Algebraic Topology[1]

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2019

Lecture 1

What is it?

Algebraic topology is the study of maps

$$\{\text{'Spaces'}\} \longrightarrow \{\text{Algebraic objects}\}$$

Or rather, 'well-behaved' such maps—they should also send maps between spaces to algebraic maps, respecting composition (so: *functors*); they should send spaces built out of smaller units to algebraic objects built out of smaller units, in a compatible way, etc.

Here, 'Spaces' roughly means topological spaces up to deformation (usually homotopy, but not always). Such equivalence classes are called *homotopy types*. 'Algebraic objects': (abelian) groups, rings, modules, chain complexes ($\cdots \to V_0 \to V_1 \to V_2 \to \cdots$).

Example 1. How can we tell if the sphere S^2 and the torus $S^1 \times S^1$ can or cannot be deformed into each other? How would you prove it cannot be done?

Example 2. For a positive example, we *can* squash $\mathbb{R}^3 \setminus \{0\} \to S^2 \hookrightarrow \mathbb{R}^3 \setminus \{0\}$, sending $x \mapsto \frac{x}{|x|}$. This map continuously deforms to the identity map. So dimension not necessarily preserved.

Example 3. Can we have $S^1 \sim S^2$?

We first need to understand how spaces are built

Topological spaces

Recall. . .

From Topology and Analysis III

Definition 1. A *topology* on a set S is a collection $\mathcal{T}$ of subsets of X such that

1. $\emptyset, X \in \mathcal{T}$
2. If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$
3. If $U_\alpha \in \mathcal{T}$, $\alpha \in I$ is an arbitrary family, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

If $U \in \mathcal{T}$ we say U is *open*. A *topological space* is a set X eqipped with a topology $\mathcal{T}$.

Example 4. Take the set of real numbers, the *Euclidean ('usual') topology* is defined by saying a set is open iff it is a union of open intervals (a, b) (including the union of no sets ie $\emptyset$).
The *discrete topology* on a set X is defined by taking every $\mathcal{T}$ to consist of all subsets. The *indiscrete topology* is defined by taking $\mathcal{T}$ to consist of just $\emptyset$ and X.

This definition is concise, but not always the best way to define a topology. We will also use *neighbourhoods*

Definition 2. A set $N \subseteq X$ is a *neighbourhood* (in a given topology $\mathcal{T}$) of a point $x \in X$ if there is an open set $U \subseteq N$ with $x \in U$.

'nhd' is a good abbreviation

Example 5. Take $\mathbb{R}$ with the Euclidean topology. $(-1, 1)$, $[-1, 1]$, $[-1, 1)$ are all neighbourhoods of every $-1 < x < 1$, but $[0, 1)$ is not a neighbourhood of 0. More complicated: $[0, 1] \cup \{2\} \cup [5, 6]$ is a nhd of all $0 < x < 1$ and $5 < x < 6$.

Example 6. Consider a metric space (X, d). The *metric topology* is defined by saying a subset $U \subseteq X$ is open iff for every $x \in U$ there is some $\varepsilon_x > 0$ with the open ball $B(x, \varepsilon_x) \subseteq U$. Open balls around x are neighbouhoods of x, as are closed balls.

Here is a more concrete approach that allows concise definitions of topologies:

Definition 3. A *neighbourhood base* $\mathcal{N}$ on a set X is a family $\{\mathcal{N}(x)\}_{x \in X}$ where each $\mathcal{N}(x)$ is a nonempty collection of subsets of X, satisfying the following, for all $x \in X$:

1. For all $N \in \mathcal{N}(x)$, $x \in N$;
2. For all $N_1, N_2 \in \mathcal{N}(x)$, there is some $N \in \mathcal{N}(x)$ with $N \subseteq N_1 \cap N_2$;
3. For all $N \in \mathcal{N}(x)$ there is a subset $U \subseteq N$ such that for all $y \in U$, there is some $V \in \mathcal{N}(y)$ such that $V \subseteq U$.

We say the sets in $\mathcal{N}(x)$ are *basic neighbourhoods* of x.

As an example: given a topological space $(X, \mathcal{T})$ defining $\mathcal{N}(x)$ to consist of all nhds of x gives a nhd base. Similarly, defining $\mathcal{N}'(x)$ to consist of all open sets containing x defines a nhd base.

Given a neighbourhood base $\mathcal{N}$ on a set X, define a subset $U \subseteq X$ to be $\mathcal{N}$*-open* iff for all $x \in U$, there is an $N \in \mathcal{N}(x)$ with $N \subseteq U$.

Proposition 1. The $\mathcal{N}$-open sets define a topology on X.

Proof. We verify the axioms for a topology on X.

1. The condition that $\emptyset$ is $\mathcal{N}$-open is vacuously true. And since $\mathcal{N}(x)$ is not empty, there is a basic nhd around every point, so X is $\mathcal{N}$-open.
2. Given U, V both $\mathcal{N}$-open, we want to show $U \cap V$ is $\mathcal{N}$-open. So take $x \in U \cap V$. We know there is $N_U, N_V \in \mathcal{N}(x)$ with $N_U \subseteq U$ and $N_V \subseteq V$, and also that $x \in N_U \cap N_V$, since it is in each of them. Thus there is some $N \in \mathcal{N}(x)$ with $N \subseteq N_U \cap N_V \subseteq U \cap V$, and this is true for all $x \in U \cap V$. Hence $U \cap V$ is $\mathcal{N}$-open.
3. Given a family U_α, $\alpha \in I$, with each U_α $\mathcal{N}$-open, we want to show $U := \bigcup_{\alpha \in I} U_\alpha$ is $\mathcal{N}$-open. Take $x \in U$, so there is some α_0 with $x \in U_{\alpha_0}$. But this set in $\mathcal{N}$-open, so there is some nhd N of x with $N \subseteq U_{\alpha_0} \subseteq U$, and this is true for all $x \in U$. So U is $\mathcal{N}$-open. □

We call the topology from this proposition the topology generated by $\mathcal{N}$. Neighbourhoods in this topology are sets that contain a basic neighbourhood: V is a neighbourhood of x if there is some $N \in \mathcal{N}(x)$ with $N \subseteq V$.

Given a neighbourhood base $\mathcal{N}$ on X, we can identify the *closure* of a set $S \subset X$ as the collection of points $x \in X$ such that for all $N \in \mathcal{N}(x)$, $\exists s \in N \cap S$.

Example 7. Given a metric space (X, d) the open balls form a nhd base on X and the topology they generate is the metric topology.

Hence many definitions you are familiar with from metric spaces work for topological spaces, if they can be phrased in terms of basic nhds. In particular, continuity!

Definition 4. Let $\mathcal{N}_X$ and $\mathcal{N}_Y$ be neighbourhood bases on sets X and Y respectively. A function $f\colon X \to Y$ is *continuous* if for every $x \in X$ and $N \in \mathcal{N}_Y(f(x))$, the set $f^{-1}(N)$ contains a basic nhd of x.

This is a big generalisation of the ε-δ definition of continuity.

Exercise 1. Show that if $f\colon (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ is continuous as just defined, it is continuous for the topologies generated on X and Y by these nhd bases.

Recall a function is continuous for topologies if $f^{-1}(U)$ is open for all open U.

As a sanity check, the identity function id_X on a space X is indeed continuous.

