Algebraic Topology[1]

David Michael Roberts

2019

What is it?

Lecture 1

Algebraic topology is the study of maps

$$\{\text{Spaces}\} \longrightarrow \{\text{Algebraic objects}\},$$

or rather, 'well-behaved' such maps. They should also send continuous functions between spaces to algebraic maps, respecting composition (so: *functors*); they should send spaces built out of simpler spaces to algebraic objects built out of simpler components, in a compatible way, etc.

Here, 'Spaces' roughly means topological spaces up to deformation (usually homotopy, but not always). Such equivalence classes are called *homotopy types*. 'Algebraic objects' means (abelian) groups, rings, modules, or even chain complexes of these .

a chain complex is a certain sequence of maps $\cdots \to V_0 \to V_1 \to V_2 \to \cdots$

Example 1. How can we tell if the sphere S^2 and the torus $S^1 \times S^1$ can or cannot be deformed into each other? How would you prove it cannot be done?

Example 2. For a positive example, we *can* squash $\mathbb{R}^3 \setminus \{0\} \to S^2 \hookrightarrow \mathbb{R}^3 \setminus \{0\}$, sending $x \mapsto \frac{x}{|x|}$. This map continuously deforms to the identity map. So dimension not necessarily preserved.

Example 3. Can we have $S^1 \sim S^2$?

We first need to understand how spaces are built

Topological spaces

Recall. . .

From Topology and Analysis III

Definition 1. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that

1. $\emptyset, X \in \mathcal{T}$
2. If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$

3. If $\{U_\alpha\}_{\alpha \in I}$ is an arbitrary family of sets in $\mathcal{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

I here is an indexing set

If $U \in \mathcal{T}$ we say U is *open*. A *topological space* is a set X eqipped with a topology $\mathcal{T}$.

Example 4. Take the set of real numbers, the *Euclidean ('usual') topology* is defined by saying a set is open iff it is a union of open intervals (a, b) (including the union of no sets ie $\emptyset$).
The *discrete topology* on a set X is defined by taking every $\mathcal{T}$ to consist of all subsets. The *indiscrete topology* is defined by taking $\mathcal{T}$ to consist of just $\emptyset$ and X.

This definition is concise, but not always the best way to define a topology. We will also use *neighbourhoods*

Definition 2. A set $N \subseteq X$ is a *neighbourhood* (in a given topology $\mathcal{T}$) of a point $x \in X$ if there is an open set $U \subseteq N$ with $x \in U$.

'nhd' is a good abbreviation

Example 5. Take $\mathbb{R}$ with the Euclidean topology. $(-1, 1)$, $[-1, 1]$, $[-1, 1)$ are all neighbourhoods of every $-1 < x < 1$, but $[0, 1)$ is not a neighbourhood of 0. More complicated: $[0, 1] \cup \{2\} \cup [5, 6]$ is a nhd of all $0 < x < 1$ and $5 < x < 6$.

Example 6. Consider a metric space (X, d). The *metric topology* is defined by saying a subset $U \subseteq X$ is open iff for every $x \in U$ there is some $\varepsilon_x > 0$ with the open ball $B(x, \varepsilon_x) \subseteq U$. Open balls around x are neighbouhoods of x, as are closed balls.

Here is a more concrete approach that allows concise definitions of topologies:

Definition 3. A *neighbourhood base* $\mathcal{N}$ on a set X is a family $\{\mathcal{N}(x)\}_{x \in X}$ where each $\mathcal{N}(x)$ is a nonempty collection of subsets of X, satisfying the following, for all $x \in X$:

1. For all $N \in \mathcal{N}(x)$, $x \in N$;
2. For all $N_1, N_2 \in \mathcal{N}(x)$, there is some $N \in \mathcal{N}(x)$ with $N \subseteq N_1 \cap N_2$;
3. For all $N \in \mathcal{N}(x)$ there is a subset $U \subseteq N$ such that $x \in U$ and for all $y \in U$, there is some $V \in \mathcal{N}(y)$ such that $V \subseteq U$.

We say the sets in $\mathcal{N}(x)$ are *basic neighbourhoods* of x.

As an example: given a topological space $(X, \mathcal{T})$ defining $\mathcal{N}(x)$ to consist of all nhds of x gives a nhd base. Similarly, defining $\mathcal{N}'(x)$ to consist of all open sets containing x defines a nhd base.

Given a neighbourhood base $\mathcal{N}$ on a set X, define a subset $U \subseteq X$ to be $\mathcal{N}$*-open* iff for all $x \in U$, there is an $N \in \mathcal{N}(x)$ with $N \subseteq U$.

Proposition 1. The $\mathcal{N}$-open sets define a topology on X.

Proof. We verify the axioms for a topology on X.

1. The condition that $\emptyset$ is $\mathcal{N}$-open is vacuously true. And since $\mathcal{N}(x)$ is not empty, there is a basic nhd around every point, so X is $\mathcal{N}$-open.

2. Given U, V both $\mathcal{N}$-open, we want to show $U \cap V$ is $\mathcal{N}$-open. So take $x \in U \cap V$. We know there is $N_U, N_V \in \mathcal{N}(x)$ with $N_U \subseteq U$ and $N_V \subseteq V$, and also that $x \in N_U \cap N_V$, since it is in each of them. Thus there is some $N \in \mathcal{N}(x)$ with $N \subseteq N_U \cap N_V \subseteq U \cap V$, and this is true for all $x \in U \cap V$. Hence $U \cap V$ is $\mathcal{N}$-open.

3. Given a family U_α, $\alpha \in I$, with each U_α $\mathcal{N}$-open, we want to show $U := \bigcup_{\alpha \in I} U_\alpha$ is $\mathcal{N}$-open. Take $x \in U$, so there is some α_0 with $x \in U_{\alpha_0}$. But this set in $\mathcal{N}$-open, so there is some nhd N of x with $N \subseteq U_{\alpha_0} \subseteq U$, and this is true for all $x \in U$. So U is $\mathcal{N}$-open. □

We call the topology from this proposition the topology generated by $\mathcal{N}$. Neighbourhoods in this topology are sets that contain a basic neighbourhood: V is a neighbourhood of x if there is some $N \in \mathcal{N}(x)$ with $N \subseteq V$.

Given a neighbourhood base $\mathcal{N}$ on X, we can identify the *closure* of a set $S \subset X$ as the collection of points $x \in X$ such that for all $N \in \mathcal{N}(x)$, $\exists s \in N \cap S$.

Example 7. Given a metric space (X, d) the open balls form a nhd base on X and the topology they generate is the metric topology.

Hence many definitions you are familiar with from metric spaces work for topological spaces, if they can be phrased in terms of basic nhds. In particular, continuity!

Definition 4. Let $\mathcal{N}_X$ and $\mathcal{N}_Y$ be neighbourhood bases on sets X and Y respectively. A function $f \colon X \to Y$ is *continuous* if for every $x \in X$ and $N \in \mathcal{N}_Y(f(x))$, the set $f^{-1}(N)$ contains a basic nhd of x.

This is a big generalisation of the ε-δ definition of continuity.

Exercise 1. Show that if $f \colon (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ is continuous as just defined, it is continuous for the topologies generated on X and Y by these nhd bases.

Recall a function is continuous for topologies if $f^{-1}(U)$ is open for all open U.

As a sanity check, the identity function id_X on a space X is indeed continuous.

You can check every function *to* an indiscrete space is continuous, as is every function *on* a discrete space

Definition 5. A continuous function $f\colon X \to Y$ is a *homeomorphism* if there is a continuous function $g\colon Y \to X$ with $g \circ f = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$. We then call X and Y *homeomorphic* if there is a homeomorphism between them.

or just isomorphic, if I'm being lazy

Now we need to show how to build new spaces, and continuous maps relating them to the original spaces.

Definition 6. Let X be a set, $(Y_\alpha, \mathcal{N}_\alpha)$, $\alpha \in I$ a family of sets with nhd bases (not necessarily all unique), and $f_\alpha\colon X \to Y_\alpha$ a family of functions. The *initial topology* on X is generated by the following nhd base: a subset of X is a basic nhd of x iff it is of the form $f_{\alpha_1}^{-1}(N_1) \cap \ldots \cap f_{\alpha_k}^{-1}(N_k)$ for some $\alpha_1, \ldots, \alpha_k$ and $N_i \in \mathcal{N}_{\alpha_i}(f_{\alpha_i}(x))$.

Exercise: verify this is a nhd base!

This generalises the product topology, which is the case that $X = Y_1 \times Y_2$, and $f_i\colon X \to Y_i$ is the projection $f_i(y_1, y_2) = y_i$, where $i = 1, 2$. But this *also* gives the subspace topology: take $f\colon X \hookrightarrow Y$ to be injective and define the initial topology on X.

Lemma 1. Giving X the initial topology, all the functions $f_\alpha\colon X \to Y_\alpha$ are continuous. Moreover, a function $k\colon Z \to X$ is continuous iff $f_\alpha \circ k\colon Z \to Y_\alpha$ is continuous for every α.

Lecture 2

Example 8. If the set of functions consists of a single *injective* map, namely $\iota\colon X \hookrightarrow Y$, with Y a space, then the initial topology is the subspace topology: basic nhds of x correspond to sets $\iota^{-1}(N)$ (basically $N \cap X$) for N a basic nhd of $\iota(x)$.

Example 9. If however we have a constant function $c_{y_0}\colon X \to Y$, sending $x \mapsto y_0 \in Y$ for all x, then for every nhd N of y_0, $c_{y_0}^{-1}(N) = X$. So the only nhd of every $x \in X$ is X itself. Thus the initial topology is indiscrete in this case.

In general, given the family of functions $f_\alpha\colon X \to Y_\alpha$, there is a function $(f_\alpha)\colon X \to \prod_\alpha Y_\alpha$. If we give $\prod_\alpha Y_\alpha$ the product topology, then the initial topology on X from the family of maps is the same as the initial topology from the map (f_α) to the product space. So if this latter map is injective, X inherits the subspace topology from the product topology. This is the major use-case we will come across for the initial topology.

Example 10. A submanifold $M \subseteq \mathbb{R}^n$ gets its topology from the coordinate functions $M \hookrightarrow \mathbb{R}^n \xrightarrow{x_i} \mathbb{R}$, and a map to M is continuous iff the composite with the maps to each factor of $\mathbb{R}^n$ are continuous.

Exercise 2. Given a set X, a space Y and a function $f\colon X \to Y$, if two points x_1, x_2 satisfy $f(x_1) = f(x_2)$, show that a subset $V \subseteq X$ is a nhd of x_1 iff it is a nhd of x_2, in the initial topology.

The following will be even more important for us, and will be new to most.

Definition 7. Let X be a set, $(Z_\beta, \mathcal{N}_\beta)$, $\beta \in J$ a family of topological spaces (not necessarily all unique), and $g_\beta\colon Z_\beta \to X$ a family of functions (note the other direction!). The *final topology* on X has open sets as following: $U \subset X$ is open iff for all $\beta \in J$, $g_\beta^{-1}(U)$ is open in Z_β.

this really is easier to describe using open sets, rather than nhds

Lemma 2. Giving X the final topology, all the functions $g_\beta\colon Z_\beta \to X$ are continuous. Moreover a function $h\colon X \to W$ is continuous for the final topology on X iff $h \circ g_\beta\colon Z_\beta \to W$ is continous for every $\beta \in J$.

We will give two special cases of this, and we will see them often.

Example 11. Let Z be a topological space, and let $\sim$ be an equivalence relation on Z, and define $X = Z/\sim$ to be the quotient by this relation. There is a function $\pi\colon Z \to X$ sending $y \mapsto [y]$. The final topology on X has as open sets those $U \subseteq X$ such that $\pi^{-1}(U)$ is open in Z.

For instance, we can give S^2 the initial topology for the maps $x_i\colon S^2 \to \mathbb{R}^3 \xrightarrow{\mathrm{pr}_i} \mathbb{R}$ (this is the usual topology on S^2), and then define the equivalence relation on S^2 generated by $x \sim -x$ for all $x \in S^2$. The quotient space is $\mathbb{RP}^2$, the real projective plane, and we give it the final topology coming from $S^2 \to \mathbb{RP}^2$. This is the topology it carries as a manifold. Incidentally, S^2 is an example of a *covering space* of $\mathbb{RP}^2$, the study of which will occupy the first section of the course.

Recall the definition of disjoint union of sets: given Z_β, $\beta \in J$, a family of sets, we have $\mathrm{in}_\gamma\colon Z_\gamma \hookrightarrow \coprod_\beta Z_\beta$ with $Z_\beta \cap Z_\gamma = \emptyset$ for $\beta \neq \gamma$. If Z_β are spaces, then we give $\coprod_\beta Z_\beta$ the final topology for the maps in_γ. This is *disjoint union* or *sum* topology, and $\coprod_\beta Z_\beta$ is sometimes called the *topological sum*. A point in $\coprod_\beta Z_\beta$ can be described by a pair (β, z), where $z \in Z_\beta$.

an important fact is that the map $\coprod_\beta X \times Z_\beta \to X \times \coprod_\beta Z_\beta$ is a homeomorphism (exercise!)

Exercise 3. Given continuous functions $h_\beta\colon Z_\beta \to W$, there is a unique continuous function $h = \langle h_\beta \rangle\colon \coprod_\beta Z_\beta \to W$ with $h_\beta = h \circ \mathrm{in}_\beta$, or in other words this diagram commutes:

$$\begin{tikzcd} Z_\gamma \arrow[r, "\mathrm{in}_\gamma"] \arrow[dr, "h_\gamma"'] & \coprod_\beta Z_\beta \arrow[d, "h"] \\ & W \end{tikzcd}$$

Lemma 3. The final topology on X for $g_\beta \colon Z_\beta \to X$ agrees with the final topology on X for $g = \langle g_\beta \rangle \colon \coprod_\beta Z_\beta \to X$, using the sum topology.

Proof. We have that $U \subseteq X$ is open iff $\forall \beta\ g_\beta^{-1}(U)$ is open iff $\forall \beta$, $(g \circ \mathrm{in}_\beta)^{-1}(U) = \mathrm{in}_\beta^{-1}\left(g^{-1}(U)\right)$ is open iff $g^{-1}(U)$ is open in the sum topology. □

The idea behind the final topology, when $g_\beta \colon Z_\beta \to X$ are jointly surjective, is that we can put an equivalence relation on $\coprod_\beta Z_\beta$ with $(\beta_1, z_1) \sim (\beta_2, z_2)$ iff $g_{\beta_1}(z_1) = g_{\beta_2}(z_2) \in X$. As a set, X is the set of equivalence classes under this relation, so you can think of it as gluing together the *underlying sets* of the spaces Z_β. The final topology on X is then the only sensible topology to described the space we get by gluing together the *spaces* Z_β.

this means $\forall x \in X, \exists \beta, x \in Z_\beta$ with $g_\beta(z) = x$

Exercise 4. Given an open cover $\{U_\alpha\}$ of a space X, then X carries the final topology for the inclusion maps $U_\alpha \hookrightarrow X$, or equivalently for the map $\coprod_\alpha U_\alpha \to X$.

Example 12. An arbitrary manifold M has the final topology arising from any choice of atlas.

Exercise 5. Given a *finite* closed cover $\{V_i\}_{i=1}^n$ of X, then X carries the final topology for $\coprod_{i=1}^n V_i \to X$.

Example 13. Any closed interval $[a,b] \subset \mathbb{R}$ with the subspace topology has the final topology arising from a collection of subintervals $[a,t_1]$, $[t_1,t_2]$,...,$[t_k,b]$, each with the subspace topology from $\mathbb{R}$.

These exercises give us what is sometimes known as the *gluing* (or *pasting*) *lemma*:

Lemma 4. Consider a space X and an arbitrary open cover $\{U_\alpha\}_{\alpha \in I}$ (respectively a finite closed cover $\{V_i\}_{i=1}^n$) and suppose Y is some other topological space. Then if a function $f \colon X \to Y$ is continuous when restricted to each U_α (resp. to each V_i) then f is continuous.

Later we'll see spaces that are built up by gluing together lots of 'simple' spaces, like disks $D^n := \{x \in \mathbb{R}^n \mod |x| \leq 1\}$ (with the subspace topology from $\mathbb{R}^n$). But what does 'simple' here mean? Roughly, "shrinkable to a point".

Homotopy

"Shrinkable" implies a kind of continuous process in time. Consider the function $I \times D^n \to D^n$. Consider the map

$$\begin{aligned} H\colon I \times D^n &\to D^n \\ (t, \mathbf{x}) &\mapsto (1-t)\mathbf{x} \end{aligned}$$

Note that this gives maps $H_0\colon D^n \to D^n$ (the identity map) and H_1 (constant at 0). The function H is continuous! How should we see this? The topology on D^n is the subspace topology $D^n \subset \mathbb{R}^n$, and $\mathbb{R}^n$ has the product topology. so the topology on D^n is also the initial topology for the coordinate functions $x_i\colon D^n \to \mathbb{R}^n \to \mathbb{R}$. So $H\colon I \times D^n \to D^n$ is continuous iff

And $I \subset \mathbb{R}$ has subspace topology

$$\begin{array}{ccccccc} I \times D^n & \xrightarrow{\mathrm{id} \times x_i} & I \times \mathbb{R} & \longrightarrow & \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (t, \mathbf{x}) & \longmapsto & & & (t, x_i) & \longmapsto & t x_i \end{array}$$

But $I \times D^n \to \mathbb{R} \times \mathbb{R}$ is continuous by definition of initial topology, and the following result:

Exercise 6. If $f\colon X \to W$ and $g\colon Y \to Z$ are continuous, then so is $f \times g\colon X \times Y \to W \times Z$. If both X and Y have at least one point each, then the reverse implication also holds.

So if we can prove that multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, then H is continuous. But the standard topology on $\mathbb{R}$ comes from the metric space structure, so can use sequential criterion for continuity. Take $(a_n, b_n) \to (a, b)$ in $\mathbb{R} \times \mathbb{R}$, then:

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a b_n + a b_n - ab| \\ &\le |a_n - a|\,|b_n| + |a|\,|b_n - b| \\ &\le |a_n - a| \sup |b_n| + |a|\,|b_n - b| \qquad \text{(as (b_n) converges, it is bounded)} \\ &\to 0 + 0 \end{aligned}$$

Hence H is continuous.

Definition 8. A space X is *contractible* if there is a point $x_0 \in X$ and a continuous function $H\colon I \times X \to X$ such that $H(0, x) = x$ and $H(1, x) = x_0$ for all $x \in X$. Such a function is called a *contraction*.

or, *contractible to* $x_0 \in X$

We have shown D^n is contractible.

Exercise 7. $\mathbb{R}$ is contractible. An arbitrary product of contractible spaces is contractible.

Example 14. Consider what it would mean if a discrete space S were contractible: there would be an element $* \in S$ and a continuous function $h\colon I \times S \to S$ such that $h(0,s) = s$ and $h(1,s) = *$. Restricting h to $I \times \{s\}$ for some given s, we get a continuous function $I \hookrightarrow I \times S \to S$, whose range includes $*$ and s. Since all functions with discrete domain are continuous, let us compose with the continuous function $\chi_{\{*\}}\colon S \to \mathbb{R}$ that sends $* \mapsto 1$ and $s \mapsto 0$ for all $s \neq *$. So we have a continuous function $\widetilde{h}\colon I \to \mathbb{R}$ with $\widetilde{h}(0) = 0$ and range contained in $\{0,1\}$. By the intermediate value theorem, we must have $\widetilde{h}(1) = \chi_{\{*\}}(h(1,s)) = 0$, so that $h(1,s) = *$, and hence $s = *$ for all $s \in S$. Thus S has exactly one element.

Question 1. If X is contractible, does the choice of point $x_0 \in X$ matter? Is X also contractible to $x \in X$ for $x \neq x_0$?

The interval can only map continuously to a discrete space if it is constant at some element, or equivalently, its image consists of a single point, and this property is important enough to have a name.

Definition 9. A space X is *connected* if every continuous map from X to a discrete space has image a single point.

If you know the 'usual' definition, this is equivalent to it

So the interval I is an example of a connected space. Even better: if a pair of points $x, y \in X$ have a *path* between them (a map $I \xrightarrow{\gamma} X$ with $\gamma(0) = x$, $\gamma(1) = y$) then any function $f\colon X \to S$ to a discrete space has $f(x) = f(y)$.

Example 15. Every contractible space is connected. This is because in a contractible space X, for every point y there is the path $t \mapsto H(t,y)$ joining y to the point x_0, so that $f(y) = f(x_0)$ for every map $X \xrightarrow{f} S$ to discrete S.

There are however lots of spaces that are connected but not contractible, but we cannot yet prove this.

This is our first example of an invariant of spaces, namely whether they are connected or not: a connected space X cannot be homeomorphic to a space Z that is not connected. But, how can we tell non-connected spaces apart?

Lecture 3

Consider
$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d] & Z \arrow[dl] \\ S & \end{tikzcd}$$
with S discrete.

Definition 10. 1. For any space X, a subset $Y \subseteq X$ is a *connected component* of X if Y is connected and for any connected $Y' \subseteq X$ such that $Y \subseteq Y'$, then $Y = Y'$.

2. Put an equivalence relation on X generated by $x_1 \sim x_2$ iff x_1 and x_2 are both contained in a connected subset $C \subseteq X$. Then define $\pi_0(X) = X/\sim$, the *set of connected components*.

Exercise: If $C, D \subseteq X$ are connected, and $\exists x \in C \cap D$, then $C \cup D$ is connected. Also show: the equivalence classes are the connected components.

Every connected space X has $\pi_0(X) = *$, but now we can tell apart non-connected spaces, by comparing their π_0. Every space that we will be consider in this course can be written as $X = \coprod_{\alpha \in \pi_0(X)} X_\alpha$, with X_α connected, and have a continuous function $X \to \pi_0(X)$ where $\pi_0(X)$ has the discrete topology. As a result, we need to try to understand *connected* spaces, though we will still *use* non-connected spaces.

Such spaces are called 'locally connected', but we will eventually be assuming a slightly stronger condition. Be warned: $\mathbb{Q}$ with the Euclidean topology is **not** locally connected, nor are many very interesting examples!

Can we get more out of the idea of contractions? Given $H\colon I \times X \to X$, we have maps H_i for $i = 0, 1$, namely $H_0 = \mathrm{id}_X$ and H_1 is constant at x_0. What if H_0 and H_1 were other sorts of continuous maps?

Example 16. Consider the annulus $A(r, R) := \{x \in \mathbb{R}^2 \mid r \leq |x| \leq R\}$, and the function $H(t, x) = ((1-t)r + tR)x/|x|$.

What if we considered general continuous maps $X \to Y$ instead of just $X \to X$?

Definition 11. A *homotopy* is a continuous function $H\colon I \times X \to Y$. If $f = H(0, -)$ and $g = H(1, -)$, we say H is a *homotopy from f to g*, and that f and g are *homotopic*, written $f \sim g$.

A useful picture is:

$$\begin{tikzcd}
\{0\} \times X \arrow[d] \arrow[dr, "f"] & \\
I \times X \arrow[r, "H"] & Y \\
\{1\} \times X \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

Example 16 gives a homotopy between the two 'retraction' maps $A(r, R) \to A(r, R)$, mapping points to the inner and outer circles respectively.

Algebraic topology most of the time considers functions *up to homotopy*, and also "spaces up to homotopy".

Definition 12. A continuous function $f\colon X \to Y$ is called a *homotopy equivalence* if there is a continuous function $g\colon Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. We then say X and Y are *homotopy equivalent*.

Example 17. A contractible space is homotopy equivalent to a one-point space.

You should think of homotopy equivalences as being 'kinda like isomorphism', but coarser. Going back to our original motivation, the maps

$$\{\text{Spaces}\} \longrightarrow \{\text{Algebraic objects}\}$$

under consideration should take homotopy equivalent spaces to isomorphic algebraic objects. To make this more rigorous we will use the language of category theory.

Here is a super-important property of homotopies we will use continuously.

Proposition 2. Given homotopies $H\colon I \times X \to Y$ and $H'\colon I \times X \to Y$ such that $H_1 = H'_0\colon X \to Y$, there is a homotopy H'' from H_0 to H_1, and a homotopy $\widetilde{H}$ from H_1 to H_0.

Proof. We will use Exercise 5 applied to the closed cover $\{[0, \frac{1}{2}] \times X, [\frac{1}{2}, 1] \times X\}$ of $I \times X$. Since $I \simeq [0, \frac{1}{2}]$ and $I \simeq [\frac{1}{2}, 1]$, H and H' give us maps $[0, \frac{1}{2}] \times X \simeq I \times X \xrightarrow{H} Y$ and $[0, \frac{1}{2}] \times X \simeq I \times X \xrightarrow{H'} Y$ respectively. By the assumption on H_1 and H'_0, we get a well-defined function $H''\colon I \times X \to Y$, which is then continuous by the Exercise. It is a simple check to see it is a homotopy from H_0 to H'_1.
For the second part, let $c\colon I \to I$ be the function $c(t) = 1 - t$. Then define H'' to be the composite $I \times X \xrightarrow{c \times \mathrm{id}_X} I \times X \xrightarrow{H} Y$, which has the required properties. □

Contractible spaces supply many homotopies.

Lemma 5. Every continuous function $f\colon X \to Y$, with Y a contractible space (say to $y_0 \in Y$), is homotopic to a function with range contained in $\{y_0\}$.

Proof. Let $H\colon I \times Y \to Y$ be a homotopy witnessing the contractility of Y. Then the composite $I \times X \xrightarrow{\mathrm{id}_I \times f} I \times Y \xrightarrow{H} Y$ is a homotopy from f to the the desired function. □

As a corollary, every pair of functions to a contractible space are homotopic. Since contractible spaces are in some sense trivial, maps to them are in the same sense trivial.

An important intermediate version of this is when we consider only the case where X is discrete, or is even just pt:

Definition 13. A space Y is *path-connected* if every map $\mathrm{pt} \to Y$ is homotopic to every other such map.

This condition is equivalent to requiring it for *all* discrete spaces in place of pt (Exercise!)

Unpacking this, we see this means that for any two points $\mathrm{pt} \to Y$ there is a path $I \to Y$ connecting them, i.e. $H\colon I \simeq I \times \mathrm{pt} \to Y$.

Proposition 3. A path-connected space is connected

Let us define $[X, Y] = \{\text{continuous } f\colon X \to Y\}/\text{homotopy}$. The set of *path components* of Y is then the set $[\mathrm{pt}, Y]$. The space Y is called *path connected* if $[\mathrm{pt}, Y] = *$.

We have been discussing topological spaces and continuous maps, but also implicitly sets and functions, not necessarily continuous, and passing between these two pictures. In both cases we have composition that is associative, and identity maps. Later we shall be using different classes of topological spaces in order to ensure the behaviour we require will hold.

Definition 14. A *category* $\mathcal{C}$ consists of a collection of *objects* $W, X, Y, Z, \ldots$ and for each pair of objects X, Y a collection of *morphisms*, denoted $\mathcal{C}(X, Y)$, together with the following data:

i) For each pair $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, a specified morphism $g \circ f \in \mathcal{C}(X, Z)$,

ii) For every object a specified morphism $\mathrm{id}_X \in \mathcal{C}(X, X)$,

such that:

1. For every triple $h \in \mathcal{C}(W, X)$, $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have $g \circ (f \circ h) = (g \circ f) \circ h$,
2. For every object X and $h \in \mathcal{C}(W, X)$, $f \in \mathcal{C}(X, Y)$ we have $\mathrm{id}_X \circ h = h$ and $f \circ \mathrm{id}_X = f$.

For $f \in \mathcal{C}(X, Y)$ we say X is the *source* of f, Y is the *target* of f, and write $X = s(f)$, $Y = t(f)$. We also write $f\colon X \to Y$ or $X \xrightarrow{f} Y$ to indicate that $f \in \mathcal{C}(X, Y)$. If $\mathcal{C}(X, Y)$ is a set for all X, Y, then $\mathcal{C}$ is called *locally small*, and each $\mathcal{C}(X, Y)$ is called a *hom-set*.

Most categories you will encounter are locally small

Many examples of categories have objects sets carrying extra structure (for instance a topology) and morphisms that are functions compatible with that structure—but not all categories. We have seen **Top**, the category of topological spaces (and continuous maps) and **Set**, the category of sets (and functions), and you implicitly already know many other examples.

Vector spaces, (abelian) groups, manifolds, rings, ...

Example 18. The category $\mathbf{Set}_*$ of pointed sets (X, x) ($x \in X$ a specified element) and pointed maps $(X, x) \to (Y, y)$ (functions $f\colon X \to Y$ with $f(x) = y$)) can be considered as consisting of algebraic objects of the weakest sort (compare homomorphisms, linear transformations, ring maps, etc, which preserve distinguised elements).

The whole point of categories is how they relate to each other, an isolated category can only tell us so much.

Definition 15. Given categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* from $\mathcal{C}$ to $\mathcal{D}$, denoted $F\colon \mathcal{C} \to \mathcal{D}$ consists of the data:

i) For every object X of $\mathcal{D}$, a specified object $F(X)$ of $\mathcal{D}$,

ii) For every morphism $f\colon X \to Y$ of $\mathcal{C}$, a specified morphism $F(f)\colon F(X) \to F(Y)$ of $\mathcal{D}$

such that for every object X of $\mathcal{C}$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and for every pair $f\colon X \to Y$ and $g\colon Y \to Z$ of morphisms of $\mathcal{C}$, $F(g \circ f) = F(g) \circ F(f)$. This latter property is called 'functoriality'. For locally small categories, the assignment on morphisms gives a function $\mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$.

We will use this notation even without making that assumption

We have already see at least four examples of functors:

- The underlying set functor $U\colon \mathbf{Top} \to \mathbf{Set}$
- The discrete topology functor $\mathrm{disc}\colon \mathbf{Set} \to \mathbf{Top}$
- The set of connected components functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$

the indiscrete topology also gives rise to a functor $\mathbf{Set} \to \mathbf{Top}$, but we won't be using it

although we haven't yet seen why π_0 is a functor. We can compose functors in the obvious way, so get functors $\mathrm{disc}U\colon \mathbf{Top} \to \mathbf{Top}$ and $\mathrm{disc}\pi_0\colon \mathbf{Top} \to \mathbf{Top}$, for instance.

Here is a trivial-seeming example (aside from the identity functor).

Let $\mathcal{C}$ be a category, and $\mathcal{D}$ a *subcategory*: a collection of some of the objects of $\mathcal{C}$ and some of the morphisms of $\mathcal{C}$ that form a category by themselves. Then the inclusion of the objects and the morphisms forms a functor $\mathcal{D} \hookrightarrow \mathcal{C}$, the *subcategory inclusion*. An important special case of this is when for every X and Y that are objects of $\mathcal{D}$, every $\mathcal{D}(X,Y) = \mathcal{C}(X,Y)$; then $\mathcal{D}$ is call a *full* subcategory. More generally we can consider a functor that is injective on objects and morphisms to define a subcategory.

Example 19. The functor $\mathrm{disc}\colon \mathbf{Set} \to \mathbf{Top}$ makes $\mathbf{Set}$ a full subcategory of $\mathbf{Top}$.

we have used and will use this result without comment

We will be later restricting attention to certain full subcategories of $\mathbf{Top}$.

Lemma 6. Let X be a connected space, and let $f: X \to Y$ be a continuous function. Then $\mathrm{im}(f) \subset Y$ is connected.

Proof. Let S be a discrete space and let $g: \mathrm{im}(f) \to S$ be a continuous function. Then the composite $X \to \mathrm{im}(f) \to S$ has image $\{s\} \subseteq S$, hence $\mathrm{im}(g) = \{s\}$ and so $\mathrm{im}(f)$ is connected. □

Proposition 4. The assignment $X \mapsto \pi_0(X)$ is a functor **Top** $\to$ **Set**.

Proof. We need to show there is an assignment $(f: X \to Y) \mapsto (\pi_0(f): \pi_0(X) \to \pi_0(Y))$, for an arbitrary continuous function f. Fix $f: X \to Y$ and let $\alpha \in \pi_0(X)$. Then this corresponds to a connected component $X_\alpha \subseteq X$, and we know $f\big|_{X_\alpha}$ has connected image. Thus this image is contained inside a single connected component of Y, and we define $\pi_0(f)(\alpha)$ to be the corresponding element of $\pi_0(Y)$.

Given another map $g: Y \to Z$, and the corresponding function $\pi_0(g): \pi_0(Y) \to \pi_0(Z)$, one can check that $\pi_0(g)\pi_0(f)(\alpha)$, for $\alpha \in \pi_0(X)$ is the same as $\pi_0(g \circ f)(\alpha)$, and $\pi_0(\mathrm{id})$ is also the identity map. This proves that π_0 is a functor **Top** $\to$ **Set**. □

Here is a bonus second proof for locally connected spaces.

Proof. We already know we have a map $X \to Y \to \pi_0(Y)$, where we give $\pi_0(Y)$ the discrete topology. This is continuous since Y is locally connected, and we want to show this *descends* along $X \to \pi_0(X)$ to a map $\pi_0(X) \to \pi_0(Y)$. Given any $\alpha \in \pi_0(X)$, it corresponds to a connected component X_α of X. Look at the restriction of $X \to Y \to \pi_0(Y)$ to X_α: since X_α is connected, its image is exactly one point in $\pi_0(Y)$. So define $\pi_0(f)(\alpha) = [f(x)]$ for an arbitrary $x \in X_\alpha$. This defines $\pi_0(f)$. Moreover, the following diagram *commutes*:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y) \end{tikzcd}$$

Since the discrete topology on $\pi_0(X)$ is the same as the quotient topology, this is a map between discrete spaces, hence continuous, but we are thinking of it as a map between sets.

Now we want to show that $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$. Given $\alpha \in \pi_0(X)$, and $x \in X_\alpha$, then $\pi_0(f)(\alpha) = [f(x)]$. To define $\pi_0(g)\,(\pi_0(f)(\alpha))$, we need to choose a point in the component $Y_{[f(x)]}$, so take it to be $f(x)$. Then $\pi_0(g)\,(\pi_0(f)(\alpha)) = [g(f(x))]$, but this is just $\pi_0(g \circ f)(\alpha)$. $\square$

Exercise 8. Show that $[\mathrm{pt}, -]\colon \mathbf{Top} \to \mathbf{Set}$ is a functor.

Or more generally, $[X, -]\colon \mathbf{Top} \to \mathbf{Set}$!

Another important example of a category is the *homotopy category* $\mathbf{hTop}$. The objects are topological spaces, but $\mathbf{hTop}(X,Y) = [X,Y]$. There is a functor $\mathbf{Top} \to \mathbf{hTop}$, which is the identity on objects, and sends a map to its homotopy class. Objects are isomorphic in $\mathbf{hTop}$ iff they are homotopy equivalent.

Exercise: prove this is a category

Proposition 5. The functor π_0 descends to a functor $\mathbf{hTop} \to \mathbf{Set}$

Lecture 4

Proof. We will prove that this is well-defined on morphism on hom-sets, the rest is routine. For $f, g\colon X \to Y$ to be homotopic via $H\colon I \times X \to Y$, we need to show that for all $\alpha \in \pi_0(X)$, $\pi_0(f)(\alpha) = \pi_0(g)(\alpha)$. Take x in the connected component X_α, then we have a map $I \to I \times X \xrightarrow{H} Y$, namely a path $f(x) \rightsquigarrow g(x)$. But I is connected, so the image of the path is connected, so that $f(x)$ and $g(x)$ are in the same connected component. As x was arbitrary $f(X_\alpha)$ and $g(X_\alpha)$ are both contained in the same connected component of Y. Thus $\pi_0(f)(\alpha) = \pi_0(g)(\alpha)$. $\square$

As a result, if $\pi_0(X) \not\cong \pi_0(Y)$, the spaces X and Y cannot be homotopy equivalent, let alone homeomorphic.

Exercise 9. Show the functor $[\mathrm{pt}, -]\colon \mathbf{Top} \to \mathbf{Set}$ descends to $\mathbf{hTop} \to \mathbf{Set}$.

Here is a useful fact about spaces.

Lemma 7. For all families X_β, $\beta \in J$, of spaces, we have isomorphisms

$$\coprod_{\beta \in J} \pi_0(X_\beta) \xrightarrow{\cong} \pi_0\Big(\coprod_{\beta \in J} X_\beta\Big) \quad \text{and} \quad \coprod_{\beta \in J} [\mathrm{pt}, X_\beta] \xrightarrow{\cong} \Big[\mathrm{pt}, \coprod_{\beta \in J} X_\beta\Big],$$

with inverses induced by the family of maps in_β. That is, π_0 and $[\mathrm{pt}, -]$ *preserve coproducts.*

Recall last time: we had functors $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ and (abusing notation) $\pi_0\colon \mathbf{hTop} \to \mathbf{Set}$.

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "\pi_0"] \arrow[d] & \mathbf{Set} \\ \mathbf{hTop} \arrow[ur, "\pi_0"'] & \end{tikzcd}$$

Example 20. If X and Y are spaces with $|\pi_0(X)| < |\pi_0(Y)|$, no continuous map $X \to Y$ is surjective.

Here is an instructive example

Example 21. The *topologist's sine curve* is the image C of $[-1,1] \sqcup (0,1] \to \mathbb{R}^2$ defined by

$$\begin{cases} y \mapsto (0,y) & y \in [-1,1] \\ x \mapsto (x, \sin(\frac{1}{x})) & x \in (0,1] \end{cases}$$

equipped with the **subspace topology**. This is a compact metric space, using the inherited Euclidean metric. Fact: *every* continuous function $f\colon C \to \{0,1\}$ is constant. If $f(1,\sin(1)) = 1$, then $f(x,\sin(x)) = 1$ for every $x \in (0,1]$ (as intervals are connected). If $f(0,0) = b \in \{0,1\}$, then $f(0,y) = b$ also, for all $y \in [-1,1]$. The sequence $(\frac{1}{n\pi}, 0)$ converges to $(0,0)$ in C, so $b = f(0,0) = \lim_{n\to\infty} f(\frac{1}{n\pi}, 0) = 1$ as f is continuous and we are in a metric space.

Hence C is connected, but there is *no* continuous function $\gamma\colon [0,1] \to C$ with $\gamma(0) = (0,0)$ and $\gamma(1) = (1,\sin(1))$. Since intervals are path connected, we can show $[\mathrm{pt}, C] = \{0,1\}$, but $\pi_0(C) = *$.

Exercise: prove this by considering $\lim_{n\to\infty} \gamma(\frac{1}{n})$

So we have two different invariants here, and there is always a surjective map $[\mathrm{pt}, X] \to \pi_0(X)$. Moreover, the following square of functions between sets always commutes, for any map $X \xrightarrow{f} Y$:

$$\begin{tikzcd} {[\mathrm{pt}, X]} \arrow[r, "{[\mathrm{pt}, f]}"] \arrow[d] & {[\mathrm{pt}, Y]} \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y) \end{tikzcd}$$

This is thus an example of a *natural transformation*.

Definition 16. Given functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a natural transformation $\alpha\colon F \Rightarrow G$ consists of the data:

i) For every object X of $\mathcal{C}$, a specified morphism $\alpha_X\colon F(X) \to G(X)$ (the *components* of α)

such that for every morphism $f\colon X \to Y$ in $\mathcal{C}$, the following square commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & F(Y) \end{tikzcd}$$

A natural transformation is called a *natural isomorphism* if all of its components are isomorphisms.

For example, there are natural transformations disc $U \Rightarrow \mathrm{id}\colon \mathbf{Top} \to \mathbf{Top}$, with component at X the identity map $\mathrm{disc}(U(X)) \to X$, and $U \Rightarrow \pi_0\colon \mathbf{Top} \to \mathbf{Top}$, with component $U(X) \to \pi_0(X)$.

We seek conditions that will define a full subcategory of **Top** such that the components $[\mathrm{pt}, X] \to \pi_0(X)$ of the natural transformation $[\mathrm{pt}, -] \Rightarrow \pi_0$ are isomorphisms for all spaces X in the subcategory.

Definition 17. A space X is *semilocally path connected* (slpc) if it has a neighbourhood base of sets N such that for any two $x, y \in N$, there is a path in X from x to y.

Note that a space is slpc iff every connected component is slpc, and if X is homeomorphic to Y, and one of them is slpc, then so is the other.

Proposition 6. If X is a semilocally path connected space, then $[\mathrm{pt}, X] \to \pi_0(X)$ is an isomorphism.

Proof. We are reduced to the case X is connected ($\pi_0(X) = *$) and slpc, by Lemma 7, and the fact the case $X = \varnothing$ is trivial. Since X is connected, take $x \in X$ and define $\chi\colon X \to \{0, 1\}$ by

$$\chi(y) = \begin{cases} 1 & \exists y \rightsquigarrow x \\ 0 & \text{otherwise} \end{cases}$$

where by $y \rightsquigarrow x$ I mean a path $\gamma\colon I \to X$ with $\gamma(0) = y$ and $\gamma(1) = x$. We will show χ is continuous. Note that χ continuous $\Leftrightarrow p^{-1}(0)$ and $p^{-1}(1)$ open $\Leftrightarrow p^{-1}(1)$ open and closed. But $p^{-1}(1) =: C_x$ is the path component containing x. Take $y \in C_x$ (so $\exists y \rightsquigarrow x$), and $V \ni y$ a path-connected nhd. Given $z \in V$, $\exists z \rightsquigarrow y$. Concatenate these paths to give $z \rightsquigarrow x$, so that $z \in C_x$. This is true for all $z \in V$, so that $V \subseteq C_x$, hence C_x contains a neighbourhood of each of its points, and so is open.

Conversely, take $y \in \overline{C_x}$, $V \ni y$ a path connected nhd. As $\exists z \in V \cap C_x \subseteq V$, $\exists z \rightsquigarrow y$. But also have $V \cap C_x \subseteq C_x$, so $\exists z \rightsquigarrow x$. Concatenate paths to get $y \rightsquigarrow x$, so that $y \in C_x$. This is true for all $y \in \overline{C_x}$, so $\overline{C_x} \subseteq C_x$ and C_x is closed. Hence χ is continuous.

But X is connected, and $\chi(x) = 1$, so that $\operatorname{im} \chi = \{1\}$, and so $C_x = \chi^{-1}(1) = X$. Thus $[\mathrm{pt}, X] \to \pi_0(X) = *$ is an isomorphism. □

So we will consider for the rest of this section of the course only slpc spaces, which form a full subcategory $\mathbf{slpcTop} \hookrightarrow \mathbf{Top}$. Note that discrete spaces are slpc, so $\mathbf{Set} \hookrightarrow \mathbf{slpcTop}$ is a subcategory.

Example 22. Any path-connected space X is slpc, since for any nhd N and points $x, y \in N$, we know there is a path $I \to X$ between x and y.

Exercise 10. Show that the product of two slpc spaces is slpc, and that any locally convex topological vector space is slpc.

