object? For which category would it be more fruitful to discuss a final object?

Definition 9. A subcategory \mathcal{D} of a category \mathcal{C} is a category with $ob(\mathcal{D}) \subset ob(\mathcal{C})$, and for each class $Mor_{\mathcal{D}}(A,B) \subset Mor_{\mathcal{C}}(A,B)$. Additionally the operation composition in \mathcal{D} is exactly the restriction of the operation in \mathcal{C} . Lastly, Each object in \mathcal{D} retains its identity morphism from \mathcal{C} .

A full subcategory is one in which only objects are removed, that is, if A and B are objects in \mathcal{D} , $Mor_{\mathcal{D}}(A, B) = Mor_{\mathcal{C}}(A, B)$.

Definition 10. We examine now the subcategory of C_A who's objects are just the injections into A. We denote this category S_A and call it the *category of subobjects* of A. An object of S_A is a *subobject* of A and it is considered up to isomorphism. That is, if f and g are isomorphic objects in S_A , they are considered to be the same subobject.

Definition 10. Similarly, we call the subcategory of C^A who's objects are just the surjections from A, the category of quotient objects of A. It is denoted Q^A . Quotient objects are also only considered up to isomorphism.

6 Range and Corange

We can now define the range of a morphism in a category \mathcal{C} .

Definition 11. Let A and B be objects of a category C, and let f be a morphism $f: B \to A$. Define S_A^B to be the full subcategory of S_A whose objects $u: X \to A$ are part of a unique factorization of f. That is, for each u there exists a unique $f': B \to X$ such that f = u(f'). This is shown below.

$$X \xrightarrow{u} A$$

$$f' \uparrow \qquad \nearrow \qquad f='u(f')=u \circ f'$$

$$B$$

Figure 4.

Definition 12. An initial object of \mathcal{S}_A^B is called the *range* of the morphism f, and is denoted Ran(f). The range is determined up to isomorphism (as are all initial objects).

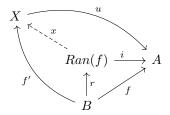


Figure 5.

Proposition 5. In the category **Set** Ran(f) is isomorphic to

$$f(B) = \{a \in A | \exists b \ s.t. \ f(b) = a\}.$$

Let X = f(B) in figure 5. Since Ran(f) is initial, there is one morphism from Ran(f) to A, namely i. Then $u \circ x = i$ and since i is injective, so must x be. Suppose now that x is not surjective. Then there is an $a \in f(B)$ such that for no $c \in Ran(f)$, x(c) = a. But for all $a \in f(B)$ there is a $b \in B$ for which f(b) = a but then u(f'(b)) = a and hence u(x(r(b))) = a. Letting c = r(b) we have derived a contradiction. So x is both injective and surjective. Since **Set** is balanced, f(B) is isomorphic to Ran(f).

Definition 13. Let A and B be objects of a category C, and let f be a morphism $f: B \to A$. Define \mathcal{Q}_A^B to be the full subcategory of \mathcal{Q}^B whose objects are surjections $u: B \to X$ which are part of a unique factorization of f. That is, for each u there exists a unique $f': X \to A$ such that $f = f' \circ u$.

Definition 14. A final object of \mathcal{Q}_A^B is called the *corange* of f. It is denoted Cor(f).