# *Algebraic Topology*[1]

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*2019*



Lecture 1

## *What is it?*

Algebraic topology is the study of maps

$$\{\text{Spaces}\} \longrightarrow \{\text{Algebraic objects}\},$$

or rather, 'well-behaved' such maps. They should also send continuous functions between spaces to algebraic maps, respecting composition (so: *functors*); they should send spaces built out of simpler spaces to algebraic objects built out of simpler components, in a compatible way, etc.

Here, 'Spaces' roughly means topological spaces up to deformation (usually homotopy, but not always). Such equivalence classes are called *homotopy types*. 'Algebraic objects' means (abelian) groups, rings, modules, or even chain complexes of these .

a chain complex is a certain sequence of maps $\cdots \to V_0 \to V_1 \to V_2 \to \cdots$

**Example 1.** How can we tell if the sphere $S^2$ and the torus $S^1 \times S^1$ can or cannot be deformed into each other? How would you prove it cannot be done?

**Example 2.** For a positive example, we *can* squash $\mathbb{R}^3 \setminus \{0\} \to S^2 \hookrightarrow \mathbb{R}^3 \setminus \{0\}$, sending $x \mapsto \frac{x}{|x|}$. This map continuously deforms to the identity map. So dimension not necessarily preserved.

**Example 3.** Can we have $S^1 \sim S^2$?

We first need to understand how spaces are built

## *Topological spaces*

Recall. . .

From Topology and Analysis III

**Definition 1.** A *topology* on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ such that

1. $\emptyset, X \in \mathcal{T}$
2. If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$

3. If $\{U_\alpha\}_{\alpha \in I}$ is an arbitrary family of sets in $\mathcal{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

$I$ here is an indexing set

If $U \in \mathcal{T}$ we say $U$ is *open*. A *topological space* is a set $X$ eqipped with a topology $\mathcal{T}$.

**Example 4.** Take the set of real numbers, the *Euclidean ('usual') topology* is defined by saying a set is open iff it is a union of open intervals $(a, b)$ (including the union of no sets ie $\emptyset$).
The *discrete topology* on a set $X$ is defined by taking every $\mathcal{T}$ to consist of all subsets. The *indiscrete topology* is defined by taking $\mathcal{T}$ to consist of just $\emptyset$ and $X$.

This definition is concise, but not always the best way to define a topology. We will also use *neighbourhoods*

**Definition 2.** A set $N \subseteq X$ is a *neighbourhood* (in a given topology $\mathcal{T}$) of a point $x \in X$ if there is an open set $U \subseteq N$ with $x \in U$.

'nhd' is a good abbreviation

**Example 5.** Take $\mathbb{R}$ with the Euclidean topology. $(-1, 1)$, $[-1, 1]$, $[-1, 1)$ are all neighbourhoods of every $-1 < x < 1$, but $[0, 1)$ is not a neighbourhood of 0. More complicated: $[0, 1] \cup \{2\} \cup [5, 6]$ is a nhd of all $0 < x < 1$ and $5 < x < 6$.

**Example 6.** Consider a metric space $(X, d)$. The *metric topology* is defined by saying a subset $U \subseteq X$ is open iff for every $x \in U$ there is some $\varepsilon_x > 0$ with the open ball $B(x, \varepsilon_x) \subseteq U$. Open balls around $x$ are neighbouhoods of $x$, as are closed balls.

Here is a more concrete approach that allows concise definitions of topologies:

**Definition 3.** A *neighbourhood base* $\mathcal{N}$ on a set $X$ is a family $\{\mathcal{N}(x)\}_{x \in X}$ where each $\mathcal{N}(x)$ is a nonempty collection of subsets of $X$, satisfying the following, for all $x \in X$:

1. For all $N \in \mathcal{N}(x)$, $x \in N$;
2. For all $N_1, N_2 \in \mathcal{N}(x)$, there is some $N \in \mathcal{N}(x)$ with $N \subseteq N_1 \cap N_2$;
3. For all $N \in \mathcal{N}(x)$ there is a subset $U \subseteq N$ such that $x \in U$ and for all $y \in U$, there is some $V \in \mathcal{N}(y)$ such that $V \subseteq U$.

We say the sets in $\mathcal{N}(x)$ are *basic neighbourhoods* of $x$.

As an example: given a topological space $(X, \mathcal{T})$ defining $\mathcal{N}(x)$ to consist of all nhds of $x$ gives a nhd base. Similarly, defining $\mathcal{N}'(x)$ to consist of all open sets containing $x$ defines a nhd base.

Given a neighbourhood base $\mathcal{N}$ on a set $X$, define a subset $U \subseteq X$ to be $\mathcal{N}$*-open* iff for all $x \in U$, there is an $N \in \mathcal{N}(x)$ with $N \subseteq U$.

**Proposition 1.** The $\mathcal{N}$-open sets define a topology on $X$.

*Proof.* We verify the axioms for a topology on $X$.

1. The condition that $\emptyset$ is $\mathcal{N}$-open is vacuously true. And since $\mathcal{N}(x)$ is not empty, there is a basic nhd around every point, so $X$ is $\mathcal{N}$-open.
2. Given $U, V$ both $\mathcal{N}$-open, we want to show $U \cap V$ is $\mathcal{N}$-open. So take $x \in U \cap V$. We know there is $N_U, N_V \in \mathcal{N}(x)$ with $N_U \subseteq U$ and $N_V \subseteq V$, and also that $x \in N_U \cap N_V$, since it is in each of them. Thus there is some $N \in \mathcal{N}(x)$ with $N \subseteq N_U \cap N_V \subseteq U \cap V$, and this is true for all $x \in U \cap V$. Hence $U \cap V$ is $\mathcal{N}$-open.
3. Given a family $U_\alpha$, $\alpha \in I$, with each $U_\alpha$ $\mathcal{N}$-open, we want to show $U := \bigcup_{\alpha \in I} U_\alpha$ is $\mathcal{N}$-open. Take $x \in U$, so there is some $\alpha_0$ with $x \in U_{\alpha_0}$. But this set in $\mathcal{N}$-open, so there is some nhd $N$ of $x$ with $N \subseteq U_{\alpha_0} \subseteq U$, and this is true for all $x \in U$. So $U$ is $\mathcal{N}$-open. □

We call the topology from this proposition the topology generated by $\mathcal{N}$. Neighbourhoods in this topology are sets that contain a basic neighbourhood: $V$ is a neighbourhood of $x$ if there is some $N \in \mathcal{N}(x)$ with $N \subseteq V$.

Given a neighbourhood base $\mathcal{N}$ on $X$, we can identify the *closure* of a set $S \subset X$ as the collection of points $x \in X$ such that for all $N \in \mathcal{N}(x)$, $\exists s \in N \cap S$.

**Example 7.** Given a metric space $(X, d)$ the open balls form a nhd base on $X$ and the topology they generate is the metric topology.

Hence many definitions you are familiar with from metric spaces work for topological spaces, if they can be phrased in terms of basic nhds. In particular, continuity!

**Definition 4.** Let $\mathcal{N}_X$ and $\mathcal{N}_Y$ be neighbourhood bases on sets $X$ and $Y$ respectively. A function $f\colon X \to Y$ is *continuous* if for every $x \in X$ and $N \in \mathcal{N}_Y(f(x))$, the set $f^{-1}(N)$ contains a basic nhd of $x$.

This is a big generalisation of the $\varepsilon$-$\delta$ definition of continuity.

**Exercise 1.** Show that if $f\colon (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ is continuous as just defined, it is continuous for the topologies generated on $X$ and $Y$ by these nhd bases.

Recall a function is continuous for topologies if $f^{-1}(U)$ is open for all open $U$.

As a sanity check, the identity function $\mathrm{id}_X$ on a space $X$ is indeed continuous.

You can check every function *to* an indiscrete space is continuous, as is every function *on* a discrete space

**Definition 5.** A continuous function $f\colon X \to Y$ is a *homeomorphism* if there is a continuous function $g\colon Y \to X$ with $g \circ f = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$. We then call $X$ and $Y$ *homeomorphic* if there is a homeomorphism between them.

or just isomorphic, if I'm being lazy

Now we need to show how to build new spaces, and continuous maps relating them to the original spaces.

**Definition 6.** Let $X$ be a set, $(Y_\alpha, \mathcal{N}_\alpha)$, $\alpha \in I$ a family of sets with nhd bases (not necessarily all unique), and $f_\alpha\colon X \to Y_\alpha$ a family of functions. The *initial topology* on $X$ is generated by the following nhd base: a subset of $X$ is a basic nhd of $x$ iff it is of the form $f_{\alpha_1}^{-1}(N_1) \cap \ldots \cap f_{\alpha_k}^{-1}(N_k)$ for some $\alpha_1, \ldots, \alpha_k$ and $N_i \in \mathcal{N}_{\alpha_i}(f_{\alpha_i}(x))$.

Exercise: verify this is a nhd base!

This generalises the product topology, which is the case that $X = Y_1 \times Y_2$, and $f_i\colon X \to Y_i$ is the projection $f_i(y_1, y_2) = y_i$, where $i = 1, 2$. But this *also* gives the subspace topology: take $f\colon X \hookrightarrow Y$ to be injective and define the initial topology on $X$.

**Lemma 1.** Giving $X$ the initial topology, all the functions $f_\alpha\colon X \to Y_\alpha$ are continuous. Moreover, a function $k\colon Z \to X$ is continuous iff $f_\alpha \circ k\colon Z \to Y_\alpha$ is continuous for every $\alpha$.

Lecture 2

**Example 8.** If the set of functions consists of a single *injective* map, namely $\iota\colon X \hookrightarrow Y$, with $Y$ a space, then the initial topology is the subspace topology: basic nhds of $x$ correspond to sets $\iota^{-1}(N)$ (basically $N \cap X$) for $N$ a basic nhd of $\iota(x)$.

**Example 9.** If however we have a constant function $c_{y_0}\colon X \to Y$, sending $x \mapsto y_0 \in Y$ for all $x$, then for every nhd $N$ of $y_0$, $c_{y_0}^{-1}(N) = X$. So the only nhd of every $x \in X$ is $X$ itself. Thus the initial topology is indiscrete in this case.

In general, given the family of functions $f_\alpha\colon X \to Y_\alpha$, there is a function $(f_\alpha)\colon X \to \prod_\alpha Y_\alpha$. If we give $\prod_\alpha Y_\alpha$ the product topology, then the initial topology on $X$ from the family of maps is the same as the initial topology from the map $(f_\alpha)$ to the product space. So if this latter map is injective, $X$ inherits the subspace topology from the product topology. This is the major use-case we will come across for the initial topology.

**Example 10.** A submanifold $M \subseteq \mathbb{R}^n$ gets its topology from the coordinate functions $M \hookrightarrow \mathbb{R}^n \xrightarrow{x_i} \mathbb{R}$, and a map to $M$ is continuous iff the composite with the maps to each factor of $\mathbb{R}^n$ are continuous.

**Exercise 2.** Given a set $X$, a space $Y$ and a function $f\colon X \to Y$, if two points $x_1, x_2$ satisfy $f(x_1) = f(x_2)$, show that a subset $V \subseteq X$ is a nhd of $x_1$ iff it is a nhd of $x_2$, in the initial topology.

The following will be even more important for us, and will be new to most.

**Definition 7.** Let $X$ be a set, $(Z_\beta, \mathcal{N}_\beta)$, $\beta \in J$ a family of topological spaces (not necessarily all unique), and $g_\beta\colon Z_\beta \to X$ a family of functions (note the other direction!). The *final topology* on $X$ has open sets as following: $U \subset X$ is open iff for all $\beta \in J$, $g_\beta^{-1}(U)$ is open in $Z_\beta$.

this really is easier to describe using open sets, rather than nhds

**Lemma 2.** Giving $X$ the final topology, all the functions $g_\beta\colon Z_\beta \to X$ are continuous. Moreover a function $h\colon X \to W$ is continuous for the final topology on $X$ iff $h \circ g_\beta\colon Z_\beta \to W$ is continous for every $\beta \in J$.

We will give two special cases of this, and we will see them often.

**Example 11.** Let $Z$ be a topological space, and let $\sim$ be an equivalence relation on $Z$, and define $X = Z/\sim$ to be the quotient by this relation. There is a function $\pi\colon Z \to X$ sending $y \mapsto [y]$. The final topology on $X$ has as open sets those $U \subseteq X$ such that $\pi^{-1}(U)$ is open in $Z$.

For instance, we can give $S^2$ the initial topology for the maps $x_i\colon S^2 \to \mathbb{R}^3 \xrightarrow{\mathrm{pr}_i} \mathbb{R}$ (this is the usual topology on $S^2$), and then define an equivalence relation on $S^2$ by $x \sim y$ iff $x = -y$. The quotient is $\mathbb{RP}^2$, the real projective plane, and we give it the final topology coming from $S^2 \to \mathbb{RP}^2$. This is the topology it carries as a manifold. Incidentally, $S^2$ is an example of a *covering space* of $\mathbb{RP}^2$, the study of which will occupy the first section of the course.

Recall the definition of disjoint union of sets: given $Z_\beta$, $\beta \in J$, a family of sets, we have $\mathrm{in}_\gamma\colon Z_\gamma \hookrightarrow \coprod_\beta Z_\beta$ with $Z_\beta \cap Z_\gamma = \emptyset$ for $\beta \neq \gamma$. If $Z_\beta$ are spaces, then we give $\coprod_\beta Z_\beta$ the final topology for the maps $\mathrm{in}_\gamma$. This is *disjoint union* or *sum* topology, and $\coprod_\beta Z_\beta$ is sometimes called the *topological sum*. A point in $\coprod_\beta Z_\beta$ can be described by a pair $(\beta, z)$, where $z \in Z_\beta$.

an important fact is that the map $\coprod_\beta X \times Z_\beta \to X \times \coprod_\beta Z_\beta$ is a homeomorphism (exercise!)

**Exercise 3.** Given continuous functions $h_\beta\colon Z_\beta \to W$, there is a unique continuous function $h = \langle h_\beta \rangle\colon \coprod_\beta Z_\beta \to W$ with $h_\beta = h \circ \mathrm{in}_\beta$, or in other words this diagram commutes:

$$\begin{tikzcd} Z_\gamma \arrow[r, "\mathrm{in}_\gamma"] \arrow[dr, "h_\gamma"'] & \coprod_\beta Z_\beta \arrow[d, "h"] \\ & W \end{tikzcd}$$

**Lemma 3.** The final topology on $X$ for $g_\beta \colon Z_\beta \to X$ agrees with the final topology on $X$ for $g = \langle g_\beta \rangle \colon \coprod_\beta Z_\beta \to X$, using the sum topology.

*Proof.* We have that $U \subseteq X$ is open iff $\forall \beta\ g_\beta^{-1}(U)$ is open iff $\forall \beta$, $(g \circ \mathrm{in}_\beta)^{-1}(U) = \mathrm{in}_\beta^{-1}\left(g^{-1}(U)\right)$ is open iff $g^{-1}(U)$ is open in the sum topology. □

The idea behind the final topology, when $g_\beta \colon Z_\beta \to X$ are jointly surjective, is that we can put an equivalence relation on $\coprod_\beta Z_\beta$ with $(\beta_1, z_1) \sim (\beta_2, z_2)$ iff $g_{\beta_1}(z_1) = g_{\beta_2}(z_2) \in X$. As a set, $X$ is the set of equivalence classes under this relation, so you can think of it as gluing together the *underlying sets* of the spaces $Z_\beta$. The final topology on $X$ is then the only sensible topology to described the space we get by gluing together the *spaces* $Z_\beta$.

this means $\forall x \in X, \exists \beta, x \in Z_\beta$ with $g_\beta(z) = x$

**Exercise 4.** Given an open cover $\{U_\alpha\}$ of a space $X$, then $X$ carries the final topology for the inclusion maps $U_\alpha \hookrightarrow X$, or equivalently for the map $\coprod_\alpha U_\alpha \to X$.

**Example 12.** An arbitrary manifold $M$ has the final topology arising from any choice of atlas.

**Exercise 5.** Given a *finite* closed cover $\{V_i\}_{i=1}^n$ of $X$, then $X$ carries the final topology for $\coprod_{i=1}^n V_i \to X$.

**Example 13.** Any closed interval $[a, b] \subset \mathbb{R}$ with the subspace topology has the final topology arising from a collection of subintervals $[a, t_1]$, $[t_1, t_2]$,$\ldots$,$[t_k, b]$, each with the subspace topology from $\mathbb{R}$.

These exercises give us what is sometimes known as the *gluing lemma*:

**Lemma 4.** Consider a space $X$ and an arbitrary open cover $\{U_\alpha\}_{\alpha \in I}$ (respectively a finite closed cover $\{V_i\}_{i=1}^n$) and suppose $Y$ is some other topological space. Then if a function $f \colon X \to Y$ is continuous when restricted to each $U_\alpha$ (resp. to each $V_i$) then $f$ is continuous.

Later we'll see spaces that are built up by gluing together lots of 'simple' spaces, like disks $D^n := \{x \in \mathbb{R}^n \mod |x| \leq 1\}$ (with the subspace topology from $\mathbb{R}^n$). But what does 'simple' here mean? Roughly, "shrinkable to a point".

## *Homotopy*

"Shrinkable" implies a kind of continuous process in time. Consider the function $I \times D^n \to D^n$. Consider the map

$$\begin{aligned} H \colon I \times D^n &\to D^n \\ (t, \mathbf{x}) &\mapsto (1-t)\mathbf{x} \end{aligned}$$

Note that this gives maps $H_0 \colon D^n \to D^n$ (the identity map) and $H_1$ (constant at 0). The function $H$ is continuous! How should we see this? The topology on $D^n$ is the subspace topology $D^n \subset \mathbb{R}^n$, and $\mathbb{R}^n$ has the product topology. so the topology on $D^n$ is also the initial topology for the coordinate functions $x_i \colon D^n \to \mathbb{R}^n \to \mathbb{R}$. So $H \colon I \times D^n \to D^n$ is continuous iff

And $I \subset \mathbb{R}$ has subspace topology

$$\begin{array}{ccccccc} I \times D^n & \xrightarrow{\mathrm{id} \times x_i} & I \times \mathbb{R} & \longrightarrow & \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (t, \mathbf{x}) & \longmapsto & & & (t, x_i) & \longmapsto & t x_i \end{array}$$

But $I \times D^n \to \mathbb{R} \times \mathbb{R}$ is continuous by definition of initial topology, and the following result:

**Exercise 6.** If $f \colon X \to W$ and $g \colon Y \to Z$ are continuous, then so is $f \times g \colon X \times Y \to W \times Z$. If both $X$ and $Y$ have at least one point each, then the reverse implication also holds.

So if we can prove that multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, then $H$ is continuous. But the standard topology on $\mathbb{R}$ comes from the metric space structure, so can use sequential criterion for continuity. Take $(a_n, b_n) \to (a, b)$ in $\mathbb{R} \times \mathbb{R}$, then:

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a b_n + a b_n - ab| \\ &\leq |a_n - a|\,|b_n| + |a|\,|b_n - b| \\ &\leq |a_n - a| \sup |b_n| + |a|\,|b_n - b| \qquad \text{(as $(b_n)$ converges, it is bounded)} \\ &\to 0 + 0 \end{aligned}$$

Hence $H$ is continuous.

**Definition 8.** A space $X$ is *contractible* if there is a point $x_0 \in X$ and a continuous function $H \colon I \times X \to X$ such that $H(0, x) = x$ and $H(1, x) = x_0$ for all $x \in X$. Such a function is called a *contraction*.

or, *contractible to* $x_0 \in X$

We have shown $D^n$ is contractible.

