

Algebraic Topology¹

David Michael Roberts

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Lecture 1

What is it?

Algebraic topology is the study of maps

$$\{\text{'Spaces'}\} \longrightarrow \{\text{Algebraic objects}\}$$

Or rather, 'well-behaved' such maps—they should also send maps between spaces to algebraic maps, respecting composition (so: *functors*); they should send spaces built out of smaller units to algebraic objects built out of smaller units, in a compatible way, etc.

Here, 'Spaces' roughly means topological spaces up to deformation (usually homotopy, but not always). Such equivalence classes are called *homotopy types*. 'Algebraic objects': (abelian) groups, rings, modules, chain complexes ($\cdots \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow \cdots$).

Example 1. How can we tell if the sphere S^2 and the torus $S^1 \times S^1$ can or cannot be deformed into each other? How would you prove it cannot be done?

Example 2. For a positive example, we *can* squash $\mathbb{R}^3 \setminus \{0\} \rightarrow S^2 \hookrightarrow \mathbb{R}^3 \setminus \{0\}$, sending $x \mapsto \frac{x}{|x|}$. This map continuously deforms to the identity map. So dimension not necessarily preserved.

Example 3. Can we have $S^1 \sim S^2$?

We first need to understand how spaces are built

Topological spaces

Recall...

From Topology and Analysis III

Definition 1. A *topology* on a set S is a collection \mathcal{T} of subsets of X such that

1. $\emptyset, X \in \mathcal{T}$
2. If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$
3. If $U_\alpha \in \mathcal{T}$, $\alpha \in I$ is an arbitrary family, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

If $U \in \mathcal{T}$ we say U is *open*. A *topological space* is a set X equipped with a topology \mathcal{T} .

Example 4. Take the set of real numbers, the *Euclidean* ('usual') topology is defined by saying a set is open iff it is a union of open intervals (a, b) (including the union of no sets ie \emptyset).

The *discrete topology* on a set X is defined by taking every \mathcal{T} to consist of all subsets. The *indiscrete topology* is defined by taking \mathcal{T} to consist of just \emptyset and X .

This definition is concise, but not always the best way to define a topology. We will also use *neighbourhoods*

Definition 2. A set $N \subseteq X$ is a *neighbourhood* (in a given topology \mathcal{T}) of a point $x \in X$ if there is an open set $U \subseteq N$ with $x \in U$.

'nhd' is a good abbreviation

Example 5. Take \mathbb{R} with the Euclidean topology. $(-1, 1)$, $[-1, 1]$, $[-1, 1)$ are all neighbourhoods of every $-1 < x < 1$, but $[0, 1)$ is not a neighbourhood of 0. More complicated: $[0, 1] \cup \{2\} \cup [5, 6]$ is a nhd of all $0 < x < 1$ and $5 < x < 6$.

Example 6. Consider a metric space (X, d) . The *metric topology* is defined by saying a subset $U \subseteq X$ is open iff for every $x \in U$ there is some $\varepsilon_x > 0$ with the open ball $B(x, \varepsilon_x) \subseteq U$. Open balls around x are neighbourhoods of x , as are closed balls.

Here is a more concrete approach that allows concise definitions of topologies:

Definition 3. A *neighbourhood base* \mathcal{N} on a set X is a family $\{\mathcal{N}(x)\}_{x \in X}$ where each $\mathcal{N}(x)$ is a nonempty collection of subsets of X , satisfying the following, for all $x \in X$:

1. For all $N \in \mathcal{N}(x)$, $x \in N$;
2. For all $N_1, N_2 \in \mathcal{N}(x)$, there is some $N \in \mathcal{N}(x)$ with $N \subseteq N_1 \cap N_2$;
3. For all $N \in \mathcal{N}(x)$ there is a subset $U \subseteq N$ such that for all $y \in U$, there is some $V \in \mathcal{N}(y)$ such that $V \subseteq U$.

We say the sets in $\mathcal{N}(x)$ are *basic neighbourhoods* of x .

As an example: given a topological space (X, \mathcal{T}) defining $\mathcal{N}(x)$ to consist of all nhds of x gives a nhd base. Similarly, defining $\mathcal{N}'(x)$ to consist of all open sets containing x defines a nhd base.

Given a neighbourhood base \mathcal{N} on a set X , define a subset $U \subseteq X$ to be *\mathcal{N} -open* iff for all $x \in U$, there is an $N \in \mathcal{N}(x)$ with $N \subseteq U$.

Proposition 1. The \mathcal{N} -open sets define a topology on X .

Proof. We verify the axioms for a topology on X .

1. The condition that \emptyset is \mathcal{N} -open is vacuously true. And since $\mathcal{N}(x)$ is not empty, there is a basic nhd around every point, so X is \mathcal{N} -open.
2. Given U, V both \mathcal{N} -open, we want to show $U \cap V$ is \mathcal{N} -open. So take $x \in U \cap V$. We know there is $N_U, N_V \in \mathcal{N}(x)$ with $N_U \subseteq U$ and $N_V \subseteq V$, and also that $x \in N_U \cap N_V$, since it is in each of them. Thus there is some $N \in \mathcal{N}(x)$ with $N \subseteq N_U \cap N_V \subseteq U \cap V$, and this is true for all $x \in U \cap V$. Hence $U \cap V$ is \mathcal{N} -open.
3. Given a family $U_\alpha, \alpha \in I$, with each U_α \mathcal{N} -open, we want to show $U := \bigcup_{\alpha \in I} U_\alpha$ is \mathcal{N} -open. Take $x \in U$, so there is some α_0 with $x \in U_{\alpha_0}$. But this set is \mathcal{N} -open, so there is some nhd N of x with $N \subseteq U_{\alpha_0} \subseteq U$, and this is true for all $x \in U$. So U is \mathcal{N} -open. \square

We call the topology from this proposition the topology generated by \mathcal{N} . Neighbourhoods in this topology are sets that contain a basic neighbourhood: V is a neighbourhood of x if there is some $N \in \mathcal{N}(x)$ with $N \subseteq V$.

