Notes on Some Set Theoretic Constructions from a Categorical Perspective

David L. Meretzky

June 22nd, 2018

These notes are based on the first chapter of Asu Vaisman's Dover book *Cohomology and Differential Forms.*

1 Categories

Definition 1. A category $\mathcal{C}$ consists of a class of objects, $Ob(\mathcal{C})$, and for each pair of objects A and B, a class of morphisms $Mor(A, B)$. For each object there is an identity morphism $1_A \in Mor(A, A)$. Lastly, for each triple of objects we have a a binary operation called composition

$$\circ : Mor(A, B) \times Mor(B, C) \to Mor(A, C)$$

If each collection of morphisms is a set, we say that the category is *locally small*. If the class of objects is a set, we say that the category is *small*.

Ex 1. The class of sets forms a category **Set** whose morphisms are functions.
Ex 2. The set of integers forms a category with morphisms given by divisibility $|$. For example, $Mor(4, 8)$ is a singleton because 4 divides 8, however, $Mor(4, 9)$ is just $\emptyset$. Each integer divides itself and lastly, we can check that composition is well defined as follows: if $a|b$ and $b|c$ then $a|c$.
Ex 3. In general, groups, rings, or any other algebraic objects form a category. The class of groups is a category **Grp** whose morphisms are group homomorphisms.
Ex 4. Every group can itself be viewed as a category with one object, whose morphisms are group elements.

2 Injectivity and Surjectivity

Definition 2. Let A, B and X be objects of a cateogry $\mathcal{C}$. A morphism $u : A \to B$ is said to be injective if for any object X, the map $'u : Mor(X, A) \to Mor(X, B)$ defined as $'u(v) = u \circ v$ is injective.

$$\begin{tikzcd}
A \arrow[r, "u"] & B \\
X \arrow[u, "v"] \arrow[ur, "u\circ v='u(v)"'] &
\end{tikzcd}$$

Figure 1.

Propositon 1. We will now verify that this definition agrees with our usual definition of injectivity in the category **Set**. Let u be a set map which satisfies the above property. Let $a_1, a_2 \in A$ such that $a_1 \neq a_2$. Let v_1, v_2 be two morphisms (both constant functions) in $Mor(X, A)$ defied as follows: $v_1 : X \to A$ such that $\forall x \in X$, $v_1(x) = a_1$ and $v_2 : X \to A$ such that $\forall x \in X$, $v_2(x) = a_2$ then because $v_1 \neq v_2$, we have $'u(v_1) \neq' u(v_2)$ and consequently, $u(a_1) \neq u(a_2)$.

Suppose now that u is injective in the usual sense. Let $v_1 \neq v_2$. This means there exists an element $x \in X$ such that $v_1(x) \neq v_2(x)$; furthermore $u(v_1(x)) \neq u(v_2(x))$ by the injectivity of u, and thus $'u(v_1) \neq' u(v_2)$.

Definition 3. A morphism $u : A \to B$ is said to be surjective, if for any object X, the map $u' : Mor(B, X) \to Mor(A, X)$ defined on $v \in Mor(B, X)$ as $u'(v) = v \circ u$ is injective.

$$\begin{tikzcd} & X \\ A \arrow[ur, "u'(v)=v\circ u"] \arrow[r, "u"'] & B \arrow[u, "v"'] \end{tikzcd}$$

Figure 2.

Proposition 2. We will now verify that this definition agrees with our usual definition of surjectivity in the category **Set**. Suppose that the above condition holds. Then for any $b \in B$ define a pair of functions $v_1, v_2 \in Mor(B, X)$ which disagree on b, but agree on every other element of B. Then since $u'(v_1) \neq u'(v_2)$, there must be an element $a \in A$ such that $u(a) = b$, if there were not such an element, then we obtain $u'(v_1) = u'(v_2)$, a contradiction.

Suppose now that u is injective. Then for any other set X take $v_1, v_2 \in Mor(B, X)$ such that $v_1 \neq v_2$. This means that there exists an element $b \in B$ for which $v_1(b) \neq v_2(b)$. Since u is surjective, there exists an element $a \in A$ such that $u(a) = b$. It follows that $v_1(u(a)) \neq v_2(u(a))$ which shows the injectivity of u'.

