Algebraic Topology[1]

David Michael Roberts

2019

[1] This document is released under a CC-By license: `creativecommons.org/licenses/by/4.0/`.

Lecture 1

What is it?

Algebraic topology is the study of maps

$$\{\text{'Spaces'}\} \longrightarrow \{\text{Algebraic objects}\}$$

Or rather, 'well-behaved' such maps—they should also send maps between spaces to algebraic maps, respecting composition (so: *functors*); they should send spaces built out of smaller units to algebraic objects built out of smaller units, in a compatible way, etc.

Here, 'Spaces' roughly means topological spaces up to deformation (usually homotopy, but not always). Such equivalence classes are called *homotopy types*. 'Algebraic objects': (abelian) groups, rings, modules, chain complexes ($\cdots \to V_0 \to V_1 \to V_2 \to \cdots$).

Example 1. How can we tell if the sphere S^2 and the torus $S^1 \times S^1$ can or cannot be deformed into each other? How would you prove it cannot be done?

Example 2. For a positive example, we *can* squash $\mathbb{R}^3 \setminus \{0\} \to S^2 \hookrightarrow \mathbb{R}^3 \setminus \{0\}$, sending $x \mapsto \frac{x}{|x|}$. This map continuously deforms to the identity map. So dimension not necessarily preserved.

Example 3. Can we have $S^1 \sim S^2$?

We first need to understand how spaces are built

Topological spaces

Recall...

From Topology and Analysis III

Definition 1. A *topology* on a set S is a collection $\mathcal{T}$ of subsets of X such that

1. $\emptyset, X \in \mathcal{T}$
2. If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$
3. If $U_\alpha \in \mathcal{T}$, $\alpha \in I$ is an arbitrary family, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

If $U \in \mathcal{T}$ we say U is *open*. A *topological space* is a set X eqipped with a topology $\mathcal{T}$.

Example 4. Take the set of real numbers, the *Euclidean ('usual') topology* is defined by saying a set is open iff it is a union of open intervals (a, b) (including the union of no sets ie $\emptyset$).
The *discrete topology* on a set X is defined by taking every $\mathcal{T}$ to consist of all subsets. The *indiscrete topology* is defined by taking $\mathcal{T}$ to consist of just $\emptyset$ and X.

This definition is concise, but not always the best way to define a topology. We will also use *neighbourhoods*

Definition 2. A set $N \subseteq X$ is a *neighbourhood* (in a given topology $\mathcal{T}$) of a point $x \in X$ if there is an open set $U \subseteq N$ with $x \in U$.

'nhd' is a good abbreviation

Example 5. Take $\mathbb{R}$ with the Euclidean topology. $(-1, 1)$, $[-1, 1]$, $[-1, 1)$ are all neighbourhoods of every $-1 < x < 1$, but $[0, 1)$ is not a neighbourhood of 0. More complicated: $[0, 1] \cup \{2\} \cup [5, 6]$ is a nhd of all $0 < x < 1$ and $5 < x < 6$.

Example 6. Consider a metric space (X, d). The *metric topology* is defined by saying a subset $U \subseteq X$ is open iff for every $x \in U$ there is some $\varepsilon_x > 0$ with the open ball $B(x, \varepsilon_x) \subseteq U$. Open balls around x are neighbouhoods of x, as are closed balls.

Here is a more concrete approach that allows concise definitions of topologies:

Definition 3. A *neighbourhood base* $\mathcal{N}$ on a set X is a family $\{\mathcal{N}(x)\}_{x \in X}$ where each $\mathcal{N}(x)$ is a nonempty collection of subsets of X, satisfying the following, for all $x \in X$:

1. For all $N \in \mathcal{N}(x)$, $x \in N$;
2. For all $N_1, N_2 \in \mathcal{N}(x)$, there is some $N \in \mathcal{N}(x)$ with $N \subseteq N_1 \cap N_2$;
3. For all $N \in \mathcal{N}(x)$ there is a subset $U \subseteq N$ such that for all $y \in U$, there is some $V \in \mathcal{N}(y)$ such that $V \subseteq U$.