Example 23. Any manifold is slpc, since every point lives in a chart homeomorphic to some $\mathbb{R}^n$, and $\mathbb{R}^n$ is path-connected.

Be warned: subspaces of slpc spaces may not be slpc, for instance the topologist's sine curve is a subspace of the contractible $\mathbb{R}^2$.

Question 2. If X is slpc and $q\colon X \to Y$ is a quotient map, then is Y slpc?

so Y has the final topology wrt q

One last technical point

Definition 18. A *pointed space* is a pair (X, x) where X is a topological space and $x \in X$. A pointed map is a pointed map between the underlying pointed sets that is continuous. These define a category $\mathbf{Top}_*$.

A *pointed homotopy* of pointed map $I \times X \to Y$, for (X, x_0) and (Y, y_0) pointed spaces, is required to satisfy $H(t, x_0) = y_0$ for all $t \in I$. Pointed homotopy classes of pointed map are denoted $[(X, x_0), (Y, y_0)]_*$. The category $\mathbf{hTop}_*$ is defined analogously to $\mathbf{hTop}$. We get a functor $\pi_0\colon \mathbf{hTop}_* \to \mathbf{Set}_*$.

Covering spaces

Sometimes when we are thinking about a particular space X, we need to construct other spaces related to X to study objects of interest.

Example 24. Take $X = \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Then the function $x \mapsto \sqrt{x}$ is not well-defined, and if we take a branch cut to give an actual function, it is not continuous on X. Even worse, if have a continuous function $f\colon \mathbb{C}^\times \to \mathbb{C}$, we may or may not have $x \mapsto f(\sqrt{x})$ continuous. However, we *do* get a continuous function if we change the domain somewhat. The problem is that the function $Z := \mathbb{C}^\times \ni z \mapsto z^2 = x \in \mathbb{C}^\times$ is not injective, so not invertible. But if we are willing to take the domain to be Z, and so pass into f the argument z (which satisfies $z^2 = x$) then we are now just dealing with a continuous function. If f is such that $f(z) = f(-z)$ for all $z \in Z$, then we get a well-defined function on X.

The properties of the map $z \mapsto z^2$ (at least away from 0) and others like z^n, $\exp(z)$, rational functions away from poles and critical points and so on, lead to the notion of covering spaces of certain domains in $\mathbb{C}$. We have a general definition for arbitrary spaces.

Definition 19. A *covering space* $Z \xrightarrow{\pi} X$ of X is a space Z equipped with a map π such that for all $x \in X$ there is a nhd $V_x \ni x$ such that $\pi^{-1}(V_x) \simeq V_x \times \pi^{-1}(x)$ *over* V_x (ie the diagram at right commutes), where $\pi^{-1}(x)$ has the discrete topology. (We will also call π itself a *covering map*.)

$$\begin{tikzcd} \pi^{-1}(V_x) \arrow[r, "\simeq"] \arrow[d, "\pi"'] & V_x \times \pi^{-1}(x) \arrow[dl, "\mathrm{pr}_1"] \\ V_x & \end{tikzcd}$$

NB: $V_x \times \pi^{-1}(x) \simeq \coprod_{\pi^{-1}(x)} V_x$, for free

For a covering space $Z \xrightarrow{\pi} X$ and $x \in X$, let $Z_x := \pi^{-1}(x)$ denotes the *fibre* over x. We will also call X the *base space*.

Examples include: $\exp\colon \mathbb{C} \to \mathbb{C}^\times$, $S^2 \to \mathbb{RP}^2$, $U(1) \xrightarrow{(-)^n} U(1)$, covers of the join ∞ of two circles.

Exercise 11. Show that if $Z \xrightarrow{\pi} Y$ is a covering map, and $Y \xrightarrow{\rho} X$ is a covering map with finite fibres (that is: Y_x is finite for all $x \in X$), then $Z \xrightarrow{\rho\pi} X$ is a covering map.

Proposition 7. For a covering space $Z \xrightarrow{\pi} X$, if $\exists x_0 \rightsquigarrow x_1$, then $Z_{x_0} \simeq Z_{x_1}$.

Lecture 5

Proof. (First proof of Proposition 7) Take $\gamma\colon I \to X$, $\gamma(i) = x_i$, and an open cover $\{U_\alpha\}$ of X over which Z trivialises. We thus get an open cover $\gamma^{-1}(U_\alpha)$ of I, which has a finite subcover $U_0, \ldots, U_N$, with $x_0 \in U_0$, $x_1 \in U_N$. The ordering is chosen so that the path enters U_i before it enters U_{i+1}, and $U_i \cap U_{i+1}$ has at least one point of the path in it.

we can shrink the cover slightly to make this ordering well-defined, if need be

We have isomorphisms $Z_{U_i} := \pi^{-1}(U_i) \xrightarrow{\phi_i} U_i \times F_i$ with discrete spaces F_i. We have $Z_{x_0} \simeq F_0$, and for all $t \in \gamma^{-1}(U_0)$, $Z_{\gamma(t)} \simeq F_0$. So for $\gamma(t) \in U_0 \cap U_1$, we have $F_0 \simeq Z_{\gamma(t)} \simeq F_1$. We can then prove by induction on N that $F_0 \simeq F_1 \simeq \cdots \simeq F_N$. $\square$

So for slpc X and each $\alpha \in \pi_0(X)$, there is associated to $Z \xrightarrow{\pi} X$ an isomorphism class of sets, the *typical fibre* over all x in the connected component $X_\alpha \subseteq X$.

Note: Fibres can be empty! But we usually don't think about this case too much. For X pointed (by $x \in X$), we can consider pointed covering spaces $(Z,x) \to (X,x)$. This is from one perspective just a choice of point $z \in Z_x$. For X connected and slpc, a pointed covering space has every fibre contain at least one point, namely the image of z under $Z_x \simeq Z_{x'}$.

We have categories $\mathbf{Cov}_X$ and $\mathbf{Cov}_{(X,x)}$ with objects covering spaces of X (resp. pointed covering spaces of (X,x)) and maps

$$\begin{tikzcd} Z_1 \arrow[rr] \arrow[dr] & & Z_2 \arrow[dl] \\ & X & \end{tikzcd}$$

and analogously in the pointed case. We will study these categories and see what they tell us about the topology of X.

Example 25. For $X = \mathbb{C} \setminus \{p_1, \ldots, p_n\}$, the study of $\mathbf{Cov}_X$ tells us about possible Riemann surfaces for holomorphic functions with critical values precisely $p_1, \ldots, p_n$.

For slpc and connected X, the fact that for a covering space Z of X, there merely *exists* some $Z_{x_0} \simeq Z_{x_1}$ for arbitrary $x_0, x_1 \in X$ can be improved. We first need a construction on covering spaces.

Definition 20. Given a covering space $Z \xrightarrow{\pi} X$ and a map $Y \xrightarrow{f} X$, the *pullback* of Z is the subspace

$$f^*Z := Y \times_X Z = \{(y,z) \in Y \times Z \mid f(y) = \pi(z)\} \subseteq Y \times Z.$$

actually π doesn't have to a covering map; the space $Y \times_X Z$ is defined for any pair of maps to X

It fits in a commutative square

$$\begin{tikzcd} f^*Z \arrow[r, "\mathrm{pr}_2"] \arrow[d, "p"'] & Z \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Proposition 8. In the setting of Definition 20:

1. $f^*Z \to Y$ is a covering space.
2. f^* is a functor $\mathbf{Cov}_X \to \mathbf{Cov}_Y$.
3. Given $Y_2 \xrightarrow{g} Y_1 \xrightarrow{f} X$ and $Z \xrightarrow{\pi} X$, there is a canonical isomorphism $(f \circ g)^*Z \simeq g^*f^*Z$ in $\mathbf{Cov}_{Y_2}$.

Of these, only 1. relies on having a covering space to start with, 2. and 3. are general facts about pullbacks, where for 2. we replace $\mathbf{Cov}_X$ by the *slice category* $\mathbf{Top}/X$, whose objects are maps to X, and morphisms are commuting triangles

Corollary 1. The fibre $(f^*Z)_y$ is canonically isomorphic to $Z_{f(y)}$.

Now given a path $\gamma\colon I \to X$ and a covering space $Z \xrightarrow{\pi} X$, we can pull back Z to get a covering space $\gamma^* Z \to I$. So let us try to understand covering spaces of I. Certainly for discrete S, the projection $S \times I \to I$ is a covering space.

Proposition 9. A covering space $Z \xrightarrow{\pi} I$ is isomorphic to the trivial covering space $\pi^{-1}(0) \times I \xrightarrow{\mathrm{pr}_2} I$ in $\mathbf{Cov}_I$.

We first need a little helper lemma

Lemma 8. A covering space of a compact space X trivialises over a *finite* cover of X by nhds.

might as well take the nhds to be open, and then consider a finite subcover

Proof. (of Proposition 9) We use the lemma to trivialise $Z \to I$ over a finite cover of I, which we can take to be by intervals $[0, t_1]$, $[s_2, t_2]$, $\ldots$, $[s_N, 1]$ for $s_1 = 0 < s_2 < t_1 < s_3 < t_2 < \cdots < s_N < t_{N-1} < 1 = t_N$. We will proceed by induction on N, but this quickly reduces to the case of $N = 2$. So take a cover of I by $[0, t]$ and $[s, 1]$, where $\tau\colon Z_0 \times [0, t] \xrightarrow{\simeq} Z_{[0,t]}$ and we are given $\sigma\colon F \times [s, 1] \xrightarrow{\simeq} Z_{[s,1]}$.

we know abstractly that $F \simeq Z_0$, but this proof will construct an isomorphism

By restriction there is the composite map

$$Z_0 \times [s, t] \xrightarrow[\simeq]{\tau|_{[s,t]}} Z_{[s,t]} \xrightarrow[\simeq]{\sigma^{-1}|_{[s,t]}} F \times [s, t] \xrightarrow{\mathrm{pr}_1} F.$$

If we fix $z \in Z_0$, we get a continuous map $\{z\} \times [s, t] \to F$, which is thus constant, say at $p_z \in F$. The function $z \mapsto p_z = \sigma^{-1}(\tau(z, s))$ is then a bijection $\phi\colon Z_0 \xrightarrow{\simeq} F$.

We thus get maps $Z_0 \times [0, t] \hookrightarrow Z \hookleftarrow F \times [s, 1] \xleftarrow{\phi\times\mathrm{id}} Z_0 \times [s, 1]$, which by construction agree on $Z_0 \times [s, t]$. There is thus a continuous map $Z_0 \times [0, 1] \to Z$. Moreover, you can check this map is a morphism of $\mathbf{Cov}_I$. There are likewise maps

$$Z_{[0,t]} \xrightarrow{\simeq} Z_0 \times [0, t] \hookrightarrow Z_0 \times I \hookleftarrow Z_0 \times [s, 1] \xleftarrow{\phi^{-1}\times\mathrm{id}} F \times [s, 1] \xleftarrow{\simeq} Z_{[s,1]}$$

which agree on $Z_{[s,t]}$, hence a continuous map $Z \to Z_0 \times I$. This map is in $\mathbf{Cov}_I$ and can be checked by pointwise evaluation to be inverse to the first one. Hence we have an isomorphism $Z \simeq Z_0 \times I$ in $\mathbf{Cov}_I$. $\square$

Corollary 2. Given a covering space $Z \xrightarrow{\pi} I$ and a point $z \in Z_0$, there is a unique path $\eta_z\colon I \to Z$ with $\eta_z(0) = z$ such that $\pi \circ \eta_z = \mathrm{id}$ (i.e. η_z is a section of π).

Proof. We can construct *a* path, given $\tau\colon Z_0 \times I \xrightarrow{\simeq} Z$, by $\eta(t) = \tau(z, t)$. Since $\pi \circ \tau = \mathrm{pr}_2$, this has the required property. Connectedness of I and discreteness of Z_0 implies that given any other path $\eta' \colon I \to Z$ with $\eta'(0) = z$ and $\pi \circ \eta' = \mathrm{id}$, we must have $\tau^{-1} \circ \eta = \tau^{-1} \circ \eta' \colon I \to Z_0 \times I$ which implies $\eta' = \eta$. □

And now we have a really important property of covering spaces

Theorem 1. Given any covering space $Z \xrightarrow{\pi} X$, path $\gamma \colon I \to X$ and point $z \in Z_{\gamma(0)}$, there is a unique lift $\widetilde{\gamma_z} \colon I \to Z$ with $\widetilde{\gamma_z}(0) = z$.

a *lift* of a path $\gamma \colon I \to X$ is a path $\widetilde{\gamma} \colon I \to Z$ with $\pi\widetilde{\gamma} = \gamma$

Proof. We can pull back Z to get $p \colon \gamma^* Z \to I$. We have unique $\eta_z \colon I \to \gamma^* Z$ so that $\eta_{(0,z)}(0) = (0, z)$. Define $\widetilde{\gamma_z} = \mathrm{pr}_2 \circ \eta_{(0,z)} \colon I \to Z$. This path satisfies $\pi \circ \widetilde{\gamma_z} = \gamma \circ p \circ \eta_{(0,z)} = \gamma$. Given any other lift $\lambda \colon I \to Z$, we get a second section of p by $t \mapsto (t, \lambda(t))$, which by uniqueness of $\eta_{(0,z)}$ has to be equal to it, so that $\lambda = \widetilde{\gamma_z}$. □

We can then give a second, more explicit proof of Proposition 7.

Corollary 3. A path $\gamma \colon I \to X$ defines a bijection $\gamma_* \colon Z_{\gamma(0)} \xrightarrow{\simeq} Z_{\gamma(1)}$, by $\gamma_*(z) = \widetilde{\gamma_z}(1)$.

Proof. We only have to start with that γ_* is a function $Z_{\gamma(0)} \to Z_{\gamma(1)}$, but the function $(-\gamma)_* \colon Z_{\gamma(1)} \to Z_{\gamma(0)}$, where $-\gamma \colon I \to X$ is the path $-\gamma(x) = \gamma(1-x)$, is inverse to γ_*. This is because the path $-\widetilde{\gamma_z}$ is a lift of $-\gamma$, hence $(-\gamma)_*(\gamma_*(z)) = \widetilde{(-\gamma)}_{\gamma_*(z)}(1) = \widetilde{\gamma_z}(0) = z$. A symmetric argument shows that $\gamma_*\left((-\gamma)_*(z)\right) = z$ for $z \in Z_{\gamma(1)}$. □

A first observation is that this bijection is invariant under reparameterisations of γ: given $\psi \colon I \xrightarrow{\simeq} I$ with $\psi(0) = 0$ and $\psi(1) = 1$, then clearly $(\gamma \circ \psi)_* = \gamma_* \colon Z_{\gamma(0)} \to Z_{\gamma(1)}$.

consider ψ as a path in I and see what happens in that case

Even better, we get a function

Lecture 6

$$\{\text{paths } x_0 \rightsquigarrow x_1 \text{ in } X\} \times Z_{x_0} \to Z_{x_1}$$

If we take $x_0 = x_1 = x$, then this is a map

we can take quotient by reparametrisations if desired, in each of these functions

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \times Z_x \to Z_x$$

such that each loop $x \rightsquigarrow x$ gives a bijection $Z_x \to Z_x$. So we can think of this instead as

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \to \mathrm{Aut}(Z_x).$$

Alternatively, if we have a pointed covering space $(Z, z) \to (X, x)$, we have a canonical function

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \to Z_x \tag{1}$$

Example 26. For $Z = S \times X$, $(\gamma)_* = \mathrm{id}_S$ always, and the image of (1) (given some $(s, x) \in Z$) is just a single point. For instance, if $X = I$, we have seen this will be the case for every covering space. But for $X = S^1$, $Z = \mathbb{R} \xrightarrow{\exp} S^1$, and taking $x = 1 \in S^1$, $z = 0 \in \mathbb{R}$, then $Z_1 = \exp^{-1}(0) = 2\pi i\mathbb{Z}$, then

$$\{\gamma\colon I \to S^1 \mid \gamma(0) = \gamma(1) = 1\} \to 2\pi i\mathbb{Z}$$

is *onto*. The path $\widetilde{\gamma}_n = 2\pi inx$ lifts the path $\gamma(x) = \exp(2\pi inx)$, and $\widetilde{\gamma}_n(0) = 0$, $\widetilde{\gamma}_n(1) = 2\pi inx$. The difference is that $\mathbb{R}$ is path connected, but $X \times S$ is not, for $|S| > 1$.

In fact, for a covering space $(Z, z) \xrightarrow{\pi} (X, x)$ with Z path connected and $z' \in Z_x$, there is $\widetilde{\gamma}\colon I \to Z$ with $\widetilde{\gamma}(0) = z$, $\widetilde{\gamma}(1) = z'$. Since $\widetilde{\gamma}$ lifts $\gamma = \pi \circ \widetilde{\gamma}$, which satsfies $\gamma(0) = x = \gamma(1)$, the map (1) is **onto**. Thus paths constrain the sizes of fibres of connected covering spaces and vice versa. Notice also that the set of loops is independent of the choice of covering space!

More generally, given points z_α in Z_x, one per path component of Z,

that is: a section of $Z \to [\mathrm{pt}, Z]$

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \times [\mathrm{pt}, Z] \simeq \{\text{loops } x \rightsquigarrow x \text{ in } X\} \times \{z_\alpha\} \to Z_x$$

is always onto. There are a huge number of paths, and reparameterisations cuts things down somewhat. But we shall go even better, and put a topology on the space of paths.

The fibres Z_x of a covering space Z are discrete spaces, but the set $\mathbf{Top}(I, X)$ of paths $I \to X$ carries a topology when X is a metric space; we can consider $C(I, X)$ with the sup metric d_∞. The aim is to give $\mathbf{Top}(I, X)$ a topology for *any* space, not necessarily metric.

Lemma 9. Let X be a topological space, fix $\gamma \in \mathbf{Top}(I, X)$ a path. Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$ be a partition of $[0, 1]$, and $U_0, \ldots, U_n \subseteq X$ a collection of basic nhds such that $\gamma([t_i, t_{i+1}]) \subseteq U_i^o$. Define the subsets

The interior V^o of a nhd V is the union of all the open sets contained in V

$$N_\gamma(t_1 < \cdots < t_n; U_0, \ldots, U_n) := \{\eta\colon I \to X \mid \forall i = 0, \ldots n,\ \eta([t_i, t_{i+1}]) \subseteq U_i^o\} \subseteq \mathbf{Top}(I, X)$$

Then define $\mathcal{N}_{co}(\gamma)$ to be the family of subsets of $\mathbf{Top}(I, X)$ consisting of the sets above, as the partition and the collection of basic nhds vary. So defined the families $\mathcal{N}_{co}(\gamma)$ give a neighbourhood base on $\mathbf{Top}(I, X)$.

Definition 21. The *path space* X^I is the set $\mathbf{Top}(I, X)$ equipped with the topology defined by Lemma 9, which we call the *compact-open topology*.

When X is a metric space, then the compact-open topology and the topology arising from the sup metric coincide. A key property of the compact-open topology is that homotopies $H\colon I \times I \to X$ give continuous paths $h\colon I \to X^I$ (defined by $h_t\colon s \mapsto H(t,s)$) and vice-versa. Moreover:

More generally, for any space Y, continuous maps $I \times Y \to X$ are in bijection with continuous maps $Y \to X^I$

Lemma 10. 1. The evaluation map $\mathrm{ev}\colon X^I \times I \to X$, $\mathrm{ev}(\gamma, t) = \gamma(t)$ is continuous, and

2. given a map $X \xrightarrow{f} Y$, the post-composition map $f_*\colon X^I \to Y^I$, $f_*(\gamma) = f \circ \gamma$, is continuous.

Then given $t \in I$, the composite map $\mathrm{ev}_t\colon X^I \simeq X^I \times \{t\} \hookrightarrow X^I \times I \xrightarrow{\mathrm{ev}} X$ is continuous. Usually we care just about the cases $t = 0, 1$. We can then look at various subspaces of X^I, for a given $x \in X$:

$$\begin{aligned} P_x X &:= \{\gamma \in X^I \mid \gamma(0) = x\} = \mathrm{ev}_0^{-1}(x) \\ P_x^y X &:= \{\gamma \in X^I \mid \gamma(0) = x,\ \gamma(1) = y\} = \mathrm{ev}_0^{-1}(x) \cap \mathrm{ev}_1^{-1}(y) \\ \Omega_x X &:= P_x^x X = \{\gamma \in X^I \mid \gamma(0) = x = \gamma(1)\} \end{aligned}$$

In particular, we have already seen the last two, albeit without their topologies. We also see that path components of these spaces have something to do with homotopy classes of paths, perhaps with constraints on endpoints.

A key property of the natural transformation $\mathrm{id} \Rightarrow \mathrm{disc}\,\pi_0\colon \mathbf{slpcTop} \to \mathbf{slpcTop}$ is that it has a universal property: given a discrete space S, an slpc space X and a continuous map $X \xrightarrow{f} S$, there is a *unique* function $\pi_0(X) \to U(S)$ such that

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S \\ \mathrm{disc}(\pi_0(X)) \arrow[ur] & \end{tikzcd}$$

commutes. Hence if we take our function

$$P_x^y X \times Z_x \to Z_y \tag{2}$$

from the previous lecture, arising from a covering space $Z \to X$, and if we can show it is continuous, we would get a factorisation

$$P_x^y X \times Z_x \to \pi_0(P_x^y X \times Z_x) \simeq \pi_0(P_x^y X) \times Z_x \to Z_y$$

where the unmarked isomorphism exist due to Z_x being discrete. If Z is path connected, a fixing some $z \in Z_x$, we get a surjective map $\pi_0(P_x^y X) \to Z_y$, which further constrains both the topology of the space of paths, and the possible fibres of $Z \to X$. However, there are two issues:

(i) We yet don't know our path lifting function is continuous

(ii) We don't know if $P_x^y X$ is slpc, hence if path components and components agree.

To address (i), the unique path lifting property from last lecture will be promoted to a *continuous function* Lift: $X^I \times_X Z \to Z^I$. Combined with $Z^I \xrightarrow{\text{ev}} Z$ we will be able to reconstruct (2) as

here $X^I \times_X Z = \{(\gamma, z) \mid \gamma(0) = \pi(z)\}$

$$P_x^y X \times Z_x \hookrightarrow X^I \times_X Z \xrightarrow{\text{Lift}} Z^I \xrightarrow{\text{ev}_1} Z$$

factors through $Z_y \subset Z$. We already have the definition of Lift, but we need to show continuity.

Theorem 2. The function Lift: $X^I \times_X Z \to Z^I$ is continuous.

Proof. We need to set up the ingredients, so take $\gamma \in X^I$, define $x = \gamma(0)$, $y = \gamma(1)$, and take $z \in Z_x$. Let $\widetilde{\gamma} = \text{Lift}(\gamma, z)$, and $z' = \widetilde{\gamma}(1) \in Z_y$. Take a basic nhd $N_{\widetilde{\gamma}} = N_{\widetilde{\gamma}}(t_1 < \cdots < t_n; U_0, \ldots, U_n)$. We want to construct a basic nhd

$$M(\gamma, z) \subseteq X^I \times_X Z$$

of (γ, z) such that $M(\gamma, z) \subseteq \text{Lift}^{-1}(N_{\widetilde{\gamma}})$.

Since $Z \xrightarrow{\pi} X$ is locally trivial and I is compact, we can find a sequence $W_0, \ldots, W_m \subseteq Z$ (with $m \geq n$) of nhds such that

- $\pi|_{W_i}: W_i \xrightarrow{\cong} \pi(W_i)$ and each $\pi(W_i)$ is a nhd in X, and
- $\forall i = 0, \ldots, m \; \exists j = j(i)$ with $W_i \subseteq U_j$.

There is then a refinement $0 < s_1 < \cdots < s_m < 1$ such that W_i is a nhd of $\widetilde{\gamma}(t)$ for all $t \in [s_i, s_{i+1}]$. The set $\widetilde{N}_{\widetilde{\gamma}} := N_{\widetilde{\gamma}}(s_1 < \cdots < s_m; W_0, \ldots, W_m) \subseteq Z$ is then contained in $N_{\widetilde{\gamma}}$.

so that $[s_i, s_{i+1}] \subseteq [t_j, t_{j+1}]$

But, defining $V_i := \pi(W_i)$, the partition $0 < \cdots s_1 < s_m < 1$ and the sets $V_0, \ldots, V_m$ satisfy the conditions required to define the basic nhd $N_\gamma(s_1 < \cdots < s_m; V_0, \ldots, V_m) \subseteq X^I$. Also note that $z = \widetilde{\gamma}(0) \in W_0$, so we can define a nhd

$$M(\gamma, z) := (N_\gamma(s_1 < \cdots < s_m; V_0, \ldots, V_m) \times W) \cap X^I \times_X Z$$

of (γ, z). By construction $\pi(\widetilde{N}_{\widetilde{\gamma}}) \subseteq N_\gamma(s_1 < \cdots < s_m; V_0, \ldots, V_m)$, but in fact $\text{Lift}(M(\gamma, z)) = \widetilde{N}_{\widetilde{\gamma}} \subseteq N_{\widetilde{\gamma}}$, as desired. $\square$

Remark. In fact, by the uniqueness of lifts, the map Lift is a bijection, and even a homeomorphism, with inverse $(\pi_*, \text{ev}_0): Z^I \to X^I \times_X Z$.

So we have a continuous map $P_x^y X \times Z_x \to Z_y$, and thus get a function $\pi_0(P_x^y X) \times Z_x \to Z_y$. But we would like to know that for any two points $\gamma, \eta \in P_x^y X$ in the same connected component, there is a path between them. Such a path, recall, is a homotopy $H\colon I \times I \to X$ satisfying $H(s,0) = x$ and $H(s,1) = y$ $\forall x \in I$. Such a homotopy between paths will be said to *fix endpoints*.

Definition 22. A space X is called *semilocally simply-connected* (or *slsc*) if every point has a basis of nhds N that are path connected, and given $x, y \in N$ and two paths $\gamma, \eta \in P_x^y N$, there is an endpoint-fixing homotopy $I \times I \to X$ from γ to η.

this is the last technical condition on spaces we require in this section of the course

Notice that if a space X is slsc, then it is slpc.

Example 27. Any manifold is slsc, since every point has a nhd homeomorphic to some $\mathbb{R}^n$, which is convex.

Example 28. The *Hawaiian earring* is the subspace

$$\bigcup_{n\in\mathbb{N}} \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, ||(x,y) - (\tfrac{1}{n}, 0)|| = \tfrac{1}{n} \right\}$$

and is not slsc. Every nhd of the point $(0,0)$ contains loops that are not contractible, and stay non-contractible in the full space.

Theorem 3 (Wada 1955, improved in Roberts 2010)**.** If the space X is semilocally simply-connected, the spaces X^I, $P_x X$ and $P_x^y X$ (hence $\Omega_x X$) are semilocally path connected.

H. Wada, "Local connectivity of mapping spaces", Duke Math. J. **22**, Number 3 (1955) pp 419–425. DMR "Fundamental bigroupoids and 2-covering spaces", Theorem 5.12.

Proof. (Non-examinable) See Handout 1. □

Lecture 7

A question that may have occurred to you is what happens with the isomorphism $\gamma_*\colon Z_x \to Z_y$ if we break the path $\gamma\colon I \to X$ into two subpaths, say $x \rightsquigarrow x' \rightsquigarrow y$, and then compose the corresponding isomorphisms $Z_x \xrightarrow{\simeq} Z_{x'} \xrightarrow{\simeq} Z_y$. Or, starting from paths $\gamma, \eta\colon I \to X$ such that $\gamma(1) = \eta(0)$ and defining the *concatenation* $\gamma\#\eta\colon I \to X$ by

$$\gamma\#\eta(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \eta(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

how do $Z_{\gamma(0)} \xrightarrow{\gamma_*} Z_{\gamma(1)} = Z_{\eta(0)} \xrightarrow{\eta_*} Z_{\eta(1)}$ and $Z_\gamma(0) \xrightarrow{(\gamma\#\eta)_*} Z_{\eta(1)}$ relate?

Lemma 11. For paths $\gamma, \eta\colon I \to X$ such that $\gamma(1) = \eta(0)$, $(\gamma\#\eta)_* = \eta_* \circ \gamma_*\colon Z_{\gamma(0)} \to Z_{\eta(1)}$.

In particular, for $\gamma, \eta \in \Omega_x X$, $\gamma \# \eta \in \Omega_x X$ and we have the map $\Omega_x X \to \mathrm{Aut}(Z_x)$, which is compatible with path concatenation. But # is not associative!

Example 29. Take $X = S^1$, and let $\gamma(t) = \exp(2\pi i t)$.

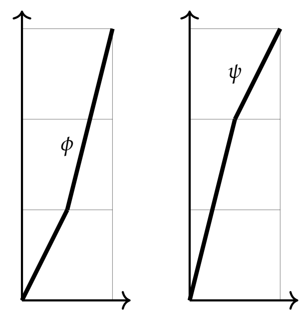
$$(\gamma\#\gamma)\#\gamma = \begin{cases} \exp(8\pi i t) & t \in [0, \frac{1}{2}] \\ \exp(4\pi i t) & t \in [\frac{1}{2}, 1] \end{cases} \quad \text{but} \quad \gamma\#(\gamma\#\gamma) = \begin{cases} \exp(4\pi i t) & t \in [0, \frac{1}{2}] \\ \exp(8\pi i t) & t \in [\frac{1}{2}, 1] \end{cases}$$

Let us re-examine how paths concatenate. Given $\gamma, \eta \colon I \to X$ such that $\gamma(1) = \eta(0)$, then we get a continuous function $\langle \gamma, \eta \rangle \colon [0,2] \to X$. The concatenation $\gamma\#\eta$ is then the precomposition of $\langle \gamma, \eta \rangle$ with the map $I = [0,1] \xrightarrow{t \mapsto 2t} [0,2]$. If we had a third map, $\lambda \colon I \to X$ with $\lambda(0) = \eta(1)$, then there is naturally a continuous function $\langle \gamma, \eta, \lambda \rangle \colon [0,3] \to X$. But the concatenations $(\gamma\#\eta)\#\lambda$ and $\gamma\#(\eta\#\lambda)$ arise from precomposing with two different maps $I = [0,1] \to [0,3]$. These are

$$\phi \colon t \mapsto \begin{cases} 4t & t \in [0, \frac{1}{2}] \\ 2t+1 & t \in [\frac{1}{2}, 1] \end{cases}$$

$$\psi \colon t \mapsto \begin{cases} 2t & t \in [0, \frac{1}{2}] \\ 4t-1 & t \in [\frac{1}{2}, 1] \end{cases}$$

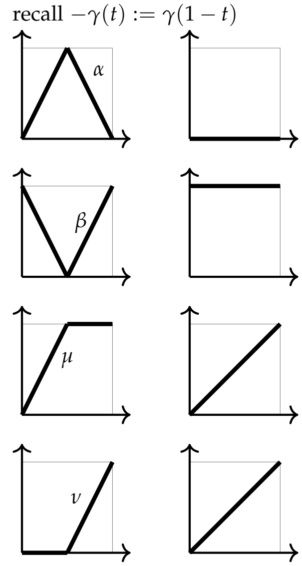
with graphs as at right.



These two paths $I \to [0,3]$ are homotopic fixing endpoints by the homotopy $h_a(s,t) = s\phi(t) + (1-s)\phi(t)$. If we then precompose $\langle \gamma, \eta, \lambda \rangle \colon [0,3] \to X$ with $h_a \colon I \times I \to [0,3]$, we get a homotopy between $(\gamma\#\eta)\#\lambda$ and $\gamma\#(\eta\#\lambda)$. Path concatenation in X is then *homotopy associative*. But what about inverses or an identity element? We will play the same trick, by considering a 'universal' case.

Given a path $\gamma \colon I \to X$, we have the reverse path $-\gamma$, and the composite $\gamma\#(-\gamma) \colon I \to X$ can be factored as $I \xrightarrow{\alpha} I \xrightarrow{\gamma} X$ for a certain path $I \xrightarrow{\alpha} I$. If we instead concatenate in the other direction, namely $(-\gamma)\#\gamma \colon I \to X$, then this factors as $I \xrightarrow{\beta} I \xrightarrow{\gamma} X$. Again β is a certain path in I. The graphs of both α and β are shown at right, and both of them are homotopic, fixing endpoints, to the constant functions at 0 and 1 respectively, by taking an affine combination as in the definition of h_a above. Then by composing the homotopies here with γ, we get homotopies between the path $\gamma\#(-\gamma)$ and the constant path at $\gamma(0)$, and also between $(-\gamma)\#\gamma$ and the constant path at $\gamma(1)$. So we have *homotopy inverses*.

recall $-\gamma(t) := \gamma(1-t)$



If we want to think about a homotopy identity element, then we should use the constant path $c_x\colon I \to X$ at a point $x \in X$, with $c_x(t) = x, \forall t \in I$. We can factor the composite $\gamma\#c_{\gamma(1)}$ as $I \xrightarrow{\mu} I \xrightarrow{\gamma} X$ for μ as shown at right, and factor $c_{\gamma(0)}\#\gamma$ as $I \xrightarrow{\nu} I \xrightarrow{\gamma} X$. As above, μ and ν are homotopic, fixing endpoints, to the identity map $I \to I$.

If we turn the five homotopies $I \times I \to X$ described above into paths $I \to X^I$, then if we start from elements of $\Omega_x X$, these homotopies correspond to paths in $\Omega_x X$. Thus $\Omega_x X$, which has a concatenation binary operator $\#\colon \Omega_x X \times \Omega_x X \to \Omega_x X$, acts like a group, except the group axioms only hold up to the existence of paths

More is true, though we won't prove it: there are homotopies assembled out of these paths for all possible cases, for instance $I \times \Omega_x X \times \Omega_x X \times \Omega_x X \to \Omega_x X$

$$
\begin{aligned}
(\gamma\#\eta)\#\lambda &\rightsquigarrow \gamma\#(\eta\#\lambda)\\
\gamma\#(-\gamma) &\rightsquigarrow c_{\gamma(0)}\\
(-\gamma)\#\gamma &\rightsquigarrow c_{\gamma(1)}\\
\gamma\#c_{\gamma(1)} &\rightsquigarrow \gamma\\
c_{\gamma(0)}\#\gamma &\rightsquigarrow \gamma
\end{aligned}
$$

in $\Omega_x X$. As a result we have proved most of

Proposition 10. Let (X, x) be a pointed space, with X slsc. The set $\pi_0(\Omega_x X)$ carries the structure of a group, its product arising from concatenation of loops and identity element represented by the constant path at x.

If we fall back on the default, namely just slpc, then we can use $[\mathrm{pt}, \Omega_x X]$ instead

Proof. To exhibit the multiplication, consider the functor π_0 applied to $\#\colon \Omega_x X \times \Omega_x X \to \Omega_x X$, giving $\pi_0(\Omega_x X \times \Omega_x X) \xrightarrow{\#} \pi_0(\Omega_x X)$. But since $\pi_0(M \times N) \xrightarrow{\simeq} \pi_0(M) \times \pi_0(N)$, for all slpc spaces M and N, we get a composite $\pi_0(\Omega_x X) \times \pi_0(\Omega_x X) \simeq \pi_0(\Omega_x X \times \Omega_x X) \to \pi_0(\Omega_x X)$. This is associative and unital, and inverses exist, by the existence of the paths above. $\square$

This requires knowing that # is continuous! See Assignment 2.

Definition 23. For (X, x) a pointed space its *fundamental group at x* is $\pi_1(X, x) := [\mathrm{pt}, \Omega_x X]$, which for X a slsc space coincides with $\pi_0(\Omega_x X)$.

Recall we also proved $[\mathrm{pt}, -]$ descends to a functor **hTop** $\to$ **Set** in Assignment 1

From the previous reasoning, we have constructed from a covering space $Z \to X$ and chosen basepoint $x \in X$ a permutation representation $\pi_1(X, x) \to \mathrm{Aut}(Z_x)$. If Z is path connected, and we choose $z \in Z_x$, we get a surjective map $\pi_1(X, x) \to Z_x$, given by $\gamma \mapsto \gamma_*(z)$. This implies we have an upper bound on the cardinality of fibres of any path connected covering space, and conversely, given a connected covering space, the fibres give a lower bound on the number of distinct homotopy classes of loops in X.

As a point of clarification, everything here works for arbitrary slpc spaces with small adjustments, but for slsc spaces the approach is slightly cleaner, as components and path components coincide for the function spaces

Example 30. The projection map $S^2 \to \mathbb{RP}^2$ is a covering space and S^2 is connected, so there exist at least two non-homotopic loops in $\mathbb{RP}^2$ at any given basepoint. One of these is the constant loop, so there exists a loop in $\mathbb{RP}^2$ not homotopic to it.

Example 31. We have the covering space $\exp(2\pi i -)\colon \mathbb{R} \to S^1$ with fibre $\mathbb{Z}$ over $1 \in S^1$, which implies $\pi_1(S^1, 1)$ is an infinite group.

Proposition 11. The loop space construction is a functor $\Omega\colon \mathbf{Top}_* \to \mathbf{Top}_*$.

Corollary 4. The fundamental group gives a functor

Exercise: This functor is naturally isomorphic to $[(S^1, 1), (X, x)]_*$

$$\pi_1 := [\mathrm{pt}, -] \circ \Omega\colon \mathbf{Top}_* \to \mathbf{Grp}.$$

which for slsc spaces is naturally isomorphic to $\pi_0 \circ \Omega$.

Lecture 8

However, as we have seen, we don't just get an action of $\pi_1(X, x)$ on the fibre Z_x of a covering space. We also get what looks like an action of paths between different points on fibres, but now points in one fibre are taken to points of another fibre. In fact, if X is not equipped with a basepoint to start with, or there are several natural options and no one of those is canonical, then we can create an even richer invariant, namely a *groupoid*.

Definition 24. A *groupoid* is a category where every morphism has an inverse.

So that we have an idea of what kinds of groupoids arise, let us consider some examples. We will be considering only *small* groupoids: those locally small groupoids Γ where there is a set Γ_0 of objects. We can then take the disjoint union of all the hom-sets to get the set $\Gamma_1 = \bigsqcup_{x,y \in \Gamma_0} \Gamma(x, y)$ of morphisms, and specify the source and target functions $s, t\colon \Gamma_1 \rightrightarrows \Gamma_0$. Groupoids and functors form a category **Gpd**.

Example 32. 1. Every set S gives a groupoid $\mathrm{disc}(S)$, by taking the set of objects to be S, and to only have identity morphisms. This gives a full subcategory inclusion $\mathrm{disc}\colon \mathbf{Set} \hookrightarrow \mathbf{Gpd}$, and such groupoids are called *discrete*.

2. Every set C also gives another groupoid $\mathrm{codisc}(C)$ with set of objects C, but with exactly one morphism from any object to any other object. The set of morphisms is $C \times C$, and every object $c \in C$ has the trivial group of automorphisms. Such groupids are called *codiscrete*.

3. Let G act on the set Y on the right. Then there is a groupoid $Y//G$ with object set Y, and set of morphisms $Y \times G$. The source and target are given by $s(y,g) = y$, $t(y,g) = yg$, and composition is $(y,g)(yg,h) = (y,gh)$.
   (a) If $G = 1$, then this recovers the first example.
   (b) If $Y = \mathrm{pt}$, then the information in the groupoid is essentially just that of the group G. Groupoids of this form will be denoted $\mathbb{B}G$, and $\mathbb{B}\colon \mathbf{Grp} \hookrightarrow \mathbf{Gpd}$ is the inclusion of a full subcategory.

A slogan people sometimes use is that a groupoid is like a group with 'many identities', but you can also usefully think of them as being a generalisation of a group action, where you have different groups acting on different parts of the set. Here is a useful lemma about the structure of groupoids.

Lemma 12. For any groupoid Γ, and given $x, y \in \Gamma_0$,

using algebraic order of composition

$$\begin{aligned} \mathrm{Ad}_a\colon \Gamma(x,x) &\xrightarrow{\simeq} \Gamma(y,y) \\ g &\mapsto a^{-1}ga \end{aligned}$$

is an isomorphism for any $a \in \Gamma(y,x)$ and the function

$(\mathrm{Ad}_a)^{-1} = \mathrm{Ad}_{a^{-1}}$

$$\begin{aligned} \Gamma(x,x) \times \Gamma(x,y) &\to \Gamma(x,y) \\ (g,a) &\mapsto ga \end{aligned}$$

transitive: $(ba^{-1}, a) \mapsto b$;
free: $ga = a$ implies $g = gaa^{-1} = aa^{-1} = \mathrm{id}_x$

defines a free and transitive action of the group $\Gamma(x,x)$.

As a reminder: a free group action $G \times S \to S$ is one where $g \cdot p = p$ implies g is the identity element, and a transitive action one where given any two elements $p, q \in S$, there is some group element $g \in G$ such that $g \cdot p = q$.

Definition 25. Given an slsc space X and a specified subset $A \subseteq X$, the *fundamental groupoid based at* A is the groupoid $\Pi_1(X,A)$ with set of objects A, and the set of morphisms from x to y is $\Pi_1(X,A)(x,y) := \pi_0(P_x^y X)$. The composition map is induced from concatenation of paths:

the definition makes sense for more general slpc spaces, using $[\mathrm{pt}, -]$ in place of π_0, but we are only consider slsc spaces here

$$\pi_0(P_x^y X) \times \pi_0(P_y^z X) \simeq \pi_0(P_x^y X \times P_y^z X) \to \pi_0(P_x^z X)$$

and constant paths are the identity morphisms.

As with other invariants, the fundamental groupoid is a functor. Define the category $\mathbf{Top}^{(2)}$ to be the category with objects pairs (X, A) where X is a topological space and $A \subseteq X$ is a subspace, and a morphism $(X,A) \to (Y,B)$ is a continuous function $f\colon X \to Y$ such that $f(A) \subseteq B$. We have a full subcategory inclusion $\mathbf{Top}_* \hookrightarrow \mathbf{Top}^{(2)}$.

Proposition 12. The fundamental groupoid gives a functor $\Pi_1 \colon \mathbf{Top}^{(2)} \to \mathbf{Gpd}$ such that

$$\begin{tikzcd} \mathbf{Top}_* \arrow[r, "\pi_1"] \arrow[d] & \mathbf{Grp} \arrow[d, "\mathbb{B}"] \\ \mathbf{Top}^{(2)} \arrow[r, "\Pi_1"'] & \mathbf{Gpd} \end{tikzcd}$$

and moreover:

$$\Pi(X \times Y, A \times B) \xrightarrow{\simeq} \Pi_1(X, A) \times \Pi_1(Y, B)$$
$$\Pi_1(X, A) \sqcup \Pi_1(Y, B) \xrightarrow{\simeq} \Pi(X \sqcup Y, A \sqcup B)$$

The product/disjoint union of groupoids is what you think it is: take the products/disjoint unions of the objects and the morphisms, respectively

We can include *unbased* spaces X into pairs, by taking (X, X), giving another fully faithful functor, $\mathbf{Top} \to \mathbf{Top}^{(2)}$. In this case, if the space X has *no* preferred basepoints whatsoever, we can still define the fundamental groupoid of X itself as $\Pi_1(X, X)$, which is a functor $\mathbf{Top} \to \mathbf{Gpd}$.

