A geometric proof of a theorem of Serre

David Michael Roberts

School of Mathematical Sciences
University of Adelaide

23 September 2014

A rather surprising conclusion

Paris, Wednesday afternoon [undated]

Dear Grothendieck,

I looked at your problem on obstructions linked to the projective group and have come to the—rather surprising—conclusion that there is no counterexample: If X is, let us say, a finite polyhedron, and $x \in \mathrm{H}^3(\mathrm{X}, \mathbf{Z})$ is a torsion class, there is a projective bundle over X whose invariant is equal to x.

The question comes down to studying the topology of the classifying space of PGL_n, as $n \to \infty$. It is however necessary to pay careful attention to the way in which the PGL_n (or the GL_n) are mapped to each other: take the

Grothendieck-Serre Correspondence, ©2004 AMS

Serre's letter proves this theorem in short order, appealing to a Postnikov decomposition of a space with divisible homotopy groups. But a geometric proof would be nicer!

Ingredients for proof

We shall use:

- gerbes
- Azumaya bundles
- Morita equivalences
- descent

Gerbes

For us, a *gerbe* on a manifold X will be a Lie groupoid over X, say

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathcal{G}_0 \arrow[d] \\ & X \end{tikzcd}$$

such that:

- $\mathcal{G}_0 \to X$ is a surjective submersion; and
- $\mathcal{G}_1 \to \mathcal{G}_0 \times_X \mathcal{G}_0 =: \mathcal{G}_0^{[2]}$ is a line bundle.

Gerbes

There is a (symmetric mononoidal) 2-groupoid of gerbes over any manifold, and in fact this gives a 2-stack on the category of manifolds. Denote this 2-stack by $2\mathsf{Line}$.

We have in particular *trivialisable* gerbes, namely those $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ for which there is a line bundle $T \to \mathcal{G}_0$ and an isomorphism of bundles $\mathcal{G}_1 \simeq s^*T \otimes t^*T^*$ over $\mathcal{G}_0^{[2]}$ compatible with the groupoid structure.

There is also a dual gerbe $\mathcal{G}^*$ for any gerbe $\mathcal{G}$, such that $\mathcal{G} \otimes \mathcal{G}^*$ is trivialisable.

Azumaya bundles

An *Azumaya bundle* on X is a fibre bundle $A \to X$ with typical fibre $M_n(\mathbb{C})$ and with transition functions algebra isomorphisms.

(Alternatively, it is an Azumaya algebra object in the category of complex vector bundles)

Denote by Az the stack of Azumaya bundles on the category of manifolds. This is a stack of symmetric monoidal groupoids with product given by tensor product.

Example: Given a vector bundle $E \to X$, the endomorphism bundle $\mathrm{End}(E)$ is an Azumaya bundle.

We can thus ask whether an Azumaya bundle arises from a vector bundle, and what the obstruction is, if any.

Brauer group

Two standard properties of Azumaya bundles:

- For any $A \to X$ there is $A^o \to X$, the Azumaya bundle with fibres the opposite algebras of the fibres of A;
- For any A we have $A \otimes A^o \simeq \mathrm{End}_{\mathbb{C}}(A)$.

The set of isomorphism classes of Azumaya bundles on X—*a priori* a commutative monoid—mod endomorphism bundles is the *Brauer group* $Br(X)$ of X.

Azumaya bundles and gerbes

Given an Azumaya bundle $A \to X$ (of rank n^2), there is a principal PGL_n-bundle $F(A)$ (the *frame bundle* of A). Recall that there is a canonical multiplicative line bundle on PGL_n.

There is a (strong monoidal) 2-functor

$$L' \colon \mathsf{Az} \to 2\mathsf{Line}$$

sending an Azumaya algebra A to the gerbe $L'(A) \rightrightarrows F(A)$. Here $L'(A) \to F(A)^{[2]}$ is the pullback of the canonical line bundle on PGL_n by the difference map $F(A)^{[2]} = F(A) \times_X F(A) \to PGL_n$.

Note also that $L'(A^o) = L'(A)^*$, and in fact L' preserves the involutions on both sides.

Azumaya bundles and gerbes

Lemma

If A is of rank n^2, then $L'(A)^{\otimes n}$ is trivialisable.

Any gerbe $\mathcal{G}$ such that $\mathcal{G}^{\otimes k}$ is trivialisable for some k is called *torsion*. These form a symmetric monoidal sub-2-category $2\mathsf{Line}_{tors} \hookrightarrow 2\mathsf{Line}$.

Azumaya bundles and gerbes

A well-known result is:

Theorem

An Azumaya bundle A is isomorphic to an endomorphism bundle if and only if $L'(A)$ is trivialisable.

One can thus think of the sequence of symmetric monoidal (2-)stacks

$$\mathsf{Line} \to \mathsf{Vect} \to \mathsf{Az} \xrightarrow{L'} 2\mathsf{Line}_{tors}$$

(take Vect with $\otimes$, not $\oplus$).

L' descends to give an *injective* homomorphism $Br(X) \hookrightarrow H^2(X, \mathbb{C}^\times)_{tors}$.

Brauer group again

Can we have a stack-like object whose set of connected components is the Brauer group, and that fits into the sequence

$$\mathsf{Line} \to \mathsf{Vect} \to ? \to 2\mathsf{Line}_{tors}$$

Two options:

1. The bigroupoid of Azumaya algebras, invertible bimodules and isomorphisms of bimodules
2. Something else $\leftarrow$ lets try (a little bit of) this today...