Given a neighbourhood base \mathcal{N} on X , we can identify the *closure* of a set $S \subset X$ as the collection of points $x \in X$ such that for all $N \in \mathcal{N}(x)$, $\exists s \in N \cap S$.

Example 7. Given a metric space (X, d) the open balls form a nhd base on X and the topology they generate is the metric topology.

Hence many definitions you are familiar with from metric spaces work for topological spaces, if they can be phrased in terms of basic nhds. In particular, continuity!

Definition 4. Let \mathcal{N}_X and \mathcal{N}_Y be neighbourhood bases on sets X and Y respectively. A function $f: X \rightarrow Y$ is *continuous* if for every $x \in X$ and $N \in \mathcal{N}_Y(f(x))$, the set $f^{-1}(N)$ contains a basic nhd of x .

This is a big generalisation of the ε - δ definition of continuity.

Exercise 1. Show that if $f: (X, \mathcal{N}_X) \rightarrow (Y, \mathcal{N}_Y)$ is continuous as just defined, it is continuous for the topologies generated on X and Y by these nhd bases.

Recall a function is continuous for topologies if $f^{-1}(U)$ is open for all open U .

As a sanity check, the identity function id_X on a space X is indeed continuous.

Definition 5. A continuous function $f: X \rightarrow Y$ is a *homeomorphism* if there is a continuous function $g: Y \rightarrow X$ with $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. We then call X and Y *homeomorphic* if there is a homeomorphism between them.

You can check every function *to* an indiscrete space is continuous, as is every function *on* a discrete space

or just isomorphic, if I'm being lazy

Now we need to show how to build new spaces, and continuous maps relating them to the original spaces.

Definition 6. Let X be a set, $(Y_\alpha, \mathcal{N}_\alpha)$, $\alpha \in I$ a family of sets with nhd bases (not necessarily all unique), and $f_\alpha: X \rightarrow Y_\alpha$ a family of functions. The *initial topology* on X is generated by the following nhd base: a subset of X is a basic nhd of x iff it is of the form $f_{\alpha_1}^{-1}(N_1) \cap \dots \cap f_{\alpha_k}^{-1}(N_k)$ for some $\alpha_1, \dots, \alpha_k$ and $N_i \in \mathcal{N}_{\alpha_i}(f_{\alpha_i}(x))$.

Exercise: verify this is a nhd base!

This generalises the product topology, which is the case that $X = Y_1 \times Y_2$, and $f_i: X \rightarrow Y_i$ is the projection $f_i(y_1, y_2) = y_i$, where $i = 1, 2$. But this *also* gives the subspace topology: take $f: X \hookrightarrow Y$ to be injective and define the initial topology on X .

Lemma 1. Giving X the initial topology, all the functions $f_\alpha: X \rightarrow Y_\alpha$ are continuous. Moreover, a function $k: Z \rightarrow X$ is continuous iff $f_\alpha \circ k: Z \rightarrow Y_\alpha$ is continuous for every α .

The following will be even more important for us, and will be new to most.

Definition 7. Let X be a set, $(Z_\beta, \mathcal{N}_\beta)$, $\beta \in J$ a family of topological spaces (not necessarily all unique), and $g_\beta: Z_\beta \rightarrow X$ a family of functions (note the other direction!). The *final topology* on X has open sets as following: $U \subset X$ is open iff for all $\beta \in J$, $g_\beta^{-1}(U)$ is open in Z_β .

Lemma 2. Giving X the final topology, all the functions $g_\beta: Z_\beta \rightarrow X$ are continuous. Moreover a function $h: X \rightarrow W$ is continuous for the final topology on X iff $h \circ g_\beta: Z_\beta \rightarrow W$ is continuous for every $\beta \in J$.

We will give two special cases of this, and we will see them often.

Example 8. Let (Z, \mathcal{T}) be a topological space, and let \sim be an equivalence relation on Z , and define $X = Z / \sim$ to be the quotient by this relation. There is a function $\pi: Z \rightarrow X$ sending $y \mapsto [y]$. The final topology on X has as open sets those $U \subseteq X$ such that $\pi^{-1}(U)$ is open in Z .

For instance, we can give S^2 the initial topology for the maps $x, y: S^2 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}$ (this is the usual topology on S^2), and then define an equivalence relation on S^2 by $x \sim y$ iff $x = -y$. The quotient is \mathbb{RP}^2 , the real projective plane, and we give it the final topology coming from $S^2 \rightarrow \mathbb{RP}^2$. This is the topology it carries as a manifold. Incidentally, S^2 is an example of a *covering space* of \mathbb{RP}^2 , the study of which will occupy the first section of the course.