You can check every function *to* an indiscrete space is continuous, as is every function *on* a discrete space

Definition 5. A continuous function $f\colon X \to Y$ is a *homeomorphism* if there is a continuous function $g\colon Y \to X$ with $g \circ f = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$. We then call X and Y *homeomorphic* if there is a homeomorphism between them.

or just isomorphic, if I'm being lazy

Now we need to show how to build new spaces, and continuous maps relating them to the original spaces.

Definition 6. Let X be a set, $(Y_\alpha, \mathcal{N}_\alpha)$, $\alpha \in I$ a family of sets with nhd bases (not necessarily all unique), and $f_\alpha\colon X \to Y_\alpha$ a family of functions. The *initial topology* on X is generated by the following nhd base: a subset of X is a basic nhd of x iff it is of the form $f_{\alpha_1}^{-1}(N_1) \cap \ldots \cap f_{\alpha_k}^{-1}(N_k)$ for some $\alpha_1, \ldots, \alpha_k$ and $N_i \in \mathcal{N}_{\alpha_i}(f_{\alpha_i}(x))$.

Exercise: verify this is a nhd base!

This generalises the product topology, which is the case that $X = Y_1 \times Y_2$, and $f_i\colon X \to Y_i$ is the projection $f_i(y_1, y_2) = y_i$, where $i = 1, 2$. But this *also* gives the subspace topology: take $f\colon X \hookrightarrow Y$ to be injective and define the initial topology on X.

Lemma 1. Giving X the initial topology, all the functions $f_\alpha\colon X \to Y_\alpha$ are continuous. Moreover, a function $k\colon Z \to X$ is continuous iff $f_\alpha \circ k\colon Z \to Y_\alpha$ is continuous for every α.

The following will be even more important for us, and will be new to most.

Definition 7. Let X be a set, $(Z_\beta, \mathcal{N}_\beta)$, $\beta \in J$ a family of topological spaces (not necessarily all unique), and $g_\beta\colon Z_\beta \to X$ a family of functions (note the other direction!). The *final topology* on X has open sets as following: $U \subset X$ is open iff for all $\beta \in J$, $g_\beta^{-1}(U)$ is open in Z_β.

Lemma 2. Giving X the final topology, all the functions $g_\beta\colon Z_\beta \to X$ are continuous. Moreover a function $h\colon X \to W$ is continuous for the final topology on X iff $h \circ g_\beta\colon Z_\beta \to W$ is continous for every $\beta \in J$.

We will give two special cases of this, and we will see them often.

Example 8. Let $(Z, \mathcal{T})$ be a topological space, and let $\sim$ be an equivalence relation on Y, and define $X = Z/\sim$ to be the quotient by this relation. There is a function $\pi\colon Z \to X$ sending $y \mapsto [y]$. The final topology on X has as open sets those $U \subseteq X$ such that $\pi^{-1}(U)$ is open in Z.

For instance, we can give S^2 the initial topology for the maps $x, y \colon S^2 \to \mathbb{R}^2 \to \mathbb{R}$ (this is the usual topology on S^2), and then define an equivalence relation on S^2 by $x \sim y$ iff $x = -y$. The quotient is $\mathbb{RP}^2$, the real projective plane, and we give it the final topology coming from $S^2 \to \mathbb{RP}^2$. This is the topology it carries as a manifold. Incidentally, S^2 is an example of a *covering space* of $\mathbb{RP}^2$, the study of which will occupy the first section of the course.

Lecture 2

Recall the definition of disjoint union of sets: given Z_β, $\beta \in J$, a family of sets, we have $\mathrm{in}_\gamma \colon Z_\gamma \hookrightarrow \coprod_\beta Z_\beta$ with $Z_\beta \cap Z_\gamma = \varnothing$ for $\beta \neq \gamma$. If Z_β are spaces, then we give $\coprod_\beta Z_\beta$ the final topology for the maps in_γ. This is *disjoint union* or *sum* topology, and $\coprod_\beta Z_\beta$ is called the *topological sum*. A point in $\coprod_\beta Z_\beta$ can be described by a pair (β, z), where $z \in Z_\beta$.

an important fact is that the map $\coprod_\beta X \times Z_\beta \to X \times \coprod_\beta Z_\beta$ is a homeomorphism (exercise!)

Exercise 2. Given functions $h_\beta \colon Z_\beta \to W$, there is a unique function $h = \langle h_\beta \rangle \colon \coprod_\beta Z_\beta \to W$ with $h_\beta = h \circ \mathrm{in}_\beta$, or in other words this diagram commutes:

$$\begin{tikzcd} Z_\gamma \arrow[r, "\mathrm{in}_\gamma"] \arrow[dr, "h_\gamma"'] & \coprod_\beta Z_\beta \arrow[d, "h"] \\ & W \end{tikzcd}$$

Lemma 3. The final topology on X for $g_\beta \colon Z_\beta \to X$ agrees with the final topology on X for $g = \langle g_\beta \rangle \colon \coprod_\beta Z_\beta \to X$, using the sum topology.

Proof. We have that $U \subseteq X$ is open iff $\forall \beta$ $g_\beta^{-1}(U)$ is open iff $\forall \beta$, $(g \circ \mathrm{in}_\beta)^{-1}(U) = \mathrm{in}_\beta^{-1}\left(g^{-1}(U)\right)$ is open iff $g^{-1}(U)$ is open in the sum topology. $\square$

The idea the behind final topology, when $g_\beta \colon Z_\beta \to X$ are jointly surjective, is that we can put an equivalence relation on $\coprod_\beta Z_\beta$ with $(\beta_1, z_1) \sim (\beta_2, z_2)$ iff $g_{\beta_1}(z_1) = g_{\beta_2}(z_2) \in X$. As a set, X is the set of equivalence classes under this relation, so you can think of it as gluing together the *underlying sets* of the spaces Z_β. The final topology on X is then the only sensible topology to described the space we get by gluing together the *spaces* Z_β.

this means $\forall x \in X, \exists \beta, x \in Z_\beta$ with $g_\beta(z) = x$

Exercise 3. Given an open cover $\{U_\alpha\}$ of a space X, then X carries the final topology for the inclusion maps $U_\alpha \hookrightarrow X$, or equivalently for the map $\coprod_\alpha U_\alpha \to X$.

Exercise 4. Given a *finite* closed cover $\{V_i\}_{i=1}^n$ of X, then X carries the final topology for $\coprod_{i=1}^n V_i \to X$.

Later we'll see spaces that are built up by gluing together lots of 'simple' spaces, like disks $D^n := \{x \in \mathbb{R}^n \mod |x| \leq 1\}$ (with the subspace topology from $\mathbb{R}^n$). But what does 'simple' here mean? Roughly, "shrinkable to a point".

Homotopy

"Shrinkable" implies a kind of continuous process in time. Consider the function $I \times D^n \to D^n$. Consider the map

$$\begin{aligned} H\colon I \times D^n &\to D^n \\ (t, \mathbf{x}) &\mapsto (1-t)\mathbf{x} \end{aligned}$$

Note that this gives maps $H_0\colon D^n \to D^n$ (the identity map) and H_1 (constant at 0). The function H is continuous! How should we see this? The topology on D^n is the subspace topology $D^n \subset \mathbb{R}^n$, and $\mathbb{R}^n$ has the product topology. It is not too difficult to see (Exercise!) that the topology on D^n is the final topology for the coordinate functions $x_i\colon D^n \to \mathbb{R}^n \to \mathbb{R}$. So $H\colon I \times D^n \to D^n$ is continuous iff

And $I \subset \mathbb{R}$ has subspace topology

$$\begin{array}{ccccccc} I \times D^n & \xrightarrow{\mathrm{id} \times x_i} & I \times \mathbb{R} & \longrightarrow & \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (t, \mathbf{x}) & \longmapsto & & & (t, x_i) & \longmapsto & tx_i \end{array}$$

But $I \times D^n \to \mathbb{R} \times \mathbb{R}$ is continuous by definition of final topology, and the following result:

Exercise 5. If $f\colon X \to W$ and $g\colon X \to Z$ are continuous, then so is $f \times g\colon X \times Y \to W \times Z$. If both X and Y have at least one point each, then the reverse implication also holds.