We haven't yet seen how to calculate the fundamental group(oid) in examples, so we will turn to that now. We need a name for spaces X that have $\Pi_1(X)$ trivial, in the sense of being codiscrete.

Definition 26. A space X that satisfies $\Pi_1(X) = \mathrm{codisc}(X)$ is called *simply-connected*.

such spaces also have $\Pi_1(X, A) = \mathrm{codisc}(A)$ for all $A \subseteq X$

If we unpack this definition, it tells us that a) given any two points $x, y \in X$, there is a (homotopy class of some) path from x to y, so that X is path-connected, and b) all paths between any two given points are endpoint-fixed homotopic, hence a unique morphism in the fundamental groupoid. As a result, $\pi_1(X, x) = \Pi_1(X)(x, x)$ is the trivial group.

Example 33. Convex subspaces $C \subseteq \mathbb{R}^n$ are simply-connected, because any two points $v, w \in C$ can be joined by a path in C, and given two paths $\gamma, \eta \colon v \rightsquigarrow w$ the map $(s, t) \mapsto s\gamma(t) + (1 - s)\eta(t)$ is a homotopy between them.

In particular, the interval I is simply-connected. The fundamental groupoid $\Pi_1(I, \{0, 1\})$ is important enough to have its own name: $\mathbf{2}$, sometimes denoted $(0 \xrightarrow{\sim} 1)$, as it has two objects $0, 1$ and a unique isomorphism between them.

Exercise 12. Define a *star-shaped region* in a (real or complex) vector space V to be a set $K \subseteq V$ such that there is a point $v_0 \in K$ such that for every $v \in K$ and $t \in I$, $tv_0 + (1 - t)v \in K$. Prove that star-shaped regions are simply-connected.

For $\mathcal{H} \subset \mathbb{C}$ the (open) upper half-plane, the set $\mathcal{H} \cup \mathbb{Q}$ is star-shaped, but not convex

Simply-connected spaces are special for the following reason.

Proposition 13. If X is a simply-connected space, then every path connected covering space $Z \xrightarrow{\pi} X$ is trivial, in the sense that π is a homeomorphism.

Proof. Recall that $\pi_1(X,x) \to Z_x$ is surjective for any $x \in X$, so X simply-connected implies $Z_x = \text{pt}$ for all x. Thus π is a bijection. The local triviality condition implies that every $x \in X$ has an open set $U \ni x$ such that $\pi^{-1}(U) \to U$ is a homeomorphism. Letting U_α range over such an cover of X, we can glue the inverses of these local homeomorphisms into an inverse for π. □

Example 34. If X is contractible then it is simply-connected. Let $H\colon I \times X \to X$ be a contraction to $x_0 \in X$. Consider the induced map $h = \Pi_1(H)\colon \Pi_1(I \times X, \{0,1\} \times X) \to \Pi_1(X,X) = \Pi_1(X)$. The domain simplifies to be $\Pi_1(I,\{0,1\}) \times \Pi_1(X) = \mathbf{2} \times \Pi_1(X)$. Consider the induced maps $\{i\} \times \Pi_1(X) \to \mathbf{2} \times \Pi_1(X) \to \Pi_1(X)$ for $i = 0,1$. Since $H\big|_{\{0\}\times X} = \mathrm{id}_X$, so $h_{\{0\}\times\Pi_1(X)} = \mathrm{id}_{\Pi_1(X)}$; and as $H\big|_{\{1\}\times X}$ is constant at x_0, so $h(0,x) = x_0$ for all $x \in X$, and $h\big|_{\{1\}\times\Pi_1(X)}$ sends every path to the constant path at x_0. We already know that X is path connected, so that for any $x,y \in X$ there is some path between them. Given a path $\gamma\colon x \rightsquigarrow y$ consider the commutative square

$$\begin{tikzcd}
(0,x) \arrow[r, "{(\mathrm{id}_0,[\gamma])}"] \arrow[d] & (0,y) \\
(1,x) \arrow[r, "{(\mathrm{id}_1,[\gamma])}"'] & (1,y) \arrow[u]
\end{tikzcd}$$

in $\mathbf{2} \times \Pi_1(X)$ (recall all morphisms are invertible). Under h this is sent to

$$\begin{tikzcd}
x \arrow[r, "{[\gamma]}"] \arrow[d] & y \\
x_0 \arrow[r, "\mathrm{id}"'] & x_0 \arrow[u]
\end{tikzcd}$$

The vertical arrows are independent of $[\gamma]$, so that every path γ in X is homotopic to the composite the long way around the square, hence to every other path.

So in some sense, we are interested in spaces that are path connected, though this is useful when building spaces out of disjoint components. Here is another way we can get information about the fundamental groupoid of a space from the fundamental groupoid of other spaces. Lecture 9

Theorem 4. Let $Z \xrightarrow{\pi} X$ be a covering space. Then $\Pi_1(Z)(z_1, z_2) \to \Pi_1(X)(\pi(z_1), \pi(z_2))$ is injective for all $z_1, z_2 \in Z$.

thus the funtor $\Pi_1(\pi)$ is *faithful*

We will prove this theorem in a little bit, but let us give an important result that follows.

Corollary 5. Given a covering space $(Z, z) \xrightarrow{\pi} (X, x)$, the induced homomorphism between fundamental groups identifies $\pi_1(Z, z)$ with a subgroup of $\pi_1(X, x)$.

This allows us, given a covering space whose fundamental groupoid we know, to place a lower bound on the size of the fundamental group of the base space. Alternatively, it places an upper bound on the size of the fundamental group of the covering space, so if $\pi_1(X, x)$ is finite, then so is $\pi_1(Z, z)$.

Proposition 14. Let $Z \to I \times X$ be a covering space. Then $Z \xrightarrow{\simeq} I \times Z_0$ over $I \times X$, where $Z_0 := Z_{\{0\} \times X}$.

Proof. The function $Z \to I \times Z_0$ is given by $(\mathrm{pr}_1 \circ \pi, \tau)$, for some $\tau \colon Z \to Z_0$, which we need to construct. The idea is similar to the situation where we constructed the trivialisation of a covering space of I, which is the special case of $X = \mathrm{pt}$. Given $x \in X$, we get a trivisalisable covering space $Z_{I \times \{x\}} \to I \times \{x\} \simeq I$, and so a function $\tau_x \colon Z_{I \times \{x\}} \to I \times Z_{(0,x)} \xrightarrow{\mathrm{pr}_2} Z_{(0,x)}$. Hence we have a (potentially discontinuous) function $Z \to Z_0$ using the various τ_x. We will write down a global version of this function using ingredients we already know to be continuous.

Given $(t, x) \in I \times X$, there is a path $(0, x) \rightsquigarrow (t, x)$ given by $\eta_{(t,x)}(s) = (ts, x)$, which we want to vary continuously with (t, x). We know that $I \times I \times X \to I \times X$, $(s, t, x) \mapsto (ts, x)$ is continuous, so that by the

$$\begin{aligned} I \times X &\to (I \times X)^I \\ (t, x) &\mapsto \eta_{(t,x)} \end{aligned}$$

is continuous. We can now define the composite

$$\begin{array}{l} \tau \colon Z \xrightarrow{\simeq} (I \times X) \times_{I \times X} Z \to (I \times X)^I \times_{I \times X} Z \xrightarrow{\mathrm{Lift}} Z^I \xrightarrow{\mathrm{ev}_1} Z \\ \quad z \mapsto (\pi(z), z) \qquad \mapsto \quad (-\eta_{\pi(z)}, z) \end{array}$$

This map factors through Z_0, as if $(t, z) := \pi(z)$, then $-\eta_z$ is a path in $I \times X$ from (t, x) to $(0, x)$ and so the evaluation of the lift of $-\eta_z$ at 1 sits over $(0, x)$. Since all the maps here are continuous, τ is continuous.

We need to supply a continuous inverse to $(\mathrm{pr}_1 \circ \pi, \tau)$, which is built the same way, except now using η_z itself to lift, rather than $-\eta_z$:

$$\sigma\colon I \times Z_0 \to (I \times X)^I \times_{I\times X} Z \xrightarrow{\text{Lift}} Z^I \xrightarrow{\mathrm{ev}_1} Z.$$

This is manifestly continuous, and one can check that this map is the required inverse by considering the composite at each point separately, where it reduces to considering Z restricted to $I \times \{x\}$. $\square$

Corollary 6. If $f, g\colon X \to Y$ are homotopic, say by $H\colon I \times X \to Y$, and $Z \to Y$ is a covering space, then $f^*Z \simeq g^*Z$ over X.

Proof. If we form $H^*Z \to I \times X$, then we have by Proposition 14 that $H^*Z \simeq I \times f^*Z$. But $g^*Z \to X$ is (isomorphic to) $(H^*Z)_{\{1\}\times X}$, hence is isomorphic to $(I \times f^*Z)_{\{1\}\times X}$, but this is isomorphic to f^*Z. $\square$

This gives us a criterion whereby we know that no interesting covering spaces exist

Corollary 7. If X is contractible, then every covering space $Z \to X$ is isomorphic to $X \times Z_x$ for any $x \in X$.

Proof. Let $H\colon I \times X \to X$ be a contraction to $x \in X$. Then for $c_x\colon X \to X$ the constant map at x, $c_x^*Z = X \times Z_x$. But H is a homotopy between id_X and c_x, and $\mathrm{id}^* Z = Z$, so by Corollary 6 we have the required isomorphism. $\square$

Exercise: such a contraction exists for all $x \in X$

Example 35. Any locally convex topological vector space has no interesting covering spaces, likewise any convex or even star-shaped region therein. The unit sphere in a separable, infinite-dimensional Hilbert space has no interesting covering spaces. The infinite-dimensional Stiefel manifolds likewise.

Corollary 8. Let $Z \xrightarrow{\pi} Y$ be a covering space, $f, g\colon X \to Y$ a pair of maps and $H\colon I \times X \to Y$ a homotopy from f to g. If $\widetilde{f}\colon \{0\} \times X \to Z$ is a lift of f, in the sense that the diagram at right commutes, then there is a unique homotopy $\widetilde{H}\colon I \times X \to Z$ lifting H from $\widetilde{f}$ to a lift of g.

$$\begin{tikzcd}
\{0\} \times X \arrow[r, "\widetilde{f}"] \arrow[d] & Z \arrow[d, "\pi"] \\
I \times X \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & Y
\end{tikzcd}$$

Proof. Since $I \times f^*Z \xrightarrow{\simeq} H^*Z$, and we have a section $X \to f^*Z$, then we get a section $I \times X \to I \times f^*Z$. Composing with the isomorphism we get a map $I \times X \to I \times f^*Z \to H^*Z \to Z$, and this both restricts to $\widetilde{f}$ on $\{0\} \times X$ and covers H. To show uniqueness, notice that $H(-, x)$ gives a path in Y for each fixed $x \in X$. Any lift $\widetilde{H}'$ of H likewise gives a path $\widetilde{H}'(-, x)$ for fixed x. Since lifts of paths are unique, the $\widetilde{H}'(-, x)$ must agree with $\widetilde{H}(-, x)$ for all x, hence $\widetilde{H}' = \widetilde{H}$. $\square$

We can now give the promised proof of Theorem 4.

Proof. (of Theorem 4) Given paths $\gamma, \eta \colon z_1 \rightsquigarrow z_2$ in Z, and an endpoint-fixing homotopy $H \colon I \times I \to X$ between $\pi \circ \gamma$ and $\pi \circ \eta$, we can lift H to give a homotopy from γ to a lift of $\pi \circ \eta$. Since H fixes endpoints, the lifts of the constant paths $H\big|_{I \times \{i\}}$, for $i = 0, 1$ are path in the fibre, discrete spaces. Hence these paths are constant, and $\widetilde{H}$ is a homotopy fixing endpoints. Since η is a lift of $\pi \circ \eta$, unique path lifting gives that $\widetilde{H}$ is in fact a homotopy (fixing endpoints) from γ to η. Thus γ and η give the same element in $\Pi_1(Z)(z_1, z_2)$, and the induced map in injective as required. $\square$

Until now, a lot of our resuls only give bounds on or estimates between the fibres of a covering space and the fundamental group of the base space. However, we can actually get an exact result, given a certain kind of covering space

Theorem 5. If $\pi \colon (Z, z) \to (X, x)$ is a covering space with Z path connected, then
$$Z_x \simeq \pi_1(X, x)/\pi_1(Z, z),$$
as sets with $\pi_1(X, x)$-action.

Lecture 10

Proof. There is in fact a canonical isomorphism, induced in the following way. For group G and any transitive G-set S, and a point $p \in S$, then the map $G \to S$, $g \mapsto g \cdot s$ induces a well-defined bijection $G/\mathrm{Stab}(s) \to S$, where $\mathrm{Stab}(s) < G$ is the subgroup of elements g such that $g \cdot s = s$ (the *stabiliser subgroup*). Notice that for *any* subgroup $H < G$, G/H inherits a G-action from the multiplication in G. And the bijection $G/\mathrm{Stab}(s) \to S$ is compatible with the G-actions.

For a group G, sets with a G-action will be called *G-sets*.

that is, *equivariant*

We apply this to the transitive $\pi_1(X, x)$-set Z_x, where we know the action is transitive as Z is path-connected. This gives an isomorphism $\pi_1(X, x)/\mathrm{Stab}(z) \xrightarrow{\simeq} Z_x$, and it remains to identify $\mathrm{Stab}(z) < \pi_1(X, x)$. But note that if for some $[\gamma] \in \pi_1(X, x)$, $\gamma_*(z) = z$, this means that the lift $\widetilde{\gamma_z}$ beginning at z also ends at z, so is a loop in Z. Thus $\mathrm{Stab}(z)$ consists of the homotopy classes of loops in X that come from loops in Z, that is, $\mathrm{Stab}(z) = \pi_1(Z, z)$. $\square$

Corollary 9. If $\pi \colon (Z, z) \to (X, x)$ is a covering space with Z simply-connected, then the map
$$\pi_1(X, x) \to Z_x$$
is an isomorphism of $\pi_1(X, x)$-sets.

Proof. Since Z is simply-connected, $\pi_1(Z,z) = 1$, and so $\pi_1(X,x) \to Z_x$ is an isomorphism of sets with $\pi_1(X,x)$-action. $\square$

Example 36. We now can say that $\pi_1(S^1,1)$ is not just infinite (see Example 31) but countable, since it is in bijection with the fibre $\mathbb{Z}$ of the simply-connected covering space $\mathbb{R} \to S^1$.

But even better, we have not just a bijection, but Corollary 9 gives a *faithful permutation representation*: given $[\gamma], [\eta] \in \pi_1(X,x)$, there is some $z \in Z_x$ such that $\gamma_*(z) \neq \eta_*(z)$, which is equivalent to $\pi_1(X,x) \to \mathrm{Aut}(Z_x)$ being injective. Thus we have represented the fundamental group of (X,x) as a permutation group, where we can do more concrete computations.

Corollary 10. For $(Z,z) \to (X,x)$ a simply-connected covering space, $\pi_1(X,x)$ acts freely on Z_x.

And now we can give the first example of an actually calculated, non-trivial fundamental group.

Theorem 6. $\pi_1(S^1,1) \simeq \mathbb{Z}$.

Proof. We have the simply-connected covering space $\mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$, with fibre over $1 \in S^1$ being the integers. The inclusion $[0,1] \to \mathbb{R}$ is a lift of the loop γ going once around the circle, and all lifts are translates of this, so that the action of γ on the fibre $\mathbb{Z}$ is translation by 1. The loop γ generates a subgroup whose action on $\mathbb{Z}$ is transitive, hence γ generates all of $\pi_1(S^1,1)$, which must then be infinite cyclic, hence $\mathbb{Z}$. $\square$

As a result, for any subset $A \subset S^1$, the fundamental groupoid $\Pi_1(S^1,A)$ has as objects the set A, for every $x \in A$, $\pi_1(S^1,x) \simeq \mathbb{Z}$, and for any two points $x,y \in S^1$, the hom-set $\Pi_1(S^1,A)(x,y)$ is isomorphic as a set to $\mathbb{Z}$.

But how do we calculate π_1 in general? Or better, Π_1? Recall that $\Pi_1(X,A) = \Pi_1(X_1, A \cap X_1) \sqcup \Pi_1(X_2, A \cap X_2)$. For instance, if X_1 and X_2 are the only path components of X, and $\exists x \in A \cap X_i$ for $i = 1,2$, then every point in X is connected by a path to a point in A. This means the fundamental goups of the two path components are captured.

Example 37. Consider $\Pi_1(S^1 \sqcup S^1, 1 \sqcup 1)$, which is a groupoid with two objects, both of which have automorphism groups given by $\mathbb{Z}$.

So we are going to focus a bit on calculating the fundamental group(oid) for path connected spaces. The easiest way to make a new connected space from two other connected spaces X, Y, say, is to take a point in each, $x \in X$, $y \in Y$, and identify x and y.

recall that we are taking spaces to be semilocally path connected, so that components and path components coincide

Definition 27. Given two pointed spaces (X, x) and (Y, y), the *join* $X \vee Y$ is the quotient space $(X \sqcup Y)/(x \sim y)$. It has a basepoint given by $* := [x] = [y]$, and the inclusion maps of $(X, x) \xrightarrow{\mathrm{in}_L} (X \vee Y, *) \xleftarrow{\mathrm{in}_R} (Y, y)$ are pointed.

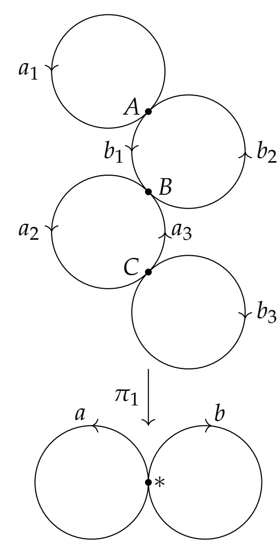
A key property of the join is that given a pointed space (M, m) and a pair of pointed maps $f\colon (X, x) \to (M, m)$, $g\colon (Y, y) \to (M, m)$, there is a unique pointed map $\langle f, g\rangle\colon (X \vee Y, *) \to (M, m)$ such that $f = \mathrm{in}_L \circ \langle f, g\rangle$ and $g = \mathrm{in}_R \circ \langle f, g\rangle$.

Since we have pointed maps, we get from functoriality of π_1 two homomorphisms $\pi_1(X, x) \to \pi_1(X \vee Y, *) \leftarrow \pi_1(Y, y)$. If we already know what the fundamental groups of X and Y are, then we can try to leverage this knowledge to tell us something about the fundamental group of the join. For instance, taking $X = Y = S^1$, we get homomorphisms

$$\mathbb{Z} \xrightarrow{\pi_1(\mathrm{in}_L)} \pi_1(S^\vee S^1, *) \xleftarrow{\pi_1(\mathrm{in}_R)} \mathbb{Z}$$

Let us define $a, b \in \pi_1(S^1 \vee S^1, *)$ to be the classes $\pi_1(\mathrm{in}_L)(1)$ and $\pi_1(\mathrm{in}_R)(1)$ respectively.

Define the covering space $Z_1 \xrightarrow{\pi_1} S^1 \vee S^1$ as at right, where $A, B, C \mapsto *$, and $a_i \mapsto a$, $b_i \mapsto b$, $i = 1, 2, 3$. Then we get a representation $\rho_1\colon \pi_1(S^1 \vee S^1, *) \to \mathrm{Aut}\{A, B, C\} \simeq S_3$. Looking at how paths representing a and b lift, we get $\rho_1(a) = (BC)$ and $\rho_1(b) = (AB)$, cycles in S_3. Calculating $\rho_1(ab)$ we get (ABC), and similarly for $\rho_1(ba)$, to get (ACB), so that $rho_1(ab) \neq \rho_1(ba)$. As a result, we must have had $ab \neq ba$ in $\pi_1(S^1 \vee S^1, *)$, or in other words, the fundamental group of $S^1 \vee S^1$ is **non-abelian**.



By a judicious choice of covering spaces, we can also prove that the two homomorphism $\mathbb{Z} \to \pi_1(S^1 \vee S^1, *)$ are injective, so that a and b generate infinite cyclic subgroups. We will later prove that $\pi_1(S^1 \vee S^1, *) \simeq \mathbb{Z} * \mathbb{Z} = F_2$, a free group on the generators a, b.

We can present $\mathbb{Z} * \mathbb{Z}$ as $\langle a, b | \rangle$, which has as elements $a^{n_1} b^{m_1} \ldots a^{n_k} b^{m_k}$ for $k \geq 1$ and $n_i, m_i \in \mathbb{Z}$, with $a^0 = e = b^0$

Lecture 11

Definition 28. The *free group on n-symbols*, F_n is any group with presentation

$$\langle x_1, \ldots, x_n \mid \rangle.$$

That is, generators $x_1, \ldots, x_n$ and no relations.

The symbols are of course arbitrary. Elements in F_n are (finite) words in x_i and x_i^{-1}, with the empty word () being the identity element, and with concatenation of words being the multiplication in F_n.

Definition 29. Given groups G and H, the *free product* $G * H$ of G and H is a group equipped with homomorphisms $i\colon G \to G * H$, $j\colon H \to G * H$, satisfying the following property: given any group K and homomorphisms $\phi\colon G \to K$, $\psi\colon H \to K$, there exists a unique homomorphism $\kappa\colon G * H \to K$ such that $\phi = i \circ \kappa$ and $\psi = j \circ \kappa$.

We can write things like this:

$$\begin{tikzcd}
& H \arrow[d, "j"] \arrow[ddr, bend left, "\psi"] & \\
G \arrow[r, "i"] \arrow[drr, bend right, "\phi"'] & G * H \arrow[dr, dashed, "\exists!", "\kappa"'] & \\
& & K
\end{tikzcd}$$

The existence of the unique κ given the data of ϕ and ψ is the *universal property* of the free product.

If $G = \langle g_1, \ldots, g_m \mid R_1, \ldots, R_n \rangle$ and $H = \langle h_1, \ldots, h_k \mid Q_1, \ldots, Q_l \rangle$ are presentations of H and G, then

here each R_i and Q_j are *relations*: equations involving the given generators of G and H respectively

$$G * H = \langle g_1, \ldots, g_m, h_1, \ldots, h_k \mid R_1, \ldots, R_n, Q_1, \ldots, Q_l \rangle$$

The free product of groups is an example of a more general construction, the *free product with amalgamation*, but this is again an example of a general construction that makes sense in an arbitrary category.

Definition 30. Let $\mathcal{C}$ be an arbitrary category. A *pushout square* is a commutative square

$$\begin{tikzcd}
W \arrow[r, "b"] \arrow[d, "a"'] & Y \arrow[d, "d"] \\
X \arrow[r, "c"'] & P
\end{tikzcd}$$

in $\mathcal{C}$ such that for any pair of morphisms $X \xrightarrow{f} Z \xleftarrow{g} Y$ such that $f \circ a = g \circ b$,

this unique existence is the *universal property* of the pushout

$$\exists!\ P \xrightarrow{k} Z \quad \text{such that} \quad f = k \circ c \text{ and } g = k \circ d.$$

Example 38. Consider a topological space X, and $U, V \subseteq X$ subspaces such that the $\{U^o, V^o\}$ is an open cover of X. Then

U and V here are 'glued together' along $U \cap V$ to give X

$$\begin{tikzcd}
U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\
U \arrow[r] & X
\end{tikzcd}$$

is a pushout square in **Top**, where all maps are the inclusions.

In the above example, we call $\{U, V\}$ a cover of X by nhds, since at least one of U and V is a nhd of each point in X.

Example 39. For any pair of pointed spaces (X, x) and (Y, y),

$$\begin{tikzcd} (\mathrm{pt}, \mathrm{pt}) \arrow[r] \arrow[d] & (Y, y) \arrow[d, "\mathrm{in}_R"] \\ (X, x) \arrow[r, "\mathrm{in}_L"'] & (X \vee Y, *) \end{tikzcd}$$

is a pushout square in $\mathbf{Top}_*$.

Example 40. For arbitrary groups G and H,

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & H \arrow[d] \\ G \arrow[r] & G * H \end{tikzcd}$$

is a pushout square in **Grp**.

Example 41. Recall the groupoid **2** with two objects, 0 and 1 and a unique arrow between any ordered pair of objects. The square

$$\begin{tikzcd} \mathrm{disc}(\{0,1\}) \arrow[r] \arrow[d] & \mathbf{2} \arrow[d, "(0 \to 1) \mapsto (\bullet \xrightarrow{1} \bullet)"] \\ \mathrm{pt} \arrow[r] & \mathbb{BZ} \end{tikzcd}$$

is a pushout in **Gpd**.

Example 42. Consider the category **Vect** of vector spaces (over some fixed field) and linear maps. The square

$$\begin{tikzcd} W \arrow[r, "L_2"] \arrow[d, "L_1"'] & V_2 \arrow[d] \\ V_1 \arrow[r] & (V_1 \oplus V_2)/J(W) \end{tikzcd}$$

with $J\colon W \to V_1 \oplus V_2$ the map $w \mapsto (L_1(w), -L_2(w))$ is a pushout.

Theorem 7 (Seifert–van Kampen theorem)**.** Let X be a space, and $\{U, V\}$ a cover by nhds. Then

A cover by nhds is equivalent to the interiors being an open cover.

$$\begin{tikzcd} \Pi_1(U \cap V) \arrow[r, "i_V"] \arrow[d, "i_U"'] & \Pi_1(V) \arrow[d] \\ \Pi_1(U) \arrow[r] & \Pi_1(X) \end{tikzcd}$$

is a pushout square in **Gpd**.

Remark. It is **not** immediate that this is a pushout just because the square of spaces is a pushout in **Top**, because we need to check the universal property for arbitrary groupoids Γ and (compatible) functors $\Pi_1(U) \to \Gamma \leftarrow \Pi_1(V)$.

Proof. We need to start with an arbitrary commutative square

$$\begin{tikzcd} \Pi_1(U \cap V) \arrow[r, "i_V"] \arrow[d, "i_U"'] & \Pi_1(V) \arrow[d, "G"] \\ \Pi_1(U) \arrow[r, "F"'] & \Gamma \end{tikzcd}$$

and construct a functor $K\colon \Pi_1(X) \to \Gamma$ compatible with F and G. That is, we need to construct a pair of functions $K_0\colon \Pi_1(X)_0 = X \to \Gamma_0$ and $K_1\colon \Pi_1(X)_1 \to \Gamma_1$ that together define a functor as needed.

Firstly, consider arbitrary $x \in X$. If $x \in U$, then define $K_0(x) = F(x)$, and if $x \in V$, define $K_0(x) = G(x)$. If $x \in U \cap V$, then since $F \circ i_U = G \circ i_V$, $F(x) = G(x)$, and so K_0 is well-defined.

We will first define K_1 on actual paths, and then show it is invariant under passing to homotopy classes. Suppose that $\gamma\colon I \to X$ factors through $U \hookrightarrow X$. Then we can define $K_1(\gamma) = F_1(\gamma)$, and similarly, if it factors through $V \hookrightarrow X$, then define $K_1(\gamma) = G_1(\gamma)$. Again, if γ lands in $U \cap V$ then it is unambiguously defined, by the commutativity of the square as given. This is compatible with source and target maps, since the start- and end-points of a path in U lie in U, and similarly for V, and F and G are functors. It is compatible with concatenation of paths that lie entirely inside U or inside V, again using the fact F and G are functors. Constant paths are sent by K_1 to identity morphisms in Γ, as needed, since they are by F and G. Also notice that if we reparametrise the path γ to $\gamma \circ \sigma$, this gives an equal morphism in $\Pi_1(U)$ or $\Pi_1(V)$ as appropriate, so that K_1 is independent of the parametrisation of the path.

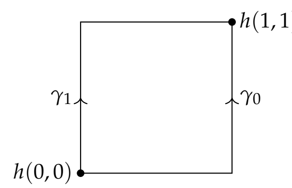
We now need to consider a general path $\gamma\colon I \to X$ and define $K_1(\gamma)$. If we pull back the open cover $\{U^o, V^o\}$ along γ to an open cover of I, we can find a partition $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = 1$ of I such that for each $i = 0, \ldots n$, $\gamma\big|_{[t_i, t_{i+1}]}$ factors through either $U \hookrightarrow X$ or $V \hookrightarrow X$ (or both). Define $\gamma_i\colon I \simeq [t_i, t_{i+1}] \to X$, so that γ is homotopic to the concatenation of all the γ_is, and in fact γ is a reparametrisation of the concatenation. We have already defined $K_1(\gamma_i)$, so let $K_1(\gamma) = K_1(\gamma_0)K_1(\gamma_1)\cdots K_1(\gamma_n) \in \Gamma_1$. Note that by the compatibility of K_1 with concatenation *inside U and V*, if we pass to a finer partition of I, we get a different sequence γ_j, but the

using the Lebesgue covering lemma

composite of the $K_1(\gamma_j)$s is equal to what we just defined. Since any two partitions have a common refinement, the definition of K_1 is independent of the choice of partition. Again, since the original given square commutes, there is no ambiguity when a given γ_i factors through $U \cap V$.

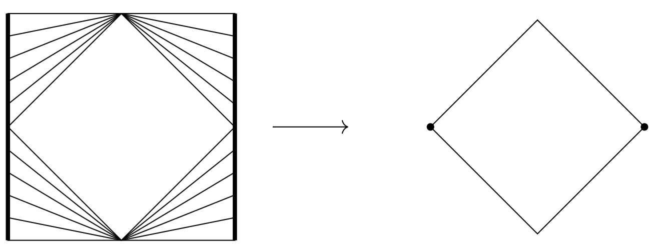
We now need to show that given an endpoint-fixing homotopy $H\colon I \times I \to X$ between paths γ and η, then K_1 maps them both to the same morphism in Γ.

Consider as a warmup, an arbitrary map $h\colon I^2 \to X$, and define paths $\gamma_0, \gamma_1\colon h(0,0) \rightsquigarrow h(1,1)$ in X as the concatenations



$$\gamma_0 := h(-,0)\#h(1,-),$$
$$\gamma_1 := h(0,-)\#h(-,1).$$

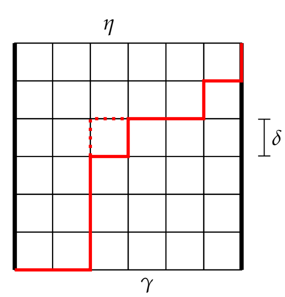
Then there is an endpoint-fixing homotopy $\gamma_0 \sim \gamma_1$. It is sufficient to define an endpoint-fixing homotopy $I \times I \to I^2$ between the two paths around the square that arise from taking h to be the identity map $I^2 \to I^2$.



Here the function is constant on the vertical edges of the square at the two vertices, and on each diagonal line as shown maps to the corresponding edges of the square on the right. Thus if h factors through one of U or V, $K_1(\gamma_0) = K_1(\gamma_1)$, since $[\gamma_0] = [\gamma_1]$ in one of $\Pi_1(U), \Pi_1(V)$.

Lecture 12

By the Lebesgue covering lemma applied (I^2, d_∞) and the open cover $\{H^{-1}(U^o), H^{-1}(V^o)\}$, there is a some $\delta > 0$ such that every square of side-length $\leq \delta$ in I^2 (a *δ-square*) is contained in one of $H^{-1}(U^o)$ and $H^{-1}(V^o)$. Thus $H|_{[a,a+\delta]\times[b,b+\delta]}$ factors through one of U or V, for any suitable $(a,b) \in I^2$. We then cover I^2 by such δ-squares, noting that this also give a partition of I into intervals such that both γ and η restricted to such intervals factor through one of U or V, so that K_1 is defined on γ and η.



Now we can use the fact about paths between opposite vertices of the square being homotopic to iteratively show that $K_1(\gamma) = K_1(\eta)$. Firstly, note that γ is homotopic to the path gotten by concatenating with the constant path up the right side of the square, and similarly, η is homopic to the path gotten by concatenating with the constant path up the left side of the square. Then the big homotopy is pasted together from homotopies that move one square at a time, each of which land in one of U or V. Then the two possible red paths shown in the figure, for example, get mapped by K_1 to the same morphism in Γ. All up, these show that $K_1(\gamma) = K_1(\eta)$, and so K_1 is well-defined on homotopy classes of paths. By the construction of K_1, it preserves composition, so is functorial, and we are done. $\square$

all homotopies here will have fixed endpoints

This is a powerful theorem, but sometimes not the best for computation, in this form. It would be good to have a version for more general $\Pi_1(X, A)$, for smaller $A \subset X$, or even $\pi_1(X, x)$. To do this, we need a general categorical lemma

Given an arbitrary category $\mathcal{C}$, we can define a category $\mathcal{C}^{\square}$ with objects commutative squares in $\mathcal{C}$, and morphisms commutative *cubes*: cubes of objects and morphisms such that every face is a commutative square.

$$\begin{tikzcd}
A_1 \arrow[rr] \arrow[dd] \arrow[dr] & & B_1 \arrow[dd] \arrow[dr] & \\
& A_2 \arrow[rr] \arrow[dd] & & B_2 \arrow[dd] \\
C_1 \arrow[rr] \arrow[dr] & & D_1 \arrow[dr] & \\
& C_2 \arrow[rr] & & D_2
\end{tikzcd}$$

A morphism from the back square to the front square in $\mathcal{C}^{\square}$

Definition 31. In a category $\mathcal{C}$, an object V is a *retract* of an object W, if there are morphisms $i\colon V \to W$ and $r\colon W \to V$ such that $r \circ i = \mathrm{id}_V$.

the morphism r is called a *retraction*

For example, in **Vect**, any subspace V of $\mathbb{R}^n$ is a retract, by taking i to be the inclusion, and r to be orthogonal projection onto V. In **Set**, given a set S and a subset $T \subseteq S$ with some chosen $t_0 \in T$ we get a retract given by the inclusion and the function $r\colon S \to T$ defined by $r(t) = t$, for $t \in T$, $r(s) = t_0$ for $s \in S \setminus T$. A more serious example is:

Example 43. Given a space X, with subspaces $A' \subseteq A$ such that every point in A is connected by a path in X to a point in A'. Then $\Pi_1(X, A')$ is a retract of $\Pi_1(X, A)$. The case we will most care about is $A = X$, and various $A' \subseteq X$.

This generalises the case from Assignment 2, where $A' = \{x\}$

We can talk about what it means for a commutative square in a category $\mathcal{C}$ to be a retract of another commutative square in $\mathcal{C}$, by looking at retracts in $\mathcal{C}^{\square}$. Recall that pushout squares are special examples of commutative squares. Also, to check that a morphism of commutative squares is a retraction, it is enough to check that it is a retraction at each vertex (that is, we have four retractions in $\mathcal{C}$, one for each vertex of the square)

Lemma 13. Retracts of pushout squares are pushout squares.

Proof. Exercise. □

We wish to apply Lemma 13 to the pushout square in **Gpd**—hence an object of $\mathbf{Gpd}^{\square}$—from the Seifert–van Kampen theorem, which involved fundamental groupoids $\Pi_1(X)$ etc. Retracts (in **Gpd**) as in Example 43 will be assembled to give a retract in $\mathbf{Gpd}^{\square}$ that is made up of smaller and more manageable groupoids.

Theorem 8 (Relative Seifert–van Kampen theorem)**.** Let X be a space, $\{U, V\}$ be a cover by nhds, and $A \subseteq X$ a given subspace. If in each of the four pairs (X, A), $(U, A \cap U)$, $(V, A \cap V)$, $(U \cap V, A \cap U \cap V)$, every point in the larger space is connected by a path (in that space) to a point in the smaller space, then

so a point in U is connected by a path in U to a point in $A \cap U$, and so on

$$\begin{tikzcd}
\Pi_1(U \cap V, A \cap U \cap V) \arrow[r] \arrow[d] & \Pi_1(V, A \cap V) \arrow[d] \\
\Pi_1(U, A \cap U) \arrow[r] & \Pi_1(X, A)
\end{tikzcd}$$

is a pushout square in **Gpd**.

Proof. The hard work involving homotopies etc is already done, we just need to exhibit the square as shown as a retract in $\mathbf{Gpd}^{\square}$ of the pushout square in the statement of the Seifert–van Kampen theorem. By Example 43, each of the groupoids in the commutative square are, individually, retracts. The inclusion functors

$$\begin{aligned}
\Pi_1(U \cap V, A \cap U \cap V) &\hookrightarrow \Pi_1(U \cap V) \\
\Pi_1(U, A \cap U) &\hookrightarrow \Pi_1(U) \\
\Pi_1(V, A \cap V) &\hookrightarrow \Pi_1(V) \\
\Pi_1(X, A) &\hookrightarrow \Pi_1(X)
\end{aligned}$$

give a morphism in $\mathbf{Gpd}^{\square}$, so we just need to construct the functors

$$\begin{aligned}
\Pi_1(U \cap V, A \cap U \cap V) &\leftarrow \Pi_1(U \cap V) \\
\Pi_1(U, A \cap U) &\leftarrow \Pi_1(U) \\
\Pi_1(V, A \cap V) &\leftarrow \Pi_1(V) \\
\Pi_1(X, A) &\leftarrow \Pi_1(X)
\end{aligned}$$

that together give a morphism of commutative squares in the other direction. To do this, we will choose, for each $x \in X$, a (homotopy class of a) path $\eta_x \colon x \rightsquigarrow a_x$, for some $a_x \in A$, such that if $x \in U$, take $a_x \in A \cap U$ and η_x a path in U; if $x \in V$, take $a_x \in A \cap V$ and η_x a path in V; and hence if $x \in U \cap V$, it follows that $a_x \in A \cap U \cap V$ and

η_x is a path in $U \cap V$. Further, if $x \in A$ already, take $a_x = x$, and η_x the constant path.

The assignment $x \mapsto a_x$, and $(x \overset{\gamma}{\rightsquigarrow} y) \mapsto (a_x \rightsquigarrow x \rightsquigarrow y \rightsquigarrow a_y)$ gives a functor $\Pi_1(X) \to \Pi_1(X, A)$, and this is a retraction. By the specific choices of a_x and η_x we made, the restrictions of this functor to the groupoids $\Pi_1(U)$, $\Pi_1(V)$, $\Pi_1(U \cap V)$ land in the corresponding subgroupoids $\Pi_1(U, A \cap U)$ etc, and again give a retraction in each case. We can check that these do indeed give us a morphism in $\mathbf{Gpd}^{\square}$, which is enough to show we have a retraction in $\mathbf{Gpd}^{\square}$. □

We would like to consider pushouts of groups, since these can be easier in some cases to compute. The statement of the relative Seifert–van Kampen theorem however involves pushouts of groupoids, so that even if we consider one-obect groupoids associated to groups we need to be careful that the universal property for the pushout in **Gpd** implies the universal property for the pushout in **Grp**. Thankfully, this is true, for abstract reasons.

Lemma 14. Let $\mathcal{C}$ be a category and let $\mathcal{D} \hookrightarrow \mathcal{C}$ be a full subcategory. Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & P \end{array}$$

be a commutative square in $\mathcal{D}$ that is a pushout square in $\mathcal{C}$. Then it is a pushout square in $\mathcal{D}$.

Proof. We will check the universal property for the pushout in $\mathcal{C}$. Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

be an arbitrary commutative square in $\mathcal{D}$. Then considering this as a commutative square in $\mathcal{C}$, we have a unique morphism $k \colon P \to D$ (in $\mathcal{D}$) compatible with the other data as in the definition of pushout square. But since $\mathcal{C}$ is a *full* subcategory, k is a morphism in $\mathcal{C}$, and moreover the commuting triangles still commute in $\mathcal{C}$. Given any other morphism $P \to D$ in $\mathcal{C}$ making the triangles commute will be equal to k in $\mathcal{D}$, and hence in $\mathcal{C}$, so the universal property for the pushout holds in $\mathcal{C}$. □

Now we can use the fact that $\mathbb{B} \colon \mathbf{Grp} \to \mathbf{Gpd}$ expresses **Grp** as a full subcategory.

Corollary 11. Let X be a path connected space, $\{U, V\}$ a cover by path connected nhds with $U \cap V$ path connected. For $x \in U \cap V$, the square

$$\begin{tikzcd} \pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(V, x) \arrow[d] \\ \pi_1(U, x) \arrow[r] & \pi_1(X, x) \end{tikzcd}$$

is a pushout square in **Grp**.

Proof. The hypotheses on X, U, V and x imply that the condition of the relative Seifert–van Kampen theorem hold, so that we have a pushout of one-object groupoids. But by the above lemma, we get a pushout of groups. ☐

So we need to know what pushouts of groups look like!

Example 44. Consider the cover of the sphere S^n, where $n > 1$, by $U = S^n \setminus \{N\}$ and $V = S^n \setminus \{S\}$, where N and S are a pair of antipodal points (North and South poles). Then $U \cap V \simeq S^{n-1} \times (-1, 1)$, and all these spaces are path connected, so we can apply the group version of Seifert–van Kampen. Take a basepoint $x \in S^{n-1} \subset U \cap V$. Using stereographic projection, we get that $U \simeq \mathbb{R}^n \simeq V$, hence both of these are contractible, and so $\pi_1(U, x) = 1 = \pi_1(V, x)$ are both the trivial group. Then by Seifert–van Kampen we know that

$$\begin{tikzcd} \pi_1(S^{n-1} \times (-1,1), x) \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & \pi_1(S^n, x) \end{tikzcd}$$

is a pushout square. If we take an arbitrary group K then to check the universal property, the data of the homomorphisms $1 \to K \leftarrow 1$ tells us nothing, the compatibility being automatically satisfied, so we need $\pi_1(S^n, x)$ to be a group such that there is a *unique* homomorphism from it to K. But the only group that has a unique homomorphism to any other group is the trivial group. Thus $\pi_1(S^n, x) = 1$ for all $n > 1$.

This argument fails for $n = 1$ since the cover as constructed in that case results in the intersection $U \cap V$ being the disjoint union of two intervals, so not path connected. Lecture 13

Definition 32. Let $G \overset{\phi}{\leftarrow} L \overset{\psi}{\rightarrow} H$ be a pair of homomorphisms. The *free product with amalgamation* $G *_L H$ is the group $G * H / \langle \phi(x)\psi(x)^{-1} \rangle$, where $\langle \phi(x)\psi(x)^{-1} \rangle$ is the smallest normal subgroup generated by the elements $\phi(x)\psi(x)^{-1}$ for all $x \in L$. There are homomorphisms $G \to G *_L H \leftarrow H$, and $G *_L H$ satisfies the universal property of the pushout in **Grp**.

