The diagonal argument redux[1]

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May 5, 2019

The following is a generalisation of the contrapositive to Lawvere's fixed point theorem,[2] from categories with finite products to a much more general setting.

[2] F.W. Lawvere, *Diagonal arguments and cartesian closed categories*, Lecture Notes in Mathematics, **92** (1969), 134–45

Define a *magmoidal category* to be a category $\mathcal{C}$ equipped with a functor $\#\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$. A magmoidal category is said to have *diagonals* if there is a natural transformation $\delta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow \#\circ\Delta_{\mathcal{C}}$. For the following we shall fix a magmoidal category with diagonals $(\mathcal{C},\#,\delta)$. Examples include categories with finite products (*cartesian* magmoidal categories) and monoidal categories with diagonals.

In the cartesian setting, a map $A\times B\to C$ can be viewed as an A-parameterised family of maps $B\to C$; in the cartesian closed case this is of course equivalent to a map $A\to C^B$. In our setting of a magmoidal category, we still want to think of a morphism $A\#B\to C$ as being an A-parametrised family of maps $B\to C$. We can ask whether, for a given A, one can find every map $B\to C$ inside *some* A-parametrised family $A\#B\to C$.

Definition. In any magmoidal category with diagonals $(C,\#,\delta)$, a map $F\colon A\#B\to C$ is an *incomplete parametrisation* of maps $B\to C$ if there exists an $f\colon B\to C$ such that for all $a\colon X\to A$, there is a $b=b_a\colon X\to B$ with $f\circ b\neq F\circ(a\#b)\circ\delta\colon X\to C$.

What this definition is saying is that there is a map $B\to C$ that differs from every map in the A-parametrised family for at least one argument. Finally, let us say that an endomorphism $\sigma\colon C\to C$ in a category is *free* if for all $c\colon X\to C$, $\sigma\circ c\neq c$.

Theorem. For $(C,\#,\delta)$ a magmoidal category with diagonals, and $\sigma\colon C\to C$ a free endomorphism, every $F\colon A\#A\to C$ is an incomplete parametrisation of maps $A\to C$.

Proof. Define $f=\sigma\circ F\circ\delta$. Then for all $a\colon X\to A$, we have $f\circ a=\sigma\circ F\circ\delta\circ a=\sigma\circ F\circ(a\#a)\circ\delta\neq F\circ(a\#a)\circ\delta$. $\square$

This covers all the cases of the diagonal argument by Lawvere, but one notable feature of the above is that it no longer relies on global elements $1\to A$ etc as in Lawvere's presentation, but arbitrary generalised elements. Also, one could conceivably generalise the definition of incomplete parametrisation so that the generalised element $b\colon X\to B$ was instead $b\colon Y\to X\to B$ for some suitable $p\colon Y\to X$ (for example some regular epimorphism) and then use the composite $p\circ a\colon Y\to A$ instead of just a.