A BRAIDED MONOIDAL CATEGORY IS SYMMETRIC IF AND ONLY IF THE PRODUCT IS BRAIDED

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The following diagram commutes, by virtue of the symmetric monoidal category axioms, and shows in that case $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is a *braided* strong monoidal functor. That $\otimes$ is a *strong* monoidal functor is due to Joyal–Street (‘Braided monoidal categories’[1], Proposition 2). The vertical composites down the sides below are the comparison maps for pre- and post-$\otimes$ products given by Joyal–Street.

$$
\begin{tikzcd}
 & (AB)(CD) \arrow[rrr, "\sigma"] \arrow[dl, "\alpha"'] & & & (CD)(AB) \arrow[d, "\alpha^{-1}"] \arrow[dr, "\alpha"] & \\
A(B(CD)) \arrow[d, "A\alpha^{-1}"'] \arrow[dr, "A\sigma"] & & & & ((CD)A)B \arrow[d, "\alpha B"] & C(D(AB)) \arrow[d, "C\alpha^{-1}"] \\
A((BC)D) \arrow[d, "A(\sigma D)"'] & A((CD)B) \arrow[rr, "\alpha^{-1}"] \arrow[d, "A\alpha"] & & (A(CD))B \arrow[ur, "\sigma B"] \arrow[d, "\alpha^{-1}B"'] & (C(DA))B \arrow[r, "\alpha"] \arrow[d, shift right=2, "(C\sigma^{-1})B"'] \arrow[d, shift left=2, "(C\sigma)B"] \arrow[d, phantom, "\overset{*}{=}"] & C((DA)B) \arrow[d, "C(\sigma B)"] \\
A((CB)D) \arrow[d, "A\alpha"'] & A(C(DB)) \arrow[dr, "\alpha^{-1}"] & & ((AC)D)B \arrow[dl, "\alpha"'] \arrow[dr, "(\sigma D)B"] & (C(AD))B \arrow[r, "\alpha"] & C((AD)B) \arrow[d, "C\alpha"] \\
A(C(BD)) \arrow[ur, "A(C\sigma)"] \arrow[dr, "\alpha^{-1}"'] & & (AC)(DB) \arrow[drr, "\sigma(DB)"] & & ((CA)D)B \arrow[u, "\alpha B"'] \arrow[d, "\alpha"] & C(A(DB)) \arrow[dl, "\alpha^{-1}"] \\
 & (AC)(BD) \arrow[ur, "(AC)\sigma"'] \arrow[rrr, "\sigma\sigma"'] & & & (CA)(DB) &
\end{tikzcd}
$$

In the diagram I have written $AB := A\otimes B$ etc, and identity arrows 1_X are written as just X. Isomorphisms arising from associators (α) are in blue, those arising from the braiding (σ) are in orange. Notice that the equality $\overset{*}{=}$ is the only place where the symmetry axiom is used.

Conversely, take $\mathcal{C}$ to be merely braided and $\otimes$ a braided functor. If $B=C=I$ the diagram above implies $(I\sigma^{-1})I = (I\sigma)I\colon (I(DA))I \to (I(AD))I$. Applying naturality squares for $IX\to X$ and $XI\to X$ gives us that $\sigma^{-1}=\sigma$, hence $\mathcal{C}$ is symmetric.

I’m sure this result is rather old, but haven’t found a reference. It follows from high-powered abstract machinery about E_∞-algebras in **Cat**, but I think this was possible to prove by the time of Mac Lane’s 1963 coherence theorem.

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[1]Available at `http://web.science.mq.edu.au/~street/JS1.pdf`