We say the sets in $\mathcal{N}(x)$ are *basic neighbourhoods* of x.

As an example: given a topological space $(X, \mathcal{T})$ defining $\mathcal{N}(x)$ to consist of all nhds of x gives a nhd base. Similarly, defining $\mathcal{N}'(x)$ to consist of all open sets containing x defines a nhd base.

Given a neighbourhood base $\mathcal{N}$ on a set X, define a subset $U \subseteq X$ to be $\mathcal{N}$-*open* iff for all $x \in U$, there is an $N \in \mathcal{N}(x)$ with $N \subseteq U$.

Proposition 1. The $\mathcal{N}$-open sets define a topology on X.

Proof. We verify the axioms for a topology on X.

1. The condition that $\emptyset$ is $\mathcal{N}$-open is vacuously true. And since $\mathcal{N}(x)$ is not empty, there is a basic nhd around every point, so X is $\mathcal{N}$-open.
2. Given U, V both $\mathcal{N}$-open, we want to show $U \cap V$ is $\mathcal{N}$-open. So take $x \in U \cap V$. We know there is $N_U, N_V \in \mathcal{N}(x)$ with $N_U \subseteq U$ and $N_V \subseteq V$, and also that $x \in N_U \cap N_V$, since it is in each of them. Thus there is some $N \in \mathcal{N}(x)$ with $N \subseteq N_U \cap N_V \subseteq U \cap V$, and this is true for all $x \in U \cap V$. Hence $U \cap V$ is $\mathcal{N}$-open.
3. Given a family U_α, $\alpha \in I$, with each U_α $\mathcal{N}$-open, we want to show $U := \bigcup_{\alpha \in I} U_\alpha$ is $\mathcal{N}$-open. Take $x \in U$, so there is some α_0 with $x \in U_{\alpha_0}$. But this set in $\mathcal{N}$-open, so there is some nhd N of x with $N \subseteq U_{\alpha_0} \subseteq U$, and this is true for all $x \in U$. So U is $\mathcal{N}$-open. □

We call the topology from this proposition the topology generated by $\mathcal{N}$. Neighbourhoods in this topology are sets that contain a basic neighbourhood: V is a neighbourhood of x if there is some $N \in \mathcal{N}(x)$ with $N \subseteq V$.

Example 7. Given a metric space (X, d) the open balls form a nhd base on X and the topology they generate is the metric topology.

So many definitions you are familiar with from metric spaces work for topological spaces, if they can be phrased in terms of basic nhds. In particular, continuity!

Definition 4. Let $\mathcal{N}_X$ and $\mathcal{N}_Y$ be neibourhood bases on sets X and Y respectively. A function $f\colon X \to Y$ is *continuous* if for every $x \in X$ and $N \in \mathcal{N}_Y(f(x))$, the set $f^{-1}(N)$ contains a basic nhd of x.

This is a big generalisation of the ε-δ definition of continuity.

Exercise 1. Show that if $f\colon (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ is continuous as just defined, it is continuous for the topologies generated on X and Y by these nhd bases.

Recall a function is continuous for topologies if $f^{-1}(U)$ is open for all open U.

As a sanity check, the identity function id_X on a space X is indeed continuous.

You can check every function *to* an indiscrete space is continuous, as is every function *on* a discrete space

Definition 5. A continuous function $f \colon X \to Y$ is a *homeomorphism* if there is a continuous function $g \colon Y \to X$ with $g \circ f = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$. We then call X and Y *homeomorphic* if there is a homeomorphism between them.

or just isomorphic, if I'm being lazy

Now we need to show how to build new spaces, and continuous maps relating them to the original spaces.

Definition 6. Let X be a set, $(Y_\alpha, \mathcal{N}_\alpha)$, $\alpha \in I$ a family of sets with nhd bases (not necessarily all unique), and $f_\alpha \colon X \to Y_\alpha$ a family of functions. The *initial topology* on X is generated by the following nhd base: a subset of X is a basic nhd of x iff it is of the form $f_{\alpha_1}^{-1}(N_1) \cap \ldots \cap f_{\alpha_k}^{-1}(N_k)$ for $N_i \in \mathcal{N}_{\alpha_i}(f_{\alpha_i}(x))$, and some $\alpha_1, \ldots, \alpha_k$.

