# Mathematical review

$X$ is a <u>Banach</u> space if

- $X$ is a vector space
- $X$ has a norm $\|\cdot\|$ operation
- the metric $d(x,y) := \|x-y\|$ is complete

If $\lim_{m,n\to\infty} d(x_m, x_n) = 0$ then $x_n \to x$ for some $x \in X$

Let $X$ be a normed vector space. A map $T: X \to X$ is a <u>contraction mapping with modulus $\gamma \in (0,1)$</u> if

$$\|Tx - Ty\| \leq \gamma \|x-y\| \quad \text{for all } x,y \in X.$$

We take $\gamma$ to be the <u>smallest</u> such constant in $(0,1)$.

Let $f: X \to X$ be a function. A <u>fixed point</u> of $f$ is a point $x \in X$ such that $f(x) = x$.

Let $\Omega$ be a set. $\mathbb{R}^\Omega$ is the set of all <u>functions</u> $f: \Omega \to \mathbb{R}$. $B\mathbb{R}^\Omega$ will denote the set of bounded functions $f: \Omega \to \mathbb{R}$. $B\mathbb{R}^\Omega \subset \mathbb{R}^\Omega$, and with

$$\|f\| := \sup_{\omega \in \Omega} |f(\omega)|$$

$B\mathbb{R}^\Omega$ is a Banach space

If $\Omega$ is finite then $B\mathbb{R}^\Omega = \mathbb{R}^\Omega$ (this is the only case of equality)

## Banach's fixed point theorem

**Theorem** Let $X$ be a Banach space, and let $T : B \to B$ be a contraction mapping with modulus $\gamma \in (0,1)$. Then $T$ has a unique fixed point $x^*$.

**Corollary** If $x_0 \in X$ is arbitrary, then the "fixed-point" algorithm

$$x_n := T x_{n-1} \qquad n = 1, 2, \ldots$$

converges to $x^*$. That is, $\|x_n - x^*\| \longrightarrow 0$ as $n \to \infty$

**Uniqueness proof** If $Tx^* = x^*$ and $Ty^* = y^*$ then

$$\|x^* - y^*\| = \|Tx^* - Ty^*\| \leq \gamma \|x^* - y^*\|$$

so $\|x^* - y^*\| \leq \gamma \|x^* - y^*\|$. Since $0 < \gamma < 1$, we must have $\|x^* - y^*\| = 0$, so $x^* = y^*$

**Corollary proof**

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|Tx_{n-1} - Tx^*\|$$

$$\leq \lim_{n\to\infty} \gamma \|x_{n-1} - x^*\| \qquad \text{(repeat the argument)}$$

$$\leq \lim_{n\to\infty} \gamma^n \underbrace{\|x_0 - x^*\|}_{\text{fixed number}}$$

$$= \|x_0 - x^*\| \cdot \lim_{n\to\infty} \gamma^n$$

$$= 0$$

Application to MDPs.

Assume $|S| < \infty$, $|A| < \infty$. Discount factor is $\gamma \in (0,1)$

The value function $v$ satisfies

Bellman equation

for all $s$, $$v(s) = \max_{a \in A} \left\{ r(s,a) + \gamma \sum_{s' \in S} v(s') P(s'|s,a) \right\}$$

$v \in \mathbb{R}^S = B\mathbb{R}^S$. Define $B_\gamma : \mathbb{R}^S \to \mathbb{R}^S$ as

$$(B_\gamma f)(s) = \max_{a \in A} \left\{ r(s,a) + \gamma \sum_{s' \in S} f(s') P(s'|s,a) \right\}$$

Bellman operator

So $B_\gamma v = v$. In fact, $B_\gamma$ is a contraction mapping with modulus $\gamma$. Thus $v$ is the unique fixed point, and from an arbitrary function $v_0 \in \mathbb{R}^S$, with $v_n := B_\gamma v_{n-1}$, $n = 1, 2, \ldots$ we have that $v_n \to v$ in the uniform norm topology.

this is value iteration.