So if we can prove that multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, then H is continuous. But the standard topology on $\mathbb{R}$ comes from the metric space structure, so can use sequential criterion for continuity. Take $(a_n, b_n) \to (a, b)$ in $\mathbb{R} \times \mathbb{R}$, then:

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n - a|\,|b_n| + |a|\,|b_n - b| \\ &\leq |a_n - a| \sup |b_n| + |a|\,|b_n - b| \qquad \text{(as } (b_n) \text{ converges, it is bounded)} \\ &\to 0 + 0 \end{aligned}$$

Hence H is continuous.

Definition 8. A space X is *contractible* if there is a point $x_0 \in X$ and a continuous function $H\colon I \times X \to X$ such that $H(0, x) = x$ and $H(1, x) = x_0$ for all $x \in X$. Such a function is called a *contraction*.

We have shown D^n is contractible.

Exercise 6. $\mathbb{R}$ is contractible. A finite product of contractible spaces is contractible.

Example 9. Consider what it would mean if a discrete space S were contractible: there would be an element $* \in S$ and a continuous function $h\colon I \times S \to S$ such that $h(0,s) = s$ and $h(1,s) = *$. Restricting h to $I \times \{s\}$ for some given s, we get a continuous function $I \hookrightarrow I \times S \to S$, whose range includes $*$ and s. Since all functions with discrete domain are continuous, let us compose with the continuous function $\chi_{\{*\}}\colon S \to \mathbb{R}$ that sends $* \mapsto 1$ and $s \mapsto 0$ for all $s \neq *$. So we have a continuous function $\widetilde{h}\colon I \to \mathbb{R}$ with $\widetilde{h}(0) = 0$ and range contained in $\{0,1\}$. By the intermediate value theorem, we must have $\widetilde{h}(1) = \chi_{\{*\}}(h(1,s)) = 0$, so that $h(1,s) = *$, and hence $s = *$ for all $s \in S$. Thus S has exactly one element.

The interval can only map continuously to a discrete space if it is constant at some element, or equivalently, its image consists of a single point, and this property is important enough to have a name.

Definition 9. A space X is *connected* if every continuous map to a discrete space has image a single point.

If you know the 'usual' definition, this is equivalent to it

This is our first example of an invariant of spaces, namely whether they are connected or not: a connected space X cannot be homeomorphic to a space Z that is not connected. But, how can we tell non-connected spaces apart?

Consider $\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d] & Z \arrow[dl] \\ S & \end{tikzcd}$ with S discrete.

Definition 10. 1. For any space X, a subset $Y \subseteq X$ is a *connected component* of X if Y is connected and the indicator function $1_Y\colon X \to \{0,1\}$ is continuous.

2. Put an equivalence relation on X with $x_1 \sim x_2$ iff x_1 and x_2 are both in a given connected component. Then define $\pi_0(X) = X/\sim$, the *set of connected components*. There is a continuous function $X \to \pi_0(X)$.

$1_Y(x) = 1$ for $x \in Y$, 0 otherwise

A connected space X has $\pi_0(X) = *$, but now we can tell apart non-connected spaces.

Lemma 4. Every space can be written as $X = \bigsqcup_{\alpha \in \pi_0(X)} X_\alpha$, with X_α connected.

As a result, we need to try to understand *connected* spaces, though we will still *use* non-connected spaces.

Can we get more out of the idea of contractions? Given $H\colon I \times X \to X$, we have maps H_i for $i = 0, 1$, namely $H_0 = \mathrm{id}_X$ and H_1 is constant at x_0. What if H_0 and H_1 were other sorts of continuous maps?

Example 10. Consider the annulus $A(r, R) := \{x \in \mathbb{R}^2 \mid r \leq |x| \leq R\}$, and the function $H(t, x) = ((1-t)r + tR)x/|x|$.

What if we considered general continuous maps $X \to Y$ instead of just $X \to X$?

Definition 11. A *homotopy* is a continuous function $H\colon I \times X \to Y$. If $f = H(0, -)$ and $g = H(1, -)$, we say H is a *homotopy from f to g*, and that f and g are *homotopic*, written $f \sim g$.

Example 10 gives a homotopy between the two 'retraction' maps $A(r, R) \to A(r, R)$, mapping points to the inner and outer circles respectively.

Algebraic topology most of the time considers functions *up to homotopy*, and also "spaces up to homotopy".

Definition 12. A continuous function $f\colon X \to Y$ is called a *homotopy equivalence* if there is a continuous function $g\colon Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. We then say X and Y are *homotopy equivalent*.

Example 11. A contractible space is homotopy equivalent to a one-point space.

You should think of homotopy equivalences as being 'kinda like isomorphism', but coarser. Going back to our original motivation, the assignment

$$\{\text{'Spaces'}\} \longrightarrow \{\text{Algebraic objects}\}$$

should take homotopy equivalent spaces to isomorphic algebraic objects. To make this more rigorous we will use the language of category theory. Lecture 3

Here is a super-important property of homotopies we will use continuously.

Proposition 2. Given homotopies $H\colon I \times X \to Y$ and $H'\colon I \times X \to Y$ such that $H_1 = H'_0\colon X \to Y$, there is a homotopy H'' from H_0 to H_1, and a homotopy $\widetilde{H}$ from H_1 to H_0.

Proof. We will use Exercise 4 applied to the closed cover $\{[0, \frac{1}{2}] \times X, [\frac{1}{2}, 1] \times X\}$ of $I \times X$. Since $I \simeq [0, \frac{1}{2}]$ and $I \simeq [\frac{1}{2}, 1]$, H and H' give

us maps $[0, \frac{1}{2}] \times X \simeq I \times X \xrightarrow{H} Y$ and $[0, \frac{1}{2}] \times X \simeq I \times X \xrightarrow{H'} Y$ respectively. By the assumption on H_1 and H'_0, we get a well-defined function $H''\colon I \times X \to Y$, which is then continuous by the Exercise. It is a simple check to see it is a homotopy from H_0 to H'_1.
For the second part, let $c\colon I \to I$ be the function $c(t) = 1 - t$. Then define H'' to be the composite $I \times X \xrightarrow{c \times \mathrm{id}_X} I \times X \xrightarrow{H} Y$, which has the required properties. □

Contractible spaces supply many homotopies.

Lemma 5. Every continuous function $f\colon X \to Y$, with Y a contractible space (say to $y_0 \in Y$), is homotopic to a function with range contained in $\{y_0\}$.

Proof. Let $H\colon I \times Y \to Y$ be a homotopy witnessing the contractility of Y. Then the composite $I \times X \xrightarrow{\mathrm{id}_I \times f} I \times Y \xrightarrow{H} Y$ is a homotopy from f to the the desired function. □

As a corollary, every pair of functions to a contractible space are homotopic. Since contractible spaces are in some sense trivial, maps to them are in the same sense trivial.