**Exercise 7.** $\mathbb{R}$ is contractible. An arbitrary product of contractible spaces is contractible.

**Example 14.** Consider what it would mean if a discrete space $S$ were contractible: there would be an element $* \in S$ and a continuous function $h\colon I \times S \to S$ such that $h(0,s) = s$ and $h(1,s) = *$. Restricting $h$ to $I \times \{s\}$ for some given $s$, we get a continuous function $I \hookrightarrow I \times S \to S$, whose range includes $*$ and $s$. Since all functions with discrete domain are continuous, let us compose with the continuous function $\chi_{\{*\}}\colon S \to \mathbb{R}$ that sends $* \mapsto 1$ and $s \mapsto 0$ for all $s \neq *$. So we have a continuous function $\widetilde{h}\colon I \to \mathbb{R}$ with $\widetilde{h}(0) = 0$ and range contained in $\{0,1\}$. By the intermediate value theorem, we must have $\widetilde{h}(1) = \chi_{\{*\}}(h(1,s)) = 0$, so that $h(1,s) = *$, and hence $s = *$ for all $s \in S$. Thus $S$ has exactly one element.

**Question 1.** If $X$ is contractible, does the choice of point $x_0 \in X$ matter? Is $X$ also contractible to $x \in X$ for $x \neq x_0$?

The interval can only map continuously to a discrete space if it is constant at some element, or equivalently, its image consists of a single point, and this property is important enough to have a name.

**Definition 9.** A space $X$ is *connected* if every continuous map to a discrete space has image a single point.

If you know the 'usual' definition, this is equivalent to it

So the interval $I$ is an example of a connected space. Even better: if a pair of points $x, y \in X$ have a *path* between them (a map $I \xrightarrow{\gamma} X$ with $\gamma(0) = x$, $\gamma(1) = y$) then any function $f\colon X \to S$ to a discrete space has $f(x) = f(y)$.

**Example 15.** Every contractible space is connected. This is because in a contractible space $X$, for every point $y$ there is the path $t \mapsto H(t,y)$ joining $y$ to the point $x_0$, so that $f(y) = f(x_0)$ for every map $X \xrightarrow{f} S$ to discrete $S$.

There are however lots of spaces that are connected but not contractible, but we cannot yet prove this.

This is our first example of an invariant of spaces, namely whether they are connected or not: a connected space $X$ cannot be homeomorphic to a space $Z$ that is not connected. But, how can we tell non-connected spaces apart?

Lecture 3

Consider $\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d] & Z \arrow[dl] \\ S & \end{tikzcd}$ with $S$ discrete.

**Definition 10.** 1. For any space $X$, a subset $Y \subseteq X$ is a *connected component* of $X$ if $Y$ is connected and for any connected $Y' \subseteq X$ such that $Y \subseteq Y'$, then $Y = Y'$.

2. Put an equivalence relation on $X$ generated by $x_1 \sim x_2$ iff $x_1$ and $x_2$ are both contained in a connected subset $C \subseteq X$. Then define $\pi_0(X) = X/\sim$, the *set of connected components*.

Exercise: If $C, D \subseteq X$ are connected, and $\exists x \in C \cap D$, then $C \cup D$ is connected. Also show: the equivalence classes are the connected components.

Every connected space $X$ has $\pi_0(X) = *$, but now we can tell apart non-connected spaces, by comparing their $\pi_0$. Every space that we will be consider in this course can be written as $X = \coprod_{\alpha \in \pi_0(X)} X_\alpha$, with $X_\alpha$ connected, and have a continuous function $X \to \pi_0(X)$ where $\pi_0(X)$ has the discrete topology. As a result, we need to try to understand *connected* spaces, though we will still *use* non-connected spaces.

Such spaces are called 'locally connected', but we will eventually be assuming a slightly stronger condition. Be warned: $\mathbb{Q}$ with the Euclidean topology is **not** locally connected, nor are many very interesting examples!

Can we get more out of the idea of contractions? Given $H\colon I \times X \to X$, we have maps $H_i$ for $i = 0, 1$, namely $H_0 = \mathrm{id}_X$ and $H_1$ is constant at $x_0$. What if $H_0$ and $H_1$ were other sorts of continuous maps?

**Example 16.** Consider the annulus $A(r, R) := \{x \in \mathbb{R}^2 \mid r \leq |x| \leq R\}$, and the function $H(t, x) = ((1-t)r + tR)x/|x|$.

What if we considered general continuous maps $X \to Y$ instead of just $X \to X$?

**Definition 11.** A *homotopy* is a continuous function $H\colon I \times X \to Y$. If $f = H(0, -)$ and $g = H(1, -)$, we say $H$ is a *homotopy from $f$ to $g$*, and that $f$ and $g$ are *homotopic*, written $f \sim g$.

A useful picture is:

$$\begin{tikzcd}
\{0\} \times X \arrow[d] \arrow[dr, "f"] & \\
I \times X \arrow[r, "H"] & Y \\
\{1\} \times X \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

Example 16 gives a homotopy between the two 'retraction' maps $A(r, R) \to A(r, R)$, mapping points to the inner and outer circles respectively.

Algebraic topology most of the time considers functions *up to homotopy*, and also "spaces up to homotopy".

**Definition 12.** A continuous function $f\colon X \to Y$ is called a *homotopy equivalence* if there is a continuous function $g\colon Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. We then say $X$ and $Y$ are *homotopy equivalent*.

**Example 17.** A contractible space is homotopy equivalent to a one-point space.

You should think of homotopy equivalences as being 'kinda like isomorphism', but coarser. Going back to our original motivation, the maps

$$\{\text{Spaces}\} \longrightarrow \{\text{Algebraic objects}\}$$

under consideration should take homotopy equivalent spaces to isomorphic algebraic objects. To make this more rigorous we will use the language of category theory.

Here is a super-important property of homotopies we will use continuously.

**Proposition 2.** Given homotopies $H\colon I \times X \to Y$ and $H'\colon I \times X \to Y$ such that $H_1 = H'_0\colon X \to Y$, there is a homotopy $H''$ from $H_0$ to $H_1$, and a homotopy $\widetilde{H}$ from $H_1$ to $H_0$.

*Proof.* We will use Exercise 5 applied to the closed cover $\{[0, \frac{1}{2}] \times X, [\frac{1}{2}, 1] \times X\}$ of $I \times X$. Since $I \simeq [0, \frac{1}{2}]$ and $I \simeq [\frac{1}{2}, 1]$, $H$ and $H'$ give us maps $[0, \frac{1}{2}] \times X \simeq I \times X \xrightarrow{H} Y$ and $[0, \frac{1}{2}] \times X \simeq I \times X \xrightarrow{H'} Y$ respectively. By the assumption on $H_1$ and $H'_0$, we get a well-defined function $H''\colon I \times X \to Y$, which is then continuous by the Exercise. It is a simple check to see it is a homotopy from $H_0$ to $H'_1$.
For the second part, let $c\colon I \to I$ be the function $c(t) = 1 - t$. Then define $H''$ to be the composite $I \times X \xrightarrow{c \times \mathrm{id}_X} I \times X \xrightarrow{H} Y$, which has the required properties. $\square$

Contractible spaces supply many homotopies.

**Lemma 5.** Every continuous function $f\colon X \to Y$, with $Y$ a contractible space (say to $y_0 \in Y$), is homotopic to a function with range contained in $\{y_0\}$.

*Proof.* Let $H\colon I \times Y \to Y$ be a homotopy witnessing the contractility of $Y$. Then the composite $I \times X \xrightarrow{\mathrm{id}_I \times f} I \times Y \xrightarrow{H} Y$ is a homotopy from $f$ to the the desired function. $\square$

As a corollary, every pair of functions to a contractible space are homotopic. Since contractible spaces are in some sense trivial, maps to them are in the same sense trivial.

An important intermediate version of this is when we consider only the case where $X$ is discrete, or is even just pt:

**Definition 13.** A space $Y$ is *path-connected* if every map $\mathrm{pt} \to Y$ is homotopic to every other such map.

This condition is equivalent to requiring it for *all* discrete spaces in place of pt (Exercise!)

Unpacking this, we see this means that for any two points $\mathrm{pt} \to Y$ there is a path $I \to Y$ connecting them, i.e. $H\colon I \simeq I \times \mathrm{pt} \to Y$.

**Proposition 3.** A path-connected space is connected

Let us define $[X, Y] = \{\text{continuous } f\colon X \to Y\}/\text{homotopy}$. The set of *path components* of $Y$ is then the set $[\mathrm{pt}, Y]$. The space $Y$ is called *path connected* if $[\mathrm{pt}, Y] = *$.

We have been discussing topological spaces and continuous maps, but also implicitly sets and functions, not necessarily continuous, and passing between these two pictures. In both cases we have composition that is associative, and identity maps. Later we shall be using different classes of topological spaces in order to ensure the behaviour we require will hold.

**Definition 14.** A *category* $\mathcal{C}$ consists of a collection of *objects* $W, X, Y, Z, \ldots$ and for each pair of objects $X, Y$ a collection of *morphisms*, denoted $\mathcal{C}(X, Y)$, together with the following data:

i) For each pair $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, a specified morphism $g \circ f \in \mathcal{C}(X, Z)$,

ii) For every object a specified morphism $\mathrm{id}_X \in \mathcal{C}(X, X)$,

such that:

1. For every triple $h \in \mathcal{C}(W, X)$, $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have $g \circ (f \circ h) = (g \circ f) \circ h$,
2. For every object $X$ and $h \in \mathcal{C}(W, X)$, $f \in \mathcal{C}(X, Y)$ we have $\mathrm{id}_X \circ h = h$ and $f \circ \mathrm{id}_X = f$.

For $f \in \mathcal{C}(X, Y)$ we say $X$ is the *source* of $f$, $Y$ is the *target* of $f$, and write $X = s(f)$, $Y = t(f)$. We also write $f\colon X \to Y$ or $X \xrightarrow{f} Y$ to indicate that $f \in \mathcal{C}(X, Y)$. If $\mathcal{C}(X, Y)$ is a set for all $X, Y$, then $\mathcal{C}$ is called *locally small*, and each $\mathcal{C}(X, Y)$ is called a *hom-set*.

Most categories you will encounter are locally small

Many examples of categories have objects sets carrying extra structure (for instance a topology) and morphisms that are functions compatible with that structure—but not all categories. We have seen **Top**, the category of topological spaces (and continuous maps) and **Set**, the category of sets (and functions), and you implicitly already know many other examples.

Vector spaces, (abelian) groups, manifolds, rings, ...

**Example 18.** The category $\mathbf{Set}_*$ of pointed sets $(X, x)$ ($x \in X$ a specified element) and pointed maps $(X, x) \to (Y, y)$ (functions $f\colon X \to Y$ with $f(x) = y$)) can be considered as consisting of algebraic objects of the weakest sort (compare homomorphisms, linear transformations, ring maps, etc, which preserve distinguised elements).

The whole point of categories is how they relate to each other, an isolated category can only tell us so much.

**Definition 15.** Given categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* from $\mathcal{C}$ to $\mathcal{D}$, denoted $F\colon \mathcal{C} \to \mathcal{D}$ consists of the data:

i) For every object $X$ of $\mathcal{D}$, a specified object $F(X)$ of $\mathcal{D}$,

ii) For every morphism $f\colon X \to Y$ of $\mathcal{C}$, a specified morphism $F(f)\colon F(X) \to F(Y)$ of $\mathcal{D}$

such that for every object $X$ of $\mathcal{C}$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and for every pair $f\colon X \to Y$ and $g\colon Y \to Z$ of morphisms of $\mathcal{C}$, $F(g \circ f) = F(g) \circ F(f)$. This latter property is called 'functoriality'. For locally small categories, the assignment on morphisms gives a function $\mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$.

We will use this notation even without making that assumption

We have already see at least four examples of functors:

- The underlying set functor $U\colon \mathbf{Top} \to \mathbf{Set}$
- The discrete topology functor $\mathrm{disc}\colon \mathbf{Set} \to \mathbf{Top}$
- The set of connected components functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$

the indiscrete topology also gives rise to a functor $\mathbf{Set} \to \mathbf{Top}$, but we won't be using it

although we haven't yet seen why $\pi_0$ is a functor. We can compose functors in the obvious way, so get functors $\mathrm{disc}U\colon \mathbf{Top} \to \mathbf{Top}$ and $\mathrm{disc}\pi_0\colon \mathbf{Top} \to \mathbf{Top}$, for instance.

Here is a trivial-seeming example (aside from the identity functor).

Let $\mathcal{C}$ be a category, and $\mathcal{D}$ a *subcategory*: a collection of some of the objects of $\mathcal{C}$ and some of the morphisms of $\mathcal{C}$ that form a category by themselves. Then the inclusion of the objects and the morphisms forms a functor $\mathcal{D} \hookrightarrow \mathcal{C}$, the *subcategory inclusion*. An important special case of this is when for every $X$ and $Y$ that are objects of $\mathcal{D}$, every $\mathcal{D}(X,Y) = \mathcal{C}(X,Y)$; then $\mathcal{D}$ is call a *full* subcategory. More generally we can consider a functor that is injective on objects and morphisms to define a subcategory.

**Example 19.** The functor $\mathrm{disc}\colon \mathbf{Set} \to \mathbf{Top}$ makes $\mathbf{Set}$ a full subcategory of $\mathbf{Top}$.

we have used and will use this result without comment

We will be later restricting attention to certain full subcategories of $\mathbf{Top}$.

**Lemma 6.** Let $X$ be a connected space, and let $f\colon X \to Y$ be a continuous function. Then $\mathrm{im}(f) \subset Y$ is connected.

*Proof.* Let $S$ be a discrete space and let $g\colon \mathrm{im}(f) \to S$ be a continuous function. Then the composite $X \to \mathrm{im}(f) \to S$ has image $\{s\} \subseteq S$, hence $\mathrm{im}(g) = \{s\}$ and so $\mathrm{im}(f)$ is connected. $\square$

**Proposition 4.** The assignment $X \mapsto \pi_0(X)$ is a functor **Top** $\to$ **Set**.

*Proof.* We need to show there is an assignment $(f\colon X \to Y) \mapsto (\pi_0(f)\colon \pi_0(X) \to \pi_0(Y))$, for an arbitrary continuous function $f$. Fix $f\colon X \to Y$ and let $\alpha \in \pi_0(X)$. Then this corresponds to a connected component $X_\alpha \subseteq X$, and we know $f|_{X_\alpha}$ has connected image. Thus this image is contained inside a single connected component of $Y$, and we define $\pi_0(f)(\alpha)$ to be the corresponding element of $\pi_0(Y)$.

Given another map $g\colon Y \to Z$, and the corresponding function $\pi_0(g)\colon \pi_0(Y) \to \pi_0(Z)$, one can check that $\pi_0(g)\pi_0(f)(\alpha)$, for $\alpha \in \pi_0(X)$ is the same as $\pi_0(g \circ f)(\alpha)$, and $\pi_0(\mathrm{id})$ is also the identity map. This proves that $\pi_0$ is a functor **Top** $\to$ **Set**. $\square$

Here is a bonus second proof for locally connected spaces.

*Proof.* We already know we have a map $X \to Y \to \pi_0(Y)$, where we give $\pi_0(Y)$ the discrete topology. This is continuous since $Y$ is locally connected, and we want to show this *descends* along $X \to \pi_0(X)$ to a map $\pi_0(X) \to \pi_0(Y)$. Given any $\alpha \in \pi_0(X)$, it corresponds to a connected component $X_\alpha$ of $X$. Look at the restriction of $X \to Y \to \pi_0(Y)$ to $X_\alpha$: since $X_\alpha$ is connected, its image is exactly one point in $\pi_0(Y)$. So define $\pi_0(f)(\alpha) = [f(x)]$ for an arbitrary $x \in X_\alpha$. This defines $\pi_0(f)$. Moreover, the following diagram *commutes*:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y) \end{tikzcd}$$

Since the discrete topology on $\pi_0(X)$ is the same as the quotient topology, this is a map between discrete spaces, hence continuous, but we are thinking of it as a map between sets.

Now we want to show that $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$. Given $\alpha \in \pi_0(X)$, and $x \in X_\alpha$, then $\pi_0(f)(\alpha) = [f(x)]$. To define $\pi_0(g)\,(\pi_0(f)(\alpha))$, we need to choose a point in the component $Y_{[f(x)]}$, so take it to be $f(x)$. Then $\pi_0(g)\,(\pi_0(f)(\alpha)) = [g(f(x))]$, but this is just $\pi_0(g \circ f)(\alpha)$. $\square$

**Exercise 8.** Show that $[\mathrm{pt}, -]\colon \mathbf{Top} \to \mathbf{Set}$ is a functor.

Or more generally, $[X, -]\colon \mathbf{Top} \to \mathbf{Set}$!

Another important example of a category is the *homotopy category* $\mathbf{hTop}$. The objects are topological spaces, but $\mathbf{hTop}(X, Y) = [X, Y]$. There is a functor $\mathbf{Top} \to \mathbf{hTop}$, which is the identity on objects, and sends a map to its homotopy class. Objects are isomorphic in $\mathbf{hTop}$ iff they are homotopy equivalent.

Exercise: prove this is a category

**Proposition 5.** The functor $\pi_0$ descends to a functor $\mathbf{hTop} \to \mathbf{Set}$

Lecture 4

*Proof.* We will prove that this is well-defined on morphism on hom-sets, the rest is routine. For $f, g\colon X \to Y$ to be homotopic via $H\colon I \times X \to Y$, we need to show that for all $\alpha \in \pi_0(X)$, $\pi_0(f)(\alpha) = \pi_0(g)(\alpha)$. Take $x$ in the connected component $X_\alpha$, then we have a map $I \to I \times X \xrightarrow{H} Y$, namely a path $f(x) \rightsquigarrow g(x)$. But $I$ is connected, so the image of the path is connected, so that $f(x)$ and $g(x)$ are in the same connected component. As $x$ was arbitrary $f(X_\alpha)$ and $g(X_\alpha)$ are both contained in the same connected component of $Y$. Thus $\pi_0(f)(\alpha) = \pi_0(g)(\alpha)$. $\square$

As a result, if $\pi_0(X) \not\cong \pi_0(Y)$, the spaces $X$ and $Y$ cannot be homotopy equivalent, let alone homeomorphic.

**Exercise 9.** Show the functor $[\mathrm{pt}, -]\colon \mathbf{Top} \to \mathbf{Set}$ descends to $\mathbf{hTop} \to \mathbf{Set}$.

Here is a useful fact about spaces.

**Lemma 7.** For all families $X_\beta$, $\beta \in J$, of spaces, we have isomorphisms

$$\coprod_{\beta \in J} \pi_0(X_\beta) \xrightarrow{\cong} \pi_0\Big(\coprod_{\beta \in J} X_\beta\Big) \quad \text{and} \quad \coprod_{\beta \in J} [\mathrm{pt}, X_\beta] \xrightarrow{\cong} \Big[\mathrm{pt}, \coprod_{\beta \in J} X_\beta\Big],$$

with inverses induced by the family of maps $\mathrm{in}_\beta$. That is, $\pi_0$ and $[\mathrm{pt}, -]$ *preserve coproducts.*

Recall last time: we had functors $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ and (abusing notation) $\pi_0\colon \mathbf{hTop} \to \mathbf{Set}$.

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "\pi_0"] \arrow[d] & \mathbf{Set} \\ \mathbf{hTop} \arrow[ur, "\pi_0"'] & \end{tikzcd}$$

**Example 20.** If $X$ and $Y$ are spaces with $|\pi_0(X)| < |\pi_0(Y)|$, no continuous map $X \to Y$ is surjective.

Here is an instructive example

**Example 21.** The *topologist's sine curve* is the image $C$ of $[-1,1] \sqcup (0,1] \to \mathbb{R}^2$ defined by

$$\begin{cases} y \mapsto (0,y) & y \in [-1,1] \\ x \mapsto (x, \sin(\frac{1}{x})) & x \in (0,1] \end{cases}$$

equipped with the **subspace topology**. This is a compact metric space, using the inherited Euclidean metric. Fact: *every* continuous function $f\colon C \to \{0,1\}$ is constant. If $f(1,\sin(1)) = 1$, then $f(x,\sin(x)) = 1$ for every $x \in (0,1]$ (as intervals are connected). If $f(0,0) = b \in \{0,1\}$, then $f(0,y) = b$ also, for all $y \in [-1,1]$. The sequence $(\frac{1}{n\pi}, 0)$ converges to $(0,0)$ in $C$, so $b = f(0,0) = \lim_{n\to\infty} f(\frac{1}{n\pi}, 0) = 1$ as $f$ is continuous and we are in a metric space.