Recall the definition of disjoint union of sets: given $Z_\beta, \beta \in J$, a family of sets, we have $\text{in}_\gamma: Z_\gamma \hookrightarrow \bigsqcup_\beta Z_\beta$ with $Z_\beta \cap Z_\gamma = \emptyset$ for $\beta \neq \gamma$. If Z_β are spaces, then we give $\bigsqcup_\beta Z_\beta$ the final topology for the maps in_γ . This is *disjoint union* or *sum* topology, and $\bigsqcup_\beta Z_\beta$ is called the *topological sum*. A point in $\bigsqcup_\beta Z_\beta$ can be described by a pair (β, z) , where $z \in Z_\beta$.

Exercise 2. Given functions $h_\beta: Z_\beta \rightarrow W$, there is a unique function $h = \langle h_\beta \rangle: \bigsqcup_\beta Z_\beta \rightarrow W$ with $h_\beta = h \circ \text{in}_\beta$, or in other words this diagram commutes:

$$\begin{array}{ccc} Z_\gamma & \xrightarrow{\text{in}_\gamma} & \bigsqcup_\beta Z_\beta \\ & \searrow h_\gamma & \downarrow h \\ & & W \end{array}$$

Lemma 3. The final topology on X for $g_\beta: Z_\beta \rightarrow X$ agrees with the final topology on X for $g = \langle g_\beta \rangle: \bigsqcup_\beta Z_\beta \rightarrow X$, using the sum topology.

Proof. We have that $U \subseteq X$ is open iff $\forall \beta \ g_\beta^{-1}(U)$ is open iff $\forall \beta, (g \circ \text{in}_\beta)^{-1}(U) = \text{in}_\beta^{-1}(g^{-1}(U))$ is open iff $g^{-1}(U)$ is open in the sum topology. \square

The idea behind final topology, when $g_\beta: Z_\beta \rightarrow X$ are jointly surjective, is that we can put an equivalence relation on $\bigsqcup_\beta Z_\beta$ with $(\beta_1, z_1) \sim (\beta_2, z_2)$ iff $g_{\beta_1}(z_1) = g_{\beta_2}(z_2) \in X$. As a set, X is the set of equivalence classes under this relation, so you can think of it as gluing together the *underlying sets* of the spaces Z_β . The final topology on X is then the only sensible topology to describe the space we get by gluing together the *spaces* Z_β .

Exercise 3. Given an open cover $\{U_\alpha\}$ of a space X , then X carries the final topology for the inclusion maps $U_\alpha \hookrightarrow X$, or equivalently for the map $\bigsqcup_\alpha U_\alpha \rightarrow X$.

Exercise 4. Given a *finite* closed cover $\{V_i\}_{i=1}^n$ of X , then X carries the final topology for $\bigsqcup_{i=1}^n V_i \rightarrow X$.

Lecture 2

an important fact is that the map $\bigsqcup_\beta X \times Z_\beta \rightarrow X \times \bigsqcup_\beta Z_\beta$ is a homeomorphism (exercise!)

this means $\forall x \in X, \exists \beta, x \in Z_\beta$ with $g_\beta(z) = x$

Later we'll see spaces that are built up by gluing together lots of 'simple' spaces, like disks $D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (with the subspace topology from \mathbb{R}^n). But what does 'simple' here mean? Roughly, "shrinkable to a point".

Homotopy

"Shrinkable" implies a kind of continuous process in time. Consider the function $I \times D^n \rightarrow D^n$. Consider the map

$$\begin{aligned} H: I \times D^n &\rightarrow D^n \\ (t, \mathbf{x}) &\mapsto (1-t)\mathbf{x} \end{aligned}$$

Note that this gives maps $H_0: D^n \rightarrow D^n$ (the identity map) and H_1 (constant at 0). The function H is continuous! How should we see this? The topology on D^n is the subspace topology $D^n \subset \mathbb{R}^n$, and \mathbb{R}^n has the product topology. It is not too difficult to see (Exercise!) that the topology on D^n is the final topology for the coordinate functions $x_i: D^n \rightarrow \mathbb{R} \rightarrow \mathbb{R}$. So $H: I \times D^n \rightarrow D^n$ is continuous iff

And $I \subset \mathbb{R}$ has subspace topology

$$\begin{aligned} I \times D^n &\xrightarrow{\text{id} \times x_i} I \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\ (t, \mathbf{x}) &\longmapsto (t, x_i) \longmapsto tx_i \end{aligned}$$

But $I \times D^n \rightarrow \mathbb{R} \times \mathbb{R}$ is continuous by definition of final topology, and the following result:

Exercise 5. If $f: X \rightarrow W$ and $g: X \rightarrow Z$ are continuous, then so is $f \times g: X \times Y \rightarrow W \times Z$. If both X and Y have at least one point each, then the reverse implication also holds.

So if we can prove that multiplication $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then H is continuous. But the standard topology on \mathbb{R} comes from the metric space structure, so can use sequential criterion for continuity. Take $(a_n, b_n) \rightarrow (a, b)$ in $\mathbb{R} \times \mathbb{R}$, then:

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n - a| |b_n| + |a| |b_n - b| \\ &\leq |a_n - a| \sup |b_n| + |a| |b_n - b| \quad (\text{as } (b_n) \text{ converges, it is bounded}) \\ &\rightarrow 0 + 0 \end{aligned}$$

Hence H is continuous.