Verify this is a nhd base!

This generalises the product topology, which is the case that $X = Y_1 \times Y_2$, and $f_i \colon X \to Y_i$ is the projection $f_i(y_1, y_2) = y_i$, where $i = 1, 2$. But this *also* gives the subspace topology: take $f \colon X \hookrightarrow Y$ to be injective and define the initial topology on X.

Lemma 1. Giving X the initial topology, all the functions $f_\alpha \colon X \to Y_\alpha$ are continuous. Moreover, a function $k \colon Z \to X$ is continuous iff $f_\alpha \circ k \colon Z \to Y_\alpha$ is continuous for every α.

The following will be even more important for us, and will be new to most.

Definition 7. Let X be a set, $(Z_\beta, \mathcal{N}_\beta)$, $\beta \in J$ a family of sets with nhd bases (not necessarily all unique), and $g_\beta \colon Z_\beta \to X$ a family of functions (note the other direction!). The *final topology* on X has open sets as following: $U \subset X$ is open iff for all $\beta \in J$, $g_\beta^{-1}(U)$ is open in Z_β.

Lemma 2. Giving X the final topology, all the functions $g_\beta \colon Z_\beta \to X$ are continuous. Moreover a function $h \colon X \to W$ is continuous for the final topology on X iff $h \circ g_\beta \colon Z_\beta \to W$ is continous for every $\beta \in J$.

EXPAND THIS TO REALLY EXPLORE THE GLUING LEMMA. EG OPEN COVERS, COVERS BY CLOSED SETS. EXAMPLE OF INTERVAL OR SQUARES OR INTERVAL TIMES SPACE.

We will give two special cases of this, and we will see them often.

Example 8. Let $(Z, \mathcal{T})$ be a topological space, and let $\sim$ be an equivalence relation on Y, and define $X = Z/\sim$ to be the quotient by this relation. There is a function $\pi \colon Z \to X$ sending $y \mapsto [y]$. The final topology on X has as open sets those $U \subseteq X$ such that $\pi^{-1}(U)$ is open in Z.

For instance, we can give S^2 the initial topology for the maps $x, y\colon S^2 \to \mathbb{R}^2 \to \mathbb{R}$ (this is the usual topology on S^2), and then define an equivalence relation on S^2 by $x \sim y$ iff $x = -y$. The quotient is $\mathbb{RP}^2$, the real projective plane, and we give it the final topology coming from $S^2 \to \mathbb{RP}^2$. This is the topology it carries as a manifold. Incidentally, S^2 is an example of a *covering space* of $\mathbb{RP}^2$, the study of which will occupy the first section of the course.

Recall the definition of disjoint union of sets: given Z_β, $\beta \in J$, a family of sets, we have $\mathrm{in}_\gamma\colon Z_\gamma \hookrightarrow \bigsqcup_\beta Z_\beta$ with $Z_\beta \cap Z_\gamma = \emptyset$ for $\beta \neq \gamma$. If Z_β are spaces, then we give $\bigsqcup_\beta Z_\beta$ the final topology for the maps in_γ. This is *disjoint union* or *sum* topology, and $\bigsqcup_\beta Z_\beta$ is called the *topological sum*. A point in $\bigsqcup_\beta Z_\beta$ can be described by a pair (β, z), where $z \in Z_\beta$.

Lecture 2

Fact. Given functions $h_\beta\colon Z_\beta \to W$, there is a unique function $h = \langle h_\beta \rangle\colon \bigsqcup_\beta Z_\beta \to W$ with $h_\beta = h \circ \mathrm{in}_\beta$, or in other words this diagram commutes:

$$\begin{tikzcd} Z_\gamma \arrow[r, "\mathrm{in}_\gamma"] \arrow[dr, "h_\gamma"'] & \bigsqcup_\beta Z_\beta \arrow[d, "h"] \\ & W \end{tikzcd}$$

Lemma 3. The final topology on X for $g_\beta\colon Z_\beta \to X$ agrees with the final topology on X for $g = \langle g_\beta \rangle\colon \bigsqcup_\beta Z_\beta \to X$, using the sum topology.

