MEASURING POLYGONAL NICENESS*

Sharmila Duppala† *and David Kraemer.*‡

Abstract.

Contents

1 Introduction

1. Paragraph introducing general problem
2. Paragraph on applications to gerrymandering problems.
3. Paragraph explaining the limitations of this application. (Perhaps this should be in the discussion?)
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† *Department of Computer Science*, sduppala@cs.stonybrook.edu

‡ *Department of Applied Mathematics*, david.kraemer@stonybrook.edu

2 Theoretical background

The area of a subset $S \subseteq \mathbb{R}^2$ of the plane is denoted by $\lambda(S)$. We will denote the boundary of S by ∂S. For two points $x, y \in \mathbb{R}^2$, the closed line segment with endpoints in x and y is denoted $[x, y]$. For $p \in [1, \infty]$, we define the p-ball as $B_p(x, \varepsilon) = \{y \in \mathbb{R}^2 : \|x - y\|_p \leq \varepsilon\}$ and we will only be interested in the closed case. Unless otherwise stated, we shall operate in the Euclidean world with $p = 2$. Throughout we shall assume that $P \subseteq \mathbb{R}^2$ is a simple bounded closed polygon. The class of such polygons is given by $\mathcal{P}$. Its perimeter is denoted by $|\partial P|$.

2.1 The α-fatness score

One approach to measuring the relative "fatness" of a polygon seeks to identify regions of the shape which behave highly unlike balls.

Definition 2.1. The α-fatness score of a polygon P is given by

$$\alpha(P) = \inf\{\frac{\lambda[P \cap B(x, \varepsilon)]}{\lambda[B(x, \varepsilon)]} : \varepsilon > 0, x \in P\}$$

where we impose that the ball $B(x, \varepsilon)$ does not contain P.

The constraint that $P \not\subseteq B(x, \varepsilon)$ ensures definiteness: otherwise, let $\varepsilon \to \infty$ and the ratio $\frac{\lambda[P \cap B(x,\varepsilon)]}{\lambda[B(x,\varepsilon)]}$ always approaches 0. An intuition for $\alpha(P)$ emerges from considering how it behaves on balls.

Proposition 2.2. *For any P, $\alpha(P) \leq \frac{1}{4}$, and if $P = B(x_0, r)$ then we have equality.*

Proof. To show that $\alpha(P) \leq \frac{1}{4}$, it suffices to show only for convex P, because $P \subseteq (P)$ implies $\lambda[P \cap B(x, \varepsilon)] \leq \lambda[(P) \cap B(x, \varepsilon)]$. Let $d = \operatorname{diam}(P)$ and fix $x, y \in \partial P$ such that $|[x, y]| = d$. Assume without loss of generality that x lies above y, and consider the ball $B(x, d)$. Then P does not meet the upper disk of $B(x, d)$, for otherwise we could choose $z \in P$ in the upper disk of $B(x, d)$ which forms a chord of P longer than d. A similar argument shows that P cannot occupy more than half of the lower disk of $B(x, d)$. Hence

$$\alpha(P) \leq \frac{\lambda[P \cap B(x, d)]}{\lambda[B(x, d)]} \leq \frac{1}{4},$$

as needed.

The fact that $\alpha(B(x_0, r)) = \frac{1}{4}$ relies on a neat result [Wan05] that shows that the L_p ellipsoid $\mathcal{E}$ with radii a, b has area

$$\lambda(\mathcal{E}) = ab \cdot 4\frac{\Gamma(1 + \frac{1}{p})^2}{\Gamma(1 + \frac{n}{p})}.$$

In our case, $\alpha(B(x_0, r))$ is achieved at a ball on a "corner" point with radius $2r$. $\square$

Hence, the α-score is greater for polygons which exhibit "niceness" characteristics. It is not affected tremendously by local nonconvexity in P so long as P "snugly fits" into a ball shape, which is a global property.

2.2 The chord-f score

A different class of "fatness" measures arises through partitioning P into two (interior disjoint) polygons $P = P' \cup P''$ via chords. A chord is a pair $x, y \in \partial P$ such that the segment $[x, y]$ is contained in the interior of P except at the endpoints x and y. Such a chord defines a partition of $P = P' \cup P''$ by orientation, so that $\partial P'$ is the arc from x to y together with $[x, y]$ and that $\partial P''$ is the arc from y to x together with $[x, y]$.

