# *Cohomology of a combinatorial Möbius strip*[1]

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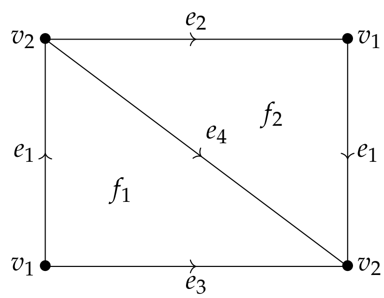
*2019*



Consider the following $\Delta$-set, $M_\bullet$:

it is instructive to check that this is indeed a $\Delta$-set!




We will calculate the cohomology with coefficients $\mathbb{Z}$ and with coefficients $\mathbb{Z}/2$.

## *$\mathbb{Z}$ coefficients*

The complex $C^\bullet(M_\bullet, \mathbb{Z})$ of abelian groups associated to $M_\bullet$ is

$$0 \to \mathbb{Z}^2 \xrightarrow{\delta_0} \mathbb{Z}^4 \xrightarrow{\delta_1} \mathbb{Z}^2 \to 0$$

where we need to describe the homomorphisms $\delta_0$ and $\delta_1$. We can take as generating sets $\{v_1, v_2\}$, $\{e_1, e_2, e_3, e_4\}$ and $\{f_1, f_2\}$, respectively,[2] and write down the two matrices with respect to these generating sets, from considering the face maps $d_0, d_1 \colon M_1 \to M_0$:

[2] identifying these with the indicator function for that element, so that, for instance $v_1(v) = 1$ if $v = v_1$ and zero othewise

$$(d_0, d_1)(e_i) = \begin{cases} (v_2, v_1) & i = 1 \\ (v_1, v_2) & i = 2 \\ (v_2, v_1) & i = 3 \\ (v_2, v_2) & i = 4 \end{cases}$$

and $d_0, d_1, d_2 \colon M_2 \to M_1$:

$$(d_0, d_1, d_2)(f_i) = \begin{cases} (e_4, e_3, e_1) & i = 1 \\ (e_1, e_4, e_2) & i = 2 \end{cases}$$

We can then calculate the matrix $D_0$ representing $\delta_0$, with entries $\delta_0(v_i)(e_j) = v_i(d_0(e_j)) - v_i(d_1(e_j))$, namely

$$D_0 := \begin{pmatrix} \delta_0(v_1)(e_1) & \delta_0(v_2)(e_1) \\ \delta_0(v_1)(e_2) & \delta_0(v_2)(e_2) \\ \delta_0(v_1)(e_3) & \delta_0(v_2)(e_3) \\ \delta_0(v_1)(e_4) & \delta_0(v_2)(e_4) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}$$

Similarly, the matrix $D_1$ with entries $\delta_1(e_i)(f_j) = e_i(d_0(f_j)) - e_i(d_2(f_j)) + e_i(d_2(f_j))$ represents $\delta_1$:

$$D_1 := \begin{pmatrix} \delta_1(e_1)(f_1) & \delta_1(e_2)(f_1) & \delta_1(e_3)(f_1) & \delta_1(e_4)(f_1) \\ \delta_1(e_1)(f_2) & \delta_1(e_2)(f_2) & \delta_1(e_3)(f_2) & \delta_1(e_4)(f_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix}$$

One can check that $D_1 D_0$ is the zero matrix. So now the integral cohomology groups of $M_\bullet$ are as follows

1. $H^0(M_\bullet, \mathbb{Z}) = \ker D_0 = (v_1 + v_2)\mathbb{Z}$.
2. $H^1(M_\bullet, \mathbb{Z}) = \ker D_1 / \operatorname{im} D_0$. But

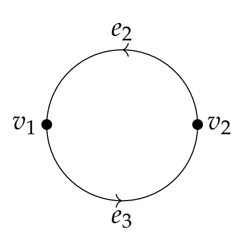
$$\begin{aligned} \ker D_1 &= \{ae_1 + be_2 + ce_3 + de_4 \in \mathbb{Z}^4 \mid b = -a + d \text{ and } c = a + d\} \\ &= \{a(e_1 - e_2 + e_3) + d(e_2 + e_3 + e_4) \mid a, d \in \mathbb{Z}\} \end{aligned}$$

   and

$$\operatorname{im} D_0 = \{a(e_1 - e_2 + e_3) \mid a \in \mathbb{Z}\},$$

   so that $H^1(M_\bullet, \mathbb{Z}) = (e_2 + e_3 + e_4)\mathbb{Z}$
3. $H^2(M_\bullet, \mathbb{Z}) = \mathbb{Z}^{M_2} / \operatorname{im} D_1 = \mathbb{Z}^2/\mathbb{Z}^2 = 0$.
4. $H^k(M_\bullet, \mathbb{Z}) = 0$ for all $k > 2$.