Morita equivalences

Definition

A *Morita equivalence* between Azumaya bundles A and B (on X) is a triple (E, F, ϕ) where E, F are vector bundles and

$$\phi\colon A \otimes \mathrm{End}(E) \to \mathrm{End}(F) \otimes B$$

is an isomorphism of Azumaya bundles.

An isomorphism of Morita equivalences $(E_1, F_1, \phi_1) \to (E_2, F_2, \phi_2)$ is a pair of isomorphisms $\lambda\colon E_1 \to E_2$ and $\kappa\colon F_1 \to F_2$ making the obvious square of algebra isomorphisms commute.

Morita equivalences

Theorem

There is a (*symmetric*) *monoidal (2,1)-category* $\mathcal{M}$ *with:*

- *Objects: Azumaya bundles*
- *1-arrows: Morita equivalences*
- *2-arrows: isomorphisms of such*

Additionally $\mathcal{M}$ *is a prestack on the category of manifolds with the open cover topology.* (*i.e. the functor* $\mathcal{M}(X) \to Desc_{\mathcal{M}}(\{U_i \to X\})$ *is fully faithful*)

Moreover, there is a (*symmetric*) *strong monoidal 2-functor*

$$L\colon \mathcal{M} \to 2\mathit{Line}_{tors},$$

extending L'*, that is locally essentially surjective.*

On taking connected components we have $\pi_0(\mathcal{M}_X) \simeq Br(X)$.

Grothendieck's question

Question

When is $Br(X) \hookrightarrow H^2(X, \mathbb{C}^\times)_{tors}$ surjective?

Answer (Serre): definitely true if X is compact (or, of the homotopy type of a finite CW-complex).

More generally, we can ask what the image is.

In the fancier language, we can ask when $L\colon \mathcal{M}_X \to 2\mathsf{Line}_{tors,X}$ is essentially surjective, or which gerbes are (equivalent to) those arising from Azumaya algebras.

Naïve first idea for new proof

If we can lift descent data for $2\mathsf{Line}_{tors}$ to descent data for $\mathcal{M}$ and the latter is a 2-stack for finite open covers, then we are done.

Can't see how to do this directly, and in any case, $\mathcal{M} \to 2\mathsf{Line}_{tors}$ is not locally fully faithful. But:

- Given descent data for $\mathcal{M}$ we *do* get descent data for $2\mathsf{Line}_{tors}$;
- A map of descent data for $\mathcal{M}$ gives an isomorphism of descent data for $2\mathsf{Line}_{tors}$.

But we can adapt a remarkable result of Gabber (from his PhD thesis) regarding covers by only *two* open sets and exploit this.

Gabber's result

Theorem

If $X = U \cup V$ is the separated union of two affine schemes, then $Br(X) \to H^2(X, \mathbb{G}_m)_{tors}$ is onto.

Gabber's result

If $X = U \cup V$ is the separated union of two affine schemes, then $Br(X) \to H^2(X, \mathbb{G}_m)_{tors}$ is onto.

Proof.

1. Show that the fact is true for affine schemes. (Nontrivial!)
2. Then given a torsion gerbe $\mathcal{G}$ on $U \cup V$, lift the restrictions $\mathcal{G}_U$, $\mathcal{G}_V$ to Azumaya bundles, and lift the descent data for the cover $\{U, V\}$ to descent data D for $\mathcal{M}$.
3. Using the fact the gerbe is torsion, can find descent data R for $\mathcal{M}$ such that $L(R)$ pastes to a trivial gerbe.
4. We can find a zig-zag of maps of descent data from $D \otimes R$ to descent data that lies completely in Az $\hookrightarrow \mathcal{M}$, and so get an Azumaya bundle by descent for Az, and this lifts the original gerbe.

$\square$

For manifolds, and induction!

All the proof works verbatim for manifolds, since gerbes are trivial over charts, for instance, and so lift to Azumaya bundles (say $\mathrm{End}(F)$ for any vector bundle F) locally.

The hardest part of the argument is point 3 (...find descent data R for $\mathcal{M}$ such that $L(R)$ pastes to a trivial gerbe), which requires some K-theoretic facts, and being able to find, up to tensoring with a trivial bundle, "$E^{-1/n}$" for a vector bundle E.

Given a compact manifold with a cover by $n+1$ opens over which a gerbe lifts to an Azumaya bundle, lift it over a compact subspace covered by n opens, then use the two-element open cover given by the last open and the union of the first n.

Thank you!

Miscellaneous points arising in the talk 1

Composition of 1-arrows in $\mathcal{M}$ is given by

$$A \xrightarrow{(E,F,\phi)} B \xrightarrow{(G,H,\psi)} C = A \xrightarrow{(E\otimes G,F\otimes H,\phi\odot\psi)} C$$

where, ignoring canonical isomorphisms of bundles, $\phi \odot \psi$ is the composite isomorphism

$$A \otimes \mathrm{End}(E \otimes G) \to \mathrm{End}(F) \otimes B \otimes \mathrm{End}(G) \to \mathrm{End}(F \otimes H) \otimes C$$

built using ϕ and ψ.

Miscellaneous points arising in the talk 2

For point 3 of the argument, we need three K-theoretic facts about any space (manifold, scheme etc) M that appears as the intersection of two charts:

- $I := \ker(\ rk\colon K(M) \to H^0(M, \mathbb{Z})\ )$ should be a nil ideal (so that $K(M)$ is complete in the I-adic topology).
- There should exist a positive integer N_1 such that for all classes $c \in K(M)$, if $rk(c) > N_1$, $c = [E]$ for some vector bundle E.
- There should exist a positive integer N_2 such that for all vector bundles E, F on M such that $[E] = [F] \in K(M)$ and $rk(E) > N_2$, there is an isomorphism $E \simeq F$.