An important intermediate version of this is when we consider only the case where X is discrete, or is even just pt:

Definition 13. A space Y is *path-connected* if every map $\mathrm{pt} \to Y$ is homotopic to every other such map.

This condition is equivalent to requiring it for *all* discrete spaces in place of pt (Exercise!)

Unpacking this, we see this means that for any two points $\mathrm{pt} \to Y$ there is a path $I \to Y$ connecting them, i.e. $H\colon I \simeq I \times \mathrm{pt} \to Y$.

Proposition 3. A path-connected space is connected

Let us define $[X, Y] = \{\text{continuous } f\colon X \to Y\}/\text{homotopy}$. The set of *path components* of Y is then the set $[\mathrm{pt}, Y]$. The space Y is called *path connected* if $[\mathrm{pt}, Y] = *$.

We have been discussing topological spaces and continuous maps, but also implicitly sets and functions, not necessarily continuous, and passing between these two pictures. In both cases we have composition that is associative, and identity maps. Later we shall be using different classes of topological spaces in order to ensure the behaviour we require will hold.

Definition 14. A *category* $\mathcal{C}$ consists of a collection of *objects* $W, X, Y, Z, \ldots$ and for each pair of objects X, Y a collection of *morphisms*, denoted $\mathcal{C}(X, Y)$, together with the following data:

i) For each pair $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, a specified morphism $g \circ f \in \mathcal{C}(X, Z)$,

ii) For every object a specified morphism $\mathrm{id}_X \in \mathcal{C}(X, X)$,

such that:

1. For every triple $h \in \mathcal{C}(W, X)$, $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have $g \circ (f \circ h) = (g \circ f) \circ h$,
2. For every object X and $h \in \mathcal{C}(W, X)$, $f \in \mathcal{C}(X, Y)$ we have $\mathrm{id}_X \circ h = h$ and $f \circ \mathrm{id}_X = f$.

For $f \in \mathcal{C}(X, Y)$ we say X is the *source* of f, Y is the *target* of f, and write $X = s(f)$, $Y = t(f)$. We also write $f \colon X \to Y$ or $X \xrightarrow{f} Y$ to indicate that $f \in \mathcal{C}(X, Y)$. If $\mathcal{C}(X, Y)$ is a set for all X, Y, then $\mathcal{C}$ is called *locally small*, and each $\mathcal{C}(X, Y)$ is called a *hom-set*.

Most categories you will encounter are locally small

Many examples of categories have objects sets carrying extra structure (for instance a topology) and morphisms that are functions compatible with that structure—but not all categories. We have seen **Top**, the category of topological spaces (and continuous maps) and **Set**, the category of sets (and functions), and you implicitly already know many other examples.

Vector spaces, (abelian) groups, manifolds, rings, ...

Example 12. The category $\mathbf{Set}_*$ of pointed sets (X, x) ($x \in X$ a specified element) and pointed maps $(X, x) \to (Y, y)$ (functions $f \colon X \to Y$ with $f(x) = y$)) can be considered as consisting of algebraic objects of the weakest sort (compare homomorphisms, linear transformations, ring maps, etc, which preserve distinguised elements).

The whole point of categories is how they relate to each other, an isolated category can only tell us so much.

Definition 15. Given categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* from $\mathcal{C}$ to $\mathcal{D}$, denoted $F \colon \mathcal{C} \to \mathcal{D}$ consists of the data:

i) For every object X of $\mathcal{D}$, a specified object $F(X)$ of $\mathcal{D}$,

ii) For every morphism $f \colon X \to Y$ of $\mathcal{C}$, a specified morphism $F(f) \colon F(X) \to F(Y)$ of $\mathcal{D}$

such that for every pair $f\colon X \to Y$ and $g\colon Y \to Z$ of morphisms of $\mathcal{C}$, $F(g \circ f) = F(g) \circ F(f)$. This latter property is called 'functoriality'. For locally small categories, the assignment on morphisms gives a function $\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$.

We will use this notation even without making that assumption

We have already see at least four examples of functors:

- The underlying set functor $U\colon \mathbf{Top} \to \mathbf{Set}$
- The discrete topology functor $\mathrm{disc}\colon \mathbf{Set} \to \mathbf{Top}$
- The indiscrete topology functor $\mathbf{Set} \to \mathbf{Top}$
- The set of connected components functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$

although we haven't yet seen why π_0 is a functor. We can compose functors in the obvious way, so get functors $\mathrm{disc}U\colon \mathbf{Top} \to \mathbf{Top}$ and $\mathrm{disc}\pi_0\colon \mathbf{Top} \to \mathbf{Top}$, for instance.

Here is a trivial-seeming example (aside from the identity functor).

Let $\mathcal{C}$ be a category, and $\mathcal{D}$ a *subcategory*: a collection of some of the objects of $\mathcal{C}$ and some of the morphisms of $\mathcal{C}$ that form a category by themselves. Then the inclusion of the objects and the morphisms forms a functor $\mathcal{D} \hookrightarrow \mathcal{C}$, the *subcategory inclusion*. An important special case of this is when for every X and Y that are objects of $\mathcal{D}$, every $\mathcal{D}(X, Y) = \mathcal{C}(X, Y)$; then $\mathcal{D}$ is call a *full* subcategory. More generally we can consider a functor that is injective on objects and morphisms to define a subcategory.

Example 13. The functor $\mathrm{disc}\colon \mathbf{Set} \to \mathbf{Top}$ makes $\mathbf{Set}$ a full subcategory of $\mathbf{Top}$.

we have used and will use this result without comment

We will be later restricting attention to certain full subcategories of $\mathbf{Top}$.

Proposition 4. The assignment $X \mapsto \pi_0(X)$ is a functor $\mathbf{Top} \to \mathbf{Set}$.

Proof. We need to show there is an assignment $(f\colon X \to Y) \mapsto (\pi_0(f)\colon \pi_0(X) \to \pi_0(Y))$, for an arbitrary continuous function f. We already know we have a function $X \to Y \to \pi_0(Y)$, and we want to show this *descends* along $X \to \pi_0(X)$. Given any $\alpha \in \pi_0(X)$, it corresponds to a connected component of X, namely X_α. Look at the restriction of $X \to Y \to \pi_0(Y)$ to X_α: since X_α is connected, its image is exactly one point in $\pi_0(Y)$. So define $\pi_0(f)(\alpha) = [f(x)]$

for an arbitrary $x \in X_\alpha$. This defines $\pi_0(f)$. Moreover, the following diagram *commutes*:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y) \end{tikzcd}$$

Now we want to show that $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$. Given $\alpha \in \pi_0(X)$, and $x \in X_\alpha$, then $\pi_0(f)(\alpha) = [f(x)]$. To define $\pi_0(g)\,(\pi_0(f)(\alpha))$, we need to choose a point in the component $Y_{[f(x)]}$, so take it to be $f(x)$. Then $\pi_0(g)\,(\pi_0(f)(\alpha)) = [g(f(x))]$, but this is just $\pi_0(g \circ f)(\alpha)$. $\square$

Exercise 7. Show that $[\mathrm{pt}, -] \colon \mathbf{Top} \to \mathbf{Set}$ is a functor.

Or more generally, $[X, -] \colon \mathbf{Top} \to \mathbf{Set}$!