Hence $C$ is connected, but there is *no* continuous function $\gamma\colon [0,1] \to C$ with $\gamma(0) = (0,0)$ and $\gamma(1) = (1,\sin(1))$. Since intervals are path connected, we can show $[\mathrm{pt}, C] = \{0,1\}$, but $\pi_0(C) = *$.

Exercise: prove this by considering $\lim_{n\to\infty} \gamma(\frac{1}{n})$

So we have two different invariants here, and there is always a surjective map $[\mathrm{pt}, X] \to \pi_0(X)$. Moreover, the following square of functions between sets always commutes, for any map $X \xrightarrow{f} Y$:

$$\begin{tikzcd} {[\mathrm{pt}, X]} \arrow[r, "{[\mathrm{pt}, f]}"] \arrow[d] & {[\mathrm{pt}, Y]} \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y) \end{tikzcd}$$

This is thus an example of a *natural transformation*.

**Definition 16.** Given functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a natural transformation $\alpha\colon F \Rightarrow G$ consists of the data:

i) For every object $X$ of $\mathcal{C}$, a specified morphism $\alpha_X\colon F(X) \to G(X)$ (the *components* of $\alpha$)

such that for every morphism $f\colon X \to Y$ in $\mathcal{C}$, the following square commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & F(Y) \end{tikzcd}$$

A natural transformation is called a *natural isomorphism* if all of its components are isomorphisms.

For example, there are natural transformations disc $U \Rightarrow \mathrm{id}\colon \mathbf{Top} \to \mathbf{Top}$, with component at $X$ the identity map $\mathrm{disc}(U(X)) \to X$, and $U \Rightarrow \pi_0\colon \mathbf{Top} \to \mathbf{Top}$, with component $U(X) \to \pi_0(X)$.

We seek conditions that will define a full subcategory of **Top** such that the components $[\mathrm{pt}, X] \to \pi_0(X)$ of the natural transformation $[\mathrm{pt}, -] \Rightarrow \pi_0$ are isomorphisms for all spaces $X$ in the subcategory.

**Definition 17.** A space $X$ is *semilocally path connected* (slpc) if it has a neighbourhood base of sets $N$ such that for any two $x, y \in N$, there is a path in $X$ from $x$ to $y$.

Note that a space is slpc iff every connected component is slpc, and if $X$ is homeomorphic to $Y$, and one of them is slpc, then so is the other.

**Proposition 6.** If $X$ is a semilocally path connected space, then $[\mathrm{pt}, X] \to \pi_0(X)$ is an isomorphism.

*Proof.* We are reduced to the case $X$ is connected ($\pi_0(X) = *$) and slpc, by Lemma 7, and the fact the case $X = \emptyset$ is trivial. Since $X$ is connected, take $x \in X$ and define $\chi\colon X \to \{0, 1\}$ by

$$\chi(y) = \begin{cases} 1 & \exists y \rightsquigarrow x \\ 0 & \text{otherwise} \end{cases}$$

where by $y \rightsquigarrow x$ I mean a path $\gamma\colon I \to X$ with $\gamma(0) = y$ and $\gamma(1) = x$. We will show $\chi$ is continuous. Note that $\chi$ continuous $\Leftrightarrow p^{-1}(0)$ and $p^{-1}(1)$ open $\Leftrightarrow p^{-1}(1)$ open and closed. But $p^{-1}(1) =: C_x$ is the path component containing $x$. Take $y \in C_x$ (so $\exists y \rightsquigarrow x$), and $V \ni y$ a path-connected nhd. Given $z \in V$, $\exists z \rightsquigarrow y$. Concatenate these paths to give $z \rightsquigarrow x$, so that $z \in C_x$. This is true for all $z \in V$, so that $V \subseteq C_x$, hence $C_x$ contains a neighbourhood of each of its points, and so is open.

Conversely, take $y \in \overline{C_x}$, $V \ni y$ a path connected nhd. As $\exists z \in V \cap C_x \subseteq V$, $\exists z \rightsquigarrow y$. But also have $V \cap C_x \subseteq C_x$, so $\exists z \rightsquigarrow x$. Concatenate paths to get $y \rightsquigarrow x$, so that $y \in C_x$. This is true for all $y \in \overline{C_x}$, so $\overline{C_x} \subseteq C_x$ and $C_x$ is closed. Hence $\chi$ is continuous.

But $X$ is connected, and $\chi(x) = 1$, so that $\operatorname{im} \chi = \{1\}$, and so $C_x = \chi^{-1}(1) = X$. Thus $[\mathrm{pt}, X] \to \pi_0(X) = *$ is an isomorphism. $\square$

So we will consider for the rest of this section of the course only slpc spaces, which form a full subcategory $\mathbf{slpcTop} \hookrightarrow \mathbf{Top}$. Note that discrete spaces are slpc, so $\mathbf{Set} \hookrightarrow \mathbf{slpcTop}$ is a subcategory.

**Example 22.** Any path-connected space $X$ is slpc, since for any nhd $N$ and points $x, y \in N$, we know there is a path $I \to X$ between $x$ and $y$.

**Exercise 10.** Show that the product of two slpc spaces is slpc, and that any locally convex topological vector space is slpc.

**Example 23.** Any manifold is slpc, since every point lives in a chart homeomorphic to some $\mathbb{R}^n$, and $\mathbb{R}^n$ is path-connected.

Be warned: subspaces of slpc spaces may not be slpc, for instance the topologist's sine curve is a subspace of the contractible $\mathbb{R}^2$.

**Question 2.** If $X$ is slpc and $q\colon X \to Y$ is a quotient map, then is $Y$ slpc?

so $Y$ has the final topology wrt $q$

One last technical point

**Definition 18.** A *pointed space* is a pair $(X, x)$ where $X$ is a topological space and $x \in X$. A pointed map is a pointed map between the underlying pointed sets that is continuous. These define a category $\mathbf{Top}_*$.

A *pointed homotopy* of pointed map $I \times X \to Y$, for $(X, x_0)$ and $(Y, y_0)$ pointed spaces, is required to satisfy $H(t, x_0) = y_0$ for all $t \in I$. Pointed homotopy classes of pointed map are denoted $[(X, x_0), (Y, y_0)]_*$. The category $\mathbf{hTop}_*$ is defined analogously to $\mathbf{hTop}$. We get a functor $\pi_0\colon \mathbf{hTop}_* \to \mathbf{Set}_*$.

## *Covering spaces*

Sometimes when we are thinking about a particular space $X$, we need to construct other spaces related to $X$ to study objects of interest.

**Example 24.** Take $X = \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Then the function $x \mapsto \sqrt{x}$ is not well-defined, and if we take a branch cut to give an actual function, it is not continuous on $X$. Even worse, if have a continuous function $f\colon \mathbb{C}^\times \to \mathbb{C}$, we may or may not have $x \mapsto f(\sqrt{x})$ continuous. However, we *do* get a continuous function if we change the domain somewhat. The problem is that the function $Z := \mathbb{C}^\times \ni z \mapsto z^2 = x \in \mathbb{C}^\times$ is not injective, so not invertible. But if we are willing to take the domain to be $Z$, and so pass into $f$ the argument $z$ (which satisfies $z^2 = x$) then we are now just dealing with a continuous function. If $f$ is such that $f(z) = f(-z)$ for all $z \in Z$, then we get a well-defined function on $X$.

The properties of the map $z \mapsto z^2$ (at least away from 0) and others like $z^n$, $\exp(z)$, rational functions away from poles and critical points and so on, lead to the notion of covering spaces of certain domains in $\mathbb{C}$. We have a general definition for arbitrary spaces.

**Definition 19.** A *covering space* $Z \xrightarrow{\pi} X$ of $X$ is a space $Z$ equipped with a map $\pi$ such that for all $x \in X$ there is a nhd $V_x \ni x$ such that $\pi^{-1}(V_x) \simeq V_x \times \pi^{-1}(x)$ *over* $V_x$ (ie the diagram at right commutes), where $\pi^{-1}(x)$ has the discrete topology. (We will also call $\pi$ itself a *covering map*.)

$$\begin{tikzcd} \pi^{-1}(V_x) \arrow[r, "\simeq"] \arrow[d, "\pi"'] & V_x \times \pi^{-1}(x) \arrow[dl, "\mathrm{pr}_1"] \\ V_x & \end{tikzcd}$$

NB: $V_x \times \pi^{-1}(x) \simeq \coprod_{\pi^{-1}(x)} V_x$, for free

For a covering space $Z \xrightarrow{\pi} X$ and $x \in X$, let $Z_x := \pi^{-1}(x)$ denotes the *fibre* over $x$. We will also call $X$ the *base space*.

Examples include: $\exp\colon \mathbb{C} \to \mathbb{C}^\times$, $S^2 \to \mathbb{RP}^2$, $U(1) \xrightarrow{(-)^n} U(1)$, covers of the join $\infty$ of two circles.

**Exercise 11.** Show that if $Z \xrightarrow{\pi} Y$ is a covering map, and $Y \xrightarrow{\rho} X$ is a covering map with finite fibres (that is: $Y_x$ is finite for all $x \in X$), then $Z \xrightarrow{\rho\pi} X$ is a covering map.

**Proposition 7.** For a covering space $Z \xrightarrow{\pi} X$, if $\exists x_0 \rightsquigarrow x_1$, then $Z_{x_0} \simeq Z_{x_1}$.

Lecture 5

*Proof.* (First proof of Proposition 7) Take $\gamma\colon I \to X$, $\gamma(i) = x_i$, and an open cover $\{U_\alpha\}$ of $X$ over which $Z$ trivialises. We thus get an open cover $\gamma^{-1}(U_\alpha)$ of $I$, which has a finite subcover $U_0, \ldots, U_N$, with $x_0 \in U_0$, $x_1 \in U_N$. The ordering is chosen so that the path enters $U_i$ before it enters $U_{i+1}$, and $U_i \cap U_{i+1}$ has at least one point of the path in it.

we can shrink the cover slightly to make this ordering well-defined, if need be

We have isomorphisms $Z_{U_i} := \pi^{-1}(U_i) \xrightarrow{\phi_i} U_i \times F_i$ with discrete spaces $F_i$. We have $Z_{x_0} \simeq F_0$, and for all $t \in \gamma^{-1}(U_0)$, $Z_{\gamma(t)} \simeq F_0$. So for $\gamma(t) \in U_0 \cap U_1$, we have $F_0 \simeq Z_{\gamma(t)} \simeq F_1$. We can then prove by induction on $N$ that $F_0 \simeq F_1 \simeq \cdots \simeq F_N$. □

So for slpc $X$ and each $\alpha \in \pi_0(X)$, there is associated to $Z \xrightarrow{\pi} X$ an isomorphism class of sets, the *typical fibre* over all $x$ in the connected component $X_\alpha \subseteq X$.

**Note:** Fibres can be empty! But we usually don't think about this case too much. For $X$ pointed (by $x \in X$), we can consider pointed covering spaces $(Z, x) \to (X, x)$. This is from one perspective just a choice of point $z \in Z_x$. For $X$ connected and slpc, a pointed covering space has every fibre contain at least one point, namely the image of $z$ under $Z_x \simeq Z_{x'}$.

We have categories $\mathbf{Cov}_X$ and $\mathbf{Cov}_{(X,x)}$ with objects covering spaces of $X$ (resp. pointed covering spaces of $(X, x)$) and maps

$$\begin{tikzcd} Z_1 \arrow[rr] \arrow[dr] & & Z_2 \arrow[dl] \\ & X & \end{tikzcd}$$

and analogously in the pointed case. We will study these categories and see what they tell us about the topology of $X$.

**Example 25.** For $X = \mathbb{C} \setminus \{p_1, \ldots, p_n\}$, the study of $\mathbf{Cov}_X$ tells us about possible Riemann surfaces for holomorphic functions with critical values precisely $p_1, \ldots, p_n$.

For slpc and connected $X$, the fact that for a covering space $Z$ of $X$, there merely *exists* some $Z_{x_0} \simeq Z_{x_1}$ for arbitrary $x_0, x_1 \in X$ can be improved. We first need a construction on covering spaces.

**Definition 20.** Given a covering space $Z \xrightarrow{\pi} X$ and a map $Y \xrightarrow{f} X$, the *pullback* of $Z$ is the subspace

$$f^*Z := Y \times_X Z = \{(y, z) \in Y \times Z \mid f(y) = \pi(z)\} \subseteq Y \times Z.$$

actually $\pi$ doesn't have to a covering map; the space $Y \times_X Z$ is defined for any pair of maps to $X$

It fits in a commutative square

$$\begin{tikzcd} f^*Z \arrow[r, "\mathrm{pr}_2"] \arrow[d, "p"'] & Z \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

**Proposition 8.** In the setting of Definition 20:

1. $f^*Z \to Y$ is a covering space.
2. $f^*$ is a functor $\mathbf{Cov}_X \to \mathbf{Cov}_Y$.
3. Given $Y_2 \xrightarrow{g} Y_1 \xrightarrow{f} X$ and $Z \xrightarrow{\pi} X$, there is a canonical isomorphism $(f \circ g)^*Z \simeq g^*f^*Z$ in $\mathbf{Cov}_{Y_2}$.

Of these, only 1. relies on having a covering space to start with, 2. and 3. are general facts about pullbacks, where for 2. we replace $\mathbf{Cov}_X$ by the *slice category* $\mathbf{Top}/X$, whose objects are maps to $X$, and morphisms are commuting triangles

**Corollary 1.** The fibre $(f^*Z)_y$ is canonically isomorphic to $Z_{f(y)}$.

Now given a path $\gamma\colon I \to X$ and a covering space $Z \xrightarrow{\pi} X$, we can pull back $Z$ to get a covering space $\gamma^* Z \to I$. So let us try to understand covering spaces of $I$. Certainly for discrete $S$, the projection $S \times I \to I$ is a covering space.

**Proposition 9.** A covering space $Z \xrightarrow{\pi} I$ is isomorphic to the trivial covering space $\pi^{-1}(0) \times I \xrightarrow{\mathrm{pr}_2} I$ in $\mathbf{Cov}_I$.

We first need a little helper lemma

**Lemma 8.** A covering space of a compact space $X$ trivialises over a *finite* cover of $X$ by nhds.

might as well take the nhds to be open, and then consider a finite subcover

*Proof.* (of Proposition 9) We use the lemma to trivialise $Z \to I$ over a finite cover of $I$, which we can take to be by intervals $[0, t_1]$, $[s_2, t_2]$, $\dots$, $[s_N, 1]$ for $s_1 = 0 < s_2 < t_1 < s_3 < t_2 < \cdots < s_N < t_{N-1} < 1 = t_N$. We will proceed by induction on $N$, but this quickly reduces to the case of $N = 2$. So take a cover of $I$ by $[0, t]$ and $[s, 1]$, where $\tau\colon Z_0 \times [0, t] \xrightarrow{\simeq} Z_{[0,t]}$ and we are given $\sigma\colon F \times [s, 1] \xrightarrow{\simeq} Z_{[s,1]}$.

we know abstractly that $F \simeq Z_0$, but this proof will construct an isomorphism

By restriction there is the composite map

$$Z_0 \times [s, t] \xrightarrow[\simeq]{\tau|_{[s,t]}} Z_{[s,t]} \xrightarrow[\simeq]{\sigma^{-1}|_{[s,t]}} F \times [s, t] \xrightarrow{\mathrm{pr}_1} F.$$

If we fix $z \in Z_0$, we get a continuous map $\{z\} \times [s, t] \to F$, which is thus constant, say at $p_z \in F$. The function $z \mapsto p_z = \sigma^{-1}(\tau(z, s))$ is then a bijection $\phi\colon Z_0 \xrightarrow{\simeq} F$.

We thus get maps $Z_0 \times [0, t] \hookrightarrow Z \hookleftarrow F \times [s, 1] \xleftarrow{\phi\times\mathrm{id}} Z_0 \times [s, 1]$, which by construction agree on $Z_0 \times [s, t]$. There is thus a continuous map $Z_0 \times [0, 1] \to Z$. Moreover, you can check this map is a morphism of $\mathbf{Cov}_I$. There are likewise maps

$$Z_{[0,t]} \xrightarrow{\simeq} Z_0 \times [0, t] \hookrightarrow Z_0 \times I \hookleftarrow Z_0 \times [s, 1] \xleftarrow{\phi^{-1}\times\mathrm{id}} F \times [s, 1] \xleftarrow{\simeq} Z_{[s,1]}$$

which agree on $Z_{[s,t]}$, hence a continuous map $Z \to Z_0 \times I$. This map is in $\mathbf{Cov}_I$ and can be checked by pointwise evaluation to be inverse to the first one. Hence we have an isomorphism $Z \simeq Z_0 \times I$ in $\mathbf{Cov}_I$. □

**Corollary 2.** Given a covering space $Z \xrightarrow{\pi} I$ and a point $z \in Z_0$, there is a unique path $\eta_z\colon I \to Z$ with $\eta_z(0) = z$ such that $\pi \circ \eta_z = \mathrm{id}$ (i.e. $\eta_z$ is a section of $\pi$).

*Proof.* We can construct *a* path, given $\tau\colon Z_0 \times I \xrightarrow{\simeq} Z$, by $\eta(t) = \tau(z, t)$. Since $\pi \circ \tau = \mathrm{pr}_2$, this has the required property. Connectedness of $I$ and discreteness of $Z_0$ implies that given any other path $\eta'\colon I \to Z$ with $\eta'(0) = z$ and $\pi \circ \eta' = \mathrm{id}$, we must have $\tau^{-1} \circ \eta = \tau^{-1} \circ \eta'\colon I \to Z_0 \times I$ which implies $\eta' = \eta$. □

And now we have a really important property of covering spaces

**Theorem 1.** Given any covering space $Z \xrightarrow{\pi} X$, path $\gamma\colon I \to X$ and point $z \in Z_{\gamma(0)}$, there is a unique lift $\widetilde{\gamma_z}\colon I \to Z$ with $\widetilde{\gamma_z}(0) = z$.

a *lift* of a path $\gamma\colon I \to X$ is a path $\widetilde{\gamma}\colon I \to Z$ with $\pi\widetilde{\gamma} = \gamma$

*Proof.* We can pull back $Z$ to get $p\colon \gamma^* Z \to I$. We have unique $\eta_z\colon I \to \gamma^* Z$ so that $\eta_{(0,z)}(0) = (0, z)$. Define $\widetilde{\gamma_z} = \mathrm{pr}_2 \circ \eta_{(0,z)}\colon I \to Z$. This path satisfies $\pi \circ \widetilde{\gamma_z} = \gamma \circ p \circ \eta_{(0,z)} = \gamma$. Given any other lift $\lambda\colon I \to Z$, we get a second section of $p$ by $t \mapsto (t, \lambda(t))$, which by uniqueness of $\eta_{(0,z)}$ has to be equal to it, so that $\lambda = \widetilde{\gamma_z}$. □

We can then give a second, more explicit proof of Proposition 7.

**Corollary 3.** A path $\gamma\colon I \to X$ defines a bijection $\gamma_*\colon Z_{\gamma(0)} \xrightarrow{\simeq} Z_{\gamma(1)}$, by $\gamma_*(z) = \widetilde{\gamma_z}(1)$.