Note that if $G = \langle g_1, \ldots, g_m \mid R_1, \ldots, R_n \rangle$ and $H = \langle h_1, \ldots, h_k \mid Q_1, \ldots, Q_l \rangle$, then

this description also works for groups that aren't finitely presented

$$G *_L H \simeq \langle g_1, \ldots, g_m, h_1, \ldots, h_k \mid R_1, \ldots R_n, Q_1, \ldots, Q_l, \phi(x)\psi(x)^{-1} = e \rangle$$

where we add a new relation for each $x \in L$, or even just each x running through a set of generators for L. Note that these relations are equivalent to $\phi(x) = \psi(x)$, so that we do indeed get a commutative square.

Example 45. Consider a *finitely generated one-relator group* $G = \langle g_1, \ldots, g_m \mid R = e \rangle$ (R is an element of the free group generated by $g_1, \ldots, g_m$). Such a group is a pushout of the form

Such groups are important in geometric group theory, and much is known about them

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "r"'] & 1 \arrow[d] \\ F_m \arrow[r] & G \end{tikzcd}$$

where $R = r(1)$.

For a more specific example, take the *surface group*

$$\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle.$$

More generally, one can write a finitely presented group as a pushout

$$\begin{tikzcd} F_n \arrow[r] \arrow[d, "r"'] & 1 \arrow[d] \\ F_m \arrow[r] & \langle g_1, \ldots, g_m \mid r(a_1) = e, \ldots r(a_n) = e \rangle \end{tikzcd}$$

where we take $F_n \langle a_1, \ldots, a_n \mid \rangle$.

Remark. Going back to free products, for a moment, a famous example is the free product $\mathbb{Z}/2 * \mathbb{Z}/3$, which is isomorphic to the *modular group*

$$PSL_2(\mathbb{Z}) = \{2 \times 2 \text{ integer matrices } A \mid \det(A) = 1\}/\{\pm I\}$$

One presentation of $PSL_2(\mathbb{Z})$ is via the generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $ST = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, which satisfy $S^2 = I$ and $(ST)^3 = I$. Note that $PSL_2(\mathbb{Z})$

It is not obvious that this even is a presentation, for a proof see Roger C. Alperin, $PSL_2(\mathbf{Z}) = \mathbf{Z}_2 * \mathbf{Z}_3$, The American Mathematical Monthly Vol. 100, No. 4 (Apr., 1993), pp. 385–386, doi:10.2307/2324963 acts by fractional linear transformations on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$, with $S \colon z \mapsto \frac{-1}{z}$ and $ST \colon z \mapsto \frac{1-z}{z}$. This action is continuous and has discrete orbits, and this is enough to make $\mathcal{H} \to \mathcal{H}/PSL_2(Z)$ a covering space.

Give the concrete treatment for the pushout of groups above (that is, as free products with amalgamation), one could hope for a similar treatment for groupoids. And indeed, one can do this, where instead of group elements being (equivalence classes of) words in the elements of the given groups, morphisms of the pushout groupoid are (equivalence classes of) words in the morphisms of the given groupoids. However, we need to be careful about what we mean by words constructed as a string of morphisms, since not all morphisms can be composed.

We will not give the most general treatment here, but show how to describe the pushout of groupoids in a special case corresponding to a situation arising from an application of the Seiert–van Kampen theorem.

Example 46. Let X be a space, $\{U, V\}$ a cover by nhds, and $A \subseteq U \cap V$ be such that every path component of U, V and $U \cap V$ contains at least one point in A. Then we can apply the Seifert–van Kampen theorem and get a pushout square

$$\begin{tikzcd} \Pi_1(U \cap V, A) \arrow[r, "i_V"] \arrow[d, "i_U"'] & \Pi_1(V, A) \arrow[d] \\ \Pi_1(U, A) \arrow[r] & \Pi_1(X, A) \end{tikzcd}$$

Note that all four groupoids have the same set of objects, and that all the functors are the identity on objects (that is: $i_U(a) = a$ and so on). From the proof of the Seifert–van Kampen theorem recall that we expressed paths in X, that is, morphisms in $\Pi_1(X)$ as a composite of paths alternating between U and V. This is the setup we are interested in calulating in general from a purely algebraic point of view. For simplicity, we will just think about the case of A finite, which is the case that turns up in calculations of 'reasonable' examples.

Suppose we are given a diagram

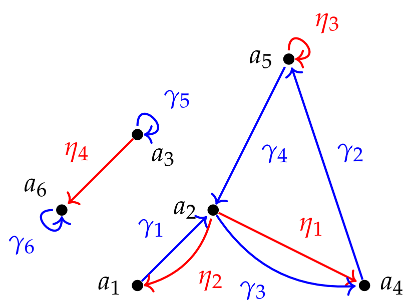
here H is the capital η

$$\begin{tikzcd} \Lambda \arrow[r, "F"] \arrow[d, "G"'] & H \\ \Gamma & \end{tikzcd}$$

in **Gpd**, where all the groupoids have the same finite set $A = \{a_1, \ldots, a_N\}$ of objects, and such that the object components of the functors F and G are all the identity function. We wish to construct a groupoid $\Gamma *_\Lambda H$ that makes this into a pushout square. Firstly, we can take the set of objects to be A again, and the functors $\Gamma \to \Gamma *_\Lambda H \leftarrow H$ will have as object component the identity function.

Given any groupoid there is a directed graph with nodes the objects of the groupoid, and as directed edges the morphism (and we are allowed edges from a node to itself, and multiple edges between nodes). And given our two groupoids Γ and H, we can form a graph $\mathcal{G}$ with set of nodes A, and the directed edges are the *disjoint union* of the morphisms of Γ (coloured blue) and H (coloured red), and with the identity morphisms removed. We also don't need to include both a morphism and its inverse, since the inverse can be gotten by traversing a directed edge against the indicated direction.

and indeed any category



in the graph shown, composites of various morphisms are omitted, for instance $\gamma_3\gamma_1\gamma_2$

Now instead of a word in group elements, as in the pushout of groups, we take a *path* in this directed graph, alternating between edges that come from Γ and edges that come from H. For instance, we could take

$$\gamma_3^{-1}\eta_1\gamma_2(\eta_3)^5\gamma_4 \qquad \text{or} \qquad \eta_4(\gamma_6)^{-3}\eta_4^{-1}$$

from the above graph. The 'empty' path consisting just of an identity arrow is also an option. However, we haven't yet actually constructed $\Gamma *_\Lambda H$, merely what we might call $\Gamma *_{\Lambda_0} H$, which is the pushout where the groupoid Λ is replaced by the trivial groupoid $\mathrm{disc}(\Lambda_0)$ with the same objects but only identity arrows. What we need to do is add 'relations', namely extra equalities between morphisms in $\Gamma *_{\Lambda_0} H$. What this means is that for each morphism λ in Λ, we identify the morphisms $F(\lambda)$ and $G(\lambda)$, or more precisely, quotient by the equivalence relation on each hom-set of $\Gamma *_{\Lambda_0} H$ generated by these identifications. For instance, if in the above graph, $\gamma_1 = F(\lambda_1)$ and $\eta_1 = G(\lambda_1)$, then we add the equality $\gamma_1 = \eta_1$. This would have

if Λ is already trivial in this sense, then we are done; this is the analogue of the free product of groups the effect of making

$$\gamma_3\eta_1\gamma_2(\eta_3)^5\gamma_4^{-1} = (\gamma_3\gamma_1\gamma_2)(\eta_3)^5\gamma_4^{-1}$$

Concatenation of strings and simplifying is the composition in $\Gamma *_\Lambda H$.

Exercise 13. Prove that this construction makes $\Gamma *_\Lambda H$ a groupoid.

If we are interested in merely looking at the group of morphisms from a single object a_i to itself, which is the case when calculating a fundamental group using the groupoid Seifert–van Kampen, then we should look at paths that start and finish at the chosen a_i.

Example 47. Let us look at an example, arising from an application of Seifert–van Kampen. Consider the circle S^1 as sitting in $\mathbb{C}$, and let $U = S^1 \setminus \{-i\}$, $V = S^1 \setminus \{i\}$. All three of these are path connected, and let us take $A = +1, -1 \subset U \cap V$. This choice of data satisfies the hypotheses of SvK. The pushout square

$$\begin{tikzcd}
\Pi_1(U \cap V, \{\pm 1\}) \arrow[r] \arrow[d] & \Pi_1(V, \{\pm 1\}) \arrow[d] \\
\Pi_1(U, \{\pm 1\}) \arrow[r] & \Pi_(S^1, \{\pm 1\})
\end{tikzcd}$$

can be simplified as follows. First, $U \simeq (-2, 2) \simeq V$, in a way that preserves $A = \{\pm 1\}$, so that

$$\Pi_1(U, \{\pm 1\}) \simeq \mathbf{2} = (-1 \underset{\gamma^{-1}}{\overset{\gamma}{\leftrightarrows}} +1) \qquad \Pi_1(V, \{\pm 1\}) \simeq \mathbf{2} = (-1 \underset{\eta}{\overset{\eta^{-1}}{\leftrightarrows}} +1)$$

and we have omitted the identity arrows. Here γ is a path that runs anticlockwise around S^1 from $+1$ to -1, and η is a path that runs anticlockwise from -1 to $+1$. Second, $\Pi(U \cap V, \{\pm 1\}) = \mathrm{disc}(\{+1, -1\})$, so we are in the easier situation as first outlined above, where no additional quotient needs to be done. The pushout then looks like

$$\begin{tikzcd}
\boxed{-1 \quad +1} \arrow[r] \arrow[d] & \boxed{-1 \underset{\eta}{\overset{\eta^{-1}}{\leftrightarrows}} +1} \arrow[d] \\
\boxed{-1 \underset{\gamma^{-1}}{\overset{\gamma}{\leftrightarrows}} +1} \arrow[r] & \Pi_1(S^1, \{\pm 1\})
\end{tikzcd}$$

The graph we need so as to generate $\Pi_1(S^1, \{\pm 1\})$ is then

$$\begin{tikzcd} -1 \arrow[r, bend right, red, "\gamma"'] & +1 \arrow[l, bend right, blue, "\eta"'] \end{tikzcd}$$

Then if we wish to consider paths from $+1$ to itself, the only options are the empty path, hence the identity arrow, or $(\eta\gamma)^n$, or $(\gamma^{-1}\eta^{-1})^n = (\eta\gamma)^{-n}$. Thus $\pi_1(S^1, +1) \simeq \mathbb{Z}$.

Exercise 14. Prove that this construction of $\Gamma *_\Lambda H$ is indeed the pushout in **Gpd**!

We will consider one more variant of Seifert–van Kampen, and this time in brief, because the details are similar to other versions. If we would like to compute the fundamental group of a join, then it is not quite enough to just consider the free product of the fundamental groups: a join does not automatically come with a cover of the sort we need for SvK. In what follows, assume: (X, x), (Y, y) are pointed spaces such that there exist nhds $x \in U \subseteq X$ and $y \in V \subseteq Y$ that are contractible to x and y respectively, with the contraction fixing the basepoint. Then $\{X \vee V, U \vee Y\}$ is a cover of $X \vee Y$ by nhds. We have retractions $X \vee V \to X$ and $U \vee Y \to Y$ that preserve the basepoints and which are also homotopy equivalences. Thus $\pi_1(X \vee V, *) \simeq \pi_1(X, x)$ and $\pi_1(U \vee Y, *) \simeq \pi_1(Y, y)$, and $\pi_1(U \vee V, *)$ is trivial. If we apply the Seifert–van Kampen theorem to the pushout square

this implies $U \vee V$ is contractible to the basepoint $* = [x] = [y]$

$$\begin{tikzcd} (U \vee V, *) \arrow[r] \arrow[d] & (U \vee Y, *) \arrow[d] \\ (X \vee V, *) \arrow[r] & (X \vee Y, *) \end{tikzcd}$$

then we get a pushout square of groups

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \pi_1(Y, y) \arrow[d] \\ \pi_1(X, x) \arrow[r] & \pi_1(X \vee Y, *) \end{tikzcd}$$

or in other words,

$$\pi_1(X \vee Y, *) = \pi_1(X, x) * \pi_1(Y, y).$$

This generalises the fact that $\pi_1(S^1 \vee S^1, *) = F_2 = \mathbb{Z} * \mathbb{Z}$ to more general spaces.

Fact. Given any presentation $G = \langle g_1, \ldots, g_m \mid R_1 = e, \ldots, R_n = e \rangle$ there is a space X arising as a pushout

$\bigvee_{j=1}^m S^1 = \underbrace{S^1 \vee \ldots \vee S^1}_{m \text{ times}}$

$$\begin{tikzcd} \coprod_{i=1}^{n} S^1 \arrow[r] \arrow[d, "\langle f_1,\ldots,f_n\rangle"'] & \coprod_{i=1}^{n} D^2 \arrow[d] \\ \bigvee_{j=1}^{m} S^1 \arrow[r] & X \end{tikzcd}$$

where $f_i(1) = R_i$, with the property that $\pi_1(X,*) \simeq G$. This space is in some sense 2-dimensional as it is gotten by gluing together 2d discs, and sometimes a manifold, though not always.

Even better, for *any* group G with any presentation, there is an appropriate pushout

$$\begin{tikzcd} \coprod_{\beta\in J} S^1 \arrow[r] \arrow[d, "\langle f_\beta\rangle"'] & \coprod_{\beta\in J} D^2 \arrow[d] \\ \bigvee_{\alpha\in I} S^1 \arrow[r] & X \end{tikzcd}$$

with the property that $\pi_1(X,*) \simeq G$. One has to be careful with the topology, and we haven't defined infinite joins, but it does work out that $\pi_1(\bigvee_{\alpha\in I} S^1, *) \simeq F_I$, the free group on the set I.

the coutable join $\bigvee_{n\in\mathbb{N}} S^1$ may seem like the Hawaiian earring, but it is in fact not compact, and *is* slsc, so they cannot be homeomorphic

Example 48. An oriented compact Riemann surface Σ_g of genus $g \geq 1$ is an example of a surface gotten by a construction as in the Fact above, and even better: it only requires one copy of D^2. For genus g, it requires doing a pushout of the form

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \\ \bigvee_{i=1}^{2g} S^1 \arrow[r] & \Sigma_g \end{tikzcd}$$

An alternative way to build this pushout is to consider a $4g$-gon, and selectively identify edges in pairs (recovering the $2g$ circles as in the pushout). The pattern of identifications is exactly that which gives rise to the one-relator group after Example 45, since the *attaching map* $S^1 \to \bigvee_{i=1}^{2g} S^1$ in the preceeding pushout is given by $\prod_{i=1}^{g}[a_i, b_i]$, for a_i and b_i the generators of the $2i-1$th and $2i$th copy of S^1 respectively.

Insert octagon picture for case $g = 2$ here

Classifying covering spaces

Recall that for a covering space $Z \xrightarrow{\pi} X$ we get a representation

Lecture 14

$$\begin{aligned} \rho_Z \colon \Pi_1(X) &\longrightarrow \mathbf{Set} \\ x &\mapsto Z_x \\ [\gamma \colon x \rightsquigarrow y] &\mapsto \left(\gamma_* \colon Z_x \xrightarrow{\simeq} Z_y\right) \end{aligned}$$

of the fundamental groupoid of X. Given a pair of covering spaces $Z_1, Z_2 \to X$ and a map between them in $\mathbf{Cov}_X$, how are the representations ρ_{Z_1} and ρ_{Z_2} related? Since the triangle at right commutes, we get for each $x \in X$ a function between the corresponding fibres, $f\big|_x \colon (Z_1)_x \to (Z_2)_x$. Notice that this is a function $\rho_{Z_1}(x) \to \rho_{Z_2}(x)$ for each object of $\Pi_1(X)$. Given a path $\gamma\colon x \rightsquigarrow y$ in X and $z \in (Z_1)_x$, we have the unique lift to Z_1 starting at z, namely $\widetilde{\gamma_z}^{-1}\colon z \rightsquigarrow \gamma_*(z)$. By composing with f, we get a path $f \circ \widetilde{\gamma_z}^{-1}\colon I \to Z_2$ from $f(z)$ to $f(\gamma_*(z))$. But by uniqueness of lifts of paths, this is the lift of γ to Z_2 starting at $f(z)$, which is a path from $f(z) \rightsquigarrow \gamma_*(f(z))$. We thus get $f(\gamma_*(z)) = \gamma_*(f(z))$ for every $z \in (Z_1)_x$, implying that the following square commutes:

$$\begin{tikzcd} Z_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & Z_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

$$\begin{tikzcd} (Z_1)_x \arrow[r, "\gamma_*"] \arrow[d, "f|_x"'] & (Z_1)_y \arrow[d, "f|_y"] \\ (Z_2)_x \arrow[r, "\gamma_*"'] & (Z_2)_y \end{tikzcd}$$

Or, in other words, the functions $f\big|_x$ define a natural transformation $\rho_{Z_1} \Rightarrow \rho_{Z_2}$. This leads us to

Proposition 15. The mapping $(Z \xrightarrow{\pi} X) \mapsto \rho_Z$ define a functor

$$\mathbf{Cov}_X \longrightarrow [\Pi_1(X), \mathbf{Set}].$$

Recall that the category $[\mathcal{C}, \mathbf{Set}]$ has as objects the functors $\mathcal{C} \to \mathbf{Set}$, and as morphisms the natural transformations

Proof. The natural transformation $\rho_Z \Rightarrow \rho_Z$ associated to the identity map $Z \to Z$ has a components the identity functions $Z_x \to Z_x$, hence is the identity natural transformation. Also, using uniqueness of path lifting one can show that given two composable maps of covering spaces, we get functoriality. $\square$

Example 49. Regard $S^1 \subset \mathbb{C}$. Recall the covering spaces $\exp(2\pi i(-))\colon \mathbb{R} \to S^1$ and $S^1 \xrightarrow{(-)^n} S^1$, for $n > 1$. There is a map of covering spaces

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "\exp(2\pi i(-)/n)"] \arrow[dr, "\exp(2\pi i(-))"'] & & S^1 \arrow[dl, "(-)^n"] \\ & S^1 & \end{tikzcd}$$

The fibres of $\mathbb{R} \to S^1$ are isomorphic to $\mathbb{Z}$, and indeed the fibre over $1 \in S^1$ *is* $\mathbb{Z}$. The fibre over 1 of $S^1 \to S^1$ is $\mathbb{Z}/n$, and we get the induced map

$$\begin{aligned} \mathbb{Z} &\to \mathbb{Z}/n \\ k &\mapsto k \pmod n \end{aligned}$$

Now note that if we focus on the point $1 \in S^1$, then a morphism in $\Pi_1(S^1)$ from 1 to itself is the homotopy class of some loop, which can identify with an integer under $\pi_1(S^1, 1) \simeq \mathbb{Z}$. Then the representation associated to $\mathbb{R} \to S^1$ is

$$\begin{aligned} \rho_{\mathbb{R}} \colon \Pi_1(S^1) &\longrightarrow \mathrm{Aut}(\mathbb{Z}) \\ m &\mapsto (k \mapsto k+m) \end{aligned}$$

whereas the representation associated to $(-)^n \colon S^1 \to S^1$ is

$$\begin{aligned} \rho_n \colon \Pi_1(S^1) &\longrightarrow \mathrm{Aut}(\mathbb{Z}/n) \simeq S_n \\ m &\mapsto (k \mapsto k+m \pmod n) \end{aligned}$$

The map $\mathbb{Z} \to \mathbb{Z}/n$ is clearly equivariant for the shift action of $\mathbb{Z}$ as shown, as a special case of the naturality of the map $\mathbb{Z} \to \mathbb{Z}/n$.

Question 3. What representations $\Pi_1(X) \to \mathbf{Set}$ can arise as ρ_Z for some covering space $Z \to X$?

Recall that it is obvious that every set is the set of connected components of some space, and while we didn't go into details, the construction of a space X with $\pi_1(X, \mathrm{pt}) \simeq G$ for any given G is a relatively uncomplicated pushout. However, given a representation $\Pi_1(X) \to \mathbf{Set}$, it is not immediately obvious how to build a covering space giving rise to it. Indeed, all the covering spaces we have seen so far are either natural examples we happen to have seen, or special toy cases chosen to illustrate some small aspect of the fundamental groupoid of a particularly simple space. We are going to look at doing some reductions to simpler cases, on both sides (the topological, $\mathbf{Cov}_X$ and the algebraic, $[\Pi_1(X), \mathbf{Set}]$) to make the task easier.

First, since we have the blanket assumption that out spaces are slpc, we can consider finding a section to the continuous map $X \to [\mathrm{pt}, X]$, namely $A \colon [\mathrm{pt}, X] \to X$, that picks out one point a_i per path component $X_i \subseteq X$. We have the full subgroupoid inclusion $\Pi_1(X, A) \hookrightarrow \Pi_1(X)$ that is additionally an equivalence. Let us denote by I the set $[\mathrm{pt}, X]$ in what follows.

by an argument as in the solutions for assignment 2, question 7

Lemma 15. If $i \colon \mathcal{C} \hookrightarrow \mathcal{D}$ is a full subcategory inclusion that is also an equivalence, then the restriction map

$$\begin{aligned} [\mathcal{D}, \mathbf{Set}] &\xrightarrow{i^*} [\mathcal{C}, \mathbf{Set}] \\ F &\mapsto F \circ i \end{aligned}$$

is an equivalence of categories.

Proof. Exercise. $\square$

Now notice also that $\Pi_1(X, A) = \Pi_1(\bigsqcup_{i\in I} X_i, \bigsqcup_{i\in I}\{a_i\}) \simeq \bigsqcup_{i\in I} \mathbb{B}\pi_1(X_i, a_i)$

Lemma 16. For any family of categories $(\mathcal{C}_i)_{i\in I}$ we have an isomorphism

$$[\bigsqcup_{i\in I} \mathcal{C}_i, \mathbf{Set}] \xrightarrow{\simeq} \prod_{i\in I}[\mathcal{C}_i, \mathbf{Set}].$$

An object of a product of family of cateories is a tuple of objects, one from each of the categories, and similar with the morphisms

Proof. Exercise. □

Given a group G, we can define the category $G\mathbf{Set}$ which has as objects sets S equipped with a G-action, $G \to \mathrm{Aut}(S)$, and with morphisms *equivariant* functions. Note that for any groupoid Γ and representation $\rho\colon \Gamma \to \mathbf{Set}$, for each object $x \in \Gamma_0$ there is a permutation representation $\Gamma(x, x) \to \mathrm{Aut}(\rho(x))$. To any natural transformation $\rho \Rightarrow \rho'$ between representations, there is an equivariant map between $\Gamma(x, x)$-sets, and this is functorial.

An equivariant function $f\colon S \to T$ satisfies $f(g \cdot p) = g \cdot f(p)$ for all $p \in S$

Lemma 17. The functor just described gives an isomorphism $[\mathbb{B}G, \mathbf{Set}] \xrightarrow{\simeq} G\mathbf{Set}$ of categories

We can put all of these lemmas together and get an equivalence of categories

$$[\Pi_1(X), \mathbf{Set}] \to [\Pi_1(X, A), \mathbf{Set}] \simeq \prod_{i\in I}[\mathbb{B}\pi_1(X_i, a_i), \mathbf{Set}] \simeq \prod_{i\in I} \pi_1(X_i, a_i)\mathbf{Set}.$$

We can compose this with the original functor we were looking at, from covering spaces to representations, to get

$$\begin{aligned} \mathbf{Cov}_X &\to \prod_{i\in I} \pi_1(X_i, a_i)\mathbf{Set} \\ (Z \to X) &\mapsto (\rho_i\colon \pi_1(X_i, a_i) \to Z_{a_i})_{i\in I} \end{aligned} \tag{3}$$

where now the codomain is much more tractable. Further, the objects in categories of the form $G\mathbf{Set}$ are not unreasonable: we can break them down into smaller parts. Each object $\rho\colon G \to \mathrm{Aut}(S)$ in $G\mathbf{Set}$ isomorphic to one of a particularly nice form, namely $S \simeq \bigsqcup_{j\in S/G} G/\mathrm{Stab}(p_j)$, where the points $p_j \in S$ are chosen so that there is one in each orbit of the G-action.

Remark. From now on, we will consider only spaces that are slsc, since this will ultimately be the case in the classification theorem, and also because X slsc implies that for every covering space $Z \to X$, the space Z is locally path connected, so that path components and components agree.

Lecture 15

Lemma 18. For a space $X = \bigsqcup_{i\in I} X_i$, there is an equivalence of categories $\mathbf{Cov}_X \simeq \prod_{i\in I} \mathbf{Cov}_{X_i}$.

Proof. A covering space $Z \to \bigsqcup_{i \in I} X_i$ gives covering spaces $Z_{X_j} := \mathrm{in}_j^* X$ for each $j \in I$, where recall the inclusion maps $\mathrm{in}_j \colon X_j \to \bigsqcup_{i \in I} X_i$. We also get, from a map of covering space $Z \to Z'$ over X, a map $Z_{X_j} \to Z'_{X_j}$ of covering spaces over X_j, for each j. This gives a functor $\mathbf{Cov}_X \to \prod_{i \in I} \mathbf{Cov}_{X_i}$.

Conversely, given a covering space $Z_i \to X_i$ for each $i \in I$, we get a covering space $\bigsqcup_{i \in I} Z_i \to \bigsqcup_{i \in I} X_i$, and for maps $Z_i \to Z'_i$ of covering spaces over X_i, there is a map $\bigsqcup_{i \in I} Z_i \to \bigsqcup_{i \in I} Z'_i$ of covering spaces over $\bigsqcup_{i \in I} X_i$. This gives a functor $\prod_{i \in I} \mathbf{Cov}_{X_i} \to \mathbf{Cov}_X$, and these two functors are an equivalence of categories. $\square$

The functor in (3) then factorises as the composite

$$\begin{array}{ccccc} \mathbf{Cov}_X & \xrightarrow{\simeq} & \prod_{i \in I} \mathbf{Cov}_{X_i} & \longrightarrow & \prod_{i \in I} \pi_1(X_i, a_i)\mathbf{Set} \\ \begin{array}{c} Z \\ \downarrow \\ X \end{array} & \mapsto & \begin{array}{c} Z_{X_i} \\ \downarrow \\ X_i \end{array} & \mapsto & (\rho_i \colon \pi_1(X_i, a_i) \to \mathrm{Aut}(Z_{a_i})) \end{array}$$

In particular, if we understand each $\mathbf{Cov}_{X_i} \to \pi_1(X_i, a_i)\mathbf{Set}$, then we are done.

So, we will consider from now on the case of a connected, slsc space X with chosen $x_0 \in X$, and the functor $\mathbf{Cov}_X \to \pi_1(X, x_0)\mathbf{Set}$.

Since X is slsc, it is locally path connected, and the local trivialisation of any covering space $Z \to X$ shows that Z is also locally path connected. We can then write $Z = \bigsqcup_{\alpha \in I} Z_\alpha \to X$, where each $Z_\alpha \to X$ is a (path) connected covering space.

there is a small subtlety here, in that we really require a basis of nhds that are path connected, I will expand on this point if pushed, but it is a technicality that can be ignored for our purposes

Example 50. Take $X = S^1$. Recall that we have various covering spaces: $S^1 \times F \simeq \bigsqcup_F S^1 \to S^1$ for discrete spaces F; the exponential map $\mathbb{R} \to S^1$; the various $S^1 \xrightarrow{(-)^n} S^1$ with $n > 1$, where we shall write S^1_n for the total space of this covering space. Then we can form a covering space

Exercise!

$$S^1 \times F \sqcup \bigsqcup \mathbb{R} \sqcup \bigsqcup S^1_2 \sqcup \bigsqcup S^1_3 \sqcup \ldots \to S^1.$$

It remains to be seen, however, if there are any covering spaces not of this form.

Recall from lecture 6 (on page 22) that given a covering space $Z \to X$, there is a surjective map

$$\Omega_{x_0} X \times \{z_\alpha \mid \alpha \in I\} \to Z_{x_0}, \qquad z_\alpha \in Z_{x_0} \cap Z_\alpha,$$

hence a surjective map

$$\pi_1(X,x_0)\times\{z_\alpha \mid \alpha\in I\}\simeq\coprod_{\alpha\in I}\pi_1(X,x_0)\times\{z_\alpha\}\to Z_{x_0}=\coprod_{\alpha\in I}Z_{x_0}\cap Z_\alpha$$

Lemma 19. Given a covering space $Z\to X$ with $x_0\in X$, then $Z_{x_0}=\coprod_{\alpha\in I}Z_{x_0}\cap Z_\alpha\simeq\coprod_{\alpha\in I}\pi_1(X,x_0)/\pi_1(Z_\alpha,z_\alpha)$ as $\pi_1(X,x_0)$-sets, for any choice of $z_\alpha\in Z_{x_0}\cap Z_\alpha$.

Proof. It is enough to show that the orbits of the $\pi_1(X,x_0)$-action are the sets $Z_{x_0}\cap Z_\alpha$, because then the general description of sets with transitive action takes over. Given $z\in Z_{x_0}\cap Z_\alpha$, there is a path $\gamma\colon z_\alpha\rightsquigarrow z$, and hence a loop $\pi\circ\gamma$ at x_0. This loop acts on Z_{x_0}, with $(\pi\circ\gamma)_*(z_\alpha)=z$. Hence $Z_{x_0}\cap Z_\alpha$ is contained in the orbit containing z_α. Conversely, given any $z\in Z_{x_0}$ and a loop η at x_0 such that $\eta_*(z_\alpha)=z$, then there is a lift $\widetilde{\eta}\colon z_\alpha\rightsquigarrow z$ of η, so that $z\in Z_\alpha$. Thus the orbit of z_α is contained in $Z_{x_0}\cap Z_\alpha$, and we are done. □

As a result, our functor $\mathbf{Cov}_X\to\pi_1(X,x_0)\mathbf{Set}$ preserves disjoint unions: it sends the covering space $\coprod_{\alpha\in I}Z_\alpha\to X$ to the disjoint union of sets with a permutation representation $\coprod_{\alpha\in I}Z_{x_0}\cap Z_\alpha$.

Remark. Given a group G and a G-set S, such that $S=\coprod_{\alpha\in I}S_\alpha$ is the partition into orbits of the action, then the representation $\rho\colon G\to\mathrm{Aut}(\coprod S_{\alpha\in I})$ factors through the subgroup $\prod_{\alpha\in I}\mathrm{Aut}(S_i)<\mathrm{Aut}(\coprod S_{\alpha\in I})$, hence $\rho=(\rho_\alpha)_{\alpha\in I}$, for $\rho_\alpha\colon G\to\mathrm{Aut}(S_\alpha)$ a transitive G-action.

Recall what our original question was: what representations $\Pi_1(X)\to\mathbf{Set}$ arise from covering spaces of X via the functor $\mathbf{Cov}_X\to[\Pi_1(X),\mathbf{Set}]$? We have now reduced this to the simpler aim of starting with a permutation representation of a fundamental group, and can reduce the input data even more. Inside $\mathbf{Cov}_X$ (for connected X) is the full subcategory $\mathbf{Cov}_X^{\mathrm{conn}}$ of *connected* covering spaces, and inside $\pi_1(X,x_0)\mathbf{Set}$ is the full subcategory $\pi_1(X,x_0)\mathbf{Set}^{\mathrm{tr}}$ of sets with a *transitive* action. Moreover, in both of these cases, arbitrary objects in $\mathbf{Cov}$ and $\pi_1(X,x_0)\mathbf{Set}$ can be gotten by disjoint union of objects in the respective subcategories. Thus it is sufficient to ask if we can get any transitive $\pi_1(X,x_0)$-set from some connected covering space of X. That is, we want to know what is the image of the functor

$$\mathbf{Cov}_X^{\mathrm{conn}}\longrightarrow\pi_1(X,x_0)\mathbf{Set}^{\mathrm{tr}}.$$

Note that in the category of transitive G-sets, every object S with a point $p\in S$ is the quotient of the underlying set of G equipped with the action by multiplication, via the map

$$\begin{aligned} G &\to S \simeq G/\mathrm{Stab}(p) \\ g &\mapsto g \cdot p \end{aligned}$$

and the G-action on G is free and transitive. More generally, given any G-set T with a free and transitive action, there is a surjective equivariant map $T \to S$ displaying S as a quotient of T.

Example 51. Given a simply-connected covering space $Z \to X$ and $x_0 \in X$, the fibre Z_{x_0} is a free and transitive $\pi_1(X, x_0)$-set.

Recall also that for a general path connected covering space $Z \to X$, $Z_{x_0} \simeq \pi_1(X, X_0)/\pi_1(Z, z)$ for any $z \in Z_{x_0}$. If we start with a pointed covering space, then we don't have to choose a point, and every connected covering space of a pointed space can be gotten by forgetting the point from some pointed covering space. So we have the final form of the question

Question 4. Given a subgroup $H < \pi_1(X, x_0)$, is there a pointed covering space $(Z, z_0) \to (X, x_0)$ such that $\pi_1(Z, z_0) = H$ as subgroups of $\pi_1(X, x_0)$?

Recall from assignment 2, where from a simply-connected space Y with free G-action, we got a covering space $Y \to Y/G$ with $\pi_1(Y/G, *) \simeq G$

Lecture 16

Recall that for a simply-connected covering space $Z^{(1)} \to X$ the fibre over x_0 is isomorphic to $\pi_1(X, x_0)$, and that our ultimate aim is to get a covering space with fibre isomorphic to $\pi_1(X, x_0)/H$. So one way to approach this is to see if there is a way to make sense of making a covering space by taking the quotient of some simply-connected covering space "fibre by fibre". It isn't possible to do this piecemeal, so we need to do this for all fibres at once.

Definition 33. Let $p\colon Y \to X$ be a map of spaces, G a group acting on Y. We say the action is *fibrewise* if $p(g \cdot y) = p(y)$ for all $y \in Y, g \in G$.

It follows from the definition that there is map $Y/G \to X$, and the fibre of this map over x is $p^{-1}(x)/G$.

there is a commutative triangle

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y/G \arrow[dl] \\ X & \end{tikzcd}$$

Proposition 16. Let $Z \xrightarrow{\pi} X$ be a covering space, and let G act fibrewise on Z. Then $Z/G \to X$ is a covering space and $Z \to Z/G$ is a map of covering spaces.

Proof. First notice that since the action of G is fibrewise, the orbits of G are subspaces of discrete spaces, and hence are discrete.

For $U \subseteq X$, we get an action of G on $Z_U \subseteq Z$, also fibrewise for the restriction $Z_U \to U$ of π. If we take U small enough nhd around

$x \in X$ then $Z_U \simeq Z_x \times U$, and moreover this homeomorphism is G-equivariant and respects the maps to U. We thus get a homeomorphism $(Z_U)/G \simeq U \times (Z_x/G)$. Then, from the question in assignment 4, $(Z/G)_U$ is homeomorphic to $(Z_U)/G$ over U, so there is a homeomorphism $(Z/G)_U \simeq U \times (Z_x/G)$ over U, and hence $Z/G \to X$ is a covering space with fibre Z_x/G. The quotient map $Z \to Z/G$ respects the maps to X by construction, so is a map of covering spaces. □

Thus if $Z^{(1)} \to X$ is a simply-connected covering space and $H < \pi_1(X, x_0)$ acts fibrewise on $Z^{(1)}$, we get a covering space $Z^{(1)}/H \to X$ with fibre $\pi_1(X, x_0)/H$ *as sets*. However, more is true.

Lemma 20. Assuming there is a simply-connected covering space $Z^{(1)} \to X$, the morphism $Z_x^{(1)} \to Z_x^{(1)}/H$ is $\pi_1(X, x)$-equvariant.

Proof. Exercise. Uses path lifing and the map $Z^{(1)} \to Z^{(1)}/H$ over X. □

We have have two problems:

1. How do we construct a simply-connected covering space of X? Or how do we know one exists?
2. Given such a thing, and an arbitrary subgroup $H < \pi_1(X, x_0)$, how do we construct a fibrewise H-action?

We can reduce the second problem to the case of finding a fibrewise $\pi_1(X, x_0)$-action, because then by restriction, any subgroup will also act fibrewise. As it turns out, the construction that will address the first point will also automatically come with the required fibrewise action of the fundamental group.

Construction 1. Fix a pointed space (X, x_0). Consider the quotient space $X^{(1)} := P_{x_0}X/\sim$ where $\gamma_1 \sim \gamma_2$ if $\gamma_1(1) = \gamma_2(1)$ and $[\gamma_1] = [\gamma_2]$ in $\Pi_1(X)$. From the definition of the equivalence relation and the quotient topology, we get a continuous pointed map $X^{(1)} \to X$ and a commutative triangle

we will denote the class of the constant path also by c_{x_0} to de-clutter the notation

$$\begin{tikzcd}
(P_{x_0}, c_{x_0}) \arrow[rr] \arrow[dr] & & (X^{(1)}, c_{x_0}) \arrow[dl] \\
& (X, x_0) &
\end{tikzcd}$$

Thus one can see this as giving a topology to a subset of the morphisms of $\Pi_1(X)$.

Proposition 17. For X semilocally simply-connected, $(X^{(1)}, c_{x_0}) \to (X, x_0)$ is a simply-connected covering space, and $\pi_1(X, x_0)$ acts by concatenation as in $\Pi_1(X)$

Proof. Notice first that the fibre $X^{(1)}_x$ of $X^{(1)} \to X$ over $x \in X$ is $\Pi_1(X)(x_0, x) \simeq P^x_{x_0}X/\sim$, and moreover, since X is slsc, $P^x_{x_0}X$ is slpc, and hence $X^{(1)}_x = [\mathrm{pt}, P^x_{x_0}X] = \pi_0(P^x_{x_0}X)$ is discrete. Thus $X^{(1)} \to X$ has discrete fibres.

Now fix $x \in X$. Since X is slsc, there is a nhd $U \ni x$ such that for every $x' \in U$ there is a path $\eta_{x'} \colon x \rightsquigarrow x'$ (in U) such that for any other path $\eta \colon x \rightsquigarrow x'$ in U, $[\eta_{x'}] = [\eta]$ in $\Pi_1(X)$. Consider $X^{(1)}_U = \{[x_0 \rightsquigarrow x'] \mid x' \in U\}$.

Claim: There is a homeomorphism

$$U \times X^{(1)}_x \xrightarrow{\simeq} X^{(1)}_U$$
$$(x', [x_0 \overset{\gamma}{\rightsquigarrow} x]) \mapsto [x_0 \overset{\gamma}{\rightsquigarrow} x \overset{\eta_{x'}}{\rightsquigarrow} x']$$

over U. We can prove this is a bijection without too much difficulty, as follows.

- Injective: If $[x_0 \overset{\gamma}{\rightsquigarrow} x \overset{\eta_{x'}}{\rightsquigarrow} x'] = [x_0 \overset{\gamma'}{\rightsquigarrow} x \overset{\eta_{x'}}{\rightsquigarrow} x']$, then we can concatenate with $-\eta_{x'}$ and the result is that $[\gamma] = [\gamma']$.
- Surjective: Suppose I have $[x_0 \overset{\gamma}{\rightsquigarrow} x'] \in X^{(1)}_U$. Then $[\gamma] = [\gamma][\eta_{x'}]^{-1}[\eta_{x'}]$.

For now, for the proof that this is a homeomorphism, see pages 64–65 of Hatcher, in the section "The Classification of Covering Spaces".

I will fill in the details here later (not examinable!)

Thus we have a local trivialisation, and hence a covering space.

We can see that $X^{(1)}$ is path connected by observing that it is the quotient of the path connected space $P_{x_0}X$, hence there is a surjective map $P_{x_0}X \to X^{(1)}$.

there is a path from c_{x_0} to any $\gamma \colon x_0 \rightsquigarrow x$, defined by $s \mapsto (t \mapsto \gamma(st))$

There is a fibrewise action of $\pi_1(X, x_0)$ on $X^{(1)}$ by $([x_0 \overset{\omega}{\rightsquigarrow} x_0], [x_0 \overset{\gamma}{\rightsquigarrow} x]) \mapsto [x_0 \overset{\omega}{\rightsquigarrow} x_0 \overset{\gamma}{\rightsquigarrow} x]$. This is continuous because it is induced from the continuous concatenation $\Omega_{x_0}X \times P_{x_0}X \to P_{x_0}X$, and the axioms for an action hold from the associativity of composition in $\Pi_1(X)$. Note especially that the $\pi_1(X, x_0)$-action is *free*, so that the stabiliser subgroups are all trivial.

Finally, since the fibre of $X^{(1)}$ at x_0 is the quotient of $\pi_1(X, x_0)$ by the stabiliser subgroup by the $\pi_1(X, x_0)$-action, and this stabiliser subgroup is $\pi_1(X^{(1)}, c_{x_0})$, we have that $\pi_1(X^{(1)}, c_{x_0})$ is trivial. Hence $X^{(1)}$ is simply-connected. □

As a result, given any subgroup $H < \pi_1(X, x_0)$, we can define the connected covering space $X^{(1)}/H \to X$, which is a pointed covering space with $\pi_1(X^{(1)}, *) = H$. Working backwards through the reasoning above, we can get *any* set with $\pi_1(X, x_0)$-action, up to isomorphism, as the fibre of some covering space of (X, x). And then, if X has multiple connected components, we can the take the disjoint union of covering spaces of each component to get a representation of the whole fundamental groupoid. Thus we have proved

Proposition 18. For an slsc space X the functor

$$\mathbf{Cov}_X \to [\Pi_1(X), \mathbf{Set}]$$

is essentially surjective: every representation $\Pi_1(X) \to \mathbf{Set}$ is isomorphic to one coming from a covering space.

We can say still more: Lecture 17

Lemma 21. Let $Z \xrightarrow{\pi} X$ be a covering space, Y path connected and $p\colon Y \to X$ be some map. Suppose we have maps $f, g\colon Y \to Z$ such that $f \circ \pi = p = g \circ \pi$. Then if $f(y_0) = g(y_0)$ for some $y_0 \in Y$, we have $f = g$.

Proof. Take $y \in Y$ arbitrary, and let $\gamma\colon y_0 \rightsquigarrow y$ be a path. Then $f \circ \gamma$ and $g \circ \gamma$ are both lifts of the path $p \circ \gamma\colon I \to X$. And since $f(\gamma(0)) = f(y_0) = g(y_0) = g(\gamma(y_0))$, we must have $f(y) = f(\gamma(1)) = g(\gamma(1)) = g(y)$. □

This applies in particular, if Y is another covering space of X.

Corollary 12. The functor $\mathbf{Cov}_X \to [\Pi_1(X), \mathbf{Set}]$ is faithful.