Another important example of a category is the *homotopy category* **Ho**. The objects are topological spaces, but $\mathbf{Ho}(X, Y) = [X, Y]$. There is a functor $\mathbf{Top} \to \mathbf{Ho}$, which is the identity on objects, and sends a map to its homotopy class. Objects are isomorphic in **Ho** iff they are homotopy equivalent.

exercise: prove this is a category

Proposition 5. The functor π_0 descends to a functor $\mathbf{Ho} \to \mathbf{Set}$

Proof. We will prove that this is well-defined on morphism on hom-sets, the rest is routine. For $f, g \colon X \to Y$ to be homotopic via $H \colon I \times X \to Y$, we need to show that for all $\alpha \in \pi_0(X)$, $\pi_0(f)(\alpha) = \pi_0(g)(\alpha)$. Take x in the connected component X_α, then we have a map $I \to I \times X \xrightarrow{H} Y \to \pi_0(Y)$ sending $0 \mapsto \pi_0(f)(\alpha)$ and $1 \mapsto \pi_0(g)(\alpha)$. But I is connected, so $\pi_0(f)(\alpha) = \pi_0(g)(\alpha)$. $\square$

As a result, if $\pi_0(X) \not\cong \pi_0(Y)$, the spaces X and Y cannot be homotopy equivalent, let alone homeomorphic.

Exercise 8. Show the functor $[\mathrm{pt}, -] \colon \mathbf{Top} \to \mathbf{Set}$ descends to $\mathbf{Ho} \to \mathbf{Set}$.

Here is a useful fact about spaces.

Lemma 6. For all families X_β, $\beta \in J$, of spaces, we have isomorphisms

$$\coprod_{\beta \in J} \pi_0(X_\beta) \xrightarrow{\simeq} \pi_0\Big(\coprod_{\beta \in J} X_\beta\Big) \quad \text{and} \quad \coprod_{\beta \in J} [\mathrm{pt}, X_\beta] \xrightarrow{\simeq} [\mathrm{pt}, \coprod_{\beta \in J} X_\beta],$$

with inverses induced by the family of maps in_β. That is, π_0 and $[\mathrm{pt}, -]$ *preserve coproducts*.

Lecture 4

Recall last time: we had functors $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ and $\pi_0\colon \mathbf{Ho} \to \mathbf{Set}$.

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "\pi_0"] \arrow[d] & \mathbf{Set} \\ \mathbf{Ho} \arrow[ur, "\pi_0"'] & \end{tikzcd}$$

Example 14. If X and Y are spaces with $|\pi_0(X)| < |\pi_0(Y)|$, no continuous map $X \to Y$ is surjective.

Here is an instructive example

Example 15. The *topologist's sine curve* is the image C of $[-1,1] \sqcup (0,1] \to \mathbb{R}^2$ defined by

$$\begin{cases} y \mapsto (0,y) & y \in [-1,1] \\ x \mapsto (x, \sin(\frac{1}{x})) & x \in (0,1] \end{cases}$$

equipped with the **subspace topology**. This is a compact metric space, using the inherited Euclidean metric. Fact: *every* continuous function $f\colon C \to \{0,1\}$ is constant. If $f(1,\sin(1)) = 1$, then $f(x,\sin(x)) = 1$ for every $x \in (0,1]$ (as intervals are connected). If $f(0,0) = b \in \{0,1\}$, then $f(0,y) = b$ also, for all $y \in [-1,1]$. The sequence $(\frac{1}{n\pi}, 0)$ converges to $(0,0)$ in C, so $b = f(0,0) = \lim_{n\to\infty} f(\frac{1}{n\pi}, 0) = 1$ as f is continuous and we are in a metric space.

Hence C is connected, but there is *no* continuous function $\gamma\colon [0,1] \to C$ with $\gamma(0) = (0,0)$ and $\gamma(1) = (1,\sin(1))$. Since intervals are path connected, we can show $[\mathrm{pt}, C] = \{0,1\}$, but $\pi_0(C) = *$.

Exercise: prove this by considering $\lim_{n\to\infty} \gamma(\frac{1}{n})$

So we have two different invariants here, and there is always a surjective map $[\mathrm{pt}, X] \to \pi_0(X)$. Moreover, the following square always commutes, for any map $X \xrightarrow{f} Y$:

$$\begin{tikzcd} {[\mathrm{pt}, X]} \arrow[r, "{[\mathrm{pt}, f]}"] \arrow[d] & {[\mathrm{pt}, Y]} \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y) \end{tikzcd}$$

This is thus an example of a *natural transformation*.

Definition 16. Given functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a natural transformation $\alpha\colon F \Rightarrow G$ consists of the data:

i) For every object X of $\mathcal{C}$, a specified morphism $\alpha_X\colon F(X) \to G(X)$ (the *components* of α)

such that for every morphism $f\colon X \to Y$ in $\mathcal{C}$, the following square commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & F(Y) \end{tikzcd}$$

A natural transformation is called a *natural isomorphism* if all of its components are isomorphisms.

For example, there are natural transformations disc $U \Rightarrow \mathrm{id}\colon \mathbf{Top} \to \mathbf{Top}$, with component at X the identity map $\mathrm{disc}(U(X)) \to X$, and $\mathrm{id} \Rightarrow \mathrm{disc}\,\pi_0\colon \mathbf{Top} \to \mathbf{Top}$, with component $X \to \pi_0(X)$.

We seek conditions that will define a full subcategory of **Top** such that the components $[\mathrm{pt}, X] \to \pi_0(X)$ are isomorphisms for all spaces X in the subcategory.

Definition 17. A space X is *locally path connected* (lpc) if it has a neighbourhood base of path-connected sets.

Note that a space is lpc iff every connected component is lpc.

Proposition 6. If X is a locally path connected space, then $[\mathrm{pt}, X] \to \pi_0(X)$ is an isomorphism.

Proof. We are reduced to the case X is connected ($\pi_0(X) = *$) and lpc, by Lemma 6, and the fact the case $X = \emptyset$ is trivial. Since X is connected, take $x \in X$ and define $\chi\colon X \to \{0,1\}$ by

$$\chi(y) = \begin{cases} 1 & \exists y \rightsquigarrow x \\ 0 & \text{otherwise} \end{cases}$$

where by $y \rightsquigarrow x$ I mean a path $\gamma\colon I \to X$ with $\gamma(0) = y$ and $\gamma(1) = x$. We will show χ is continuous. Note that χ continuous $\Leftrightarrow p^{-1}(0)$ and $p^{-1}(1)$ open $\Leftrightarrow p^{-1}(1)$ open and closed. But $p^{-1}(1) =: C_x$ is the path component containing x. Take $y \in C_x$ (so $\exists y \rightsquigarrow x$), and $V \ni y$ a path-connected nhd. Given $z \in V$, $\exists z \rightsquigarrow y$. Concatenate these paths to give $z \rightsquigarrow x$, so that $z \in C_x$. This is true for all $z \in V$, so that $V \subseteq C_x$, hence C_x contains a neighbourhood of each of its points, and so is open.

Conversely, take $y \in \overline{C_x}$, $V \ni y$ a path connected nhd. As $\exists z \in V \cap C_x \subseteq V$, $\exists z \rightsquigarrow y$. But also have $V \cap C_x \subseteq C_x$, so $\exists z \rightsquigarrow x$. Concatenate paths to get $y \rightsquigarrow x$, so that $y \in C_x$. This is true for all $y \in \overline{C_x}$, so $\overline{C_x} \subseteq C_x$ and C_x is closed. Hence χ is continuous.