Definition 8. A space X is *contractible* if there is a point $x_0 \in X$ and a continuous function $H: I \times X \rightarrow X$ such that $H(0, x) = x$ and $H(1, x) = x_0$ for all $x \in X$. Such a function is called a *contraction*.

We have shown D^n is contractible.

Exercise 6. \mathbb{R} is contractible. A finite product of contractible spaces is contractible.

Example 9. Consider what it would mean if a discrete space S were contractible: there would be an element $*$ $\in S$ and a continuous function $h: I \times S \rightarrow S$ such that $h(0, s) = s$ and $h(1, s) = *$. Restricting h to $I \times \{s\}$ for some given s , we get a continuous function $I \hookrightarrow I \times S \rightarrow S$, whose range includes $*$ and s . Since all functions with discrete domain are continuous, let us compose with the continuous function $\chi_{\{*\}}: S \rightarrow \mathbb{R}$ that sends $*$ to 1 and s to 0 for all $s \neq *$. So we have a continuous function $\tilde{h}: I \rightarrow \mathbb{R}$ with $\tilde{h}(0) = 0$ and range contained in $\{0, 1\}$. By the intermediate value theorem, we must have $\tilde{h}(1) = \chi_{\{*\}}(h(1, s)) = 0$, so that $h(1, s) = *$, and hence $s = *$ for all $s \in S$. Thus S has exactly one element.


The interval can only map continuously to a discrete space if it is constant at some element, or equivalently, its image consists of a single point, and this property is important enough to have a name.

Definition 9. A space X is *connected* if every continuous map to a discrete space has image a single point.

If you know the ‘usual’ definition, this is equivalent to it

This is our first example of an invariant of spaces, namely whether they are connected or not: a connected space X cannot be homeomorphic to a space Z that is not connected. But, how can we tell non-connected spaces apart?

Consider $X \xrightarrow{\sim} Z$ with S discrete.



Definition 10. 1. For any space X , a subset $Y \subseteq X$ is a *connected component* of X if Y is connected and the indicator function $1_Y: X \rightarrow \{0, 1\}$ is continuous.

$$1_Y(x) = 1 \text{ for } x \in Y, 0 \text{ otherwise}$$

2. Put an equivalence relation on X with $x_1 \sim x_2$ iff x_1 and x_2 are both in a given connected component. Then define $\pi_0(X) = X / \sim$, the *set of connected components*. There is a continuous function $X \rightarrow \pi_0(X)$.

A connected space X has $\pi_0(X) = *$, but now we can tell apart non-connected spaces.

Lemma 4. Every space can be written as $X = \bigsqcup_{\alpha \in \pi_0(X)} X_\alpha$, with X_α connected.

As a result, we need to try to understand *connected* spaces, though we will still *use* non-connected spaces.

Can we get more out of the idea of contractions? Given $H: I \times X \rightarrow X$, we have maps H_i for $i = 0, 1$, namely $H_0 = \text{id}_X$ and H_1 is constant at x_0 . What if H_0 and H_1 were other sorts of continuous maps?

Example 10. Consider the annulus $A(r, R) := \{x \in \mathbb{R}^2 \mid r \leq |x| \leq R\}$, and the function $H(t, x) = ((1-t)r + tR)x/|x|$.

What if we considered general continuous maps $X \rightarrow Y$ instead of just $X \rightarrow X$?

Definition 11. A *homotopy* is a continuous function $H: I \times X \rightarrow Y$. If $f = H(0, -)$ and $g = H(1, -)$, we say H is a *homotopy from f to g* , and that f and g are *homotopic*, written $f \sim g$.

Example 10 gives a homotopy between the two ‘retraction’ maps $A(r, R) \rightarrow A(r, R)$, mapping points to the inner and outer circles respectively.

Algebraic topology most of the time considers functions *up to homotopy*, and also “spaces up to homotopy”.

Definition 12. A continuous function $f: X \rightarrow Y$ is called a *homotopy equivalence* if there is a continuous function $g: Y \rightarrow X$ such that $g \circ f \sim \text{id}_X$ and $f \circ g \sim \text{id}_Y$. We then say X and Y are *homotopy equivalent*.

Example 11. A contractible space is homotopy equivalent to a one-point space.

You should think of homotopy equivalences as being ‘kinda like isomorphism’, but coarser. Going back to our original motivation, the assignment

$$\{\text{‘Spaces’}\} \longrightarrow \{\text{Algebraic objects}\}$$

should take homotopy equivalent spaces to isomorphic algebraic objects. To make this more rigorous we will use the language of category theory.

Lecture 3

Here is a super-important property of homotopies we will use continuously.

Proposition 2. Given homotopies $H: I \times X \rightarrow Y$ and $H': I \times X \rightarrow Y$ such that $H_1 = H'_0: X \rightarrow Y$, there is a homotopy H'' from H_0 to H_1 , and a homotopy \tilde{H} from H_1 to H_0 .