Proof. We can reduce to the case of X connected, and can choose a basepoint $x_0 \in X$ to get a functor $\mathbf{Cov}_X \to [\Pi_1(X), \mathbf{Set}] \to \pi_1(X, x_0)\mathbf{Set}$. This functor will be faithful if and only if the original functor is (after the reduction to X connected.) Now suppose we have two covering spaces $Z_i \xrightarrow{\pi_i} X$ ($i = 1, 2$) and a pair of maps $f, g\colon Z_1 \to Z_2$ between them (over X). Then if $f\big|_x = g\big|_x\colon (Z_1)_{x_0} \to (Z_2)_{x_0}$, in particular for each path component of Z_1, there is a point z such that $f(z) = f\big|_x(z) = g\big|_x(z) = g(z)$, so that f and g coincide on each path component, and hence agree everywhere. □

In fact, we have the following landmark result:

Theorem 9. For slsc X, $\mathbf{Cov}_X \to [\Pi_1(X), \mathbf{Set}]$ is an equivalence of categories.

The proof reduces to connected and pointed X, then shows that given $(Z_1)_x \to (Z_2)_x$ a map of $\pi_1(X,x)$-sets, there is a map $Z_1 \to Z_2$ between covering spaces that induces it. This shows that the functor is full, hence together with the previous results, that it is an equivalence.

Higher homotopy groups

So far, we have invariants π_0, $[\mathrm{pt}, -]$ and π_1 of a space. Notice that $\pi_1 = [(S^1, 1), -]_*$, and in fact for pointed spaces, $[\mathrm{pt}, -] = [(S^0, 1), -]_*$, since $S^0 = \{1, -1\}$, and a pointed map $(S^0, 1) \to (X, x)$ is specified completely by where it sends -1. This leads naturally to the question of what should π_n be?

treat it as a fluke of low dimensions that there are two invariants that aim to capture what is meant by "components" of a space

Definition 34. Given a pointed space (X, x), define the n^{th} *homotopy group of* X *based at* x to be $[(S^n, 1), (X, x)]_*$.

This is automatically a functor $\mathbf{Top}_* \to \mathbf{Set}$, though we shall soon see we can refine this. It follows quickly from the definition that we have $\pi_n(X \times Y, (x, y)) \simeq \pi_n(X, x) \times \pi_n(Y, y)$.

An important observation is that S^n here can be treated in a number of different ways that all lead to the same definition. Notice particularly that $S^n \simeq I^n/\partial I^n \simeq D^n/\partial D^n$. We can take the basepint in S^n to be the image of the collapsed subspace, so that continuous maps of pairs $(I^n, \partial I^n) \to (X, x)$ are in bijection with maps $(S^n, 1) \to (X, x)$, and similarly for maps $(D^n, \partial D^n) \to (X, x)$. The relation of homotopy of maps $(Y, A) \to (X, x)$ out of a pair generalises that of a pointed homotopy, and demands that the homotopy $H \colon I \times Y \to X$ maps $I \times A$ to x. With this definition of relative homotopy, we have that $[(S^n, 1), (X, x)]_* = [(I^n, \partial I^n), (X, x)] = [(D^n, \partial D^n), (X, x)]$.

For a pair (Y, A), we define the quotient Y/A of Y by the subspace A to be the quotient space $Y/(a_1 \sim a_2, \forall a_1, a_2 \in A)$

Further, given a map $f \colon (I^n, \partial I^n) \to (X, x)$, we get a continuous map $I^{n-1} \to \Omega_x X$, and the boundary condition on f implies that ∂I^{n-1} is mapped to the constant loop c_x. Thus $\pi_n(X, x) = [(I^n, \partial I^n), (X, x)] \simeq [(I^{n-1}, \partial I^{n-1}), (\Omega_x X, c_x)] = \pi_{n-1}(\Omega_x X, c_x)$. Since we know path concatenation is continuous, we have the map $\Omega_x X \times \Omega_x X \to \Omega_x X$. This allows us to define a binary operation

$$\pi_n(X,x) \times \pi_n(X,x) \simeq \pi_{n-1}(\Omega_x X, c_x) \times \pi_{n-1}(\Omega_x X, c_x) \simeq \pi_{n-1}(\Omega_x X \times \Omega_x X, (c_x, c_x)) \to \pi_{n-1}(\Omega_x X, c_x) \simeq \pi_n(X, x)$$

The up-to-homotopy associativity of the concatenation of loops, together with inverses and identity element up to homotopy, means that $\pi_n(X, x)$ is in fact a group, and so we have a functor $\pi_n \colon \mathbf{Top}_* \to \mathbf{Grp}$. The following is a famous later abstraction of a result that originally arose when the higher homotopy groups were first defined in 1932 (or so).

Lemma 22 (Eckmann–Hilton argument)**.** Let M be a set with two unital binary operations $\circ, \# \colon M \times M \to M$, with units $1_\circ$ and $1_\#$, such that for all $a, b, c, d \in M$,

$$(a \circ b)\#(c \circ d) = (a\#c) \circ (b\#d).$$

Then $1_\circ = 1_\#$, $a \circ b = a\#b$, $a\#b = b\#a$ and moreover the binary operation is associative, making M an abelian group

Proof. 1. First, $1_\circ = 1_\circ \circ 1_\circ = (1_\circ\#1_\#) \circ (1_\#\#1_\circ) = (1_\circ \circ 1_\#)\#(1_\# \circ 1_\circ) = 1_\#\#1_\# = 1_\#$, so that the unit elements agree, and we can just denote $1_\circ = 1_\# =: 1$.

2. Then $a \circ b = (a\#1) \circ (1\#b) = (a \circ 1)\#(1 \circ b) = a\#b$, so we can write $ab := a \circ b = a\#b$ for the single binary operation.

3. Now $ab = (1a)(b1) = (1b)(a1) = ba$, so that the binary operation is commutative.

4. Finally, $(ab)c = (ab)(1c) = (a1)(bc) = a(bc)$, so that the binary operation is associative, and thus we have an abelian group structure.

□

Example 52. If G is a Lie group then $\pi_1(G, e)$ is abelian, because we have the concatenation operation, and the operation of pointwise multiplication of loops (which passes down to homotopy classes of loops). The constant loop is the identity element for both of these operations, so we get the first part of the previous lemma for free. The only thing that needs checking is that pointwise multiplication and concatenation distribute as needed for the Eckmann–Hilton argument, but this is not difficult to check.

or even just a topological group

Now let us go back to our observation that $\mathbf{Top}_*((S^n, 1), (X, x)) \simeq \mathbf{Top}_*((S^{n-1}, 1), (\Omega_x X, c_x))$. We can iterate this, to get $\mathbf{Top}_*((S^{n-1}, 1), (\Omega_x X, c_x)) \simeq \mathbf{Top}_*((S^{n-2}, 1), (\Omega_{c_x}\Omega_x X, c_{c_x}))$. This makes sense because in the definition of the based loop space $\Omega_y Y$, the space Y is arbitrary. Let us write $\Omega^2_x X$ for $\Omega_{c_x}\Omega_x X$, and always assume it has the basepoint c_{c_x}. Then there are two continuous concatenation operations

$$\Omega^2 X \times \Omega^2 X \to \Omega^2 X,$$

arising from concatenating in either the first or the second parameter. Given $f_1, f_2 \in \Omega^2 X$, that is, functions $I^2 \to X$ such that ∂I^2 is mapped to x, we have

$$(f_1 \#_1 f_2)(s,t) = \begin{cases} f_1(2s,t) & \forall t \in I,\ s \in [0, \frac{1}{2}] \\ f_2(2s-1,t) & \forall t \in I,\ s \in [\frac{1}{2}, 1] \end{cases}$$

$$(f_1 \#_2 f_2)(s,t) = \begin{cases} f_1(s,2t) & \forall t \in [0, \frac{1}{2}],\ s \in I \\ f_1(s,2t-1) & \forall t \in [\frac{1}{2}, 1],\ s \in I \end{cases}$$

And, moreover, $(f_1 \#_1 f_2) \#_2 (f_3 \#_1 f_4) = (f_1 \#_2 f_3) \#_1 (f_2 \#_2 f_4)$, where each of these is defined on one quadrant of the ^{2} subdivided into four squares. Thus, from the Eckmann–Hilton argument,

Proposition 19. The group $\pi_n(X,x)$ is abelian for all $n \geq 2$.

The assignment Lecture 18

$$(X,x) \mapsto \pi_n(X,x)$$
$$(f\colon (X,x) \to (Y,y)) \mapsto (f_*\colon \pi_1(X,x) \to \pi_n(Y,y))$$

is then a functor $\mathbf{Top}_* \to \mathbf{Ab}$.

Example 53. For T a discrete space, $\pi_n(T,*) = 1$, since all $S^n \to T$ are constant maps (S^n is connected!)

Example 54. If X is contractible (eg a star-shaped domain in a topological vector space) then $\pi_n(X,x) = 1$

Recall: Given a covering space $Z \to X$ there is an associated representation $\Pi_1(X) \to \mathbf{Set}$. But there is nothing special here about the category $\mathbf{Set}$, we can have other categories, for instance: the category $\mathbf{Fin}$ of finite sets, the category $\mathbf{Vect}$ of vector spaces, the category $\mathbf{Ab}$ of abelian groups, or more generally the category $R\mathbf{Mod}$ of R-modules (R here is a given ring).

Proposition 20. Fix the space X. The assignment $x \mapsto \pi_n(X,x)$ is the object component of a representation $\Pi_1(X) \to \mathbf{Ab}$.

Proof. (Sketch) Given $\gamma\colon [0,1] \to X$, $x \rightsquigarrow y$, and $\alpha\colon (I^n, \partial I^n) \to (X,x)$ representing a class in $\pi_n(X,x)$, we need to construct a class in $\pi_n(X,y)$, in such a way that this gives a group isomorphism $\pi_n(X,x) \xrightarrow{\simeq} \pi_n(X,y)$. For this construction, consider the interval $I = [-\frac{1}{2}, \frac{1}{2}]$. Fix an orientation-preserving homeomorphism $I = [-\frac{1}{2}, \frac{1}{2}] \xrightarrow{\simeq} [-\frac{1}{4}, \frac{1}{4}]$, and thus a map $i\colon [-\frac{1}{4}, \frac{1}{4}]^n \simeq I^n$. Also fix orientation-preserving $j\colon [\frac{1}{4},\frac{1}{2}] \xrightarrow{\simeq} [0,1]$. Define $\alpha^\gamma\colon I^n \to X$ to be the piecewise defined function

$$\alpha^\gamma(\mathbf{x}) = \begin{cases} \alpha(i(\mathbf{x})) & \mathbf{x} \in [-\frac{1}{4},\frac{1}{4}]^n \\ \gamma(j(|\mathbf{x}|)) & \mathbf{x} \in [-\frac{1}{2},\frac{1}{2}]^n \setminus [-\frac{1}{4},\frac{1}{4}]^n \end{cases}$$

By the pasting lemma this is continuous, as $\alpha(\mathbf{x}) = x$ for all $\mathbf{x} \in \partial I^n$, and $\gamma(0) = x$. Moreover, $\alpha^\gamma(\mathbf{x}) = y$ for all $\mathbf{x} \in \partial I^n$, and so $\alpha^\gamma\colon (I^n, \partial I^n) \to (X, y)$. The homotopy class of α^γ is independent of the choice of γ and α as representatives for their respective classes in $\Pi_1(X)(x,y)$ and $\pi_n(X,x)$, as we can use homotopies between these and other representatives to create a homotopy between maps $(I^n, \partial I^n) \to (X, y)$. We thus get a function $\pi_n(X,x) \to \pi_n(X,y)$ for each $[\gamma] \in \Pi_1(X)(x,y)$.

As functions between sets, this is functorial by a reparametrisation argument, so that $[\alpha^{\gamma\#\eta}] = [(\alpha^\gamma)^\eta]$, and $[\alpha^{c_x}] = [\alpha]$. The last thing that needs to be checked is that $[\alpha] \mapsto [\alpha^\gamma]$ is a group homomorphism. This is a mildly fiddly, but overall unenlightening argument. □

there is a proof in Hatcher, §4.1 on page 341

Note that if we consider the case $n = 1$, then we have already seen a version of this: given a path $\gamma\colon x \rightsquigarrow y$, there is an isomorphism $\pi_1(X,x) \xrightarrow{\simeq} \pi_1(X,y)$. If we specialise to the case of $x = y$, then γ is a loop, and the resulting automorphism $\pi_1(X,x)$ is conjugation by γ.

Remark. If X is simply-connected, then we get *canonical* isomorphisms $\pi_n(X,x) \simeq \pi_n(X,y)$ for all pairs of points $x, y \in X$. For this reason, many authors omit basepoints when talking about homotopy groups when it is not important.

there is an analogue when we talk about just a simply-connected path component of X and points in it

Remark. Since, for each n we get a representation $\Pi_1(X) \to \mathbf{Ab}$ from the collection of higher homotopy groups $\pi_n(X,x)$, there is a representation $\Pi_1(X) \to \mathbf{Ab} \to \mathbf{Set}$ gotten by composing with the underlying set functor $\mathbf{Ab} \to \mathbf{Set}$. From this we get a covering space of X, whose fibre at $x \in X$ is $\pi_n(X,x)$. In fact this covering space 'remembers' the fact that each higher homotopy group is actually a group, in that it is a continuously-varying family of groups, not just of sets.

Lemma 23. Given a pair of pointed spaces (X,x) and (Y,y), there is an isomorphism $\pi_n(X \times Y, (x,y)) \xrightarrow{\simeq} \pi_n(X,x) \times \pi_n(Y,y)$.

natural, even

Just as we saw that homotopic maps should be considered the same from the point of view of fundamental groups and covering spaces, the same is true for higher homotopy groups

Lemma 24. Let $f, g\colon X \to Y$ be homotopic, say via $H\colon I \times X \to Y$. Then there is a commutative triangle

$$\begin{tikzcd} & \pi_n(Y, f(x)) \arrow[dd, "\simeq"] \\ \pi_n(X, x) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\ & \pi_n(Y, g(x)) \end{tikzcd}$$

Proof. The result follows for the special case of $Y = I \times X$ with $H = \mathrm{id}_{I\times X}$ the homotopy between the inclusion maps $f\colon X \simeq \{0\} \times X \hookrightarrow I \times X$ and $g\colon X \simeq \{1\} \times X \hookrightarrow I \times X$.

□

Proposition 21. If $f\colon X \to Y$ is a homotopy equivalence, then $f_*\colon \pi_n(X, x) \xrightarrow{\simeq} \pi_n(Y, f(x))$ is an isomorphism.

Proof. (Idea) Since f is a homotopy equivalence, there is a map $g\colon Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. We apply the lemma to these homotopies. □

It is very hard to calculate homotopy groups, and one needs to use all kinds of tricks and sometimes even results from differential topology to even calculate them for relatively simple spaces, like spheres. In fact we don't know all the homotopy groups for *any* sphere S^n with $n > 1$. However, once there are a few homotopy groups that we know for 'standard' spaces, then we can use the following objects in order to generate relations between homotopy groups of different spaces, and thus calculate more of them.

like Sard's theorem, for example

Definition 35. A *fibre bundle* on a space X is a space P together with a map $\pi\colon P \to X$ such that for every $x \in X$ there is a nhd $U \ni x$ and an isomorphism $\pi^{-1}(U) \xrightarrow{\simeq} U \times F$ over U for some space F. In this setting we call X the *base space*, P the *total space* and F the *fibre*.

recall that this means that this diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

It is not obvious, but it follows that for X path connected, we do not need to assume that for every point the space F is some fixed space, as all the fibres $\pi^{-1}(x)$ are automatically (non-canonically) homeomorphic.

This is a big generalisation of the notion of covering space, in that the space F no longer needs to be discrete.

Example 55. Every covering space is a fibre bundle.

Example 56. Let $S^3 \subset \mathbb{C}^2$ be the unit sphere, consisting of pairs of points (z, w) such that $|z|^2 + |w|^2 = 1$. There is a projection map $S^3 \to \mathbb{CP}^1$ sending the point (z, w) to the point of the complex projective line $\mathbf{CP}^1$ with homogeneous coordinates $[z : w]$. By the defining equation, this is well-defined, since we don't have z and w simultaneously vanishing. The preimage of a point $[z : w]$ is homeomorphic to a copy of the unit complex numbers $U(1) \subset \mathbb{C}$. This is a famous fibre bundle, called the *Hopf bundle*. There are very few fibre bundles whose base space, total space and fibre are all spheres, so this is a somewhat atypical object, but very concrete and a good test case for trying out new ideas.

We need to relate homotopy groups of different spaces not just via the functoriality we already know about, but even homotopy groups in different dimensions.

Definition 36. A sequence

$$\cdots \to A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \to \cdots$$

of (abelian) groups and homomorphisms is called *exact at A_n* if $\ker(f_n) = \operatorname{im}(f_{n-1})$. It is called *exact* if it is exact at A_n for all n.

Example 57. A special case is where we have three successive nonzero terms:

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

Such an exact sequence is called a *short exact sequence*. It has the properties that: α is injective, β is surjective, and $\ker(\beta) = \operatorname{im}(\alpha)$.

An even more trivial-seeming example, that still turns up in practice

Example 58. A sequence $0 \to A \xrightarrow{\phi} B \to 0$ is exact if and only if ϕ is an isomorphism.

Remark. The definition of exact sequence still makes sense (just!) for pointed sets and functions: given $(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$ pointed functions, this is exact at (B, b) if $\operatorname{im}(f) = g^{-1}(c)$. The reason we care about this is that if (X, x) is a pointed space $[\mathrm{pt}, X] \simeq [S^0, (X, x)]_*$ is a pointed set.

Given a fibre bundle $\pi \colon P \to X$, we say it is pointed if we are given $x \in X$ and some $p \in F = \pi^{-1}(F)$.

Theorem 10. For a pointed fibre bundle $q\colon (P,p) \to (X,x)$, there is an exact sequence

$$\cdots \to \pi_n(F,p) \xrightarrow{i_*} \pi_n(P,p) \xrightarrow{q_*} \pi_n(X,x) \xrightarrow{\delta} \pi_{n-1}(F,p) \xrightarrow{i_*} \cdots \to \pi_1(F,p) \xrightarrow{i_*} \pi_1(P,p) \xrightarrow{q_*} \pi_1(X,x) \xrightarrow{\delta} [\mathrm{pt},F] \xrightarrow{i_*} [\mathrm{pt},P] \xrightarrow{q_*} [\mathrm{pt},X]$$

where $i\colon F \hookrightarrow P$ is the inclusion.

For the proof of this theorem, see Hatcher, Theorem 4.41. The exact sequence in the theorem is sometimes called the 'long exact sequence' to distinguish it from various short exact sequences that arise.

Remark. A few words are in order about the exactness at $\pi_1(X,x)$ and further terms that are only pointed sets. If P is path connected, then we have $\cdots \to \pi_1(P,p) \to \pi_1(X,x) \to [\mathrm{pt},F] \to *$. If $\pi_1(X,x) \to [\mathrm{pt},F]$ *were* a group homomorphism, this would be enough to tell us that $[\mathrm{pt},F] \simeq \pi_1(X,x)/q_*(\pi_1(P,p))$, but we aren't even guaranteed that $\pi_1(P,p)$ is normal in $\pi_1(X,x)$. So we should interpret exactness to be defined as $[\mathrm{pt},F] \simeq \pi_1(X,x)/q_*(\pi_1(P,p))$ *as pointed sets*. In general, if P is not path connected, we should take exactness here to mean that $\pi_1(X,x)/q_*(\pi_1(P,p)) \simeq i_*^{-1}(p) \subset [\mathrm{pt},F]$. Finally, even though $\delta\pi_1(X,x) \to [\mathrm{pt},F]$ is not a homomorphism, we have that $q_*(\pi_1(P,p))$ is the preimage of the basepoint under δ.

Example 59. For a pointed covering space $(Z,z) \to (X,x)$, the long exact sequence breaks up, because of the fibre being discrete. There are sections of the form

$$\cdots \to 0 = \pi_n(Z_x,z) \to \pi_n(Z,z) \to \pi_n(X,x) \to \pi_{n-1}(Z_x,z) = 0 \to \cdots \qquad n > 1$$

implying that $\pi_n(Z,z) \simeq \pi_n(X,x)$ for $n > 1$. The exact sequence ends on

$$\cdots \to 0 = \pi_1(Z_x,z) \to \pi_1(Z,z) \to \pi_1(X,x) \to [\mathrm{pt},Z_x] = Z_x \to [\mathrm{pt},Z] \to [\mathrm{pt},X]$$

So we see again that $\pi_1(Z,z) \to \pi_1(X,x)$ is injective, as we had before. Similarly, if $[\mathrm{pt},Z] = *$, that is, Z is path connected, then $\pi_1(X,x) \to Z_x$ is surjective.

This example should serve to highlight the fact that the long exact sequence of homotopy groups is a big generalisation of the relation between the fundamental group of X and those of its covering spaces. However, we can still get some new things using covering spaces here.

Example 60. If X is an slsc space with *contractible* universal covering space, then $\pi_n(X) = 0$ for $n > 1$. This is true for any torus $\mathbb{T}^n$, for instance, and the circle S^1 in particular.

Let's now apply the long exact sequence to our other example

Example 61. Consider the Hopf bundle $S^3 \to S^2$. We have that spheres are connected, and even $\pi_1(S^n) = 0$ for $n > 1$, as well . So we get

$$\cdots \to 0 = \pi_n(S^1) \to \pi_n(S^3) \to \pi_n(S^2) \to \pi_{n-1}(S^1) = 0 \to \cdots \qquad n > 2$$

and

$$\cdots \to 0 = \pi_2(S^1) \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) = \mathbb{Z} \to 0 \to \cdots$$

Thus we see that for $n > 2$, $\pi_n(S^3) \simeq \pi_n(S^2)$, and that there is a short exact sequence

$$0 \to \pi_2(S^3) \to \pi_2(S^2) \to \mathbb{Z} \to 0$$

This implies that $\pi_2(S^2)$ is an infinite abelian group, and $\pi_2(S^2)/\pi_2(S^3) \simeq \mathbb{Z}$.

Allowing ourselves some black box results, then it is true that $\pi_2(S^3) = 0$, and $\pi_3(S^3) \simeq \mathbb{Z}$. This then implies that $\pi_2(S^2) = \mathbb{Z}$, and $\pi_3(S^2) = \mathbb{Z}$.

Remark. In fact, $\pi_k(S^n) = 0$ for *all* $0 < k < n$. These is a standard result, but the proof is beyond the techniques of the course so far.

Here is a bonus example, presented with no construction.

Example 62. There is also a fibre bundle $S^7 \to S^4$ with fibre S^3. Applying the long exact sequence we get a section of it that looks like

$$\cdots \to 0 = \pi_4(S^7) \to \pi_4(S^4) \to \pi_3(S^3) \to \pi_3(S^7) = 0 \to \cdots$$

Since we know $\pi_3(S^3) = \mathbb{Z}$, then $\pi_4(S^4) = \mathbb{Z}$ also.

Remark. And in fact it's a classical result that $\pi_n(S^n) = \mathbb{Z}$ for *all* n. Again, the proof is beyond the scope of the course so far.

Even knowing the homotopy groups of sphere is hard, which is why Serre was awarded a Fields medal (in part) for developing tools that could calulate that, for example, all homotopy groups of spheres are finite except for $\pi_n(S^n) = \mathbb{Z}$, and $\pi_{4n-1}(S^{2n}) = \mathbb{Z} \oplus A$ where A is finite abelian. Odd and unexpected stuff happens, for instance (to pick an example at random) $\pi_{25}(S^6) = \mathbb{Z}/1056\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.

A large source of fibre bundles to which the long exact sequence can be applied arise from Lie groups. Lecture 19

Let G be a Lie group, and let H be a closed subgroup. Then the quotient map $G \to G/H$ for the multiplication action of H on G is a fibre bundle with fibres isomorphic to H.

For instance: the groups $GL(n), O(n), SO(n), U(n), SU(n)$ of $n \times n$ invertible, orthogonal, special orthogonal, unitary and special unitary matrices respectively, where 'special' means the determinant is 1. The smaller groups are closed subgroups of the bigger ones as the subgroups of block submatrices extended by the identity matrix

Example 63. There are fibre bundles on spheres arising in this way: namely $SO(n+1) \to SO(n+1)/SO(n) \simeq S^n$ with fibre $SO(n)$, and $SU(n+1) \to SU(n+1)/SU(n) \simeq S^{2n+1}$ with fibre $SU(n)$.

Example 64. The Hopf bundle turns out to be an example of this form, namely $SU(2) \to SU(2)/U(1) \simeq S^2$, where $U(1) \hookrightarrow SU(2)$ as the diagonal matrices with entries z and $\overline{z}$.

Trivial-seeming, but useful, is the result that $SU(2) = SU(2)/SU(1) \simeq S^3$, as $SU(1)$, the group of 1×1 unitary matrices with determinant 1, is trivial. Another very useful fact is that there is a surjective homomorphism $SU(2) \to SO(3)$ whose kernel is the centre $\{\pm I\} < SU(2)$. This means that $SO(3) = SU(2)/\{\pm I\}$, and hence there is a fibre bundle $SU(2) \to SO(3)$ (in fact a covering space).

this arises because there is an isomorphism *of Lie groups* of $SU(2)$ with the group of quaternions of unit length, and unit quaternions act by conjugation on the subspace of pure imaginary quaternions as rotations

Since $SU(2)$ is topologically S^3, it is simply-connected, and since $\{\pm I\}$ acts freely with quotient $SO(3)$, $\pi_1(SO(3), I) = \{\pm I\} \simeq \mathbb{Z}/2\mathbb{Z}$. Now let's consider what the long exact sequence of homotopy groups associated to the fibre bundle $SO(n+1) \to S^n$ tells us. We will only look at the tail end of it at present, assuming ≥ 3:

$$\cdots \to \pi_2(S^n, *) = 0 \to \pi_1(SO(n), I) \to \pi_1(SO(n+1), I) \to \pi_1(S^n, *) = 0 \to \cdots$$

where we have used $\pi_2(S^n) = 0$ as remarked before. Exactness tells us that $\pi_1(SO(n), I) \to \pi_1(SO(n+1), I)$ is an isomorphism for $n \geq 3$, and so by induction $\pi_1(SO(n), I) \simeq \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$. If we look at the case $n = 2$, we see an interesting phenomenon, namely that we have the exact sequence

$$\cdots \to \pi_2(SO(3), I) \to \pi_2(S^2, *) = \mathbb{Z} \to \pi_1(SO(2), I) \to \pi_1(SO(3), I) \to \pi_1(S^2, *) = 0 \to \cdots$$

Now $SO(2)$ is the group of 2×2 rotation matrices, which is homeomorphic to S^1, and we know well that $\pi_1(S^1, 1) = \mathbb{Z}$. Making the substitutions for the known groups we get the exact sequence

$$\cdots \to \pi_2(SO(3), I) \xrightarrow{\pi_*} \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z}/2\mathbb{Z} \to 0$$

Exactness tells us that i_* is surjective, so that its kernel is $2\mathbb{Z}$, and this is then the image of δ is also $2\mathbb{Z}$. Thus $\delta\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by 2! Moreover, this also tells us that the homomorphism $\pi_*\colon \pi_2(SO(3), I) \to \pi_2(S^2, *) = \mathbb{Z}$ is the zero map. In this way, we don't just get information about the homotopy groups, but about the induced maps between them, which is just as important. If we go one more step up the sequence, and consider

$$\cdots \to \pi_2(SO(2), I) \xrightarrow{i_*} \pi_2(SO(3), I) \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \to 0$$

then we know that $\pi_2(SO(2), I) \simeq \pi_2(S^1, *) = 0$, so we know that i_* must have trivial image. But this is the kernel of $\pi_* \colon \pi_2(SO(3), I) \to \pi_2(S^2, *)$, namely all of $\pi_2(SO(3), I)$, and so $\pi_2(SO(3)) = 0$. We could have gotten this result a different way, using $0 = \pi_2(S^3) \simeq \pi_2(SU(2)) \simeq \pi_2(SO(3))$, as $SU(2) \to SO(3)$ is a covering space, but here we didn't need to already know that $\pi_2(S^3) = 0$.

Remark. It is a hard fact that for *all* finite-dimensional Lie groups G, $\pi_2(G, e) = 0$.

Earlier in the course, we said that algebraic topology was about finding invariants of spaces up to continuous deformation, and that continuous deformation meant homotopy equivalence. However, that was not quite the whose story. If we take the collection of all homotopy groups to be our collection of invariants, then if given a pair of spaces X, Y and a map $f \colon X \to Y$, if f induces isomorphisms between all homotopy groups, then there is no way to tell X and Y apart. This leads to the following definition

Definition 37. A map $f \colon X \to Y$ of spaces is called a *weak homotopy equivalence* if $\Pi_1(X) \to \Pi_1(Y)$ is an equivalence of groupoids, and if for all $x \in X$ and $n > 1$, the induced map $f_* \colon \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism.

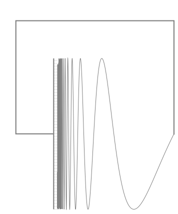
this is equivalent to asking that $[\text{pt}, X] \xrightarrow{\simeq} [\text{pt}, Y]$ and $\pi_1(X, x) \xrightarrow{\simeq} \pi_1(Y, f(x))$ for all $x \in X$

Example 65. All homotopy equivalences $f \colon X \to Y$ are weak homotopy equivalences.

But there are examples of weak homotopy equivalences that are not homotopy equivalences

Example 66. Consider the topologist's sine curve C. Equip the set $\{a, b\}$ with the discrete topology, and take a map $\{a, b\} \to C$ that picks out a basepoint in each path component. This is a weak homotopy equivalence as both spaces have trivial homotopy groups π_n for $n \geq 1$ and both have two path components, but there is no surjective map $C \to \{a, b\}$, as C is connected.

Example 67. The *Warsaw circle* W is constructed from the topologist's sine curve by adding an arc from the 'free endpoint' of the oscillating path component to the other path component on the y-axis. Then W is path connected, but still has all homotopy groups trivial. Thus the



more formally, it is the quotient space $C \sqcup [0, 1] \to W$ where we identify 0 and 1 with the appropriate points on C

map $W \to \mathrm{pt}$ is a weak homotopy equivalence, but there is no contraction of W as this would ultimately give rise to a path contained in the topologist's sine curve from one path component to the other.

Another class of examples illustrates why making a general assumption that our spaces are slpc is harmless from the point of view of homotopy theory.

Definition 38. For any space X, define the quotient topology on $[\mathrm{pt}, X]$ coming from the map $X \to [\mathrm{pt}, X]$. We can also consider the discrete space $\mathrm{disc}[\mathrm{pt}, X]$, which comes with a bijective map $\mathrm{disc}[\mathrm{pt}, X] \to [\mathrm{pt}, X]$. Define the space $\mathrm{slpc}(X)$ to be the pullback in the square

$$\begin{tikzcd}
\mathrm{disc}[\mathrm{pt}, X] \times_{[\mathrm{pt}, X]} X \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{disc}[\mathrm{pt}, X] \arrow[r] & [\mathrm{pt}, X]
\end{tikzcd}$$

One way to think of this is that we put a new topology on the underlying set of X so that it becomes the disjoint union of its path components.

Lemma 25. The assignment $X \mapsto \mathrm{slpc}(X)$ is the object component of a functor **Top** $\to$ **Top** landing inside the full subcategory of slpc spaces.

Proof. Exercise. □

By construction, there is a continuous, bijective map $\mathrm{slpc}(X) \to X$ that is the identity map on the underlying sets.

Lemma 26. The map $\mathrm{slpc}(X) \to X$ is a weak homotopy equivalence, and is not a homotopy equivalence if X is not already slpc.

Thus if we cannot tell apart spaces that are weakly homotopy equivalent, it is relatively harmless to work only with slpc spaces.

Complexes

Recall the Euler characteristic of a polyhedron P:

$$\chi(P) = \underbrace{\#(\text{vertices})}_{\text{0-dim}} - \underbrace{\#(\text{edges})}_{\text{1-dim}} + \underbrace{\#(\text{faces})}_{\text{2-dim}}$$

This definition doesn't require the polyhedron to be convex, simply-connected or even connected. It's not even restricted to surfaces, if we define

$$\chi(P) = \sum_{d=0}^{\dim P} (-1)^d \#(d\text{-dim faces})$$

However, χ is not functorial in any way and so we cannot relate in any obvious way the Euler characteristics of different polyhedra. Ideally, we would have a functor from which we can then reconstruct the Euler characteristic. The key idea is to replace the vertex, edge, etc count by the dimension of some vector space. The vertx, edge, etc counts can be reconstructed from the vector space, but now we could in principle have an infinite-dimensional vector space, which would arise in the case that we have an *infinite* polyhedron

for instance a triangulation of an infinite-genus surface

Definition 39. A *complex* $A_\bullet$ (of vector spaces, abelian groups, R-modules) is a sequence

$$\cdots \to A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \to \cdots$$

(of vector spaces, abelian groups, R-modules) such that $d_n \circ d_{n-1} = 0$ for all n

equivalently, $\mathrm{im}(d_{n-1}) \subseteq \ker(d_n)$

Example 68. Any exact sequence, of abelian groups say, gives a complex.

As for exact sequences, we can have a section of the complex that consists of nontrivial groups, vector spaces etc and the rest of the complex can be trivial. In this case we can restrict attention to the nontrivial section.

Example 69. The following is a complex:

$$0 \to \mathbb{Z} \xrightarrow{\times 4} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \to 0$$

Now we have an injective map on the left, and a surjective map on the right, but we only have that $4\mathbb{Z} < 2\mathbb{Z}$, not an equality of the kernel and the image.

Example 70. Let A and B be a pair of $n \times n$ real matrices such that BA is the zero matrix, considered as linear maps. Then

$$0 \to \mathbb{R}^n \xrightarrow{A} \mathbb{R}^n \xrightarrow{B} \mathbb{R}^n \to 0$$

is a complex. There is no way for this to be exact, because then B would have to be surjective, hence invertible, and A would have to be injective, hence invertible, but then we cannot have $BA = 0$.

Here is an example which should be familiar from multivariable calculus. For $U \subset \mathbb{R}^n$ an open subset, let $C^\infty(U)$ denote the vector space of smooth functions on U and $C^\infty(U, \mathbb{R}^3)$ be the vector space of vector fields on U.

Example 71. The various derivative operators ∇ (gradient), $\nabla \times -$ (curl) and $\nabla \cdot -$ (divergence) give acomplex denoted $\Omega^\bullet(U)$:

$$C^\infty(U) \xrightarrow{\nabla} C^\infty(U, \mathbb{R}^3) \xrightarrow{\nabla \times -} C^\infty(U, \mathbb{R}^3) \xrightarrow{\nabla \cdot -} C^\infty(U) \to 0$$

because the curl of a gradient is zero, and the divergence of a curl is zero.

Definition 40. A map of complexes $A_\bullet \to B_\bullet$ consists of a sequence of maps $f_n \colon A_n \to B_n$ such that all the squares

sometimes called a *chain map*

$$\begin{tikzcd} A_{n-1} \arrow[r, "d^A_{n-1}"] \arrow[d, "f_{n-1}"'] & A_n \arrow[d, "f_n"] \\ B_{n-1} \arrow[r, "d^B_{n-1}"'] & B_n \end{tikzcd}$$

The category of complexes of R-modules and chain maps is denoted $\mathbf{Cplx}_R$.

Example 72. Given the matrices A and B from the previous example, there is map of complexes

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & \mathbb{R}^n \arrow[r, "A"] \arrow[d] & \mathbb{R}^n \arrow[r, "B"] \arrow[d] & \mathbb{R}^n \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ 0 \arrow[r] & \ker(A) \arrow[r] & 0 \arrow[r] & \mathbb{R}^n / \operatorname{im}(A) \arrow[r] & \mathbb{R}^n \arrow[r] & 0 \end{tikzcd}$$

Example 73. There is a map of complexes $\Omega^\bullet(\mathbb{R}^3) \to \Omega^\bullet(U)$

$$\begin{tikzcd} C^\infty(\mathbb{R}^3) \arrow[r, "\nabla"] \arrow[d] & C^\infty(\mathbb{R}^3, \mathbb{R}^3) \arrow[r, "\nabla\times-"] \arrow[d] & C^\infty(\mathbb{R}^3, \mathbb{R}^3) \arrow[r, "\nabla\cdot-"] \arrow[d] & C^\infty(\mathbb{R}^3) \arrow[r] \arrow[d] & 0 \arrow[d] \\ C^\infty(U) \arrow[r, "\nabla"'] & C^\infty(U, \mathbb{R}^3) \arrow[r, "\nabla\times-"'] & C^\infty(U, \mathbb{R}^3) \arrow[r, "\nabla\cdot-"'] & C^\infty(U) \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps restrict functions and vector fields.

Now the complex $\Omega^\bullet(\mathbb{R}^3)$ is exact, because a vector field $\mathbf{v}$ on $\mathbb{R}^3$ satisfies $\nabla \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{v} = \nabla f$ for some function f, $\nabla \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \nabla \times \mathbf{w}$ for some vector field $\mathbf{w}$, and any function $\mathbb{R}^3 \to \mathbb{R}$ is the divergence of some vector field. Ultimately, this is because $\mathbb{R}^3$ is contractible. But if we take $U = \mathbb{R}^3 \setminus \{0\}$, then there are

by the Poincaré lemma, or more prosaically by standard multivariable calculus vector fields $\mathbf{v}$ on U with zero divergence that are not the curl of a vector field on U. This is because $\mathbb{R}^3 \setminus \{0\}$ is homotopy equivalent to S^2, and so the complex $\Omega^\bullet(\mathbb{R}^3 \setminus \{0\})$ can 'see' the nontrivial topological structure also captured by $\pi_2(S^2) = \mathbb{Z}$. This is measured by the fact that the kernel of $\nabla \cdot -$ and the image of $\nabla \times -$ do not coincide.

Similarly, one can take $U = \mathbb{R}^3 \setminus \ell$ where $\ell \subset \mathbb{R}^3$ is a 1-dimensional subspace. We have a homotopy equivalence between $\mathbb{R}^3 \setminus \ell$ and S^1. And this is detected by the fact there are vector fields in $\mathbb{R}^3 \setminus \ell$ whose curl vanishes but which are not the gradient of any function. Thus the kernel of $\nabla \times -$ and the image ∇ are different. Ultimately, it is not the specific complex that captures the information of interest, because $\Omega^\bullet(\mathbb{R}^n)$ is made up of enormous infinite-dimensional vector spaces, but $\mathbb{R}^n$ itself is homotopically uninteresting.

for instance $\ell =$ the z-axis

the keen-eyed will have noticed that the kernel of ∇ consists of the constant functions, so is isomorphic to $\mathbb{R}$. This one-dimensional vector space accounts for the fact $[\mathrm{pt}, \mathbb{R}^3] = *$

We will look at much smaller examples to properly warm up, to illustrate the type of algebra that will turn up.

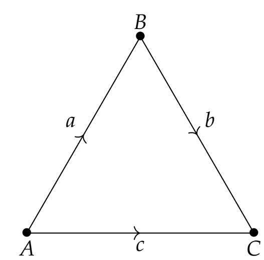
Definition 41. A *simple directed graph* consists of a set V of *vertices*, a set E of edges and two functions $d_0, d_1 \colon E \to V$ such that $(d_0, d_1)\colon E \to V \times V$ is injective, and $\mathrm{im}(d_0, d_1) \cap \Delta(V) = \varnothing$

$\Delta(V) = \{(v, v) \in V^2\}$ is the diagonal

Such a directed graph has no self-loops from a vertex to itself, and no more than one edge between any two vertices. The idea is that an edge e points from $d_1(e)$ to $d_0(e)$.

Example 74. Let $V = \{A, B, C\}$ and $E = \{a, b, c\}$, with

$$(d_1, d_0)\colon \begin{cases} a & \mapsto (A, B) \\ b & \mapsto (B, C) \\ c & \mapsto (A, C) \end{cases}$$



This graph is denoted $\partial\Delta[2]$, for reasons that will become clearer below.

To define a complex from a directed simple graph, for any given set S and ring R, let R^S denote the R-module of functions $f\colon S \to R$. The following definition is made using $R = \mathbb{Z}$, but it works for any ring R more generally.

if $R = \mathbb{Z}$, this is the product of $|S|$-many copies of $\mathbb{Z}$, and if R is a field, it's the vector space of functions on S

Definition 42. Let $d_0, d_1 \colon E \rightrightarrows V$ be a simple directed graph. Define the complex

$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z}^V & \xrightarrow{\delta} & \mathbb{Z}^E & \to & 0 \\ & & f & \mapsto & f \circ d_0 - f \circ d_1 & & \end{array}$$

Here are a bunch of concrete examples

Lecture 20

Example 75. Consider the trivial graph with one vertex and no edges. The complex is $0 \to \mathbb{Z} \xrightarrow{\delta} 0 \to 0$, and clearly $\ker(\delta) = \mathbb{Z}$ and $\operatorname{coker}(\delta) = 0$.

Example 76. The complex that arises from $\partial\Delta[2]$ in Example 74 is $0 \to \mathbb{Z}^3 \xrightarrow{\delta} \mathbb{Z}^3 \to 0$. We can identify a generating set of $\mathbb{Z}^3 = \mathbb{Z}^V$, namely $\underline{X}\colon \{A, B, C\} \to \mathbb{Z}$ for $X \in \{A, B, C\}$ with

$$\underline{A}(v) = \begin{cases} 1 & v = A \\ 0 & \text{else} \end{cases}$$

and similarly for $\underline{B}$ and $\underline{C}$. We can calculate

$$\begin{aligned} \delta(\underline{A})(e) &= \underline{A}(d_0(e)) - \underline{A}(d_1(e)) \\ &= \begin{cases} -1 & e = a \\ 0 & e = b \\ -1 & e = c \end{cases} \end{aligned}$$

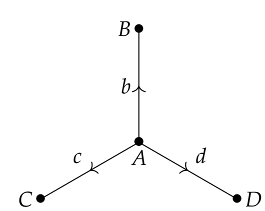
We can see this from the graph itself, in that the vertex A is the source of the edges a and c, and isn't incident with the edge b. To contrast, $\delta(\underline{B})(a) = 1$, as B is the target of the edge a. We can then write down the matrix D representing δ, namely

$$D = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

We can calculate that $\ker(D)$ is torsion-free and generated by $\underline{A} + \underline{B} - \underline{C}$, hence is isomorphic to $\mathbb{Z}$. Similarly, the image of D is generated by $\underline{A} - \underline{B}$ and $\underline{B} - \underline{C}$, and, incidentally, $\mathbb{Z}^E$ is generated by these two functions together with $\underline{B}$, so that the $\operatorname{coker}(D) \simeq \mathbb{Z}$. As a final remark, note that the Euler characteristic $\chi = 0$.

Remark. For any *finite* graph (and later, more general finite combinatorial objects), we can take the vertices and the edges to represent generating sets for $\mathbb{Z}^V$ and $\mathbb{Z}^E$. But for infinite graphs, this is not the case, and we'd have to be more careful.