Definition 2.3. Let $f : \mathcal{P} \to \mathbb{R}$. For a given $P \in \mathcal{P}$, its *chord-f score* is given by

$$s_f(P) = \inf\{\max(f(P'), f(P'')) : x, y \in \partial P\}$$

where the chord $[x, y]$ partitions $P = P' \cup P''$.

The intuition for f is some global measure of "cost" associated with respect to P. For example, if $f(P) = |\partial P|$, then s_f identifies the partition (or limiting sequence of partitions) which minimizes the maximum subpolygon perimeter. Indeed, measures such as perimeter or area are the typical choices of f, but any suitable property of P may be employed.

The chord-f score can be understood relatively simply in the context of a minimax game. Max, given a polygon with a chord partitioning it, always chooses the subpolygon associated with a greater f (i.e., the worst cost of the partition). Min, who plays first, tries to find a chord with the least-bad worst cost with respect to f.

The member of this class of scores under this present investigation is $s_{|\partial\ \cdot\ |}$, the so-called *chord-arc* score. It is important to note that the length of the chord $[x, y]$ which partitions P is included in the estimation of $s_{|\partial\ \cdot\ |}$. One implication of this is that the chord-arc score need not be bounded above by $\frac{|\partial P|}{2}$. We have several facts about $s_{|\partial\ \cdot\ |}$ which develop further intuition for the score.

Proposition 2.4. *1. Let R be a rectangle with height h and length ℓ. Assume that $h \leq \ell$. Then $s_{|\partial\ \cdot\ |}(R) = 2h + \ell$.*

2. Let P be convex with perimeter $|\partial P| = w$. Then $s_{|\partial\ \cdot\ |}(P) \leq \frac{w}{2} + \mathrm{diam}(P)$.

3. Let C be a circle with "perimeter" $2\pi r$. Then $s_{|\partial\ \cdot\ |}(C) = (\pi + 2)r$.

Proof. 1. That the optimal partition bisects R is readily apparent, for this minimizes the maximum perimeter length of either sub-rectangle. A case analysis confirms that the preferred strategy for Min is to choose a chord along the smaller dimension (i.e., h) to bisect R, and in this case the maximum arc length is $2h + 2\left(\frac{\ell}{2}\right) = 2h + \ell$, as needed.

2. For P convex, a perimeter-bisecting chord, whose length is at most $\mathrm{diam}(P)$, always exists.

3. This follows from part 2 by noticing that the length of *any* perimeter-bisecting chord is $\mathrm{diam}(C) = 2r$. $\square$

3 Overview of empirical design

In principal of computing the α and chord-f scores relies on sweeping the continuum of points of P and evaluating the aforementioned measurements. To estimate this process in practice, we employ the following discretizing scheme. Let P be determined by the vertices $[v_1, \ldots, v_n]$ given in counterclockwise order. Then, given $\delta > 0$, loop through the vertices of P, checking if $\|v_{i+1} - v_i\| > \delta$. In the case where this holds, we insert $v_i' = \frac{v_i + v_{i+1}}{2}$ after v_i into the polygon and resume the loop. This ensures that for any two adjacent vertices v_i and v_{i+1} in the representation of P we have $\|v_{i+1} - v_i\| \leq \delta$. As $\delta \to 0$ this estimates ∂P more exactly. All computations are done with respect to the L_∞ norm. The implementation of this procedure is given in Algorithm 3.

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procedure RefineBy(P, δ)
  $P = [v_1, v_2, v_3, \ldots, v_n]$;
  $v = v_1$
  while $\mathit{next}(v) \neq v_1$ **do**
    if $\|\mathit{next}(v) - v\|_2 > \delta$ **then**
      Insert $\mathit{midpoint}(v, \mathit{next}(v))$ into P after v ▷ Now $\mathit{next}(v)$ is the midpoint

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Given a δ-refined polygon P, the computation of $\alpha(P)$ is relatively straightforward, with the simplification that points x on which the ratio is computed lie on ∂P. The computation steps through $[v_1, \ldots, v_n]$ and for each v_i the radius ε of the minimum-enclosing ∞-ball centered at v_i is computed. Then for $\varepsilon_k = \frac{k}{100}\varepsilon$, $k = 1, \ldots, 100$, the ratio

$$\frac{\lambda[P \cap B_\infty(v_i, \varepsilon_k)]}{B_\infty(v_i, \varepsilon_k)}$$

is computed and the smallest such ratio is kept. The minimum over all v_i is given as $\alpha(P)$. The implementation of the α score is given in Algorithm 3. **[alg:alpha]**

