The diagonal argument redux[1]

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The following is a generalisation of the contrapositive to Lawvere's fixed point theorem,[2] from categories with finite products to a much more general setting.

[2] F.W. Lawvere, *Diagonal arguments and cartesian closed categories*, Lecture Notes in Mathematics, **92** (1969), 134–45

Define a *magmoidal category* to be a category $\mathcal{C}$ equipped with a functor $\#\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. A magmoidal category is said to have *diagonals* if there is a natural transformation $\delta\colon \mathrm{id}_C \Rightarrow \# \circ \Delta_{\mathcal{C}}$. For the following we shall fix a magmoidal category with diagonals $(\mathcal{C}, \#, \delta)$. Examples include categories with finite products (*cartesian* magmoidal categories) and monoidal categories with diagonals.

In the cartesian setting, a map $A \times B \to C$ can be viewed as an A-parameterised family of maps $B \to C$; in the cartesian closed case this is of course equivalent to a map $A \to C^B$. In our setting of a magmoidal category, we still want to think of a morphism $A\#B \to C$ as being an A-parametrised family of maps $B \to C$. We can ask whether, for a given A, one can find *every* map $B \to C$ inside some given A-parametrised family $A\#B \to C$.

Definition. In any magmoidal category with diagonals $(C, \#, \delta)$, a map $F\colon A\#B \to C$ is an *incomplete parametrisation* of maps $B \to C$ if there exists an $f\colon B \to C$ such that for all $a\colon X \to A$, there is a $b = b_a\colon X \to B$ with $f \circ b \neq F \circ (a\#b) \circ \delta\colon X \to C$.

What this definition is saying is that there is a map $B \to C$ that differs from every map in the A-parametrised family for at least one argument. Finally, let us say that an endomorphism $\sigma\colon C \to C$ in a category is *free* if for all $c\colon X \to C$, $\sigma \circ c \neq c$.

Theorem. For $(C, \#, \delta)$ a magmoidal category with diagonals, and $\sigma\colon C \to C$ a free endomorphism, every $F\colon A\#A \to C$ is an incomplete parametrisation of maps $A \to C$.

Proof. Define $f = \sigma \circ F \circ \delta$. Then for all $a\colon X \to A$, we have $f \circ a = \sigma \circ F \circ \delta \circ a = \sigma \circ F \circ (a\#a) \circ \delta \neq F \circ (a\#a) \circ \delta$. $\square$

This covers all the cases of the diagonal argument by Lawvere, but one notable feature of the above is that it no longer relies on global elements $1 \to A$ etc as in Lawvere's presentation, but arbitrary generalised elements. Also, one could conceivably generalise the definition of incomplete parametrisation so that the generalised element $b\colon X \to B$ was instead $b\colon Y \to X \to B$ for some suitable $p\colon Y \to X$ (for example some regular epimorphism) and then use the composite $p \circ a\colon Y \to A$ instead of just a.