Proof. We will use Exercise 4 applied to the closed cover $\{[0, \frac{1}{2}] \times X, [\frac{1}{2}, 1] \times X\}$ of $I \times X$. Since $I \simeq [0, \frac{1}{2}]$ and $I \simeq [\frac{1}{2}, 1]$, H and H' give

us maps $[0, \frac{1}{2}] \times X \simeq I \times X \xrightarrow{H} Y$ and $[0, \frac{1}{2}] \times X \simeq I \times X \xrightarrow{H'} Y$ respectively. By the assumption on H_1 and H'_0 , we get a well-defined function $H'' : I \times X \rightarrow Y$, which is then continuous by the Exercise. It is a simple check to see it is a homotopy from H_0 to H'_1 . For the second part, let $c : I \rightarrow I$ be the function $c(t) = 1 - t$. Then define H'' to be the composite $I \times X \xrightarrow{c \times \text{id}_X} I \times X \xrightarrow{H} Y$, which has the required properties. \square

Contractible spaces supply many homotopies.

Lemma 5. Every continuous function $f : X \rightarrow Y$, with Y a contractible space (say to $y_0 \in Y$), is homotopic to a function with range contained in $\{y_0\}$.

Proof. Let $H : I \times Y \rightarrow Y$ be a homotopy witnessing the contractility of Y . Then the composite $I \times X \xrightarrow{\text{id}_I \times f} I \times Y \xrightarrow{H} Y$ is a homotopy from f to the the desired function. \square

As a corollary, every pair of functions to a contractible space are homotopic. Since contractible spaces are in some sense trivial, maps to them are in the same sense trivial.

An important intermediate version of this is when we consider only the case where X is discrete, or is even just pt :

Definition 13. A space Y is *path-connected* if every map $\text{pt} \rightarrow Y$ is homotopic to every other such map.

This condition is equivalent to requiring it for *all* discrete spaces in place of pt (Exercise!)

Unpacking this, we see this means that for any two points $\text{pt} \rightarrow Y$ there is a path $I \rightarrow Y$ connecting them, i.e. $H : I \simeq I \times \text{pt} \rightarrow Y$.

Proposition 3. A path-connected space is connected

Let us define $[X, Y] = \{\text{continuous } f : X \rightarrow Y\} / \text{homotopy}$. The set of *path components* of Y is then the set $[\text{pt}, Y]$. If equipped with the final topology arising from the surjective function $Y \rightarrow [\text{pt}, Y]$, we denote the resulting topological space by $[\text{pt}, Y]^{\text{top}}$.

We have been discussing topological spaces and continuous maps, but also implicitly sets and functions, not necessarily continuous, and passing between these two pictures. In both cases we have composition that is associative, and identity maps. Later we shall be using different classes of topological spaces in order to ensure the behaviour we require will hold.

Definition 14. A *category* \mathcal{C} consists of a collection of *objects* W, X, Y, Z, \dots and for each pair of objects X, Y a collection of *morphisms*, denoted $\mathcal{C}(X, Y)$, together with the following data:

- i) For each pair $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, a specified morphism $g \circ f \in \mathcal{C}(X, Z)$,
- ii) For every object a specified morphism $\text{id}_X \in \mathcal{C}(X, X)$,

such that:

1. For every triple $h \in \mathcal{C}(W, X)$, $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have $g \circ (f \circ h) = (g \circ f) \circ h$,
2. For every object X and $h \in \mathcal{C}(W, X)$, $f \in \mathcal{C}(X, Y)$ we have $\text{id}_X \circ h = h$ and $f \circ \text{id}_X = f$.

For $f \in \mathcal{C}(X, Y)$ we say X is the *source* of f , Y is the *target* of f , and write $X = s(f)$, $Y = t(f)$. We also write $f: X \rightarrow Y$ or $X \xrightarrow{f} Y$ to indicate that $f \in \mathcal{C}(X, Y)$. If $\mathcal{C}(X, Y)$ is a set for all X, Y , then \mathcal{C} is called *locally small*, and each $\mathcal{C}(X, Y)$ is called a *hom-set*.

Most categories you will encounter are locally small

Many examples of categories have objects sets carrying extra structure (for instance a topology) and morphisms that are functions compatible with that structure—but not all categories. We have seen **Top**, the category of topological spaces (and continuous maps) and **Set**, the category of sets (and functions), and you implicitly already know many other examples.

Vector spaces, (abelian) groups, manifolds, rings, ...

Example 12. The category **Set**_{*} of pointed sets (X, x) ($x \in X$ a specified element) and pointed maps $(X, x) \rightarrow (Y, y)$ (functions $f: X \rightarrow Y$ with $f(x) = y$) can be considered as consisting of algebraic objects of the weakest sort (compare homomorphisms, linear transformations, ring maps, etc, which preserve distinguished elements).

The whole point of categories is how they relate to each other, an isolated category can only tell us so much.

Definition 15. Given categories \mathcal{C} and \mathcal{D} , a *functor* from \mathcal{C} to \mathcal{D} , denoted $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of the data:

- i) For every object X of \mathcal{C} , a specified object $F(X)$ of \mathcal{D} ,
- ii) For every morphism $f: X \rightarrow Y$ of \mathcal{C} , a specified morphism $F(f): F(X) \rightarrow F(Y)$ of \mathcal{D}

such that for every pair $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ of morphisms of \mathcal{C} , $F(g \circ f) = F(g) \circ F(f)$. This latter property is called ‘functoriality’. For locally small categories, the assignment on morphisms gives a function $\mathcal{C}(X, Y) \rightarrow \mathcal{D}(F(X), F(Y))$.

