Measuring polygonal niceness

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[1]AMS 545: Computational Geometry
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May 1, 2018

Preliminaries

- $\lambda(A)$ is the area of a set $A \subseteq \mathbb{R}^2$.
- Let $P \subseteq \mathbb{R}^2$ denote a simple bounded closed polygon.
- ∂P is the boundary of P.
- $|\partial P|$ is the perimeter of P.
- $[x, y]$ is the closed line segment bounded by $x, y \in \mathbb{R}^2$.

Measures of niceness

Definition (α-fatness)

The **α-fatness score** is given by

$$\alpha(P) = \inf\{\frac{\lambda(B(x,\rho)\cap P)}{\lambda(B(x,\rho))} : \rho > 0\}$$

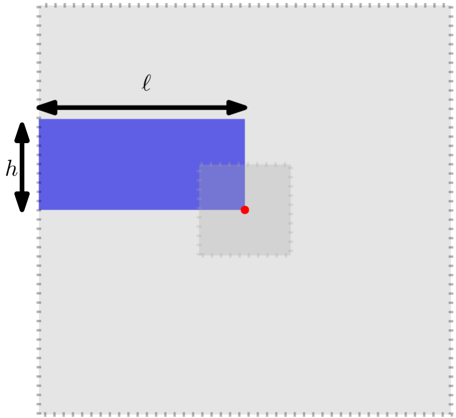
where $x \in P$, and $B(x,\rho)$ is a ball centered at x not containing P.

- The "minimizing" ball might contain P.
- Find the smallest such proportion. This is $\alpha(P)$.
- It's much easier when the ball is a square!

Measures of niceness

- Let R be a rectangle with length ℓ and height h. (WLOG, $\ell \geq h$.) Then

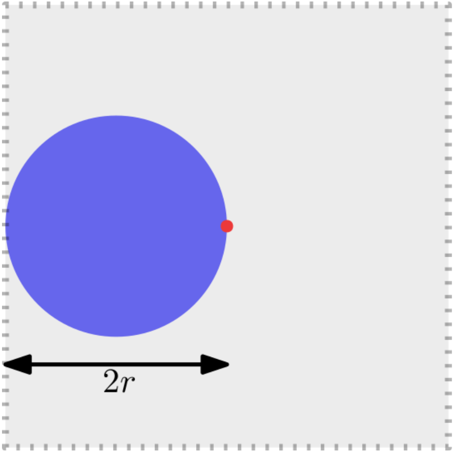
$$\alpha(R) = \frac{h}{4\ell}.$$



Measures of niceness

- Let C be a circle with radius r. Then

$$\alpha(C) = \frac{\pi r^2}{16r^2} = \frac{\pi}{16} < \frac{1}{4}.$$



Measures of niceness

Definition (Chord-arc)

The **chord-arc score** is given by

$$s_p(P) = \inf\{\max(|\partial P'|, |\partial P''|) : x, y \in \partial P\},$$

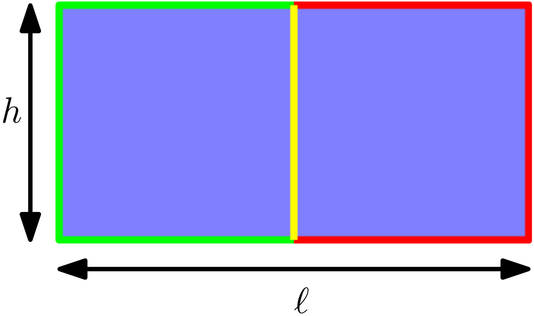
where the chord $[x, y]$ partitions $P = P' \cup P''$.

- This is a "minimax" definition. We want the least bad resulting split.
- The polygon perimeter is computed by summing the "lengths" of each boundary edge.
- Here p indicates a norm. Ideally $p = 2$, but for computation purposes we choose $p = 1$ or $p = \infty$.

Measures of niceness

- Let R be a rectangle with length ℓ and height h. (WLOG, $\ell \geq h$.) Then

$$s_p(R) = 2h + \ell$$



- This holds for many p.

Stray observations

- The α fatness score seems to penalize oblong polygons and reward "squarely compact" polygons.
- The chord-arc score seems to penalize local nonconvexity.
- Remember that we want a large α score but a small chord-arc score.

Implementation

- The measurements were implemented in C++ using CGAL with exact arithmetic kernel.
- We used a δ-boundary discretizing scheme: the length of $[x_k, x_{k+1}]$ is at most $\delta > 0$ for consecutive boundary vertices.
- Generating useful test polygons was tricky.
- We ran our measurements on
  - Special internally generated nice polygons,
  - Randomly generated "typical" polygons (courtesy of Professor Mitchell),
  - (Simplified) US state boundaries.

Randomly generated polygons

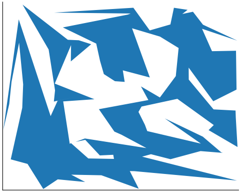
α fatness rank: 6
Arc L_∞ rank: 5

α fatness: 0.1160
Arc L_∞: 0.4460

α fatness rank: 5
Arc L_∞ rank: 2

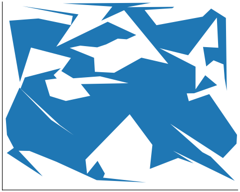
α fatness: 0.1260
Arc L_∞: 0.3977

α fatness rank: 4
Arc L_∞ rank: 6



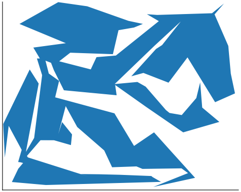
α fatness: 0.1278
Arc L_∞: 0.5619

α fatness rank: 3
Arc L_∞ rank: 4



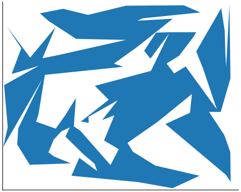
α fatness: 0.1394
Arc L_∞: 0.4334

α fatness rank: 2
Arc L_∞ rank: 1



α fatness: 0.1424
Arc L_∞: 0.2334

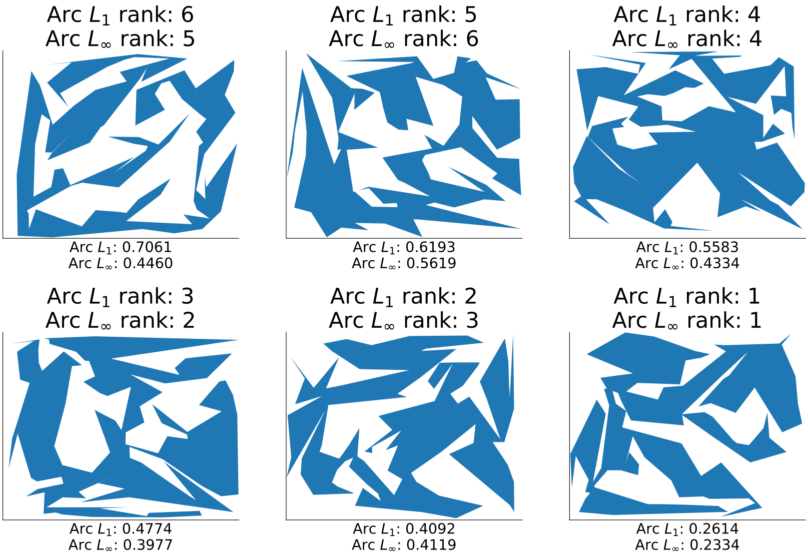
α fatness rank: 1
Arc L_∞ rank: 3



α fatness: 0.1492
Arc L_∞: 0.4119

$(\delta = 0.05)$

Randomly generated polygons



$(\delta = 0.05)$