*Proof.* We only have to start with that $\gamma_*$ is a function $Z_{\gamma(0)} \to Z_{\gamma(1)}$, but the function $(-\gamma)_*\colon Z_{\gamma(1)} \to Z_{\gamma(0)}$, where $-\gamma\colon I \to X$ is the path $-\gamma(x) = \gamma(1 - x)$, is inverse to $\gamma_*$. This is because the path $-\widetilde{\gamma_z}$ is a lift of $-\gamma$, hence $(-\gamma)_*(\gamma_*(z)) = \widetilde{(-\gamma)}_{\gamma_*(z)}(1) = \widetilde{\gamma_z}(0) = z$. A symmetric argument shows that $\gamma_*\left((-\gamma)_*(z)\right) = z$ for $z \in Z_{\gamma(1)}$. □

A first observation is that this bijection is invariant under reparameterisations of $\gamma$: given $\psi\colon I \xrightarrow{\simeq} I$ with $\psi(0) = 0$ and $\psi(1) = 1$, then clearly $(\gamma \circ \psi)_* = \gamma_*\colon Z_{\gamma(0)} \to Z_{\gamma(1)}$.

consider $\psi$ as a path in $I$ and see what happens in that case

Even better, we get a function

Lecture 6

$$\{\text{paths } x_0 \rightsquigarrow x_1 \text{ in } X\} \times Z_{x_0} \to Z_{x_1}$$

If we take $x_0 = x_1 = x$, then this is a map

we can take quotient by reparametrisations if desired, in each of these functions

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \times Z_x \to Z_x$$

such that each loop $x \rightsquigarrow x$ gives a bijection $Z_x \to Z_x$. So we can think of this instead as

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \to \mathrm{Aut}(Z_x).$$

Alternatively, if we have a pointed covering space $(Z, z) \to (X, x)$, we have a canonical function

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \to Z_x \tag{1}$$

**Example 26.** For $Z = S \times X$, $(\gamma)_* = \mathrm{id}_S$ always, and the image of (1) (given some $(s,x) \in Z$) is just a single point. For instance, if $X = I$, we have seen this will be the case for every covering space. But for $X = S^1$, $Z = \mathbb{R} \xrightarrow{\exp} S^1$, and taking $x = 1 \in S^1$, $z = 0 \in \mathbb{R}$, then $Z_1 = \exp^{-1}(0) = 2\pi i\mathbb{Z}$, then

$$\{\gamma\colon I \to S^1 \mid \gamma(0) = \gamma(1) = 1\} \to 2\pi i\mathbb{Z}$$

is *onto*. The path $\widetilde{\gamma}_n = 2\pi i n x$ lifts the path $\gamma(x) = \exp(2\pi i n x)$, and $\widetilde{\gamma}_n(0) = 0$, $\widetilde{\gamma}_n(1) = 2\pi i n x$. The difference is that $\mathbb{R}$ is path connected, but $X \times S$ is not, for $|S| > 1$.

In fact, for a covering space $(Z,z) \xrightarrow{\pi} (X,x)$ with $Z$ path connected and $z' \in Z_x$, there is $\widetilde{\gamma}\colon I \to Z$ with $\widetilde{\gamma}(0) = z$, $\widetilde{\gamma}(1) = z'$. Since $\widetilde{\gamma}$ lifts $\gamma = \pi \circ \widetilde{\gamma}$, which satsfies $\gamma(0) = x = \gamma(1)$, the map (1) is **onto**. Thus paths constrain the sizes of fibres of connected covering spaces and vice versa. Notice also that the set of loops is independent of the choice of covering space!

More generally, given points $z_\alpha$ in $Z_x$, one per path component of $Z$,

that is: a section of $Z \to [\mathrm{pt}, Z]$

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \times [\mathrm{pt}, Z] \simeq \{\text{loops } x \rightsquigarrow x \text{ in } X\} \times \{z_\alpha\} \to Z_x$$

is always onto. There are a huge number of paths, and reparameterisations cuts things down somewhat. But we shall go even better, and put a topology on the space of paths.

The fibres $Z_x$ of a covering space $Z$ are discrete spaces, but the set $\mathbf{Top}(I,X)$ of paths $I \to X$ carries a topology when $X$ is a metric space; we can consider $C(I,X)$ with the sup metric $d_\infty$. The aim is to give $\mathbf{Top}(I,X)$ a topology for *any* space, not necessarily metric.

**Lemma 9.** Let $X$ be a topological space, fix $\gamma \in \mathbf{Top}(I,X)$ a path. Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$ be a partition of $[0,1]$, and $U_0, \ldots, U_n \subseteq X$ a collection of basic nhds such that $\gamma([t_i, t_{i+1}]) \subseteq U_i^o$. Define the subsets

The interior $V^o$ of a nhd $V$ is the union of all the open sets contained in $V$

$$N_\gamma(t_1 < \cdots < t_n; U_0, \ldots, U_n) := \{\eta\colon I \to X \mid \forall i = 0, \ldots n,\ \eta([t_i, t_{i+1}]) \subseteq U_i^o\} \subseteq \mathbf{Top}(I,X)$$

Then define $\mathcal{N}_{co}(\gamma)$ to be the family of subsets of $\mathbf{Top}(I,X)$ consisting of the sets above, as the partition and the collection of basic nhds vary. So defined the families $\mathcal{N}_{co}(\gamma)$ give a neighbourhood base on $\mathbf{Top}(I,X)$.

**Definition 21.** The *path space* $X^I$ is the set $\mathbf{Top}(I,X)$ equipped with the topology defined by Lemma 9, which we call the *compact-open topology*.

When $X$ is a metric space, then the compact-open topology and the topology arising from the sup metric coincide. A key property of the compact-open topology is that homotopies $H\colon I \times I \to X$ give continuous paths $h\colon I \to X^I$ (defined by $h_t\colon s \mapsto H(t,s)$) and vice-versa. Moreover:

**Lemma 10.** 1. The evaluation map $\mathrm{ev}\colon X^I \times I \to X$, $\mathrm{ev}(\gamma, t) = \gamma(t)$ is continuous, and

2. given a map $X \xrightarrow{f} Y$, the post-composition map $f_*\colon X^I \to Y^I$, $f_*(\gamma) = f \circ \gamma$, is continuous.

Then given $t \in I$, the composite map $\mathrm{ev}_t\colon X^I \simeq X^I \times \{t\} \hookrightarrow X^I \times I \xrightarrow{\mathrm{ev}} X$ is continuous. Usually we care just about the cases $t = 0, 1$. We can then look at various subspaces of $X^I$, for a given $x \in X$:

$$\begin{aligned} P_x X &:= \{\gamma \in X^I \mid \gamma(0) = x\} = \mathrm{ev}_0^{-1}(x) \\ P_x^y X &:= \{\gamma \in X^I \mid \gamma(0) = x,\ \gamma(1) = y\} = \mathrm{ev}_0^{-1}(x) \cap \mathrm{ev}_1^{-1}(y) \\ \Omega_x X &:= P_x^x X = \{\gamma \in X^I \mid \gamma(0) = x = \gamma(1)\} \end{aligned}$$

In particular, we have already seen the last two, albeit without their topologies. We also see that path components of these spaces have something to do with homotopy classes of paths, perhaps with constraints on endpoints.

A key property of the natural transformation $\mathrm{id} \Rightarrow \mathrm{disc}\,\pi_0\colon \mathbf{slpcTop} \to \mathbf{slpcTop}$ is that it has a universal property: given a discrete space $S$, an slpc space $X$ and a continuous map $X \xrightarrow{f} S$, there is a *unique* function $\pi_0(X) \to U(S)$ such that

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S \\ \mathrm{disc}(\pi_0(X)) \arrow[ur] & \end{tikzcd}$$

commutes. Hence if we take our function

$$P_x^y X \times Z_x \to Z_y \tag{2}$$

from the previous lecture, arising from a covering space $Z \to X$, and if we can show it is continuous, we would get a factorisation

$$P_x^y X \times Z_x \to \pi_0(P_x^y X \times Z_x) \simeq \pi_0(P_x^y X) \times Z_x \to Z_y$$

where the unmarked isomorphism exist due to $Z_x$ being discrete. If $Z$ is path connected, a fixing some $z \in Z_x$, we get a surjective map $\pi_0(P_x^y X) \to Z_y$, which further constrains both the topology of the space of paths, and the possible fibres of $Z \to X$. However, there are two issues:

(i) We yet don't know our path lifting function is continuous

(ii) We don't know if $P_x^y X$ is slpc, hence if path components and components agree.

To address (i), the unique path lifting property from last lecture will be promoted to a *continuous function* Lift: $X^I \times_X Z \to Z^I$. Combined with $Z^I \xrightarrow{\text{ev}} Z$ we will be able to reconstruct (2) as

here $X^I \times_X Z = \{(\gamma, z) \mid \gamma(0) = \pi(z)\}$

$$P_x^y X \times Z_x \hookrightarrow X^I \times_X Z \xrightarrow{\text{Lift}} Z^I \xrightarrow{\text{ev}_1} Z$$

factors through $Z_y \subset Z$. We already have the definition of Lift, but we need to show continuity.

**Theorem 2.** The function Lift: $X^I \times_X Z \to Z^I$ is continuous.

*Proof.* We need to set up the ingredients, so take $\gamma \in X^I$, define $x = \gamma(0)$, $y = \gamma(1)$, and take $z \in Z_x$. Let $\widetilde{\gamma} = \text{Lift}(\gamma, z)$, and $z' = \widetilde{\gamma}(1) \in Z_y$. Take a basic nhd $N_{\widetilde{\gamma}} = N_{\widetilde{\gamma}}(t_1 < \cdots < t_n; U_0, \ldots, U_n)$. We want to construct a basic nhd

$$M(\gamma, z) \subseteq X^I \times_X Z$$

of $(\gamma, z)$ such that $M(\gamma, z) \subseteq \text{Lift}^{-1}(N_{\widetilde{\gamma}})$.

Since $Z \xrightarrow{\pi} X$ is locally trivial and $I$ is compact, we can find a sequence $W_0, \ldots, W_m \subseteq Z$ (with $m \geq n$) of nhds such that

- $\pi\big|_{W_i} : W_i \xrightarrow{\cong} \pi(W_i)$ and each $\pi(W_i)$ is a nhd in $X$, and
- $\forall i = 0, \ldots, m\ \exists j = j(i)$ with $W_i \subseteq U_j$.

There is then a refinement $0 < s_1 < \cdots < s_m < 1$ such that $W_i$ is a nhd of $\widetilde{\gamma}(t)$ for all $t \in [s_i, s_{i+1}]$. The set $\widetilde{N}_{\widetilde{\gamma}} := N_{\widetilde{\gamma}}(s_1 < \cdots < s_m; W_0, \ldots, W_m) \subseteq Z$ is then contained in $N_{\widetilde{\gamma}}$.

so that $[s_i, s_{i+1}] \subseteq [t_j, t_{j+1}]$

But, defining $V_i := \pi(W_i)$, the partition $0 < \cdots s_1 < s_m < 1$ and the sets $V_0, \ldots, V_m$ satisfy the conditions required to define the basic nhd $N_\gamma(s_1 < \cdots < s_m; V_0, \ldots, V_m) \subseteq X^I$. Also note that $z = \widetilde{\gamma}(0) \in W_0$, so we can define a nhd

$$M(\gamma, z) := (N_\gamma(s_1 < \cdots < s_m; V_0, \ldots, V_m) \times W) \cap X^I \times_X Z$$

of $(\gamma, z)$. By construction $\pi(\widetilde{N}_{\widetilde{\gamma}}) \subseteq N_\gamma(s_1 < \cdots < s_m; V_0, \ldots, V_m)$, but in fact $\text{Lift}(M(\gamma, z)) = \widetilde{N}_{\widetilde{\gamma}} \subseteq N_{\widetilde{\gamma}}$, as desired. ☐

**Remark.** In fact, by the uniqueness of lifts, the map Lift is a bijection, and even a homeomorphism, with inverse $(\pi_*, \text{ev}_0) : Z^I \to X^I \times_X Z$.

So we have a continuous map $P_x^y X \times Z_x \to Z_y$, and thus get a function $\pi_0(P_x^y X) \times Z_x \to Z_y$. But we would like to know that for any two points $\gamma, \eta \in P_x^y X$ in the same connected component, there is a path between them. Such a path, recall, is a homotopy $H \colon I \times I \to X$ satisfying $H(s,0) = x$ and $H(s,1) = y \; \forall x \in I$. Such a homotopy between paths will be said to *fix endpoints*.

**Definition 22.** A space $X$ is called *semilocally simply-connected* (or *slsc*) if every point has a basis of nhds $N$ that are path connected, and given $x, y \in N$ and two paths $\gamma, \eta \in P_x^y N$, there is an endpoint-fixing homotopy $I \times I \to X$ from $\gamma$ to $\eta$.

this is the last technical condition on spaces we require in this section of the course

Notice that if a space $X$ is slsc, then it is slpc.

**Example 27.** Any manifold is slsc, since every point has a nhd homeomorphic to some $\mathbb{R}^n$, which is convex.

**Example 28.** The *Hawaiian earring* is the subspace

$$\bigcup_{n \in \mathbb{N}} \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, ||(x,y) - (\tfrac{1}{n}, 0)|| = \tfrac{1}{n} \right\}$$

and is not slsc. Every nhd of the point $(0,0)$ contains loops that are not contractible, and stay non-contractible in the full space.

**Theorem 3** (Wada 1955, improved in Roberts 2010)**.** If the space $X$ is semilocally simply-connected, the spaces $X^I$, $P_x X$ and $P_x^y X$ (hence $\Omega_x X$) are semilocally path connected.

H. Wada, "Local connectivity of mapping spaces", Duke Math. J. **22**, Number 3 (1955) pp 419–425. DMR "Fundamental bigroupoids and 2-covering spaces", Theorem 5.12.

*Proof.* (Non-examinable) See Handout 1. □

Lecture 7

A question that may have occurred to you is what happens with the isomorphism $\gamma_* \colon Z_x \to Z_y$ if we break the path $\gamma \colon I \to X$ into two subpaths, say $x \rightsquigarrow x' \rightsquigarrow y$, and then compose the corresponding isomorphisms $Z_x \xrightarrow{\simeq} Z_{x'} \xrightarrow{\simeq} Z_y$. Or, starting from paths $\gamma, \eta \colon I \to X$ such that $\gamma(1) = \eta(0)$ and defining the *concatenation* $\gamma \# \eta \colon I \to X$ by

$$\gamma \# \eta(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \eta(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

how do $Z_{\gamma(0)} \xrightarrow{\gamma_*} Z_{\gamma(1)} = Z_{\eta(0)} \xrightarrow{\eta_*} Z_{\eta(1)}$ and $Z_\gamma(0) \xrightarrow{(\gamma\#\eta)_*} Z_{\eta(1)}$ relate?

**Lemma 11.** For paths $\gamma, \eta \colon I \to X$ such that $\gamma(1) = \eta(0)$, $(\gamma \# \eta)_* = \eta_* \circ \gamma_* \colon Z_{\gamma(0)} \to Z_{\eta(1)}$.

In particular, for $\gamma, \eta \in \Omega_x X$, $\gamma \# \eta \in \Omega_x X$ and we have the map $\Omega_x X \to \mathrm{Aut}(Z_x)$, which is compatible with path concatenation. But # is not associative!

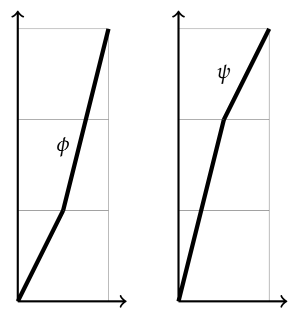
**Example 29.** Take $X = S^1$, and let $\gamma(t) = \exp(2\pi i t)$.

$$(\gamma\#\gamma)\#\gamma = \begin{cases} \exp(8\pi i t) & t \in [0, \frac{1}{2}] \\ \exp(4\pi i t) & t \in [\frac{1}{2}, 1] \end{cases} \quad \text{but} \quad \gamma\#(\gamma\#\gamma) = \begin{cases} \exp(4\pi i t) & t \in [0, \frac{1}{2}] \\ \exp(8\pi i t) & t \in [\frac{1}{2}, 1] \end{cases}$$

Let us re-examine how paths concatenate. Given $\gamma, \eta \colon I \to X$ such that $\gamma(1) = \eta(0)$, then we get a continuous function $\langle \gamma, \eta \rangle \colon [0,2] \to X$. The concatenation $\gamma\#\eta$ is then the precomposition of $\langle \gamma, \eta \rangle$ with the map $I = [0,1] \xrightarrow{t \mapsto 2t} [0,2]$. If we had a third map, $\lambda \colon I \to X$ with $\lambda(0) = \eta(1)$, then there is naturally a continuous function $\langle \gamma, \eta, \lambda \rangle \colon [0,3] \to X$. But the concatenations $(\gamma\#\eta)\#\lambda$ and $\gamma\#(\eta\#\lambda)$ arise from precomposing with two different maps $I = [0,1] \to [0,3]$. These are

$$\phi \colon t \mapsto \begin{cases} 4t & t \in [0, \frac{1}{2}] \\ 2t + 1 & t \in [\frac{1}{2}, 1] \end{cases}$$

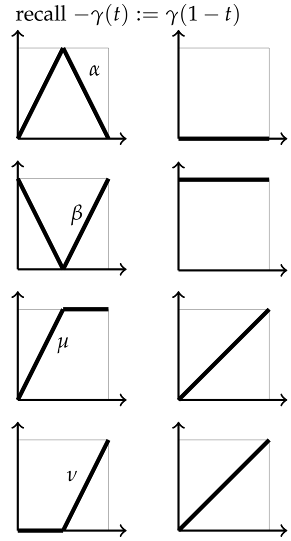
$$\psi \colon t \mapsto \begin{cases} 2t & t \in [0, \frac{1}{2}] \\ 4t - 1 & t \in [\frac{1}{2}, 1] \end{cases}$$




with graphs as at right.

These two paths $I \to [0,3]$ are homotopic fixing endpoints by the homotopy $h_a(s,t) = s\phi(t) + (1-s)\phi(t)$. If we then precompose $\langle \gamma, \eta, \lambda \rangle \colon [0,3] \to X$ with $h_a \colon I \times I \to [0,3]$, we get a homotopy between $(\gamma\#\eta)\#\lambda$ and $\gamma\#(\eta\#\lambda)$. Path concatenation in $X$ is then *homotopy associative*. But what about inverses or an identity element? We will play the same trick, by considering a 'universal' case.

recall $-\gamma(t) := \gamma(1-t)$




Given a path $\gamma \colon I \to X$, we have the reverse path $-\gamma$, and the composite $\gamma\#(-\gamma) \colon I \to X$ can be factored as $I \xrightarrow{\alpha} I \xrightarrow{\gamma} X$ for a certain path $I \xrightarrow{\alpha} I$. If we instead concatenate in the other direction, namely $(-\gamma)\#\gamma \colon I \to X$, then this factors as $I \xrightarrow{\beta} I \xrightarrow{\gamma} X$. Again $\beta$ is a certain path in $I$. The graphs of both $\alpha$ and $\beta$ are shown at right, and both of them are homotopic, fixing endpoints, to the constant functions at 0 and 1 respectively, by taking an affine combination as in the definition of $h_a$ above. Then by composing the homotopies here with $\gamma$, we get homotopies between the path $\gamma\#(-\gamma)$ and the constant path at $\gamma(0)$, and also between $(-\gamma)\#\gamma$ and the constant path at $\gamma(1)$. So we have *homotopy inverses*.

If we want to think about a homotopy identity element, then we should use the constant path $c_x\colon I \to X$ at a point $x \in X$, with $c_x(t) = x$, $\forall t \in I$. We can factor the composite $\gamma\# c_{\gamma(1)}$ as $I \xrightarrow{\mu} I \xrightarrow{\gamma} X$ for $\mu$ as shown at right, and factor $c_{\gamma(0)}\#\gamma$ as $I \xrightarrow{\nu} I \xrightarrow{\gamma} X$. As above, $\mu$ and $\nu$ are homotopic, fixing endpoints, to the identity map $I \to I$.

