MEASURING POLYGONAL NICENESS*

Sharmila Duppala[†] *and David Kraemer.*[‡]

Abstract.

Contents

1 Introduction

1. Paragraph introducing general problem
2. Paragraph on applications to gerrymandering problems.
3. Paragraph explaining the limitations of this application. (Perhaps this should be in the discussion?)
4. Paragraph outlining the paper.

---

*This paper comprises the completion of the final project for AMS 545 "Computational Geometry".

[†] *Department of Computer Science*, sduppala@cs.stonybrook.edu
[‡] *Department of Applied Mathematics*, david.kraemer@stonybrook.edu

2 Theoretical background

The area of a subset $S \subseteq \mathbb{R}^2$ of the plane is denoted by $\lambda(S)$. We will denote the boundary of S by ∂S. For two points $x, y \in \mathbb{R}^2$, the closed line segment with endpoints in x and y is denoted $[x, y]$. For $p \in [1, \infty]$, we define the p-ball as $B_p(x, \varepsilon) = \{y \in \mathbb{R}^2 : \|x - y\|_p \leq \varepsilon\}$ and we will only be interested in the closed case. Unless otherwise stated, we shall operate in the Euclidean world with $p = 2$. Throughout we shall assume that $P \subseteq \mathbb{R}^2$ is a simple bounded closed polygon. The class of such polygons is given by $\mathcal{P}$. Its perimeter is denoted by $|\partial P|$.

2.1 The α-fatness score

One approach to measuring the relative "fatness" of a polygon seeks to identify regions of the shape which behave highly unlike balls.

Definition 2.1. The α-fatness score of a polygon P is given by

$$\alpha(P) = \inf\{\frac{\lambda[P \cap B(x, \varepsilon)]}{\lambda[B(x, \varepsilon)]} : \varepsilon > 0, x \in P\}$$

where we impose that the ball $B(x, \varepsilon)$ does not contain P.

The constraint that $P \not\subseteq B(x, \varepsilon)$ ensures definiteness: otherwise, let $\varepsilon \to \infty$ and the ratio $\frac{\lambda[P \cap B(x,\varepsilon)]}{\lambda[B(x,\varepsilon)]}$ always approaches 0. An intuition for $\alpha(P)$ emerges from considering how it behaves on balls.

Proposition 2.2. *For any P, $\alpha(P) \leq \frac{1}{4}$, and if $P = B(x_0, r)$ then we have equality.*

Proof. To show that $\alpha(P) \leq \frac{1}{4}$, it suffices to show only for convex P, because $P \subseteq (P)$ implies $\lambda[P \cap B(x, \varepsilon)] \leq \lambda[(P) \cap B(x, \varepsilon)]$. Let $d = \mathrm{diam}(P)$ and fix $x, y \in \partial P$ such that $|[x, y]| = d$. Assume without loss of generality that x lies above y, and consider the ball $B(x, d)$. Then P does not meet the upper disk of $B(x, d)$, for otherwise we could choose $z \in P$ in the upper disk of $B(x, d)$ which forms a chord of P longer than d. A similar argument shows that P cannot occupy more than half of the lower disk of $B(x, d)$. Hence

$$\alpha(P) \leq \frac{\lambda[P \cap B(x, d)]}{\lambda[B(x, d)]} \leq \frac{1}{4},$$

as needed.

The fact that $\alpha(B(x_0, r)) = \frac{1}{4}$ relies on a neat result [Wan05] that shows that the L_p ellipsoid $\mathcal{E}$ with radii a, b has area

$$\lambda(\mathcal{E}) = ab \cdot 4\frac{\Gamma(1 + \frac{1}{p})^2}{\Gamma(1 + \frac{n}{p})}.$$

In our case, $\alpha(B(x_0, r))$ is achieved at a ball on a "corner" point with radius $2r$. $\square$

Hence, the α-score is greater for polygons which exhibit "niceness" characteristics. It is not affected tremendously by local nonconvexity in P so long as P "snugly fits" into a ball shape, which is a global property.