But X is connected, and $\chi(x) = 1$, so that $\operatorname{im}\chi = \{1\}$, and so $C_x = \chi^{-1}(1) = X$. Thus $[\mathrm{pt}, X] \to \pi_0(X) = *$ is an isomorphism. $\square$

So we will consider for the rest of this section of the course only lpc spaces, which form a full subcategory **lpcTop** $\hookrightarrow$ **Top**. Note that discrete spaces are lpc, so **Set** $\hookrightarrow$ **lpcTop** is a subcategory.

Exercise 9. Show that the product of lpc spaces is lpc, and that any locally convex topological vector space is lpc.

Be warned: subspaces of lpc spaces may not be lpc, for instance the topologist's sine curve is a subspace of the contractible $\mathbb{R}^2$. Here is a condition that can be used to transfer lpc-ness.

Definition 18. A *local homeomorphism* $X \xrightarrow{p} Y$ is a map of spaces such that for all $x \in X$ $\exists$ nhds $U \ni x$ and $V \ni p(x)$ such that $p\big|_U : U \xrightarrow{\sim} V$.

Examples include: maps $\sqcup U_\alpha \to X$ induced by open an open cover $\{U_\alpha\}$ of X, the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^\times$, the n^{th}-power map $U(1) \xrightarrow{(-)^n} U(1)$ and the projection $S \times X \to X$ for S discrete.

A trivial example is the inclusion of an open subspace

Lemma 7. If $X \xrightarrow{p} Y$ is a local homeomorphism, then Y lpc implies X is lpc. If p is surjective then X lpc implies Y lpc.

One last technical point

Definition 19. A *pointed space* is a pair (X, x) where X is a topological space and $x \in X$. A pointed map is a pointed map between the underlying pointed sets that is continuous. These define a category $\mathbf{Top}_*$.

A *pointed homotopy* of pointed map $I \times X \to Y$, for (X, x_0) and (Y, y_0) pointed spaces, is required to satisfy $H(t, x_0) = y_0$ for all $t \in I$. Pointed homotopy classes of pointed map are denoted $[(X, x_0), (Y, y_0)]_*$. The category $\mathbf{Ho}_*$ is defined analogously to **Ho**. We get a functor $\pi_0\colon \mathbf{Ho}_* \to \mathbf{Set}_*$.

Covering spaces

Given a local homeomorphism $X \xrightarrow{p} Y$, the *fibre* $p^{-1}(y) \subset X$ over y has the discrete topology for every $y \in Y$. We have no idea how $p^{-1}(\gamma(t))$ varies along a path $\gamma\colon I \to Y$. For example, given an arbitrary collection $B(x_\alpha, r_\alpha) \subset \mathbb{R}^2$ of open balls, $\sqcup_\alpha B(x_\alpha, r_\alpha) \to \mathbb{R}^2$ is a local homeomorphism, but the fibres can jump in size arbitrarily. We would like fibres 'close' to a given $p^{-1}(y)$ to 'vary continuously'. For spaces 'vary continuously' really means homotopy equivalence. But for discrete spaces, homotopy equivalence is isomorphism.

Definition 20. A *covering space* $Z \xrightarrow{\pi} X$ of X is a map such that for all $x \in X$ there is a nhd $V_x \ni x$ such that $\pi^{-1}(V_x) \simeq V_x \times \pi^{-1}(x)$ over V_x. We will also call π a *covering map*.

$$\begin{tikzcd} \pi^{-1}(V_x) \arrow[r, "\simeq"] \arrow[d, "\pi"'] & V_x \times \pi^{-1}(x) \arrow[dl, "\mathrm{pr}_1"] \\ V_x & \end{tikzcd}$$

For a covering space $Z \xrightarrow{\pi} X$ and $x \in X$, let $Z_x := \pi^{-1}(x)$. We will also call X the *base space*.

Examples include: $\exp\colon \mathbb{C} \to \mathbb{C}^\times$, $S^2 \to \mathbb{RP}^2$, $U(1) \xrightarrow{(-)^n} U(1)$, covers of the join ∞ of two circles.

Exercise 10. Show that a covering map is a local homeomorphism.

Proposition 7. For a covering space $Z \xrightarrow{\pi} X$, if $\exists x_0 \rightsquigarrow x_1$, then $Z_{x_0} \simeq Z_{x_1}$.

Proof. (First proof of Proposition 7) Take $\gamma\colon I \to X$, $\gamma(i) = x_i$, and an open cover $\{U_\alpha\}$ of X over which Z trivialises. We thus get an open cover $\gamma^{-1}(U_\alpha)$ of I, which has a finite subcover $U_0, \ldots, U_N$, with $x_0 \in U_0$, $x_1 \in U_N$. The ordering is chosen so that the path enters U_i before it enters U_{i+1}, and $U_i \cap U_{i+1}$ has at least one point of the path in it.

we can shrink the cover slightly to make this ordering well-defined, if need be

We have isomorphisms $Z_{U_n} := \pi^{-1}(U_n) \xrightarrow{\phi_n} U_n \times F_n$ with discrete spaces F_n. We have $Z_{x_0} \simeq F_0$, and for all $t \in \gamma^{-1}(U_0)$, $Z_{\gamma(t)} \simeq F_0$. So for $\gamma(t) \in U_0 \cap U_1$, we have $F_0 \simeq Z_{\gamma(t)} \simeq F_1$. We can then prove by induction on N that $F_0 \simeq F_1 \simeq \cdots \simeq F_N$. $\square$

So for lpc X and each $\alpha \in \pi_0(X)$, there is associated to $Z \xrightarrow{\pi} X$ an isomorphism class of sets, the *typical fibre* over all x in the connected component $X_\alpha \subseteq X$.

Note: Fibres can be empty! But we usually don't think about this case too much. For X pointed (by $x \in X$), we can consider pointed covering spaces $(Z, x) \to (X, x)$. This is from one perspective just a choice of point $z \in Z_x$. For X connected and lpc, a pointed covering space has every fibre contain at least one point, namely the image of z under $Z_x \simeq Z_{x'}$.

We have categories $\mathbf{Cov}_X$ and $\mathbf{Cov}_{(X,x)}$ with objects covering spaces of X (resp. pointed covering spaces of (X, x)) and maps

$$\begin{tikzcd} Z_1 \arrow[rr] \arrow[dr] & & Z_2 \arrow[dl] \\ & X & \end{tikzcd}$$

and analogously in the pointed case. We will study these categories and see what they tell us about the topology of X.

For lpc and connected X, the fact that for a covering space Z of X, $Z_{x_0} \simeq Z_{x_1}$ for arbitrary $x_0, x_1 \in X$ can be improved. We first need a construction on covering spaces.

Lecture 5

Definition 21. Given a covering space $Z \xrightarrow{\pi} X$ and a map $Y \xrightarrow{f} X$, the *pullback* of Z is the subspace

$$f^*Z := Y \times_X Z = \{(y,z) \in Y \times Z \mid f(y) = \pi(z)\}.$$

this construction works for any pair of maps, not just one of them a covering space

It fits in a commutative square

$$\begin{tikzcd} f^*Z \arrow[r] \arrow[d, "p"'] & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

Proposition 8. In the setting of Definition 21

1. $f^*Z \to Y$ is a covering space.
2. f^* is a functor $\mathbf{Cov}_X \to \mathbf{Cov}_Y$.
3. Given $Y_2 \xrightarrow{g} Y_1 \xrightarrow{f} X$ and $Z \xrightarrow{\pi} X$, there is a canonical isomorphism $(f \circ g)^*Z \simeq g^*f^*Z$ in $\mathbf{Cov}_{Y_2}$.