We will use this notation even without making that assumption

We have already see at least four examples of functors:

- The underlying set functor $U: \mathbf{Top} \rightarrow \mathbf{Set}$
- The discrete topology functor $\text{disc}: \mathbf{Set} \rightarrow \mathbf{Top}$
- The indiscrete topology functor $\mathbf{Set} \rightarrow \mathbf{Top}$
- The set of connected components functor $\pi_0: \mathbf{Top} \rightarrow \mathbf{Set}$

although we haven’t yet seen why π_0 is a functor. We can compose functors in the obvious way, so get functors $\text{disc}U: \mathbf{Top} \rightarrow \mathbf{Top}$ and $\text{disc}\pi_0: \mathbf{Top} \rightarrow \mathbf{Top}$, for instance.

Here is a trivial-seeming example (aside from the identity functor).

Let \mathcal{C} be a category, and \mathcal{D} a *subcategory*: a collection of some of the objects of \mathcal{C} and some of the morphisms of \mathcal{C} that form a category by themselves. Then the inclusion of the objects and the morphisms forms a functor $\mathcal{D} \hookrightarrow \mathcal{C}$, the *subcategory inclusion*. An important special case of this is when for every X and Y that are objects of \mathcal{D} , every $\mathcal{D}(X, Y) = \mathcal{C}(X, Y)$; then \mathcal{D} is call a *full* subcategory. More generally we can consider a functor that is injective on objects and morphisms to define a subcategory.

Example 13. The functor $\text{disc}: \mathbf{Set} \rightarrow \mathbf{Top}$ makes \mathbf{Set} a full subcategory of \mathbf{Top} .

we have used and will use this result without comment

We will be later restricting attention to certain full subcategories of \mathbf{Top} .

Proposition 4. The assignment $X \mapsto \pi_0(X)$ is a functor $\mathbf{Top} \rightarrow \mathbf{Set}$.

Proof. We need to show there is an assignment $(f: X \rightarrow Y) \mapsto (\pi_0(f): \pi_0(X) \rightarrow \pi_0(Y))$, for an arbitrary continuous function f . We already know we have a function $X \rightarrow Y \rightarrow \pi_0(Y)$, and we want to show this *descends* along $X \rightarrow \pi_0(X)$. Given any $\alpha \in \pi_0(X)$, it corresponds to a connected component of X , namely X_α . Look at the restriction of $X \rightarrow Y \rightarrow \pi_0(Y)$ to X_α : since X_α is connected, its image is exactly one point in $\pi_0(Y)$. So define $\pi_0(f)(\alpha) = [f(x)]$

for an arbitrary $x \in X_\alpha$. This defines $\pi_0(f)$. Moreover, the following diagram *commutes*:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \pi_0(X) & \xrightarrow{\pi_0(f)} & \pi_0(Y) \end{array}$$

Now we want to show that $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$. Given $\alpha \in \pi_0(X)$, and $x \in X_\alpha$, then $\pi_0(f)(\alpha) = [f(x)]$. To define $\pi_0(g)(\pi_0(f)(\alpha))$, we need to choose a point in the component $Y_{[f(x)]}$, so take it to be $f(x)$. Then $\pi_0(g)(\pi_0(f)(\alpha)) = [g(f(x))]$, but this is just $\pi_0(g \circ f)(\alpha)$. \square

Exercise 7. Show that $[\text{pt}, -]: \mathbf{Top} \rightarrow \mathbf{Set}$ is a functor.

Or more generally, $[X, -]: \mathbf{Top} \rightarrow \mathbf{Set}$!

Another important example of a category is the *homotopy category* \mathbf{Ho} .

The objects are topological spaces, but $\mathbf{Ho}(X, Y) = [X, Y]$. There is a functor $\mathbf{Top} \rightarrow \mathbf{Ho}$, which is the identity on objects, and sends a map to its homotopy class. Objects are isomorphic in \mathbf{Ho} iff they are homotopy equivalent.

exercise: prove this is a category

Proposition 5. The functor π_0 descends to a functor $\mathbf{Ho} \rightarrow \mathbf{Set}$

Proof. We will prove that this is well-defined on morphism on hom-sets, the rest is routine. For $f, g: X \rightarrow Y$ be homotopic via $H: I \times X \rightarrow Y$, we need to show that for all $\alpha \in \pi_0(X)$, $\pi_0(f)(\alpha) = \pi_0(g)(\alpha)$. Take x in the connected component X_α , then we have a map $I \rightarrow I \times X \xrightarrow{H} Y \rightarrow \pi_0(Y)$ sending $0 \mapsto \pi_0(f)(\alpha)$ and $1 \mapsto \pi_0(g)(\alpha)$. But I is connected, so $\pi_0(f)(\alpha) = \pi_0(g)(\alpha)$. \square

As a result, if $\pi_0(X) \not\cong \pi_0(Y)$, the spaces X and Y cannot be homotopy equivalent.

Exercise 8. Show the functor $[\text{pt}, -]: \mathbf{Top} \rightarrow \mathbf{Set}$ descends to $\mathbf{Ho} \rightarrow \mathbf{Set}$.

Here is a useful fact about spaces.