A morphism which is both injective and surjective is said to be *bijective*.

3 Isomorphisms and Balance

Definition 4. In a category, two objects A and B are said to be isomorphic if there exist morphisms $f : A \to B$ and $g : B \to A$ so that $g \circ f = 1_A$ and $f \circ g = 1_B$.

Exercise 2. Prove that the composition of injections, surjections, and isomorphisms are again injections, surjections and isomorphisms.

A category in which the bijections are exactly the isomorphisms is said to be *balanced.*

Ex 5. When considered as a set map, the inclusion of the integers in the rationals is obviously not surjective. However, in the category of rings, **Ring**, it is a simple matter to show that this inclusion does define a surjection in **Ring**.

Exercise 3. Is **Set** balanced? Is **Ring** balanced?

4 Initial, Final, and Zero Objects

Very often, when investigating some mathematical object we look at its subobjects or quotient objects. Similarly when looking at a map between two objects it is useful to understand some of the subobjects involved and the way in which the morphism behaves on those subobjects.

We now introduce a pair of definitions and a pair of categories which will give us the proper setting to discuss these constructions.

Definition 5. Given a category $\mathcal{C}$ if there is an object $X \in ob(\mathcal{C})$ such that for every other object $C \in ob(\mathcal{C})$, the class $Mor(X, C)$ is a singleton, we say that X is an *initial object* for the category $\mathcal{C}$. That is, there is exactly one morphism from X to every other object in the category.

Definition 6. Given a category $\mathcal{C}$ if there is an object $X \in ob(\mathcal{C})$ such that for every other object $C \in ob(\mathcal{C})$, the class $Mor(C, X)$ is a singleton, we say that X is a $final$ *object* for the category $\mathcal{C}$. That is, there is exactly one morphism to X from every other object in the category.

Proposition 3. If a category $\mathcal{C}$ has an initial object X, it is unique up to isomorphism. That is, if there is any other object Y which is also initial, it must be isomorphic to X. To see this, note that there is a unique morphism from X to Y and from Y to X. Composing these morphisms we obtain a morphism from X to X or from Y to Y depending on the order of composition. However, $Mor(X, X)$ and $Mor(Y, Y)$ are both singletons and by the definition of a category must contain 1_X and 1_Y respectively. This does not leave much choice for what the compositions are.

Proposition 4. If a category $\mathcal{C}$ has a final object X, it is unique up to isomorphism. The proof is similar to the one given in proposition 3.

Definition 7. If a category has both an initial and a final object and they are equal, then that object is called a *zero object* for the category. Note that if a category has a zero object $*$ then between any two objects A and B there is a morphism from A to B which passes through $*$ and is denoted $0_{A,B}$. It is defined to be the composite map $A \to * \to B$ which exists because $*$ is both initial and final.

5 Subobjects and Quotient Objects

Definition 8. Given an object A in a category $\mathcal{C}$, define the category $\mathcal{C}_A$ as follows. The objects of this category are morphisms in $\mathcal{C}$ going into A. Morphisms in this category are given by precomposing the morphisms going into A with other morphisms of $\mathcal{C}$. Similarly define a category $\mathcal{C}^A$ who's objects are morphisms in $\mathcal{C}$ coming out of A and who's morphisms are given by post composition.

Let X and Y be objects of $\mathcal{C}$ and let $f : X \to A$ and $g : Y \to A$ be morphisms of $\mathcal{C}$. Notice also that f and g are objects of $\mathcal{C}_A$. Let $u : Y \to X$ be a morphism of $\mathcal{C}$. If $f \circ u = g$, then u is a morphism in $\mathcal{C}_A$. This is shown in the figure below:

$$\begin{tikzcd} X \arrow[r, "f"] & A \\ Y \arrow[u, "u"] \arrow[ur, "g='u(f)"'] & \end{tikzcd}$$

Figure 3.