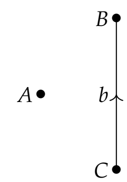
Example 77. Consider the graph with four vertices $\{A, B, C, D\}$ and three edges $\{b, c, d\}$ as at right. The complex that arises is $0\mathbb{Z}^4 \xrightarrow{\delta} \mathbb{Z}^3 \to 0$, where δ is represented by



$$D = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

with respect to the generating set as in Remark . The image is all of $\mathbb{Z}^3$, hence the cokernel is trivial, and the kernel is $\mathbb{Z}$. For this example we have $\chi = 1$.

We don't have to use connected graphs!



Example 78. Consider the graph with vertex set $\{A, B, C\}$, edge set $\{b\}$ such that $d_0(b) = B$, $d_1(b) = C$. The complex is $0 \to \mathbb{Z}^3 \xrightarrow{\delta} \mathbb{Z} \to 0$, with δ onto, hence $\operatorname{coker}(\delta) = 0$. The kernel of δ is $\mathbb{Z}^2$, and the Euler characteristic is 2.

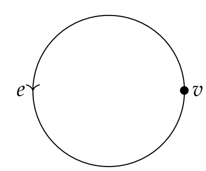
Exercise 15. 1. Given any two simple directed graphs G_1, G_2, we can define the disjoint union $G_1 \sqcup G_2$ by taking the disjoint union of their edges and vertices and taking the induced functions d_0, d_1. Show $\ker \delta_{G_1 \sqcup G_2} \simeq \ker \delta_{G_1} \oplus \ker \delta_{G_2}$.

2. For any directed graph G with underlying shape a polyhedron, show that the kernel and cokernel of the associated map δ_G both isomorphic to $\mathbb{Z}$.

The first of these two exercises prove that $\dim \ker \delta$ counts the number of connected components, and the second strongly suggest that $\dim \operatorname{coker} \delta$ counts the number of loops.

Exercise 16. Calculate the kernel and cokernel of a connected simple directed graph with two cycles.

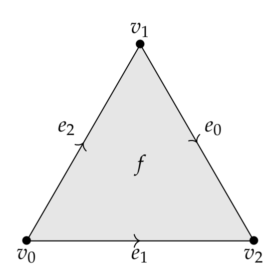
Remark. It is entirely possible to work not just with simple directed graphs, but general directed graphs, since the definition of the complex associated to a graph as above does not use the injectivity of (d_0, d_1) or disjointness from the diagonal. Thus we can consider a directed graph with one vertex and one edge—a combinatorial model of the circle—and get the complex $0 \to \mathbb{Z} \xrightarrow{\delta=0} \mathbb{Z} \to 0$, where now $\ker \delta = \mathbb{Z}$ and $\operatorname{coker} \delta = \mathbb{Z}$, as in Example 77.



However, as much fun as this is, we really need to think about more than just 1-dimensional objects. This leads to the question of what the two-dimensional version of a directed graph is. One option is the following: take sets of vertices, edges and triangular faces, and specify how they fit together, by means of functions analogous to d_0 and d_1 from before.

I would call this (2-skeletal) semisimplicial set, but Mike Hopkins called such a thing a *combinatorial Δ-complex*

Example 79. Consider just a single, filled triangle. Let the vertices be called v_0, v_1 and v_2. There are edges e_0, e_1 and e_2, and one face, f.



Note that in this triangle, $d_0(e_2) = d_1(e_0)$ and so on, where e_i is the edge opposite the vertex v_i. The combinatorics of how the edges and vertices fit together are captured in the following definition.

Definition 43. A *combinatorial surface* $X_\bullet$ consists of sets of vertices, edges and faces, denoted X_0, X_1 and X_2 respectively together with functions $d_i^n \colon X_n \to X_{n-1}$ for all $0 \le i \le n$, $0 < n \le 2$ such that

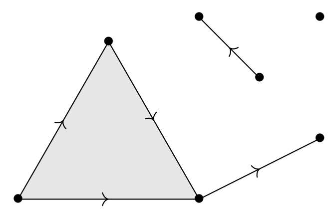
$$\begin{aligned} d_0^1 \circ d_2^2 &= d_1^1 \circ d_0^2, \\ d_0^1 \circ d_1^2 &= d_0^1 \circ d_0^2, \\ d_1^1 \circ d_2^2 &= d_1^1 \circ d_1^2. \end{aligned} \tag{4}$$

so that X_i is the set of i-dimensional 'faces'; the functions d_i^n are called *face maps*

the easiest way to remember these identities is to draw the triangle in Example 79

When the context is clear, we can usually the superscripts, as the dimension can be inferred from the data. The *1-skeleton* of $X_\bullet$ is the underlying directed graph gotten by forgetting X_2.

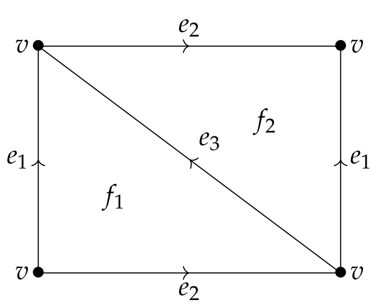
Now note that 'surface' here is really a stand-in for 'at most 2-dimensional'. There is nothing in the definition that requires that $X_2 \neq \emptyset$. And, moreover, it is possible to have 0- or 1-dimensional 'components' in a surface, much as a graph is generically 1-dimensional, but can have isolated vertices, or even consist purely of vertices and no edges. In this way, a combinatorial surface could be so degenerate it has no triangles and no edges, but if there is at least one triangle, then there must be at least one edge (and at least one vertex).



Example 80. The triangle from Example 79 is denoted $\Delta[2]$, and has $\Delta[2]_0 = \{v_0, v_1, v_2\}$, $\Delta[2]_1 = \{e_0, e_1, e_2\}$ and $\Delta[2]_2 = \{f\}$, $d_i^2(f) = e_i$ and $d_0^1, d_1^1 \colon \Delta[2]_1 \to \Delta[2]_0$ as in Example 74, up to relabelling. The 1-skeleton of $\Delta[2]$ is the directed graph $\partial\Delta[2]$.

As noted above, we do not need to restrict to the case where the 1-skeleton is a simple directed graph, and so it can be any directed graph.

Example 81. We can provide a combinatorial model of a torus, $T_\bullet$, by taking $T_0 = \{v\}$, $T_1 = \{e_1, e_2, e_3\}$ and $T_2 = \{f_1, f_2\}$ fitting together as



Thus, $d_0^2(f_1) = e_3$, $d_1^2(f_1) = e_1$ and $d_2^2(f_1) = e_2$; $d_0^2(f_2) = e_2$, $d_1^2(f_2) = e_1$, $d_2^2(f_2) = e_3$, and $d_0^1(e_i) = v = d_1^1(e_i)$ for $e = 1, 2, 3$.

Given a combinatorial surface $X_\bullet$, we can define a sequence $C^\bullet(X_\bullet)$ of abelian groups in a similar way as for a directed graph:

here taking the ring $R = \mathbb{Z}$ for simplicty, the general definition works the same

$$0 \to \mathbb{Z}^{X_0} \xrightarrow{\delta_0} \mathbb{Z}^{X_1} \xrightarrow{\delta_1} \mathbb{Z}^{X_2} \to 0$$

where δ_0 is defined the same way as for a directed graph: $\delta_0(g) = gd_0^1 - gd_1^1 \colon X_1 \to \mathbb{Z}$ for $g \in \mathbb{Z}^{X_0}$ and $\delta_1(g') = g'd_0^2 - g'd_1^2 + g'd_2^2 = \sum_{i=0}^2 g'd_i \colon X_2 \to \mathbb{Z}$ for $g' \in \mathbb{Z}^{X_1}$. We will see below that this is indeed a complex.

Remark. If the combinatorial surface has X_i finite for $i = 0, 1, 2$, then we can take as basis for $\mathbb{Z}^{X_i}$ the set X_i, where we identify an element $x \in X_i$ with the function $X_i \to \mathbb{Z}$ that is equal to 1 when evaluated on x, and otherwise is 0. It was remarked in class that some sources consider functions that are only nonzero on finitely many elements of X_n, but this is not what we are doing here, and even the maps δ_i cease to become well-defined, as infinitely many edges might share a vertex, for example. For infinite complexes we need to rely on more abstract means to describe the maps δ_i, if we want them explicity in terms of a basis.

Example 82. Given the combinatorial surface $T_\bullet$ from Example 81, the resulting sequence is

$$0 \to \mathbb{Z} \xrightarrow{\delta_0} \mathbb{Z}^3 \xrightarrow{\delta_1} \mathbb{Z}^2 \to 0$$

where $\delta_0(g)(e) = g(d_0(e)) - g(d_1^2(e)) = g(v) - g(v) = 0$ for any edge $e \in X_1$, and so is the zero map. If we take the basis as in the previous Remark, the map δ_1 is represented by the matrix

$$\begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

and the composite $\delta_1\delta_0$ is clearly the zero map, so this is a complex.

Lemma 27. For any combinatorial surface $X_\bullet$, the sequence $C^\bullet(X_\bullet)$ is a complex.

Proof. We need to prove that for any $g \colon X_0 \to \mathbb{Z}$, and any $x \in X_2$, we have $\delta_1(\delta_0(g))(x) = 0$.

$$\begin{aligned} \delta_1(\delta_0(g))(x) &= \delta_1(gd_0^2 - gd_1^2)(x) \\ &= \delta_1(gd_0^2)(x) - \delta_1(gd_1^2)(x) \\ &= gd_0^1d_0^2(x) - gd_0^1d_1^2(x) + gd_0^1d_2^2(x) \\ &- (gd_1^1d_2^2(x) - gd_1^1d_1^2(x) + gd_1^1d_2^2(x)) \\ &= 0 \end{aligned}$$

where in the last step we use the equations (4). $\square$

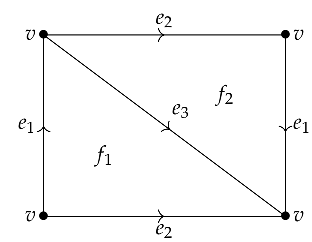
The combinatorial torus above has Euler characteristic $\chi = 1 - 3 + 2 = 0$, but if we look at the failure of it to be exact, we get $\ker \delta_0 \simeq \mathbb{Z}$, $\operatorname{coker} \delta_1 \simeq \mathbb{Z}$, and $\ker \delta_1 / \operatorname{im} \delta_0 \simeq \mathbb{Z}^2$.

Remark. We could have taken an arbitrary (commutative, unital) ring R instead of $\mathbb{Z}$ in the above example, and the resulting $\ker \delta_0$ etc would be R-modules. For $R = \mathbb{Z}$ these are $\mathbb{Z}$-modules, hence abelian groups, but for example taking $R = \mathbb{Z}/2$ we get abelian groups, but they have more structure as $\mathbb{Z}/2$-modules.

Exercise 17. Calculate the complex of abelian groups arising from a tetrahedron considered as a combinatorial surface, and the groups $\ker \delta_0$, $\ker \delta_1 / \ker \delta_0$.

Hint: label the vertices $0, 1, 2, 3$, order the edges from from lower to higher labels, and then define the maps d_0, d_1, d_2 for faces using $\Delta[2]$ as a model

Exercise 18. Consider a combinatorial model of the Klein bottle $K_\bullet$, with two triangles, three edges and one vertex, as in the sketch at right, where $d_0(f_1) = e_3$, $d_1(f_1) = e_2$ and $d_2(f_1) = e_1$, and $d_0(f_2) = e_1$, $d_1(f_2) = e_2$ and $d_2(f_2) = e_2$. And, as for the combinatorial torus, $d_0(e_i) = x = d_1(e_i)$ for $i = 1, 2, 3$. (The triangles are considered to be filled, despite the lack of shading.)



Definition 44. Given a complex $A_\bullet$ of R-modules, and an integer n, the n^{th} cohomology $H^n(A_\bullet)$ is the R-module

$$\frac{\ker A_n \xrightarrow{d_n} A_{n+1}}{\operatorname{im} A_{n-1} \xrightarrow{d_{n-1}} A_n}$$

Lemma 28. For all integers n, $H^n \colon \mathbf{Cplx}_R \to R\mathbf{Mod}$ is a functor.

Proof. Exercise. □

Here is the big idea: spaces and their homotopy groups are hard, so to study one, turn it into a complex, then look at the cohomology groups instead. And, we want this to be a functor. We have seen so far that

$$\pi_i(S^2, *) = \begin{cases} * & i = 0 \text{ connected} \\ 0 & i = 1 \text{ by Seifert–van Kampen} \\ \mathbb{Z} & i = 2 \text{ without proof!} \\ \mathbb{Z} & i = 3 \text{ also without proof!} \\ \vdots & \end{cases}$$

However, we can calculate the cohomology groups of the combinatorial surface $T_\bullet$ using just linear algebra, and these seem to capture at least some of this information, with much less work. There are some caveats, in that we haven't actually constructed a functor assigning to a combinatorial surface its cohomology groups, and, worse, there's no guarantee that a combinatorial tetrahedron (as opposed to an actual space!) really captures the topology of S^2. But it should be suggestive as to a different approach.

Remark. This lecture started with a long recap of the previous lecture, and I have gone back and incorporated some of this material just above, in the section labelled lecture 20.

Lecture 21

Note that given a combinatorial surface $X_\bullet$ and a ring R we have cohomology groups

$$H^0(X_\bullet, R) := \ker \delta_0$$
$$H^1(X_\bullet, R) := \frac{\ker \delta_1}{\operatorname{im} \delta_0}$$
$$H^2(X_\bullet, R) := \frac{R^{X_2}}{\operatorname{im} \delta_1} = \operatorname{coker} \delta_1$$

arising from the complex $0 \to R^{X_0} \to R^{X_1} \to R^{X_2} \to 0$. We technically have $H^n(X_\bullet, R)$ for all integers n, but these are all the zero module. If $X_2 = \emptyset$, then $R^{X_2} = R^\emptyset = \{0\}$, and so $H^2(X_\bullet, R) = \{0\}$.

We calculated the cohomology with $R = \mathbb{Z}$ for the combinatorial torus $T_\bullet$ above to be

$$H^n(T_\bullet, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^2 & n = 1 \\ \mathbb{Z} & n = 2 \end{cases}$$

But, the combinatorial surface $T_\bullet$ is very definitely not the topological space $S^1 \times S^1$! So we need a way to relate actual topological spaces to this combinatorial data. If we go back to the directed graph $\partial\Delta[2]$, it looks like it should be a circle (topologically, at least). We could make an actual circle by taking the disjoint union of three intervals $[0, 1]$ and forming a quotient space so the endpoints are appropriately identified. This idea is called geometric realisation. The following does not work, but gives an idea of how the real version might go.

Construction. (Attempt 1) Given a directed graph $X_1 \rightrightarrows X_0$ We could try to take the quotient of $\coprod_{X_1} I$, where we identify the endpoints of the different copies of I according to the incidence of edges in the graph. But, this information is recorded in which edges map to the same vertex under the maps d_0, d_1, so we should involve these functions somehow. A bigger problem is what to do with isolated vertices! They certainly are not given by gluing intervals in any sense. So we need to use X_0 as well.

Construction 2. (Attempt 2) We could instead take the discrete space on the set of vertices of a directed graph, and then attach the intervals to them. In this sense, any isolated vertices turn up in the construction, and any edge only needs to know what vertices it is incident with, which is indeed the case by the definition of directed graph. To get this working we need some topological ingredients. Consider the functions $\partial_0, \partial_1 \colon \mathrm{pt} \to I$ with $\partial_0(\mathrm{pt}) = 1$ and $\partial_1(\mathrm{pt}) = 0$. Then the geometric realisation of a directed graph $X_1 \rightrightarrows X_0$ should be $(\coprod_{X_0} \mathrm{pt} \sqcup \coprod_{X_1} I)/\sim$ where the equivalence relation identifies $(e, \partial_i(v)) \sim (d_i(e), v)$. That is, the i-endpoint of the interval indexed by the element $e \in X_1$ should be identified with the point indexed by the vertex v.

these are in one sense 'dual' to the combinatorial endpoint functions d_0, d_1

This rough construction will soon be superceded by a more formal and systematic definition, but it should capture the idea.

Example 83. Consider the directed graph given by three vertices $\{v_1, v_2, v_3\}$ and two edges $\{e_1, e_2\}$ with $d_1(e_1) = v_1$, $d_0(e_1) = v_2 = d_1(e_2)$ and $d_0(e_2) = v_3$. Its geometric realisation is homeomorphic to $(\{v_1, v_2, v_3\} \sqcup [0,1] \sqcup [2,3])/\sim$ where $0 \sim v_1$, $1 \sim v_2 \sim 2$ and $3 \sim v_3$. Thus it is homeomorphic to $([0,1] \sqcup [2,3])/(1 \sim 2) \simeq [0,2] \simeq [0,1]$.

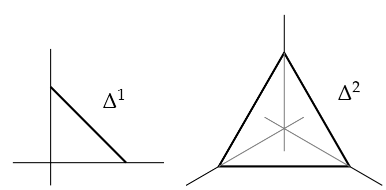
Now, how do so something similar for combinatorial surfaces? Now we must make some definitions that will generalise more easily down the track, that are less ad hoc.

Definition 45. The *standard n-simplex* is the subspace of $\mathbb{R}^{n+1}$ given by

$$\Delta^n := \{(v_0, v_1, \ldots, v_n) \in \mathbb{R}^{n+1} \mid \forall\, 0 \le i \le n, v_i \ge 0 \text{ and } v_0 + \cdots + v_n = 1\}$$

There are inclusion maps between the standard n-simplices, namely $\partial_i \colon \Delta^n \to \Delta^{n+1}$ defined to be $\partial_i(v_0, \ldots, v_n) = (v_0, \ldots, v_{i-1}, 0, v_i, \ldots, v_n)$, where $0 \le i \le n+1$.

Example 84. So the standard 0-simplex is a single point, namely $v_0 = 1 \in \mathbb{R}$, the standard 1-simplex is the interval $\Delta^1 = \{(v_0, v_1 \in \mathbb{R}^2 \mid v_0, v_1 \ge 0,\ v_0 + v_1 = 1\}$, and the standard 2-simplex is the portion of the hyperplane $v_0 + v_1 + v_2 = 1$ with the positive octant in $\mathbb{R}^3$.



Example 85. We have the two maps $\partial_0, \partial_1 \colon \Delta^0 \to \Delta^1$ from before, $\partial_0(v_0) = (0, v_0) = (0,1)$ and $\partial_1(v_0) = (v_0, 0) = (1,0)$. And now, imporantly, maps $\partial_i \colon \Delta^1 \to \Delta^2$, $i = 0,1,2$ with

$$\partial_i(v_0, v_1) = \begin{cases} (0, v_0, v_1) & i = 0 \\ (v_0, 0, v_1) & i = 1 \\ (v_0, v_1, 0) & i = 2 \end{cases}$$

Definition 46. The *geometric realisation* of a combinatorial surface $X_\bullet$ is the quotient space

$$|X_\bullet| := \left(\coprod_{n=0}^{2} \mathrm{disc}(X_n) \times \Delta^n \right) \Big/ \sim$$

where the equivalence relation is generated by $(d_i(x), \mathbf{v}) \sim (x, \partial_i(\mathbf{v}))$.

For instance, the geometric realisation of $\Delta[2]$ is

$$\Big((\Delta^0 \sqcup \Delta^0 \sqcup \Delta^0) \sqcup (\Delta^1 \sqcup \Delta^1 \sqcup \Delta^1) \sqcup \Delta^2 \Big) / \sim \;\simeq\; (\partial\Delta^2 \sqcup \Delta^2)/\sim \;\simeq\; \Delta^2$$

And, more degenerately, $|\Delta[1]| = \Delta^1$ and $|\Delta[0]| = \Delta^0$

Let us, for the sake of the following definition, be generous with the definition of 'surface'; it should include at least all topological manifolds of dimension 2 or lower, and even of mixed dimension (eg the disjoint union of a circle and a torus). The important thing is that a surface here is a topological space, not a combinatorial object.

Definition 47. A *triangulation* of a surface Σ is a combinatorial surface $X_\bullet$ equipped with a homeomorphism $\Sigma \simeq |X_\bullet|$.

we will usually leave the homeomorphism implicit in what follows

Example 86. 1. Δ^2 is triangulated by $\Delta[2]$.

2. More generally, $|X_\bullet|$ is triangulated by $X_\bullet$.

3. S^2 is triangulated by $\partial\Delta[3]$.

4. S^1 is triangulated by $\partial\Delta[2]$, but also by any directed graph in the shape of a polygon, or even the directed graph with one vertex and one edge.

5. $S^1 \times S^1$ is triangulated by the combinatorial torus $T_\bullet$.

Ultimately, of course, we want some kind of functorial behaviour, so we need maps between combinatorial surfaces. The idea is that vertices get mapped to vertices, edges to edges, and triangles to triangles, in a compatible way.

Lecture 22

Definition 48. Given combinatorial surfaces $X_\bullet$ and $Y_\bullet$, a map $f \colon X_\bullet \to Y_\bullet$ is a triple of functions $f_n \colon X_n \to Y_n$, $n = 0, 1, 2$ such that $d_i f_n = f_{n-1} d_i$ for $0 < n \leq 2$, $0 \leq i \leq n$.

Such maps are very rigid, in the sense that there are 'obvious' functions between the intended geometric objects that don't come from a map between given combinatorial surfaces. This definition also includes map between directed graphs, if we take the set of triangles to be empty, and maps from a directed graph to a non-degenerate combinatorial surface.

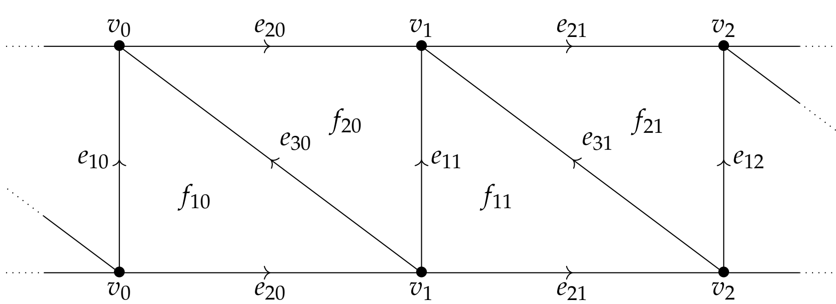
Example 87. Given a combinatorial surface $X_\bullet$, and its 1-skeleton $\mathrm{sk}_1 X_\bullet$, there is a morphism $\mathrm{sk}_1 X_\bullet \to X_\bullet$ that is the identity on X_0 and X_1, and the only possible function $(\mathrm{sk}_1 X_\bullet)_2 = \emptyset \to X_2$.

Example 88. We can include a single triangle into a combinatorial surface $X_\bullet$, via $\Delta[2] \to X_\bullet$. For instance, $\Delta[2] \to \partial\Delta[3]$.

Example 89. If $L_\bullet$ is the directed graph with one vertex and one edge, and $P_\bullet$ is any polygonal directed graph (for instance $\partial\Delta[2]$), then there is a map $P_\bullet \to L_\bullet$, sending all vertices to the single vertex of $L_\bullet$, and all edges to the singe edge. We can even triangulate $\mathbb{R}$ by taking an infinite directed graph $R_\bullet$ with vertices indexed by $\mathbb{Z}$ and an edge from k to $k+1$, and then define a map $R_\bullet \to L_\bullet$ in a similar way.

a polygonal directed graph is a finite directed graph with the same number of vertices and edges, the vertices are cyclicly ordered, and there is an edge between adjacent vertices, in either direction

Example 90. For a similar but nondegenerate example, we can define an infinite combinatorial surface that models an infinite cylinder, with set of faces $\{f_{1i}, f_{2i} \mid i \in \mathbb{Z}\}$, set of edges $\{e_{1i}, e_{2i}, e_{3i} \mid i \in \mathbb{Z}\}$ and set of vertices $\{v_i \mid i \in \mathbb{Z}\}$ as in the following picture,



mapping to the combinatorial torus $T_\bullet$ via $v_i \mapsto v$, $e_{ai} \mapsto e_a$, $a = 1,2,3$ and $f_{bi} \mapsto f_b$, $b = 1,2$ (in fact the face maps d_i for the infinite combinatorial cylinder can be reconstructed from the defintion of this map).

Lemma 29. A map $f\colon X_\bullet \to Y_\bullet$ between combinatorial surfaces gives rise to a continuous map $|f|\colon |X_\bullet| \to |Y_\bullet|$ between their geometric realisations, and this construction is functorial.

Proof. First, the map f gives rise to a continuous map

$$\widetilde{|f|} := \sqcup_n f_n \times \mathrm{id}_{\Delta^n} \colon \bigsqcup_{n=0}^{2} \mathrm{disc}(X_n) \times \Delta^n \to \bigsqcup_{n=0}^{2} \mathrm{disc}(Y_n) \times \Delta^n.$$

We can check that this respects the relation that defines the quotients $|X_\bullet|$ and $|Y_\bullet|$: take $(x, \partial_i(\mathbf{v})) \in \mathrm{disc}(X_n) \times \Delta^n$, so that $(x, \partial_i(\mathbf{v})) \sim$

$(d_i(x), \mathbf{v})$, and then

$$\begin{aligned} \widetilde{|f|}(x, \partial_i(\mathbf{v})) &= (f_n(x), \partial_i(\mathbf{v})) \\ &\sim (d_i(f_n(x)), \mathbf{v}) \\ &= (f_{n-1}(d_i(x)), \mathbf{v}) \\ &= \widetilde{|f|}(d_i(x), \mathbf{v}) \end{aligned}$$

Hence there is a unique map $|f|\colon |X_\bullet| \to |Y_\bullet|$ making the following diagram commute:

$$\begin{tikzcd} \coprod_{n=0}^{2} \operatorname{disc}(X_n) \times \Delta^n \arrow[r, "\widetilde{|f|}"] \arrow[d] & \coprod_{n=0}^{2} \operatorname{disc}(Y_n) \times \Delta^n \arrow[d] \\ |X_\bullet| \arrow[r, "|f|"'] & |Y_\bullet| \end{tikzcd}$$

The uniqueness of this map means that $|g \circ f| = |g| \circ |f|$, for any composable pair of maps f, g of combinatorial surfaces. $\square$

For example, the infinite combinatorial cylinder in Example 90 has as geometric realisation the cylinder $\mathbb{R} \times S^1$, and the map in that example gives rise to the map $\exp \times \mathrm{id}\colon \mathbb{R} \times S^1 \to S^1 \times S^1$.

Given a topological surface Σ that admits a triangulation $\Sigma \simeq |X_\bullet|$, we could in principle define its cohomology modules by $H^n(\Sigma, R) := H^n(X_\bullet, R)$—but this is a terrible definition. It is only functorial in an extremely limited way, because there's nothing that guarantees that different choices of triangulation give rise to the same cohomology modules, which means this is really a definition for *triangulated* surfaces, those equipped with a triangulation; also, only those continuous functions that arise from the geometric realisation of a map of combinatorial surfaces give rise to a linear map of cohomology modules.

the functoriality of cohomology modules with respect to maps of combinatorial surfaces will be subsumed by a definition to be given below

Moreover, what about other spaces? We clearly don't want to restrict ourselves to surfaces when studying topology! Here is a general combinatorial definition with which we can examine the notion of cohomology and calculate interesting examples.

Definition 49. A Δ*-set* is a sequence of sets X_n, $n = 0, 1, 2, \ldots$ of *n-simplices* together with *face maps* $d_i^n\colon X_n \to X_{n-1}$ for $n > 0$ and $0 \leq i \leq n$, such that

$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n \qquad \text{for } 0 \leq i < j \leq n$$

(Eventually we will drop the superscripts, as it becomes clear from context what the superscripts should be)

Example 91. The combinatorial n-simplex $\Delta[n]$ has

$$\Delta[n]_0 = \{0, 1, 2, \ldots, n\} =: \mathbf{n+1},$$

$$\Delta[n]_1 = \text{set of 2-element subsets of } \mathbf{n+1} =: \binom{\mathbf{n+1}}{2},$$

$$\vdots$$

$$\Delta[n]_k = \text{set of } (k+1)\text{-element subsets of } \mathbf{n+1} =: \binom{\mathbf{n+1}}{k+1}, \qquad k < n$$

$$\vdots$$

$$\Delta[n]_n = \{\mathbf{n+1}\} = \{\text{top face}\},$$

$$\Delta[n]_k = \emptyset, \qquad k > n.$$

Each of the subsets in these sets is ordered, and the function $d_i^k \colon \binom{\mathbf{n+1}}{k+1} \to \binom{\mathbf{n+1}}{k}$ discards the i^{th} element from each subset, where the indexing starts from 0.

Example 92. The boundary $\partial\Delta[n]$ is defined so that $\partial\Delta[n]_k = \Delta[n]_k$ for $k < n$ empty otherwise. So for instance, the combinatorial surface $\partial\Delta[3]$ has vertices, edges and triangles (that is: 0-, 1-, and 2-simplices) but no 3-dimensional simplex filling it.

More generally, given any Δ-set $X_\bullet$, we can truncate it to its *k-skeleton* $\operatorname{sk}_k X_\bullet$ which has

$$\operatorname{sk}_m X_k = \begin{cases} X_k & k \leq m \\ \emptyset & k > m \end{cases}$$

If a Δ-set $X_\bullet$ has $x \in X_n$ for some n, then $X_m \neq \emptyset$ for all $0 \leq m < n$. We call a Δ-set n-dimensional if n is the largest integer such that it has an n-simplex, and if no such integer exists, we call it infinite-dimensional. If $X_\bullet$ is n-dimensional and $0 \leq m < n$ (or $X_\bullet$ is infinite-dimensional), then $\operatorname{sk}_m X_\bullet$ is m-dimensional. Hence $\Delta[n]$ is n-dimensional and $\partial\Delta[n] = \operatorname{sk}_{n-1} \Delta[n]$ is $n-1$-dimensional. A combinatorial surface, as defined above, has dimension ≤ 2.

The defintitions of geometric realisation, maps and triangulations generalise from the 2-dimensional case to general Δ-sets.

Definition 50. The geometric realisation of a Δ-set $X_\bullet$ is the quotient space

$$|X_\bullet| := \left(\coprod_{n=0}^{\infty} \operatorname{disc}(X_n) \times \Delta^n \right) \Big/ \sim$$

by the equivalence relation generated by $(d_i(x), \mathbf{v}) \sim (x, \partial_i(\mathbf{v}))$.

Definition 51. A map of Δ-sets $f\colon X_\bullet \to Y_\bullet$ is a sequence of functions $f_n\colon X_n \to Y_n$, $n = 0, 1, 2, \ldots$ such that $d_i^n \circ f_n = f_{n-1} \circ d_i^n$ for $0 < n$ and $0 \leq i \leq n$. We thus get a category $\Delta\mathbf{Set}$.

Example 93. There is always an inclusion map $\mathrm{sk}_m X_\bullet \to X_\bullet$, and even $\mathrm{sk}_m X_\bullet \to \mathrm{sk}_l X_\bullet$ for all $0 \leq m \leq l$. Moreover, these maps are natural in the sense that given $f\colon X_\bullet \to Y_\bullet$, there is a commutative square

$$\begin{tikzcd} \mathrm{sk}_m X_\bullet \arrow[r, "\mathrm{sk}_m f"] \arrow[d] & \mathrm{sk}_m Y_\bullet \arrow[d] \\ X_\bullet \arrow[r, "f"'] & Y_\bullet \end{tikzcd}$$

and this triangle commutes:

$$\begin{tikzcd} \mathrm{sk}_m X_\bullet \arrow[r] \arrow[dr] & \mathrm{sk}_l X_\bullet \arrow[d] \\ & X_\bullet \end{tikzcd}$$

Example 94. Given any n-simplex $x \in X_n$ in a Δ-set $X_\bullet$, there is a map $\ulcorner x \urcorner\colon \Delta[n] \to X_\bullet$ taking the unique top face of $\Delta[n]$ to x.

More generally, given subsets $Y_n \subseteq X_n$ for all $n = 0, 1, 2, \ldots$ such that the face maps of $X_\bullet$ restrict to functions $d_i^n\colon Y_n \to Y_{n-1}$, we get a Δ-set $Y_\bullet$ and an inclusion $Y_\bullet \hookrightarrow X_\bullet$. If $X_\bullet$ is k-dimensional, then any subset $Y_k \subseteq X_k$ gives rise to a Δ-set by taking the union of the images $d_i^k(Y_k) \subseteq X_{k-1}$, the union of the images $d_j^{k-1} d_i^k(Y_k) \subseteq X_{k-2}$ and so on down to X_0.

Lemma 30. Geometric relisation defines a functor $|-|\colon \Delta\mathbf{Set} \to \mathbf{Top}$.

Definition 52. A triangulation of a topological space X is a Δ-set $X_\bullet$ equipped with a homeomorphism $X \simeq |X_\bullet|$.

As in the 2-dimensional case, the geometric realisation $|X_\bullet|$ is triangulated by $X_\bullet$ together with the identity map.

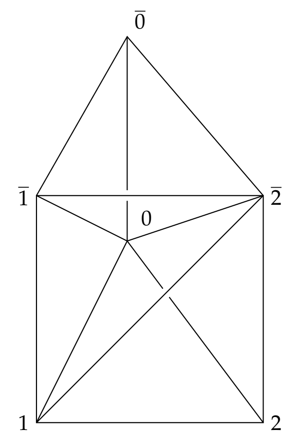
Given a triangulation of a subspace $Y \subset X$ of a topological space X (say by the Δ-set $Y_\bullet$), we can sometimes need to extend this to a triangulation of X. In good situations we can do this by finding a Δ-set $X_\bullet$ such that $Y_\bullet \subset X_\bullet$.

Example 95. The standard topological n-simplex Δ^n is triangulated by the combinatrial n-simplex $\Delta[n]$, and the canonical isomorphism induced by $\coprod_k \mathrm{disc}(\Delta[k]) \times \Delta^k \to \Delta^n$.

For a mildly nontrivial example, consider the product $I \times \Delta^2$. We know that $\{i\} \times \Delta^2$ is triangulated by $\Delta[2]$ for $i = 0, 1$; let us call the vertices of the copy of $\Delta[2]$ corresponding to $\{0\} \times \Delta^2$, 0, 1 and 2, and the vertices of the copy of $\Delta[2]$ corresponding to $\{1\} \times \Delta^2$, $\overline{0}$, $\overline{1}$ and $\overline{2}$.

Then there is a Δ-set $P_\bullet$ with three 3-simplices labelled by the sets

$$0,1,2,\overline{2}$$
$$0,1,\overline{1},\overline{2}$$
$$0,\overline{0},\overline{1},\overline{2}$$



with the 2-simplices given by three-element subsets of these, the 1-simplices given by two-element subsets of these, and six 0-simplices, $0,1,2,\overline{0},\overline{1}$ and $\overline{2}$. This can visualised as at right.

More generally, given any n, we can define a triangulation of the space $I \times \Delta^n$ using an analogous recipe: the desired Δ-set has $n+1$ $(n+1)$-simplices labelled by the lists

$$\begin{array}{l} 0,1,\ldots,n,\overline{n} \\ 0,1,\ldots,n-1,\overline{n-1},\overline{n} \\ \vdots \\ 0,\overline{0},\overline{1},\ldots,\overline{n} \end{array}$$

such that the $n+2$ vertices of each simplex are ordered as shown, and the lower-dimensional simplices are labelled by the lists arising from applying the face maps d_i that omit the i^{th} element from the list.

Given a Δ-set $X_\bullet$ we can define a sequence of R-modules

$$\cdots \to R^{X_n} \xrightarrow{\delta_n} R^{X_{n+1}} \to \cdots$$

where for $g\colon X_n \to R$,

$$\delta_n(g) = \sum_{i=0}^{n} (-1)^i g \circ d_i^{n+1}\colon X_{n+1} \to R$$

Lemma 31. This sequence is a complex, so that $\delta_{n+1} \circ \delta_n = 0$.

Proof. Exercise! $\square$

We denote this complex by $C^\bullet(X_\bullet, R)$, and call it the *simplicial cochain complex* of the Δ-set $X_\bullet$.

Now notice that given a function $\alpha\colon A \to B$ of sets, there is an R-linear map $\alpha^*\colon R^B \to R^A$ defined on $(g\colon B \to R) \mapsto (g \circ \alpha\colon A \to R)$. And, given another function $\beta\colon B \to C$, we have $\alpha^* \circ \beta^* = (\beta \circ \alpha)^*\colon R^C \to R^A$. Thus we have a kind of functoriality, but where the source and target get flipped, and the order of composition likewise.

This is summed up by saying we have a functor $R^{(-)} \colon \mathbf{Set}^{op} \to \mathbf{Mod}_R$, where the ^{op} is a reminder that everything gets flipped on applying the functor.

Lemma 32. There is a functor $C^\bullet(-,R) \colon \Delta\mathbf{Set}^{op} \to \mathbf{Cplx}_R$.

Proof. One just needs to check that given $f \colon X_\bullet \to Y_\bullet$, we have $\delta_n f_n^* = f_{n+1}^* \delta_n$, which is a direct computation. □

As a result, we can define the cohomology modules of a Δ-set.

Definition 53. The n^{th} cohomology module $H^n(X_\bullet, R)$ with coefficients in R is the n^{th} cohomology of the complex $C^\bullet(X_\bullet, R)$, and so is a functor $H^n(-,R) \colon \Delta\mathbf{Set}^{op} \to \mathbf{Mod}_R$.

Note that since $R^\emptyset = 0$, the trivial module, it is immediate that for an n-dimensional Δ-set $X_\bullet$ we have $H^k(X_\bullet, R) = 0$ for all $k > n$. Infinite-dimensional Δ-sets may or may not have nontrivial cohomology modules in infinitely-many dimensions.

Another immediate corollary of this definition is that a finite Δ-set has finitely-generated cohomology modules.

or more generally, a Δ-set with finitely many simplices in dimension n has H^n finitely-generated

Remark. In practice, the only rings R we will consider are $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{Z}/2$.

Remark. Given a finite Δ-set, with all of the face maps explicitly described, to calulate its cohomology modules is purely a calculational effort in combinatorics and linear algebra. However, the effort required may be significant, so there are tools we shall develop that will assist. Further, for an Δ-set that is not finite, for instance, being infinite-dimensional or having infinitely-many n-simplices for a given n, we cannot rely on simple linear algebra to help us much. This will become important later, when we define the cohomology modules of a general topological space, without using Δ-sets.

Given a map of rings $\alpha \colon R \to S$ and a set A, recall that there is an R-linear map $R^A \to S^A$ given by $g \mapsto \alpha \circ g$. This is R-linear as we can consider the S-module S^A to be an R-module via α. As a result, for a Δ-set $X_\bullet$, we can apply this to the singular cochain complex of $X_\bullet$ at each slot.

Lemma 33. For a map of rings $\alpha \colon R \to S$ there is a map of complexes $C^\bullet(X_\bullet, R) \to C^\bullet(X_\bullet, S)$, and for fixed $X_\bullet$ this is functorial in α.

Since a map of complexes gives a map between cohomology modules, we get from α as above an R-linear map $H^n(X_\bullet, R) \to H^n(X_\bullet, S)$, the *change of coefficients* map.

Example 96. Consider the inclusion $\mathbb{Z} \to \mathbb{R}$, which gives rise to maps $H^n(X_\bullet, \mathbb{Z}) \to H^n(X_\bullet, \mathbb{R})$. Notice that the domain is an abelian group, and in particular can contain torsion subgroups, whereas the codomain is a real vector space. The kernel of this map is precisely the torsion subgroup $H^n(X_\bullet, \mathbb{Z})_{\text{tors}} < H^n(X_\bullet, \mathbb{Z})$, and the image is a lattice in $H^n(X_\bullet, \mathbb{R})$.

Example 97. We have the quotient maps $\mathbb{Z} \to \mathbb{Z}/p$ for p a prime, and so get maps $H^n(X_\bullet, \mathbb{Z}) \to H^n(X_\bullet, \mathbb{Z}/p)$, where the codomain is now a $\mathbb{F}_p$-vector space. Now this map destroys any torsion coprime to p, so can be useful in trying to focus on specific phenomena relating to a specific prime number.

Remark. At the dawn of algebraic topology, the focus was largely on objects similar to finite Δ-complexes and in that case, one could define the dimension of the vector spaces $H^n(X_\bullet, \mathbb{R})$, called the *Betti numbers* of $X_\bullet$, and consider the orders of the cyclic subgroups that defined the finite group $H^n(X_\bullet, \mathbb{Z})_{\text{tors}}$ (under the classification of finitely-generated abelian groups), called the *torsion coefficients*. These were the way mathematicians at the time unpacked the information that went into the Euler characteristic, but still these numbers were not functorial. It took Emmy Noether and others in the 1920s to emphasise that having invariants that are themselves algebraic objects was more important than considering just their dimensions or other numeric invariants.

Lecture 23

Going back to thinking about functoriality with respect to maps of Δ-sets, consider for $x \in X_n$ ($X_\bullet$ a given Δ-set) the map in Example 94, $\Delta[n] \to X_\bullet$ We can consider the induced map on complexes near dimension n:

$$\begin{tikzcd}
{} \arrow[r] & R^{X_n} \arrow[r, "\delta"] \arrow[d] & R^{X_{n+1}} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & R^{\Delta[n]_n} \arrow[r] & R^{\Delta[n]_{n+1}} \arrow[r] & {}
\end{tikzcd}$$

but $R^{\Delta[n]_n} = R^1 = R$ (as a module) and $R^{\Delta[n]_{n+1}} = R^{\emptyset} = 0$, so we get an R-linear map $C^n(X_\bullet, R) = R^{X_{n+1}} \to R$. This map is precisely evaluation at $x \in X_n$. An important case for us is when we have a chosen basepoint $x \in X_0$. Then we get a map $C^\bullet(X_\bullet, R) = R^{X_0} \to C^\bullet(\Delta[0], R)$, and the codomain here is a complex of the form $0 \to R \to 0 \to \cdots$. This map on passing to cohomology gives a map $H^0(X_\bullet, R) \to R$ of R-modules. Such a map on an

this really is, under the indended geometric interpretation, a point

R-module M is called an *augmentation* of M, and $M \to R$ is called an *augmented module*. The assignment $(X_\bullet) \mapsto (H^0(X_\bullet, R) \to R)$ is functorial for maps of Δ-set respecting the chosen basepoints, and where augmentations are preserved on the R-module side. The extra geometric structure contained in the choice of basepoint is reflected by the augmentation.

Remark. While Δ-sets and their cohomology are not at present helping to define or calculate cohomology of topological spaces, the tools we are developing will come in handy when we get to that point. So for the present we will continue to focus on Δ-sets and their associated complexes.

What sort of general results help us to calculate cohomology of Δ-sets? Just as for space, let us consider the simplest method for constructing a new object from old: disjoint union.