Corollary 1. The fibre $(f^*Z)_y$ is canonically isomorphic to $Z_{f(y)}$.

Now given a path $\gamma \colon I \to X$ and a covering space $Z \xrightarrow{\pi} X$, we can pull back Z to get a covering space $\gamma^*Z \to I$. So let us try to understand covering spaces of I. Certainly for discrete S, the projection $S \times I \to I$ is a covering space.

Proposition 9. A covering space $Z \xrightarrow{\pi} I$ is isomorphic to $\pi^{-1}(0) \times I \xrightarrow{\mathrm{pr}_2} I$ in $\mathbf{Cov}_I$.

We first need a little helper lemma

Lemma 8. A covering space of a compact space X trivialises over a *finite* cover of X by nhds.

Proof. (of Proposition 9) We use the lemma to trivialise $Z \to I$ over a finite cover of I, which we can take to be by intervals $[0, t_1]$, $[s_2, t_2]$, $\dots$, $[s_N, 1]$ for $s_1 = 0 < s_2 < t_1 < s_3 < t_2 < \cdots < s_N < t_{N-1} < 1 = t_N$. We will proceed by induction on N, but this quickly reduces to the case of $N = 2$. So take a cover of I by $[0,t]$ and $[s,1]$, where $\tau\colon Z_0 \times [0,t] \xrightarrow{\simeq} Z_{[0,t]}$ and we are given $\sigma\colon F \times [s,1] \xrightarrow{\simeq} Z_{[s,1]}$.

we know abstractly that $F \simeq Z_0$, but this proof will construct an isomorphism

By restriction there is the composite map

$$Z_0 \times [s,t] \xrightarrow[\simeq]{\tau|_{[s,t]}} Z_{[s,t]} \xrightarrow[\simeq]{\sigma^{-1}|_{[s,t]}} F \times [s,t] \xrightarrow{\mathrm{pr}_1} F.$$

If we fix $z \in Z_0$, we get a continuous map $\{z\} \times [s,t] \to F$, which is thus constant, say at $p_z \in F$. The function $z \mapsto p_z = \sigma^{-1}(\tau(z,s))$ is then a bijection $\phi\colon Z_0 \xrightarrow{\simeq} F$.

We thus get maps $Z_0 \times [0,t] \hookrightarrow Z \hookleftarrow F \times [s,1] \xleftarrow{\phi\times\mathrm{id}} Z_0 \times [s,1]$, which by construction agree on $Z_0 \times [s,t]$. There is thus a continuous map $Z_0 \times [0,1] \to Z$. Moreover, you can check this map is a morphism of $\mathbf{Cov}_I$. There are likewise maps

$$Z_{[0,t]} \xrightarrow{\simeq} Z_0 \times [0,t] \hookrightarrow Z_0 \times I \hookleftarrow Z_0 \times [s,1] \xleftarrow{\phi^{-1}\times\mathrm{id}} F \times [s,1] \xleftarrow{\simeq} Z_{[s,1]}$$

which agree on $Z_{[s,t]}$, hence a continuous map $Z \to Z_0 \times I$. This map is in $\mathbf{Cov}_I$ and cen be checked by pointwise evaluation to be inverse to the first one. Hence We have an isomorphism $Z \simeq Z_0 \times I$ in $\mathbf{Cov}_I$. □

Corollary 2. Given a covering space $Z \xrightarrow{\pi} I$ and a point $z \in Z_0$, there is a unique path $\eta_z\colon I \to Z$ with $\eta_z(0) = z$ such that $\pi \circ \eta_z = \mathrm{id}$ (i.e. η_z is a section of π)..

Proof. We can construct *a* path, given $\tau\colon Z_0 \times I \xrightarrow{\simeq} Z$, by $\eta(x) = \tau(z,x)$. Since $\pi \circ \tau = \mathrm{pr}_2$, this has the required property. Connectedness of I and discreteness of Z_0 implies that given any other path $\eta'\colon I \to Z$ with $\eta'(0) = z$ and $\pi \circ \eta' = \pi$, we must have $\tau^{-1} \circ \eta = \tau^{-1} \circ \eta'\colon I \to Z_0 \times I$ which implies $\eta' = \eta$. □

And now we have a really important property of covering spaces

Proposition 10. Given any covering space $Z \xrightarrow{\pi} X$, path $\gamma\colon I \to X$ and point $z \in Z_{\gamma(0)}$, there is a unique lift $\widetilde{\gamma_z}\colon I \to Z$ with $\widetilde{\gamma_z}(0) = z$.

a *lift* of a path $\gamma\colon I \to X$ is a path $\widetilde{\gamma}\colon I \to Z$ with $\pi\widetilde{\gamma} = \gamma$

Proof. We can pull back Z to get $p\colon \gamma^*Z \to I$. We have unique $\eta_z\colon I \to \gamma^*Z$ so that $\eta_z(0) = (0,z)$. Define $\widetilde{\gamma_z} = \mathrm{pr}_2 \circ \eta_z\colon I \to Z$. This path satisfies $\pi\widetilde{\gamma_z} = \gamma p \eta_z = \gamma$. Given any other lift $\lambda\colon I \to Z$, we get a second section of p by $x \mapsto (x, \lambda(x))$, which by uniqueness of η_z has to be equal to it, so that $\lambda = \widetilde{\gamma_z}$. □

We can then give a second proof of Proposition 7.

Corollary 3. A path $\gamma\colon I \to X$ defines a bijection $[\gamma]\colon Z_{\gamma(0)} \xrightarrow{\simeq} Z_{\gamma(1)}$, by $[\gamma](z) = \widetilde{\gamma_z}(1)$.

Proof. We only have to start with that $[\gamma]$ is a function $Z_{\gamma(0)} \to Z_{\gamma(1)}$, but the function $[-\gamma]\colon Z_{\gamma(1)} \to Z_{\gamma(0)}$, where $-\gamma\colon I \to X$ is the path $-\gamma(x) = \gamma(1-x)$, is inverse to $[\gamma]$. This is because the path $-\widetilde{\gamma_z}$ is a lift of $-\gamma$, hence $[-\gamma]([\gamma](z)) = \widetilde{-\gamma_{[\gamma](z)}}(1) = -\widetilde{\gamma_z}(0) = z$. A symmetric argument shows that $[\gamma]([-\gamma](z)) = z$ for $z \in Z_{\gamma(1)}$. $\square$

A first observation is that this bijection is invariant under reparameterisations of γ: given $\psi\colon I \xrightarrow{\simeq} I$ with $\psi(0) = 0$ and $\psi(1) = 1$, then clearly $[\gamma \circ \psi] = [\gamma]\colon Z_{\gamma(0)} \to Z_{\gamma(1)}$.

consider ψ as a path in I and see what happens in that case

Even better, we get a function

$$(\{\text{paths } x_0 \rightsquigarrow x_1 \text{ in } X\}/\text{param}) \times Z_{x_0} \to Z_{x_1}$$

If we take $x_0 = x_1 = x$, then this is a map

we can take quotient by reparams if desired

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \times Z_x \to Z_x$$

such that each loop $x \rightsquigarrow x$ gives a bijection $Z_x \to Z_x$. So we can think of this instead as

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \to \text{Aut}(Z_x).$$

Alternatively, if we have a pointed covering space $(Z, z) \to (X, x)$, we have a canonical function

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \to Z_x \tag{1}$$

Example 16. For $Z = S \times X$, $[\gamma] = \text{id}_S$ always, and the image of (1) (given some $(s, x) \in Z$) is just a single point. For instance, if $X = I$, we have seen this will be the case for every covering space. But for $X = S^1$, $Z = \mathbb{R} \xrightarrow{\exp} S^1$, and taking $x = 1 \in S^1$, $z = 0 \in \mathbb{R}$, then $Z_1 = \exp^{-1}(0) = 2\pi i\mathbb{Z}$, then

$$\{\gamma\colon I \to S^1 \mid \gamma(0) = \gamma(1) = 1\} \to 2\pi i\mathbb{Z}$$

is *onto*. The path $\widetilde{\gamma}_n = 2\pi i n x$ lifts the path $\gamma(x) = \exp(2\pi i n x)$, and $\widetilde{\gamma}_n(0) = 0$, $\widetilde{\gamma}_n(1) = 2\pi i n x$. The difference is that $\mathbb{R}$ is path connected, but $X \times S$ is not, for $|S| > 1$.

