Fatness of Simple Polygons

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1 Problem Statement

Professor Mitchell's description. Computing the Niceness of a Polygonal Shape. There are various notions of quantifying how "nice" or how "fat" a simple polygon P is. A "nicest" polygon might be a regular n-gon, which most closely approximates a circular disk. This project seeks to implement some precise metrics for niceness, and compare them on simple polygons (possibly moused in by a user or read in from a file, etc). To make it simple and discrete, I propose that you discretize the boundary of P with discrete points p_i (additional vertices), with a prescribed spacing, δ. (That is, for edges of P longer than δ, add vertices along the edge so that the new sub-segments are of length at most δ, for a user-specified parameter δ.)

Then, consider discrete choices of radii $r = \rho, 2\rho, 3\rho, \ldots$ for disks, $B(p_i, r)$, centered at the discrete boundary points p_i, for each radius r. The (discrete) fatness of P is given by the smallest value of the ratio $\lambda(B(p_i, r) \cap P)/\lambda(B(p_i, r))$, over all choices of p_i and $r = \rho, 2\rho, 3\rho, \ldots$ such that P is not contained fully inside $B(p_i, r)$.

Implement and experiment with this fatness measure. It may be possible to use the algorithm to assist a project at Harvard on Gerrymandering, where the goal is to quantify how "compact" polygonal election districts are. (So I hope that at least a couple of people choose to do this project! We can discuss further.)

(Theoretically, we are interested in finding an algorithm to compute the "exact" fatness of P (which allows disks centered anywhere inside P, of any radius such that the disk does not contain all of P), without resorting to the simple discretization. Or, can we compute a provable approximation to the exact fatness?)

2 Definitions

Our present objective is to develop rigorous measures of the "niceness" of simple bounded closed planar polygons with the intuitive hierarchy that regular n-gons are "nicest" followed by convex n-gons and proceeded by "fat" polygons—i.e., polygons which retain their structure under smoothing filters.

Throughout we assume that $P \subseteq \mathbb{R}^2$ is a simple bounded closed polygon. We shall denote by ∂P the boundary of P, with $|\partial P|$ denoting the perimeter and $\lambda(P)$ denoting the area. For given points $x, y \in \mathbb{R}$, denote the closed line segment bounded by x and y by $[x, y]$. The class of such polygons is denoted by $\mathcal{P}$.

One class of "fatness" measures arises through partitioning P into two (interior disjoint) polygons $P = P' \cup P''$ via chords. A chord is a pair $x, y \in \partial P$ such that the segment $[x, y]$ is contained by the relative interior of P except at the endpoints x and y. Such a chord $[x, y]$ defines a partition of $P = P' \cup P''$ by orientation, so that $\partial P'$ is the arc from x to y with $[y, x]$ and that $\partial P''$ is the arc from y to x with $[x, y]$.

Definition 2.1. Let $f : \mathcal{P} \to \mathbb{R}$. For a given $P \in \mathcal{P}$, its *chord-f score* is given by

$$s_f(P) = \inf_{x,y \in \partial P} \max(f(P'), f(P''))$$

where the chord $[x, y]$ partitions $P = P' \cup P''$.

Intuitively, f represents a polygon's cost with respect to a certain measurement. For example, if f indicates the perimeter of P, then s_f is associated with a partition of P such that the maximum perimeter of either subpolygon is minimized. Indeed, measures such as perimeter or area are the typical choices of f, but any suitable property of the polygon may be employed.

The visibility kernel of P, denoted by $\ker P$, is the subset of P which "sees" all of P. That is, if $x \in \ker P$ and $y \in P$, then $[x, y] \subseteq P$. Another proposed measure of polygonal "niceness" utilizes the visibility kernel.

Definition 2.2. The *visibility kernel score* is given by

$$s_{\mathrm{vis}}(P) = \frac{\lambda(\ker P)}{\lambda(P)}.$$

If P is a regular n-gon or convex $s_{\mathrm{vis}}(P) = 1$, since trivially $\ker P = P$. On the other hand, if P is not star shaped, then $s_{\mathrm{vis}}(P) = 0$.

Equipped with an L_p norm ($p \in [1, \infty]$), denote by $B(x, \rho)$ the closed ball of radius $\rho > 0$ centered at $x \in \mathbb{R}^2$. The natural choice is simply the Euclidean norm, but L_∞ provides a setting for which computing $B(x, \rho) \cap P$ resolves to polygonal intersection. By topological equivalence, we lose no specificity. A final measure of "fatness" measures local nonconvexities of P and identifies the most offending ratio.

Definition 2.3. The *α-fatness score* is given by

$$\alpha(P) = \inf\{\frac{\lambda(B(x, \rho) \cap P)}{\lambda(B(x, \rho))} : x \in \partial P, \rho > 0, P \not\subseteq B(x, \rho)\}$$

For a given $S \subseteq \mathbb{R}^2$, let $\mathcal{C}(x, S)$ denote the connected component of S containing x. An alternative formulation of α-fatness replaces $B(x, \rho) \cap P$ with its connected component: $\mathcal{C}(x, B(x, \rho) \cap P)$. In principle, $\alpha(P)$ is computed by sweeping continuously, but for simplicity we can merely discretize ∂P with a parameter $\delta > 0$ indicating the maximal distance between points in ∂P.

3 Implementation

We plan to implement these measures of polygonal "fatness" and compare them for assorted collections of polygons, from theoretical and randomly-generated examples to historical electoral map data. We hope to determine the relative strengths of these measures as heuristics for polygonal "fatness" individually or in an ensemble on this data.