Lemma 6. For all families $X_\beta, \beta \in J$, of spaces, we have isomorphisms

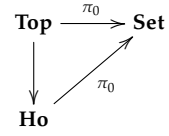
$$\bigsqcup_{\beta \in J} \pi_0(X_\beta) \xrightarrow{\cong} \pi_0\left(\bigsqcup_{\beta \in J} X_\beta\right) \quad \text{and} \quad \bigsqcup_{\beta \in J} [\text{pt}, X_\beta] \xrightarrow{\cong} [\text{pt}, \bigsqcup_{\beta \in J} X_\beta],$$

with inverses induced by the family of maps in_β . That is, π_0 and $[\text{pt}, -]$ preserve coproducts.

Lecture 4

Recall last time: we had functors $\pi_0: \mathbf{Top} \rightarrow \mathbf{Set}$ and $\pi_0: \mathbf{Ho} \rightarrow \mathbf{Set}$.

Example 14. If X and Y are spaces with $|\pi_0(X)| < |\pi_0(Y)|$, no continuous map $X \rightarrow Y$ is surjective.



Here is an instructive example

Example 15. The *topologist's sine curve* is the image C of $[-1, 1] \sqcup (0, 1] \rightarrow \mathbb{R}^2$ defined by

$$\begin{cases} y \mapsto (0, y) & y \in [-1, 1] \\ x \mapsto (x, \sin(\frac{1}{x})) & x \in (0, 1] \end{cases}$$

equipped with the **subspace topology**. This is a compact metric space, using the inherited Euclidean metric. Fact: *every* continuous function $f: C \rightarrow \{0, 1\}$ is constant. If $f(1, \sin(1)) = 1$, then $f(x, \sin(x)) = 1$ for every $x \in (0, 1]$ (as intervals are connected). If $f(0, 0) = b \in \{0, 1\}$, then $f(0, y) = b$ also, for all $y \in [-1, 1]$. The sequence $(\frac{1}{n\pi}, 0)$ converges to $(0, 0)$ in C , so $b = f(0, 0) = \lim_{n \rightarrow \infty} f(\frac{1}{n\pi}, 0) = 1$ as f is continuous and we are in a metric space.

Hence C is connected, but there is *no* continuous function $\gamma: [0, 1] \rightarrow C$ with $\gamma(0) = (0, 0)$ and $\gamma(1) = (1, \sin(1))$. Since intervals are path connected, we can show $[\text{pt}, C] = \{0, 1\}$, but $\pi_0(C) = *$.

Exercise: prove this by considering $\lim_{n \rightarrow \infty} \gamma(\frac{1}{n})$

So we have two different invariants here, and there is always a surjective map $[\text{pt}, X] \rightarrow \pi_0(X)$. Moreover, the following square always commutes, for any map $X \xrightarrow{f} Y$:

$$\begin{array}{ccc} [\text{pt}, X] & \xrightarrow{[\text{pt}, f]} & [\text{pt}, Y] \\ \downarrow & & \downarrow \\ \pi_0(X) & \xrightarrow{\pi_0(f)} & \pi_0(Y) \end{array}$$

This is thus an example of a *natural transformation*.

Definition 16. Given functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$, a natural transformation $\alpha: F \Rightarrow G$ consists of the data:

- i) For every object X of \mathcal{C} , a specified morphism $\alpha_X: F(X) \rightarrow G(X)$ (the *components* of α)

such that for every morphism $f: X \rightarrow Y$ in \mathcal{C} , the following square

commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ G(X) & \xrightarrow{G(f)} & F(Y) \end{array}$$

A natural transformation is called a *natural isomorphism* if all of its components are isomorphisms.

For example, there are natural transformations $\text{disc } U \Rightarrow \text{id}: \mathbf{Top} \rightarrow \mathbf{Top}$, with component at X the identity map $\text{disc}(U(X)) \rightarrow X$, and $\text{id} \Rightarrow \text{disc } \pi_0(X): \mathbf{Top} \rightarrow \mathbf{Top}$, with component $X \rightarrow \pi_0(X)$.

We seek conditions that will define a full subcategory of \mathbf{Top} such that the components $[\text{pt}, X] \rightarrow \pi_0(X)$ are isomorphisms for all spaces X in the subcategory.

Definition 17. A space X is *locally path connected* (lpc) if it has a neighbourhood base of path-connected sets.

Note that a space is lpc iff every connected component is lpc.

Proposition 6. If X is a locally path connected space, then $[\text{pt}, X] \rightarrow \pi_0(X)$ is an isomorphism.

Proof. We are reduced to the case X is connected ($\pi_0(X) = *$) and lpc, by Lemma 6, and the fact the case $X = \emptyset$ is trivial. Since X is connected, take $x \in X$ and define $\chi: X \rightarrow \{0, 1\}$ by

$$\chi(y) = \begin{cases} 1 & \exists y \rightsquigarrow x \\ 0 & \text{otherwise} \end{cases}$$

where by $y \rightsquigarrow x$ I mean a path $\gamma: I \rightarrow X$ with $\gamma(0) = y$ and $\gamma(1) = x$. We will show χ is continuous. Note that χ continuous $\Leftrightarrow p^{-1}(0)$ and $p^{-1}(1)$ open $\Leftrightarrow p^{-1}(1)$ open and closed. But $p^{-1}(1) =: C_x$ is the path component containing x . Take $y \in C_x$ (so $\exists y \rightsquigarrow x$), and $V \ni y$ a path-connected nhd. Given $z \in V$, $\exists z \rightsquigarrow y$. Concatenate these paths to give $z \rightsquigarrow x$, so that $z \in C_x$. This is true for all $z \in V$, so that $V \subseteq C_x$, hence C_x contains a neighbourhood of each of its points, and so is open.