Proof. We have that $U \subseteq X$ is open iff $\forall \beta\ g_\beta^{-1}(U)$ is open iff $\forall \beta$, $(g \circ \mathrm{in}_\beta)^{-1}(U) = \mathrm{in}_\beta^{-1}\left(g^{-1}(U)\right)$ is open iff $g^{-1}(U)$ is open in the sum topology. $\square$

Idea behind final topology, when $g_\beta\colon Z_\beta \to X$ are jointly surjective is that we can put an equivalence relation on $\bigsqcup_\beta Z_\beta$ with $(\beta_1, z_1) \sim (\beta_2, z_2)$ iff $g_{\beta_1}(z_1) = g_{\beta_2}(z_2) \in X$. As a set, X is the set of equivalence classes under this relation, so you can think of it as gluing together the underlying sets of the spaces Z_β. The final topology on X is then the only sensible topology to described the space we get by gluing together the *spaces* Z_β. Leter we'll see spaces that are built uop by gluing together lots of 'simple' spaces, like disks $D^n := \{x \in \mathbb{R}^n \bmod |x| \leq 1\}$ (with the subspace topology from $\mathbb{R}^n$). But what does 'simple' here mean? Roughly, "shrinkable to a point".

this means $\forall x \in X, \exists \beta, x \in Z_\beta$ with $g_\beta(z) = x$

Homotopy

"Shrinkable" implies a kind of continuous process in time. Consider the function $I \times D^n \to D^n$. Consider the map

$$\begin{aligned} H\colon I \times D^n &\to D^n \\ (t, \mathbf{x}) &\mapsto (1-t)\mathbf{x} \end{aligned}$$

Note that this gives maps $H_0\colon D^n \to D^n$ (the identity map) and H_1 (constant at 0). The function H is continuous! How should we see this? The topology on D^n is the subspace topology $D^n \subset \mathbb{R}^n$, and $\mathbb{R}^n$ has the product topology. It is not too difficult to see (Exercise!) that the topology on D^n is the final topology for the coordinate functions $x_i\colon D^n \to \mathbb{R}^n \to \mathbb{R}$. So $H\colon I \times D^n \to D^n$ is continuous iff [[FIX LINE SPACING]]

And $I \subset \mathbb{R}$ has subspace topology

$$\begin{tikzcd} I \times D^n \arrow[r, "\mathrm{id} \times x_i"] & I \times \mathbb{R} \arrow[r] & \mathbb{R} \times \mathbb{R} \arrow[r] & \mathbb{R} \\ (t, \mathbf{x}) \arrow[rr, mapsto] & & (t, x_i) \arrow[r, mapsto] & t x_i \end{tikzcd}$$

But $I \times D^n \to \mathbb{R} \times \mathbb{R}$ is continuous by definition of final topology, and the following fact:

Fact. If $f\colon X \to W$ and $g\colon X \to Z$ are continuous, then so is $f \times g\colon X \times Y \to W \times Z$. If both X and Y have at least one point each, then the reverse implication also holds.

So if we can prove that multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, then H is continuous. But the standard topology on $\mathbb{R}$ comes from the metric space structure, so can use sequential criterion for continuity. Take $(a_n, b_n) \to (a, b)$ in $\mathbb{R} \times \mathbb{R}$, then:

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a b_n + a b_n - ab| \\ &\leq |a_n - a|\,|b_n| + |a|\,|b_n - b| \\ &\leq |a_n - a| \sup |b_n| + |a|\,|b_n - b| \qquad \text{(as } (b_n) \text{ converges, it is bounded)} \\ &\to 0 + 0 \end{aligned}$$

Hence H is continuous.

Definition 8. A space X is *contractible* if there is a point $x_0 \in X$ and a continuous function $H\colon I \times X \to X$ such that $H(0, x) = x$ and $H(1, x) = x_0$ for all $x \in X$. Such a function is called a *contraction*.