2.2 The chord-f score

A different class of "fatness" measures arises through partitioning P into two (interior disjoint) polygons $P = P' \cup P''$ via chords. A chord is a pair $x, y \in \partial P$ such that the segment $[x, y]$ is contained in the interior of P except at the endpoints x and y. Such a chord defines a partition of $P = P' \cup P''$ by orientation, so that $\partial P'$ is the arc from x to y together with $[x, y]$ and that $\partial P''$ is the arc from y to x together with $[x, y]$.

Definition 2.3. Let $f : \mathcal{P} \to \mathbb{R}$. For a given $P \in \mathcal{P}$, its *chord-f score* is given by

$$s_f(P) = \inf\{\max(f(P'), f(P'')) : x, y \in \partial P\}$$

where the chord $[x, y]$ partitions $P = P' \cup P''$.

The intuition for f is some global measure of "cost" associated with respect to P. For example, if $f(P) = |\partial P|$, then s_f identifies the partition (or limiting sequence of partitions) which minimizes the maximum subpolygon perimeter. Indeed, measures such as perimeter or area are the typical choices of f, but any suitable property of P may be employed.

The chord-f score can be understood relatively simply in the context of a minimax game. Max, given a polygon with a chord partitioning it, always chooses the subpolygon associated with a greater f (i.e., the worst cost of the partition). Min, who plays first, tries to find a chord with the least-bad worst cost with respect to f.

The member of this class of scores under this present investigation is $s_{|\partial\ \cdot\ |}$, the so-called *chord-arc* score. It is important to note that the length of the chord $[x, y]$ which partitions P is included in the estimation of $s_{|\partial\ \cdot\ |}$. One implication of this is that the chord-arc score need not be bounded above by $\frac{|\partial P|}{2}$. We have several facts about $s_{|\partial\ \cdot\ |}$ which develop further intuition for the score.

Proposition 2.4.

1. *Let R be a rectangle with height h and length ℓ. Assume that $h \leq \ell$. Then $s_{|\partial\ \cdot\ |}(R) = 2h + \ell$.*
2. *Let P be convex with perimeter $|\partial P| = w$. Then $s_{|\partial\ \cdot\ |}(P) \leq \frac{w}{2} + \mathrm{diam}(P)$.*
3. *Let C be a circle with "perimeter" $2\pi r$. Then $s_{|\partial\ \cdot\ |}(C) = (\pi + 2)r$.*

Proof.

1. That the optimal partition bisects R is readily apparent, for this minimizes the maximum perimeter length of either sub-rectangle. A case analysis confirms that the preferred strategy for Min is to choose a chord along the smaller dimension (i.e., h) to bisect R, and in this case the maximum arc length is $2h + 2\left(\frac{\ell}{2}\right) = 2h + \ell$, as needed.
2. For P convex, a perimeter-bisecting chord, whose length is at most $\mathrm{diam}(P)$, always exists.
3. This follows from part 2 by noticing that the length of *any* perimeter-bisecting chord is $\mathrm{diam}(C) = 2r$. $\square$

3 Overview of empirical design

1. Paragraph explaining discretization procedure.
2. Pseudocode and explanation for the alpha fatness.
3. Pseudocode and explanation for the chord f score.
4. Formulate "random" polygons.
5. Explain U.S. state boundary data collection.

4 Empirical results

1. Show how changing δ affects scores.
2. Compare scores (by ranking) on random polygons.
3. Compare scores (by ranking) on U.S. state boundary data.

5 Discussion

1. Comment on the results.[The18]
2. Talk about applications to Gerrymandering.[GW18]

References

[Wan05] Xianfu Wang. "Volumes of Generalized Unit Balls". In: *Mathematics Magazine* 78.5 (2005), pp. 390–395. ISSN: 0025570X, 19300980. URL: `http://www.jstor.org/stable/30044198`.

[GW18] Geert-Jan Giezeman and Wieger Wesselink. "2D Polygons". In: *CGAL User and Reference Manual*. 4.12. CGAL Editorial Board, 2018. URL: `https://doc.cgal.org/4.12/Manual/packages.html#PkgPolygon2Summary`.

[The18] The CGAL Project. *CGAL User and Reference Manual*. 4.12. CGAL Editorial Board, 2018. URL: `https://doc.cgal.org/4.12/Manual/packages.html`.