Conversely, take $y \in \overline{C_x}$, $V \ni y$ a path connected nhd. As $\exists z \in V \cap C_x \subseteq V$, $\exists z \rightsquigarrow y$. But also have $V \cap C_x \subseteq C_x$, so $\exists z \rightsquigarrow x$. Concatenate paths to get $y \rightsquigarrow x$, so that $y \in C_x$. This is true for all $y \in \overline{C_x}$, so $\overline{C_x} \subseteq C_x$ and C_x is closed. Hence χ is continuous.

But X is connected, and $\chi(x) = 1$, so that $\text{im } \chi = \{1\}$, and so $C_x = \chi^{-1}(1) = X$. Thus $[\text{pt}, X] \rightarrow \pi_0(X) = *$ is an isomorphism. \square

So we will consider for the rest of this section of the course only lpc spaces, which form a full subcategory $\mathbf{lpcTop} \hookrightarrow \mathbf{Top}$.

Exercise 9. Show that the product of lpc spaces is lpc, and that any locally convex topological vector space is lpc.

Be warned: subspaces of lpc spaces may not be lpc, for instance the topologist's sine curve is a subspace of the contractible \mathbb{R}^2 . Here is a condition that can be used to transfer lpc-ness.

Definition 18. A *local homeomorphism* $X \xrightarrow{p} Y$ is a map of spaces such that for all $x \in X \ni$ nhds $U \ni x$ and $V \ni p(x)$ such that $p|_U: U \xrightarrow{\sim} V$.

Examples include: maps $\sqcup U_\alpha \rightarrow X$ induced by open an open cover $\{U_\alpha\}$ of X , the exponential map $\exp: \mathbb{C} \rightarrow \mathbb{C}^\times$, the n^{th} -power map $U(1) \xrightarrow{(-)^n} U(1)$ and the projection $S \times X \rightarrow X$ for S discrete.

A trivial example is the inclusion of an open subspace

Lemma 7. If $X \xrightarrow{p} Y$ is a local homeomorphism, then Y lpc implies X is lpc. If p is surjective then X lpc implies Y lpc.

One last technical point

Definition 19. A *pointed space* is a pair (X, x) where X is a topological space and $x \in X$. A *pointed map* is a pointed map between the underlying pointed sets that is continuous. These define a category \mathbf{Top}_* .

A *pointed homotopy* of pointed map $I \times X \rightarrow Y$, for (X, x_0) and (Y, y_0) pointed spaces, is required to satisfy $H(t, x_0) = y_0$ for all $t \in I$. Pointed homotopy classes of pointed map are denoted $[(X, x_0), (Y, y_0)]_*$. The category \mathbf{Ho}_* is defined analogously to \mathbf{Ho} . We get a functor $\pi_0: \mathbf{Ho}_* \rightarrow \mathbf{Set}_*$.

Covering spaces

Given a local homeomorphism $X \xrightarrow{p} Y$, the *fibre* $p^{-1}(y) \subset X$ over y has the discrete topology for every $y \in Y$. We have no idea how $p^{-1}(\gamma(t))$ varies along a path $\gamma: I \rightarrow Y$. For example, given an arbitrary collection $B(x_\alpha, r_\alpha) \subset \mathbb{R}^2$ of open balls, $\sqcup_\alpha B(x_\alpha, r_\alpha) \rightarrow \mathbb{R}^2$ is a local homeomorphism, but the fibres can jump in size arbitrarily. We would like fibres 'close' to a given $p^{-1}(y)$ to 'vary continuously'. For spaces 'vary continuously' really means homotopy equivalence. But for discrete spaces, homotopy equivalence is isomorphism.

Definition 20. A *covering space* $Z \xrightarrow{\pi} X$ of X is a map such that for all $x \in X$ there is a nhd $V_x \ni x$ such that $\pi^{-1}(V_x) \simeq V_x \times \pi^{-1}(x)$ over V_x [[expand, maybe?]]

Examples include: $\exp: \mathbb{C} \rightarrow \mathbb{C}^\times$, $S^2 \rightarrow \mathbb{RP}^2$, $U(1) \xrightarrow{(-)^n} U(1)$, covers of the join ∞ of two circles.

Proposition 7. For a covering space $Z \xrightarrow{\pi} X$, if $\exists x_0 \rightsquigarrow x_1$, then $\pi^{-1}(x_0) \simeq \pi^{-1}(x_1)$.

Proof. (First proof) Take $\gamma: I \rightarrow X$, $\gamma(i) = x_i$, and an open cover $\{U_\alpha\}$ of X over which Z trivialises. We thus get an open cover $\gamma^{-1}(U_\alpha)$ of I , which has a finite subcover U_0, \dots, U_N , with $x_0 \in U_0$, $x_1 \in U_N$. We have isomorphisms $Z_{U_n} := \pi^{-1}(U_n) \xrightarrow{\phi_n} U_n \times F_n$ with discrete spaces F_n . We have $\pi^{-1}(x_0) \simeq F_0$, and for all $t \in \gamma^{-1}(U_0)$, $\pi^{-1}(\gamma(t)) \simeq F_0$.

□