Some observations:

1. For sets P and Q and a ring R, there is a natural isomorphism $R^{P \sqcup Q} \xrightarrow{\simeq} R^Q \oplus R^Q$ of R-modules.

In fact $R^{\sqcup P_\alpha} \simeq \prod R^{P_\alpha}$ for any sets P_α

2. From Δ-sets $X_\bullet$ and $Y_\bullet$ we can make a new Δ-set $X_\bullet \sqcup Y_\bullet$ with set of n-simplices $X_n \sqcup Y_n$.

3. From complexes $A_\bullet$ and $B_\bullet$ of R-modules, we can make a new complex $A_\bullet \oplus B_\bullet$, namely

$$\cdots \to A_n \oplus B_n \xrightarrow{\delta_n^A \oplus \delta_n^B} A_{n+1} \oplus B_{n+1} \to \cdots$$

   the *direct sum* of $A_\bullet$ and $B_\bullet$.

4. Given a direct sum of complexes $A_\bullet \oplus B_\bullet$, we have an natural isomorphism $H^n(A_\bullet \oplus B_\bullet) \xrightarrow{\simeq} H^n(A_\bullet) \oplus H^n(B_\bullet)$.

If we put these ingredients together, we get:

Lemma 34. There is a natural isomorphism

$$C^\bullet(X_\bullet \sqcup Y_\bullet, R) \xrightarrow{\simeq} C^\bullet(X_\bullet, R) \oplus C^\bullet(Y_\bullet, R)$$

of complexes of R-modules.

Corollary 13. There is a natural isomorphism

this is a version of this for infinite disjoint union, where the direct sum is replaced by product, and this follows from the fact infinite products commute with taking images, quotients and kernels

$$H^(X_\bullet \sqcup Y_\bullet, R) \xrightarrow{\simeq} H^n(X_\bullet, R) \oplus H^n(Y_\bullet, R)$$

of R-modules, for $n = 0, 1, 2, \ldots$.

Recalling the situation for the fundamental groupoid Π_1, we had the result that $\Pi_1(X \sqcup Y) \simeq \Pi_1(X) \sqcup \Pi_1(Y)$. So, in this case we can reduce computations to Δ-sets that are not disjoint unions of smaller Δ-sets. One thing to notice is that the lemma is legitimately stronger than its corollary, since there might be an induced isomorphism between the cohomology modules while the complexes are not isomorphism.

The next step up from calculating the fundamental groupoid of a disjoint union is to calculate the fundamental groupoid of a pushout of spaces: $X = U \cup V$ for neighbourhoods $U, V \subset X$. More precisely there was a way to get information about $\Pi_1(X)$ from $\Pi_1(U)$, $\Pi_1(V)$ and $\Pi_1(U \cap V)$. Things are not so simple now, even ignoring the fact we are working with Δ-sets. The following two examples should be in some sense motivational for the big tool we are about to develop.

Example 98. Consider the combinatorial surface $\partial\Delta[3]$, with vertices $0, 1, 2, 3$, and n-simplices given by $(n+1)$-element subsets of this. Define two sub-Δ-sets $U_\bullet$ and $V_\bullet$ as follows:

1. $U_\bullet$ has the same vertices as $\partial\Delta[3]$ but only two 2-simplices: $\{0, 1, 2\}$ and $\{0, 1, 3\}$, and all the 1-simplices of $\partial\Delta[3]$ *except* $\{2, 3\}$.
2. $V_\bullet$ also has the same vertices as $\partial\Delta[3]$ but now the pair of 2-simplices $\{1, 2, 3\}$ and $\{0, 2, 3\}$, and all the 1-simplices *except* $\{0, 1\}$.

The intersection $U_\bullet \cap V_\bullet$ is then 1-dimensional—that is a directed graph—with vertices $\{0, 1, 2, 3\}$ and edges $\{0, 2\}$, $\{0, 3\}$, $\{1, 2\}$ and $\{1, 3\}$ (ordered from lower to higher label). Now, in principle, we already know, or suspect we know, the cohomology modules of $U_\bullet$, $V_\bullet$ and $U_\bullet \cap V_\bullet$ as the first two triangulate I^2, and the latter triangulates a circle. Then it would be nice if we could calculate the cohomology of $\partial\Delta[3]$ just from this information.

defined to have as set of n-simplices $U_n \cap V_n \subset X_n$

this geometrically realises to a circle

From the functoriality of $C^\bullet(-, R)$ we get restriction maps, which in each dimension look like

$$\begin{tikzcd} R^{\partial\Delta[3]_n} \arrow[r] \arrow[d] & R^{U_n} \arrow[d] \\ R^{V_n} \arrow[r] & R^{U_n \cap V_n} \end{tikzcd}$$

and this square commutes. However, we are in a more of a linear, sequency mood, so will turn this into the sequence

$$\begin{aligned} 0 \to R^{\partial\Delta[3]_n} &\to R^{U_n} \oplus R^{V_n} \to R^{U_n \cap V_n} \to 0 \\ g &\mapsto (g|_{U_n}, g|_{V_n}) \\ &\qquad (f,h) \mapsto f|_{U_n \cap V_n} - h|_{U_n \cap V_n} \end{aligned}$$

It is a short and simple exercise to check that in fact this sequence is in fact exact. This then gives a short exact sequence of complexes

$$0 \to C^\bullet(\partial\Delta[3], R) \to C^\bullet(U_\bullet, R) \oplus C^\bullet(V_\bullet, R) \to C^\bullet(U_\bullet \cap V_\bullet, R) \to 0$$

We want to know the cohomology of the leftmost non-zero complex, but we (in principle) have only calculated the cohomology of the other two complexes.

The above argument works perfectly well for an arbitrary Δ-set $X_\bullet$ and sub-Δ-sets $U_\bullet$ and $V_\bullet$ such that $X_n = U_n \cup V_n$, to give a short exact sequence of complexes of R-modules

$$0 \to C^\bullet(X_\bullet, R) \to C^\bullet(U_\bullet, R) \oplus C^\bullet(V_\bullet, R) \to C^\bullet(U_\bullet \cap V_\bullet, R) \to 0$$

For the second example, we want to consider how we might calculate the cohomology of a quotient from the cohomology of the original Δ-set and that of the sub-Δ-set that gets squashed.

Example 99. Consider now a Δ-set $X_\bullet$ together with a sub-Δ-set $A_\bullet \subset X_\bullet$. Morally speaking, we might have $X_\bullet$ triangulating some space, and $A_\bullet$ triangulating a subspace. We can form the quotient space $|X_\bullet|/|A_\bullet|$, but it is not immediately clear that we can form a sensible Δ-set $X_\bullet/A_\bullet$ that is a quotient of $X_\bullet$ so that this triangulates the topological quotient. Assume for now that there *is* a Δ-set $X_\bullet/Y_\bullet$ with set of n-simplices X_n/Y_n, and a quotient map $X_\bullet \to X_\bullet/Y_\bullet$ of Δ-sets. Can we calculate $H^n(X_\bullet/Y_\bullet, R)$ from $H^n(X_\bullet, R)$ and $H^n(Y_\bullet, R)$?

that is, a *pair* $(X_\bullet, A_\bullet)$

Notice that given a set X and a subset $i\colon Y \hookrightarrow X$, we get a set $X/Y := X/(y_1 \sim y_2)$ for all $y_i \in Y$, and there is a function $q\colon X \to X/Y$. The set X/Y has a canonical basepoint $\mathrm{pt} = [y] \in X/Y$ for any $y \in Y$. We get R-linear maps given by precomposition:

$$R^{X/Y} \xrightarrow{q^*} R^X \xrightarrow{i^*} R^Y$$

where the left map is injective, and the right map is surjective. However, this is not even a complex, as the image of q^* is not contained in the kernel of i^*!

Examining the situation, we see that the image of q^* consists of those functions $X \to R$ that are constant on Y, whereas the kernel of i^*

consists of those functions that are 0 on Y. Moreover, recalling that X/Y has a canonical basepoint, the module $R^{X/Y}$ has an augmentation, namely evaluation on that basepoint: $\mathrm{ev}_{\mathrm{pt}} \colon R^{X/Y} \to R$. The kernel of this map includes into R^X as precisely those functions that are in the kernel of i^*! This example may not be telling us something deep, other than to get a complex that plays well with quotient we may need to play around with the kernel a bit: in one sense the 'correct' module of functions on X/A is really $\ker i^*$, so as to get an exact sequence

$$0 \to \ker i^* \to R^X \to R^A \to 0$$

Phrased this way, we don't even need to consider the quotient set X/A in order to get a module from it. And this also helps with the issue above, in that it's not clear to what extent $X_\bullet/Y_\bullet$ is a good construction.

Hence, given a pair $(X_\bullet, A_\bullet)$, with inclusion function $i \colon A_\bullet \hookrightarrow X_\bullet$, we can consider the (surjective) restriction map Lecture 24

$$C^\bullet(X_\bullet, R) \xrightarrow{i^*} C^\bullet(A_\bullet, R) \to 0$$

and its kernel $\ker(i^*)$ acts like virtual functions on $X_\bullet/A_\bullet$, without having to define this Δ-set. Moreover, we then have a short exact sequence of complexes of R-modules, analogously to the previous example.

Before continuing, it is worth noting that in fact the last construction does make sense

Lemma 35. Given a map of complexes $\varphi \colon A_\bullet \to B_\bullet$, the degreewise kernels $\ker(\varphi_n) \subseteq A_n$ assemble into a complex $\ker(\varphi)$, using the restriction of $A_n \to A_{n+1}$ to $\ker(\varphi_n)$.

Using this lemma, we can define a complex associated to the pair $(X_\bullet, A_\bullet)$.

Definition 54. Given a pair $(X_\bullet, A_\bullet)$ denote by $C^\bullet(X_\bullet, A_\bullet; R)$ the complex $\ker(i^*)$ as above, the simplicial relative cochain complex of the pair.

Both of these examples are special cases of a general principle: given a short exact sequence of complexes

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

(of R-modules, say) then we might wish to calculate $H^n(A_\bullet)$, but only know $H^n(B_\bullet)$ and $H^n(C_\bullet)$. Or we might know $H^n(A_\bullet)$ and

$H^n(C_\bullet)$, and want to calculate $H^n(B_\bullet)$. In the finite setting, this might be merely an issue of computational efficiency, but in general we need to deal with infinitely-generated R-modules, where simple linear algebra techniques start to break down. So we will prove a general result using *homological algebra* that relates all these cohomology groups.

Homological algebra is the area of algebra that deals with the interaction of sequences, maps of sequences, commutative diagrams of algebraic objects with certain 'exactness' properties, and how one can calculate various objects including (co)homology groups

Remark. If you get anything out of this section of the course, the following result is probably it, because you can apply it to your own setting to get a long exact sequence. Or it might be the case there is a standard long exact sequence in your area, and it probably arose from this theorem, so it's a good idea to understand how the abstract proof goes. Together with the long exact sequence of homotopy groups associated to a fibre bundle, this is one of the major computational tools until you get to spectral sequences, which are super powerful, but also much less intuitive.

or, in the case of K-theory, a long exact sequence that folds back on itself

Theorem 11. Given a short exact sequence

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{\pi} C_\bullet \to 0$$

of complexes of R-modules, there is a long exact sequence

$$\cdots \xrightarrow{\delta^{k-1}} H^k(A_\bullet) \xrightarrow{H^k(i)} H^k(B_\bullet) \xrightarrow{H^k(\pi)} H^k(C_\bullet) \xrightarrow{\delta^k} H^{k+1}(A_\bullet) \xrightarrow{H^{k+1}(i)} (B_\bullet) \to \cdots$$

of R-modules.

I call this the *algebraic Mayer–Vietoris* theorem, but it is also known as the *zig-zag lemma*

The proof of this theorem we will give uses a famous lemma in homological algebra, the *Snake Lemma*.

there is a small zoo of lemmas named after animals, other examples being the Salamander Lemma and the Snail Lemma

Lemma 36 (Snake Lemma)**.** Given a commutative diagram

$$\begin{tikzcd}
& A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "\pi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"'] & B' \arrow[r, "\pi'"'] & C &
\end{tikzcd}$$

of R-modules where the rows are exact, there is an exact sequence

$$\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma$$

of R-modules (see Figure 1).

This is a major tool, so the full proof is rather lengthy with lots of details, but at each stage there is generally only one or two things to try.

$$\begin{tikzcd}
 & \ker\alpha \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & \ker\gamma \arrow[d] \arrow[dddll, "\delta"] & \\
 & A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "\pi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"] \arrow[d] & B' \arrow[r, "\pi'"] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma &
\end{tikzcd}$$

Figure 1: The classic Snake Lemma diagram

Proof. We need to do a number of things:

1. Construct the function $\delta\colon \ker\gamma \to \operatorname{coker}\alpha$
2. Prove this is an R-module homomorphism
3. Show $\operatorname{im}(\ker\alpha \to \ker\beta) = \ker(\ker\beta \to \ker\gamma)$
4. Show $\operatorname{im}(\ker\beta \to \ker\gamma) = \ker(\delta)$
5. Show $\operatorname{im}(\delta) = \ker(\operatorname{coker}\alpha \to \operatorname{coker}\beta)$
6. Show $\operatorname{im}(\operatorname{coker}\alpha \to \operatorname{coker}\beta) = \ker(\operatorname{coker}\beta \to \operatorname{coker}\gamma)$

For the present, I will do 1., 4. and 5. Items 3. and 6. are a bit more straightforward, as they don't involve δ. And, given the techniques here, item 2. should be a not-too-challenging exercise. The main technique here is called 'diagram chasing', as it involves starting with an element in one module, and applying homomorphisms or exactness properties to cook up elements of other nearby modules, and repeat, chasing the new elements until we find one in a module we are interested in.

Since we want a function $\delta\colon \ker\gamma \to \operatorname{coker}\alpha$, we will start with a given element $c \in \ker\gamma \subseteq C$ and aim to end up with an element in $\operatorname{coker}\alpha$ Since we know π is surjective, there is a $b \in B$ such that $\pi(b) = c$. Then consider $\beta(b) \in B'$: applying π' we get $\pi'(\beta(b)) = \gamma(\pi(b)) = \gamma(c) = 0$, so that $\beta(b) = i'(a'_b)$ for a unique $a'_b \in A'$. Then we have $[a'_b] \in \operatorname{coker}\alpha$. But is it unique? I hear you ask. Well, what choices did we make along the way? Only the fact that there is not a unique b such that $\pi(b) = c$. So consider another $\tilde{b} \in B$ such that $\pi(\tilde{b}) = c$. We get $\beta(\tilde{b})$, as before, and there is a unique $a'_{\tilde{b}} \in A'$ such that $i'(a'_{\tilde{b}}) = \beta(\tilde{b})$. So we get another element $[a'_{\tilde{b}}] \in \operatorname{coker}\alpha$. But, $\pi(b - \tilde{b}) = \pi(b) - \pi(\tilde{b}) = c - c = 0$, and $\ker \pi = \operatorname{im} i$, so there is some $\underline{a} \in A$ such that $i(\underline{a}) = b - \tilde{b}$, or rather, $b = \tilde{b} + i(\underline{a})$. So now apply β to get $i'(a_b) = \beta(b) = \beta(\tilde{b} + i(\underline{a})) = \beta(\tilde{b}) + \beta(i(\underline{a})) = i'(a'_{\tilde{b}}) + i'(\alpha(\underline{a})) = i'(a'_{\tilde{b}} + \alpha(\underline{a}))$. But i' is injective, so that $a_b = a'_{\tilde{b}} + \alpha(\underline{a})$. But this means that $[a'_b] = [a'_{\tilde{b}}]$, and so from $c \in \ker\gamma$ we have found a unique element of $\operatorname{coker}\alpha$. Thus we have a function $\delta\colon \ker\gamma \to \operatorname{coker}\alpha$, as required.

One needs to then check this is an R-module homomorphism, by comparing $\delta(c_1 + c_2)$ and $\delta(c_1) + \delta(c_2)$ etc, using exactness in various ways as in the previous paragraph. This is an exercise for the keen reader.

We can now check the exactness of the sequence. Assume $c \in \ker\gamma$ such that $\delta(c) = 0$, that is, $[a'_b] = 0 \in \operatorname{coker}\alpha$ for some $b \in B$ such that $\pi(b) = c$. But this means $a'_b = \alpha(\underline{a})$. But since $\beta(b) = i'(a'_b) = i'(\alpha(\underline{a})) = \beta(i(\underline{a}))$, we get $b - i(\underline{a}) \in \ker\beta$. And $\pi(b - i(\underline{a})) = \pi(b) - \pi(i(\underline{a})) = c$. Thus $c \in \operatorname{im}(\ker\beta \to \ker\gamma)$ and $\ker\delta \subseteq \operatorname{im}(\ker\beta \to \ker\gamma)$

Conversely, assume that $c = \pi(b)$ where $b \in \ker\beta$. Then $\delta(c) = [a'_b]$ for $i'(a'_b) = \beta(b)$, but as i' is injective, $a'_b = 0$, hence $\delta(c) = 0$. Thus $\operatorname{im}(\ker\beta \to \ker\gamma) \subseteq \ker\delta$, and so $\operatorname{im}(\ker\beta \to \ker\gamma) = \ker\delta$.

Now consider an arbitrary $c \in \ker\gamma$, and the image of $\delta(c)$ in $\operatorname{coker}\beta$. This is precisely $[i'(a'_b)] = [\beta(b)] = 0$, so that $\operatorname{im}\delta \subseteq \ker(\operatorname{coker}\alpha \to \operatorname{coker}\beta)$. Now consider a $[a'] \in \operatorname{coker}\alpha$ such that $[i'(a')] = 0 \in \operatorname{coker}\beta$. But then this means that $i'(a') = \beta(b)$ for some $b \in B$. If we take $c := \pi(b)$, then $\delta(c) = [a']$, so that $\operatorname{im}\delta = \ker(\operatorname{coker}\alpha \to \operatorname{coker}\beta)$.

The proof that we have exactness in the other positions is left for the keen reader. □

To apply the Snake Lemma to the proof of Theorem 11, we need to cook up a diagram with the appropriate properties. Despite the temptation to apply the Snake Lemma to (two rows of) the short exact sequence of complexes, this is not the correct thing to do, since then the kernels and cokernels are not the cohomology groups in the Theorem.

Lemma 37. The commutative diagram

$$\begin{tikzcd} & A_k/\delta^A_{k-1}(A_{k-1}) \arrow[r] \arrow[d, "\delta^A_k"] & B_k/\delta_{k-1}(B_{k-1}) \arrow[r] \arrow[d, "\delta^B_k"] & C_k/\delta^C_{k-1}(C_{k-1}) \arrow[r] \arrow[d, "\delta^C_k"] & 0 \\ 0 \arrow[r] & \ker(\delta^A_{k+1}) \arrow[r] & \ker(\delta^B_{k+1}) \arrow[r] & \ker(\delta^C_{k+1}) & \end{tikzcd}$$

satisfies the hypotheses of the Snake Lemma, that is, the rows are exact.

Proof. Exercise, for now. □

Proof. (of Theorem 11)

First notice that $\ker(A_k/\delta^A_{k-1}(A_{k-1}) \to \ker\delta^A_{k+1}) = \ker(A_k/\delta^A_{k-1}(A_{k-1}) \to A_{k+1})$. But this is isomorphic to $\ker(A_k \to A_{k+1})/\delta^A_{k-1}(A_{k-1}) = H^k(A_\bullet)$. Similarly, we have $\operatorname{im}(A_k/\delta^A_{k-1}(A_{k-1}) \to \ker\delta^A_{k+1}) = \operatorname{im}(A_k \to \ker\delta^A_{k+1}$ and hence the cokernel is $H^{k+1}(A_\bullet)$.

exercise!

Using Lemma 37, we get an exact sequence

$$H^k(A_\bullet) \to H^k(B_\bullet) \to H^k(C_\bullet) \xrightarrow{\delta^k} H^{k+1}(A_\bullet) \to H^{k+1}(B_\bullet) \to H^{k+1}(C_\bullet)$$

for each k. We can put these together as k varies to get one long exact sequence as in the statement of the theorem. □

For a collection of exact sequences $L_{k-1} \to M_{k-1} \to N_{k-1} \to L_k \to M_k \to N_k$, $k \in \mathbb{Z}$ (where the maps are re-used) there is a long exact sequence $\cdots \to N_{k-2} \to L_{k-1} \to M_{k-1} \to N_{k-1} \to L_k \to M_k \to N_k \to L_{k+1} \to \cdots$ (Exercise)

One nice result is that given two diagrams of the sort that go into the Snake Lemma, and maps between each of the corresponding modules making all the possible cubes commute, there are maps between the kernels and cokernels that appear in the exact sequence, giving a map between the complexes. This means that given two short exact sequences of complexes, and maps between *them*, there is a map between the long exact sequences. This might happen, for instance, if one is changing the coefficient ring in the cohomology of Δ-sets, and one is in the situation of one of the two motivational examples. Or, one might have a map of pairs $(X_\bullet, A_\bullet) \to (Y_\bullet, B_\bullet)$ of Δ-sets, so that each of them gives rise to a short exact sequence of complexes, and the map between the pairs induces a map between the short exact sequences.

the Snake Lemma is thus 'natural'

The cohomology of the complex of relative simplicial cochains turns out to be quite important, and also gives more flexibility in the definition of cohomology of a Δ-set. We recover the complex $C^\bullet(X_\bullet, R)$ by taking $A_\bullet = \emptyset$, hence the complex $C^\bullet(X_\bullet, \emptyset; R)$.

Lecture 25

Definition 55. Given a pair $(X_\bullet, A_\bullet)$ of Δ-sets, its *relative cohomology* is $H^k(X_\bullet, A_\bullet; R) := H^k(C^\bullet(X_\bullet, A_\bullet; R))$.

Example 100. Given $X_\bullet$ a finite-dimensional Δ-set of dimension n, then $(X_\bullet, \operatorname{sk}_{n-1} X_\bullet)$ is a pair, and so we get the relative cochain complex

$$0 \to C^1(X_\bullet, \operatorname{sk}_{n-1} X_\bullet; R) \to \cdots \to C^{n-1}(X_\bullet, \operatorname{sk}_{n-1} X_\bullet; R) \to C^n(X_\bullet, \operatorname{sk}_{n-1} X_\bullet; R) \to 0$$

but for $k < n$, $\operatorname{sk}_{n-1} X_k = X_k$, so that $C^k(X_\bullet, \operatorname{sk}_{n-1} X_\bullet) = \ker(\mathrm{id}_{R^{X_k}}) = 0$ Moreover, $\operatorname{sk}_{n-1} X_n = \varnothing$, so that $C^n(X_\bullet, \operatorname{sk}_{n-1} X_\bullet; R) = \ker(R^{X_n} \to R^{\varnothing} = 0) = R^{X_n}$. Hence the complex $C^\bullet(X_\bullet, \operatorname{sk}_{n-1} X_\bullet; R)$ consists entirely of copies of the zero R-module, except at position n, where it is the module of R-valued functions on X_n. Thus

This is often denoted $R^{X_n}[n]$ in homological algebra

$$H^k(X_\bullet, \operatorname{sk}_{n-1} X_\bullet; R) = \begin{cases} 0 & k \neq n \\ R^{X_n} & k = n \end{cases}$$

The proof of the following lemma follows immediately from the definitions.

Lemma 38. For $k < n$, $H^k(\operatorname{sk}_n X_\bullet, R) \simeq H^k(X_\bullet, R)$.

As a result, every cohomology module of a given Δ-set can be calculated as a cohomology module of a finite-dimensional Δ-set, albeit the dimension of the Δ grows with the dimension the cohomology module sits in.

A more sophisticated result shows the cohomology modules can be approximated, in a precise way, by cohomology modules of *finite* Δ-sets. This will take us too far afield to cover now.

Proposition 22. Given a pair $(X_\bullet, A_\bullet)$ of Δ-sets, there is a long exact sequence of R-modules

$$\begin{tikzcd} 0 \arrow[r] & H^0(X_\bullet, A_\bullet; R) \arrow[r] & H^0(X_\bullet, R) \arrow[r] & H^0(A_\bullet, R) \arrow[dll, rounded corners] \\ & H^1(X_\bullet, A_\bullet; R) \arrow[r] & H^1(X_\bullet, R) \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} & & \cdots \arrow[r] & H^{k-1}(A_\bullet, R) \arrow[dll, rounded corners] \\ & H^k(X_\bullet, A_\bullet; R) \arrow[r] & H^k(X_\bullet, R) \arrow[r] & \cdots \end{tikzcd}$$

Example 101. Consider for instance the pair $(\Delta[n], \partial\Delta[n])$ We have $H^k(\Delta[n], \partial\Delta[n]; R) = 0$ for $k \neq n$, and since $\Delta[n]$ has a single n-simplex, $H^n(\Delta[n], \partial\Delta[n]; R) = R$. The long exact sequence breaks up into small pieces, namely

A special case of Example 100, as $\partial\Delta[n] = \operatorname{sk}_{n-1}\Delta[n]$

$$0 \to H^k(\Delta[n], R) \xrightarrow{\simeq} H^k(\partial\Delta[n], R) \to 0$$

for $k < n-1$, which we already knew on general grounds from the above example, and

$$0 \to H^{n-1}(\Delta[n], R) \to H^{n-1}(\partial\Delta[n], R) \to R \to H^n(\Delta[n], R) \to 0$$

From this we can see that the cohomology module $H^n(\Delta[n], R)$ is a quotient of the rank-one module R, so it is not so big.

in fact it is trivial, but we haven't proved that yet!

If we think of relative cohomology as a kind of cohomology of a 'virtual quotient' by a sub-Δ-set, then if take the pair to be $(X_\bullet, \{x\})$, where $x \in X_0$, then we really can take the quotient squashing $\{x\}$ to a point: it changes nothing! But the relative cohomology really is different from the ordinary cohomology, so the naive idea that it looks like a kind of quotient really needs a bit more subtle interpretation.

Example 102. Consider a really simple Δ-set, namely $\partial\Delta[1]$, which has two 0-simplices, and nothing else. Call these x_0 and x_1, and look at the relative cochain complex. The only non-zero module is

$$C^0(\partial\Delta[1], \{x_0\}; R) = \ker(R^{\{x_0,x_1\}} = R^2 \xrightarrow{\mathrm{pr}_2} R^{\{x_0\}} = R) = R$$

and so $H^0(\partial\Delta[1], \{x_0\}; R) = R$ (and all other H^k are 0). Compare this to $H^0(\partial\Delta[1], R) = R^2$.

More generally, given a 0-dimensional Δ-set with $n+1$ 0-simplices and a chosen basepoint, the relative cohomology will be a rank n free R-module. So this counts the number of points *apart from the specified basepoint*.

This case comes up often enough that it warrants a special name. We call a Δ-set with a specified 0-simplex a *pointed Δ-set*.

Definition 56. Given a pointed Δ-set $(X_\bullet, x)$, the *reduced cohomology* is the relative cohomology $H^k(X_\bullet, x; R)$.

Example 103. Define the Δ-set $Pt_\bullet$ to be $Pt_n = *$ for all $n \geq 0$, with all face maps the identity function. Thus $C^n(Pt_\bullet, R) = R$ for all n. We need to calculate the maps $\delta_n \colon R \to R$ that appear in the complex. Firstly, we think of the elements of R as given by functions $* \to R$ (of sets), so precomposition with $d_i = \mathrm{id}_*$ becomes the identity function on R. Thus

this is infinite-dimensional, with one n-simplex for every $n \in \mathbb{N}$

$$g \overset{\delta_n}{\mapsto} \sum_{i=0}^{n+1} (-1)^i g = \begin{cases} 0 & n \text{ even} \\ g & n \text{ odd} \end{cases}$$

Thus the complex is

$$0 \to R \xrightarrow{0} R \xrightarrow{\mathrm{id}} R \xrightarrow{0} R \xrightarrow{\mathrm{id}} R \to \cdots$$

and so $H^0(Pt_\bullet, R) = R$, but $H^k(Pt_\bullet, R) = 0$ for all $k > 0$. But $Pt_\bullet$ has a canonical basepoint, and $H^k(Pt_\bullet, *; R) = 0$ for all k.

You can think of $Pt_\bullet$ in the last example as a kind of infinite-dimensional fat point, or perhaps a kind of contractible 'space', even though we haven't got a notion of continuous deformation of Δ-sets. This is more of a combinatorial analogue, or, better, and algebraic one, since Δ-sets are really just a way to construct examples of complexes, which are our simpler, algebraic, versions of spaces. We can define an analogue of a map of complexes being weak homotopy equivalence as follows.

Definition 57. A map of complexes $f\colon A_\bullet \to B_\bullet$ is called a *quasi-isomorphism* if $H^k(f)\colon H^k(A_\bullet) \to H^k(B_\bullet)$ is an isomorphism for all k.

Example 104. The map $C^\bullet(Pt_\bullet, R) \to C^\bullet(\mathrm{sk}_0\, Pt_\bullet, R)$ induced by the inclusion is a quasi-isomorphism, despite the domain being nontrivial in all non-negative positions, and the latter being concentrated in a single position.

We will need just one more homological algebra lemma that is very useful in practice.

not named after an animal this time!

Lemma 39 (5 Lemma)**.** Given a diagram of R-modules

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "k"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \arrow[r, "k'"'] & E'
\end{tikzcd}$$

where the rows are exact, then if α is surjective, β and δ are isomorphisms, and ε is injective, then γ is an isomorphism.

Proof. The proof is, as usual, a diagram chase. We split the proof into two steps, each of which only uses half of the assumptions:

1. If ε is injective, and β and δ are surjective, then γ is surjective. Consider $c' \in C'$, and choose some $d \in D$ such that $\delta(d) = h'(c')$. Consider $\varepsilon(k(d)) = k'(\delta(d)) = k'(h'(c')) = 0$. Since ε is injective, this means $k(d) = 0$, and since the top row is exact, this means that $d \in \ker(k) = \mathrm{im}(h)$, so that there exists $c \in C$ such that $d = h(c)$. It's not immediately true that $\gamma(c) = c'$, so let us compare them: $h'(c' - \gamma(c)) = h'(c') - h'(\gamma(c)) = \delta(d) - \delta(h(c)) = 0$. By exactness of the bottom row, this means that $c' - \gamma(c) = g'(b')$ for some $b' \in B$. But as β is surjective, $b' = \beta(b)$ for some $b \in B$. That is, $c' - \gamma(c) = g'(\beta(b)) = \gamma(g(b))$. We can rearrange this so that $c' = \gamma(c) + \gamma(g(b)) = \gamma(c + g(b))$, and hence γ is surjective.

2. If α is surjective, and β and δ are injective, then γ is injective. This is an exercise in dualising the above steps. $\square$

Now to revisit the idea of Euler characteristic of a finite Δ-set, which is, recall, the sum

$$\chi(X_\bullet) = \sum_{d=0}^{\infty} (-1)^d |X_d|$$

the sum terminates, as $|X_d| = 0$ for all large enough d

A key idea introduced at the beginning of this section was that we wanted to replace numerical invariants, such as cardinality of finite sets by vector spaces, or more generally modules, so that dimension replaced cardinality. We now have a different way to construct a numerical invariant from a Δ-set, namely using the dimensions of the cohomology modules, in the case when we take $R = \mathbb{R}$, say. Thus we can define the *cohomological Euler characteristic* to be

any characteristic zero field would do

$$\chi^{coh}(X_\bullet) := \sum_{d=0}^{\infty} (-1)^d \dim H^d(X_\bullet, \mathbb{R})$$

as long as this sum exists. While now this implies that $H^d(X_\bullet, \mathbb{R}) = 0$ for all large enough d, we certainly don't need to have $X_\bullet$ finite, or even finite-dimensional, as the example of $Pt_\bullet$ shows. In that case, $\chi^{coh}(X_\bullet) = 1$. We can also consider finite-dimensional but infinite Δ-sets, for instance a triangulation of $\mathbb{R}$ by 1-simplices.

Example 105. Let $L_\bullet$ be the directed graph with $L_0 = \mathbb{Z}$ and $L_1 = \mathbb{Z}$, where $d_0(n) = n + 1$ and $d_1(n) = n$. This has an edge from n to $n + 1$ for each n. Given $g \in \mathbb{R}^{L_0} = \mathbb{R}^{\mathbb{Z}}$, $\delta_0(g)(n) = g(n+1) - g(n)$. So $\delta(g) = 0$ precisely if g is constant, hence $\ker(\delta_0) = H^0(L_\bullet, \mathbb{R}) = \mathbb{R}$.

And, given $h \in \mathbb{R}^{L_1} = \mathbb{R}^{\mathbb{Z}}$, define $g(0) = 0$, and then use $g(n+1) = h(n) + g(n)$ to define $g\colon \mathbb{Z} \to \mathbb{R}$ for all nonzero $n \in \mathbb{Z}$, so that $\delta_0(g) = h$. Thus $\mathrm{coker}(\delta_0) = H^1(L_\bullet, \mathbb{R}) = 0$. All other real-coefficient cohomology vector spaces are trivial, so that $\chi^{coh}(L_\bullet) = 1$.

However, now we have two numerical invariants of a finite Δ-set, namely χ and χ^{coh}, and it is not immediately obvious how they relate. Thankfully, they coincide, and so we can just call this the Euler characteristic

Proposition 23. For a finite Δ-set $X_\bullet$, we have $\chi(X_\bullet) = \chi^{coh}(X_\bullet)$

Proof. First, $|X_d| = \dim \mathbb{R}^{X_d}$, and consider the part of the complex near there:

$$\mathbb{R}^{X_{d-1}} \xrightarrow{\delta_{d-1}} \mathbb{R}^{X_d} \xrightarrow{\delta_d} \mathbb{R}^{X_{d+1}}$$

We can, using the standard inner product, break $\mathbb{R}^{X_d}$ into a direct sum:

$$\begin{aligned}\mathbb{R}^{X_d} &= \ker \delta_d \oplus \operatorname{im} \delta_d \\ &= \begin{cases} H^d(X_\bullet, \mathbb{R}) \oplus \operatorname{im} \delta_{d-1} \oplus \operatorname{im} \delta_d & d > 0 \\ H^0(X_\bullet, \mathbb{R}) \oplus \operatorname{im} \delta_0 & d = 0 \end{cases}\end{aligned}$$

We can unify these two cases if we agree that $H^{-1}(X_\bullet, \mathbb{R}) = 0$. Then we have

$$\dim \mathbb{R}^{X_d} = \dim H^d(X_\bullet, \mathbb{R}) + \dim \operatorname{im} \delta_{d-1} + \dim \delta_d$$

and so

$$\begin{aligned}\chi(X_\bullet) &= \sum_{d=0}^{\infty} (-1)^d |X_\bullet| \\ &= \sum_{d=0}^{\infty} (-1)^d \dim \mathbb{R}^{X_d} \\ &= \sum_{d=0}^{\infty} (-1)^d \dim H^d(X_\bullet, \mathbb{R}) + \sum_{d=0}^{\infty} (-1)^d \left(\dim \operatorname{im} \delta_{d-1} + \dim \operatorname{im} \delta_d\right) \\ &= \sum_{d=0}^{\infty} (-1)^d \dim H^d(X_\bullet, \mathbb{R}) \\ &= \chi^{coh}(X_\bullet)\end{aligned}$$

□

We thus can drop the superscript on χ^{coh} and just talk about **the** Euler characteristic of a Δ-set.

Remark. This proof can be adapted pretty much verbatim to show that for a complex $V_\bullet$ of finite-dimensional vector spaces of finite length, say

$$0 \to V_m \to V_{m+1} \to \cdots \to V_{m+N} \to 0$$

then

$$\sum_{d=m}^{m+N} (-1) \dim V_d = \sum_{d=m}^{m+N} (-1)^d H^d(V_\bullet).$$

Recall the geometric realisation of a Δ-set $X_\bullet$: Lecture 26

$$|X_\bullet| = \left(\coprod_{n=0}^{\infty} \operatorname{disc}(X_n) \times \Delta^n \right) \Big/ \sim$$

This space has a set of distinguished maps $\Delta^n \to |X_\bullet|$, namely for a given $x \in X_n$, we have the composite

$$\Delta^n \to \operatorname{disc}(X_n) \times \Delta^n \hookrightarrow \coprod_{n=0}^{\infty} \operatorname{disc}(X_n) \times \Delta^n \to |X_\bullet|$$

Note also that precomposing this map with $\partial_i \colon \Delta^{n-1} \hookrightarrow \Delta^n$ gives another map in the distinguised class, corresponding to $d_i(x) \in X_{n-1}$.

Note also that if we have a smooth manifold M (for instance an open set of $\mathbb{R}^n$) and ω is a differential k-form on M, then this determines a function

by a smooth function on Δ^k here it is enough to assume it is smooth on the interior and extends continuously to the boundary

$$C^\infty(\Delta^k, M) \to \mathbb{R}$$
$$(f \colon \Delta^k \to M) \mapsto \int_{\Delta^k} f^*\omega$$

Thus this gives a function from k-forms on M to the vector space $\mathbb{R}^{C^\infty(\Delta^k, M)}$. This function also interacts well with the exterior derivative and, by Stokes' theorem, the restriction of the primitive of an exact form to the boundary.

For a general topological space X, we are somewhere in the neighbourhood of these two ideas: since we do not have distinguished maps $\Delta^n \to X$, we should consider *all* maps and then functions on the set of these.

Definition 58. Let X be a topological space. The *singular cochain complex* of X with coefficients in the ring R, denoted $C^\bullet(X, R)$ is given by

$$0 \to R^{\mathbf{Top}(\Delta^0, X)} \xrightarrow{\delta} R^{\mathbf{Top}(\Delta^1, X)} \xrightarrow{\delta} R^{\mathbf{Top}(\Delta^2, X)} \to \cdots$$

where given $g \colon \mathbf{Top}(\Delta^n, X) \to R$ and $f \colon \Delta^{n+1} \to X$, $\delta(g)(f) = \sum_{i=0}^{n+1} (-1)^i g(f \circ \partial_i)$. This defines a functor $\mathbf{Top}^{op} \to \mathbf{Cplx}_R$. Given a map $f \colon X \to Y$, where the induced map $R^{\mathbf{Top}(\Delta^k, Y)} \to R^{\mathbf{Top}(\Delta^k, X)}$ is given by precomposing with the induced $\mathbf{Top}(\Delta^k, X) \to \mathbf{Top}(\Delta^k Y)$.

The *singular cohomology* of X with coefficients in R is the cohomology of this complex:

$$H^n(X, R) := H^n(C^\bullet(X, R))$$

and hence gives functors $H^n(-, R) \colon \mathbf{Top}^{op} \to \mathbf{Mod}_R$.

These modules are *huge* (in general). For example, take $X = I$ and $R = \mathbb{Z}/2$, and then $|C^1(I, \mathbb{Z}/2)| = 2^{|\mathbf{Top}(\Delta^1, I)|} = 2^{|\mathbb{R}|}$, but, as we shall see in a moment, $H^1(I, \mathbb{Z}/2) = 0$. Hence we *must* rely on theorems to calculate the singular cohomology, unlike the much easier case of cohomology of Δ-sets.

However, here is the (more of less) only example we can calculate from the definition

Example 106. Let $X = \mathrm{pt}$. Then $\mathbf{Top}(\Delta^k, \mathrm{pt}) = *$, so that the singular cochain complex is

we've seen this before!

$$0 \to R \xrightarrow{0} R \xrightarrow{\mathrm{id}} R \xrightarrow{0} R \xrightarrow{\mathrm{id}} \cdots$$

which has cohomology $H^0(\mathrm{pt}, R) = R$ and $H^k(\mathrm{pt}, R) = 0$ for $k > 0$.

Just as for Δ-sets, we have relative cohomology, which is useful for the additional flexibility it affords.

Definition 59. For (X, A) a pair of spaces, the *relative singular cochain complex* $C^\bullet(X, A; R)$ is the kernel of $i^*\colon C^\bullet(X, R) \to C^\bullet(A, R)$ where $i\colon A \hookrightarrow X$ is the inclusion. This gives a functor $\mathbf{Top}^{(2),op} \to \mathbf{Cplx}_R$. We then define the *relative singular cohomology* $H^k(X, A; R) = H^k(C^\bullet(X, A; R))$, which is functorial for maps of pairs of spaces.

Note that we recover ordinary singular cohomology of X as the relative cohomology of the pair $(X, \emptyset)$. Using the same argument as for relative cohomology of Δ-sets, we get

Proposition 24. Given a pair (X, A) of spaces, there is a long exact sequence

$$0 \to H^0(X, A; R) \to H^0(X, R) \to H^0(A, R) \to H^1(X, A; R) \to H^1(X, R) \to \cdots$$

of R-modules.

Proof. There is a short exact sequence of complexes

$$0 \to C^\bullet(X, A; R) \to C^\bullet(X, R) \to C^\bullet(A, R) \to 0$$

and then apply Theorem 11. $\square$

Let (X, x) be a pointed space, and consider the long exact sequence of the relative cohomology of the pair $(X, \mathrm{pt}) = (X, \{x\})$. Since $H^k(\mathrm{pt}, R) = 0$ for positive k, the long exact sequence breaks up into

$$0 \to H^0(X, \mathrm{pt}; R) \to H^0(X, R) \to H^0(\mathrm{pt}, R) = R \to H^1(X, \mathrm{pt}; R) \to H^1(X, R) \to 0$$

and $0 \to H^k(X, \mathrm{pt}; R) \xrightarrow{\simeq} H^k(X, R) \to 0$ for $k > 1$. From the fragment at the start of the exact sequence, we see that $H^0(X, \mathrm{pt}; R) \to H^0(X, R)$ is injective, and this is the inclusion of the kernel of the map $H^0(X, R) \to R$. Thus $H^0(X, \{x\}; R) = \ker(H^0(X, R) \to R)$. For simplicity, we denote $H^k(X, \{x\}; R)$ by $H^k(X, x; R)$, and call it the *reduced cohomology* of the pointed space (X, x).

Remark. We also have the result that $H^1(X, R)$ is the quotient of $H^1(X, x; R)$ by the image of $R \to H^1(X, x; R)$, but to say definitively what this is we would need to study the construction of this map.

Example 107. For a discrete space S with chosen basepoint $p \in S$, then $H^0(S, p; R) \simeq R^{S \setminus \{p\}}$. In particular, $H^0(\mathrm{pt}, \mathrm{pt}; R) = R^{\varnothing} = 0$.

Since maps of pairs $(X, x) \to (Y, y)$ are just pointed maps, we have that reduced cohomology is a functor $H^k(-; R) \colon \mathbf{Top}_*^{op} \to \mathbf{Mod}_R$. Note particularly that we only have functoriality for pointed maps.

Exercise!