If we turn the five homotopies $I \times I \to X$ described above into paths $I \to X^I$, then if we start from elements of $\Omega_x X$, these homotopies correspond to paths in $\Omega_x X$. Thus $\Omega_x X$, which has a concatenation binary operator $\#\colon \Omega_x X \times \Omega_x X \to \Omega_x X$, acts like a group, except the group axioms only hold up to the existence of paths

More is true, though we won't prove it: there are homotopies assembled out of these paths for all possible cases, for instance $I \times \Omega_x X \times \Omega_x X \times \Omega_x X \to \Omega_x X$

$$
\begin{aligned}
(\gamma\#\eta)\#\lambda &\rightsquigarrow \gamma\#(\eta\#\lambda) \\
\gamma\#(-\gamma) &\rightsquigarrow c_{\gamma(0)} \\
(-\gamma)\#\gamma &\rightsquigarrow c_{\gamma(1)} \\
\gamma\# c_{\gamma(1)} &\rightsquigarrow \gamma \\
c_{\gamma(0)}\#\gamma &\rightsquigarrow \gamma
\end{aligned}
$$

in $\Omega_x X$. As a result we have proved most of

**Proposition 10.** Let $(X, x)$ be a pointed space, with $X$ slsc. The set $\pi_0(\Omega_x X)$ carries the structure of a group, its product arising from concatenation of loops and identity element represented by the constant path at $x$.

If we fall back on the default, namely just slpc, then we can use $[\mathrm{pt}, \Omega_x X]$ instead

*Proof.* To exhibit the multiplication, consider the functor $\pi_0$ applied to $\#\colon \Omega_x X \times \Omega_x X \to \Omega_x X$, giving $\pi_0(\Omega_x X \times \Omega_x X) \xrightarrow{\#} \pi_0(\Omega_x X)$. But since $\pi_0(M \times N) \xrightarrow{\simeq} \pi_0(M) \times \pi_0(N)$, for all slpc spaces $M$ and $N$, we get a composite $\pi_0(\Omega_x X) \times \pi_0(\Omega_x X) \simeq \pi_0(\Omega_x X \times \Omega_x X) \to \pi_0(\Omega_x X)$. This is associative and unital, and inverses exist, by the existence of the paths above. $\square$

This requires knowing that # is continuous! See Assignment 2.

**Definition 23.** For $(X, x)$ a pointed space its *fundamental group at $x$* is $\pi_1(X, x) := [\mathrm{pt}, \Omega_x X]$, which for $X$ a slsc space coincides with $\pi_0(\Omega_x X)$.

Recall we also proved $[\mathrm{pt}, -]$ descends to a functor **hTop** $\to$ **Set** in Assignment 1

From the previous reasoning, we have constructed from a covering space $Z \to X$ and chosen basepoint $x \in X$ a permutation representation $\pi_1(X, x) \to \mathrm{Aut}(Z_x)$. If $Z$ is path connected, and we choose $z \in Z_x$, we get a surjective map $\pi_1(X, x) \to Z_x$, given by $\gamma \mapsto \gamma_*(z)$. This implies we have an upper bound on the cardinality of fibres of any path connected covering space, and conversely, given a connected covering space, the fibres give a lower bound on the number of distinct homotopy classes of loops in $X$.

As a point of clarification, everything here works for arbitrary slpc spaces with small adjustments, but for slsc spaces the approach is slightly cleaner, as components and path components coincide for the function spaces

**Example 30.** The projection map $S^2 \to \mathbb{RP}^2$ is a covering space and $S^2$ is connected, so there exist at least two non-homotopic loops in $\mathbb{RP}^2$ at any given basepoint. One of these is the constant loop, so there exists a loop in $\mathbb{RP}^2$ not homotopic to it.

**Example 31.** We have the covering space $\exp(2\pi i -)\colon \mathbb{R} \to S^1$ with fibre $\mathbb{Z}$ over $1 \in S^1$, which implies $\pi_1(S^1, 1)$ is an infinite group.

**Proposition 11.** The loop space construction is a functor $\Omega\colon \mathbf{Top}_* \to \mathbf{Top}_*$.

**Corollary 4.** The fundamental group gives a functor

Exercise: This functor is naturally isomorphic to $[(S^1, 1), (X, x)]_*$

$$\pi_1 := [\mathrm{pt}, -] \circ \Omega\colon \mathbf{Top}_* \to \mathbf{Grp}.$$

which for slsc spaces is naturally isomorphic to $\pi_0 \circ \Omega$.

Lecture 8

However, as we have seen, we don't just get an action of $\pi_1(X, x)$ on the fibre $Z_x$ of a covering space. We also get what looks like an action of paths between different points on fibres, but now points in one fibre are taken to points of another fibre. In fact, if $X$ is not equipped with a basepoint to start with, or there are several natural options and no one of those is canonical, then we can create an even richer invariant, namely a *groupoid*.

**Definition 24.** A *groupoid* is a category where every morphism has an inverse.

So that we have an idea of what kinds of groupoids arise, let us consider some examples. We will be considering only *small* groupoids: those locally small groupoids $\Gamma$ where there is a set $\Gamma_0$ of objects. We can then take the disjoint union of all the hom-sets to get the set $\Gamma_1 = \coprod_{x,y \in \Gamma_0} \Gamma(x, y)$ of morphisms, and specify the source and target functions $s, t\colon \Gamma_1 \rightrightarrows \Gamma_0$. Groupoids and functors form a category **Gpd**.

**Example 32.** 1. Every set $S$ gives a groupoid $\mathrm{disc}(S)$, by taking the set of objects to be $S$, and to only have identity morphisms. This gives a full subcategory inclusion $\mathrm{disc}\colon \mathbf{Set} \hookrightarrow \mathbf{Gpd}$, and such groupoids are called *discrete*.

2. Every set $C$ also gives another groupoid $\mathrm{codisc}(C)$ with set of objects $C$, but with exactly one morphism from any object to any other object. The set of morphisms is $C \times C$, and every object $c \in C$ has the trivial group of automorphisms. Such groupids are called *codiscrete*.

3. Let $G$ act on the set $Y$ on the right. Then there is a groupoid $Y//G$ with object set $Y$, and set of morphisms $Y \times G$. The source and target are given by $s(y,g) = y$, $t(y,g) = yg$, and composition is $(y,g)(yg,h) = (y,gh)$.
   (a) If $G = 1$, then this recovers the first example.
   (b) If $Y = \mathrm{pt}$, then the information in the groupoid is essentially just that of the group $G$. Groupoids of this form will be denoted $\mathbb{B}G$, and $\mathbb{B}\colon \mathbf{Grp} \hookrightarrow \mathbf{Gpd}$ is the inclusion of a full subcategory.

A slogan people sometimes use is that a groupoid is like a group with 'many identities', but you can also usefully think of them as being a generalisation of a group action, where you have different groups acting on different parts of the set. Here is a useful lemma about the structure of groupoids.

**Lemma 12.** For any groupoid $\Gamma$, and given $x, y \in \Gamma_0$,

using algebraic order of composition

$$\begin{aligned} \mathrm{Ad}_a\colon \Gamma(x,x) &\xrightarrow{\simeq} \Gamma(y,y) \\ g &\mapsto a^{-1}ga \end{aligned}$$

is an isomorphism for any $a \in \Gamma(y,x)$ and the function

$(\mathrm{Ad}_a)^{-1} = \mathrm{Ad}_{a^{-1}}$

$$\begin{aligned} \Gamma(x,x) \times \Gamma(x,y) &\to \Gamma(x,y) \\ (g,a) &\mapsto ga \end{aligned}$$

transitive: $(ba^{-1}, a) \mapsto b$;
free: $ga = a$ implies $g = gaa^{-1} = aa^{-1} = \mathrm{id}_x$

defines a free and transitive action of the group $\Gamma(x,x)$.

As a reminder: a free group action $G \times S \to S$ is one where $g \cdot p = p$ implies $g$ is the identity element, and a transitive action one where given any two elements $p, q \in S$, there is some group element $g \in G$ such that $g \cdot p = q$.

**Definition 25.** Given an slsc space $X$ and a specified subset $A \subseteq X$, the *fundamental groupoid based at $A$* is the groupoid $\Pi_1(X,A)$ with set of objects $A$, and the set of morphisms from $x$ to $y$ is $\Pi_1(X,A)(x,y) := \pi_0(P_x^y X)$. The composition map is induced from concatenation of paths:

the definition makes sense for more general slpc spaces, using $[\mathrm{pt}, -]$ in place of $\pi_0$, but we are only consider slsc spaces here

$$\pi_0(P_x^y X) \times \pi_0(P_y^z X) \simeq \pi_0(P_x^y X \times P_y^z X) \to \pi_0(P_x^z X)$$

and constant paths are the identity morphisms.

As with other invariants, the fundamental groupoid is a functor. Define the category $\mathbf{Top}^{(2)}$ to be the category with objects pairs $(X,A)$ where $X$ is a topological space and $A \subseteq X$ is a subspace, and a morphism $(X,A) \to (Y,B)$ is a continuous function $f\colon X \to Y$ such that $f(A) \subseteq B$. We have a full subcategory inclusion $\mathbf{Top}_* \hookrightarrow \mathbf{Top}^{(2)}$.

**Proposition 12.** The fundamental groupoid gives a functor $\Pi_1\colon \mathbf{Top}^{(2)} \to \mathbf{Gpd}$ such that

$$\begin{tikzcd} \mathbf{Top}_* \arrow[r, "\pi_1"] \arrow[d] & \mathbf{Grp} \arrow[d, "\mathbb{B}"] \\ \mathbf{Top}^{(2)} \arrow[r, "\Pi_1"'] & \mathbf{Gpd} \end{tikzcd}$$

and moreover:

$$\Pi(X \times Y, A \times B) \xrightarrow{\simeq} \Pi_1(X, A) \times \Pi_1(Y, B)$$
$$\Pi_1(X, A) \sqcup \Pi_1(Y, B) \xrightarrow{\simeq} \Pi(X \sqcup Y, A \sqcup B)$$

The product/disjoint union of groupoids is what you think it is: take the products/disjoint unions of the objects and the morphisms, respectively

We can include *unbased* spaces $X$ into pairs, by taking $(X, X)$, giving another fully faithful functor, $\mathbf{Top} \to \mathbf{Top}^{(2)}$. In this case, if the space $X$ has *no* preferred basepoints whatsoever, we can still define the fundamental groupoid of $X$ itself as $\Pi_1(X, X)$, which is a functor $\mathbf{Top} \to \mathbf{Gpd}$.

We haven't yet seen how to calculate the fundamental group(oid) in examples, so we will turn to that now. We need a name for spaces $X$ that have $\Pi_1(X)$ trivial, in the sense of being codiscrete.

**Definition 26.** A space $X$ that satisfies $\Pi_1(X) = \text{codisc}(X)$ is called *simply-connected*.

such spaces also have $\Pi_1(X, A) = \text{codisc}(A)$ for all $A \subseteq X$

If we unpack this definition, it tells us that a) given any two points $x, y \in X$, there is a (homotopy class of some) path from $x$ to $y$, so that $X$ is path-connected, and b) all paths between any two given points are endpoint-fixed homotopic, hence a unique morphism in the fundamental groupoid. As a result, $\pi_1(X, x) = \Pi_1(X)(x, x)$ is the trivial group.

**Example 33.** Convex subspaces $C \subseteq \mathbb{R}^n$ are simply-connected, because any two points $v, w \in C$ can be joined by a path in $C$, and given two paths $\gamma, \eta\colon v \rightsquigarrow w$ the map $(s, t) \mapsto s\gamma(t) + (1 - s)\eta(t)$ is a homotopy between them.

In particular, the interval $I$ is simply-connected. The fundamental groupoid $\Pi_1(I, \{0, 1\})$ is important enough to have its own name: **2**, sometimes denoted $(0 \xrightarrow{\sim} 1)$, as it has two objects $0, 1$ and a unique isomorphism between them.

**Exercise 12.** Define a *star-shaped region* in a (real or complex) vector space $V$ to be a set $K \subseteq V$ such that there is a point $v_0 \in K$ such that for every $v \in K$ and $t \in I$, $tv_0 + (1 - t)v \in K$. Prove that star-shaped regions are simply-connected.

For $\mathcal{H} \subset \mathbb{C}$ the (open) upper half-plane, the set $\mathcal{H} \cup \mathbb{Q}$ is star-shaped, but not convex

Simply-connected spaces are special for the following reason.

**Proposition 13.** If $X$ is a simply-connected space, then every path connected covering space $Z \xrightarrow{\pi} X$ is trivial, in the sense that $\pi$ is a homeomorphism.

*Proof.* Recall that $\pi_1(X, x) \to Z_x$ is surjective for any $x \in X$, so $X$ simply-connected implies $Z_x = \text{pt}$ for all $x$. Thus $\pi$ is a bijection. The local triviality condition implies that every $x \in X$ has an open set $U \ni x$ such that $\pi^{-1}(U) \to U$ is a homeomorphism. Letting $U_\alpha$ range over such an cover of $X$, we can glue the inverses of these local homeomorphisms into a into an inverse for $\pi$. □

**Example 34.** If $X$ is contractible then it is simply-connected. Let $H \colon I \times X \to X$ be a contraction to $x_0 \in X$. Consider the induced map $h = \Pi_1(H) \colon \Pi_1(I \times X, \{0,1\} \times X) \to \Pi_1(X, X) = \Pi_1(X)$. The domain simplifies to be $\Pi_1(I, \{0,1\}) \times \Pi_1(X) = \mathbf{2} \times \Pi_1(X)$. Consider the induced maps $\{i\} \times \Pi_1(X) \to \mathbf{2} \times \Pi_1(X) \to \Pi_1(X)$ for $i = 0, 1$. Since $H\big|_{\{0\} \times X} = \mathrm{id}_X$, so $h_{\{0\} \times \Pi_1(X)} = \mathrm{id}_{\Pi_1(X)}$; and as $H\big|_{\{1\} \times X}$ is constant at $x_0$, so $h(0, x) = x_0$ for all $x \in X$, and $h\big|_{\{1\} \times \Pi_1(X)}$ sends every path to the constant path at $x_0$. We already know that $X$ is path connected, so that for any $x, y \in X$ there is some path between them. Given a path $\gamma \colon x \rightsquigarrow y$ consider the commutative square

$$\begin{tikzcd}
(0,x) \arrow[r, "{(\mathrm{id}_0, [\gamma])}"] \arrow[d] & (0,y) \\
(1,x) \arrow[r, "{(\mathrm{id}_1, [\gamma])}"'] & (1,y) \arrow[u]
\end{tikzcd}$$

in $\mathbf{2} \times \Pi_1(X)$ (recall all morphisms are invertible). Under $h$ this is sent to

$$\begin{tikzcd}
x \arrow[r, "{[\gamma]}"] \arrow[d] & y \\
x_0 \arrow[r, "\mathrm{id}"'] & x_0 \arrow[u]
\end{tikzcd}$$

The vertical arrows are independent of $[\gamma]$, so that every path $\gamma$ in $X$ is homotopic to the composite the long way around the square, hence to every other path.

So in some sense, we are interested in spaces that are path connected, though this is useful when building spaces out of disjoint components. Here is another way we can get information about the fundamental groupoid of a space from the fundamental groupoid of other spaces. Lecture 9

**Theorem 4.** Let $Z \xrightarrow{\pi} X$ be a covering space. Then $\Pi_1(Z)(z_1, z_2) \to \Pi_1(X)(\pi(z_1), \pi(z_2))$ is injective for all $z_1, z_2 \in Z$.

thus the funtor $\Pi_1(\pi)$ is *faithful*

We will prove this theorem in a little bit, but let us give an important result that follows.

**Corollary 5.** Given a covering space $(Z, z) \xrightarrow{\pi} (X, x)$, the induced homomorphism between fundamental groups identifies $\pi_1(Z, z)$ with a subgroup of $\pi_1(X, x)$.

This allows us, given a covering space whose fundamental groupoid we know, to place a lower bound on the size of the fundamental group of the base space. Alternatively, it places an upper bound on the size of the fundamental group of the covering space, so if $\pi_1(X, x)$ is finite, then so is $\pi_1(Z, z)$.

**Proposition 14.** Let $Z \to I \times X$ be a covering space. Then $Z \xrightarrow{\simeq} I \times Z_0$ over $I \times X$, where $Z_0 := Z_{\{0\} \times X}$.

*Proof.* The function $Z \to I \times Z_0$ is given by $(\mathrm{pr}_1 \circ \pi, \tau)$, for some $\tau\colon Z \to Z_0$, which we need to construct. The idea is similar to the situation where we constructed the trivialisation of a covering space of $I$, which is the special case of $X = \mathrm{pt}$. Given $x \in X$, we get a trivisalisable covering space $Z_{I \times \{x\}} \to X$, and so a function $\tau_x\colon Z_{I \times \{x\}} \to I \times Z_{(0,x)} \xrightarrow{\mathrm{pr}_2} Z_{(0,x)}$. Hence we have a (potentially discontinuous) function $Z \to Z_0$ using the various $\tau_x$. We will write down a global version of this function using ingredients we already know to be continuous.

Given $(t, x) \in I \times X$, there is a path $(0, x) \rightsquigarrow (t, x)$ given by $\eta_{(t,x)}(s) = (ts, x)$, which we want to vary continuously with $(t, x)$. We know that $I \times I \times X \to I \times X$, $(s, t, x) \mapsto (ts, x)$ is continuous, so that by the

$$\begin{aligned} I \times X &\to (I \times X)^I \\ (t, x) &\mapsto \eta_{(t,x)} \end{aligned}$$

is continuous. We can now define the composite

$$\begin{array}{ccccc} \tau\colon Z \xrightarrow{\simeq} (I \times X) \times_{I \times X} Z & \to & (I \times X)^I \times_{I \times X} Z \xrightarrow{\mathrm{Lift}} Z^I \xrightarrow{\mathrm{ev}_1} Z \\ z \mapsto (\pi(z), z) & \mapsto & (-\eta_{\pi(z)}, z) \end{array}$$

This map factors through $Z_0$, as if $(t, z) := \pi(z)$, then $-\eta_z$ is a path in $I \times X$ from $(t, x)$ to $(0, x)$ and so the evaluation of the lift of $-\eta_z$ at 1 sits over $(0, x)$. Since all the maps here are continuous, $\tau$ is continuous.

We need to supply a continuous inverse to $(\mathrm{pr}_1 \circ \pi, \tau)$, which is built the same way, except now using $\eta_z$ itself to lift, rather than $-\eta_z$:

$$\sigma\colon I \times Z_0 \to (I \times X)^I \times_{I\times X} Z \xrightarrow{\text{Lift}} Z^I \xrightarrow{\mathrm{ev}_1} Z.$$

This is manifestly continuous, and one can check that this map is the required inverse by considering the composite at each point separately, where it reduces to considering $Z$ restricted to $I \times \{x\}$. $\square$

**Corollary 6.** If $f, g\colon X \to Y$ are homotopic, say by $H\colon I \times X \to Y$, and $Z \to Y$ is a covering space, then $f^*Z \simeq g^*Z$ over $X$.

*Proof.* If we form $H^*Z \to I \times X$, then we have by Proposition 14 that $H^*Z \simeq I \times f^*Z$. But $g^*Z \to X$ is (isomorphic to) $(H^*Z)_{\{1\}\times X}$, hence is isomorphic to $(I \times f^*Z)_{\{1\}\times X}$, but this is isomorphic to $f^*Z$. $\square$

This gives us a criterion whereby we know that no interesting covering spaces exist

**Corollary 7.** If $X$ is contractible, then every covering space $Z \to X$ is isomorphic to $X \times Z_x$ for any $x \in X$.

*Proof.* Let $H\colon I \times X \to X$ be a contraction to $x \in X$. Then for $c_x\colon X \to X$ the constant map at $x$, $c_x^*Z = X \times Z_x$. But $H$ is a homotopy between $\mathrm{id}_X$ and $c_x$, and $\mathrm{id}^* Z = Z$, so by Corollary 6 we have the required isomorphism. $\square$

Exercise: such a contraction exists for all $x \in X$

**Example 35.** Any locally convex topological vector space has no interesting covering spaces, likewise any convex or even star-shaped region therein. The unit sphere in a separable, infinite-dimensional Hilbert space has no interesting covering spaces. The infinite-dimensional Stiefel manifolds likewise.