Exercise 3. Verify that $\mathcal{C}_A$ and $\mathcal{C}^A$ are a categories. For which category would it be more fruitful to discuss an initial object? For which category would it be more fruitful to discuss a final object?

Definition 9. A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is a category with $ob(\mathcal{D}) \subset ob(\mathcal{C})$, and for each class $Mor_{\mathcal{D}}(A, B) \subset Mor_{\mathcal{C}}(A, B)$. Additionally the operation composition in $\mathcal{D}$ is exactly the restriction of the operation in $\mathcal{C}$. Lastly, Each object in $\mathcal{D}$ retains its identity morphism from $\mathcal{C}$.

A *full subcategory* is one in which only objects are removed, that is, if A and B are objects in $\mathcal{D}$, $Mor_{\mathcal{D}}(A, B) = Mor_{\mathcal{C}}(A, B)$.

Definition 10. We examine now the subcategory of $\mathcal{C}_A$ who's objects are just the injections into A. We denote this category $\mathcal{S}_A$ and call it the *category of subobjects* of A. An object of $\mathcal{S}_A$ is a *subobject* of A and it is considered up to isomorphism. That is, if f and g are isomorphic objects in $\mathcal{S}_A$, they are considered to be the same subobject.

Definition 10. Similarly, we call the subcategory of $\mathcal{C}^A$ who's objects are just the surjections from A, the category of quotient objects of A. It is denoted $\mathcal{Q}^A$. Quotient objects are also only considered up to isomorphism.

6 Range and Corange

We can now define the range of a morphism in a category $\mathcal{C}$.

Definition 11. Let A and B be objects of a category $\mathcal{C}$, and let f be a morphism $f : B \to A$. Define $\mathcal{S}_A^B$ to be the full subcategory of $\mathcal{S}_A$ whose objects $u : X \to A$ are part of a unique factorization of f. That is, for each u there exists a unique $f' : B \to X$ such that $f = u(f')$. This is shown below.

$$\begin{tikzcd}
X \arrow[r, "u"] & A \\
B \arrow[u, "f'"] \arrow[ur, "f='u(f')=u\circ f'"'] &
\end{tikzcd}$$

Figure 4.

Definition 12. An initial object of $\mathcal{S}_A^B$ is called the *range* of the morphism f, and is denoted $Ran(f)$. The range is determined up to isomorphism (as are all initial objects).

$$\begin{tikzcd}
X \arrow[rrd, bend left, "u"] & & \\
& Ran(f) \arrow[ul, dashed, "x"'] \arrow[r, "i"] & A \\
& B \arrow[u, "r"'] \arrow[ur, "f"'] \arrow[uul, bend left, "f'"] &
\end{tikzcd}$$

Figure 5.

Proposition 5. In the category **Set** $Ran(f)$ is isomorphic to

$$f(B) = \{a \in A |\ \exists b\ s.t.\ f(b) = a\}.$$

Let $X = f(B)$ in figure 5. Since $Ran(f)$ is initial, there is one morphism from $Ran(f)$ to A, namely i. Then $u \circ x = i$ and since i is injective, so must x be. Suppose now that x is not surjective. Then there is an $a \in f(B)$ such that for no $c \in Ran(f)$, $x(c) = a$. But for all $a \in f(B)$ there is a $b \in B$ for which $f(b) = a$ but then $u(f'(b)) = a$ and hence $u(x(r(b))) = a$. Letting $c = r(b)$ we have derived a contradiction. So x is both injective and surjective. Since **Set** is balanced, $f(B)$ is isomorphic to $Ran(f)$.

Definition 13. Let A and B be objects of a category $\mathcal{C}$, and let f be a morphism $f : B \to A$. Define $\mathcal{Q}_A^B$ to be the full subcategory of $\mathcal{Q}^B$ whose objects are surjections $u : B \to X$ which are part of a unique factorization of f. That is, for each u there exists a unique $f' : X \to A$ such that $f = f' \circ u$.

Definition 14. A final object of $\mathcal{Q}_A^B$ is called the *corange* of f. It is denoted $Cor(f)$.