We have shown D^n is contractible.

Example 9. $\mathbb{R}$ is contractible. A finite product of contractible spaces is contractible.

Example 10. Consider what it would mean if a discrete space S were contractible: there would be an element $* \in S$ and a continuous function $h\colon I \times S \to S$ such that $h(0,s) = s$ and $h(1,s) = *$. Restricting h to $I \times \{s\}$ for some given s, we get a continuous function $I \hookrightarrow I \times S \to S$, whose range includes $*$ and s. Since all functions with discrete domain are continuous, let us compose with the continuous function $\chi_{\{*\}}\colon S \to \mathbb{R}$ that sends $* \mapsto 1$ and $s \mapsto 0$ for all $s \neq *$. So we have a continuous function $\widetilde{h}\colon I \to \mathbb{R}$ with $\widetilde{h}(0) = 0$ and range contained in $\{0,1\}$. By the intermediate value theorem, we must have $\widetilde{h}(1) = \chi_{\{*\}}(h(1,s)) = 0$, so that $h(1,s) = *$, and hence $s = *$ for all $s \in S$. Thus S has exactly one element.

The interval can only map continuously to a discrete space if it is constant at some element, or equivalently, its image consists of a single point, and this property is important enough to have a name.

Definition 9. A space X is *connected* if the only continuous maps to a discrete space have image a single point.

If you know the 'usual' definition, this is equivalent to it

This is our first example of an invariant of spaces, namely whether they are connected or not: a connected space X cannot be homeomorphic to a space Z that is not connected. But, how can we tell non-connected spaces apart?

Consider
$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d] & Z \arrow[dl] \\ S & \end{tikzcd}$$
with S discrete.

Definition 10. 1. For any space X, a subset $Y \subseteq X$ is a *connected component* of X if Y is connected and the indicator function $\chi_Y\colon X \to \{0,1\}$ is continuous.

2. Put an equivalence relation on X with $x_1 \sim x_2$ iff x_1 and x_2 are both in a given connected component. Then define $\pi_0(X) = X/\sim$, the *set of connected components*. There is a continuous function $X \to \pi_0(X)$.

A connected space X has $\pi_0(X) = *$, but now we can tell apart non-connected spaces.

Lemma 4. Every space can be written as $X = \bigsqcup_{\alpha \in \pi_0(X)} X_\alpha$, with X_α connected.

As a result, we need to try to understand *connected* spaces, though we will still *use* non-connected spaces.

Can we get more out of the idea of contractions? Given $H\colon I \times X \to X$, we have maps $_i$ for $i = 0, 1$, namely $H_0 = \mathrm{id}_X$ and H_1 is constant at x_0. What if H_0 and H_1 were other sorts of continuous maps?

[example: retraction of annulus to inner and outer circles]

What if we considered general continuous maps $X \to Y$ instead of just $X \to X$?

Definition 11. A *homotopy* is a continuous function $H\colon I \times X \to Y$. If $f = H(0, -)$ and $g = H(1, -)$, we say H is a *homotopy from f to g*, and that f and g are *homotopic*, written $f \sim g$.

Algebraic topology most of the time considers functions *up to homotopy*, and also "spaces up to homotopy".

Definition 12. A continuous function $f\colon X \to Y$ is called a *homotopy equivalence* if there is a continuous function $g\colon Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. We then say X and Y are *homotopy equivalent*.

Example 11. A contractible space is homotopy equivalent to a one-point space.

You should think of homotopy equivalences as being 'kinda like isomorphism', but coarser. Going back to our original motivation, the assignment

$$\{\text{'Spaces'}\} \longrightarrow \{\text{Algebraic objects}\}$$

should take homotopy equivalent spaces to isomorphic algebraic objects. To make this more rigorous we will use the language of category theory. Lecture 3

Here is a super-important property of homotopies.

Proposition 2. Homotopy between maps is an equivalence relation.

Proof. sorry. □

Here is a supply of many homotopies.