Here's a first result that would help calculate (relative) cohomology

Proposition 25. Given pairs of spaces (X, A) and (Y, B), there is a canonical isomorphism

$$H^k(X \sqcup Y, A \sqcup B; R) \xrightarrow{\simeq} H^k(X, A : R) \oplus H^k(Y, B; R)$$

for all k. Even better: there is a canonical isomorphism of complexes

$$C^\bullet(X \sqcup Y, A \sqcup B; R) \xrightarrow{\simeq} C^\bullet(X, A : R) \oplus C^\bullet(Y, B; R)$$

that, on passing to cohomology, give the previous isomorphisms.

Proof. This is because $\mathbf{Top}(\Delta^k, X \sqcup Y) = \mathbf{Top}(\Delta^k, X) \sqcup \mathbf{Top}(\Delta^k, Y)$, and the earlier observation that $R^{P \sqcup Q} \simeq R^P \oplus R^Q$ for any sets P and Q. $\square$

Of course, we get the analogous result for plain cohomology by looking at pairs $(X, \varnothing)$ and $(Y, \varnothing)$.

Here is a much more powerful and difficult result. Recall that we write f^* generically for $H^k(f)$.

Theorem 12. If the maps $f, g \colon X \to Y$ are homotopic, then

$$f^* = g^* \colon H^k(Y, R) \to H^k(X, R)$$

for all k.

I will give a few corollaries of this before discussing what goes into the proof, and how we get the above theorem from a stronger statement about complexes. The proofs of the following are applications of functoriality and the above theorem.

Corollary 14. If X and Y are homotopy equivalent, via $f \colon X \leftrightarrows Y : g$, say, then $f^* = (g^*)^{-1}$ and $H^k(X, R)$ and $H^k(Y, R)$ are isomorphic for all k.

Corollary 15. If a space X is contractible, then $H^k(X, R) = 0$ for $k > 0$ and $H^k(X, R) \simeq R$. More precisely, if X is contractible to $x \in X$, then $H^k(X, R) \to H^k(\{x\}, R)$ is an isomorphism for all k, and hence $H^k(X, x; R) = 0$ for all k.

Corollary 16. Given a pointed space X with a path $\gamma\colon x \rightsquigarrow x'$, the two induced maps $H^0(X, R) \to H^0(\mathrm{pt}, R) = R$ given by the inclusion of x and x' are equal, so that $H^0(X, x; R) = H^0(X, x'; R)$. Thus reduced cohomology only depends on the path component of the basepoint, not the basepoint specifically.

We saw earlier the concept of quasi-isomorphism, which is an analogue for complexes of weak homotopy equivalence. But for spaces we have the stronger notion of homotopy of maps, and this should be reflected by some construction for complexes.

Definition 60. Let $f, g\colon A_\bullet \to B_\bullet$ be maps of complexes. A *cochain homotopy* from f to g is a collection of functions $\{h_n\colon A_n \to B_{n-1}\}$ satisfying the identities

$$\delta^B_{n-1} h_n + h_{n+1}\delta^A_n = f_n - g_n$$

Lemma 40. If there is a cochain homotopy from f to g, both maps $A_\bullet \to B_\bullet$, then $H^k(f) = H^k(g)$.

Proof. An element in $H^k(A_\bullet)$ is the equivalence class of some $c \in A_k$ such that $\delta^A_k(c) = 0$ so

$$\begin{aligned}
H^k(f)([c]) &= [f_k(c)] \\
&= [g_k(c) + \delta^B_{k-1}(h_k(c)) + h_{k+1}(\delta^A_k(x))] \\
&= [g_k(c)] + [\delta^B_{k-1}(h_k(c))] \\
&= [g_k(c)] \\
&= H^k(g)([c])
\end{aligned}$$

□

Here is a stronger version of Theorem 12:

Theorem 13. If the maps $f, g\colon X \to Y$ are homotopic, then there is a cochain homotopy between the two induced maps $C^\bullet(Y, R) \to C^\bullet(X, R)$.

The proof is reasonably detailed, but constructs an actual such cochain homotopy. Matters are simplified somewhat because one can immediately reduce to the case $Y = I \times X$ and the two functions being the inclusions $X \simeq \{i\} \times X \to I \times X$ for $i = 0, 1$. This is because of functoriality and the given homotopy $H\colon I \times X \to Y$. Further, one can reduce a lot of the work to the case $X = \Delta^k$, and an explicit triangulation of $I \times \Delta^k$, so that the inclusion maps $\Delta^k \to I \times \Delta^k$ come from maps of Δ-sets. Then it is messy combinatorics to make sure the required identity holds.

Lecture 27

Let us consider for a short time again the reduced cohomology, which as noted above is functorial for pointed maps. For an arbitrary space X, there is of course a canonical map $X \xrightarrow{!_X} \mathrm{pt}$, which induces a map in cohomology $R = H^0(\mathrm{pt}, R) \to H^0(X, R)$. Moreover, since for any map $f\colon X \to Y$ we have $!_Y \circ f = !_X$, the induced map in cohomology $H^0(Y, R) \to H^0(X, R)$ commutes with these maps from R. This is somewhat reminiscent of the situation with reduced cohomology, except now we have a map *from* R, not *to* R. But what is this map?

Exercise 19. Given a space X, $H^0(X, R) \simeq R^{[\mathrm{pt}, X]}$, that is, functions that are constant on path-components. Moreover, given $f\colon X \to Y$, the induced map $H^0(Y, R) \to H^0(X, R)$ is given by precomposition with $[\mathrm{pt}, X] \to [\mathrm{pt}, Y]$.

Thus $R = H^0(\mathrm{pt}, R) \to H^0(X, R)$ sends $r \in R$ to the constant function on X with value r—assuming X is not empty—so we shall denote it by const. Further, given $x \in X$, the map ! is a retraction to $x\colon \mathrm{pt} \to X$, so that $!_X \circ x = \mathrm{id}_X$. Thus we have $R \xrightarrow{\mathrm{const}} H^0(X, R) \xrightarrow{\mathrm{ev}_x} R$ is the identity map on R, and so const is injective.

Definition 61. A *left splitting* of a short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ of R-modules is a map $r\colon B \to A$ such that $r \circ i = \mathrm{id}_A$.

Thus we have a short exact sequence $0 \to R \to H^0(X, R) \to \mathrm{coker}(\mathrm{const}) \to 0$, and ev_x is a left splitting.

Lemma 41. Given a left splitting r of a short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$, the map $(r, \pi)\colon B \to A \oplus C$ is an isomorphism, and $C \simeq \ker(r)$.

As a result, from a choice $x \in X$ we get an isomorphism $\mathrm{coker}(\mathrm{const}) \simeq H^0(X, x; R)$. Since the map const is canonical, and doesn't depend on the choice of x, it turns out that reduced cohomology is essentially independent of the choice of basepoint. Thus we can redefine reduced cohomology to be $\widetilde{H}^k(X, R) := \mathrm{coker}(H^k(\mathrm{pt}, R) \to H^k(X, R))$; for $k > 0$, $\widetilde{H}^k(X, R) \simeq H^k(X, R)$, but otherwise $H^0(X, R) \simeq \widetilde{H}^0(X, R) \oplus R$. Further, this is functorial for all maps of spaces, not pointed maps.

the submodule $H^0(X, x; R) \subseteq H^0(X, R)$ can be different for different choices of $x \in X$, though

Example 108. Given any path-connected space X we get $\widetilde{H}^0(X, R) = 0$ for all k. In particular, if X is contractible, then $\widetilde{H}^k(X, R) = 0$ for all k.

Example 109. For the 0-sphere S^0, we have $\widetilde{H}^0(S^0, R) = R$ and $\widetilde{H}^k(S^0, R) = 0$ for $k > 0$.

Since we are in the realm of looking at long exact sequences, let us consider the topological space version of Mayer–Vietoris for a union of two subspaces. Take $\mathcal{U} = \{U, V\}$ an open cover of the space X. There is a pushout diagram

$$\begin{tikzcd} U \cap V \arrow[r, "i_V"] \arrow[d, "i_U"'] & V \arrow[d, "j_V"] \\ U \arrow[r, "j_U"'] & X \end{tikzcd}$$

We can define a map

$$C^\bullet(U, R) \oplus C^\bullet(V, R) \to C^\bullet(U \cap V, R) \tag{5}$$
$$(f, g) \mapsto i_U^* f - i_V^* g \tag{6}$$

which turns out to be onto: given $\widetilde{f} \colon \mathbf{Top}(\Delta^n, U \cap V) \to R$, we can define a function $f \colon \mathbf{Top}(\Delta^n, U) \to R$ by extension by zero, as $\mathbf{Top}(\Delta^n, U \cap V)$ is naturally a subset of $\mathbf{Top}(\Delta^n, U)$. Then $(f, 0) \mapsto \widetilde{f}$. So if we define $C^\bullet_{\mathcal{U}}(X, R)$ as the kernel of (5), we get a short exact sequence of complexes

$$0 \to C^\bullet_{\mathcal{U}}(X, R) \to C^\bullet(U, R) \oplus C^\bullet(V, R) \to C^\bullet(U \cap V, R) \to 0$$

We can identify this kernel as something concrete, namely

$$C^k_{\mathcal{U}}(X, R) := \{f \colon \mathbf{Top}(\Delta^k, X) \to R \mid f(\sigma) = 0 \text{ if } \sigma \colon \Delta^k \to X \text{ doesn't factor through } U \text{ or } V\}$$

The following result is key, but also has a very long and complicated proof

Proposition 26. The inclusion $C^\bullet_{\mathcal{U}}(X, R) \to C^\bullet(X, R)$ is a quasi-isomorphism, so that

$$H^k(C^\bullet_{\mathcal{U}}(X, R)) \xrightarrow{\simeq} H^k(X, R).$$

Proof. This follows from reasoning similar to Hatcher's Proposition 2.21, albeit using cohomology, not homology. □

The idea of the proof is that given $\Delta^k \to X$, one can interatively retriangulate Δ^k by more and smaller simplices so that eventually you can break up Δ^k into a collection of functions on small simplices, each of which lands (by an application of the Lebesgue covering lemma) inside one of the open subsets U or V. This is formally similar to how one can integrate over a simplex in a manifold by covering a manifold by charts, by breaking the simplex up into parts each of which land inside a chart, and then integrate each bit and add them up.

Theorem 14. (Mayer–Vietoris) Given an open cover $\{U, V\}$ of the space X, there is a long exact sequence

$$0 \to H^0(X,R) \to H^0(U,R) \oplus H^0(V,R) \to H^0(U \cap V,R) \to H^1(X,R) \to H^1(U,R) \oplus H^1(V,R) \to \cdots$$

of R-modules, and similarly with reduced cohomology, starting

$$0 \to \widetilde{H}^0(X,R) \to \widetilde{H}^0(U,R) \oplus \widetilde{H}^0(V,R) \to \widetilde{H}^0(U \cap V,R) \to H^1(X,R) \to \cdots$$

Here is a key example.

Example 110. Cover the sphere S^n ($n \geq 1$) by two open sets D^n_+ and D^n_-, both homeomorphic to discs (hence contractible). Their intersection is homeomorphic to $S^{n-1} \times J$, for J a small open interval, hence homotopic to S^{n-1}. We thus get by the Mayer–Vietoris theorem a long exact sequence

$$0 \to \widetilde{H}^0(S^n,R) \to \widetilde{H}^0(D^n_+,R) \oplus \widetilde{H}^0(D^n_-,R) \to \widetilde{H}^0(S^{n-1} \times J,R) \to H^1(S^n,R) \to H^1(D^n_+,R) \oplus H^1(D^n_-,R) \to H^1(S^{n-1} \times J,R) \to \cdots$$

but since $D^n_\pm$ are contractible, this breaks up into pieces. Firstly, $0 \to \widetilde{H}^0(S^n,R) \to 0$, hence $\widetilde{H}^0(S^n,R) = 0$, which we knew already, as S^n is path connected. Then for $k > 0$ we have

$$0 \to \widetilde{H}^0(S^{n-1},R) \to H^1(S^n,R) \to 0$$

and

$$0 \to H^{k-1}(S^{n-1},R) \to H^k(S^n,R) \to 0$$

for all $k > 1$. Hence we can attack this problem by induction as (combining the two cases)

$$\widetilde{H}^{k-1}(S^{n-1},R) \simeq \widetilde{H}^k(S^n,R). \tag{7}$$

If we take $k = n$, then this gives $\widetilde{H}^{n-1}(S^{n-1},R) \simeq \widetilde{H}^n(S^n,R)$,

By Example 109 we know the reduced cohomology of S^0, namely $\widetilde{H}^0(S^0,R) = R$, so that $\widetilde{H}^n(S^n,R) = H^n(S^n,R) = R$ for all $n \geq 1$.

Lecture 28

From the calculation of the connected components and the fundamental group, we know that S^0 and S^1 can't be contractible, but the spheres S^n for $n \geq 2$ are simply-connected, so Π_1 cannot tell us that they aren't contractible. But from this result on H^n we have proved that all S^n are not contractible. We can also calculate the rest of the cohomology of S^n. We use equation (7) in the following two cases.

of course, we have seen the claim that $\pi_n(S^n) = \mathbb{Z}$, but we didn't calculate this ourselves

- Take $k = n + l$ for $l \geq 1$ to get $\widetilde{H}^{n+l}(S^n, R) \simeq \widetilde{H}^l(S^0, R) = 0$. Thus $H^m(S^n, R) = 0$ for all $m > n$. This result implies that S^n is not homotopy equivalent to S^m for $m > n$.

In the combinatorial world of Δ-sets, where S^n is triangulated by $\partial\Delta[n+1]$, this is obvious as the sets of m-simplices are empty for $m > n$, but here $\mathbf{Top}(\Delta^m, S^n)$ is uncountable.

- Given $n > 1$, take $0 \leq k \leq n - 2$, and then we get $\widetilde{H}^{k+1}(S^n, R) \simeq \widetilde{H}^1(S^{n-k}, R)$. As $n - k > 1$, this means we are reduced to calculating $\widetilde{H}^1(S^n, R)$ for all $n > 1$. If we examine the start of the long exact sequence above, we have

$$0 = \widetilde{H}^0(S^{n-1}, R) \to \widetilde{H}^1(S^n, R) \to \widetilde{H}^1(D^n_+, R) \oplus \widetilde{H}^1(D^n_-, R) = 0$$

so that $\widetilde{H}^1(S^n, R) = 0$.

Proposition 27. Putting these all together, we have calculated all the cohomology modules of S^n, for all $n \geq 0$:

$$\widetilde{H}^k(S^n, R) = \begin{cases} R & k = n \\ 0 & k \neq n \end{cases}$$

Here's an application of the above calculation to a problem in pure topology. We know that a linear isomorphism between finite-dimensional vector spaces over $\mathbb{R}$ is automatically an isomorphism of topological vector spaces: it is linear, continuous, and with linear and continuous inverse. Since two such vector spaces are isomorphic if they have the same dimension, this shows that there is a linear homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}^m$ precisely when $n = m$. For arbitrary $n, m > 0$ then we know that $|\mathbb{R}^n| = |\mathbb{R}^m| = 2^{\aleph_0}$, so there is no cardinality obstruction to the existence of a *nonlinear* homeomorphism $\mathbb{R}^n \simeq \mathbb{R}^m$ for different positive n, m.

arbitrary continuous maps $\mathbb{R}^n \to \mathbb{R}^m$ can be quite wild, for instance space-filling curves

Proposition 28. $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$

Proof. We will prove the forward implication, the other is trivial. We can assume that a homeomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^m$ satisfies $\phi(0) = 0$, since otherwise we can compose with the translation by $-\phi(0)$, which is also a homeomorphism. We can restrict ϕ to $\mathbb{R}^n \setminus \{0\}$ to get a homeomorphism $\mathbb{R}^n \setminus \{0\} \simeq \mathbb{R}^m \setminus \{0\}$, and so we get an isomorphism in cohomology, $\widetilde{H}^{n-1}(\mathbb{R}^n \setminus \{0\}, R) \simeq \widetilde{H}^{n-1}(\mathbb{R}^m \setminus \{0\}, R)$. But since $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1} for all $n \geq 1$, we get an isomorphism $R \simeq \widetilde{H}^{n-1}(S^{n-1}, R) \simeq \widetilde{H}^{n-1}(S^{m-1}, R)$. Thus $m - 1 = n - 1$, hence the result. □

The following property is rather difficult to prove, and requires non-trivial topological input.

this is evidenced by the fact the hypothesis involves closures and interiors of the input subspaces. The hypothesis on $Z \subset A$ automatically holds if A is open and Z is closed.

Theorem 15. (Excision) Let (X, A) be a pair of spaces, and $Z \subset A$ a subspace such that $\overline{Z}$ is contained in the interior of A. Then the inclusion map $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism

$$H^k(X, A; R) \xrightarrow{\simeq} H^k(X \setminus Z, A \setminus Z; R)$$

for all k.

If we think of relative cohomology of (X, A) as telling us about the cohomology of X/A, then the above theorem reflects the homeomorphism $(X \setminus Z)/(A \setminus Z) \simeq X/A$ that exists for all (X, A) and $Z \subset A$ satsifying the hypotheses of the theorem.

Not mentioned earlier, but $X \setminus \emptyset := X \sqcup \mathrm{pt}$, where the basepoint of $X \setminus \emptyset$ is the new disjoint point.

This theorem allows us to prove that relative cohomology, at least in certain cases, really *is* the cohomology of the quotient.

Theorem 16. Let (X, A) be a pair such that A is a non-empty closed subspace, and that there exists a nhd $U \supset A$ so that A is a deformation retract of U. The quotient map $(X, A) \to (X/A, A/A) = (X/A, *)$ induces an isomorphism

$$H^k(X, A; R) \xrightarrow{\simeq} H^k(X/A, *; R) = \widetilde{H}^k(X/A, R)$$

for all k.

Proof. The proof follows from Hatcher's Proposition 2.22. □

An example of such a pair arises from a pair Δ-sets: $(X_\bullet, A_\bullet)$ gives a pair of space $(|X_\bullet|, |A_\bullet|)$ and Hatcher's Proposition A.4 can be used to prove that there is a nhd U as in the theorem.

Corollary 17. Given (X, A) as in Theorem 16, there is a long exact sequence

$$0 \to \widetilde{H}^0(X/A, R) \to H^0(X, R) \to H^0(A, R) \to H^1(X/A, R) \to \cdots$$

Proof. Apply Theorem 16 to the long exact sequence of a pair that appears in Proposition 24. □

A canonical example of a pair of Δ-sets arises as follows. Given $X_\bullet$ n-dimensional, we can consider $\mathrm{sk}_{n-1} X_\bullet \subset X_\bullet$. There are homeomorphisms

$$\begin{aligned}|X_\bullet|/|\operatorname{sk}_{n-1}X_\bullet| &\simeq \left(\coprod_{X_n}\Delta^n\right) / \left(\coprod_{X_n}\partial\Delta^n\right)\\ &\simeq \left(\coprod_{X_n}(\Delta^n/\partial\Delta^n)\right) / \left(\coprod_{X_n} *\right)\\ &\simeq \left(\coprod_{X_n} S^n\right) / \operatorname{disc}(X_n)\\ &=: \bigvee_{X_n} S^n\end{aligned}$$

The last item is the join of $|X_n|$ copies of S^n: since X_n may be infinite, this is defined in a slightly different way to the finite case.

Definition 62. Let $\{(X_\alpha, x_\alpha)\}_{\alpha\in J}$ be a family of pointed spaces. The *(infinite) join* $\bigvee_{\alpha\in J} X_\alpha$ is the quotient $\left(\coprod_{\alpha\in J} X_\alpha\right) / \operatorname{disc}(J)$, where $J \hookrightarrow \coprod X_\alpha$ is defined as $\alpha \mapsto x_\alpha$. It has a canonical basepoint given by the image of $\operatorname{disc}(J)$.

Given n-dimensional Δ-set $X_\bullet$, there is a long exact sequences which reads, in part

$$\cdots \to H^{k-1}(|\operatorname{sk}_{n-1}X_\bullet|, R) \to H^k(|X_\bullet|, |\operatorname{sk}_{n-1}X_\bullet|; R) \to H^k(|X_\bullet|, R) \to H^k(|\operatorname{sk}_{n-1}X_\bullet|, R) \to \cdots$$

We can apply Theorem 16 to get

$$H^k(|X_\bullet|, |\operatorname{sk}_{n-1}X_\bullet|; R) \simeq \widetilde{H}^k(\bigvee_{X_n} S^n, R)$$

and the right hand side is something we can acually calculate. Thus $H^k(|X_\bullet|, R)$ is built up from the cohomology of the smaller-dimensional $|\operatorname{sk}_{n-1}X)\bullet|$ and something more explicit.

Proposition 29. Given a family $\{(X_\alpha, x_\alpha)\}_{\alpha\in J}$ of pointed spaces, the inclusions $(X_\alpha, x_\alpha) \to \bigvee_{\alpha\in J} X_\alpha$ induce an isomorphism

$$\widetilde{H}^k(\bigvee_{\alpha\in J} X_\alpha) \xrightarrow{\simeq} \prod_{\alpha\in J} \widetilde{H}^k(X_\alpha, R)$$

for all k.

Example 111. Given an arbitrary set J,

$$\widetilde{H}^k(\bigvee_J S^n, R) = \begin{cases}\prod_J R \simeq R^J & k = n\\ 0 & k \neq n\end{cases}$$

Remark. Given any Δ-set $X_\bullet$, recall the space $|X_\bullet|$ has distinguished maps $\Delta^n \to |X_\bullet|$. These give an inclusion map

$$C^\bullet(X_\bullet, R) \hookrightarrow C^\bullet(|X_\bullet|, R)$$

where on the left we have the combinatorially-defined complex associated to a Δ-set, and on the right we have the topologically-defined complex associated to a space. We shall see shortly how the cohomology modules of these two complexes relate.

Lecture 29

Lemma 42. Given a map $X_\bullet \to Y_\bullet$ of Δ-sets, the following square commutes:

$$\begin{tikzcd} C^\bullet(Y_\bullet, R) \arrow[r] \arrow[d] & C^\bullet(|Y_\bullet|, R) \arrow[d] \\ C^\bullet(X_\bullet, R) \arrow[r] & C^\bullet(|X_\bullet|, R) \end{tikzcd}$$

If we consider the inclusion map $\mathrm{sk}_{n-1} X_\bullet \to X_\bullet$, this induces a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & C^\bullet(X_\bullet, \mathrm{sk}_{n-1} X_\bullet; R) \arrow[r] \arrow[d] & C^\bullet(X_\bullet, R) \arrow[r] \arrow[d] & C^\bullet(\mathrm{sk}_{n-1} X_\bullet, R) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^\bullet(|X_\bullet|, |\mathrm{sk}_{n-1} X_\bullet|; R) \arrow[r] & C^\bullet(|X_\bullet|, R) \arrow[r] & C^\bullet(|\mathrm{sk}_{n-1} X_\bullet|, R) \arrow[r] & 0 \end{tikzcd}$$

and so we get map of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & H^{k-1}(\mathrm{sk}_{n-1} X_\bullet, R) \arrow[r] \arrow[d] & H^k(X_\bullet, \mathrm{sk}_{n-1} X_\bullet; R) \arrow[r] \arrow[d, "(*)"] & H^k(X_\bullet, R) \arrow[r] \arrow[d] & H^k(\mathrm{sk}_{n-1} X_\bullet, R) \arrow[r] \arrow[d] & H^{k+1}(X_\bullet, \mathrm{sk}_{n-1} X_\bullet; R) \arrow[r] \arrow[d, "(*)"] & \cdots \\ \cdots \arrow[r] & H^{k-1}(|\mathrm{sk}_{n-1} X_\bullet|, R) \arrow[r] & H^k(|X_\bullet|, |\mathrm{sk}_{n-1} X_\bullet|; R) \arrow[r] & H^k(|X_\bullet|, R) \arrow[r] & H^k(|\mathrm{sk}_{n-1} X_\bullet|, R) \arrow[r] & H^{k+1}(|X_\bullet|, |\mathrm{sk}_{n-1} X_\bullet|; R) \arrow[r] & \cdots \end{tikzcd}$$

Fact. For $X_\bullet$ an n-dimensional Δ-set, the induced map $H^k(X_\bullet, \mathrm{sk}_{n-1} X_\bullet; R) \to \widetilde{H}^k(\bigvee_{X_n} S^n, R)$ is an isomorphism for all k. Why? If $k \neq n$, then both are zero, and the result is trivial. If $k = n$, then the map is

$$\prod_{X_n} R \simeq H^n(X_\bullet, \mathrm{sk}_{n-1} X_\bullet; R) \to \widetilde{H}^n(\bigvee_{X_n} S^n, R) \simeq \prod_{X_n} \widetilde{H}^k(S^n, R) \simeq \prod_{X_n} R$$

and moreover this map is the product of X_n-many maps $R \to R$ all arising as $H^n(\Delta[n], \partial\Delta[n]; R) \to H^n(S^n, R)$. Hatcher explicitly calculates this map to be an isomorphism.

Thus both of the downward maps labelled as $(*)$ in the big diagram above are isomorphisms. The relation between the combinatorial and the topological cohomologies is in fact the best possible:

Theorem 17. For a finite-dimensional Δ-set $X_\bullet$, $H^k(X_\bullet, R) \xrightarrow{\simeq} H^k(|X_\bullet|, R)$ for all k.

Proof. Use induction on the dimension of $X_\bullet$, as this allows us to assume that for all k, the result holds for $\operatorname{sk}_{n-1} X_\bullet$, and then we can apply the 5 Lemma to the map of long exact sequences. □

Remark. The result is also true for relative cohomology, although the induction is a bit tricker, and we have to set up the map of long exact sequences differently. As a result, it is also true for reduced cohomology.

Corollary 18. For *all* Δ-sets $X_\bullet$,

$$H^k(X_\bullet) \xrightarrow{\simeq} H^k(|X_\bullet|, R) \tag{8}$$

for all k..

Proof. We saw earlier that $H^k(X_\bullet, R) \xrightarrow{\simeq} H^k(\operatorname{sk}_n X_\bullet, R)$ for all $k < n$, and similarly for the relative cohomology, so the cohomology on the combinatorial side can be calculated using a finite-dimensional sub-Δ-set. On the topological side, it is in fact true that every $\Delta^k \to |X_\bullet|$ factors through some $|\operatorname{sk}_n X|$. However, it might not be the same n-dimensional approximation for every map. However, one can stil show the desired result using the technology of *filtered colimits*, which would take us too far for the present purposes. □

Remark. 1. The LHS of (8) is only functorial for maps of Δ-sets but the RHS is functorial for all continuous maps, so this is not an isomorphism of the cohomology functors, which have different domains.

2. However, to calculate the cohomology of an *individual* space, it is very useful.

3. If the two Δ-sets $X_\bullet$, $Y_\bullet$ have homeomorphism geometric realisations $|X_\bullet|$, $|Y_\bullet|$ (and there may be no map of Δ-sets in either direction!), then

$$H^k(X_\bullet, R) \simeq H^k(|X_\bullet|, R) \simeq H^k(|Y_\bullet|, R) \simeq H^k(Y_\bullet, R)$$

4. More generally, if $|X_\bullet|$ and $|Y_\bullet|$ are merely homotopy equivalent, then $X_\bullet$ and $Y_\bullet$ have isomorphic cohomology modules.

To properly state the last big theorem for this section, we need a minor digression on a certain class of spaces that includes all triangulable spaces, but is more general. Recall that the geometric realisation of a Δ-set is defined as a quotient of $\coprod_{n=0}^{\infty} \mathrm{disc}(X_n) \times \Delta^n$, and moreover, one has the sequence of subspaces $|sk_n X_\bullet| \subseteq |X_\bullet|$, such that $|sk_{n+1} X_\bullet|$ can be obtained by a quotient of $\left(\mathrm{disc}(X_{n+1}) \times \Delta^{n+1}\right) \sqcup |\,\mathrm{sk}_n X_\bullet|$. However, simplices are very rigid in how they are glued together, in that the combinatorics of all the lower-dimensional faces have to agree. The following definition takes a more flexible approach, that captures many more examples.

recall that Δ^n is homeomorphic to D^n and $\partial\Delta^{n+1}$ is homeomorphic to S^n

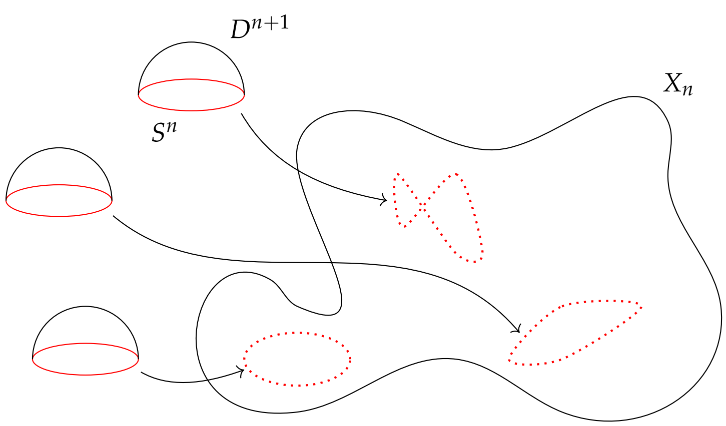
Definition 63. A *CW-complex* structure on a space is a homeomorphism to one built of the form $\bigcup_{n=0}^{\infty} X_n$ where $\cdots \hookrightarrow X_n \hookrightarrow X_{n+1} \hookrightarrow \cdots$ are a sequence of subspace inclusions where:

given such subspace inclusions $i_n \colon X_n \to X_{n+1}$ for all n, define $\bigcup_n X_n$ to be $(\coprod_n X_n) / \sim$ where for $x \in X_n$, $x \sim i_n(x)$

- X_0 is a discrete space
- for all $n \geq 1$ there is a set J_n, a map $j_n \colon \mathrm{disc}(J_n) \times S^n \to X_n$ such that the following is a pushout square

$$\begin{tikzcd}
\mathrm{disc}(J_n) \times S^n \arrow[r, "j_n"] \arrow[d, "\mathrm{id} \times \iota_n"'] & X_n \arrow[d] \\
\mathrm{disc}(J_n) \times D^{n+1} \arrow[r] & X_{n+1}
\end{tikzcd}$$

for $\iota_n \colon S^n \hookrightarrow D^{n+1}$ the boundary inclusion. That is, $X_{n+1} := (\mathrm{disc}(J_n) \times D^{n+1} \sqcup X_n)/\sim$.



For simplicity, any space constructed as in the definition together with the data constructing it is called a CW-complex. Note that for each $\alpha \in J_n$, we get a map $S^n \hookrightarrow \mathrm{disc}(J_n) \times S^n \to X_n$, and so we can consider the maps j_n as encoding a family of maps $S^n = \partial D^{n+1} \to X_n$ which we want to use to attach copies of D^{n+1} to X_n along their boundary. These maps $S^n \to X_n$ are called *attaching maps*. We define the category *CW* to consist of spaces with a CW-complex structure and with arbitrary continuous maps between them.

Example 112. Any triangulation $X \simeq |X_\bullet|$ gives a CW-complex structure on X.

Thus the geometric realisation functor is really $|-|\colon \Delta\mathbf{Set} \to CW$.

Example 113. Any compact manifold of dimension not 4 has a CW-complex structure, and moreover every compact manifolds is homotopy equivalent to a CW-complex.

Since we want to talk about relative cohomology, we have a certain class of pairs (X, A) that we are interested in.

Definition 64. A *CW-pair* (X, A) consists of a CW-complex X together with a subspace $A \subseteq X$ that is also a CW-complex built by considering subsets $K_n \subseteq J_n$ of the attaching maps for X.

One can prove by induction that for a CW-pair, $A \subseteq X$ is a closed subspace.

Example 114. Given a pair $(X_\bullet, A_\bullet)$ of Δ-sets, the geometric realisation $(|X_\bullet|, |A_\bullet|)$ is a CW-pair.

There is a category $CW^{(2)}$ whose objects are CW-pairs and whose maps are maps of pairs: $(X, A) \to (Y, B)$ is a map $f\colon X \to Y$ such that $f(A) \subseteq B$. The category CW includes into $CW^{(2)}$ via $X \mapsto (X, \varnothing)$. CW-pairs (X, A) have the property that there is a nhd $U \supset A$ in X of which A is a deformation retract, hence we have isomorphisms $\widetilde{H}^k(X/A, R) \xrightarrow{\simeq} H^k(X, A; R)$ for all k.

We can of course talk about homotopy of maps between spaces with CW-complex structure, and even homotopies between maps of pairs, which are required to be maps of pairs at each intermediate point of the homotopy. Thus we can define a category $hCW^{(2)}$ where morphisms are homotopy equivalence classes of maps of pairs (and the analogous category hCW where we don't take pairs).

Theorem 18. (Eilenberg–Steenrod 1945) Let

$$h^k\colon \left(hCW^{(2)}\right)^{op} \to \mathbf{Mod}_R$$

be a sequence of functors, for $k \in \mathbb{Z}$, such that

define $h^k(X) := h^k(X, \varnothing)$

1. For every family $\{X_\alpha\}_{\alpha \in J}$ of CW complexes, $h^k(\coprod_{\alpha \in J} X_\alpha) \xrightarrow{\simeq} \prod_{\alpha \in J} h^k(X_\alpha)$ for all k;
2. For all CW-pairs (X, A) there is a natural map $h^k(A) \to h^{k+1}(X, A)$ and a long exact sequence

$$\cdots \to h^k(X, A) \to h^k(X) \to h^k(A) \to h^{k+1}(X, A) \to \cdots$$

3. Given a CW-pair (X, A) and a subspace $Z \subset A$ such that $\overline{Z} \subset \operatorname{int}(A)$, the inclusion induces an isomorphism $h^k(X, A) \xrightarrow{\simeq} h^k(X \setminus Z, A \setminus Z)$ for all k;

4. $h^0(\mathrm{pt}) \simeq R$ and $h^k(\mathrm{pt}) = 0$ for $k \neq 0$;

then there is a natural isomorphism $h^k \simeq H^k(-,-;R)$, where $H^k(-,-;R)$ is the restriction of relative cohomology of spaces to CW-pairs.

or rather the induced functor on the homotopy category

Note that we can get a reduced version $\widetilde{h}^k$ of h^k, and in this case $\widetilde{h}^k \simeq \widetilde{H}^k$ as well. We can derive, from the above axioms, all of the exact sequences and properties of cohomology, so the construction of singular cohomology can be seen as an existence proof of a series of functors satisfying the Eilenberg–Steenrod theorem. After that, one can usually just work with the abstract properties.

Classical applications

Lecture 30

In the remaining time, we will give some applications of the techniques we now have to classical problems, namely the existence of fixed points of maps $D^n \to D^n$, the proof of the fundamental theorem of algebra, and the existence of non-vanishing vector fields on spheres.

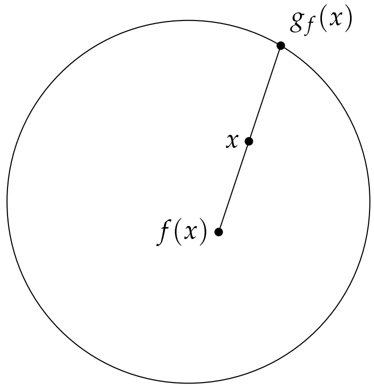
Recall that the contraction mapping theorem implies that for any endomorphism $f\colon D^n \to D^n$ with the property that $|f(x) - f(y)| \leq C|x - y|$, for some uniform constant $C \in (0, 1)$, there is a fixed point $x_0 \in D^n$: $f(x_0) = x_0$. But there are many endomorphisms that aren't contractions, and which have fixed points; for instance, rotations about 0. Alternatively, the function f could be constant in some small region (hence with many fixed points), but move other nearby points far apart. So what happens in general? This is answered by the following theorem. Let us call an endomorphism $f\colon X \to X$ of any space X *free* if $f(x) \neq x$ for all $x \in X$.

and in fact exactly one fixed point

we are of course only considering continuous endomorphisms

Theorem 19. (Brouwer fixed-point theorem) No endomorphism $f\colon D^n \to D^n$ is free.

Proof. Define the map $g_f\colon D^n \to \partial D^n = S^{n-1}$ by the following picture



Then g_f is continuous[2] and moreover if $x \in \partial D^n$ then $g_f(x) = x$. If i denotes the inclusion $\partial D^n \hookrightarrow D^n$, then we have $g_f \circ i = \mathrm{id}_{\partial D^n}$. Apply the funtor $H^{n-1}(-, \mathbb{Z})$ to get

[2] Exercise! Try using a sequential characterisation

$$\mathbb{Z} \simeq H^{n-1}(\partial D^n, \mathbb{Z}) \xrightarrow{i^*} H^{n-1}(D^n, \mathbb{Z}) \xrightarrow{g_f^*} H^{n-1}(\partial D^n, \mathbb{Z}) \simeq \mathbb{Z},$$

which is the identity map on $\mathbb{Z}$. But $H^{n-1}(D^n, \mathbb{Z}) = 0$, hence a contradiction, and so there is so such endomorphism. □

Remark. Usually this is stated as "every endomorphism has a fixed point", though the proof directly shows that no endomorphism can fail to have a fixed point. The location of the fixed point can jump discontinuously given a continuous family of endomorphisms, and so there is no 'method' to construct the fixed point, unlike in the case of the contraction mapping theorem. In that case, the proof constructs a Cauchy sequence converging to the (unique) fixed point, but here the fixed point set can have rather wild behaviour.

the distinction is important in both non-classical logic and numerical/computational settings

Another even more classical result is the fundamental theorem of algebra. The proof requires *some* topological input and here we will implicitly use the fundamental group of the space of non-zero complex numbers $\mathbb{C}^\times$, which is isomorphic to $\mathbb{Z}$. I will again state this in a slightly non-standard way, reflecting the actual content of the proof. Recall that a *monic* polynomial is one with the coefficient of the leading term equal to 1. We will consider polynomials with complex coefficients.

the minimum required seems to be that of a *real-closed field* k: ordered, and every odd-degree polynomial has a root. Given such a field, the extension $k[\sqrt{-1}]$ is then algebraically closed. The intermediate value theorem guarantees this for $k = \mathbb{R}$.

Theorem 20. (Fundamental Theorem of Algebra) A non-constant monic polynomial function $p\colon \mathbb{C} \to \mathbb{C}$ cannot factor through the inclusion $\mathbb{C}^\times \hookrightarrow \mathbb{C}$.

Proof. For fixed $R \gg 0$, then $p_R(\theta) := p(Re^{i\theta}) \approx R^n e^{in\theta} =: q(\theta)$. Even better, there is a homotopy between p_R and q, as functions $S^1 \to \mathbb{C}^\times$.

Note that q is not homotopic to a constant function, using a path-lifting argument through the covering map $\exp\colon \mathbb{C} \to \mathbb{C}^\times$. Now assume p factors through $\mathbb{C}^\times$. There is then a homotopy between p_R and the constant function at $p(0)$, as functions $S^1 \to \mathbb{C}^\times$, via $(t, \theta) \mapsto p((1-t)Re^{i\theta})$. Thus we get a contraction, as this would imply q is homotopic to the constant function at $p(0)$, and so p cannot land in $\mathbb{C}^\times$. □

The last result belongs to the area of differential topology, which is the intersection between differential geometry and topology. Recall that there is a nonvanishing vector field on S^1 given by translating the unit tangent vector at 1. More concretely, we can take $S^1 \subset \mathbb{R}^2$, and the tangent vector at (x, y) to be $(-y, x)$. One can ask if it is possible to find a nonvanishing vector field on other spheres . The nickname of the following theorem comes from the case of S^2, where one can visualise a vector field as being given by little hairs, the tangency condition corresponding to asking the hair be combed flat.

or even how many, which is a harder problem!

Theorem 21. (Hairy sphere theorem) There exists a nonvanishing vector field on S^n if and only if n is odd.

The proof requires some additional technology, which I will give without proofs. The only difficult part is the lemma below, a proof can be found in Hatcher. Note that we can assume the vector field on has length 1 everywhere, and this allows us to think of the vector field as a map $S^n \to S^n$, as we can identify the unit sphere in $T_pS^n \subset \mathbb{R}^{n+1}$ with S^n. Then, given such a map, it induces a homomorphism $f^*\colon \mathbb{Z} \simeq H^n(S^n, \mathbb{Z}) \to H^n(S^n, \mathbb{Z}) \simeq \mathbb{Z}$.

Definition 65. Given a function $f\colon S^n \to S^n$, define the *degree* $\deg(f)$ of f to be the integer $f^*(1) \in H^n(S^n, \mathbb{Z}) \simeq \mathbb{Z}$.

The degree of a map has the following properties, which follow quickly from the definition.

1. If f is not surjective, then $\deg(f) = 0$, as it factors through a chart, which is contractible;
2. $\deg(\mathrm{id}_{S^n}) = 1$;
3. $\deg(g \circ f) = \deg(g)\deg(f)$;
4. If f is homotopic to g, then $\deg(f) = \deg(g)$;

As a result, deg gives a map of monoids $[S^n, S^n] \to \mathrm{End}(\mathbb{Z}) \simeq (\mathbb{Z}, \times)$

Lemma 43. Define the coordinate reflection map $r_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1})$. Then $\deg(r_i) = -1$.

Hatcher gives a proof using an explicit calculation with a triangulation of S^n using two copies of $\Delta[n]$ that are swapped under r_i.

if you have seen de Rham cohomology, then the degree of this map can be seen as arising from the reversal of orientation of S^n, and the resulting sign change in the global volume form

Corollary 19. For the antipodal map $-\operatorname{id}_{S^n}$, $\deg(-\operatorname{id}_{S^n}) = (-1)^{n+1}$.

Proof. (of Theorem 21) Assume we have $v \colon S^n \to S^n$ corresponding to a nonvanishing vector field. We can think of this as a map $v \colon S^n \to §^n \hookrightarrow \mathbb{R}^{n+1}$ satisfying $v(x) \cdot x = 0$ for all $x \in S^n$. Define the map

$$\begin{aligned} h \colon I \times S^n &\to \mathbb{R}^{n+1} \\ (t, x) &\mapsto \cos(\pi t)x + \sin(\pi t)v(x) \end{aligned}$$

Then $h(t,x) \cdot h(t,x) = 1$, so is a map $I \times S^n \to S^n$. Moreover, $h(0,x) = x$, and $h(1,x) = -x$ so that h is a homotopy between id_{S^n} and $-\operatorname{id}_{S^n}$. Sinc degree is a homotopy invariant, this implies that $1 = \deg(\operatorname{id}_{S^n}) = \deg(-\operatorname{id}_{S^n}) = (-1)^{n+1}$. Thus a nonvanishing vector field can only exist if n is odd.

Conversely, if $n = 2k-1$, for $x \in S^{2k-1} \subset \mathbb{R}^{2k}$ define

$$v(x) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{2k}, x_{2k=1})$$

which gives a map $S^{2k-1} \to S^{2k-1}$ such that $v(x) \cdot x = 1$, hence is a nonvanishing vector field. This looks like the vector field on the circle above on each circle $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ (k times).

$\square$