In fact, for a covering space $(Z, z) \xrightarrow{\pi} (X, x)$ with Z path connected and $z' \in Z_x$, there is $\widetilde{\gamma}\colon I \to Z$ with $\widetilde{\gamma}(0) = z$, $\widetilde{\gamma}(1) = z'$. Since $\widetilde{\gamma}$ lifts $\gamma = \pi \circ \widetilde{\gamma}$, which satsfies $\gamma(0) = x = \gamma(1)$, the map (1) is **onto**. Thus paths constrain the sizes of fibres of connected covering spaces and vice versa. Notice also that the set of loops is independent of the choice of covering space!

More generally, given points z_α in Z_x, one per path component of Z,

that is: a section of $Z \to \pi_0(Z)$

$$\{\text{loops } x \rightsquigarrow x \text{ in } X\} \times \{z_\alpha\} \to Z_x$$

is always onto. There are a huge number of paths, and reparameterisations cuts things down somewhat. But we shall go even better, and put a topology on the space of paths.

Lecture 6

The fibres Z_x are discrete spaces, but the set $\mathbf{Top}(I, X)$ of paths $I \to X$ carries a topology when X is a metric space; we can consider $C(I, X)$ with the sup metric d_∞. The aim is to give $\mathbf{Top}(I, X)$ a topology for *any* space, not necessarily metric.

Lemma 9. Let X be a topological space, $\gamma \in \mathbf{Top}(I, X)$ a path, $0 = t_0 < t_1 < \ldots < t_N < t_{N+1} = 1$ a partition of $[0, 1]$, and $U_0, \ldots, U_N \subseteq X$ a collection of open sets such that $\gamma([t_i, t_{i+1}]) \subseteq U_i$. Define the subsets

$$N_\gamma(t_1 < \ldots < t_N; U_0, \ldots, U_n) := \{\eta \colon I \to X \mid \forall i = 0, \ldots N,\ \eta([t_i, t_{i+1}]) \subseteq U_i\} \subseteq \mathbf{Top}(I, X)$$

Then these sets give a neighbourhood base on $\mathbf{Top}(I, X)$

Definition 22. The *path space* X^I is the set $\mathbf{Top}(I, X)$ equipped with the topology defined by Lemma 9, which we call the *compact-open topology*.

When X is a metric space, then the compact-open topology and the topology arising from the sup metric coincide. A key property of the compact-open topology is that homotopies $H \colon I \times I \to X$ give continuous paths $h \colon I \to X^I$ (defined by $h_t \colon s \mapsto H(t, s)$) and vice-versa. Moreover:

Lemma 10. The evaluation map $\mathrm{ev} \colon X^I \times I \to X$, $\mathrm{ev}(\gamma, x) = \gamma(x)$ is continuous.

Then given $t \in I$, the composite map $\mathrm{ev}_t \colon X^I \simeq X^I \times \{t\} \hookrightarrow X^I \times I \xrightarrow{\mathrm{ev}} X$ is continuous. Usually we care just about the cases $t = 0, 1$. We can then look at various subspaces of X^I, for a given $x \in X$:

$$\begin{aligned} P_x X &:= \{\gamma \in X^I \mid \gamma(0) = x\} = \mathrm{ev}_0^{-1}(x) \\ P_x^y X &:= \{\gamma \in X^I \mid \gamma(0) = x,\ \gamma(1) = y\} = \mathrm{ev}_0^{-1}(x) \cap \mathrm{ev}_1^{-1}(y) \\ \Omega_x X &:= P_x^x X = \{\gamma \in X^I \mid \gamma(0) = x = \gamma(1)\} \end{aligned}$$

In particular, we have already seen the last two, albeit without their topologies. We also see that path components of these spaces have something to do with homotopy classes of paths, perhaps with constraints on endpoints.

A key property of the natural transformation $\mathrm{id} \Rightarrow \mathrm{disc}\,\pi_0 \colon \mathbf{lpcTop} \to \mathbf{lpcTop}$ is that it has a universal property: given a discrete space S, an lpc space X and a continuous map $X \xrightarrow{f} S$, there is a *unique* $\pi_0(X) \to S$ such that

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S \\ \pi_0(X) \arrow[ur] & \end{tikzcd}$$

commutes. Hence if we take our function

$$P_x^y \times Z_x \to Z_y \tag{2}$$

from the previous lecture, arising from a covering space $Z \to X$, and if we can show it is continuous, we would get a factorisation

$$P_x^y \times Z_x \to \pi_0(P_x^y \times Z_x) \simeq \pi_0(P_x^y) \times Z_x \to Z_y$$

where the unmarked isomorphism exist due to Z_x being discrete. If Z is path connected, a fixing some $z \in Z_x$, we get a surjective map $\pi_0(P_x^y) \to Z_y$, which further constrains both the topology of the space of paths, and the possible fibres of $Z \to X$. However, there are two issues:

(1) We yet don't know our path lifting function is continuous

(2) We don't know if P_x^y is lpc, hence if path components and components agree.

To address (1), the unique path lifting property from last lecture will be promoted to a *continuous function* $\mathrm{Lift}\colon X^I \times_X Z \to Z^I$. Combined with $Z^I \xrightarrow{\mathrm{ev}} Z$ we will be able to reconstruct (2) as

$X^I \times_X Z = \{(\gamma, z) \mid \gamma(0) = \pi(z)\}$

$$P_x^y \times Z_x \hookrightarrow X^I \times_X Z \xrightarrow{\mathrm{Lift}} Z^I \xrightarrow{\mathrm{ev}_1} Z$$

factors through $Z_y \subset Z$. We already have the definition of Lift, but we need to show continuity.

Theorem 1. The function $\mathrm{Lift}\colon X^I \times_X Z \to Z$ is continuous.

Proof. We need to set up the ingredients, so take $\gamma \in X^I$, define $x = \gamma(0)$, $y = \gamma(1)$, and take $z \in Z_x$. Let $\widetilde{\gamma} = \mathrm{Lift}(\gamma, z)$, and $z' = \widetilde{\gamma}(1) \in Z_y$. Take a basic nhd $N_{\widetilde{\gamma}} = N_{\widetilde{\gamma}}(t_1 < \cdots < t_n; U_0, \ldots, U_n)$. We want to construct a basic nhd

$$M(\gamma, z) := (N_\gamma(s_1 < \cdots < s_m; V_0, \ldots, V_m) \times W) \cap X^I \times_X Z$$

of (γ, z) such that $M(\gamma, z) \subseteq \mathrm{Lift}^{-1}(N_{\widetilde{\gamma}})$.

[...]

$\square$