Lemma 5. Every continuous function $f\colon X \to Y$, with Y a contractible space (say to $y_0 in Y$), is homotopic to a function with range contained in $\{y_0\}$.

Proof. □

As a corollary, every pair of functions to a contractible space are homotopic. Since contractible spaces are in some sense trivial, maps to them are in the same sense trivial.

NEED: distributivity (prove in assignment 1)

We have been discussing topological spaces and continuous maps, but also implicitly sets and functions, not necessarily continuous, and passing between these two pictures. In both cases we have composition that is associative, and identity maps.

Definition 13. A *category* $\mathcal{C}$ consists of a collection of *objects* $W, X, Y, Z, \dots$ and for each pair of objects X, Y a collection of *morphisms*, denoted $\mathcal{C}(X, Y)$, together with the following data:

i) For each pair $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, a specified morphism $g \circ f \in \mathcal{C}(X, Z)$,

ii) For every object a specified morphism $\mathrm{id}_X \in \mathcal{C}(X, X)$,

such that:

1. For every triple $h \in \mathcal{C}(W, X)$, $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have $g \circ (f \circ h) = (g \circ f) \circ h$,
2. For every object X and $h \in \mathcal{C}(W, X)$, $f \in \mathcal{C}(X, Y)$ we have $\mathrm{id}_X \circ h = h$ and $f \circ \mathrm{id}_X = f$.

For $f \in \mathcal{C}(X, Y)$ we say X is the *source* of f, Y is the *target* of f, and write $X = s(f)$, $Y = t(f)$. We also write $f \colon X \to Y$ or $X \xrightarrow{f} Y$ to indicate that $f \in \mathcal{C}(X, Y)$.

Many examples of categories have objects sets carrying extra structure (for instance a topology) and morphisms that are functions compatible with that structure—but not all categories. We have seen **Top**, the category of topological spaces (and continuous maps) and **Set**, the category of sets (and functions), and you implicitly already know many other examples. The whole point of categories is how they relate to each other, an isolated category can only tell us so much.

Vector spaces, (abelian) groups, manifolds, rings, ...

Definition 14. Given categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* from $\mathcal{C}$ to $\mathcal{D}$, denoted $F \colon \mathcal{C} \to \mathcal{D}$ consists of the data:

i) For every object X of $\mathcal{D}$, a specified object $F(X)$ of $\mathcal{D}$,

ii) For every morphism $f\colon X \to Y$ of $\mathcal{C}$, a specified morphism $F(f)\colon F(X) \to F(Y)$ of $\mathcal{D}$

such that for every pair $f\colon X \to Y$ and $g\colon Y \to Z$ of morphisms of $\mathcal{C}$, $F(g \circ f) = F(g) \circ F(f)$. This latter property is called 'functoriality'.

We have already see at least four examples of functors:

- The underlying set functor $U\colon \mathbf{Top} \to \mathbf{Set}$
- The discrete topology functor $\mathbf{Set} \to \mathbf{Top}$
- The indiscrete topology functor $\mathbf{Set} \to \mathbf{Top}$
- The set of connected components functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$

although we haven't yet seen why π_0 is a functor.

Proposition 3. The assignment $X \mapsto \pi_0(X)$ is a functor $\mathbf{Top} \to \mathbf{Set}$.

Proof. We need to show there is an assignment $(f\colon X \to Y) \mapsto (\pi_0(f)\colon \pi_0(X) \to \pi_0(Y))$, for an arbitrary continuous function f. We already know we have a map $X \to Y \to \pi_0(Y)$, and we want to show this *descends* along $X \to \pi_0(X)$. Given any $\alpha \in \pi_0(X)$, it corresponds to a connected component of X, namely X_α. Look at the restriction of $X \to Y \to \pi_0(Y)$ to X_α: since X_α is connected, its image is exactly one point in $\pi_0(Y)$. So define $\pi_0(f)(\alpha) = f(x)$ for an arbitrary $x \in X_\alpha$. This defines $\pi_0(f)$. Moreover, the following diagram *commutes*:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y) \end{tikzcd}$$

Now we want to show that $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.

$\square$