**Corollary 8.** Let $Z \xrightarrow{\pi} Y$ be a covering space, $f, g\colon X \to Y$ a pair of maps and $H\colon I \times X \to Y$ a homotopy from $f$ to $g$. If $\widetilde{f}\colon \{0\} \times X \to Z$ is a lift of $f$, in the sense that the diagram at right commutes, then there is a unique homotopy $\widetilde{H}\colon I \times X \to Z$ lifting $H$ from $\widetilde{f}$ to a lift of $g$.

$$\begin{tikzcd}
\{0\} \times X \arrow[r, "\widetilde{f}"] \arrow[d] & Z \arrow[d, "\pi"] \\
I \times X \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & Y
\end{tikzcd}$$

*Proof.* Since $I \times f^*Z \xrightarrow{\simeq} H^*Z$, and we have a section $X \to f^*Z$, then we get a section $I \times X \to I \times f^*Z$. Composing with the isomorphism we get a map $I \times X \to I \times f^*Z \to H^*Z \to Z$, and this both restricts to $\widetilde{f}$ on $\{0\} \times X$ and covers $H$. To show uniqueness, notice that $H(-, x)$ gives a path in $Y$ for each fixed $x \in X$. Any lift $\widetilde{H}'$ of $H$ likewise gives a path $\widetilde{H}'(-, x)$ for fixed $x$. Since lifts of paths are unique, the $\widetilde{H}'(-, x)$ must agree with $\widetilde{H}(-, x)$ for all $x$, hence $\widetilde{H}' = \widetilde{H}$. $\square$

We can now give the promised proof of Theorem 4.

*Proof.* (of Theorem 4) Given paths $\gamma, \eta \colon z_1 \rightsquigarrow z_2$ in $Z$, and an endpoint-fixing homotopy $H \colon I \times I \to X$ between $\pi \circ \gamma$ and $\pi \circ \eta$, we can lift $H$ to give a homotopy from $\gamma$ to a lift of $\pi \circ \eta$. Since $H$ fixes endpoints, the lifts of the constant paths $H\big|_{I \times \{i\}}$, for $i = 0, 1$ are path in the fibre, discrete spaces. Hence these paths are constant, and $\widetilde{H}$ is a homotopy fixing endpoints. Since $\eta$ is a lift of $\pi \circ \eta$, unique path lifting gives that $\widetilde{H}$ is in fact a homotopy (fixing endpoints) from $\gamma$ to $\eta$. Thus $\gamma$ and $\eta$ give the same element in $\Pi_1(Z)(z_1, z_2)$, and the induced map in injective as required. $\square$

Until now, a lot of our resuls only give bounds on or estimates between the fibres of a covering space and the fundamental group of the base space. However, we can actually get an exact result, given a certain kind of covering space

**Theorem 5.** If $\pi \colon (Z, z) \to (X, x)$ is a covering space with $Z$ path connected, then

$$Z_x \simeq \pi_1(X, x)/\pi_1(Z, z),$$

as sets with $\pi_1(X, x)$-action.

Lecture 10

*Proof.* There is in fact a canonical isomorphism, induced in the following way. For group $G$ and any transitive $G$-set $S$, and a point $p \in S$, then the map $G \to S$, $g \mapsto g \cdot s$ induces a well-defined bijection $G/\mathrm{Stab}(s) \to S$, where $\mathrm{Stab}(s) < G$ is the subgroup of elements $g$ such that $g \cdot s = s$ (the *stabiliser subgroup*). Notice that for *any* subgroup $H < G$, $G/H$ inherits a $G$-action from the multiplication in $G$. And the bijection $G/\mathrm{Stab}(s) \to S$ is compatible with the $G$-actions.

For a group $G$, sets with a $G$-action will be called *G-sets*.

that is, *equivariant*

We apply this to the transitive $\pi_1(X, x)$-set $Z_x$, where we know the action is transitive as $Z$ is path-connected. This gives an isomorphism $\pi_1(X, x)/\mathrm{Stab}(z) \xrightarrow{\simeq} Z_x$, and it remains to identify $\mathrm{Stab}(z) < \pi_1(X, x)$. But note that if for some $[\gamma] \in \pi_1(X, x)$, $\gamma_*(z) = z$, this means that the lift $\widetilde{\gamma_z}$ beginning at $z$ also ends at $z$, so is a loop in $Z$. Thus $\mathrm{Stab}(z)$ consists of the homotopy classes of loops in $X$ that come from loops in $Z$, that is, $\mathrm{Stab}(z) = \pi_1(Z, z)$. $\square$

**Corollary 9.** If $\pi \colon (Z, z) \to (X, x)$ is a covering space with $Z$ simply-connected, then the map

$$\pi_1(X, x) \to Z_x$$

is an isomorphism of $\pi_1(X, x)$-sets.

*Proof.* Since $Z$ is simply-connected, $\pi_1(Z,z) = 1$, and so $\pi_1(X,x) \to Z_x$ is an isomorphism of sets with $\pi_1(X,x)$-action. □

**Example 36.** We now can say that $\pi_1(S^1,1)$ is not just infinite (see Example 31) but countable, since it is in bijection with the fibre $\mathbb{Z}$ of the simply-connected covering space $\mathbb{R} \to S^1$.

But even better, we have not just a bijection, but Corollary 9 gives a *faithful permutation representation*: given $[\gamma],[\eta] \in \pi_1(X,x)$, there is some $z \in Z_x$ such that $\gamma_*(z) \neq \eta_*(z)$, which is equivalent to $\pi_1(X,x) \to \mathrm{Aut}(Z_x)$ being injective. Thus we have represented the fundamental group of $(X,x)$ as a permutation group, where we can do more concrete computations.

**Corollary 10.** For $(Z,z) \to (X,x)$ a simply-connected covering space, $\pi_1(X,x)$ acts freely on $Z_x$.

And now we can give the first example of an actually calculated, non-trivial fundamental group.

**Theorem 6.** $\pi_1(S^1,1) \simeq \mathbb{Z}$.

*Proof.* We have the simply-connected covering space $\mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$, with fibre over $1 \in S^1$ being the integers. The inclusion $[0,1] \to \mathbb{R}$ is a lift of the loop $\gamma$ going once around the circle, and all lifts are translates of this, so that the action of $\gamma$ on the fibre $\mathbb{Z}$ is translation by 1. The loop $\gamma$ generates a subgroup whose action on $\mathbb{Z}$ is transitive, hence $\gamma$ generates all of $\pi_1(S^1,1)$, which must then be infinite cyclic, hence $\mathbb{Z}$. □

As a result, for any subset $A \subset S^1$, the fundamental groupoid $\Pi_1(S^1,A)$ has as objects the set $A$, for every $x \in A$, $\pi_1(S^1,x) \simeq \mathbb{Z}$, and for any two points $x,y \in S^1$, the hom-set $\Pi_1(S^1,A)(x,y)$ is isomorphic as a set to $\mathbb{Z}$.

But how do we calculate $\pi_1$ in general? Or better, $\Pi_1$? Recall that $\Pi_1(X,A) = \Pi_1(X_1, A\cap X_1) \sqcup \Pi_1(X_2, A \cap X_2)$. For instance, if $X_1$ and $X_2$ are the only path components of $X$, and $\exists x \in A \cap X_i$ for $i = 1,2$, then every point in $X$ is connected by a path to a point in $A$. This means the fundamental goups of the two path components are captured.

**Example 37.** Consider $\Pi_1(S^1 \sqcup S^1, 1 \sqcup 1)$, which is a groupoid with two objects, both of which have automorphism groups given by $\mathbb{Z}$.

So we are going to focus a bit on calculating the fundamental group(oid) for path connected spaces. The easiest way to make a new connected space from two other connected spaces $X, Y$, say, is to take a point in each, $x \in X$, $y \in Y$, and identify $x$ and $y$.

recall that we are taking spaces to be semilocally path connected, so that components and path components coincide

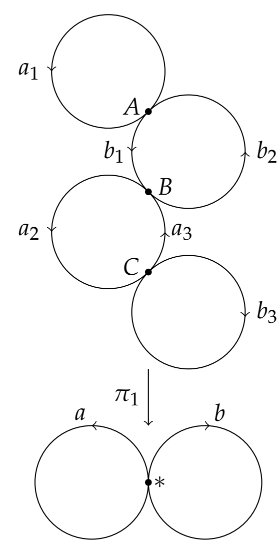
**Definition 27.** Given two pointed spaces $(X, x)$ and $(Y, y)$, the *join* $X \vee Y$ is the quotient space $(X \sqcup Y)/(x \sim y)$. It has a basepoint given by $* := [x] = [y]$, and the inclusion maps of $(X, x) \xrightarrow{\mathrm{in}_L} (X \vee Y, *) \xleftarrow{\mathrm{in}_R} (Y, y)$ are pointed.

A key property of the join is that given a pointed space $(M, m)$ and a pair of pointed maps $f\colon (X, x) \to (M, m)$, $g\colon (Y, y) \to (M, m)$, there is a unique pointed map $\langle f, g\rangle\colon (X \vee Y, *) \to (M, m)$ such that $f = \mathrm{in}_L \circ \langle f, g\rangle$ and $g = \mathrm{in}_R \circ \langle f, g\rangle$.

Since we have pointed maps, we get from functoriality of $\pi_1$ two homomorphisms $\pi_1(X, x) \to \pi_1(X \vee Y, *) \leftarrow \pi_1(Y, y)$. If we already know what the fundamental groups of $X$ and $Y$ are, then we can try to leverage this knowledge to tell us something about the fundamental group of the join. For instance, taking $X = Y = S^1$, we get homomorphisms

$$\mathbb{Z} \xrightarrow{\pi_1(\mathrm{in}_L)} \pi_1(S^\vee S^1, *) \xleftarrow{\pi_1(\mathrm{in}_R)} \mathbb{Z}$$

Let us define $a, b \in \pi_1(S^1 \vee S^1, *)$ to be the classes $\pi_1(\mathrm{in}_L)(1)$ and $\pi_1(\mathrm{in}_R)(1)$ respectively.




Define the covering space $Z_1 \xrightarrow{\pi_1} S^1 \vee S^1$ as at right, where $A, B, C \mapsto *$, and $a_i \mapsto a$, $b_i \mapsto b$, $i = 1, 2, 3$. Then we get a representation $\rho_1\colon \pi_1(S^1 \vee S^1, *) \to \mathrm{Aut}\{A, B, C\} \simeq S_3$. Looking at how paths representing $a$ and $b$ lift, we get $\rho_1(a) = (BC)$ and $\rho_1(b) = (AB)$, cycles in $S_3$. Calculating $\rho_1(ab)$ we get $(ABC)$, and similarly for $\rho_1(ba)$, to get $(ACB)$, so that $rho_1(ab) \neq \rho_1(ba)$. As a result, we must have had $ab \neq ba$ in $\pi_1(S^1 \vee S^1, *)$, or in other words, the fundamental group of $S^1 \vee S^1$ is **non-abelian**.

By a judicious choice of covering spaces, we can also prove that the two homomorphism $\mathbb{Z} \to \pi_1(S^1 \vee S^1, *)$ are injective, so that $a$ and $b$ generate infinite cyclic subgroups. We will later prove that $\pi_1(S^1 \vee S^1, *) \simeq \mathbb{Z} * \mathbb{Z} = F_2$, a free group on the generators $a, b$.

We can present $\mathbb{Z} * \mathbb{Z}$ as $\langle a, b | \,\rangle$, which has as elements $a^{n_1} b^{m_1} \ldots a^{n_k} b^{m_k}$ for $k \geq 1$ and $n_i, m_i \in \mathbb{Z}$, with $a^0 = e = b^0$

Lecture 11

**Definition 28.** The *free group on n-symbols*, $F_n$ is any group with presentation

$$\langle x_1, \ldots, x_n \mid \rangle.$$

That is, generators $x_1, \ldots, x_n$ and no relations.

The symbols are of course arbitrary. Elements in $F_n$ are (finite) words in $x_i$ and $x_i^{-1}$, with the empty word $()$ being the identity element, and with concatenation of words being the multiplication in $F_n$.

**Definition 29.** Given groups $G$ and $H$, the *free product* $G * H$ of $G$ and $H$ is a group equipped with homomorphisms $i\colon G \to G * H$, $j\colon H \to G * H$, satisfying the following property: given any group $K$ and homomorphisms $\phi\colon G \to K$, $\psi\colon H \to K$, there exists a unique homomorphism $\kappa\colon G * H \to K$ such that $\phi = i \circ \kappa$ and $\psi = j \circ \kappa$.

We can write things like this:

$$\begin{tikzcd} & H \arrow[d, "j"] \arrow[ddr, bend left, "\psi"] & \\ G \arrow[r, "i"] \arrow[drr, bend right, "\phi"'] & G * H \arrow[dr, dashed, "\exists!", "\kappa"'] & \\ & & K \end{tikzcd}$$

The existence of the unique $\kappa$ given the data of $\phi$ and $\psi$ is the *universal property* of the free product.

If $G = \langle g_1, \ldots, g_m \mid R_1, \ldots, R_n \rangle$ and $H = \langle h_1, \ldots, h_k \mid Q_1, \ldots, Q_l \rangle$ are presentations of $H$ and $G$, then

here each $R_i$ and $Q_j$ are *relations*: equations involving the given generators of $G$ and $H$ respectively

$$G * H = \langle g_1, \ldots, g_m, h_1, \ldots, h_k \mid R_1, \ldots, R_n, Q_1, \ldots, Q_l \rangle$$

The free product of groups is an example of a more general construction, the *free product with amalgamation*, but this is again an example of a general construction that makes sense in an arbitrary category.

**Definition 30.** Let $\mathcal{C}$ be an arbitrary category. A *pushout square* is a commutative square

$$\begin{tikzcd} W \arrow[r, "b"] \arrow[d, "a"'] & Y \arrow[d, "d"] \\ W \arrow[r, "c"'] & P \end{tikzcd}$$

is $\mathcal{C}$ such that for any pair of morphisms $X \xrightarrow{f} Z \xleftarrow{g} Y$ such that $f \circ a = g \circ d$,

this unique existence is the *universal property* of the pushout

$$\exists!\ P \xrightarrow{k} Z \quad \text{such that} \quad f = k \circ c \text{ and } g = k \circ d.$$

**Example 38.** Consider a topological space $X$, and $U, V \subseteq X$ subspaces such that the $\{U^o, V^o\}$ is an open cover of $X$. Then

$U$ and $V$ here are 'glued together' along $U \cap V$ to give $X$

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

is a pushout square in **Top**, where all maps are the inclusions.

In the above example, we call $\{U, V\}$ a cover of $X$ by nhds, since at least one of $U$ and $V$ is a nhd of each point in $X$.

**Example 39.** For any pair of pointed spaces $(X, x)$ and $(Y, y)$,

$$\begin{tikzcd} (\mathrm{pt}, \mathrm{pt}) \arrow[r] \arrow[d] & (Y, y) \arrow[d, "\mathrm{in}_R"] \\ (X, x) \arrow[r, "\mathrm{in}_L"'] & (X \vee Y, *) \end{tikzcd}$$

is a pushout square in $\mathbf{Top}_*$.

**Example 40.** For arbitrary groups $G$ and $H$,

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & H \arrow[d] \\ G \arrow[r] & G * H \end{tikzcd}$$

is a pushout square in **Grp**.

**Example 41.** Recall the groupoid **2** with two objects, 0 and 1 and a unique arrow between any ordered pair of objects. The square

$$\begin{tikzcd} \mathrm{disc}(\{0,1\}) \arrow[r] \arrow[d] & \mathbf{2} \arrow[d, "(0 \to 1) \mapsto (\bullet \xrightarrow{1} \bullet)"] \\ \mathrm{pt} \arrow[r] & \mathbb{B}\mathbb{Z} \end{tikzcd}$$

is a pushout in **Gpd**.

**Example 42.** Consider the category **Vect** of vector spaces (over some fixed field) and linear maps. The square

$$\begin{tikzcd} W \arrow[r, "L_2"] \arrow[d, "L_1"'] & V_2 \arrow[d] \\ V_1 \arrow[r] & (V_1 \oplus V_2)/J(W) \end{tikzcd}$$

with $J \colon W \to V_1 \oplus V_2$ the map $w \mapsto (L_1(w), -L_2(w))$ is a pushout.

**Theorem 7** (Seifert–van Kampen theorem)**.** Let $X$ be a space, and $\{U, V\}$ a cover by nhds. Then

$$\begin{tikzcd} \Pi_1(U \cap V) \arrow[r, "i_V"] \arrow[d, "i_U"'] & \Pi_1(V) \arrow[d] \\ \Pi_1(U) \arrow[r] & \Pi_1(X) \end{tikzcd}$$

is a pushout square in **Gpd**.

**Remark.** It is **not** immediate that this is a pushout just because the square of spaces is a pushout in **Top**, because we need to check the universal property for arbitrary groupoids $\Gamma$ and (compatible) functors $\Pi_1(U) \to \Gamma \leftarrow \Pi_1(V)$.

*Proof.* We need to start with an arbitrary commutative square

$$\begin{tikzcd} \Pi_1(U \cap V) \arrow[r, "i_V"] \arrow[d, "i_U"'] & \Pi_1(V) \arrow[d, "G"] \\ \Pi_1(U) \arrow[r, "F"'] & \Gamma \end{tikzcd}$$

and construct a functor $K\colon \Pi_1(X) \to \Gamma$ compatible with $F$ and $G$. That is, we need to construct a pair of functions $K_0\colon \Pi_1(X)_0 = X \to \Gamma_0$ and $K_1\colon \Pi_1(X)_1 \to \Gamma_1$ that together define a functor as needed.

Firstly, consider arbitrary $x \in X$. If $x \in U$, then define $K_0(x) = F(x)$, and if $x \in V$, define $K_0(x) = G(x)$. If $x \in U \cap V$, then since $F \circ i_U = G \circ i_V$, $F(x) = G(x)$, and so $K_0$ is well-defined.

We will first define $K_1$ on actual paths, and then show it is invariant under passing to homotopy classes. Suppose that $\gamma\colon I \to X$ factors through $U \hookrightarrow X$. Then we can define $K_1(\gamma) = F_1(\gamma)$, and similarly, if it factors through $V \hookrightarrow X$, then define $K_1(\gamma) = G_1(\gamma)$. Again, if $\gamma$ lands in $U \cap V$ then it is unambiguously defined, by the commutativity of the square as given. This is compatible with source and target maps, since the start- and end-points of a path in $U$ lie in $U$, and similarly for $V$, and $F$ and $G$ are functors. It is compatible with concatenation of paths that lie entirely inside $U$ or inside $V$, again using the fact $F$ and $G$ are functors. Constant paths are sent by $K_1$ to identity morphisms in $\Gamma$, as needed, since they are by $F$ and $G$. Also notice that if we reparametrise the path $\gamma$ to $\gamma \circ \sigma$, this gives an equal morphism in $\Pi_1(U)$ or $\Pi_1(V)$ as appropriate, so that $K_1$ is independent of the parametrisation of the path.

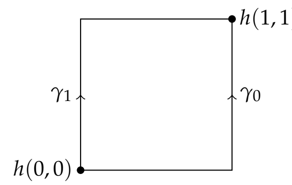
We now need to consider a general path $\gamma\colon I \to X$ and define $K_1(\gamma)$. If we pull back the open cover $\{U^o, V^o\}$ along $\gamma$ to an open cover of $I$, we can find a partition $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = 1$ of $I$ such that for each $i = 0, \ldots n$, $\gamma\big|_{[t_i,t_{i+1}]}$ factors through either $U \hookrightarrow X$ or $V \hookrightarrow X$ (or both). Define $\gamma_i\colon I \simeq [t_i, t_{i+1}] \to X$, so that $\gamma$ is homotopic to the concatenation of all the $\gamma_i$s, and in fact $\gamma$ is a reparametrisation of the concatenation. We have already defined $K_1(\gamma_i)$, so let $K_1(\gamma) = K_1(\gamma_0)K_1(\gamma_1)\cdots K_1(\gamma_n) \in \Gamma_1$. Note that by the compatibility of $K_1$ with concatenation *inside $U$ and $V$*, if we pass to a finer partition of $I$, we get a different sequence $\gamma_j$, but the

using the Lebesgue covering lemma

composite of the $K_1(\gamma_j)$s is equal to what we just defined. Since any two partitions have a common refinement, the definition of $K_1$ is independent of the choice of partition. Again, since the original given square commutes, there is no ambiguity when a given $\gamma_i$ factors through $U \cap V$.