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function αScore(P)
  $P = [v_1, v_2, v_3, \ldots, v_n]$;
  min $= \infty$
  for $v \in P$ **do**
    $\varepsilon =$ radius of minimum enclosing ball of P at v
    for $k = 1, 2, \ldots, 100$ **do** ▷ 100 is arbitrary
      $\varepsilon_k = \frac{k}{100}\varepsilon$
      area $= \frac{\lambda[P \cap B_\infty(v, \varepsilon_k)]}{B_\infty(v, \varepsilon_k)}$
      if min > area **then**
        min = area
  return min

Similarly, given a δ-refined polygon P, the chord-f score is computed by sweeping all possible chords of P. For each point $v_i, v_j \in \partial P$ with $i < j$, if $[v_i, v_j]$ is indeed a valid chord of P, the partition $P = P' \cup P''$ is computed and the maximum of $f(P')$ and $f(P'')$ is

Figure 1: Effect of sweeping δ on measurements

kept. The minimum such value is given as $s_f(P)$. The implementation of the chord-f score is given in Algorithm 3.

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function ChordScore(P,f)
  $P = [v_1, v_2, v_3, \ldots, v_n]$;
  for $v \in P$ **do**
    for $w \in [next(v), \ldots, v_n]$ **do**
      if $[v, w]$ is a chord of P **then**
        Partition $P = P' \cup P''$ by $[v, w]$
        submax $= max(f(P'), f(P''))$
        if min $>$ submax **then**
          min $=$ submax
  return min

All of these algorithms are implemented in CGAL [The18] with the extensive use of the 2D Polygon library [GW18].

4 Empirical results

We evaluate our measurements on randomly generated polygons with vertices in the unit square $[0,1] \times [0,1]$. We consider collections of such polygons of 10, 50, and 100 vertices. In Figure 1, we measure the effect of sweeping δ between 0.05 and 1 on the resulting measurements. Figure 3 shows a comparison of α-fatness and $s_{|\partial\ \cdot\ |_\infty}$ scores for polygons on 10 vertices. Figure **??** shows a similar comparison of α-fatness and $s_{|\partial\ \cdot\ |_1}$ scores for polygons on 50 vertices.

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1. Show how changing δ affects scores.
2. Compare scores (by ranking) on random polygons.
3. Compare scores (by ranking) on U.S. state boundary data.

5 Discussion

1. Comment on the results.
2. Talk about applications to Gerrymandering.

References

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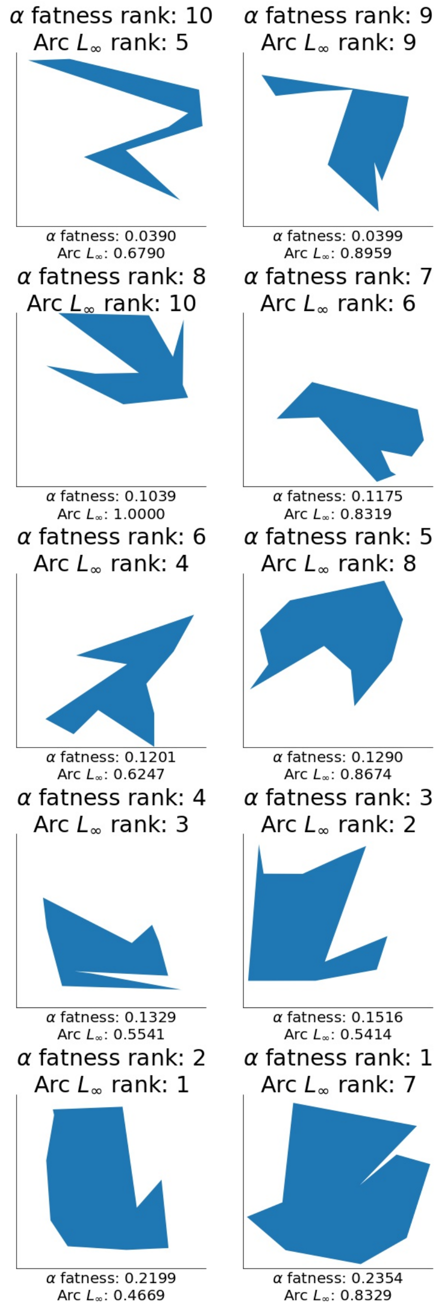

Figure 2: α scores and $s_{|\partial\,\cdot\,|_\infty}$ scores for randomly generated polygons on 10 vertices.

Figure 3: α scores and $s_{|\partial\,\cdot\,|_\infty}$ scores for randomly generated polygons on 50 vertices.