We can consider also the inclusion map of $\Delta$-sets $i \colon C_\bullet \hookrightarrow M_\bullet$ where the sub-$\Delta$-set $C_\bullet$ is given by




There are induced maps on cohomology $H^k(M_\bullet, \mathbb{Z}) \to H^k(C_\bullet, \mathbb{Z})$ for $k = 0, 1, 2$. Let's calculate both $H^k(C_\bullet, \mathbb{Z})$ and these maps. Since $C_\bullet$ is

really for all $k$, but these are clearly trivial for $k > 2$

a directed graph, hence 1-dimensional, we only need to calculate $H^0$ and $H^1$ of the complex

$$0 \to \mathbb{Z}^2 \xrightarrow{\delta_0} \mathbb{Z}^2 \to 0$$

or in other words, $\ker \delta_0$ and $\operatorname{coker} \delta_0$. The face maps $d_0, d_1 \colon C_1 \to C_0$ are given by

$$(d_0, d_1)(e_i) = \begin{cases} (v_1, v_2) & i = 2 \\ (v_2, v_1) & i = 3 \end{cases}$$

so $\delta_0$ is represented by the matrix

$$D := \begin{pmatrix} \delta_0(v_1)(e_2) & \delta_0(v_2)(e_2) \\ \delta_0(v_1)(e_3) & \delta_0(v_2)(e_3) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

with

1. $H^0(C_\bullet, \mathbb{Z}) = \ker D = (v_1 + v_2)\mathbb{Z}$
2. $H^1(C_\bullet, \mathbb{Z}) = \mathbb{Z}^2 / \operatorname{im} D = \mathbb{Z}^2 / (-e_2 + e_3)\mathbb{Z} \simeq (e_2 + e_3)\mathbb{Z}$

So now let's calculate the map of complexes that gives rise to the map on cohomology, namely

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^2 \arrow[r] \arrow[d, "i_0^*"'] & \mathbb{Z}^4 \arrow[r] \arrow[d, "i_1^*"'] & \mathbb{Z}^2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

Since $i_0 \colon C_0 = \{v_1, v_2\} \to \{v_1, v_2\} = M_0$ is the identity map, $i_0^* \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is the identity map, so that $\ker D_0 \to \ker D$ is also the identity map. This is to be expected, since both $\Delta$-sets have one "connected component". However, the map induced by $i_1^*$ could potentially be more interesting. The homomorphism $i_1^*$ is a projection, represented by the matrix

$$I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We then consider the map $\ker D_1 \to \mathbb{Z}^2 = \mathbb{Z}^{C_1}$, namely

$$\ker D_1 \ni a(e_1 - e_2 + e_3) + d(e_2 + e_3 + e_4) \mapsto a(-e_2 + e_3) + d(e_2 + e_3) \in \mathbb{Z}^{C_1}$$

So then the induced map $H^1(M_\bullet, \mathbb{Z}) \to H^1(C_\bullet, \mathbb{Z})$ is also an isomorphism

$$(e_1 - e_2 + e_3)\mathbb{Z} \xrightarrow{\simeq} (e_2 + e_3)\mathbb{Z}$$

sending $(e_1 - e_2 + e_3) \mapsto (e_2 + e_3)$. Thus the map $C_\bullet \to M_\bullet$ induces an isomorphism $i^* \colon H^k(M_\bullet, \mathbb{Z}) \xrightarrow{\simeq} H^k(C_\bullet, \mathbb{Z})$ for all $k$, meaning cohomology with integer coefficients cannot tell these two combinatorial surfaces apart.

## $\mathbb{Z}/2$ *coefficients*

Now I will calculate the cohomology of $M_\bullet$ using coefficients in $\mathbb{Z}/2$. A lot of the hard work has been done, because the complex $C^\bullet(M_\bullet, \mathbb{Z}/2)$ of $\mathbb{Z}/2$-modules looks the same as the complex $C^\bullet(M_\bullet, \mathbb{Z})$, but where we now look at the entries of the matrices $D_0$ and $D_1$ mod 2. Call these matrices $D_0'$ and $D_1'$:

$$D_0' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad D_1' = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

So now

1. $H^0(M_\bullet, \mathbb{Z}/2) = \ker D_0' = \{av_1 + bv_2 \mid a, b \in \mathbb{Z}/2,\ a + b = 0\} = (v_1 + v_2)\mathbb{Z}/2$
2. $H^1(M_\bullet, \mathbb{Z}/2) = \ker D_1' / \operatorname{im} D_0'$, where

   $$\begin{aligned} \ker D_1' &= \{ae_1 + be_2 + ce_3 + de_4 \in (\mathbb{Z}/2)^4 \mid b = a + d \text{ and } c = a + d\} \\ &= \{a(e_1 + e_2 + e_3) + d(e_2 + e_3 + e_4) \mid a, d \in \mathbb{Z}/2\} \end{aligned}$$

   and $\operatorname{im} D_0' = (e_1 + e_2 + e_3)\mathbb{Z}/2$, so

   $$H^1(M_\bullet, \mathbb{Z}/2) = (e_2 + e_3 + e_4)\mathbb{Z}/2.$$

3. $H^2(M_\bullet, \mathbb{Z}/2) = (\mathbb{Z}/2)^{M_2} / \operatorname{im} D_1' = (\mathbb{Z}/2)^2 / (\mathbb{Z}/2)^2 = 0.$

In this case, the cohomology $\mathbb{Z}/2$-modules are the mod 2 reduction of the integer-coefficient cohomology groups, but this **not always the case!**