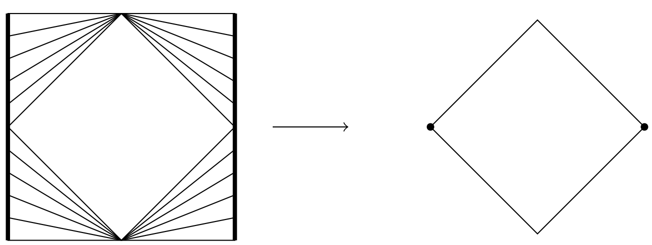
We now need to show that given an endpoint-fixing homotopy $H \colon I \times I \to X$ between paths $\gamma$ and $\eta$, then $K_1$ maps them both to the same morphism in $\Gamma$.

Consider as a warmup, an arbitrary map $h \colon I^2 \to X$, and define paths $\gamma_0, \gamma_1 \colon h(0,0) \rightsquigarrow h(1,1)$ in $X$ as the concatenations




$$\gamma_0 := h(-,0)\#h(1,-),$$
$$\gamma_1 := h(0,-)\#h(-,1).$$

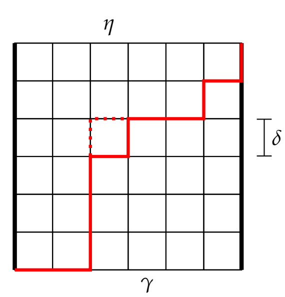
Then there is an endpoint-fixing homotopy $\gamma_0 \sim \gamma_1$. It is sufficient to define an endpoint-fixing homotopy $I \times I \to I^2$ between the two paths around the square that arise from taking $h$ to be the identity map $I^2 \to I^2$.



Here the function is constant on the vertical edges of the square at the two vertices, and on each diagonal line as shown maps to the corresponding edges of the square on the right. Thus if $h$ factors through one of $U$ or $V$, $K_1(\gamma_0) = K_1(\gamma_1)$, since $[\gamma_0] = [\gamma_1]$ in one of $\Pi_1(U), \Pi_1(V)$.

Lecture 12

By the Lebesgue covering lemma applied $(I^2, d_\infty)$ and the open cover $\{H^{-1}(U^o), H^{-1}(V^o)\}$, there is a some $\delta > 0$ such that every square of side-length $\leq \delta$ in $I^2$ (a *$\delta$-square*) is contained in one of $H^{-1}(U^o)$ and $H^{-1}(V^o)$. Thus $H|_{[a,a+\delta]\times[b,b+\delta]}$ factors through one of $U$ or $V$, for any suitable $(a,b) \in I^2$. We then cover $I^2$ by such $\delta$-squares, noting that this also give a partition of $I$ into intervals such that both $\gamma$ and $\eta$ restricted to such intervals factor through one of $U$ or $V$, so that $K_1$ is defined on $\gamma$ and $\eta$.




Now we can use the fact about paths between opposite vertices of the square being homotopic to iteratively show that $K_1(\gamma) = K_1(\eta)$. Firstly, note that $\gamma$ is homotopic to the path gotten by concatenating with the constant path up the right side of the square, and similarly, $\eta$ is homopic to the path gotten by concatenating with the constant path up the left side of the square. Then the big homotopy is pasted together from homotopies that move one square at a time, each of which land in one of $U$ or $V$. Then the two possible red paths shown in the figure, for example, get mapped by $K_1$ to the same morphism in $\Gamma$. All up, these show that $K_1(\gamma) = K_1(\eta)$, and so $K_1$ is well-defined on homotopy classes of paths. By the construction of $K_1$, it preserves composition, so is functorial, and we are done. $\square$

all homotopies here will have fixed endpoints

This is a powerful theorem, but sometimes not the best for computation, in this form. It would be good to have a version for more general $\Pi_1(X, A)$, for smaller $A \subset X$, or even $\pi_1(X, x)$. To do this, we need a general categorical lemma

Given an arbitrary category $\mathcal{C}$, we can define a category $\mathcal{C}^\square$ with objects commutative squares in $\mathcal{C}$, and morphisms commutative *cubes*: cubes of objects and morphisms such that every face is a commutative square.

$$\begin{tikzcd}
A_1 \arrow[rr] \arrow[dr] \arrow[dd] & & B_1 \arrow[dr] \arrow[dd] & \\
& A_2 \arrow[rr] \arrow[dd] & & B_2 \arrow[dd] \\
C_1 \arrow[rr] \arrow[dr] & & D_1 \arrow[dr] & \\
& C_2 \arrow[rr] & & D_2
\end{tikzcd}$$

A morphism from the back square to the front square in $\mathcal{C}^\square$

**Definition 31.** In a category $\mathcal{C}$, an object $V$ is a *retract* of an object $W$, if there are morphisms $i\colon V \to W$ and $r\colon W \to V$ such that $r \circ i = \mathrm{id}_V$.

the morphism $r$ is called a *retraction*

For example, in **Vect**, any subspace $V$ of $\mathbb{R}^n$ is a retract, by taking $i$ to be the inclusion, and $r$ to be orthogonal projection onto $V$. In **Set**, given a set $S$ and a subset $T \subseteq S$ with some chosen $t_0 \in T$ we get a retract given by the inclusion and the function $r\colon S \to T$ defined by $r(t) = t$, for $t \in T$, $r(s) = t_0$ for $s \in S \setminus T$. A more serious example is:

**Example 43.** Given a space $X$, with subspaces $A' \subseteq A$ such that every point in $A$ is connected by a path in $X$ to a point in $A'$. Then $\Pi_1(X, A')$ is a retract of $\Pi_1(X, A)$. The case we will most care about is $A = X$, and various $A' \subseteq X$.

This generalises the case from Assignment 2, where $A' = \{x\}$

We can talk about what it means for a commutative square in a category $\mathcal{C}$ to be a retract of another commutative square in $\mathcal{C}$, by looking at retracts in $\mathcal{C}^\square$. Recall that pushout squares are special examples of commutative squares. Also, to check that a morphism of commutative squares is a retraction, it is enough to check that it is a retraction at each vertex (that is, we have four retractions in $\mathcal{C}$, one for each vertex of the square)

**Lemma 13.** Retracts of pushout squares are pushout squares.

*Proof.* Exercise. □

We wish to apply Lemma 13 to the pushout square in **Gpd**—hence an object of $\mathbf{Gpd}^{\square}$—from the Seifert–van Kampen theorem, which involved fundamental groupoids $\Pi_1(X)$ etc. Retracts (in **Gpd**) as in Example 43 will be assembled to give a retract in $\mathbf{Gpd}^{\square}$ that is made up of smaller and more manageable groupoids.

**Theorem 8** (Relative Seifert–van Kampen theorem)**.** Let $X$ be a space, $\{U, V\}$ be a cover by nhds, and $A \subseteq X$ a given subspace. If in each of the four pairs $(X, A)$, $(U, A \cap U)$, $(V, A \cap V)$, $(U \cap V, A \cap U \cap V)$, every point in the larger space is connected by a path (in that space) to a point in the smaller space, then

so a point in $U$ is connected by a path in $U$ to a point in $A \cap U$, and so on

$$\begin{tikzcd}
\Pi_1(U \cap V, A \cap U \cap V) \arrow[r] \arrow[d] & \Pi_1(V, A \cap V) \arrow[d] \\
\Pi_1(U, A \cap U) \arrow[r] & \Pi_1(X, A)
\end{tikzcd}$$

is a pushout square in **Gpd**.

*Proof.* The hard work involving homotopies etc is already done, we just need to exhibit the square as shown as a retract in $\mathbf{Gpd}^{\square}$ of the pushout square in the statement of the Seifert–van Kampen theorem. By Example 43, each of the groupoids in the commutative square are, individually, retracts. The inclusion functors

$$\begin{aligned}
\Pi_1(U \cap V, A \cap U \cap V) &\hookrightarrow \Pi_1(U \cap V) \\
\Pi_1(U, A \cap U) &\hookrightarrow \Pi_1(U) \\
\Pi_1(V, A \cap V) &\hookrightarrow \Pi_1(V) \\
\Pi_1(X, A) &\hookrightarrow \Pi_1(X)
\end{aligned}$$

give a morphism in $\mathbf{Gpd}^{\square}$, so we just need to construct the functors

$$\begin{aligned}
\Pi_1(U \cap V, A \cap U \cap V) &\leftarrow \Pi_1(U \cap V) \\
\Pi_1(U, A \cap U) &\leftarrow \Pi_1(U) \\
\Pi_1(V, A \cap V) &\leftarrow \Pi_1(V) \\
\Pi_1(X, A) &\leftarrow \Pi_1(X)
\end{aligned}$$

that together give a morphism of commutative squares in the other direction. To do this, we will choose, for each $x \in X$, a (homotopy class of a) path $\eta_x \colon x \rightsquigarrow a_x$, for some $a_x \in A$, such that if $x \in U$, take $a_x \in A \cap U$ and $\eta_x$ a path in $U$; if $x \in V$, take $a_x \in A \cap V$ and $\eta_x$ a path in $V$; and hence if $x \in U \cap V$, it follows that $a_x \in A \cap U \cap V$ and

$\eta_x$ is a path in $U \cap V$. Further, if $x \in A$ already, take $a_x = x$, and $\eta_x$ the constant path.

The assignment $x \mapsto a_x$, and $(x \overset{\gamma}{\rightsquigarrow} y) \mapsto (a_x \rightsquigarrow x \rightsquigarrow y \rightsquigarrow a_y)$ gives a functor $\Pi_1(X) \to \Pi_1(X, A)$, and this is a retraction. By the specific choices of $a_x$ and $\eta_x$ we made, the restrictions of this functor to the groupoids $\Pi_1(U)$, $\Pi_1(V)$, $\Pi_1(U \cap V)$ land in the corresponding subgroupoids $\Pi_1(U, A \cap U)$ etc, and again give a retraction in each case. We can check that these do indeed give us a morphism in $\mathbf{Gpd}^{\square}$, which is enough to show we have a retraction in $\mathbf{Gpd}^{\square}$. $\square$

We would like to consider pushouts of groups, since these can be easier in some cases to compute. The statement of the relative Seifert–van Kampen theorem however involves pushouts of groupoids, so that even if we consider one-obect groupoids associated to groups we need to be careful that the universal property for the pushout in **Gpd** implies the universal property for the pushout in **Grp**. Thankfully, this is true, for abstract reasons.

**Lemma 14.** Let $\mathcal{C}$ be a category and let $\mathcal{D} \hookrightarrow \mathcal{C}$ be a full subcategory. Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & P \end{tikzcd}$$

be a commutative square in $\mathcal{D}$ that is a pushout square in $\mathcal{C}$. Then it is a pushout square in $\mathcal{C}$.

*Proof.* We will check the universal property for the pushout in $\mathcal{C}$. Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

be an arbitrary commutative square in $\mathcal{D}$. Then considering this as a commutative square in $\mathcal{C}$, we have a unique morphism $k \colon P \to D$ (in $\mathcal{D}$) compatible with the other data as in the definition of pushout square. But since $\mathcal{C}$ is a *full* subcategory, $k$ is a morphism in $\mathcal{C}$, and moreover the commuting triangles still commute in $\mathcal{C}$. Given any other morphism $P \to D$ in $\mathcal{C}$ making the triangles commute will be equal to $k$ in $\mathcal{D}$, and hence in $\mathcal{C}$, so the universal property for the pushout holds in $\mathcal{C}$. $\square$

Now we can use the fact that $\mathbb{B} \colon \mathbf{Grp} \to \mathbf{Gpd}$ expresses **Grp** as a full subcategory.

**Corollary 11.** Let $X$ be a path connected space, $\{U, V\}$ a cover by path connected subspaces with $U \cap V$ path connected. For $x \in U \cap V$, the square

$$\begin{tikzcd} \pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(V, x) \arrow[d] \\ \pi_1(U, x) \arrow[r] & \pi_1(X, x) \end{tikzcd}$$

is a pushout square in **Grp**.

*Proof.* The hypotheses on $X$, $U$, $V$ and $x$ imply that the condition of the relative Seifert–van Kampen theorem hold, so that we have a pushout of one-object groupoids. But by the above lemma, we get a pushout of groups. □

So we need to know what pushouts of groups look like!

**Example 44.** Consider the cover of the sphere $S^n$, where $n > 1$, by $U = S^n \setminus \{N\}$ and $V = S^n \setminus \{S\}$, where $N$ and $S$ are a pair of antipodal points (North and South poles). Then $U \cap V \simeq S^{n-1} \times (-1, 1)$, and all these spaces are path connected, so we can apply the group version of Seifert–van Kampen. Take a basepoint $x \in S^{n-1} \subset U \cap V$. Using stereographic projection, we get that $U \simeq \mathbb{R}^n \simeq V$, hence both of these are contractible, and so $\pi_1(U, x) = 1 = \pi_1(V, x)$ are both the trivial group. Then by Seifert–van Kampen we know that

$$\begin{tikzcd} \pi_1(S^{n-1} \times (-1, 1), x) \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & \pi_1(S^n, x) \end{tikzcd}$$

is a pushout square. If we take an arbitrary group $K$ then to check the universal property, the data of the homomorphisms $1 \to K \leftarrow 1$ tells us nothing, the compatibility being automatically satisfied, so we need $\pi_1(S^n, x)$ to be a group such that there is a *unique* homomorphism from it to $K$. But the only group that has a unique homomorphism to any other group is the trivial group. Thus $\pi_1(S^n, x) = 1$ for all $n > 1$.

This argument fails for $n = 1$ since the cover as constructed in that case results in the intersection $U \cap V$ being the disjoint union of two intervals, so not path connected. Lecture 13

**Definition 32.** Let $G \xleftarrow{\phi} L \xrightarrow{\psi} H$ be a pair of homomorphisms. The *free product with amalgamation* $G *_L H$ is the group $G * H / \langle \phi(x)\psi(x)^{-1} \rangle$, where $\langle \phi(x)\psi(x)^{-1} \rangle$ is the smallest normal subgroup generated by the elements $\phi(x)\psi(x)^{-1}$ for all $x \in L$. There are homomorphisms $G \to G *_L H \leftarrow H$, and $G *_L H$ satisfies the universal property of the pushout in **Grp**.

Note that if $G = \langle g_1, \dots, g_m \mid R_1, \dots, R_n \rangle$ and $H = \langle h_1, \dots, h_k \mid Q_1, \dots, Q_l \rangle$, then

this description also works for groups that aren't finitely presented

$$G *_L H \simeq \langle g_1, \dots, g_m, h_1, \dots, h_k \mid R_1, \dots R_n, Q_1, \dots, Q_l, \phi(x)\psi(x)^{-1} = e \rangle$$

where we add a new relation for each $x \in L$, or even just each $x$ running through a set of generators for $L$. Note that these relations are equivalent to $\phi(x) = \psi(x)$, so that we do indeed get a commutative square.

**Example 45.** Consider a *finitely generated one-relator group* $G = \langle g_1, \dots, g_m \mid R = e \rangle$ ($R$ is an element of the free group generated by $g_1, \dots, g_m$). Such a group is a pushout of the form

Such groups are important in geometric group theory, and much is known about them

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & 1 \\ {\scriptstyle r}\downarrow & & \downarrow \\ F_m & \longrightarrow & G \end{array}$$

where $R = r(1)$.

For a more specific example, take the *surface group*

$$\langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle.$$

More generally, one can write a finitely presented group as a pushout

$$\begin{array}{ccc} F_n & \longrightarrow & 1 \\ {\scriptstyle r}\downarrow & & \downarrow \\ F_m & \longrightarrow & \langle g_1, \dots, g_m \mid r(a_1) = e, \dots r(a_n) = e \rangle \end{array}$$

where we take $F_n \langle a_1, \dots, a_n \mid \rangle$.

**Remark.** Going back to free products, for a moment, a famous example is the free product $\mathbb{Z}/2 * \mathbb{Z}/3$, which is isomorphic to the *modular group*

$$PSL_2(\mathbb{Z}) = \{2 \times 2 \text{ integer matrices } A \mid \det(A) = 1\}/\{\pm I\}$$

One presentation of $PSL_2(\mathbb{Z})$ is via the generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $ST = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, which satisfy $S^2 = I$ and $(ST)^3 = I$. Note that $PSL_2(\mathbb{Z})$

It is not obvious that this even is a presentation, for a proof see Roger C. Alperin, $PSL_2(\mathbf{Z}) = \mathbf{Z}_2 * \mathbf{Z}_3$, The American Mathematical Monthly Vol. 100, No. 4 (Apr., 1993), pp. 385–386, doi:10.2307/2324963 acts by fractional linear transformations on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$, with $S\colon z \mapsto \frac{-1}{z}$ and $ST\colon z \mapsto \frac{1-z}{z}$. This action is continuous and has discrete orbits, and this is enough to make $\mathcal{H} \to \mathcal{H}/PSL_2(Z)$ a covering space.

Give the concrete treatment for the pushout of groups above (that is, as free products with amalgamation), one could hope for a similar treatment for groupoids. And indeed, one can do this, where instead of group elements being (equivalence classes of) words in the elements of the given groups, morphisms of the pushout groupoid are (equivalence classes of) words in the morphisms of the given groupoids. However, we need to be careful about what we mean by words constructed as a string of morphisms, since not all morphisms can be composed.

We will not give the most general treatment here, but show how to describe the pushout of groupoids in a special case corresponding to a situation arising from an application of the Seiert–van Kampen theorem.

**Example 46.** Let $X$ be a space, $\{U, V\}$ a cover by nhds, and $A \subseteq U \cap V$ be such that every path component of $U$, $V$ and $U \cap V$ contains at least one point in $A$. Then we can apply the Seifert–van Kampen theorem and get a pushout square

$$\begin{tikzcd} \Pi_1(U \cap V, A) \arrow[r, "i_V"] \arrow[d, "i_U"'] & \Pi_1(V, A) \arrow[d] \\ \Pi_1(U, A) \arrow[r] & \Pi_1(X, A) \end{tikzcd}$$

Note that all four groupoids have the same set of objects, and that all the functors are the identity on objects (that is: $i_U(a) = a$ and so on). From the proof of the Seifert–van Kampen theorem recall that we expressed paths in $X$, that is, morphisms in $\Pi_1(X)$ as a composite of paths alternating between $U$ and $V$. This is the setup we are interested in calulating in general from a purely algebraic point of view. For simplicity, we will just think about the case of $A$ finite, which is the case that turns up in calculations of ‘reasonable’ examples.

Suppose we are given a diagram

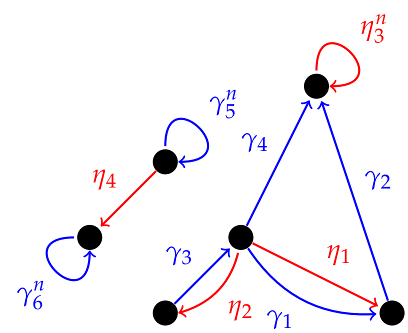
here $H$ is the capital $\eta$

$$\begin{tikzcd} \Lambda \arrow[r, "F"] \arrow[d, "G"'] & H \\ \Gamma & \end{tikzcd}$$

in **Gpd**, where all the groupoids have the same finite set $A = \{a_1, \ldots, a_N\}$ of objects, and such that the object components of the functors $F$ and $G$ are all the identity function. We wish to construct a groupoid $\Gamma *_\Lambda H$ that makes this into a pushout square. Firstly, we can take the set of objects to be $A$ again, and the functors $\Gamma \to \Gamma *_\Lambda H \leftarrow H$ will have as object component the identity function.

Given any groupoid there is a directed graph with nodes the objects of the groupoid, and as directed edges the morphism (and we are allowed edges from a node to itself, and multiple edges between nodes). And given our two groupoids $\Gamma$ and $H$, we can form a graph $\mathcal{G}$ with set of nodes $A$, and the directed edges are the *disjoint union* of the morphisms of $\Gamma$ (coloured blue) and $H$ (coloured red), and with the identity morphisms removed. We also don't need to include both a morphism and its inverse, since the inverse can be gotten by traversing a directed edge against the indicated direction.

and indeed any category




in the graph shown, composites of various morphisms are omitted, for instance $\gamma_3\gamma_1\gamma_2$

Now instead of a word in group elements, as in the pushout of groups, we take a *path* in this directed graph, alternating between edges that come from $\Gamma$ and edges that come from $H$. For instance, we could take

$$\gamma_3\eta_1\gamma_2(\eta_3)^5\gamma_4^{-1} \qquad \text{or} \qquad \eta_4(\gamma_6)^{-3}\eta_4^{-1}$$

from the above graph. The 'empty' path consisting just of an identity arrow is also an option. However, we haven't yet actually constructed $\Gamma *_\Lambda H$, merely what we might call $\Gamma *_{\Lambda_0} H$, which is the pushout where the groupoid $\Lambda$ is replaced by the trivial groupoid $\mathrm{disc}(\Lambda_0)$ with the same objects but only identity arrows. What we need to do is add 'relations', namely extra equalities between morphisms in $\Gamma *_{\Lambda_0} H$. What this means is that for each morphism $\lambda$ in $\Lambda$, we identify the morphisms $F(\lambda)$ and $G(\lambda)$, or more precisely, quotient by the equivalence relation on each hom-set of $\Gamma *_{\Lambda_0} H$ generated by these identifications. For instance, if in the above graph, $\gamma_1 = F(\lambda_1)$

if $\Lambda$ is already trivial in this sense, then we are done; this is the analogue of the free product of groups and $\eta_1 = G(\lambda_1)$, then we add the equality $\gamma_1 = \eta_1$. This would have the effect of making

$$\gamma_3\eta_1\gamma_2(\eta_3)^5\gamma_4^{-1} = (\gamma_3\gamma_1\gamma_2)(\eta_3)^5\gamma_4^{-1}$$

Concatenation of strings and simplifying is the composition in $\Gamma *_\Lambda H$.

**Exercise 13.** Prove that this construction gives a groupoid $\Gamma *_\Lambda H$.

If we are interested in merely looking at the group of morphisms from a single object $a_i$ to itself, which is the case when calculating a fundamental group using the groupoid Seifert–van Kampen, then we should look at paths that start and finish at the chosen $a_i$.

**Example 47.** Let us look at an example, arising from an application of Seifert–van Kampen. Consider the circle $S^1$ as sitting in $\mathbb{C}$, and let $U = S^1 \setminus \{-i\}$, $V = S^1 \setminus \{i\}$. All three of these are path connected, and let us take $A = +1, -1 \subset U \cap V$. This choice of data satisfies the hypotheses of SvK. The pushout square

$$\begin{tikzcd} \Pi_1(U \cap V, \{\pm 1\}) \arrow[r] \arrow[d] & \Pi_1(V, \{\pm 1\}) \arrow[d] \\ \Pi_1(U, \{\pm 1\}) \arrow[r] & \Pi_(S^1, \{\pm 1\}) \end{tikzcd}$$

can be simplified as follows. First, $U \simeq (-2, 2) \simeq V$, in a way that preserves $A = \{\pm 1\}$, so that

$$\Pi_1(U, \{\pm 1\}) \simeq \mathbf{2} = (-1 \underset{\gamma^{-1}}{\overset{\gamma}{\leftrightarrows}} +1) \qquad \Pi_1(V, \{\pm 1\}) \simeq \mathbf{2} = (-1 \underset{\eta}{\overset{\eta^{-1}}{\leftrightarrows}} +1)$$

and we have omitted the identity arrows. Here $\gamma$ is a path that runs anticlockwise around $S^1$ from $+1$ to $-1$, and $\eta$ is a path that runs anticlockwise from $-1$ to $+1$. Second, $\Pi(U \cap V, \{\pm 1\}) = \mathrm{disc}(\{+1, -1\})$, so we are in the easier situation as first outlined above, where no additional quotient needs to be done. The pushout then looks like

$$\begin{tikzcd} \boxed{-1 \quad +1} \arrow[r] \arrow[d] & \boxed{-1 \underset{\eta}{\overset{\eta^{-1}}{\leftrightarrows}} +1} \arrow[d] \\ \boxed{-1 \underset{\gamma^{-1}}{\overset{\gamma}{\leftrightarrows}} +1} \arrow[r] & \Pi_1(S^1, \{\pm 1\}) \end{tikzcd}$$

The graph we need so as to generate $\Pi_1(S^1, \{\pm 1\})$ is then

$$\begin{tikzcd} -1 \arrow[r, bend right, "\gamma"'] & +1 \arrow[l, bend right, "\eta"'] \end{tikzcd}$$

Then if we wish to consider paths from $+1$ to itself, the only options are the empty path, hence the identity arrow, or $(\eta\gamma)^n$, or $(\gamma^{-1}\eta^{-1})^n = (\eta\gamma)^{-n}$. Thus $\pi_1(S^1, +1) \simeq \mathbb{Z}$.

**Exercise 14.** Prove that this construction of $\Gamma *_\Lambda H$ is indeed the pushout in **Gpd**!

We will consider one more variant of Seifert–van Kampen, and this time in brief, because the details are similar to other versions. If we would like to compute the fundamental group of a join, then it is not quite enough to just consider the free product of the fundamental groups: a join does not automatically come with a cover of the sort we need for SvK. In what follows, assume: $(X, x)$, $(Y, y)$ are pointed spaces such that there exist nhds $x \in U \subseteq X$ and $y \in V \subseteq Y$ that are contractible to $x$ and $y$ respectively, with the contraction fixing the basepoint. Then $\{X \vee V, U \vee Y\}$ is a cover of $X \vee Y$ by nhds. We have retractions $X \vee V \to X$ and $U \vee Y \to Y$ that preserve the basepoints and which are also homotopy equivalences. Thus $\pi_1(X \vee V, *) \simeq \pi_1(X, x)$ and $\pi_1(U \vee Y, *) \simeq \pi_1(Y, y)$, and $\pi_1(U \vee V, *)$ is trivial. If we apply the Seifert–van Kampen theorem to the pushout square

this implies $U \vee V$ is contractible to the basepoint $* = [x] = [y]$

$$\begin{tikzcd} (U \vee V, *) \arrow[r] \arrow[d] & (U \vee Y, *) \arrow[d] \\ (X \vee V, *) \arrow[r] & (X \vee Y, *) \end{tikzcd}$$

then we get a pushout square of groups

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \pi_1(Y, y) \arrow[d] \\ \pi_1(X, x) \arrow[r] & \pi_1(X \vee Y, *) \end{tikzcd}$$

or in other words,

$$\pi_1(X \vee Y, *) = \pi_1(X, x) * \pi_1(Y, y).$$

This generalises the fact that $\pi_1(S^1 \vee S^1, *) = F_2 = \mathbb{Z} * \mathbb{Z}$ to more general spaces.

**Fact.** Given any presentation $G = \langle g_1, \ldots, g_m \mid R_1 = e, \ldots, R_n = e \rangle$ there is a space $X$ arising as a pushout

$\bigvee_{j=1}^m S^1 = \underbrace{S^1 \vee \ldots \vee S^1}_{m \text{ times}}$

$$\begin{tikzcd} \coprod_{i=1}^n S^1 \arrow[r] \arrow[d, "\langle f_1,\dots,f_n\rangle"'] & \coprod_{i=1}^n D^2 \arrow[d] \\ \bigvee_{j=1}^m S^1 \arrow[r] & X \end{tikzcd}$$

where $f_i(1) = R_i$, with the property that $\pi_1(X,*) \simeq G$. This space is in some sense 2-dimensional as it is gotten by gluing together 2d discs, and sometimes a manifold, though not always.

Even better, for *any* group $G$ with any presentation, there is an appropriate pushout

$$\begin{tikzcd} \coprod_{\beta\in J} S^1 \arrow[r] \arrow[d, "\langle f_\beta\rangle"'] & \coprod_{\beta\in J} D^2 \arrow[d] \\ \bigvee_{\alpha\in I} S^1 \arrow[r] & X \end{tikzcd}$$

with the property that $\pi_1(X,*) \simeq G$. One has to be careful with the topology, and we haven't defined infinite joins, but it does work out that $\pi_1(\bigvee_{\alpha\in I} S^1, *) \simeq F_I$, the free group on the set $I$.

the coutable join $\bigvee_{n\in\mathbb{N}} S^1$ may seem like the Hawaiian earring, but it is in fact not compact, and *is* slsc, so they cannot be homeomorphic

**Example 48.** An oriented compact Riemann surface $\Sigma_g$ of genus $g \geq 1$ is an example of a surface gotten by a construction as in the Fact above, and even better: it only requires one copy of $D^2$. For genus $g$, it requires doing a pushout of the form

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \\ \bigvee_{i=1}^{2g} S^1 \arrow[r] & \Sigma_g \end{tikzcd}$$

An alternative way to build this pushout is to consider a $4g$-gon, and selectively identify edges in pairs (recovering the $2g$ circles as in the pushout). The pattern of identifications is exactly that which gives rise to the one-relator group after Example 45, since the *attaching map* $S^1 \to \bigvee_{i=1}^{2g} S^1$ in the preceeding pushout is given by $\prod_{i=1}^{g}[a_i, b_i]$, for $a_i$ and $b_i$ the generators of the $2i-1$th and $2i$th copy of $S^1$ respectively.

Insert octagon picture for case $g = 2$ here

## *Classifying covering spaces*

Recall that for a covering space $Z \xrightarrow{\pi} X$ we get a representation

Lecture 14

$$\begin{aligned} \rho_Z \colon \Pi_1(X) &\longrightarrow \mathbf{Set} \\ x &\mapsto Z_x \\ [\gamma\colon x \rightsquigarrow y] &\mapsto \left(\gamma_*\colon Z_x \xrightarrow{\simeq} Z_y\right) \end{aligned}$$

of the fundamental groupoid of $X$. Given a pair of covering spaces $Z_1, Z_2 \to X$ and a map between them in $\mathbf{Cov}_X$, how are the representations $\rho_{Z_1}$ and $\rho_{Z_2}$ related? Since the triangle at right commutes, we get for each $x \in X$ a function between the corresponding fibres, $f\big|_x \colon (Z_1)_x \to (Z_2)_x$. Notice that this is a function $\rho_{Z_1}(x) \to \rho_{Z_2}(x)$ for each object of $\Pi_1(X)$. Given a path $\gamma \colon x \rightsquigarrow y$ in $X$ and $z \in (Z_1)_x$, we have the unique lift to $Z_1$ starting at $z$, namely $\widetilde{\gamma_z}^{1} \colon z \rightsquigarrow \gamma_*(z)$. By composing with $f$, we get a path $f \circ \widetilde{\gamma_z}^{1} \colon I \to Z_2$ from $f(z)$ to $f(\gamma_*(z))$. But by uniqueness of lifts of paths, this is the lift of $\gamma$ to $Z_2$ starting at $f(z)$, which is a path from $f(z) \rightsquigarrow \gamma_*(f(z))$. We thus get $f(\gamma_*(z)) = \gamma_*(f(z))$ for every $z \in (Z_1)_x$, implying that the following square commutes:

$$\begin{tikzcd}
Z_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & Z_2 \arrow[dl, "\pi_2"] \\
& X &
\end{tikzcd}$$

$$\begin{tikzcd}
(Z_1)_x \arrow[r, "\gamma_*"] \arrow[d, "f|_x"'] & (Z_1)_y \arrow[d, "f|_y"] \\
(Z_2)_x \arrow[r, "\gamma_*"'] & (Z_2)_y
\end{tikzcd}$$

Or, in other words, the functions $f\big|_x$ define a natural transformation $\rho_{Z_1} \Rightarrow \rho_{Z_2}$. This leads us to

**Proposition 15.** The mapping $(Z \xrightarrow{\pi} X) \mapsto \rho_Z$ define a functor

$$\mathbf{Cov}_X \longrightarrow [\Pi_1(X), \mathbf{Set}].$$

Recall that the category $[\mathcal{C}, \mathbf{Set}]$ has as objects the functors $\mathcal{C} \to \mathbf{Set}$, and as morphisms the natural transformations

*Proof.* The natural transformation $\rho_Z \Rightarrow \rho_Z$ associated to the identity map $Z \to Z$ has a components the identity functions $Z_x \to Z_x$, hence is the identity natural transformation. Also, using uniqueness of path lifting one can show that given two composable maps of covering spaces, we get functoriality. □

**Example 49.** Regard $S^1 \subset \mathbb{C}$. Recall the covering spaces $\exp(2\pi i(-)) \colon \mathbb{R} \to S^1$ and $S^1 \xrightarrow{(-)^n} S^1$, for $n > 1$. There is a map of covering spaces

$$\begin{tikzcd}
\mathbb{R} \arrow[rr, "\exp(2\pi i(-)/n)"] \arrow[dr, "\exp(2\pi i(-))"'] & & S^1 \arrow[dl, "(-)^n"] \\
& S^1 &
\end{tikzcd}$$

The fibres of $\mathbb{R} \to S^1$ are isomorphic to $\mathbb{Z}$, and indeed the fibre over $1 \in S^1$ *is* $\mathbb{Z}$. The fibre over 1 of $S^1 \to S^1$ is $\mathbb{Z}/n$, and we get the induced map

$$\begin{aligned}
\mathbb{Z} &\to \mathbb{Z}/n \\
k &\mapsto k \pmod n
\end{aligned}$$

Now note that if we focus on the point $1 \in S^1$, then a morphism in $\Pi_1(S^1)$ from 1 to itself is the homotopy class of some loop, which can identify with an integer under $\pi_1(S^1, 1) \simeq \mathbb{Z}$. Then the representation associated to $\mathbb{R} \to S^1$ is

$$\begin{aligned} \rho_{\mathbb{R}} \colon \Pi_1(S^1) &\longrightarrow \mathrm{Aut}(\mathbb{Z}) \\ m &\mapsto (k \mapsto k + m) \end{aligned}$$

whereas the representation associated to $(-)^n \colon S^1 \to S^1$ is

$$\begin{aligned} \rho_n \colon \Pi_1(S^1) &\longrightarrow \mathrm{Aut}(\mathbb{Z}/n) \simeq S_n \\ m &\mapsto (k \mapsto k + m \pmod{n}) \end{aligned}$$

The map $\mathbb{Z} \to \mathbb{Z}/n$ is clearly equivariant for the shift action of $\mathbb{Z}$ as shown, as a special case of the naturality of the map $\mathbb{Z} \to \mathbb{Z}/n$.

**Question 3.** What representations $\Pi_1(X) \to \mathbf{Set}$ can arise as $\rho_Z$ for some covering space $Z \to X$?

Recall that it is obvious that every set is the set of connected components of some space, and while we didn't go into details, the construction of a space $X$ with $\pi_1(X, \mathrm{pt}) \simeq G$ for any given $G$ is a relatively uncomplicated pushout. However, given a representation $\Pi_1(X) \to \mathbf{Set}$, it is not immediately obvious how to build a covering space giving rise to it. Indeed, all the covering spaces we have seen so far are either natural examples we happen to have seen, or special toy cases chosen to illustrate some small aspect of the fundamental groupoid of a particularly simple space. We are going to look at doing some reductions to simpler cases, on both sides (the topological, $\mathbf{Cov}_X$ and the algebraic, $[\Pi_1(X), \mathbf{Set}]$) to make the task easier.

First, since we have the blanket assumption that out spaces are slpc, we can consider finding a section to the continuous map $X \to [\mathrm{pt}, X]$, namely $A \colon [\mathrm{pt}, X] \to X$, that picks out one point $a_i$ per path component $X_i \subseteq X$. We have the full subgroupoid inclusion $\Pi_1(X, A) \hookrightarrow \Pi_1(X)$ that is additionally an equivalence. Let us denote by $I$ the set $[\mathrm{pt}, X]$ in what follows.

by an argument as in the solutions for assignment 2, question 7

**Lemma 15.** If $i \colon \mathcal{C} \hookrightarrow \mathcal{D}$ is a full subcategory inclusion that is also an equivalence, then the restriction map

$$\begin{aligned} [\mathcal{D}, \mathbf{Set}] &\xrightarrow{i^*} [\mathcal{C}, \mathbf{Set}] \\ F &\mapsto F \circ i \end{aligned}$$

is an equivalence of categories.

*Proof.* Exercise. $\square$

Now notice also that $\Pi_1(X, A) = \Pi_1(\bigsqcup_{i\in I} X_i, \bigsqcup_{i\in I}\{a_i\}) \simeq \bigsqcup_{i\in I} \mathbb{B}\pi_1(X_i, a_i)$

**Lemma 16.** For any family of categories $(\mathcal{C}_i)_{i\in I}$ we have an isomorphism

$$[\bigsqcup_{i\in I} \mathcal{C}_i, \mathbf{Set}] \xrightarrow{\simeq} \prod_{i\in I}[\mathcal{C}_i, \mathbf{Set}].$$

An object of a product of family of cateories is a tuple of objects, one from each of the categories, and similar with the morphisms

*Proof.* Exercise. □

Given a group $G$, we can define the category $G\mathbf{Set}$ which has as objects sets $S$ equipped with a $G$-action, $G \to \mathrm{Aut}(S)$, and with morphisms *equivariant* functions. Note that for any groupoid $\Gamma$ and representation $\rho\colon \Gamma \to \mathbf{Set}$, for each object $x \in \Gamma_0$ there is a permutation representation $\Gamma(x, x) \to \mathrm{Aut}(\rho(x))$. To any natural transformation $\rho \Rightarrow \rho'$ between representations, there is an equivariant map between $\Gamma(x, x)$-sets, and this is functorial.

An equivariant function $f\colon S \to T$ satisfies $f(g \cdot p) = g \cdot f(p)$ for all $p \in S$

**Lemma 17.** The functor just described gives an isomorphism $[\mathbb{B}G, \mathbf{Set}] \xrightarrow{\simeq} G\mathbf{Set}$ of categories

We can put all of these lemmas together and get an equivalence of categories

$$[\Pi_1(X), \mathbf{Set}] \to [\Pi_1(X, A), \mathbf{Set}] \simeq \prod_{i\in I}[\mathbb{B}\pi_1(X_i, a_i), \mathbf{Set}] \simeq \prod_{i\in I} \pi_1(X_i, a_i)\mathbf{Set}.$$

We can compose this with the original functor we were looking at, from covering spaces to representations, to get

$$\begin{aligned} \mathbf{Cov}_X &\to \prod_{i\in I} \pi_1(X_i, a_i)\mathbf{Set} \\ (Z \to X) &\mapsto (\rho_i\colon \pi_1(X_i, a_i) \to Z_{a_i})_{i\in I} \end{aligned}$$

where now the codomain is much more tractable. Further, the objects in categories of the form $G\mathbf{Set}$ are not unreasonable: we can break them down into smaller parts. Each object $\rho\colon G \to \mathrm{Aut}(S)$ in $G\mathbf{Set}$ isomorphic to one of a particularly nice form, namely $S \simeq \bigsqcup_{j\in S/G} G/\mathrm{Stab}(p_j)$, where the points $p_j \in S$ are chosen so that there is one in each orbit of the $G$-action.

**Remark.** From now on, we will consider only spaces that are slsc, since this will ultimately be the case in the classification theorem, and also because $X$ slsc implies that for every covering space $Z \to X$, the space $Z$ is locally path connected, so that path components and components agree.